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On the first nontrivial strand of syzygies of projective schemes and condition ND(ℓ)

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Let $X \subset \mathbb{P}^{n+e}$ be any n-dimensional closed subscheme. We are mainly interested in two notions related to syzygies: one is the property $\mathbf{N}_{d,p}$ ($d \geq 2$, $p \geq 1$), which means that X is d-regular up to p-th step in the minimal free resolution and the other is a new notion ND(ℓ) which generalizes the classical "being nondegenerate" to the condition that requires a general finite linear section not to be contained in any hypersurface of degree ℓ.

First, we introduce condition ND(ℓ) and consider examples and basic properties deduced from the notion. Next we prove sharp upper bounds on the graded Betti numbers of the first nontrivial strand of syzygies, which generalize results in the quadratic case to higher degree case, and provide characterizations for the extremal cases. Further, after regarding some consequences of property $\mathbf{N}_{d,p}$, we characterize the resolution of X to be d-linear arithmetically Cohen–Macaulay as having property $\mathbf{N}_{d,e}$ and condition ND($d-1$) at the same time. From this result, we obtain a syzygetic rigidity theorem which suggests a natural generalization of syzygetic rigidity on 2-regularity due to Eisenbud, Green, Hulek and Popescu to a general d-regularity.

1. Introduction

Since the foundational paper on syzygy computation by Green [1984], there has been a great deal of interest and progress in understanding the structure of the Betti tables of algebraic varieties during the past decades. In particular, the first nontrivial linear strand starting from quadratic equations has been

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	0	1	$\cdots$	p	$p+1$	$\cdots$	e	$e+1$	$\cdots$	$n+e$
0	1	-	-	-	-	-	-	-	-	-
1	-	*	*	*	*	*	*	*	*	*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$d-2$	-	*	*	*	*	*	*	*	*	*
$d-1$	-	*	*	*	*	*	*	*	*	*
d	-	-	-	-	*	*	*	*	*	*
$d+1$	-	-	-	-	*	*	*	*	*	*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

	0	1	$\cdots$	$\cdots$	$e-1$	e	$e+1$	$\cdots$	$n+e$
0	1	-	-	-	-	-	-	-	-
1	-	-	-	-	-	-	-	-	-
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$\ell-1$	-	-	-	-	-	-	-	-	-
ℓ	-	*	*	*	*	*	*	*	*
$\ell+1$	-	*	*	*	*	*	*	*	*
$\ell+2$	-	*	*	*	*	*	*	*	*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Figure 1. Two typical Betti tables $\mathbb{B}(X)$ of $X \subset \mathbb{P}^{n+e}$ with property $\mathbf{N}_{d,p}$ and with condition ND(ℓ). Note that the shape of $\mathbb{B}(X)$ with ND(ℓ) is preserved under taking general hyperplane sections and general linear projections.

intensively studied by several authors [Castelnuovo 1893; Green 1984; Green and Lazarsfeld 1988; Eisenbud et al. 2005; 2006; Ein and Lazarsfeld 2015; Han and Kwak 2015].

Let X be any nondegenerate n-dimensional closed subscheme X in a projective space $\mathbb{P}^{n+e}$ defined over an algebraically closed field $\Bbbk$ of any characteristic and $R = \Bbbk[x_0, \dots, x_{n+e}]$. In this article, we are mainly interested in two notions related to syzygies of X. One notion is the property $\mathbf{N}_{d,p}(d \geq 2,\ p \geq 1)$, which was first introduced in [Eisenbud et al. 2005] and means that X is d-regular up to p-th step in the minimal free resolution. To be precise, X is said to satisfy property $\mathbf{N}_{d,p}$ if the following condition holds:

$$\beta_{i,j}(X) := \dim_{\Bbbk} \operatorname{Tor}_i^R(R/I_X, \Bbbk)_{i+j} = 0 \quad \text{for } i \leq p \text{ and } j \geq d.$$

The other one is a new notion *condition* ND(ℓ), which generalizes the classical “being nondegenerate” in degree one to cases of higher degrees. More precisely, it means that a general linear section $X \cap \Lambda$ is not contained in any hypersurface of degree ℓ of Λ, where Λ is a general linear subspace of each dimension $\geq e$. So, for irreducible varieties the classical nondegenerate condition is equivalent to condition ND(1) by Bertini-type theorem. We give many examples and basic properties on condition ND(ℓ).

With this notion, we obtain a new angle to study syzygies of high degrees in the Betti table $\mathbb{B}(X)$. Especially, it turns out to be very effective to understand the first nontrivial ℓ-th linear strand arising from equations of degree $\ell+1$ and also to answer many interesting questions which can be raised as compared to the classical quadratic case.

To review previous results for the quadratic case, let us begin by recalling the well known theorems due to Castelnuovo and Fano. Let $X \subset \mathbb{P}^{n+e}$ be any “nondegenerate” irreducible variety:

- (Castelnuovo, 1889) $h^0(\mathcal{I}_X(2)) \leq \binom{e+1}{2}$ and “=” holds if and only if X is a variety of minimal degree.
- (Fano, 1894) Unless X is a variety of minimal degree, $h^0(\mathcal{I}_X(2)) \leq \binom{e+1}{2} - 1$ and “=” holds if and only if X is a del Pezzo variety (i.e.,, arithmetically Cohen–Macaulay and $\deg X = e+2$).

A few years ago, Han and Kwak [2015] developed an inner projection method to compare syzygies of X with those of its projections by using the theory of mapping cone and partial elimination ideals. As applications, over any algebraically closed field $\Bbbk$ of arbitrary characteristic, they proved the sharp upper bounds on the ranks of higher linear syzygies by quadratic equations, and characterized the extremal and next-to-extremal cases, which generalized the results of Castelnuovo and Fano:

- [Han and Kwak 2015] $\beta_{i,1}(X) \le i\binom{e+1}{i+1}$, $i \ge 1$ and the equality holds for some $1 \le i \le e$ if and only if X is a variety of minimal degree (abbr. VMD).
- Unless X is a variety of minimal degree, then $\beta_{i,1}(X) \le i\binom{e+1}{i+1} - \binom{e}{i-1} \forall i \le e$ and the equality holds for some $1 \le i \le e-1$ if and only if X is a del Pezzo variety.

Thus, the theorem above by Han and Kwak can be thought of as a syzygetic characterization of varieties of minimal degree and del Pezzo varieties.

It is worth to note here that the condition $(I_X)_1 = 0$ (i.e., to be "nondegenerate") implies not only an upper bound for the number of quadratic equations $h^0(\mathcal{I}_X(2)) \le \binom{e+1}{2}$ as we reviewed, but also on the degree of X via the so-called "basic inequality" $\deg X \ge \binom{e+1}{1}$. Thus, for "more" nondegenerate varieties, it seems natural to raise a question as follows: For any irreducible variety X with $(I_X)_2 = 0$ (i.e., having no linear and quadratic forms vanishing on X)

does it hold that $h^0(\mathcal{I}_X(3)) \le \binom{e+2}{3}$ *and* $\deg X \ge \binom{e+2}{2}$?

But, there is a counterexample for this question: the Veronese surface $S \subset \mathbb{P}^4$ $(e = 2)$ i.e., an isomorphic projection of $\nu_2(\mathbb{P}^2)$, one of the Severi varieties classified by Zak, where S has no quadratic equations on it, but $h^0(\mathcal{I}_S(3)) = 7 \nleq \binom{2+2}{3}$ and $\deg X = 4 \ngeq \binom{2+2}{2}$. One reason for the failure is that a general hyperplane section of S sits on a quadric hypersurface while S itself does not. It leads us to consider the notion of condition ND(ℓ).

Under condition ND(ℓ) it can be easily checked that the degree of X satisfies the expected bound $\deg X \ge \binom{e+\ell}{\ell}$ (see Remark 2.1). Further, one can see that condition ND(ℓ) is determined by the *injectivity* of the restriction map $H^0(\mathcal{O}_\Lambda(\ell)) \to H^0(\mathcal{O}_{X\cap\Lambda}(\ell))$ for a general point section $X \cap \Lambda$ which can happen in larger degree for a given ℓ, while the problem on "imposing independent conditions on ℓ-forms (or ℓ-normality)" concerns surjectivity of the above map in degree at most $\binom{e+\ell}{\ell}$. The latter has been intensively studied in many works in the literature (see e.g., [Cook et al. 2018]), but the former has not been considered well.

With this notion, we can also obtain sharp upper bounds on the numbers of defining equations of degree $\ell + 1$ and the graded Betti numbers for their higher linear syzygies. As in the quadratic case, we prove that the extremal cases for these Betti numbers are only arithmetically Cohen–Macaulay (abbr. ACM) varieties with $(\ell+1)$-linear resolution (we call a variety $X \subset \mathbb{P}^N$ ACM if its homogeneous coordinate ring R_X is arithmetically Cohen–Macaulay i.e., $\operatorname{depth}(R_X) = \dim X + 1$).

Now, we present our first main result.

Theorem 1.1. *Let X be any closed subscheme of codimension e satisfying condition* $\mathrm{ND}(\ell)$ *for some* $\ell \geq 1$ *in* $\mathbb{P}^{n+e}$ *over an algebraically closed field* $\Bbbk$ *with* $\mathrm{ch}(\Bbbk) = 0$. *Then, we have*:

(a) $\beta_{i,\ell}(X) \leq \binom{i+\ell-1}{\ell}\binom{e+\ell}{i+\ell}$ *for all* $i \geq 1$.

(b) *The following are equivalent*:

(i) $\beta_{i,\ell}(X) = \binom{i+\ell-1}{\ell}\binom{e+\ell}{i+\ell}$ *for all* $i \geq 1$.

(ii) $\beta_{i,\ell}(X) = \binom{i+\ell-1}{\ell}\binom{e+\ell}{i+\ell}$ *for some i among* $1 \leq i \leq e$.

(iii) *X is arithmetically Cohen–Macaulay with $(\ell+1)$-linear resolution.*

In particular, if X satisfies one of equivalent conditions then X has a minimal degree $\binom{e+\ell}{\ell}$.

We would like to note that if $\ell = 1$, then this theorem recovers the previous results on the linear syzygies by quadrics for the case of integral varieties (see also Remark 2.8). In general, the set of closed subschemes satisfying $\mathrm{ND}(1)$ is much larger than that of *nondegenerate* irreducible varieties; see [Ahn and Han 2015, Section 1] for details. Furthermore, a closed subscheme X (with possibly many components) has condition $\mathrm{ND}(\ell)$ if so does the top-dimensional part of X. Note that the Betti table $\mathbb{B}(X)$ is usually very sensitive for addition some components to X (e.g., when we add points to a rational normal curve, Betti table can be totally changed; see, e.g., [Ahn and Kwak 2015, Example 3.10]). But condition $\mathrm{ND}(\ell)$ has been still preserved under such addition of low dimensional components (thus, we could make many examples with condition $\mathrm{ND}(\ell)$ in this way).

On the other hand, if X satisfies property $\mathbf{N}_{d,e}$, then the degree of X is at most $\binom{e+d-1}{d-1}$ and the equality happens only when X has ACM d-linear resolution. We prove this by establishing a syzygetic Bézout theorem (Theorem 3.1), a geometric implication of property $\mathbf{N}_{d,p}$ using projection method. We also investigate an effect of $\mathbf{N}_{d,p}$ on loci of d-secant lines (Theorem 3.3).

Furthermore, if two notions - condition $\mathrm{ND}(d-1)$ and property $\mathbf{N}_{d,e}$ on X - meet together, then the degree of X should be equal to $\binom{e+d-1}{d-1}$ and X has ACM d-linear resolution (in particular, X is d-regular). From this point of view, we can obtain another main result, a syzygetic rigidity for d-regularity as follows:

Theorem 1.2 (syzygetic rigidity for d-regularity). *Let X be any algebraic set of codimension e in* $\mathbb{P}^{n+e}$ *satisfying condition* $\mathrm{ND}(d-1)$ *for* $d \geq 2$. *If X has property* $\mathbf{N}_{d,e}$, *then X is d-regular* (*more precisely*, *X has ACM d-linear resolution*).

Note that if $d = 2$, for nondegenerate algebraic sets this theorem recovers the syzygetic rigidity for 2-regularity due to Eisenbud, Green, Hulek and Popescu [Eisenbud et al. 2005, Corollary 1.8] where the condition $\mathrm{ND}(1)$ was implicitly used. In [Eisenbud et al. 2005], the rigidity for 2-regularity was obtained using the classification of so-called "small" schemes in the category of algebraic sets in [Eisenbud et al. 2006]. But, for next 3 and higher d-regular algebraic sets, it seems out of reach to get such classifications at this moment. From this point of view, Theorem 1.2 is a natural generalization and gives a more direct proof for the rigidity.

We would like to also remark that for a generalization of this syzygetic rigidity into higher d, one needs somewhat a sort of "higher nondegeneracy condition" such as the condition $\mathrm{ND}(\ell)$, because there

	0	1	$\cdots$	$\cdots$	$e-1$	e	$e+1$	$\cdots$	N
0	1	-	-	-	-	-	-	-	-
1	-	-	-	-	-	-	-	-	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$d-2$	-	-	-	-	-	-	-	-	-
$d-1$	-	*	*	*	*	*	*	*	*
d	-	-	-	-	-	-	*	*	*
$d+1$	-	-	-	-	-	-	*	*	*
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

	0	1	$\cdots$	$\cdots$	$e-1$	e	$e+1$	$\cdots$	N
0	1	-	-	-	-	-	-	-	-
1	-	-	-	-	-	-	-	-	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$d-2$	-	-	-	-	-	-	-	-	-
$d-1$	-	*	*	*	*	*	-	-	-
d	-	-	-	-	-	-	-	-	-
$d+1$	-	-	-	-	-	-	-	-	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure 2. $\mathbf{N}_{d,e}$ meets only $(I_X)_{\leq d-1} = 0$ (left-hand side) and $\mathbf{N}_{d,e}$ meets condition ND($d-1$) (right-hand side). Condition ND($d-1$) and property $\mathbf{N}_{d,e}$ implies ACM d-linear resolution in the category of algebraic sets.

exist some examples where Theorem 1.2 does not hold without condition ND(ℓ) even though the given X is an irreducible variety and there is no forms of degree ℓ vanishing on X (see Figure 2 and Example 3.6).

In the final Section 4, we present relevant examples and more consequences of our theory (see, e.g., Corollary 4.3) and raise some questions for further development.

2. Condition ND(ℓ) and syzygies

2A. ***Condition* ND(ℓ)*: basic properties and examples.*** Throughout this section, we assume that the base field is algebraically closed and $\mathrm{ch}(\Bbbk) = 0$ (see Remark 2.10 for finite characteristics).

As before, let X be a n-dimensional closed subscheme of codimension e in $\mathbb{P}^N$ over $\Bbbk$. Let I_X be $\bigoplus_{m=0}^{\infty} H^0(\mathcal{I}_{X/\mathbb{P}^N}(m))$, the defining ideal of X in the polynomial ring $R = \Bbbk[x_0, x_1, \ldots, x_N]$. We mean (co)dimension and degree of $X \subset \mathbb{P}^N$ by the definition deduced from the Hilbert polynomial of R/I_X.

Let us begin this study by introducing the definition of condition ND(ℓ) as follows:

Definition (condition ND(ℓ)). Let $\Bbbk$ be any algebraically closed field. We say that a closed subscheme $X \subset \mathbb{P}^N_{\Bbbk}$ satisfies *condition* ND(ℓ) if

$$H^0(\mathcal{I}_{X\cap\Lambda/\Lambda}(\ell)) = 0 \quad \text{for a } \textit{general} \text{ linear section } \Lambda \text{ of each dimension} \geq e.$$

We sometimes call a subscheme with condition ND(ℓ) a ND(ℓ)-*subscheme* as well.

Remark 2.1. We would like to make some remarks on this notion as follows:

(a) First of all, if $X \subset \mathbb{P}^N$ satisfies condition ND(ℓ), then every general linear section of $X \cap \Lambda$ also has the condition (i.e., condition ND(ℓ) is preserved under taking general hyperplane sections). Further, from the definition, condition ND(ℓ) on X is completely determined by a general point section of X.

(b) (Basic degree bound) If X is a closed subscheme of codimension e in $\mathbb{P}^{n+e}$ satisfying condition ND(ℓ), then from the sequence $0 \to H^0(\mathcal{I}_{X\cap\Lambda/\Lambda}(\ell)) \to H^0(\mathcal{O}_\Lambda(\ell)) \to H^0(\mathcal{O}_{X\cap\Lambda}(\ell))$ it can be easily proved that $\deg X \geq \binom{e+\ell}{\ell}$.

(c) A general linear projection of ND(ℓ)-subscheme is also an ND(ℓ)-subscheme.

(d) Any nondegenerate variety (i.e., irreducible and reduced) satisfies condition ND(1) due to Bertini-type theorem; see, e.g., [Eisenbud 2005, Lemma 5.4].

(e) If a closed subscheme $X \subset \mathbb{P}^N$ has top dimensional components satisfying ND (ℓ), then X also satisfies condition ND(ℓ) whatever X takes as a lower-dimensional component.

(f) (Maximal ND-index) From the definition, it is easy to see that

$$X: \text{ not satisfying condition } \mathrm{ND}(\ell) \Rightarrow X: \text{ neither having } \mathrm{ND}(\ell+1).$$

Thus, it is natural to regard a notion like

$$\mathrm{index}_{\mathrm{ND}}(X) := \max\{\ell \in \mathbb{Z}_{\geq 0} : X \text{ satisfies condition } \mathrm{ND}(\ell)\} \tag{1}$$

which is a new projective invariant of a given subscheme $X \subset \mathbb{P}^N$.

(g) From the viewpoint (a), one can restate the definition of condition ND(ℓ) as the *injectivity* of the restriction map $H^0(\mathcal{O}_\Lambda(\ell)) \to H^0(\mathcal{O}_{X\cap\Lambda}(\ell))$ for a general point section $X \cap \Lambda$, while many works in the literature have focused on *surjectivity* (or imposing independent conditions) to study dimensions of linear systems in relatively small degree.

Example 2.2. We list some first examples achieving condition ND(ℓ):

(a) If $X \subset \mathbb{P}^{n+e}$ is an ACM subscheme with $H^0(\mathcal{I}_X(\ell)) = 0$, then X is an ND(ℓ)-subscheme.

(b) Every linearly normal curve with no quadratic equation is a ND(2)-curve. Further, a variety X is ND(2) if a general curve section $X \cap \Lambda$ is linearly normal.

(c) (From a projection of Veronese embedding) We can also find examples of non-ACM ND(ℓ)-variety using projections. For instance, if we consider the case of $v_3(\mathbb{P}^2) \subset \mathbb{P}^9$ and its general projection into $\mathbb{P}^4$ (say $\pi(v_3(\mathbb{P}^2))$), then $\deg \pi(v_3(\mathbb{P}^2)) = 9 \geq \binom{2+2}{2}$ and all the quadrics disappear after this projection. This is a ND(2)-variety by Proposition 2.6 (see also Remark 2.5).

In general, it is not easy to determine whether a given closed subscheme X satisfies condition ND(ℓ) or not. The following proposition tells us a way to verify condition ND(ℓ) by aid of computation the generic initial ideal of X; see, e.g., [Bigatti et al. 2005, Section 1] for the theory of generic initial ideal and Borel fixed property.

In what follows, for a homogeneous ideal I in R, we denote by Gin(I) the generic initial ideal of I with respect to the *degree reverse lexicographic order*.

Proposition 2.3 (a characterization of condition ND(ℓ)). *Let X be a closed subscheme of codimension e in $\mathbb{P}^{n+e}$. Then the followings are equivalent*:

(a) *X satisfies condition* ND(ℓ).

(b) $\mathrm{Gin}(I_X) \subset (x_0, \ldots, x_{e-1})^{\ell+1}$.

Proof. Let Λ be a general linear space of dimension e and let Γ be the zero-dimensional intersection of X with Λ.

$(a) \Rightarrow (b)$ For a monomial $T \in \mathrm{Gin}(I_X)$, decompose T as a product of two monomials N and M such that

$$\mathrm{Supp}(N) \subset \{x_0, \dots, x_{e-1}\} \quad \text{and} \quad \mathrm{Supp}(M) \subset \{x_e, \dots, x_{n+e}\}.$$

By the Borel fixed property, we see that $N x_e^{\deg(M)} \in \mathrm{Gin}(I_X)$. Then, it follows from [Ahn and Han 2015, Theorem 2.1] that

$$\mathrm{Gin}(I_{\Gamma/\Lambda}) = \left[\frac{(\mathrm{Gin}(I_X), x_{e+1}, \dots, x_{n+e})}{(x_{e+1}, \dots, x_{n+e})}\right]^{\mathrm{sat}} = \left[\frac{(\mathrm{Gin}(I_X), x_{e+1}, \dots, x_{n+e})}{(x_{e+1}, \dots, x_{n+e})}\right]_{x_e \to 1},$$

which implies $N \in \mathrm{Gin}(I_{\Gamma/\Lambda})$. By the assumption that X satisfies ND(ℓ), we see that $\deg(N) \geq \ell + 1$, and thus $N \in (x_0, \dots, x_{e-1})^{\ell+1}$. Therefore $T = NM \in (x_0, \dots, x_{e-1})^{\ell+1}$ as we wished.

$(a) \Leftarrow (b)$ Conversely, assume that $\mathrm{Gin}(I_X) \subset (x_0, \dots, x_{e-1})^{\ell+1}$. Then,

$$\mathrm{Gin}(I_{\Gamma/\Lambda}) = \left[\frac{(\mathrm{Gin}(I_X), x_{e+1}, \dots, x_{n+e})}{(x_{e+1}, \dots, x_{n+e})}\right]^{\mathrm{sat}} \subset \left[\frac{((x_0, \dots, x_{e-1})^{\ell+1}, x_{e+1}, \dots, x_{n+e})}{(x_{e+1}, \dots, x_{n+e})}\right]_{x_e \to 1}.$$

Note that the rightmost ideal is identified with the ideal $(x_0, \dots, x_{e-1})^{\ell+1}$ in the polynomial ring $\Bbbk[x_0, \dots, x_e]$. Therefore $(I_{\Gamma/\Lambda})_\ell = 0$ and thus X satisfies condition ND(ℓ). □

Beyond the first examples in Example 2.2, one can raise a question as "Is there a higher-dimensional ND(ℓ)-variety X which is linearly normal (i.e., not coming from isomorphic projections) but also non-ACM?". We can construct such an example as a toric variety which is 3-dimensional and has depth 3 as follows.

Example 2.4 (a linearly normal and non-ACM ND(3)-variety). Consider a matrix

$$A = \begin{bmatrix} 3 & -5 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

and we consider the toric ideal induced by the matrix A. Using `Macaulay 2` [Macaulay2], we compute the defining ideal as

$$I_A = (x_1x_2^2x_3 - x_0x_4x_5^2,\ x_2x_3^3x_4 - x_0^3x_1x_5,\ x_0^2x_1^2x_2 - x_3^2x_4^2x_5,\ x_2^3x_3^4 - x_0^4x_5^3,\ x_0x_1^3x_2^3 - x_3x_4^3x_5^3,\ x_0^5x_1^3 - x_3^5x_4^3,\ x_1^4x_2^5 - x_4^4x_5^5).$$

Then the generic initial ideal of I_A with respect to degree reverse lexicographic order is

$$\mathrm{Gin}(I_A) = (x_0^4, x_0^3x_1^2, x_0^2x_1^3, x_0x_1^5, x_1^6, x_0x_1^4x_2^2, x_1^5x_2^2, x_0^3x_1x_2^4, x_0^2x_1^2x_2^5).$$

Hence, I_A defines a 3-dimensional toric variety $X \subset \mathbb{P}^5$ with $\mathrm{depth}(X) = 3$, which satisfies condition ND(ℓ) for $\ell \leq 3$ by Proposition 2.3. Note that I_A is linearly normal but not ACM.

Finally, we would like to remark that condition ND(ℓ) is expected to be generally satisfied in the following manner.

Remark 2.5 (ND(ℓ) in a relatively large degree). For a given codimension e, fixed ℓ, and any *general* closed subscheme X in $\mathbb{P}^{n+e}$, it is expected that

$$X \to \mathrm{ND}(\ell) \quad \text{as } \deg X \to \infty \tag{2}$$

under the condition $H^0(\mathcal{I}_{X/\mathbb{P}^{n+e}}(\ell)) = 0$ and exceptional cases do appear with some special geometric properties (e.g., such as projected Veronese surface), because the failure of ND(ℓ) means that any general point section $X \cap \Lambda$ sits in a hypersurface of degree ℓ, which is not likely to happen for a sufficiently large $\deg X$. For instance, the "expectation" (2) can have an explicit form in case of codimension two in the following proposition (see Section 4 for further discussion).

Proposition 2.6 (ND(ℓ) in codimension two). *Let $X \subset \mathbb{P}^N$ be any nondegenerate integral variety of codimension two over an algebraically closed field $\Bbbk$ with* $\mathrm{ch}(\Bbbk) = 0$. *Say $d = \deg X$. Suppose that $H^0(\mathcal{I}_{X/\mathbb{P}^N}(\ell)) = 0$ for some $\ell \geq 2$. Then, any such X satisfies condition* ND(ℓ) *if $d > \ell^2 + 1$.*

Proof of Proposition 2.6. For the proof, we would like to recall a result for the "lifting problem" (for the literature, see, e.g., [Chiantini and Ciliberto 1993; Bonacini 2015]) as follows:

> *Let $X \subset \mathbb{P}^N$ be any nondegenerate reduced irreducible scheme of codimension two over an algebraically closed field $\Bbbk$ with* $\mathrm{ch}(\Bbbk) = 0$ *and let X_H be the general hyperplane section of X. Suppose that X_H is contained in a hypersurface of degree ℓ in $\mathbb{P}^{N-1}$ for some $\ell \geq 2$. If $d > \ell^2 + 1$, then X is contained in a hypersurface of degree ℓ in $\mathbb{P}^N$.*

Say $n = \dim X$ and suppose that $X \subset \mathbb{P}^N$ does not satisfy ND(ℓ). Then for some r with $2 \leq r \leq n+1$, the $(r-2)$-dimensional general linear section of X, $X \cap \Lambda^r$ lies on a hypersurface of degree ℓ in Λ^r (i.e., $H^0(\mathcal{I}_{X\cap\Lambda^r/\Lambda^r}(\ell)) \neq 0$). By above lifting theorem, this implies $H^0(\mathcal{I}_{X\cap\Lambda^{r+1}/\Lambda^{r+1}}(\ell)) \neq 0$ for the $(r-1)$-dimensional general linear section $X \cap \Lambda^{r+1}$. By repeating the argument, we obtain that $H^0(\mathcal{I}_{X/\mathbb{P}^N}(\ell)) \neq 0$, which is a contradiction. □

Example 2.7 (general curves in $\mathbb{P}^3$). Suppose that $C \subset \mathbb{P}^3$ be a general curve of degree $d \geq g+3$ with nonspecial line bundle $\mathcal{O}_C(1)$, where g is the genus of C. When $g \geq 3$, then by the maximal rank theorem due to Ballico and Ellia [1985], the natural restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_C(2))$$

is injective. So there is no quadric containing C. Further, from Proposition 2.6 we see that a general point section $C \cap H$ also has no quadric. Thus C satisfies condition ND(2). In a similar manner, we can show that if $g \geq 8$ then such curve satisfies ND(3) and in general it has condition ND(ℓ) in case of $d \geq \max\{g+3, \ell^2+2\}$.

2B. ***Sharp upper bounds on Betti numbers of the first nontrivial strand.*** From now on, we proceed to prove Theorem 1.1, which is one of our main results.

Theorem 1.1(a). *Let X be any closed subscheme of codimension e in $\mathbb{P}^{n+e}$ satisfying condition* ND(ℓ) *for some $\ell \geq 1$ and let I_X be the (saturated) defining ideal of X. Then we have*

$$\beta_{i,\ell}(X) \leq \binom{i+\ell-1}{\ell}\binom{e+\ell}{i+\ell} \quad \textit{for all } i \geq 1. \tag{3}$$

A proof of Theorem 1.1(a). First, recall that by [Green 1998, Corollary 1.21] we have

$$\beta_{i,j}(X) \leq \beta_{i,j}(R/\operatorname{Gin}(I_X)) \quad \text{for all } i, j \geq 0. \tag{4}$$

By the assumption that X satisfies condition ND(ℓ) for a given $\ell > 0$, we see that $\operatorname{Gin}(I_X)_d = 0$ for $d \leq \ell$. Moreover, by Proposition 2.3, we have

$$\operatorname{Gin}(I_X) \subset (x_0, \ldots, x_{e-1})^{\ell+1}. \tag{5}$$

For a monomial ideal I, we write $\mathcal{G}(I)$ for the set of minimal monomial generators and $\mathcal{G}(I)_{j+1}$ for the subset of degree $j+1$ part. We denote $\max\{a : k_a > 0\}$ for a given monomial $T = x_0^{k_0} \cdots x_n^{k_n}$ by $\max(T)$. Then, for any Borel fixed ideal $J \subset R$ we have a formula as

$$\beta_{i,j}(R/J) = \sum_{T \in \mathcal{G}(J)_{j+1}} \binom{\max(T)}{i-1} \quad \text{for every } i, j \tag{6}$$

from the result of Eliahou and Kervaire; see e.g., [Ahn and Han 2015, Theorem 2.3].

(i) Let $0 \leq i \leq e$. Consider the ideal $J_0 = (x_0, \ldots, x_{e-1})^{\ell+1}$ which is Borel-fixed. We see that J_0 is generated by the maximal minors of $(\ell+1) \times (\ell+e)$ matrix:

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{e-1} & 0 & \cdots & 0 & 0 \\ 0 & x_0 & x_1 & \cdots & x_{e-1} & 0 & \cdots & 0 \\ & & & \cdots & & & & \\ 0 & \cdots & 0 & x_0 & x_1 & x_2 & \cdots & x_{e-1} \end{pmatrix}$$

So, the graded Betti numbers of R/J_0 are those given by the Eagon–Northcott resolution of the maximal minors of a generic matrix of size $(\ell+1) \times (\ell+e)$; see [Geramita et al. 2013, Remark 2.11]. This implies that

$$\beta_{i,\ell}(R/J_0) = \binom{i+\ell-1}{\ell}\binom{e+\ell}{i+\ell}. \tag{7}$$

By relation (5), we see $\mathcal{G}(\operatorname{Gin}(I_X))_{\ell+1} \subset \mathcal{G}(J_0)_{\ell+1}$. So, above formula (6) implies $\beta_{i,\ell}(R/\operatorname{Gin}(I_X)) \leq \beta_{i,\ell}(R/J_0)$. Consequently, for each $0 \leq i \leq e$ we conclude that

$$\beta_{i,\ell}(X) \leq \beta_{i,\ell}(R/\operatorname{Gin}(I_X)) \leq \beta_{i,\ell}(R/J_0) = \binom{i+\ell-1}{\ell}\binom{e+\ell}{i+\ell},$$

as we wished.

(ii) Let $e < i$. By (5), we see that if $T \in \mathrm{Gin}(I_X)_{\ell+1}$ then $\max(T) \le e-1$. Then, from (6) it follows

$$\beta_{i,\ell}(R/\,\mathrm{Gin}(I_X)) = \sum_{T \in \mathcal{G}(\mathrm{Gin}(I_X))_{\ell+1}} \binom{\max(T)}{i-1} = 0 \quad \text{for all } i > e.$$

Hence, we get $\beta_{i,\ell}(X) = 0$ by (4). □

Theorem 1.1(b). *Let X be any closed subscheme of codimension e in $\mathbb{P}^{n+e}$ satisfying condition* $\mathrm{ND}(\ell)$ *for some $\ell \ge 1$ and let I_X be the (saturated) defining ideal of X. Then, the followings are all equivalent:*

(i) $\beta_{i,\ell}(X) = \binom{i+\ell-1}{\ell}\binom{e+\ell}{i+\ell}$ *for all* $1 \le i \le e$.

(ii) $\beta_{i,\ell}(X) = \binom{i+\ell-1}{\ell}\binom{e+\ell}{i+\ell}$ *for some* $1 \le i \le e$.

(iii) $\mathrm{Gin}(I_X) = (x_0, x_1, \dots, x_{e-1})^{\ell+1}$.

(iv) *X is an ACM variety with $(\ell+1)$-linear resolution.*

In this case, X has minimal degree, i.e., $\deg X = \binom{e+\ell}{\ell}$.

A proof of Theorem 1.1(b). (i) ⇒ (ii) This is trivial.

(ii) ⇒ (iii) Suppose that there exists an index i such that $1 \le i \le e$ and $\beta_{i,\ell}(X) = \binom{i+\ell-1}{\ell}\binom{e+\ell}{i+\ell}$. Recall that $J_0 = (x_0, \dots, x_{e-1})^{\ell+1}$ has the Borel fixed property. By (6), we have

$$\begin{aligned}\beta_{i,\ell}(R/J_0) &= \sum_{T \in \mathcal{G}(J_0)_{\ell+1}} \binom{\max(T)}{i-1} \\ &= \sum_{j=i-1}^{e-1} \binom{j}{i-1} |\{T \in \mathcal{G}(J_0)_{\ell+1} \mid \max(T) = j\}| \\ &= \sum_{j=i-1}^{e-1} \binom{j}{i-1} \dim_{\Bbbk} x_j \cdot \Bbbk[x_0, \dots, x_j]_\ell \\ &= \sum_{j=i-1}^{e-1} \binom{j}{i-1}\binom{j+\ell}{\ell}.\end{aligned}$$

Hence we see from (7) that the following binomial identity holds:

$$\binom{i+\ell-1}{\ell}\binom{e+\ell}{i+\ell} = \sum_{j=i-1}^{e-1} \binom{j}{i-1}\binom{j+\ell}{\ell}. \tag{8}$$

By the assumption that $\beta_{i,\ell}(X) = \binom{i+\ell-1}{\ell}\binom{e+\ell}{i+\ell} = \beta_{i,\ell}(R/J_0)$ and the binomial identity (8), we have

$$\begin{aligned}\beta_{i,\ell}(R/\operatorname{Gin}(I_X)) &= \sum_{T\in\mathcal{G}(\operatorname{Gin}(I_X))_{\ell+1}} \binom{\max(T)}{i-1}\\ &= \sum_{j=i-1}^{e-1}\binom{j}{i-1}|\{T\in\mathcal{G}(\operatorname{Gin}(I_X))_{\ell+1}\mid \max(T)=j\}|\\ &\le \sum_{j=i-1}^{e-1}\binom{j}{i-1}\dim_{\Bbbk} x_j\cdot\Bbbk[x_0,\dots,x_j]_\ell\\ &= \sum_{j=i-1}^{e-1}\binom{j}{i-1}\binom{j+\ell}{\ell}\\ &= \beta_{i,\ell}(R/I_X).\end{aligned}$$

Thus, by the cancellation principle (4), we conclude that $\beta_{i,\ell}(R/\operatorname{Gin}(I_X)) = \beta_{i,\ell}(R/I_X)$. This implies that, for each j with $i-1\le j\le e-1$,

$$\{T\in\mathcal{G}(\operatorname{Gin}(I_X))_{\ell+1}\mid \max(T)=j\} = x_j\cdot\Bbbk[x_0,\dots,x_j]_\ell.$$

In particular, when $j=e-1$, we obtain that $x_{e-1}^{\ell+1}\in\operatorname{Gin}(I_X)$ and it follows from Borel fixed property that

$$\operatorname{Gin}(I_X)_{\ell+1} = (J_0)_{\ell+1}.$$

Now, since X satisfies condition ND(ℓ), by Proposition 2.3 we have that $\operatorname{Gin}(I_X)\subset J_0$. Because J_0 is generated in degree $\ell+1$, this implies that $\operatorname{Gin}(I_X)=J_0$.

(iii) $\Rightarrow$ (iv) Note that if $\operatorname{Gin}(I_X) = (x_0,\dots,x_{e-1})^{\ell+1}$, then $R/\operatorname{Gin}(I_X)$ has ℓ-linear resolution. By cancellation principle [Green 1998, Corollary 1.12], the minimal free resolution of I_X is obtained from that of $\operatorname{Gin}(I_X)$ by canceling some adjacent terms of the same shift in the free resolution. This implies that the betti table of R/I_X are the same as that of $R/\operatorname{Gin}(I_X)$, because $R/\operatorname{Gin}(I_X)$ has ℓ-linear resolution. This means R/I_X is arithmetically Cohen–Macaulay with ℓ-linear resolution.

(iv) $\Rightarrow$ (i) This follows directly from [Eisenbud and Goto 1984, Proposition 1.7]. □

Remark 2.8. For the case of $\ell=1$, Theorem 1.1 was proved in [Han and Kwak 2015] for any nondegenerate variety X over any algebraically closed field (recall that every nondegenerate variety satisfies ND(1)). Thus, this theorem is a generalization of the previous result to cases of $\ell\ge 2$.

Further, we would also like to remark that for $\ell=1$ a given X satisfies all the consequences of Theorem 1.1(b) once the degree inequality $\deg X\ge\binom{e+\ell}{\ell}$ attains equality (i.e., the case of classical minimal degree), since they are all 2-regular and arithmetically Cohen–Macaulay. But, for higher $\ell\ge 2$, this is no more true (see Example 4.8). If one does hope to establish a "converse" in Theorem 1.1(b), then it is necessary to impose some additional conditions on components of those ND(ℓ)-schemes of "minimal degree of ℓ-th kind" (i.e., $\deg X=\binom{e+\ell}{\ell}$).

As a consequence of Theorem 1.1, using the upper bound for $\beta_{i,\ell}(X)$ we can obtain a generalization of a part of Green's $K_{p,1}$-theorem on the linear strand by quadrics of nondegenerate varieties in [Green 1984] to case of the first nontrivial linear strand by higher degree equations of any ND(ℓ)-schemes as follows.

Corollary 2.9 ($K_{p,\ell}$-theorem for ND(ℓ)-subscheme). *Let X be any closed subscheme of codimension e in $\mathbb{P}^{n+e}$ satisfying condition* ND(ℓ). *Then, $\beta_{i,j}(X) = 0$ for each $i > e$, $j \leq \ell$.*

Remark 2.10 (characteristic p case). Although we made the assumption that the base field $\Bbbk$ has characteristic zero at the beginning of this section, most of results in the section still hold outside of low characteristics; see [Eisenbud 1995, Theorem 15.23]. For instance, Theorem 1.1 holds for any characteristic p such that $p > \operatorname{reg}(I_X)$, where $\operatorname{reg}(I_X)$ is equal to the maximum of degrees of monomial generators in $\operatorname{Gin}(I_X)$ with respect to the degree reverse lexicographic order.

3. Property $\mathbf{N}_{d,p}$ and Syzygies

3A. *Geometry of property* $\mathbf{N}_{d,p}$. In this subsection, we assume that the base field $\Bbbk$ is algebraically closed of any characteristic. We obtain two geometric implications of property $\mathbf{N}_{d,p}$ via projection method and the elimination mapping cone sequence; see [Ahn and Kwak 2015; Han and Kwak 2015]. For the remaining of the paper, we call a reduced projective scheme $X \subset \mathbb{P}^N$ an *algebraic set*; see also [Eisenbud 2005, Chapter 5].

Theorem 3.1 (syzygetic Bézout theorem). *Let $X \subset \mathbb{P}^{n+e}$ be a nondegenerate algebraic set of dimension n satisfying $\mathbf{N}_{d,p}$ with $2 \leq d$ and $p \leq e$. Suppose that $L \subset \mathbb{P}^{n+e}$ is any linear space of dimension p whose intersection with X is zero-dimensional. Then*:

(a) $\operatorname{length}(L \cap X) \leq \binom{d-1+p}{p}$.

(b) *Moreover, if* $\operatorname{length}(L \cap X) = \binom{d-1+p}{p}$, *then for $1 \leq k \leq d-1$ the base locus of a linear system $|H^0(\mathcal{I}_{X/\mathbb{P}^{n+e}}(k))|$ contains the multisecant space L.*

Remark 3.2. We would like to make some remarks on this result as follows:

(a) If $p = 1$ then it is straightforward by Bézout's theorem. Thus, Theorem 3.1 can be regarded as a syzygetic generalization to multisecant linear spaces when $p \geq 2$.

(b) Note that in the theorem the length bound itself can be also obtained from [Eisenbud et al. 2005, Theorem 1.1]. We provide an alternative proof on it using geometric viewpoint of projection and further investigate the situation in which the equality holds.

Proof of Theorem 3.1. (a) It is obvious when $p = 1$. Now, let X be an algebraic set satisfying the property $\mathbf{N}_{d,p}$, $p \geq 2$ and suppose that $L \subset \mathbb{P}^{n+e}$ is a linear space of dimension p whose intersection with X is zero-dimensional.

Choose a linear subspace $\Lambda \subset L$ of dimension $p-1$ with homogeneous coordinates $x_0, x_1, \ldots, x_{p-1}$ such that $X \cap \Lambda = \varnothing$. Consider a projection $\pi_\Lambda : X \to \pi_\Lambda(X) \subset \mathbb{P}^{n+e-p}$. Then, $L \cap X$ is a fiber of π_Λ at the point $\pi_\Lambda(L \setminus \Lambda) \in \pi_\Lambda(X)$. The key idea is to consider the syzygies of R/I_X as an $S_p = \Bbbk[x_p, \dots, x_{n+e}]$-module which is the coordinate ring of $\mathbb{P}^{n+e-p}$. By [Ahn and Kwak 2015, Corollary 2.4], R/I_X satisfies $\mathbf{N}^{S_p}_{d,0}$ as an $S_p = \Bbbk[x_p, \dots, x_{n+e}]$-module, i.e., we have the following surjection

$$S_p \oplus S_p(-1)^p \oplus S_p(-2)^{\beta^{S_p}_{0,2}} \oplus \cdots \oplus S_p(-d+1)^{\beta^{S_p}_{0,d-1}} \xrightarrow{\varphi_0} R/I_X \to 0. \tag{9}$$

Sheafifying (9), we have

$$\cdots \to \mathcal{O}_{\mathbb{P}^{n+e-p}} \oplus \mathcal{O}_{\mathbb{P}^{n+e-p}}(-1)^p \oplus \mathcal{O}_{\mathbb{P}^{n+e-p}}(-2)^{\beta^{S_p}_{0,2}} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{n+e-p}}(-d+1)^{\beta^{S_p}_{0,d-1}} \xrightarrow{\widetilde{\varphi_p}} \pi_{\Lambda_*}\mathcal{O}_X \to 0.$$

Say $q = \pi_\Lambda(L \setminus \Lambda)$. By tensoring $\mathcal{O}_{\mathbb{P}^{n+e-p}}(d-1) \otimes \Bbbk(q)$, we have the surjection on vector spaces:

$$\Big[\bigoplus_{0\le i\le d-1} \mathcal{O}_{\mathbb{P}^{n+e-p}}(d-1-i)^{\beta^{S_p}_{0,i}}\Big] \otimes \Bbbk(q) \twoheadrightarrow H^0(\langle \Lambda, q\rangle, \mathcal{O}_{\pi_\Lambda{}^{-1}(q)}(d-1)). \tag{10}$$

Note that by [loc. cit., Corollary 2.5] $\beta^{S_p}_{0,i} \le \binom{p-1+i}{i} = h^0(\mathcal{O}_\Lambda(i))$ for $0 \le i \le d-1$ in (10). So we have

$$\dim_\Bbbk H^0(\langle \Lambda, q\rangle, \mathcal{O}_{\pi_\Lambda{}^{-1}(q)}(d-1)) = \operatorname{length}(L \cap X) \le \sum_{i=0}^{d-1} \beta^{S_p}_{0,i} \le \sum_{i=0}^{d-1} \binom{p-1+i}{i} = \binom{d-1+p}{p}.$$

(b) Now assume that $\operatorname{length}(L \cap X) = \binom{d-1+p}{p}$. From the above inequalities, we see that $\beta^{S_p}_{0,i} = \binom{p-1+i}{i}$ for every i. Hence the map in (10) is an isomorphism. Thus, there is no equation of degree $d-1$ vanishing on $\pi_\Lambda{}^{-1}(q) \subset L = \langle \Lambda, q\rangle$ (i.e., $H^0(\mathcal{I}_{\pi_\Lambda{}^{-1}(q)/L}(d-1)) = 0$). So, if $F \in H^0(\mathcal{I}_{X/\mathbb{P}^{n+e}}(k))$ for $2 \le k \le d-1$, then $F|_L$ vanishes on $\pi_\Lambda{}^{-1}(q) \subset L$ and this implies that $F|_L$ is identically zero. Thus, L is contained in $Z(F)$, the zero locus of F as we claimed. □

Now, we think of another effect of property $\mathbf{N}_{d,p}$ on loci of d-secant lines. For this purpose, let us consider an outer projection $\pi_q : X \to \pi_q(X) \subset \mathbb{P}^{n+e-1} = \operatorname{Proj}(S_1)$, $S_1 = \Bbbk[x_1, x_2, \dots, x_{n+e}]$ from a point $q = (1, 0, \dots, 0) \in (\operatorname{Sec} X \cup \operatorname{Tan} X) \setminus X$. We are going to consider the locus on X engraved by d-secant lines passing through q via partial elimination ideals (abbr. PEIs) theory as below.

When $f \in (I_X)_m$ has a leading term $\operatorname{in}(f) = x_0^{d_0} \cdots x_{n+e}^{d_{n+e}}$ in the lexicographic order, we set $d_{x_0}(f) = d_0$, the leading power of x_0 in f. Then it is well known (e.g., [Han and Kwak 2015, Section 2.1]) that $K_0(I_X) := \bigoplus_{m\ge 0}\{f \in (I_X)_m \mid d_{x_0}(f) = 0\} = I_X \cap S_1$ is the saturated ideal defining $\pi_q(X) \subset \mathbb{P}^{n+e-1}$.

Let us recall some definitions and basic properties of partial elimination ideals; see also, e.g., [Green 1998, Chapter 6] or [Han and Kwak 2015] for details.

Definition (partial elimination ideal). Let $I \subset R$ be a homogeneous ideal and let

$$\tilde{K}_i(I) = \bigoplus_{m\ge 0}\{f \in I_m \mid d_{x_0}(f) \le i\}.$$

If $f \in \tilde{K}_i(I)$, we may write uniquely $f = x_0^i \bar{f} + g$ where $d_{x_0}(g) < i$ and define $K_i(I)$ by the image of $\tilde{K}_i(I)$ in S_1 under the map $f \mapsto \bar{f}$. We call $K_i(I)$ the i-th partial elimination ideal of I.

Note that $K_0(I) = I \cap S_1$ and there is a short exact sequence as graded S_1-modules

$$0 \to \frac{\tilde{K}_{i-1}(I)}{\tilde{K}_0(I)} \to \frac{\tilde{K}_i(I)}{\tilde{K}_0(I)} \to K_i(I)(-i) \to 0. \tag{11}$$

In addition, we have the filtration on partial elimination ideals of I:

$$K_0(I) \subset K_1(I) \subset K_2(I) \subset \cdots \subset K_i(I) \subset \cdots \subset S_1 = \Bbbk[x_1, x_2, \ldots, x_{n+e}].$$

It is well-known that for $i \geq 1$, the i-th partial elimination ideal $K_i(I_X)$ set-theoretically defines

$$Z_{i+1} := \{y \in \pi_q(X) \mid \text{mult}_y(\pi_q(X)) \geq i+1\};$$

e.g., [Green 1998, Proposition 6.2]. Using this PEIs theory, we can describe the d-secant locus

$$\Sigma_d(X) := \{x \in X \mid {\pi_q}^{-1}(\pi_q(x)) \text{ has length d}\}$$

as a hypersurface F of degree d in the linear span $\langle F, q\rangle$ provided that X satisfies $\mathbf{N}_{d,2} (d \geq 2)$.

Theorem 3.3 (locus of d-secant lines). *Let $X \subset \mathbb{P}^{n+e}$ be a nondegenerate integral variety of dimension n satisfying $\mathbf{N}_{d,2} (d \geq 2)$. For a projection $\pi_q : X \to \pi_q(X) \subset \mathbb{P}^{n+e-1}$ where $q \in (\text{Sec}\, X \cup \text{Tan}\, X) \setminus X$, consider the d-secant locus $\Sigma_d(X)$. Then, we have*:

(a) *$\Sigma_d(X)$ is either empty or a hypersurface F of degree d in the linear span $\langle F, q\rangle$.*

(b) *$Z_d = \pi_q(\Sigma_d(X))$ is either empty or a linear subspace in $\pi_q(X)$ parametrizing the locus of d-secant lines through q.*

(c) *For a point $q \in \text{Sec}\, X \setminus (\text{Tan}\, X \cup X)$, there is a unique d-secant line through q if $Z_d \neq \varnothing$.*

Proof. (a) Since R/I_X satisfies $\mathbf{N}_{d,2}$, it also satisfies $\mathbf{N}_{d,1}$ as an S_1-module and we have the following exact sequence:

$$\to \cdots \to \bigoplus_{j=1}^{d-1} S_1(-1-j)^{\beta^{S_1}_{1,j}} \xrightarrow{\varphi_1} \bigoplus_{i=0}^{d-1} S_1(-i) \xrightarrow{\varphi_0} R/I_X \to 0.$$

Furthermore, $\ker \varphi_0$ is just $\tilde{K}_{d-1}(I_X)$ and we have a surjection

$$\cdots \to \bigoplus_{j=1}^{d-1} S_1(-1-j)^{\beta^{S_1}_{1,j}} \xrightarrow{\varphi_1} \tilde{K}_{d-1}(I_X) \to 0.$$

Therefore, $\tilde{K}_{d-1}(I_X)$ is generated by elements of at most degree d.

Now consider the following commutative diagram of S_1-modules with $K_0(I_X) = I_X \cap S_1$:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_0(I_X) & \longrightarrow & S_1 & \longrightarrow & S_1/K_0(I_X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \tilde{\alpha}} & & \\
0 & \longrightarrow & \tilde{K}_{d-1}(I_X) & \longrightarrow & \bigoplus_{i=0}^{d-1} S_1(-i) & \xrightarrow{\varphi_0} & R/I_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{K}_{d-1}(I_X)/K_0(I_X) & \longrightarrow & \bigoplus_{i=1}^{d-1} S_1(-i) & \longrightarrow & \operatorname{coker}\tilde{\alpha} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{12}
$$

From the left column sequences in the diagram (12), $\tilde{K}_{d-1}(I_X)/K_0(I_X)$ is also generated by at most degree d elements. On the other hands, we have a short exact sequence from (11)

$$0 \to \frac{\tilde{K}_{d-2}(I_X)}{K_0(I_X)} \to \frac{\tilde{K}_{d-1}(I_X)}{K_0(I_X)} \to K_{d-1}(I_X)(-d+1) \to 0, \tag{13}$$

Hence, $K_{d-1}(I_X)$ is generated by at most linear forms. So, Z_{d-1} is either empty or a linear space. Since $\pi_q : \Sigma_d(X) \twoheadrightarrow Z_d \subset \pi_q(X)$ is a $d:1$ morphism, $\Sigma_q(X)$ is a hypersurface of degree d in $\langle Z_{d-1}, q\rangle$. For a proof of (c), if $\dim \Sigma_d(X)$ is positive, then clearly, $q \in \operatorname{Tan}\ \Sigma_q(X) \subset \operatorname{Tan} X$. So, $\dim \Sigma_d(X) = \dim Z_d = 0$ and there is a unique d-secant line through q. □

In particular, in the case of $d = 2$, *entry locus* of X (i.e., locus of 2-secant lines through an outer point) is a quadric hypersurface, which was very useful to classify nonnormal del Pezzo varieties by Brodmann and Park [2010].

3B. ***Syzygetic rigidity for d-regularity.*** In particular, if $p = e$ then we have the following corollary of Theorem 3.1 with characterization of the extremal cases.

Corollary 3.4. *Let $X \subset \mathbb{P}^{n+e}$ be any nondegenerate algebraic set over an algebraically closed field $\Bbbk$ of characteristic zero. Suppose that X satisfies $\mathbf{N}_{d,e}$ for some $d \geq 2$. Then, we have*

$$\deg X \leq \binom{d-1+e}{e}$$

and the following are equivalent:

(a) $\deg X = \binom{d-1+e}{e}$.

(b) *X is arithmetically Cohen–Macaulay* (*ACM*) *with d-linear resolution.*

Proof. It suffices to show that (a) implies (b). By the assumption that $\deg X$ is maximal, $\operatorname{length}(L \cap X) = \binom{d-1+e}{e}$ for a generic linear space Λ of dimension e. From a proof of Theorem 3.1, we see that there is no equation of degree $d-1$ vanishing on $\pi_\Lambda{}^{-1}(q) \subset L = \langle \Lambda, q\rangle$ (i.e., $H^0(\mathcal{I}_{\pi_\Lambda{}^{-1}(q)/L}(d-1)) = 0$). This means X satisfies ND$(d-1)$ condition. In particular, it follows from Theorem 1.1(a) that $\beta_{e,d-1}(X) \le \binom{e+d-2}{d-1}$.

We also see from [Ahn and Kwak 2015, Corollary 2.4] that $\beta^{S_e}_{0,d-1} \le \beta^R_{e,d-1} = \beta_{e,d-1}(X)$ because X satisfies $\mathbf{N}_{d,e}$. Note that $\beta^{S_e}_{0,d-1} = \binom{e+d-2}{d-1} = h^0(\mathcal{O}_\Lambda(d-1))$ in (10). Therefore,

$$\beta_{e,d-1}(X) = \binom{e+d-2}{d-1}.$$

So, we conclude from Theorem 1.1(b) that X is ACM with d-linear resolution. □

Remark 3.5. The above corollary can also be proved by the generalized version of the multiplicity conjecture which was shown by Boij and Söderberg [2012]. Not relying on Boij–Söderberg theory, here we give a geometric proof for the multiplicity conjecture in this special case.

As a consequence of previous results, now we can derive a syzygetic rigidity for d-regularity as follows:

Theorem 1.2 (syzygetic rigidity for d-regularity). *Let $X \subset \mathbb{P}^{n+e}$ be any algebraic set of codimension e over an algebraically closed field $\Bbbk$ of* $\mathrm{ch}(\Bbbk) = 0$ *satisfying condition* ND$(d-1)$ *for some $d \ge 2$. If X has property $\mathbf{N}_{d,e}$, then X is d-regular* (*more precisely*, *X has ACM d-linear resolution*).

Proof. By Theorem 1.1 and Corollary 3.4, if X satisfies both condition ND$(d-1)$ and property $\mathbf{N}_{d,e}$, then the degree of X should be equal to $\binom{d-1+e}{e}$ and this implies that X has ACM d-linear resolution (in particular, X is d-regular). □

We would like to note that Theorem 1.2 does not hold without condition ND(ℓ) even though the given X is an *irreducible* variety.

Example 3.6 (syzygetic rigidity fails without condition ND(ℓ)). Let $\mathbf{N} = (d_0, \dots, d_s)$ be a strictly increasing sequence of integers and $\mathbb{B}(\mathbf{N})$ be the pure Betti table associated to $\mathbf{N}$; see [Boij and Söderberg 2012]. Due to Boij–Söderberg theory, we can construct a Betti table $\mathbb{B}_0$ as given by

	0	1	2	3	4
0	1	-	-	-	-
1	-	-	-	-	-
2	-	-	-	-	-
3	-	18	32	16	-
4	-	-	-	-	1

from the linear combination $\frac{4}{5}\mathbb{B}((0,4,5,6)) + \frac{1}{5}\mathbb{B}((0,4,5,6,8))$. This $\mathbb{B}_0$ expects a curve C of degree 16 and genus 13 in $\mathbb{P}^4$ with $h^1(\mathcal{O}_C(1)) = 1$ (i.e., $e = 3$), which satisfies property $\mathbf{N}_{4,e}$, but not 4-regular (i.e., Theorem 1.2 fails). This Betti table can be realized as the one of a projection C into $\mathbb{P}^4$ of a canonically embedded genus 13 general curve $\widetilde{C} \subset \mathbb{P}^{12}$ from random 8 points of $\widetilde{C}$. Note that C is irreducible (in fact, smooth) and has no defining equations of degree less than 4, but is not ND(3)-curve because $\deg(C) = 16 \ngeq \binom{3+3}{3} = 20$. Here is a `Macaulay 2` code for this:

```
loadPackage("RandomCanonicalCurves",Reload=>true);
setRandomSeed("alpha");
g=13; k=ZZ/32003;
S=k[x_0..x_(g-1)];
I=(random canonicalCurve)(g,S);
for i from 0 to 7 do P_i=randomKRationalPoint I;
L=intersect apply(8,i->P_i); R=k[y_0..y_4];
f=map(S,R,super basis(1,L));
RI=preimage(f,I); betti res RI
```

4. Comments and further questions

In the final section, we present some relevant examples and discuss a few open questions related to our main results in this paper.

4A. ***Certificates of condition* ND(ℓ).** First of all, from the perspective of this article, it would be very interesting to provide more situations to guarantee condition ND(ℓ). As one way of thinking, one may ask where condition ND(ℓ) does hold largely. For instance, as discussed in Remark 2.5, we can consider this problem as follows:

Question 4.1. For given $e, \ell > 0$, is there a function $f(e, \ell)$ such that any $X \subset \mathbb{P}^{n+e}$ of codimension e is ND(ℓ)-subscheme if $\deg X > f(e, \ell)$ and $H^0(\mathcal{J}_{X/\mathbb{P}^{n+e}}(\ell)) = 0$?

We showed that there are positive answers for this question in case of codimension two in Proposition 2.6 and Example 2.7. What about in *higher codimensional* case? (Recall that a key ingredient for Proposition 2.6 is "lifting theorem" which is well-established in codimension 2.)

The following example tells us that for Question 4.1 one needs to assume irreducibility or some conditions on irreducible components of X in general.

Example 4.2 (a non-ND(2) reduced scheme of arbitrarily large degree). Consider a closed subscheme $X \subset \mathbb{P}^3$ of codimension 2 defined by the monomial ideal $I_X = (x_0^3, x_0^2x_1, x_0x_1^2, x_1^t, x_0^2x_2)$ for any positive integer $t \geq 4$. Note that $h^0(\mathcal{J}_{X/\mathbb{P}^3}(2)) = 0$ and $\deg X = t + 2 \geq 6 = \binom{e+2}{2}$. Since I_X is a Borel fixed monomial ideal, we see that $I_{X \cap L/\mathbb{P}^3} = (x_0^2, x_0x_1^2, x_1^t)$ for a general linear form L, which implies that X does not satisfy ND(2).

If we consider a sufficiently generic distraction $D_{\mathcal{L}}(I_X)$ of I_X (see [Bigatti et al. 2005] for details of distraction), then it is of the form

$$D_{\mathcal{L}}(I_X) = \left(L_1L_2L_3, L_1L_2L_4, L_1L_4L_5, \prod_{j=1}^{t} M_j, L_1L_2L_7\right),$$

where L_i and M_j are generic linear forms for each $1 \leq i \leq 7$ and $1 \leq j \leq t$. Then $D_{\mathcal{L}}(I_X)$ defines the union of $t+2$ lines and 3 points. Using this, we can construct an example of non-ND(2) algebraic set of arbitrarily large degree.

4B. ***Condition* ND(ℓ) *and nonnegativity of h-vector.*** For any closed subscheme $X \subset \mathbb{P}^{n+e}$ of dimension n, the Hilbert series of $R_X := \Bbbk[x_0, \dots, x_{n+e}]/I_X$ can be written as

$$H_{R_X}(t) = \sum (\dim_{\Bbbk}(R_X)_i) t^i = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1-t)^{n+1}} \tag{14}$$

and the h-vector $h_0, h_1, \dots, h_s$ usually contains much information on the coordinate ring R_X and on geometric properties of X. One of the interesting questions on the h-vector is the one to ask about *nonnegativity* of the h_i and it is well-known that every $h_i \geq 0$ if R_X is Cohen–Macaulay (i.e., X is ACM). Recently, a relation between Serre's condition (S_ℓ) on R_X and nonnegativity of h-vector has been focused as answering such a question as:

Does Serre's condition (S_ℓ) *imply* $h_0, h_1, \dots, h_\ell \geq 0$?

This was checked affirmatively in case of I_X being a square-free monomial ideal by Murai and Terai [2009]. More generally, Dao, Ma and Varbaro [Dao et al. 2019] proved the above question is true under some mild singularity conditions on X (to be precise, X has Du Bois singularity in $\mathrm{ch}(\Bbbk) = 0$ or R_X is F-pure in $\mathrm{ch}(\Bbbk) = p$). Here, we present an implication of condition ND(ℓ) on this question as follows.

Corollary 4.3 (ND(ℓ) implies nonnegativity of h-vector). *Let* $X = \mathrm{Proj}(R_X)$ *be any closed subscheme of codimension e in* $\mathbb{P}^{n+e}$ *over an algebraically closed field* $\Bbbk$ *with* $\mathrm{ch}(\Bbbk) = 0$ *and* h_i*'s be the h-vector of* R_X *in* (14). *Suppose that X has condition* ND($\ell - 1$). *Then,* $h_0, h_1, \dots, h_\ell \geq 0$.

Proof. Say $r_i = \dim_{\Bbbk}(R_X)_i$. First of all, by (14), we have

$$(1-t)^{n+1}(r_0 + r_1 t + r_2 t^2 + \cdots) = h_0 + h_1 t + h_2 t^2 + \cdots,$$

which implies that $h_0 = r_0, h_1 = \binom{n+1}{1}(-1)r_0 + r_1, \dots, h_j = \sum_{i=0}^{j} \binom{n+1}{i}(-1)^i r_{j-i}$ for any j. Since $r_{j-i} = \binom{n+e+j-i}{j-i} - \dim_{\Bbbk}(I_X)_{j-i}$, it holds that

$$\begin{aligned} h_j &= \sum_{i=0}^{j} \binom{n+1}{i}(-1)^i \binom{n+e+j-i}{j-i} - \sum_{i=0}^{j} \binom{n+1}{i}(-1)^i \dim_{\Bbbk}(I_X)_{j-i} \\ &= \binom{e+j-1}{j} - \sum_{i=0}^{j} \binom{n+1}{i}(-1)^i \dim_{\Bbbk}(I_X)_{j-i} \cdots, \end{aligned} \tag{$*$}$$

where the last equality comes from comparing j-th coefficients in both sides of the identity

$$(1-t)^{n+1}\left[\sum_{i \geq 0} \binom{n+e+i}{i} t^i\right] = \frac{1}{(1-t)^e}.$$

Now, by Theorem 1.1(a), we know that $\dim_{\Bbbk}(I_X)_0 = \dim_{\Bbbk}(I_X)_1 = \cdots = \dim_{\Bbbk}(I_X)_{\ell-1} = 0$ and $\dim_{\Bbbk}(I_X)_\ell \leq \binom{e+\ell-1}{\ell}$. So, for any $j \leq \ell - 1$, by $(*)$ we see that $h_j = \binom{e+j-1}{j} \geq 0$. Similarly, we obtain that $h_\ell = \binom{e+\ell-1}{\ell} - \dim_{\Bbbk}(I_X)_\ell \geq 0$ as we wished. □

Hence, it is natural to ask:

Question 4.4. How are Serre's (S_ℓ) on R_X and condition ND(ℓ) on X related to each other?

For example, it would be nice if one could find some implications between the notions under reasonable assumptions on singularities or connectivity of components.

4C. ***Geometric classification/characterization of ACM d-linear varieties.*** For further development, it is natural and important to consider the boundary cases in Theorem 1.1 from a *geometric* viewpoint. When $\ell = 1$, due to del Pezzo–Bertini classification, we completely understand the extremal case, that is ACM 2-linear varieties, geometrically; (a cone of) quadric hypersurface, Veronese surface in $\mathbb{P}^5$ or rational normal scrolls. It is also done in category of algebraic sets in [Eisenbud et al. 2006]. What about ACM varieties having 3-*linear* resolution? or *higher d-linear* resolution? The followings are first examples of variety with ACM 3-linear resolution.

Example 4.5 (varieties having ACM 3-linear resolution). We have:

(a) Cubic hypersurface ($e = 1$).

(b) 3-minors of 4×4 generic symmetric matrix (i.e., the secant line variety $Sec(\upsilon_2(\mathbb{P}^3)) \subset \mathbb{P}^9$).

(c) 3-minors of $3 \times (e+2)$ sufficiently generic matrices (e.g., secant line varieties of rational normal scrolls).

(d) $\mathrm{Sec}(\upsilon_3(\mathbb{P}^2))$; $\mathrm{Sec}(\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1)$.

Most of above examples come from taking secants. Unless a hypersurface, are they all the secant varieties of relatively small degree varieties? Recall that any secant variety Sec X not equal to the ambient space is always "singular" because $\mathrm{Sing}(\mathrm{Sec}\, X) \supset X$. But, we can construct examples of *smooth* 3-linear ACM of low dimension as follows:

Example 4.6 (nonsingular varieties with ACM 3-linear resolution). We have:

(a) (A nonhyperelliptic low degree curve of genus 3 in $\mathbb{P}^3$) For a smooth plane quartic curve C of genus $g = 3$. One can reembed C into $\mathbb{P}^9$ using the complete linear system $|\mathcal{O}_C(3)|$. Say this image as $\widetilde{C}$. For $\deg \widetilde{C} = 12$, $\widetilde{C} \subset \mathbb{P}^9$ satisfies at least property $\mathbf{N}_5$ by the Green–Lazarsfeld theorem. We also know that

$$H^0(\mathcal{I}_{\widetilde{C}}(2)) = H^0(\mathcal{O}_{\mathbb{P}^9}(2)) - H^0(\mathcal{O}_{\widetilde{C}}(2)) = \binom{9+2}{2} - (2 \cdot 12 + 1 - 3) = 55 - 22 = 33.$$

Now, take any 6 smooth points on $\widetilde{C}$ and consider inner projection of $\widetilde{C}$ from these points into $\mathbb{P}^3$. Denote this image curve in $\mathbb{P}^3$ by $\overline{C}$. From [Han and Kwak 2012, Proposition 3.6], we obtain that

$$H^0(\mathcal{I}_{\overline{C}}(2) = H^0(\mathcal{I}_{\widetilde{C}}(2)) - (8+7+6+5+4+3) = 33 - 33 = 0.$$

In other words, there is no quadric which cuts out $\overline{C}$ in $\mathbb{P}^3$. Since C is nonhyperelliptic, $\overline{C}$ is projectively normal (i.e., ACM). Therefore, $\overline{C}$ is a smooth ND(2)-curve in $\mathbb{P}^3$ and has $\deg \overline{C} = 6$ which is equal to $2g$. Using `Macaulay 2` [Macaulay2], we can also check all these computations including the minimal resolution of $\overline{C} \subset \mathbb{P}^3$. $\overline{C}$ has ACM 3-linear resolution such as:

	0	1	2
0	1	-	-
1	-	-	-
2	-	4	3

(b) (A surface in $\mathbb{P}^6$) Consider a rational normal surface scroll $X = S(4, 4)$ in $\mathbb{P}^9$. Its secant line variety $Y = \operatorname{Sec} X$ is a 5-fold and has a minimal free resolution as

	0	1	2	3	4
0	1	-	-	-	-
1	-	-	-	-	-
2	-	20	45	36	10

which is ACM 3-linear. Even though Y is singular, as we cut Y by three general hyperplanes H_1, H_2, H_3 we obtain a smooth surface $S = Y \cap H_1 \cap H_2 \cap H_3$ of degree 15 in $\mathbb{P}^6$ whose resolution is same as above; one can check all the computations using [Macaulay2].

It is interesting to observe that every variety of dimension ≥ 2 in Examples 4.5 and 4.6 has a *determinantal* presentation for its defining ideal.

Question 4.7. Can we give a geometric classification or characterization of ACM d-linear varieties for $d \geq 3$? Do they all come from (a linear section of) secant construction except very small (co)dimension? In particular, does it always have a determinantal presentation if X is ACM 3-linear variety and $\dim X \geq 2$?

Finally, we present some example as we discussed in Remark 2.8.

Example 4.8 (minimal degree of ℓ-th kind ($\ell \geq 2$) does not guarantee ACM linear resolution). In contrast with $\ell = 1$ case, a converse of Theorem 1.1(b):

The equality $\deg X = \binom{e+\ell}{\ell}$ *with* $\mathrm{ND}(\ell)$ *implies that* X *has ACM* $(\ell+1)$*-linear resolution,*

does not hold for $\ell \geq 2$ (note that, in the case of classical minimal degree, the statement does hold under ND(1)-condition once we assume irreducibility or some connectivity condition on components of X such as "linearly joined" in [Eisenbud et al. 2006]).

By manipulating Gin ideals and distraction method, one could generate many reducible examples of such kind. Even though X is irreducible, we can construct a counterexample. As a small example, using [Macaulay2] we can verify that a smooth rational curve C in $\mathbb{P}^3$ of degree 6, a (isomorphic) projection of a rational normal curve in $\mathbb{P}^6$ from 3 random points, has Betti table as in Figure 3.

Note that C satisfies condition ND(2) and is of minimal degree of 2nd kind $\big($i.e., $\deg(C) = \binom{2+2}{2}\big)$, but its resolution is still not 3-linear.

	0	1	2	3
0	1	-	-	-
1	-	-	-	-
2	-	1	-	-
3	-	6	9	3

	0	1	2
0	1	-	-
1	-	-	-
2	-	4	3
3	-	-	-

Figure 3. Betti tables of C and $C \cap H$.

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Spectral reciprocity via integral representations

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We prove a spectral reciprocity formula for automorphic forms on GL(2) over a number field that is reminiscent of one found by Blomer and Khan. Our approach uses period representations of L-functions and the language of automorphic representations.

1. Introduction

In the past few years, some attention has been given to spectral reciprocity formulae. By this we mean identities of the shape

$$\sum_{\pi\in\mathcal{F}}\mathcal{L}(\pi)\mathcal{H}(\pi)=\sum_{\pi\in\widetilde{\mathcal{F}}}\widetilde{\mathcal{L}}(\pi)\widetilde{\mathcal{H}}(\pi), \tag{1}$$

where $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are families of automorphic representations, $\mathcal{L}(\pi)$ and $\widetilde{\mathcal{L}}(\pi)$ are certain L-values associated to π, and $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are some weight functions.

The term *spectral reciprocity* first appeared in this context in a paper by Blomer, Li and Miller [Blomer et al. 2019] but such identities have been around at least since Motohashi's formula [1993] connecting the fourth moment of the Riemann zeta-function to the cubic moment of L-functions of cusp forms for GL(2).

The more recent results concern the cases where the families $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are the same or nearly the same. Most commonly, these families are taken to be formed by automorphic representations of GL(2).

There are at least two reasons that help understand the appeal of such formulae. The first one is that they give a somewhat conceptual way of summarizing a technique often used in dealing with problems on families of GL(2) L-functions in which one uses the Kuznetsov formula on both directions in order to estimate a moment of L-values. The second one comes from their satisfying intrinsic nature relating objects that have no a priori reason to be linked.

The first versions of these GL(2) spectral reciprocity formulae [Blomer and Khan 2019a; 2019b; Andersen and Kıral 2018] used classical techniques such as the Voronoi summation formula and the Kuznetsov formula. Starting from [Zacharias 2021], it became clear that an adelic approach could be of interest. Not only does this render generalization to number fields almost immediate, it can also avoid some of the combinatorial difficulties that arise when applying the Voronoi formula.

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Blomer and Khan [2019a] have shown a reciprocity formula which is the main inspiration for the present work: Let Π be a fixed automorphic representation of GL(3) over $\mathbb{Q}$. Let q and ℓ be coprime integers. We write

$$\mathcal{M}(q,\ell;h) := \frac{1}{q}\sum_{\mathrm{cond}(\pi)=q} \frac{L\left(\frac{1}{2},\Pi\times\pi\right)L\left(\frac{1}{2},\pi\right)}{L(1,\mathrm{Ad},\pi)}\frac{\lambda_\pi(\ell)}{\ell^{1/2}}h(t_\pi)+(\cdots),$$

where

- π runs over cuspidal automorphic representations of PGL(2),
- $\lambda_\pi(\ell)$ is the eigenvalue of the Hecke operator T_ℓ on π,
- t_π is the spectral parameter,
- h is a *fairly general* smooth function, and
- $(\cdots)$ denotes the contribution of the Eisenstein part, the terms of lower conductor and some degenerate terms.

Blomer and Khan have showed that

$$\mathcal{M}(q,\ell,h) = \mathcal{M}(\ell,q,\check{h}),$$

where $h \mapsto \check{h}$ is given by an explicit integral transformation. When Π corresponds to an Eisenstein series, this has an application to subconvexity: Let π be a cuspidal automorphic representation for GL(2) over $\mathbb{Q}$ of *squarefree* conductor. Then

$$L\left(\tfrac{1}{2},\pi\right) \ll_\epsilon (\mathrm{cond}(\pi))^{\frac{1}{4}-\frac{1}{24}(1-2\vartheta)+\epsilon}, \tag{2}$$

where ϑ is an admissible exponent towards the Ramanujan conjecture (we know that $\frac{7}{64}$ is admissible and $\vartheta = 0$ corresponds to the conjecture). This was then the best-known bound of its kind but it was later superseded by the one in [Blomer et al. 2020].

In this article we use the theory of adelic automorphic representations and integral representations of Rankin–Selberg L-functions to deduce a result on number fields of similar flavor to that of [Blomer and Khan 2019a, Theorem 1].

With respect to Blomer and Khan's result, our result has the advantage of being valid for any number field. On the other hand we need to make some technical restrictions that prevent us from having a full generalization of their reciprocity formula. For the moment our results only work when the fixed GL(3) form is cuspidal and our formula only contemplates forms that are spherical at every infinity place. The first restriction is made for analytic reasons and is due to the fact that unlike cusp forms, the Eisenstein series are not of rapid decay. This can probably be resolved by means of a suitable notion of regularized integrals. As for the second restriction, this seems to be of a more representation-theoretic nature. It requires showing analyticity of certain local factors for nonunitary representations of GL(2). We hope to address both of these technical issues in future work.

1A. ***Statement of results.*** Let F be a number field, with ring of integers $\mathfrak{o}_F$. Let Π be a cuspidal automorphic representation of GL(3) over F. For each automorphic representation π of GL(2), we consider the completed L-functions

$$\Lambda(s,\pi), \quad \Lambda(s,\mathrm{Ad},\pi) \quad \text{and} \quad \Lambda(s,\Pi\times\pi).$$

These are, respectively, the Hecke L-function and the adjoint L-function of π, and the Rankin–Selberg L-function of $\Pi\times\pi$, where for the Rankin–Selberg L-functions we take the naive definition (20). These coincide with the local L-functions à la Langlands at all the unramified places but might differ at the ramified ones. Notice that this might also affect the values of $L(s,\mathrm{Ad},\pi)$.

Let ξ_F denote the completed Dedekind zeta function of F, and let $\xi_F^*(1)$ denote its residue at 1. Let $\Phi \simeq \bigotimes_v \Phi_v$ be a vector in the representation space of Π. Let s and w be complex numbers, and let H denote the weight function given by (29). We consider the sums

$$\mathcal{C}_{s,w}(\Phi) := \sum_{\pi\in C(S)} \frac{\Lambda(s,\Pi\times\pi)\Lambda(w,\pi)}{\Lambda(1,\mathrm{Ad},\pi)} H(\pi)$$

and

$$\mathcal{E}_{s,w}(\Phi) := \sum_{\omega\in\Xi(S)} \int_{-\infty}^{\infty} \frac{\Lambda(s,\Pi\times\pi(\omega,it))\Lambda(w,\pi(\omega,it))}{\Lambda^*(1,\mathrm{Ad},\pi(\omega,it))} H(\pi(\omega,it)) \frac{\mathrm{d}t}{2\pi}, \tag{3}$$

where S is any finite set of places containing all the archimedean ones and those for which Φ_v is ramified, $C(S)$ (resp. $\Xi(S)$) denotes the collection of cuspidal automorphic representations of GL(2) (resp. unitary normalized idele characters) over F that are unramified everywhere outside S. Finally, $\pi(\omega,it)$ denotes a normalized induced representation as in Section 3A1 and $\Lambda^*(1,\mathrm{Ad},\pi)$ denotes the first nonzero Laurent coefficient of $\Lambda(s,\mathrm{Ad},\pi)$ at $s=1$. The main object of study in this work is the following "moment":

$$\mathcal{M}_{s,w}(\Phi) := \mathcal{C}_{s,w}(\Phi) + \mathcal{E}_{s,w}(\Phi). \tag{4}$$

We remark that the values of $\Lambda(s,\Pi\times\pi(\omega,it))$, $\Lambda(w,\pi(\omega,it))$ and $\Lambda^*(1,\mathrm{Ad},\pi(\omega,it))$ can be given in terms of simpler L-functions as follows:

$$\begin{aligned}
\Lambda(s,\Pi\times\pi(\omega,it)) &= \Lambda(s+it,\Pi\times\omega)\Lambda(s-it,\Pi\times\bar{\omega}),\\
\Lambda(w,\pi(\omega,it)) &= \Lambda(w+it,\omega)\Lambda(w-it,\bar{\omega}),\\
\Lambda^*(1,\mathrm{Ad},\pi(\omega,it)) &= \mathrm{Res}_{s=1}[\Lambda(s+2it,\omega^2)\Lambda(s-2it,\bar{\omega}^2)\xi_F(s)]\\
&= \Lambda(1+2it,\omega^2)\Lambda(1-2it,\bar{\omega}^2)\xi_F^*(1) \qquad (t\neq 0),
\end{aligned}$$

where $\Lambda(s,\Pi\times\omega)$ and $\Lambda(s,\omega)$ are the (completed) Rankin–Selberg L-function of $\Pi\times\omega$ and Dirichlet L-function of ω, respectively.

We start with the following result which can be seen as a preliminary reciprocity formula.

Theorem 1.1. *Let $s,w\in\mathbb{C}$ and define*

$$(s',w') := \left(\tfrac{1}{2}(1+w-s), \tfrac{1}{2}(3s+w-1)\right). \tag{5}$$

Let H be as in (29) *and $\breve{H}$ be given by* (31). *Suppose the real parts of s, w, s' and w' are sufficiently large. Then we have the relation*

$$\mathcal{M}_{s,w}(\Phi)+\mathcal{D}_{s,w}(\Phi)=\mathcal{M}_{s',w'}(\breve{\Phi})+\mathcal{D}_{s',w'}(\breve{\Phi}),$$

where $\mathcal{D}_{s,w}(\Phi)$ is given by (30).

Theorem 1.1 is a completely symmetrical formula but only holds when the real part of the parameters s, w, s' and w' are sufficiently large. In order to obtain a formula that also holds at the central point $s=w=s'=w'=\frac{1}{2}$, we need to analytically continue the term $\mathcal{E}_{s,w}(\Phi)$. This is done in Section 9 under a technical condition enclosed in Hypothesis 1.

Spectral reciprocity at the central point. Let Π be an everywhere unramified cuspidal automorphic representation for GL(3) over F. This means that $\Pi\simeq\bigotimes_v'\Pi_v$, where for each v, Π_v is isomorphic to the isobaric sum

$$|\cdot|_v^{it_{1,v}}\boxplus|\cdot|_v^{it_{2,v}}\boxplus|\cdot|_v^{it_{3,v}}.$$

We say that Π is θ-tempered if for all v and $i=1,2,3$, we have $|\mathrm{Re}(t_{i,v})|\leq\theta$. It follows from a result of Luo, Rudnick and Sarnak [Luo et al. 1999] that every automorphic representation of GL(n) is θ-tempered for some $\theta<\frac{1}{2}$. Therefore we can, and will, let $\theta=\theta(\Pi)<\frac{1}{2}$ be such that Π is θ-tempered.

Suppose that Φ_v is spherical for every archimedean place v. The reason for this restriction is twofold. The first and main reason is that this leads to weight functions satisfying Hypothesis 1. The second is that this trivializes the transformation $H_v\to\breve{H}_v$ on the local archimedean weights. It would be very interesting to have a better understanding of this transformation. In particular it would be interesting to have an understanding of $\breve{H}_v$ when H_v is taken to be a bump function selecting spectral parameters of a certain size.

Let $s,w\in\mathbb{C}$, let $\mathfrak{q}$ and $\mathfrak{l}$ be coprime ideals with absolute norms q and ℓ, respectively, and write $U_\infty:=\prod_{v|\infty}\{y\in F_v^\times:|y_v|=1\}$. Suppose that $\Phi=\Phi^{\mathfrak{q},\mathfrak{l}}$ is the vector given in Section 7. It follows from (34) and Propositions 7.1 and 7.2 that

$$H(\pi)=\delta_\infty(\pi)\frac{\widehat{\lambda}_\pi(\mathfrak{l},w)}{(N\mathfrak{l})^w}\frac{\varphi(q)}{q^2}h_{\mathfrak{q}}(s,w;\Pi,\pi),$$

where $\delta_\infty(\pi)$ is the characteristic function of representations that are unramified at every archimedean place, $\widehat{\lambda}_\pi(\mathfrak{l},w)$ are modified Hecke eigenvalues given by

$$\widehat{\lambda}_\pi(\mathfrak{l},w):=\prod_{\substack{\mathfrak{p}^{n_\mathfrak{p}}\|\mathfrak{l}\\ n_\mathfrak{p}\geq 1}}\left(\lambda_\pi(\mathfrak{p}^{n_\mathfrak{p}})-\frac{\lambda_\pi(\mathfrak{p}^{n_\mathfrak{p}-1})}{(N\mathfrak{p})^w}\right),\tag{6}$$

φ is the Euler function, and finally, $h_{\mathfrak{q}}(s,w;\Pi,\pi)=1$ if $\mathrm{cond}(\pi)=\mathfrak{q}$, $h_{\mathfrak{q}}(s,w;\Pi,\pi)\ll q^{\theta+\epsilon}$ if $\mathrm{cond}(\pi)\mid\mathfrak{q}$, and it vanishes otherwise. Thus, choosing Φ as above, we get

$$\mathcal{M}_{s,w}(\Phi)=\mathcal{M}_0(\Pi,s,w,\mathfrak{q},\mathfrak{l}),$$

where

$$\mathcal{M}_0(\Pi, s, w, \mathfrak{q}, \mathfrak{l}) := \mathcal{C}_0(\Pi, s, w, \mathfrak{q}, \mathfrak{l}) + \mathcal{E}_0(\Pi, s, w, \mathfrak{q}, \mathfrak{l}),$$

with

$$\mathcal{C}_0(\Pi, s, w, \mathfrak{q}, \mathfrak{l}) = \frac{\varphi(q)}{q^2} \sum_{\substack{\pi \text{ cusp}^0 \\ \text{cond}(\pi)|\mathfrak{q}}} \frac{\Lambda(s, \Pi \times \pi)\Lambda(w, \pi)}{\Lambda(1, \text{Ad}, \pi)} \frac{\widehat{\lambda}_\pi(\mathfrak{l}, w)}{\ell^w} h_{\mathfrak{q}}(s, w; \Pi, \pi)$$

and

$$\mathcal{E}_0(\Pi, s, w, \mathfrak{q}, \mathfrak{l}) = \frac{\varphi(q)}{q^2} \sum_{\substack{\omega \in \widehat{F^\times U_\infty \backslash \mathbb{A}^\times_{(1)}} \\ \text{cond}(\omega)^2|\mathfrak{q}}} \int_{-\infty}^{\infty} \frac{\Lambda(s, \Pi \times \pi(\omega, it))\Lambda(w, \pi(\omega, it))}{\Lambda^*(1, \text{Ad}, \pi(\omega, it))} \times \frac{\widehat{\lambda}_{\pi(\omega,it)}(\mathfrak{l}, w)}{\ell^w} h_{\mathfrak{q}}(s, w; \Pi, \pi(\omega, it)) \frac{dt}{2\pi}.$$

The notation cusp^0 denotes that we are restricting to forms that are unramified at every archimedean place and the analogous role in the Eisenstein part is played by quotienting by U_∞. Finally, we let

$$\mathcal{N}_0(\Pi, s, w, \mathfrak{q}, \mathfrak{l}) := \mathcal{D}_{s',w'}(\check{\Phi}) + \mathcal{R}_{s',w'}(\check{\Phi}) - \mathcal{D}_{s,w}(\Phi) - \mathcal{R}_{s,w}(\Phi),$$

where $\mathcal{D}$ is given by (30) and $\mathcal{R}$ is given by (53).

Theorem 1.2. *Let* Π *be an everywhere unramified cuspidal automorphic representation of* GL(3) *over* F. *Suppose* $\mathfrak{q}$ *and* $\mathfrak{l}$ *are coprime ideals with absolute norms* q *and* ℓ, *respectively, and that* $\frac{1}{2} \le \text{Re}(s) \le \text{Re}(w) < \frac{3}{4}$. *Then we have*

$$\mathcal{M}_0(\Pi, s, w, \mathfrak{q}, \mathfrak{l}) = \mathcal{N}_0(\Pi, s, w, \mathfrak{q}, \mathfrak{l}) + \mathcal{M}_0(\Pi, s', w', \mathfrak{l}, \mathfrak{q}),$$

where s' *and* w' *are as in* (5). *Moreover, in this same region,* $\mathcal{N}_0$ *satisfies*

$$\mathcal{N}_0(\Pi, s, w, \mathfrak{q}, \mathfrak{l}) \ll_{s,w,\epsilon} \min(q, \ell)^{\theta-1+\epsilon}. \tag{7}$$

As an application, we may deduce a nonvanishing result which is similar in spirit to [Khan 2012, Theorem 1.2]: we prove an asymptotic formula for a family of forms of prime level $\mathfrak{p}$, and let $N\mathfrak{p}$ tend to infinity. It may be worth mentioning that although the results are similar, Khan's result concerns modular forms of sufficiently large weight k for GL(2) over the field of rationals, while our result holds for everywhere unramified forms for GL(2) over an arbitrary number field.

Corollary 1.3. *Let* Π *be an unramified cuspidal automorphic representation of* GL(3) *over* F, *and let* $\mathfrak{p}$ *be a prime ideal of* $\mathfrak{o}_F$ *with absolute norm* q. *Then, for every* $\epsilon > 0$,

$$\frac{\varphi(q)}{q} \sum_{\substack{\pi \text{ cusp}^0 \\ \text{cond}(\pi)=\mathfrak{p}}} \frac{\Lambda\left(\frac{1}{2}, \Pi \times \pi\right)\Lambda\left(\frac{1}{2}, \pi\right)}{\Lambda(1, \text{Ad}, \pi)} = \frac{4\Lambda(1, \Pi)\Lambda(0, \Pi)}{\xi_F(2)} + O_\epsilon(q^{\vartheta-\frac{1}{2}+\epsilon}).$$

In particular, for q *sufficiently large, there is at least one automorphic representation* π *of conductor* $\mathfrak{p}$, *unramified for every archimedean place and such that* $\Lambda\left(\frac{1}{2}, \Pi \times \pi\right)$ *and* $\Lambda\left(\frac{1}{2}, \pi\right)$ *are both nonzero.*

Plan of the paper. In Section 2 we lay down our first conventions on number fields and local fields. In Section 3 we recall the notion of automorphic representations for $\mathrm{GL}(n)$ over F and some of its properties. Special attention is given to the case $n = 2$ where, in particular, we recall the construction of Eisenstein series and write down an explicit spectral decomposition. In Section 4, we introduce the Whittaker models and their relation to periods of Rankin–Selberg L-functions for $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$. We work in complete generality but only use the results in the cases $n = 1$ and $n = 2$.

