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Height bounds and the Siegel property


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# Correction to the article Height bounds and the Siegel property

Martin Orr and Christian Schnell

This is a correction to the paper "Height bounds and the Siegel property" (*Algebra Number Theory* **12**:2 (2018), 455–478). We correct an error in the proof of Theorem 4.1. Theorem 4.1 as stated in the original paper is correct, but the correction affects additional information about the theorem which is important for applications.

There is an error in the proof of [Orr 2018, Theorem 4.1]. The statement of Theorem 4.1 is correct, but [loc. cit., Lemma 4.4] is incorrect under the conditions on $K_{\boldsymbol{G}}$ stated above it.

Subsequent applications [Bakker et al. 2020, Theorem 1.1(2); Daw and Orr 2021, Lemma 2.3] have required greater control of the maximal compact subgroup $K_{\boldsymbol{G}}$ than is given by the statement of [Orr 2018, Theorem 4.1]. As a result of the error in the proof, the choice of $K_{\boldsymbol{G}}$ is more constrained than it appears in [loc. cit.]. We therefore state a version of [loc. cit., Theorem 4.1], extended to correctly describe the constraints on $K_{\boldsymbol{G}}$.

**Theorem 1.** *Let $\boldsymbol{G}$ and $\boldsymbol{H}$ be reductive $\mathbb{Q}$-algebraic groups, with $\boldsymbol{H} \subset \boldsymbol{G}$. Let $\mathfrak{S}_{\boldsymbol{H}}$ be a Siegel set in $\boldsymbol{H}(\mathbb{R})$ with respect to the Siegel triple $(\boldsymbol{P_H}, \boldsymbol{S_H}, K_{\boldsymbol{H}})$. Let $K_{\boldsymbol{G}} \subset \boldsymbol{G}(\mathbb{R})$ be a maximal compact subgroup such that*

(i) *$K_{\boldsymbol{H}} \subset K_{\boldsymbol{G}}$; and*

(ii) *the Cartan involution of $\boldsymbol{G}$ associated with $K_{\boldsymbol{G}}$ stabilises $\boldsymbol{S_H}$.*

*Then there exist subgroups $\boldsymbol{P_G}, \boldsymbol{S_G} \subset \boldsymbol{G}$ forming a Siegel triple $(\boldsymbol{P_G}, \boldsymbol{S_G}, K_{\boldsymbol{G}})$, a Siegel set $\mathfrak{S}_{\boldsymbol{G}} \subset \boldsymbol{G}(\mathbb{R})$ with respect to this Siegel triple, and a finite set $C \subset \boldsymbol{G}(\mathbb{Q})$ such that*

$$\mathfrak{S}_{\boldsymbol{H}} \subset C.\mathfrak{S}_{\boldsymbol{G}}.$$

*Furthermore, $R_u(\boldsymbol{P_H}) \subset R_u(\boldsymbol{P_G})$ and $\boldsymbol{S_H} = \boldsymbol{S_G} \cap \boldsymbol{H}$.*

**Remark 2.** In the setting of Theorem 1, let $\Theta$ be the Cartan involution of $\boldsymbol{G}$ associated with $K_{\boldsymbol{G}}$. We now compare (ii) with:

(ii′) $\Theta$ stabilises $\boldsymbol{H}$.

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If (i) and (ii′) are satisfied, then the restriction $\Theta_{|\boldsymbol{H}}$ is the Cartan involution of $\boldsymbol{H}$ associated with $K_{\boldsymbol{H}}$. Hence, by the definition of Siegel triple, (ii) is satisfied. However, if (i) and (ii) are satisfied, then (ii′) does not necessarily hold. This may be seen in the example $\boldsymbol{G} = \mathrm{SL}_2$, $\boldsymbol{H} = \left\{\left(\begin{smallmatrix} a & b \\ db & a \end{smallmatrix}\right) : a^2 - db^2 = 1\right\}$ where $d$ is a nonsquare positive rational number, $K_{\boldsymbol{G}} = \mathrm{SO}_2(\mathbb{R})$, $\boldsymbol{S_H} = \{1\}$, $K_{\boldsymbol{H}} = \{1\}$.

