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Degree growth for tame automorphisms of an affine quadric threefold

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We consider the degree sequences of the tame automorphisms preserving an affine quadric threefold. Using some valuative estimates derived from the work of Shestakov and Umirbaev and the action of this group on a CAT(0), Gromov-hyperbolic square complex constructed by Bisi, Furter and Lamy, we prove that the dynamical degrees of tame elements avoid any value strictly between 1 and $\frac{4}{3}$. As an application, these methods allow us to characterize when the growth exponent of the degree of a random product of finitely many tame automorphisms is positive.

Introduction

Fix a projective variety X of dimension n defined over an algebraically closed field k of characteristic zero and a rational map f on X. We are interested in the complexity associated to the dynamical system induced by f, more precisely on the growth of the degrees of the p-fold composition $f^p = f \circ \cdots \circ f$. This general problem was addressed in the work of Russakovski and Shiffman [1997] when $X = \mathbb{P}^n$ in which they related the asymptotic behavior of the images by f of the linear subvarieties of $\mathbb{P}^n$ with the degree sequences. The asymptotic ratios of these sequences, denoted $\lambda_i(f)$ for $i \leqslant n$, and referred as dynamical degrees, control the topological entropy of those maps [Dinh and Sibony 2005] and are crucial for the construction of an invariant measure of maximal entropy [Bedford and Smillie 1992; Bedford and Diller 2005; Guedj 2005].

When f is a birational surface map, the situation is completely classified [Blanc and Cantat 2016; Cantat 2011; Diller and Favre 2001; Gizatullin 1980]. For general rational maps on surfaces, the behavior of the degree is known for morphisms of the affine plane [Favre and Jonsson 2011] and when $\lambda_1(f)^2 > \lambda_2(f)$ [Boucksom et al. 2008].

From the dimension three on, only the degree growth of monomial maps [Favre and Wulcan 2012; Lin 2012], regular morphisms, pseudoautomorphisms [Bedford 2015; Oguiso and Truong 2014; 2015; Truong 2016; 2017], birational maps on hyperkähler varieties [Lo Bianco 2019], dominant rational maps satisfying $\lambda_1(f)^2 > \lambda_2(f)$ [Dang and Favre 2021] and sporadic examples [Abarenkova et al. 1999a; 1999b; Anglès d'Auriac et al. 2006; Bedford and Truong 2010; Bedford and Kim 2014] were studied. Recently, new constraints on slow degree growth appeared for polynomial maps of the affine space [Urech

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2018], birational mappings [Cantat and Xie 2020] and nonregularizable birational transformations [Lonjou and Urech 2021], however the general problem of understanding the degree of the iterates of birational transformations of $\mathbb{P}^3$ remains open. The main reason is that we usually rely on the construction of a good birational model (e.g., an algebraically stable model in the sense of Fornaess and Sibony [1995]) to find the degree sequences, but the structure of the set of birational models of threefolds is far more complicated than its analog for surfaces. It is thus natural to ask whether we can find a large class of birational transformations of $\mathbb{P}^3$ for which this sequence is fully understood.

A first natural choice would be the group of polynomial automorphisms of the three dimensional affine space. Even though there has been some recent work on particular subgroups of this group [Lamy 2019; Lamy and Przytycki 2021; Wright 2015], their dynamical degrees were computed explicitly for degree 2 maps [Maegawa 2001], for degree 3 maps [Blanc and van Santen 2019]. Recently, the author proved with C. Favre that the dynamical degrees of polynomial automorphisms of $\mathbb{A}^3$ are all algebraic numbers [Dang and Favre 2021]. However, the problem of classifying all the dynamical degrees and all the possible degree growths remain open. We have thus turned our attention to a simpler situation, namely the subgroup of tame automorphisms of the affine quadric threefold.

We denote by (x, y, z, t) the affine coordinates in $\mathbb{A}^4$ and consider the affine quadric $\mathcal{Q}$ given by

$$\mathcal{Q} = V(xt - yz - 1).$$

Observe that the Picard group of the closure $\overline{\mathcal{Q}}$ of $\mathcal{Q}$ in $\mathbb{P}^4$ is generated by $H = c_1(\mathcal{O}(1)_{|\overline{\mathcal{Q}}})$ so that one can define the algebraic degree of an automorphism by

$$\deg(f) := \deg_1(f) = (\pi_1^* H^2 \cdot \pi_2^* H),$$

where π_1 and π_2 are the projections of the graph of the birational map induced by f in $\overline{\mathcal{Q}} \times \overline{\mathcal{Q}}$ onto the first and the second factor respectively. Observe that by definition $\deg_2(f) = (\pi_1^* H \cdot \pi_2^* H^2) = \deg(f^{-1})$ since f is an automorphism.

The group of automorphism naturally contains the subgroup $O_4 \subset GL_4(k)$ of linear maps of $\mathbb{A}^4$ preserving the quadric Q. The subgroup of tame automorphisms, denoted Tame(Q), is defined as the subgroup generated by O_4 and transformations induced by

$$(x, y, z, t) \mapsto (x, y, z + xP(x, y), t + yP(x, y)),$$

with $P \in k[x, y]$.

Theorem 1. *Let f be a tame automorphism, then one of the following possibilities occur*:

(i) *The sequence $(\deg(f^n), \deg(f^{-n}))$ is bounded. Moreover, f is conjugated to an element of O_4 or f^2 is conjugated to an automorphism of the form*

$$(x, y, z, t) \mapsto (ax, by + xR(x), b^{-1}z + xP(x, y), a^{-1}(t + yP(x, y) + zR(x) + xR(x)P(x, y)))$$

with $a, b \in k^*$, $P \in k[x, y]$ and $R \in k[x]$.

(ii) *There exists a constant* $C > 0$ *such that*

$$\tfrac{1}{C}n \leqslant \deg(f^{\epsilon n}) \leqslant Cn,$$

for all $\epsilon \in \{+1, -1\}$ *and* f *is conjugated to an automorphism of the form*

$$(x, y, z, t) \mapsto (ax, b^{-1}(z + xR(x)), b(y + xP(x)z), a^{-1}(t + z^2P(x) + yR(x) + xzP(x)R(x))),$$

with $a, b \in k^*$, $R \in k[x]$ *and* $P \in k[x] \setminus k$.

(iii) *The sequences* $\deg(f^n)$ *and* $\deg(f^{-n})$ *grow at least exponentially and there exists a constant* $C(f) > 0$ *such that*

$$\min(\deg(f^{-n}), \deg(f^n)) \geqslant C(f)\left(\tfrac{4}{3}\right)^n.$$

Theorem 1 is a first step towards an understanding of the dynamical degrees of these particular automorphisms.

Corollary 2. *The following inclusion is satisfied*:

$$\{\lambda_1(f) \mid f \in \mathrm{Tame(Q)}\} \subset \{1\} \cup \left[\tfrac{4}{3}, +\infty\right[.$$

This result is reminiscent of a theorem of Blanc and Cantat [2016, Corollary 2.7] stating that the set of first dynamical degrees of any birational surface maps is included in $\{1\} \cup [\lambda_L, \infty)$ where $\lambda_L \simeq 1.176280$ denotes the Lehmer number. We conjecture however that the gap should be bigger and that there should be no dynamical degree of Tame(Q) in the interval $]1, 2[$. The verification of such a conjecture would suggest that the dynamical degrees of tame automorphisms of the quadric are always integers.

Another immediate consequence of Theorem 1 is the following corollary.

Corollary 3. *Any tame automorphism* $f \in \mathrm{Tame(Q)}$ *satisfying* $\lambda_1(f) = 1$ *preserves a fibration or belongs to* O_4 *and both sequences* $\deg(f^n)$, $\deg(f^{-n})$ *are either bounded or linear.*

The above result gives a positive answer to a question by Urech [2018, Question 4] in this special situation.

The proof of Theorem 1 exploits extensively the structure of the group of tame automorphisms. We use the natural action of Tame(Q) on a square complex $\mathcal{C}$ which was introduced and studied by Bisi, Furter and Lamy [Bisi et al. 2014]. This action is faithful, transitive on squares, and isometric. The complex $\mathcal{C}$ plays the same role for Tame(Q) as the Bass–Serre tree for Aut$[k^2]$.

One of the main result of [Bisi et al. 2014] is that $\mathcal{C}$ is a geodesic space which is both CAT(0) and Gromov-hyperbolic. As a result, a tame automorphism induces an action on the complex which is rather constrained: either it is elliptic and fixes a vertex in the complex $\mathcal{C}$; or it is hyperbolic and acts by translation on an invariant geodesic line.

Using an explicit description of the stabilizer subgroups of each vertices, we compute the degree sequences of all elliptic tame automorphisms.

The crucial point of the proof is the study in Section 5 of the degree growth of hyperbolic automorphisms. In this case, we show that the sequence of degrees is bounded from below by $C\left(\frac{4}{3}\right)^n$ for some positive constant $C > 0$ and where n depends on the distance of translation on an invariant geodesic line. Let us state a weaker statement which summarizes the overall idea of our proof and which relates the degree with the displacement by f of a vertex v_0 fixed by the linear group.

Theorem 4. *For any tame automorphisms $f \in \mathrm{Tame}(\mathrm{Q})$ for which f is not in O_4, the following inequality holds*:

$$\log(\deg(f)) \geqslant \frac{\log(4/3)}{2\sqrt{2}} d_{\mathcal{C}}(f\cdot v_0, v_0) - 2\log\left(\tfrac{4}{3}\right),$$

where $d_{\mathcal{C}}$ denotes the distance in the complex.

This phenomenon already appears in the case of plane automorphisms since one can bound from below the logarithm of the degree of a plane automorphism by log(2) multiplied by the distance between two vertices in the Bass–Serre tree associated to the group $\mathrm{Aut}(\mathbb{A}^2)$. Also in the case of $\mathrm{Bir}(\mathbb{P}^2)$, there is a relationship between the degree and the distance on a suitable hyperbolic space. The above result does not imply Theorem 1 and one needs to prove a more refined statement to obtain that the degree of f^n is indeed larger than $\left(\frac{4}{3}\right)^n$. Let us explain how this is done.