In Section 5 we prove an identity between periods which we call abstract reciprocity. This is connected to the actual reciprocity via a spectral decomposition which is performed in Section 6. In Section 7 we make some explicit computation for the local weights. Section 8 is dedicated to analyzing the degenerate term $\mathcal{D}_{s,w}(\Phi)$ and we show the meromorphic continuation of the spectral moment in Section 9, thus introducing the term $\mathcal{R}_{s,w}(\Phi)$. Theorem 1.1 only uses the results up to Section 6 and a few observations from Section 8. On the other hand, Theorem 1.2 requires the full power of the results in Sections 7, 8 and 9 and its proof is given in Section 10 along with that of Corollary 1.3.

2. Notation

Number fields and completions. Throughout the paper, F will denote a fixed number field with ring of integers $\mathfrak{o}_F$ and discriminant d_F. For v a place of F, we let F_v be the completion of F at the place v. If v is nonarchimedean, we write $\mathfrak{o}_v$ for the ring of integers in F_v, $\mathfrak{m}_v$ for its maximal ideal and ϖ_v for its uniformizer. The adele ring of F is denoted by $\mathbb{A}$, its unit group is denoted by $\mathbb{A}^\times$, and finally, $\mathbb{A}^\times_{(1)}$ denotes the ideles of norm 1. We also fix, once and for all, an isomorphism $\mathbb{A}^\times \simeq \mathbb{A}^\times_{(1)} \times \mathbb{R}_{>0}$.

Additive characters. We let $\psi = \bigotimes_v \psi_v$ be the additive character $\psi = \psi_{\mathbb{Q}} \circ \mathrm{Tr}_{F/\mathbb{Q}}$, where $\mathrm{Tr}_{F/\mathbb{Q}}$ is the trace map and $\psi_{\mathbb{Q}}$ is the additive character on $\mathbb{A}_{\mathbb{Q}}$ which is trivial on $\mathbb{Q}$ and such that $\psi(x) = e^{2\pi i x}$ for $x \in \mathbb{R}$. Let d_v be the conductor of ψ_v, i.e., the smallest nonnegative integer such that ψ_v is trivial on $\mathfrak{m}_v^{-d_v}$. Notice that $d_v = 0$ for every finite place not dividing the discriminant and we have the relation $d_F = \prod_v p_v^{d_v}$, where $p_v := |\mathfrak{o}_v/\mathfrak{m}_v|$.

Measures. In the group $\mathbb{A}$ we use a product measure $\mathrm{d}x = \prod_v \mathrm{d}x_v$, where for real v, $\mathrm{d}x_v$ is the Lebesgue measure on $\mathbb{R}$, for complex v, $\mathrm{d}x_v$ is twice the Lebesgue measure on $\mathbb{C}$ and for each finite v, $\mathrm{d}x_v$ is a Haar measure on F_v giving measure $p_v^{-\frac{1}{2}d_v}$ to the compact subgroup $\mathfrak{o}_v$. As for the multiplicative group $\mathbb{A}^\times$, we also take a product measure $\mathrm{d}^\times x = \prod \mathrm{d}^\times x$, where $\mathrm{d}^\times x_v = \zeta_v(1)(\mathrm{d}x_v/|x_v|)$ for infinite or unramified v and we take $\mathrm{d}^\times x_v := p_v^{\frac{1}{2}d_v} \xi_{F_v}(1)(\mathrm{d}x_v/|x_v|)$ for ramified v so that for any finite v, we are giving measure 1 to $\mathfrak{o}_v^\times$. Such measures can naturally give rise to measures on the quotient spaces $F\backslash\mathbb{A}$ and $F^\times\backslash\mathbb{A}^\times_{(1)}$ such that

$$\mathrm{vol}(F\backslash\mathbb{A}) = 1 \quad \text{and} \quad \mathrm{vol}(F^\times\backslash\mathbb{A}^\times_{(1)}) = d_F^{\frac{1}{2}} \xi_F^*(1).$$

The first can be found in Tate's thesis [1950] and the second is [Lang 1994, Proposition XIV.13] (the factor $d_F^{\frac{1}{2}}$ comes from our different normalization of the multiplicative measure).

3. Preliminaries on automorphic representations

In the course of studying automorphic forms in $\mathrm{GL}(n)$, it will be important to distinguish a few of its subgroups. For any unitary ring R with group of invertible elements given by $R^\times$, we let $Z_n(R)$ denote the group of central matrices (i.e., nontrivial multiples of the identity) and $N_n(R)$ denote the maximal unipotent group formed by matrices with entries 1 on the diagonal and 0 below the diagonal, and we let $A_n(R)$ denote the diagonal matrices with lower-right entry 1.

We extend our additives character to N_n in the following way: If $n = (x_{i,j})_{1\le i,j\le n} \in \mathrm{N}_n(\mathbb{A})$, then $\psi(n) := \psi(x_{1,2} + \cdots + x_{n-1,n})$ and similarly for ψ_v. We can extend the measures on the local fields F_v and their unit groups $F_v^\times$ to measures on the groups $Z_n(F_v)$, $N_n(F_v)$ and $A_n(F_v)$ using the obvious isomorphisms $Z_n(R) \simeq R^\times$, $N_n(R) \simeq R^{\frac{1}{2}n(n-1)}$ and $A_n(R) \simeq (R^\times)^{n-1}$.

Moreover, let K_v denote a maximal compact subgroup of $\mathrm{GL}_n(F_v)$ given by

$$K_v := \begin{cases} O(n) & \text{if } F_v = \mathbb{R}, \\ U(n) & \text{if } F_v = \mathbb{C}, \\ \mathrm{GL}_n(\mathfrak{o}_v) & \text{for } v < \infty. \end{cases}$$

We can now define a Haar measure on $\mathrm{GL}_n(F_v)$ by appealing to the Iwasawa decomposition. Let $\mathrm{d}k$ be a Haar probability measure on K_v and consider the surjective map

$$Z_n(F_v) \times N_n(F_v) \times A_n(F_v) \times K_v \to \mathrm{GL}_n(F_v), \quad (z,n,a,k) \mapsto znak,$$

and let $\mathrm{d}g_v$ be the pullback by this map of the measure

$$\Delta(a)^{-1} \prod_{k=1}^{n-1} y_k^{-k(n-k)} \times \mathrm{d}z \times \mathrm{d}n \times \mathrm{d}a \times \mathrm{d}k,$$

where

$$\Delta\begin{pmatrix} y_1 & & & \\ & \ddots & & \\ & & y_{n-1} & \\ & & & 1 \end{pmatrix} = \prod_{j=1}^{n-1} |y_j|^{n+1-2j}.$$

In particular, for GL_2,

$$\int_{\mathrm{GL}_2(F_v)} f(g_v)\,\mathrm{d}g_v = \int_{K_v} \int_{F_v^\times} \int_{F_v} \int_{F_v^\times} f\big(z(u)n(x)a(y)k\big)\,\mathrm{d}^\times u\,\mathrm{d}x\,\frac{\mathrm{d}^\times y}{|y|_v}\,\mathrm{d}k,$$

where

$$z(u) = \begin{pmatrix} u & \\ & u \end{pmatrix}, \quad n(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \quad a(y) = \begin{pmatrix} y & \\ & 1 \end{pmatrix}.$$

Similarly, we shall consider measures on the quotients

$$N_n(F_v)\backslash \mathrm{GL}_n(F_v) \quad \text{and} \quad \mathrm{PGL}_n(F_v) := Z_n(F_v)\backslash \mathrm{GL}_n(F_v)$$

by omitting the terms $\mathrm{d}n$ and $\mathrm{d}z$ respectively. Now, given a group G for which we have attached Haar measures $\mathrm{d}g_v$ to $G(F_v)$, we attach to $G(\mathbb{A})$ the product measure $\mathrm{d}g = \prod_v \mathrm{d}g_v$. Since $\mathrm{PGL}_2(F) \hookrightarrow \mathrm{PGL}_2(\mathbb{A})$ discretely, we may use the measure of $\mathrm{PGL}_2(\mathbb{A})$ to define one on

$$X := \mathrm{PGL}_2(F)\backslash\mathrm{PGL}_2(\mathbb{A}) = Z_2(\mathbb{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),$$

which turns out to have finite total measure $\mathrm{vol}(X) < +\infty$.

3A. ***Automorphic representations for* GL(2).** Consider the Hilbert space $L^2(X)$ with an action of $\mathrm{GL}_2(\mathbb{A})$ given by right multiplication and a $\mathrm{GL}_2(\mathbb{A})$-invariant inner product given by

$$\langle \phi_1, \phi_2 \rangle_{L^2(X)} = \int_X \phi_1(g)\overline{\phi_2(g)}\,\mathrm{d}g. \tag{8}$$

It is well known that this space decomposes as

$$L^2(X) = L^2_{\mathrm{cusp}}(X) \oplus L^2_{\mathrm{res}}(X) \oplus L^2_{\mathrm{cont}}(X), \tag{9}$$

where $L^2_{\mathrm{cusp}}(X)$ denotes the closed subspace of cuspidal functions given by the functions satisfying the relation

$$\int_{N_2(F)\backslash N_2(\mathbb{A})} \phi(ng)\,\mathrm{d}n = 0,$$

$L^2_{\mathrm{res}}(X)$ is the residual spectrum consisting of all the one-dimensional subrepresentations of $L^2(X)$, and $L^2_{\mathrm{cont}}(X)$ is expressed in terms of Eisenstein series which we discuss further below. Moreover, $L^2_{\mathrm{cusp}}(X)$ decomposes as a direct sum of irreducible representations, which are called the cuspidal automorphic representations.

3A1. *Induced representations and Eisenstein series.* Given a character $\omega : F^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ (not necessarily unitary), we denote by $\pi(\omega)$ the isobaric sum $\omega \boxplus \omega^{-1}$, i.e., the space of *measurable* functions f on $\mathrm{GL}_2(\mathbb{A})$ such that

$$f\left(\begin{pmatrix} a & b \\ & d \end{pmatrix} g\right) = |a/d|^{\frac{1}{2}}\omega(a)\omega^{-1}(d)f(g), \quad \langle f, f\rangle_{\mathrm{Ind}} < +\infty,$$

where $|\cdot|$ denotes the adelic norm, $K := \prod_v K_v$, and $\langle f_1, f_2\rangle_{\mathrm{Ind}} < \infty$, where we put

$$\begin{aligned}\langle f_1, f_2\rangle_{\mathrm{Ind}} &:= \int_{F^\times\backslash\mathbb{A}^\times_{(1)}\times K} f_1(a(y)k)\overline{f_2(a(y)k)}\,\mathrm{d}^\times y\,\mathrm{d}k \\ &= \mathrm{vol}(F^\times\backslash\mathbb{A}^\times_{(1)}) \int_K f_1(k)\overline{f_2(k)}\,\mathrm{d}k. \end{aligned}\tag{10}$$

Given such ω and $f \in \pi(\omega)$, we define an Eisenstein series by a process of analytic continuation. It is given by the following series, as long as it converges:

$$\mathrm{Eis}(f)(g) := \sum_{\gamma \in B_2(F)\backslash\mathrm{GL}_2(F)} f(\gamma g),$$

where for a ring R,

$$B_2(R) := \left\{\begin{pmatrix} a & b \\ & d \end{pmatrix} : a, d \in R^\times, b \in R\right\}.$$

Suppose $\omega \neq 1$. We define further the *normalized Eisenstein series* by taking

$$\widetilde{\mathrm{Eis}}(f) := L(1, \omega^2)\,\mathrm{Eis}(f).$$

It will be convenient to define the inner product of two normalized Eisenstein series in terms of the inner product in the induced model of the functions used for generating it. In other words, for $f_1, f_2 \in \pi(\omega)$ and $\phi_i = \widetilde{\mathrm{Eis}}(f_i)$, where $i = 1, 2$, we write

$$\langle \phi_1, \phi_2 \rangle_{\widetilde{\mathrm{Eis}}} := |L(1, \omega^2)|^2 \langle f_1, f_2 \rangle_{\mathrm{Ind}} = d_F^{\frac{1}{2}} \Lambda^*(1, \mathrm{Ad}\,\pi(\omega)) \int_K f_1(k) \overline{f_2(k)}\,\mathrm{d}k. \tag{11}$$

Finally, for a complex parameter s, we use the notation $\pi(\omega, s) := \pi(\omega \times |\cdot|^s)$. For a character ω_v of F_v we can similarly define the induced representation $\pi_v(\omega_v, s)$ so that if $\omega \simeq \bigotimes_v' \omega_v$, we have $\pi(\omega, s) \simeq \bigotimes_v' \pi_v(\omega_v, s)$.

3A2. *Spectral decomposition for smooth functions.* We already encountered a decomposition of $L^2(\boldsymbol{X})$ in (9), but in practice we will encounter functions in $L^2(\boldsymbol{X})$ which are right-invariant by a large compact subgroup $K_0 \subset \mathrm{GL}_2(\mathbb{A})$ and moreover we will need more uniformity than simply L^2-convergence. In the following, we write down a more precise form of this decomposition for functions in $\mathcal{C}^\infty(\boldsymbol{X}/K^S)$, where for a finite set S of places of F containing the archimedean ones, K^S is the compact group given by

$$K^S := \prod_{v \notin S} K_v.$$

The only intervening representations are those that are unramified outside S. That means $\pi \in C(S)$, $\pi = \pi(\omega, it)$ for $\omega \in \Xi(S)$ or $\pi = \omega \circ \det$ for $\omega \in \Xi(S)$. For each cuspidal automorphic representation π, we let $\mathcal{B}_c(\pi)$ denote an orthonormal basis of the realization of π in $L^2(\boldsymbol{X})$ with respect to the inner product $\langle \cdot, \cdot \rangle_{L^2(\boldsymbol{X})}$. Similarly, for an induced representation $\pi = \pi(\omega)$, we define $\mathcal{B}_e(\pi)$ to be a basis of normalized Eisenstein series (not vectors in the induced models!) with respect to $\langle \cdot, \cdot \rangle_{\widetilde{\mathrm{Eis}}}$. We may therefore state the following version of the spectral theorem:

Proposition 3.1. *Let $F \in \mathcal{C}^\infty(\boldsymbol{X}/K^S)$ be of rapid decay. Then*

$$F(g) = \sum_{\pi \in C(S)} \sum_{\phi \in \mathcal{B}_c(\pi)} \langle F, \phi \rangle \phi(g) + \mathrm{vol}(\boldsymbol{X})^{-1} \sum_{\substack{\omega \in \Xi(S) \\ \omega^2 = 1}} \langle F, \omega \circ \det \rangle \omega(\det g) + \frac{1}{4\pi} \sum_{\omega \in \Xi(S)} \int_{-\infty}^{\infty} \sum_{\phi \in \mathcal{B}_e(\pi(\omega, it))} \langle F, \phi \rangle \phi(g)\,\mathrm{d}t,$$

where $\langle \cdot, \cdot \rangle$ denotes the same integral as in the definition of $\langle \cdot, \cdot \rangle_{L^2(\boldsymbol{X})}$ and convergence is absolute and uniform for g on any compact subset of $\boldsymbol{X}$.

The result, for pseudo-Eisenstein series, follows from (4.21) and (4.25) in [Gelbart and Jacquet 1979] and by extending the inner product $(a_1(iy), a_2(iy))$ with respect to an orthogonal basis of $L^2(F^\times \backslash \mathbb{A}^\times_{(1)} \times K)$. The general result is a consequence of the fact that the space of cusp forms decomposes discretely and spans the orthogonal complement to the space of pseudo-Eisenstein series.

4. Whittaker models and periods

In this section, we consider irreducible automorphic representations π of $\mathrm{GL}_n(\mathbb{A})$ and the period integrals related to some Rankin–Selberg L-functions. We will only be concerned with the generic representations, which are those admitting a Whittaker model. This is done for arbitrary n but only the cases $n = 2$ and $n = 1$ are used in the sequel. Finally, we shall not distinguish between a representation π and its space of smooth vectors V_π^∞. An automorphic form ϕ will always denote a smooth vector in an irreducible automorphic representation.

4A. ***Whittaker functions.*** Let π be a generic automorphic representation of $\mathrm{GL}_n(\mathbb{A})$, and let $\phi \in \pi$ be an automorphic form. Let $W_\phi : \mathrm{GL}_n(\mathbb{A}) \to \mathbb{C}$ be the Whittaker function of ϕ given by

$$W_\phi(g) = \int_{N_n(F)\backslash N_n(\mathbb{A})} \phi(ng)\overline{\psi(n)}\,\mathrm{d}n. \tag{12}$$

It satisfies $W_\phi(ng) = \psi(n)W_\phi(g)$ for all $n \in \mathrm{N}_n(\mathbb{A})$.

Given a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$, we might write down an isomorphism $\pi \simeq \bigotimes_v' \pi_v$ where for each v, π_v is a local generic admissible representation of $\mathrm{GL}_n(F_v)$, and we might define Whittaker functions for each local representation such that for every $\phi \in \pi$ with $\phi = \bigotimes_v' \phi_v$ through the above isomorphism, we have

$$W_\phi(g) = \prod_v W_{\phi_v}(g_v), \quad g = (g_v)_v \in \mathrm{GL}_n(\mathbb{A}). \tag{13}$$

In fact, the map $\phi \mapsto W_\phi$ is an intertwiner between π and its image, denoted by $\mathcal{W}(\pi, \psi)$, the so-called Whittaker model of π. We similarly define the local Whittaker models $\mathcal{W}(\pi_v, \psi_v)$. Later on, it will be convenient to exchange freely between a representation and its associated Whittaker model. The importance of the latter comes from its close relation to local Rankin–Selberg L-functions, as we will see in Section 4B.

There is a similar story for noncuspidal forms but in this case it is better to work with *normalized* Eisenstein series. As we will only need this for $n = 2$, we shall restrict to this case. Let $f \in \pi(\omega)$ and suppose that f is factorable, i.e., $f = \bigotimes_v' f_v$ with $f_v \in \pi_v(\omega_v)$. Then it follows by analytic continuation and Bruhat decomposition that

$$W_{\widetilde{\mathrm{Eis}}(f)}(g) = L(1, \omega^2)\int_{N_2(\mathbb{A})} f(wng)\overline{\psi(n)}\,\mathrm{d}n = \prod_v W_{f_v}^J(g_v),$$

where $W_{f_v}^J$ is the normalized Jacquet integral, given by

$$W_{f_v}^J(g_v) = L(1, \omega_v^2)\int_{N_2(F_v)} f_v(wng)\overline{\psi(n)}\,\mathrm{d}n.$$

By putting $\phi = \widetilde{\mathrm{Eis}}(f)$ and $W_{\phi_v} := W_{f_v}^J$, we see that (13) also holds in this case.

It is also important to consider Whittaker functions with respect to the inverse character $\psi' = \overline{\psi}$, so we analogously define W_ϕ' and W_{ϕ_v}' by replacing ψ_v by $\psi_v' = \overline{\psi}_v$ and ψ by $\psi' = \prod_v \psi_v'$ in all the previous definitions. It follows from uniqueness of local Whittaker functions that we may take

$$W'_{\phi_v} = \overline{W_{\phi_v}} \quad \text{for all places } v \text{ of } F. \tag{14}$$

If a local generic admissible representation π_v is unramified for some finite place v, this mean that in π_v there exists a vector which is right-invariant by the action of $\mathrm{GL}_n(\mathfrak{o}_v)$. Such a vector is called *spherical* and spherical vectors are unique up to multiplication by scalars. Among the spherical vectors we shall distinguish a certain one which we call *normalized spherical*. If v is unramified, the normalized spherical vector will be the one for which $W_{\phi_v}(e) = 1$, where $e \in \mathrm{GL}_n(F_v)$ denotes the identity element. For the finite ramified places we simply define it to be the newform (defined in Section 4C). This avoids repetition and is justified by the fact that the two notions also coincide for unramified primes.

4B. *Integral representations of* $\mathbf{GL}_{n+1} \times \mathbf{GL}_n$ *L-functions.* This theory is an outgrowth of Hecke's theory of L-functions for GL_2 and has been developed by Jacquet, Piatetski-Shapiro and Shalika. We start with Π and π irreducible automorphic representations of $\mathrm{GL}_{n+1}(\mathbb{A})$ and $\mathrm{GL}_n(\mathbb{A})$, respectively, and let $\Phi \in \Pi$ and $\phi \in \pi$ be automorphic forms. Suppose momentarily that Φ is a cusp form and hence rapidly decreasing. We can thus consider for every $s \in \mathbb{C}$ the integral

$$I(s, \Phi, \phi) := \int_{\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A})} \Phi\begin{pmatrix} h & \\ & 1 \end{pmatrix} \phi(h) |\det h|^{s-\frac{1}{2}} \, \mathrm{d}h.$$

It follows from the Whittaker decomposition of cusp forms (see [Cogdell 2007, Theorem 1.1]) that if Φ is a cuspidal function, then

$$I(s, \Phi, \phi) = \Psi(s, W_\Phi, W'_\phi) \qquad (\mathrm{Re}(s) \gg 1), \tag{15}$$

where

$$\Psi(s, W, W') := \int_{N_n(\mathbb{A})\backslash \mathrm{GL}_n(\mathbb{A})} W\begin{pmatrix} h & \\ & 1 \end{pmatrix} W'(h) |\det h|^{s-\frac{1}{2}} \, \mathrm{d}h. \tag{16}$$

Our next result gives some of the good properties of $\Psi(s, W, W')$, namely, convergence and the fact that it factors into local integrals whenever Φ and ϕ also factor.

Proposition 4.1. *Let Π and π be automorphic representations of* $\mathrm{GL}(n+1)$ *and* $\mathrm{GL}(n)$ *over F, respectively. Let $\Phi = \bigotimes'_v \Phi_v \in \Pi$ and $\phi = \bigotimes'_v \phi_v \in \pi$ be automorphic forms. Let W_{Φ_v} and W'_{ϕ_v} be as in Section 4A. Then, for* $\mathrm{Re}(s) \gg 1$, *$\Psi(s, W_\Phi, W'_\phi)$ converges and we have the factorization*

$$\Psi(s, W_\Phi, W'_\phi) = \prod_v \Psi_v(s, W_{\Phi_v}, W'_{\phi_v}),$$

where

$$\Psi_v(s, W, W') := \int_{N_n(F_v)\backslash \mathrm{GL}_n(F_v)} W\begin{pmatrix} h_v & \\ & 1 \end{pmatrix} W'(h_v) |\det h_v|^{s-\frac{1}{2}} \, \mathrm{d}h_v. \tag{17}$$

Moreover, if v is finite and both Π_v and π_v are unramified and Φ_v and ϕ_v are normalized spherical,

$$\Psi_v(s, W_{\Phi_v}, W'_{\phi_v}) = p_v^{d_v l_n(s)} L(s, \Pi_v \times \pi_v), \quad \textit{where } \; l_n(s) = \tfrac{1}{2}n(n+1)s - \tfrac{1}{12}n(n+1)(2n+1).$$

Proof. The first part follows from gauge estimates for Whittaker functions (see [Jacquet et al. 1979, §2]). It is an important fact that this part does not require the representations to be cuspidal. The reason is that, in some sense, the integral representation using Whittaker functions only sees the nonconstant terms. For the local computation this is well known when p_v is unramified (see, e.g., [Cogdell 2007, Theorem 3.3]). In general we may restrict to the unramified situation by following the computation in the proof of [Cogdell and Piatetski-Shapiro 1994, Lemma 2.1]. □

Remark. When $n = 1$, we write $I(s, \phi)$, $\Psi(s, W_\phi)$ and $\Psi_v(s, W_{\phi_v})$ instead of $I(s, \phi, \mathbf{1})$, $\Psi(s, W_\phi, W_{\mathbf{1}_v})$ and $\Psi_v(s, W_{\phi_v}, W_{\mathbf{1}})$, where $\mathbf{1}$ and $\mathbf{1}_v$ denote the constant functions on $\mathrm{GL}_1(\mathbb{A})$ and $\mathrm{GL}_1(F_v)$ respectively.

4C. ***Newforms and ramified L-factors.*** For a finite place v and any admissible irreducible generic representation of $\mathrm{GL}_n(F_v)$, not necessarily unramified, we define a distinguished vector in its Whittaker model, called *newform*. This was first introduced by Casselman [1973] when $n = 2$ by translating the results of Atkin and Lehner to the representation-theoretic language. This was later generalized by Jacquet, Piatetski-Shapiro and Shalika [Jacquet et al. 1981] for general n by requiring that they are good test vectors for representing L-functions via Rankin–Selberg periods as in Section 4B. Moreover, when π_v is unramified, these coincide with normalized spherical vectors.

The fact that these newvectors are test vectors for Rankin–Selberg L-functions can be rephrased by relating their values to the Langlands parameters of the representation. This was carefully carried out in [Miyauchi 2014]. In order to quote these results we introduce the following notation: for $\underline{\nu} = (\nu_1, \dots, \nu_{n-1}) \in \mathbb{Z}^{n-1}$, let $s(\underline{\nu}) = \sum_{i=1}^{n-1} \frac{1}{2} i(n-i)\nu_i$, and for $y \in F_v^\times$, we write

$$a(\underline{\nu}) := \begin{pmatrix} \varpi_v^{\nu_1+\cdots+\nu_{n-1}} & & & \\ & \varpi_v^{\nu_2+\cdots+\nu_{n-1}} & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

The main result of [Miyauchi 2014] states that if ψ_v is unramified, then $W_{\pi_v}(a(\underline{\nu})) = p^{s(\nu)}\lambda_\pi(\underline{\nu})$, where

$$\lambda_{\pi_v}(\underline{\nu}) = 0 \quad \text{unless } \nu_1, \dots, \nu_{n-1} \geq 0, \tag{18}$$

and the $\lambda_{\pi_v}(\underline{\nu})$ are in general given by Schur polynomials evaluated on the Langlands parameters of π_v (see [Miyauchi 2014] for details).

When ψ_v is ramified, this has to be modified. First, we write $\psi_v(x) = \psi_{F_v}(\lambda x)$ for some $\lambda \in F_v^\times$, where ψ_{F_v} is an unramified additive character of F_v, and let $d = v(\lambda)$. We then define the *newvector* by taking

$$W_{\pi_v}(g) = W_{\pi_v}^{\mathrm{unr}}(a(\iota_n(d))g),$$

where $\iota_n(d) = (d, d, \dots, d) \in \mathbb{Z}^{n-1}$ and $W_{\pi_v}^{\mathrm{unr}}$ denotes the newvector for the unramified character ψ_{F_v}. The term $a(\iota_n(d))$ is responsible for the change in the additive character.

In addition to Proposition 4.1, we shall also need to compute L-functions for certain ramified local representations. In particular, we require the following computation that appears, for instance, in the work of Booker, Krishnamurthy and Lee [Booker et al. 2020, proof of Lemma 3.1]: Let $n > m$, and let Π_v (resp. π_v) be an irreducible admissible generic representation of $\mathrm{GL}_n(F_v)$ (resp. $\mathrm{GL}_m(F_v)$) with Langlands parameters $(\gamma_{\Pi_v}^{(i)})_{i=1}^n$ (resp. $(\gamma_{\pi_v}^{(j)})_{j=1}^m$). Supposing further that π_v is ramified, one then has

$$\sum_{\substack{\underline{\nu}\in\mathbb{Z}^{m-1}\\ \nu_1,\dots,\nu_{m-1}\geq 0}} \frac{\lambda_{\Pi_v}(\underline{\nu},0,\dots,0)\lambda_{\pi_v}(\underline{\nu})}{p_v^{(\nu_1+2\nu_2+\cdots+(m-1)\nu_{m-1})s}} = L(s,\Pi_v\times\pi_v), \tag{19}$$

where

$$L(s,\Pi_v\times\pi_v) := \prod_{\substack{1\leq i\leq n\\ 1\leq j\leq m}} (1-\gamma_{\Pi_v}^{(i)}\gamma_{\pi_v}^{(j)} p_v^{-s})^{-1}. \tag{20}$$

This coincides with Langlands local L-function when Π_v is unramified, which we shall suppose.

4D. *Relation between inner products on* GL(2). Let π be a generic automorphic representation of GL(2) over F with trivial central character. We define a $\mathrm{GL}_2(\mathbb{A})$-invariant inner product on the representation space of π as follows: If π is cuspidal, then we may see V_π embedded in $L^2(\boldsymbol{X})$ and therefore π may inherit the inner product from $L^2(\boldsymbol{X})$ given by (8). If π is Eisenstein we cannot see the representation space of π inside $L^2(\boldsymbol{X})$ and hence we equip it with the inner product given by (11).

There is however another way of defining an inner product for factorable vectors in these representations which is independent of whether π is cuspidal or Eisenstein. This is done by using the Whittaker model as follows: For each place v, we have a $\mathrm{GL}_2(F_v)$-invariant inner form on $\mathcal{W}(\pi_v,\psi_v)$ by letting

$$\vartheta_v(W_1,W_2) = \frac{\int_{F_v^\times} W_1(a(y_v))\overline{W_2(a(y_v))}\,\mathrm{d}^\times y_v,}{\zeta_v(1)L_v(1,\mathrm{Ad}\,\pi)/\zeta_v(2)}. \tag{21}$$

The fact that the numerator of (21) is indeed right $\mathrm{GL}_2(F_v)$-invariant follows from the theory of the Kirillov model and the inclusion of the denominator is to ensure the following property: Whenever π_v and ψ_v are unramified and W is normalized spherical, we have $\vartheta_v(W,W) = 1$. Finally, letting $\phi_1 = \bigotimes\phi_{1,v}$ and $\phi_2 = \bigotimes\phi_{2,v}$ be either cusp forms or normalized Eisenstein series, we define the *canonical* inner product by the formula

$$\langle\phi_1,\phi_2\rangle_{\mathrm{can}} := 2d_F^{\frac{1}{2}}\Lambda^*(1,\mathrm{Ad}\,\pi)\times\prod_v \vartheta_v(W_{\phi_{1,v}},W_{\phi_{2,v}}). \tag{22}$$

Since every two $\mathrm{GL}_2(\mathbb{A})$-invariant inner products in π must be equal up to multiplication by some scalar, it follows that we can compare the canonical inner product with the ones introduced earlier for cuspidal and Eisenstein representations. Indeed, Rankin–Selberg theory in the cuspidal case and a direct computation in the Eisenstein case gives us the following relation:

$$\langle\phi_1,\phi_2\rangle_{\mathrm{can}} = \begin{cases} \langle\phi_1,\phi_2\rangle_{L^2(\boldsymbol{X})} & \text{if } \pi \text{ is cuspidal},\\ 2\langle\phi_1,\phi_2\rangle_{\mathrm{Eis}} & \text{if } \pi \text{ is Eisenstein}.\end{cases} \tag{23}$$

The computation in the Eisenstein case follows from [Wu 2014, Lemma 2.8]. For the cusp forms we combine the proof of [Wu 2014, Proposition 2.13] with the value of the residue of an Eisenstein series computed in [Michel and Venkatesh 2010, (4.6)].[1]

5. Abstract reciprocity

In this section we show an identity between two periods. At this point we make no attempt to relate them to moments of L-functions. The proof is a rather simple matrix computation.

Suppose $\Phi \in \mathcal{C}^\infty\big(Z_3(\mathbb{A})\mathrm{GL}_3(F)\backslash \mathrm{GL}_3(\mathbb{A})\big)$ is such that for every $h \in \mathrm{GL}_2(\mathbb{A})$, the integral

$$\mathcal{A}_s\Phi(h) := |\det h|^{s-\frac{1}{2}} \int_{F^\times\backslash\mathbb{A}^\times} \Phi\begin{pmatrix} z(u)h & \\ & 1\end{pmatrix} |u|^{2s-1}\, d^\times u \tag{24}$$

converges and such that $y \mapsto \mathcal{A}_s\Phi\big(\big(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\big)h\big)$ is of rapid decay as $|y| \to 0$ or $+\infty$.

Proposition 5.1. *Let Φ be as above, and let $I(w, \cdot)$ be as in the remark on page* 1392. *Then, for every $s, w \in \mathbb{C}$, we have the reciprocity relation*

$$I(w, \mathcal{A}_s\Phi) = I(w', \mathcal{A}_{s'}\check{\Phi}),$$

where (s', w') are as in (5) *and*

$$\check{\Phi}(g) := \Phi(g w_{23}), \quad w_{23} = \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}. \tag{25}$$

Proof. By definition,

$$I(w, \mathcal{A}_s\Phi) = \int_{F^\times\backslash\mathbb{A}^\times}\int_{F^\times\backslash\mathbb{A}^\times} \Phi\begin{pmatrix} z(u)a(y) & \\ & 1\end{pmatrix} |u|^{2s-1}|y|^{s+w-1}\,\mathrm{d}^\times u\,\mathrm{d}^\times y. \tag{26}$$

Now, since Φ is left-invariant by $Z_3(\mathbb{A})\mathrm{GL}_3(F)$, we see that for every $u, y \in \mathbb{A}^\times$, one has

$$\Phi\begin{pmatrix} uy & & \\ & u & \\ & & 1\end{pmatrix} = \Phi\left(\begin{pmatrix} u & & \\ & u & \\ & & u\end{pmatrix} w_{23}\begin{pmatrix} y & & \\ & u^{-1} & \\ & & 1\end{pmatrix} w_{23}\right) = \Phi\left(\begin{pmatrix} y & & \\ & u^{-1} & \\ & & 1\end{pmatrix} w_{23}\right) = \check{\Phi}\begin{pmatrix} y & & \\ & u^{-1} & \\ & & 1\end{pmatrix}.$$

Applying this to (26) and making the change of variables $(u, y) = (u'^{-1}, u'y')$ gives the result. □

6. Spectral expansion of the period

In this section we will give a spectral decomposition of the period $I(w, \mathcal{A}_s\Phi)$. Let Π be an automorphic cuspidal representation for GL(3) over F, and let $\Phi = \bigotimes_v \Phi_v \in \Pi$ be a cusp form. Let S be a finite set of places containing all archimedean places and all the places for which Φ is not normalized spherical. Since Φ is of rapid decay, then the same holds for $\mathcal{A}_s\Phi$. More precisely this follows by combining the

[1] In [Wu 2014], a factor $d_F^{\frac{1}{2}}$ seems to be missing in the computation of this residue.

rapid decay of Whittaker functions with the action of the Weyl group of GL(3). We can thus spectrally decompose it as in Proposition 3.1:

$$\mathcal{A}_s\Phi(h) = \sum_{\pi\in C(S)}\sum_{\phi\in\mathcal{B}_c(\pi)} \langle \mathcal{A}_s\Phi,\phi\rangle\phi(h) + \mathrm{vol}(X)^{-1}\sum_{\substack{\omega\in\Xi(S)\\ \omega^2=1}} \langle \mathcal{A}_s\Phi, \omega\circ\det\rangle\omega(\det g) + \frac{1}{4\pi}\sum_{\omega\in\Xi(S)}\int_{-\infty}^{\infty}\sum_{\phi\in\mathcal{B}_e(\pi(\omega,it))}\langle \mathcal{A}_s\Phi,\phi\rangle\phi(h)\,\mathrm{d}t.$$

By integrating both sides of the above expression against an additive character and over the compact set $N_2(F)\backslash N_2(\mathbb{A})$, we get the following relation for Whittaker functions:

$$W_{\mathcal{A}_s\Phi}(h) = \sum_{\pi\in C(S)}\sum_{\phi\in\mathcal{B}_c(\pi)} \langle \mathcal{A}_s\Phi,\phi\rangle W_\phi(h) + \frac{1}{4\pi}\sum_{\pi\in\Xi(S)}\int_{-\infty}^{\infty}\sum_{\phi\in\mathcal{B}_e(\pi(\omega,it))}\langle \mathcal{A}_s\Phi,\phi\rangle W_\phi(h)\,\mathrm{d}t.$$

Notice that since the one-dimensional representations are not generic, they do not contribute to the above expression. Now, because of rapid decay of the Whittaker functions W_ϕ as $|y|\to+\infty$, if we take $\mathrm{Re}(w)$ sufficiently large, we get

$$\Psi(w,\mathcal{A}_s\Phi) = \sum_{\pi\in C(S)}\sum_{\phi\in\mathcal{B}_c(\pi)} \langle \mathcal{A}_s\Phi,\phi\rangle \Psi(w,W_\phi) + \frac{1}{4\pi}\sum_{\omega\in\Xi(S)}\int_{-\infty}^{\infty}\sum_{\phi\in\mathcal{B}_e(\pi(\omega,it))}\langle \mathcal{A}_s\Phi,\phi\rangle \Psi(w,W_\phi)\,\mathrm{d}t. \tag{27}$$

By using the Fourier decomposition of $\mathcal{A}_s\Phi$, we see that

$$\int_{\mathbb{A}^\times} W_{\mathcal{A}_s\Phi}(a(y))|y|^{w-\frac{1}{2}}\,\mathrm{d}^\times y = I(w,\mathcal{A}_s\Phi) - I(w,(\mathcal{A}_s\Phi)_0),$$

where for any ϕ on $\mathcal{C}^\infty(X)$, ϕ_0 is given by

$$\phi_0(h) := \int_{F\backslash\mathbb{A}} \phi(n(x)h)\,\mathrm{d}x.$$

The next step is to realize the terms $\langle \mathcal{A}_s\Phi,\phi\rangle$ and $\Psi(w,W_\phi)$ as a product of local integrals. First, it follows from Proposition 4.1 that if $\phi = \bigotimes_v \phi_v$ is decomposable,

$$\Psi(w,W_\phi) = \prod_v \Psi_v(w,W_{\phi_v}).$$

Moreover, from the definition of $\mathcal{A}_s\Phi$, we deduce, after changing variables, that

$$\langle \mathcal{A}_s\Phi,\phi\rangle = I(s,\Phi,\bar{\phi}).$$

Since Φ is a cusp form on GL(3), it follows from (15) and Proposition 4.1 that for $\mathrm{Re}(s)$ sufficiently large and factorable ϕ,

$$I(s,\Phi,\bar{\phi}) = \Psi(s,W_\Phi,W'_{\bar{\phi}}) = \prod_v \Psi_v(s,W_{\Phi_v},W'_{\bar{\phi}_v}),$$

where $\Psi_v(s, W, W')$ is given by (17). As a consequence, we have

$$I(w, W_{\mathcal{A}_s\Phi}) = I(w, (\mathcal{A}_s\Phi)_0) + \sum_{\pi\in C(S)} \sum_{\phi\in\mathcal{B}_c(\pi)} \prod_v \Psi_v(s, W_{\Phi_v}, W'_{\bar{\phi}_v}) \prod_v \Psi_v(s, W_{\phi_v})$$
$$+ \frac{1}{4\pi} \sum_{\omega\in\Xi(S)} \int_{-\infty}^{\infty} \sum_{\phi\in\mathcal{B}_e(\pi(\omega,it))} \prod_v \Psi_v(s, W_{\Phi_v}, W'_{\bar{\phi}_v}) \prod_v \Psi_v(s, W_{\phi_v})\, \mathrm{d}t. \quad (28)$$

For each generic automorphic representation π we will now construct an orthonormal basis for V_π which is formed exclusively by factorable vectors: We start by choosing for each place v, an orthogonal basis $\mathcal{B}^W(\pi_v)$ of the space $\mathcal{W}(\pi_v, \psi_v)$. Consider now the elements $\phi = \bigotimes_v \phi_v$ such that for every finite v, W_{ϕ_v} lies in $\mathcal{B}^W(\pi_v)$, and for all but finitely many v, W_{ϕ_v} is normalized spherical. This provides us with an *orthogonal* basis for V_π. In order to get an *orthonormal* basis we multiply these vectors by the correcting factors coming from (23). Applying these steps to (28) leads to the following (the slightly awkward normalization is justified by the last part of Proposition 4.1):

Proposition 6.1. *Let Π be a cuspidal automorphic representation, and let $\Phi = \bigotimes_v \Phi_v \in \Pi$ be a cusp form. Then, for complex numbers s and w with sufficiently large real parts, we have*

$$2d_F^{\frac{7}{2}-3s-w} I(w, \mathcal{A}_s\Phi) = \mathcal{M}_{s,w}(\Phi) + \mathcal{D}_{s,w}(\Phi),$$

where

$$H(\pi) = \prod_v H_v(\pi_v), \quad H_v(\pi_v) := p_v^{d_v(3-3s-w)} \sum_{W\in\mathcal{B}^W(\pi_v)} \frac{\Psi_v(s, W_{\Phi_v}, \overline{W})\Psi_v(w, W)}{L(s, \Pi_v\times\pi_v)L(w, \pi_v)}, \quad (29)$$

$\mathcal{M}_{s,w}(\Phi)$ is as in (4), *and*

$$\mathcal{D}_{s,w}(\Phi) := 2d_F^{\frac{7}{2}-3s-w} \int_{F^\times\backslash\mathbb{A}^\times} \int_{F\backslash\mathbb{A}} \int_{F^\times\backslash\mathbb{A}^\times} \Phi\begin{pmatrix} z(u)n(x)a(y) & \\ & 1 \end{pmatrix} |u|^{2s-1} |y|^{s+w-1}\, \mathrm{d}^\times u\, \mathrm{d}x\, \mathrm{d}^\times y. \quad (30)$$

We will refer to the function H given by (29) where $\Phi = \bigotimes \Phi_v \in \Pi$ as the (s, w)-weight function of kernel Φ. If s and w and Φ are clear from the context, we shall refer to it simply as the weight function of kernel Φ.

Finally, given $s, w \in \mathbb{C}$, if H is the (s, w)-weight function with kernel Φ, we let $\check{H}$ be the (s', w')-weight function associated to $\check{\Phi}$, where s' and w' are as in (5) and $\check{\Phi}$ is as in (25). In other words,

$$\check{H}(\pi) = \prod_v \check{H}_v(\pi_v), \quad \check{H}_v(\pi) := p_v^{d_v(3-3s'-w')} \sum_{W\in\mathcal{B}^W(\pi_v)} \frac{\Psi_v(s', W_{\check{\Phi}_v}, \overline{W})\Psi_v(w', W)}{L(s', \Pi_v\times\pi_v)L(w', \pi_v)}. \quad (31)$$

7. Local computations

Let Π be an unramified cuspidal automorphic representation of PGL(3) over F. For all v, we let Φ_v^0 correspond to the normalized spherical vector in the Whittaker model, that is, $W_{\Phi_v^0} = W_{\Pi_v}$. Let $\mathfrak{q}$ and $\mathfrak{l}$

be two coprime integral ideals of F. Finally, let $\Phi^{\mathfrak{q},\mathfrak{l}} = \bigotimes_v \Phi_v^{\mathfrak{q},\mathfrak{l}}$, where, for all $v \nmid \mathfrak{q}\mathfrak{l}$, we put $\Phi_v^{\mathfrak{q},\mathfrak{l}} = \Phi_v^0$, for $v \mid \mathfrak{q}$,

$$\Phi_v^{\mathfrak{q},\mathfrak{l}}(g) := \frac{1}{p_v^n} \sum_{\beta \in \mathfrak{m}_v^{-n}/\mathfrak{o}_v} \Phi_v^0\left(g\begin{pmatrix}1 & & \beta\\ & 1 & \\ & & 1\end{pmatrix}\right), \tag{32}$$

where $n = v(\mathfrak{q})$, and, for $v \mid \mathfrak{l}$,

$$\Phi_v^{\mathfrak{q},\mathfrak{l}}(g) := \frac{1}{p_v^m} \sum_{\beta \in \mathfrak{m}_v^{-m}/\mathfrak{o}_v} \Phi_v^0\left(g\begin{pmatrix}1 & \beta & \\ & 1 & \\ & & 1\end{pmatrix}\right),$$

with $m = v(\mathfrak{l})$.

We will now proceed to the calculation of H_v for $\Phi = \Phi^{\mathfrak{q},\mathfrak{l}}$. First notice that if, for some compact group K'_v of $\mathrm{GL}_2(F_v)$, we have that Φ_v is invariant on the right by matrices of the shape $\left(\begin{smallmatrix}k & \\ & 1\end{smallmatrix}\right)$, where $k \in K'_v$, then we may restrict the sum over the basis $\mathcal{B}^W(\pi_v)$ to a sum over a basis of the right K'_v-invariant vectors. In particular, if $v < \infty$ and $v \nmid \mathfrak{q}\mathfrak{l}$, this basis will have only one element, which can be taken to be normalized spherical. Thus, by Proposition 4.1, we see that $H_v(\pi_v) = 1$ in those cases. We divide the remaining cases in three subcategories: $v \mid \mathfrak{l}$, $v \mid \mathfrak{q}$ and $v \mid \infty$ and treat them in that order.

7A. ***Nonarchimedean case I: $v \mid \mathfrak{l}$.*** Even though this is not obvious at first glance, we will show that H_v vanishes unless π_v is unramified. First, notice that by right GL(2)-invariance of the Whittaker norm, we have that for every orthonormal basis $\mathcal{B}$ of $\mathcal{W}(\pi_v, \psi_v)$, one may construct another one by taking $\mathcal{B}' := \{\pi_v(h)W, W \in \mathcal{B}\}$. Applying this for $h = \left(\begin{smallmatrix}1 & \beta\\ & 1\end{smallmatrix}\right)$ for $\beta \in \mathfrak{m}_v^{-m}/\mathfrak{o}_v$, changing variables in the $\mathrm{GL}(3)\times\mathrm{GL}(2)$ Rankin–Selberg integral and summing over β, we deduce that

$$H_v(\pi_v) = p_v^{d_v(3-3s-w)} \sum_{W \in \mathcal{B}^W(\pi_v)} \frac{\Psi_v(s, W_{\Pi_v}, \overline{W})\Psi_v(w, W^{(m)})}{L(s, \Pi_v \times \pi_v)L(w, \pi_v)},$$

where

$$W^{(m)}(h) := p^{-m} \sum_{\beta \in \mathfrak{m}_v^{-m}/\mathfrak{o}_v} W\left(h\begin{pmatrix}1 & -\beta\\ & 1\end{pmatrix}\right).$$

Now, since W_{Π_v} is spherical, we may restrict the sum over $\mathcal{B}^W(\pi_v)$ to only one term for which W is the normalized spherical vector. Now, by Proposition 4.1, $\Psi_v(s, W_{\Pi_v}, \overline{W}_{\pi_v}) = p_v^{d_v(3s-\frac{5}{2})}L(s, \Pi_v \times \pi_v)$ and

$$\begin{aligned}\Psi_v(w, W_{\pi_v}^{(m)}) &= \int_{F_v^\times} \delta_{v(y)\geq m-d} W_{\pi_v}\begin{pmatrix}y & \\ & 1\end{pmatrix}|y|^{w-\frac{1}{2}}\,\mathrm{d}^\times y,\\ &= p_v^{d_v(w-\frac{1}{2})} \sum_{\mu \geq m} \frac{\lambda_{\pi_v}(\mu)}{p_v^{\mu w}}, = p_v^{-mw}\left(\lambda_{\pi_v}(m) - \frac{\lambda_{\pi_v}(m-1)}{p_v^w}\right)p_v^{d_v(w-\frac{1}{2})}L(w, \pi_v).\end{aligned} \tag{33}$$

Hence we have that

$$H_v(\pi_v) = p_v^{-mw}\left(\lambda_{\pi_v}(m) - \frac{\lambda_{\pi_v}(m-1)}{p_v^w}\right). \tag{34}$$

7B. ***Nonarchimedean case II: v | q.*** We will show that $H_v(\pi_v)$ vanishes unless $c(\pi_v) \le n$ and that $H_v(\pi_v) \ll_\epsilon p_v^{n(\theta-1+\epsilon)}$, and if $c(\pi_v) = n$, then $H_v(\pi_v) = \varphi(p_v^n) p_v^{-2n}$.

We first notice that by a result of Casselman [1973], if we let $W_0 = W_{\pi_v}$ be the newvector and for each $j \ge 0$, we let

$$W_j := \pi_v \begin{pmatrix} 1 & \\ & \varpi_v^j \end{pmatrix} W_0. \tag{35}$$

Then, for each $j \ge 0$, $\{W_0, W_1, \dots, W_j\}$ is a basis for the $K_v[n_0+j]$-invariant vectors in $\mathcal{W}(\pi_v, \psi_v)$, where $n_0 = c(\pi_v)$. We now construct an *orthonormal* basis by employing the Gram–Schmidt process. This is the local counterpart of the method in [Blomer and Milićević 2015].

Let $\lambda_{\pi_v} = \lambda_{\pi_v}(1)$ be as in Section 4C and $\delta_{\pi_v} = \delta_{n_0=0}$, and take $\alpha_{\pi_v} := \lambda_{\pi_v}/(\sqrt{p_v}(1 + \delta_{\pi_v}/p_v))$. We put

$$\xi_{\pi_v}(0,0) = 1, \quad \xi_{\pi_v}(1,1) = \frac{1}{\sqrt{1-\alpha_{\pi_v}^2}}, \quad \xi_{\pi_v}(1,0) = -\alpha_{\pi_v} p_v^{\frac{1}{2}} \xi_{\pi_v}(1,1),$$

and

$$\xi_{\pi_v}(j,j) = \frac{1}{\sqrt{1-\alpha_{\pi_v}^2}\sqrt{1-\delta_{\pi_v}/p_v^2}}, \quad \xi_{\pi_v}(j,j-1) = -\lambda_{\pi_v}\xi_{\pi_v}(j,j), \quad \xi_{\pi_v}(j,j-2) = \delta_{\pi_v}\xi_{\pi_v}(j,j),$$

and $\xi_{\pi_v}(j,k) = 0$ for $k \le j-2$. If one assumes any nontrivial bound towards the Ramanujan conjecture $\lambda_{\pi_v} \ll p_v^{\vartheta}$, with $\vartheta < \frac{1}{2}$, one has that $|\alpha_{\pi_v}|$ is uniformly bounded by some constant $C_\vartheta < 1$ and therefore

$$\xi_{\pi_v}(j,k) \ll p_v^{j\epsilon} p_v^{(j-k)\vartheta}. \tag{36}$$

More importantly, for $j \ge 0$, $\{\widetilde{W_0}, \widetilde{W_1}, \dots, \widetilde{W_j}\}$ is an orthonormal basis for the space of $K_v[n_0+j]$-vectors in $\mathcal{W}(\pi_v, \psi_v)$, where

$$\widetilde{W_j} := \frac{1}{\langle W_0, W_0\rangle^{1/2}} \sum_{k=1}^{j} \xi_{\pi_v}(j,k) p_v^{\frac{1}{2}(k-j)} W_k. \tag{37}$$

To see this we first compute $\langle W_{k_1}, W_{k_2}\rangle$, which, by (35) and the definition of λ_{π_j}, equals

$$p^{-\frac{1}{2}|k_2-k_1|} S_{|k_2-k_1|},$$

where, for $t \ge 0$,

$$S_t = \frac{\zeta_v(2)}{L_v(1, \pi_v \times \overline{\pi}_v)} \sum_{\nu \ge 0} \frac{\lambda_{\pi_v}(\nu)\lambda_{\pi_v}(\nu+t)}{p_v^{\nu}}.$$

It follows from the Hecke relations for $\lambda_{\pi_v}(\nu)$ that

$$S_t = \lambda_{\pi_v} S_{t-1} - \delta_{\pi_v} S_{t-2} \quad \text{for } t \ge 2 \qquad \text{and} \qquad S_1 = \alpha_{\pi_v} p_v^{\frac{1}{2}} S_0,$$

from which the claim follows.

By definition, we have

$$W_{\Phi_v}\begin{pmatrix} a & b & \\ c & d & \\ & & 1 \end{pmatrix} = \frac{1}{p_v^n} \sum_{\beta \in \mathfrak{m}_v^{-n}/\mathfrak{o}_v} \psi(\beta c) W_{\Pi_v}\begin{pmatrix} a & b & \\ c & d & \\ & & 1 \end{pmatrix} = \delta_{v(c)\geq n-d_v} W_{\Pi_v}\begin{pmatrix} a & b & \\ c & d & \\ & & 1 \end{pmatrix}.$$

Hence, if we write $h = z(u)n(x)a(y)k$, with $x \in F_v$, $u, y \in F_v^\times$ and $k = (k_{ij}) \in K_v$, then W_{Φ_v} vanishes unless $v(uk_{21}) \geq n - d_v$. Letting $d_1 := \min(n, v(u) + d_v)$ and $d_2 := n - d_1$, we see that this is equivalent to k belonging to $K_v[d_2]$. This allows us to write

$$\Psi_v(s, W_{\Phi_v}, \overline{W}) = p_v^{d_v(2s-1)} \sum_{d_1+d_2=n} \sum_{\min(\nu_1,n)=d_1} p_v^{-2\nu_1(s-\frac{1}{2})} \Psi_{\nu_1,d_2}(W), \tag{38}$$

where

$$\Psi_{\nu_1,d_2}(W) = \int_{F_v^\times} \int_{K_v[d_2]} W_{\Pi_v}\big(z(\varpi_v^{\nu_1-d_v})a(y)\big) \overline{W}(a(y)k) |y|^{s-\frac{3}{2}} \, \mathrm{d}^\times y \, \mathrm{d}k.$$

Now, if $W = \widetilde{W_j}$ is an element of our basis, given by (37), then it follows that

$$\int_{K_v[f]} \widetilde{W_j}(hk) \, \mathrm{d}k = \begin{cases} \mathrm{vol}(K_v[f]) \widetilde{W_j}(h) & \text{if } j + n_0 \leq f, \\ 0 & \text{otherwise.} \end{cases} \tag{39}$$

We reason as follows: On the one hand, for every j, $\widetilde{W_j}$ is $K_v[n_0+j]$-invariant and is orthogonal to $\mathcal{W}(\pi_v, \psi_v)^{K_v[n_0+j-1]}$. On the other, the operator

$$W \mapsto \frac{1}{\mathrm{vol}(K_v[f])} \int_{K_v[f]} \pi_v(k) W \, \mathrm{d}k$$

is the orthogonal projection into the space of $K_v[f]$-invariant vectors.

Applying (38) and (39) to the definition of $H_v(\pi_v)$ and changing order of summation, we are led to

$$H_v(\pi_v) = \frac{p_v^{d_v(2-s-w)}}{L(s, \Pi_v \times \pi_v) L(w, \pi_v)} \sum_{d_1+d_2=n} \mathrm{vol}(K_v[d_2]) \sum_{j \leq d_2 - n_0} \sum_{\min(\nu_1,n)=d_1} p_v^{-2\nu_1(s-\frac{1}{2})}$$
$$\times \int_{F_v^\times} W_{\Pi_v}\begin{pmatrix} z(\varpi_v^{\nu_1-d_v})a(y) & \\ & 1 \end{pmatrix} \overline{\widetilde{W_j}}(a(y)) |y|^{s-\frac{3}{2}} \, \mathrm{d}^\times y \, \Psi_v(w, \widetilde{W_j}). \tag{40}$$

By letting $\lambda_{\pi_v,j}(\mu) = \widetilde{W_j}(a(\varpi_v^{\mu-d_v})) p^{\frac{1}{2}\mu}$ and using (18), we see that

$$\int_{F_v^\times} W_{\Pi_v}\begin{pmatrix} z(\varpi_v^{\nu_1-d_v})a(y) & \\ & 1 \end{pmatrix} \overline{\widetilde{W_j}}(a(y)) |y|^{s-\frac{3}{2}} \, \mathrm{d}^\times y$$
$$= p_v^{d_v(s-\frac{3}{2})} \sum_{\nu_2 \geq 0} \lambda_{\pi_v}(\nu_2, \nu_1) \overline{\lambda_{\pi_v,j}(\nu_2)} p_v^{-\nu_1} p_v^{-\nu_2 s} \tag{41}$$

and also

$$\Psi_v(w, \widetilde{W_j}) = p_v^{d_v(w-\frac{1}{2})} \sum_{\mu \geq 0} \lambda_{\pi_v,j}(\mu) p_v^{-\mu w}. \tag{42}$$

Inserting (41) and (42) in (40), we deduce that

$$H_v(\pi_v)=\frac{1}{L(s,\Pi_v\times\pi_v)L(w,\pi_v)}\times\sum_{d_1+d_2=n}\mathrm{vol}(K_v[d_2])\sum_{j\le d_2-n_0}\sum_{\min(\nu_1,n)=d_1}\sum_{\nu_2\ge 0}\sum_{\mu\ge 0}\frac{\lambda_{\Pi_v}(\nu_2,\nu_1)\overline{\lambda_{\pi_v,j}(\nu_2)}\lambda_{\pi_v,j}(\mu)}{p_v^{(2\nu_1+\nu_2)s}p_v^{\mu w}}.$$

Combining (35) and (37), we get

$$\lambda_{\pi_v,j}(\nu)=\langle W_0,W_0\rangle^{-\frac{1}{2}}\sum_{k=1}^{\min(j,k)}\xi_{\pi_v}(j,k)p_v^{k-\frac{1}{2}j}\lambda_{\pi_v}(\nu-k)\delta_{\nu\ge k}.$$

As a consequence, we deduce, after changing variables, that

$$\begin{aligned}H_v(\pi_v)&=\frac{\langle W_0,W_0\rangle^{-1}}{L(s,\Pi_v\times\pi_v)L(w,\pi_v)}\\&\times\sum_{d_1+d_2=n}\mathrm{vol}(K_v[d_2])\sum_{j\le d_2-n_0}p_v^{-j}\sum_{k_1,k_2\le j}\xi_{\pi_v}(j,k_1)\xi_{\pi_v}(j,k_2)p_v^{k_1(1-w)}p_v^{k_2(1-s)}p_v^{-2d_1s}\\&\times\sum_{\min(\nu_1,d_2)=0}\sum_{\nu_2\ge 0}\frac{\lambda_{\Pi_v}(\nu_2+k_2,\nu_1+d_1)\overline{\lambda_{\pi_v}(\nu_2)}}{p_v^{(2\nu_1+\nu_2)s}}\sum_{\mu\ge 0}\frac{\lambda_{\pi_v}(\mu)}{p_v^{\mu w}}.\end{aligned}$$

We recognize the last sum as $L(w,\pi_v)$, so that

$$\begin{aligned}H_v(\pi_v)=\frac{\langle W_0,W_0\rangle^{-1}}{L(s,\Pi_v\times\pi_v)}&\sum_{d_1+d_2=n}\mathrm{vol}(K_v[d_2])\sum_{j\le d_2-n_0}p_v^{-j}\sum_{k_1,k_2\le j}\xi_{\pi_v}(j,k_1)\xi_{\pi_v}(j,k_2)\\&\times p_v^{k_1(1-w)}p_v^{k_2(1-s)}p_v^{-2d_1s}\sum_{\min(\nu_1,d_2)=0}\sum_{\nu_2\ge 0}\frac{\lambda_{\Pi_v}(\nu_2+k_2,\nu_1+d_1)\overline{\lambda_{\pi_v}(\nu_2)}}{p_v^{(2\nu_1+\nu_2)s}}.\end{aligned}\tag{43}$$

We are now ready to prove the following

Proposition 7.1. *Let* $\Phi_v=\Phi_v^{\mathfrak{q},\mathfrak{l}}$ *be as in* (32), *and let* $n=v(\mathfrak{q})$. *Then for every* $\epsilon>0$ *there exists* $\delta>0$ *such that*

- (i) $H_v(\pi_v)$ *vanishes if* $c(\pi_v)>n$,
- (ii) $H_v(\pi_v)=\varphi(p_v^n)p_v^{-2n}$ *if* $c(\pi_v)=n$,
- (iii) $H_v(\pi_v)\ll_\epsilon p_v^{n(\theta-1+\epsilon)}$ *in general for* $\mathrm{Re}(s),\mathrm{Re}(w)>\frac{1}{2}-\delta$ *and* $\delta>0$ *sufficiently small.*

The first assertion follows by observing that if $n_0=c(\pi_v)>n$ then the sum over j in (43) will vanish independently of the value of d_2.

The second one holds because if $n_0=n$, we automatically have $d_2=n$ and $d_1=j=k_1=k_2=0$ and moreover

$$\langle W_0,W_0\rangle=\frac{\zeta_v(2)}{L_v(1,\pi_v\times\overline{\pi}_v)}\sum_{n\ge 1}\frac{|\lambda_{\pi_v}(n)|^2}{p_v^n}=\begin{cases}1 & \text{if } n_0=0,\\ \zeta_v(2) & \text{otherwise.}\end{cases}$$

Hence,

$$H_v(\pi_v) = \frac{\varphi(p_v^n) p_v^{-2n}}{L(s, \Pi_v \times \pi_v)} \sum_{\nu_2 \geq 0} \lambda_{\Pi_v}(\nu_2, 0) \lambda_{\pi_v}(\nu_2),$$

and we conclude by (19).

Finally, in order to show (iii), we apply the estimate in (36) and the bounds

$$\lambda_{\Pi_v}(\nu_1, \nu_2) \ll p_v^{(\nu_1+\nu_2)\theta}, \quad \lambda_{\pi_v}(\nu) \ll p_v^{\nu\vartheta}$$

to (43), which gives

$$H_v(\pi_v) \ll_\epsilon p_v^{n(-1+\epsilon)} \sum_{d_1+d_2=n} p^{2(d_1+j)\delta} \sum_{j=0}^{d_2-n_0} \sum_{k_1,k_2=0}^{j} p_v^{(k_1+k_2-2j)(\frac{1}{2}-\vartheta)} p_v^{(d_1+k_2)\theta}$$

for $\Re(s) > \frac{1}{2}\theta, \theta+\vartheta$ and it follows from the results in [Luo et al. 1999] and the Kim–Sarnak bound [2003, Appendix 2] that one has $\theta + \vartheta < \frac{1}{2}$. We conclude by taking δ sufficiently small.

7C. *Local computations, the archimedean case.* The analysis of the archimedean weight functions is of a somewhat different nature from the nonarchimedean case. For those places, we make the simplest choice imaginable. Namely we impose that Π_v is unramified and Φ_v is normalized spherical for every archimedean place v. As a consequence it easily follows that $H_v(\pi_v)$ vanishes unless π_v is itself unramified, in which case we may choose a basis of $B^W(\pi_v)$ such that each term corresponds to a different K-type, and then there will be at most one element W of $\mathcal{B}^W(\pi_v)$ for which the period $\Psi_v(s, W_{\Phi_v}, W)$ is nonvanishing, and it must be a spherical vector for π_v. Moreover, it follows from Stade's formula [2001, Theorem 3.4] that

$$\Psi_v(s, W_{\Phi_v}, \overline{W}) = L_v(s, \Pi_v \times \pi_v), \quad \Psi_v(w, W) = L_v(w, \pi_v).$$

where $W = W_{\pi_v} \in \mathcal{W}(\pi_v, \psi_v)$ is spherical and such that $\vartheta_v(W, W) = 1$. In particular, the following holds:

Proposition 7.2. *Let v be an archimedean place of F. Let Π_v be an irreducible admissible generic representation for $\mathrm{GL}_3(F_v)$. Then there exists a vector $\Phi_v \in \Pi_v$ such that for every irreducible admissible generic representation for $\mathrm{GL}_2(F_v)$, we have*

$$H(\pi_v) = \begin{cases} 1 & \text{if } \pi_v \text{ is unramified,} \\ 0 & \text{otherwise.} \end{cases}$$

7D. *Meromorphic continuation with respect to the spectral parameter.* Let $\Phi^{\mathfrak{q},\mathfrak{l}}$ be as in (32), and let H be the (s, w)-weight function of kernel $\Phi^{\mathfrak{q},\mathfrak{l}}$. Our goal in this section is to find that for any unitary character ω of $F_v^\times$ there is a domain of $\mathbb{C}^3$ on which the function

$$(s, w, t) \mapsto H_v(\pi_v)$$

is meromorphic with respect to all three variables with only finitely many polar divisors, where $\pi_v = \pi_v(\omega_v, it)$ (see Section 3A1).

From our computations so far, we know that $H_v(\pi_v) = 1$ unless $v \mid \mathfrak{l}$ or $v \mid \mathfrak{q}$. Moreover, in the first of these cases, we saw that

$$H_v(\pi_v) = p_v^{-mw}\left(\lambda_{\pi_v}(m) - \frac{\lambda_{\pi_v}(m-1)}{p_v^w}\right), \quad m = v(\mathfrak{l}),$$

which is clearly an entire function with respect to s, w and t, since it is a combination of terms of the shape $p_v^{\alpha w+\beta t}$, where $\alpha, \beta \in \mathbb{C}$.

We are now left with the case where $v \mid \mathfrak{q}$. It follows from the proof of Proposition 7.1 that given $\eta < \frac{1}{2}$, there exists $\delta > 0$ such that the right-hand side of (43) converges in the region

$$|\mathrm{Im}(t)| < \eta, \quad \mathrm{Re}(s), \mathrm{Re}(w) > \tfrac{1}{2} - \delta,$$

and defines in it a holomorphic function in the variables s and w. We observe that $H_v(\pi_v)$ is a linear combination of terms of the shape

$$L_{\omega,k_2,d_1,d_2}(s,t) := \frac{1}{L(s, \Pi_v \times \pi_v)} \sum_{\min(\nu_1,d_2)=0} \sum_{\nu_2 \geq 0} \frac{\lambda_{\Pi_v}(\nu_2+k_2, \nu_1+d_1)\overline{\lambda_{\pi_v}(\nu_2)}}{p_v^{(2\nu_1+\nu_2)s}},$$

with coefficients given by meromorphic functions in the variables s, w and t. The only possible polar divisors occur for t satisfying $\omega(\varpi_v)^2 p_v^{2it} = p_v^{\pm 1}$, due to the term $(1-\alpha_{\pi_v}^2)^{-1}$ appearing as a factor of $\xi_{\pi_v}(j,k_1)\xi_{\pi_v}(j,k_2)$. Moreover, it follows from [Blomer and Khan 2019a, Lemma 14], applied to the tuple $(M, d, g_1, g_2, q) = (p_v^{k_2}, p_v^{d_1}, 1, p_v^{d_2}, p_v^n)$, that for any $\epsilon > 0$, there exists $\delta > 0$ such that $L_{\omega,k_2,d_1,d_2}(s,t)$ admits a holomorphic continuation to the region

$$\mathrm{Re}(s) > \tfrac{1}{4} - \delta, \quad \mathrm{Re}(s) \pm \mathrm{Im}(t) > -\delta. \tag{44}$$

Moreover, using again the Ramanujan bound for $\lambda_{\Pi_v}(\nu_2,\nu_2)$ and recalling that $\pi_v = \pi_v(\omega_v, it)$, so that $\lambda_{\pi_v}(\nu) \ll p_v^{\nu|\mathrm{Im}(t)|+\epsilon}$, we see that in the region (44) we have

$$L_{\omega,k_2,d_1,d_2}(s,w,t) \ll p_v^{(d_1+k_2)(\theta+\epsilon)}. \tag{45}$$

As a consequence, $H_v(\pi_v)$ admits meromorphic continuation to (44). Now, suppose $\omega = \mathbf{1}$ is the trivial character, and let

$$D_v(s,w) := H_v(\pi_v)|_{t=(1-w)/i}. \tag{46}$$

From what we have just seen, $D_v(s,w) = 1$ unless $v \mid \mathfrak{l}$ or $v \mid \mathfrak{q}$. In the first case, it is clear that $D_v(s,w)$ is entire with respect to both s and w. Also, when $\frac{1}{2} \leq \mathrm{Re}(s), \mathrm{Re}(w) < 1$, we have $\lambda_{\pi_v}(m) \ll p_v^{m(1-\mathrm{Re}(w))}$, and thus by (34), we see that

$$D_v(s,w) \ll 1.$$

Finally, if $v \mid \mathfrak{q}$, then for sufficiently small $\delta > 0$, D_v is meromorphic in the region

$$\tfrac{1}{2} - \delta < \mathrm{Re}(s), \mathrm{Re}(w) \leq 1, \tag{47}$$

where the only possible polar divisors are at the values of w such that $p_v^{2-2w} = p_v$. We will now show that such poles cannot occur. To see this, let

$$E_{j,k_2}(w) := \xi_{\pi_v}(j,k_2) \sum_{k_1=0}^{j} \xi_{\pi_v}(j,k_1) p_v^{k_1(1-w)}.$$

An easy computation shows that

$$E_{j,k_2}(w) = \begin{cases} 1 & \text{if } j=0, \\ t_{j,k_2}(w)/(1-p_v^{2w-3}) & \text{if } j=1, \\ 0 & \text{otherwise,} \end{cases}$$

where t_{j,k_2} is an entire function. This and the fact that $L_{\omega,k,d_1,d_2}(s,w,(1-w)/i)$ is holomorphic in (47) are enough to guarantee that $H_v(\pi_v)$ is holomorphic in the same region. Moreover, we may argue analogously to Proposition 7.1(iii), appealing to (45), to deduce that for $\frac{1}{2} \le \mathrm{Re}(s), \mathrm{Re}(w) < 1$, we have the inequality

$$D_v(s,w) \ll_{s,w,\epsilon} p_v^{n(-1+\theta+\epsilon)} \sum_{d_1+d_2=n} \sum_{j=0}^{d_2-n_0} p_v^{j(1-2\,\mathrm{Re}(w))} + p_v^{j(1-\mathrm{Re}(s)-\mathrm{Re}(w))} \ll_\epsilon p_v^{n(-1+\theta+\epsilon)}.$$

We now summarize what we obtained in this subsection as follows:

Proposition 7.3. *Let $\omega = \bigotimes_v' \omega_v$ be an unitary character of $F^\times\backslash\mathbb{A}^\times$, and let H_v be given by* (29) *with $\Phi_v = \Phi_v^{\mathfrak{q},\mathfrak{l}}$, with $\Phi_v^{\mathfrak{q},\mathfrak{l}}$ given by* (32)*. Then $(s,w,t) \mapsto H_v(\pi_v(\omega_v, it))$ admits meromorphic continuation to the region* (44) *with possible polar divisor of the form $t = t_0$, where t_0 is a solution to $\omega(\varpi_v)^2 p_v^{2it_0} = p_v^{\pm 1}$. Moreover, if D_v is given by* (46)*, then it admits a holomorphic continuation to the region* (47) *and if $\frac{1}{2} \le \mathrm{Re}(s), \mathrm{Re}(w) < 1$, it satisfies $D_v(s,w) = 1$ unless $v \mid \mathfrak{q}\mathfrak{l}$, in which case*

$$D_v(s,w) \ll_{s,w,\epsilon} \begin{cases} p_v^{m\epsilon} & \text{if } v \mid \mathfrak{l}, \\ p_v^{n(-1+\theta+\epsilon)} & \text{if } v \mid \mathfrak{q}. \end{cases}$$

8. The degenerate term

In this section we study the term $\mathcal{D}_{s,w}(\Phi)$ given by (30) and its companion $\mathcal{D}_{s',w'}(\breve{\Phi})$. First, by rapid decay of Whittaker functions and the action of the Weyl group of GL(3), we may see that both converge for any values of $s, w \in \mathbb{C}$. This is all that is needed to know with respect to these terms for Theorem 1.1.

Let us now turn to their use in Theorem 1.2. Here we make the specialization to $\Phi = \Phi^{\mathfrak{q},\mathfrak{l}}$. It turns out that is easier to study first the term $\mathcal{D}_{s',w'}(\breve{\Phi})$, so we start with this one and later deduce an analogous result for the other by using their symmetry. First, we recall that

$$\mathcal{D}_{s',w'}(\breve{\Phi}) = 2d_F^{\frac{7}{2}-3s'-w'} \int_{F^\times\backslash\mathbb{A}^\times} \int_{F\backslash\mathbb{A}} \int_{F^\times\backslash\mathbb{A}^\times} \breve{\Phi}\begin{pmatrix} z(u)n(x)a(y) & \\ & 1 \end{pmatrix} |u|^{2s'-1} |y|^{s'+w'-1} \, \mathrm{d}^\times u \, \mathrm{d}x \, \mathrm{d}^\times y. \quad (48)$$

We will show that, in the region

$$\mathrm{Re}(3s+w) > 1, \quad \mathrm{Re}(s+w), \quad \mathrm{Re}(2s) > \theta, \quad (49)$$

$\mathcal{D}_{s',w'}(\breve{\Phi}) \ll_{s,w,\epsilon} \ell^{\theta-\mathrm{Re}(s)-\mathrm{Re}(w)+\epsilon}$, where ℓ is the absolute norm of $\mathfrak{l}$.

We begin by noticing that, using the definition of $\check{\Phi}$, reversing the change of variables used in the proof of Proposition 5.1 and changing the order of summation, we see that the integral in (48) equals

$$\int_{(F^\times\backslash\mathbb{A}^\times)^2}\left(\int_{F\backslash\mathbb{A}}\Phi\left(\begin{pmatrix}1&&x\\&1&\\&&1\end{pmatrix}\begin{pmatrix}z(u)a(y)&\\&1\end{pmatrix}\right)\mathrm{d}x\right)|u|^{2s-1}|y|^{s+w-1}\,\mathrm{d}^\times u\,\mathrm{d}^\times y.$$

By the Whittaker expansion of Φ, the inner integral is

$$\int_{F\backslash\mathbb{A}}\sum_{\gamma\in N_2(F)\backslash\mathrm{GL}_2(F)}W_\Phi\left(\begin{pmatrix}\gamma&\\&1\end{pmatrix}\begin{pmatrix}1&&x\\&1&\\&&1\end{pmatrix}\begin{pmatrix}z(u)a(y)&\\&1\end{pmatrix}\right)\mathrm{d}x,$$

which, by elementary manipulations and changing the order of summation and integration, becomes

$$\sum_{\gamma\in N_2(F)\backslash\mathrm{GL}_2(F)}W_\Phi\begin{pmatrix}\gamma z(u)a(y)&\\&1\end{pmatrix}\int_{F\backslash\mathbb{A}}\psi(\gamma_{21}x)\,\mathrm{d}x,$$

where γ_{21} is the lower left entry of γ. Since $\gamma_{21}\in F$, the inner integral vanishes unless $\gamma_{21}=0$, in which case, it equals one. In other words, we may change the sum over $N_2(F)\backslash\mathrm{GL}_2(F)$ into a sum over $N_2(F)\backslash B_2(F)$, which can be parametrized by $Z_2(F)A_2(F)$. Altogether, this implies that

$$\begin{aligned}\mathcal{D}_{s',w'}(\check{\Phi})&=2d_F^{\frac{7}{2}-3s'-w'}\int_{(F^\times\backslash\mathbb{A}^\times)^2}\sum_{\gamma\in Z_2(F)A_2(F)}W_\Phi\begin{pmatrix}\gamma z(u)a(y)&\\&1\end{pmatrix}|u|^{2s-1}|y|^{s+w-1}\,\mathrm{d}^\times u\,\mathrm{d}^\times y\\&=2d_F^{\frac{7}{2}-3s'-w'}\int_{(\mathbb{A}^\times)^2}W_\Phi\begin{pmatrix}z(u)a(y)&\\&1\end{pmatrix}|u|^{2s-1}|y|^{s+w-1}\,\mathrm{d}^\times u\,\mathrm{d}^\times y.\qquad(50)\end{aligned}$$

Suppose that $\mathrm{Re}(s)$ and $\mathrm{Re}(w)$ are sufficiently large. We are now in a fairly advantageous position, as the integral above can be factored into local ones. These local integrals are

$$\mathcal{J}_v=\int_{(F_v^\times)^2}W_{\Phi_v}\begin{pmatrix}z(u)a(y)&\\&1\end{pmatrix}|u|^{2s-1}|y|^{s+w-1}\,\mathrm{d}^\times u\,\mathrm{d}^\times y.$$

We notice that for a finite place v for which Π_v is unramified, this equals

$$p_v^{d_v(3s+w-2)}\sum_{\nu_1,\nu_2\geq 0}\frac{\lambda_{\Pi_v}(\nu_1,\nu_2)}{p_v^{\nu_1(s+w)+2\nu_2 s}},$$

whose inner sum we recognize as being the local factor of Bump's double Dirichlet series (see, e.g., [Goldfeld 2006, §6.6]). In particular, it follows that for $\mathrm{Re}(s+w),\mathrm{Re}(2s)>\theta$ (recall the bound $\lambda_{\Pi_v}(\nu_1,\nu_2)\ll p_v^{(\nu_1+\nu_2)\theta}$) the above equals

$$\mathcal{J}_v^0:=p_v^{d_v(3s+w-2)}\frac{L(s+w,\Pi_v)L(2s,\overline{\Pi}_v)}{\zeta_v(3s+w)}\asymp_{s,w}1.\qquad(51)$$

As for the remaining places, we first observe that for $v\mid\mathfrak{q}$, the unipotent averaging has no effect on the values of the Whittaker function at diagonal element. Thus, it follows that the local integral $\mathcal{J}_v$ will also coincide with (51). Furthermore, (51) also holds for archimedean v. For real places this is done in [Bump 1984] and for the complex places this is [Bump and Friedberg 1989, Theorem 1]. Finally, for $v \mid \mathfrak{l}$,

$$W_{\Phi_v}\begin{pmatrix} z(u)a(y) & \\ & 1 \end{pmatrix} = \delta_{v(y)\geq m-d_v} W_{\Pi_v}\begin{pmatrix} z(u)a(y) & \\ & 1 \end{pmatrix},$$

where $m = v(\mathfrak{l})$. Hence, in this case, the local factor is

$$\mathcal{J}_v = p_v^{d_v(3s+w-2)} \sum_{\nu_1\geq m, \nu_2\geq 0} \frac{\lambda_{\Pi_v}(\nu_1,\nu_2)}{p_v^{\nu_1(s+w)+2\nu_2 s}},$$

which, by using yet again the Ramanujan bound for $\lambda_{\Pi_v}(\nu_1,\nu_2)$, we may see converges in the region $\mathrm{Re}(s+w), \mathrm{Re}(2s) > \theta$, where it satisfies

$$\mathcal{J}_v \ll_\epsilon p_v^{m(\theta-\mathrm{Re}(s)-\mathrm{Re}(w)+\epsilon)}.$$

In particular, if $\mathfrak{l} = 1$,

$$\mathcal{D}_{\frac{1}{2},\frac{1}{2}}(\check{\Phi}) = 2d_F^{\frac{3}{2}} \frac{\Lambda(1,\Pi)\Lambda(1,\overline{\Pi})}{\xi_F(2)}. \tag{52}$$

Now, notice that $\mathcal{D}_{s,w}(\Phi)$ is the same as $\mathcal{D}_{s',w'}(\check{\Phi})$ but with $(\mathfrak{q},\mathfrak{l},s,w)$ replaced by $(\mathfrak{l},\mathfrak{q},s',w')$. This allows us to immediately reuse our efforts in this section to study the latter function as well. We record the results for both these functions in a weaker form in the following proposition.

Proposition 8.1. *Let $\mathcal{D}_{s,w}(\Phi)$ and $\mathcal{D}_{s,w}(\Phi)$ be as defined in* (30) *with $\Phi = \Phi^{\mathfrak{q},\mathfrak{l}}$ and $\check{\Phi}$ given by* (25). *Then they are entire functions of s and w, and in the region*

$$\mathrm{Re}(s), \mathrm{Re}(w), \mathrm{Re}(s'), \mathrm{Re}(w') > \tfrac{1}{4},$$

they satisfy

$$\mathcal{D}_{s,w}(\Phi) \ll_{s,w,\epsilon} q^{\theta-\mathrm{Re}(s+w)+\epsilon} \quad \textit{and} \quad \mathcal{D}_{s',w'}(\check{\Phi}) \ll_{s,w,\epsilon} \ell^{\theta-\mathrm{Re}(s+w)+\epsilon}.$$

9. Analytic continuation of the Eisenstein part

The conclusion of our next proposition will be subject to the following hypothesis, whose verification when $\Phi = \Phi^{\mathfrak{q},\mathfrak{l}}$ follows from the main results of Section 7:

Hypothesis 1. There exists $\delta > 0$ such that for every idele character ω, the function

$$(s,w,t) \mapsto H(\pi(\omega,it))$$

is holomorphic in the region

$$\mathrm{Re}(s), \mathrm{Re}(w) > \tfrac{1}{2} - \delta, \quad |\mathrm{Im}(t)| < \delta.$$

Moreover, $H(\pi(\omega,(1-w))$ admits a holomorphic continuation to the region

$$\tfrac{1}{2} - \delta < \mathrm{Re}(s), \mathrm{Re}(w) < 1.$$

We will show that the term $\mathcal{E}_{s,w}(\Phi)$ admits meromorphic continuation for values of s and w with real parts smaller than 1. The proof follows the same lines as those of Blomer and Khan [2019a, Lemma 16; 2019b, Lemma 3].

Proposition 9.1. *Suppose that* Π *is a cuspidal automorphic representation, and let* $\Phi \in \Pi$ *be an automorphic form such that the associated weight function* H *satisfies Hypothesis 1 for some* $\delta > 0$. *Let* $\mathcal{E}_{s,w}(\Phi)$ *be given by* (3), *defined initially for* $\mathrm{Re}(s), \mathrm{Re}(w) \gg 1$. *It admits a meromorphic continuation to* $\mathrm{Re}(s), \mathrm{Re}(w) \geq \frac{1}{2} - \epsilon$ *for some* $\epsilon > 0$ *with at most finitely many polar divisors. If* $\frac{1}{2} \leq \mathrm{Re}(s), \mathrm{Re}(w) < 1$, *its analytic continuation is given by* $\mathcal{E}_{s,w}(\Phi) + \mathcal{R}_{s,w}(\Phi)$, *where*

$$\mathcal{R}_{s,w}(\Phi) = \sum_{\pm} \operatorname*{res}_{t=\pm(1-w)/i} (\pm i) \frac{\Lambda(s+it,\Pi)\Lambda(s-it,\Pi)\xi_F(w+it)\xi_F(w-it)}{\xi_F^*(1)\xi_F(1+2it)\xi_F(1-2it)} H(\pi(\mathbf{1},it)). \tag{53}$$

Proof. Let $\delta > 0$ to be chosen later. We use nonvanishing of completed Dirichlet L-functions $\Lambda(s,\omega)$ at $\mathrm{Re}(s) = 1$ and continuity to define a continuous function $\sigma : \mathbb{R} \mapsto (0,\delta)$ so that neither $\Lambda(1-2\sigma-2it,\omega^2)$ nor $H(\pi(\omega, it+\sigma))$ have poles for $0 \leq \sigma < \sigma(t)$.

We start by noticing that we can *analytically* continue $\mathcal{E}_{s,w}(\Phi)$ to $\mathrm{Re}(s), \mathrm{Re}(w) > 1$, since in that region, one does not encounter any poles of $\Lambda(w, \pi(\mathbf{1}, it))$. Now, suppose that

$$1 < \mathrm{Re}(s) < 1 + \sigma(\mathrm{Im}(s)) \quad \text{and} \quad 1 < \mathrm{Re}(w) < 1 + \sigma(\mathrm{Im}(w)).$$

We shift the contour of the integral defining $\mathcal{E}_{s,w}(\Phi)$ down to $\mathrm{Im}\, t = -\sigma(\mathrm{Re}(t))$. We pick up a pole of $\Lambda(w-it,\omega)$ when ω is the trivial character and $w - it = 1$.

We observe that in view of our choice for σ, the resulting integral defines a holomorphic function in the region

$$\begin{cases} 1 - \sigma(\mathrm{Im}(s)) < \mathrm{Re}(s) < 1 + \sigma(\mathrm{Im}(s)), \\ 1 - \sigma(\mathrm{Im}(t)) < \mathrm{Re}(w) < 1 + \sigma(\mathrm{Im}(w)). \end{cases}$$

Take now s and w satisfying $1 - \sigma(\mathrm{Im}(s)) < \mathrm{Re}(s) < 1$ and $1 - \sigma(\mathrm{Im}(t)) < \mathrm{Re}(w) < 1$. We may shift the contour back to the real line and pick a new pole when ω is trivial and at $w + it = 1$. This proves the desired formula for $1 - \sigma(\mathrm{Im}(s)) < \mathrm{Re}(s) < 1$ and $1 - \sigma(\mathrm{Im}(t)) < \mathrm{Re}(w)$ and it follows in general by analytic continuation to all s, w such that $\frac{1}{2} - \delta < \mathrm{Re}(s), \mathrm{Re}(w) < 1$ by Proposition 7.3. ☐

10. Conclusion

In this section we put together the results of the last three sections and deduce Theorem 1.2. We have seen in Proposition 6.1 that for sufficiently large values of $\mathrm{Re}(s)$ and $\mathrm{Re}(w)$, we have the relation

$$2d_F^{\frac{7}{2}-3s-w} I(w, \mathcal{A}_s\Phi) = \mathcal{M}_{s,w}(\Phi) + \mathcal{D}_{s,w}(\Phi).$$

If we assume that H satisfies Hypothesis 1, then we may apply Proposition 9.1, and deduce that, for $\frac{1}{2} - \delta < \mathrm{Re}(s), \mathrm{Re}(w) < 1$,

$$2d_F^{\frac{7}{2}-3s-w} I(w, \mathcal{A}_s\Phi) = \mathcal{M}_{s,w}(\Phi) + \mathcal{D}_{s,w}(\Phi) + \mathcal{R}_{s,w}(\Phi). \tag{54}$$

Now suppose that $\check{H}$ also satisfies Hypothesis 1 and that $\frac{1}{2} < \mathrm{Re}(s) \le \mathrm{Re}(w) \le \frac{3}{4}$. The last assertion implies that

$$\tfrac{1}{2} \le \mathrm{Re}(s'), \mathrm{Re}(w') < 1.$$

Thus, we may deduce that (54) also holds with H, s and w replaced by $\check{H}$, s' and w', respectively. The main equality in Theorem 1.2 is now a direct consequence of Proposition 5.1 and the description of the local weights $H_v(\pi_v)$ from Section 7. In particular, we showed in Section 7D that the weight function associated to $\Phi^{\mathfrak{q},\mathfrak{l}}$ satisfies Hypothesis 1. As for the inequality (7), it follows from (53) and Propositions 7.3 and 8.1.