In this note, we explain how to correct the proof of [Orr 2018, Theorem 4.1] and prove Theorem 1. We also give examples showing that condition (ii) of Theorem 1 cannot be deleted from the statement of the theorem: first an example in which $\boldsymbol{H}$ is a torus, then a more sophisticated example in which $\boldsymbol{H}$ is semisimple. At the end of the note, we correct some unrelated minor errors in [loc. cit.].

**A.** ***Correction to proof of*** **[Orr 2018, Theorem 4.1].** On [Orr 2018, page 470], item (2) (the choice of $K_{\boldsymbol{G}}$) should be replaced by:

(2) $K_{\boldsymbol{G}}$, a maximal compact subgroup of $\boldsymbol{G}(\mathbb{R})$ containing $K_{\boldsymbol{H}}$, such that the Cartan involution of $\boldsymbol{G}$ associated with $K_{\boldsymbol{G}}$ stabilises $\boldsymbol{S_H}$.

Paragraph 1 of the proof of [loc. cit., Lemma 4.4] is incorrect: neither the original constraint on $K_{\boldsymbol{G}}$, nor the corrected constraint, are sufficient to guarantee that $\Theta$ restricts to an involution of $\boldsymbol{H}$ (see Remark 2). With the corrected constraint, that paragraph can be ignored and paragraph 2 of the proof of [loc. cit., Lemma 4.4] is valid. Hence the lemma is true under the corrected constraint on $K_{\boldsymbol{G}}$.

The remainder of the proof of [loc. cit., Theorem 4.1] is valid without any changes related to the choice of $K_{\boldsymbol{G}}$ (but see unrelated minor corrections in Section E of this note). No further conditions are imposed on $K_{\boldsymbol{G}}$, so this proves Theorem 1.

In order to establish [loc. cit., Theorem 4.1], it is necessary to verify the existence of $K_{\boldsymbol{G}}$ satisfying (2) above. To show this, choose a faithful representation $\rho \colon \boldsymbol{G}_{\mathbb{R}} \to \mathrm{GL}(V)$ for some real vector space $V$. By [Mostow 1955, Theorem 7.3], there exists a positive definite symmetric form $\psi$ on $V$ with respect to which the groups $K_{\boldsymbol{H}} \subset \boldsymbol{H}(\mathbb{R}) \subset \boldsymbol{G}(\mathbb{R}) \subset \mathrm{GL}(V)$ are simultaneously self-adjoint. In other words, if $\Theta$ denotes the Cartan involution of $\mathrm{GL}(V)$ associated with $\psi$, then $\Theta$ restricts to Cartan involutions of $\boldsymbol{G}$, $\boldsymbol{H}$ and $K_{\boldsymbol{H}}$.

Letting $K_{\boldsymbol{G}}$ denote the stabiliser of $\psi$ in $\boldsymbol{G}(\mathbb{R})$, we obtain $K_{\boldsymbol{H}} \subset K_{\boldsymbol{G}}$. By Remark 2, $\Theta$ stabilises $\boldsymbol{S_H}$.

**B.** ***Counterexample in which condition (ii) of Theorem 1 is not satisfied: a torus.*** Let $\boldsymbol{G} = \mathrm{SL}_2$ and let $(\boldsymbol{P}_0, \boldsymbol{S}_0, K_{\boldsymbol{G}})$ be the standard Siegel triple for $\boldsymbol{G}$, that is, $\boldsymbol{P}_0$ is the subgroup of upper triangular matrices in $\boldsymbol{G}$, $\boldsymbol{S}_0$ is the subgroup of diagonal matrices in $\boldsymbol{G}$ and $K_{\boldsymbol{G}} = \mathrm{SO}_2(\mathbb{R})$.

Let

$$\boldsymbol{H} = \left\{ \begin{pmatrix} x & x^{-1} - x \\ 0 & x^{-1} \end{pmatrix} \right\} \subset \boldsymbol{G}.$$

This is a $\mathbb{Q}$-split torus so it possesses a unique Siegel triple, namely $\boldsymbol{P_H} = \boldsymbol{S_H} = \boldsymbol{H}$, $K_{\boldsymbol{H}} = \{\pm 1\}$, and a unique Siegel set, $\mathfrak{S}_{\boldsymbol{H}} = \boldsymbol{H}(\mathbb{R})$.