Let $f \in \mathrm{Tame}(\mathrm{Q})$ be any hyperbolic automorphism. First we show that by conjugating with an appropriate automorphism, we can suppose that v_0 lies at distance ≤ 2 of an f-invariant geodesic line. Suppose that v_0 is contained in an invariant geodesic of f. Our goal is to prove that

$$\deg(f^n) \geqslant \left(\tfrac{4}{3}\right)^{d_{\mathcal{C}}(v_0, f^n\cdot v_0)} \text{ for all } n \in \mathbb{N}. \tag{1}$$

The sequence of large squares (i.e., isometric to $[0, 2]^2$) cut by the geodesic segment $[v_0, f^n\cdot v_0]$ allows us to write

$$f^n = g_p \circ g_{p-1} \cdots \circ g_1 \tag{2}$$

as a composition of elementary automorphisms and linear transformations (in O_4) preserving the quadric. This decomposition is not unique in general and ideally, one would hope to prove that the degree is multiplicative so that $\deg(f^n) \geq \prod_{i=1}^{p} \deg(g_i)$. The obstruction to this property is the presence of resonances, which are explained as follows. Two regular functions $P, R \in k[Q]$ are *resonant* if there exists $\lambda \in k^*$ and two integers p, q such that $\deg(P^p - \lambda R^q) < p\deg(P) = q\deg(R)$ and they are called *critical* if $p = 1$ or $q = 1$.

When these resonances are not *critical*, we show that one can apply the so-called parachute inequalities (recalled in Section 4E) to deduce (1). These inequalities are elementary valuative estimates on the values of partial derivatives of suitable polynomials, and are derived from the proof of Nagata's conjecture by Shestakov and Umirbaev; see [Kuroda 2016; Lamy and Vénéreau 2013; Shestakov and Umirbaev 2003].

To get around the appearance of critical resonances, we exploit the structure of the tame group to prove that f^n always admits an appropriate factorization for which the parachute inequalities can be applied inductively. In other words, we write $f^n = g'_p \circ \cdots \circ g'_1$ where g'_i are tame automorphisms such that for each $i \leqslant p$, g'_{i+1} and $(g'_i \circ \cdots \circ g'_1)$ do not have critical resonances.

Using the correspondence between the factorizations of f and the sequences of large squares cut out by the invariant geodesic, we are reduced to proving that one can modify inductively our initial sequence of large squares to avoid critical resonances. The essential point is to choose a valuation ν of monomial type (i.e., with different weights on the coordinate axis x, y, z, t) such that one of the vertex of our initial large square has ν-value strictly less than the three others. The dissymmetry induced by ν will be propagated along any sequence of squares following our geodesic. We then argue that this minimality property on each large square allows us to choose another square with no critical resonances. As a result, the core of our approach relies deeply on the structure of the tame group which is reflected by the geometric properties of the square complex. Our proof is presented using purely combinatorial arguments.

In the last part of this paper, we shall give a random version of Theorem 1. Consider a finitely generated subgroup G of the tame group and an atomic probability measure μ on G such that

$$\int_G \log(\deg(g))\, d\mu(g) < +\infty.$$

The random walk on G with transition law μ is the Markov chain starting at Id with transition law μ. The state of the Markov chain g_n at the time n is equal to the product of n independent, identically distributed random variable on G with distribution law μ. Its distribution law ν_n is the n-fold convolution of μ. Since the degree is submultiplicative, Kingman's subadditivity asserts that the degree exponents given by

$$\lambda_1(\mu) := \limsup_{n\to+\infty} \frac{1}{n} \int_G \log(\deg(g))\, d\nu_n(g) \quad \text{and} \quad \lambda_2(\mu) := \limsup_{n\to+\infty} \frac{1}{n} \int_G \log(\deg(g^{-1}))\, d\nu_n(g)$$

are finite. These numbers measure the complexity of our random walk and one recovers the first and second dynamical degrees of f when μ is equal to the Dirac measure at f.

Since the degree is equal to the norm of the pullback operator induced by f on the Neron–Severi group of the quadric, these quantities play the same role as the Lyapounov exponents of a random products of matrices [Furstenberg 1963; Furstenberg and Kesten 1960] for this group and the existence of these exponents can thus be interpreted as a law of large number [Benoist and Quint 2016, Theorem 0.6].

We now state the following result on the behavior of any symmetric random walks on this particular group.

Theorem 5. *Let G be a finitely generated subgroup of the tame group and let μ be a symmetric atomic measure on G satisfying the condition*

$$\int_G \log(\deg(g))\, d\mu(g) < +\infty.$$

Then the degree exponents $\lambda_1(\mu) = \lambda_2(\mu)$ are positive if and only if G contains two automorphisms with dynamical degree strictly larger than 1 *generating a free group of rank* 2*.*

Moreover, we also obtain the following classification.

Corollary 6. *When $\lambda_1(\mu) = \lambda_2(\mu) = 0$ then G satisfies one of the following properties*:

(i) *The group G is conjugated to a subgroup of the linear group O_4.*

(ii) *There exists a G-equivariant morphism $\varphi : Q \to \mathbb{A}^2 \setminus \{(0,0)\}$ where G acts on $\mathbb{A}^2 \setminus \{(0,0)\}$ linearly.*

(iii) *The group G contains an automorphism h with $\lambda_1(h) > 1$ and there exists an integer M such that any automorphism $f \in G$ can be decomposed into $g \circ h^p$ where p is an integer and g has a degree bounded by M.*

(iv) *There exists a G-equivariant morphism $\varphi : Q \to \mathbb{A}^1$ where G acts on $\mathbb{A}^1$ by multiplication and any automorphism of G has dynamical degree* 1.

In other words, the degree exponents detect whenever the random walk has a chaotic behavior.

These last two results essentially follow from a classification of the finitely generated subgroups of the tame group and a theorem due to Maher and Tiozzo [2018, Theorem 1.2] which asserts that a random walk on a subgroup G of isometries of a CAT(0) space will drift to the boundary whenever G contains two noncommuting hyperbolic elements. When this happens, we obtain using Theorem 4 that the degree exponent is bounded below by a multiplicative factor of the drift and is thus positive. Otherwise, we prove that G preserves a vertex in the complex or a geodesic line. We then determine the degree sequences explicitly and conclude.

If we pursue the analogy with the random walk on groups, it is natural to ask whether one can obtain a central limit theorem analog to the one for random products of matrices [Benoist and Quint 2016, Theorem 0.7] or for random products of mapping classes [Dahmani and Horbez 2018]. We state it as follows.

Conjecture 7. *Take μ a symmetric atomic measure on the tame group. Then the limit*

$$\sigma^2 := \lim_{n \to +\infty} \frac{1}{n} \int_G (\log \deg(g) - \lambda(\mu) n)^2 \, d\nu_n(g)$$

exists where $\nu_n = \mu^{*n}$ denotes the n-fold convolution of μ and the sequence of random variables

$$\frac{\log \deg(g_n) - \lambda(\mu) n}{\sqrt{n}}$$

converges to the normal distribution law $\mathcal{N}(0, \sigma^2)$.

Structure of the paper. In Section 1, we recall some general facts on the tame group and then review in Section 2 the construction of the associated square complex. In Section 3, we focus on the global properties of the complex and exploit them to describe the degree sequences of particular automorphisms whose action fix a vertex on the complex. We then state in Section 4 the main valuative estimates needed for our proofs of Theorems 1 and 4 which are presented in Section 5. Finally, we apply the previous result to deduce Theorem 5 and Corollary 6 in the last section.

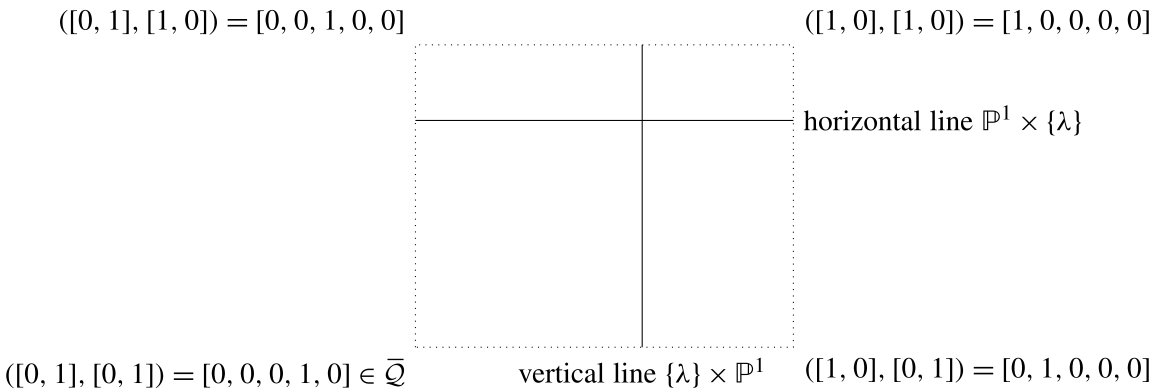


Figure 1. Vertical and horizontal lines in H_∞.

1. General facts on the tame group of the quadric

We work over an algebraically closed field k of characteristic zero. Take some affine coordinates $(x, y, z, t) \in \mathbb{A}^4$ and consider the smooth affine quadric threefold $\mathcal{Q}$ given by

$$\mathcal{Q} := V(xt - yz - 1) \subset \mathbb{A}^4.$$

Let us also fix an open embedding $\mathbb{A}^4 \subset \mathbb{P}^4$ so that $\mathbb{A}^4 = \mathbb{P}^4 \setminus V(w)$ in the homogeneous coordinates $[x, y, z, t, w] \in \mathbb{P}^4$.

In this section, we briefly describe the geometry of the affine quadric and give some preliminary properties of its elementary and orthogonal group of automorphism.

1A. ***The geometry of a quadric threefold and its compactification in $\mathbb{P}^4$.*** The affine variety $\mathcal{Q} \subset \mathbb{A}^4$ is a smooth quadric threefold. The Zariski closure $\overline{\mathcal{Q}}$ of the affine quadric is also smooth in $\mathbb{P}^4$ and has Picard rank one by Lefschetz hyperplane theorem. A birational map from $\overline{\mathcal{Q}}$ to $\mathbb{P}^3$ is given by choosing a point $p_0 \in \overline{\mathcal{Q}}$ and sending a point $p \in \overline{\mathcal{Q}}$ to the intersection of the line (pp_0) with a hyperplane in $\mathbb{P}^4$ which does not contain p_0.

We denote by $H_\infty := \overline{\mathcal{Q}} \setminus \mathcal{Q}$ the hyperplane section at infinity. It is a smooth quadric surface given in homogeneous coordinates by

$$H_\infty := V(xt - yz, w) \subset \mathbb{P}^4.$$

We identify H_∞ with $\mathbb{P}^1 \times \mathbb{P}^1$ by the isomorphism induced by the composition of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ with the inclusion $\mathbb{P}^3 = V(w) \hookrightarrow \mathbb{P}^4$. In homogeneous coordinates, it is given by

$$([\xi_0, \xi_1], [\eta_0, \eta_1]) \mapsto [\xi_0\eta_0, \xi_0\eta_1, \xi_1\eta_0, \xi_1\eta_1, 0]. \tag{3}$$

Any line in H_∞ of the form $\{\lambda\} \times \mathbb{P}^1$ (resp. $\mathbb{P}^1 \times \{\lambda\}$) where $\lambda \in \mathbb{P}^1$ is said to be vertical (resp. horizontal); see Figure 1.