10A. ***Proof of Corollary 1.3.*** We use Theorem 1.2 with $s = w = \frac{1}{2}$, $\mathfrak{l} = \mathfrak{o}_F$ and $\mathfrak{q} = \mathfrak{p}$, a prime ideal. We obtain that

$$\mathcal{M}(\Phi) = \mathcal{D}(\check{\Phi}) + \mathcal{R}(\check{\Phi}) - \mathcal{D}(\Phi) - \mathcal{R}(\Phi) + \mathcal{M}(\check{\Phi}),$$

where we dropped the $\frac{1}{2}, \frac{1}{2}$ from the index for brevity. It follows from Proposition 9.1 and the fact that $\widehat{\lambda}_{\pi(\mathbf{1},\pm\frac{1}{2})}\big(\frac{1}{2}, \mathfrak{q}\big) = 1$ for any ideal $\mathfrak{q}$ that

$$\mathcal{R}(\check{\Phi}) = 2\frac{\Lambda(1,\Pi)\Lambda(0,\Pi)}{\xi_F(2)}.$$

Furthermore, we have from (52) that

$$\mathcal{D}(\check{\Phi}) = 2d_F^{\frac{3}{2}}\frac{\Lambda(1,\Pi)\Lambda(1,\overline{\Pi})}{\xi_F(2)} = 2\frac{\Lambda(1,\Pi)\Lambda(0,\Pi)}{\xi_F(2)}.$$

Moreover, in view of Propositions 7.1 and 8.1, we may obtain that

$$\mathcal{R}(\Phi), \mathcal{D}(\Phi) \ll (N\mathfrak{p})^{\theta-1+\epsilon}$$

and

$$\mathcal{M}(\Phi) = \frac{\varphi(q)}{q^2} \sum_{\substack{\pi \text{ cusp}^0 \\ \mathrm{cond}(\pi)=\mathfrak{p}}} \frac{\Lambda\big(\frac{1}{2}, \Pi\times\pi\big)\Lambda\big(\frac{1}{2},\pi\big)}{\Lambda(1,\mathrm{Ad},\pi)} + O(q^{\theta-1}\mathcal{M}^*),$$

where

$$\mathcal{M}^* := \sum_{\substack{\pi \text{ cusp}^0 \\ \mathrm{cond}(\pi)=\mathfrak{o}_F}} \frac{\big|\Lambda\big(\frac{1}{2}, \Pi\times\pi\big)\Lambda\big(\frac{1}{2},\pi\big)\big|}{|\Lambda(1,\mathrm{Ad},\pi)|} + \sum_{\substack{\omega\in\widehat{F^\times U_\infty\backslash\mathbb{A}^\times_{(1)}} \\ \mathrm{cond}(\omega)=\mathfrak{o}_F}} \int_{-\infty}^{\infty} \frac{\big|\Lambda\big(\frac{1}{2}, \Pi\times\pi(\omega,it)\big)\Lambda\big(\frac{1}{2},\pi(\omega,it)\big)\big|}{|\Lambda^*(1,\mathrm{Ad},\pi(\omega,it))|}\frac{\mathrm{d}t}{2\pi}.$$

Finally, it is easy to see that we also have the bound

$$\mathcal{M}(\check{\Phi}) \ll q^{\vartheta-\frac{1}{2}}\mathcal{M}^*.$$

Corollary 1.3 will follow provided that one is able to show $\mathcal{M}^* \ll 1$. In other words, we just need to ensure that it converges since it is clearly independent of $\mathfrak{p}$. To see that, we notice that the finite part of $\Lambda\bigl(\frac{1}{2}, \Pi \times \pi\bigr)\Lambda\bigl(\frac{1}{2}, \pi\bigr)/\Lambda(1, \mathrm{Ad}, \pi)$ is bounded polynomially in terms of the eigenvalues of π_v for archimedean v and that, by Stirling's formula, we have, for archimedean v,

$$\frac{L\bigl(\frac{1}{2}, \Pi_v \times \pi_v\bigr)L\bigl(\frac{1}{2}, \pi_v\bigr)}{L(1, \mathrm{Ad}, \pi_v)} \ll |t_{\pi_v}|^C e^{-2c_{F_v}\pi|t_{\pi_v}|}$$

for $\pi_v = \pi_v(\mathbf{1}, i t_{\pi_v})$, where

$$c_{F_v} = \begin{cases} 1 & \text{if } F_v = \mathbb{R}, \\ 2 & \text{if } F_v = \mathbb{C}. \end{cases}$$

This implies that the factor $\Lambda\bigl(\frac{1}{2}, \Pi \times \pi\bigr)\Lambda(\frac{1}{2}, \pi)/\Lambda(1, \mathrm{Ad}, \pi)$ decays exponentially as the t_{π_v} grow and the convergence of $\mathcal{M}^*$ follows by appealing to the Weyl law for GL(2) over number fields (see [Palm 2012, Theorem 3.2.1]).

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Quadratic points on intersections of two quadrics

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We prove that a smooth complete intersection of two quadrics of dimension at least 2 over a number field has index dividing 2, i.e., that it possesses a rational 0-cycle of degree 2.

1. Introduction

The *index* of a variety over a field k is the greatest common divisor of the degrees $[\boldsymbol{k}(x) : k]$ ranging over the residue fields $\boldsymbol{k}(x)$ of the (zero-dimensional) closed points x of the variety. Equivalently, the index is the smallest positive degree of a k-rational 0-cycle.

Let $X \subset \mathbb{P}^n_k$ be a smooth complete intersection of two quadrics over a field k of characteristic not equal to 2. Then the index of X necessarily divides 4, because intersecting with a plane yields a 0-cycle of degree 4. In general, this is the best possible bound. Indeed, there are examples with index 4 over local and global fields when $n = 3$ [Lang and Tate 1958, Theorem 7] and over fields of characteristic 0 when $n = 4$, as we show in Theorem 7.6.

Our main result is the following sharp bound on the index when $n \geq 4$ and k is a number field or a local field.

Theorem 1.1. *Let X be a smooth complete intersection of two quadrics in $\mathbb{P}^n_k$ with $n \geq 4$ and assume that k is either a number field or a local field. Then the index of X divides* 2.

This result allows us to complete the list of integers which occur as the index of a del Pezzo surface over a local field or a number field (see Section 7D). It also allows us to deduce nontrivial index bounds for other interesting classes of varieties. In particular, if C/k is a genus 2 curve over a number field with a rational Weierstrass point, then it follows from the result above that any torsor of period 2 under the Jacobian of C has index dividing 8 (see Theorem 7.7) and the corresponding Kummer variety, which is an intersection of 3 quadrics in $\mathbb{P}^5$, has index dividing 4 (see Remarks 7.8). Again, these results fail for arbitrary fields (see Remarks 7.8). Theorem 1.2 below shows that Theorem 1.1 also holds for global function fields of odd characteristic when $n \geq 5$ and conditionally in a number of cases when $n \geq 4$.

Theorems of Amer [1976], Brumer [1978] and Springer [1956] show that, for X as above, index 1 is equivalent to the existence of a k-rational point. Analogously one can ask if index 2 implies the existence of a closed point of degree 2. Colliot-Thélène has recently sketched an argument that if X is a smooth complete intersection of two quadrics in $\mathbb{P}^4$ over a field of characteristic 0 and X has index 2, then X has

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a closed point of degree 14, 6 or 2. Our next result identifies conditions under which we can prove that a smooth intersection of two quadrics in $\mathbb{P}^n$ has a closed point of degree 2. In order to state it we introduce the following notation: We say that a global field k satisfies ($\star$) if Brauer–Manin is the only obstruction to the Hasse principle for del Pezzo surfaces of degree 4 over all quadratic extensions of k.

Theorem 1.2. *Let $n \geq 4$ and let $X \subset \mathbb{P}^n_k$ be a smooth complete intersection of two quadrics over a field k. In any of the following cases there is a quadratic extension K/k such that $X(K) \neq \varnothing$:*

(1) *k is a local field and $n \geq 4$.*

(2) *k is a global function field and $n \geq 5$.*

(3) *k is a global function field of characteristic 2 and $n = 4$.*

(4) *k is a number field that satisfies Schinzel's hypothesis and $n \geq 5$.*

(5) *k is a global field that satisfies ($\star$) or a number field that satisfies Schinzel's hypothesis, $n = 4$ and the following holds: for any quadratic field extension L/k and rank 4 quadric $\mathcal{Q} \subset \mathbb{P}^4_L$ such that $X = \bigcap_{\sigma \in \mathrm{Gal}(L/k)} \sigma(\mathcal{Q})$ and $\mathrm{Norm}_{L/k}(\mathrm{disc}(\mathcal{Q})) \in k^{\times 2}$, we have that $\mathcal{Q}$ fails to have smooth local points at an* ***even*** *number of primes of L.*

When $n = 4$, there are exactly five rank 4 quadrics in the pencil of quadrics containing X (see Section 4 for details). The condition in case (5) holds for most intersections of quadrics and can be easily checked. In particular, it is satisfied if there is no pair of Galois conjugate rank 4 quadrics in the pencil or if X has points everywhere locally (for then any quadric containing X will have points over all completions). In fact, if X is assumed everywhere locally solvable, the proofs of our main results become much easier (see Corollary 3.4 and Remark 4.8). For further details of the cases covered (and not covered) in case (5), see Remark 6.2 and Section 7A.

Theorem 1.2(5) naturally raises the question of whether the parity condition is necessary. We have constructed many examples that fail this parity condition, but in each we have found an ad hoc proof that ($\star$) implies the existence of a quadratic point. Based on our results and this extensive numerical evidence, we expect the following question to have a positive answer.

Question 1.3. Does every complete intersection of 2 quadrics $X \subset \mathbb{P}^4_k$ over a number field k possess a K-rational point for some quadratic extension K/k?

One can also pose this question for other classes of fields, e.g., C_r fields. Over C_3 fields, the question has a negative answer (see Section 7C for examples), but it is open for C_2 fields.

1A. ***Obstructions to index 1 over local and global fields.*** Over local and global fields, necessary and sufficient conditions for an intersection of two quadrics to have index 1 (equivalently, to have a rational point) have been well studied. When k is a local field and $n \leq 7$ there are examples with $X(k) = \varnothing$ (which necessarily have index greater than 1), while for $n \geq 8$ and k a p-adic field, $X(k) \neq \varnothing$ [Demyanov 1956]. For k a number field, Colliot-Thélène, Sansuc and Swinnerton-Dyer conjecture that a smooth complete intersection of quadrics in $\mathbb{P}^n_k$ satisfies the Hasse principle as soon as $n \geq 5$ [Colliot-Thélène et al. 1987b, Section 16]. For $n \geq 8$, the conjecture is proven in [Colliot-Thélène et al. 1987a; 1987b] and this has been extended to $n \geq 7$ by Heath-Brown [2018]. The analogue of this conjecture over global function fields of odd characteristic has been established by Tian [2017], allowing us to deduce case (2) from case (1) of Theorem 1.2.

When $n = 4$ (in which case X is a del Pezzo surface of degree 4), the Hasse principle can fail [Birch and Swinnerton-Dyer 1975]. Colliot-Thélène and Sansuc [1980] have conjectured that this failure is always explained by the Brauer–Manin obstruction. This conjecture implies that all number fields satisfy the condition ($\star$) appearing in Theorem 1.2(5). Most cases of the $n = 4$ conjecture have been proven conditionally on Schinzel's hypothesis and the finiteness of Tate–Shafarevich groups of elliptic curves by Wittenberg [2007]. This also gives a conditional proof of the Hasse principle when $n \geq 5$ as this can be reduced to cases of the $n = 4$ conjecture which are covered by Wittenberg's result.

1B. ***Outline of the proof of Theorems 1.1 and 1.2.*** Using an argument of Wittenberg [2007] (which we review in Section 6B), we can reduce to the case $n = 4$, when X is a del Pezzo surface of degree 4.

In Section 2 we prove that any del Pezzo surface of degree 4 over a local field must have points over some quadratic extension, which proves Theorem 1.2(1) and the local case of Theorem 1.1. Our approach uses the theorems of Amer, Brumer, and Springer to reduce to the case where no integral model of X has a special fiber that is split (i.e., contains a geometrically integral open subscheme) over a quadratic extension. We then use semistable models of degree 4 del Pezzo surfaces, introduced by Tian [2017], to directly show that the remaining types of degree 4 del Pezzo surfaces obtain points over *every* ramified quadratic extension of k.

In Section 2F, we give an easy generalization of a result in [Dolgachev and Duncan 2018], showing that, for k a field of characteristic 2, any del Pezzo surface of degree 4 obtains a point over $k^{1/2}$. For local and global fields of characteristic 2 we have $[k^{1/2} : k] = 2$, so this proves Theorem 1.2(3) and gives an alternate proof of Theorem 1.2(1) in characteristic 2. Thus, for the remainder of the paper, it suffices to assume that k is of characteristic different from 2.

Over a global field, the results of Section 2 show that after base change to a suitable quadratic extension X becomes everywhere locally solvable. While it is also true that the Brauer group of X becomes constant after a suitable quadratic extension (this can be deduced from the explicit calculation of $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ in [Várilly-Alvarado and Viray 2014]), one cannot deduce that Theorem 1.2 holds for fields k satisfying ($\star$) directly from case (1) in this way because, in general, there is no quadratic extension K/k for which X_K is locally solvable and the Brauer group of X_K is trivial modulo constant algebras (see Example 6.4).

To obtain our results when k is a global field of characteristic not equal to 2 we study the arithmetic of the symmetric square of X, which is birational to the variety $\mathcal{G}$ parametrizing lines on the quadrics in the pencil of quadrics in $\mathbb{P}^4_k$ containing X (see Section 4 for more details). In Section 5, we develop the main tools for studying the arithmetic of $\mathcal{G}$ over a global field. We determine explicit central simple algebras over the function field of $\mathcal{G}$ representing the Brauer group of $\mathcal{G}$ modulo constant algebras and then develop techniques to calculate the evaluation maps of these central simple algebras at several types of local points.

Theorem 1.2(1) implies that $\mathcal{G}$ is everywhere locally solvable. The results of Section 5 are used in Section 6 to show further that there is always an adelic 0-cycle of degree 1 on $\mathcal{G}$ orthogonal to the Brauer group and, under the hypothesis of Theorem 1.2(5), that there is an adelic point on $\mathcal{G}$ orthogonal to the Brauer group. This is perhaps surprising given that in this case the Brauer group of $\mathcal{G}$ can contain nonconstant algebras and in general can obstruct weak approximation on $\mathcal{G}$ (see Corollary 6.3 and Example 6.4).

The variety of lines on a smooth quadric 3-fold is a Severi–Brauer 3-fold, so the arithmetic of $\mathcal{G}$ is amenable to the fibration method, as first observed in [Colliot-Thélène and Sansuc 1982]. Results of Colliot-Thélène and Swinnerton-Dyer [1994] show that, in the number field case, the vanishing of the Brauer–Manin obstruction on $\mathcal{G}$ implies the existence of a 0-cycle of degree 1 on $\mathcal{G}$ and, conditionally on Schinzel's hypothesis, a k-rational point on $\mathcal{G}$. This yields a 0-cycle of degree 2 on X and, under the hypothesis of Theorem 1.2(5), a quadratic point on X if we assume Schinzel's hypothesis. To the best of our knowledge the function field analogue of these results based on the fibration method have not been established. This prevents us from considering global function fields in the $n = 4$ case of Theorem 1.1.

One can ask whether $\operatorname{index}(\mathcal{G}) = 1$ always implies that $\mathcal{G}$ has a rational point (when k is a global field this is equivalent to Question 1.3). Our results do not answer this question, but they do show that a stronger condition on 0-cycles fails over p-adic fields. Namely, $\mathcal{G}$ can contain 0-cycles of degree 1 that are *not* rationally equivalent to a rational point (see Remarks 7.4(1)).

To deduce the results in case (5) of Theorem 1.2 assuming that k satisfies ($\star$) (without assuming Schinzel), we make use of Proposition 3.6, which may be of interest in its own right. It relates the Brauer–Manin obstruction on the symmetric square of a variety that has finite Brauer group (modulo constant algebras) to the Brauer–Manin obstruction over quadratic extensions. (More generally, in Section 3 we collect results relating the Brauer–Manin obstruction on a nice variety Y to the Brauer–Manin over an extension which may also be of independent interest.) In a similar spirit, we answer a question posed in [Colliot-Thélène and Poonen 2000] concerning Brauer–Manin obstructions over extensions (see Remarks 7.4(2)) and give an example of a del Pezzo surface of degree 4 defined over $\mathbb{Q}$ which, for any finite extension $k/\mathbb{Q}$, has a Brauer–Manin obstruction to the existence of k-points if and only if k is of odd degree over $\mathbb{Q}$ (see Section 7B).

Notation. For a field k we use $\bar{k}$ to denote a separable closure and $G_k := \operatorname{Gal}(\bar{k}/k)$ to denote the absolute Galois group of k. In Sections 2 and 3, we allow k of arbitrary characteristic; in the remainder of the paper we restrict to k of characteristic different from 2. For k-schemes $Y \to \operatorname{Spec}(k)$ and $S \to \operatorname{Spec}(k)$ we define

$$Y_S := Y \times_{\operatorname{Spec}(k)} S \quad \text{and} \quad \overline{Y} = Y \times_{\operatorname{Spec} k} \operatorname{Spec}(\bar{k}).$$

When $S = \operatorname{Spec}(A)$ is the spectrum of a k-algebra A, we use the notation $Y_A := Y_{\operatorname{Spec}(A)}$. A *quadratic point* on Y is a morphism of k-schemes $\operatorname{Spec}(K) \to Y$, where K is an étale k-algebra of degree 2. In particular, $K = k \times k$ is allowed in which case $Z_K \simeq Z \times Z$ for any k-subscheme $Z \subset Y$.

The Brauer group of a scheme Y is the étale cohomology group $\mathrm{Br}(Y) := \mathrm{H}^2_{\text{ét}}(Y, \mathbb{G}_m)$; when $Y = \mathrm{Spec}(R)$ is the spectrum of a ring R we define $\mathrm{Br}(R) := \mathrm{Br}(\mathrm{Spec}\, R)$. If $s_Y : Y \to \mathrm{Spec}(k)$ is a k-scheme, then $\mathrm{Br}_0(Y) \subset \mathrm{Br}(Y)$ is the image of the pullback map $s_Y^* : \mathrm{Br}(k) \to \mathrm{Br}(Y)$. We use $\mathrm{Br}_1(Y)$ to denote the kernel of the map $\mathrm{Br}(Y) \to \mathrm{Br}(\overline{Y})$. We recall that there is a canonical injective map $\mathrm{Br}_1(Y)/\mathrm{Br}_0(Y) \to \mathrm{H}^1(k, \mathrm{Pic}(\overline{Y}))$ coming from the Hochschild–Serre spectral sequence [Colliot-Thélène and Skorobogatov 2021, Proposition 4.3.2] and that this map is an isomorphism if $\mathrm{H}^3(k, \mathbb{G}_m) = 0$.

An element $\beta \in \mathrm{Br}(Y)$ may be evaluated at a k-point $y : \mathrm{Spec}(k) \to Y$ by pulling back along y to obtain $\beta(y) := y^*\beta \in \mathrm{Br}(k)$. For a finite locally free morphism of schemes $Y \to Z$ we use $\mathrm{Cor}_{Y/Z} \colon \mathrm{Br}(Y) \to \mathrm{Br}(Z)$ to denote the corestriction map. When $Y = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$ are affine schemes this is also denoted by $\mathrm{Cor}_{A/B} \colon \mathrm{Br}(A) \to \mathrm{Br}(B)$.

A variety over k is a separated scheme of finite type over k. A variety is called *nice* if it is smooth, projective and geometrically integral and is called *split* if it contains an open subscheme that is geometrically integral.

If Y is an integral k-variety, $\boldsymbol{k}(Y)$ denotes its function field. More generally, if Y is a finite union of integral k-varieties Y_i, then $\boldsymbol{k}(Y) := \prod \boldsymbol{k}(Y_i)$ is the ring of global sections of the sheaf of total quotient rings. In particular, if a finite dimensional étale k-algebra A decomposes as a product $A \simeq \prod k_j$ of finite field extensions of k and Y is a reduced k-variety, then $\boldsymbol{k}(Y_A) \simeq \prod \boldsymbol{k}(Y_{k_j})$, and $\mathrm{Cor}_{\boldsymbol{k}(Y_A)/\boldsymbol{k}(Y)} = \sum \mathrm{Cor}_{\boldsymbol{k}(Y_{k_j})/\boldsymbol{k}(Y)}$.

For a global field k, we use Ω_k to denote the set of primes of k. For a prime $v \in \Omega_k$ we use k_v to denote the corresponding completion and for a k-scheme Y we set $Y_v := Y_{k_v}$. We use $\mathbb{A}_k$ to denote the adele ring of k. For a subgroup $B \subset \mathrm{Br}(Y)$, $Y(\mathbb{A}_k)^B \subset Y(\mathbb{A}_k)$ denotes the set of adelic points orthogonal to B, i.e.,

$$Y(\mathbb{A}_k)^B = \left\{ (y_v) \in Y(\mathbb{A}_k) : \forall \beta \in B, \sum_{v \in \Omega_k} \mathrm{inv}_v(\beta(y_v)) = 0 \right\}.$$

We define $Y(\mathbb{A}_k)^{\mathrm{Br}} := Y(\mathbb{A}_k)^{\mathrm{Br}(Y)}$.

2. Intersections of quadrics in $\mathbb{P}^4$ over local fields

Theorem 2.1. *Let $X \subset \mathbb{P}^4_k$ be a smooth complete intersection of two quadrics over a local field k. There is a quadratic extension K/k such that $X(K) \neq \varnothing$.*

Outline of proof of Theorem 2.1. In Section 2A, we prove that if there exists an integral model $\mathcal{X} \subset \mathbb{P}^4$ with split special fiber, then $X(k) \neq \varnothing$. We use this result to reduce to the case that the special fiber is a union of four planes permuted transitively by the Galois group. We then use the geometric classification results in Section 2C together with the existence of semistable models proved by Tian [2017] (following Kollár [1997]) to give explicit models of the remaining cases in Section 2D. Next, we study these explicit models and show directly that over every ramified quadratic extension there is a change of coordinates so that the model has split special fiber. Thus, by the results of Section 2A, these models have points over every ramified quadratic extension. The details of how the ingredients come together are in Section 2E.

Remark 2.2. The methods of this proof are fairly flexible, but it does rely on two key properties of finite fields:

(1) There is a unique quartic extension of any finite field, it is Galois, and the Galois group is cyclic.

(2) Every split variety over a finite field has index 1.

If k is a complete field with respect to a discrete valuation and its residue field satisfies the above two properties, then Theorem 2.1 holds over k.

As mentioned in the introduction, we also give alternate proofs of Theorem 2.1 which work in the case that k has odd residue characteristic (Section 4B) and in the case that k has characteristic 2 (Section 2F); this latter proof also holds for global fields of characteristic 2.

2A. *Intersections of quadrics with split special fiber.*

Proposition 2.3. *Let k be a nonarchimedean local field, let $\mathcal{O}$ denote the valuation ring of k, and let X/k smooth complete intersection of quadrics in $\mathbb{P}^4_k$. Assume there exists an integral model $\mathcal{X}/\mathcal{O}$ such that the special fiber is **split** (i.e., contains a geometrically integral open subscheme). Then $X(k) \neq \varnothing$.*

Proof. Since the special fiber is split, it contains a geometrically integral open subscheme $U^\circ/\mathbb{F}$. By the Hasse–Weil bounds, U° contains a smooth $\mathbb{F}'$-point for all extensions with sufficiently large cardinality. In particular, there exists an extension $\mathbb{F}'/\mathbb{F}$ of *odd* degree where U° has a smooth $\mathbb{F}'$-point. Thus, by Hensel's Lemma, X has a k'-point for k'/k an unramified extension of odd degree. Since X is an intersection of two quadrics, the theorems of Amer [1976], Brumer [1978] and Springer [1956] then imply that $X(k) \neq \varnothing$. (In characteristic 2, see [Elman et al. 2008, Corollary 18.5 and Theorem 17.14] for proofs of the Amer, Brumer and Springer theorems; the Amer and Brumer theorem in characteristic 2 is attributed to an unpublished preprint of Leep.) □

2B. *Ranks of a quadratic forms in arbitrary characteristic.*

Let q be a quadratic form on a vector space V over a field F. Then (by definition) the mapping $B_q : V \times V \to F$ given by $B_q(x, y) = q(x+y) - q(x) - q(y)$ is bilinear. We say that q is *regular* if the set $\{x \in V : q(x) = 0 \text{ and } \forall y \in V, B_q(x, y) = 0\}$ contains only the zero vector in V. (If the characteristic of F is not 2, then the condition $q(x) = 0$ is superfluous.) We say that q is *geometrically regular* if its base change to the algebraic closure of F is regular. Such forms are called *nondegenerate* in [Elman et al. 2008, Definition 7.17]. A quadratic form q on a vector space of dimension at least 2 is geometrically regular if and only if the quadric $\mathcal{Q}$ in $\mathbb{P}(V)$ defined by the vanishing of q is geometrically regular or, equivalently, smooth; see [Elman et al. 2008, Proposition 22.1].

The *rank* of a quadratic form q is the largest integer m such that there is a subspace $W \subset V$ of dimension m such that the restriction of q to W is geometrically regular, i.e., such that the intersection of $\mathcal{Q}$ with the linear space corresponding to W is smooth. The rank of a quadric in $\mathbb{P}^n$ is defined to be the rank of any quadratic form defining it. If F has characteristic different from 2, then the rank of q is the same as the rank of a symmetric matrix associated to B_q.

If $\mathrm{char}(F) = 2$, then the rank of q is not necessarily equal to the rank of (a matrix associated to) B_q, but the definition yields the lower bound $\mathrm{rank}(B_q) \leq \mathrm{rank}(q)$. The possible discrepancy between these two ranks is due to the fact that $B_q(x, x) = q(2x) - 2q(x) = 0$ for all $x \in V$. Thus a matrix associated to B_q has zeros along the diagonal and so is skew-symmetric (and symmetric). Skew symmetric matrices always have even rank, but quadratic forms can have odd rank (e.g., $q = x^2$ has rank 1).

Over an algebraically closed field a quadratic form q has rank $2n$ if and only if there is a change of coordinates such that $q = x_1x_2 + x_3x_4 + \cdots + x_{2n-1}x_{2n}$, and $\mathrm{rank}(q) = 2n + 1$ if and only if there is a change of coordinates such that $q = x_0^2 + x_1x_2 + x_3x_4 + \cdots + x_{2n-1}x_{2n}$; see [Elman et al. 2008, Propositions 7.29 and 7.31 and Example 7.34].[1] It follows from this characterization that a quadric in $\mathbb{P}^n$ of rank 1 with $n \geq 1$ is not geometrically reduced and a quadric in $\mathbb{P}^n$ of rank 2 with $n \geq 2$ is not geometrically irreducible.

It also follows that, for a quadratic form q over an algebraically closed field, the rank is the smallest integer r such that there exists a linear change of variables under which q becomes a quadratic form in the variables $x_1, \ldots, x_r$ alone. This is the definition of rank used in [Heath-Brown 2018]. We will only require the equivalence of these definitions over algebraically closed fields, but we note that they are also equivalent if the field is not of characteristic 2 (by the well known fact that q can be diagonalized) or if the field is perfect of characteristic 2 (as follows from [Elman et al. 2008, Proposition 7.31] using that in this case $c_1x_1^2 + \cdots + c_sx_s^2 = (c_1^{1/2}x_1 + \cdots + c_s^{1/2}x_s)^2$). In general, the two notions differ as seen by considering the rank 1 form $x_1^2 + tx_2^2 = (x_1 + t^{1/2}x_2)^2$ over $\mathbb{F}_2(t)$ for which there is no $\mathbb{F}_2(t)$-linear change of variables writing it as a form in 1 variable.

Lemma 2.4. *Suppose q and $\tilde{q}$ are quadratic forms of rank $r(q)$ and $r(\tilde{q})$, respectively, over a field F. Then $r(q \perp \tilde{q}) = r(q) + r(\tilde{q})$ except when $\mathrm{char}(F) = 2$ and $r(q)$ and $r(\tilde{q})$ are both odd, in which case $r(q \perp \tilde{q}) = r(q) + r(\tilde{q}) - 1$.*

Proof. For $\mathrm{char}(F) \neq 2$ see [Elman et al. 2008, Proposition 7.29]. For $\mathrm{char}(F) = 2$ this follows from [Elman et al. 2008, Proposition 7.31 and Remark 7.21] and the fact that an orthogonal direct sum of rank 1 forms has rank 1; see [Elman et al. 2008, Remark 7.24]. □

2C. *Intersections of two quadrics with many irreducible components.*

Lemma 2.5. *Let $X \subset \mathbb{P}^4$ be a reduced complete intersection of two quadrics over an algebraically closed field. If X is reducible, then X contains a 2-plane or an irreducible quadric surface. In addition*:

1. *If X contains an irreducible quadric surface, then X is the union of two quadric surfaces (with one possibly reducible) and X is contained in a rank 2 quadric.*
2. *If X contains two distinct 2-planes P_1, P_2, then X is either the union of four distinct 2-planes or the union of P_1 and P_2 with an irreducible quadric surface.*

[1] This characterization shows that, in general, the rank of the symmetric bilinear form can only differ from the rank of the quadratic form by 1, namely that $\mathrm{rank}(B_q) \leq \mathrm{rank}(q) \leq \mathrm{rank}(B_q) + 1$.

Proof. The components of this proof can be found in [Colliot-Thélène et al. 1987a, Section 1] and [Heath-Brown 2018, Proof of Lemma 3.2]. We repeat them here for the reader's convenience.

The degrees of the irreducible components of X sum to 4, so we consider the partitions

$$3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

In any of these cases, X contains a surface of degree 1 (i.e., a 2-plane) or a surface of degree 2 (i.e., a quadric surface). To complete the proof, it remains to show that if X contains an irreducible quadric surface, then X is contained in a rank 2 quadric, and in the case of the partition $2+1+1$, the union of the two planes is a quadric surface, i.e., is contained in a hyperplane.

Assume that X contains an irreducible quadric surface, given by the vanishing of a quadratic form q and a linear form ℓ. The rank of q cannot be 1 because X is reduced and the rank of q cannot be 2 because the quadric surface is irreducible. So q must have rank at least 3. Then the quadratic forms defining X must be of the form $cq+\ell\ell'$, for some constant c and some linear form ℓ'. There will be some linear combination of these where $c=0$, and so X is cut out by the ideal

$$\langle \ell\ell', q+\ell\ell''\rangle = \langle \ell, q\rangle \cdot \langle \ell', q+\ell\ell''\rangle,$$

for some linear forms ℓ', ℓ''. The first factor gives our original quadric surface, the residual factor will give a (possibly reducible) quadric surface, and $V(\ell\ell')$ is a rank 2 quadric hypersurface containing X. □

Lemma 2.6. *Let $X \subset \mathbb{P}^4$ be a complete intersection of two quadrics over an algebraically closed field. If X is the union of* 4 *distinct planes, then X is a cone and X is contained in a quadric hypersurface of rank* 2. *If, in addition, X has a unique cone point and there is cyclic subgroup of* $\operatorname{Aut}(X)$ *acting transitively on the irreducible components of X, then, up to an automorphism of $\mathbb{P}^4$, $X = V(x_0x_1, x_2x_3) \subset \mathbb{P}^4$.*

Proof. After a change of coordinates, we may assume that one of the planes is $V(x_0, x_1)$. If all pairs of the planes meet in a line, then we may assume that one of the other planes is $V(x_0, x_2)$. Thus, X must be defined by $x_0\ell = x_0\widetilde{\ell} + x_1x_2 = 0$ for some linear forms $\ell, \widetilde{\ell}$. Note that $x_0\ell$ has rank 2. If $x_0, x_1, x_2, \ell, \widetilde{\ell}$ are linearly dependent, then X is a cone. If $x_0, x_1, x_2, \ell, \widetilde{\ell}$ are linearly independent, then, without loss of generality, we may assume that $\ell = x_3$ and $\widetilde{\ell} = x_4$, so

$$X = V(x_0x_3, x_0x_4 + x_1x_2) = V(x_0, x_1) \cup V(x_0, x_2) \cup V(x_3, x_0x_4 + x_1x_2).$$

This is not a union of four planes, so we have a contradiction.

If any pair of the planes meet in a point (in which case any cone point would be unique), then we may instead assume that one of the other planes is $V(x_2, x_3)$. Under these assumptions X must be the intersection of $V(a_ix_0x_2 + b_ix_0x_3 + c_ix_1x_2 + d_ix_1x_3)$ for $i = 0, 1$ and some a_i, b_i, c_i, d_i. In particular, X is a cone. In addition, if (a_0, d_0), (a_1, d_1) are linearly independent, then one of the defining equations can be taken to be a rank 2 quadric divisible by x_i, and similarly if (b_0, c_0), (b_1, c_1) are linearly independent. Thus, it remains to consider the case that $X = V(ax_0x_2 + dx_1x_3, bx_0x_3 + cx_1x_2)$, with $abcd \neq 0$. Then

$$bc(ax_0x_2 + dx_1x_3) + \sqrt{abcd}(bx_0x_3 + cx_1x_2) = (\sqrt{a}bx_0 + \sqrt{bcd}x_1)(\sqrt{a}cx_2 + \sqrt{bcd}x_3),$$

and so X is contained in a rank 2 quadric.

It remains to show that if X has a unique cone point and admits a transitive cyclic action on its irreducible components, then, up to an automorphism of $\mathbb{P}^4$, $X = V(x_0x_1, x_2x_3) \subset \mathbb{P}^4$. Without loss of generality, we may assume the cone point is $[0:0:0:0:1]$, and so X is a cone over an intersection of quadrics in $\mathbb{P}^3$, which is a curve Z of arithmetic genus 1. Since by assumption X is a union of 4 planes, Z must be the union of 4 lines. Furthermore, since X has a unique cone point, the four lines of Z cannot all meet. This combined with the transitive $\mathbb{Z}/4\mathbb{Z}$-action then implies that any triple of the lines cannot meet. By enumerating the possible intersection configurations, one can check that the only arrangement of lines with a transitive $\mathbb{Z}/4\mathbb{Z}$-action, with no triple meeting, and whose union is a curve of genus 1 is a 4-gon, i.e., a cycle of rational curves, where each curve meets exactly two of the others. After a change of coordinates, we may assume that the intersections are

$$P_1 \cap P_2 = V(x_0, x_1, x_2), \quad P_2 \cap P_3 = V(x_0, x_1, x_3), \quad P_3 \cap P_4 = V(x_0, x_2, x_3), \quad P_4 \cap P_1 = V(x_1, x_2, x_3),$$

so

$$X = V(x_0x_2, x_1x_3). \qquad \square$$

Corollary 2.7. *Let $X \subset \mathbb{P}^4_k$ be a geometrically reduced complete intersection of two quadrics over a field k. If X is nonsplit, then X is contained in a rank 2 quadric.*

Proof. Assume X is nonsplit. Since X is geometrically reduced and nonsplit, it must be geometrically reducible, and so reducible over a separable closure. Thus the absolute Galois group of k acts on the geometric components. Since X is nonsplit, none of the components are fixed by Galois, and so, by Lemma 2.5, X is geometrically either the union of two irreducible quadric surfaces or the union of four planes. In the first case, Lemma 2.5(1) gives the result, and in the second Lemma 2.6 does. $\square$

2D. ***Semistable models.*** Following work of Kollár [1997] in the case of hypersurfaces, Tian [2017, Section 2.1] has defined a notion of semistability for intersections of two quadrics over discrete valuation rings. This notion of semistability allows one to find a model of X whose special fiber is fairly well controlled.

Before stating our results, we first review some of the definitions from Tian's semistability machinery. Suppose k is a nonarchimedean local field with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$. We will use π to denote a uniformizer. Let $\mathcal{X} \subset \mathbb{P}^4_{\mathcal{O}}$ be an intersection of two quadrics. Given $\mathcal{X}$, we can associate a 2×15 matrix A such that each row is the coefficient vector of the corresponding defining equation for $\mathcal{X}$. Note that changing the defining equations corresponds to multiplying A on the left by an element of $\mathrm{GL}_2(\mathcal{O})$. Thus, up to this GL_2-action, we have a well-defined matrix $A_{\mathcal{X}}$.

Given an nonnegative integer weight vector $\mathbf{w} \in \mathbb{N}^5$, we define the change of coordinates $f_{\mathbf{w}} \colon \mathbb{P}^4_{\mathcal{O}} \to \mathbb{P}^4_{\mathcal{O}}$, $x_i \mapsto \pi^{w_i} x_i$. Then we define the *multiplicity of $\mathcal{X}$ with respect to* $\mathbf{w}$ to be

$$\mathrm{mult}_{\mathbf{w}}(\mathcal{X}) := \min\{v(m) : m \text{ is a } 2 \times 2 \text{ minor of } A_{f_{\mathbf{w}}^*\mathcal{X}}\},$$

where v denotes the valuation on $\mathcal{O}$. Then $\mathcal{X}$ is said to be *semistable* if for all weight vectors $\mathbf{w}$ and all automorphisms $g \in \mathrm{Aut}(\mathbb{P}^4_{\mathcal{O}}) = \mathrm{PGL}_5(\mathcal{O})$, we have

$$\mathrm{mult}_{\mathbf{w}}(g(\mathcal{X})) \leq \tfrac{4}{5}\biggl(\sum_{i=0}^{4} w_i\biggr).$$

By [Tian 2017, Theorem 2.7], any smooth intersection of two quadrics $X \subset \mathbb{P}^4_k$ has a semistable integral model. For more details, see [Tian 2017, Section 2.1 and 2.4].

We will also make use of the following results from [Tian 2017].

Lemma 2.8. *Let k be a nonarchimedean local field, let $\mathcal{O}$ denote the valuation ring of k, let $\mathbb{F}$ denote the residue field of k, and let $X \subset \mathbb{P}^4_k$ be a smooth complete intersection of two quadrics. Let $\mathcal{X} \subset \mathbb{P}^4_{\mathcal{O}}$ be a semistable model of X (which exists by* [Tian 2017, Theorem 2.7]). *Then*:

(1) [Tian 2017, Lemma 2.9] *The special fiber of $\mathcal{X}$ is a complete intersection of two quadrics.*

(2) [Tian 2017, Lemma 2.22(1)] *The special fiber is not contained in a reducible quadric hypersurface defined over $\mathbb{F}$.*

(3) [Tian 2017, Lemma 2.22(2)] *The special fiber does not contain a plane defined over $\mathbb{F}$.*

(4) [Tian 2017, Lemma 2.22(4)] *The special fiber is reduced.*

Remark 2.9. In [Tian 2017, Sections 2.2–2.4], Tian works over local function fields, but as noted in [Tian 2017, beginning of Section 2.2], the proofs go through essentially verbatim for any nonarchimedean local field. In [Tian 2017, Section 2.4] (in which [Tian 2017, Lemma 2.22] is stated and proved), Tian adds the hypothesis that the residue field has odd characteristic, and so freely interchanges smooth and nonsingular. However, no assumption on the residue characteristic is needed for the proofs of [Tian 2017, Lemma 2.22(1), (2), and (4)]. For the sake of completeness, we repeat Tian's proof of Lemma 2.8(2)–(4).

Proof. If the special fiber is contained in a reducible quadric hypersurface defined over $\mathbb{F}$, then, after possibly changing variables, one of the quadrics defining $\mathcal{X}$ must be of the form $x_0x_1 + \pi\tilde{q}$, in which case $\mathrm{mult}_{(1,0,0,0,0)}(\mathcal{X}) \geq 1$. However, since $\mathcal{X}$ is assumed to be semistable we must have $\mathrm{mult}_{(1,0,0,0,0)}(\mathcal{X}) \leq \frac{4\cdot(1)}{5}$, resulting in a contradiction. This proves (2). Similarly, if the special fiber contains a linear subspace of dimension 2 defined over $\mathbb{F}$, which we may assume is $V(x_0, x_1)$, then $\mathrm{mult}_{(1,1,0,0,0)}(\mathcal{X}) \geq 2$. However, the semistability hypothesis implies that $\mathrm{mult}_{(1,1,0,0,0)}(\mathcal{X}) \leq \frac{4\cdot(1+1)}{5} = \frac{8}{5}$, giving a contradiction. Thus, we conclude (3).

Now we prove (4). By [Tian 2017, Lemma 2.9], the special fiber is a complete intersection, so the special fiber is reduced if and only if all geometric irreducible components are reduced. Assume that the special fiber has a nonreduced geometric irreducible component. Since the special fiber has degree 4 and contains no plane defined over $\mathbb{F}$, the only possibilities are:

(a) A quadric surface of multiplicity 2.

(b) A union of two conjugate planes, each with multiplicity 2.

Note that in case (b), the two planes must meet in a line, as otherwise a general hyperplane section would be the union of two skew double lines, which is not possible. Thus, case (b) is subsumed by case (a), and so the reduced special fiber is given by the vanishing of a linear form ℓ and a quadratic form q. Hence, the special fiber is defined by quadratic forms of the form $\ell\ell_1$, $\ell\ell_2+q$ for some linear forms ℓ_1, ℓ_2, which contradicts (2). □

Proposition 2.10. *Let k be a nonarchimedean local field, let $\mathcal{O}$ denote the valuation ring of k, let $\mathbb{F}$ denote the residue field of k, and let $X \subset \mathbb{P}^4_k$ be smooth complete intersection of two quadrics. Let $\mathcal{X} \subset \mathbb{P}^4_{\mathcal{O}}$ be a semistable model of X (which exists by* [Tian 2017, Theorem 2.7]). *Assume that the special fiber of $\mathcal{X}/\mathcal{O}$ is geometrically the union of four 2-planes and that the Galois group acts transitively on the four 2-planes. Then, for any choice of uniformizer π, X must be given by the vanishing of quadratic forms of the shape*

$$q(x_0,\dots,x_3)+\pi^m x_4\ell(x_0,\dots,x_3) \quad \text{and} \quad \tilde{q}(x_0,\dots,x_3)+\pi x_4^2+\pi^n x_4\tilde{\ell}(x_0,\dots,x_3), \tag{2-1}$$

(*with m, n positive integers, and $q, \tilde{q}$ quadratic forms such that every $\bar{\mathbb{F}}$-linear combination of q and $\tilde{q}$ modulo π has rank at least* 2); *or*

$$\begin{aligned} &g(x_0,x_1,x_2)+\pi h(x_3,x_4)+\pi^a x_3\ell_3(x_0,x_1,x_2)+\pi^b x_4\ell_4(x_0,x_1,x_2), \quad \text{and}\\ &\tilde{g}(x_0,x_1,x_2)+\pi\tilde{h}(x_3,x_4)+\pi^c x_3\tilde{\ell}_3(x_0,x_1,x_2)+\pi^d x_4\tilde{\ell}_4(x_0,x_1,x_2), \end{aligned} \tag{2-2}$$

(*with a, b, c, d positive integers, ℓ_i, $\tilde{\ell}_i$ linear forms and $g, \tilde{g}, h, \tilde{h}$ quadratic forms such that every $\bar{\mathbb{F}}$-linear combination of g and $\tilde{g}$ modulo π has rank at least* 2 *and every $\bar{\mathbb{F}}$-linear combination of h and $\tilde{h}$ modulo π has rank at least* 1).

Proof. By Lemma 2.6, the special fiber must be isomorphic (over $\bar{\mathbb{F}}$) to $V(x_0x_1, x_2x_3)$ or a cone over a complete intersection of two quadrics in $\mathbb{P}^2$ (i.e., a complete intersection of two conics).

Let us first assume that the special fiber is geometrically isomorphic to $V(x_0x_1, x_2x_3)$. Note that this variety has a unique singular point, the cone point, so it must be defined over $\mathbb{F}$. After a change of coordinates, we may assume the cone point reduces to $V(x_0, x_1, x_2, x_3)$ and hence X is given by

$$q(x_0,\dots,x_3)+\pi^m x_4\ell(x_0,\dots,x_4) \quad \text{and} \quad \tilde{q}(x_0,\dots,x_3)+\pi^n x_4\tilde{\ell}(x_0,\dots,x_4),$$

for some integers $m, n \geq 1$, quadratic forms $q, \tilde{q}$ and linear forms $\ell, \tilde{\ell}$ that are nonzero modulo π. We will first use the semistability of $\mathcal{X}$ for the weight vector $\mathbf{w} := (1, 1, 1, 1, 0)$ to show that one of $\pi^m\ell$ or $\pi^n\tilde{\ell}$ must evaluate to a uniformizer at $[0:0:0:0:1]$. Note that $A_{f_{\mathbf{w}}^*\mathcal{X}}$ has the following form

$$\begin{pmatrix} \pi^2\operatorname{coefs}(q) & \pi^{m+1}\ell_0 & \pi^{m+1}\ell_1 & \pi^{m+1}\ell_2 & \pi^{m+1}\ell_3 & \pi^m\ell_4\\ \pi^2\operatorname{coefs}(\tilde{q}) & \pi^{n+1}\tilde{\ell}_0 & \pi^{n+1}\tilde{\ell}_1 & \pi^{n+1}\tilde{\ell}_2 & \pi^{n+1}\tilde{\ell}_3 & \pi^n\tilde{\ell}_4 \end{pmatrix},$$

where $\ell = \sum_i \ell_i x_i$, $\tilde{\ell} = \sum_i \tilde{\ell}_i x_i$ and $\operatorname{coefs}(q)$, $\operatorname{coefs}(\tilde{q})$ denote the coefficient vectors of $q, \tilde{q}$ respectively. Hence, using the strong triangle equality and the definition of multiplicity, one can compute that $\operatorname{mult}_{\mathbf{w}}(\mathcal{X}) \geq \min(4, 2+m+v(\ell_4), 2+n+v(\tilde{\ell}_4))$. However, the semistability assumption implies that $\operatorname{mult}_{\mathbf{w}}(\mathcal{X}) \leq \frac{4\cdot(1+1+1+1)}{5} = \frac{16}{5}$, and so $\min(m+v(\ell_4), n+v(\tilde{\ell}_4)) = 1$. Thus, after renaming $q, \tilde{q}$ and $\ell, \tilde{\ell}$

and possibly scaling the equations, we may assume the equations are of the form

$$q(x_0,\dots,x_3)+\pi^m x_4\ell(x_0,\dots,x_3) \quad\text{and}\quad \tilde{q}(x_0,\dots,x_3)+\pi^n x_4\widetilde{\ell}(x_0,\dots,x_3)+\pi x_4^2.$$

To see that every $\bar{\mathbb{F}}$-linear combination of q and $\tilde{q}$ modulo π is rank at least 2, recall that the variety defined by q and $\tilde{q}$ modulo π is geometrically isomorphic to $V(x_0x_1, x_2x_3)$ and note that $ax_0x_1+bx_2x_3$ has rank 4 for all $a, b \neq 0$.

Now assume that the special fiber is a cone over a complete intersection of two quadrics in $\mathbb{P}^2$. Then, up to a change of variables, $\mathcal{X}$ must be given by quadratic forms of the shape

$$\begin{aligned}&g(x_0,x_1,x_2)+\pi^m h(x_3,x_4)+\pi^a x_3\ell_3(x_0,x_1,x_2)+\pi^b x_4\ell_4(x_0,x_1,x_2) \quad\text{and}\\ &\tilde{g}(x_0,x_1,x_2)+\pi^{\tilde{m}}\tilde{h}(x_3,x_4)+\pi^c x_3\widetilde{\ell}_3(x_0,x_1,x_2)+\pi^d x_4\widetilde{\ell}_4(x_0,x_1,x_2),\end{aligned}$$

where $a, b, c, d, m, \tilde{m}$ are positive integers, $g, \tilde{g}, h, \tilde{h}$ are quadratic forms, and $\ell_i, \tilde{\ell}_i$ are linear forms. Since, by assumption, the special fiber is reduced, the complete intersection in $\mathbb{P}^2_{\bar{\mathbb{F}}}$ defined by the vanishing of g and $\tilde{g}$ modulo π must also be reduced. This complete intersection is therefore, geometrically, a set of 4 noncolinear points in $\mathbb{P}^2_{\bar{\mathbb{F}}}$. These points are not contained in any quadric of rank 1 so every $\bar{\mathbb{F}}$-linear combination of g and $\tilde{g}$ modulo π has rank at least 2.

To complete the proof, we need to show that $m=\tilde{m}=1$ and that $h, \tilde{h}$ are linearly independent modulo π. We will again use our semistability hypothesis. Consider the weight vector $\mathbf{w}=(1,1,1,0,0)$. One can compute that $\mathrm{mult}_{\mathbf{w}}(\mathcal{X})$ is at least $\min\{4, m+\tilde{m}, 2+m, 2+\tilde{m}\}$ and, in addition, if h and $\tilde{h}$ are linearly dependent modulo π, then $\mathrm{mult}_{\mathbf{w}}(\mathcal{X}) \geq \min\{4, m+\tilde{m}+1, 2+m, 2+\tilde{m}\}$. However, the semistability assumption implies that $\mathrm{mult}_{\mathbf{w}}(\mathcal{X}) \leq \frac{4\cdot(1+1+1)}{5} = \frac{12}{5}$. Thus, we must have that h and $\tilde{h}$ are linearly independent modulo π, and $m+\tilde{m}=2$, which implies that $m=\tilde{m}=1$. ☐

2E. *Proof of Theorem 2.1.* If k is archimedean, then $[\bar{k}:k]\leq 2$ so the result is immediate. Henceforth we assume that k is nonarchimedean, and we write $\mathcal{O}$ for the valuation ring of k and $\mathbb{F}$ for the residue field of k. By [Tian 2017, Theorem 2.7], there is a linear change of coordinates on $\mathbb{P}^4_k$ such that the resulting integral model $\mathcal{X}\subset\mathbb{P}^4_{\mathcal{O}}$ of X is semistable. In particular, by Lemma 2.8, the special fiber of $\mathcal{X}$ is a reduced complete intersection of quadrics.

If the special fiber of $\mathcal{X}$ is split, then the desired result follows from Proposition 2.3. If the special fiber of $\mathcal{X}$ is not split, but becomes split over the quadratic extension of $\mathbb{F}$, then we may apply Proposition 2.3 over k', the unique quadratic unramified extension of k, and conclude that $X(k')\neq\varnothing$.

Thus, we have reduced to the case that the special fiber $\mathcal{X}^\circ$ of $\mathcal{X}$ is nonsplit and remains nonsplit over the unique quadratic extension $\mathbb{F}'/\mathbb{F}$. Since $\mathbb{F}$ is perfect and $\mathcal{X}^\circ$ is reduced, $\mathcal{X}^\circ$ must be geometrically reduced. Therefore $\mathcal{X}^\circ$ must be geometrically reducible and $\mathrm{Gal}(\bar{\mathbb{F}}/\mathbb{F}')$ must act nontrivially on the components. By Lemma 2.5, this is possible only if $\mathcal{X}^\circ_{\bar{\mathbb{F}}}$ is the union of four 2-planes. Furthermore, the current assumptions imply that $\mathrm{Gal}(\bar{\mathbb{F}}/\mathbb{F})$ must act transitively on the four 2-planes. Thus, by Proposition 2.10, we may assume that X is given by quadrics as in (2-1) or (2-2).

Consider a ramified quadratic extension k'/k and let ϖ be a uniformizer of k'. First assume that $\mathcal{X}$ is given by equations of the form (2-1). Over k' we may absorb a ϖ into x_4 and obtain the model $\mathcal{X}'/\mathcal{O}'$ (where $\mathcal{O}'$ is the valuation ring of k'):

$$q(x_0,\ldots,x_3)+u^r\varpi^{2r-1}x_4\ell(x_0,\ldots,x_3) \quad \text{and} \quad \tilde{q}(x_0,\ldots,x_3)+u^n\varpi^{2n-1}x_4\widetilde{\ell}(x_0,\ldots,x_3)+ux_4^2, \tag{2-3}$$

where u is the unit such that $u\varpi^2=\pi$. Every $\bar{\mathbb{F}}$-linear combination of the forms in (2-3) modulo ϖ is an orthogonal sum of an $\bar{\mathbb{F}}$-linear combination of q and $\tilde{q}$ modulo ϖ (which has rank at least 2 by (2-1)) with a quadratic form of rank 1. It follows from Lemma 2.4 that every $\bar{\mathbb{F}}$-linear combination of the forms in (2-3) modulo ϖ has rank at least 3. Thus, by Corollary 2.7, the special fiber of $\mathcal{X}'$ is split, so, by Proposition 2.3, $\mathcal{X}'$ has a k'-point.

Now assume that X is given by equations of the form (2-2). Then, we may absorb a ϖ into x_3 and x_4 and obtain the model $\mathcal{X}'/\mathcal{O}$ given by

$$\begin{aligned} &g(x_0,x_1,x_2)+uh(x_3,x_4)+u^a\varpi^{2a-1}x_3\ell_3(x_0,x_1,x_2)+u^b\varpi^{2b-1}x_4\ell_4(x_0,x_1,x_2) \quad \text{and} \\ &\tilde{g}(x_0,x_1,x_2)+u\tilde{h}(x_3,x_4)+u^c\varpi^{2c-1}x_3\widetilde{\ell}_3(x_0,x_1,x_2)+u^d\varpi^{2d-1}x_4\widetilde{\ell}_4(x_0,x_1,x_2), \end{aligned} \tag{2-4}$$

where u is the unit such that $u\varpi^2=\pi$. Then every $\bar{\mathbb{F}}$-linear combination of the forms in (2-4) modulo ϖ is an orthogonal direct sum of an $\bar{\mathbb{F}}$-linear combination of g and $\tilde{g}$ modulo ϖ (which is a form of rank 2 or 3) with an $\bar{\mathbb{F}}$-linear combination of h and $\tilde{h}$ modulo ϖ (which is a form of rank 1 or 2). Thus, by Lemma 2.4, every $\bar{\mathbb{F}}$-linear combination of the forms in (2-4) modulo ϖ has rank at least 3. Hence, by Corollary 2.7, the special fiber of $\mathcal{X}'$ is split, and so X has a k'-point, by Proposition 2.3. □

2F. *Alternate proof in characteristic* 2. The following is a slight generalization of [Dolgachev and Duncan 2018, Theorem 4.4].

Proposition 2.11. *Suppose k is a field of characteristic 2 and $X\subset\mathbb{P}^4_k$ is smooth complete intersection of two quadrics. Then $X(k^{1/2})\neq\varnothing$. In particular, if k is a local or global field of characteristic 2, then X contains a point defined over the quadratic extension $k^{1/2}$ of k.*

Proof. By [Dolgachev and Duncan 2018, Theorem 1.1], X can be defined by the vanishing of quadratic forms of the form

$$a_0x_0^2+a_1x_1^2+a_2x_2^2+x_3\ell_1+x_4\ell_2 \quad \text{and} \quad b_0x_0^2+b_1x_1^2+b_2x_2^2+x_3\ell_3+x_4\ell_4$$

where $a_i,b_i\in k$ and $\ell_i\in k[x_0,\ldots,x_4]$ are linear forms. In particular, the intersection of X with the plane $V(x_3,x_4)$ is an intersection of two conics in $\mathbb{P}^2$ neither of which is geometrically reduced. The reduced subschemes of the base changes of these conics to the algebraic closure are the lines $V(a_0^{1/2}x_0+a_1^{1/2}x_1+a_2^{1/2}x_2)$ and $V(b_0^{1/2}x_0+b_1^{1/2}x_1+b_2^{1/2}x_2)$, which are defined over $k^{1/2}$. Their intersection yields a $k^{1/2}$-point on X.

It remains to show that $[k^{1/2}:k]=2$ when k is a local or global field of characteristic 2. If k is local, then $k=\mathbb{F}((t))$ with $\mathbb{F}$ a finite field of characteristic 2 and $k^{1/2}=\mathbb{F}((t^{1/2}))$ which is clearly an extension of degree 2. Similarly, if $k=\mathbb{F}(t)$ is a global function field of genus 0 and characteristic 2, then

$k^{1/2} = \mathbb{F}(t^{1/2})$ is clearly a degree 2 extension. For a general global field k of characteristic 2, which is necessarily a finite extension of $k_0 = \mathbb{F}(t)$ with $\mathbb{F}$ finite characteristic 2, we may reduce to the genus 0 case as follows; see [Becker and MacLane 1940, Theorem 3]. Frobenius gives an isomorphism $F : k^{1/2} \to k$ which restricts to an isomorphism $k_0^{1/2} \to k_0$, and so $[k^{1/2} : k_0^{1/2}] = [k : k_0]$. Since k and $k_0^{1/2}$ are both intermediate fields of the extension $k_0 \subset k^{1/2}$ we have

$$[k^{1/2} : k][k : k_0] = [k^{1/2} : k_0^{1/2}][k_0^{1/2} : k_0].$$

Taken together these observations show that $[k^{1/2} : k] = [k_0^{1/2} : k_0]$. □

3. Brauer–Manin obstructions over extensions

In this section, we prove some general results relating the Brauer–Manin obstruction on a nice variety Y to the Brauer–Manin obstruction over an extension. Moreover, for quadratic extensions, we relate the Brauer–Manin obstruction on (a desingularization of) the symmetric square to the Brauer–Manin obstruction over quadratic extensions.

Lemma 3.1. *Let Y/k be a nice variety over a global field k, let K/k be a finite extension, and let B be a subset of* $\mathrm{Br}(Y_K)$. *Then $Y(\mathbb{A}_k)^{\mathrm{Cor}_{K/k}(B)} \subset Y_K(\mathbb{A}_K)^B$. In particular,*

(1) *if $Y(\mathbb{A}_k)^{\mathrm{Br}} \neq \varnothing$, then $Y_K(\mathbb{A}_K)^{\mathrm{Br}} \neq \varnothing$, and*

(2) *for any $d \mid [K : k]$, $Y(\mathbb{A}_k) \subset Y_K(\mathbb{A}_K)^{\mathrm{Res}_{K/k} \mathrm{Br}(Y)[d]}$.*

Proof. By [Colliot-Thélène and Skorobogatov 2021, Proposition 3.8.1], for any $\alpha \in \mathrm{Br}(Y_K)$ and for any local point $P_v \in Y(k_v)$, we have $(\mathrm{Cor}_{Y_K/Y}(\alpha))(P_v) = \mathrm{Cor}_{K_v/k_v}(\alpha(P_v))$, where $K_v = K \otimes_k k_v$. Thus, for $(P_v) \in Y(\mathbb{A}_k)$,

$$\sum_{v\in\Omega_k} \mathrm{inv}_v(\mathrm{Cor}_{Y_K/Y}(\alpha)(P_v)) = \sum_{v\in\Omega_k} \mathrm{inv}_v(\mathrm{Cor}_{K_v/k_v}(\alpha(P_v))) = \sum_{v\in\Omega_k} \sum_{w\in\Omega_K, w \mid v} \mathrm{inv}_w(\alpha(P_v))$$

(where the last equality follows from the equality of maps $\mathrm{inv}_w = \mathrm{inv}_v \circ \mathrm{Cor}_{K_w/k_v}$ for any prime $w \mid v$), and so $Y(\mathbb{A}_k)^{\mathrm{Cor}_{K/k}(\alpha)} \subset Y(\mathbb{A}_K)^{\alpha}$. The general statement follows by considering the intersection of $Y(\mathbb{A}_K)^{\alpha}$ for all $\alpha \in B$.

It remains to prove statements (1) and (2). The first follows from taking $B = \mathrm{Br}(Y_K)$ and observing that $Y(\mathbb{A}_k)^{\mathrm{Br}(Y)} \subset Y(\mathbb{A}_k)^{\mathrm{Cor}_{K/k}(\mathrm{Br}(Y_K))}$, and the second follows from taking $B = \mathrm{Res}_{K/k} \mathrm{Br}(Y)[d]$ and using that $\mathrm{Cor}_{K/k} \circ \mathrm{Res}_{K/k} = [K : k]$. □

Remark 3.2. Yang Cao has given an alternative proof of Lemma 3.1(1) which also yields a similar statement for the étale-Brauer obstruction. This will appear in forthcoming work of Yang Cao and Yongqi Liang [2022].

The following lemma and corollary extend techniques of Kanevsky [1987] in the case of cubic surfaces.

Lemma 3.3. *Let Y be a nice variety over a field k such that $\mathrm{H}^3(k, \mathbb{G}_m) = 0$. Assume that*

(1) $\mathrm{Pic}(\overline{Y})$ *is finitely generated and torsion free*,

(2) $\mathrm{Br}(\overline{Y})$ *is finite*, *and*

(3) $\mathrm{Br}(Y) \to \mathrm{Br}(\overline{Y})^{G_k}$ *is surjective.*

Then there is a finite Galois extension k_1/k such that for all extensions K/k linearly disjoint from k_1 the map $\mathrm{Res}_{K/k}\colon \mathrm{Br}(Y)/\mathrm{Br}_0(Y) \to \mathrm{Br}(Y_K)/\mathrm{Br}_0(Y_K)$ is surjective.

Proof. The assumption $\mathrm{H}^3(k, \mathbb{G}_m) = 0$ implies that the injective map $\mathrm{Br}_1(Y)/\mathrm{Br}_0(Y) \to \mathrm{H}^1(k, \mathrm{Pic}(\overline{Y}))$ coming from the Hochschild–Serre spectral sequence [Colliot-Thélène and Skorobogatov 2021, Proposition 4.3.2] is an isomorphism. Assumption (1) implies that $\mathrm{H}^1(k, \mathrm{Pic}(\overline{Y})) \simeq \mathrm{H}^1(k_0/k, \mathrm{Pic}(\overline{Y}))$ for some finite Galois extension k_0/k. By assumption (2), there is a finite Galois extension k_1/k_0 such that $\mathrm{Res}_{\bar{k}/k_1}\colon \mathrm{Br}(Y_{k_1}) \to \mathrm{Br}(\overline{Y})$ is surjective. Now suppose K/k is linearly disjoint from k_1. In particular, K is linearly disjoint from k_0, so $\mathrm{Res}_{K/k}\colon \mathrm{Br}_1(Y)/\mathrm{Br}_0(Y) \simeq \mathrm{H}^1(k, \mathrm{Pic}(\overline{Y})) \to \mathrm{H}^1(K, \mathrm{Pic}(\overline{Y})) \simeq \mathrm{Br}_1(Y_K)/\mathrm{Br}_0(Y_K)$ is an isomorphism. So it will suffice to show that $\mathrm{Br}(Y)$ and $\mathrm{Br}(Y_K)$ have the same image in $\mathrm{Br}(\overline{Y})$. Since $\mathrm{Br}(Y_{k_1}) \to \mathrm{Br}(\overline{Y})$ is surjective, the image of $\mathrm{Br}(Y_K) \to \mathrm{Br}(\overline{Y})$ is contained in $\mathrm{Br}(\overline{Y})^{G_K} \cap \mathrm{Br}(\overline{Y})^{G_{k_1}}$, which is equal to $\mathrm{Br}(\overline{Y})^{G_k}$, since k_1 and K are linearly disjoint. Thus, by assumption (3), $\mathrm{Br}(Y)$ and $\mathrm{Br}(Y_K)$ have the same image in $\mathrm{Br}(\overline{Y})$. □

Corollary 3.4. *If Y is a nice variety over a global field k such that $Y(\mathbb{A}_k) \neq \varnothing$ and $\mathrm{Br}(Y)/\mathrm{Br}_0(Y)$ is generated by the image of $\mathrm{Br}(Y)[d]$, then for any extension K/k of degree d, $Y_K(\mathbb{A}_K)^{\mathrm{Res}_{K/k}(\mathrm{Br}(Y))} \neq \varnothing$. Moreover, if Y satisfies the conditions of Lemma 3.3, then there is a finite extension k_1/k such that for any degree d extension K/k which is linearly disjoint from k_1 we have $Y_K(\mathbb{A}_K)^{\mathrm{Br}} \neq \varnothing$.* □

Proof. For a global field k we have $\mathrm{H}^3(k, \mathbb{G}_m) = 0$. So the corollary follows immediately from Lemmas 3.1(2) and 3.3. □

Remark 3.5. If $Y \subset \mathbb{P}^4_k$ is smooth complete intersection of two quadrics over a global field k of characteristic not equal to 2 and Y is everywhere locally solvable, then the corollary applies with $d = 2$. This gives a proof of the $n = 4$ case of Theorem 1.2(5) under the additional hypothesis of local solubility. Note that local solubility is used here in two distinct ways. First it ensures that $\mathrm{Br}(Y)/\mathrm{Br}_0(Y)$ is generated by the image of $\mathrm{Br}(Y)[2]$ (which is not the case in general even though $\mathrm{Br}(Y)/\mathrm{Br}_0(Y)$ is 2-torsion) [Várilly-Alvarado and Viray 2014, Theorem 3.4]. Second, it implies that the canonical maps $\mathrm{Br}(k) \to \mathrm{Br}_0(Y)$ are isomorphisms, locally and globally. This is used implicitly in the proof of Lemma 3.1. In general, $\mathrm{Br}(k) \to \mathrm{Br}_0(Y)$ need not be injective (see Lemma 5.9 for a description of the kernel when Y is a del Pezzo surface of degree 4) and so $\mathrm{Res}_{K/k}$ does not necessarily annihilate $[K : k]$-torsion elements of $\mathrm{Br}_0(Y)$. Consequently, the exact sequence

$$0 \to \mathrm{Br}(k) \to \bigoplus \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z} \to 0$$

of global class field theory has no analogue for $\mathrm{Br}_0(Y)$.

The following proposition relates the Brauer–Manin obstruction over quadratic extensions to the Brauer–Manin obstruction on the symmetric square. Note that while the symmetric square $\operatorname{Sym}^2(Y)$ is singular if Y has dimension at least 2, there exists a smooth projective model $Y^{(2)}$ over any field of characteristic different from 2 (see the proof of part (1) of the proposition for details).

Proposition 3.6. *Let k be a field of characteristic different from 2, let Y/k be a nice variety of dimension at least 2 with torsion free geometric Picard group, and let $Y^{(2)}$ be a smooth projective model of the symmetric square of Y over k:*

(1) *The rational map $Y^2 \to \operatorname{Sym}^2(Y) \dashrightarrow Y^{(2)}$ induces a corestriction map*

$$\operatorname{Cor}_{Y^2/Y^{(2)}} \colon \operatorname{Br}(Y^2) \to \operatorname{Br}(Y^{(2)})$$

on the Brauer groups of the varieties. Furthermore, if π_1 denotes projection onto the first factor of $Y^2 = Y \times Y$, then the composition $\operatorname{Cor}_{Y^2/Y^{(2)}} \circ \pi_1^* : \operatorname{Br}(Y) \to \operatorname{Br}(Y^{(2)})$ induces an injective map

$$\phi \colon \frac{\operatorname{Br}_1(Y)}{\operatorname{Br}_0(Y)} \hookrightarrow \frac{\operatorname{Br}_1(Y^{(2)})}{\operatorname{Br}_0(Y^{(2)})}.$$

(2) *Let $\alpha \in \operatorname{Br}(Y)$ and let $\beta = \operatorname{Cor}_{Y^2/Y^{(2)}} \circ \pi_1^*(\alpha) \in \operatorname{Br}(Y^{(2)})$. There exists a dense open $U \subset Y^{(2)}$ such that for any $y \in U$, y corresponds to a quadratic point $\tilde{y} \colon \operatorname{Spec}(K) \to Y$ for some degree 2 étale $\boldsymbol{k}(y)$-algebra K and we have $\beta(y) = \operatorname{Cor}_{K/\boldsymbol{k}(y)}(\alpha(\tilde{y}))$.*

(3) *Suppose k is a global field, $\operatorname{Br}(Y)/\operatorname{Br}_0(Y)$ is finite and let $B \subset \operatorname{Br}(Y^{(2)})/\operatorname{Br}_0(Y^{(2)})$ denote the image of $\operatorname{Cor}_{Y^2/Y^{(2)}} \circ \pi_1^*$ modulo constant algebras. If there exists a quadratic extension K/k such that $Y_K(\mathbb{A}_K)^{\operatorname{Res}_{K/k}(\operatorname{Br}(Y))} \neq \varnothing$, then $Y^{(2)}(\mathbb{A}_k)^B \neq \varnothing$.*

(4) *Let $B \subset \operatorname{Br}(Y^{(2)})/\operatorname{Br}_0(Y^{(2)})$ denote the image of $\operatorname{Cor}_{Y^2/Y^{(2)}} \circ \pi_1^*$ modulo constant algebras. Suppose that k is a global field, that $Y^{(2)}(\mathbb{A}_k)^B \neq \varnothing$, and that Y satisfies the hypotheses of Lemma 3.3. Then there exists a finite set $S \subset \Omega_k$, degree 2 étale k_v-algebras K_w/k_v for $v \in S$ and a finite extension k_1/k such that for any quadratic extension K/k that is linearly disjoint from k_1 and such that $K \otimes k_v \simeq K_w$ for $v \in S$ we have $Y_K(\mathbb{A}_K)^{\operatorname{Br}} \neq \varnothing$. In particular, there are infinitely many quadratic extensions K/k such that $Y_K(\mathbb{A}_K)^{\operatorname{Br}} \neq \varnothing$.*

Proof. (1) Let $\Delta = \{(y, y) : y \in Y\} \subset Y^2$ denote the diagonal subscheme and let $\operatorname{Bl}_\Delta Y^2$ denote the blow-up of Y along Δ. Observe that the S_2-action on Y^2 extends to an action on $\operatorname{Bl}_\Delta Y^2$ whose fixed locus is the exceptional divisor E_Δ; we claim that the quotient $(\operatorname{Bl}_\Delta Y^2)/S_2$ is smooth (equivalently geometrically regular). Since $\operatorname{Bl}_\Delta Y^2$ is smooth, the quotient $(\operatorname{Bl}_\Delta Y^2)/S_2$ is automatically smooth away from the branch locus. Let $y \in E_\Delta$. Since E_Δ is a divisor, the involution acts as a pseudoreflection on the geometric tangent space of y. Since the order of the group acting is not divisible by the characteristic of k, the Chevalley–Shephard–Todd theorem (see, e.g., [Smith 1985]) implies that the dimensions of the geometric tangent spaces of y and its image in the quotient are equal. Hence the quotient is smooth at the image of y.

Consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Bl}_\Delta Y^2 & \longrightarrow & (\mathrm{Bl}_\Delta Y^2)/S_2 \\ \downarrow & & \downarrow \\ Y^2 & \longrightarrow & \mathrm{Sym}^2 Y \end{array}$$

The left vertical map is birational by definition, and since $Y^2 \to \mathrm{Sym}^2 Y$ is generically degree 2, the right vertical map is also birational. The top horizontal map is flat of degree 2 [Stacks 2005–, Tag 00R4], so we have a corestriction morphism $\mathrm{Br}(\mathrm{Bl}_\Delta Y^2) \to \mathrm{Br}((\mathrm{Bl}_\Delta Y^2)/S_2)$ that extends to the corestriction map on function fields [Colliot-Thélène and Skorobogatov 2021, Section 3.8]. Since the Brauer group of smooth projective varieties is a birational invariant (and pullback along any birational map gives an isomorphism) [Colliot-Thélène and Skorobogatov 2021, Corollary 6.2.11], this yields the first claim.

It remains to prove injectivity of the induced map ϕ on the quotient $\mathrm{Br}_1(Y)/\mathrm{Br}_0(Y)$. Since $\boldsymbol{k}(Y^2)$ is Galois over $\boldsymbol{k}(Y^{(2)})$ with Galois group generated by the involution σ interchanging the factors of $Y \times Y$, by [Gille and Szamuely 2006, Chapter 3, Exercise 3], the composition

$$\mathrm{Res}_{\boldsymbol{k}(Y^2)/\boldsymbol{k}(Y^{(2)})} \circ \mathrm{Cor}_{\boldsymbol{k}(Y^2)/\boldsymbol{k}(Y^{(2)})} \colon \mathrm{Br}(\boldsymbol{k}(Y^2)) \to \mathrm{Br}(\boldsymbol{k}(Y^2))$$

is given by $x \mapsto x + \sigma(x)$. We may then deduce that the same formula holds for the composition $\mathrm{Res}_{Y^2/Y^{(2)}} \circ \mathrm{Cor}_{Y^2/Y^{(2)}} \colon \mathrm{Br}(Y^2) \to \mathrm{Br}(Y^2)$ by evaluating at generic points [Colliot-Thélène and Skorobogatov 2021, Theorem 3.5.4]. Therefore, the composition $\mathrm{Res} \circ \mathrm{Cor} \circ \pi_1^*$ is equal to the diagonal map $\mathrm{Br}(Y) \to \mathrm{Br}(Y) \oplus \mathrm{Br}(Y) \to \mathrm{Br}(Y^2)$ sending α to $\pi_1^*\alpha + \sigma(\pi_1^*\alpha) = \pi_1^*\alpha + \pi_2^*\alpha$.

If $\mathrm{Pic}(\overline{Y})$ is torsion free, then $\mathrm{Pic}(\overline{Y}) \oplus \mathrm{Pic}(\overline{Y}) \simeq \mathrm{Pic}(\overline{Y}^2)$; see [Skorobogatov and Zarhin 2014, Proposition 1.7]. So the diagonal map together with the Hochschild–Serre spectral sequence gives a commutative diagram:

$$\begin{array}{ccccc} \mathrm{H}^1(k, \mathrm{Pic}(\overline{Y})) & \longrightarrow & \mathrm{H}^1(k, \mathrm{Pic}(\overline{Y}))^{\oplus 2} & = & \mathrm{H}^1(k, \mathrm{Pic}(\overline{Y}^2)) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Br}_1(Y)/\mathrm{Br}_0(Y) & \longrightarrow & (\mathrm{Br}_1(Y)/\mathrm{Br}_0(Y))^{\oplus 2} & \longrightarrow & \mathrm{Br}_1(Y^2)/\mathrm{Br}_0(Y^2) \end{array}$$

As the composition along the top row is injective, the same must be true of the composition along the bottom row. This composition is induced by $\mathrm{Res} \circ \mathrm{Cor} \circ \pi_1^*$ and it factors through the map ϕ in the last statement of (1), so ϕ must also be injective.

(2) Since $Y^{(2)}$ is birational to $\mathrm{Sym}^2 Y$, there is an open set $U \subset Y^{(2)}$ that is isomorphic to an open set of the regular locus of $\mathrm{Sym}^2 Y$, i.e., the image of $Y^2 - \Delta$. For $y \in U$, we obtain $\tilde{y}$ by taking the preimage of y under $Y^2 \to \mathrm{Sym}^2 Y \dashrightarrow Y^{(2)}$. The points y and $\tilde{y}$ fit into a commutative diagram displayed on the left

below. This induces the diagram displayed on the right. Commutativity of the latter gives the result:

$$\begin{array}{ccc} \operatorname{Spec}(\boldsymbol{k}(y)) & \xrightarrow{\;y\;} & Y^{(2)} \\ \uparrow & & \uparrow{\scriptstyle f} \\ f^{-1}(y) & \longrightarrow & Y \times Y \\ \| & & \downarrow{\scriptstyle \pi_1} \\ \operatorname{Spec}(K) & \xrightarrow{\;\tilde{y}\;} & Y \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \operatorname{Br}(\boldsymbol{k}(y)) & \xleftarrow{\;y^*\;} & \operatorname{Br}(Y^{(2)}) \\ {\scriptstyle \mathrm{Cor}}\uparrow & & \uparrow{\scriptstyle \mathrm{Cor}} \\ \operatorname{Br}(f^{-1}(y)) & \longleftarrow & \operatorname{Br}(Y \times Y) \\ \| & & \uparrow{\scriptstyle \pi_1^*} \\ \operatorname{Br}(K) & \xleftarrow[\;\tilde{y}^*\;]{} & \operatorname{Br}(Y) \end{array}$$

(3) Suppose that K/k is a quadratic extension, $Y_K(\mathbb{A}_K)^{\operatorname{Res}_{K/k}(\operatorname{Br}(Y))} \neq \varnothing$ and that $\beta = \operatorname{Cor}_{Y^2/Y^{(2)}}(\pi_1^*(\alpha)) \in \operatorname{Br}(Y^{(2)})$ represents a class in B that is the image of $\alpha \in \operatorname{Br}(Y)$. Since $\operatorname{Br}(Y)/\operatorname{Br}_0(Y)$ is finite, $Y_K(\mathbb{A}_K)^{\operatorname{Res}_{K/k}(\operatorname{Br}(Y))}$ is an open subset of $Y_K(\mathbb{A}_K)$ in the adelic topology which we have assumed is nonempty. So for any $v \in \Omega_k$ the image of the projection map $Y_K(\mathbb{A}_K)^{\operatorname{Res}_{K/k}(\operatorname{Br}(Y))} \to \prod_{w \mid v} Y_K(K_w) = Y(K \otimes k_v)$ is a nonempty open subset and therefore contains a quadratic point $\tilde{y}_v : \operatorname{Spec}(K \otimes k_v) \to Y$ corresponding to a point $y_v \in U(k_v)$, where U is the open set from (2). For the case that v does not split in K we are using the fact that $Y(K_w) \neq Y(k_v)$ since k_v is a local field; see, e.g., [Liu and Lorenzini 2018, Proposition 8.3]. By (2) we have

$$\sum_{v \in \Omega_k} \operatorname{inv}_v \beta(y_v) = \sum_{v \in \Omega_k} \operatorname{inv}_v \operatorname{Cor}_{K \otimes k_v/k_v}(\alpha(\tilde{y}_v)) = 0.$$

So the adelic point $y = (y_v) \in Y^{(2)}(\mathbb{A}_k)$ is orthogonal to β.

(4) Suppose $Y^{(2)}(\mathbb{A}_k)^B \neq \varnothing$. The hypothesis in Lemma 3.3 implies that $\operatorname{Br}(Y)/\operatorname{Br}_0(Y)$ and, hence, B is finite. Thus, $Y^{(2)}(\mathbb{A}_k)^B$ is open and, arguing as in (3), we see that, for each $v \in \Omega_k$, its image in $Y(k_v)$ contains a point $y_v \in U(k_v)$ corresponding to a quadratic point $\tilde{y}_v\colon \operatorname{Spec}(K_v) \to Y$, where K_v is an étale k_v-algebra of degree 2. Moreover, by (2) if $\alpha \in \operatorname{Br}(Y)$ and $\beta = \operatorname{Cor}_{Y^2/Y^{(2)}}(\pi_1^*(\alpha))$, then $\beta(y_v) = \operatorname{Cor}_{K_v/k_v}(\alpha(\tilde{y}_v))$. By assumption $\sum_{v \in \Omega_k} \operatorname{inv}_v \beta(y_v) = 0$, so $(\tilde{y}_v)_{v \in \Omega_k}$ is an effective adelic 0-cycle of degree 2 on Y which is orthogonal to the Brauer group of Y. Under the additional hypotheses of (4), $\operatorname{Br}(Y)/\operatorname{Br}_0(Y)$ is finite and, by Lemma 3.3, there is an extension k_1/k such that for K/k linearly disjoint from k_1, $\operatorname{Res}_{K/k}\colon \operatorname{Br}(Y) \to \operatorname{Br}(Y_K)/\operatorname{Br}_0(Y_K)$ is surjective. Moreover, for any set $\alpha_1, \dots, \alpha_n \in \operatorname{Br}(Y)$ of representatives for $\operatorname{Br}(Y)/\operatorname{Br}_0(Y)$, there is a finite set $S \subset \Omega_k$ such that for all $i = 1, \dots, n$ and all $v \notin S$ the evaluation maps $\operatorname{inv}_v \circ \alpha_i\colon Y(k_v) \to \mathbb{Q}/\mathbb{Z}$ are constant; see [Colliot-Thélène and Skorobogatov 2013, Lemma 1.2 and Theorem 3.1]. In particular $Y(k_v) \neq \varnothing$ for $v \notin S$. Let K/k be a quadratic extension linearly disjoint from k_1 and such that $K \otimes k_v \simeq K_w$ for $v \in S$. By weak approximation on $k^\times$ the map $k^\times/k^{\times 2} \to \prod_{v \in S'} k_v^\times/k_v^{\times 2}$ is surjective for any finite set of primes $S' \subset \Omega_k$, so such extensions K/k do in fact exist. Any adelic point $(x_w)_{w \in \Omega_K} \in Y_K(\mathbb{A}_K)$ such that $\tilde{y}_v = \sum_{w \mid v} x_w$ for $v \in S$ will be orthogonal to $\operatorname{Br}(Y_K)$. □

4. Pencils of quadrics in $\mathbb{P}^4$ and associated objects

Let $\mathcal{Q} \subset \mathbb{P}^4 \times \mathbb{P}^1$ be a pencil of quadrics, i.e., the zero locus of a bihomogeneous polynomial Q of degree (2, 1), defined over a field k of characteristic different from 2. If the projection map $\mathcal{Q} \to \mathbb{P}^1$ is generically smooth, then we may naturally associate three objects. First, we may consider the base locus $X = X_{\mathcal{Q}} \subset \mathbb{P}^4$ of the pencil of quadrics, i.e., $\cap_{t \in \mathbb{P}^1} \mathcal{Q}_t$, where $\mathcal{Q}_t \subset \mathbb{P}^4$ denotes the fiber over $t \in \mathbb{P}^1$. This is a degree 4 projective surface. Second, we may consider the subscheme $\mathcal{S} \subset \mathbb{P}^1$ parametrizing the singular quadrics in the pencil. If Q is any degree (2, 1) form defining $\mathcal{Q}$, then $\mathcal{S}$ is given by the vanishing of $\det(M_Q)$, where M_Q denotes the symmetric matrix corresponding to Q considered as a quadratic form whose coefficients are linear polynomials in the homogeneous coordinate ring of $\mathbb{P}^1$. Since $\mathcal{Q} \to \mathbb{P}^1$ is generically smooth, $\mathcal{S} \subset \mathbb{P}^1$ is a degree 5 subscheme. Third, we may consider the fourfold $\mathcal{G} = \mathcal{G}_{\mathcal{Q}} \to \mathbb{P}^1$ that parametrizes lines on quadrics in the pencil; the generic fiber of $\mathcal{G}$ is a Severi–Brauer variety with index dividing 4 and order dividing 2 [Elman et al. 2008, Example 85.4].

Remark 4.1. Over a field of characteristic 2, $\det(M_Q)$ is identically 0 since M_Q is a $5x5$ skew-symmetric matrix, and so the correspondences between these objects already fails. Due to this, the assumption that k has characteristic different from 2 will remain in force for the remainder of the paper.

Each of these objects has been well-studied, and their conditions for smoothness are known to be closely related.

Proposition 4.2. *Let $\mathcal{Q} \subset \mathbb{P}^4 \times \mathbb{P}^1$ be a pencil of quadrics over a field of characteristic different from* 2. *Then the following are equivalent*:

(1) *The base locus X is smooth and purely of dimension* 2, *in which case X is a del Pezzo surface of degree* 4.

(2) *The degree* 5 *subscheme $\mathcal{S} \subset \mathbb{P}^1$ is reduced.*

(3) *For every $s \in \mathcal{S}$, the fiber $\mathcal{Q}_s$ is rank* 4 *and the vertex of $\mathcal{Q}_s$ does not lie on any other quadric in the pencil.*

(4) *The fourfold $\mathcal{G}$ is smooth*, *the map $\mathcal{G} \to \mathbb{P}^1$ is smooth away from $\mathcal{S}$*, *and above $\mathcal{S}$ the fibers are geometrically reducible.*

Proof. The equivalence of conditions (1), (2), and (3) is given by [Reid 1972, Proposition 2.1]. The equivalence of (4) with any (equivalently all) of the others is given by [Reid 1972, Theoerem 1.10]. □

Definition 4.3. A pencil of quadrics $\mathcal{Q}$ over a field of characteristic different from 2 satisfies (†) if any of the equivalent conditions in Proposition 4.2 hold. Given a pencil $\mathcal{Q}$ satisfying (†), we define $\varepsilon_{\mathcal{S}} \in \boldsymbol{k}(\mathcal{S})/\boldsymbol{k}(\mathcal{S})^{\times 2}$ to be the discriminant of a smooth hyperplane section of $\mathcal{Q}_{\mathcal{S}}$; note that the square class of the discriminant does not depend on the choice of hyperplane, nor on the choice of a defining equation for $\mathcal{Q}_{\mathcal{S}}$.

Given a pencil of quadrics satisfying (†), there are even stronger connections among these three objects.

Proposition 4.4. *Let $\mathcal{Q}$ be a pencil of quadrics satisfying* (†). *Let $X = X_{\mathcal{Q}}, \mathcal{G} = \mathcal{G}_{\mathcal{Q}}$, and $(\mathcal{S}, \varepsilon_{\mathcal{S}}) = (\mathcal{S}_{\mathcal{Q}}, \varepsilon_{\mathcal{S}_{\mathcal{Q}}})$:*

(1) *The variety $\mathcal{G}$ is birational to the symmetric square $\mathrm{Sym}^2(X)$ of X. Moreover, $\mathcal{G}(k) \neq \varnothing$ if and only if $X(K) \neq \varnothing$ for some quadratic extension K/k.*

(2) *The residues of the Brauer class $[\mathcal{G}_{\boldsymbol{k}(\mathbb{P}^1)}] \in \mathrm{Br}\,\boldsymbol{k}(\mathbb{P}^1)$ are*

$$\varepsilon_{\mathcal{S}} \in \boldsymbol{k}(\mathcal{S})^\times/\boldsymbol{k}(\mathcal{S})^{\times 2} \simeq \mathrm{H}^1\big(\boldsymbol{k}(\mathcal{S}), \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\big) \subset \bigoplus_{t \in (\mathbb{P}^1)^{(1)}} \mathrm{H}^1(\boldsymbol{k}(t), \mathbb{Q}/\mathbb{Z}).$$

In particular, $\mathrm{Norm}_{\boldsymbol{k}(\mathcal{S})/k}(\varepsilon_{\mathcal{S}}) \in k^{\times 2}$.

(3) *Given a pair $(\mathcal{S}', \varepsilon_{\mathcal{S}'})$ where $\mathcal{S}' \subset \mathbb{P}^1$ is a reduced degree 5 subscheme and a class $\varepsilon_{\mathcal{S}'} \in \boldsymbol{k}(\mathcal{S}')^\times/\boldsymbol{k}(\mathcal{S}')^{\times 2}$ of square norm, there exists a unique (up to isomorphism) pencil of quadrics $\mathcal{Q}$ such that $(\mathcal{S}', \varepsilon_{\mathcal{S}'}) = (\mathcal{S}_{\mathcal{Q}}, \varepsilon_{\mathcal{S}_{\mathcal{Q}}})$. Thus, for any $t \in \mathbb{P}^1 - \mathcal{S}$, $[\mathcal{G}_t] \in \mathrm{Br}(\boldsymbol{k}(t))$ is determined by $(\mathcal{S}, \varepsilon_{\mathcal{S}})$.*

Remark 4.5. The second statement of Part (2) provides an alternate proof of a proposition by Wittenberg [2007, Proposition 3.39].