Clearly $K_{\boldsymbol{H}} = \{\pm 1\} \subset K_{\boldsymbol{G}}$. Thus $K_{\boldsymbol{G}}$ satisfies condition (i) of Theorem 1. However by [Orr 2018, Lemma 2.1], $\boldsymbol{S}_0$ is the only $\mathbb{Q}$-split torus in $\boldsymbol{P}_0$ stabilised by the Cartan involution of $\boldsymbol{G}$ associated with $K_{\boldsymbol{G}}$.

Hence this Cartan involution does not stabilise $\boldsymbol{S_H}$. In other words, $K_{\boldsymbol{G}}$ does not satisfy condition (ii) of Theorem 1.

Now we shall show that this $\mathfrak{S}_{\boldsymbol{H}}$ and $K_{\boldsymbol{G}}$ do not satisfy the conclusion of Theorem 1. Suppose for contradiction that there exist subgroups $\boldsymbol{P_G}, \boldsymbol{S_G} \subset \boldsymbol{G}$ forming a Siegel triple $(\boldsymbol{P_G}, \boldsymbol{S_G}, K_{\boldsymbol{G}})$, a Siegel set $\mathfrak{S}_{\boldsymbol{G}} \subset \boldsymbol{G}(\mathbb{R})$ with respect to this Siegel triple, and a finite set $C \subset \boldsymbol{G}(\mathbb{Q})$ such that $\mathfrak{S}_{\boldsymbol{H}} \subset C.\mathfrak{S}_{\boldsymbol{G}}$.

By [Borel and Tits 1965, Théorème 4.13], there exists $g \in \boldsymbol{G}(\mathbb{Q})$ such that $\boldsymbol{P}_0 = g\boldsymbol{P_G}g^{-1}$. Writing $g = pk$ where $p \in \boldsymbol{P}_0(\mathbb{R})$ and $k \in K_{\boldsymbol{G}}$, $(\boldsymbol{P}_0, k\boldsymbol{S_G}k^{-1}, K_{\boldsymbol{G}})$ is a Siegel triple and $g\mathfrak{S}_{\boldsymbol{G}}$ is a Siegel set with respect to $(\boldsymbol{P}_0, k\boldsymbol{S_G}k^{-1}, K_{\boldsymbol{G}})$. Hence we can replace $\boldsymbol{P_G}$ by $\boldsymbol{P}_0$, $\boldsymbol{S_G}$ by $k\boldsymbol{S_G}k^{-1}$, $\mathfrak{S}_{\boldsymbol{G}}$ by $g\mathfrak{S}_{\boldsymbol{G}}$ and $C$ by $Cg^{-1}$. We can thus assume that $\boldsymbol{P_G} = \boldsymbol{P}_0$. By the uniqueness of the torus in a Siegel triple, this implies that $\boldsymbol{S_G} = \boldsymbol{S}_0$ and $\mathfrak{S}_{\boldsymbol{G}}$ is a standard Siegel set in $\boldsymbol{G}(\mathbb{R})$.

The image of $\mathfrak{S}_{\boldsymbol{H}} = \boldsymbol{S_H}(\mathbb{R})$ in $\boldsymbol{G}(\mathbb{R})/K_0$, identified with the upper half-plane, is the ray

$$R = \{(1-y) + yi : y \in \mathbb{R}_{>0}\}.$$

Write $\mathcal{F}_{\boldsymbol{G}}$ for the image of $\mathfrak{S}_{\boldsymbol{G}}$ in the upper half-plane.

Since $R \subset C\mathcal{F}_{\boldsymbol{G}}$ and $C$ is finite, there exists $\gamma \in C \subset \boldsymbol{G}(\mathbb{Q})$ such that $R \cap \gamma\mathcal{F}_{\boldsymbol{G}}$ contains points $z$ where both $\operatorname{Im} z$, $|\operatorname{Re} z| \to \infty$. But this is impossible because:

(i) If $\gamma \notin \boldsymbol{P}_0(\mathbb{Q})$, then $\gamma\mathcal{F}_{\boldsymbol{G}}$ lies below a horizontal line.

(ii) If $\gamma \in \boldsymbol{P}_0(\mathbb{Q})$, then $\gamma\mathcal{F}_{\boldsymbol{G}}$ lies within a vertical strip of finite width.