The two projection maps $\pi_x : Q \to \mathbb{A}^1$ and $\pi_y : Q \to \mathbb{A}^1$ given by

$$\pi_x : (x, y, z, t) \in Q \mapsto x,$$
$$\pi_y : (x, y, z, t) \in Q \mapsto y,$$

induce algebraic fibrations which are trivial over $\mathbb{A}^1 \setminus \{0\}$ such that $\pi_x^{-1}(\mathbb{A}^1 \setminus \{0\})$ and $\pi_y^{-1}(\mathbb{A}^1 \setminus \{0\})$ are isomorphic to $\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2$. Observe that the fibers over 0 are both isomorphic to $\mathbb{A}^1 \times \mathbb{A}^1 \setminus \{0\}$ so that the fibrations are not locally trivial over a neighborhood of the origin. Observe that the intersection with H_∞ of the closure of the fiber over 0 in $\overline{Q}$ is the union of a vertical line and a horizontal line. The projection on the two components

$$\pi_{x,y} : (x, y, z, t) \to (x, y)$$

induces a surjective morphism $\pi_{x,y} : Q \to \mathbb{A}^2 \setminus (0, 0)$ which is also trivial over $\mathbb{A}^2 \setminus \{x = 0\}$.

The affine quadric Q carries naturally a volume form Ω which is the Poincaré residue of the rational 4-form $dx \wedge dy \wedge dz \wedge dt/f$ along Q. More explicitly, Ω is defined by

$$\Omega = \frac{dx \wedge dy \wedge dz}{x}\Big|_Q = \frac{dy \wedge dz \wedge dt}{t}\Big|_Q = \frac{dx \wedge dz \wedge dt}{z}\Big|_Q.$$

One checks that Ω extends as a rational 3-form $\overline{\Omega}$ on $\overline{Q}$ such that its divisors of poles and zeros satisfies

$$\operatorname{div}(\overline{\Omega}) = -3[H_\infty].$$

1B. ***The orthogonal group.*** A regular automorphism f of Q is determined by a morphism $f^\sharp$ of the k-algebra $k[Q]$ and hence by its image on the four regular functions x, y, z, t. If we denote by $f_x, f_y, f_z, f_t \in \mathrm{k}[Q]$ the image of x, y, z, t by $f^\sharp$, it is convenient to adopt a matrix-like notation for f as follows:

$$f = \begin{pmatrix} f_x & f_y \\ f_z & f_t \end{pmatrix}.$$

Observe that $f_x f_t - f_z f_y = 1$ since $f^\sharp$ is a morphism of the k-algebra $k[Q]$ and that any such automorphism preserves the volume form Ω (up to a constant).

Denote by $q(x, y, z, t) = xt - yz$ the quadratic form defined on the vector space $V = k^4$. The group O_4 is the subgroup of linear automorphisms of k^4 which leave the quadratic form q invariant

$$O_4 = \{f \in \mathrm{GL}_4(k) \mid q \circ f = q\}.$$

An element of O_4 naturally defines an automorphism of the quadric hypersurface Q. As a consequence, we have that for any $f \in O_4$,

$$f^*\Omega = \epsilon(f)\Omega,$$

where $\epsilon : O_4 \to k^*$ is a morphism of groups. Since Ω is the Poincaré residue of the form $dx \wedge dy \wedge dz \wedge dt/(xt - yz - 1)$ to Q, this implies that for any $f \in O_4$, $\epsilon(f)$ is equal to the determinant of the endomorphism of k^4 associated to f, hence $\epsilon(f) \in \{+1, -1\}$. The subgroup SO_4 is the kernel of ϵ and has index 2 in O_4.

Observe that every element of O_4 extends as regular automorphism of $\overline{Q}$ which leaves the hyperplane at infinity invariant. In particular, the restriction map onto H_∞ induces a morphism of groups from O_4 onto $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$.

The main properties of O_4 and SO_4 are summarized in the following proposition.

Proposition 1.1. *The following properties are satisfied*:

(i) *The group* SO_4 *acts transitively on the set of horizontal and vertical lines at infinity respectively*, *and on the set of points at infinity.*

(ii) *Any element of* $f \in O_4$ *which does not belong to* SO_4 *exchanges the horizontal lines at infinity with the vertical lines at infinity.*

(iii) *The following sequence is exact*:

$$1 \to \{+1, -1\} \to O_4 \to \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) \to 1.$$

(iv) *For any element* $f \in O_4$, *we have*

$$f^*\Omega = \epsilon(f)\Omega,$$

where $\epsilon(f) \in \{+1, -1\}$ *and* $\mathrm{Ker}(\epsilon) = \mathrm{SO}_4$.

Proof. Observe that (iii) follows directly from the following exact sequence:

$$1 \to \{+1, -1\} \to O_4 \to \mathrm{PSO}_4 \to 1,$$

and the fact that $\mathrm{PSO}_4 \simeq \mathrm{PGL}_2 \times \mathrm{PGL}_2$ which is given in [Fulton and Harris 1991, Section 23.1].

In particular, (iii) directly implies (i). ☐

1C. *Elementary transformations.* The group E_V (resp. E_H) of vertical (resp. horizontal) elementary transformations is defined by

$$E_V := \left\{ \begin{pmatrix} ax & by \\ b^{-1}(z + xP(x, y)) & a^{-1}(t + yP(x, y)) \end{pmatrix} \;\middle|\; P \in k[x, y], a, b \in k^* \right\},$$
$$E_H := \left\{ \begin{pmatrix} ax & b(y + xP(x, z)) \\ b^{-1}z & a^{-1}(t + zP(x, z)) \end{pmatrix} \;\middle|\; P \in k[x, y], a, b \in k^* \right\}.$$

The terminology comes from the fact that these transformations are restrictions to the quadric of transformations of $\mathbb{A}^4$ of the form

$$(x, y, z, t) \to (x, y + P(x), z + R(x, y), t + S(x, y, z))$$

where $P \in k[x]$, $R \in k[x, y]$, $S \in k[x, y, z]$, which are elementary in the sense of [Shestakov and Umirbaev 2003].

Any automorphism in E_V fix the two fibrations $\pi_x : (x, y, z, t) \to x$ and $\pi_y : (x, y, z, t) \to y$ and this geometric property characterizes the group E_V; see [Dang 2018, Proposition 3.2.3.1]. An explicit computation proves that any elementary automorphism f preserves the volume form Ω:

$$f^*\Omega = \Omega.$$

We will not focus on the action of these elementary transformations on the compactification $\overline{\mathcal{Q}}$. For more details on the study of the birational transformations induced by these transformations, we refer to [Dang 2018, Chapter 3, Section 3.2.3].

2. The square complex associated to the tame group

The tame group, denoted Tame(Q), is the subgroup of $\mathrm{Aut}(\mathcal{Q})$ generated by E_V and O_4. It is naturally included in $\mathrm{Bir}(\mathbb{P}^3)$ since the variety $\overline{\mathcal{Q}}$ is rational.

Observe that any tame automorphism f fixes the volume form Ω up to a sign, i.e., there exists a group morphism $\epsilon : \mathrm{Tame(Q)} \to \{+1, -1\}$ such that

$$f^*\Omega = \epsilon(f)\Omega.$$

This allows us to identify the kernel $\mathrm{STame}(Q)$ of ϵ as the group generated by SO_4 and E_V. It has index 2 in Tame(Q).

The tame group Tame(Q) is a strict subgroup of $\mathrm{Aut}(\mathcal{Q})$ [Lamy and Vénéreau 2013] and satisfies the Tits alternative; see [Bisi et al. 2014, Theorem C]. The proof of this last fact is due to Bisi, Furter and Lamy and relies on the construction of a square complex on which the group acts by isometry.

The plan of this section is as follows. In Section 2A we detail the construction of the square complex due to Bisi, Furter and Lamy. Then, following the presentation in [Bisi et al. 2014] we shall review in Sections 2B, 2C and 2D the properties of the stabilizer of each vertex of this complex. We will focus particularly on the stabilizer of the vertices which we call of type I in Sections 2C and 2D, for which the analysis is more involving. Finally, we state in Section 2E five technical lemmas on how four squares glue together near each vertices. As before, the situation is also more delicate near the vertices of type I and we need to introduce more terminology to describe the local geometry at those vertices. For a more detailed explanation of the results in this section, we refer to [Bisi et al. 2014, Sections 2, and 3.1] and to [Dang 2018, Chapter 3, Section 3.3].

2A. ***Construction of the square complex.*** The square complex, denoted $\mathcal{C}$, is a 2-dimensional polyhedral complex where the cells of dimension 2 are squares and where the cells of dimension 0 and 1 have some special markings.

We say that a regular function $f_1 \in \mathrm{k}[\mathcal{Q}]$ is a component of an automorphism if there exists $f_2, f_3, f_4 \in \mathrm{k}[\mathcal{Q}]$ such that $f = (f_1, f_2, f_3, f_4)$ defines an automorphism of the quadric. One similarly defines the notion of components for a pair (f_1, f_2) or for a triple (f_1, f_2, f_3) of regular functions on $\mathcal{Q}$ when they can be completed to a 4-tuple defining an automorphism of the affine quadric.

We distinguish three types of vertices for the complex $\mathcal{C}$:

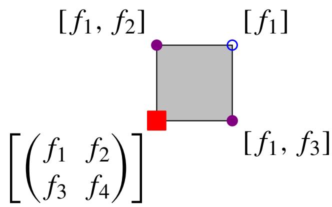


Figure 2. A cell of dimension 2.

- Type I vertices are equivalence classes of components $f_1 \in \mathrm{k}[\mathcal{Q}]$ of a tame automorphism, where two components f_1 and f_2 are identified if there exists an element $a \in k^*$ such that $f_1 = af_2$. A vertex induced by a component $f_1 \in \mathrm{k}[\mathcal{Q}]$ is denoted by $[f_1]$.
- Type II vertices are equivalence class of components (f_1, f_2) of an automorphism where $f_1 = x \circ f$, $f_2 = y \circ f \in \mathrm{k}[\mathcal{Q}]$ for $f \in \mathrm{Tame(Q)}$ and where one identifies two components (f_1, f_2) with (g_1, g_2) if $(g_1, g_2) = (af_1 + bf_2, cf_1 + df_2)$ for some matrix
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2 .$$
A vertex induced by a component (f_1, f_2) is denoted by $[f_1, f_2]$. Denote by $f_3 = z \circ f$ and $f_4 = t \circ f$, the vertices $[f_1, f_2]$, $[f_1, f_3]$, $[f_2, f_4]$, $[f_3, f_4]$ are well-defined since the automorphisms (f_1, f_3, f_2, f_4), $(-f_2, -f_4, f_1, f_3)$ and $(-f_3, f_4, -f_1, f_2)$ are also tame. Moreover, given a component (f_1, f_2) and an invertible matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2$, there exists an automorphism g such that $x \circ g = af_1 + bf_2$ and $y \circ g = cf_1 + df_2$. Let us insist on the fact that on the contrary, there are no vertices of the form $[f_1, f_4]$ or $[f_2, f_3]$.
- Type III vertices are equivalence classes of automorphisms $f \in \mathrm{Tame(Q)}$ where two tame automorphisms f and g are equivalent if there exists $h \in O_4$ such that $f = h \circ g$. An equivalence class of $f \in \mathrm{Tame(Q)}$ is denoted by $[f]$.