Proof. (1) Consider a point $(x, x') \in X \times X - \Delta$, where Δ denotes the diagonal image of X, and let $\ell_{\{x,x'\}}$ be the line joining them. For generic (x, x') the line $\ell_{\{x,x'\}}$ is not contained in X, in which case we claim that $\ell_{\{x,x'\}}$ lies on a quadric in the pencil containing X. This quadric will be unique since a line that is contained in more than one quadric in the pencil lies on X. To see that $\ell_{\{x,x'\}}$ is contained in some quadric note that the intersections $\mathcal{Q}_t \cap \ell_{\{x,x'\}}$ determine a nonzero pencil of binary quadrics (i.e., quadrics in $\mathbb{P}^1$) that all contain x and x'. The singular binary quadrics of this pencil are rank at most 1 and contain the distinct points x and x' so they must be identically 0 on $\ell_{\{x,x'\}}$.

Therefore, we have a rational map

$$f\colon X \times X \dashrightarrow \mathcal{G}, \quad (x, x') \mapsto (t_{\{x,x'\}}, \ell_{\{x,x'\}}),$$

defined on the locus of pairs $(x, x') \in X \times X - \Delta$ such that the line $\ell_{\{x,x'\}}$ is not contained in X, where $t_{\{x,x'\}} \in \mathbb{P}^1$ is such that $\ell_{\{x,x'\}} \subset \mathcal{Q}_{t_{\{x,x'\}}}$. Noting that a line $\ell \subset \mathcal{Q}_t$ which is not contained in X intersects X in 0-dimensional scheme of degree 2 we see that f is dominant, generically of degree 2, and factors through the symmetric square of X. Thus, the induced map $\mathrm{Sym}^2 X \dashrightarrow \mathcal{G}$ is birational.

If $\mathcal{G}(k) \neq \varnothing$, then the Lang–Nishimura theorem (see, e.g., [Poonen 2017, Theorem 3.6.11]) (which applies since $\mathcal{G}$ is smooth) implies that $\mathrm{Sym}^2(X)(k) \neq \varnothing$ and, consequently, that there is a quadratic point on X. In particular, there is a quadratic extension K/k with $X(K) \neq \varnothing$. Conversely, if $X(K) \neq \varnothing$ for some quadratic extension K/k, then $X(K)$ is infinite by [Salberger and Skorobogatov 1991, Theorem (0.1)]. The line through any Galois stable pair of distinct points gives a k-rational point on $\mathcal{G}$.

(2) Let $t \in \mathbb{P}^1$. By [Reid 1972, Theorems 1.2 and 1.8], the fiber $\mathcal{G}_t$ is smooth and geometrically irreducible exactly when $\mathcal{Q}_t$ has rank 5. Thus, for all $t \in \mathbb{P}^1 - \mathcal{S}$, the class $[\mathcal{G}_{\boldsymbol{k}(\mathbb{P}^1)}]$ has trivial residue at t. By Proposition 4.2 and assumption (†), if $t \in \mathcal{S}$, then $\mathcal{Q}_t$ has rank 4. If $\mathcal{Q}_t$ is rank 4 and has square

discriminant, then by [Reid 1972, Theorem 1.8] the fiber $\mathcal{G}_t$ is reducible and split over $\boldsymbol{k}(t)$. If $\mathcal{Q}_t$ is rank 4 and has nonsquare discriminant, then the same result of Reid says that $\mathcal{G}_t$ is irreducible and nonsplit over $\boldsymbol{k}(t)$, but becomes split over the quadratic discriminant extension. Thus, the residue of $[\mathcal{G}_{\boldsymbol{k}(\mathbb{P}^1)}]$ at t is the discriminant of $\mathcal{Q}_t$ [Frossard 1997, Proposition 2.3]. By definition of $\varepsilon_{\mathcal{S}}$, this gives the first statement. The second statement now follows from the Faddeev exact sequence for $\operatorname{Br}\boldsymbol{k}(\mathbb{P}^1)$; see [Gille and Szamuely 2006, Theorem 6.4.5] or (5-4).

(3) The first statement is a theorem of Flynn [2009] which was expanded upon by Skorobogatov [2010]. The second statement follows from the first together with the Faddeev exact sequence for $\operatorname{Br}(\boldsymbol{k}(\mathbb{P}^1))$; see [Gille and Szamuely 2006, Theorem 6.4.5] or (5-4). □

The proceeding proposition together with Theorem 2.1 yields the following.

Corollary 4.6. *Assume k is a local field of characteristic not equal to 2. For any pencil of quadric threefolds $\mathcal{Q}\to\mathbb{P}^1$ satisfying (†), $\mathcal{G}_{\mathcal{Q}}(k)\neq\varnothing$.* □

4A. *Notation.* For a pencil of quadrics that satisfies (†) we will move freely between the objects $\mathcal{Q}$, $X = X_{\mathcal{Q}}$, $\mathcal{G} = \mathcal{G}_{\mathcal{Q}}$, and $(\mathcal{S}, \varepsilon_{\mathcal{S}}) = (\mathcal{S}_{\mathcal{Q}}, \varepsilon_{\mathcal{S}_{\mathcal{Q}}})$. We will assume that $\mathcal{S}\subset\mathbb{A}^1=\mathbb{P}^1-\infty$. This can be arranged by an automorphism of $\mathbb{P}^1$, provided k has at least 5 elements. We will write $k[T]$ for the coordinate ring of $\mathbb{A}^1$ and let $f(T)$ be the unique monic polynomial whose vanishing defines $\mathcal{S}$.

Let $Q_{\mathbb{A}^1}\in k(T)[x_0,\dots,x_4]$ be a quadratic form whose coefficients are linear polynomials in $k[T]$ and whose vanishing defines $\mathcal{Q}_{\mathbb{A}^1}$ on $\mathbb{A}^1\subset\mathbb{P}^1$. While $Q_{\mathbb{A}^1}$ is only defined up to multiplication by an element of $k^\times$, none of our results depend on this choice. For a (possibly reducible) subscheme $\mathcal{T}\subset\mathbb{A}^1=\operatorname{Spec}(k[T])$, the canonical map $k[T]\to\boldsymbol{k}(\mathcal{T})$ can be applied to the coefficients of $Q_{\mathbb{A}^1}$ to obtain a quadratic form $Q_{\mathcal{T}}$ over the k-algebra $\boldsymbol{k}(\mathcal{T})$ whose vanishing defines $\mathcal{Q}_{\mathcal{T}}=\mathcal{Q}\times_{\mathbb{P}^1}\mathcal{T}$. In particular, for $a\in k=\mathbb{A}^1(k)$, the form Q_a is obtained by evaluating the coefficients of $Q_{\mathbb{A}^1}$ at a. We define $Q_\infty = Q_1 - Q_0$, so that $Q_{\mathbb{A}^1} = Q_0 + TQ_\infty$.

We will write θ for the image of T in $\boldsymbol{k}(\mathcal{S}) = k[T]/\langle f(T)\rangle$. For a subscheme $\mathcal{T}\subset\mathcal{S}$ we use $\varepsilon_{\mathcal{T}}\in \boldsymbol{k}(\mathcal{T})^\times/\boldsymbol{k}(\mathcal{T})^{\times 2}\subset\boldsymbol{k}(\mathcal{S})^\times/\boldsymbol{k}(\mathcal{S})^{\times 2}$ to denote the discriminant corresponding to $\mathcal{Q}_{\mathcal{T}}$. We will use N to denote any map induced in an obvious way by the norm map $\operatorname{Norm}_{\boldsymbol{k}(\mathcal{S})/k}\colon \boldsymbol{k}(\mathcal{S})\to k$. Note that $\operatorname{Norm}_{\boldsymbol{k}(\mathcal{T})/k}(\varepsilon_{\mathcal{T}}) = \operatorname{Norm}_{\boldsymbol{k}(\mathcal{S})/k}(\varepsilon_{\mathcal{T}}) = \mathrm{N}(\varepsilon_{\mathcal{T}})$.

4B. *Alternate proof of Theorem 2.1 for odd residue characteristic.* We now give an alternate proof of Theorem 2.1 (valid for local fields of odd residue characteristic) which avoids the classification of reducible special fibers.

Proposition 4.7. *Let $X\subset\mathbb{P}^4_k$ be a smooth complete intersection of two quadrics over a local field k of characteristic not equal to 2. Then X has index dividing 2. If the residue characteristic of k is odd, then there is a quadratic extension K/k such that X has a K-point.*

Proof. First let us prove that X has a quadratic point assuming that $s\in\mathcal{S}(k)\neq\varnothing$. After a change of coordinates on the $\mathbb{P}^1$ parametrizing the pencil and a change of coordinates on $\mathbb{P}^4$, we may assume that

$s = 0$, that $Q_0 = Q_0(x_0, x_1, x_2, x_3)$, and that $Q_\infty = \tilde{Q}_\infty(x_0, x_1, x_2, x_3) + x_4^2$. If Q_0 contains a smooth k-point, then the line joining the vertex of Q_0 and this point will intersect X in a degree 2 subscheme, which shows that X has a quadratic point. Thus, we may restrict to the case that Q_0 has no smooth k-points.

Projection away from the vertex of $Q_0 \subset \mathbb{P}^4_k$ gives a double cover $X \to Y := Q_0 \cap V(x_4)$ onto the quadric surface Y. Since Q_0 has no smooth k-points, $Y(k) = \varnothing$. We will prove that, in this case, the branch curve C of the double cover $X \to Y$ has a quadratic point. Note that by definition of the double cover, $C = X \cap V(x_4)$ and so is a degree 4 genus 1 curve that is the base locus of the pencil of quadric surfaces $\mathcal{Q}' \to \mathbb{P}^1$ with $\mathcal{Q}'_t = \mathcal{Q}_t \cap V(x_4)$. Moreover, C is a 2-covering of the degree 2 genus one curve C' given by the equation $y^2 = \det(M)$ where M is the 4×4 symmetric matrix with entries in $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(1))$ corresponding to a defining equation for $\mathcal{Q}'$; see [An et al. 2001].

Consider the fiber of $C' \to \mathbb{P}^1$ above 0. By definition of $\mathcal{Q}'$, this is given by the equation $y^2 = \operatorname{disc}(Q_0 \cap V(x_4))$. By assumption, $Q_0 \cap V(x_4)$ has no k-points. Since there is (up to isomorphism) a unique rank 4 quadric over the local field k that is anisotropic and it has square discriminant, we conclude that $\operatorname{disc}(Q_0 \cap V(x_4))$ is a square and so $C'(k) \neq \varnothing$. Consequently, $C' \simeq \operatorname{Jac}(C)$ and so the order of C in $\mathrm{H}^1(k, \operatorname{Jac}(C))$ divides 2. By a result of Lichtenbaum [1968, Theorems 3 and 4] it follows that C has a point defined over some quadratic extension of the local field k. The aforementioned result of Lichtenbaum is stated for k a p-adic field, but the proof works for any local field due to Milne's extension of Tate's local duality results to positive characteristic [Milne 2006, Corollary I.3.4, Remark I.3.5, Theorem III.7.8].

Now we can deduce the statement in the proposition. The scheme $\mathcal{S} \subset \mathbb{P}^1_k$ parametrizing singular quadrics in the pencil has degree 5, so there is an odd degree extension k'/k such that $\mathcal{S}(k') \neq \varnothing$. By what we have shown above, X has a K-rational point for some quadratic extension K/k'. It follows that X has index at most 2. If the residue characteristic is odd, then the inclusion $k \subset k'$ induces an isomorphism $k^\times/k^{\times 2} \simeq k'^\times/k'^{\times 2}$, so K contains a quadratic extension k_2/k as an odd index subfield. By the theorems of Amer [1976], Brumer [1978] and Springer [1956], we have $X(K) \neq \varnothing \Rightarrow X(k_2) \neq \varnothing$, so X has a k_2-point. □

Remark 4.8. The preceding proof can be adapted to give an easy proof that a locally solvable del Pezzo surface of degree 4 over a global field over a field of characteristic different from 2 must have index dividing 2. Indeed, over some odd degree extension X may be written as a double cover of a quadric surface, which is known to satisfy the Hasse principle. Hence X obtains a rational point over some extension of degree $2m$ with m odd.

5. Arithmetic of the space of lines on the quadrics in the pencil

In this section we develop the main tools to prove Theorems 1.1 and 1.2 over global fields of characteristic not equal to 2. We maintain the notation defined in Section 4A. Specifically, $\mathcal{Q} \to \mathbb{P}^1$ is a pencil of quadrics in $\mathbb{P}^4_k$ over a field k of characteristic not equal to 2 which satisfies (†), and we let $X = X_{\mathcal{Q}}$, $\mathcal{G} = \mathcal{G}_{\mathcal{Q}}$ and $(\varepsilon_{\mathcal{S}}, \mathcal{S}) = (\varepsilon_{\mathcal{S}_{\mathcal{Q}}}, \mathcal{S}_{\mathcal{Q}})$.

In Section 5A, we compute $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$ and construct explicit representatives in $\mathrm{Br}(\mathcal{G})$, denoted by $\beta_{\mathcal{T}}$, which are determined by subsets $\mathcal{T}\subset\mathcal{S}$ such that $\mathrm{N}(\varepsilon_{\mathcal{T}})\in k^{\times 2}$. In Section 5B, we study the rank 4 quadrics $\mathcal{Q}_{\mathcal{T}}$ corresponding to subsets $\mathcal{T}\subset\mathcal{S}$ such that $\mathrm{N}(\varepsilon_{\mathcal{T}})\in k^{\times 2}$. We use Clifford algebras associated to these rank 4 quadrics to define constant Brauer classes $\mathcal{C}_{\mathcal{T}}\in\mathrm{Br}(k)$ and we show how these are related to the kernel of the canonical map $\mathrm{Br}(k)\to\mathrm{Br}(X)$. The two constructions come together in Sections 5C where we show how the $\mathcal{C}_{\mathcal{T}}$ arise when evaluating $\beta_{\mathcal{T}}$ at certain local points of $\mathcal{G}$ (see Lemmas 5.11 and 5.14). Finally, in Section 5D, we deduce consequences for the evaluation of $\beta_{\mathcal{T}}$ at adelic points of $\mathcal{G}$.

5A. *The Brauer group of $\mathcal{G}$.* It follows from the Faddeev exact sequence (see [Gille and Szamuely 2006, Theorem 6.4.5]) that the homomorphism

$$\gamma'\colon \boldsymbol{k}(\mathcal{S})^{\times}\ni\varepsilon\mapsto \mathrm{Cor}_{\boldsymbol{k}(\mathcal{S})/k}(\varepsilon, T-\theta)\in\mathrm{Br}(\boldsymbol{k}(\mathbb{P}^1))$$

induces an isomorphism

$$\gamma\colon \ker\left(\mathrm{N}\colon \frac{\boldsymbol{k}(\mathcal{S})^{\times}}{\boldsymbol{k}(\mathcal{S})^{\times 2}}\to\frac{k^{\times}}{k^{\times 2}}\right)\simeq \ker(\mathrm{Br}(\mathbb{P}^1-\mathcal{S})[2]\xrightarrow{\infty^*}\mathrm{Br}\,k[2]), \tag{5-1}$$

where ∞^* denotes evaluation of the Brauer class at $\infty\in\mathbb{P}^1-\mathcal{S}$. Recall that $\mathrm{N}(\varepsilon_{\mathcal{S}})\in k^{\times 2}$ by Proposition 4.4(2).

Define $\beta=\pi^*\gamma\colon \ker(\mathrm{N}\colon \boldsymbol{k}(\mathcal{S})^{\times}/\boldsymbol{k}(\mathcal{S})^{\times 2}\to k^{\times}/k^{\times 2})\to\mathrm{Br}(\boldsymbol{k}(\mathcal{G}))$. For $\mathcal{T}\subset\mathcal{S}$ such that $\mathrm{N}(\varepsilon_{\mathcal{T}})\in k^{\times 2}$, we set $\beta_{\mathcal{T}}:=\beta(\varepsilon_{\mathcal{T}})$.

Proposition 5.1. *The map β induces a homomorphism*

$$\ker\left(\mathrm{N}\colon \bigoplus_{s\in\mathcal{S}}\langle\varepsilon_s\rangle\to k^{\times}/k^{\times 2}\right)\xrightarrow{\beta}\mathrm{Br}(\mathcal{G}), \tag{5-2}$$

whose image surjects onto $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$. *Furthermore*, $\beta_{\mathcal{S}}=[\mathcal{G}_{\infty}]\in\mathrm{Br}_0(\mathcal{G})$, *and for all* $\mathcal{T}\subset\mathcal{S}$ *with* $\mathrm{N}(\varepsilon_{\mathcal{T}})\in k^{\times 2}$ *and* $\varepsilon_{\mathcal{T}}\neq\varepsilon_{\mathcal{S}}\in\boldsymbol{k}(\mathcal{S})^{\times}/\boldsymbol{k}(\mathcal{S})^{\times 2}$, *we have*

$$\beta_{\mathcal{T}}\in\mathrm{Br}_0(\mathcal{G})\subset\mathrm{Br}(\mathcal{G})\iff\beta_{\mathcal{T}}=0\in\mathrm{Br}(\mathcal{G})\iff\varepsilon_{\mathcal{T}}\in\boldsymbol{k}(\mathcal{T})^{\times 2}.$$

Corollary 5.2. (1) *Every nontrivial element of* $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$ *is represented by $\beta_{\mathcal{T}}$ for some degree* 2 *subscheme $\mathcal{T}\subset\mathcal{S}$ with* $\mathrm{N}(\varepsilon_{\mathcal{T}})\in k^{\times 2}$.

(2) $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})\simeq(\mathbb{Z}/2\mathbb{Z})^n$ *for some* $n\in\{0,1,2\}$.

(3) *If* $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$ *is not cyclic, then every degree* 2 *subscheme $\mathcal{T}\subset\mathcal{S}$ with* $\mathrm{N}(\varepsilon_{\mathcal{T}})\in k^{\times 2}$ ***must*** *be reducible.*

(4) *Let $s_0\in\mathcal{S}(k)$ be such that there exists an $s'\in\mathcal{S}(k)$ with* $\beta_{\{s_0,s'\}}\in\mathrm{Br}(\mathcal{G})-\mathrm{Br}_0(\mathcal{G})$. *Then* $\{\beta_{\{s_0,s\}}: s\in\mathcal{S}(k),\ \mathrm{N}(\varepsilon_{\{s_0,s\}})\in k^{\times 2}\}$ *generates* $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$.

(5) *There is a collection $\mathbb{T}$ of degree* 2 *subschemes of $\mathcal{S}$ and an element $\varepsilon\in k^{\times}$, such that*
- $\mathrm{N}(\varepsilon_{\mathcal{T}})\in k^{\times 2}$ *for all* $\mathcal{T}\in\mathbb{T}$;
- $\{\beta_{\mathcal{T}}:\mathcal{T}\in\mathbb{T}\}$ *generates* $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$;
- *for all* $s\in\cup_{\mathcal{T}\in\mathbb{T}}\mathcal{T}$, *the image of ε in* $\boldsymbol{k}(s)^{\times}/\boldsymbol{k}(s)^{\times 2}$ *is equal to* ε_s; *and*

- *for any extension L/k and any $s \in \cup_{\mathcal{T}\in\mathbb{T}}\mathcal{T}$, $\varepsilon \in \boldsymbol{k}(s_L)^{\times 2}$ if and only if $\varepsilon \in \boldsymbol{k}(s'_L)^{\times 2}$ for all $s' \in \bigcup_{\mathcal{T}\in\mathbb{T}}\mathcal{T}$.*

(6) $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G}) \simeq \mathrm{H}^1(k, \mathrm{Pic}(\overline{X}))$.

(7) *If k is a local or global field, then the injective map $\mathrm{Br}(X)/\mathrm{Br}_0(X) \to \mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$ given by Proposition 3.6(1) is an isomorphism.*

Proof of Corollary 5.2. The proposition implies that $\beta_{\mathcal{S}} \in \mathrm{Br}_0(\mathcal{G})$ and, for any $\mathcal{T} \subset \mathcal{S}$ such that $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in \boldsymbol{k}(\mathcal{T})^{\times 2}$, that $\beta_{\mathcal{T}} = \beta_{\mathcal{S}-\mathcal{T}} \in \mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$. Since $\mathcal{S}$ has degree 5, it follows that every nontrivial element in $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$ is represented by some $\beta_{\mathcal{T}}$ with $\deg(\mathcal{T}) \leq 2$. But if $\mathcal{T}$ has degree 1, then $\varepsilon_{\mathcal{T}} = \mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$ and $\beta_{\mathcal{T}} = 0$. Thus we have (1). In particular, if $\mathrm{Br}(\mathcal{G}) \neq \mathrm{Br}_0 \mathcal{G}$, then $\{\deg s : s \in \mathcal{S}\}$ must be $\{3, 2\}$, $\{3, 1, 1\}$, $\{2, 2, 1\}$, $\{2, 1, 1, 1\}$, or $\{1, 1, 1, 1, 1\}$. Now a straightforward case by case analysis of the possible relations on $\oplus_{s\in\mathcal{S}}\langle\varepsilon_s\rangle \cong \oplus_{s\in\mathcal{S}}\mathbb{Z}/2\mathbb{Z}$ allows one to deduce statements (2)–(4). Given this characterization of $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$ in terms of degree 2 subschemes $\mathcal{T} \subset \mathcal{S}$, (5) can be established using [Várilly-Alvarado and Viray 2014, Lemma 3.1] for the existence of $\varepsilon \in k^\times$. For (6), we observe that [Várilly-Alvarado and Viray 2014, proof of Theorem 3.4] gives a description of $\mathrm{H}^1(k, \mathrm{Pic}\,\overline{X})$ in terms of degree two subschemes $\mathcal{T} \subset \mathcal{S}$ and the square classes $\varepsilon_{\mathcal{T}}$; comparing this description with (5) gives the desired isomorphism. Finally, when k is a local or global field, the injective map $\mathrm{Br}_1(X)/\mathrm{Br}_0(X) \to \mathrm{H}^1(k, \mathrm{Pic}(\overline{X}))$ coming from the Hochschild–Serre spectral sequence [Colliot-Thélène and Skorobogatov 2021, Proposition 4.3.2] is an isomorphism, so (6) implies that the injective map $\mathrm{Br}(X)/\mathrm{Br}_0(X) \to \mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$ from Proposition 3.6(1) is also surjective. □

Remark 5.3. If $\mathcal{T} \subset \mathcal{S}$ is a degree 2 subscheme with $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$ such that the quadric $\mathcal{Q}_{\mathcal{T}}$ has a smooth $\boldsymbol{k}(\mathcal{T})$-point, then [Várilly-Alvarado and Viray 2014, Corollary 3.5] yields a rational map $\rho\colon X \dashrightarrow \mathbb{P}^1$ such that $\rho^*\gamma(\varepsilon_{\mathcal{T}}) \in \mathrm{Br}(X)$. One can show that the image of $\rho^*\gamma(\varepsilon_{\mathcal{T}})$ under the map $\mathrm{Br}(X)/\mathrm{Br}_0(X) \to \mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$ given by Proposition 3.6(1) is equal to the class of $\beta_{\mathcal{T}}$.

Proof of Proposition 5.1. Let $\eta \in \mathbb{P}^1$ be the generic point. Since $\mathcal{G}$ is smooth, $\mathrm{Br}(\mathcal{G})$ injects into $\mathrm{Br}(\mathcal{G}_\eta)$. Further, by the Hochschild–Serre spectral sequence, we have an exact sequence

$$0 \to \mathrm{Pic}(\mathcal{G}_\eta) \to (\mathrm{Pic}(\overline{\mathcal{G}_\eta}))^{G_{\boldsymbol{k}(\eta)}} \to \mathrm{Br}(\boldsymbol{k}(\eta)) \to \ker(\mathrm{Br}(\mathcal{G}_\eta) \to \mathrm{Br}(\overline{\mathcal{G}_\eta})) \to \mathrm{H}^1(G_{\boldsymbol{k}(\eta)}, \mathrm{Pic}(\overline{\mathcal{G}_\eta})).$$

Since $\mathcal{G}_\eta$ is a Severi–Brauer variety, $\mathrm{Pic}(\overline{\mathcal{G}_\eta}) \simeq \mathbb{Z}$ with trivial Galois action, and $\mathrm{Br}(\overline{\mathcal{G}_\eta}) = 0$. Hence, the exact sequence simplifies to

$$\mathbb{Z} \to \mathrm{Br}(\boldsymbol{k}(\eta)) \xrightarrow{\pi^*} \mathrm{Br}(\mathcal{G}_\eta) \to 0, \tag{5-3}$$

where the first map sends 1 to $[\mathcal{G}_\eta] \in \mathrm{Br}(\boldsymbol{k}(\eta))$. Thus, to determine $\mathrm{Br}(\mathcal{G})$, it suffices to determine $\mathrm{Br}(\mathcal{G}) \cap \pi^*\mathrm{Br}(\boldsymbol{k}(\eta))$.

The projection map $\pi : \mathcal{G} \to \mathbb{P}^1$ induces the following commutative diagram of exact sequences where the top row is the Faddeev exact sequence [Gille and Szamuely 2006, Theorem 6.4.5]:

$$\begin{array}{ccccccc}
\mathrm{Br}(k) & \hookrightarrow & \mathrm{Br}(\boldsymbol{k}(\eta)) & \xrightarrow{(\partial_t)} & \bigoplus_{t\in(\mathbb{P}^1)^{(1)}} \mathrm{H}^1(\boldsymbol{k}(t), \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\sum_t \mathrm{Cor}_{\boldsymbol{k}(t)/k}} & \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \\
\downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*_{\mathrm{Br}}} & & \downarrow{\scriptstyle \pi^*_{\mathrm{H}^1}} & & \\
\mathrm{Br}(\mathcal{G}) & \hookrightarrow & \mathrm{Br}(\mathcal{G}_\eta) & \xrightarrow{(\partial_x)} & \bigoplus_{t\in\mathbb{P}^1} \bigoplus_{\substack{x\in\mathcal{G}^{(1)} \\ \pi(x)=t}} \mathrm{H}^1(\boldsymbol{k}(x), \mathbb{Q}/\mathbb{Z}), & &
\end{array} \tag{5-4}$$

If $t \in \mathbb{P}^1 - \mathcal{S}$, then the fiber $\mathcal{G}_t$ is geometrically irreducible by Proposition 4.2 and hence

$$\pi^*_{\mathrm{H}^1} \colon \mathrm{H}^1(\boldsymbol{k}(t), \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(\boldsymbol{k}(\mathcal{G}_t), \mathbb{Q}/\mathbb{Z})$$

is an injection. For $t \in \mathcal{S}$, the fiber $\mathcal{G}_t$ consists of two split components that are conjugate over $\boldsymbol{k}(t)(\sqrt{\varepsilon_t})$.

Therefore, for $t \in \mathcal{S}$, the kernel of $\pi^*\colon \mathrm{H}^1(\boldsymbol{k}(t), \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(\boldsymbol{k}(\mathcal{G}_t), \mathbb{Q}/\mathbb{Z})$ is the 2-torsion cyclic subgroup corresponding to the extension $\bar{k} \cap \boldsymbol{k}(\mathcal{G}_t) = \boldsymbol{k}(t)(\sqrt{\varepsilon_t})$. Moreover, the residues of the kernel of π^*_{Br} are $(\partial_t)(\ker(\pi^*_{\mathrm{Br}})) = (\varepsilon_t)_{t\in\mathcal{S}} = \varepsilon_{\mathcal{S}} \in \boldsymbol{k}(\mathcal{S})/\boldsymbol{k}(\mathcal{S})^{\times 2}$. Thus, the commutativity of the above diagram shows that

$$\ker \pi^*_{\mathrm{H}^1} \bigcap \ker \sum_t \mathrm{Cor}_{\boldsymbol{k}(t)/k} \simeq \ker\Bigl(\mathrm{N}\colon \bigoplus_{t\in\mathcal{S}} \langle \varepsilon_t \rangle \to k^\times/k^{\times 2}\Bigr).$$

In particular, the image under β of $\ker\bigl(\mathrm{N}\colon \bigoplus_{t\in\mathcal{S}} \langle\varepsilon_t\rangle \to k^\times/k^{\times 2}\bigr)$ is contained inside of $\mathrm{Br}(\mathcal{G})$. Further, since π^*_{Br} is surjective, the image of $\ker\bigl(\mathrm{N}\colon \bigoplus_{t\in\mathcal{S}} \langle\varepsilon_t\rangle \to k^\times/k^{\times 2}\bigr)$ under β generates $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$.

It remains to understand which subsets $\mathcal{T} \subset \mathcal{S}$ give rise to $\beta_{\mathcal{T}} \in \mathrm{Br}_0(\mathcal{G})$. If $\beta_{\mathcal{T}} \in \mathrm{Br}_0(\mathcal{G})$, then by definition of $\mathrm{Br}_0(\mathcal{G})$ there exists $\mathcal{A} \in \mathrm{Br}(k)$ such that $\gamma(\varepsilon_{\mathcal{T}}) - \mathcal{A} \in \ker \pi^*_{\mathrm{Br}}$. By (5-3), the kernel of π^*_{Br} is generated by $[\mathcal{G}_\eta]$. Thus, $\gamma(\varepsilon_{\mathcal{T}}) = [\mathcal{G}_\eta] + \mathcal{A}$ or $\gamma(\varepsilon_{\mathcal{T}}) = \mathcal{A}$, where both equalities are in $\mathrm{Br}(\mathbb{P}^1 - \mathcal{S})$. The final statement of the proposition follows from these equalities after computing residues and evaluating at ∞. □

Recall that B_{Q_t} denotes the bilinear form corresponding to Q_t.

Lemma 5.4. *Let $f\colon \mathrm{Sym}^2(X) \dashrightarrow \mathcal{G}$ be the birational map given in Proposition 4.4 and let $\{x, x'\} \in \mathrm{Sym}^2(X) - \mathrm{Indet}(f)$. Suppose that $f(\{x, x'\}) = (t, \ell) =: y \in \mathcal{G}(k)$. Then $\pi(y) = t = [B_{Q_0}(x, x') : -B_{Q_\infty}(x, x')] \in \mathbb{P}^1(k)$. If, moreover, $\mathcal{T} \subset \mathcal{S}$ is such that $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$ and $\pi(y) \notin \mathcal{T} \cup \{\infty\}$, then*

$$\beta_{\mathcal{T}}(y) = \mathrm{Cor}_{\boldsymbol{k}(\mathcal{T})/k}\biggl(\varepsilon_{\mathcal{T}}, -\frac{B_{Q_{\mathcal{T}}}(x, x')}{B_{Q_\infty}(x, x')}\biggr).$$

Proof. Observe that for any point $ax + bx'$ on the line $\ell_{\{x,x'\}}$ through x and x' we have

$$Q_t(ax + bx') = B_{Q_t}(ax + bx', ax + bx') = a^2 Q_t(x) + b^2 Q_t(x') + 2ab B_{Q_t}(x, x') = 2ab B_{Q_t}(x, x').$$

Therefore, the line $\ell_{\{x,x'\}}$ is contained in the quadric $\mathcal{Q}_t$ precisely when $B_{Q_t}(x, x') = 0$. If $B_{Q_0}(x, x') = B_{Q_\infty}(x, x') = 0$, then $\ell_{\{x,x'\}} \subset X$ in which case f is not defined at $\{x, x'\}$. Otherwise, the relation $B_{Q_t}(x, x') := B_{Q_0}(x, x') + t B_{Q_\infty}(x, x') = 0$ shows that $t = \pi(y) \in \mathbb{P}^1$ must be equal to $[B_{Q_0}(x, x') : -B_{Q_\infty}(x, x')]$.

For the second statement recall that $\beta_{\mathcal{T}} = \pi^*\gamma(\varepsilon_{\mathcal{T}}) = \pi^* \operatorname{Cor}_{\boldsymbol{k}(\mathcal{S})/k}(\varepsilon_{\mathcal{T}}, T-\theta)$, where T is the coordinate on $\operatorname{Spec}(k[T]) = \mathbb{A}^1 \subset \mathbb{P}^1$ and θ is the image of T in $\boldsymbol{k}(\mathcal{S})$. We have

$$\pi(y) - \theta = -\frac{B_{Q_0}(x, x') + \theta B_{Q_\infty}(x, x')}{B_{Q_\infty}(x, x')} = -\frac{B_{Q_{\mathcal{S}}}(x, x')}{B_{Q_\infty}(x, x')} \in \boldsymbol{k}(\mathcal{S}).$$

As $\gamma(\varepsilon_{\mathcal{T}})$ is unramified away from $\mathcal{T}$ we may evaluate at $\pi(y)$ to obtain

$$\beta_{\mathcal{T}}(y) = \gamma(\varepsilon_{\mathcal{T}})(\pi(y)) = \operatorname{Cor}_{\boldsymbol{k}(\mathcal{S})/k}\left(\varepsilon_{\mathcal{T}}, -\frac{B_{Q_{\mathcal{S}}}(x, x')}{B_{Q_\infty}(x, x')}\right).$$

The projections of $\varepsilon_{\mathcal{T}} \in \bigoplus_{s\in\mathcal{S}} \boldsymbol{k}(s)^\times/\boldsymbol{k}(s)^{\times 2}$ onto the factors corresponding to $s \in \mathcal{S} - \mathcal{T}$ are trivial. So

$$\beta_{\mathcal{T}}(y) = \operatorname{Cor}_{\boldsymbol{k}(\mathcal{S})/k}\left(\varepsilon_{\mathcal{T}}, -\frac{B_{Q_{\mathcal{S}}}(x, x')}{B_{Q_\infty}(x, x')}\right) = \operatorname{Cor}_{\boldsymbol{k}(\mathcal{T})/k}\left(\varepsilon_{\mathcal{T}}, -\frac{B_{Q_{\mathcal{T}}}(x, x')}{B_{Q_\infty}(x, x')}\right). \qquad \square$$

5B. ***Clifford algebras and Brauer classes.*** For a quadratic form F over a field of characteristic not equal to 2 we use $\operatorname{Clif}(F)$ to denote the Clifford algebra of the restriction of F to a maximal regular subspace, and $\operatorname{Clif}_0(F)$ to denote the corresponding even subalgebra. By Witt's theorem [Lam 2005, Chapter I, Theorems 4.2 and 4.3], these do not depend on the choice of maximal regular subspace. If F has even rank, then $\operatorname{Clif}(F)$ is a central simple algebra, which will be identified with its class in the Brauer group. This extends to quadratic forms over finite étale algebras in the natural way, i.e., factor by factor.

In particular, we will consider $\operatorname{Clif}(Q_{\mathcal{T}}) \in \operatorname{Br}(\boldsymbol{k}(\mathcal{T}))$ where $Q_{\mathcal{T}}$ is a quadratic form defining the quadric $\mathcal{Q}_{\mathcal{T}}$ corresponding to a subscheme $\mathcal{T} \subset \mathcal{S}$. This depends on the choice of quadratic form as indicated by the following lemma.

Lemma 5.5. *Let $s \in \mathcal{S}$ and $c \in \boldsymbol{k}(s)^\times$. Then*

$$\operatorname{Clif}(cQ_s) = \operatorname{Clif}(Q_s) + (\varepsilon_s, c) \in \operatorname{Br}(\boldsymbol{k}(s)).$$

Proof. This follows from a short calculation using [Lam 2005, Chapter V, Corollary 2.7]. $\square$

For a rank 4 quadric $\mathcal{Q}_s$, $s \in \mathcal{S}$ with $\varepsilon_s \in \boldsymbol{k}(s)^{\times 2}$, any quadratic form Q_s defining $\mathcal{Q}_s$ is a constant multiple of the reduced norm form of a quaternion algebra whose class in $\operatorname{Br}(\boldsymbol{k}(s))$ is equal to $\operatorname{Clif}(Q_s)$ [Elman et al. 2008, Proposition 12.4]. The following lemma gives a description of $\operatorname{Clif}(Q_s)$ in cases when $\varepsilon_s \notin \boldsymbol{k}(s)^{\times 2}$.

Lemma 5.6. *Assume that there exists some $s \in \mathcal{S}$ with $\varepsilon_s \notin \boldsymbol{k}(s)^{\times 2}$ such that $\mathcal{Q}_s$ has a smooth $\boldsymbol{k}(s)$-point. Let Q_s be a quadratic form whose vanishing defines $\mathcal{Q}_s$. Then for any $\operatorname{Gal}(\boldsymbol{k}(s))$-stable pair $\{x, x'\} \subset \mathcal{Q}_s(\bar{k})$ and any $\boldsymbol{k}(s)$-linear form ℓ defining a hyperplane tangent to $\mathcal{Q}_s$ at a smooth point with $\ell(x)\ell(x') \neq 0$ we have the following equality in $\operatorname{Br}(\boldsymbol{k}(s))$:*

$$\operatorname{Clif}(Q_s) = \left(\varepsilon_s, -\frac{B_{Q_s}(x, x')}{\ell(x)\ell(x')}\right),$$

where B_{Q_s} denotes the bilinear form corresponding to Q_s.

Proof. By [Várilly-Alvarado and Viray 2014, Lemma 2.1], for any $\ell = \ell_0$ tangent to $\mathcal{Q}_s$ at a smooth point, the quadric $\mathcal{Q}_s$ is defined by the vanishing of $Q_s = c(\ell_0\ell_1 - \ell_2^2 + \varepsilon_s\ell_3^2)$, for some linear forms ℓ_1, ℓ_2, ℓ_3 and some $c \in \boldsymbol{k}(s)^\times$. In particular, we have $\ell_0(x)\ell_1(x) = \ell_2(x)^2 - \varepsilon_s\ell_3(x)^2$ and similarly for x'. Thus, we may compute

$$\begin{aligned} -\frac{B_{Q_s}(x,x')}{\ell(x)\ell(x')} &= -c\cdot\frac{\ell_0(x)\ell_1(x')+\ell_0(x')\ell_1(x)-2\ell_2(x)\ell_2(x')+2\varepsilon_s\ell_3(x)\ell_3(x')}{\ell_0(x)\ell_0(x')} \\ &= -c\left(\frac{\ell_2(x')^2-\varepsilon_s\ell_3(x')^2}{\ell_0(x')^2}+\frac{\ell_2(x)^2-\varepsilon_s\ell_3(x)^2}{\ell_0(x)^2}-2\frac{\ell_2(x)\ell_2(x')}{\ell_0(x)\ell_0(x')}+2\varepsilon_s\frac{\ell_3(x)\ell_3(x')}{\ell_0(x)\ell_0(x')}\right) \\ &= -c\left[\left(\frac{\ell_2(x)}{\ell_0(x)}-\frac{\ell_2(x')}{\ell_0(x')}\right)^2-\varepsilon_s\left(\frac{\ell_3(x)}{\ell_0(x)}-\frac{\ell_3(x')}{\ell_0(x')}\right)^2\right], \end{aligned}$$

which shows that $(\varepsilon_s, -B_{Q_s}(x,x')/(\ell(x)\ell(x'))) = (\varepsilon_s, -c)$. Thus, it remains to relate the quaternion algebra $(\varepsilon_s, -c)$ to the Clifford algebra of Q_s. By [Lam 2005, Chapter V, Corollary 2.7],

$$\mathrm{Clif}(Q_s) \simeq \mathrm{Clif}(Q_s|_{\langle\ell_0,\ell_1\rangle}) \otimes \mathrm{Clif}(c^2\cdot Q_s|_{\langle\ell_2,\ell_3\rangle}) \simeq \mathrm{M}_2(k)\otimes(-c, c\varepsilon_s).$$

To complete the proof, we observe that $(-c, c\varepsilon_s) = (-c, \varepsilon_s) = (\varepsilon_s, -c) \in \mathrm{Br}(k)$. □

Definition 5.7. Given $\mathcal{T} \subset \mathcal{S}$ such that $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$, define

$$\mathcal{C}_{\mathcal{T}} := \mathrm{Cor}_{\boldsymbol{k}(\mathcal{T})/k}(\mathrm{Clif}(Q_{\mathcal{T}})) \in \mathrm{Br}(k).$$

Remark 5.8. Even though $\mathrm{Clif}(Q_{\mathcal{T}})$ may depend on the choice of quadratic form defining the pencil, the condition $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$ ensures that the class $\mathcal{C}_{\mathcal{T}}$ does not. Indeed, if one computes $\mathcal{C}_{\mathcal{T}}$ using instead a form $cQ_{\mathcal{T}}$ which differs from $Q_{\mathcal{T}}$ by $c \in k^\times$, Lemma 5.5 shows that the result will differ by $\mathrm{Cor}_{\boldsymbol{k}(\mathcal{T})/k}(\varepsilon_{\mathcal{T}}, c) = (\mathrm{N}(\varepsilon_{\mathcal{T}}), c)$, which is trivial whenever $\mathrm{N}(\varepsilon_{\mathcal{T}})$ is a square.

Lemma 5.9. *The kernel of the canonical map* $\mathrm{Br}(k) \to \mathrm{Br}(X)$ *is generated by*

$$\{\mathcal{C}_s : s \in \mathcal{S} \textit{ such that } \varepsilon_s \in \boldsymbol{k}(s)^{\times 2}\}.$$

Proof. By the exact sequence of low degree terms coming from the Hochschild–Serre spectral sequence [Colliot-Thélène and Skorobogatov 2021, Proposition 4.3.2], the kernel of $\mathrm{Br}(k) \to \mathrm{Br}(X)$ is the image of the cokernel $\mathrm{Pic}(X) \to \mathrm{Pic}(\overline{X})^{G_k}$. By [Várilly-Alvarado and Viray 2014, Proposition 2.3] (which relies on results from [Kunyavskiĭ et al. 1989]), $\mathrm{Pic}(\overline{X})^{G_k}$ is freely generated by the hyperplane section and, for every $s \in \mathcal{S}$ such that $\varepsilon_s \in \boldsymbol{k}(s)^{\times 2}$, the divisor class $\mathrm{Norm}_{\boldsymbol{k}(s)/k}([C_s])$ where C_s is obtained by intersecting X with a plane contained in $\mathcal{Q}_s$. Since the hyperplane section is k-rational, the cokernel of $\mathrm{Pic}(X) \to \mathrm{Pic}(\overline{X})^{G_k}$ is generated by

$$\{\mathrm{Norm}_{\boldsymbol{k}(s)/k}([C_s]) : s \in \mathcal{S} \text{ such that } \varepsilon_s \in \boldsymbol{k}(s)^{\times 2} \text{ and } \mathcal{Q}_s \text{ contains no } k\text{-rational planes}\}.$$

By definition, the image of $[C_s]$ in $\mathrm{Br}(\boldsymbol{k}(s))$ is the Severi–Brauer variety whose points parametrize representatives of the class $[C_s]$. Since ε_s is a square, by [Colliot-Thélène and Skorobogatov 1993, Theorem 2.5], $\mathcal{Q}_s$ is a cone over the surface $Z_s \times Z_s$, where Z_s is a smooth conic obtained by intersecting

$\mathcal{Q}_s$ with a general 2-plane. Since planes in a fixed ruling on $\mathcal{Q}_s$ correspond to fibers in a projection $Z_s \times Z_s \to Z_s$, we deduce that $[C_s] \mapsto Z_s \in \mathrm{Br}(\boldsymbol{k}(s))$. By [Elman et al. 2008, Proposition 12.4] we also have that $\mathrm{Clif}(Q_s) = Z_s \in \mathrm{Br}(\boldsymbol{k}(s))$. Hence,

$$\mathrm{Norm}_{\boldsymbol{k}(s)/k}([C_s]) = \mathrm{Cor}_{\boldsymbol{k}(s)/k}(\mathrm{Clif}(Q_s)) = \mathcal{C}_s. \qquad \square$$

5C. *Local evaluation maps.*

Lemma 5.10. *If there exists a degree* 2 *subscheme* $\mathcal{T} \subset \mathcal{S}$ *such that for all* $t \in \mathcal{T}$, $\varepsilon_t \in \boldsymbol{k}(t)^{\times 2}$ *and* $\mathcal{Q}_t$ *has a smooth* $\boldsymbol{k}(t)$*-point, then* $X(k) \neq \varnothing$.

Proof. Let $\mathcal{T}(\bar{k}) = \{t_1, t_2\}$. The assumptions in the lemma imply that there are $k(t_i)$-rational planes contained in $\mathcal{Q}_{t_i}$. The intersection of one with X gives a $k(t_i)$-rational conic C_i on X. If $t_1 \notin \mathcal{T}(k)$, then we replace C_2 with the conjugate of C_1. Thus, the pair $\{C_1, C_2\}$ are Galois invariant. As computed in [Várilly-Alvarado and Viray 2014, proof of Proposition 2.2] we have $C_1.C_2 = 1$. (We note that our C_2 may be either C_2 or C_2' in the notation of [Várilly-Alvarado and Viray 2014], but both have the same intersection number with C_1.) Therefore the intersection of these divisors produces a k-point on X. $\square$

Lemma 5.11. *Assume that* k *is a local field of characteristic not equal to* 2 *and let* $\mathcal{T} \subset \mathcal{S}$ *be a degree* 2 *subscheme such that* $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$. *Then, for any quadratic extension* K/k *with* $\varepsilon_{\mathcal{T}} \in \boldsymbol{k}(\mathcal{T}_K)^{\times 2}$ *and* $K \neq \boldsymbol{k}(\mathcal{T})$, *there exists* $y \in \mathcal{G}(k)$ *corresponding to a quadratic point* Spec $K \to X$. *Moreover, for such* y,

$$\beta_{\mathcal{T}}(y) = \begin{cases} \mathcal{C}_{\mathcal{T}} & \text{if } \varepsilon_{\mathcal{T}} \notin \boldsymbol{k}(\mathcal{T})^{\times 2}, \\ 0 & \text{if } \varepsilon_{\mathcal{T}} \in \boldsymbol{k}(\mathcal{T})^{\times 2}. \end{cases}$$

Proof. If $X(k) \neq \varnothing$, then for any nontrivial extension K/k we have $X(k) \subsetneq X(K)$ because k is local; see, e.g., [Liu and Lorenzini 2018, Proposition 8.3]. Then any pair of $\mathrm{Gal}(K/k)$-conjugate points on X will give the required $y \in \mathcal{G}(k)$. Now we prove the first statement in the case where $X(k) = \varnothing$. Over any local field, there is a unique rank 4 quadric (up to isomorphism) that fails to have a point, and it has square discriminant. Furthermore, this anisotropic quadric has a point over any quadratic extension of $\boldsymbol{k}(t)$.

If $\varepsilon_t \in \boldsymbol{k}(t)^{\times 2}$ for some $t \in \mathcal{T}$ (equivalently, for all $t \in \mathcal{T}$ by Corollary 5.2(5)), then $\mathcal{Q}_t$ may not have a smooth $\boldsymbol{k}(t)$-point, but it will have a smooth point over any quadratic extension of $\boldsymbol{k}(t)$. If K/k is a quadratic extension different from $\boldsymbol{k}(\mathcal{T})/k$, then $\boldsymbol{k}(\mathcal{T}_K)$ will be a quadratic extension of $\boldsymbol{k}(\mathcal{T})$ and hence we may apply Lemma 5.10. Moreover, since $\varepsilon_{\mathcal{T}} \in \boldsymbol{k}(\mathcal{T})^{\times 2}$, by definition, $\beta_{\mathcal{T}} = 0 \in \mathrm{Br}(\mathcal{G})$.

Now we consider the case when $\varepsilon_t \notin \boldsymbol{k}(t)^{\times 2}$, so Q_t has nonsquare discriminant, and thus is isotropic. Hence, Lemma 5.10 gives the existence of K-points on X for any K such that $\varepsilon_{\mathcal{T}} \in \boldsymbol{k}(\mathcal{T}_K)^{\times 2}$.

Now suppose y corresponds to the line joining the K/k-conjugate points $x, x' \in X(K)$, with K satisfying the conditions of the lemma. By continuity of the evaluation map, we may reduce to the case where $\pi(y) \neq \infty$, $\pi(y) \notin \mathcal{T}$. By Lemma 5.4 we have

$$\begin{aligned}
\beta_{\mathcal{T}}(y) &= (\varepsilon_{\mathcal{T}}, -B_{Q_{\mathcal{T}}}(x,x')/B_{Q_\infty}(x,x')) \\
&= \mathrm{Cor}_{\boldsymbol{k}(\mathcal{T})/k}(\varepsilon_{\mathcal{T}}, -B_{Q_{\mathcal{T}}}(x,x')) + (\mathrm{N}_{\boldsymbol{k}(\mathcal{T})/k}(\varepsilon_{\mathcal{T}}), B_{Q_\infty}(x,x')) \\
&= \mathrm{Cor}_{\boldsymbol{k}(\mathcal{T})/k}(\varepsilon_{\mathcal{T}}, -B_{Q_{\mathcal{T}}}(x,x')) && (\text{since } N(\varepsilon_{\mathcal{T}}) \in k^{\times 2}) \\
&= \mathrm{Cor}_{\boldsymbol{k}(\mathcal{T})/k}[(\varepsilon_{\mathcal{T}}, \ell_{\mathcal{T}}(x)\ell_{\mathcal{T}}(x')) + \mathrm{Clif}(Q_{\mathcal{T}})] && (\text{by Lemma 5.6}) \\
&= \mathrm{Cor}_{\boldsymbol{k}(\mathcal{T}_K)/k}(\varepsilon_{\mathcal{T}}, \ell_{\mathcal{T}}(x)) + \mathrm{Cor}_{\boldsymbol{k}(\mathcal{T})/k}\mathrm{Clif}(Q_{\mathcal{T}}) \\
&= \mathrm{Cor}_{\boldsymbol{k}(\mathcal{T})/k}\mathrm{Clif}(Q_{\mathcal{T}}) && (\text{since } \varepsilon_{\mathcal{T}} \in \boldsymbol{k}(\mathcal{T}_K)^{\times 2}) \\
&= \mathcal{C}_{\mathcal{T}} && (\text{by Definition 5.7}). \qquad \square
\end{aligned}$$

Lemma 5.12. *Assume that k is a local field of characteristic not equal to 2. Suppose $s \in \mathcal{S}(k)$ is such that $\mathcal{Q}_s$ has a smooth k-point and let v_s denote the vertex of $\mathcal{Q}_s$. For any $t \in \mathbb{A}^1(k) - \{s\}$ sufficiently close to s, we have*

$$\mathcal{G}_t(k) \neq \varnothing \iff (\varepsilon_s, t-s) = \mathrm{Clif}(Q_s) + (\varepsilon_s, -Q_\infty(v_s)) \text{ in } \mathrm{Br}(k).$$

Remark 5.13. Note that by Lemma 5.5, the sum $\mathrm{Clif}(Q_s) + (\varepsilon_s, -Q_\infty(v_s))$ appearing on the right-hand side above does not depend on the choice of quadratic form defining the pencil.

Proof. Since $t \in \mathbb{A}^1(k) - \{s\}$ is sufficiently close to s and $\mathcal{S}$ is closed, we have $t \notin \mathcal{S}$ and Q_t has rank 5. So by [Elman et al. 2008, Example 85.4] the Severi–Brauer variety $\mathcal{G}_t$ and the even Clifford algebra $\mathrm{Clif}_0(Q_t)$ (which is a central simple k-algebra) determine the same class in $\mathrm{Br}(k)$. In particular, $\mathcal{G}_t(k) \neq \varnothing$ if and only if $\mathrm{Clif}_0(Q_t) = 0$ in $\mathrm{Br}(k)$.

Since X is smooth, $Q_t(v_s) \neq 0$. So the quadratic forms Q_t and $Q_t|_{\langle v_s\rangle^\perp} \oplus Q_t|_{\langle v_s\rangle}$ are equivalent by [Lam 2005, Chapter I, Corollary 2.5]. Therefore,

$$\begin{aligned}
\mathrm{Clif}_0(Q_t) &= \mathrm{Clif}(-Q_t(v_s) \cdot Q_t|_{\langle v_s\rangle^\perp}) && (\text{by [Lam 2005, Chapter V, Corollary 2.9]}) \\
&= \mathrm{Clif}(Q_t|_{\langle v_s\rangle^\perp}) + (\mathrm{disc}(Q_t|_{\langle v_s\rangle^\perp}), -Q_t(v_s)) && (\text{by Lemma 5.5}).
\end{aligned}$$

For t sufficiently close to s, the quadratic forms $Q_t|_{\langle v_s\rangle^\perp}$ and $Q_s|_{\langle v_s\rangle^\perp}$ will be equivalent. For such t, $\mathrm{Clif}(Q_t|_{\langle v_s\rangle^\perp}) = \mathrm{Clif}(Q_s) \in \mathrm{Br}(k)$ and $\mathrm{disc}(Q_t|_{\langle v_s\rangle^\perp}) = \mathrm{disc}(Q_s) \in k^\times/k^{\times 2}$. Hence

$$\mathrm{Clif}_0(Q_t) = \mathrm{Clif}(Q_s) + (\varepsilon_s, -Q_t(v_s)).$$

To complete the proof, we note that $Q_t(v_s) = (Q_s + (t-s)Q_\infty)(v_s) = (t-s)Q_\infty(v_s)$. $\square$

Lemma 5.14. *Assume that k is a local field of characteristic not equal to 2 and $\mathcal{T} \subset \mathcal{S}$ is a degree 2 subscheme with $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$ and $\varepsilon_{\mathcal{T}} \notin \boldsymbol{k}(\mathcal{T})^{\times 2}$. Then there exists $y \in \mathcal{G}_{\mathcal{T}}(\boldsymbol{k}(\mathcal{T}))$ such that $\pi(y) = \mathcal{T}$. Moreover, for any such y,*

$$\mathrm{Cor}_{\boldsymbol{k}(\mathcal{T})/k}(\beta_{\mathcal{T}}(y)) = \mathcal{C}_{\mathcal{T}} + (\varepsilon, -\Delta_{\mathcal{T}}\mathrm{N}(Q_\infty(v_{\mathcal{T}}))) \in \mathrm{Br}(k),$$

where $\varepsilon \in k^\times$ is an element whose image in $\boldsymbol{k}(\mathcal{T})^\times/\boldsymbol{k}(\mathcal{T})^{\times 2}$ represents $\varepsilon_{\mathcal{T}}$, $\Delta_{\mathcal{T}}$ is the discriminant of $\boldsymbol{k}(\mathcal{T})/k$ (which we take to be 1 if $\mathcal{T}$ is reducible), and $v_{\mathcal{T}}$ is the vertex of $\mathcal{Q}_{\mathcal{T}}$.

Proof. By [Várilly-Alvarado and Viray 2014, Lemma 3.1] there exists $\varepsilon \in k^\times$ such that $\varepsilon \cdot \varepsilon_t \in \boldsymbol{k}(t)^{\times 2}$ for all $t \in \mathcal{T}$. Fix a closed point $s \in \mathcal{T}$, and let s' be the unique $\boldsymbol{k}(s)$ point in $\mathcal{T}_{\boldsymbol{k}(s)} - \{s\}$.

Since $\varepsilon_s \notin \boldsymbol{k}(s)^{\times 2}$, $\mathcal{Q}_s$ is a cone over an isotropic quadric and as such contains smooth $\boldsymbol{k}(s)$-points and $\boldsymbol{k}(s)$-rational lines (passing through the vertex). Hence $\mathcal{G}_s(\boldsymbol{k}(s))$ is nonempty. By the implicit function theorem, we can find $t \in (\mathbb{P}^1 - \{s\})(\boldsymbol{k}(s))$ arbitrarily close to s such that $\mathcal{G}_t(\boldsymbol{k}(s)) \neq \varnothing$. In addition, by Lemma 5.12 and the fact that the evaluation map $\beta\colon \mathcal{G}(\boldsymbol{k}(s)) \to \operatorname{Br}(\boldsymbol{k}(s))$ is locally constant and constant on the fibers of $\pi : \mathcal{G} \to \mathbb{P}^1$ (because $\beta_{\mathcal{T}}$ is the pullback of a element of $\operatorname{Br}(\boldsymbol{k}(\mathbb{P}^1))$), we may choose such a t sufficiently close to s so that

(1) $(\varepsilon, t-s) = \operatorname{Clif}(Q_s) + (\varepsilon, -Q_\infty(v_s)) \in \operatorname{Br}(\boldsymbol{k}(s))$,

(2) $\beta_{\mathcal{T}}(\mathcal{G}_s(\boldsymbol{k}(s))) = \beta_{\mathcal{T}}(\mathcal{G}_t(\boldsymbol{k}(s))) \in \operatorname{Br}(\boldsymbol{k}(s))$, and

(3) $t-s'$ and $s-s'$ represent the same class in $\boldsymbol{k}(s)^{\times 2}$.

Then for $y_1 \in \mathcal{G}_s(\boldsymbol{k}(s))$ and $y_1' \in \mathcal{G}_t(\boldsymbol{k}(s))$, we have

$$\beta_{\mathcal{T}}(y_1) = \beta_{\mathcal{T}}(y_1') = (\varepsilon, (t-s)(t-s')) = \operatorname{Clif}(Q_s) + (\varepsilon, -Q_\infty(v_s)(s-s')) \in \operatorname{Br}(\boldsymbol{k}(s)).$$

Suppose $y\colon \operatorname{Spec}(\boldsymbol{k}(\mathcal{T})) \to \mathcal{G}$ is such that $\pi(y) = \mathcal{T}$. Then, because $\varepsilon \in k^\times$ we have $\operatorname{Cor}_{\boldsymbol{k}(\mathcal{T})/k}(\varepsilon, s-s') = (\varepsilon, \operatorname{Norm}_{\boldsymbol{k}(\mathcal{T})/k}(s-s')) = (\varepsilon, (s-s')(s'-s))$. It follows that

$$\begin{aligned}\operatorname{Cor}_{\boldsymbol{k}(\mathcal{T})/k}(\beta_{\mathcal{T}}(y)) &= \operatorname{Cor}_{\boldsymbol{k}(\mathcal{T})/k}[\operatorname{Clif}(Q_{\mathcal{T}}) + (\varepsilon, -Q_\infty(v_{\mathcal{T}})) + (\varepsilon, s-s')]\\ &= \operatorname{Cor}_{\boldsymbol{k}(\mathcal{T})/k}(\operatorname{Clif}(Q_{\mathcal{T}})) + (\varepsilon, \mathrm{N}(Q_\infty(v_{\mathcal{T}}))) + (\varepsilon, (s-s')(s'-s))\\ &= \mathcal{C}_{\mathcal{T}} + (\varepsilon, \mathrm{N}(Q_\infty(v_{\mathcal{T}}))) + (\varepsilon, -\Delta_{\mathcal{T}}).\end{aligned}$$ □

5D. *Evaluation of Brauer classes on $\mathcal{G}(\mathbb{A}_k)$.*

Definition 5.15. Let k be a global field of characteristic not equal to 2. Given $\mathcal{T} \subset \mathcal{S}$ define

$$R_{\mathcal{T}} := \{v \in \Omega_k : \varepsilon_{\mathcal{T}_v} \in \boldsymbol{k}(\mathcal{S}_v)^{\times 2} \text{ and } \mathcal{C}_{\mathcal{T}_v} \neq 0\}.$$

Theorem 5.16. *Assume that k is a global field of characteristic different from* 2:

(1) *There exists $(y_v) \in \mathcal{G}(\mathbb{A}_k)$ such that for all degree* 2 *subschemes $\mathcal{T} \subset \mathcal{S}$ with $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$, we have $\sum_{v \in \Omega_k} \operatorname{inv}_v(\beta_{\mathcal{T}}(y_v)) = \#R_{\mathcal{T}}/2 \in \mathbb{Q}/\mathbb{Z}$.*

(2) *For all $t \in \mathcal{S}(k)$ there exists $(y_v) \in \mathcal{G}(\mathbb{A}_k)$ such that for all degree* 2 *subschemes $\mathcal{T} \subset \mathcal{S}$ with $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$ and $t \in \mathcal{T}$, we have $\sum_{v \in \Omega_k} \operatorname{inv}_v(\beta_{\mathcal{T}}(y_v)) = \#R_t/2 \in \mathbb{Q}/\mathbb{Z}$.*

Proof. (1) It suffices to prove the result for $\{\beta_{\mathcal{T}} : \mathcal{T} \in \mathbb{T}\}$, where $\mathbb{T}$ is a collection of degree 2 subschemes of $\mathcal{S}$ as in Corollary 5.2(5), with corresponding $\varepsilon \in k^\times$ simultaneously representing the discriminants of all $\mathcal{T} \in \mathbb{T}$.

We define an adelic point $(y_v) \in \mathcal{G}(\mathbb{A}_k)$ as follows. For $v \in \Omega_k$ such that $\varepsilon \in \boldsymbol{k}(\mathcal{T}_v)^{\times 2}$ for some $\mathcal{T} \in \mathbb{T}$ (equivalently, for all $\mathcal{T} \in \mathbb{T}$ by Corollary 5.2(5)), let $y_v \in \mathcal{G}(k_v)$ be any point (which exists by Corollary 4.6). Note that if $\varepsilon \in \boldsymbol{k}(\mathcal{T}_v)^{\times 2}$, then $\beta_{\mathcal{T}} \otimes k_v = 0$ by Proposition 5.1. For each $v \in \Omega_k$ with $\varepsilon \notin \boldsymbol{k}(\mathcal{T}_v)^{\times 2}$ for some (equivalently all) $\mathcal{T} \in \mathbb{T}$, let $y_v \in \mathcal{G}(k_v)$ be a point corresponding to a $k_v(\sqrt{\varepsilon})$-point on X, as provided by Lemma 5.11. Note that Lemma 5.11 further implies that for such y_v, $\beta_{\mathcal{T}}(y_v) = \mathcal{C}_{\mathcal{T}_v}$ for all $\mathcal{T} \in \mathbb{T}$. Thus, for any $\mathcal{T} \in \mathbb{T}$ we have

$$\sum_{v \in \Omega_k} \mathrm{inv}_v(\beta_{\mathcal{T}}(y_v)) = \sum_{\varepsilon \notin \boldsymbol{k}(\mathcal{T}_v)^{\times 2}} \mathrm{inv}_v(\mathcal{C}_{\mathcal{T}_v}) = \sum_{\varepsilon \in \boldsymbol{k}(\mathcal{T}_v)^{\times 2}} \mathrm{inv}_v(\mathcal{C}_{\mathcal{T}_v}) = \frac{\# R_{\mathcal{T}}}{2} \in \mathbb{Q}/\mathbb{Z},$$

where the penultimate equality follows from quadratic reciprocity.

(2) Let $t \in \mathcal{S}(k)$ and set $\varepsilon := \varepsilon_t$. If t is not contained in any degree 2 subschemes $\mathcal{T} \subset \mathcal{S}$ with $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$, then we need only show that $\mathcal{G}(\mathbb{A}_k) \neq \varnothing$, which follows from Corollary 4.6. Thus, we may assume there is some degree 2 subscheme $\mathcal{T} \subset \mathcal{S}$ containing t such that $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$. For any such $\mathcal{T}$ we have $\varepsilon_{\mathcal{T}} = (\varepsilon, \varepsilon) \in \boldsymbol{k}(\mathcal{T})^{\times}/\boldsymbol{k}(\mathcal{T})^{\times 2} \simeq k^{\times}/k^{\times 2} \times k^{\times}/k^{\times 2}$.

We define an adelic point $(y_v) \in \mathcal{G}(\mathbb{A}_k)$ as follows. For $v \in \Omega_k$ such that $\varepsilon \in k_v^{\times 2}$, take y_v to be any point of $\mathcal{G}(k_v)$ (which exists by Corollary 4.6). For $v \in \Omega_k$ such that $\varepsilon \notin k_v^{\times 2}$ we take $y_v \in \mathcal{G}(k_v)$ to be any point such that $\pi(y_v) \in \mathbb{P}^1(k_v)$ is close enough t so that Lemma 5.12 applies (note that $\mathcal{Q}_t$ is a cone over an isotropic quadric surface so the hypothesis of the Lemma 5.12 is satisfied) and so that, for all $s \in \mathcal{S}(k) - \{t\}$ with $\varepsilon\varepsilon_s \in k^{\times 2}$, $(\pi(y_v) - s)$ and $(t - s)$ have the same class in $k_v^{\times}/k_v^{\times 2}$.

Suppose $\mathcal{T} = \{s, t\} \subset \mathcal{S}(k)$ is such that $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$. For $v \in \Omega_k$ such that $\varepsilon \in k_v^{\times 2}$, we have $\mathrm{inv}_v(\beta_{\mathcal{T}}(y_v)) = 0$. For $v \in \Omega_k$ such that $\varepsilon \notin k_v^{\times 2}$ we have

$$\begin{aligned} \mathrm{inv}_v(\beta_{\mathcal{T}}(y_v)) &= \mathrm{inv}_v(\varepsilon, (\pi(y_v) - t)(\pi(y_v) - s)) \\ &= \mathrm{inv}_v(\varepsilon, \pi(y_v) - t) + \mathrm{inv}_v(\varepsilon, t - s) \\ &= \mathrm{inv}_v(\mathrm{Clif}(Q_t)) + \mathrm{inv}_v(\varepsilon, -Q_\infty(v_t)) + \mathrm{inv}_v(\varepsilon, t - s) \quad \text{(By Lemma 5.12)}. \end{aligned}$$

Since $(\varepsilon, -Q_\infty(v_t)(t - s))$ is an element of $\mathrm{Br}(k)$, its local invariants sum to 0. Furthermore, for all $v \in \Omega_k$ with $\varepsilon \in k_v^{\times 2}$, $(\varepsilon, -Q_\infty(v_t)(t - s))$ has trivial invariant. Thus,

$$\sum_{v \in \Omega_k} \mathrm{inv}_v(\beta_{\mathcal{T}}(y_v)) = \sum_{\varepsilon \notin k_v^{\times 2}} \mathrm{inv}_v(\beta_{\mathcal{T}}(y_v)) = \sum_{\varepsilon \notin k_v^{\times 2}} \mathrm{inv}_v(\mathrm{Clif}(Q_t)) = \sum_{\varepsilon \in k_v^{\times 2}} \mathrm{inv}_v(\mathrm{Clif}(Q_t)).$$

where the last equality follows from the fact that the local invariants of $\mathrm{Clif}(Q_t) \in \mathrm{Br}(k)$ sum to 0. For $v \in \Omega_k$ such that $\varepsilon_t \in k_v^{\times 2}$ we have $\mathrm{inv}_v(\mathrm{Clif}(Q_t)) = \mathrm{inv}_v(\mathcal{C}_{t_v})$. Hence,

$$\sum_{v \in \Omega_k} \mathrm{inv}_v(\beta_{\mathcal{T}}(y_v)) = \sum_{\varepsilon \in k_v^{\times 2}} \mathrm{inv}_v(\mathcal{C}_{t_v}) = \frac{\# R_t}{2}, \in \mathbb{Q}/\mathbb{Z}. \qquad \square$$

The following lemma relates the set $R_{\mathcal{T}}$ to the condition given in (5) of Theorem 1.2.

Lemma 5.17. *Let k be a global field of characteristic not equal to 2 and $\mathcal{T} \subset \mathcal{S}$ a degree 2 subscheme such that $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$. Then $v \in R_{\mathcal{T}}$ if and only if there are an odd number of components of $\mathcal{Q}_{\mathcal{T}_v} = \cup_{t_v \in \mathcal{T}_v} \mathcal{Q}_{t_v}$ which have no smooth $\boldsymbol{k}(t_v)$-point.*

Proof. Let $v \in \Omega_k$. First suppose that $\varepsilon_{\mathcal{T}_v} \notin \boldsymbol{k}(\mathcal{S}_v)^{\times 2}$. Then $v \notin R_{\mathcal{T}}$ by definition. Note also that for all $t_v \in \mathcal{T}_v$, $\varepsilon_{t_v} \notin \boldsymbol{k}(t_v)^{\times 2}$ (a priori this must hold for some $t_v \in \mathcal{T}_v$; the stronger conclusion holds because $\mathcal{T}$

has degree 2 and $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$). Recall that there is a unique anisotropic quadratic form of rank 4 over any local field and that it has square discriminant. Hence, when $\varepsilon_{\mathcal{T}_v} \notin \boldsymbol{k}(t_v)^{\times 2}$, all components $\mathcal{Q}_{t_v}$ have smooth $\boldsymbol{k}(t_v)$-points.

Now suppose that $\varepsilon_{\mathcal{T}_v} \in \boldsymbol{k}(\mathcal{S}_v)^{\times 2}$. As above $\varepsilon_{t_v} \in \boldsymbol{k}(t_v)^{\times 2}$, for each $t_v \in \mathcal{T}_v$. Then the rank 4 quadratic forms Q_{t_v} are equivalent to constant multiples of the norm forms of the quaternion algebras $\mathrm{Clif}(Q_{t_v})$; see [Elman et al. 2008, Proposition 12.4]. In particular, $\mathcal{Q}_{t_v}$ has a smooth $\boldsymbol{k}(t_v)$-point if and only if $\mathrm{Clif}(Q_{t_v}) = 0 \in \mathrm{Br}(\boldsymbol{k}(t_v))$. The corestriction maps $\mathrm{Cor}_{\boldsymbol{k}(t_v)/k_v} : \mathrm{Br}(\boldsymbol{k}(t_v)) \to \mathrm{Br}(k_v)$ are isomorphisms, so $\mathcal{C}_{\mathcal{T}_v} = \sum_{t_v \in \mathcal{T}_v} \mathrm{Cor}_{\boldsymbol{k}(t_v)/k_v} \mathrm{Clif}(Q_{t_v})$ is nonzero if and only if there are an odd number of components of $\mathcal{Q}_{\mathcal{T}_v}$ with no smooth $\boldsymbol{k}(t_v)$-points. By definition $v \in R_{\mathcal{T}}$ if and only if $\mathcal{C}_{\mathcal{T}_v} \neq 0$. ☐

Lemma 5.18. *Assume that k is a global field of characteristic different from 2 and suppose $\mathcal{T} \subset \mathcal{S}$ is irreducible of degree 2 such that $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$. For any $t \in \mathcal{T}(\boldsymbol{k}(\mathcal{T}))$, the cardinalities of the sets*

$$R_{\mathcal{T}} \subset \Omega_k \quad \textit{and} \quad R_t \subset \Omega_{\boldsymbol{k}(\mathcal{T})}$$

have the same parity.

Proof. For a prime $v \in \Omega_k$, we have $\varepsilon_{\mathcal{T}} \in \boldsymbol{k}(\mathcal{T}_v)^{\times 2}$ if and only if $\varepsilon_t \in \boldsymbol{k}(t)_w^{\times 2}$ for all (equivalently some) $w \in \Omega_{\boldsymbol{k}(\mathcal{T})}$ with $w \mid v$. For such v we have

$$\mathrm{inv}_v(\mathcal{C}_{\mathcal{T}_v}) = \mathrm{inv}_v(\mathrm{Cor}_{\boldsymbol{k}(\mathcal{T})/k}(\mathrm{Clif}(Q_{\mathcal{T}}))) = \sum_{w \mid v} \mathrm{inv}_w(\mathrm{Clif}(Q_t)) = \sum_{w \mid v} \mathrm{inv}_w(\mathcal{C}_{t_w}).$$

In particular, $\mathcal{C}_{\mathcal{T}_v} \neq 0$ if and only if there are an odd number of primes $w \mid v$ with $\mathcal{C}_{t_w} \neq 0$. ☐

6. Proofs of the main theorems

6A. *Corollaries of Theorem 5.16.*

Corollary 6.1. *Assume that k is a global field of characteristic not equal to 2 and that either of the following conditions hold*:

(1) *Every nontrivial element of $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$ can be represented by $\beta_{\mathcal{T}}$ for some degree 2 subscheme $\mathcal{T} \subset \mathcal{S}$ such that $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$ and $\#R_{\mathcal{T}}$ even.*

(2) *Every nontrivial element of $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$ can be represented by $\beta_{\mathcal{T}}$ for some degree 2 subscheme $\mathcal{T} = \{t_1, t_2\} \subset \mathcal{S}(k)$ such that $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$.*

Then $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} \neq \varnothing$.

Proof. If condition (1) holds, then the corollary follows from Theorem 5.16(1). Now assume condition (2) holds and (1) fails. Then there exists a nontrivial element of $\mathrm{Br}(\mathcal{G})$ of the form $\beta_{\{t,t'\}}$ with $t, t' \in \mathcal{S}(k)$ such that $R_{\{t,t'\}}$ has odd cardinality. Note that $R_{\{t,t'\}}$ is the symmetric difference of R_t and $R_{t'}$. Thus, interchanging t and t' if needed we may assume R_t has even cardinality. Theorem 5.16(2) then gives an adelic point orthogonal to all $\beta_{\mathcal{T}}$ such that $\mathcal{T}$ has degree 2, contains t and $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$. The result follows since Corollary 5.2(4) shows that such $\beta_{\mathcal{T}}$ generate $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$. ☐

Remark 6.2. If both conditions of Corollary 6.1 fail, then $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G}) \simeq \mathbb{Z}/2\mathbb{Z}$ by Corollary 5.2 and any $\beta_{\mathcal{T}}$ with $\mathcal{T}$ of degree 2 which represents the nontrivial class must have $\mathcal{T}$ irreducible. Thus, $\mathcal{S}$ must contain an irreducible degree 2 subscheme $\mathcal{T}$ such that

- $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$,
- $\varepsilon_{\mathcal{T}} \notin \boldsymbol{k}(\mathcal{T})^{\times 2}$,
- if $\#(\mathcal{S} - \mathcal{T})(k) = 3$, then $\varepsilon_t \notin k^{\times 2}$ for all $t \in \mathcal{S} - \mathcal{T}$, and
- $\#R_{\mathcal{T}}$ is odd, which in particular implies that $\mathcal{Q}_{\mathcal{T}}$ has no smooth $\boldsymbol{k}(\mathcal{T})$-points.

Corollary 6.3. *Assume that k is a global field of characteristic not equal to* 2. *Suppose there is a degree* 2 *subscheme* $\mathcal{T} \subset \mathcal{S}$ *with* $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$ *such that* $R_{\mathcal{T}}$ *has odd cardinality. Then* $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} \neq \mathcal{G}(\mathbb{A}_k)$ *and there exists a quadratic extension* K/k *such that* $X_K(\mathbb{A}_K) \neq X_K(\mathbb{A}_K)^{\mathrm{Br}} = \varnothing$. *In particular,* $\mathcal{G}$ *does not satisfy weak approximation and there exists quadratic extension* K/k *such that* X_K *has a Brauer–Manin obstruction to the Hasse principle.*

Proof. The first statement follows immediately from Theorem 5.16(1). For the second statement we construct K by approximating fixed quadratic extensions of k_v for the primes $v \in S := \{v : X(k_v) = \varnothing \text{ or } \mathrm{inv}_v \circ \beta_{\mathcal{T}} : \mathcal{G}(k_v) \to \mathbb{Q}/\mathbb{Z} \text{ is nonzero}\}$. (In particular, by Lemma 5.11 and the definition of $\mathcal{C}_{\mathcal{T}}$, we will approximate K at every prime where $\mathcal{C}_{\mathcal{T}}$ ramifies.) For such v, if $\varepsilon \notin k_v^{\times 2}$, then we fix $K_v := k_v(\sqrt{\varepsilon})$. If v is such that $\varepsilon \in k_v^{\times 2}$, then we let K_v be any quadratic extension such that $X(K_v) \neq \varnothing$. Then Lemma 5.11 implies that for every $v \in S$, there exists a $y_v \in \mathcal{G}(k_v)$ corresponding to a quadratic point $\mathrm{Spec}\, K_v \to X$ and for all such y_v, $\beta_{\mathcal{T}}(y_v) = \mathcal{C}_{\mathcal{T}_v}$ if $\varepsilon \notin k_v^{\times 2}$ and $\beta_{\mathcal{T}}(y_v) = 0$ otherwise. Furthermore, for $v \notin S$ (which necessarily means that $\mathcal{C}_{\mathcal{T}_v} = 0$), our assumptions imply that $X(K_v) \neq \varnothing$ and that $\beta_{\mathcal{T}}(y_v) = 0$ for all $y_v \in \mathcal{G}(k_v)$. Thus, for all $(y_v) \in \mathcal{G}(\mathbb{A}_k)$ corresponding to an adelic quadratic point $\mathrm{Spec}(\mathbb{A}_K) \to X$ we have

$$\sum_v \mathrm{inv}_v\, \beta_{\mathcal{T}}(y_v) = \sum_{\varepsilon \notin k_v^{\times 2}} \mathrm{inv}_v\, \beta_{\mathcal{T}}(y_v) = \sum_{\varepsilon \notin k_v^{\times 2}} \mathrm{inv}_v\, \mathcal{C}_{\mathcal{T}_v} = \sum_{\varepsilon \in k_v^{\times 2}} \mathrm{inv}_v\, \mathcal{C}_{\mathcal{T}_v} = \frac{\#R_{\mathcal{T}}}{2} \in \mathbb{Q}/\mathbb{Z}.$$

By Proposition 3.6(2) and Corollary 5.2(7), this implies that $X_K(\mathbb{A}_K)^{\mathrm{Br}} = \varnothing$. □

Example 6.4. Let $\mathcal{G} \to \mathbb{P}^1$ be the fibration of Severi–Brauer threefolds corresponding to the pencil containing the quadrics given by the vanishing of the rank 4 forms

$$Q_0 = x_0 x_1 - x_2^2 + \varepsilon x_3^2 \quad \text{and} \quad Q_1 = a x_0^2 + b x_1^2 - ab x_2^2 - \varepsilon x_4^2$$

where $a, b, \varepsilon \in k^{\times}$. Then $\mathcal{T} = \{0, 1\} \subset \mathcal{S}$ is a degree 2 subscheme with $\varepsilon_{\mathcal{T}} = (\varepsilon, \varepsilon)$. Hence, Corollaries 5.2 and 6.1 imply that $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} \neq \varnothing$. Note that $\mathcal{Q}_0$ has smooth k-points, so $R_{\mathcal{T}} = R_1 = \{v \in \Omega_k : \varepsilon \in k_v^{\times 2} \text{ and } \mathrm{inv}_v(a, b) \neq 0\}$. Clearly one can choose a, b, ε so that $R_{\mathcal{T}}$ has odd cardinality (e.g., for $k = \mathbb{Q}$, $a = 3, b = 7, \varepsilon = 2$ we have $R_{\mathcal{T}} = \{7\}$), in which case $\mathcal{G}$ has a Brauer–Manin obstruction to weak approximation and the base locus X of the pencil is a counterexample to the Hasse principle over some quadratic extension by Corollary 6.3.

If $4-ab \in k_v^{\times 2} - k^{\times 2}$ for some prime $v \in R_{\mathcal{T}}$ (which holds for the values indicated above), then there exists no quadratic extension K/k such that X_K is everywhere locally solvable and $\operatorname{Br}(X_K)=\operatorname{Br}_0(X_K)$. To see this first observe that $4-ab=\varepsilon_t$ is the discriminant of the rank 4 quadric $Q_t=(Q_1-abQ_0)/(1-ab)$ (here $t=1/(1-ab)\in \mathcal{S}(k)$). Now note that if a prime $v\in R_{\mathcal{T}}$ splits in a quadratic extension K, then X_K is not locally solvable because $\mathcal{Q}_1$ has no smooth K_w-points for the primes $w \mid v$. On the other hand, Proposition 5.1 shows that $\beta_{\mathcal{T}}\otimes K \in \operatorname{Br}(X_K)$ lies in the subgroup $\operatorname{Br}_0(X_K)$ if and only if $\varepsilon \in K^{\times 2}$ (in which case $K=k(\sqrt{\varepsilon})$ and all primes of $R_{\mathcal{T}}$ split in K) or $\varepsilon_{\mathcal{S}-\mathcal{T}} \in \boldsymbol{k}(\mathcal{S}_K)^{\times 2}$ (in which case $K=k(\sqrt{4-ab})$ and so some prime of $R_{\mathcal{T}}$ splits in K by assumption). We conclude that if K/k is a quadratic extension such that $\beta_{\mathcal{T}}\otimes K\in \operatorname{Br}_0(X_K)$, then $X_K(\mathbb{A}_K)=\varnothing$.

Corollary 6.5. *Assume that k is a global field of characteristic not equal to* 2. *There is an adelic* 0*-cycle of degree* 1 *on $\mathcal{G}$ orthogonal to* $\operatorname{Br}(\mathcal{G})$.

Proof. We may assume that $\mathcal{G}(\mathbb{A}_k)^{\operatorname{Br}}=\varnothing$ (for otherwise the Corollary holds immediately) and hence, that the hypothesis of Corollary 6.1 fails. As explained in Remark 6.2, this implies that there is an irreducible degree 2 subscheme $\mathcal{T}\subset\mathcal{S}$ such that $\mathrm{N}(\varepsilon_{\mathcal{T}})\in k^{\times 2}$, $\varepsilon_{\mathcal{T}}\notin \boldsymbol{k}(\mathcal{T})^{\times 2}$ and $R_{\mathcal{T}}$ has odd cardinality. By Corollary 5.2(3), the existence of such an irreducible $\mathcal{T}$ implies that $\operatorname{Br}(\mathcal{G})/\operatorname{Br}_0(\mathcal{G})$ has order 2. Moreover, if $t\in\mathcal{T}(\boldsymbol{k}(\mathcal{T}))$ then, by Lemma 5.18, the set $R_t\subset \Omega_{\boldsymbol{k}(\mathcal{T})}$ has odd cardinality. Thus, by Theorem 5.16 applied over $\boldsymbol{k}(\mathcal{T})$ we obtain an effective adelic 0-cycle of degree 2 over k, denote it by (z_v), such that $\sum_{v\in\Omega_k}\operatorname{inv}_v(\beta_{\mathcal{T}}(z_v))=1/2$. If $(y_v)\in\mathcal{G}(\mathbb{A}_k)$ is any adelic point (which exists by Corollary 4.6), then (z_v-y_v) is an adelic 0-cycle of degree 1 and, since $\mathcal{G}(\mathbb{A}_k)^{\operatorname{Br}}=\mathcal{G}(\mathbb{A}_k)^{\beta_{\mathcal{T}}}=\varnothing$, we have $\sum_{v\in\Omega_k}\operatorname{inv}_v(\beta_{\mathcal{T}}(z_v-y_v))=\sum_{v\in\Omega_k}\operatorname{inv}_v(\beta_{\mathcal{T}}(z_v))-\sum_{v\in\Omega_k}\operatorname{inv}_v(\beta_{\mathcal{T}}(y_v))=1/2-1/2=0$. □

Remark 6.6. In the cases not already covered by Corollary 6.1, the proof above hinges on constructing an adelic 0-cycle of degree 2 on $\mathcal{G}$ that is not orthogonal to the Brauer group. Lemma 5.14 can be used to give an alternative construction of such a 0-cycle. See Section 7A.

6B. ***Proof of Theorem 1.1.*** Let $X'\subset\mathbb{P}^n_k$ be a smooth complete intersection of two quadrics over k. By Bertini's theorem the intersection of X' with a suitable linear subspace will yield a smooth complete intersection of two quadrics $X\subset\mathbb{P}^4_k$. If k is a local field, then the result follows from Theorem 2.1. It remains to consider the case that k is a number field. By Corollary 6.5, $\mathcal{G}$ has an adelic 0-cycle of degree 1 orthogonal to the Brauer group. Since $\mathcal{G}$ is a pencil of Severi–Brauer varieties, [Colliot-Thélène and Swinnerton-Dyer 1994, Theorem 5.1] shows that $\mathcal{G}$ has a 0-cycle of degree 1. By Proposition 4.4 this gives a 0-cycle of degree 2 on X.

6C. ***Proof of Theorem 1.2.*** Let $X'\subset\mathbb{P}^n_k$ be a smooth complete intersection of two quadrics over k. As noted above, X' contains a smooth of two quadrics in $\mathbb{P}^4_k$. Thus, Theorem 2.1 implies that X' contains a quadratic point when k is local. This proves Theorem 1.2(1). Similarly, Theorem 1.2(3) follows from Proposition 2.11.

Now assume k is global of characteristic not equal to 2. Theorem 2.1 implies that there is a quadratic extension K/k such that X'_K is everywhere locally solvable. If k is a global function field and $n \geq 5$, then X'_K satisfies the Hasse principle by [Tian 2017]. This proves Theorem 1.2(2).

We claim that (under the hypotheses of the theorem) X' contains a smooth del Pezzo surface X of degree 4 such that the corresponding Severi–Brauer pencil $\mathcal{G}$ has $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} \neq \varnothing$. If $n \geq 5$, then by [Wittenberg 2007, Section 3.5] the intersection of X' with an appropriate linear subspace is a smooth del Pezzo surface X of degree 4 with $\mathrm{Br}(X) = \mathrm{Br}_0(X)$. Corollary 5.2(7) implies that the corresponding $\mathcal{G}$ has $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G}) = 0$, so $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} \neq \varnothing$ by Corollary 6.1. When $n = 4$, $X' = X$ is itself a smooth del Pezzo surface of degree 4. If all of the nontrivial elements of $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$ can be represented by some $\beta_{\mathcal{T}}$ with $\mathcal{T}$ reducible, then Corollary 6.1(1) implies that $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} \neq \varnothing$. Otherwise, by Corollary 5.2, the order of $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$ divides 2 and any nontrivial element can be represented by $\beta_{\mathcal{T}}$ with $\mathcal{T}$ irreducible and $N(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$. Any such element determines a quadratic extension $L = \boldsymbol{k}(\mathcal{T})$ and the assumption in case (5) of the theorem is that the geometric components of $\mathcal{Q}_{\mathcal{T}}$ (which are defined over L) each fail to have smooth local points at an even number of primes of L. By Lemma 5.17 this implies that $R_{\mathcal{T}}$ has even cardinality and so $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} \neq \varnothing$ by Corollary 6.1(2).

If k is a number field for which Schinzel's hypothesis holds, then it is a result of Serre that the Brauer–Manin obstruction is the only obstruction to the Hasse principle for fibrations of Severi–Brauer varieties (Serre's result is unpublished, but a more general result [Colliot-Thélène and Swinnerton-Dyer 1994, Theorem 4.2] implies this result of Serre). In this case we obtain a k-point on $\mathcal{G}$ and, consequently by Proposition 4.4, a quadratic point on X. To prove the result assuming k satisfies ($\star$) it is enough to find a quadratic extension K/k such that $X_K(\mathbb{A}_K)^{\mathrm{Br}} \neq \varnothing$. The existence of such a K follows from Proposition 3.6(4), since as noted in Corollary 5.2(7) the map $\mathrm{Br}(X)/\mathrm{Br}_0(X) \to \mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$ given by Proposition 3.6(1) is an isomorphism.

7. Complements and remarks

7A. ***Remarks on the cases not covered by Theorem 1.2.*** Suppose X is a del Pezzo surface of degree 4 over a global field k of characteristic not equal to 2 with corresponding Severi–Brauer pencil $\mathcal{G}$ such that the conditions of Corollary 6.1 are not satisfied. As noted in Remark 6.2 this implies that $\mathrm{Br}(\mathcal{G})/\mathrm{Br}_0(\mathcal{G})$ is cyclic of order 2, with the nontrivial class represented by $\beta_{\mathcal{T}}$ for an irreducible subscheme $\mathcal{T} \subset \mathcal{S}$ of degree 2 with $\mathrm{N}(\varepsilon_{\mathcal{T}}) \in k^{\times 2}$ for which $\#R_{\mathcal{T}}$ is odd. By Corollary 6.3, $\beta_{\mathcal{T}}$ obstructs weak approximation and so $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} \neq \varnothing$ if and only if there exists a prime $v \in \Omega_k$ such that the evaluation map $\beta_{\mathcal{T}} : \mathcal{G}(k_v) \to \mathrm{Br}(k_v)$ is not constant.