**C. *Counterexample in which condition (ii) of Theorem 1 is not satisfied: a semisimple subgroup.*** Let $\boldsymbol{G} = \mathrm{SL}_3$ and let $(\boldsymbol{P}_0, \boldsymbol{S}_0, K_{\boldsymbol{G}})$ be the standard Siegel triple for $\boldsymbol{G}$. Let

$$\boldsymbol{H}_0 = \mathrm{SO}_3(J) \quad \text{where } J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let $Q_J$ denote the quadratic form on $\mathbb{R}^3$ represented by $J$. This form is negative definite on the 1-dimensional subspace $L = \mathbb{R}(1, 0, -1)^t \subset \mathbb{R}^3$ and positive definite on the 2-dimensional subspace $M = \mathbb{R}(0, 1, 0)^t + \mathbb{R}(1, 0, 1)^t$. Let

$$K_{\boldsymbol{H}} = \{h \in \boldsymbol{H}_0(\mathbb{R}) : h(L) = L \text{ and } h(M) = M\}.$$

This is a maximal compact subgroup of $\boldsymbol{H}_0(\mathbb{R})$ and is isomorphic to $\mathrm{O}_2(\mathbb{R})$ via restriction to its action on $M$.

Let $c \in \mathbb{Q} \setminus \{0, \pm 1\}$. Let $\eta \in \mathrm{GL}_3(\mathbb{Q})$ be the linear map which acts as multiplication by $c$ on $L$ and as the identity on $M$. Explicitly,

$$\eta = \begin{pmatrix} \frac{1}{2}(1+c) & 0 & \frac{1}{2}(1-c) \\ 0 & 1 & 0 \\ \frac{1}{2}(1-c) & 0 & \frac{1}{2}(1+c) \end{pmatrix}.$$

Let

$$\boldsymbol{H} = \eta\boldsymbol{H}_0\eta^{-1} = \mathrm{SO}_3(\eta J \eta^t).$$

By construction, $\eta$ centralises $K_{\boldsymbol{H}}$. It follows that $\eta K_{\boldsymbol{H}}\eta^{-1} = K_{\boldsymbol{H}} = K_{\boldsymbol{G}} \cap \boldsymbol{H}(\mathbb{R})$ and $K_{\boldsymbol{H}}$ is a maximal compact subgroup of $\boldsymbol{H}(\mathbb{R})$.

Let $Q_0$ denote the standard quadratic form on $\mathbb{R}^3$. The spaces $L$ and $M$ are orthogonal with respect to $Q_0$ and $Q_{0\,|\,M} = Q_{J\,|\,M}$. Hence $K_{\boldsymbol{H}} \subset \mathrm{SO}_3(Q_0) = K_{\boldsymbol{G}}$. Thus condition (i) of Theorem 1 is satisfied.

Let $\boldsymbol{P_H} = \eta(\boldsymbol{P}_0 \cap \boldsymbol{H}_0)\eta^{-1}$ and $\boldsymbol{S_H} = \eta(\boldsymbol{S}_0 \cap \boldsymbol{H}_0)\eta^{-1}$. As in [Borel 1969, 11.16], $\boldsymbol{P}_0 \cap \boldsymbol{H}_0$ is a minimal $\mathbb{Q}$-parabolic subgroup of $\boldsymbol{H}_0$ so $(\boldsymbol{P_H}, \boldsymbol{S_H}, K_{\boldsymbol{H}})$ is a Siegel triple in $\boldsymbol{H}$. Let $\mathfrak{S}_{\boldsymbol{H}} = \Omega_{\boldsymbol{H}} A_{\boldsymbol{H},t} K_{\boldsymbol{H}}$ be a Siegel set in $\boldsymbol{H}(\mathbb{R})$ with respect to this Siegel triple.

We shall show that $\mathfrak{S}_{\boldsymbol{H}}$ and $K_{\boldsymbol{G}}$ do not satisfy the conclusion of Theorem 1. Suppose for contradiction that there exist subgroups $\boldsymbol{P_G}, \boldsymbol{S_G} \subset \boldsymbol{G}$ forming a Siegel triple $(\boldsymbol{P_G}, \boldsymbol{S_G}, K_{\boldsymbol{G}})$, a Siegel set $\mathfrak{S}_{\boldsymbol{G}} \subset \boldsymbol{G}(\mathbb{R})$ with respect to this Siegel triple, and a finite set $C \subset \boldsymbol{G}(\mathbb{Q})$ such that $\mathfrak{S}_{\boldsymbol{H}} \subset C.\mathfrak{S}_{\boldsymbol{G}}$. By the same argument as in Section B, we may assume that $\boldsymbol{P_G} = \boldsymbol{P}_0$ and $\boldsymbol{S_G} = \boldsymbol{S}_0$.