The edges of the complex $\mathcal{C}$ are of two types:

- Type I edges join a vertex of type I of the form $[f_1]$ with a vertex of type II of the form $[f_1, f_2]$ where (f_1, f_2) are the components of a tame automorphism.
- Type III edges join a vertex of type II of the form $[f_1, f_2]$ with a vertex of type III $[f]$ where (f_1, f_2) are the components of the automorphism given by f.

The cells of dimension 2 are squares containing two type II vertices of the form $[f_1, f_2]$, $[f_1, f_3]$, one vertex of type I given by $[f_1]$ and one vertex of type III given by $[f]$ where (f_1, f_2, f_3) are the components of the automorphism $f \in \mathrm{Tame(Q)}$. We have the figure of a square given in Figure 2. As in [Bisi et al. 2014], we adopt the following convention for the pictures: the vertices of type I, II and III are represented by the symbol $\circ$, $\bullet$ and ■ respectively.

The square complex $\mathcal{C}$ is obtained by the quotient of the disjoint union of all cells by the equivalence relation $\sim$ where any two cells C_1, C_2 are identified along $C_1 \cap C_2$.

Each square of the complex is endowed with the euclidean metric d so that each square is isometric to $[0, 1] \times [0, 1]$. For any points p and q in $\mathcal{C}$, define by

$$d_{\mathcal{C}}(p, q) = \inf\left\{ \sum_{i=0}^{N} d(p_i, p_{i+1}) \right\},$$

where the infimum is taken over all sequence of points $p_0 = p, \dots, p_N = q$ where p_i and p_{i+1} lie on the same square in $\mathcal{C}$. As any cell of the complex $\mathcal{C}$ has only finitely many isometries, we may apply a general result from [Bridson and Haefliger 1999, Section I.7] and conclude that the function $d_{\mathcal{C}}$ induces a metric on the complex and turns $(\mathcal{C}, d_{\mathcal{C}})$ into a complete metric space. We will explain in Section 3 the global properties on the complex induced by this metric.

Let us define the action of the tame group Tame(Q) on the complex $\mathcal{C}$. Pick any two automorphisms $f, g \in \mathrm{Tame(Q)}$. We define the action of g on the each vertices of the complex by setting

$$\begin{aligned} g \cdot [f_1] &:= [f_1 \circ g^{-1}], \\ g \cdot [f_1, f_2] &:= [f_1 \circ g^{-1}, f_2 \circ g^{-1}], \\ g \cdot [f] &:= [f \circ g^{-1}]. \end{aligned}$$

The action on vertices induces a morphism of the square complex which preserves the type of vertices and edges and preserves the distance.

Remark 2.1. Although, this is not clear at this stage, the action of the tame group on this complex will act transitively on the set of squares and the precise study of the stabilizer of type III vertices will result from the geometry of the complex near a vertex of type III done in Proposition 2.3. As a result of the study, a square is determined uniquely once it contains a vertex of type III and a vertex of type I, or by three vertices.

Recall that the subgroup $\mathrm{STame}(Q)$ generated by SO_4 and elementary transformations has index 2 in Tame(Q).

Definition 2.2. An edge E of the complex is called *horizontal* (resp. *vertical*) if there exists an element $f \in \mathrm{STame}(Q)$ such that $f \cdot E$ is equal to the edge joining $[x, y]$ with $[x]$ (resp. $[x, z]$ with $[x]$) or to the edge between [Id] and $[x, z]$ (resp. [Id] and $[x, y]$).

We will see that the set of vertical and horizontal edges form a partition of the set of edges (see (iii) and (iv) of Proposition 2.7).

2B. ***Stabilizer of vertices of type III, II and the properties of the action.*** In this section, we shall first review the properties of the stabilizer of type II and III vertices then deduce from these the global properties of the action of the group on this complex. To do so, we shall exploit the relationship between the local geometry near each vertices and their respective stabilizer subgroups. The geometry near a given vertex v

is encoded in its link $\mathcal{L}(v)$ which is constructed as follows. The vertices of $\mathcal{L}(v)$ are in bijection with the vertices v' such that $[v, v']$ is an edge of the complex $\mathcal{C}$. And we draw an edge joining v' and v'' in $\mathcal{L}(v)$ if the vertices v, v', v'' belong to the same square.

Observe that the action of the tame group on the vertices of type III is transitive. As a result, we shall focus on the stabilizer subgroup of the vertex [Id], which is by construction O_4. Its action on the complex induces an action on the link $\mathcal{L}([\mathrm{Id}])$.

Proposition 2.3. *The link $\mathcal{L}([\mathrm{Id}])$ is a complete bipartite graph and there exists an O_4-equivariant bijection between the set of vertices of the link $\mathcal{L}([\mathrm{Id}])$ to the set of lines at infinity such that the vertices which belong to a vertical (resp. horizontal) edge of type* III *are mapped to vertical (resp. horizontal) lines at infinity in H_∞. Moreover, this bijection induces an O_4-equivariant bijection from the edges of $\mathcal{L}([\mathrm{Id}])$ to the set of points at infinity H_∞.*

Remark 2.4. Observe that Proposition 1.1 and Proposition 2.3 imply that the group O_4 acts faithfully and transitively on the link $\mathcal{L}([\mathrm{Id}])$.

Proof. We identify two types of vertices in the link of [Id], the vertices which belong to a horizontal edge containing [Id] or those which are contained in a vertical edge containing [Id].

We define a map φ from the vertices of the link $\mathcal{L}([\mathrm{Id}])$ to the set of lines in H_∞. Take a vertex v in the link $\mathcal{L}([\mathrm{Id}])$ and a component (f_1, f_2) such that $[f_1, f_2] = v$. By definition, there exists an element $f \in O_4$ such that $f_1 = x \circ f$ and $f_2 = y \circ f$ since the stabilizer of [Id] is O_4. The zero locus $V(f_1) \cap V(f_2) \cap H_\infty$ in $\overline{\mathcal{Q}}$ is the line at infinity corresponding to the preimage of $\{x = y = 0\} \cap \overline{\mathcal{Q}}$ by f. Observe that the line $V(f_1) \cap V(f_2) \cap H_\infty$ does not depend on the choice of representative of the equivalence class v since any two other component in the same class defines the same homogeneous ideal $\langle f_1, f_2, xt - yz - w^2\rangle$. We thus define $\varphi(v)$ to be the line $V(f_1) \cap V(f_2) \cap H_\infty$. Observe that if v is a vertex of type II such that the edge containing v and [Id] is vertical, then $f \in \mathrm{SO}_4$. Hence the line at infinity $V(x \circ f) \cap V(y \circ f) \cap H_\infty$ is vertical. Observe also that φ is naturally O_4-equivariant. The same argument holds for the vertices of type II which belong to horizontal edges containing [Id].

Let us prove that the map φ is surjective. Consider a vertical line $L \subset H_\infty$ at infinity, then there exists by Proposition 1.1(i) an automorphism f in SO_4 such that the image of the vertical line at infinity given by $[0, 1] \times \mathbb{P}^1$ is L. Since $\varphi([x, y])$ corresponds to the line $[0, 1] \times \mathbb{P}^1$, the vertex of type II $[x \circ f, y \circ f]$ defines a component of an automorphism which belongs to the link $\mathcal{L}([\mathrm{Id}])$ such that $\varphi([x \circ f, y \circ f]) = L$. Hence, φ is surjective.

Let us prove that φ is injective. Consider two vertices v_1, v_2 such that their image by φ is equal, we prove that $v_1 = v_2$. Consider two components $(f_1, f_2), (g_1, g_2)$ such that $[f_1, f_2] = v_1$ and $[g_1, g_2] = v_2$. We must prove that (f_1, f_2) and (g_1, g_2) belong to the same equivalence class. By symmetry, we can suppose that the line $\varphi(v_1)$ is vertical. Hence, there exists $f, g \in \mathrm{SO}_4$ such that $f_1 = x \circ f$, $g_1 = x \circ g$, $f_2 = y \circ f$ and $g_2 = y \circ g$. In particular, this implies that $f \circ g^{-1}$ fixes the vertical line at infinity given by $\{[0, 1]\} \times \mathbb{P}^1$.

Using Proposition 1.1(iii), we conclude that $f\circ g^{-1}$ is of the form

$$f\circ g^{-1}=\begin{pmatrix} ax+by & cx+dy \\ a'z+b't & c'z+d't \end{pmatrix},$$

where the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} d' & -b' \\ -c' & a' \end{smallmatrix}\right)\in M_2(k)$ satisfy

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\begin{pmatrix} d' & -b' \\ -c' & a' \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In particular, this implies that the components (f_1, f_2) and (g_1, g_2) are equivalent since $f_1 = ag_1+bg_2$, $f_2 = cg_1+dg_2$.

One similarly defines a bijection from the edges of the link $\mathcal{L}([\mathrm{Id}])$ to H_∞. The link is complete since a horizontal and a vertical line in H_∞ always intersect at a point in H_∞, hence for any vertices v_1, v_2 in $\mathcal{L}([\mathrm{Id}])$ which are mapped by φ to a vertical and a horizontal line respectively, there exists an edge joining v_1 and v_2. □

Proposition 2.5. *The following properties are satisfied*:

(1) *The stabilizer of a vertex of type* III *in* $\mathrm{STame}(Q)$ *is conjugated in* $\mathrm{STame}(Q)$ *to* SO_4.

(2) *The stabilizer of an edge of type* III *is conjugated in* Tame(Q) *to the subgroup*

$$A\cdot\begin{pmatrix} x & y \\ z & t \end{pmatrix}\cdot B^t,$$

where $A\in\mathrm{SL}_2(k)$ *is a lower triangular matrix and* $B\in\mathrm{SL}_2(k)$.