Let $\mathcal{C}'_{\mathcal{T}} := \mathcal{C}_{\mathcal{T}} + (\varepsilon, -\Delta_{\mathcal{T}}\,\mathrm{N}(Q_\infty(v_{\mathcal{T}}))) \in \mathrm{Br}(k)$ be the class from Lemma 5.14 and define

$$R'_{\mathcal{T}} := \{v \in \Omega_k : \varepsilon_{\mathcal{T}} \notin \boldsymbol{k}(\mathcal{S}_v)^{\times 2} \text{ and } \mathrm{inv}_v(\mathcal{C}'_{\mathcal{T}}) \neq 0\}.$$

Since $R_{\mathcal{T}}$ has odd cardinality, so too must $R'_{\mathcal{T}}$. In particular, $R'_{\mathcal{T}}$ is nonempty. If $\mathcal{T}_v$ is reducible for a prime $v \in R'_{\mathcal{T}}$, then Lemma 5.14 shows that the evaluation map $\beta_{\mathcal{T}} \colon \mathcal{G}(k_v) \to \mathrm{Br}(k_v)$ is nonconstant and so $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} = \mathcal{G}(\mathbb{A}_k)^{\beta_{\mathcal{T}}} \neq \varnothing$. If $\mathcal{T}_v$ is irreducible at $v \in R'_{\mathcal{T}}$, then Lemma 5.14 shows that $\beta_{\mathcal{T}} \otimes \boldsymbol{k}(\mathcal{T}_v)\colon \mathcal{G}(\boldsymbol{k}(\mathcal{T}_v)) \to \mathrm{Br}(\boldsymbol{k}(\mathcal{T}_v))$ is nonconstant. Indeed the lemma gives a $\boldsymbol{k}(\mathcal{T}_v)$-point where $\beta_{\mathcal{T}} \otimes \boldsymbol{k}(\mathcal{T}_v)$ takes the nonzero value $\mathcal{C}'_{\mathcal{T}_v}$, but $\beta_{\mathcal{T}} \otimes \boldsymbol{k}(\mathcal{T}_v)$ takes the value 0 at any elements in the subset $\mathcal{G}(k_v) \subset \mathcal{G}(\boldsymbol{k}(\mathcal{T}_v))$. Unfortunately, this is not enough to conclude that $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} \neq \varnothing$ because $\beta_{\mathcal{T}}\colon \mathcal{G}(k_v) \to \mathrm{Br}(k_v)$ can still be constant. Using the lemma below one can check that this occurs at $v = 5$ for the pencil of quadrics defined by

$$Q_0 = -55x_1^2 + 2x_1x_2 + x_3^2 + 5x_4^2 \quad \text{and} \quad Q_\infty = 33x_0^2 - 5x_1^2 - x_2^2 + 10x_3x_4.$$

We note, however, that in this example (and in all others with $R'_{\mathcal{T}} \neq \varnothing$ that we have considered) there is some prime $w \in \Omega_k$ (in this case $w = 2$) for which the evaluation map $\beta_{\mathcal{T}}\colon \mathcal{G}(k_w) \to \mathrm{Br}(k_w)$ is not constant and, hence, $\mathcal{G}(\mathbb{A}_k)^{\mathrm{Br}} \neq \varnothing$.

Lemma 7.1. *If $v \in R'_{\mathcal{T}}$ is such that $\mathcal{T}_v$ is irreducible, k_v has odd residue characteristic and $X(\boldsymbol{k}(\mathcal{T}_v)) = \varnothing$, then $\beta_{\mathcal{T}}\colon \mathcal{G}(k_v) \to \mathrm{Br}(k_v)$ is constant.*

Proof. Suppose $X(\boldsymbol{k}(\mathcal{T}_v)) = \varnothing$ and let $y \in \mathcal{G}(k_v)$. Then y corresponds to a quadratic point $\mathrm{Spec}(K) \to X$, with K/k_v a quadratic field extension such that $K \neq \boldsymbol{k}(\mathcal{T}_v)$. Since k_v has odd residue characteristic, $\boldsymbol{k}(\mathcal{T}_L)$ is the compositum of all quadratic extensions of k_v. In particular, it must contain a square root of $\varepsilon_{\mathcal{T}}$ (since $\varepsilon_{\mathcal{T}} \in k_v^\times \boldsymbol{k}(\mathcal{T}_v)^{\times 2}$). Therefore, Lemma 5.11 applies, and its conclusion shows that $\beta_{\mathcal{T}}(y)$ does not depend on y. □

In contrast, the following lemma shows that for X (in place of $\mathcal{G}$) nonconstancy of an evaluation map over an extension of k_v does imply nonconstancy over k_v.

Lemma 7.2. *Let X be a del Pezzo surface of degree* 4 *over a local field k of characteristic not equal to* 2 *such that $X(k) \neq \varnothing$. If $\alpha \in \mathrm{Br}(X)$ is such that $\mathrm{inv}_k \circ \alpha\colon X(k) \to \mathbb{Q}/\mathbb{Z}$ is constant, then for all finite extensions K/k, $\mathrm{inv}_K \circ \alpha_K\colon X(K) \to \mathbb{Q}/\mathbb{Z}$ is constant and equal to $[K:k](\mathrm{inv}_k \circ \alpha)$.*

Remark 7.3. In the case that k_v has odd residue characteristic, $\mathcal{T}_v$ is irreducible, $\varepsilon_{\mathcal{T}_v} \in k_v^{\times 2}$ and $\mathrm{inv}_v(\mathcal{C}'_{\mathcal{T}}) \neq 0$, Lemma 7.2 can be used to prove the converse of Lemma 7.1. Namely, if $\beta_{\mathcal{T}}$ is constant on k_v-points, then $X(\boldsymbol{k}(\mathcal{T}_v))$ must be empty.

Proof. Let $P \in X(k)$. By [Salberger and Skorobogatov 1991, Lemma 4.4] (which follows from [Colliot-Thélène and Coray 1979, Theorem C]), every 0-cycle of degree 0 on X is linearly equivalent to $Q - P$ for some $Q \in X(k)$. Therefore, for any closed point R on X, there is some $Q \in X(k)$ such that $R \sim Q + (\deg(R) - 1)P$. Since evaluation of Brauer classes factors through rational equivalence and by assumption $\alpha(P) = \alpha(Q)$, we see that $\mathrm{inv}_K \circ \alpha_K = [K:k](\mathrm{inv}_k \circ \alpha)$ for any extension K/k. □

Remarks 7.4. (1) The result of [Colliot-Thélène and Coray 1979] used in the proof above shows that every 0-cycle of degree 1 on a conic bundle with 5 or fewer degenerate fibers is rationally equivalent to a rational point. The example mentioned just before Lemma 7.1 shows that this does not extend to more general Severi–Brauer bundles (at least over a p-adic fields). Indeed, evaluation of Brauer classes factors through rational equivalence and in the example there is a Brauer class on $\mathcal{G}$ which is nonconstant on 0-cycles of degree 1, but is constant on rational points. An example of a Severi–Brauer bundle (in fact a conic bundle) with a 0-cycle of degree 1 but no rational point was constructed by Colliot-Thélène and Coray [Colliot-Thélène and Coray 1979, Section 5]; in this example the conic bundle has 6 singular fibers.

(2) If X/k is a del Pezzo surface of degree 4 over a number field which is a counterexample to the Hasse principle explained by the Brauer–Manin obstruction, then as shown in [Colliot-Thélène and Poonen 2000, Section 3.5] there exists $\alpha \in \mathrm{Br}(X)$ such that $X(\mathbb{A}_k)^\alpha = \varnothing$ (a priori multiple elements of $\mathrm{Br}(X)$ might be required to give the obstruction). An immediate consequence of Lemma 7.2 is that over any odd degree extension K/k the same Brauer class will give an obstruction, i.e., $X_K(\mathbb{A}_K)^{\alpha_K} = \varnothing$. This answers a question posed in [Colliot-Thélène and Poonen 2000, Remark 3, page 95]. In particular, this shows that the conjecture that all failures of the Hasse principle for del Pezzo surfaces of degree 4 are explained by the Brauer–Manin obstruction is compatible with the theorems of Amer [1976], Brumer [1978] and Springer [1956] which imply that an intersection of two quadrics with index 1 has a rational point.

7B. *A degree 4 del Pezzo surface with obstructions only over odd degree extensions.*

Proposition 7.5. *Let $X/\mathbb{Q}$ be the del Pezzo surface of degree 4 given by the vanishing of*

$$Q_0 = (x_0 + x_1)(x_0 + 2x_1) - x_2^2 + 5x_4^2 \quad \text{and} \quad Q_1 = 2(x_0x_1 - x_2^2 + 5x_3^2).$$

For any finite extension $K/\mathbb{Q}$ we have $X_K(\mathbb{A}_K)^{\mathrm{Br}} = \varnothing$ if and only if $[K:\mathbb{Q}]$ is odd.

Proof. This surface was considered by Birch and Swinnerton-Dyer [1975] who showed that X is a counterexample to the Hasse principle explained by the Brauer–Manin obstruction. It follows from Lemma 7.2 that for any K with $[K:\mathbb{Q}]$ odd, X_K is also a counterexample to the Hasse principle explained by the Brauer–Manin obstruction.

Since X is locally solvable over $\mathbb{Q}$, $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is generated by the image of $\mathrm{Br}(X)[2]$. The singular quadrics in the pencil lie above $\mathcal{S}(\overline{\mathbb{Q}}) = \{0, \pm 1, \pm 4\sqrt{2} + 5/7\} \subset \mathbb{P}^1$ and the corresponding discriminants satisfy $\varepsilon_0 = \varepsilon_1 = 5$, $\varepsilon_{-1} = -1$ and $\mathrm{N}(\varepsilon_{(\pm 4\sqrt{2}+5)/7}) = -1$. For any $K/\mathbb{Q}$ linearly disjoint from $k_1 = \mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{5})$, the restriction map induces an isomorphism $\mathrm{Br}(X)/\mathrm{Br}_0(X) \simeq \mathrm{Br}(X_K)/\mathrm{Br}_0(X_K)$ and so $X_K(\mathbb{A}_K)^{\mathrm{Br}} \neq \varnothing$ by Lemma 3.1(2). On the other hand, if $K/\mathbb{Q}$ is not linearly disjoint from k_1, we can check directly that $X(K) \neq \varnothing$. Indeed, K must contain $\mathbb{Q}(\sqrt{d})$ for some $d \in \{-1, \pm 2, \pm 5, \pm 10\}$. Over these quadratic fields one can exhibit points

$$(1:1:1:0:\sqrt{-1}), \quad (1:-2:2\sqrt{2}:\sqrt{2}:1), \quad (4:9:6:0:5\sqrt{-2}), \quad (0:0:\sqrt{5}:1:1),$$
$$(5:0:0:0:\sqrt{-5}), \quad (2\sqrt{10}:-\sqrt{10}:0:2:0), \quad (0:\sqrt{-10}:0:0:2). \qquad \square$$

7C. *A degree 4 del Pezzo surface with index 4.*

Theorem 7.6. *There exists a del Pezzo surface X of degree 4 over a field k of characteristic 0 such that X has index 4.*

Proof. Let k_0 be an algebraically closed field of characteristic 0. For $i = 1, \dots, 2g$, set $k_i := k_{i-1}((t_i))$ and set $k := k_{2g}$. By a result of Lang and Tate [1958, page 678], if A/k_0 is an abelian variety of dimension g and n is an integer, then there exists a torsor under $A_k = A \times_{k_0} \operatorname{Spec}(k)$ of period n and index n^{2g}. In particular, if C/k_0 is any genus 2 curve, then there exists a torsor under the Jacobian $J = \operatorname{Jac}(C_k)$ of C_k of period 2 and index 16. Since C is defined over the algebraically closed field k_0, it has a rational Weierstrass point over k. As observed by Flynn [2009], and worked out in detail by Skorobogatov [2010], if J_λ is a 2-covering $\pi_\lambda : J_\lambda \to J$ (i.e., a twist of $[2] : J \to J$ corresponding to $\lambda \in \mathrm{H}^1(k, J[2])$), then there are morphisms

$$J_\lambda \leftarrow \tilde{J}_\lambda \to Z_\lambda \to X_\lambda,$$

where $\tilde{J}_\lambda \to J_\lambda$ is the blow up of J_λ at $\pi_\lambda^{-1}(0_J)$, Z_λ is the desingularized Kummer variety associated to J_λ and $Z_\lambda \to X_\lambda$ is a double cover of a del Pezzo surface of degree 4. In particular, there is a degree 4 morphism $\tilde{J}_\lambda \to X_\lambda$. So the index of X_λ is at least $\operatorname{index}(J_\lambda)/4$, which will equal 4 for suitable choice of λ by the aforementioned result of Lang and Tate. □

Theorem 7.7. *Suppose k is a number field and Y is a torsor of period* 2 *under the Jacobian of a genus* 2 *curve over k with a rational Weierstrass point. The index of Y divides* 8.

Proof. As in the proof of the previous theorem, the index of Y divides $4 \operatorname{index}(X)$ for some del Pezzo surface X of degree 4. The result follows from Theorem 1.1. □

Remarks 7.8. (1) The conclusion of Theorem 7.7 was known to hold by work of Clark [2004, Theorems 2 and 3] when k is a p-adic field and when k is a number field and Y is locally solvable.

(2) Arguing as in the proof of the theorem we see that the Kummer variety Z_λ has index dividing 4 when k is a local or global field. This is lower than one would expect, given that Z_λ is an intersection of 3 quadrics in $\mathbb{P}^5_k$.

(3) The result of Lang–Tate quoted in the proof above shows that over general fields of characteristic 0, there are examples where Z_λ and Y have index 8 and 16, respectively.

(4) In response to a preliminary report on this work by the authors, John Ottem suggested the following alternate proof of Theorem 7.6 which gives an example over the C_3 field $\boldsymbol{k}(\mathbb{P}^3_{\mathbb{C}})$. Let $D_1, D_2 \subset \mathbb{P}^3 \times \mathbb{P}^4$ be two general (2, 2) divisors over $\mathbb{C}$, and let $Y = D_1 \cap D_2$. Then, by the Lefschetz hyperplane theorem (applied twice), restriction gives an isomorphism $\mathrm{H}^4(\mathbb{P}^3 \times \mathbb{P}^4, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^4(D_1, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^4(Y, \mathbb{Z})$. Note that the generic fiber of the first projection is a del Pezzo surface of degree 4 over $\boldsymbol{k}(\mathbb{P}^3)$. Hence any threefold $V \subset Y$ can be expressed as $aH_1^2 + bH_1H_2 + cH_2^2$, where H_i denotes the pullback of $\mathcal{O}(1)$ under the projection π_i. Then the degree of $V \to \mathbb{P}^3$ is given by

$$V.H_1^3 = V.H_1^3.X = (aH_1^2 + bH_1H_2 + cH_2^2).H_1^3.(2H_1 + 2H_2)^2,$$

which must be divisible by 4. Thus $Y_{\boldsymbol{k}(\mathbb{P}^3)}$ has index 4. Note that to apply the Lefschetz hyperplane theorem, we need $\dim D_i > 5$, so this argument does not extend to $\boldsymbol{k}(\mathbb{P}^2_{\mathbb{C}})$.

This construction suggested by Ottem generalizes to arbitrary complete intersections. Namely, given a sequence of degrees $(d_1, \ldots, d_r)$ and an ambient dimension N, one can consider an intersection of general $(d_1, d_1), (d_2, d_2), \ldots, (d_r, d_r)$ hypersurfaces in $\mathbb{P}^M \times \mathbb{P}^N$. If $M > N - r$, then the same argument as above yields a $(d_1, \ldots, d_r)$ smooth complete intersection $Y \subset \mathbb{P}^N_{\boldsymbol{k}(\mathbb{P}^M)}$ with index $d_1 d_2 \cdots d_r$.

(5) After viewing an early draft of this paper, Olivier Wittenberg [2013] shared a correspondence of his that provides yet another construction that proves Theorem 7.6. Let k be any field of characteristic different 2 such that there exists a quadric surface Q with no k-points that remains pointless after a quadratic extension k'/k. Wittenberg's construction gives an example over the field $k((t))$.

Let f be a general rank 2 quadric in $\mathbb{P}^4$ that splits over k'. Then for a general quadric g, the intersection $Q \cap V(f + tg)$ is a smooth del Pezzo surface of degree 4 that has index 4 over $k((t))$. Indeed, the smooth locus of the special fiber has index 4 by construction, so (for general enough g), the general fiber must also have index 4. This construction of Wittenberg extends to give complete intersections of n quadrics with index 2^n (over fields of larger transcendence degree).

7D. ***The index of a degree d del Pezzo surface.*** The following table gives sharp upper bounds for the indices of degree d del Pezzo surfaces over local fields, number fields and arbitrary fields of characteristic 0. The entries in the column $d = 4$ are a consequence of the results in this paper, while for $d \neq 4$, they can be deduced fairly easily from known results as described below:

d	9	8	7	6	5	4	3	2	1
k arbitrary	3	4	1	6	1	4 [Theorem 7.6]	3	2	1
k a number field	3	2	1	6	1	2 [Theorem 1.1]	3	2	1
k a local field	3	2	1	2 or 3	1	2 [Theorem 1.1]	3	2	1

When $d = 9$, Y is a Severi–Brauer surface and so the index of Y divides 3 and examples of index 3 exist whenever $\mathrm{Br}(k)$ contains an element of order 3.

When $d = 8$, $Y = \mathrm{Res}_{L/k}(C)$ is the restriction of scalars of a conic C/L defined over a degree 2 étale algebra L/k [Poonen 2017, Proposition 9.4.12]. Since the conic has a point over some quadratic extension L'/L, the index of Y divides 4 and over general fields there are examples with index 4. Over local and global fields however, the index must divide 2. Indeed, in this case C will have a point over a quadratic extension L'/L of the form $L' = k' \otimes_k L$ for some quadratic extension k'/k. The universal property of restriction of scalars then gives $Y(k') \neq \varnothing$, showing that the index divides 2.

When $d = 7$, $Y(k) \neq \varnothing$ over any field k and so the index is always equal to 1. The same applies to $d = 1, 5$; see, e.g., [Poonen 2017, Theorem 9.4.8 and Section 9.4.11].

For $d = 6$, Y is determined by a $\mathrm{Gal}(L/k)$-stable triple of geometric points on a Severi–Brauer surface S/L over a quadratic étale algebra L/k such that if $S \not\simeq \mathbb{P}^2_L$ then the class of S in the Brauer group does not lie in the image of $\mathrm{Br}(k) \to \mathrm{Br}(L)$ [Corn 2005]. If k is a local field and L is a quadratic field extension, then the map $\mathrm{Br}(k) \to \mathrm{Br}(L)$ is an isomorphism, so either $S = \mathbb{P}^2_L$ (in which case Y has index dividing 2) or $L = k \times k$ in which case the index of Y divides 3. One can construct examples of index 6 over number fields, by arranging to have index 2 at one completion and index 3 at another.

For $d = 3$ and k local, index 1 implies the existence of a k-rational point [Coray 1976], and so a cubic surface without points over some local field has index 3. This gives examples of index 3 over number fields as well.

For $d = 2$, index 2 examples can be obtained by blowing up a degree 4 del Pezzo surface of index 2 at a quadratic point. By Theorems 1.1 and 1.2, any del Pezzo surface of degree 4 without points over a local field gives such an example. The surface considered in Section 7B gives an example over a number field.

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A p-adic Simpson correspondence for rigid analytic varieties

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We establish a p-adic Simpson correspondence in the spirit of Liu and Zhu for rigid analytic varieties X over $\mathbb{C}_p$ with a liftable good reduction by constructing a new period sheaf on $X_{\text{proét}}$. To do so, we use the theory of cotangent complexes described by Beilinson and Bhatt. Then we give an integral decompletion theorem and complete the proof by local calculations. Our construction is compatible with the previous works of Faltings and Liu and Zhu.

1. Introduction

In the theory of complex geometry, for a compact Kähler manifold X, Simpson [1992] established a tensor equivalence between the category of semisimple flat vector bundles on X and the category of polystable Higgs bundles with vanishing Chern classes. Nowadays, such a correspondence is known as nonabelian Hogde theory or the Simpson correspondence. There is a well-established theory of the Simpson correspondence for smooth varieties in characteristic $p > 0$ admitting a lifting modulo p^2 (see [Ogus and Vologodsky 2007]). This leads us to ask for a p-adic analogue of Simpson's correspondence.

The first step is due to Deninger and Werner [2005]. They gave a partial analogue of classical Narasimhan–Seshadri theory by studying parallel transport for vector bundles of curves. At the same time, Faltings [2005] constructed an equivalence between the category of small generalised representations and the category of small Higgs bundles for schemes $\mathfrak{X}_0$ with toroidal singularities over $\mathcal{O}_k$, the ring of integers of some p-adic local field k, under a certain deformation assumption. His method was elaborated and generalised by Abbes, Gros and Tsuji [Abbes et al. 2016] and related to integral p-adic Hodge theory by Morrow and Tsuji [2020]. When X is a rigid analytic space over k, Liu and Zhu [2017] related a

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Higgs bundle on $X_{\hat{\bar{k}},\text{ét}}$ to each $\mathbb{Q}_p$-local system on $X_{\text{ét}}$ and proved that the resulting Higgs field must be nilpotent (see [Liu and Zhu 2017, Theorem 2.1]). Their work was generalised to the logarithmic case in [Diao et al. 2023b]. However, their Higgs functor is not an equivalence, so it is still open to classify Higgs bundles coming from representations. For smooth rigid spaces X over $\hat{\bar{k}}$, Heuer [2022] established an equivalence between the category of one-dimensional $\hat{\bar{k}}$-representations of the fundamental group $\pi_1(X)$ and the category of pro-finite-étale Higgs bundles. Using his method, Heuer, Mann and Werner [Heuer et al. 2023] constructed a Simpson correspondence for abeloids over $\hat{\bar{k}}$.

In this paper, we establish an equivalence between the category of small generalised representations (Definition 5.1) and the category of small Higgs bundles (Definition 5.2) for rigid analytic varieties X with liftable (see the notation section) good reductions $\mathfrak{X}$ over $\mathcal{O}_{\mathbb{C}_p}$ in the spirit of the work of Liu and Zhu. Our construction is global and the main ingredient is a new overconvergent period sheaf $\mathcal{O}\mathbb{C}^\dagger$ endowed with a canonical Higgs field Θ on $X_{\text{proét}}$, which can be viewed as a kind of p-adic complete version of the period sheaf $\mathcal{O}\mathbb{C}$ due to Hyodo [1989]. The main theorem is stated as follows:

Theorem 1.1 (Theorem 5.3). *Assume $a \geq 1/(p-1)$. Let $\mathfrak{X}$ be a liftable smooth formal scheme over $\mathcal{O}_{\mathbb{C}_p}$ of relative dimension d with the rigid generic fibre X and $\nu : X_{\text{proét}} \to \mathfrak{X}_{\text{ét}}$ be the natural projection of sites. Then there is an overconvergent period sheaf $\mathcal{O}\mathbb{C}^\dagger$ endowed with a canonical Higgs field Θ such that the following assertions are true*:

(1) *For any a-small generalised representation $\mathcal{L}$ of rank l on $X_{\text{proét}}$, let $\Theta_{\mathcal{L}} := \mathrm{id}_{\mathcal{L}} \otimes \Theta$ be the induced Higgs field on $\mathcal{L} \otimes_{\widehat{\mathcal{O}}_X} \mathcal{O}\mathbb{C}^\dagger$; then $\mathrm{R}\nu_*(\mathcal{L} \otimes_{\widehat{\mathcal{O}}_X} \mathcal{O}\mathbb{C}^\dagger)$ is discrete. Define $\mathcal{H}(\mathcal{L}) := \nu_*(\mathcal{L} \otimes_{\widehat{\mathcal{O}}_X} \mathcal{O}\mathbb{C}^\dagger)$ and $\theta_{\mathcal{H}(\mathcal{L})} = \nu_*\Theta_{\mathcal{L}}$. Then $(\mathcal{H}(\mathcal{L}), \theta_{\mathcal{H}(\mathcal{L})})$ is an a-small Higgs bundle of rank l.*

(2) *For any a-small Higgs bundle $(\mathcal{H}, \theta_{\mathcal{H}})$ of rank l on $\mathfrak{X}_{\text{ét}}$, let $\Theta_{\mathcal{H}} := \mathrm{id}_{\mathcal{H}} \otimes \Theta + \theta_{\mathcal{H}} \otimes \mathrm{id}_{\mathcal{O}\mathbb{C}^\dagger}$ be the induced Higgs field on $\mathcal{H} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}\mathbb{C}^\dagger$ and define*

$$\mathcal{L}(\mathcal{H}, \theta_{\mathcal{H}}) = (\mathcal{H} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}\mathbb{C}^\dagger)^{\Theta_{\mathcal{H}}=0}.$$

Then $\mathcal{L}(\mathcal{H}, \theta_{\mathcal{H}})$ is an a-small generalised representation of rank l.

(3) *The functor $\mathcal{L} \mapsto (\mathcal{H}(\mathcal{L}), \theta_{\mathcal{H}(\mathcal{L})})$ induces an equivalence from the category of a-small generalised representations to the category of a-small Higgs bundles, whose quasi-inverse is given by $(\mathcal{H}, \theta_{\mathcal{H}}) \mapsto \mathcal{L}(\mathcal{H}, \theta_{\mathcal{H}})$. The equivalence preserves tensor products and dualities and identifies the Higgs complexes*

$$\mathrm{HIG}(\mathcal{L} \otimes_{\widehat{\mathcal{O}}_X} \mathcal{O}\mathbb{C}^\dagger, \Theta_{\mathcal{L}}) \simeq \mathrm{HIG}(\mathcal{H}(\mathcal{L}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}\mathbb{C}^\dagger, \Theta_{\mathcal{H}(\mathcal{L})}).$$

(4) *Let $\mathcal{L}$ be an a-small generalised representation with associated Higgs bundle $(\mathcal{H}, \theta_{\mathcal{H}})$. Then there is a canonical quasi-isomorphism*

$$\mathrm{R}\nu_*(\mathcal{L}) \simeq \mathrm{HIG}(\mathcal{H}, \theta_{\mathcal{H}}),$$

where $\mathrm{HIG}(\mathcal{H}, \theta_{\mathcal{H}})$ is the Higgs complex induced by $(\mathcal{H}, \theta_{\mathcal{H}})$. In particular, $\mathrm{R}\nu_*(\mathcal{L})$ is a perfect complex of $\mathcal{O}_{\mathfrak{X}}[\frac{1}{p}]$-modules concentrated in degree $[0, d]$.

(5) *Assume $f : \mathfrak{X} \to \mathfrak{Y}$ is a smooth morphism between liftable smooth formal schemes over $\mathcal{O}_{\mathbb{C}_p}$. Let $\widetilde{\mathfrak{X}}$ and $\widetilde{\mathfrak{Y}}$ be the fixed A_2-liftings of $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. Assume f lifts to an A_2-morphism $\tilde{f} : \widetilde{\mathfrak{X}} \to \widetilde{\mathfrak{Y}}$. Then the equivalence in* (3) *is compatible with the pull-back along f.*

Note that when $\mathcal{L} = \widehat{\mathcal{O}}_X$, we get $(\mathcal{H}(\widehat{\mathcal{O}}_X), \theta_{\mathcal{H}(\widehat{\mathcal{O}}_X)}) = \big(\mathcal{O}_{\mathfrak{X}}\big[\frac{1}{p}\big], 0\big)$. So our result can be viewed as a generalisation of [Scholze 2013b, Proposition 3.23]. Theorem 1.1(3) also provides a way to compute the pro-étale cohomology for a small generalised representation $\mathcal{L}$. More precisely, we get a quasi-isomorphism

$$\mathrm{R}\Gamma(X_{\text{proét}}, \mathcal{L}) \simeq \mathrm{R}\Gamma(\mathfrak{X}_{\text{ét}}, \mathrm{HIG}(\mathcal{H}(\mathcal{L}), \theta_{\mathcal{H}(\mathcal{L})})).$$

If, in addition, $\mathfrak{X}$ is proper, then we get a finiteness result on pro-étale cohomology of small generalised representations.

Corollary 1.2. *Keep the notation as in Theorem 1.1 and assume $\mathfrak{X}$ is proper. Then for any a-small generalised representation $\mathcal{L}$, $\mathrm{R}\Gamma(X_{\text{proét}}, \mathcal{L})$ is concentrated in degree* $[0, 2d]$ *and has cohomologies as finite dimensional $\mathbb{C}_p$-spaces.*

The overconvergent period sheaf $\mathcal{O}\mathbb{C}^\dagger$ (with respect to a certain lifting of $\mathfrak{X}$) has $\mathcal{O}\mathbb{C}$ as a subsheaf. Indeed, it is a direct limit of certain p-adic completions of $\mathcal{O}\mathbb{C}$. In particular, when $\mathfrak{X}$ comes from a scheme $\mathfrak{X}_0$ over $\mathcal{O}_k$ and the generalised representation $\mathcal{L}$ comes from a $\mathbb{Z}_p$-local system on the rigid generic fibre X_0 of $\mathfrak{X}_0$, our construction coincides with the work of Liu and Zhu (Remark 5.6). On the other hand, $\mathcal{O}\mathbb{C}^\dagger$ is related to an obstruction class $\mathrm{cl}(\mathcal{E}^+)$ solving a certain deformation problem (Remark 2.10 and Proposition 2.14). Since the class $\mathrm{cl}(\mathcal{E}^+)$ is exactly the one used to establish the Simpson correspondence in [Faltings 2005], our construction is compatible with the works of Faltings and Abbes, Gros and Tsuji (Remark 5.5). These answer a question appearing in [Liu and Zhu 2017, Remark 2.5]. Another answer, using a different method, was announced in [Yang and Zuo 2020].

Since we need to take p-adic completions of $\mathcal{O}\mathbb{C}$, we have to find its integral models. Note that $\mathcal{O}\mathbb{C}$ is a direct limit of symmetric products of Faltings' extension, which was constructed for varieties by Faltings [1988] at first and revisited by Scholze [2013a] in the rigid analytic case. So we are reduced to finding an integral version of Faltings' extension. To do so, we use the method of cotangent complexes which was established and developed in [Quillen 1970; Illusie 1971; 1972; Gabber and Ramero 2003], and was systematically used in the p-adic theory by [Scholze 2012; Beilinson 2012; Bhatt 2012]. The proof of Theorem 1.1 is based on some explicit local calculations, especially an integral decompletion theorem (Theorem 3.4) for small representations, which can be regarded as a generalisation of [Diao et al. 2023b, Appendix A].

Notation. Let k be a complete discrete valuation field of mixed characteristics $(0, p)$ with ring of integers $\mathcal{O}_k$ and perfect residue field κ. We normalise the valuation on k by setting $\nu_p(p) = 1$ and the associated norm is given by $\|\cdot\| = p^{-\nu_p(\cdot)}$. We denote by $k_0 = \mathrm{Frac}(\mathrm{W}(\kappa))$ the maximal absolutely unramified subfield of k. Denote by $\mathcal{D}_k = \mathcal{D}_{k/k_0}$ the relative differential ideal of $\mathcal{O}_k$ over $\mathrm{W}(\kappa)$.

Let $\bar{k}$ be a fixed algebraic closure of k and $\mathbb{C}_p = \hat{\bar{k}}$ be its p-adic completion. We denote by $\mathcal{O}_{\mathbb{C}_p}$ (resp. $\mathfrak{m}_{\mathbb{C}_p}$) the ring of integers of $\mathbb{C}_p$ (resp. the maximal ideal of $\mathcal{O}_{\mathbb{C}_p}$). In this paper, when we write $p^a A$ for some $\mathcal{O}_{\mathbb{C}_p}$-module A, we always assume $a \in \mathbb{Q}$. An $\mathcal{O}_{\mathbb{C}_p}$-module M is called *almost vanishing* if it is $\mathfrak{m}_{\mathbb{C}_p}$-torsion, and in this case we write $M^{\mathrm{al}} = 0$. A morphism $f : M \to N$ of $\mathcal{O}_{\mathbb{C}_p}$-modules is *almost injective* (resp. *almost surjective*) if $\mathrm{Ker}(f)^{\mathrm{al}} = 0$ (resp. $\mathrm{Coker}(f)^{\mathrm{al}} = 0$). A morphism is an *almost isomorphism* if it is both almost injective and almost surjective.

We choose a sequence $\{1, \zeta_p, \dots, \zeta_{p^n}, \dots\}$ such that ζ_{p^n} is a primitive p^n-th root of unity in $\bar{k}$ satisfying $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for every $n \geq 0$. For every $\alpha \in \mathbb{Z}\left[\frac{1}{p}\right] \cap (0, 1)$, one can (uniquely) write $\alpha = (t(\alpha))/p^{n(\alpha)}$ with $\gcd(t(\alpha), p) = 1$ and $n(\alpha) \geq 1$. Then we define $\zeta^\alpha := \zeta_{p^{n(\alpha)}}^{t(\alpha)}$ when $\alpha \neq 0$ and $\zeta^\alpha := 1$ when $\alpha = 0$.

We always fix an element $\rho_k \in \mathbb{C}_p$ with $\nu_p(\rho_k) = \nu_p(\mathcal{D}_k) + 1/(p-1)$. Let $\mathrm{A}_{\mathrm{inf},k} = \mathrm{W}(\mathcal{O}_{\mathbb{C}_p^\flat}) \otimes_{\mathrm{W}(\kappa)} \mathcal{O}_k$ be the period ring of Fontaine. Then there is a surjective homomorphism $\theta_k : \mathrm{A}_{\mathrm{inf},k} \to \mathcal{O}_{\mathbb{C}_p}$ whose kernel is a principal ideal by [Fargues and Fontaine 2018, Proposition 3.1.9]. We fix a generator ξ_k of $\mathrm{Ker}(\theta_k)$. For instance, if $k = k_0$ is absolutely unramified, then we choose $\rho_k = \zeta_p - 1$ and $\xi_k = ([\epsilon]-1)/([\epsilon]^{1/p}-1)$ for $\epsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}_{\mathbb{C}_p}^\flat$. Put $\mathrm{A}_2 = \mathrm{A}_{\mathrm{inf},k}/\xi_k^2$ and denote Fontaine's p-adic analogue of $2\pi i$ by $t = \log[\epsilon]$.

For a p-adic formal scheme $\mathfrak{X}$ over $\mathcal{O}_{\mathbb{C}_p}$, we say it is *smooth* if it is formally smooth and locally of topologically finite type. We say $\mathfrak{X}$ is *liftable* if it admits a lifting $\widetilde{\mathfrak{X}}$ to $\mathrm{Spf}(\mathrm{A}_2)$. In this paper, we always assume $\mathfrak{X}$ is liftable. Let X be the rigid analytic generic fibre of $\mathfrak{X}$ and denote by $\nu : X_{\mathrm{pro\acute{e}t}} \to \mathfrak{X}_{\mathrm{\acute{e}t}}$ the natural projection of sites. Let $\widehat{\mathcal{O}}_X^+$ and $\widehat{\mathcal{O}}_X$ be the completed structure sheaves on $X_{\mathrm{pro\acute{e}t}}$ in the sense of [Scholze 2013a, Definition 4.1]. Both of them can be viewed as $\mathcal{O}_{\mathfrak{X}}$-algebras via the projection ν.

Let K be an object in the derived category of complexes of $\mathbb{Z}_p$-modules. We denote by $\hat{K}$ the derived p-adic completion $\mathrm{R}\varprojlim_n K \otimes_{\mathbb{Z}_p} \mathbb{Z}_p/p^n$. In particular, for a morphism $A \to B$ of $\mathbb{Z}_p$-algebras, we denote the derived p-adic completion of cotangent complex $\mathrm{L}_{B/A}$ by $\widehat{\mathrm{L}}_{B/A}$. In this paper, for two complexes K_1 and K_2 of (sheaves of) modules, we write $K_1 \simeq K_2$ if they are quasi-isomorphic. For two modules or sheaves M_1 and M_2, we write $M_1 \cong M_2$ if they are isomorphic.

Organisation. In Section 2, we construct the integral Faltings' extension by using p-complete cotangent complexes and explaining how it is related to deformation theory. At the end of this section we construct the desired overconvergent sheaf. In Section 3, we prove an integral decompletion theorem for small representations. In Section 4, we establish a local Simpson correspondence. We first consider the trivial representation and then reduce the general case to this special case. Finally, in Section 5, we state and prove our main theorem. The Appendix specifies some notation and includes some elementary facts that were used in previous sections.

2. Integral Faltings' extension and period sheaves

We construct the overconvergent period sheaf $\mathcal{O}\mathbb{C}^\dagger$ in this section. To do so, we have to construct an integral version of Faltings' extension.

Integral Faltings' extension. We first discuss the properties of the cotangent complex. The following lemmas are well known, but for the convenience of readers, we include their proofs here.

Lemma 2.1. *Let A be a ring. Suppose that $(f_1, \dots, f_n)$ is a regular sequence in A and generates the ideal $I = (f_1, \dots, f_n)$. Then $\mathrm{L}_{(A/I)/A} \simeq (I/I^2)[1]$.*

Proof. Regard A as a $\mathbb{Z}[X_1, \dots, X_n]$-algebra by mapping X_i to f_i for every i. Since $f_1, \dots, f_n$ is a regular sequence in A, for any $i \geq 1$, we have

$$\mathrm{Tor}_i^{\mathbb{Z}[X_1,\dots,X_n]}(\mathbb{Z}, A) = 0.$$

It follows from [Weibel 1994, 8.8.4] that

$$\mathrm{L}_{(A/I)/A} \simeq \mathrm{L}_{\mathbb{Z}/\mathbb{Z}[X_1,\dots,X_n]} \otimes^L_{\mathbb{Z}[X_1,\dots,X_n]} A.$$

So we may assume $A = \mathbb{Z}[X_1, \dots, X_n]$ and $I = (X_1, \dots, X_n)$. From homomorphisms $\mathbb{Z} \to A \to A/I$ of rings, we get an exact triangle

$$\mathrm{L}_{A/\mathbb{Z}} \otimes^L A/I \longrightarrow \mathrm{L}_{(A/I)/\mathbb{Z}} \longrightarrow \mathrm{L}_{(A/I)/A} \longrightarrow .$$

The middle term is trivial since $A/I = \mathbb{Z}$ and hence we deduce that

$$\mathrm{L}_{(A/I)/A} \simeq (\mathrm{L}_{A/\mathbb{Z}} \otimes^L_A \mathbb{Z})[1] \simeq (I/I^2)[1]. \qquad \square$$

Lemma 2.2. (1) *The map* $\mathrm{dlog} : \mu_{p^\infty} \to \Omega^1_{\mathcal{O}_{\bar{k}}/\mathcal{O}_k}$, $\zeta_{p^n} \mapsto d\zeta_{p^n}/\zeta_{p^n}$ *induces an isomorphism*

$$\mathrm{dlog} : \bar{k}/\rho_k^{-1}\mathcal{O}_{\bar{k}} \otimes \mathbb{Z}_p(1) \to \Omega^1_{\mathcal{O}_{\bar{k}}/\mathcal{O}_k},$$

where $\mathbb{Z}_p(1)$ denotes the Tate twist.

(2) $\mathrm{L}_{\mathcal{O}_{\bar{k}}/\mathcal{O}_k} \simeq \Omega^1_{\mathcal{O}_{\bar{k}}/\mathcal{O}_k}[0]$.

(3) $\widehat{\mathrm{L}}_{\mathcal{O}_{\mathbb{C}_p}/\mathcal{O}_k} \simeq (1/\rho_k)\mathcal{O}_{\mathbb{C}_p}(1)[1]$.

Proof. (1) This is [Fontaine 1982, Théorème 1’].

(2) This is [Beilinson 2012, Theorem 1.3].

(3) This follows from (1) and (2) after taking derived p-completions on both sides. $\square$

Corollary 2.3. (1) $\widehat{\mathrm{L}}_{\mathcal{O}_{\mathbb{C}_p}/A_{\mathrm{inf},k}}[-1] \simeq (1/\rho_k)\mathcal{O}_{\mathbb{C}_p}(1)[0] \simeq \xi_k A_{\mathrm{inf},k}/\xi_k^2 A_{\mathrm{inf},k}[0]$.

(2) $\widehat{\mathrm{L}}_{\mathcal{O}_{\mathbb{C}_p}/A_2} \simeq (1/\rho_k)\mathcal{O}_{\mathbb{C}_p}(1)[1] \oplus (1/\rho_k^2)\mathcal{O}_{\mathbb{C}_p}(2)[2]$.

Proof. (1) Considering the morphisms $\mathcal{O}_k \to A_{\mathrm{inf},k} \to \mathcal{O}_{\mathbb{C}_p}$ of rings, we have an exact triangle

$$\mathrm{L}_{A_{\mathrm{inf},k}/\mathcal{O}_k} \widehat{\otimes}^L_{A_{\mathrm{inf},k}} \mathcal{O}_{\mathbb{C}_p} \to \widehat{\mathrm{L}}_{\mathcal{O}_{\mathbb{C}_p}/\mathcal{O}_k} \to \widehat{\mathrm{L}}_{\mathcal{O}_{\mathbb{C}_p}/A_{\mathrm{inf},k}} \to .$$

Since

$$\widehat{\mathrm{L}}_{A_{\mathrm{inf},k}/\mathcal{O}_k} \simeq \mathrm{L}_{A_{\mathrm{inf}}/\mathrm{W}(\kappa)} \widehat{\otimes}^L_{\mathrm{W}(\kappa)} \mathcal{O}_k = 0,$$

the first quasi-isomorphism follows from Lemma 2.2(3). Now, the second quasi-isomorphism is straightforward from Lemma 2.1.

(2) Considering the morphisms $A_{\mathrm{inf},k}\to A_2\to \mathcal{O}_{\mathbb{C}_p}$ of rings, we have the exact triangle

$$\mathrm{L}_{A_2/A_{\mathrm{inf},k}}\widehat{\otimes}^L_{A_2}\mathcal{O}_{\mathbb{C}_p}\to \widehat{\mathrm{L}}_{\mathcal{O}_{\mathbb{C}_p}/A_{\mathrm{inf},k}}\to \widehat{\mathrm{L}}_{\mathcal{O}_{\mathbb{C}_p}/A_2}\to .$$

Combining Lemma 2.1 with (1), the above exact triangle reduces to

$$\xi_k^2A_{\mathrm{inf},k}/\xi_k^4A_{\mathrm{inf},k}\otimes_{A_2}\mathcal{O}_{\mathbb{C}_p}[1]\to \xi_kA_{\mathrm{inf},k}/\xi_k^2A_{\mathrm{inf},k}[1]\to \widehat{\mathrm{L}}_{\mathcal{O}_{\mathbb{C}_p}/A_2}\to .$$

Now we complete the proof by noting that the first arrow is trivial. □

We identify $\mathcal{O}_{\mathbb{C}_p}(1)$ with $\mathcal{O}_{\mathbb{C}_p}t$, where t is Fontaine's p-adic analogue of $2\pi i$. It follows from Lemma 2.2(1) that the sequence $\{\mathrm{dlog}(\zeta_{p^n})\}_{n\geq 0}$ can be identified with the element $t\in(1/\rho_k)\mathcal{O}_{\mathbb{C}_p}(1)$. If we regard $A_{\mathrm{inf},k}$ as a subring of $\mathrm{B}_{\mathrm{dR}}^+$ and identify $t\mathrm{B}_{\mathrm{dR}}^+/t^2\mathrm{B}_{\mathrm{dR}}^+$ with $\mathbb{C}_p(1)$, then Corollary 2.3 says that t and $\rho_k\xi_k$ in $\mathbb{C}_p(1)$ differ by a p-adic unit in $\mathcal{O}_{\mathbb{C}_p}^\times$.

Remark 2.4. The corollary is still true if one replaces $\mathbb{C}_p$ by any closed subfield $K\subset\mathbb{C}_p$ containing μ_{p^∞}. All results in this paper hold for K instead of $\mathbb{C}_p$.

Now we construct the integral Faltings' extension in the local case. We fix some notation as follows. Let $\mathfrak{X}=\mathrm{Spf}(R^+)$ be a smooth formal scheme over $\mathrm{Spf}(\mathcal{O}_{\mathbb{C}_p})$ endowed with an étale morphism

$$\square:\mathfrak{X}\to\hat{\mathbb{G}}_m^d=\mathrm{Spf}(\mathcal{O}_{\mathbb{C}_p}\langle\underline{T}^{\pm1}\rangle),$$

where $\mathcal{O}_{\mathbb{C}_p}\langle\underline{T}^{\pm1}\rangle=\mathcal{O}_{\mathbb{C}_p}\langle T_1^{\pm1},\dots,T_d^{\pm1}\rangle$. We say $\mathfrak{X}$ is *small* in this case. Let $X=\mathrm{Spa}(R,R^+)$ be the rigid analytic generic fibre of $\mathfrak{X}$ and $X_\infty=\mathrm{Spa}(\widehat{R}_\infty,\widehat{R}_\infty^+)$ be the affinoid perfectoid space associated to the base-change of X along the Galois cover

$$\mathbb{G}_{m,\infty}^d=\mathrm{Spa}(\mathbb{C}_p\langle\underline{T}^{\pm\frac{1}{p^\infty}}\rangle,\mathcal{O}_{\mathbb{C}_p}\langle\underline{T}^{\pm\frac{1}{p^\infty}}\rangle)\to\mathbb{G}_m^d=\mathrm{Spa}(\mathbb{C}_p\langle\underline{T}^{\pm1}\rangle,\mathcal{O}_{\mathbb{C}_p}\langle\underline{T}^{\pm1}\rangle).$$

Denote by Γ the Galois group of the cover $X_\infty\to X$ and let γ_i be in Γ satisfying

$$\gamma_i(T_j^{\frac{1}{p^n}})=\zeta_{p^n}^{\delta_{ij}}T_j^{\frac{1}{p^n}}\tag{2-1}$$

for any $1\leq i,j\leq d$ and $n\geq 0$. Here, δ_{ij} denotes the Kronecker delta. Then $\Gamma\cong\mathbb{Z}_p\gamma_1\oplus\cdots\oplus\mathbb{Z}_p\gamma_d$. Let $\widetilde{R}^+$ be a lifting of R^+ along $A_2\to\mathcal{O}_{\mathbb{C}_p}$. Then the morphisms $\widetilde{R}^+\to R^+\to\widehat{R}^+_\infty$ of rings give an exact triangle of p-complete cotangent complexes

$$\mathrm{L}_{R^+/\widetilde{R}^+}\widehat{\otimes}^L_{R^+}\widehat{R}^+_\infty\to\widehat{\mathrm{L}}_{\widehat{R}^+_\infty/\widetilde{R}^+}\to\widehat{\mathrm{L}}_{\widehat{R}^+_\infty/R^+}\to .\tag{2-2}$$

The first term is easy to handle. Indeed, combining [Weibel 1994, 8.8.4] with Corollary 2.3(2), we deduce that

$$\mathrm{L}_{R^+/\widetilde{R}^+}\widehat{\otimes}^L_{R^+}\widehat{R}^+_\infty\simeq\frac{1}{\rho_k}\widehat{R}^+_\infty(1)[1]\oplus\frac{1}{\rho_k^2}\widehat{R}^+_\infty(2)[2].$$

Now we compute the third term of (2-2).

Lemma 2.5. *We have* $\widehat{\mathrm{L}}_{\widehat{R}^+_\infty/R^+}\simeq\widehat{\Omega}^1_{R^+}\otimes_{R^+}\widehat{R}^+_\infty[1]$, *where* $\widehat{\Omega}^1_{R^+}$ *denotes the module of formal differentials of* R^+ *over* $\mathcal{O}_{\mathbb{C}_p}$.

Proof. Since R^+ is étale over $\mathcal{O}_{\mathbb{C}_p}\langle \underline{T}^{\pm 1}\rangle$, thanks to [Bhatt et al. 2018, Lemma 3.14], we are reduced to the case $R^+ = \mathcal{O}_{\mathbb{C}_p}\langle \underline{T}^{\pm 1}\rangle$. For any $n \geq 0$, put $A_n^+ = \mathcal{O}_{\mathbb{C}_p}[\underline{T}^{\pm \frac{1}{p^n}}]$ and define $A_\infty^+ = \varinjlim_n A_n^+$. Since all rings involved are p-torsion free, we get

$$\widehat{\mathrm{L}}_{\widehat{R}^+_\infty/R^+} \simeq \widehat{\mathrm{L}}_{A^+_\infty/A^+_0}.$$

By [Illusie 1971, Chapitre II(1.2.3.4)], we see that

$$\mathrm{L}_{A^+_\infty/A^+_0} = \varinjlim_n \mathrm{L}_{A^+_n/A^+_0}.$$

Since all A_n^+'s are smooth over $\mathcal{O}_{\mathbb{C}_p}$, from the exact triangle

$$\mathrm{L}_{A^+_0/\mathcal{O}_{\mathbb{C}_p}} \otimes^L_{A^+_0} A^+_n \to \mathrm{L}_{A^+_n/\mathcal{O}_{\mathbb{C}_p}} \to \mathrm{L}_{A^+_n/A^+_0} \to,$$

we deduce that

$$\mathrm{L}_{A^+_n/A^+_0} \simeq A^+_n \otimes_{A^+_0} \frac{1}{p^n}\Omega^1_{A^+_0}/\Omega^1_{A^+_0}[0],$$

where we identify $\Omega^1_{A^+_n}$ with $A^+_n \otimes_{A^+_0} (1/p^n)\Omega^1_{A^+_0}$. Therefore, we get

$$\mathrm{L}_{A^+_\infty/A^+_0} \simeq A^+_\infty \otimes_{A^+_0} \Omega^1_{A^+_0} \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p)[0].$$

Now the result follows by taking p-completions. □

Since R^+ admits a lifting $\widetilde{R}^+$ to A_2, the composition

$$\widehat{\mathrm{L}}_{\widehat{R}^+_\infty/R^+} \simeq \widehat{\mathrm{L}}_{\mathrm{A}_2(\widehat{R}^+_\infty)/\widetilde{R}^+} \widehat{\otimes}^L_{\mathrm{A}_2(\widehat{R}^+_\infty)} \widehat{R}^+_\infty \to \widehat{\mathrm{L}}_{\widehat{R}^+_\infty/\widetilde{R}^+}$$

defines a section of $\widehat{\mathrm{L}}_{\widehat{R}^+_\infty/\widetilde{R}^+} \to \widehat{\mathrm{L}}_{\widehat{R}^+_\infty/R^+}$. Since the exact triangle (2-2) is Γ-equivariant, by taking cohomologies along (2-2), we get the following proposition.

Proposition 2.6. *There exists a Γ-equivariant short exact sequence of $\widehat{R}^+_\infty$-modules*

$$0 \to \frac{1}{\rho_k}\widehat{R}^+_\infty(1) \to E^+ \to \widehat{R}^+_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+} \to 0, \tag{2-3}$$

where $E^+ = \mathrm{H}^{-1}(\widehat{\mathrm{L}}_{\widehat{R}^+_\infty/\widetilde{R}^+})$. The above exact sequence admits a (non-Γ-equivariant) section such that $E^+ \cong (1/\rho_k)\widehat{R}^+_\infty(1) \oplus \widehat{R}^+_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}$ as $\widehat{R}^+_\infty$-modules.

Remark 2.7. When R^+ is the base-change of some formal smooth $\mathcal{O}_k$-algebra R_0^+ of topologically finite type along $\mathcal{O}_k \to \mathcal{O}_{\mathbb{C}_p}$, it admits a canonical lifting $\widetilde{R}^+ = R_0^+ \widehat{\otimes}_{\mathcal{O}_k} A_2$. After inverting p, the resulting E^+ becomes the usual Faltings' extension and the corresponding sequence (2-3) is even $\mathrm{Gal}(\bar{k}/k)$-equivariant.

We describe the Γ-action on E^+. For any $1 \leq i \leq d$, by the proof of Lemma 2.5, the compatible sequence $\{\mathrm{dlog}(T_i^{1/p^n})\}_{n\geq 0}$ defines an element $x_i \in E^+$, which goes to $\mathrm{dlog}\, T_i$ via the projection $E^+ \to \widehat{R}^+_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}$. Since Γ acts on T_i's via (2-1), we deduce that, for any $1 \leq i, j \leq d$,

$$\gamma_i(x_j) = x_j + \delta_{ij}.$$

In summary, we have the following proposition.

Proposition 2.8. *The $\widehat{R}^+_\infty$-module E^+ is free of rank $d+1$ and has a basis $t/\rho_k, x_1, \dots, x_d$ such that*

(1) *for any $1 \le i \le d$, x_i is a lifting of $\mathrm{dlog}(T_i) \in \widehat{R}^+_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}$ and that*

(2) *for any $1 \le i, j \le d$, $\gamma_i(x_j) = x_j + \delta_{ij} t$.*

Also, let $c : \Gamma \to \mathrm{Hom}_{R^+}\big(\widehat{\Omega}^1_{R^+}, (1/\rho_k)\widehat{R}^+_\infty(1)\big)$ be the map carrying γ_i to $c(\gamma_i)$, which sends $\mathrm{dlog}(T_j)$ to $\delta_{ij} t$. Then the cocycle determined by c in $\mathrm{H}^1\big(\Gamma, \mathrm{Hom}_{R^+}(\widehat{\Omega}^1_{R^+}, (1/\rho_k)\widehat{R}^+_\infty(1))\big)$ coincides with the extension class represented by E^+ in $\mathrm{Ext}^1_\Gamma(\widehat{R}^+_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}, (1/\rho_k)\widehat{R}^+_\infty(1))$ via the canonical isomorphism

$$\mathrm{H}^1\left(\Gamma, \mathrm{Hom}_{R^+}\left(\widehat{\Omega}^1_{R^+}, \frac{1}{\rho_k}\widehat{R}^+_\infty(1)\right)\right) \cong \mathrm{Ext}^1_\Gamma\left(\widehat{R}^+_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}, \frac{1}{\rho_k}\widehat{R}^+_\infty(1)\right).$$

Proof. It remains to prove the "also" part. By (1), the extension class of E^+ is represented by the cocycle

$$f : \Gamma \to \mathrm{Hom}_{\widehat{R}^+_\infty}\left(\widehat{R}^+_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}, \frac{1}{\rho_k}\widehat{R}^+_\infty(1)\right) \cong \mathrm{Hom}_{R^+}\left(\widehat{\Omega}^1_{R^+}, \frac{1}{\rho_k}\widehat{R}^+_\infty(1)\right)$$

such that $f(\gamma)(\mathrm{dlog}(T_i)) = \gamma(x_i) - x_i$ for any $\gamma \in \Gamma$ and any i. However, by (2), f is exactly c. We are done. □

Now we extend the above construction to the global case. Let $\mathfrak{X}$ be a smooth formal scheme over $\mathcal{O}_{\mathbb{C}_p}$ with a fixed lifting $\widetilde{\mathfrak{X}}$ to A_2. Denote by X its rigid analytic generic fibre over $\mathbb{C}_p$. We regard both $\mathcal{O}_{\mathfrak{X}}$ and $\mathcal{O}_{\widetilde{\mathfrak{X}}}$ as sheaves on $X_{\mathrm{pro\acute{e}t}}$ via the projection $\nu : X_{\mathrm{pro\acute{e}t}} \to \mathfrak{X}_{\mathrm{\acute{e}t}}$ (note that $\mathfrak{X}$ and $\widetilde{\mathfrak{X}}$ have the same étale site). Considering morphisms of sheaves of rings $\mathcal{O}_{\widetilde{\mathfrak{X}}} \to \mathcal{O}_{\mathfrak{X}} \to \widehat{\mathcal{O}}^+_X$, we get an exact triangle

$$\mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\widetilde{\mathfrak{X}}}} \widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{O}}^+_X \to \widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\widetilde{\mathfrak{X}}}} \to \mathrm{L}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathfrak{X}}} \to . \tag{2-4}$$

Similar to the local case, the first term becomes

$$\mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\widetilde{\mathfrak{X}}}} \widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{O}}^+_X \simeq \mathrm{L}_{\mathcal{O}_{\mathbb{C}_p}/A_2} \otimes^L_{\mathcal{O}_{\mathbb{C}_p}} \widehat{\mathcal{O}}^+_X$$

and the composition

$$\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\widetilde{\mathfrak{X}}}} \simeq \widehat{\mathrm{L}}_{\mathrm{A}_2(\widehat{\mathcal{O}}^+_X)/\mathcal{O}_{\widetilde{\mathfrak{X}}}} \widehat{\otimes}^L_{\mathrm{A}_2(\widehat{\mathcal{O}}^+_X)} \widehat{\mathcal{O}}^+_X \to \widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\widetilde{\mathfrak{X}}}}$$

defines a section of $\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\widetilde{\mathfrak{X}}}} \to \mathrm{L}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathfrak{X}}}$.

We claim that

$$\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathfrak{X}}} \simeq \widehat{\mathcal{O}}^+_X \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}}[1]. \tag{2-5}$$

Granting this and taking cohomologies along (2-4), we get the following theorem.

Theorem 2.9. *There is an exact sequence of sheaves of $\widehat{\mathcal{O}}^+_X$-modules*

$$0 \to \frac{1}{\rho_k}\widehat{\mathcal{O}}^+_X(1) \to \mathcal{E}^+ \to \widehat{\mathcal{O}}^+_X \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}} \to 0, \tag{2-6}$$

where $\mathcal{E}^+ = \mathrm{H}^{-1}(\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\widetilde{\mathfrak{X}}}})$.

Remark 2.10. Apply $\mathrm{RHom}(-, (1/\rho_k)\widehat{\mathcal{O}}_X^+(1))$ to the exact triangle (2-4) and consider the induced long exact sequence

$$\cdots \to \mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\widetilde{\mathfrak{X}}}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}_X^+, \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\Big) \xrightarrow{\partial} \mathrm{Ext}^2\Big(\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/\mathcal{O}_{\mathfrak{X}}}, \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\Big) \to \cdots$$

and the commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}^1\big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\widetilde{\mathfrak{X}}}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}_X^+, \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\big) & \xrightarrow{\partial} & \mathrm{Ext}^2\big(\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/\mathcal{O}_{\mathfrak{X}}}, \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\big) \\
\downarrow \cong & & \downarrow \cong \\
\mathrm{Hom}\big(\frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1), \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\big) & \xrightarrow{\partial} & \mathrm{Ext}^1\big(\widehat{\mathcal{O}}_X^+\otimes_{\mathcal{O}_{\mathfrak{X}}}\widehat{\Omega}^1_{\mathcal{O}_{\mathfrak{X}}}, \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\big)
\end{array}$$

Then the extension class $[\mathcal{E}^+]$ associated to $\mathcal{E}^+$ is the image of the natural inclusion $(1/\rho_k)\mathcal{O}_{\mathfrak{X}}(1) \to (1/\rho_k)\widehat{\mathcal{O}}_X^+(1)$ via the connecting map ∂. By construction, it is the obstruction class to lift $\widehat{\mathcal{O}}_X^+$ (as a sheaf of $\mathcal{O}_{\mathfrak{X}}$-algebras) to a sheaf of $\mathcal{O}_{\widetilde{\mathfrak{X}}}$-algebras in the sense of [Illusie 1971, III Proposition 2.1.2.3]. In particular, $\mathcal{E}^+$ depends on the choice of $\widetilde{\mathfrak{X}}$. When $\mathfrak{X}$ comes from a smooth formal scheme $\mathfrak{X}_0$ over $\mathcal{O}_k$ and $\widetilde{\mathfrak{X}}$ is the base-change of $\mathfrak{X}_0$ along $\mathcal{O}_k \to A_2$, the $\mathcal{E}^+$ coincides with the usual Faltings' extension after inverting p. So we call $\mathcal{E}^+$ the *integral Faltings's extension* (with respect to the lifting $\widetilde{\mathfrak{X}}$).

It remains to prove the claim (2-5).

Lemma 2.11. *With the notation as above, we have*

$$\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/\mathcal{O}_{\mathfrak{X}}} \simeq \widehat{\mathcal{O}}_X^+ \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}}.$$

Proof. Since the problem is local on $X_{\text{proét}}$, by the proof of [Scholze 2013a, Corollary 4.7], we may assume $\mathfrak{X} = \mathrm{Spf}(R)$ is small and are reduced to showing, for any perfectoid affinoid space $U = \mathrm{Spa}(S, S^+) \in X_{\text{proét}}/X_\infty$,

$$\widehat{\mathrm{L}}_{S^+/R^+} \simeq S^+ \otimes_{R^+} \widehat{\Omega}^1_{R^+}. \tag{2-7}$$

Since both S^+ and $\widehat{R}^+_\infty$ are perfectoid rings, by [Bhatt et al. 2018, Lemma 3.14], we have a quasi-isomorphism

$$\widehat{\mathrm{L}}_{\widehat{R}^+_\infty/R^+}\widehat{\otimes}_{\widehat{R}^+_\infty} S^+ \to \widehat{\mathrm{L}}_{S^+/R^+}.$$

Combining this with Lemma 2.5, we get (2-7) as desired. □

Faltings' extension as obstruction class. In this subsection, we shall give another description of the integral Faltings' extension from the perspective of deformation theory. To make the notation clear, in this subsection, for a sheaf S of A_2-algebras, we always identify $\xi_k A_2$ with $(1/\rho_k)S(1)$. Before moving on, we recall some basic results due to Illusie. Although their statements are given in terms of rings, all results still hold for ring topoi.

Let A be a ring with an ideal $I \lhd A$ satisfying $I^2 = 0$. Put $\overline{A} = A/I$ and fix a flat $\overline{A}$-algebra $\overline{B}$. A natural question is whether there exists a flat A-algebra B whose reduction modulo I is $\overline{B}$.

Theorem 2.12 [Illusie 1971, III Proposition 2.1.2.3]. *There is an obstruction class* $\mathrm{cl}\in\mathrm{Ext}^2(\mathrm{L}_{\overline{B}/\overline{A}}, \overline{B}\otimes_{\overline{A}} I)$ *such that* $\overline{B}$ *lifts to some flat* A*-algebra* B *if and only if* $\mathrm{cl}=0$. *In this case, the set of isomorphism classes of such deformations forms a torsor under* $\mathrm{Ext}^1(\mathrm{L}_{\overline{B}/\overline{A}}, \overline{B}\otimes_{\overline{A}} I)$ *and the group of automorphisms of a fixed deformation is* $\mathrm{Hom}(\mathrm{L}_{\overline{B}/\overline{A}}, \overline{B}\otimes_{\overline{A}} I)$.

If B and C are flat A-algebras with reductions $\overline{B}$ and $\overline{C}$, respectively, and if $\bar{f}:\overline{B}\to\overline{C}$ is a morphism of $\overline{A}$-algebras, then one can ask whether there exists a deformation $f:B\to C$ of $\bar{f}$ along $A\to\overline{A}$.

Theorem 2.13 [Illusie 1971, III Proposition 2.2.2]. *There is an obstruction class* $\mathrm{cl}\in\mathrm{Ext}^1(\mathrm{L}_{\overline{B}/\overline{A}}, \overline{C}\otimes_{\overline{A}} I)$ *such that* $\bar{f}$ *lifts to a morphism* $f:B\to C$ *if and only if* $\mathrm{cl}=0$. *In this case, the set of all lifts forms a torsor under* $\mathrm{Hom}(\mathrm{L}_{\overline{B}/\overline{A}}, \overline{C}\otimes_{\overline{A}} I)$.

We only focus on the case where $(A,I)=(A_2,(\xi))$. Let $\mathfrak{X}$ be a smooth formal scheme over $\mathcal{O}_{\mathbb{C}_p}$ and denote by

$$\mathrm{ob}(\mathfrak{X})\in\mathrm{Ext}^2\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\mathbb{C}_p}}, \frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big)$$

the obstruction class to lift $\mathfrak{X}$ to a flat A_2-scheme (see, for example, [Illusie 1971, III Théorème 2.1.7]). Consider the exact triangle

$$\mathrm{L}_{\mathcal{O}_{\mathbb{C}_p}/A_2}\widehat{\otimes}^L_{\mathcal{O}_{\mathbb{C}_p}}\mathcal{O}_{\mathfrak{X}}\to\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/A_2}\to\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\mathbb{C}_p}}$$

and the induced long exact sequence

$$\cdots\to\mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/A_2}, \frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big)\to\mathrm{Ext}^1\Big(\mathrm{L}_{\mathcal{O}_{\mathbb{C}_p}/A_2}\widehat{\otimes}^L_{\mathcal{O}_{\mathbb{C}_p}}\mathcal{O}_{\mathfrak{X}}, \frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big)\xrightarrow{\partial}\mathrm{Ext}^2\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\mathbb{C}_p}}, \frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big)\to\cdots.$$

The obstruction class $\mathrm{ob}(\mathfrak{X})$ is the image of the identity morphism of $(1/\rho_k)\mathcal{O}_{\mathfrak{X}}(1)$ under ∂ via the canonical isomorphism

$$\mathrm{Ext}^1\Big(\mathrm{L}_{\mathcal{O}_{\mathbb{C}_p}/A_2}\widehat{\otimes}^L_{\mathcal{O}_{\mathbb{C}_p}}\mathcal{O}_{\mathfrak{X}}, \frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big)\cong\mathrm{Hom}\Big(\frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1), \frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big).$$

If $\mathfrak{X}$ is also liftable and $\widetilde{\mathfrak{X}}$ is such a lifting, then $\mathrm{ob}(\mathfrak{X})=0$ and $\widetilde{\mathfrak{X}}$ defines a class

$$[\widetilde{\mathfrak{X}}]\in\mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/A_2}, \frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big)$$

which goes to the identity map of $(1/\rho_k)\mathcal{O}_{\mathfrak{X}}(1)$. Indeed, $[\widetilde{\mathfrak{X}}]$ is the image of the identity map of $(1/\rho_k)\mathcal{O}_{\mathfrak{X}}(1)$ via the morphism

$$\mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\widetilde{\mathfrak{X}}}}, \frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big)\to\mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/A_2}, \frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big).$$

We also consider the similar deformation problem for $\widehat{\mathcal{O}}_X^+$. Since $\widehat{\mathcal{O}}_X^+$ is locally perfectoid, thanks to [Bhatt et al. 2018, Lemma 3.14], $\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/\mathcal{O}_{\mathbb{C}_p}}=0$ and hence we get a quasi-isomorphism

$$\mathrm{L}_{\mathcal{O}_{\mathbb{C}_p}/A_2}\widehat{\otimes}^L_{\mathcal{O}_{\mathbb{C}_p}}\widehat{\mathcal{O}}_X^+\simeq\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/A_2}.$$

In particular, we have an isomorphism

$$\mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/A_2}, \frac{1}{\rho_k}\widehat{\mathcal{O}}^+_X(1)\Big) \cong \mathrm{Hom}\Big(\frac{1}{\rho_k}\widehat{\mathcal{O}}^+_X(1), \frac{1}{\rho_k}\widehat{\mathcal{O}}^+_X(1)\Big).$$

Therefore, $\widehat{\mathcal{O}}^+_X$ admits a canonical lifting, which turns out to be $\mathrm{A}_2(\widehat{\mathcal{O}}^+_X)$ and there is a unique class

$$[X] \in \mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/A_2}, \frac{1}{\rho_k}\widehat{\mathcal{O}}^+_X(1)\Big)$$

corresponding to the identity map of $(1/\rho_k)\widehat{\mathcal{O}}^+_X(1)$.

Regard $[\widetilde{\mathfrak{X}}]$ and $[X]$ as classes in $\mathrm{Ext}^1(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/A_2}, (1/\rho_k)\widehat{\mathcal{O}}^+_X(1))$ via the canonical morphisms induced by $(1/\rho_k)\mathcal{O}_{\mathfrak{X}}(1) \to (1/\rho_k)\widehat{\mathcal{O}}^+_X(1)$ and $\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/A_2} \to \widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/A_2}$, respectively. Then as shown in [Illusie 1971, III Proposition 2.2.4], the difference

$$\mathrm{cl}(\mathcal{E}^+) := [\widetilde{\mathfrak{X}}] - [X]$$

belongs to

$$\mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\mathbb{C}_p}}, \frac{1}{\rho_k}\widehat{\mathcal{O}}^+_X(1)\Big) \cong \mathrm{Ext}^1\Big(\widehat{\Omega}^1_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\mathbb{C}_p}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{O}}^+_X, \frac{1}{\rho_k}\widehat{\mathcal{O}}^+_X(1)\Big)$$

via the injection

$$\mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\mathbb{C}_p}}, \frac{1}{\rho_k}\widehat{\mathcal{O}}^+_X(1)\Big) \to \mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/A_2}, \frac{1}{\rho_k}\widehat{\mathcal{O}}^+_X(1)\Big),$$

and $\mathrm{cl}(\mathcal{E}^+)$ is the obstruction answering whether there is an A_2-morphism from $\mathcal{O}_{\widetilde{\mathfrak{X}}}$ to $\mathrm{A}_2(\widehat{\mathcal{O}}^+_X)$ which lifts the $\mathcal{O}_{\mathbb{C}_p}$-morphism $\mathcal{O}_{\mathfrak{X}} \to \widehat{\mathcal{O}}^+_X$ as described in Theorem 2.13.

Recall we have another obstruction class $[\mathcal{E}^+]$ described in Remark 2.10. We claim that it coincides with the class $\mathrm{cl}(\mathcal{E}^+)$ constructed above.

Proposition 2.14. $\mathrm{cl}(\mathcal{E}^+) = [\mathcal{E}^+]$.

Proof. Note that we have a commutative diagram of morphisms of cotangent complexes

$$\begin{array}{ccccccc}
\mathrm{L}_{\mathcal{O}_{\widetilde{\mathfrak{X}}}/\mathrm{A}_2}\widehat{\otimes}^L_{\mathcal{O}_{\widetilde{\mathfrak{X}}}}\widehat{\mathcal{O}}^+_X & \longrightarrow & \mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/\mathrm{A}_2}\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}^+_X & \xrightarrow{\alpha} & \mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\widetilde{\mathfrak{X}}}}\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}^+_X & \xrightarrow{+1} & \\
\| & & \downarrow{\scriptstyle\beta} & & \downarrow & & \\
\mathrm{L}_{\mathcal{O}_{\widetilde{\mathfrak{X}}}/\mathrm{A}_2}\widehat{\otimes}^L_{\mathcal{O}_{\widetilde{\mathfrak{X}}}}\widehat{\mathcal{O}}^+_X & \longrightarrow & \widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/\mathrm{A}_2} & \longrightarrow & \widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\widetilde{\mathfrak{X}}}} & \xrightarrow{+1} & \\
 & \underset{-1}{\overset{\simeq}{\searrow}} & \downarrow & & \downarrow & & \\
 & & \widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathfrak{X}}} & = & \widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathfrak{X}}} & & \\
 & & \downarrow{\scriptstyle +1} & & \downarrow{\scriptstyle +1} & &
\end{array} \tag{2-8}$$

where the notation "+1" and "−1" denote the shifts of dimensions.

Consider the resulting diagram from applying $\mathrm{RHom}(-,(1/\rho_k)\widehat{\mathcal{O}}_X^+(1))$ to (2-8). Denote the identity map of $(1/\rho_k)\widehat{\mathcal{O}}_X^+(1)$ by id. By construction, $[\mathcal{E}^+]$ is the image of id via the connecting map induced by the triangle

$$\mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\widetilde{\mathfrak{X}}}}\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}_X^+ \to \widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/\mathcal{O}_{\widetilde{\mathfrak{X}}}} \to \widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/\mathcal{O}_{\mathfrak{X}}}.$$

By the commutativity of diagram (2-8), $[\mathcal{E}^+]$ is also the image of $\alpha^*(\mathrm{id})$ via the connecting map ∂ induced by the triangle

$$\mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/A_2}\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}_X^+ \to \widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/A_2} \to \widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/\mathcal{O}_{\mathfrak{X}}}.$$

On the other hand, by the constructions of $[\widetilde{\mathfrak{X}}]$ and $[X]$, as elements in

$$\mathrm{Ext}^1\bigg(\mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/\mathrm{A}_2}\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}_X^+,\ \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\bigg),$$

we have $[\widetilde{\mathfrak{X}}]=\alpha^*(\mathrm{id})$ and $[X]=\beta^*(\mathrm{id})$; here, for the second equality, we identify

$$\mathrm{Hom}\bigg(\frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1),\ \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\bigg)=\mathrm{Ext}^1\bigg(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathbb{C}_p}/A_2}\widehat{\otimes}^L_{\mathcal{O}_{\mathbb{C}_p}}\widehat{\mathcal{O}}_X^+,\ \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\bigg)$$

with $\mathrm{Ext}^1(\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/A_2}\widehat{\otimes}^L_{\mathcal{O}_{\mathbb{C}_p}}\widehat{\mathcal{O}}_X^+,(1/\rho_k)\widehat{\mathcal{O}}_X^+(1))$. So we have

$$\mathrm{cl}(\mathcal{E}^+)=\alpha^*(\mathrm{id})-\beta^*(\mathrm{id})\in\mathrm{Ext}^1\bigg(\mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/A_2}\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}_X^+,\ \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\bigg).$$

However, the diagram

$$\begin{array}{ccc} \widehat{\mathrm{L}}_{\mathrm{A}_2(\widehat{\mathcal{O}}_X^+)/\mathcal{O}_{\widetilde{\mathfrak{X}}}} & \xrightarrow{\qquad +1 \qquad} & \mathrm{L}_{\mathcal{O}_{\widetilde{\mathfrak{X}}}/\mathrm{A}_2}\widehat{\otimes}^L_{\mathcal{O}_{\widetilde{\mathfrak{X}}}}\widehat{\mathcal{O}}_X^+ \\ \downarrow & & \downarrow \\ \widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/\mathcal{O}_{\mathfrak{X}}} \xrightarrow{+1} & \mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/\mathrm{A}_2}\widehat{\otimes}^L_{\mathcal{O}_{\widetilde{\mathfrak{X}}}}\widehat{\mathcal{O}}_X^+ \longrightarrow & \mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\mathbb{C}_p}}\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}_X^+ \end{array}$$

induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1\big(\mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\mathbb{C}_p}}\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}_X^+,\tfrac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\big) & \xrightarrow{\subset} & \mathrm{Ext}^1\big(\mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/A_2}\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}_X^+,\tfrac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\big) \\ & \searrow^{\cong} & \downarrow{\partial} \\ & & \mathrm{Ext}^2\big(\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/\mathcal{O}_{\mathfrak{X}}},\tfrac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\big) \end{array}$$

In particular, as elements in $\mathrm{Ext}^1(\mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\mathbb{C}_p}}\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}_X^+,(1/\rho_k)\widehat{\mathcal{O}}_X^+(1))$, we have

$$\mathrm{cl}(\mathcal{E}^+)=\partial(\alpha^*(\mathrm{id})-\beta^*(\mathrm{id}))=\partial(\alpha^*(\mathrm{id}))=[\mathcal{E}^+] \qquad \square$$

Remark 2.15. When $\mathfrak{X}$ is small affine and comes from a formal scheme over $\mathcal{O}_k$, the obstruction class $\mathrm{cl}(\mathcal{E}^+)$ was considered as a *Higgs–Tate extension associated to* $\widetilde{\mathfrak{X}}$ in [Abbes et al. 2016, I. 4.3].

Example 2.16. *Let* $R^+ = \mathcal{O}_{\mathbb{C}_p}\langle \underline{T}^{\pm 1}\rangle$ *and* $\widetilde{R}^+ = A_2\langle \underline{T}^{\pm 1}\rangle$ *for simplicity. Consider the* A_2*-morphism* $\widetilde{\psi} : \widetilde{R}^+ \to \mathrm{A}_2(\widehat{R}^+_\infty)$, *which sends* T_i *to* $[T_i^\flat]$ *for all* i, *where* $T_i^\flat \in \widehat{R}^{\flat+}_\infty$ *is determined by the compatible sequence* $(T_i^{1/p^n})_{n\geq 0}$. *Then* $\widetilde{\psi}$ *is a lifting of the inclusion* $R^+ \to \widehat{R}^+_\infty$, *but is not* Γ*-equivariant. For any* $\gamma \in \Gamma$, $\gamma \circ \widetilde{\psi}$ *is another lifting. By Theorem 2.13, their difference* $c(\gamma) := \gamma \circ \widetilde{\psi} - \widetilde{\psi}$ *belongs to* $\mathrm{Hom}_{R^+}(\widehat{\Omega}^1_{R^+}, (1/\rho_k)\widehat{R}^+_\infty(1))$. *One can check that, for any* $1 \leq i, j \leq 1$,

$$c(\gamma_i)(\mathrm{dlog}(T_j)) = \frac{(\gamma_i - 1)([T_j^\flat])}{T_j} = \delta_{ij}([\epsilon] - 1) = \delta_{ij} t,$$

where the last equality follows from the fact that $[\epsilon] - 1 - t \in t^2\mathrm{B}^+_{\mathrm{dR}}$. *By construction, the cocycle* $c : \Gamma \to \mathrm{Hom}_{R^+}(\widehat{\Omega}^1_{R^+}, (1/\rho_k)\widehat{R}^+_\infty(1))$ *is exactly the class* $\mathrm{cl}(\mathcal{E}^+)$. *Comparing this with Proposition 2.8, we deduce that* $\mathrm{cl}(\mathcal{E}^+) = [\mathcal{E}^+]$ *in this case.*

As an application of Proposition 2.14, we study the behaviour of integral Faltings' extension under the pull-back.

Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a formally smooth morphism of liftable smooth formal schemes. Fix liftings $\widetilde{\mathfrak{X}}$ and $\widetilde{\mathfrak{Y}}$ of $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. Denote by $\mathcal{E}^+_X$ and $\mathcal{E}^+_Y$ the corresponding integral Faltings' extensions. Then the pull-back of $\mathcal{E}^+_X$ along the injection

$$f^*\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{O}}^+_X \to \widehat{\Omega}^1_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{O}}^+_X$$

defines an extension $\mathcal{E}^+_1$ of $\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\mathcal{O}}^+_X \cong f^*\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{O}}^+_X$ by $(1/\rho_k)\widehat{\mathcal{O}}^+_X(1)$.[1] We denote its extension class by

$$\mathrm{cl}_1 \in \mathrm{Ext}^1\left(\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\mathcal{O}}^+_X, \frac{1}{\rho_k}\widehat{\mathcal{O}}^+_X(1)\right).$$

On the other hand, the tensor product $\mathcal{E}^+_2 = \mathcal{E}^+_Y \otimes_{\widehat{\mathcal{O}}^+_Y} \widehat{\mathcal{O}}^+_X$ induced by applying $- \otimes_{\widehat{\mathcal{O}}^+_Y} \widehat{\mathcal{O}}^+_X$ to

$$0 \to \frac{1}{\rho_k}\widehat{\mathcal{O}}^+_Y(1) \to \mathcal{E}^+_Y \to \widehat{\mathcal{O}}^+_Y \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\Omega}^1_{\mathfrak{Y}} \to 0$$

is also an extension of $\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\mathcal{O}}^+_X$ by $(1/\rho_k)\widehat{\mathcal{O}}^+_X(1)$ and we denote the associated extension class by

$$\mathrm{cl}_2 \in \mathrm{Ext}^1\left(\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\mathcal{O}}^+_X, \frac{1}{\rho_k}\widehat{\mathcal{O}}^+_X(1)\right).$$

Then it is natural to ask whether $\mathcal{E}^+_1 \cong \mathcal{E}^+_2$ (equivalently, $\mathrm{cl}_1 = \mathrm{cl}_2$).

Proposition 2.17. *Keep the notation as above. If* $f : \mathfrak{X} \to \mathfrak{Y}$ *lifts to an* A_2*-morphism* $\tilde{f} : \widetilde{\mathfrak{X}} \to \widetilde{\mathfrak{Y}}$, *then* $\mathrm{cl}_1 = \mathrm{cl}_2$.

We are going to prove this proposition in the rest of this subsection.

[1] Here, the tensor product $\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\mathcal{O}}^+_X$ should be understood as $f^{-1}\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{f^{-1}\mathcal{O}_{\mathfrak{Y}}} \widehat{\mathcal{O}}^+_X$. The same applies to sheaves like $\mathcal{O}^+_X \otimes_{\widehat{\mathcal{O}}^+_Y} \mathcal{E}^+_Y$, $\widehat{\mathcal{O}}^+_X \otimes_{\widehat{\mathcal{O}}^+_Y} \mathcal{O}\mathbb{C}^+_{Y,\rho}$, $\widehat{\mathcal{O}}^+_X \otimes_{\widehat{\mathcal{O}}^+_Y} \mathcal{O}\widehat{\mathbb{C}}^+_{Y,\rho}$, $\widehat{\mathcal{O}}^+_X \otimes_{\widehat{\mathcal{O}}^+_Y} \mathcal{O}\mathbb{C}^{\dagger,+}_{Y,\rho}$.

By Theorem 2.13, there exists an obstruction class

$$\mathrm{cl}(f) \in \mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{Y}}/\mathcal{O}_{\mathbb{C}_p}}, \frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big)$$

to lift f along the surjection $A_2 \to \mathcal{O}_{\mathbb{C}_p}$. Before moving on, let us recall the definition of $\mathrm{cl}(f)$.

Let $[\widetilde{\mathfrak{X}}]$ and $[\widetilde{\mathfrak{Y}}]$ be classes defined as before and regard them as elements in $\mathrm{Ext}^1(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{Y}}/A_2}, (1/\rho_k)\mathcal{O}_{\mathfrak{X}}(1))$ via the obvious morphisms. Then similar to the construction of $\mathrm{cl}(\mathcal{E}^+)$, one can check that

$$\mathrm{cl}(f) = [\widetilde{\mathfrak{X}}] - [\widetilde{\mathfrak{Y}}]$$

via the injection

$$\mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{Y}}/\mathcal{O}_{\mathbb{C}_p}}, \frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big) \to \mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{Y}}/A_2}, \frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big).$$

For simplicity, we still denote by $\mathrm{cl}(f)$ its image in

$$\mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{Y}}/\mathcal{O}_{\mathbb{C}_p}}, \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\Big) \cong \mathrm{Ext}^1\Big(\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\mathcal{O}}_X^+, \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\Big)$$

via the natural map $(1/\rho_k)\mathcal{O}_{\mathfrak{X}}(1) \to (1/\rho_k)\widehat{\mathcal{O}}_X^+(1)$. Then the following proposition is true.

Proposition 2.18. $\mathrm{cl}(f) = \mathrm{cl}_1 - \mathrm{cl}_2.$

Proof. By the constructions of $\mathcal{E}_1^+$ and $\mathcal{E}_2^+$, we see that cl_1 is the image of $\mathrm{cl}(\mathcal{E}_X^+)$ via the morphism

$$\mathrm{Ext}^1\Big(\widehat{\Omega}^1_{\mathfrak{X}}, \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\Big) \to \mathrm{Ext}^1\Big(\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \mathcal{O}_{\mathfrak{X}}, \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\Big)$$

induced by

$$\mathrm{L}_{\mathcal{O}_{\mathfrak{Y}}/\mathcal{O}_{\mathbb{C}_p}} \widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{Y}}} \mathcal{O}_{\mathfrak{X}} \to \widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\mathbb{C}_p}},$$

and that cl_2 is the image of $\mathrm{cl}(\mathcal{E}_Y^+)$ via the morphism

$$\mathrm{Ext}^1\Big(\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\mathcal{O}}_Y^+, \frac{1}{\rho_k}\widehat{\mathcal{O}}_Y^+(1)\Big) \to \mathrm{Ext}^1\Big(\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\mathcal{O}}_X^+, \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\Big)$$

induced by the inclusion $(1/\rho_k)\widehat{\mathcal{O}}_Y^+(1) \to (1/\rho_k)\widehat{\mathcal{O}}_X^+(1)$.

Now by Proposition 2.14, we have

$$\mathrm{cl}_1 - \mathrm{cl}_2 = \mathrm{cl}(\mathcal{E}_X^+) - \mathrm{cl}(\mathcal{E}_Y^+) = ([\widetilde{\mathfrak{X}}] - [\widetilde{\mathfrak{Y}}]) - ([X] - [Y]).$$

However, the inclusion $\widehat{\mathcal{O}}_Y^+ \to \widehat{\mathcal{O}}_X^+$ admits a canonical A_2-lifting, namely $\mathrm{A}_2(\widehat{\mathcal{O}}_Y^+) \to \mathrm{A}_2(\widehat{\mathcal{O}}_X^+)$. So we deduce that $[X] - [Y] = 0$, which completes the proof. □

Now, Proposition 2.17 is a special case of Proposition 2.18.

Corollary 2.19. *Assume $f : \mathfrak{X} \to \mathfrak{Y}$ admits a lifting along $A_2 \to \mathcal{O}_{\mathbb{C}_p}$. Then there is an exact sequence of sheaves of $\widehat{\mathcal{O}}_X^+$-modules*

$$0 \to \widehat{\mathcal{O}}_X^+ \otimes_{\widehat{\mathcal{O}}_Y^+} \mathcal{E}_Y^+ \to \mathcal{E}_X^+ \to \widehat{\mathcal{O}}_X^+ \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} \to 0, \tag{2-9}$$

where $\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}$ is the module of relative differentials.

Proof. This follows from the Proposition 2.17 combined with the definitions of $\mathcal{E}_1^+$ and $\mathcal{E}_2^+$. □

Period sheaves. Now, we define the desired period sheaf $\mathcal{O}\mathbb{C}^\dagger$ as mentioned in Section 1. The construction generalises the previous work of Hyodo [1989].