Let $\sigma_s = \mathrm{diag}(s, 1, s^{-1})$ for $s \in \mathbb{R}_{>0}$. Now

$$\{\eta\sigma_s\eta^{-1} : s \geq t\} = A_{\boldsymbol{H},t} \subset \mathfrak{S}_{\boldsymbol{H}} \subset C\mathfrak{S}_{\boldsymbol{G}}.$$

Since $C$ is finite, there exists some $\gamma \in C$ such that $\gamma\mathfrak{S}_{\boldsymbol{G}}$ contains elements of the form $\eta\sigma_s\eta^{-1}$ for arbitrarily large $s$. Consequently $\eta^{-1}\gamma\mathfrak{S}_{\boldsymbol{G}}\eta$ contains $\sigma_s$ for arbitrarily large $s$. Furthermore the standard Siegel set $\mathfrak{S}_{\boldsymbol{G}}$ contains $\{\sigma_s : s \geq t'\}$ for some $t' \in \mathbb{R}_{>0}$.

Let $\chi_1, \chi_2$ denote the simple roots of $\boldsymbol{G}$ with respect to $\boldsymbol{S}_0$, using the ordering induced by $\boldsymbol{P}_0$. Then $\chi_1(\sigma_s) = \chi_2(\sigma_s) = s$ so the previous paragraph shows that $\mathfrak{S}_{\boldsymbol{G}} \cap \eta^{-1}\gamma\mathfrak{S}_{\boldsymbol{G}}\eta$ contains elements $\sigma_s$ with arbitrarily large values for $\chi_1$ and $\chi_2$. Applying Lemma 3 below (with $\Omega_{\boldsymbol{G}} = K_{\boldsymbol{G}} \cup K_{\boldsymbol{G}}\eta^{-1}$), we deduce that $\eta^{-1}\gamma$ is contained in the standard parabolic subgroup ${}_{\mathbb{Q}}\boldsymbol{P}_{0,\varnothing} = \boldsymbol{P}_0$.

Let $\boldsymbol{U}_0 = R_u(\boldsymbol{P}_0)$. Write the Iwasawa decomposition of $\eta^{-1}$ as

$$\eta^{-1} = \mu\alpha\kappa \quad \text{where } \mu \in \boldsymbol{U}_0(\mathbb{R}), \alpha \in \boldsymbol{S}_0(\mathbb{R}), \kappa \in K_{\boldsymbol{G}}.$$

For arbitrarily large real numbers $s$, we have

$$\sigma_s\mu\sigma_s^{-1}.\sigma_s\alpha.\kappa = \sigma_s\eta^{-1} \in \mathfrak{S}_{\boldsymbol{G}}\eta^{-1} \cap \eta^{-1}\gamma\mathfrak{S}_{\boldsymbol{G}} \subset \eta^{-1}\gamma\mathfrak{S}_{\boldsymbol{G}}.$$

By the definition of Siegel sets and since $\eta^{-1}\gamma \in \boldsymbol{P}_0(\mathbb{R})$, the $\boldsymbol{U}_0(\mathbb{R})$-component in the Iwasawa decomposition of every element of $\eta^{-1}\gamma\mathfrak{S}_{\boldsymbol{G}}$ is bounded. Thus $\sigma_s\mu\sigma_s^{-1}$ lies in a bounded set for arbitrarily large real numbers $s$. By direct calculation, this implies that $\mu = 1$. (This is the opposite situation to [Borel 1969, Lemme 12.2], adapted to our conventions about Siegel sets.) Hence $\eta^{-1} = \alpha\kappa \in \boldsymbol{S}_0(\mathbb{R})K_{\boldsymbol{G}}$.

It follows that $\eta^t\eta = (\alpha^{-1})^t(\kappa^{-1})^t\kappa^{-1}\alpha^{-1} = \alpha^{-2}$ is diagonal. But $\eta^t\eta$ is not diagonal, as can be seen either by direct calculation or by noting that $\eta$ is symmetrical so $\eta^t\eta = \eta^2$ has $L$ as a 1-dimensional eigenspace yet $L$ is not a coordinate axis.