(3) *The stabilizer of a* 1×1 *square is conjugated in* Tame(Q) *to*

$$\left\{\begin{pmatrix} ax & b(y+cx) \\ b^{-1}(z+dx) & a^{-1}(t+cz+dy+dcx) \end{pmatrix}\;\middle|\;(a,b,c,d)\in k^*\times k^*\times k\times k\right\}\rtimes\left\{\begin{pmatrix} x & z \\ y & t \end{pmatrix}, \mathrm{Id}\right\}$$

(4) *The pointwise stabilizer of the union of the four squares containing* [Id] *and* $[x]$, $[y]$, $[z]$ *and* $[t]$ *respectively is equal to*

$$\left\{\begin{pmatrix} ax & by \\ b^{-1}z & a^{-1}t \end{pmatrix}\;\middle|\;a,b\in k^*\right\}.$$

Proof. Observe that (i) follows directly from the definition of the definition. Moreover, the next assertions (ii), (iii) and (iv) are exactly the content of [Bisi et al. 2014, Lemmas 2.5(2), 2.7 and 2.11]. □

We focus on the stabilizer subgroups of vertices of type II. For that, we also define some special subgroups of E_V, E_H where the constant are all 1. Set $\tilde{E}_H$ the subgroup of E_H of elements of the form

$$\begin{pmatrix} x & y+xP(x,z) \\ z & t+zP(x,z) \end{pmatrix},$$

with $P\in k[x,y]$ and respectively elements in $\tilde{E}_V$ are of the form

$$\begin{pmatrix} x & y \\ z+xP(x,y) & t+yP(x,y) \end{pmatrix}.$$

Proposition 2.6. *The following properties are satisfied*:

(i) *The stabilizer of a vertex of type* II *in* Tame(Q) *is conjugated in* Tame(Q) *to the semidirect product* $\tilde{E}_V \rtimes \mathrm{GL}_2$ *where the group* GL_2 *is identified with the elements of the form*

$$\begin{pmatrix} ax+by & cx+dy \\ a'z+b't & c'z+d't \end{pmatrix}$$

where $a, b, c, d, a', b', c', d' \in k$ *such that* $\det\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \neq 0$ *and*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{a'd'-b'c'} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

(ii) *The stabilizer of a vertical edge of type* I *is conjugated in* $\mathrm{STame}(Q)$ *to the subgroup*

$$\tilde{E}_H \rtimes \left\{ \begin{pmatrix} ax & d^{-1}y \\ dz+cx & a^{-1}t+ca^{-1}d^{-1}y \end{pmatrix} \,\middle|\, (a, c, d) \in k^* \times k \times k^* \right\}.$$

(iii) *The pointwise stabilizer of the geodesic segment of length* 2 *joining the vertices* $[f_1]$, $[f_3]$ *and* $[f_1, f_3]$ *where* $f = (f_1, f_2, f_3, f_4) \in \mathrm{STame}(Q)$ *is conjugated in* $\mathrm{STame}(Q)$ *to*

$$\tilde{E}_H \rtimes \left\{ \begin{pmatrix} ax & by \\ b^{-1}z & a^{-1}t \end{pmatrix}, a, b \in k^* \right\}.$$

Proof. Assertion (i), (ii) and (iii) are given in [Bisi et al. 2014, Lemmas 2.3, 2.5(1) and 2.6(1)] respectively. □

From the description of the previous stabilizer subgroups, we state the following consequences on the action of the group on this complex.

Proposition 2.7. *The tame group* Tame(Q) *acts by isometry on the complex* $\mathcal{C}$ *and this action satisfies the following properties*:

(i) *The action preserves the types of vertices and the types of edges.*

(ii) *The action is faithful and transitive on the set of vertices of type* I , II *and* III *respectively.*

(iii) *The subgroup* $\mathrm{STame}(Q)$ *acts transitively on the set of vertical* (*resp. horizontal*) *edges of type* I *and* III.

(iv) *Any automorphism* $f \in$ Tame(Q) *which does not belong to the subgroup* $\mathrm{STame}(Q)$ *sends a vertical edge to a horizontal edge of the same type.*

(v) *The subgroup* $\mathrm{STame}(Q)$ *acts transitively on the set of* 1×1 *squares.*

(vi) *The group* Tame(Q) *acts transitively on the union of* 4 *squares which is isometric to* $[0, 2] \times [0, 2]$ *and which contains a common vertex of type* III.

Proof. The transitive of the action on the set of vertices of type I, II and III and assertions (i) and (iv) follow from [Bisi et al. 2014, Lemmas 2.1 and 2.4].

To prove (ii), we need to explain why the action is also faithful. Observe that if a tame automorphism fixes every vertices of type III, or type II or type I, then it fixes the whole complex since every vertex of type III (resp. type II or I) is the middle point of a geodesic segment joining type I or type II points. Then the faithfulness follows from the faithfulness of the action on the link $\mathcal{L}([\mathrm{Id}])$.

The assertions (iii), (v) and (vi) are exactly the content of [Bisi et al. 2014, Lemmas 2.4 and 2.7, Corollary 2.10] respectively. □

2C. ***Bass–Serre tree associated to plane automorphisms.*** We consider the field $K = k(x)$. We define the graph $\mathcal{T}_{k(x)}$ which is a bipartite metric graph:

(1) Vertices of type I are equivalence classes of components $f_1 \in k(x)[y, z]$ of plane automorphisms where one identifies two components f_1 and g_1 if there exists $a \in k(x)^*$ and $b \in k(x)$ such that $f_1 = ag_1 + b$. An equivalence class induced by a component f_1 is denoted $[f_1]$.

(2) Vertices of type II are equivalence classes of automorphisms f of $\mathbb{A}^2_{k(x)}$ where one identifies two automorphisms $f = (f_1, f_2)$ and $g = (g_1, g_2)$ if there exists an affine automorphism $h \in \mathbb{A}^2_{k(x)}$ whose coefficients are given by the matrix
$$\begin{pmatrix} a & b & c \\ a' & b' & c' \\ 0 & 0 & 1 \end{pmatrix}$$
such that $(f_1, f_2) = (ag_1 + bg_2 + c, a'g_1 + b'g_2 + c)$. An equivalence class induced by a plane automorphism $f = (f_1, f_2)$ is denoted $[f_1, f_2]$.

(3) Edges link a vertex v_1 of type I with a vertex v_2 of type II if there exists a polynomial automorphism $f = (f_1, f_2)$ such that $[f_1] = v_1$ and $[f_1, f_2] = v_2$.

We endow this graph $\mathcal{T}_{k(x)}$ with the distance such that each edge is of length 1. This graph $\mathcal{T}_{k(x)}$ is thus a complete geodesic metric space.

The action of an automorphism $g \in \mathbb{A}^2_{k(x)}$ on $\mathcal{T}_{k(x)}$ is defined as follows
$$g \cdot [f_1] = [f_1 \circ g^{-1}] \quad \text{and} \quad g \cdot [f_1, f_2] = [f_1 \circ g^{-1}, f_2 \circ g^{-1}]$$
for any automorphism $f = (f_1, f_2) \in \mathrm{Aut}(\mathbb{A}^2_{k(x)})$.

A classical theorem from Jung [1942] proves that the graph $\mathcal{T}_{k(x)}$ is a tree and that the group of plane automorphism acts faithfully, by isometry and transitively on the set of type I and II vertices respectively.

2D. ***Link over a vertex of type I.*** In this subsection, we study the link over the vertex of type I given by $[x]$. Observe that the stabilizer subgroup of the vertex $[x]$ acts naturally in the link of the vertex $[x]$.

Lemma 2.8. *The group* $\mathrm{Stab}([x])$ *acts transitively, faithfully on the set of vertices in the link of* $[x]$ *induced by the edges joining* $[x]$ *and the vertices of type* II *of* $\mathcal{C}$.

Proof. By Proposition 2.7(v), the group $\mathrm{STame}(Q)$ acts transitively on the set of 1×1 squares and since a 1×1 square containing $[x]$ defines an edge in the link $\mathcal{L}([x])$, the induced action of $\mathrm{Stab}([x])$ is transitive on the edges of the link $\mathcal{L}([x])$. Observe also that the involution $\sigma : (x, y, z, t) \mapsto (x, z, y, t)$ induces an action on the link which exchanges the vertices $[x, y]$, $[x, z]$ in the link and fixes the edge between these two vertices. This proves that the action of the stabilizer $\mathrm{Stab}([x])$ is transitive on the link of $[x]$.

Let us prove that the action is faithful. Suppose $f \in \mathrm{Stab}([x])$ acts by the identity map in the link over $[x]$, then in particular, f must fix pointwise the square containing [Id] and $[x]$. By Proposition 2.5(iii), f is of the form

$$f = \begin{pmatrix} ax & d^{-1}(y+bx) \\ d(z+cx) & a^{-1}(t+cy+bz+bcx) \end{pmatrix},$$

where $a, d \in k^*$ and $b, c \in k$. Since f must also fix the vertices of type II $[x, y+xP(x)]$ and $[x, z+xP(x)]$ where $P \in k[x]$, we have that $a = d = 1$ and $c = b = 0$ as required. □

In the following arguments, we will use the fact that the link $\mathcal{L}([x])$ is connected [Bisi et al. 2014, Lemma 3.2], which is a highly nontrivial argument which relies deeply on the reduction theory inspired by the work of Shestakov and Umirbaev; see [Bisi et al. 2014, Corollary 1.5].

Recall that the general fiber of the projection $\pi_x : \mathcal{Q} \to \mathbb{A}^1$ defined in Section 1C is isomorphic to $\mathbb{A}^2$. We fix an identification of $\pi_x^{-1}(\mathbb{A}^1 \setminus \{0\})$ with $\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2$ given by

$$(x, y, z) \mapsto (x, y, z, (yz+1)/x). \tag{4}$$

The relationship between the stabilizer of the vertex $[x]$ and $\mathrm{Aut}(\mathbb{A}^2_{k(x)})$ is realized explicitly as follows.

Denote by $\mathcal{L}([x])'$ the first barycentric division of $\mathcal{L}([x])$. We shall define a simplicial map $\pi : \mathcal{L}([x])' \to \mathcal{T}_{k(x)}$ as follows.

Let v be a vertex of type II in $\mathcal{C}$ which defines a vertex in the link of $[x]$, then since the action of $\mathrm{Stab}([x])$ on the link $\mathcal{L}([x])$ is transitive by Lemma 2.8, there exists an element $f \in \mathrm{Stab}([x])$ such that $f \cdot [x, z] = v$. Since f naturally fixes the fibration π_x, under the identification $\pi_x^{-1}(\mathbb{A}^1 \setminus \{0\}) \simeq \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2$ given by (4), the regular map f is given by

$$(x, y, z) \mapsto (x \circ f, y \circ f, z \circ f).$$

Under this identification, $(y \circ f, z \circ f)$ induces an element of $\mathbb{A}^2_{k(x)}$. We thus define

$$\pi(v) = [z \circ f^{-1}] \in \mathcal{T}_{k(x)}.$$

Observe that $\pi(v)$ does not depend on the choice of f. Indeed, if $g \in \mathrm{Stab}([x])$ is another automorphism such that $g \cdot [x, z] = f \cdot [x, z]$, there are $a, b \in k^*$ such that $x \circ f^{-1} = ax$, $x \circ g^{-1} = bx$ and $[x, z \circ f^{-1}] = [x, z \circ g^{-1}]$. Then $z \circ g^{-1} = cz \circ f^{-1} + dx$ for some $c \in k^*$, $d \in k$. We obtain that $[z \circ g^{-1}] = [z \circ f^{-1}] \in \mathcal{T}_{k(x)}$.