Let $\mathfrak{X} = \mathrm{Spf}(R^+)$ be a small smooth formal scheme and $\widetilde{\mathfrak{X}} = \mathrm{Spf}(\widetilde{R}^+)$ be a fixed A_2-lifting. Let E^+ be the integral Faltings' extension introduced in Proposition 2.6. Define $E^+_{\rho_k} = \rho_k E^+(-1)$. Then it fits into the exact sequence

$$0 \to \widehat{R}^+_\infty \to E^+_{\rho_k} \to \rho_k \widehat{R}^+_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1) \to 0.$$

For any $\rho \in \rho_k \mathcal{O}_{\mathbb{C}_p}$, denote by E^+_ρ the pull-back of $E^+_{\rho_k}$ along the inclusion

$$\rho \widehat{R}^+_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1) \to \rho_k \widehat{R}^+_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1).$$

Then it fits into the Γ-equivariant exact sequence

$$0 \to \widehat{R}^+_\infty \to E^+_\rho \to \rho \widehat{R}^+_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1) \to 0. \tag{2-10}$$

By Proposition 2.8, E^+_ρ admits an $\widehat{R}^+_\infty$-basis $1, (\rho x_1)/t, \dots, (\rho x_d)/t$. Let $E = E^+_\rho\left[\frac{1}{p}\right]$, which fits into the induced exact sequence

$$0 \to \widehat{R}_\infty \to E \to \widehat{R}_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1) \to 0.$$

Then it is independent of the choice of ρ and has E^+_ρ as a sub-$\widehat{R}^+_\infty$-module. Also, it admits an $\widehat{R}_\infty$-basis

$$1, y_1 = \frac{x_1}{t}, \dots, y_d = \frac{x_d}{t}$$

such that $\gamma_i(y_j) = y_j + \delta_{ij}$ for any $1 \le i, j \le d$. Define $S_\infty = \varinjlim_n \mathrm{Sym}^n_{\widehat{R}_\infty} E$. Then by similar arguments used in [Hyodo 1989, Section I], we have the following result.

Proposition 2.20. *There exists a canonical Higgs field*

$$\Theta : S_\infty \to S_\infty \otimes_{\widehat{R}_\infty} \widehat{\Omega}^1_{R^+}(-1)$$

on S_∞ such that the induced Higgs complex is a resolution of $\widehat{R}_\infty$. The Higgs field Θ is induced by taking alternative sum along the projection $E \to \widehat{R}_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1)$ and if we denote by Y_i the image of y_i in S_∞, then there is a Γ-equivariant isomorphism

$$\iota : S_\infty \xrightarrow{\cong} \widehat{R}_\infty[Y_1, \dots, Y_d]$$

such that $\Theta = \sum_{i=1}^d (\partial/\partial Y_i) \otimes ((\mathrm{dlog}\, T_i)/t)$ via this isomorphism, where $\widehat{R}_\infty[Y_1, \dots, Y_d]$ is the polynomial ring on free variables Y_i's over $\widehat{R}_\infty$.

Since we have $\widehat{R}^+_\infty$-lattices E^+_ρ's of E, inspired by Proposition 2.20, we make the following definition.

Definition 2.21. For any $\rho \in \rho_k \mathcal{O}_{\mathbb{C}_p}$, define

(1) $S^+_{\infty,\rho} = \varinjlim_n \mathrm{Sym}^n_{\widehat{R}^+_\infty} E^+_\rho$;

(2) $\widehat{S}^+_{\infty,\rho} = \varprojlim_n S^+_{\infty,\rho}/p^n$;

(3) $S^{\dagger,+}_\infty = \varinjlim_{\nu_p(\rho) > \nu_p(\rho_k)} \widehat{S}^+_{\infty,\rho}$ and $S^\dagger_\infty = S^{\dagger,+}_\infty\left[\frac{1}{p}\right]$.

For any $\rho_1, \rho_2 \in \rho_k\mathcal{O}_{\mathbb{C}_p}$ satisfying $\nu_p(\rho_1) \geq \nu_p(\rho_2)$, we have $E^+_{\rho_1} \subset E^+_{\rho_2} \subset E$. So Proposition 2.20 implies that $S^+_{\infty,\rho_1} \subset S^+_{\infty,\rho_2} \subset S_\infty$. Moreover, the restriction of Θ to $S^+_{\infty,\rho}$ (for $\rho \in \rho_k\mathcal{O}_{\mathbb{C}_p}$) induces a Higgs field on $S^+_{\infty,\rho}$, which is identified with $\widehat{R}^+_\infty[\rho Y_1, \ldots, \rho Y_d]$ via the canonical isomorphism ι. In this case, we still have $\Theta = \sum_{i=1}^d (\partial/\partial Y_i) \otimes ((\mathrm{dlog}\, T_i)/t)$. Since Θ is continuous, it extends to $\widehat{S}^+_{\infty,\rho}$ and thus we have the following corollary.

Corollary 2.22. *For any $\rho \in \rho_k\mathcal{O}_{\mathbb{C}_p}$, there exists a canonical Higgs field*

$$\Theta : \widehat{S}^+_{\infty,\rho} \to \widehat{S}^+_{\infty,\rho} \otimes_{\widehat{R}^+_\infty} \widehat{\Omega}^1_{R^+}(-1)$$

on $\widehat{S}^+_{\infty,\rho}$. Additionally, there is a Γ-equivariant isomorphism

$$\iota : \widehat{S}^+_{\infty,\rho} \xrightarrow{\cong} \widehat{R}^+_\infty\langle \rho Y_1, \ldots, \rho Y_d\rangle$$

such that

$$\Theta = \sum_{i=1}^d \frac{\partial}{\partial Y_i} \otimes \frac{\mathrm{dlog}\, T_i}{t}$$

via this isomorphism, where $\widehat{R}^+_\infty\langle \rho Y_1, \ldots, \rho Y_d\rangle$ is the p-adic completion of $\widehat{R}^+_\infty[\rho Y_1, \ldots, \rho Y_d]$.

After taking the inductive limit of $\{\rho \in \rho_k\mathcal{O}_{\mathbb{C}_p} | \nu_p(\rho) > \nu_p(\rho_k)\}$, we get the following corollary.

Corollary 2.23. *There exists a canonical Higgs field*

$$\Theta : S^{\dagger,+}_\infty \to S^{\dagger,+}_\infty \otimes_{\widehat{R}^+_\infty} \widehat{\Omega}^1_{R^+}(-1)$$

on $S^{\dagger,+}_\infty$. Additionally, there is a Γ-equivariant isomorphism

$$\iota : S^{\dagger,+}_\infty \xrightarrow{\cong} \varinjlim_{\nu_p(\rho) > \nu_p(\rho_k)} \widehat{R}^+_\infty\langle \rho Y_1, \ldots, \rho Y_d\rangle$$

such that $\Theta = \sum_{i=1}^d (\partial/\partial Y_i) \otimes ((\mathrm{dlog}\, T_i)/t)$ via this isomorphism. After inverting p, the induced Higgs complex

$$\mathrm{HIG}(S^\dagger_\infty, \Theta) : \quad S^\dagger_\infty \xrightarrow{\Theta} S^\dagger_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1) \xrightarrow{\Theta} S^\dagger_\infty \otimes_{R^+} \widehat{\Omega}^2_{R^+}(-2) \to \cdots \tag{2-11}$$

is a resolution of $\widehat{R}_\infty$.

Proof. It remains to prove the Higgs complex $\mathrm{HIG}(S^\dagger_\infty, \Theta)$ is a resolution of $\widehat{R}_\infty$. For any $\rho \in \rho_k\mathcal{O}_{\mathbb{C}_p}$, consider the Higgs complexes

$$\mathrm{HIG}(\widehat{S}^+_{\infty,\rho}, \Theta) : \quad \widehat{S}^+_{\infty,\rho} \xrightarrow{\Theta} \widehat{S}^+_{\infty,\rho} \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1) \xrightarrow{\Theta} \widehat{S}^+_{\infty,\rho} \otimes_{R^+} \widehat{\Omega}^2_{R^+}(-2) \to \cdots$$

and

$$\mathrm{HIG}(S^{\dagger,+}_\infty, \Theta) : \quad S^{\dagger,+}_\infty \xrightarrow{\Theta} \widehat{S}^{\dagger,+}_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1) \xrightarrow{\Theta} S^{\dagger,+}_\infty \otimes_{R^+} \widehat{\Omega}^2_{R^+}(-2) \to \cdots.$$

Then we have

$$\mathrm{HIG}(S^\dagger_\infty, \Theta) = \mathrm{HIG}(S^{\dagger,+}_\infty, \Theta)\Big[\frac{1}{p}\Big] = \varinjlim_{\nu_p(\rho) > \nu_p(\rho_k)} \mathrm{HIG}(\widehat{S}^+_{\infty,\rho}, \Theta)\Big[\frac{1}{p}\Big].$$

By Corollary 2.22, $\mathrm{HIG}(\widehat{S}^+_{\infty,\rho},\Theta)$ is computed by the Koszul complex

$$\mathrm{K}\Big(\widehat{R}^+_\infty\langle\rho Y_1,\dots,\rho Y_d\rangle;\frac{\partial}{\partial Y_1},\dots,\frac{\partial}{\partial Y_d}\Big)\simeq\mathrm{K}\Big(\widehat{R}^+_\infty\langle\rho Y_1\rangle;\frac{\partial}{\partial Y_1}\Big)\widehat{\otimes}^L_{\widehat{R}^+_\infty}\cdots\widehat{\otimes}^L_{\widehat{R}^+_\infty}\mathrm{K}\Big(\widehat{R}^+_\infty\langle\rho Y_d\rangle;\frac{\partial}{\partial Y_d}\Big),$$

via the canonical isomorphism ι. Note that, for any j,

$$\mathrm{H}^i\Big(\mathrm{K}\Big(\widehat{R}^+_\infty\langle\rho Y_j\rangle;\frac{\partial}{\partial Y_j}\Big)\Big)=\begin{cases}\widehat{R}^+_\infty, & i=0,\\ \widehat{R}^+_\infty\langle\Lambda_{j,\rho}\rangle/\widehat{R}^+_\infty\langle\Lambda_{j,\rho},I,+\rangle, & i=1,\\ 0, & i\geq 2,\end{cases}$$

is derived p-complete by Proposition A.2, where $\widehat{R}^+_\infty\langle\Lambda_{j,\rho}\rangle$ and $\widehat{R}^+_\infty\langle\Lambda_{j,\rho},I,+\rangle$ are defined as in Definition A.1 for $\Lambda_{j,\rho}=\{\rho^nY_j^n\}_{n\geq 0}$ and $I=\{\nu_p(n+1)\}_{n\geq 0}$. We deduce that, for any $i\geq 0$,

$$\mathrm{H}^i\Big(\mathrm{K}\Big(\widehat{R}^+_\infty\langle\rho Y_1,\dots,\rho Y_d\rangle;\frac{\partial}{\partial Y_1},\dots,\frac{\partial}{\partial Y_d}\Big)\Big)=\bigwedge^i_{\widehat{R}^+_\infty}\Big(\bigoplus_{j=1}^d\widehat{R}^+_\infty\langle\Lambda_{j,\rho}\rangle/\widehat{R}^+_\infty\langle\Lambda_{j,\rho},I,+\rangle\Big).$$

In particular, we get

$$\mathrm{H}^0(\mathrm{HIG}(S^{\dagger,+}_\infty,\Theta))=\varinjlim_{\nu_p(\rho)>\nu_p(\rho_k)}\mathrm{H}^0(\mathrm{HIG}(\widehat{S}^+_{\infty,\rho},\Theta))=\widehat{R}^+_\infty.$$

It remains to show that, for any $i\geq 1$,

$$\varinjlim_{\nu_p(\rho)>\nu_p(\rho_k)}\mathrm{H}^i(\mathrm{HIG}(\widehat{S}^+_{\infty,\rho},\Theta))\cong\varinjlim_{\nu_p(\rho)>\nu_p(\rho_k)}\bigwedge^i_{\widehat{R}^+_\infty}\Big(\bigoplus_{j=1}^d\widehat{R}^+_\infty\langle\Lambda_{j,\rho}\rangle/\widehat{R}^+_\infty\langle\Lambda_{j,\rho},I,+\rangle\Big)$$

is p^∞-torsion. To do so, it suffices to prove that for any $\nu_p(\rho_1)>\nu_p(\rho_2)>\nu_p(\rho_k)$, there is an $N\geq 0$ such that

$$p^N\widehat{R}^+_\infty\langle\Lambda_{j,\rho_1}\rangle\subset\widehat{R}^+_\infty\langle\Lambda_{j,\rho_2},I,+\rangle.$$

By Remark A.3, we only need to find an N such that the following conditions hold:

(1) For any $i\geq 0$, $N+i\nu_p(\rho_1)-i\nu_p(\rho_2)-\nu_p(i+1)\geq 0$.

(2) $\lim_{i\to+\infty}(N+i\nu_p(\rho_1)-i\nu_p(\rho_2)-\nu_p(i+1))=+\infty$.

Since $\nu_p(\rho_1)>\nu_p(\rho_2)$, such an N exists. This completes the proof. □

Remark 2.24. (1) In the proof of Corollary 2.23, we have seen that for any $\rho\in\rho_k\mathcal{O}_{\mathbb{C}_p}$, the Higgs complex $\mathrm{HIG}\big(S^+_{\infty,\rho}\big[\frac{1}{p}\big],\Theta\big)$ is not a resolution of $\widehat{R}_\infty$.

(2) For any $1\leq i\leq d$, the p^∞-torsion of $\mathrm{H}^i(\mathrm{HIG}(S^{\dagger,+}_\infty,\Theta))$ is unbounded.

Remark 2.25. Since for any $1\leq i,j\leq d$, $\gamma_i(Y_j)=Y_j+\delta_{ij}$, one can check that $\partial/\partial Y_i=\log\gamma_i$ on $S^\dagger_\infty$. So the Higgs field is $\Theta=\sum_{i=1}^d\log\gamma_i\otimes((\mathrm{dlog}\,T_i)/t)$.

Remark 2.26. A similar local construction of $S^{\dagger,+}_\infty$ also appeared in [Abbes et al. 2016, I.4.7].

There is a global analogue by using Theorem 2.9 instead of Proposition 2.6. Put $\mathcal{E}^+_{\rho_k} = \rho_k\mathcal{E}^+(-1)$ and for any $\rho \in \rho_k\mathcal{O}_{\mathbb{C}_p}$, denote by $\mathcal{E}^+_\rho$ the pull-back of $\mathcal{E}^+_{\rho_k}$ along the inclusion

$$\rho\widehat{\mathcal{O}}^+_X \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}}(-1) \to \rho_k\widehat{\mathcal{O}}^+_X \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}}(-1).$$

Then it fits into the exact sequence

$$0 \to \widehat{\mathcal{O}}^+_X \to \mathcal{E}^+_\rho \to \rho\widehat{\mathcal{O}}^+_X \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}}(-1) \to 0. \tag{2-12}$$

As an analogue of Definition 2.21 in the local case, we define period sheaves as follows:

Definition 2.27. For any $\rho \in \rho_k\mathcal{O}_{\mathbb{C}_p}$, define

(1) $\mathcal{O}\mathbb{C}^+_\rho = \varinjlim_n \operatorname{Sym}^n_{\widehat{\mathcal{O}}^+_X} \mathcal{E}^+_\rho$;

(2) $\mathcal{O}\widehat{\mathbb{C}}^+_\rho = \varprojlim_n \mathcal{O}\mathbb{C}^+_\rho / p^n$;

(3) $\mathcal{O}\mathbb{C}^{\dagger,+} = \varinjlim_{\nu_p(\rho)>\nu_p(\rho_k)} \mathcal{O}\widehat{\mathbb{C}}^+_\rho$ and $\mathcal{O}\mathbb{C}^\dagger = \mathcal{O}\mathbb{C}^{\dagger,+}\big[\frac{1}{p}\big]$.

Theorem 2.28. *There is a canonical Higgs field Θ on $\mathcal{O}\mathbb{C}^{\dagger,+}$ such that the induced Higgs complex*

$$\mathrm{HIG}(\mathcal{O}\mathbb{C}^\dagger, \Theta): \ \mathcal{O}\mathbb{C}^\dagger \xrightarrow{\Theta} \mathcal{O}\mathbb{C}^\dagger \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}}(-1) \xrightarrow{\Theta} \mathcal{O}\mathbb{C}^\dagger \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^2_{\mathfrak{X}}(-2) \to \cdots \tag{2-13}$$

is a resolution of $\widehat{\mathcal{O}}_X$. Additionally, when $\mathfrak{X} = \mathrm{Spf}(R^+)$ is small affine, there is an isomorphism

$$\iota : \mathcal{O}\mathbb{C}^{\dagger,+}_{|X_\infty} \to \varinjlim_{\nu_p(\rho)>\nu_p(\rho_k)} \widehat{\mathcal{O}}^+_X\langle \rho Y_1, \dots, \rho Y_d\rangle_{|X_\infty}$$

such that the Higgs field Θ equals $\sum_{i=1}^d (\partial/\partial Y_i) \otimes ((\mathrm{dlog}\, T_i)/t)$.

Proof. Since the problem is local, we are reduced to Corollary 2.23. □

Finally, we describe the relative version of the above constructions. We assume that $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of liftable smooth formal schemes and lifts to an A_2-morphism $\tilde{f} : \widetilde{\mathfrak{X}} \to \widetilde{\mathfrak{Y}}$. Then by Corollary 2.19, for any $\rho \in \rho_k\mathcal{O}_{\mathbb{C}_p}$, we have the exact sequence

$$0 \to \widehat{\mathcal{O}}^+_X \otimes_{\widehat{\mathcal{O}}^+_Y} \mathcal{E}^+_{\rho,Y} \to \mathcal{E}^+_{\rho,X} \to \widehat{\mathcal{O}}^+_X \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}(-1) \to 0.$$

By construction of period sheaves in Definition 2.27, we get morphisms of sheaves $\widehat{\mathcal{O}}^+_X \otimes_{\widehat{\mathcal{O}}^+_Y} \mathcal{F}_Y \to \mathcal{F}_X$ for $\mathcal{F} \in \{\mathcal{O}\mathbb{C}^+_\rho, \mathcal{O}\widehat{\mathbb{C}}^+_\rho, \mathcal{O}\mathbb{C}^{\dagger,+}\}$. Also, the natural projection $\mathcal{E}^+_{\rho,X} \to \widehat{\mathcal{O}}^+_X \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}(-1)$ induces relative Higgs fields

$$\Theta_{X/Y} : \mathcal{F}_X \to \mathcal{F}_X \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}(-1)$$

for $\mathcal{F} \in \{\mathcal{O}\mathbb{C}^+_\rho, \mathcal{O}\widehat{\mathbb{C}}^+_\rho, \mathcal{O}\mathbb{C}^{\dagger,+}\}$. Using similar arguments as above, we get the following proposition.

Proposition 2.29. *Assume that $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of liftable smooth formal schemes and lifts to an A_2-morphism $\tilde{f} : \widetilde{\mathfrak{X}} \to \widetilde{\mathfrak{Y}}$. The induced relative Higgs complex*

$$\mathrm{HIG}(\mathcal{O}\mathbb{C}^\dagger_X, \Theta_{X/Y}): \ \mathcal{O}\mathbb{C}^\dagger_X \xrightarrow{\Theta_{X/Y}} \mathcal{O}\mathbb{C}^\dagger_X \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}(-1) \xrightarrow{\Theta_{X/Y}} \mathcal{O}\mathbb{C}^\dagger_X \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^2_{\mathfrak{X}/\mathfrak{Y}}(-2) \to \cdots$$

is a resolution of $\varinjlim_{\rho,\nu_p(\rho)>\nu_p(\rho_k)}(\widehat{\mathcal{O}}_X^+\widehat{\otimes}_{\widehat{\mathcal{O}}_Y^+}\mathcal{O}\widehat{\mathbb{C}}_{\rho,Y}^+)\big[\frac{1}{p}\big]$ *and makes the diagram*

$$\begin{array}{ccc}
f^*\mathcal{O}\mathbb{C}_Y^\dagger & \xrightarrow{f^*\Theta_Y} & f^*\mathcal{O}\mathbb{C}_Y^\dagger\otimes_{\mathcal{O}_{\mathfrak{Y}}}\widehat{\Omega}^1_{\mathfrak{Y}}(-1)\longrightarrow\cdots \\
\downarrow & & \downarrow \\
\mathcal{O}\mathbb{C}_X^\dagger & \xrightarrow{\Theta_X} & \mathcal{O}\mathbb{C}_X^\dagger\otimes_{\mathcal{O}_{\mathfrak{X}}}\widehat{\Omega}^1_{\mathfrak{X}}(-1)\longrightarrow\cdots \\
\downarrow{\scriptstyle\Theta_{X/Y}} & & \downarrow{\scriptstyle\Theta_{X/Y}} \\
\mathcal{O}\mathbb{C}_X^\dagger\otimes_{\mathcal{O}_{\mathfrak{X}}}\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}(-1) & \xrightarrow{\Theta_{X/Y}} & \mathcal{O}\mathbb{C}_X^\dagger\otimes_{\mathcal{O}_{\mathfrak{X}}}\widehat{\Omega}^2_{\mathfrak{X}/\mathfrak{Y}}(-2)\longrightarrow\cdots \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array} \tag{2-14}$$

commute, where $f^*\mathcal{O}\mathbb{C}_Y^\dagger=\widehat{\mathcal{O}}_X\otimes_{\widehat{\mathcal{O}}_Y}\mathcal{O}\mathbb{C}_Y^\dagger$ *and* $f^*\Theta_Y=\mathrm{id}\otimes\Theta_Y$.

Proof. Put $\mathcal{C}:=\varinjlim_{\rho,\nu_p(\rho)>\nu_p(\rho_k)}(\widehat{\mathcal{O}}_X^+\widehat{\otimes}_{\widehat{\mathcal{O}}_Y^+}\mathcal{O}\widehat{\mathbb{C}}_{\rho,Y}^+)\big[\frac{1}{p}\big]$. Since f admits a lifting $\tilde{f}$, for any $\rho\in\rho_k\mathcal{O}_{\mathbb{C}_p}$, we have a morphism $\widehat{\mathcal{O}}_X^+\otimes_{\widehat{\mathcal{O}}_Y^+}\mathcal{O}\mathbb{C}_{\rho,Y}^+\to\mathcal{O}\mathbb{C}_{\rho,X}^+$ and hence morphisms $f^*\mathcal{O}\mathbb{C}_Y^\dagger\to\mathcal{C}\to\mathcal{O}\mathbb{C}_X^\dagger$. It remains to show the relative Higgs complex $\mathrm{HIG}(\mathcal{O}\mathbb{C}_X^\dagger,\Theta_{X/Y})$ is a resolution of $\mathcal{C}$ and that the diagram (2-14) commutes. Since the problem is local, we may assume $\mathfrak{Y}=\mathrm{Spf}(S^+)$ and $\mathfrak{X}=\mathrm{Spf}(R^+)$ are both small affine such that the morphism $f:\mathfrak{X}\to\mathfrak{Y}$ is induced by a morphism $S^+\to R^+$ which makes the diagram

$$\begin{array}{ccc}
\mathcal{O}_{\mathbb{C}_p}\langle T_1^{\pm1},\dots,T_d^{\pm1}\rangle & \xrightarrow{\subset} & \mathcal{O}_{\mathbb{C}_p}\langle T_1^{\pm1},\dots,T_d^{\pm1},T_{d+1}^{\pm1},\dots,T_{d+r}^{\pm1}\rangle \\
\downarrow & & \downarrow \\
S^+ & \longrightarrow & R^+
\end{array}$$

commute, where d is the dimension of $\mathfrak{Y}$ over $\mathcal{O}_{\mathbb{C}_p}$, r is the dimension of $\mathfrak{X}$ over $\mathfrak{Y}$ and both vertical maps are étale. Let $\widehat{S}_\infty^+$ and $\widehat{R}_\infty^+$ be the perfectoid rings corresponding to the base-changes of S^+ and R^+ along morphisms

$$\mathcal{O}_{\mathbb{C}_p}\langle T_1^{\pm1},\dots,T_d^{\pm1}\rangle\to\mathcal{O}_{\mathbb{C}_p}\langle T_1^{\pm\frac{1}{p^\infty}},\dots,T_d^{\pm\frac{1}{p^\infty}}\rangle$$

and

$$\mathcal{O}_{\mathbb{C}_p}\langle T_1^{\pm1},\dots,T_d^{\pm1},T_{d+1}^{\pm1},\dots,T_{d+r}^{\pm1}\rangle\to\mathcal{O}_{\mathbb{C}_p}\langle T_1^{\pm\frac{1}{p^\infty}},\dots,T_d^{\pm\frac{1}{p^\infty}},T_{d+1}^{\pm\frac{1}{p^\infty}},\dots,T_{d+r}^{\pm\frac{1}{p^\infty}}\rangle,$$

respectively. Put $Y_\infty=\mathrm{Spa}(\widehat{S}_\infty,\widehat{S}_\infty^+)$ and $X_\infty=\mathrm{Spa}(\widehat{R}_\infty,\widehat{R}_\infty^+)$ with $\widehat{S}_\infty=\widehat{S}_\infty^+\big[\frac{1}{p}\big]$ and $\widehat{R}_\infty=\widehat{R}_\infty^+\big[\frac{1}{p}\big]$. For any $\rho\in\rho_k\mathcal{O}_{\mathbb{C}_p}$, since $\mathcal{E}_{\rho,Y}^+$ fits into the exact sequence

$$0\to\widehat{\mathcal{O}}_X^+\to\widehat{\mathcal{O}}_X^+\otimes_{\widehat{\mathcal{O}}_Y^+}\mathcal{E}_{\rho,Y}^+\to\rho\widehat{\Omega}^1_{\mathfrak{Y}}\otimes_{\mathcal{O}_{\mathfrak{Y}}}\widehat{\mathcal{O}}_X^+(-1)\to 0,$$

we see that $(\widehat{\mathcal{O}}_X^+\otimes_{\widehat{\mathcal{O}}_Y^+}\mathcal{E}_Y^+)(X_\infty)(\subset\mathcal{E}_{\rho,X}^+(X_\infty))$ coincides with $\widehat{R}_\infty^+\otimes_{\widehat{S}_\infty^+}\mathcal{E}_{\rho,Y}^+(Y_\infty)$. This implies that

$$(\widehat{\mathcal{O}}_X^+\otimes_{\widehat{\mathcal{O}}_Y^+}\mathcal{O}\mathbb{C}_{\rho,Y}^+)(X_\infty)\cong\widehat{R}_\infty^+[\rho Y_1,\dots,\rho Y_d]$$

such that the induced Higgs field is given by $\sum_{i=0}^{d}(\partial/\partial Y_i)\otimes((\mathrm{dlog}\,T_i)/t)$. On the other hand, we have

$$\mathcal{O}\mathbb{C}^+_{\rho,X}(X_\infty)\cong\widehat{R}^+_\infty[\rho Y_1,\dots,\rho Y_{d+r}]$$

such that the induced Higgs field is given by $\sum_{i=0}^{d+r}(\partial/\partial Y_i)\otimes((\mathrm{dlog}\,T_i)/t)$. So the morphism

$$\widehat{\mathcal{O}}^+_X\otimes_{\widehat{\mathcal{O}}^+_Y}\mathcal{O}\mathbb{C}^+_{\rho,Y}\to\mathcal{O}\mathbb{C}^+_{\rho,X}$$

is compatible with Higgs fields for any $\rho\in\rho_k\mathcal{O}_{\mathbb{C}_p}$. Therefore, for any $\rho\in\rho_k\mathcal{O}_{\mathbb{C}_p}$, we have morphisms of sheaves

$$\widehat{\mathcal{O}}^+_X\otimes_{\widehat{\mathcal{O}}^+_Y}\mathcal{O}\mathbb{C}^+_{\rho,Y}\to\widehat{\mathcal{O}}^+_X\otimes_{\widehat{\mathcal{O}}^+_Y}\mathcal{O}\widehat{\mathbb{C}}^+_{\rho,Y}\to\widehat{\mathcal{O}}^+_X\widehat{\otimes}_{\widehat{\mathcal{O}}^+_Y}\mathcal{O}\widehat{\mathbb{C}}^+_{\rho,Y}\to\mathcal{O}\widehat{\mathbb{C}}^+_{\rho,X}$$

which are all compatible with Higgs fields. After taking direct limits and inverting p, we get morphisms

$$f^*\mathcal{O}\mathbb{C}^\dagger_Y\to\mathcal{C}\to\mathcal{O}\mathbb{C}^\dagger_X$$

of sheaves which are compatible with Higgs fields. In particular, the top two rows of (2-14) form a commutative diagram.

To complete the proof, we have to show that $\mathrm{HIG}(\mathcal{O}\mathbb{C}^\dagger_X,\Theta_{X/Y})$ is a resolution of $\mathcal{C}$. Since we do have a morphism $\mathcal{C}\to\mathrm{HIG}(\mathcal{O}\mathbb{C}^\dagger_X,\Theta_{X/Y})$, we can conclude by checking the exactness locally.

By the "additionally" part of Corollary 2.23, we obtain that

$$\mathcal{O}\mathbb{C}^\dagger_X(X_\infty)=\Big(\varinjlim_{\rho,\nu_p(\rho)>\nu_p(\rho_k)}\widehat{R}^+_\infty\langle\rho Y_1,\dots,\rho Y_{d+r}\rangle\Big)\Big[\frac{1}{p}\Big]$$

with $\Theta_X=\sum_{i=1}^{d+r}(\partial/\partial Y_i)\otimes((\mathrm{dlog}\,T_i)/t)$. A similar argument also shows that

$$\Theta_{X/Y}=\sum_{i=d+1}^{d+r}\frac{\partial}{\partial Y_i}\otimes\frac{\mathrm{dlog}\,T_i}{t}.$$

So the rest of (2-14) commutes. Note that $\mathcal{C}(X_\infty)=(\varinjlim_{\rho,\nu_p(\rho)>\nu_p(\rho_k)}\widehat{R}^+_\infty\langle\rho Y_1,\dots,\rho Y_d\rangle)\big[\frac{1}{p}\big]$. By a similar argument in the proof of Corollary 2.23, we see that $\mathrm{HIG}(\mathcal{O}\mathbb{C}^\dagger_X,\Theta_{X/Y})$ is a resolution of $\mathcal{C}$. □

3. An integral decompletion theorem

In this section, we generalise results in [Diao et al. 2023b, Appendix A] to an integral case which will be used to simplify local calculations. Let $\mathfrak{X}=\mathrm{Spf}(R^+)$, $\widehat{R}^+_\infty$ and Γ be as in the previous section. Throughout this section, we put $\pi=\zeta_p-1$, $r=\nu_p(\pi)=1/(p-1)$ and $c=p^r$. Recall $\nu_p(\rho_k)\geq r$. We begin with some definitions.

Definition 3.1. (1) By a *Banach $\mathcal{O}_{\mathbb{C}_p}$-algebra*, we mean a flat $\mathcal{O}_{\mathbb{C}_p}$-algebra A such that $A\big[\frac{1}{p}\big]$ is a Banach $\mathbb{C}_p$-algebra, and that $A=\big\{a\in A\big[\frac{1}{p}\big]\mid\|a\|\leq 1\big\}$.

(2) Assume A is a Banach $\mathcal{O}_{\mathbb{C}_p}$-algebra. For an A-module M, we say it is a *Banach A-module* if $M\big[\frac{1}{p}\big]$ is a Banach $A\big[\frac{1}{p}\big]$-module, and $M=\big\{m\in M\big[\frac{1}{p}\big]\mid\|m\|\leq 1\big\}$.

There are some typical examples.

Example 3.2. (1) *If A is a Banach $\mathcal{O}_{\mathbb{C}_p}$-algebra, then any topologically free A-module endowed with the supreme norm is a Banach A-module.*

(2) *The rings R^+ and $\widehat{R}^+_\infty$ are Banach $\mathcal{O}_{\mathbb{C}_p}$-algebras.*

(3) *The $\widehat{R}^+_\infty/R^+$ is a Banach R^+-module.*

Now, we make the definition of (a-trivial) Γ-representations.

Definition 3.3. Assume $a > r$ and $A \in \{R^+, \widehat{R}^+_\infty\}$.

(1) By an *A-representation of* Γ of rank l, we mean a finite free A-module M of rank l endowed with a continuous semilinear Γ-action.

(2) Let M be a representation of Γ of rank l over A. We say M is *a-trivial*, if $M/p^a \cong (A/p^a)^l$ as representations of Γ over A/p^a.

(3) Let M be a representation of Γ of rank l over R^+. We say M is *essentially $(a+r)$-trivial* if M is a-trivial and $M \otimes_{R^+} \widehat{R}^+_\infty$ is $(a+r)$-trivial.

The goal of this section is to prove the following integral decompletion theorem.

Theorem 3.4. *Assume $a > r$. Then the functor $M \mapsto M \otimes_{R^+} \widehat{R}^+_\infty$ induces an equivalence from the category of $(a+r)$-trivial R^+-representations of Γ to the category of $(a+r)$-trivial $\widehat{R}^+_\infty$-representations of Γ. The equivalence preserves tensor products and dualities.*

The first difficulty is to construct the quasi-inverse, namely the decompletion functor, of the functor in Theorem 3.4. To do so, we need to generalise the method adapted in [Diao et al. 2023b] to the small integral case. However, their method only shows the decompletion functor takes values in the category of essentially $(a+r)$-trivial representations. So, the second difficulty is to show the resulting representation is actually $(a+r)$-trivial. The trivialness condition is crucial to overcome both difficulties.

Construction of decompletion functor. We first construct the decompletion functor. From now on, we use $\mathrm{R}\Gamma(\Gamma, M)$ to denote the continuous group cohomology of a p-adically completed R^+-module endowed with a continuous Γ-action. By virtues of [Bhatt et al. 2018, Lemma 7.3], $\mathrm{R}\Gamma(\Gamma, M) = \mathrm{R}\varprojlim_k \mathrm{R}\Gamma(\Gamma, M/p^k)$ can be calculated by the Koszul complex

$$\mathrm{K}(M; \gamma_1 - 1, \dots, \gamma_d - 1): \; M \xrightarrow{(\gamma_1-1,\dots,\gamma_d-1)} M^d \to \cdots .$$

Proposition 3.5. *Assume $a > r$. Let M_∞ be an $(a+r)$-trivial $\widehat{R}^+_\infty$-representation of Γ. Then there exists a finite free R^+-submodule $M \subset M_\infty$ such that the following assertions are true*:

(1) *The finite free A-module M is an essentially $(a+r)$-trivial R^+-representation of Γ such that the natural inclusion $M \hookrightarrow M_\infty$ induces an isomorphism $M \otimes_{R^+} \widehat{R}^+_\infty \cong M_\infty$ of $\widehat{R}^+_\infty$-representations of Γ.*

(2) *The induced morphism $\mathrm{R}\Gamma(\Gamma, M) \to \mathrm{R}\Gamma(\Gamma, M_\infty)$ identifies the former as a direct summand of the latter, whose complement is concentrated in positive degrees and killed by π.*

Remark 3.6. The finite free A-module M is unique up to isomorphism and the functor $M_\infty \mapsto M$ turns out to be the quasi-inverse of the functor $M \mapsto M \otimes R_\infty^+$ described in Theorem 3.4.

Now we prove Proposition 3.5 by using similar arguments in [Diao et al. 2023b]. Since we work on the integral level, so we need to control (p-adic) norms carefully. We start with the following result.

Lemma 3.7. *For any cocycle $f \in \mathrm{C}^\bullet(\Gamma, \widehat{R}_\infty/R)$, there exists a cochain $g \in \mathrm{C}^{\bullet-1}(\Gamma, \widehat{R}_\infty/R)$ such that $dg = f$ and $\|g\| \le c\,\|f\|$.*

Proof. The result follows from the same argument used in the proof of [Diao et al. 2023b, Proposition A.2.2.1], especially the part for checking the condition (3) of [Diao et al. 2023b, Definition A.1.6], by using [Scholze 2013a, Lemma 5.5] instead of [Diao et al. 2023a, Lemma 6.1.7]. □

Since the norm on R (resp. $\widehat{R}_\infty$) is induced by that on R^+ (resp. $\widehat{R}_\infty^+$), there exists a norm-preserving embedding of complexes

$$\mathrm{C}^\bullet(\Gamma, \widehat{R}_\infty^+/R^+) \to \mathrm{C}^\bullet(\Gamma, \widehat{R}_\infty/R).$$

We shall apply Lemma 3.7 via this embedding.

Lemma 3.8. *For any cocycle $f \in \mathrm{C}^\bullet(\Gamma, \widehat{R}_\infty^+/R^+)$, there is a cochain $g \in \mathrm{C}^{\bullet-1}(\Gamma, \widehat{R}_\infty^+/R^+)$ such that $\|g\| \le \|f\|$ and $dg = \pi f$.*

Proof. Regard $\mathrm{C}^\bullet(\Gamma, \widehat{R}_\infty^+/R^+)$ as a subcomplex of $\mathrm{C}^\bullet(\Gamma, \widehat{R}_\infty/R)$ as above. Applying Lemma 3.7 to πf, we get a cochain $g \in \mathrm{C}^{\bullet-1}(\Gamma, \widehat{R}_\infty/R)$ such that $\|g\| \le c\,\|\pi f\|$ and $dg = \pi f$. But $c\,\|\pi f\| = \|f\| \le 1$, so we see $g \in \mathrm{C}^{\bullet-1}(\Gamma, \widehat{R}_\infty^+/R^+)$. □

Lemma 3.9. *Let $(\mathrm{C}^\bullet, d)$ be a complex of Banach modules over a Banach $\mathcal{O}_{\mathbb{C}_p}$-algebra A. Suppose that for every degree s and every cocycle $f \in \mathrm{C}^s$, there exists a $g \in \mathrm{C}^{s-1}$ such that $\|g\| \le \|f\|$ and $dg = \pi f$. Then, for any cochain $f \in \mathrm{C}^s$, there exists an $h \in \mathrm{C}^{s-1}$ such that $\|h\| \le \max(\|f\|/c, \|df\|)$ and $\|\pi^2 f - dh\| \le \|df\|/c$.*

Proof. By assumption, one can choose a $g \in \mathrm{C}^s$ such that $dg = \pi df$ and that $\|g\| \le \|df\|$. Then $(g - \pi f) \in \mathrm{C}^s$ is a cocycle. Using this assumption again, there is an $h \in \mathrm{C}^{s-1}$ satisfying $\|h\| \le \|g - \pi f\|$ and $dh = \pi(g - \pi f)$. Then $\|\pi^2 f - dh\| \le \|g\|/c \le \|df\|/c$ and $\|h\| \le \max(\|df\|, \|f\|/c)$. □

The following lemma is a consequence of Lemmas 3.8 and 3.9.

Lemma 3.10. *For any cochain $f \in \mathrm{C}^\bullet(\Gamma, \widehat{R}_\infty^+/R^+)$, there is a cochain $h \in \mathrm{C}^{\bullet-1}(\Gamma, \widehat{R}_\infty^+/R^+)$ such that $\|h\| \le \max(\|f\|/c, \|df\|)$ and $\|\pi^2 f - dh\| \le \|df\|/c$.*

The following lemma can be viewed as an integral version of [Diao et al. 2023b, Lemma A.1.12].

Lemma 3.11. *We denote $(R^+, \widehat{R}_\infty^+/R^+)$ by (A, M) for simplicity.*

Let $L = \bigoplus_{i=1}^n Ae_i$ be a Banach A-module (with the supreme norm) endowed with a continuous Γ-action. Assume there exists an $R > 1$ such that, for each $\gamma \in \Gamma$ and each i, $\|(\gamma - 1)(e_i)\| \le 1/(Rc)$. Then the following assertions are true:

(1) *For any cocycle* $f \in \mathrm{C}^\bullet(\Gamma, L\otimes_A M)$, *there is a cochain* $g \in \mathrm{C}^{\bullet-1}(\Gamma, L\otimes_A M)$ *such that* $\|g\| \le \|f\|$ *and* $dg = \pi f$.

(2) *For any cochain* $f \in \mathrm{C}^\bullet(\Gamma, L\otimes_A M)$, *there exists an* $h \in \mathrm{C}^\bullet(\Gamma, L\otimes_A M)$ *such that* $\|h\| \le \max(\|f\|/c, \|df\|)$ *and* $\|\pi^2 f - dh\| \le \|df\|/c$.

Proof. We only prove (1) and then (2) follows from Lemma 3.9 directly.

Now, let $f = \sum_{i=1}^n e_i \otimes f_i$ be a cocycle with $f_j \in \mathrm{C}^s(\Gamma, M)$ for all $1 \le j \le n$. Then $\|f\| \le 1$. For any $\gamma_1, \gamma_2, \dots, \gamma_{s+1} \in \Gamma$, we have

$$\Big(\sum_{i=1}^n e_i \otimes df_i\Big)(\gamma_1,\dots,\gamma_{s+1}) = \Big(\sum_{i=1}^n e_i \otimes df_i\Big)(\gamma_1,\dots,\gamma_{s+1}) - df(\gamma_1,\dots,\gamma_{s+1})$$
$$= \sum_{i=1}^n (1-\gamma_1)(e_i) \otimes f_i(\gamma_2,\dots,\gamma_{s+1}).$$

It follows that $\big\|\sum_{i=1}^n e_i \otimes df_i\big\| \le \|f\|/(Rc)$. In other words, for each $1 \le j \le n$, $\|df_j\| \le \|f\|/(Rc)$. By Lemma 3.10, for every j, there is a $g_j \in \mathrm{C}^{s-1}(\Gamma, M)$ such that $\|g_j\| \le \max(\|f_j\|/c, \|df_j\|) \le \|f_j\|/c$ and $\|\pi^2 f_j - dg_j\| \le \|df_j\|/c \le \|f\|/(Rc^2)$.

Now, put $g = \sum_{i=1}^n e_i \otimes g_i$. Then $\|g\| \le \|f\|/c$. On the other hand, we have

$$\pi^2 f - dg = \sum_{i=1}^n e_i \otimes (\pi^2 f_i - dg_i) + \Big(\sum_{i=1}^n e_i \otimes (dg_i - dg)\Big).$$

The first term on the right-hand side is bounded by $\|f\|/(Rc^2)$ and the second term is bounded by $\|g\|/(Rc) \le \|f\|/(Rc^2)$. Thus $\|\pi^2 f - dg\|$ is bounded by $\|f\|/(Rc^2)$. Then $h_1 := g/\pi$ belongs to $\mathrm{C}^{s-1}(\Gamma, (L\otimes_A M))$ such that $\|h_1\| \le \|f\|$ and that $\|\pi f - dh_1\| \le \|f\|/(Rc)$.

Assume we have already $h_1, h_2, \dots, h_t \in \mathrm{C}^{s-1}(\Gamma, L\otimes_A M)$ satisfying

$$\|h_j\| \le \frac{\|f\|}{R^{j-1}} \quad \text{and} \quad \Big\|\pi f - \sum_{i=1}^j dh_i\Big\| \le \frac{\|f\|}{R^j c}, \qquad \text{for all } 1 \le j \le t.$$

Then $f - \pi^{-1}\sum_{i=1}^t dh_i \in \mathrm{C}^s(\Gamma, L\otimes_A M)$ with norm $\|f - \pi^{-1}\sum_{i=1}^t dh_i\| \le \|f\|/R^t$. Replacing f by $f - \pi^{-1}\sum_{i=1}^t dh_i$ and proceeding as above, we get an $h_{t+1} \in \mathrm{C}^{s-1}(\Gamma, L\otimes_A M)$ with norm $\|h_{t+1}\| \le \big\|f - \pi^{-1}\sum_{i=1}^t dh_i\big\| \le \|f\|/R^t$ such that

$$\Big\|\pi f - \sum_{i=1}^t dh_i - dh_{t+1}\Big\| \le \frac{\big\|f - \pi^{-1}\sum_{i=1}^t dh_i\big\|}{Rc} \le \frac{\|f\|}{R^{t+1}c}.$$

Then $\sum_{i=1}^{+\infty} h_i$ converges to an element $h \in \mathrm{C}^{s-1}(\Gamma, L\otimes_A M)$ such that $\pi f = dh$ and that $\|h\| \le \sup_{j\ge1}(\|h_j\|) \le \|f\|$. This implies (1). □

The following lemma is a generalisation of [Diao et al. 2023b, Lemma A.1.14] whose proof is similar.

Lemma 3.12. *Let $A \to B$ be an isometry of Banach $\mathcal{O}_{\mathbb{C}_p}$-algebras. Suppose the natural projection $\mathrm{pr}: B \to B/A$ admits an isometric section $s: B/A \to B$ as Banach modules over A. Then, for all $b_1, b_2 \in B$, we have*

$$\|\mathrm{pr}(b_1b_2)\| \le \max(\|b_1\|\,\|\mathrm{pr}(b_2)\|, \|b_2\|\,\|\mathrm{pr}(b_1)\|)$$

We shall apply this lemma to the inclusion $R^+ \to \widehat{R}^+_\infty$.

Lemma 3.13. *Denote the triple $(R^+, \widehat{R}^+_\infty)$ by (A, B) for simplicity. Let f be a cocycle in $\mathrm{C}^1(\Gamma, \mathrm{GL}_n(B))$. Suppose there exists an $R > 1$ such that $\|f(\gamma) - 1\| \le 1/(Rc)$ for all $\gamma \in \Gamma$. Let $\bar{f}$ be the image of f in $\mathrm{C}^1(\Gamma, \mathrm{M}_n(B/A))$ (which is not necessary a cocycle). If $\|\bar{f}\| \le 1/(Rc^2)$, then there exists a cocycle $f' \in \mathrm{C}^1(\Gamma, \mathrm{GL}_n(A))$ which is equivalent to f such that $\|f'(\gamma) - 1\| \le 1/(Rc)$ for all $\gamma \in \Gamma$.*

Proof. We proceed as in the proof of [Diao et al. 2023b, Lemma A.1.15]. It is enough to show that there exists an $h \in \mathrm{M}_n(B)$ with $\|h\| \le c\,\|\bar{f}\|$ such that the cocycle

$$g : \gamma \mapsto \gamma(1+h)f(\gamma)(1+h)^{-1}$$

satisfies $\|g(\gamma) - 1\| \le 1/(Rc)$ for all $\gamma \in \Gamma$ and $\|\bar{g}\| \le \|\bar{f}\|/R$ in $\mathrm{C}^1(\Gamma, \mathrm{M}_n(B/A))$.

Granting the claim, by iterating this process, we can find a sequence $h_1, h_2, \dots$ in $\mathrm{M}_n(B)$ with $\|h_n\| \le (c\,\|\bar{f}\|)/R^{n-1} \le 1/(cR^n)$ such that

$$\overline{\gamma\Big(\prod_{i=1}^{n}(1+h_i)\Big)f(\gamma)\Big(\prod_{i=1}^{n}(1+h_i)\Big)^{-1}} \le \frac{\|\bar{f}\|}{R^n}.$$

Set $h = \prod_{i=1}^{+\infty}(1+h_i) \in \mathrm{GL}_n(B)$. Then we have a cocycle

$$f' : \gamma \mapsto \gamma(h)f(\gamma)h^{-1}$$

taking values in $\mathrm{M}_n(A) \cap \mathrm{GL}_n(B)$ such that $\|f'(\gamma) - 1\| \le 1/(Rc)$ for every $\gamma \in \Gamma$. Thus $f' \in \mathrm{GL}_n(A)$ and we prove the lemma.

Now, we prove the claim. Since $f \in \mathrm{C}^1(\Gamma, \mathrm{GL}_n(B))$ is a cocycle, for all $\gamma_1, \gamma_2 \in \Gamma$, we have $f(\gamma_1\gamma_2) = \gamma_1(f(\gamma_2))f(\gamma_1)$. Using Lemma 3.12, we get

$$\begin{aligned} \|d\bar{f}(\gamma_1, \gamma_2)\| &= \|\overline{\gamma_1 f(\gamma_2) + f(\gamma_1) - f(\gamma_1\gamma_2)}\| \\ &= \|\overline{(\gamma_1 f(\gamma_2) - 1)(f(\gamma_1) - 1) - 1}\| \\ &= \|\overline{(\gamma_1 f(\gamma_2) - 1)(f(\gamma_1) - 1)}\| \le \frac{\|\bar{f}\|}{Rc}. \end{aligned} \tag{3-1}$$

Since $\|\bar{f}\| \le 1/(Rc^2)$, we can apply Lemma 3.10 to $\pi^{-2}\bar{f}$ and get an $\bar{h} \in \mathrm{M}_n(B/A)$ such that

$$\|\bar{h}\| \le \max\bigg(\frac{\|\pi^{-2}\bar{f}\|}{c}, \|\pi^{-2}d\bar{f}\|\bigg) \le \max(c\|\bar{f}\|, c^2\|d\bar{f}\|) \le c\|\bar{f}\| \le \frac{1}{Rc}.$$

and that

$$\|\bar{f} - d\bar{h}\| \le \frac{\|\pi^{-2}d\bar{f}\|}{c} \le c\,\|d\bar{f}\| \le \frac{\|\bar{f}\|}{R}. \tag{3-2}$$

By assumption, we can lift $\bar{h}$ to an $h \in \mathrm{M}_n(B)$ such that $\|h\| = \|\bar{h}\| \le c\,\|\bar{f}\|$. It follows that for all $\gamma \in \Gamma$, we have

$$\|\gamma(1+h)f(\gamma)(1+h)^{-1} - f(\gamma)\| \le \|h\| \le \frac{1}{Rc}$$

and, therefore,

$$\|\gamma(1+h)f(\gamma)(1+h)^{-1} - 1\| \le \frac{1}{Rc}.$$

Moreover, we have

$$\|\overline{\gamma(1+h)f(\gamma)(1+h)^{-1} - \gamma(1+h)f(\gamma)(1-h)}\| \le \|\bar{h}^2\| \le \frac{c\|\bar{f}\|}{Rc} = \frac{\|\bar{f}\|}{R}. \tag{3-3}$$

By Lemma 3.12, we have

$$\begin{aligned}\|\overline{\gamma(1+h)f(\gamma)(1-h)} - \bar{f}(\gamma) - \gamma(\bar{h}) + \bar{h}\| \\ = \|\overline{\gamma(h)(f(\gamma)-1)} - \overline{(f(\gamma)-1)h} - \overline{\gamma(h)f(\gamma)h}\| \le \frac{\|\bar{f}\|}{R}.\end{aligned} \tag{3-4}$$

Combining (3-2), (3-3) and (3-4), we conclude that

$$\|\overline{\gamma(1+h)f(\gamma)(1+h)^{-1}}\| \le \frac{\|\bar{f}\|}{R}$$

which proves the claim as desired. □

Now we are able to prove Proposition 3.5.

Proof of Proposition 3.5. (1) Since $a > r$, we may choose $s > 1$ such that $\|p^{a+r}\| = 1/(sc^2)$. By our assumptions, a basis $\{e_1, e_2, \dots, e_n\}$ of M_∞ determines a cocycle $f \in \mathrm{C}^1(\Gamma, \mathrm{GL}_n(\widehat{R}^+_\infty))$ satisfying $\|f(\gamma)-1\| \le 1/(sc^2)$. In particular, f satisfies the hypothesis of Lemma 3.13. Thus there exists a cocycle $f' \in \mathrm{C}^1(\Gamma, R^+)$ which is equivalent to f such that

$$\|f'(\gamma) - 1\| \le \frac{1}{sc}, \quad \text{for all } \gamma \in \Gamma.$$

Then the cocycle f' defines a finite free sub-R^+-module M of rank n such that

$$M \otimes_{R^+} \widehat{R}^+_\infty \cong M_\infty.$$

(2) By (1), we have $M_\infty \cong M \oplus M \otimes_{R^+} (\widehat{R}^+_\infty/R^+)$. Applying Lemma 3.11(1) to $L = M$, we deduce that $\mathrm{H}^i(\Gamma, M \otimes_{R^+} \widehat{R}^+_\infty/R^+)$ is killed by π for every $i \ge 0$. But $\mathrm{H}^0(\Gamma, M_\infty) = M_\infty^\Gamma$ is π-torsion free, so we get

$$\mathrm{H}^0(\Gamma, M_\infty) = \mathrm{H}^0(\Gamma, M)$$

and complete the proof. □

Up to now, we have constructed a decompletion functor from the category of $(a+r)$-trivial $\widehat{R}^+_\infty$-representations of Γ to the category of essentially $(a+r)$-trivial R^+-representations of Γ. Now Theorem 3.4 follows from the next proposition directly.

Proposition 3.14. *Every essentially $(a+r)$-trivial R^+-representation of Γ is $(a+r)$-trivial.*

We give the proof of this proposition in the next subsection.

Essentially* $(a+r)$*-trivial representation is* $(a+r)$*-trivial. Throughout this subsection, we always assume $a > r$. For any R^+-module N with a continuous Γ-action, we denote $\mathrm{H}^i(\Gamma, N)$ by $\mathrm{H}^i(N)$ for simplicity.

Now for a fixed essentially $(a+r)$-trivial R^+-representation M of Γ of rank n, we define

$$M_\infty = M \otimes_{R^+} \widehat{R}^+_\infty.$$

Then it is $(a+r)$-trivial and of the form $M_\infty = M \oplus M_{\mathrm{cp}}$ for $M_{\mathrm{cp}} = M \otimes_{R^+} \widehat{R}^+_\infty / R^+$. Since M is a-trivial, by Lemma 3.11, we see that $\mathrm{R}\Gamma(\Gamma, M_{\mathrm{cp}})$ is concentrated in positive degrees and is killed by π. As a consequence, for any $h \geq r$, we have

$$\mathrm{R}\Gamma(\Gamma, M_{\mathrm{cp}}/p^h) \simeq \mathrm{R}\Gamma(\Gamma, M_{\mathrm{cp}})[1].$$

In particular, $\mathrm{R}\Gamma(\Gamma, M_{\mathrm{cp}}/p^h)$ is killed by π. So we deduce that

$$\pi \mathrm{H}^0(M_\infty/p^h) \cong \pi \mathrm{H}^0(M/p^h).$$

Replacing M by $(\widehat{R}^+_\infty)^l$, we get

$$\pi \mathrm{H}^0(\widehat{R}^+_\infty/p^h)^n \cong \pi \mathrm{H}^0(R^+/p^h)^n = (\pi R^+/p^h)^n.$$

Since M_∞ is $(a+r)$-trivial, choose $h = a + r$ and we get

$$\pi \mathrm{H}^0(M/p^{a+r}) \cong \pi \mathrm{H}^0(M_\infty/p^{a+r}) \cong \pi \mathrm{H}^0(\widehat{R}^+_\infty/p^{a+r})^n \cong (\pi R^+/p^{a+r})^n \cong (R^+/p^a)^n.$$

Thus, $\pi \mathrm{H}^0(M/p^{a+r})$ is a free R^+/p^a-module of rank n.

Choose $g_1, \dots g_n \in \mathrm{H}^0(M/p^{a+r})$ such that $\pi g_1, \dots, \pi g_n$ is an R^+/p^a-basis of $\pi \mathrm{H}^0(M/p^{a+r})$. We claim that the sub-$R^+/p^{a+r}$-module

$$\sum_{i=1}^n R^+/p^{a+r} \cdot g_i \subset \mathrm{H}^0(M/p^{a+r})$$

is free. For any i, let $\tilde{g}_i \in M$ be a lifting of g_i. Assume $x_1, \dots, x_n \in R^+$ such that

$$\sum_{i=1}^n x_i \tilde{g}_i \equiv 0 \mod p^{a+r}.$$

Then

$$\sum_{i=1}^n x_i \pi \tilde{g}_i \equiv 0 \mod p^{a+r}.$$

By the choice of g_i's, we deduce that $x_i \in p^a R^+$ for any i. Write $x_i = \pi y_i$ for some $y_i \in R^+$. Then

$$\sum_{i=1}^n y_i \pi \tilde{g}_i \equiv 0 \mod p^{a+r}.$$

So $y_i \in p^a R^+$ and hence $x_i \in p^{a+r} R^+$ for all i. This proves the claim.

It remains to prove $\tilde{g}_1, \dots, \tilde{g}_n$ is an R^+-basis of M. Let $e_1, \dots, e_n$ be an R^+-basis of M. Since M is a-trivial, we get

$$M/p^a = \mathrm{H}^0(M/p^a) = \sum_{i=1}^{n} R^+/p^a e_i.$$

So $\pi e_1, \dots, \pi e_n$ is an R^+/p^{a-r}-basis of $\pi M/p^a$. However, by the choice of $\tilde{g}_i$'s, $\pi \tilde{g}_1, \dots, \pi \tilde{g}_n$ is also an R^+/p^{a-r}-basis of $\pi M/p^a$. Since $a > r$, we deduce that $\tilde{g}_i$'s generate M as an R^+-module. This completes the proof.

4. Local Simpson correspondence

In this section, we establish an equivalence between the category of a-small representations of Γ over $\widehat{R}^+_\infty$ and the category of a-small Higgs modules over R^+. This is a local p-adic Simpson correspondence. Throughout this section, put $r = 1/(p-1)$.

Definition 4.1. Assume $a > r$ and $A \in \{R^+, \widehat{R}^+_\infty\}$. We say a representation M of Γ over A is *a-small* if it is $(a+\nu_p(\rho_k))$-trivial in the sense of Definition 3.3.

Definition 4.2. By a *Higgs module* over R^+, we mean a finite free R^+-module H together with an R^+-linear morphism $\theta : H \to H \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1)$ such that $\theta \wedge \theta = 0$. A Higgs module (H, θ) is called *a-small*, if θ is divided by $p^{a+\nu_p(\rho_k)}$; that is,

$$\mathrm{Im}(\theta) \subset p^{a+\nu_p(\rho_k)} H \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1).$$

Let $S^{\dagger,+}_\infty$ with the canonical Higgs field Θ be as in Corollary 2.23. For an a-small representation M over $\widehat{R}^+_\infty$, define

$$\Theta_M = \mathrm{id}_M \otimes \Theta : M \otimes_{\widehat{R}^+_\infty} S^{\dagger,+}_\infty \to M \otimes_{\widehat{R}^+_\infty} S^{\dagger,+}_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1). \tag{4-1}$$

Then it is a Higgs field on $M \otimes_{\widehat{R}^+_\infty} S^{\dagger,+}_\infty$. We denote the induced Higgs complex by $\mathrm{HIG}(H \otimes_{R^+} S^{\dagger,+}_\infty, \Theta_H)$. For an a-small Higgs module (H, θ_H), define

$$\Theta_H = \theta_H \otimes \mathrm{id} + \mathrm{id}_H \otimes \Theta : H \otimes_{R^+} S^{\dagger,+}_\infty \to H \otimes_{R^+} S^{\dagger,+}_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1). \tag{4-2}$$

Then Θ_H is a Higgs field on $H \otimes_{R^+} S^{\dagger,+}_\infty$. We denote the induced Higgs complex by $\mathrm{HIG}(H \otimes_{R^+} S^{\dagger,+}_\infty, \Theta_H)$. The main theorem in this section is the following local Simpson correspondence.

Theorem 4.3 (local Simpson correspondence). *Assume $a > r$.*

(1) *Let M be an a-small $\widehat{R}^+_\infty$-representation of Γ of rank l. Let $H(M) := (M \otimes_{\widehat{R}^+_\infty} S^{\dagger,+}_\infty)^\Gamma$ and $\theta_{H(M)}$ be the restriction of Θ_M to $H(M)$. Then $(H(M), \theta_{H(M)})$ is an a-small Higgs module of rank l.*

(2) *Let (H, θ_H) be an a-small Higgs module of rank l over R^+. Put $M(H, \theta_H) = (H \otimes_{R^+} S^{\dagger,+}_\infty)^{\Theta_H = 0}$. Then $M(H, \theta_H)$ is an a-small $\widehat{R}^+_\infty$-representation of Γ of rank l.*

(3) *The functor* $M\mapsto (H(M),\theta_{H(M)})$ *induces an equivalence from the category of a-small* $\widehat{R}^+_\infty$*-representations of* Γ *to the category of a-small Higgs modules over* R^+*, whose quasi-inverse is given by* $(H,\theta_H)\mapsto M(H,\theta_H)$*. The equivalence preserves tensor products and dualities.*

(4) *Let* M *be an a-small* $\widehat{R}^+_\infty$*-representation of* Γ *and* (H,θ_H) *be the corresponding Higgs module. Then there is a canonical* Γ*-equivariant isomorphism of Higgs complexes*

$$\mathrm{HIG}(H\otimes_{R^+}S^{\dagger,+}_\infty,\Theta_H)\to \mathrm{HIG}(M\otimes_{\widehat{R}^+_\infty}S^{\dagger,+}_\infty,\Theta_M).$$

Also, there is a canonical quasi-isomorphism

$$\mathrm{R}\Gamma\Big(\Gamma, M\Big[\frac{1}{p}\Big]\Big)\simeq \mathrm{HIG}\Big(H\Big[\frac{1}{p}\Big],\theta_H\Big),$$

where $\mathrm{HIG}\big(H\big[\frac{1}{p}\big],\theta_H\big)$ *is the Higgs complex induced by* (H,θ_H)*.*

The following corollary follows from Theorems 3.4 and 4.3 directly.

Corollary 4.4. *Assume* $a>r$*. The following categories are equivalent*:

(1) *The category of a-small representations of* Γ *over* R^+*.*

(2) *The category of a-small representations of* Γ *over* $\widehat{R}^+_\infty$*.*

(3) *The category of a-small Higgs modules over* R^+*.*

In order to prove the theorem, we need to compute $\mathrm{R}\Gamma(\Gamma, M\otimes_{\widehat{R}^+_\infty}S^{\dagger,+}_\infty)$. By Corollary 2.23, we are reduced to computing $\mathrm{R}\Gamma(\Gamma, M\otimes_{\widehat{R}^+_\infty}\widehat{R}^+_\infty\langle\rho Y_1,\dots,\rho Y_d\rangle)$ for any $\rho\in\rho_k\mathcal{O}_{\mathbb{C}_p}$. So before we move on, let us fix some notation to simplify the calculation.

For any $n\geq 0$, define

$$F_n(Y)=n!\binom{Y}{n}=Y(Y-1)\cdots(Y-n+1)\in\mathbb{Z}[Y].$$

For any $\alpha\in\mathbb{N}\big[\frac{1}{p}\big]\cap(0,1)$, define $\epsilon_\alpha=1-\zeta^{-\alpha}$. Then $\nu_p(\rho_k)\geq r\geq\nu_p(\epsilon_\alpha)$.

Calculation in trivial representation case. We are going to compute $\mathrm{R}\Gamma(\Gamma,\widehat{R}^+_\infty\langle\rho Y_1,\dots,\rho Y_d\rangle)$ in this subsection. We assume $d=1$ first. In this case, $\Gamma=\mathbb{Z}_p\gamma$ and acts on $\widehat{R}^+_\infty\langle\rho Y\rangle$ via $\gamma(Y)=Y+1$. Note that $\{\rho^nF_n\}_{n\geq 0}$ is a set of topological $\widehat{R}^+_\infty$-basis of $\widehat{R}^+_\infty\langle\rho Y\rangle$ and, for any $n\geq 0$,

$$\gamma(\rho^nF_n)=\rho^nF_n+n\rho\cdot\rho^{n-1}F_{n-1}.$$

So we get a γ-equivariant decomposition

$$\widehat{R}^+_\infty\langle\rho Y\rangle=\widehat{\bigoplus}_{\alpha\in\mathbb{N}[\frac{1}{p}]\cap[0,1)}R^+\langle\rho Y\rangle\cdot T^\alpha.$$

So it suffices to compute $\mathrm{R}\Gamma(\Gamma, R^+\langle\rho Y\rangle\cdot T^\alpha)$ for any α. We only need to consider the Koszul complex

$$\mathrm{K}(R^+\langle\rho Y\rangle\cdot T^\alpha;\gamma-1):\ \ R^+\langle\rho Y\rangle\cdot T^\alpha\xrightarrow{\gamma-1}R^+\langle\rho Y\rangle\cdot T^\alpha.$$

Note that for any α, $\{\rho^n F_n T^\alpha\}_{n\geq 0}$ is a set of topological R^+-basis of $R^+\langle\rho Y\rangle T^\alpha$. So we have

$$(\gamma-1)(\rho^n F_n T^\alpha) = \begin{cases} n\rho\cdot\rho^{n-1}F_{n-1}, & \alpha=0,\\ \zeta^\alpha\epsilon_\alpha T^\alpha\big(\rho^n F_n + n\frac{\rho}{\epsilon_\alpha}\rho^{n-1}F_{n-1}\big), & \alpha\neq 0.\end{cases} \tag{4-3}$$

Put $\Lambda_\rho=\{\rho^n F_n\}_{n\geq 0}$ and $I_\rho=\{\nu_p(\rho(n+1))\}_{n\geq 0}$. Let $R^+\langle\Lambda_\rho\rangle$ and $R^+\langle\Lambda_\rho, I_\rho, +\rangle$ be as in Definition A.1. Then by (4-3), we see that

$$(\gamma-1)(R^+\langle\rho Y\rangle) = R^+\langle\Lambda_\rho, I_\rho, +\rangle$$

and that

$$(\gamma-1)(R^+\langle\rho Y\rangle T^\alpha) \sim \left\{\zeta^\alpha\epsilon_\alpha\left(\rho^n F_n + n\frac{\rho}{\epsilon_\alpha}\rho^{n-1}F_{n-1}\right)\right\}_{n\geq 0}$$

in the sense of Definition A.4. By Proposition A.5, we get

$$(\gamma-1)(R^+\langle\rho Y\rangle T^\alpha) = \epsilon_\alpha(R^+\langle\rho Y\rangle T^\alpha).$$

In summary, we see that for $\alpha\neq 0$, $\mathrm{H}^1(\mathbb{Z}_p\gamma, R^+\langle\rho Y\rangle T^\alpha)$ is killed by ϵ_α and that for $\alpha=0$,

$$\mathrm{H}^1(\mathbb{Z}_p\gamma, R^+\langle\rho Y\rangle) = R^+\langle\rho Y\rangle / R^+\langle\Lambda_\rho, I_\rho, +\rangle.$$

So, keeping the notation as above, we have the following lemma.

Lemma 4.5. (1) *The inclusion* $R^+\langle\rho Y\rangle \hookrightarrow \widehat{R}^+_\infty\langle\rho Y\rangle$ *identifies* $\mathrm{R}\Gamma(\Gamma, R^+\langle\rho Y\rangle)$ *with a direct summand of* $\mathrm{R}\Gamma(\mathbb{Z}_p\gamma, \widehat{R}^+_\infty\langle\rho Y\rangle)$ *whose complement is concentrated in degree* 1 *and is killed by* ζ_p-1.

(2) $\mathrm{H}^0(\Gamma, R^+\langle\rho Y\rangle) = R^+$ *is independent of* ρ.

(3) $\mathrm{H}^1(\Gamma, R^+\langle\rho Y\rangle) = R^+\langle\rho Y\rangle/R^+\langle\Lambda_\rho, I_\rho, +\rangle$ *is the derived* p*-adic completion of*

$$\bigoplus_{i\geq 0} R^+/(i+1)\rho R^+.$$

Proof. It remains to compute $\mathrm{H}^0(\Gamma, R^+\langle\rho Y\rangle T^\alpha)$.

When $\alpha\neq 0$, assume $\sum_{n\geq 0} a_n\rho^n F_n T^\alpha$ is γ-invariant. Then we have

$$\sum_{n\geq 0}\zeta^\alpha\epsilon_\alpha\left(a_n + \frac{\rho}{\epsilon_\alpha}(n+1)a_{n+1}\right)\rho^n F_n T^\alpha = 0.$$

This implies that, for any $n\geq 0$ and any $m\geq 0$,

$$a_n = (-1)^m\prod_{j=1}^m\left(\frac{\rho}{\epsilon_\alpha}(n+j)\right)a_{n+m}.$$

In particular, $\nu_p(a_n)\geq\sum_{j=1}^m\nu_p(n+j)$ for any $m\geq 0$. This forces $a_n=0$ for any $n\geq 0$.

When $\alpha=0$, assume $\sum_{n\geq 0} a_n\rho^n F_n$ is γ-invariant. Then we have

$$\sum_{n\geq 0}(n+1)\rho a_{n+1}\rho^n F_n = 0,$$

which implies $a_n=0$ for any $n\geq 1$. So we have $R^+\langle\rho Y\rangle^\Gamma = R^+$. □

Now we are able to handle the higher dimensional case.

Lemma 4.6. *Identify $\widehat{S}^+_{\infty,\rho}$ with $\widehat{R}^+_\infty\langle\rho Y_1,\dots,\rho Y_d\rangle$.*

(1) *The inclusion $R^+\langle\rho\underline{Y}\rangle\hookrightarrow\widehat{S}^+_{\infty,\rho}$ identifies $\mathrm{R}\Gamma(\Gamma,R^+\langle\rho\underline{Y}\rangle)$ with a direct summand of $\mathrm{R}\Gamma(\Gamma,\widehat{S}^+_{\infty,\rho})$ whose complement is concentrated in degree ≥ 1 and is killed by ζ_p-1.*

(2) *For any $i\geq 0$, we have*

$$\mathrm{H}^i(\Gamma,R^+\langle\rho\underline{Y}\rangle)=\bigwedge\nolimits^i_{R^+}\Big(\bigoplus_{j=1}^d R^+\langle\rho Y_j\rangle/R^+\langle\Lambda_{\rho,j},I_\rho,+\rangle\Big)$$

for $\Lambda_{\rho,j}=\{\rho^nF_n(Y_j)\}$ and $I_\rho=\{\nu_p((n+1)\rho)\}_{n\geq 0}$.

Proof. Note that $\mathrm{R}\Gamma(\Gamma,\widehat{R}^+_\infty\langle\rho Y_1,\dots,\rho Y_d\rangle)$ is presented by the Koszul complex

$$\mathrm{K}(\widehat{R}^+_\infty\langle\rho Y_1,\dots,\rho Y_d\rangle;\gamma_1-1,\dots,\gamma_d-1)\simeq\mathrm{K}(\widehat{R}^+_\infty\langle\rho Y_1\rangle;\gamma_1-1)\widehat{\otimes}^L_{\widehat{R}^+_\infty}\cdots\widehat{\otimes}^L_{\widehat{R}^+_\infty}\mathrm{K}(\widehat{R}^+_\infty\langle\rho Y_d\rangle;\gamma_d-1).$$

Since $R^+\langle\rho Y_j\rangle/R^+\langle\Lambda_{\rho,j},I_\rho,+\rangle$ is already derived p-complete, the lemma follows from Lemma 4.5 directly. □

Proposition 4.7. (1) $(S^{\dagger,+}_\infty)^\Gamma=R^+$.

(2) *For any $i\geq 1$, $\mathrm{H}^i(\Gamma,S^{\dagger,+}_\infty)$ is p^∞-torsion.*

Proof. We only need to show, for any $i\geq 1$,

$$\varinjlim_{\nu_p(\rho)>\nu_p(\rho_k)}\mathrm{H}^i(\Gamma,\widehat{S}^+_{\infty,\rho})$$

is p^∞-torsion. However, by Lemma 4.6, this follows from a similar argument as in the proof of Corollary 2.23. □

Calculation in general case. Now, by virtues of Theorem 3.4, we may assume that M is an a-small representation of Γ over R^+. Let $e_1,\dots,e_l$ be an R^+-basis of M and A_j be the matrix of γ_j with respect to the chosen basis for all $1\leq j\leq d$; that is,

$$\gamma_j(e_1,\dots,e_l)=(e_1,\dots,e_l)A_j.$$

Put $B_j=A_j-I$. It is the matrix of γ_j-1 and has p-adic valuation $\nu_p(B_j)\geq a+\nu_p(\rho_k)$ by a-smallness of M. Similar to the trivial representation case, we are reduced to computing $\mathrm{R}\Gamma(\Gamma,M\otimes_{R^+}\widehat{R}^+_\infty\langle\rho Y_1,\dots,\rho Y_d\rangle)$. Note that we still have a Γ-equivariant decomposition

$$M\otimes_{R^+}\widehat{R}^+_\infty\langle\rho Y_1,\dots,\rho Y_d\rangle=\widehat{\bigoplus}_{\underline{\alpha}\in(\mathbb{N}[\frac{1}{p}]\cap[0,1))^d}M\otimes_{R^+}R^+\langle\rho Y_1,\dots,\rho Y_d\rangle\underline{T}^{\underline{\alpha}},$$

where $\underline{T}^{\underline{\alpha}}$ denotes $T_1^{\alpha_1}\cdots T_d^{\alpha_d}$ for any $\underline{\alpha}=(\alpha_1,\dots,\alpha_d)$.

Assume $\underline{\alpha}\neq 0$ at first. Without loss of generality, we assume $\alpha_d\neq 0$. Note that

$$\{e_{i,n}:=e_i\rho^nF_n(Y_d)\underline{T}^{\underline{\alpha}}\}_{1\leq i\leq l,n\geq 0}$$

is a set of topological basis of $M\otimes_{R^+}R^+\langle\rho Y_1,\dots,\rho Y_d\rangle\underline{T}^{\underline{\alpha}}$ over $R^+\langle\rho Y_1,\dots,\rho Y_{d-1}\rangle$. We have

$$(\gamma_d-1)(e_{1,n},\dots,e_{l,n})=\zeta^{\alpha_d}\epsilon_{\alpha_d}\Big((e_{1,n},\dots,e_{l,n})\cdot(\epsilon_{\alpha_d}^{-1}B_d+I)+(e_{1,n-1},\dots,e_{l,n-1})\cdot n\frac{\rho}{\epsilon_{\alpha_d}}A_d\Big).$$

Similar to the trivial representation case, using Proposition A.6, we deduce that

$$\mathrm{R}\Gamma(\mathbb{Z}_p\gamma_d,M\otimes_{R^+}R^+\langle\rho Y_1,\dots,\rho Y_d\rangle\underline{T}^{\underline{\alpha}})\simeq M\otimes_{R^+}R^+\langle\rho Y_1,\dots,\rho Y_d\rangle\underline{T}^{\underline{\alpha}}/\epsilon_{\alpha_d}[-1].$$

Using the Hochschild–Serre spectral sequence, we have the following lemma.

Lemma 4.8. *Assume $\underline{\alpha}\neq 0$. Then the complex $\mathrm{R}\Gamma(\Gamma,M\otimes_{R^+}R^+\langle\rho Y_1,\dots,\rho Y_d\rangle\underline{T}^{\underline{\alpha}})$ is concentrated in positive degrees and is killed by ζ_p-1.*

Now, we focus on the $\underline{\alpha}=0$ case and prove the following proposition.

Proposition 4.9. *Keep the notation as above. Assume $\nu_p(\rho)<a+\nu_p(\rho_k)-r$. Define*

$$H(M):=(M\otimes_{R^+}R^+\langle\rho Y_1,\dots,\rho Y_d\rangle)^{\Gamma}.$$

Then the following assertions are true:

(1) *$H(M)$ is a finite free R^+-module of rank l and is independent of the choice of ρ. More precisely*, *if we define*

$$(h_1,\dots,h_l)=(e_1,\dots,e_l)\sum_{n_1,\dots,n_d\geq 0}\prod_{i=1}^{d}\frac{(-A_i^{-1}B_i)^{n_i}}{n_i!}F_{n_i}(Y_i),$$

then $h_1,\dots,h_l$ is an R^+-basis of $H(M)$.

(2) *The inclusion $H(M)\hookrightarrow M\otimes_{R^+}R^+\langle\rho Y_1,\dots,\rho Y_d\rangle$ induces a Γ-equivariant isomorphism*

$$H(M)\otimes_{R^+}R^+\langle\rho Y_1,\dots,\rho Y_d\rangle\cong M\otimes_{R^+}R^+\langle\rho Y_1,\dots,\rho Y_d\rangle.$$

Proof. We first consider the $d=1$ case. In this case, $\Gamma=\mathbb{Z}_p\gamma$ acts on $R^+\langle\rho Y\rangle$ via $\gamma(Y)=Y+1$. Let $e_1,\dots,e_l$ be a basis of M and A be the matrix of γ associated to the chosen basis. Put $B=A-I$ and then $\nu_p(B)\geq a+\nu_p(\rho_k)>\nu_p(\rho)+r$. Note that $\{\rho^nF_n(Y)\}_{n\geq 0}$ is a set of topological basis of $R^+\langle\rho Y\rangle$.

(1) Assume $x=\sum_{n\geq 0}\underline{e}X_n\rho^nF_n(Y)\in H(M)$, where $X_n\in(R^+)^l$ for any $n\geq 0$ and $\underline{e}$ denotes $(e_1,\dots,e_l)$. Since $\gamma(x)=x$, we deduce that, for any $n\geq 0$,

$$BX_n=-(n+1)\rho AX_{n+1}.$$

In other words, we have

$$X_n=\frac{-A^{-1}B}{n\rho}X_{n-1}=\frac{(-A^{-1}B)^n}{\rho^nn!}X_0.$$

Note that $\nu_p((A^{-1}B)^n/(\rho^nn!))\geq(a+\nu_p(\rho_k)-r-\nu_p(\rho))n$. So we get $(A^{-1}B)^n/(\rho^nn!)\in\mathrm{M}_l(R^+)$ and hence X_n is uniquely determined by X_0. In particular, we have

$$x=\underline{e}\sum_{n\geq 0}\frac{(-A^{-1}B)^n}{\rho^nn!}\rho^nF_n(Y)X_0=\underline{e}\sum_{n\geq 0}\frac{(-A^{-1}B)^n}{n!}F_n(Y)X_0. \tag{4-4}$$

Conversely, any $x \in M \otimes_{R^+} R^+\langle \rho Y_1, \dots, \rho Y_d\rangle$ which is of the form (4-4) for some $X_0 \in (R^+)^l$ is γ-invariant. So we are done.

(2) From the proof of (1), we see that $\sum_{n\geq 0}((-A^{-1}B)^n/(\rho^n n!))\rho^n F_n(Y) \in \mathrm{GL}_l(R^+\langle \rho Y\rangle)$. Thus the h_i's form an $R^+\langle \rho Y\rangle$-basis of $M \otimes_{R^+} R^+\langle \rho Y\rangle$ as desired.

Now, we handle the case for any $d \geq 1$. By what we have proved and by iterating, we get

$$\begin{aligned}\underline{e}(R^+\langle \rho Y_1, \dots, \rho Y_d\rangle)^l &= \underline{e}\sum_{n_d\geq 0}\frac{(-A_d^{-1}B_d)^{n_d}}{n_d!}F_{n_d}(Y_d)(R^+\langle \rho Y_1, \dots, \rho Y_d\rangle)^l \\ &= \underline{e}\sum_{n_{d-1},n_d\geq 0}\frac{(-A_{d-1}^{-1}B_{d-1})^{n_{d-1}}}{n_{d-1}!}F_{n_{d-1}}(Y_{d-1})\frac{(-A_d^{-1}B_d)^{n_d}}{n_d!}F_{n_d}(Y_d)(R^+\langle \rho Y_1, \dots, \rho Y_d\rangle)^l \\ &= \cdots \\ &= \underline{e}\sum_{n_1,\dots,n_d\geq 0}\prod_{i=1}^d\frac{(-A_i^{-1}B_i)^{n_i}}{n_i!}F_{n_i}(Y_i)(R^+\langle \rho Y_1, \dots, \rho Y_d\rangle)^l.\end{aligned}$$

Since $\underline{e}\sum_{n_1,\dots,n_d\geq 0}\prod_{i=1}^d((-A_i^{-1}B_i)^{n_i}/n_i!)F_{n_i}(Y_i)$ forms a Γ-invariant basis, the result follows from $(R^+\langle \rho Y_1, \dots, \rho Y_d\rangle)^\Gamma = R^+$. □

Remark 4.10. Note that if $\nu_p(z) > r$, then

$$(1+z)^Y = \sum_{n\geq 0}\frac{z^n}{n!}F_n(Y).$$

Therefore, for M and ρ as above, as $\nu_p(A_i^{-1}B_j) \geq a > r$, the operator $\prod_{i=1}^d \gamma_i^{-Y_i}$, whose matrix is given by $\sum_{n_1,\dots,n_d\geq 0}\prod_{i=1}^d((-A_i^{-1}B_i)^{n_i}/n_i!)F_{n_i}(Y_i)$, is well defined on $M\otimes_{R^+} R^+\langle \rho Y_1, \dots, \rho Y_d\rangle$. Then the above proposition says that we have $H(M) = \prod_{i=1}^d \gamma_i^{-Y_i}M$. Since $\log(1+z)(1+z)^Y = \sum_{n\geq 0}(z^n/n!)F_n'(Y)$ when $\nu_p(z) > r$, for any $\underline{e}\vec{m} \in M$ with $\vec{m} \in (R^+)^l$ and $1 \leq j \leq d$, we get

$$\begin{aligned}\frac{\partial}{\partial Y_j}\Big(\prod_{i=1}^d \gamma_i^{-Y_i}\underline{e}\vec{m}\Big) &= \underline{e}\frac{\partial}{\partial Y_j}\Big(\sum_{n_1,\dots,n_d\geq 0}\prod_{i=1}^d\frac{(-A_i^{-1}B_i)^{n_i}}{n_i!}F_{n_i}(Y_i)\vec{m}\Big) \\ &= \underline{e}\sum_{n_1,\dots,n_d\geq 0}\frac{(-A_j^{-1}B_j)^{n_j}}{n_j!}F_{n_j}'(Y_j)\prod_{\substack{1\leq i\leq d\\ i\neq j}}\frac{(-A_i^{-1}B_i)^{n_i}}{n_i!}F_{n_i}(Y_i)\vec{m} \\ &= \underline{e}(-\log(A_j)\sum_{n_1,\dots,n_d\geq 0}\prod_{i=1}^d\frac{(-A_i^{-1}B_i)^{n_i}}{n_i!}F_{n_i}(Y_i)\vec{m}) \\ &= -\log\gamma_j\prod_{i=1}^d\gamma_i^{-Y_i}\underline{e}\vec{m}.\end{aligned}$$

Corollary 4.11. *Keep the notation as above.*

(1) *Denote by $\theta_{H(M)}$ the restriction of Θ to $H(M)$. Then $(H(M), \theta_{H(M)})$ is an a-small Higgs module. Also,* $\theta_{H(M)} = \sum_{i=1}^{d} -\log \gamma_i \otimes ((\mathrm{dlog}\, T_i)/t)$.

(2) *The inclusion $H(M) \to M \otimes_{R^+} S_\infty^{\dagger,+}$ induces a Γ-equivariant isomorphism*

$$H(M) \otimes_{R^+} S_\infty^{\dagger,+} \cong M \otimes_{R^+} S_\infty^{\dagger,+}$$

and identifies the corresponding Higgs complexes

$$\mathrm{HIG}(H(M) \otimes_{R^+} S_\infty^{\dagger,+}, \Theta_{H(M)}) \cong \mathrm{HIG}(M \otimes_{R^+} S_\infty^{\dagger,+}, \Theta_M).$$

Proof. (1) Since $\Theta = \sum_{i=1}^{d} (\partial/\partial Y_i) \otimes ((\mathrm{dlog}\, T_i)/t)$, the "Also" part follows from Remark 4.10. Since $\nu_p(B_i) \geq a + \nu_p(\rho_k)$ for all j and $\log \gamma_j = -\sum_{n\geq 1} (-B_j)^n/n$, we see the a-smallness of $(H(M), \theta_{H(M)})$ as $\nu_p(B_i^n/n) \geq a + \nu_p(\rho_k)$ for all n.

(2) This follows from Proposition 4.9(2) and the definition of $\theta_{H(M)}$. □

We have seen how to achieve an a-small Higgs module from an a-small representation. It remains to construct an a-small representation of Γ from an a-small Higgs module.

Proposition 4.12. *Assume $a > r$. Let (H, θ_H) be an a-small Higgs module of rank l over R^+. Put* $M = (H \otimes_{R^+} S_\infty^{\dagger,+})^{\Theta_H = 0}$.

(1) *The restricted Γ-action on M makes it an a-small $\widehat{R}_\infty^+$-representation of rank l. Also, if $\theta_H = \sum_{i=1}^{d} \theta_i \otimes ((\mathrm{dlog}\, T_i)/t)$, then γ_i acts on M via* $\exp(-\theta_i)$.

(2) *The inclusion $M \hookrightarrow H \otimes_{R^+} S_\infty^{\dagger,+}$ induces a Γ-equivariant isomorphism*

$$M \otimes_{\widehat{R}_\infty^+} S_\infty^{\dagger,+} \cong H \otimes_{R^+} S_\infty^{\dagger,+}$$

and identifies the corresponding Higgs complexes

$$\mathrm{HIG}(M \otimes_{\widehat{R}_\infty^+} S_\infty^{\dagger,+}, \Theta_M) \cong \mathrm{HIG}(H \otimes_{R^+} S_\infty^{\dagger,+}, \Theta_H).$$

Proof. (1) The argument is similar to the proof of Proposition 4.9.

Assume $\rho \in \rho_k \mathfrak{m}_{\mathbb{C}_p}$ such that $a + \nu_p(\rho_k) > \nu_p(\rho) + r$. Let $e_1, \ldots, e_l$ be an R^+-basis of H. We claim that $M = (H \otimes_{R^+} \widehat{R}_\infty^+ \langle \rho Y_1, \ldots, \rho Y_d \rangle)^{\Theta_H = 0}$.

In fact, if $\vec{G} = (G_1, \ldots, G_l)^t \in (\widehat{R}_\infty^+ \langle \rho Y_1, \ldots, \rho Y_d \rangle)^l$ such that $m = \sum_{i=1}^{l} e_i G_i \in M$, then we see that, for any $1 \leq i \leq d$,

$$\theta_i \vec{G} + \frac{\partial \vec{G}}{\partial Y_i} = 0.$$

This forces $\vec{G} = \prod_{i=1}^{d} \exp(-\theta_i Y_i) \vec{a}$ for some $\vec{a} \in (\widehat{R}_\infty^+)^l$. Since $\nu_p(\theta_j) \geq a + \nu_p(\rho_k)$, the matrix $\prod_{i=1}^{d} \exp(-\theta_i Y_i)$ is well defined in $\mathrm{GL}_l(\widehat{R}_\infty^+ \langle \rho Y_1, \ldots, \rho Y_d \rangle)$. This shows that M is finite free of rank l and is independent of the choice of ρ.

Note that $\gamma_i(Y_j) = Y_j + \delta_{ij}$. We see γ_i acts on M via $\exp(-\theta_i)$. Since $\nu_p(\theta_i) \geq a + \nu_p(\rho_k)$, using $\exp(-\theta_i Y_i) = \sum_{n\geq 0} ((-\theta_i)^n/n!) Y_i^n$, we deduce that M is a-small.

(2) This follows from the fact that $\prod_{i=1}^d \exp(-\theta_i Y_i) \in \mathrm{GL}_l(\widehat{R}^+_\infty\langle \rho Y_1, \dots, \rho Y_d\rangle)$ and the definition of Γ-action on M. □

Finally, we complete the proof of Theorem 4.3.

Proof of Theorem 4.3. Part (1) was given in Corollary 4.11. Part (2) was proved in Proposition 4.12. The equivalence part of (3) follows from Corollary 4.11(2) (as the θ_i's act via the $-\log\gamma_i$'s) together with Proposition 4.12(2) (as the γ_i's act via the $\exp(-\theta_i)$'s). Elementary linear algebra shows that the equivalence preserves tensor products and dualities. So we only need to prove the "Also" part of (4).