**D.** ***Siegel sets with noncompact intersection.*** In this section, we prove a generalisation of [Borel 1969, Proposition 12.6], replacing a Siegel set $\mathfrak{S} = \Omega_{\boldsymbol{P}} A_t K$ by a set of the form $\Omega_{\boldsymbol{P}} A_t \Omega_{\boldsymbol{G}}$ where $\Omega_{\boldsymbol{G}}$ may be any compact subset of $\boldsymbol{G}(\mathbb{R})$. This generalisation was used in Section C.

Let $\boldsymbol{G}$ be a reductive $\mathbb{Q}$-algebraic group. Let $\boldsymbol{P}$ be a minimal parabolic $\mathbb{Q}$-subgroup of $\boldsymbol{G}$ and let $\boldsymbol{U}$ be the unipotent radical of $\boldsymbol{P}$. Let $\boldsymbol{S}$ be a maximal $\mathbb{Q}$-split torus in $\boldsymbol{S}$ and let $\boldsymbol{M}$ be the maximal $\mathbb{Q}$-anisotropic subgroup of $Z_{\boldsymbol{G}}(\boldsymbol{S})$. Let $t$ be a positive real number and let $A_t$ be the subset of $\boldsymbol{S}(\mathbb{R})$ defined in [Orr 2018, Section 2B]. Let $\mathfrak{g}$ and $\mathfrak{u}$ denote the Lie algebras of $\boldsymbol{G}$ and $\boldsymbol{U}$ respectively (over $\mathbb{R}$).

Let $\Delta$ be the set of simple roots of $\boldsymbol{G}$ with respect to $\boldsymbol{S}$, using the ordering induced by $\boldsymbol{P}$. For $\theta \subset \Delta$, let $\Psi_\theta$ denote the set of roots $\phi$ such that the expression of $\phi$ as a linear combination of elements of $\Delta$ has a positive coefficient for at least one element of $\theta$.

For each character $\chi \in X^*(\boldsymbol{S})$, there is a unique continuous group homomorphism $\boldsymbol{P}(\mathbb{R}) \to \mathbb{R}_{>0}$, which we denote $f_\chi$, with the properties $f_\chi(s) = |\chi(s)|$ for all $s \in \boldsymbol{S}(\mathbb{R})$ and $f_\chi = 1$ on $\boldsymbol{U}(\mathbb{R})\boldsymbol{M}(\mathbb{R})$. (This is because $\boldsymbol{S}(\mathbb{R}) \cap \boldsymbol{U}(\mathbb{R})\boldsymbol{M}(\mathbb{R})$ is finite, so $|\chi(s)| = 1$ for all $s \in \boldsymbol{S}(\mathbb{R}) \cap \boldsymbol{U}(\mathbb{R})\boldsymbol{M}(\mathbb{R})$, and $\boldsymbol{S}$ normalises $\boldsymbol{U}\boldsymbol{M}$.) Choose a maximal compact subgroup $K \subset \boldsymbol{G}(\mathbb{R})$. Then $f_\chi(\boldsymbol{P}(\mathbb{R}) \cap K)$ is a compact subgroup of $\mathbb{R}_{>0}$, so is trivial. Therefore we can extend $f_\chi$ to a continuous function $\boldsymbol{G}(\mathbb{R}) = \boldsymbol{P}(\mathbb{R})K \to \mathbb{R}_{>0}$ by setting $f_\chi(pk) = f_\chi(p)$ for all $p \in \boldsymbol{P}(\mathbb{R})$ and $k \in K$. These functions $f_\chi$ are not necessarily "of type $(\boldsymbol{P}, \chi)$" as defined in [Borel 1969, 14.1] because $\chi \in X^*(\boldsymbol{S})$ might not extend to a character of $\boldsymbol{P}$, but the argument in [loc. cit., 14.2(c)] still applies to the functions $f_\chi$.