Let $m \in \mathcal{L}([x])'$ be the middle point of an edge E of $\mathcal{L}([x])$ and let m_0 be the middle point of the geodesic joining $[x, y]$ and $[x, z]$ in $\mathcal{L}([x])'$. Since the action of $\mathrm{Stab}([x])$ in the link $\mathcal{L}([x])$ is transitive by Lemma 2.8, there exists an element $f \in \mathrm{Stab}([x])$ such that $f \circ m_0 = m$. Since f naturally fixes the fibration π_x, it induces an automorphism of $\pi_x^{-1}(\mathbb{A}^1 \setminus \{0\})$ and under the identification given by (4), it is of the form

$$(x, y, z) \mapsto (x \circ f, y \circ f, z \circ f).$$

We thus define

$$\pi(m) = [y \circ f^{-1}, z \circ f^{-1}].$$

Observe also that $\pi(m)$ does not depend on the choice of f. If $g \in \mathrm{Stab}([x])$ such that $g \cdot m_0 = m$, then $g^{-1} \circ f$ belongs to the subgroup

$$\left\{ \begin{pmatrix} ax & b(y+cx) \\ b^{-1}(z+dx) & a^{-1}(t+dy+cz+cdx) \end{pmatrix} \,\middle|\, a, b \in k^*, c, d \in k \right\} \rtimes \left\{ \mathrm{Id}, \begin{pmatrix} x & z \\ y & t \end{pmatrix} \right\},$$

hence $[y \circ g^{-1}, z \circ g^{-1}] = [y \circ f^{-1}, z \circ f^{-1}] \in \mathcal{T}_{k(x)}$ and $\pi(m)$ is well-defined.

If E is an edge of $\mathcal{L}([x])'$ of length 1, then we define the image of E by π as the geodesic joining the image of the endpoints of E by π. As a result,the map π is a simplicial map between $\mathcal{L}([x])'$ and $\mathcal{T}_{k(x)}$ such that the action of $\mathrm{Stab}([x])$ descends into an action on the image $\pi(\mathcal{L}([x])') \subset \mathcal{T}_{k(x)}$ (one can prove that $\pi : \mathcal{L}([x])' \to \pi(\mathcal{L}([x])')$ is the unique $\mathrm{Stab}([x])$-equivariant map for which $\pi([x, y]) = [y]$ and $\pi([x, z]) = [z]$). To simplify the next statement, we denote by $\mathcal{T}_{\pi,k(x)}$ the image in $\mathcal{T}_{k(x)}$ of $\mathcal{L}([x])'$ by π.

Definition 2.9. The subgroup $A^S_{[x]}$ of $\mathrm{Stab}([x])$ is the intersection of $\mathrm{STame}(Q)$ with the stabilizer of the vertices $[y], [z], [y, z]$ in $\mathcal{T}_{\pi,k(x)}$ where S is the standard 2×2 square containing $[x], [y], [z], [t]$. More generally, if v is any vertex of type I contained in a 2×2 square S', the subgroup $A^{S'}_v$ is equal to $gA^S_{[x]}g^{-1}$ where $g \in \mathrm{STame}(Q)$ such that $g \cdot S = S'$ and $g \cdot [x] = v$.

Proposition 2.10. *Denote by $m \in \mathcal{L}([x])'$ the middle point between the point $[x, y]$ and $[x, z]$. The simplicial map $\pi : \mathcal{L}([x])' \to \mathcal{T}_{\pi,k(x)}$ satisfies the following properties*:

(i) *The image of the edge between the point $[x, y]$ and m by π is a fundamental domain of $\mathcal{T}_{\pi,k(x)}$.*

(ii) *The image $\pi(\mathcal{L}([x])') = \mathcal{T}_{\pi,k(x)}$ is a subtree of $\mathcal{T}_{k(x)}$.*

(iii) *The preimage by π of the segment of length 2 joining $[z]$ and $[y]$ is a bipartite graph.*

(iv) *The subgroup $A^S_{[x]} \subset \mathrm{Stab}([x]) \cap \mathrm{STame}(Q)$ is the set of elements of the form*

$$\begin{pmatrix} ax & b(y + xP(x)) \\ b^{-1}(z + xT(x)) & a^{-1}(t + zP(x) + yT(x) + xP(x)T(x)) \end{pmatrix},$$

where $P, T \in k[x]$ and $a, b \in k^*$.

(v) *The group $\mathrm{Stab}([x])$ is the amalgamated product $\tilde{E} * \tilde{A}$ along their intersection where $\tilde{E}$ is the group generated by elements of the form*

$$\begin{pmatrix} ax & b(y + xP(x, z)) \\ b^{-1}(z + xT(x)) & a^{-1}(t + zP(x, z) + yT(x)) \end{pmatrix}, \tag{5}$$

where $a, b \in k^*$, $P \in k[x, z]$, $T \in k[x]$, and $\tilde{A}$ is the group generated by $A^S_{[x]}$ and the involution $(x, y, z, t) \mapsto (x, z, y, t)$.

Remark 2.11. In fact, the action of Stab([x]) on $\mathcal{T}_{\pi,k(x)}$ can be extended to an action on the whole tree $\mathcal{T}_{k(x)}$. One can view this since the fundamental domain of the tree $\mathcal{T}_{k(x)}$ is the image of an edge of $\mathcal{L}([x])'$ by π. Another way is to identify the general fiber Q_η using (4) with $\mathbb{A}^2_{k(x)} = \mathrm{Spec}(k(x)[y,z])$. So for every $g, h \in k(x)[y,z]$ such that (g,h) defines an automorphism, we can define $f \cdot [g] = [g \circ f^{-1}]$ and $f \cdot [g,h] = [g \circ f^{-1}, h \circ f^{-1}]$ for any $f \in \mathrm{Stab}([x])$.

Proof. Assertion (i), (ii), (iii) and (v) are the content of [Bisi et al. 2014, Lemmas 3.4(1), 3.5(1) and (2), Proposition 4.11] respectively.

Let us prove statement (iv). Let us denote by $\phi : \mathrm{Stab}([x]) \to \mathrm{Aut}(\mathcal{T}_{\pi,k(x)})$ the morphism of groups induced by the simplicial map $\pi : \mathcal{L}([x])' \to \mathcal{T}_{\pi,k(x)}$ where $\mathrm{Aut}(\mathcal{T}_{\pi,k(x)})$ denotes the induced simplicial map on the tree. It is clear that any element of the form

$$\begin{pmatrix} ax & b(y+xP(x)) \\ b^{-1}(z+xT(x)) & a^{-1}(t+zP(x)+yT(x)+xP(x)T(x)) \end{pmatrix},$$

where $P, T \in k[x]$ and $a, b \in k^*$ induces an action which preserves the vertices $[y]$, $[z]$ and $[y,z]$ on $\mathcal{T}_{\pi,k(x)}$. Conversely, we prove that any element of $A^S_{[x]}$ has this form. Pick $g \in A^S_{[x]}$, since $\phi(g)$ fixes the vertices $[y]$, $[z]$ and $[y,z]$ of $\mathcal{T}_{\pi,k(x)}$, As $\phi(g)$ fixes every vertex of type I and since it belongs to the image of ϕ, the components in x, y, z of the automorphism g must be of the form

$$g = (x, y, z) \to (ax, b(y+xP(x)), c(z+xT(x))),$$

where $P, T \in k[x]$ and where $a, b, c \in k^*$. In particular, as $g \in \mathrm{Tame(Q)}$, $b = c^{-1}$ and g is of the form

$$\begin{pmatrix} ax & b(y+xP(x)) \\ b^{-1}(z+xT(x)) & a^{-1}(t+zP(x)+yT(x)+xP(x)T(x)) \end{pmatrix},$$

proving (iv). □

Proposition 2.12. *Any element of* Stab([x]) *whose action on* $\mathcal{T}_{\pi,k(x)}$ *is hyperbolic is conjugated to a composition of automorphisms of the form*

$$\begin{pmatrix} ax & b(z+xP(x,y)) \\ b^{-1}(y+xR(x)) & a^{-1}(t+zR(x)+yP(x,y)+xP(x,y)R(x)) \end{pmatrix},$$

where $R \in k[x]$ *and* $P \in k[x,y]$ *such that* $\deg_y(P) \geqslant 2$.

Proof. Let us fix an element $f \in \mathrm{Stab}([x])$ whose action on $\mathcal{T}_{\pi,k(x)}$ is loxodromic. Using assertion (v) of Proposition 2.10, up to conjugation, we can assume f is decomposed into

$$f = a_1 \circ e_1 \circ \cdots \circ a_n \circ e_n,$$

where $a_i \in \tilde{A} \setminus \tilde{E}$ and $e_i \in \tilde{E} \setminus \tilde{A}$. Let σ be the involution $(x,y,z,t) \mapsto (x,z,y,t)$. Every element of $\tilde{A}$ preserve the set of vertices $[y]$, $[z]$ in $\mathcal{T}_{\pi,k(x)}$. Observe that the elements in $A^S_{[x]}$ fix each vertex $[y]$, $[z]$ in $\mathcal{T}_{\pi,k(x)}$ and belong to $\tilde{E}$. Moreover, an element belongs to $\tilde{A} \cap \tilde{E}$ if it preserves the vertex $[z]$, hence it must be in $A^S_{[x]}$. We deduce that the elements a_i are of the form

$$\begin{pmatrix} ax & b(z + xT(x)) \\ b^{-1}(y + xP(x)) & a^{-1}(t + zP(x) + yT(x) + xP(x)T(x)) \end{pmatrix}, \tag{6}$$

where $a, b \in k^*$, $P, T \in k[x]$. This shows that we can decompose a_i into $a_i = \sigma \circ a'_i$ where $a'_i \in A^S_{[x]} \subset \tilde{A} \cap \tilde{E}$. We can thus absorb the element a'_i in the term in $\tilde{E}$:

$$f = \sigma \circ (a'_1 \circ e_1) \circ \sigma \circ (a'_2 \circ e_2) \circ \cdots \circ \sigma \circ (a'_n \circ e_n).$$

We then conclude since the product $\sigma \circ (a'_i \circ e_i)$ is of the required form. $\square$

2E. ***Five technical consequences on the local geometry at each vertex.*** We say that a subset $S \subset \mathcal{C}$ is a 2×2 square of $\mathcal{C}$ if S is the union of four distinct 1×1 squares such that S isometric to $[0, 2] \times [0, 2]$. Moreover, we say that a 2×2 square is centered on a vertex v if the vertex v corresponds to the image of the point $(1, 1)$ by an isometry from $[0, 2] \times [0, 2]$ to S.