Let M be an a-small representation of Γ over $\widehat{R}^+_\infty$ and (H, θ_H) be the corresponding Higgs module over R^+. By Corollary 2.23, we have quasi-isomorphisms of complexes over $\widehat{R}_\infty$

$$M\Big[\frac{1}{p}\Big] \xrightarrow{\simeq} \mathrm{HIG}(M \otimes_{\widehat{R}^+_\infty} S^\dagger_\infty, \Theta_M) \simeq \mathrm{HIG}(H \otimes_{R^+} S^\dagger_\infty, \Theta_H).$$

Applying $\mathrm{R}\Gamma(\Gamma, \cdot)$, we get a quasi-isomorphism

$$\mathrm{R}\Gamma\Big(\Gamma, M\Big[\frac{1}{p}\Big]\Big) \to \mathrm{R}\Gamma(\Gamma, \mathrm{HIG}(H \otimes_{R^+} S^\dagger_\infty, \Theta_H)).$$

However, it follows from Proposition 4.7 that

$$\mathrm{R}\Gamma(\Gamma, S^\dagger_\infty) \simeq R[0].$$

So we get

$$\mathrm{R}\Gamma(\Gamma, \mathrm{HIG}(H \otimes_{R^+} S^\dagger_\infty, \Theta_H)) \simeq \mathrm{HIG}\Big(H\Big[\frac{1}{p}\Big], \theta_H\Big).$$

Therefore, we conclude the desired quasi-isomorphism

$$\mathrm{R}\Gamma\Big(\Gamma, M\Big[\frac{1}{p}\Big]\Big) \simeq \mathrm{HIG}\Big(H\Big[\frac{1}{p}\Big], \theta_H\Big).$$ □

Finally, it is worth pointing out that all results in Theorem 4.3 still hold for $\widehat{S}^+_{\infty,\rho_k}$ instead of $S^{\dagger,+}_\infty$ except the "Also" part of (4) because $\mathrm{HIG}\big(\widehat{S}^+_{\infty,\rho_k}\big[\frac{1}{p}\big], \Theta\big) \neq \widehat{R}_\infty[0]$ and $\mathrm{R}\Gamma\big(\Gamma, \widehat{S}^+_{\infty,\rho_k}\big[\frac{1}{p}\big]\big) \neq R[0]$. For the future use, we give the following proposition.

Proposition 4.13. *Keep the notation as in Theorem 4.3.*

1. *Let M be an a-small $\widehat{R}^+_\infty$-representation of Γ of rank l. Then $H(M) = (M \otimes_{\widehat{R}^+_\infty} \widehat{S}^+_{\infty,\rho_k})^\Gamma$ and $\theta_{H(M)}$ is the restriction of Θ_M to $H(M)$.*
2. *Let (H, θ_H) be an a-small Higgs module of rank l over R^+. Then $M(H, \theta_H) = (H \otimes_{R^+} \widehat{S}^+_{\infty,\rho_k})^{\Theta_H=0}$.*
3. *Let M be an a-small $\widehat{R}^+_\infty$-representation of Γ and (H, θ_H) be the corresponding Higgs module. Then there is a canonical Γ-equivariant isomorphism of Higgs complexes*

$$\mathrm{HIG}(H \otimes_{R^+} \widehat{S}^+_{\infty,\rho_k}, \Theta_H) \to \mathrm{HIG}(M \otimes_{\widehat{R}^+_\infty} \widehat{S}^+_{\infty,\rho_k}, \Theta_M).$$

Proof. By Corollary 2.22, we have a Γ-equivariant decomposition

$$\widehat{S}^+_{\infty,\rho_k} = \widehat{\bigoplus}_{\underline{\alpha}\in(\mathbb{N}\cap[0,1))^d} R^+\langle \rho_k Y_1, \dots, \rho_k Y_d\rangle \underline{T}^{\underline{\alpha}}.$$

Let N be the a-small R^+-representation of Γ corresponding to M in the sense of Theorem 3.4. Then $M = N \otimes_{R^+} \widehat{R}^+_\infty$.

(1) Thanks to Lemma 4.8, we have

$$(M \otimes_{\widehat{R}^+_\infty} \widehat{S}^+_{\infty,\rho_k})^\Gamma = (N \otimes_{R^+} R^+\langle \rho_k Y_1, \dots, \rho_k Y_d\rangle)^\Gamma.$$

Since $a > r$, it is automatic that $\nu_p(\rho_k) < a + \nu_p(\rho_k) - r$. So (1) is a consequence of Proposition 4.9.

(2) This follows from the proof of Proposition 4.12(1) directly (because $\nu_p(\rho_k) < a + \nu_p(\rho_k) - r$).

(3) This follows from (1), (2) and Theorem 4.3(4) via the base-change along $S^{\dagger,+}_\infty \to \widehat{S}^+_{\infty,\rho_k}$. □

5. A p-adic Simpson correspondence

Statement and preliminaries. Now, we want to globalise the local Simpson correspondence established in the last section for a liftable smooth formal scheme $\mathfrak{X}$. We fix such an $\mathfrak{X}$ together with an A_2-lifting $\widetilde{\mathfrak{X}}$. Then we have the corresponding integral Faltings' extension $\mathcal{E}^+$ and overconvergent period sheaf $\mathcal{O}\mathbb{C}^{\dagger,+}$. Let X be the rigid analytic generic fibre of $\mathfrak{X}$ and $\nu : X_{\text{proét}} \to \mathfrak{X}_{\text{ét}}$ be the projection of sites. Throughout this section, we assume $r = 1/(p-1)$.

Definition 5.1. Assume $a \geq r$. By an *a-small generalised representation* of rank l on $X_{\text{proét}}$, we mean a sheaf $\mathcal{L}$ of locally finite free $\widehat{\mathcal{O}}_X$-modules of rank l which admits a p-complete sub-$\widehat{\mathcal{O}}^+_X$-module $\mathcal{L}^+$ such that there is an étale covering $\{\mathfrak{X}_i \to \mathfrak{X}\}_{i\in I}$ and rationals $b_i > b > a$ such that, for any i,

$$(\mathcal{L}^+/p^{b_i+\nu_p(\rho_k)})^{\text{al}}_{|X_i} \cong ((\widehat{\mathcal{O}}^+_X/p^{b_i+\nu_p(\rho_k)})^l)^{\text{al}}_{|X_i}$$

is an isomorphism of $(\widehat{\mathcal{O}}^{+\text{al}}_X/p^{b_i+\nu_p(\rho_k)})_{|X_i}$-modules, where $\widehat{\mathcal{O}}^{+\text{al}}_X$ is the almost integral structure sheaf[2] and X_i denotes the rigid analytic generic fibre of $\mathfrak{X}_i$.

Definition 5.2. Assume $a \geq r$. By an *a-small Higgs bundle* of rank l on $\mathfrak{X}_{\text{ét}}$, we mean a sheaf $\mathcal{H}$ of locally finite free $\mathcal{O}_{\mathfrak{X}}[\frac{1}{p}]$-modules of rank l together with an $\mathcal{O}_{\mathfrak{X}}[\frac{1}{p}]$-linear operator $\theta_{\mathcal{H}} : \mathcal{H} \to \mathcal{H} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}}(-1)$ satisfying $\theta_{\mathcal{H}} \wedge \theta_{\mathcal{H}} = 0$ such that it admits a $\theta_{\mathcal{H}}$-preserving $\mathcal{O}_{\mathfrak{X}}$-lattice $\mathcal{H}^+$ — i.e., $\mathcal{H}^+ \subset \mathcal{H}$ is a subsheaf of locally free $\mathcal{O}_{\mathfrak{X}}$-modules with $\mathcal{H}^+[\frac{1}{p}] = \mathcal{H}$ — satisfying the condition

$$\theta_{\mathcal{H}}(\mathcal{H}^+) \subset p^{b+\nu_p(\rho_k)}\mathcal{H}^+ \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}}(-1)$$

for some $b > a$.

For any a-small generalised representation, define

$$\Theta_{\mathcal{L}} = \text{id}_{\mathcal{L}} \otimes \Theta : \mathcal{L} \otimes_{\widehat{\mathcal{O}}_X} \mathcal{O}\mathbb{C}^\dagger \to \mathcal{L} \otimes_{\widehat{\mathcal{O}}_X} \mathcal{O}\mathbb{C}^\dagger \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{O}}^1_{\mathfrak{X}}(-1).$$

[2]This is the presheaf on $X_{\text{proét}}$ sending each affinoid perfectoid space $U = \text{Spa}(R, R^+)$ to the almost $\mathcal{O}_{\mathbb{C}_p}$-module $R^{+\text{al}}$ in the sense of [Scholze 2012, Section 4]. Since $X_{\text{proét}}$ admits a basis of affinoid perfectoid spaces, the proof of [Scholze 2012, Proposition 7.13] shows that $\widehat{\mathcal{O}}^{+\text{al}}_X$ is a sheaf.

Then $\Theta_{\mathcal{L}}$ is a Higgs field on $\mathcal{L}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\mathbb{C}^{\dagger}$. Denote the induced Higgs complex by $\mathrm{HIG}(\mathcal{L}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\mathbb{C}^{\dagger},\Theta_{\mathcal{L}})$. For any a-small Higgs field $(\mathcal{H},\theta_{\mathcal{H}})$, put

$$\Theta_{\mathcal{H}}=\theta_{\mathcal{H}}\otimes\mathrm{id}+\mathrm{id}_{\mathcal{H}}\otimes\Theta:\mathcal{H}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^{\dagger}\to\mathcal{H}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^{\dagger}\otimes_{\mathcal{O}_{\mathfrak{X}}}\widehat{\Omega}^1_{\mathfrak{X}}(-1).$$

Then $\Theta_{\mathcal{H}}$ is a Higgs field on $\mathcal{H}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^{\dagger}$. Denote the induced Higgs complex by $\mathrm{HIG}(\mathcal{H}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^{\dagger},\Theta_{\mathcal{H}})$. Then our main theorem is the following p-adic Simpson correspondence.

Theorem 5.3 (p-adic Simpson correspondence). *Keep the notation as above.*

(1) *For any a-small generalised representation $\mathcal{L}$ of rank l on $X_{\mathrm{pro\acute{e}t}}$, $\mathrm{R}\nu_*(\mathcal{L}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\mathbb{C}^{\dagger})$ is discrete. Define $\mathcal{H}(\mathcal{L}):=\nu_*(\mathcal{L}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\mathbb{C}^{\dagger})$ and $\theta_{\mathcal{H}(\mathcal{L})}=\nu_*\Theta_{\mathcal{L}}$. Then $(\mathcal{H}(\mathcal{L}),\theta_{\mathcal{H}(\mathcal{L})})$ is an a-small Higgs bundle of rank l.*

(2) *For any a-small Higgs bundle $(\mathcal{H},\theta_{\mathcal{H}})$ of rank l on $\mathfrak{X}_{\mathrm{\acute{e}t}}$, put*

$$\mathcal{L}(\mathcal{H},\theta_{\mathcal{H}})=(\mathcal{H}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^{\dagger})^{\Theta_{\mathcal{H}}=0}.$$

Then $\mathcal{L}(\mathcal{H})$ is an a-small generalised representation of rank l.

(3) *The functor $\mathcal{L}\mapsto(\mathcal{H}(\mathcal{L}),\theta_{\mathcal{H}(\mathcal{L})})$ induces an equivalence from the category of a-small generalised representations to the category of a-small Higgs bundles, whose quasi-inverse is given by $(\mathcal{H},\theta_{\mathcal{H}})\mapsto\mathcal{L}(\mathcal{H},\theta_{\mathcal{H}})$. The equivalence preserves tensor products and dualities and identifies the Higgs complexes*

$$\mathrm{HIG}(\mathcal{L}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\mathbb{C}^{\dagger},\Theta_{\mathcal{L}})\simeq\mathrm{HIG}(\mathcal{H}(\mathcal{L})\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^{\dagger},\Theta_{\mathcal{H}(\mathcal{L})}).$$

(4) *Let $\mathcal{L}$ be an a-small generalised representation with associated Higgs bundle $(\mathcal{H},\theta_{\mathcal{H}})$. Then there is a canonical quasi-isomorphism*

$$\mathrm{R}\nu_*(\mathcal{L})\simeq\mathrm{HIG}(\mathcal{H},\theta_{\mathcal{H}}),$$

where $\mathrm{HIG}(\mathcal{H},\theta_{\mathcal{H}})$ is the Higgs complex induced by $(\mathcal{H},\theta_{\mathcal{H}})$. In particular, $\mathrm{R}\nu_*(\mathcal{L})$ is a perfect complex of $\mathcal{O}_{\mathfrak{X}}[\frac{1}{p}]$-modules concentrated in degree $[0,d]$, where d denotes the dimension of $\mathfrak{X}$ relative to $\mathcal{O}_{\mathbb{C}_p}$.

(5) *Assume $f:\mathfrak{X}\to\mathfrak{Y}$ is a smooth morphism between liftable smooth formal schemes over $\mathcal{O}_{\mathbb{C}_p}$. Let $\widetilde{\mathfrak{X}}$ and $\widetilde{\mathfrak{Y}}$ be the fixed A_2-liftings of $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. Assume f lifts to an A_2-morphism $\tilde{f}:\widetilde{\mathfrak{X}}\to\widetilde{\mathfrak{Y}}$. Then the equivalence in* (3) *is compatible with the pull-back along f.*

Remark 5.4. Assume $\mathcal{L}$ is a sheaf of locally free $\widehat{\mathcal{O}}_X$-modules which becomes a-small after a finite étale base-change $f:\mathfrak{Y}\to\mathfrak{X}$. By étale descent, $\mathrm{R}\nu_*(\mathcal{L}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\mathbb{C}^{\dagger})$ is well defined and discrete. Also, $\nu_*(\mathcal{L}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\mathbb{C}^{\dagger})$ is a Higgs bundle which becomes an a-small Higgs bundle via pull-back along f. Conversely, if $(\mathcal{H},\theta_{\mathcal{H}})$ is a Higgs bundle on $\mathfrak{X}$ which becomes a-small after taking pull-back along a finite étale morphism f, by pro-étale descent for $\widehat{\mathcal{O}}_X$-bundles, $(\mathcal{H}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^{\dagger})^{\Theta_{\mathcal{H}}=0}$ is a well defined $\widehat{\mathcal{O}}_X$-bundle. Also, it becomes a-small via the pull-back along f. Therefore, one can establish a p-adic Simpson correspondence in this case.

Remark 5.5. Assume $\mathfrak{X}$ comes from a smooth formal scheme $\mathfrak{X}_0$ over $\mathbb{Z}_p$ and admits an A_2-lifting $\widetilde{\mathfrak{X}}$. Note that Faltings [2005, Definition 2] used Breuil–Kisin twists to define Higgs fields while we use Tate twists, so our smallness conditions on Higgs fields differ from his by a multiplication of (ζ_p-1). By

Proposition 2.14, after choosing a covering $\{\mathfrak{X}_i \to \mathfrak{X}\}_{i\in I}$, the cocycle $\{\theta_{ij}\}_{i,j\in I}$ corresponding to the integral Faltings' extension is exactly the one used in [Faltings 2005, Section 4]. Note that locally we define Higgs fields by $\theta = -\log\gamma$ (Corollary 4.11) while Faltings [2005, Remark(ii)] defined $\theta = \log\gamma$. So our construction is compatible with [Faltings 2005] up to a sign on Higgs fields.

Remark 5.6. Suppose $\mathfrak{X}$ comes from a smooth formal scheme $\mathfrak{X}_0$ over $\mathcal{O}_k$ and $\widetilde{\mathfrak{X}}$ is the base-change of $\mathfrak{X}_0$ along $\mathcal{O}_k \to A_2$. Let $\mathcal{O}\mathbb{C}^\dagger$ be the associated overconvergent period sheaf. By its construction, there is a natural inclusion $\mathcal{O}\mathbb{C} \hookrightarrow \mathcal{O}\mathbb{C}^\dagger$. Now assume $\mathbb{L}$ is a $\mathbb{Z}_p$-local system on $\mathfrak{X}_{\text{ét}}$ and $\mathcal{L} = \mathbb{L}\otimes_{\mathbb{Z}_p}\widehat{\mathcal{O}}_X$ is the corresponding $\widehat{\mathcal{O}}_X$-bundle on $X_{\text{proét}}$. Since the resulting Higgs field is nilpotent by [Liu and Zhu 2017, Theorem 2.1], it can be seen from the proof of Theorem 5.3 that the morphism

$$\nu_*(\mathcal{L}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\mathbb{C}) \to \nu_*(\mathcal{L}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\mathbb{C}^\dagger)$$

is an isomorphism. So our construction is compatible with the work of [Liu and Zhu 2017] in this case.

We do some preparations before proving Theorem 5.3.

Lemma 5.7. *Let $U \in X_{\text{proét}}$ be an affinoid perfectoid and $\mathcal{M}^+$ be a sheaf of p-torsion free $\widehat{\mathcal{O}}_X^+$-modules satisfying one of the following conditions*:

(*a*) *$\mathcal{M}^+_{|U}$ is a sheaf of free $\widehat{\mathcal{O}}^+_{X|U}$-modules.*

(*b*) *$\mathcal{M}^+$ is p-complete and there is an almost isomorphism*

$$(\mathcal{M}^+_{|U}/p^c)^{\text{al}} \cong ((\widehat{\mathcal{O}}^+_{X|U}/p^c)^r)^{\text{al}}$$

for some $c > 0$.

Then the following assertions are true:

(1) *For any $i \geq 1$ and $a > 0$, $H^i(U,\mathcal{M}^+)^{\text{al}} \cong H^i(U,\mathcal{M}^+/p^a)^{\text{al}} = 0$.*

(2) *For any $b > a > 0$, the image of $(\mathcal{M}^+/p^b)(U)$ in $(\mathcal{M}^+/p^a)$ is $\mathcal{M}^+(U)/p^a$.*

(3) *Put $\widehat{\mathcal{M}}^+ = \varprojlim_n \mathcal{M}^+/p^n$. Then $\widehat{\mathcal{M}}^+(U) = \varprojlim_n \mathcal{M}^+(U)/p^n$ and for any $i \geq 1$, $H^i(U,\hat{\mathcal{M}}^+)^{\text{al}} = 0$.*

Proof. By [Scholze 2013a, Lemma 4.10], both (1) and (2) hold for free $\widehat{\mathcal{O}}_X^+$-modules. So we only focus on $\mathcal{M}^+$'s satisfying the second condition.

(1) It is enough to show that for any $i \geq 1$, $H^i(U,\mathcal{M}^+)^{\text{al}} = 0$. Granting this, the rest can be deduced from the long exact sequence induced by

$$0 \to \mathcal{M}^+ \xrightarrow{\times p^a} \mathcal{M}^+ \to \mathcal{M}^+/p^a \to 0.$$

Since $(\mathcal{M}^+_{|U}/p^c)^{\text{al}} \cong ((\widehat{\mathcal{O}}^+_{X|U}/p^c)^r)^{\text{al}}$, by [Scholze 2013a, Lemma 4.10(v)], we deduce that

$$H^i(U,\mathcal{M}^+/p^c)^{\text{al}} = 0$$

for any $i \geq 1$. Consider the exact sequence

$$0 \to \mathcal{M}^+/p^c \xrightarrow{p^{(n-1)c}} \mathcal{M}^+/p^{nc} \to \mathcal{M}^+/p^{(n-1)c} \to 0.$$

By induction on n, we see that for any $i \geq 1$, $H^i(U, \mathcal{M}^+/p^{nc})^{\mathrm{al}} = 0$. Now, the desired result follows from [Scholze 2013a, Lemma 3.18].

(2) Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M}^+ & \xrightarrow{p^b} & \mathcal{M}^+ & \longrightarrow & \mathcal{M}^+/p^b & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \times p^{b-a}} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{M}^+ & \xrightarrow{p^a} & \mathcal{M}^+ & \longrightarrow & \mathcal{M}^+/p^a & \longrightarrow & 0 \end{array}$$

Then by (1), we get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M}^+(U)/p^b & \longrightarrow & (\mathcal{M}^+/p^b)(U) & \xrightarrow{\delta_b} & H^1(U, \mathcal{M}^+) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \times p^{b-a}} & & \\ 0 & \longrightarrow & \mathcal{M}^+(U)/p^a & \longrightarrow & (\mathcal{M}^+/p^a)(U) & \xrightarrow{\delta_a} & H^1(U, \mathcal{M}^+) & \longrightarrow & 0 \end{array}$$

Since the multiplication by p^{b-a} is zero on $H^1(U, \mathcal{M}^+)$, the image of $(\mathcal{M}^+/p^b)(U)$ in $(\mathcal{M}^+/p^a)(U)$ is contained in the kernel of δ_a. In other words, $(\mathcal{M}^+/p^b)(U)$ takes values in $\mathcal{M}^+(U)/p^a$. Now, the result follows.

(3) When $\mathcal{M}^+$ is p-complete, there is nothing to prove. Now, assume $\mathcal{M}^+$ is a free $\widehat{\mathcal{O}}_X^+$-module. The first part follows from (2) and the second part follows from the same argument used in (1). □

Remark 5.8. In this paper, we say a module (or a sheaf of $\widehat{\mathcal{O}}_X^+$-modules) M is p-complete, if $M \cong \mathrm{Rlim}_n\, M \otimes^L_{\mathbb{Z}_p} \mathbb{Z}_p/p^n$. This is different from that $M = \lim_n M/p^n$ in general. However, as mentioned in the paragraph below [Bhatt et al. 2019, Lemma 4.6], if M has bounded p^∞-torsion; that is, $M[p^\infty] = M[p^N]$ for some $N \geq 0$, then saying M is p-complete amounts to saying $M = \lim_n M/p^n$. Indeed, in this case, the pro-systems $\{M \otimes^L_{\mathbb{Z}_p} \mathbb{Z}_p/p^n\}_{n\geq 0}$ and $\{M/p^n\}_{n\geq 0}$ are pro-isomorphic. So we obtain that

$$\operatorname*{Rlim}_n M \otimes^L_{\mathbb{Z}_p} \mathbb{Z}_p/p^n \simeq \operatorname*{Rlim}_n M/p^n.$$

Lemma 5.9. *Assume $\mathfrak{X} = \mathrm{Spf}(R^+)$ is small. Define X_∞, $\widehat{R}_\infty^+$ as before. Let $\mathcal{L}^+$ be a sheaf of p-complete and p-torsion free $\widehat{\mathcal{O}}_X^+$-modules such that*

$$(\mathcal{L}^+/p^a)^{\mathrm{al}} \cong ((\widehat{\mathcal{O}}_X^+/p^a)^l)^{\mathrm{al}}$$

for some $a > 0$. Put $M = \mathcal{L}^+(X_\infty)$. Then:

(1) *M is a finite free $\widehat{R}_\infty^+$-module of rank l.*

(2) *For any $0 < b < a$, there is a Γ-equivariant isomorphism $M/p^b \cong (\widehat{R}_\infty^+/p^b)^l$.*

Proof. By Lemma 5.7, we have Γ-equivariant almost isomorphisms

$$M/p^a \xrightarrow{\approx} (\mathcal{L}^+/p^a)(X_\infty) \approx (\widehat{\mathcal{O}}_X^+/p^a)^l(X_\infty) \xleftarrow{\approx} (\widehat{R}_\infty^+/p^a)^l. \tag{5-1}$$

In particular, we get an almost isomorphism $M/p^a \approx (\widehat{R}^+_\infty/p^a)^l$. Denote by $e_1, \dots, e_l$ the standard basis of $(\widehat{R}^+_\infty)^l$.

(1) As mentioned in the paragraph after [Scholze 2013a, Definition 2.2], for any $\epsilon \in \mathbb{Q}_{>0}$, one can find $\mathcal{O}_{\mathbb{C}_p}$-morphisms

$$f : M/p^a \to (\widehat{R}^+_\infty/p^a)^l \quad \text{and} \quad g : (\widehat{R}^+_\infty/p^a)^l \to M/p^a$$

such that $f \circ g = p^\epsilon$ and $g \circ f = p^\epsilon$. In particular, the image of g is $p^\epsilon M/p^a$ and the kernel of g is killed by p^ϵ.

For any i, choose $x_i \in M$ such that

$$x_i \equiv g(e_i) \mod p^a M.$$

Then the x_i's generate

$$p^\epsilon M/p^a \cong M/p^{a-\epsilon}.$$

We claim the x_i's are linear independent over $\widehat{R}^+_\infty/p^{a-\epsilon}$. Granting this, we see $M/p^{a-\epsilon}$ is a finite free $\widehat{R}^+_\infty/p^{a-\epsilon}$-module. Since M is p-torsion free and p-complete by Lemma 5.7(3), by choosing $\epsilon < a$, we deduce that M is finite free of rank l as desired.

So we are reduced to proving the claim. Assume $\lambda_i \in \widehat{R}^+_\infty$ such that $\sum_{i=1}^l \lambda_i x_i \in p^a M$, that is, $g\big(\sum_{i=1}^l \lambda_i e_i\big) \in p^a M$. So $\sum_{i=1}^l \lambda_i e_i \in \operatorname{Ker}(g)$ and thus is killed by p^ϵ. In other words, $p^\epsilon \sum_{i=1}^l \lambda_i e_i \in p^a(\widehat{R}^+_\infty)^l$. This forces $\lambda_i \in p^{a-\epsilon}\widehat{R}^+_\infty$ for any i. So we are done.

(2) By [Scholze 2012, Proposition 4.4], the almost isomorphism $M/p^a \approx (\widehat{R}^+_\infty/p^a)^l$ induces an isomorphism

$$\iota : \mathfrak{m}_{\mathbb{C}_p} \otimes_{\mathcal{O}_{\mathbb{C}_p}} (\widehat{R}^+_\infty/p^a)^l \to \mathfrak{m}_{\mathbb{C}_p} \otimes_{\mathcal{O}_{\mathbb{C}_p}} M/p^a.$$

Since (5-1) is Γ-equivariant, so is ι. Since $\mathfrak{m}_{\mathbb{C}_p}$ is flat over $\mathcal{O}_{\mathbb{C}_p}$, this amounts to a Γ-equivariant isomorphism

$$h : (\mathfrak{m}_{\mathbb{C}_p}\widehat{R}^+_\infty/p^a\mathfrak{m}_{\mathbb{C}_p}\widehat{R}^+_\infty)^l \to \mathfrak{m}_{\mathbb{C}_p}M/p^a\mathfrak{m}_{\mathbb{C}_p}M.$$

Now, for any $\epsilon > 0$, choose $x_{i,\epsilon} \in \mathfrak{m}_{\mathbb{C}_p}M$ such that, for any i,

$$x_{i,\epsilon} \equiv h(p^\epsilon e_i) \mod p^a M.$$

Note that $x_{i,\epsilon}$ is unique modulo $p^a M$. So for $0 < \epsilon' < \epsilon$, we have

$$p^{\epsilon-\epsilon'} x_{i,\epsilon'} \equiv x_{i,\epsilon} \mod p^a M.$$

Assume $\epsilon < a$, we see that $p^{\epsilon-\epsilon'}$ divides $x_{i,\epsilon}$ for any ϵ'. By [Bhatt et al. 2018, Lemma 8.10], R^+ is a topologically free $\mathcal{O}_{\mathbb{C}_p}$-module; therefore, so is $\widehat{R}^+_\infty$. As we have seen that M is a finite free $\widehat{R}^+_\infty$-module, it is also topologically free over $\mathcal{O}_{\mathbb{C}_p}$. This forces that $x_{i,\epsilon}$ is divided by p^ϵ. So we may assume $x_{i,\epsilon} = p^\epsilon y_{i,\epsilon}$ for some $y_{i,\epsilon} \in M$. By construction, $y_{i,\epsilon}$ is unique modulo $p^{a-\epsilon}M$.

Now define $H_\epsilon : (\widehat{R}^+_\infty/p^{a-\epsilon})^l \to M/p^{a-\epsilon}$ by sending e_i to $y_{i,\epsilon}$. By construction of H_ϵ, we see that it is the unique $\widehat{R}^+_\infty$-morphism from $(\widehat{R}^+_\infty/p^{a-\epsilon})^l$ to $M/p^{a-\epsilon}$ whose restriction to $(\mathfrak{m}_{\mathbb{C}_p}\widehat{R}^+_\infty/p^{a-\epsilon})^l$ coincides with h.

We need to show H_ϵ is an isomorphism. However, since M is also finite free, after interchanging M and $(\widehat{R}^+_\infty)^l$ and proceeding as above, we get a unique $G_\epsilon : M/p^{a-\epsilon} \to (\widehat{R}^+_\infty/p^{a-\epsilon})^l$, whose restriction to $\mathfrak{m}_{\mathbb{C}_p}M/p^{a-\epsilon}$ coincides with h^{-1}. Now, a similar argument shows that $H_\epsilon \circ G_\epsilon = \mathrm{id}$ and $G_\epsilon \circ H_\epsilon = \mathrm{id}$. So H_ϵ is an isomorphism.

Finally, since h is Γ-equivariant, by the uniqueness of H_ϵ, we deduce that H_ϵ is also Γ-equivariant. Since ϵ is arbitrary, we are done. □

The following corollary is a special case of Lemma 5.9.

Corollary 5.10. *Assume $\mathfrak{X} = \mathrm{Spf}(R^+)$ is small affine. Let $\mathcal{L}$ be an a-small generalised representation with a sub-$\widehat{\mathcal{O}}^+_X$-sheaf $\mathcal{L}^+$ satisfying $(\mathcal{L}^+/p^{b+\nu_p(\rho_k)})^{\mathrm{al}} \cong ((\widehat{\mathcal{O}}^+_X/p^{b+\nu_p(\rho_k)})^l)^{\mathrm{al}}$ for some $b > a$. Then $\mathcal{L}^+(X_\infty)$ is a b'-small $\widehat{R}^+_\infty$-representation of Γ for any $a < b' < b$.*

Lemma 5.11. *Assume $\mathfrak{X} = \mathrm{Spf}(R^+)$ is affine small. Let $\mathcal{L}^+$ be a sheaf of p-complete and p-torsion free $\widehat{\mathcal{O}}^+_X$-modules such that*

$$(\mathcal{L}^+/p^c)^{\mathrm{al}} \cong ((\widehat{\mathcal{O}}^+_X/p^c)^l)^{\mathrm{al}}$$

for some $c > 0$. Then for any $\mathcal{P}^+ \in \{\mathcal{O}\mathbb{C}^+_\rho, \mathcal{O}\widehat{\mathbb{C}}^+_\rho, \mathcal{O}\mathbb{C}^{\dagger,+}\}$ and for each $i \geq 0$, the natural map

$$H^i(\Gamma, (\mathcal{L}^+ \otimes_{\widehat{\mathcal{O}}^+_X} \mathcal{P}^+)(X_\infty)) \to H^i(X_{\text{proét}}/\mathfrak{X}, \mathcal{L}^+ \otimes_{\widehat{\mathcal{O}}^+_X} \mathcal{P}^+)$$

is an almost isomorphism. When $i = 0$, it is an isomorphism.

Proof. The proof is similar to [Scholze 2013a, Lemma 5.6; Liu and Zhu 2017, Lemma 2.7]. Denote by $X^{m/X}_\infty$ the m-fold fibre product of X_∞ over X. As X_∞ is a Galois cover of X with Galois group Γ, we have $X^{m/X}_\infty \simeq X_\infty \times \Gamma^{m-1}$. Note that $\widehat{\mathcal{O}}^+_X/p^c$ comes from the étale sheaf $\mathcal{O}^+_X/p^c$ on $X_{\text{ét}}$ and that $(\mathcal{L}^+/p^c)^{\mathrm{al}} \cong ((\widehat{\mathcal{O}}^+_X/p^c)^l)^{\mathrm{al}}$. By [Scholze 2013a, Lemma 3.16], for any $i \geq 0$ and $m \geq 1$, we have almost isomorphisms

$$\mathrm{Hom}_{\mathrm{cts}}(\Gamma^{m-1}, H^i(X_\infty, \mathcal{L}^+ \otimes_{\widehat{\mathcal{O}}^+_X} \mathcal{P}^+/p^c)) \to H^i(X^{m/X}_\infty, \mathcal{L}^+ \otimes_{\widehat{\mathcal{O}}^+_X} \mathcal{P}^+/p^c).$$

By induction on n, we have almost isomorphisms

$$\mathrm{Hom}_{\mathrm{cts}}(\Gamma^{m-1}, H^i(X_\infty, \mathcal{L}^+ \otimes_{\widehat{\mathcal{O}}^+_X} \mathcal{P}^+/p^{nc})) \to H^i(X^{m/X}_\infty, \mathcal{L}^+ \otimes_{\widehat{\mathcal{O}}^+_X} \mathcal{P}^+/p^{nc}),$$

for any $n \geq 1$. By letting n go to $+\infty$, we get almost isomorphisms

$$\mathrm{Hom}_{\mathrm{cts}}(\Gamma^{m-1}, H^i(X_\infty, \mathcal{L}^+ \otimes_{\widehat{\mathcal{O}}^+_X} \mathcal{P}^+)) \to H^i(X^{m/X}_\infty, \mathcal{L}^+ \otimes_{\widehat{\mathcal{O}}^+_X} \mathcal{P}^+)$$

for $\mathcal{P}^+ \in \{\mathcal{O}\mathbb{C}^{+,\leq r}_\rho, \mathcal{O}\widehat{\mathbb{C}}^+_\rho\}$, where $\mathcal{O}\mathbb{C}^{+,\leq r}_\rho$ denotes the subsheaf of

$$\mathcal{O}\mathbb{C}^+_\rho \cong \widehat{\mathcal{O}}^+_X[\rho Y_1, \dots, \rho Y_d]$$

consisting of polynomials of degrees $\leq r$. By the coherence of restricted pro-étale topos, $H^i(X_\infty^{m/X}, -)$ commutes with direct limits for all i. Since $\mathcal{O}\mathbb{C}_\rho^+ = \bigcup_{r\geq 0}\mathcal{O}\mathbb{C}_\rho^{+,\leq r}$, we also get desired almost isomorphisms for $\mathcal{P}^+ = \mathcal{O}\mathbb{C}_\rho^+$. A similar argument also works for $\mathcal{P}^+ = \mathcal{O}\mathbb{C}^{\dagger,+} = \bigcup_{\rho,\nu_p(\rho)>\nu_p(\rho_k)}\mathcal{O}\widehat{\mathbb{C}}_\rho^+$. When $i=0$, since both sides are $\mathfrak{m}_{\mathbb{C}_p}$-torsion free, so we get injections.

Now applying the Cartan–Leray spectral sequence to the Galois cover $X_\infty \to X$ and using Lemma 5.7, we conclude that the map

$$H^i(\Gamma_\infty, (\mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{P}^+)(X_\infty)) \to H^i(X_{\text{proét}}/\mathfrak{X}, \mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{P}^+)$$

is an almost isomorphism for every $i \geq 0$.

For $i=0$, we know $H^0(X_{\text{proét}}/\mathfrak{X}, \mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{P}^+)$ is the (0, 0)-term of the Cartan–Leray spectral sequence at the E_2-page, which is the kernel of the map

$$(\mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{P}^+)(X_\infty) \to (\mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{P}^+)(X_\infty^{2/X}).$$

On the other hand, $H^0(\Gamma, (\mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{P}^+)(X_\infty))$ is the kernel of the map

$$(\mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{P}^+)(X_\infty) \to \mathrm{Hom}_{\mathrm{cts}}(\Gamma, (\mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{P}^+)(X_\infty)).$$

So the result follows from the injectivity of the map

$$\mathrm{Hom}_{\mathrm{cts}}(\Gamma, (\mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{P}^+)(X_\infty)) \to (\mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{P}^+)(X_\infty^{2/X}). \qquad \square$$

Proof of Theorem 5.3. Now we are prepared to prove Theorem 5.3.

(1) Let $\mathcal{L}$ be an a-small generalised representation of rank l and $\mathcal{L}^+$ be the sub-$\widehat{\mathcal{O}}_X^+$-sheaf as described in Definition 5.1. Define $\mathcal{H}^+ := \nu_*(\mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{O}\mathbb{C}^{\dagger,+})$. It suffices to show that $\mathrm{R}^i\nu_*(\mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{O}\mathbb{C}^{\dagger,+})$ is p^∞-torsion for any $i\geq 1$ and that $\mathcal{H}^+$ satisfies conditions in Definition 5.2. Let $b>a$ and $\{\mathfrak{X}_i\to\mathfrak{X}\}_{i\in I}$ be as in Definition 5.1. Since the problem is local on $\mathfrak{X}_{\text{ét}}$, we are reduced to showing that for any $i\in I$, if we write $\mathfrak{X}_i = \mathrm{Spf}(R_i^+)$, then $H^n(X_{\text{proét}}/\mathfrak{X}_i, \mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{O}\mathbb{C}^{\dagger,+})$ is p^∞-torsion for any $n\geq 1$ and is a b_i-small Higgs module over R_i^+ for $n=0$ in the sense of Definition 4.2 for some $b_i>b$. So we only need to deal with the case for $\mathfrak{X}$ small affine.

Now we may assume $\mathfrak{X}=\mathrm{Spf}(R^+)$ is affine small itself and that

$$(\mathcal{L}^+/p^{b'})^{\mathrm{al}} \cong ((\widehat{\mathcal{O}}_X^+)^l/p^{b'})^{\mathrm{al}}$$

for some $b'>b$. Let X_∞, $\widehat{R}_\infty^+$ and Γ be as before. By Lemma 5.11, the natural morphism

$$H^i(\Gamma, \mathcal{L}^+(X_\infty)\otimes_{\widehat{R}_\infty^+}S_\infty^{\dagger,+}) \to H^i(X_{\text{proét}}/\mathfrak{X}, \mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{O}\mathbb{C}^{\dagger,+})$$

is an almost isomorphism for $i\geq 1$ and is an isomorphism for $i=0$. So we are reduced to showing $\mathrm{R}\Gamma(\Gamma, \mathcal{L}^+(X_\infty)\otimes_{\widehat{R}_\infty^+}S_\infty^\dagger)$ is discrete after inverting p and $H^0(\Gamma, \mathcal{L}^+(X_\infty)\otimes_{\widehat{R}_\infty^+}S_\infty^\dagger)$ is a b''-small Higgs module for some $b''>b$.

However, by Corollary 5.10, $\mathcal{L}^+(X_\infty)$ is a b''-small $\widehat{R}_\infty^+$-representation of Γ for some fixed $b''>b$. So the result follows from Theorem 4.3(1).

(2) Let $(\mathcal{H},\theta_{\mathcal{H}})$ be an a-small Higgs bundle of rank l and $\mathcal{H}^+$ be the $\mathcal{O}_{\mathfrak{X}}$-lattice as described in Definition 5.2. Fix an a' satisfying $a<a'<b$. Define $\mathcal{L}^+=(\mathcal{H}^+\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^{\dagger,+})^{\Theta_{\mathcal{H}}=0}$. Then it is a subsheaf of $\mathcal{L}=(\mathcal{H}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^{\dagger})^{\Theta_{\mathcal{H}}=0}$ and hence p-torsion free. We claim that the inclusion $\mathcal{O}\mathbb{C}^{\dagger,+}\to\mathcal{O}\widehat{\mathbb{C}}^{+}_{\rho_k}$ induces a natural isomorphism

$$(\mathcal{H}^+\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^{\dagger,+})^{\Theta_{\mathcal{H}}=0}\to(\mathcal{H}^+\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\widehat{\mathbb{C}}^{+}_{\rho_k})^{\Theta_{\mathcal{H}}=0}.$$

Indeed, this is a local problem and therefore follows from Proposition 4.13. As $\mathcal{H}^+\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\widehat{\mathbb{C}}^{+}_{\rho_k}$ is p-complete, by continuity of $\Theta_{\mathcal{H}}$, so is $\mathcal{L}^+$. It remains to prove that $\mathcal{L}^+$ is locally almost trivial modulo $p^{a'+\nu_p(\rho_k)}$.

Assume $\mathfrak{X}=\mathrm{Spf}(R^+)$ is small affine and let X_∞, $\widehat{R}^+_\infty$ and Γ be as before. Shrinking $\mathfrak{X}$ if necessary, we may assume $(\mathcal{H}^+,\theta_{\mathcal{H}})$ is induced by a b'-small Higgs module over R^+ for some $b'>a'$. Then by Theorem 4.3, $\mathcal{L}^+(X_\infty)$ is a b'-small $\widehat{R}^+_\infty$-representation of Γ.

Let us go back to the global case. Choose an étale covering $\{\mathfrak{X}_i\to\mathfrak{X}\}$ of $\mathfrak{X}$ by small affine $\mathfrak{X}_i=\mathrm{Spf}(R_i^+)$ such that on each $\mathfrak{X}_i$, $(\mathcal{H}^+,\theta_{\mathcal{H}^+})$ is induced by a b_i-small Higgs module over R_i^+ for some $b_i>a'$. Denote by $X_{i,\infty}$ the corresponding "X_∞" for $\mathfrak{X}_i$ instead of $\mathfrak{X}$. As above, we have

$$\mathcal{L}^+(X_{i,\infty})/p^{b_i}\cong(\widehat{\mathcal{O}}^+_X(X_{i,\infty})/p^{b_i})^l.$$

Therefore, by the proof of [Scholze 2013a, Lemma 4.10(i)], we get an almost isomorphism

$$(\mathcal{L}^+/p^{b_i})^{\mathrm{al}}_{|X_i}\cong((\widehat{\mathcal{O}}^+_X/p^{b_i})^l)^{\mathrm{al}}_{|X_i}$$

with $b_i>a'>a$ as desired.

(3) Let $\mathcal{L}$ be an a-small generalised representation. There exists a natural morphism of Higgs complexes

$$\iota:\mathrm{HIG}(\mathcal{H}(\mathcal{L})\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^{\dagger},\Theta_{\mathcal{H}(\mathcal{L})})\to\mathrm{HIG}(\mathcal{L}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\mathbb{C}^{\dagger},\Theta_{\mathcal{L}}).$$

By construction of $(\mathcal{H}(\mathcal{L}),\theta_{\mathcal{H}(\mathcal{L})})$, it follows from Theorem 4.3(4) that ι is an isomorphism. Since $\mathcal{O}\mathbb{C}^{\dagger}$ is a resolution of $\widehat{\mathcal{O}}_X$ by Theorem 2.28, we see that $\mathcal{L}(\mathcal{H}(\mathcal{L}),\theta_{\mathcal{H}(\mathcal{L})})=\mathcal{L}$. The isomorphism

$$(\mathcal{H},\theta_{\mathcal{H}})\to(\mathcal{H}(\mathcal{L}(\mathcal{H})),\theta_{\mathcal{H}(\mathcal{L}(\mathcal{H}))})$$

can be deduced in a similar way. So we get the equivalence as desired.

It remains to show the equivalence preserves products and dualities. But this is a local problem, so we are reduced to Theorem 4.3(3).

(4) This follows from the same arguments in the proof of Theorem 4.3(4). Indeed, combining Theorem 2.28 and the item (3), we have a quasi-isomorphism

$$\mathcal{L}\to\mathrm{HIG}(\mathcal{L}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\mathbb{C}^{\dagger},\Theta_{\mathcal{L}})\simeq\mathrm{HIG}(\mathcal{H}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^{\dagger},\Theta_{\mathcal{H}}).$$

On the other hand, it follows from (1) that there exists a quasi-isomorphism

$$\mathrm{R}\nu_*(\mathrm{HIG}(\mathcal{H}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^{\dagger},\Theta_{\mathcal{H}}))\simeq\mathrm{HIG}(\mathcal{H},\theta_{\mathcal{H}}).$$

So we get a quasi-isomorphism

$$\mathrm{R}\nu_*(\mathcal{L}) \simeq \mathrm{HIG}(\mathcal{H}, \theta_{\mathcal{H}})$$

as desired.

(5) Since $f : \mathfrak{X} \to \mathfrak{Y}$ admits an A_2-lifting $\tilde{f}$, by Proposition 2.29, we get a morphism $f^*\mathcal{O}\mathbb{C}_Y^\dagger \to \mathcal{O}\mathbb{C}_X^\dagger$ which is compatible with Higgs fields.

Assume $(\mathcal{H}, \theta_{\mathcal{H}})$ is an a-small Higgs field on $\mathfrak{Y}_{\text{ét}}$. Denote by $(f^*\mathcal{H}, f^*\theta_{\mathcal{H}})$ its pull-back along f. By (3), we get the following isomorphisms, which are compatible with Higgs fields:

$$\begin{aligned}
\mathcal{L}(f^*\mathcal{H}, f^*\theta_{\mathcal{H}}) \otimes_{\widehat{\mathcal{O}}_X} \mathcal{O}\mathbb{C}_X^\dagger &\cong f^*\mathcal{H} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}\mathbb{C}_X^\dagger \\
&\cong f^*(\mathcal{H} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \mathcal{O}\mathbb{C}_Y^\dagger) \otimes_{f^*\mathcal{O}\mathbb{C}_Y^\dagger} \mathcal{O}\mathbb{C}_X^\dagger \\
&\cong f^*(\mathcal{L}(\mathcal{H}, \theta_{\mathcal{H}}) \otimes_{\widehat{\mathcal{O}}_Y} \mathcal{O}\mathbb{C}_Y^\dagger) \otimes_{f^*\mathcal{O}\mathbb{C}_Y^\dagger} \mathcal{O}\mathbb{C}_X^\dagger \\
&\cong f^*\mathcal{L}(\mathcal{H}, \theta_{\mathcal{H}}) \otimes_{\widehat{\mathcal{O}}_X} \mathcal{O}\mathbb{C}_X^\dagger.
\end{aligned}$$

After taking kernels of Higgs fields, we obtain that

$$\mathcal{L}(f^*\mathcal{H}, f^*\theta_{\mathcal{H}}) \cong f^*\mathcal{L}(\mathcal{H}, \theta_{\mathcal{H}}).$$

So the functor $(\mathcal{H}, \theta_{\mathcal{H}}) \to \mathcal{L}(\mathcal{H}, \theta_{\mathcal{H}})$ in (2) is compatible with the pull-back along f. But we have shown it is an equivalence, so its quasi-inverse must commute with the pull-back along f. This completes the proof.

Corollary 5.12. *Assume $\mathfrak{X}$ is a liftable proper smooth formal scheme of relative dimension d over $\mathcal{O}_{\mathbb{C}_p}$. For any small generalised representation $\mathcal{L}$, $\mathrm{R}\Gamma(X_{\text{proét}}, \mathcal{L})$ is concentrated in degree $[0, 2d]$, whose cohomologies are finite dimensional $\mathbb{C}_p$-spaces.*

Proof. Since we have assumed $\mathfrak{X}$ is proper smooth, this follows from Theorem 5.3(4) directly. □

Remark 5.13. Except the item (4), all results in Theorem 5.3 are still true by using $\mathcal{O}\widehat{\mathbb{C}}_{\rho_k}^+$ instead of $\mathcal{O}\mathbb{C}^{\dagger,+}$.

Remark 5.14. In Corollary 5.12, one can also deduce that $\mathrm{R}\Gamma(X_{\text{proét}}, \mathcal{L})$ is concentrated in degree $[0, 2d]$ when $\mathfrak{X}$ is just quasi-compact of relative dimension d over $\mathcal{O}_{\mathbb{C}_p}$. Indeed, in this case, we have

$$\mathrm{R}\Gamma(X_{\text{proét}}, \mathcal{L}) \simeq \mathrm{R}\Gamma(\mathfrak{X}_{\text{ét}}, \mathrm{HIG}(\mathcal{H}, \theta_{\mathcal{H}})) \simeq \mathrm{R}\Gamma(X_{\text{ét}}, \mathrm{HIG}(\mathcal{H}, \theta_{\mathcal{H}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_{\text{ét}}}),$$

where $\mathrm{HIG}(\mathcal{H}, \theta_{\mathcal{H}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_{\text{ét}}}$ denotes the induced Higgs complex on $X_{\text{ét}}$. On the other hand, by étale descent, the category of étale vector bundles on $X_{\text{ét}}$ is equivalent to the category of analytic vector bundles on X_{an}, where X_{an} denotes the analytic site of X. So the Higgs complex $\mathrm{HIG}(\mathcal{H}, \theta_{\mathcal{H}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_{\text{ét}}}$ upgrades to an analytic Higgs complex $\mathrm{HIG}(\mathcal{H}_{\text{an}}, \theta_{\mathcal{H}})$ such that

$$\mathrm{HIG}(\mathcal{H}_{\text{an}}, \theta_{\mathcal{H}}) \otimes_{\mathcal{O}_{X_{\text{an}}}} \mathcal{O}_{X_{\text{ét}}} = \mathrm{HIG}(\mathcal{H}, \theta_{\mathcal{H}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_{\text{ét}}}.$$

By analytic-étale comparison (see [Fresnel and van der Put 2004, Proposition 8.2.3]), for any coherent $\mathcal{O}_{X_{\mathrm{an}}}$-module $\mathcal{M}$, there is a canonical quasi-isomorphism

$$\mathrm{R}\Gamma(X_{\mathrm{an}}, \mathcal{M}) \simeq \mathrm{R}\Gamma(X_{\mathrm{\acute{e}t}}, \mathcal{M} \otimes_{\mathcal{O}_{X_{\mathrm{an}}}} \mathcal{O}_{X_{\mathrm{\acute{e}t}}}).$$

So by considering corresponding spectral sequences of these complexes, we get a quasi-isomorphism

$$\mathrm{R}\Gamma(X_{\mathrm{an}}, \mathrm{HIG}(\mathcal{H}_{\mathrm{an}}, \theta_{\mathcal{H}})) \simeq \mathrm{R}\Gamma(X_{\mathrm{\acute{e}t}}, \mathrm{HIG}(\mathcal{H}, \theta_{\mathcal{H}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_{\mathrm{\acute{e}t}}}).$$

Now, the quasi-compactness of $\mathfrak{X}$ implies that X is a noetherian space. So the result follows from Grothendieck's vanishing theorem [1957, Théorème 3.6.5] directly. The author thanks the anonymous referees for pointing this out.

6. Appendix

We prove some elementary facts used in this paper. Throughout this section, we always assume A is a p-complete flat $\mathcal{O}_{\mathbb{C}_p}$-algebra.

Definition A.1. Let $\Lambda = \{\alpha\}_{\alpha\in\Lambda}$ be an index set and $I = \{i_\alpha\}_\alpha$ be a set of nonnegative real numbers indexed by Λ. Define

(1) $A[\Lambda] = \bigoplus_{\alpha\in\Lambda} A$;

(2) $A\langle\Lambda\rangle = \varprojlim_m A[\Lambda]/p^m A[\Lambda]$;

(3) $A[\Lambda, I] = \bigoplus_{\alpha\in\Lambda} p^{i_\alpha} A$;

(4) $A\langle\Lambda, I\rangle = \varprojlim_m (A[\Lambda, I] + p^m A[\Lambda])/p^m A[\Lambda]$;

(5) $A\langle\Lambda, I, +\rangle = \varprojlim_m A[\Lambda, I]/p^m A[\Lambda, I]$.

Proposition A.2. (1) *$A\langle\Lambda\rangle/A\langle\Lambda, I\rangle$ is the classical p-completion of* $A[\Lambda]/A[\Lambda, I]$.

(2) *$A\langle\Lambda\rangle/A\langle\Lambda, I, +\rangle$ is the derived p-completion of* $A[\Lambda]/A[\Lambda, I]$.

Proof. Since $A\langle\Lambda, I\rangle$ is the closure of $A\langle\Lambda, I, +\rangle$ in $A\langle\Lambda\rangle$ with respect to the p-adic topology, the item (1) follows from (2) directly. So we are reduced to proving (2).

Consider the short exact sequence

$$0 \longrightarrow A[\Lambda, I] \longrightarrow A[\Lambda] \longrightarrow A[\Lambda]/A[\Lambda, I] \longrightarrow 0.$$

For any $n \geq 0$, we get an exact triangle

$$A[\Lambda, I] \otimes^L_{\mathbb{Z}_p} \mathbb{Z}_p/p^n \to A[\Lambda] \otimes^L_{\mathbb{Z}_p} \mathbb{Z}_p/p^n \to (A[\Lambda]/A[\Lambda, I]) \otimes^L_{\mathbb{Z}_p} \mathbb{Z}_p/p^n \to .$$

Applying Rlim_n to this exact triangle and using p-complete flatness of A, we get the exact triangle

$$A\langle\Lambda, I, +\rangle[0] \to A\langle\Lambda\rangle[0] \to K \to,$$

where K denotes the derived p-completion of $A[\Lambda]/A[\Lambda, I]$. Now, the item (2) follows from the injectivity of the map $A\langle\Lambda, I, +\rangle \to A\langle\Lambda\rangle$. □

Remark A.3. For any $(\lambda_\alpha)_{\alpha\in\Lambda}\in\prod_{\alpha\in\Lambda}A$, we write $\lambda_\alpha\xrightarrow{\nu_p}0$, if for any $M>0$ the set $\{\alpha\in\Lambda\,|\,\nu_p(\lambda_\alpha)\leq M\}$ is finite. Then we have

$$A\langle\Lambda, I\rangle=\left\{(\lambda_\alpha)_{\alpha\in\Lambda}\mid \nu_p\Big(\frac{\lambda_\alpha}{p^{i_\alpha}}\Big)\geq 0\right\}$$

and

$$A\langle\Lambda, I, +\rangle=\left\{(\lambda_\alpha)_{\alpha\in\Lambda}\mid \nu_p\Big(\frac{\lambda_\alpha}{p^{i_\alpha}}\Big)\geq 0,\ \frac{\lambda_\alpha}{p^{i_\alpha}}\xrightarrow{\nu_p}0\right\}.$$

Definition A.4. Assume M is a (topologically) free A-module. Let Σ_1 and Σ_2 be two subsets of M.

(1) We write $\Sigma_1\sim\Sigma_2$, if they (topologically) generate the same sub-A-module of M.

(2) We write $\Sigma_1\approx\Sigma_2$, if both of them are sets of (topological) basis of M. In this case, we also write $M\approx\Sigma_1$ if no ambiguity appears.

Proposition A.5. *Fix $\epsilon,\omega\in\mathcal{O}_{\mathbb{C}_p}$. Let M be a (topologically) free A-module with basis $\{x_i\}_{i\geq 0}$. If $N\subset M$ is a submodule such that*

$$N\sim\{\omega(x_i+i\epsilon x_{i-1})\mid i\geq 0\},$$

where $x_{-1}=0$, then $N=\omega M$.

Proof. Put $y_i=x_i+i\epsilon x_{i-1}$ for all i. Then we see that

$$(y_0,y_1,y_2,y_3,\dots)=(x_0,x_1,x_2,x_3,\dots)\cdot X$$

with

$$X=\begin{pmatrix}1&\epsilon&0&0&\cdots\\0&1&2\epsilon&0&\cdots\\0&0&1&3\epsilon&\cdots\\0&0&0&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}$$

and that

$$(x_0,x_1,x_2,x_3,\dots)=(y_0,y_1,y_2,y_3,\dots)\cdot Y$$

with

$$Y=\begin{pmatrix}1&-\epsilon&2\epsilon^2&-6\epsilon^3&\cdots\\0&1&-2\epsilon&6\epsilon^2&\cdots\\0&0&1&-3\epsilon&\cdots\\0&0&0&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}$$

The (i,j)-entry of Y is δ_{ij} if $i\geq j$ and is $(-\epsilon)^{j-i}((j-1)!/(i-1)!)$ if $i<j$. Then the proposition follows from the fact $XY=YX=\mathrm{Id}$. □

The following proposition can be proved in the same way.

Proposition A.6. *Fix $\Theta \in \mathrm{M}_l(A)$. Let M be a (topologically) free A-module with basis $\{x_i\}_{i\geq 0}$. Let N be a finite free R-module of rank l with a basis $\{e_1, \dots, e_l\}$. For every $1 \leq j \leq l$ and $i \geq 0$, put $f_{j,i} \in N \otimes_A M$ satisfying*

$$(f_{1,i}, \dots, f_{l,i}) = (e_1 \otimes x_i, \dots, e_l \otimes x_i) + i(e_1 \otimes x_{i-1}, \dots, e_l \otimes x_{i-1})\Theta,$$

where $x_{-1} = 0$. Then $N \otimes_A M \approx \{f_{j,i} \mid 1 \leq j \leq l, i \geq 0\}$.

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On moment map and bigness of tangent bundles of G-varieties

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Let G be a connected algebraic group and let X be a smooth projective G-variety. We prove a sufficient criterion to determine the bigness of the tangent bundle TX using the moment map $\Phi_X^G : T^*X \to \mathfrak{g}^*$. As an application, the bigness of the tangent bundles of certain quasihomogeneous varieties are verified, including symmetric varieties, horospherical varieties and equivariant compactifications of commutative linear algebraic groups. Finally, we study in details the Fano manifolds X with Picard number 1 which is an equivariant compactification of a vector group $\mathbb{G}_a^n$. In particular, we will determine the pseudoeffective cone of $\mathbb{P}(T^*X)$ and show that the image of the projectivised moment map along the boundary divisor D of X is projectively equivalent to the dual variety of the variety of minimal rational tangents of X at a general point.

1. Introduction

Throughout we work over the field of complex numbers $\mathbb{C}$.

Since the seminal works of Mori and Siu-Yau on the solutions to the Hartshorne conjecture and the Frankel conjecture [27; 36], it becomes apparent that making an assumption about the positivity of the tangent bundle TX of a projective manifold X, or equivalently the positivity of the tautological divisor Λ of the projectivisation $\mathbb{P}(T^*X)$ (in the geometric sense), allows us to derive a particularly rich geometry of X. While the situation where TX is ample or nef has been intensively studied in the literature (see [27; 3; 4; 28]), the case where TX is big is much less understood. The main difficulty in investigating the bigness of TX in the general case is the lack of numerical characterisations in terms of invariants of X even in low dimensions. As far as we know, there are three main tools which are used to prove or disprove the bigness of TX. The first one is the (*projectivised*) *moment map*, i.e., the rational map defined by certain subspaces of $|\Lambda|$. The second one is the existence of *twisted symmetric vector fields*, i.e.,

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nonvanishing of $H^0(X, \mathrm{Sym}^m TX \otimes \mathscr{O}_X(-A))$ with A being a big divisor. The third one is to determine the cohomological class of the *total dual variety of minimal rational tangents* $\check{\mathcal{C}} \subseteq \mathbb{P}(T^*X)$. We give in the following an overview of a few of the varieties which have already been studied and also the method used to prove or disprove the bigness of their tangent bundles:

- (Projectivised) moment map:
  - Rational homogeneous spaces [33].
- Twisted symmetric vector fields:
  - Toric varieties [14].
  - Intersection of two quadrics in $\mathbb{P}^4$ and cubic surfaces in $\mathbb{P}^3$ [24].
  - Hypersurfaces in $\mathbb{P}^n$ ($n \geq 3$) [13].
- Total dual variety of minimal rational tangents:
  - del Pezzo surfaces and del Pezzo threefolds [13].
  - Fano manifolds with Picard number 1 and with zero-dimensional variety of minimal rational tangents [12].
  - Moduli spaces $\mathrm{SU}_C(r, d)$ of stable vector bundles of rank r and degree d over a projective curve C of genus g such that $r \geq 3$, $g \geq 4$ and $(r, d) = 1$ [8].
  - Fano threefolds with Picard number 2 [21].

The main body of this paper will be devoted to pursue furthermore the criterion for the bigness of TX via moment map. Let G be a connected algebraic group with Lie algebra $\mathfrak{g}$ and let X be a smooth projective G-variety. Then the *moment map* $\Phi_X^G : T^*X \to \mathfrak{g}^*$ is defined as follows: for a point $x \in X$, the map $T_x^*X \to \mathfrak{g}^*$ is defined as the cotangent map of the orbit map $\mu_x : G \to Gx$ at x; see Section 2C for more details. We denote by $\mathcal{M}_X^G \subseteq \mathfrak{g}^*$ the closure of the image of Φ_X^G. The starting point of this paper is the following criterion for the bigness of TX, which is proved by combining the moment map method with the approach via twisted symmetric vector fields.

Proposition 1.1. *Let G be a connected algebraic group and let X be a smooth projective G-variety. Then TX is big if there exists an effective big divisor A such that*

$$\dim(\Phi_X^G(T^*X|_{\mathrm{Supp}(A)})) < \dim(\mathcal{M}_X^G).$$

We refer the reader to Section 3A for discussion on how to verify the conditions in the criterion. As the first application of Proposition 1.1, the following theorem confirms the bigness of the tangent bundles of certain interesting smooth projective quasihomogeneous varieties.

Theorem 1.2. *Let X be a projective manifold. Then TX is big if X is isomorphic to one of the following varieties*:

(1) *A spherical G-variety with a G-stable affine open subset*, *e.g.*, *symmetric varieties.*

(2) *A horospherical G-variety.*

(3) *A quasihomogeneous G-variety with G a commutative linear algebraic group.*

We refer the reader to Section 3B for the definitions of spherical varieties, symmetric varieties and horospherical varieties. Our initial motivation for the present work is trying to produce more examples of Fano manifolds with Picard number 1 and with big tangent bundle, while the only previous known nonhomogeneous examples, up to our knowledge, are the quintic del Pezzo threefold V_3 [13] and the horospherical G_2-variety X^5 [32]. As the second application of Proposition 1.1, we derive infinitely many (nonhomogeneous) examples of Fano manifolds with Picard number 1 and with big tangent bundle, which are summarised in the following:

- Rational homogeneous spaces G/P with Picard number 1 [33]; see Theorem 2.10 and Example 3.9.
- Nonhomogeneous projective symmetric varieties (and their degenerations) [34]:
  - The Cayley Grassmannian CG [25].
  - The double Cayley Grassmannian DG [26].
  - A smooth hyperplane section of the third row of the *geometric Freudenthal's magic square* $\mathrm{Gr}_\omega(\mathbb{A}^3, \mathbb{A}^6)$, where $\mathbb{A}$ is a complex composition algebra (i.e., the complexification of $\mathbb{R}$, $\mathbb{C}$, the quaternions $\mathbb{H}$, or the octonions $\mathbb{O}$) [23].

  See Corollary 3.11 and Remark 3.12.
- Nonhomogeneous smooth projective horospherical varieties [31]:
  - $X^1(m) := (B_m, \omega_{m-1}, \omega_m)$ $(m \geq 3)$.
  - $X^2 := (B_3, \omega_1, \omega_3)$.
  - $X^3(m, i) := (C_m, \omega_i, \omega_{i+1})$ $(m \geq 2, 1 \leq i \leq m-1)$.
  - $X^4 := (F_4, \omega_2, \omega_3)$.
  - $X^5 := (G_2, \omega_2, \omega_1)$.

  The varieties $X^3(m, i)$ are the odd symplectic Grassmannians and we refer the reader to [31] for the notations; see Proposition 3.13 and Remark 3.14.
- A smooth linear section V_k of $\mathrm{Gr}(2, 5) \subseteq \mathbb{P}^9$ with codimension $k \leq 3$ in its Pücker embedding [12; 13]. The variety V_1 is isomorphic to the horospherical variety $X^3(2, 1)$; see Proposition 3.15 and Example 4.5.
- A smooth linear section S_k of the 10-dimensional spinor variety $\mathbb{S}_5 \subseteq \mathbb{P}^{15}$ with codimension $k \leq 3$ in its minimal embedding. The variety S_1 is the horospherical variety X^2; see Proposition 3.15, Example 4.5 and Corollary 4.18.
- The smooth projective two-orbits F_4-variety $\boldsymbol{X}_1$ given in [31, Definition 2.11]. Note that the smooth projective two-orbits $G_2 \times \mathrm{PGL}_2$-variety $\boldsymbol{X}_2$ is isomorphic to the general codimension 2 linear section S_2^g of $\mathbb{S}_5$; see [1, Proposition 4.8; 31, Definition 2.12] and Proposition 4.21.

An interesting class of examples belonging to case (3) of Theorem 1.2 is the equivariant compactifications of vector groups. Recall that an *equivariant compactification* of an algebraic group G is a pair (X, x), where X is a normal complete algebraic variety equipped with a regular action $G \times X \to X$ and $x \in X$ is a point with the trivial stabiliser such that the orbit Gx is open and dense in X. We find it quite remarkable that the moment map of a Fano manifold X with Picard number 1 which is an equivariant compactification of a vector group $\mathbb{G}_a^n$ exhibits many interesting geometric properties and it has a surprising connection with the variety of minimal rational tangents (VMRT, for short) of X. In particular, in this situation, as the last application of Proposition 1.1, we have a complete description of the pseudoeffective cone of $\mathbb{P}(T^*X)$ and we can relate the criterion given in Proposition 1.1 to the criterion given by total dual VMRT in [8; 13]. We refer the reader to Section 4A for the explicit definitions of dual varieties, codegree, VMRT and total dual VMRT.

Theorem 1.3. *Let X be a Fano manifold with Picard number* 1, *different from projective spaces*, *which is an equivariant compactification of the vector group $G = \mathbb{G}_a^n$ with an open orbit $O \subseteq X$. Denote by D the complementary $X \setminus O$ and by $\mathcal{M}_D^G$ the closure of the image of the restricted map $\Phi_X^G|_D : T^*X|_D \to \mathfrak{g}^*$. Then the following statements hold*:

(1) *The pseudoeffective cone $\overline{\mathrm{Eff}}(\mathbb{P}(T^*X))$ is generated by π^*D and all the prime divisors $D_{\mathcal{H}}$ (see Notation 3.2), where $\pi : \mathbb{P}(T^*X) \to X$ is the natural projection and $\mathcal{H}$ is a reduced and irreducible hypersurface in $\mathbb{P}(\mathfrak{g}^*)$ containing $\mathbb{P}(\mathcal{M}_D^G)$.*

(2) *If the VMRT $\mathcal{C}_x \subseteq \mathbb{P}(T_xX)$ at a point $x \in O$ is smooth, then $\mathbb{P}(\mathcal{M}_D^G) \subseteq \mathbb{P}(\mathfrak{g}^*)$ is projectively equivalent to the dual variety of $\mathcal{C}_x \subseteq \mathbb{P}(T_xX)$.*

(3) *If the VMRT $\mathcal{C}_x \subseteq \mathbb{P}(T_xX)$ at a point $x \in O$ is smooth and not dual defective, then we have*

$$D_{\mathcal{H}} = \check{\mathcal{C}}, \quad \overline{\mathrm{Eff}}(\mathbb{P}(T^*X)) = \langle D_{\mathcal{H}}, \pi^*D \rangle \quad \text{and} \quad D_{\mathcal{H}} \sim a\Lambda - 2\pi^*D$$

where $\mathcal{H} = \mathbb{P}(\mathcal{M}_D^G) \subseteq \mathbb{P}(\mathfrak{g}^*)$, the variety $\check{\mathcal{C}} \subseteq \mathbb{P}(T^*X)$ is the total dual VMRT, the coefficient a is the codegree of the VMRT and Λ is the tautological divisor of $\mathbb{P}(T^*X)$.

Remark 1.4. (1) For projective spaces, there exist nonisomorphic equivariant compactification structures of vector groups and they are classified by the so-called *Hassett–Tschinkel correspondence* proved in [11]; see also [6]. In particular, Theorem 1.3 above holds for the simplest equivariant compactification structure on projective spaces (see Example 4.6), however the statement (2) is no longer true for others. Indeed, since the VMRT of a projective space is the whole projectivised tangent space, its total dual VMRT is an empty set. Therefore, if the statement (2) holds, then $\mathcal{M}_D^G$ is the origin of $\mathfrak{g}^*$. This implies that the points in D are fixed under the action of G.

(2) For the known examples of equivariant compactifications of vector groups with Picard number 1 (see Example 4.5), we will determine in Table 1 the dual defect and the codegree of their VMRTs, i.e., the value of a.

2. Notation, conventions, and facts used

Let X be a projective manifold. Denote by $N^1(X)_{\mathbb{R}}$ the finite-dimensional $\mathbb{R}$-vector space of numerical equivalence classes of $\mathbb{R}$-divisors. The *pseudoeffective cone* $\overline{\mathrm{Eff}}(X) \subseteq N^1(X)_{\mathbb{R}}$ is the closure of the convex cone spanned by the classes of effective $\mathbb{R}$-divisors. It is known the interior of $\overline{\mathrm{Eff}}(X)$ is the *big cone* $\mathrm{Big}(X)$ of X; that is, the open cone generated by big divisors on X.

2A. *Positivity of vector bundles.* Let E be a vector bundle over a smooth projective variety X. We denote by $\mathbb{P}(E)$ the natural (not the Grothendieck) projectivisation of E; that is, we have

$$\mathbb{P}(E) := \mathrm{Proj}\Bigg(\bigoplus_{m\geq 0} \mathrm{Sym}^m E^*\Bigg),$$

where E^* is the dual bundle of E.

Definition 2.1. Let E be a vector bundle over a projective manifold X. We say that E is big (resp. ample, nef, pseudoeffective) if the tautological line bundle $\mathscr{O}_{\mathbb{P}(E^*)}(1)$ of the projective bundle $\mathbb{P}(E^*)$ is big (resp. ample, nef, pseudoeffective).

Example 2.2. Let $E \cong \mathscr{O}_{\mathbb{P}^1}(a_1)\oplus\cdots\oplus\mathscr{O}_{\mathbb{P}^1}(a_r)$ be a vector bundle of rank r over $\mathbb{P}^1$ such that $a_1 \geq \cdots \geq a_r$. Then we have

$$E \text{ is } \begin{cases} \text{ample} & \text{if and only if } a_r > 0;\\ \text{nef} & \text{if and only if } a_r \geq 0;\\ \text{big} & \text{if and only if } a_1 > 0;\\ \text{pseudoeffective} & \text{if and only if } a_1 \geq 0. \end{cases}$$

We have the following simple but useful criterion for bigness of vector bundles:

Lemma 2.3 (equivalent definitions of bigness, [13, Lemma 2.3]). *Let E be a vector bundle over a projective manifold X. Denote by $\pi : \mathbb{P}(E^*) \to X$ the natural projection and by Λ the tautological divisor class of $\mathbb{P}(E^*)$. Then the following statements are equivalent*:

(1) *The vector bundle E is big.*

(2) *There exists a big divisor A on X and a positive integer m such that $m\Lambda - \pi^*A$ is big.*

(3) *There exists a big divisor A on X and a positive integer m such that $m\Lambda - \pi^*A$ is pseudoeffective.*

Proof. The implication (1)⟹(2) follows from the openness of bigness, and the implication (2)⟹(3) is trivial. Finally the implication (3)⟹(1) follows from [13, Lemma 2.3]. □

Lemma 2.4. *Let F and E be two vector bundles over a projective manifold X such that there exists a generically injective map $\sigma : F \to E$. If F is big, then E is big.*

Proof. By Lemma 2.3, there exists a big divisor A on X and a positive integer m such that $m\Lambda_F - \pi_F^*A$ is big, where Λ_F is the tautological divisor of $\mathbb{P}(F^*)$ and $\pi_F : \mathbb{P}(F^*) \to X$ is the natural projective. Then, after replacing m by its large enough multiple $m'm$ and replacing A by $m'A$, we may assume that $|m\Lambda_F - \pi_F^*A|$ is nonempty. In particular, we have $H^0(X, \mathrm{Sym}^m F \otimes \mathscr{O}_X(-A)) \neq \varnothing$. This implies that $H^0(X, \mathrm{Sym}^m E \otimes \mathscr{O}_X(-A)) \neq \varnothing$. In other words, we have $|m\Lambda_E - \pi_E^*A| \neq \varnothing$ and hence E is big by Lemma 2.3, where Λ_E is the tautological divisor of $\mathbb{P}(E^*)$ and $\pi_E : \mathbb{P}(E^*) \to X$ is the natural projection. □

Example 2.5. Let X be a projective manifold and let L be a big line bundle over X. Set $E = L \oplus L^*$. Then the tangent bundle of $\mathbb{P}(E^*)$ is big. Indeed, we note that E is big by Lemma 2.4 as L is big. In particular, the relative tangent bundle of $\mathbb{P}(E^*) \to X$ is big as it is isomorphic to the line bundle $\mathscr{O}_{\mathbb{P}(E^*)}(2)$. Consequently, the tangent bundle $T\mathbb{P}(E^*)$ itself is big by Lemma 2.4.

2B. ***Pseudoeffective cone of divisors on G-varieties.*** Let D be a $\mathbb{Q}$-Weil divisor on a normal projective variety X and let $\bar{D}$ be an irreducible component of $\mathrm{supp}(D)$. We denote by $m_{\bar{D}}(D)$ the multiplicity of D along $\bar{D}$, i.e., the coefficient of $\bar{D}$ in D.

Lemma 2.6. *Let X be a projective manifold. Let D and D' be two effective $\mathbb{Q}$-Weil divisors on X such that* $\mathrm{supp}(D) = \mathrm{supp}(D')$. *Then D is big if and only if D' is big.*

Proof. By symmetry, it suffices to prove that if D is big, then D' is big. Let us denote by $D_1, \dots, D_r$ the irreducible components of $\mathrm{supp}(D) = \mathrm{supp}(D')$. We define

$$m := \max\{m_{D_i}(D) \mid i = 1, \dots, r\} \quad \text{and} \quad m' := \min\{m_{D_i}(D') \mid i = 1, \dots, r\}.$$

Let n be a positive integer such that $nm' \geq m$. Then $nD' - D$ is effective. In particular, it follows that nD' is big and hence so is D' itself. □

In general, the pseudoeffective cone of a projective manifold may be very complicated to describe. However, if X admits a G-action for some solvable linear algebraic group G, then we have the following very useful result concerning $\overline{\mathrm{Eff}}(X)$.

Theorem 2.7 [2, Théorème 1.3]. *Let G be a connected solvable linear algebraic group and let X be a smooth projective G-variety. Then every effective cycle on X is rationally equivalent to a G-stable effective cycle. In particular, the pseudoeffective cone $\overline{\mathrm{Eff}}(X)$ of X is generated by G-stable divisors.*

As an immediate application of the theorem above, one can easily derive the following criterion for bigness of G-equivariant vector bundles over smooth projective G-varieties.

Proposition 2.8 (criterion for bigness of G-equivariant vector bundles). *Let G be a connected solvable linear algebraic group. Let E be a G-equivariant vector bundle over a smooth projective G-variety X. Denote by $\pi : \mathbb{P}(E^*) \to X$ the natural projection and by Λ the tautological divisor class of $\mathbb{P}(E^*)$. Then E is big if and only if there exist G-stable effective integral divisors Δ on $\mathbb{P}(E^*)$ and D on X satisfying*:

(1) *There exists a positive integer $m > 0$ such that $\Delta \in |m\Lambda|$.*

(2) *The divisor D is big and $\Delta - \pi^* D \geq 0$.*

Proof. One direction is clear by Lemma 2.3. Thus we may assume that E is big. By Lemma 2.3, there exists a big divisor A on X and a positive integer m_1 such that $m_1\Lambda - \pi^* A$ is big. On the other hand, by Theorem 2.7, there exists a G-stable effective divisor D' on X such that $A \sim_{\mathbb{Q}} r_1 D'$ and a G-stable effective divisor Δ' on $\mathbb{P}(E^*)$ such that $m_1\Lambda - \pi^* A \sim_{\mathbb{Q}} r_2\Delta'$ for some rational numbers $r_1, r_2 > 0$. Set $\Delta = m_2(r_2\Delta' + r_1\pi^* D')$ for some sufficiently divisible positive integer m_2. Then we conclude by letting $m = m_1 m_2$ and $D = m_2 r_1 D'$. □

2C. *Moment map of G-varieties.* In this subsection, we briefly recall basic facts concerning moment maps of G-varieties and we refer the reader to [40] for more details. Let X be an n-dimensional smooth algebraic variety. Then there exists a standard symplectic structure on the cotangent bundle T^*X of X, which is given by a 2-form $\omega = d\boldsymbol{x} \wedge d\boldsymbol{y} = \sum dx_i \wedge dy_i$, where $\boldsymbol{x} = (x_1, \dots, x_n)$ is a tuple of local coordinates on X and $\boldsymbol{y} = (y_1, \dots, y_n)$ is an impulse, i.e., a tuple of dual coordinates in a cotangent space. If X is a G-variety, then the symplectic structure on T^*X is G-invariant and, for every $\xi \in \mathfrak{g}$, the velocity field of ξ on T^*X has a Hamiltonian $H_\xi = \xi_*$, the respective velocity field on X considered as a linear function on T^*X. Furthermore, the action of G on T^*X is Hamiltonian, i.e., the map $\xi \mapsto H_\xi$ is a homomorphism of $\mathfrak{g}$ to the Poisson algebra of functions on T^*X. The dual morphism $\Phi_X^G : T^*X \to \mathfrak{g}^*$ defined as following

$$\langle \Phi_X^G(w), \xi \rangle = H_\xi(w) = \langle w, H_\xi(x) \rangle, \forall w \in T_x^*X, \xi \in \mathfrak{g}, \tag{2.8.1}$$

is called the *moment map*. We denote by $\mathcal{M}_X^G \subseteq \mathfrak{g}^*$ the closure of the image of the moment map. If $Z \subseteq X$ is a closed (maybe reducible) subvariety, we denote by $\mathcal{M}_X^G(Z) \subseteq \mathfrak{g}^*$ the closure of the image $\Phi_X^G(T^*X|_Z)$. Let $T^{\mathfrak{g}}X \subseteq X \times \mathfrak{g}^*$ be the closure of the image of the following map

$$\pi \times \Phi_X^G : T^*X \to X \times \mathfrak{g}^*,$$

where $\pi : T^*X \to X$ is the natural projection. Then clearly the moment map factors as

$$\Phi_X^G : T^*X \to T^{\mathfrak{g}}X \xrightarrow{\widehat{\Phi}_X^G} \mathfrak{g}^*.$$

The morphism $\widehat{\Phi}_X^G$ is called the *localised moment map*. The general fibres of $\widehat{\Phi}_X^G$ are the cotangent spaces $\mathfrak{g}_x^\perp = T_x^*Gx$ to general orbits and the induced map $T_x^{\mathfrak{g}}X \to \mathfrak{g}^*$ is exactly the cotangent map of the orbit map μ_x at x. Here $\mathfrak{g}_x$ is the Lie algebra of the isotropy subgroup G_x of G at x.

The moment map Φ_X^G is equivariant with respect to the natural $\mathbb{C}^*$-actions on T^*X and $\mathfrak{g}^*$. This implies that the Φ_X^G induces a *projectivised moment map*

$$\bar{\Phi}_X^G : \mathbb{P}(T^*X) \dashrightarrow \mathbb{P}(\mathfrak{g}^*).$$

Then the closure of the image of $\bar{\Phi}_X^G$ is exactly $\mathbb{P}(\mathcal{M}_X^G)$ and $\mathcal{M}_X^G$ is the affine cone of $\mathbb{P}(\mathcal{M}_X^G)$. Moreover, denote by $V \subseteq H^0(X, TX)$ the subspace of Hamiltonians. Then V can be naturally identified to a linear system $\overline{V} \subseteq |\mathscr{O}_{\mathbb{P}(T^*X)}(1)|$ and the rational map $\bar{\Phi}_X^G$ is exactly the rational map defined by the linear system $\overline{V}$. Indeed, firstly note that we have a natural surjective linear map $\mathfrak{g} \to V$ whose kernel consists of elements $\xi \in \mathfrak{g}$ such that $H_\xi = 0$. Let $x \in X$ be a general point and $\omega \in T_x^*X$ be a general element. Then the rational map $\Phi_{\overline{V}} : \mathbb{P}(T^*X) \dashrightarrow \mathbb{P}(V^*) \subseteq \mathbb{P}(\mathfrak{g}^*)$ defined by the linear system $\overline{V}$ sends the point $[\omega] \in \mathbb{P}(T_x^*X)$ to $[V_x^\perp] \in \mathbb{P}(V^*)$, where V_x is the codimension one subspace of V defined as

$$V_x := \{H_\xi \in V \mid \langle \omega, H_\xi(x) \rangle = 0\}.$$

Comparing this with (2.8.1), one can easily derive that $\bar{\Phi}_X^G([\omega]) = \Phi_{\overline{V}}([\omega])$; that is, the map $\bar{\Phi}_X^G$ coincides with $\Phi_{\overline{V}}$.

Finally, we note that the moment map Φ_X^G is G-equivariant and $\mathcal{M}_X^G$ is a G-birational invariant of X. In particular, after passing to a smooth G-stable open subset, we can define the moment map for singular G-varieties. Moreover, the moment map also induces a homomorphism of filtered algebras

$$\operatorname{gr}\Phi^* : \operatorname{Sym}^\bullet \mathfrak{g} = \mathbb{C}[\mathfrak{g}^*] \to H^0(X, \operatorname{Sym}^\bullet TX) \subseteq \mathbb{C}[T^*X], \tag{2.8.2}$$

where $\mathbb{C}[\mathfrak{g}^*]$ (resp. $\mathbb{C}[T^*X]$) is the algebra of regular functions on $\mathfrak{g}^*$ (resp. T^*X).

If G is a reductive linear algebraic group, the dimension of $\mathcal{M}_X^G$ is given by Knop [22] in terms of two numerical invariants of X related to the action of a Borel subgroup of G: the complexity and the rank; see for instance Example 3.8.

Definition 2.9 (complexity and rank). Let G be a connected reductive linear algebraic group with a fixed Borel subgroup B, and let X be an algebraic G-variety:

(1) The *complexity* $c(X)$ of the action G on X is the codimension of a general B-orbit in X.

(2) The *rank* $r(X)$ of the action G on X is the rank of $\Lambda(X)$, where $\Lambda(X)$ is the set of weights of all rational B-eigenfunctions on X.

We recall that for any linear algebraic group G acting on a variety X, the set of rational G-eigenfunctions on X are defined as

$$\mathbb{C}(X)^{(G)} = \{f \in \mathbb{C}(X) \setminus \{0\} \mid \exists \chi \in \mathscr{X}(G) \text{ s.t. } \forall g \in G, g \cdot f = \chi(g) f\},$$

where $\mathbb{C}(X)$ is the field of rational functions on X and $\mathscr{X}(G)$ is the group of characters of G.

Theorem 2.10 (dimension formula of $\mathcal{M}_X^G$, [22, Satz 7.1]). *Let G be a connected reductive linear algebraic group and let X be a projective G-variety. Then we have*

$$\dim(\mathcal{M}_X^G) = \dim(\mathbb{P}(\mathcal{M}_X^G)) + 1 = 2\dim(X) - 2c(X) - r(X).$$

3. Criteria for bigness and proof of Theorem 1.2

In this section, firstly we shall prove Proposition 1.1 which gives a sufficient condition to guarantee the bigness of tangent bundles of smooth projective G-varieties via its moment map along effective big divisors. Next we will discuss several situations where the conditions in Proposition 1.1 hold automatically. Finally, we apply these criteria to prove Theorem 1.2, which confirms the bigness of tangent bundles of certain quasihomogeneous spaces, including symmetric varieties, spherical varieties and equivariant compactifications of commutative linear algebraic groups.

3A. ***Criterion for bigness via moment map.*** Let m be a positive integer. For every $\xi \in \operatorname{Sym}^m \mathfrak{g}$, we denote by $H_\xi \in H^0(X, \operatorname{Sym}^m TX)$ the image of ξ under the map $\operatorname{gr}\Phi^*$; see (2.8.2). Then for any point $x \in X$, the evaluation $H_\xi(x)$ of H_ξ at x can be regarded as a homogeneous polynomial of degree m over the cotangent space T_x^*X. We note that $H_\xi(x)$ may be identically zero over T_x^*X and the following observation relates the vanishing locus of $H_\xi(x)$ in T_x^*X to the zero locus of ξ in $\mathfrak{g}^*$.

Lemma 3.1. *Let G be a connected algebraic group and let X be a smooth projective G-variety. Given a positive integer m, an element $\xi \in \operatorname{Sym}^m \mathfrak{g}$ and a point $w \in T^*_x X$, then $H_\xi(x)$ vanishes at w if and only if ξ vanishes at $\Phi^G_X(w) \in \mathfrak{g}^*$.*

Proof. Note that the moment map Φ^G_X restricted to $T^*_x X$ is just the following composition

$$\Phi^G_{X,x} : T^*_x X \to T^*_x O_x = \mathfrak{g}_x^\perp \hookrightarrow \mathfrak{g}^*,$$

where O_x is the G-orbit of x. In particular, the form $H_\xi(x)$ is the composition $\xi \circ \Phi^G_{X,x}$, where ξ is regarded as a function over $\mathfrak{g}^*$. □

Given a positive integer m and a Weil divisor A on X, recall that we have the following natural isomorphism

$$H^0(\mathbb{P}(T^*X), \mathscr{O}_{\mathbb{P}(T^*X)}(m) \otimes \mathscr{O}_{\mathbb{P}(T^*X)}(-\pi^*A)) \to H^0(X, \operatorname{Sym}^m T_X \otimes \mathscr{O}_X(-A)). \tag{3.1.1}$$

Let $\sigma \in H^0(\mathbb{P}(T^*X), \mathscr{O}_{\mathbb{P}(T^*X)}(m))$ be a section and denote by $H_\sigma \in H^0(X, \operatorname{Sym}^m T_X)$ the corresponding symmetric vector field on X. Then, for a prime divisor A on X, the section σ vanishes along π^*A if and only if the corresponding form H_σ vanishes along A.

Proof of Proposition 1.1. By Lemma 2.6, we may assume that A is reduced. Since both $\mathcal{M}^G_X(A)$ and $\mathcal{M}^G_X$ are invariant under the dilation action of $\mathbb{C}^*$ on $\mathfrak{g}^*$, by assumption, we have

$$\dim(\mathbb{P}(\mathcal{M}^G_X(A))) < \dim(\mathbb{P}(\mathcal{M}^G_X)).$$

In particular, as $\mathcal{M}^G_X$ is irreducible, there is a hypersurface in $\mathbb{P}(\mathfrak{g}^*)$ defined by a homogeneous polynomial $\xi \in \operatorname{Sym}^m \mathfrak{g}$ of degree m such that it contains $\mathbb{P}(\mathcal{M}^G_X(A))$ but not $\mathbb{P}(\mathcal{M}^G_X)$. Let $H_\xi \in H^0(X, \operatorname{Sym}^m T_X)$ be the corresponding symmetric vector field on X. Then we have $H_\xi \neq 0$ and H_ξ vanishes identically along A.

Let $\sigma \in H^0(\mathbb{P}(T^*X), \mathscr{O}_{\mathbb{P}(T^*X)}(m))$ be the global section such that $H_\sigma = H_\xi$. Then according to Lemma 3.1 and the discussion before the proof, the section σ vanishes identically along π^*A. In particular, the following divisor

$$m\Lambda - \pi^*A \sim \operatorname{div}(\sigma) - \pi^*A \geq 0$$

is pseudoeffective. Thus, as A is big, it follows from Lemma 2.3 that TX is big. □

Notation 3.2. Given a (maybe nonreduced and reducible) hypersurface $\mathcal{H} \subseteq \mathbb{P}(\mathfrak{g}^*)$ defined by $\xi \in \operatorname{Sym}^m \mathfrak{g}$. We will denote by $D_\xi \in |\mathscr{O}_{\mathbb{P}(T^*X)}(m)|$ the divisor corresponding to ξ and the divisor $D_{\mathcal{H}}$ is defined as the horizontal part of D_ξ. Let $\pi : \mathbb{P}(T^*X) \to X$ be the natural projection. Then we have

$$D_\xi = D_{\mathcal{H}} + \sum \operatorname{mult}_{\pi^*D}(D_\xi)\pi^*D,$$

where D runs over all the prime divisor in X such that $\mathbb{P}(\mathcal{M}^G_X(D))$ is contained in $\mathcal{H}$.