**Lemma 3.** *Let $\Omega_{\boldsymbol{P}}$ and $\Omega_{\boldsymbol{G}}$ be compact subsets of $\boldsymbol{P}(\mathbb{R})$ and $\boldsymbol{G}(\mathbb{R})$ respectively. Let $\gamma \in \boldsymbol{G}(\mathbb{R})$. If $\Omega_{\boldsymbol{P}} A_t \Omega_{\boldsymbol{G}} \cap \gamma \Omega_{\boldsymbol{P}} A_t \Omega_{\boldsymbol{G}}$ is noncompact, then $\gamma$ is contained in a proper parabolic $\mathbb{Q}$-algebraic subgroup of $\boldsymbol{G}$ containing $\boldsymbol{P}$. More precisely, let*

$$\theta = \{\chi \in \Delta : f_\chi \text{ is bounded above on } \Omega_{\boldsymbol{P}} A_t \Omega_{\boldsymbol{G}} \cap \gamma \Omega_{\boldsymbol{P}} A_t \Omega_{\boldsymbol{G}}\}.$$

*Then $\gamma$ lies in the standard parabolic subgroup ${}_{\mathbb{Q}}\boldsymbol{P}_\theta$ in the notation of* [Borel and Tits 1965, 5.12].

*Proof.* Let

$$\Omega = \Bigl(\bigcup_{a \in A_t} a^{-1} \Omega_{\boldsymbol{P}} a\Bigr) \Omega_{\boldsymbol{G}} \subset \boldsymbol{G}(\mathbb{R}).$$

By [Borel 1969, Lemme 12.2], $\Omega$ is compact. From the definitions, $\Omega_{\boldsymbol{P}} A_t \Omega_{\boldsymbol{G}} \subset A_t \Omega$. Hence, for all $\chi \in \Delta \setminus \theta$, $f_\chi$ is unbounded on $A_t\Omega \cap \gamma A_t \Omega$.

Let ${}_{\mathbb{Q}}\boldsymbol{U}_\theta$ denote the unipotent radical of ${}_{\mathbb{Q}}\boldsymbol{P}_\theta$ and let ${}_{\mathbb{Q}}\mathfrak{u}_\theta = \mathrm{Lie}({}_{\mathbb{Q}}\boldsymbol{U}_\theta)$. Let

$$Y = \{v \in \mathfrak{g} : (\mathrm{Ad}\, \xi_n^{-1}) v \to 0 \text{ for some sequence } (\xi_n) \text{ in } A_t\Omega \cap \gamma A_t \Omega\}.$$

Let $\langle Y \rangle$ denote the subspace of $\mathfrak{g}$ generated by $Y$. We shall show that

$${}_{\mathbb{Q}}\mathfrak{u}_\theta \subset \langle Y \rangle \subset (\mathrm{Ad}\,\gamma)\mathfrak{u}. \tag{1}$$

To prove the first inclusion of (1), note that ${}_{\mathbb{Q}}\mathfrak{u}_\theta$ is the direct sum of the root spaces $\mathfrak{u}_\phi$ for $\phi \in \Psi_{\Delta\setminus\theta}$, so it suffices to prove that $\mathfrak{u}_\phi \subset Y$ for each $\phi \in \Psi_{\Delta\setminus\theta}$.

Let $\phi \in \Psi_{\Delta\setminus\theta}$ and write $\phi$ as a linear combination of simple roots: $\phi = \sum_{\psi \in \Delta} m_\psi \psi$. By the definition of $\Psi_{\Delta\setminus\theta}$, there exists some $\chi \in \Delta \setminus \theta$ such that $m_\chi > 0$.

By the definition of $\theta$, $f_\chi$ is unbounded on $\Omega_{\boldsymbol{P}} A_t \Omega_{\boldsymbol{G}} \cap \gamma \Omega_{\boldsymbol{P}} A_t \Omega_{\boldsymbol{G}} \subset A_t \Omega \cap \gamma A_t \Omega$. Choose a sequence $(\xi_n)$ in $A_t\Omega \cap \gamma A_t \Omega$ such that $f_\chi(\xi_n) \to +\infty$. Write $\xi_n = \alpha_n \kappa_n$ where $\alpha_n \in A_t$ and $\kappa_n \in \Omega$.

The argument of [Borel 1969, 14.2(c)] shows that $f_\chi(\xi_n)/f_\chi(\alpha_n)$ is bounded both above and below independently of $n$. Hence

$$|\chi(\alpha_n)| = f_\chi(\alpha_n) \to +\infty.$$

Since $\phi$ is a positive root, $m_\psi \geq 0$ for all $\psi \in \Delta$. Since $\alpha_n \in A_t$ and $m_\chi > 0$, it follows that $\phi(\alpha_n) \to +\infty$.