Two 1×1 (resp. 2×2) squares S, S' are said to be *adjacent* if their union $S \cup S'$ is isometric to $[0, 2] \times [0, 1]$ (resp. $[0, 4] \times [0, 2]$). Two 1×1 squares S and S' are adjacent along a vertical (resp. horizontal) edge if they are adjacent and their intersection $S \cap S'$ is a vertical edge (resp. horizontal). Similarly, two 2×2 squares are said to be *adjacent* along a horizontal (resp. vertical) if they intersect along a boundary segment (i.e., $\{0\} \times [0, 2]$, $\{2\} \times [0, 2]$, $[0, 2] \times \{0\}$ or $[0, 2] \times \{2\}$) isometric to $[0, 2]$ which is the union of two horizontal edges (resp. vertical edges).

Two 1×1 (resp. 2×2) squares S_1 and S_2 are said to be *adherent* if they are not adjacent but their intersection is reduced to a vertex which is in a corner of each respective square (i.e., one of the extremal point of each square). If a vertex $v \in \mathcal{C}$ belongs to the intersection of two adherent squares $S_1 \cap S_2$, then S_1 and S_2 are said to be adherent along the vertex v.

We say that two 1×1 (resp. 2×2) squares S, S' are *flat* if there exists two 1×1 (resp. 2×2) squares S_1, S_2 such that the union $S_1 \cup S_2 \cup S \cup S'$ is isometric to $[0, 2] \times [0, 2]$ (resp. $[0, 4] \times [0, 4]$). Similarly, three 1×1 (resp. 2×2) squares are flat if we can find another 1×1 (resp. 2×2) square such that their union is isometric to $[0, 2] \times [0, 2]$ (resp. $[0, 4] \times [0, 4]$). Once Lemma 2.16 and Lemma 2.15 are obtained, they will allow us to work only with 2×2 squares centered around vertices of type III, instead of 1×1 squares.

We will prove that three 1×1 squares S_1, S_2, S_3 such that S_1 and S_2, S_2 and S_3 are adjacent and contain a common vertex of type II or III are necessarily flat; see Lemmas 2.15 and 2.16 below. However, this property does not necessarily hold when the squares contain a common vertex of type I (see Lemma 2.17 below), we prove that they are either flat or contained in a spiral staircase. We explain this terminology below.

A collection (S, S') of 1×1 or 2×2 squares is contained in a *spiral staircase around* v (see 2.14 for an example) if they contain a common vertex v of type I and such that any minimal sequence $S_1 = S, \dots, S_k = S'$ of squares containing v and connecting S to S' satisfies the following conditions:

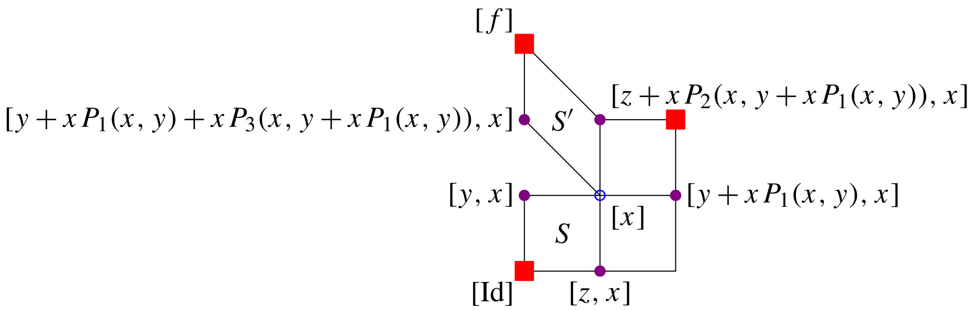


Figure 3. Example of horizontal spiral staircase.

(1) For all integer $i \leqslant k-1$, the squares S_i and S_{i+1} are alternatively adjacent along a vertical or horizontal edge containing v.

(2) Any three consecutive squares (S_i, S_{i+1}, S_{i+2}) for $i \leqslant k-2$ is not flat.

When the first two squares S_1 and S_2 are adjacent along a horizontal edge (containing v), we say that the *spiral staircase around* v is *vertical*. Otherwise the squares S_1 and S_2 are adjacent along a vertical edge and we have a *horizontal spiral staircase around* v.

Remark 2.13. We conjecture that there cannot be any spiral staircase that is both vertical and horizontal but do not need such a statement in our proofs.

When two squares S, S' are flat, then the collection (S, S') is not contained in a spiral staircase.

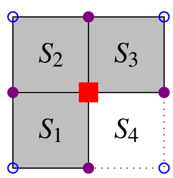
Example 2.14. Consider $P_1, P_2, P_3 \in k[x, y] \setminus k[x]$, denote by S the square containing $[x]$ and $[\mathrm{Id}]$ and S' the square containing $[x]$ and $[f]$ where $f \in \mathrm{Tame(Q)}$ is given by

$$f = \begin{pmatrix} x & y + x P_1(x, y) + x P_3(x, z + x P_2(x, y + x P_1(x, y))) \\ z + x P_2(x, y + x P_1(x, y)) & f_4 \end{pmatrix},$$

where $f_4 = t + y(P_1(x, y) + P_3(x, z + x P_2(x, y + x P_1(x, y)))) + y P_2(x, y + x P_1(x, y)) + x(P_1(x, y) + P_3(x, z + x P_2(x, y + x P_1(x, y)))) P_2(x, y + x P_1(x, y)))$. Then the pair (S, S') is contained in a horizontal spiral staircase and one has the Figure 3.

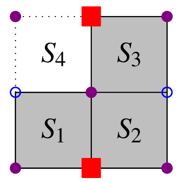
The next lemmas describe when three squares containing a common vertex are flat.

Lemma 2.15. *Let v be a vertex of type* III *and let S_1, S_2, S_3 be three distinct 1×1 squares such that S_1 is adjacent to S_2 along an edge containing v, and S_2 is adjacent to S_3 along an edge containing v. Then the three squares can be completed into a 2×2 square centered along v:*



Proof. Since the group acts transitively on the vertices of type III by Proposition 2.7, we can reduce by conjugating by a tame element to the case where the vertex [Id] is a common point of the three squares. By Proposition 2.3(i) and (ii), the three squares determine 3 distinct points p_1, p_2, p_3 at infinity such that p_1 and p_2 are on a same line at infinity L_{12}, and p_2, p_3 lie on another line L_{23} which is transverse to L_{12}. Denote by p_4 the intersection of the line passing through p_1 transverse to L_{12} with the line passing through p_3 transverse to L_{23}. This point determines a unique square S_4 containing [Id] by Proposition 2.3(ii) and the union $S_1 \cup S_2 \cup S_3 \cup S_4$ is isometric to $[0, 2] \times [0, 2]$ since p_1, p_2, p_3 and p_4 lie on a cycle of four lines at infinity. □

Lemma 2.16. *Let v be a vertex of type* II *and let S_1, S_2, S_3 be three distinct 1×1 squares such that S_1 is adjacent to S_2 along an edge containing v, and S_2 is adjacent to S_3 along an edge containing v. Then the three squares can be completed into a 2×2 square centered along v:*



Proof. Since the tame group and PGL_2 act transitively on the vertices of type III and on the pairs of points on $\mathbb{P}^1$ respectively, we are reduced by conjugating with an appropriate tame automorphism to the situation where the squares S_1 and S_2 contain [Id] and the points $[y]$ and $[x]$ respectively. Take $f \in \mathrm{STame}(Q)$ a tame automorphism such that the vertex $f \cdot S_2 = S_3$. Note that since S_2 and S_3 are adjacent, f preserve the edge $S_2 \cap S_3$. By Proposition 2.6(ii), f belongs to

$$E_V \rtimes \left\{ \begin{pmatrix} ax & dy + cx \\ d^{-1}z & a^{-1}t + ca^{-1}d^{-1}z \end{pmatrix} \,\middle|\, (a, c, d) \in k^* \times k \times k^* \right\}.$$

In particular, f can be decomposed as $f = g \circ e$ where $e \in E_V$ and g belongs to the subgroup

$$\left\{ \begin{pmatrix} ax & dy + cx \\ d^{-1}z & a^{-1}t + ca^{-1}d^{-1}z \end{pmatrix} \,\middle|\, (a, c, d) \in k^* \times k \times k^* \right\}.$$

The vertex of type III in S_3 is determined by $[f] = O_4 f = O_4 g \circ e = O_4 e = [e]$. Set $S_4 = e^{-1} \cdot [S_1]$, by construction, S_4 is adjacent to S_1 and since it also contains the vertices $e^{-1} \cdot [\mathrm{Id}] = [e]$ and $[x, y]$, it is adjacent to S_3. We have thus proved that S_1, S_2, S_3 are flat. □

The important consequence of the above two lemmas is the following: Assume S is a 2×2 square centered around a vertex of type III and that $\tilde{S}'$ is a 1×1 square adjacent to S along an edge of type I. Then using Lemma 2.16 and Lemma 2.15, there exists another 2×2 square S' containing $\tilde{S}'$, centered on a vertex of type III, adjacent to S. This is synthesized in Figure 4.

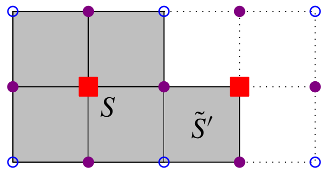
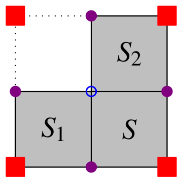
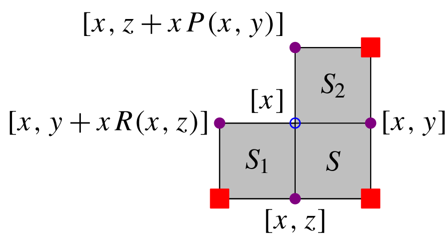


Figure 4. A 2×2 square S centered around a type III vertex with a 1×1 square $\tilde{S}'$ adjacent to S along a type I edge where $\tilde{S}'$ is contained in another 2×2 square S'.