3A1. *Criteria for bigness of boundary divisors.* To apply Proposition 1.1, firstly we need to find an effective big divisor A in X. In most cases considered in this paper, the variety X will be a smooth quasihomogeneous projective variety and it is natural to choose A to be the complement of the unique open orbit.

Proposition 3.3 [10, Section I, Proposition 1 and Theorem 1]. *Let X be a projective manifold and let $U \subseteq X$ be an affine dense open subset of X. Then the complement $D := X \setminus U$ has pure codimension* 1 *and the line bundle $\mathscr{O}_X(D)$ is big.*

Proof. The first statement follows from [10, Proposition 1]. For the second statement, since U is affine, by [10, Theorem 1], there is a closed subvariety Z of D and a blowing-up $\varphi : \overline{X} \to X$ with the centre $Z \subseteq D$ such that $\varphi^{-1}(D)$ is the support of an effective ample divisor A on $\overline{X}$. In particular, the push-forward $\varphi_* A$ is a big Weil divisor with support D. Then it follows from Lemma 2.6 that D itself is big. □

Let G be connected linear algebraic group. A closed subgroup H of G is said to be *regularly embedded* in G if $\mathrm{Rad}_u(H) \subseteq \mathrm{Rad}_u(G)$, where $\mathrm{Rad}_u(H)$ (resp. $\mathrm{Rad}_u(G)$) is the unipotent radical of H (resp. G). For example, if there is no parabolic subgroup of G containing H, then H is regularly embedded in G [16, 30.3].

Lemma 3.4 (criteria for bigness of boundary). *Let G be a connected linear algebraic group and let X be a smooth projective G-variety with a Zariski open dense orbit O. Then the complement $D := X \setminus O$ is a big divisor if one of the following holds*:

(1) *The group G is solvable.*

(2) *For a point $x \in O$, the isotropy subgroup G_x of G at x is regularly embedded in G.*

(3) *For a point $x \in O$, the isotropy subgroup G_x of G at x is reductive.*

Proof. By Proposition 3.3, it is enough to show that O is an affine variety and the latter follows from certain known criteria for affineness of homogeneous spaces; see for instance [39, Theorems 3.5, 3.7 and 3.8]. □

3A2. *Image of moment map along boundary divisors.* Once we have an effective big divisor D on a projective G-variety X, to apply Proposition 1.1, we need to control the dimension of $\mathcal{M}_X^G(D_{\mathrm{red}})$. In the following we consider the case where D is G-stable. Let G be a connected algebraic group and H be a closed subgroup. Let F be an H-variety. Then H acts on $G \times F$ by $h(g, f) = (gh^{-1}, h \cdot f)$ and we denote by $G *_H F$ the quotient set $(G \times F)/H$, which is a homogeneous fibre bundle over G/H.

Lemma 3.5. *Let G be a connected algebraic group and let $X = G/H$ be a homogeneous variety. Denote by N the normaliser $N_G(H)$ of H in G and by $\mathfrak{n}$ its Lie algebra. Then we have $T^*X = G *_H \mathfrak{h}^\perp$ and the moment map Φ_X^G factors as*

$$G *_H \mathfrak{h}^\perp \to G *_N \mathfrak{h}^\perp \to \mathfrak{g}^*,$$

where N acts on $\mathfrak{h}^\perp$ by coadjoint action. In particular, we have

$$\dim(\mathcal{M}_X^G) \leq \dim(X) + \dim(\mathfrak{g}) - \dim(\mathfrak{n}). \tag{3.5.1}$$

Proof. This is a standard fact about homogeneous spaces. Let us recall the proof for the reader's convenience. There is a canonical isomorphism $T_x X \cong \mathfrak{g}/\mathfrak{g}_x$ for any point $x \in X$. In particular, we get a canonical embedding $T^*X \to X \times \mathfrak{g}^*$. Then the isomorphism $\Psi : G *_H \mathfrak{h}^\perp \to T^*X$ is given as follows:

$$\Psi(g, w) = ([gH], \mathrm{Ad}_g^*(w)),$$

where $g \in G$, $w \in \mathfrak{h}^\perp$ and $\mathrm{Ad}_g^* \in \mathrm{GL}(\mathfrak{g}^*)$ is the coadjoint representation of $\mathfrak{g}$. In particular, under the isomorphism Ψ, the moment map $\Phi_X^G : T^*X \to \mathfrak{g}^*$ can be written as

$$\Phi_X^G(g * w) = \mathrm{Ad}_g^*(w),$$

where $g \in G$ and $w \in \mathfrak{h}^\perp$. For any $n \in N$, we have $\mathrm{Ad}_n^*(\mathfrak{h}^\perp) = \mathfrak{h}^\perp$ by definition. This immediately implies that Φ_X^G factors through $G *_N \mathfrak{h}^\perp \to \mathfrak{g}^*$. The inequality (3.5.1) then follows from the fact that $\dim(\mathcal{M}_X^G) \leq \dim(G *_N \mathfrak{h}^\perp) = \dim(G/N) + \dim(\mathfrak{h}^\perp)$. □

Lemma 3.6. *Let G be a connected linear algebraic group and let X be a smooth projective G-variety. Let D be a G-stable prime divisor in X. Then we have*

$$\dim(\mathcal{M}_X^G(D)) = \dim(\mathcal{M}_D^G) \leq \dim(X) + \dim(\mathfrak{g}) - \dim(\mathfrak{n}) - 1,$$

where $\mathfrak{n}$ is the Lie algebra of the normaliser $N_G(H)$ of the isotropy subgroup of G at a general point $x \in D$.

Proof. Since D is G-stable, the restriction of the moment map Φ_X^G to D factors as

$$T^*X|_{D_{\mathrm{reg}}} \to T^*D_{\mathrm{reg}} \to \mathfrak{g}^*,$$

where D_{reg} is the smooth locus of D. In particular, we have $\mathcal{M}_X^G(D) = \mathcal{M}_D^G$. Let O_x be the orbit of a general point $x \in D_{\mathrm{reg}}$. Then we also have

$$\dim(\mathcal{M}_D^G) = \dim(\widehat{\Phi}_D^G(T^{\mathfrak{g}} D_{\mathrm{reg}})) = \dim(D) - \dim(O_x) + \dim(\mathcal{M}_{O_x}^G).$$

Then we conclude by applying Lemma 3.5 to the homogeneous space $O_x = G/G_x$. □

Lemma 3.7. *Let G be a connected reductive linear algebraic group and let X be a smooth projective G-variety. Let D be a G-stable prime divisor in X. Then*

$$\dim(\mathcal{M}_X^G(D)) = \dim(\mathcal{M}_D^G) < \dim(\mathcal{M}_X^G).$$

if and only if

$$c(X) = c(D) \quad \textit{and} \quad r(X) = r(D) + 1.$$

Proof. This follows directly from Knop's dimension formula, see Theorem 2.10. Here we remark that we have always $c(D) \leq c(X)$, $r(D) \leq r(X)$ and the equality holds if and only if $D = X$ [39, Theorem 5.7]. □

Example 3.8 (quintic del Pezzo threefold). Let $X = V_3$ be the smooth quintic del Pezzo threefold, e.g., a smooth codimension 3 linear section of $\mathrm{Gr}(2, 5) \subseteq \mathbb{P}^9$. Then TX is big by [13, Theorem 1.5]. Denote by H the ample generator of $\mathrm{Pic}(X)$. Recall that there is an SL_2-action on X with three orbits [29]: a unique open orbit, a unique 2-dimensional orbit whose closure D is linearly equivalent to $2H$, a unique 1-dimensional orbit which is a rational normal curve of degree 6. Then under the SL_2-action, we have

$$c(X) = r(X) = 1, \quad c(D) = 0 \quad \text{and} \quad r(D) = 1.$$

Hence, by Theorem 2.10, we obtain that $\mathcal{M}_X^G = \mathcal{M}_D^G = \mathfrak{g}^*$. This shows that the converse of Proposition 1.1 is false in general.

3B. ***Proof of Theorem 1.2.*** In this subsection, we aim to apply the criteria proved in the previous subsection to prove the bigness of the tangent bundles of certain interesting quasihomogeneous G-varieties. In particular, we shall finish the proof of Theorem 1.2.

3B1. *Spherical varieties.* Let G be a connected reductive linear algebraic group and let X be a normal G-variety. Then X is said to be *spherical* if $c(X) = 0$. In particular, there is an open G-orbit $G/H \subseteq X$. Let Y be a G-orbit in X. Denote by $\mathcal{V}_Y(X)$ the set of G-stable prime divisors in X containing Y and by $\mathcal{D}_Y(X)$ the set of B-stable but not G-stable prime divisors in X containing Y. Write $\mathbb{X}_{\mathbb{Q}}^{\vee}$ the tensor product of the dual lattice of $\Lambda(X)$ with $\mathbb{Q}$; see Definition 2.9. For any prime divisor D in X there is an associated valuation ν_D and also an associated element $\rho(D)$ in $\mathbb{X}_{\mathbb{Q}}^{\vee}$. Denote by $\mathcal{C}_Y^{\vee}(X) \subseteq \mathbb{X}_{\mathbb{Q}}^{\vee}$ the cone generated by the images of divisors in $\mathcal{V}_Y(X)$ and $\mathcal{D}_Y(X)$.

Example 3.9. We collect some typical examples of spherical varieties:

(1) Recall that a toric variety is a normal variety X with a dense orbit of a torus $T = \mathbb{G}_m^r$ such that the points in the dense T-orbit have a trivial stabiliser in T. The variety X is spherical for $G = T$ with $H = \{e\}$. Moreover, we have $r(X) = \dim(T) = \dim(X)$. Recall that it is shown in [14, Corollary 1.3] that the tangent bundle of a smooth projective toric variety is big.

(2) For G a connected semisimple linear algebraic group and P a parabolic subgroup containing a maximal torus T of G, the quotient G/P is a projective rational homogeneous space and the Bruhat decomposition implies that G/P is a spherical G-variety. Moreover, we have $r(X) = 0$. Thus the moment map $\bar{\Phi}_X^G$ is generically finite (see Theorem 2.10) and hence TX is big, see also [33].

(3) Let G be a connected semisimple linear algebraic group equipped with a nonidentical involution $\theta \in \mathrm{Aut}(G)$. Let H be a closed subgroup of G such that $G^{\theta} \subseteq H \subseteq N_G(G^{\theta})$. Then G/H is said to be a *symmetric homogeneous space* and G/H-embeddings are called *symmetric varieties*. The symmetric varieties are spherical.

Proposition 3.10. *Let G be a connected reductive linear algebraic group and let X be smooth projective spherical G-variety. Let D be a G-stable prime divisor in X. Then we have $c(X) = c(D) = 0$ and $r(D) = r(X) - 1$. In particular, if there exists a G-stable affine open subset O of X, then TX is big.*

Proof. Since X is a spherical G-variety, we have $c(D) \leq c(X) = 0$. On the other hand, let Y be the unique open G-orbit in D. Then one can easily obtain that $\mathcal{V}_Y(X) = \{D\}$ and $\mathcal{D}_Y(X) = \varnothing$. This implies that the cone $\mathcal{C}_Y^\vee(X) \subseteq \mathbb{X}_{\mathbb{Q}}^\vee$ is 1-dimensional. In particular, by [39, Proposition 15.14], we obtain

$$r(D) = r(Y) = r(X) - \dim(\mathcal{C}_Y^\vee) = r(X) - 1.$$

Now we assume that there exists a G-stable affine open subset O of X. By Proposition 3.3, the complement $D := X \setminus O$ is a big divisor on X. Then the bigness of TX follows from Proposition 1.1 and Lemma 3.7. □

Corollary 3.11. *Let G be a connected reductive linear algebraic group and let X be a smooth projective spherical G-variety. Then TX is big if one of the following holds*:

(1) *The variety X is a symmetric variety.*

(2) *The variety X has Picard number* 1 *and contains a G-stable prime divisor.*

Proof. Firstly we assume that X is a symmetric variety and denote by O the unique open G-orbit of X. Then O is isomorphic to a symmetric homogeneous variety G/H. On the other hand, thanks to [37, Section 8], the subgroup H is reductive. Therefore, by Lemma 3.4, the boundary divisor $D := X \setminus O$ is a big divisor and it follows from Proposition 3.10 above that TX is big.

Next we assume that X has Picard number 1 and there exists a G-stable prime divisor D in X. Then D is ample and it is well known that $O := X \setminus D$ is an affine variety. Hence it follows again from Proposition 3.10 that TX is big. □

Remark 3.12. The smooth projective symmetric varieties with Picard number 1 are classified by Ruzzi [34] and there are exactly six nonhomogeneous ones, including the Cayley Grassmannian CG, the double Cayley Grassmannian DG, a general hyperplane section of $\mathrm{Gr}_\omega(\mathbb{A}^3, \mathbb{A}^6)$, where $\mathbb{A}$ is a complex composition algebra. In particular, by Semicontinuity Theorem, the tangent bundle of any smooth hyperplane section of $\mathrm{Gr}_\omega(\mathbb{A}^3, \mathbb{A}^6)$ is big.

3B2. *Horospherical varieties.* Let G be a connected reductive linear algebraic group. A closed subgroup H of G is said to be *horospherical* if it contains the unipotent radical of a Borel subgroup of G. In this case we shall say that the homogeneous space G/H is *horospherical*. Denote by P the normaliser $N_G(H)$ of H in G. Then P is a parabolic subgroup of G such that P/H is a torus and G/H is a torus bundle over the flag variety G/P. A normal G-variety is said to be *a horospherical variety* if G has an open orbit isomorphic to G/H for some horospherical subgroup H. Horospherical varieties are spherical and their ranks are equal to the rank of the torus P/H.

Proposition 3.13. *Let G be a connected reductive linear algebraic group and let X be a smooth projective horospherical G-variety. Then TX is big.*

Proof. Let D be a G-stable prime divisor in X. As shown in the proof of Proposition 3.10, we have $r(D) = r(X) - 1$ and, by Lemma 3.7, we obtain $\dim(\mathcal{M}_X^G(D)) < \dim(\mathcal{M}_X^G)$. Let $O = G/H$ be the unique open G-orbit in X with H a horospherical subgroup of G. Denote by $P = N_G(H)$ the normaliser of H in G. Then, by Lemma 3.5, the restriction of the moment map $\Phi_X^G : T^*X \to \mathfrak{g}^*$ to O factors as

$$G *_H \mathfrak{h}^\perp \xrightarrow{\pi_A} G *_P \mathfrak{h}^\perp \xrightarrow{\varphi} \mathfrak{g}^*.$$

Moreover, it is known that φ is generically finite onto its image $\mathcal{M}_X^G$; see Theorem 2.10. Let D be a B-stable but not G-stable prime divisor in X. Then $D \cap O \neq \varnothing$ and $D \cap O$ is the inverse image by the torus fibration $G/H \to G/P$ of a Schubert divisor D' of the flag variety G/P. As a consequence, the Zariski closure of the image $\Phi_X^G(T^*X|_D)$ is equal to the Zariski closure of the image $\varphi(p^{-1}(D'))$, where $p : G *_P \mathfrak{h}^\perp \to G/P$ is the natural projection. However, as φ is generically finite onto $\mathcal{M}_X^G$, we get

$$\dim(\varphi(p^{-1}(D'))) < \dim(G *_P \mathfrak{h}^\perp) = \dim(\mathcal{M}_X^G).$$

As a consequence, our argument above shows that for every B-stable prime divisor D in X, we have always $\dim(\mathcal{M}_X^G(D)) < \dim(\mathcal{M}_X^G)$. On the other hand, let O_B be the open B-orbit of X. Then O_B is an affine variety and the complement $D := X \setminus O_B$ is big divisor by Lemma 3.4. Then it follows from Proposition 1.1 that TX is big. □

Remark 3.14. Smooth projective horospherical varieties with Picard number 1 are classified by Pasquier [31]. With the same notations as in [31], there are five classes of nonhomogeneous ones, including $X^1(m) = (B_m, \omega_{m-1}, \omega_m)$ $(m \geq 3)$, $X^2 = (B_3, \omega_1, \omega_3)$, $X^3(m, i) = (C_m, \omega_i, \omega_{i+1})$ $(m \geq 2, 1 \leq i \leq m-1)$, $X^4 = (F_4, \omega_2, \omega_3)$ and $X^5 = (G_2, \omega_2, \omega_1)$.

3B3. *Quasihomogeneous G-varieties with G commutative.* The following result confirms the bigness of the tangent bundles of equivariant compactifications of connected commutative linear algebraic groups.

Proposition 3.15. *Let G be a connected commutative linear algebraic group and let X be a smooth projective G-variety with an open G-orbit O. Then TX is big. In particular, the tangent bundle of an equivariant compactification of G is big.*

Proof. Let O be the unique open G-orbit in X and let $D := X \setminus O$ be the complement of O. Since G is solvable, by Lemma 3.4, the divisor D is big. Moreover, as G is commutative, for any subgroup H, we have always $N_G(H) = G$. In particular, Lemma 3.5 implies that $\dim(\mathcal{M}_X^G) \leq \dim(X)$. On the other hand, as G has an open orbit O in X, we must have $\dim(\mathcal{M}_X^G) \geq \dim(O) = \dim(X)$. Hence, we obtain $\dim(\mathcal{M}_X^G) = \dim(X)$. Let D_i be an irreducible component of D. As G is commutative, Lemma 3.6 yields

$$\dim(\mathcal{M}_X^G(D_i)) = \dim(\mathcal{M}_{D_i}^G) \leq \dim(X) - 1 < \dim(\mathcal{M}_X^G).$$

Hence, the tangent bundle TX is big by Proposition 1.1. □

Proof of Theorem 1.2. It follows from Propositions 3.10, 3.13 and 3.15 and Corollary 3.11. □

Remark 3.16. Recall that a connected commutative linear algebraic group is known to be isomorphic to $\mathbb{G}_m^r \times \mathbb{G}_a^s$ with some nonnegative integers r and s:

(1) If $s = 0$, then $G = \mathbb{G}_m^r$ is a torus and an equivariant compactification of G is a toric variety. In particular, our result above recovers the bigness of tangent bundles of smooth projective toric varieties [14, Corollary 1.3].

(2) If $r = 0$, then $G = \mathbb{G}_a^s$ is a vector group and the equivariant compactifications of vector groups are studied actively during the past decades. A full classification of all Fano threefolds admitting an equivariant compactification structure of the vector group $\mathbb{G}_a^3$ is given in [15, Main Theorem], including 14 toric ones and 5 nontoric ones. In particular, this allows us to give a different proof of the bigness of the tangent bundles of the Fano threefolds № 28, № 30 and № 31 in [21, Table 1], which are proved there using total dual VMRT. In higher dimension, a classification of Fano manifolds admitting an equivariant compactification structure of vector groups is available only for varieties with high index; see [9], Examples 4.5 and 4.6. The equivariant compactifications of vector groups with Picard number 1 are of special interests and we will discuss them in details in the next section.

4. Equivariant compactification of vector groups

In this section, we will investigate the Fano manifolds with Picard number 1 which is an equivariant compactification of a vector group $\mathbb{G}_a^n$. The study of equivariant compactification of vector groups is started in [11], where a classification of them in dimension 3 and with Picard number 1 is obtained. Nevertheless, it seems difficult to obtain a full classification in higher dimension; see [6; 7; 9] for more details. The main goal of this section is to show that the image of the projectivised moment map $\bar{\Phi}_X^G$ along the boundary divisor of an equivariant compactification of a vector group is projectively equivalent to the dual variety of its VMRT. In particular, this allows us to relate the criterion for the bigness of tangent bundles given in Proposition 1.1 via moment map to the previous approach to the bigness of tangent bundles via total dual VMRT initiated in [13]; see also [8; 12; 21] and Theorem 4.1 below.

4A. ***VMRT and its dual variety.*** Let X be a uniruled projective manifold. An irreducible component $\mathcal{K}$ of the space of rational curves on X is called a *minimal rational component* if the subscheme $\mathcal{K}_x$ of $\mathcal{K}$ parametrising curves passing through a general point $x \in X$ is nonempty and proper. Curves parametrised by $\mathcal{K}$ will be called *minimal rational curves*. Let $q : \mathcal{U} \to \mathcal{K}$ be the universal family and by $\mu : \mathcal{U} \to X$ the evaluation map. The tangent map $\tau : \mathcal{U} \dashrightarrow \mathbb{P}(TX)$ is defined by

$$\tau(u) = [T_{\mu(u)}\mu(q^{-1}q(u))] \in \mathbb{P}(T_{\mu(x)}X).$$

The closure $\mathcal{C} \subseteq \mathbb{P}(TX)$ of its image is the *total variety of minimal rational tangents* (total VMRT for short) of X. The projection $\mathcal{C} \to X$ is a proper and surjective morphism, and a general fibre $\mathcal{C}_x \subseteq \mathbb{P}(T_x X)$ is called the *variety of minimal rational tangents* of X at the point $x \in X$. A general minimal rational curve l passing through a general point x is standard; that is, if $f : \mathbb{P}^1 \to l$ is the normalisation, we have

$$f^*TX \cong \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus p} \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus(n-p-1)},$$

where $p = \dim(\mathcal{C}_x)$. Moreover, the projectivised tangent space $\boldsymbol{T}_{[T_x l]}\mathcal{C}_x$ of $\mathcal{C}_x$ at $[T_x l]$ is the linear subspace of $\mathbb{P}(T_x X)$ corresponding to the positive factors of f^*TX at $x \in l$.

Let $Z \subseteq \mathbb{P}(V)$ be a projective variety. The *dual variety* $\check{Z} \subseteq \mathbb{P}(V^*)$ is defined as the Zariski closure of the set of hyperplanes in $\mathbb{P}(V)$ which are tangent to Z at some smooth point. The *dual defect* of Z is

$\operatorname{codim}(\check{Z})-1$ and Z is called *dual defective* if its dual variety $\check{Z}$ is not a hypersurface. The *codegree* of Z is defined as the degree of $\check{Z}$. The *total dual VMRT* $\check{\mathcal{C}} \subseteq \mathbb{P}(T^*X)$ of a uniruled projective manifold X is defined as the closure of the union of dual varieties $\check{\mathcal{C}}_x \subseteq \mathbb{P}(T^*_xX)$ of the VMRT $\mathcal{C}_x \subseteq \mathbb{P}(T_xX)$ at general points. The total dual VMRT $\check{\mathcal{C}}$ is dominated by a family U of rational curves such that $\Lambda \cdot C = 0$ for a general element $[C] \in U$, where Λ is the tautological divisor of $\mathbb{P}(T^*X)$; see [30, Section 4.1] and [13, Section 2.B] for more details. The importance of the total dual VMRT in the study of the bigness of tangent bundles is illustrated in the following theorem.

Theorem 4.1 [8, Theorem 3.4; 12, Proposition 5.8]. *Let X be a Fano manifold with Picard number* 1 *and denote by H the ample generator of* $\operatorname{Pic}(X)$. *Assume that the VMRT of X at a general point is not dual defective and denote by $a \in \mathbb{Z}_{>0}$, $b \in \mathbb{Z}$ the unique integers such that*

$$[\check{\mathcal{C}}] \equiv a\Lambda + b\pi^*H,$$

where Λ is the tautological divisor class of $\mathbb{P}(T^*X)$ and $\pi : \mathbb{P}(T^*X) \to X$ is the natural projection. Then TX is big if and only if $b < 0$.

The following result suggests that there may exist some interesting relations between the criterion given in Proposition 1.1 via moment map and that given in Theorem 4.1 via total dual VMRT.

Lemma 4.2. *Let G be a connected algebraic group and let X be a smooth projective uniruled G-variety. Fix a minimal rational component $\mathcal{K}$ on X with total dual VMRT $\check{\mathcal{C}} \subseteq \mathbb{P}(T^*X)$. Then for any reduced big divisor D in X we have*

$$\bar{\Phi}^G_X(\check{\mathcal{C}}) \subseteq \bar{\Phi}^G_X(\mathbb{P}(T^*X|_D)) = \mathbb{P}(\mathcal{M}^G_X(D)).$$

Proof. The inclusion is clear if $\mathcal{M}^G_X(D) = \mathcal{M}^G_X$. Thus we may assume that $\mathcal{M}^G_X(D) \neq \mathcal{M}^G_X$. Let $\mathcal{H} \subseteq \mathbb{P}(\mathfrak{g}^*)$ be an arbitrary reduced (maybe reducible) hypersurface of degree m containing $\mathbb{P}(\mathcal{M}^G_X(D))$, but not containing $\mathbb{P}(\mathcal{M}^G_X)$. Following Notation 3.2, consider the divisor $D_{\mathcal{H}} \subseteq \mathbb{P}(T^*X)$. Then there exists an effective big divisor $D' \geq D$ such that $D_{\mathcal{H}} + \pi^*D' \sim m\Lambda$, where Λ is the tautological divisor of $\pi : \mathbb{P}(T^*X) \to X$. On the other hand, we note that the total dual VMRT $\check{\mathcal{C}}$ is dominated by a family U of π-horizontal rational curves with $\Lambda \cdot C = 0$ for a general element $[C] \in U$. Thus, the restriction $D_{\mathcal{H}}|_{\check{\mathcal{C}}}$ is not pseudoeffective since D' is big and $\pi^*D' \cdot C > 0$ for a general element $[C] \in U$. In particular, the total dual VMRT $\check{\mathcal{C}}$ is contained in the support of the divisor $D_{\mathcal{H}}$. By the construction of $D_{\mathcal{H}}$, we must have $\bar{\Phi}^G_X(D_{\mathcal{H}}) \subseteq \mathcal{H}$ and therefore $\bar{\Phi}^G_X(\check{\mathcal{C}}) \subseteq \mathcal{H}$. As $\mathcal{H}$ is an arbitrary reduced hypersurface containing $\mathbb{P}(\mathcal{M}^G_X(D))$, it follows that $\bar{\Phi}^G_X(\check{\mathcal{C}}) \subseteq \mathbb{P}(\mathcal{M}^G_X(D))$. □

Remark 4.3. The assumption on the bigness of D cannot be removed, see Example 4.6 below. Moreover, the following inclusion is in general strict:

$$\bar{\Phi}^G_X(\check{\mathcal{C}}) \subseteq \bigcap_{D \text{ effective big divisor}} \mathbb{P}(\mathcal{M}^G_X(D)).$$

Let $X = G/P$ be a rational homogeneous space with Picard number 1. Then the moment map $\Phi^G_X : T^*X \to \mathfrak{g}^*$ is a generically finite dominant map to its image $\mathcal{M}^G_X$, which is the closure of a nilpotent orbit. Moreover, the projectivised moment map $\bar{\Phi}_X^G : \mathbb{P}(T^*X) \to \mathbb{P}(\mathcal{M}_X^G)$ is everywhere well-defined. Let $\mathbb{P}(T^*X) \xrightarrow{\varepsilon} \mathbb{P}(\widetilde{\mathcal{M}}_X^G) \to \mathbb{P}(\mathcal{M}_X^G)$ be the Stein factorisation; see [8, Seciton 5.A]. Then ε is actually the birational morphism defined by $|m\Lambda|$ with $m \gg 1$, where Λ is the tautological divisor of $\mathbb{P}(T^*X)$. As in the proof of Lemma 4.2, since $\check{\mathcal{C}}$ is dominated by rational curves C with $\Lambda \cdot C = 0$, the total dual VMRT $\check{\mathcal{C}}$ is contained in the exceptional locus E of ε. Thanks to [8, Theorem 5.5], we have

$$\bigcap_{D \text{ effective ample divisor}} \varepsilon(\mathbb{P}(T^*X|_{\mathrm{supp}(D)})) = \varepsilon(E).$$

Consequently, if $\bar{\Phi}_X^G$ is birational and if E is a divisor and $\check{\mathcal{C}}$ is not a divisor, then $\bar{\Phi}_X^G(\check{\mathcal{C}})$ is a proper subvariety of $\bar{\Phi}_X^G(E)$; see [8, Proposition 5.4, Definition 5.6 and Table 2].

4B. ***Geometry of equivariant compactifications.*** In this subsection, we collect some basic facts about equivariant compactifications of vector groups. Recall that for a smooth projective variety X, an *EC-structure* on X is an algebraic action $\mathbb{G}_a^n \times X \to X$ which makes X an equivariant compactification of $\mathbb{G}_a^n$.

Proposition 4.4 [7, Proposition 5.4]. *Let X be a Fano manifold with Picard number* 1 *which is an equivariant compactification of $\mathbb{G}_a^n$. Denote by D the complement of the unique open $\mathbb{G}_a^n$-orbit $O \subseteq X$. Let $\mathcal{K}$ be a covering family of minimal rational curves on X and denote by $\mathcal{C} \subseteq \mathbb{P}(TX)$ its total VMRT. Then the following statements hold*:

(1) *The closed subvariety D is an irreducible divisor such that* $\mathrm{Pic}(X) \cong \mathbb{Z}D$.

(2) *If the points in D are fixed by $\mathbb{G}_a^n$, then X is isomorphic to $\mathbb{P}^n$.*

(3) *For any point $x \in O$, the VMRT $\mathcal{C}_x \subseteq \mathbb{P}(T_xX)$ is irreducible and is independent of x up to projective equivalence.*

(4) *If the VMRT is smooth, then a member C of $\mathcal{K}_x$, for $x \in O$, is the closure of the image of a* 1*-dimensional subspace in $\mathbb{G}_a^n$ and $D \cdot C = 1$.*

Proof. The first statement is proved in [11, Theorem 2.5] and the second statement is proved in [11, Corollary 2.9]. The first part of the third statement is proved in [6, Proposition 2.2] and the second part follows from the fact that the total VMRT $\mathcal{C} \subseteq \mathbb{P}(TX)$ is preserved by the natural action of $\mathbb{G}_a^n$ on $\mathbb{P}(TX)$. The last statement follows from [7, Proposition 5.4]. □

Let $Z \subseteq \mathbb{P}(V)$ be a nondegenerate submanifold and let $W \subseteq V$ be a subspace such that $\mathbb{P}(W) \subseteq Z$. Denote by $(V/W)^* \subset V^*$ the set of linear functionals on V annihilating W such that $\mathbb{P}((V/W)^*)$ parametrises the set of hyperplanes in $\mathbb{P}(V)$ containing $\mathbb{P}(W)$. Then a general member of $\mathbb{P}((V/W)^*)$ is called a $\mathbb{P}(W)$-*general* hyperplane in $\mathbb{P}(V)$. More generally, a linear subspace of codimension k in $\mathbb{P}(V)$ is $\mathbb{P}(W)$-*general* if it is defined by a general member of $\mathrm{Gr}(k, (V/W)^*)$.

Example 4.5. Up to our knowledge, the known examples of Fano manifolds with Picard number 1 which are equivariant compactifications of vector groups are as follows:

- The irreducible Hermitian symmetric spaces.
- The odd Lagrangian Grassmannians $X^3(m, m-1)$ ($m \geq 2$); see Remark 3.14.
- A smooth linear section V_k of $\mathrm{Gr}(2, 5) \subseteq \mathbb{P}^9$ with codimension $k \leq 2$.
- A smooth $\mathbb{P}^4$-general linear section S_k^a of $\mathbb{S}_5 \subseteq \mathbb{P}^{15}$ with codimension $k \leq 3$.

The following example shows that there are many smooth equivariant compactifications of vector groups with higher Picard number.

Example 4.6 [7, Example 2.2]. Let $[x_0 : \cdots : x_n]$ be the homogeneous coordinates of the n-dimensional projective space $\mathbb{P}^n$. Let $H = \mathbb{P}^{n-1} \subseteq \mathbb{P}^n$ be the hyperplane defined by the equation $x_0 = 0$. Then there is a natural EC-structure $\Psi : \mathbb{G}_a^n \times \mathbb{P}^n \to \mathbb{P}^n$ on $\mathbb{P}^n$ with the unique open orbit $\mathbb{P}^n \setminus H$. More precisely, for a point $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{G}_a^n$, we define an automorphism $\Psi_{\boldsymbol{y}} : \mathbb{P}^n \to \mathbb{P}^n$ as follows:

$$[x_0 : x_1 : \cdots : x_n] \mapsto [x_0 : x_1 + y_1 x_0 : \cdots : x_n + y_n x_0].$$

Clearly this gives an EC-structure on $\mathbb{P}^n$ such that the induced action on the hyperplane H is trivial. Let $S \subseteq H$ be a smooth irreducible projective variety, and let

$$\nu : Z := \mathrm{Bl}_S \mathbb{P}^n \to \mathbb{P}^n$$

be the blowing-up of $\mathbb{P}^n$ along S with exceptional divisor $E = \mathbb{P}(N_{S/\mathbb{P}^n})$. Then the EC-structure Ψ on $\mathbb{P}^n$ can be naturally lifted to be an EC-structure Ψ_Z on Z such that μ is equivariant. Denote by $W \subseteq \mathbb{P}(N_{S/\mathbb{P}^n})$ the subvariety $\mathbb{P}(N_{S/H})$ of E. Then it is clear that the induced action of Φ_Z on W is trivial and for each point $s \in S$, the fibre E_s of $E \to S$ over s is invariant such that Φ_Z is transitive over the open subset $E_s \setminus W_s$, where W_s is the fibre of $W \to S$ over s.

Denote by $\widetilde{H}$ the strict transform of H in Z. Then the induced $\mathbb{G}_a^n$-action on $\widetilde{H}$ is trivial. In particular, the image $\Phi_Z^{\mathbb{G}_a^n}(T^*Z|_{\widetilde{H}})$ is the origin $0 \in \mathfrak{g}^*$. Let $\mathcal{K}$ be the irreducible component of the space of rational curves in Z parametrising the strict transforms of lines in $\mathbb{P}^n$ meeting S. Then $\mathcal{K}$ is a minimal rational component on Z such that its VMRT is projectively equivalent to $S \subseteq \mathbb{P}^{n-1}$. For any point $z \in Z \setminus (\widetilde{H} \cup E)$, we have $\Phi_Z^{\mathbb{G}_a^n}(T_z^*Z) = \mathfrak{g}^*$. Since the members in $\mathcal{K}$ have $\widetilde{H}$-degree 0, the divisor $\widetilde{H}$ is not big. This shows that the assumption on the bigness of D in Lemma 4.2 cannot be removed.

The proof of the following result is communicated to me by Baohua Fu.

Proposition 4.7. *Notations as in Example 4.6. Let X be a Fano manifold with Picard number* 1 *which is an equivariant compactification of $\mathbb{G}_a^n$, different from projective spaces. Denote by $D \subseteq X$ the boundary divisor. Assume that there exists a covering family $\mathcal{K}$ of minimal rational curves on X such that its VMRT $\mathcal{C}_x \subseteq \mathbb{P}(T_x X)$ at a general point $x \in X$ is projectively equivalent to a smooth projective variety $S \subseteq \mathbb{P}^{n-1}$. Then there exist $\mathbb{G}_a^n$-stable proper subvarieties $D_0 \subseteq D$ and $E_0 \subseteq E$ such that there exists a $\mathbb{G}_a^n$-equivariant isomorphism*

$$\Phi : X \setminus D_0 \to \mathrm{Bl}_S(\mathbb{P}^n) \setminus (\widetilde{H} \cup E_0).$$

Proof. Let $\mathcal{K}'$ be the covering family of minimal rational curves on $\mathrm{Bl}_S(\mathbb{P}^n)$ parametrising the strict transform of lines in $\mathbb{P}^n$ meeting S. Denote by O and O' the open $\mathbb{G}_a^n$-orbits of X and $\mathrm{Bl}_S(\mathbb{P}^n)$, respectively. Fix two points $o \in O$ and $o' \in O'$. Denote by $\nu_o : \mathbb{G}_a^n \to O$ and $\nu_{o'} : \mathbb{G}_a^n \to O'$ the orbit maps, respectively. For a general point $x' \in \mathrm{Bl}_S(\mathbb{P}^n)$, we denote by $\mathcal{C}'_{x'} \subseteq \mathbb{P}(T_{x'}\,\mathrm{Bl}_S(\mathbb{P}^n))$ the VMRT of $\mathcal{K}'$ at x'. Note that the VMRT of $\mathcal{K}$ at a point $x \in O'$ is projectively equivalent to $S \subseteq \mathbb{P}^{n-1}$. In particular, applying [6, Proposition 2.4] shows that there exists a group automorphism F of $\mathbb{G}_a^n$ such that the biholomorphic map $\varphi : O \to O'$ defined by $\varphi := \nu_{o'} \circ F \circ \nu_o^{-1}$ satisfying:

(1) $\varphi(o) = o'$.

(2) $\varphi(g \cdot o) = F(g) \cdot \varphi(o')$ for any $g \in \mathbb{G}_a^n$.

(3) the differential map $d\varphi : \mathbb{P}(TO) \to \mathbb{P}(TO')$ sends $\mathcal{C}_x$ to $\mathcal{C}'_{\varphi(x)}$ for all $x \in O$.

The last statement (3) implies that general members in $\mathcal{K}$ are sent to general members in $\mathcal{K}'$ by φ. Denote by $\Phi : X \dashrightarrow \mathrm{Bl}_S(\mathbb{P}^n)$ the rational map defined by φ. Let $D_0 \subseteq D$ be the closed subvariety such that Φ is an isomorphism over $X \setminus D_0$. We note that $\Phi(D)$ is a divisor in $\mathrm{Bl}_S(\mathbb{P}^n)$. In fact, a general minimal rational curve in $\mathcal{K}$ is disjoint from the indeterminacy locus of Φ and it meets D as X has Picard number 1. Thus, if $\Phi(D)$ has codimension 2 in $\mathrm{Bl}_S(\mathbb{P}^n)$, then every minimal rational curves in $\mathcal{K}'$ passes through the codimension 2 subvariety $\Phi(D)$, which is impossible. Thus $\Phi(D)$ is a divisor and this yields that the map Φ is a local isomorphism at general points of D. As a consequence, the closed subvariety D_0 is a proper subvariety of D and hence has codimension at least 2 in X as D is irreducible. On the other hand, the statement (2) shows that the rational map Φ is $\mathbb{G}_a^n$-equivariant and it follows that D_0 is $\mathbb{G}_a^n$-stable.

Next we consider the inverse map $\Phi^{-1} : \mathrm{Bl}_S(\mathbb{P}^n) \dashrightarrow X$. Note that the points in the prime divisor $\widetilde{H}$ are fixed by $\mathbb{G}_a^n$. We claim that $\Phi(\widetilde{H})$ has codimension at least 2 in X. Otherwise, we must have $\Phi(\widetilde{H}) = D$. In particular, the points in D are fixed by $\mathbb{G}_a^n$ and X is isomorphic to the projective space $\mathbb{P}^n$ by Proposition 4.4, which contradicts our assumption. Hence, the divisor $\widetilde{H}$ is contracted by Φ^{-1} and we have $\Phi^{-1}(E) = D$. In particular, the map Φ^{-1} is a local isomorphism at general points of E and consequently there exists a closed proper $\mathbb{G}_a^n$-stable subvariety E_0 of E such that Φ^{-1} is an isomorphism over the Zariski open subset $\mathrm{Bl}_S(\mathbb{P}^n) \setminus (\widetilde{H} \cup E_0)$. □

4C. *Pseudoeffective cone of* $\mathbb{P}(T^*X)$. In this subsection, we will finish the proof of the first statement of Theorem 1.3. Let G be a connected linear algebraic group and let X be an equivariant compactification of G. Fix a point o in the unique open orbit. We define

$$\mathfrak{D} := \{\Delta \subseteq \mathbb{P}(T^*X) \mid \Delta \text{ is a } G\text{-stable } \pi\text{-horizontal prime divisor}\}$$

and

$$\mathfrak{H} := \{\mathcal{H} \subseteq \mathbb{P}(T_o^*X) \mid \mathcal{H} \text{ is a reduced but maybe reducible hypersurface}\}.$$

We can naturally identify $\mathfrak{g}^*$ to T_o^*X via the cotangent map of the orbit map $\mu_o : G \to Go = O$ at the identity $e \in G$. In particular, we shall also regard the set $\mathfrak{H}$ as the set of reduced but maybe reducible hypersurfaces in $\mathbb{P}(\mathfrak{g}^*)$; see Notation 3.2.

Lemma 4.8. *Let $\Delta \in \mathfrak{D}$ and let $\mathcal{H} \in \mathfrak{H}$ be the intersection $\Delta \cap \mathbb{P}(T_o^* X)$. If G is commutative, then $\Delta = D_{\mathcal{H}}$.*

Proof. Since both Δ and $D_{\mathcal{H}}$ are π-horizontal, it is enough to show that the equality $\Delta = D_{\mathcal{H}}$ holds over the open subset $\mathbb{P}(T^* O)$. For an arbitrary point $o' \in O$, there exists a unique element $g \in G$ such that $o' = go$. Moreover, as Δ is G-stable, we must have

$$\Delta \cap \mathbb{P}(T_{o'}^* X) = d\mu_g|_o(d\mu_o|_e(\mathcal{H})),$$

where $d\mu_g$ is the tangent map of the map $\mu_g : X \to X$, $x \mapsto gx$, at x. Consider the following diagram

$$\begin{array}{ccc} G & \xrightarrow{\ \mu_o\ } & Go = O \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \mu_g} \\ G & \xrightarrow[\ \mu_{o'}\ ]{} & Go' = O \end{array}$$

Since G is commutative, for any $g' \in G$, we have

$$\mu_g(\mu_o(g')) = g(g'o) = (g'g)o = g'(go) = \mu_{o'}(g').$$

In particular, the diagram above is commutative. This yields

$$\Delta \cap \mathbb{P}(T_{o'}^* X) = d\mu_g|_o \circ d\mu_o|_e(\mathcal{H}) = d\mu_{o'}|_e(\mathcal{H}) = D_{\mathcal{H}} \cap \mathbb{P}(T_{o'}^* X).$$

Thus, we have $\Delta = D_{\mathcal{H}}$ over $\mathbb{P}(T^* O)$ and hence $\Delta = D_{\mathcal{H}}$. □

The first statement in Theorem 1.3 is a special case of the following more general result.

Proposition 4.9. *Let G be a connected commutative linear algebraic group, and let X be a smooth equivariant compactification of G with the unique open orbit O. Then the pseudoeffective cone $\overline{\mathrm{Eff}}(\mathbb{P}(T^* X))$ of $\mathbb{P}(T^* X)$ is generated by following divisors:*

(1) *the divisors $\pi^* D$, where $\pi : \mathbb{P}(T^* X) \to X$ is the natural projection and D is an irreducible component of the complement $X \setminus O$.*

(2) *the prime divisors $D_{\mathcal{H}}$, where $\mathcal{H}$ is an irreducible reduced hypersurface in $\mathbb{P}(\mathfrak{g}^*)$.*

Proof. The action of G on X can be naturally lifted to an action on $\mathbb{P}(T^* X)$. Thus, according to Theorem 2.7, the pseudoeffective cone of $\mathbb{P}(T^* X)$ is generated by G-stable prime divisors. Let Δ be a prime G-stable prime divisor in $\mathbb{P}(T^* X)$. Then we have the following possibilities for Δ:

- The divisor Δ is π-vertical.
- The divisor Δ is π-horizontal.

If the prime divisor Δ is π-vertical, then there exists a G-stable prime divisor D in X such that $\pi^* D = \Delta$ because the projection π is G-equivariant. This implies that D is an irreducible component of $X \setminus O$.

Now we assume that Δ is π-horizontal. Fix a point o in the open orbit O. Taking intersection with $\mathbb{P}(T_o^* X)$ yields an injection $\mathfrak{D} \to \mathfrak{H}$. Let $\Delta \in \mathfrak{D}$ be an arbitrary element and let $\mathcal{H} \in \mathfrak{H}$ be the corresponding hypersurface. Then we have $\Delta = D_{\mathcal{H}}$ by Lemma 4.8. Note that if $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$ as divisors in $\mathbb{P}(\mathfrak{g}^*)$, then clearly we have $D_{\mathcal{H}} = D_{\mathcal{H}_1} + D_{\mathcal{H}_2}$ as divisors in $\mathbb{P}(T^*X)$. Thus we may assume that the hypersurfaces in $\mathfrak{H}$ are irreducible. □

Remark 4.10. The commutativity of G is necessary in Proposition 4.9. In general, if G is not commutative, then Lemma 4.8 is false and it is possible that there are G-stable prime divisors in $\mathbb{P}(T^*X)$ which are not of the form $D_{\mathcal{H}}$. For example, for the quintic del Pezzo threefold $X = V_3$ (see Example 3.8), its total dual VMRT $\check{\mathcal{C}} \subseteq \mathbb{P}(T^*X)$ is an SL_2-invariant prime divisor such that the intersection $\check{\mathcal{C}} \cap \mathbb{P}(T^*_xX)$ for a point x contained in the open orbit is a union of three hyperplanes; see [13, Section 5]. In particular, if $\check{\mathcal{C}}$ is of the form $D_{\mathcal{H}}$, then $\mathcal{H}$ is also a union of three hyperplanes in $\mathbb{P}(\mathfrak{g}^*)$ and consequently $D_{\mathcal{H}}$ is a reducible divisor containing three irreducible components, which contradicts the irreducibility of $\check{\mathcal{C}}$. Actually, the pseudoeffective cone of $\mathbb{P}(T^*V_3)$ is generated by $\check{\mathcal{C}}$ and π^*D, where D is the closure of the unique 2-dimensional SL_2-orbit.

4D. ***Pseudoeffective slope.*** In this subsection, we will pursue further the study of the pseudoeffective cone of $\mathbb{P}(T^*X)$ for X being a smooth equivariant compactification with Picard number 1 of a vector group.

Definition 4.11 (pseudoeffective slope of vector bundles). Let E be a vector bundle over a normal projective variety X, and let A be a big $\mathbb{R}$-Cartier divisor on X. The pseudoeffective slope of E with respect to A is defined as

$$\mu(E, A) := \sup\{\varepsilon \in \mathbb{R} \mid \Lambda - \varepsilon\pi^*A \text{ is pseudoeffective}\},$$

where $\pi : \mathbb{P}(E^*) \to X$ is the natural projection and Λ is the tautological divisor of $\mathbb{P}(E^*)$.

Remark 4.12. The invariant $\mu(E, A)$ is also called *pseudoeffective threshold of E with respect to A* in [8; 35] and E is big if and only if $\mu(E, A) > 0$ for some big divisor A on X. On the other hand, if X is a projective manifold with Picard number 1, then the pseudoeffective cone of $\mathbb{P}(T^*X)$ is generated by $\Lambda - \mu(E, A)\pi^*A$ and π^*A, where A is an ample divisor on X.

4D1. *Behaviour under deformation.* Let $p : \mathcal{X} \to \Delta$ be smooth family of projective manifolds over a disk Δ. By Semicontinuity Theorem, if the tangent bundle of the fibre $\mathcal{X}_t$ is big for $t \neq 0$, then the tangent bundle of the central fibre $\mathcal{X}_0$ is also big. Thus one may expect to get more examples of Fano manifolds with Picard number 1 and with big tangent bundle by degenerating known examples. Nevertheless, it turns out that this may be not so successful. Recall that a smooth projective X is *rigid* if for any smooth deformation $\mathcal{X} \to \Delta$ with $\mathcal{X}_t \cong X$ for any $t \neq 0$, we have $\mathcal{X}_0 \cong X$:

- The rational homogeneous spaces with Picard number 1 are rigid except the orthogonal Grassmannian $B_3/P_2 = \mathrm{Gr}_q(2, 7)$ by a series works of Hwang and Mok [18, Main Theorem] and the latter one has a degeneration to X^5 [32, Proposition 2.3].
- The odd symplectic Grassmannians $X^3(m, i)$ $(m \geq 2, 1 \leq i \leq m-1)$ are rigid by [17, Theorem 1.7].
- The codimension $k(\leq 3)$ linear section V_k of $\mathrm{Gr}(2, 5) \subseteq \mathbb{P}^9$ and the smooth $\mathbb{P}^4$-general linear section S^a_k of $\mathbb{S}_5 \subseteq \mathbb{P}^{15}$ with $1 \leq k \leq 3$ are rigid by the classification of Fano manifolds with coindex at most 3.

Question 4.13 [20; 26]. Let X be either a nonhomogeneous smooth projective symmetric variety with Picard number 1 or a nonhomogeneous smooth projective horospherical variety with Picard number 1, different from the odd symplectic Grassmannians. Is X rigid?

Conversely, if the tangent bundle of $\mathcal{X}_0$ is big, then it is not clear for us if $T_{\mathcal{X}_t}$ is big for t small enough and up to our knowledge there are no known counterexamples yet. However, in certain special cases, we can show that the bigness of tangent bundles is preserved under small deformation.

Lemma 4.14. *Let $E \to \Delta$ be a vector bundle over the disk Δ and let $\mathcal{S} \subseteq \mathbb{P}(E^*)$ be a smooth family of embedded smooth projective varieties over Δ. Then the codegrees and the dual defects of the fibres $\mathcal{S}_t \subseteq \mathbb{P}(E_t^*)$ are independent of t.*

Proof. Let us consider the total conormal variety $\mathcal{I} \subseteq \mathbb{P}(E^*) \times_\Delta \mathbb{P}(E)$ of $\mathcal{S}$; that is, the variety defined as follows:

$$\{(t, s, [H]) \mid t \in \Delta, s \in \mathcal{S}_t, [H] \in \mathbb{P}(E_t) \text{ is a hyperplane tangent to } \mathcal{S}_t \text{ at } s\}$$

Then the total dual variety $\check{\mathcal{S}} \subseteq \mathbb{P}(E)$ is the image of the natural projection $\mathcal{I} \to \mathbb{P}(E)$. Moreover, it is clear that the fibre of $\check{\mathcal{S}} \to \Delta$ over t is just the dual variety of the fibre of $\mathcal{S} \to \Delta$ over t. Since Δ is one-dimensional, the family $\check{\mathcal{S}} \to \Delta$ is flat. In particular, the degrees and the dimensions of the fibres of $\check{\mathcal{S}} \to \Delta$ are independent of t and so are the codegrees and the dual defects of the fibres of $\mathcal{S} \to \Delta$. □

Remark 4.15. The result is false without the smoothness assumption. This can be shown by considering a family of smooth hypersurfaces in $\mathbb{P}^n$ degenerating to a dual defective singular hypersurface in $\mathbb{P}^n$.

Proposition 4.16. *Let $p : \mathcal{X} \to \Delta$ be a smooth family of Fano manifolds with Picard number* 1. *Let $\mathcal{K}$ be an irreducible component of the relative Chow variety* Chow$(\mathcal{X}/\Delta)$ *such that $\mathcal{K}_t$ is a minimal rational component of $\mathcal{X}_t$ for any $t \in \Delta$. Assume moreover that the VMRT of $\mathcal{K}_t$ at general points of $\mathcal{X}_t$ is smooth for every $t \in \Delta$. If the VMRT of $\mathcal{X}_0$ is not dual defective and $T\mathcal{X}_0$ is big, then $T\mathcal{X}_t$ is big for any $t \in \Delta$ and we have*

$$\mu(T\mathcal{X}_t, -K_{\mathcal{X}_t}) = \mu(T\mathcal{X}_0, -K_{\mathcal{X}_0}), \quad \forall t \in \Delta.$$

Proof. Let $\sigma : \Delta \to \mathcal{X}$ be a general section passing through a general point in $\mathcal{X}_0$. Then the normalised Chow space $\mathcal{K}_{\sigma(t)}$ along this section gives a family of smooth projective varieties. On the other hand, since the VMRT of $\mathcal{X}_t$ is smooth for any t, it follows that the VMRTs $\mathcal{C}_{\sigma(t)} \subseteq \mathbb{P}(T_{\sigma(t)}\mathcal{X}_t)$ along $\sigma(\Delta)$ is a smooth family of embedded projective varieties. Then by Lemma 4.14, the VMRT $\mathcal{C}_{\sigma(t)} \subseteq \mathbb{P}(T_{\sigma(t)}\mathcal{X}_t)$ is not dual defective for any $t \in \Delta$. In particular, the relative total dual VMRT $\check{\mathcal{C}}_{\mathcal{X}} \subseteq \mathbb{P}(T^*(\mathcal{X}/\Delta))$ of the relative total VMRT $\mathcal{C}_{\mathcal{X}} \subseteq \mathbb{P}(T(\mathcal{X}/\Delta))$ is a prime divisor, where $T(\mathcal{X}/\Delta)$ is the relative tangent bundle of p. Since the fibration $\mathcal{X} \to \Delta$ has relative Picard number 1, there are two unique real numbers a and b such that

$$\check{\mathcal{C}}_{\mathcal{X}} \sim_p a\Lambda_{\mathcal{X}} + b\pi^* K_{\mathcal{X}/\Delta},$$

where $\Lambda_{\mathcal{X}}$ is the tautological divisor of $\mathbb{P}(T^*(\mathcal{X}/\Delta))$. Then it is clear that a is equal to the codegree of the VMRT of $\mathcal{X}_0$ (see Lemma 4.14) and

$$\mu(T\mathcal{X}_t, -K_{\mathcal{X}_t}) = \tfrac{b}{a}, \quad \forall t \in \Delta.$$

As $T\mathcal{X}_0$ is big, we have $b > 0$ by Theorem 4.1. Hence, the tangent bundle $T\mathcal{X}_t$ is big for any $t \in \Delta$. □

Remark 4.17. Recall that a smooth projective variety X is said *locally rigid* if for any smooth deformation $\mathcal{X} \to \Delta$ with $\mathcal{X}_0 \cong X$, we have $\mathcal{X}_t \cong X$ for t in a small analytic neighbourhood of 0:

- Smooth projective horospherical varieties with Picard number 1 and the smooth projective two-orbits varieties $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are locally rigid except the horospherical G_2-variety X^5 [32, Theorem 0.5 and Proposition 2.3].
- Smooth projective symmetric varieties with Picard number 1 are locally rigid; see [1; 20; 26].
- Smooth equivariant compactifications of vector groups with Picard number 1 may be not locally rigid. Among all the known examples (see Example 4.5), the only locally nonrigid ones are the smooth $\mathbb{P}^4$-general linear sections $\mathbb{S}_k^a$ of $\mathbb{S}_5 \subseteq \mathbb{P}^{15}$ with codimension $k = 2$ or 3; see [1].

Corollary 4.18. *Let S_k be a smooth codimension k linear section of $\mathbb{S}_5 \subseteq \mathbb{P}^{15}$. Then TS_k is big if $k \leq 3$.*

Proof. Recall that the VMRT of $\mathbb{S}_5$ is the Grassmannian $\mathrm{Gr}(2, 5) \subseteq \mathbb{P}^9$ in its Plücker embedding which is self-dual. In particular, its dual defect is 2. Moreover, the VMRT of S_k is a smooth codimension k linear section of $\mathrm{Gr}(2, 5) \subseteq \mathbb{P}^9$. In particular, the VMRT of S_k is dual defective if and only if $k = 1$. Moreover, if $k \leq 3$, a codimension k smooth $\mathbb{P}^4$-general linear section S_k^a of $\mathbb{S}_5$ is an equivariant compactification of a vector group. As a consequence, if $k = 2$ or 3, by Theorem 1.2 and Proposition 4.16, the tangent bundle TS_k is big. On the other hand, if $k = 1$, then there is only one class of S_1 up to projective equivalence. Hence, the tangent bundle TS_1 is also big. □

Remark 4.19. The variety S_9 is a smooth curve of genus 7 and S_8 is a smooth K3 surface. In particular, their tangent bundles are even not pseudoeffective. For S_6 and S_7, their VMRTs are 0-dimensional and it follows from [12, Theorem 1.1] that their tangent bundles are not big. Thus the only remaining unknown cases are S_4 and S_5. On the other hand, there are exactly two isomorphic classes of S_2. The special one S_2^a is an equivariant compactification of $\mathbb{G}_a^8$, while the general one S_2^g is the $G_2 \times \mathrm{PSL}_2$-variety $\boldsymbol{X}_2$ given in [31, Theorem 0.2 and Definition 2.12]; see [1, Proposition 4.8].

Question 4.20. Are the tangent bundles of S_4 and S_5 big?

In the following we apply Proposition 1.1 to treat the two-orbits F_4-variety $\boldsymbol{X}_1$. Let us give a brief geometric description of $\boldsymbol{X}_1$ [31, proof of Proposition 2.2 and Definition 2.11]. Set $G = F_4$. Let $G/H = O \subseteq \boldsymbol{X}_1$ be the unique open G-orbit and by Z its complement, which is the unique closed G-orbit. Then Z has codimension 3. Let P be a parabolic subgroup of G containing H and minimal for this property. Then $R(P) \subseteq H$ and P is the maximal parabolic subgroup $P(\omega_1)$ of F_4. Let

$$\varphi : G/H = O \to Y = G/P$$

be the natural projection. Let F be an arbitrary fibre of φ. Then P acts transitively over F. Denote by Q the quotient $P/R(P)$. Then Q is a semisimple group of type C_3 and Q acts transitively over F. Moreover, the Q-variety F has an equivariant compactification $\mathrm{Gr}(2, \mathbb{C}^6) \subseteq \mathbb{P}(\wedge^2\mathbb{C}^6)$ whose boundary divisor is a closed Q-orbit isomorphic to $\mathrm{Gr}_\omega(2, 6)$.

Proposition 4.21. *The tangent bundle of X_1 is big.*

Proof. Firstly we show that the image $\mathcal{M}_F^P \subseteq \mathfrak{p}^*$ has dimension $2\dim(F) - 1$. Indeed, note that the Q-variety F has complexity 0 and rank 1. It follows from Theorem 2.10 that the variety $\mathcal{M}_F^Q \subseteq \mathfrak{q}^*$ has dimension $2\dim(F) - 1$. Let $\bar{\mathfrak{h}}$ be the Lie algebra of the image $\overline{H}$ of H in Q and let $\iota : \mathfrak{q}^* \to \mathfrak{p}^*$ be the natural inclusion induced by $P \to Q$. As $R(P) \subseteq H$, we have $\iota(\bar{\mathfrak{h}}^\perp) = \mathfrak{h}^\perp$. Consider the following commutative diagram:

$$\begin{array}{ccc} T^*F = P *_H \mathfrak{h}^\perp & \xrightarrow{\Phi_F^P} & \mathfrak{p}^* \\ \sigma \downarrow & & \uparrow \iota \\ T^*F = Q *_{\overline{H}} \bar{\mathfrak{h}}^\perp & \xrightarrow[\Phi_F^Q]{} & \mathfrak{q}^* \end{array}$$

where σ is an isomorphism. This yields $\mathcal{M}_F^P = \iota(\mathcal{M}_F^Q)$ and consequently $\mathcal{M}_F^P$ has dimension $2\dim(F) - 1$.

Next we show that the image $\mathcal{M}_{X_1}^G(F)$ has dimension $\dim(X_1) + \dim(F) - 1$. To see this, let us consider the following commutative diagram:

$$\begin{array}{ccccc} N^*_{F/X_1} & \longrightarrow & T^*X_1|_F & \longrightarrow & T^*F \\ & \searrow^{\nu} & \downarrow \Phi_{X_1}^G & & \downarrow \Phi_F^P \\ & & \mathfrak{g}^* & \xrightarrow{\eta} & \mathfrak{p}^* \end{array}$$

As $\eta(\mathcal{M}_{X_1}^G(F)) = \eta(\Phi_{X_1}^G(T^*X|_F)) = \Phi_F^P(T^*F) = \mathcal{M}_F^P$, it follows that we have

$$\dim(\mathcal{M}_{X_1}^G(F)) \leq \dim(\mathcal{M}_F^P) + \dim(\mathfrak{g}^*) - \dim(\mathfrak{p}^*) = \dim(X_1) + \dim(F) - 1.$$

Finally, note that the G-variety X_1 has complexity 0 and rank 1. The image $\mathcal{M}_{X_1}^G$ has dimension $2\dim(X_1) - 1$ by Theorem 2.10. Hence, we obtain

$$2\dim(X_1) - 1 = \dim(\mathcal{M}_{X_1}^G) \leq \dim(Y) + \dim(\mathcal{M}_{X_1}^G(F)) \leq 2\dim(X_1) - 1.$$

This implies that $\dim(\mathcal{M}_{X_1}^G(F)) = \dim(X_1) + \dim(F) - 1$. Let A be a prime ample divisor in Y. Then the closure of φ^*A in X_1 is an ample prime divisor as X_1 has Picard number 1. On the other hand, note that we have

$$\dim(\mathcal{M}_{X_1}^G(\varphi^*A)) \leq \dim(A) + \dim(\mathcal{M}_{X_1}^G(F)) = 2\dim(X) - 2.$$

Hence, according to Proposition 1.1, the tangent bundle of X_1 is big. $\square$

4D2. *Pseudoeffective slope of equivariant compactifications.* Recall from Notation 3.2 that if $\mathcal{H} \subseteq \mathbb{P}(\mathfrak{g}^*)$ is an irreducible reduced hypersurface defined by $\xi \in \operatorname{Sym}^m \mathfrak{g}$, then we have

$$D_\xi = D_{\mathcal{H}} + \sum \operatorname{mult}_{\pi^* D}(D_\xi)\pi^* D,$$

where D runs over all the prime divisor in X such that $\mathbb{P}(\mathcal{M}_X^G(D))$ is contained in $\mathcal{H}$ and $\pi : \mathbb{P}(T^*X) \to X$ is the natural projection.

Notation 4.22. Let C be a smooth projective curve and let E be a vector bundle of rank n over C. Assume that there exists a nonzero map $\varphi : E \to V^r$, where V^r is the trivial vector bundle of rank r over C. Let $p : \mathbb{P}(V^r) = C \times \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$ be the second projection. Let ξ be a homogeneous polynomial of degree d over $\mathbb{P}^{r-1}$. Then for any point $c \in C$, denote by E_c the fibre of E over c. Then the restricted linear map $\varphi_c := \varphi|_{E_c} : E_c \to \mathbb{C}^r$ induces a homogeneous polynomial $\varphi_c^*\xi$ on the fibre $F = \mathbb{P}(E_c)$ of $\mathbb{P}(E) \to C$ over c, which is either zero or of degree d. In particular, if $\varphi_{c'}^*\xi$ is nonzero for some $c' \in C$, then we can define the *multiplicity* $m_F(\varphi, \xi)$ *along* F as the multiplicity of the pull-back $(p \circ \varphi)^*\xi$ along F. Clearly we have $m_F(\varphi, \xi) = m_F(\varphi, a\xi)$ for any nonzero constant a.

Let us give a geometric explanation for the notation $m_F(\varphi, \xi)$. Denote by $\bar{\varphi}$ the induced rational map $\mathbb{P}(E) \dashrightarrow \mathbb{P}(V^r)$. Then the composition $p \circ \bar{\varphi} : \mathbb{P}(E) \dashrightarrow \mathbb{P}^{r-1}$ is defined by the following linear system

$$\overline{V} := \operatorname{Image}(H^0(C, (V^r)^*) \to H^0(C, E^*)) \subseteq H^0(C, E^*) = H^0(\mathbb{P}(E), \mathscr{O}_{\mathbb{P}(E)}(1)).$$

Let D be the fixed part of $|\overline{V}|$. Then we have

$$(p \circ \bar{\varphi})^* \mathscr{O}_{\mathbb{P}^{r-1}}(1) \otimes \mathscr{O}_{\mathbb{P}(E)}(D) \cong \mathscr{O}_{\mathbb{P}(E)}(1).$$

Let H_ξ be the degree d divisor in $\mathbb{P}^{r-1}$ corresponding to ξ. Then the assumption $\varphi_{c'}^*\xi \neq 0$ means that the pull-back $(p \circ \bar{\varphi})^* H_\xi$ is a well-defined divisor in $\mathbb{P}(E)$. Moreover, we have

$$m_F(\varphi, \xi) = \operatorname{mult}_F((p \circ \bar{\varphi})^* H_\xi) + d \operatorname{mult}_F(D).$$

Lemma 4.23. *Let $\mathcal{H} \subset \mathbb{P}(\mathfrak{g}^*)$ be a hypersurface defined by a homogeneous polynomial ξ. Let D be a prime divisor in X such that $\mathbb{P}(\mathcal{M}_X^G(D)) \subseteq \mathcal{H}$. Fix a general point $x \in D$. Let C be an irreducible curve passing through x such that $C \not\subseteq D$ and $\mathbb{P}(\mathcal{M}_X^G(C)) \not\subseteq \operatorname{supp}(\mathcal{H})$. Denote by $f : \widetilde{C} \to C$ its normalisation. Then we have*

$$\operatorname{mult}_{\pi^* D}(D_\xi) = m_F(\varphi, \xi),$$

where the map $\varphi : f^*(T^*X|_C) \to \widetilde{C} \times \mathfrak{g}^*$ is naturally induced by the moment map Φ_X^G and F is a fibre of $\mathbb{P}(f^*(T^*X|_C)) \to \widetilde{C}$ over a point c such that $f(c) = x \in D$.

Proof. This follows directly from the definition of D_ξ and the fact that the multiplicity $\operatorname{mult}_{\pi^* D}(D_\xi)$ only depends on the multiplicity of D_ξ along general fibres of $\pi^* D \to D$. □

Now we are in the position to finish the proof of Theorem 1.3.

Proof of Theorem 1.3. The first statement follows from Proposition 4.9. For the second statement, as the VMRT of X is smooth, following the notations in Proposition 4.7, there exist closed subvarieties $E_0 \subseteq \mathrm{Bl}_S(\mathbb{P}^n)$ and $D_0 \subseteq X$ of codimension at least 2 such that the following morphism

$$\Phi : X \setminus D_0 \to \mathrm{Bl}_S(\mathbb{P}^n) \setminus (\widetilde{H} \cup E_0)$$

is an isomorphism. Moreover, the induced rational map $E \dashrightarrow D$ is birational and $\mathbb{G}_a^n$-equivariant. Note that the $\mathbb{G}_a^n$-orbits on $E = \mathbb{P}(N_{S/\mathbb{P}^n}) \setminus W = \mathbb{P}(N_{S/H})$ are just the fibres of the natural projection $E \setminus W \to S$ (see Example 4.6). Thus the general $\mathbb{G}_a^n$-orbits on D have dimension $n-p-1$, where p is the dimension of the VMRT $S \subseteq \mathbb{P}^{n-1}$ of X.

Fix a point $o \in X$. If l is a general minimal rational curve passing through o, then l is the strict transform of a line l' in $\mathbb{P}^n$ passing through o. Moreover, we may also assume that the strict transform of l in $\mathrm{Bl}_S(\mathbb{P}^n)$ is disjoint from $\widetilde{H} \cup E_0$. In particular, the curve l meets D at a smooth point $z \in D \setminus D_0$. Since l is standard, we have

$$f^*TX \cong \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus p} \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus n-p-1}.$$

where $f : \mathbb{P}^1 \to l \subseteq X$ is the natural embedding. Denote by T_l^+X the positive factor of f^*TX and by T_o^+l the fibre of T_l^+X over o. By Proposition 4.4, there exists a 1-dimensional subspace V_l of $\mathbb{G}_a^n$ such that l is the closure of the V_l-orbit of o. Moreover, the subbundle T_l^+X is preserved by the V_l-action.

Denote by $\mathfrak{g}_l$ the subspace of $\mathfrak{g}$ corresponding to the subspace $T_o^+l \subseteq T_oX \cong \mathfrak{g}$. Then the V_l-action induces a map of vector bundles

$$\Psi : l \times \mathfrak{g} \to f^*TX$$

such that the induced map $l \times \mathfrak{g}_l \to T_l^+X$ is nondegenerate along $l \setminus \{z\}$. Moreover, as $\mathbb{G}_a^n$ is commutative, the dual map $\Psi^* : f^*T^*X \to l \times \mathfrak{g}^*$ is exactly the restriction of the map

$$\pi \times \Phi_X^{\mathbb{G}_a^n} : T^*X \to X \times \mathfrak{g}^*.$$

From the splitting type of T_l^+X, one can derive that the linear map $\{z\} \times \mathfrak{g}_l \to T_z^+l$ is zero. As the $\mathbb{G}_a^n$-orbit of z has dimension $n-p-1$, the rank of the linear map

$$\{z\} \times \mathfrak{g} \to T_zX$$

is $n-p-1$. This implies $\Phi_X^{\mathbb{G}_a^n}(T_z^*X) = \mathfrak{g}_l^\perp$. On the other hand, as $\mathbb{P}(T_o^+l) \subseteq \mathbb{P}(T_oX)$ is the projectivised tangent bundle of the VMRT $\mathcal{C}_o \subseteq \mathbb{P}(T_oX)$ at $[T_ol]$, thus we may regard $\mathfrak{g}_l^\perp$ as the set of hyperplanes in $\mathbb{P}(T_oX)$ which are tangent to $\mathcal{C}_o$ at $[T_ol]$.

For a general point $z \in D$, the $\mathbb{G}_a^n$-orbit O_z of z is the image of a fibre of $E \setminus W \to S$ over some point $s \in S$. In particular, the strict transform of the line connecting o and s is a minimal rational curve l on X passing through o and meeting O_z at a point z'. As $\mathbb{G}_a^n$ is commutative, we have $\Phi_X^{\mathbb{G}_a^n}(T^*X|_{O_z}) = \Phi_X^{\mathbb{G}_a^n}(T_{z'}^*X) = \mathfrak{g}_l^\perp$, where $\mathfrak{g}_l$ is the subspace of $\mathfrak{g}$ corresponding to T_o^+l. As a consequence, the image $\mathbb{P}(\mathcal{M}_D^{\mathbb{G}_a^n}) \subseteq \mathbb{P}(\mathfrak{g}^*)$ is the closure of the following

$$\bigcup_{[l]\in\mathcal{K}_x \text{ general}} \mathbb{P}(\mathfrak{g}_l^{\perp}) = \bigcup_{[l]\in\mathcal{K}_x \text{ general}} \mathbb{P}((T_o^{+}l)^{\perp}) \subseteq \mathbb{P}(T_o^* X)$$

which is exactly the dual variety of the VMRT $\mathcal{C}_o \subseteq \mathbb{P}(T_o X)$ by definition.

Finally, we assume that the VMRT is smooth and not dual defective. Then $\mathbb{P}(\mathcal{M}_D^G)$ is a hypersurface in $\mathbb{P}(\mathfrak{g}^*)$ defined by a homogeneous polynomial $\xi \in \mathrm{Sym}^a\,\mathfrak{g}$. where a is the degree of $\mathcal{H}$, i.e., the codegree of $\mathcal{C}_o$. For simplicity, we denote it by $\mathcal{H}$. Now we want to determine the cohomological class of the divisor $D_{\mathcal{H}} \subseteq \mathbb{P}(T^*X)$. By Lemma 4.2, we have $\bar{\Phi}_X^{\mathbb{G}_a^n}(\check{\mathcal{C}}) \subseteq \mathcal{H}$. This implies that $\check{\mathcal{C}}$ is contained in $\mathrm{supp}(D_\xi)$. On the other hand, since both $\check{\mathcal{C}}$ and $D_{\mathcal{H}}$ are π-horizontal prime divisors, we obtain $\check{\mathcal{C}} = D_{\mathcal{H}}$. In particular, by Theorem 4.1, it remains to prove $D_{\mathcal{H}} \sim a\Lambda - 2\pi^* D$.

Choose a general minimal rational curve l on X meeting D at z. Fix a general point $o \in l$ and identify $\mathcal{H} \subseteq \mathbb{P}(\mathfrak{g}^*)$ to the dual variety of $\mathcal{C}_o \subseteq \mathbb{P}(T_o X)$. Consider the following map $\Psi^* : f^*T^*X \to l \times \mathfrak{g}^*$. By Lemma 4.23, we only need to calculate $\mathrm{mult}_F(\Psi^*, \xi)$, where F is the fibre of $f^*T^*X \to l$ over z. Fix a coordinate t around $z \in \mathbb{A}^1 \subseteq l$. Then after choosing suitable trivialisation of f^*T^*X, the map $\Psi^* : \mathbb{A}^1 \times \mathbb{C}^n \to \mathbb{A}^1 \times \mathbb{C}^n$ can be written in coordinates as follows:

$$(x, v_0, v_1, \dots, v_p, v_{p+1}, v_{n-1}) \mapsto (x, t^2 v_0, t v_1, \dots, t v_p, v_{p+1}, \dots, v_{n-1}),$$

where the first coordinate v_0 corresponds to the cotangent bundle $\mathscr{O}_{\mathbb{P}^1}(-2)$ of l and the first $p+1$ coordinates correspond to the negative factors of f^*T^*X. Given a general point y on $\bar{\Phi}_X^{\mathbb{G}_a^n}(\mathbb{P}(F)) \subseteq \mathcal{H}$, then we may assume that $\mathcal{H}$ is smooth at y as l is general. In particular, by biduality theorem, the projectivised tangent bundle of $\mathcal{H}$ at y corresponds to the point $[T_o l] \in \mathcal{C}_o$. This implies that the linear part of the local equation of $\mathcal{H}$ at y only consists of the first coordinate v_0. In particular, the local description above shows that multiplicity $\mathrm{mult}_F(\Psi^*, \xi)$ is 2. Hence, we have $D_\xi = D_{\mathcal{H}} + 2\pi^* D$ and the result follows as $D_\xi \in |a\Lambda|$. □

Remark 4.24. Let us give a more geometric description of the linear map $\{z\} \times \mathfrak{g} \to T_z X$. Let $\pi_o : \mathbb{P}^n \setminus \{o\} \to H$ be the projection from o. Then π_o induces a natural projection

$$p_o : \mathfrak{g} \setminus \{0\} \xrightarrow{d\mu_o} \boldsymbol{T}_o\mathbb{P}^n \setminus \{o\} = \mathbb{P}^n \setminus \{o\} \xrightarrow{\pi_o} H,$$

where $\boldsymbol{T}_o\mathbb{P}^n$ is the projectivised tangent space of $\mathbb{P}^n$ at o. For a general point $s \in S$, for a point $z' \in E_s \setminus W_s$, the Lie algebra $\mathfrak{g}_{z'}$ of the isotropy subgroup $G_{z'}$ of z' is exactly the linear subspace of the inverse image $p_o^{-1}(\boldsymbol{T}_s S) = \mathfrak{g}_l$, where $\boldsymbol{T}_s S \subseteq H$ is the projectivised tangent bundle of S at s. Thus the map $\{z\} \times \mathfrak{g} \to T_z X$ is the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{g}_l$.

4D3. *Codegree of VMRT.* Now we proceed to calculate the codegree of the VMRT of the equivariant compactifications X of vector groups given in Example 4.5. If X is an irreducible Hermitian symmetric space, the pseudoeffective cone of $\mathbb{P}(T^*X)$ and hence the value $\mu(TX, -K_X)$ are determined in [35] and [8]. In particular, if the VMRT is not dual defective, it turns out that its codegree is equal to the rank of X in the sense of [35, Definition 4.6]. Here we remark that the definition of rank in [35, Definition 4.6]

X	VMRT	embedding	defect	codegree
$X^3(m,m-1)$, $(m\geq 2)$	$\mathbb{P}(\mathscr{O}_{\mathbb{P}^{m-1}}(-1)\oplus\mathscr{O}_{\mathbb{P}^{m-1}}(-2))$	$\lvert\mathscr{O}(1)\rvert$	0	$m+1$
V_2	$\mathbb{P}^1$	$\lvert\mathscr{O}(3)\rvert$	0	4
S_1^a	V_1	$\lvert\mathscr{O}(1)\rvert$	1	–
S_2^a	V_2	$\lvert\mathscr{O}(1)\rvert$	0	5
S_3^a	V_3	$\lvert\mathscr{O}(1)\rvert$	0	10

Table 1. Known examples of nonhomogeneous EC-structure.

of X is different from that given in Definition 2.9. For the remaining nonhomogeneous examples, we summarise the results in Table 1.

Here we recall that V_2 is the codimension two smooth linear section of $\mathrm{Gr}(2,5)\subseteq\mathbb{P}^9$ in its Plücker embedding and S_i^a is a codimension i smooth $\mathbb{P}^4$-general linear section of the spinor tenfold $\mathbb{S}_5\subseteq\mathbb{P}^{15}$ in its minimal embedding (see Example 4.5).

Odd Lagrangian Grassmannians. Let $\boldsymbol{a}:=(a_0,\dots,a_r)$ be a sequence of integers such that $0\leq a_0\leq\cdots\leq a_r$ with $a_r>1$. Denote by $E_m(\boldsymbol{a})$ the following vector bundle over $\mathbb{P}^m$:

$$\bigoplus_{i=0}^{r}\mathscr{O}_{\mathbb{P}^m}(-a_i).$$

Then the tautological linear bundle $\mathscr{O}_{\mathbb{P}(E_m(\boldsymbol{a}))}(1)$ of $\mathbb{P}(E_m(\boldsymbol{a}))$ is globally generated and defines a morphism

$$\Phi_m(\boldsymbol{a}):\mathbb{P}(E_m(\boldsymbol{a}))\to\mathbb{P}^{N(m,\boldsymbol{a})}.$$

This map is birational because $a_r>0$. Write $S_m(\boldsymbol{a})$ for the image of this map. Note that if $a_0>0$, the morphism $\Phi_m(\boldsymbol{a})$ is an embedding.

According to [17, Section 6] (see also [18, Proposition 3.5.2]), the VMRT of the odd symplectic Grassmannian $X^3(m,i)$ is projectively equivalent to

$$S_{m-1}(1^{2m-2i-1},2)\subseteq\mathbb{P}^{N(m,1^{2m-2i-1},2)}.$$

In particular, the codegree of the VMRT of odd Lagrangian Grassmannians $X^3(m,m-1)$ can be derived from the following general result.

Proposition 4.25. *The dual variety of the scroll $S_m(1^r,2)\subseteq\mathbb{P}^{N(m,1^r,2)}$ is a hypersurface of degree $m+r+1$ if $m\geq r$.*

Proof. Denote by Λ a hyperplane section of $S_m(1^r,2)$. Firstly we recall that the projective variety $S_m(1^r,2)$ is isomorphic to the blowing-up of the projective space $\mathbb{P}^{m+r}$ along a linear subspace $\mathbb{L}\cong\mathbb{P}^{r-1}$, see for instance [5, Section 9.3.2]. Denote this blowing-up $S_m(1^r,2)\to\mathbb{P}^{m+r}$ by μ and let E be the exceptional divisor. Then we have an isomorphism

$$\mathscr{O}_{S_m(1^r,2)}(\Lambda)\cong\mu^*\mathscr{O}_{\mathbb{P}^{m+r}}(2)\otimes\mathscr{O}_{S_m(1^r,2)}(-E).$$

In particular, taking push-forward yields a linear isomorphism

$$p : |\Lambda| \to |\mathscr{O}_{\mathbb{P}^{m+r}}(2) \otimes \mathscr{I}_{\mathbb{L}}| = H_{\mathbb{L}},$$

where $\mathscr{I}_{\mathbb{L}}$ is the ideal sheaf of $\mathbb{L}$. Denote by $\check{S} \subseteq |\Lambda|$ the dual variety of $S_m(1^r, 2)$, i.e., the closure of the set of singular elements. For a general point $[Q] \in \check{S}$, we may assume that the singular locus of Q is not contained in E. This implies that the push-forward p_*Q is a singular element in the linear system $|\mathscr{O}_{\mathbb{P}^{m+r}}(2)|$ which contains $\mathbb{L}$.

Conversely, as $r \leq m$, a general element in $H_{\mathbb{L}}$ is smooth and a general singular element $[Q] \in H_{\mathbb{L}}$ is a quadric hypersurface containing $\mathbb{L}$ such that it is a cone with a single point $p \in \mathbb{P}^{m+r} \setminus \mathbb{L}$ as vertex and hence $p^{-1}[Q]$ is contained in $\check{S}$. As a consequence, the map p induces a dominant map

$$\bar{p} : \check{S} \to \check{X} \cap H_{\mathbb{L}} \subseteq |\mathscr{O}_{\mathbb{P}^{m+r}}(2)|,$$

where $\check{X}$ is the dual variety of the Veronese embedding $X = \nu_2(\mathbb{P}^{m+r}) \subseteq |\mathscr{O}_{\mathbb{P}^{m+r}}(2)|$. As $\mathbb{P}^{m+r}$ is homogeneous, the variety $\check{X} \cap H_{\mathbb{L}}$ is an irreducible proper subvariety of $H_{\mathbb{L}}$. In particular, the map $\bar{p}$ is an isomorphism. Note that $\check{X}$ is a hypersurface of degree $m+r+1$ by Boole formula [38, Example 6.4], hence $\check{S} \subseteq |\Lambda|$ is a hypersurface of degree $m+r+1$. □

Remark 4.26. Let $[Q] \in \check{S}$ be a general singular hyperplane section of S. If $m < r$, then the divisor p_*Q is a quadric cone containing $\mathbb{L}$ with vertex $\mathbb{L}' \subseteq \mathbb{P}^{m+r}$, which is a $(r-m)$-dimensional linear subspace such that $\dim(\mathbb{L} \cap \mathbb{L}') = r - m - 1$. In particular, the singular locus of Q has dimension $r-m$ and this implies that the scroll $S_m(1^r, 2)$ has dual defect $r-m$ [38, Theorem 7.21].

Linear section V_k of the Grassmannian Gr(2, 5). The VMRT of the Grassmannian Gr(2, 5) is projectively equivalent to the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \subseteq \mathbb{P}^5$. Moreover, for $k \leq 3$, there is only one isomorphic class of codimension k linear section V_k of Gr(2, 5). This implies that the VMRT of V_k is projectively equivalent to a general linear section of $\mathbb{P}^1 \times \mathbb{P}^2$ with codimension k. Then an easy computation shows that the VMRT of V_2 is the twisted cubic in $\mathbb{P}^3$ whose dual variety is a quartic surface; see for instance [38, Example 10.3].

Linear section S_k of the spinor tenfold $\mathbb{S}_5$. The VMRT of the 10-dimensional spinor variety $\mathbb{S}_5$ is the Grassmannian Gr(2, 5) $\subseteq \mathbb{P}^9$ in its Plücker embedding. Hence, the VMRT of the codimension k linear section S_k of $\mathbb{S}_5$ is projectively equivalent to the smooth codimension k linear section $V_k \subseteq \mathbb{P}^{9-k}$ of the Grassmannian Gr(2, 5). As Gr(2, 5) $\subseteq \mathbb{P}^9$ has dual defect 2, the linear section $V_k \subseteq \mathbb{P}^{9-k}$ has dual defect $\max\{0, 2-k\}$ [38, Theorem 5.3]. In the following we will compute the codegree of $Z = V_k$, $k = 2$ or 3, using the Katz–Kleiman formula [38, Theorem 6.2]:

$$\operatorname{codeg}(Z) = \sum_{i=0}^{\dim(Z)} (i+1) c_{\dim(Z)-i}(T^*Z) \cdot H^i,$$

where H is the hyperplane section. To calculate the Chern classes of Z, firstly we write the total Chern classes of Gr(2, 5) as

$$\begin{pmatrix} 1 & & & \\ 5 & 12 & & \\ 11 & 30 & 25 & \\ 15 & 35 & 30 & 33 \end{pmatrix}$$

where the rows and columns are labelled from 0, and the (i, j)-th element is the coefficient of the Schubert cycles $\sigma_{i,j}$. From the tangent sequence of Z we have

$$c(Z) := c(V_k) = \frac{c(\mathrm{Gr}(2,5))}{(1+\sigma_1)^k},$$

where $\sigma_1 := \sigma_{1,0}$ denotes the ample generator of the Picard group, i.e., the hyperplane section. Using the Pieri's formula, that is,

$$\sigma_{a,b} \cdot \sigma_1 = \sigma_{a+1,b} + \sigma_{a,b+1},$$

a routine computation then yields:

$$c(V_2) = \begin{pmatrix} 1 & & & \\ 3 & 5 & & \\ 4 & 6 & 4 & \\ 4 & 2 & * & * \end{pmatrix} \quad \text{and} \quad c(V_3) = \begin{pmatrix} 1 & & & \\ 2 & 3 & & \\ 2 & 1 & * & \\ 2 & * & * & * \end{pmatrix}$$

Using again the Pieri's formula and our computations of the degrees of the Schubert classes we deduce that the codegree of V_2 and V_3 are 5 and 10, respectively.

There is also an alternative more geometric way to see that the codegree of V_2 is 5. Since the Grassmannian $\mathrm{Gr}(2,5) \subseteq \mathbb{P}^9$ is self-dual, the dual variety of $V_k \subseteq \mathbb{P}^{9-k}$, $k = 1$ or 2, is projectively equivalent to the image of Gr(2, 5) under the projection $\pi_{\mathbb{L}} : \mathbb{P}^9 \dashrightarrow \mathbb{P}^{9-k}$ from a general linear subspace $\mathbb{L} \subseteq \mathbb{P}^9$ of dimension $k-1$; see for instance [38, Theorem 5.3]. As $\mathbb{L}$ is general, the restriction of $\pi_{\mathbb{L}}$ to Gr(2, 5) is a birational morphism. Since Gr(2, 5) is of degree 5 and with dimension 6, it follows that the dual variety of V_2 is a hypersurface in $\mathbb{P}^7$ with degree 5.

Remark 4.27. (1) Let X be a Fano manifold with Picard number 1. Then the *anticanonical pseudoeffective slope* $\mu(TX, -K_X)$ *of* TX is bounded by the *maximal slope* of TX with respect to $-K_X$; see [8, Lemma 2.8]. In particular, if TX is semistable, then $\mu(TX, -K_X)$ is bounded by $1/\dim(X)$. Actually, it is expected that this should hold without the semistability assumption; see [8, Conjecture 1.3]. On the other hand, while the semistability of TX is confirmed in many cases, [19, Theorem 0.3] says that the tangent bundles of the horospherical varieties $X^1(m)$ ($m \geq 4$) and X^4 are not semistable. Thus it is natural and interesting to ask if their anticanonical pseudoeffective slopes are (strictly) dominated by the reciprocal of their dimensions.

(2) For odd Lagrangian Grassmannians $X = X^3(m, m-1)$ ($m \geq 2$), according to Table 1, we have

$$\mu(TX, -K_X) = \frac{2}{(m+1)(m+2)} < \frac{1}{\dim(X)} = \frac{2}{m(m+3)}.$$

(3) By [35, Corollary 1.4; 8, Theorem 1.14] and Table 1 above, the anticanonical pseudoeffective slope $\mu(TX, -K_X)$ of the varieties in Example 4.5 are determined except the hyperplane section S_1 of $\mathbb{S}_5$. As the VMRT of S_1 is dual defective, maybe we need a different treatment.

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