Hence for every $v \in \mathfrak{u}_\phi$, we have $(\operatorname{Ad} \alpha_n^{-1})v \to 0$. Since $\Omega$ is compact, after replacing $(\xi_n)$ by a subsequence, we may assume that $\kappa_n$ converges, say to $\kappa \in \Omega$. Then $(\operatorname{Ad} \xi_n^{-1})v \to (\operatorname{Ad} \kappa)^{-1} 0 = 0$. Thus $\mathfrak{u}_\phi \subset Y$.

To prove the second inclusion of (1), consider an element $v \in Y$. Let $(\xi_n)$ be a sequence in $A_t\Omega \cap \gamma A_t \Omega$ such that $(\operatorname{Ad} \xi_n^{-1})v \to 0$. Write $\xi_n = \gamma \beta_n \lambda_n$ with $\beta_n \in A_t$, $\lambda_n \in \Omega$. Since $\Omega$ is compact, after replacing $(\xi_n)$ by a subsequence, we may assume that $\lambda_n$ converges, say to $\lambda \in \Omega$. Then

$$(\operatorname{Ad} \beta_n^{-1})(\operatorname{Ad} \gamma^{-1})v = (\operatorname{Ad} \lambda_n)(\operatorname{Ad} \xi_n^{-1})v \to (\operatorname{Ad} \lambda)0 = 0.$$

Hence, when we decompose $(\operatorname{Ad} \gamma^{-1})v$ using the root space decomposition of $\mathfrak{g}$, nonzero components can occur only for those roots $\phi$ satisfying $|\phi(\beta_n)| \to +\infty$. Since $\beta_n \in A_t$, such roots $\phi$ must be positive roots. Thus $(\operatorname{Ad} \gamma^{-1})v \in \bigoplus_{\phi \in \Phi^+} \mathfrak{u}_\phi = \mathfrak{u}$.

We have proved both parts of (1). Passing from Lie algebras to groups, we obtain

$$_{\mathbb{Q}}\boldsymbol{U}_\theta \subset \gamma \boldsymbol{U} \gamma^{-1} \subset \gamma \boldsymbol{P} \gamma^{-1} \subset \gamma \, _{\mathbb{Q}}\boldsymbol{P}_\theta \, \gamma^{-1}.$$

By [Borel and Tits 1965, Corollaire 4.5], it follows that $_{\mathbb{Q}}\boldsymbol{P}_\theta = \gamma \, _{\mathbb{Q}}\boldsymbol{P}_\theta \, \gamma^{-1}$. Since a parabolic subgroup of $\boldsymbol{G}$ is its own normaliser, we conclude that $\gamma \in {_{\mathbb{Q}}\boldsymbol{P}_\theta}(\mathbb{R})$. ☐

**E.** ***Additional minor corrections to*** **[Orr 2018].** The following are additional corrections to [Orr 2018]:

- (page 461, Section 2D) (F2) should begin "For every $g \in \boldsymbol{G}(\mathbb{Q})$."
- (page 474, proof of Proposition 4.7) On the fifth line from the end, should say "$\chi_{|\boldsymbol{S}_{\boldsymbol{H}}} \in \Phi_\alpha \cup \{0\}$." instead of "$\chi_{|\boldsymbol{S}_{\boldsymbol{H}}} \in \Phi_\alpha$".
- (page 474, proof of Lemma 4.10) The first paragraph should say "Let $\boldsymbol{T}_{\boldsymbol{G}}$ be a maximal $\mathbb{R}$-split torus in $\boldsymbol{G}$ which contains $\boldsymbol{S}_{\boldsymbol{G}}$ and is stabilised by the Cartan involution of $\boldsymbol{G}$ associated with $K_{\boldsymbol{G}}$." This is necessary to apply [Borel and Tits 1965, Section 14].

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Orr is grateful to Dave Witte Morris for informing him of the errors in [Orr 2018, page 474] which are corrected in Section E.

We are grateful to the referee for very careful reading of this note and for suggesting alternative arguments for Sections C and D, including a more general statement for Lemma 3 than in the original version.

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