Lemma 2.17. *Let v be a vertex of type* I *and let S, S_1, S_2 be three distinct 1×1 squares such that S is adjacent to S_1 along an edge containing v, and S is adjacent to S_2 along an edge containing v. Let g_1 and $g_2 \in \mathrm{STame}(Q)$ such that $g_1 S = S_1$ and $g_2 S = S_2$. Then the three squares can be completed into a 2×2 square centered along v if and only if g_1 or g_2 belongs to A_v^S:*



Proof. Since the group $\mathrm{STame}(Q)$ is transitive on the set of 1×1 squares, we can suppose that the common vertex v is $[x]$ and that S contains the vertex $[\mathrm{Id}]$. We are thus in the following situation:



where $P, R \in k[x, y]$.

Let us prove the reverse implication ($\Leftarrow$). Assume g_1 or $g_2 \in A_{[x]}^S$. Let us assume that $g_1 \in A_{[x]}^S$, then this implies that $R(x, z) \in k[x, z]$ and $P(x) \in k[x]$ and g_1, g_2 can be taken to be

$$g_1 = \begin{pmatrix} x & y - xR(x) \\ z & t - zR(x) \end{pmatrix} \quad \text{and} \quad g_2 = \begin{pmatrix} x & y \\ z - xP(x, y) & t - yP(x, y) \end{pmatrix}.$$

Set $S' = g_1 g_2 S$, since $g_1 \circ g_2 = g_2 \circ g_1$, we obtain that S' is adjacent to S_2 and S_1, and contains $[x]$ because $g_1 \in A_{[x]}^S \subset \mathrm{Stab}([x])$. In particular, S, S_1, S_2 are flat.

Let us prove the first implication ($\Rightarrow$). Suppose that the squares S_1, S_2, S_3 are flat. Then there exists a component $f_4 \in \mathrm{k}[Q]$ such that the element f given by

$$f = \begin{pmatrix} x & y + xR(x, z) \\ z + xP(x, y) & f_4 \end{pmatrix}$$

belongs to Tame(Q). In particular, it must fix the volume form Ω, this implies that

$$\partial_y P(x, y)\partial_z R(x, z) = 0 \in \mathrm{k}[\mathcal{Q}]\,.$$

This implies that $\partial_y P(x, y) = 0$ or $\partial_z R(x, z) = 0$ hence g_1 or g_2 belongs to $A^S_{[x]}$ as required. □

Lemma 2.18. *Take S and S' two 2×2 squares centered at a vertex of type* III *which are adherent along a vertex of type* I. *Then S and S' satisfy one of the following properties*:

(i) *Either the pair (S, S') is flat.*

(ii) *Either the pair of squares (S, S') is contained in a horizontal or vertical spiral staircase.*

Proof. Consider two squares S, S' such that the pair of square (S, S') is not flat. Up to a conjugation by an element of STame(Q), we can suppose that S and S' are adherent along the vertex $[x]$. Since the group Tame(Q) acts transitively on the set of 2×2 squares centered on type III vertices by Proposition 2.7(vi), there exists an element $g \in$ STame(Q) such that $g \cdot S = S'$. Using the fact that the link of type I vertices is connected, we can choose any minimal sequence S_i of adjacent 2×2 squares centered along a vertex of type III all containing $[x]$ such that $S_1 = S, \ldots, S_k = S'$. Observe that we can always construct such a sequence by taking a sequence of $\tilde{S}_1, \ldots, \tilde{S}_k$ of 1×1 squares containing the same vertex of type I and such that $\tilde{S}_1 \subset S$, $\tilde{S}_k \subset S'$. We apply Lemmas 2.16 and 2.15 inductively to $S_i, \tilde{S}_{i+1}$ for $i = 1, \ldots, k-1$ and we obtain that there exists a square S_{i+1} containing $\tilde{S}_{i+1}$ centered around a vertex of type III, which is adjacent to S_i for each $i = 1, \ldots, k-1$.

Since the sequence of square $S_1, \ldots, S_k$ is minimal, we claim that the squares S_i and S_{i+2} are adherent along the vertex $[x]$ but the sequence S_i, S_{i+1}, S_{i+2} is not flat. Indeed, if it were the case, there exists a 2×2 square $\tilde{S}_i$ containing $[x]$ such that $\tilde{S}_i$ is adjacent to S_i and S_{i+2} and such that the union $S_i, S_{i+1}, S_{i+2}, \tilde{S}_i$ is isometric to $[0, 4] \times [0, 4]$. Observe that the edge $S_i \cap S_{i-1}$ and $S_i \cap \tilde{S}_i$ are equal so the sequence $S_1, \ldots, S_{i-1}, \tilde{S}_i, S_{i+2}, \ldots, S_k$ is a sequence of squares connecting S and S' of length $k-1$. This contradicts the fact that the sequence $S_1, \ldots, S_k$ is minimal.

Moreover, the squares S_i and S_{i+1} are alternatively adjacent along vertical and horizontal edges. Hence the pair (S, S') is contained in a horizontal or vertical spiral staircase, as required. □

In practice, we will use the following explicit characterization to determine whether two squares adherent along a vertex of type I are flat.

Lemma 2.19. *Consider two 2×2 adjacent squares S_1, S_2 along a horizontal edge containing $[x_1], [y_1]$ and a polynomial $P \in k[x, y] \setminus k$. Denote by $[z_1], [t_1]$ the other vertices of S_1 such that $[x_1], [z_1]$ belong to a vertical edge of S_1 and by $[z_1 + x_1 P(x_1, y_1)], [t_1 + y_1 P(x_1, y_1)]$ the two other vertices of S_2. Let g be the tame automorphism defined by*

$$g = \begin{pmatrix} x & y \\ z + xP(x, y) & t + yP(x, y) \end{pmatrix}$$

so that $g \cdot S_1 = S_2$.

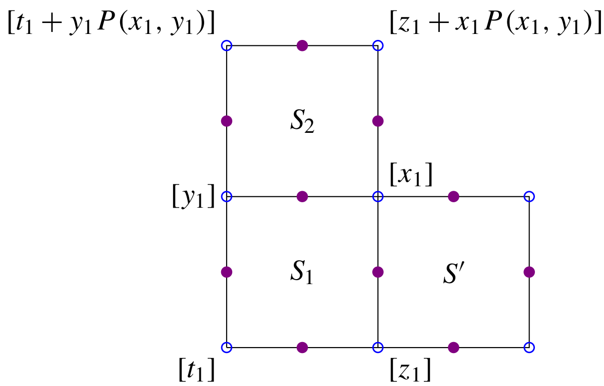


Figure 5. The initial situation of Lemma 2.19.

The following assertions hold:

(i) *We have* $g \in A^{S_1}_{[x_1]}$ *if and only if* $P \in k[x] \setminus k$.

(ii) *For any square* S' *adjacent to* S_1 *along the vertical edge containing* $[x_1]$, $[z_1]$, *the squares* S_1, S', S_2 *are flat if and only if* $P \in k[x] \setminus k$.

Figure 5 summarizes the initial situation in the previous lemma.

Proof. By conjugation, we can suppose that $x_1 = x$, $y_1 = y$, $z_1 = z$ and $t_1 = t$. Assertion (i) follows directly from the definition of $A^{S_1}_{[x]}$.

Let us prove assertion (ii). Choose a square S' such that $g'S_1 = S'$ where $g' \notin A^{S_1}_{[x]}$. Using successively Lemma 2.17, Lemma 2.16 and Lemma 2.15, we obtain that the squares S_1, S_2, S' are flat if and only if g or g' are in $A^{S_1}_{[x]}$. Since $g' \notin A^{S_1}_{[x]}$, then we deduce that S_1, S_2, S' are flat if and only if $g \in A^{S_1}_{[x]}$. Finally the condition $g \in A^{S_1}_{[x]}$ is equivalent to the fact that $P \in k[x] \setminus k$ by assertion (i). ☐

3. Global geometry of the complex

In this section, we first review the results due to Bisi, Furter and Lamy regarding the global geometric properties of the metric square complex $(\mathcal{C}, d_{\mathcal{C}})$ introduced in Section 2. We then describe the degree of iterates of a tame automorphism fixing a vertex of the complex.

3A. ***Gromov curvature and Gromov-hyperbolicity.*** Recall that a map $\gamma : [0, l] \to (\mathcal{C}, d_{\mathcal{C}})$ defines a geodesic segment of length l if γ induces an isometry from $[0, l]$ to $\gamma([0, l])$. A map $\gamma : \mathbb{R} \to \mathcal{C}$ which is an isometry onto its image is called a geodesic line and a map $\gamma : \mathbb{R}^+ \to \mathcal{C}$ which is an isometry onto its image is called a geodesic half-line. Recall also that $\gamma : I \to \mathcal{C}$ where $I = [0, l]$ or $I = \mathbb{R}, \mathbb{R}^+$ is a quasigeodesic if there exists $\lambda > 0$, $M > 0$ such that for any $s, s' \in I$, the following inequality is satisfied:

$$\tfrac{1}{\lambda}|s - s'| - M \leqslant d_{\mathcal{C}}(\gamma(s), \gamma(s')) \leqslant \lambda |s - s'| + M.$$

As a result, a geodesic line is also a quasigeodesic. When any two points on a metric space can be joined by a geodesic segment, we say that the space is a geodesic metric space.

A geodesic space (X, d) is CAT(0) (see [Bridson and Haefliger 1999, Section II.1]) if its triangles are thinner than euclidean triangles. In other words, (X, d) satisfies the following condition. For any three points p, q, r in X, take a triangle in the euclidean plane $(\mathbb{R}^2, \|\cdot\|)$ with vertices $\bar{p}, \bar{q}, \bar{r} \in \mathbb{R}^2$ such that $d(p, q) = \|\bar{p} - \bar{q}\|$, $d(q, r) = \|\bar{q} - \bar{r}\|$ and $d(r, p) = \|\bar{r} - \bar{p}\|$. Then for any point $m_1 \in X$ and $m_2 \in X$ in the geodesic segment $[p, q]$ and $[q, r]$ respectively, one has

$$d(m_1, m_2) \leqslant \|\bar{m}_1 - \bar{m}_2\|,$$

where $\bar{m}_1$ and $\bar{m}_2$ are the unique points on the segments $[\bar{p}, \bar{q}]$ and $[\bar{q}, \bar{r}]$ respectively such that $d(m_1, p) = \|\bar{p} - \bar{m}_1\|$ and $d(r, m_2) = \|\bar{r} - \bar{m}_2\|$.

Let us recall the notion of Gromov-hyperbolic metric space. Let $\delta > 0$ be a positive real number. A metric space (X, d) is δ-hyperbolic if for any geodesic triangle $T = [p, q] \cup [q, r] \cup [r, p]$ in X and for any point $m \in [p, q]$, we have

$$d(m, [q, r] \cup [r, p]) \leqslant \delta$$

Theorem 3.1 [Bisi et al. 2014, Theorem A]. *The square complex $\mathcal{C}$, endowed with the distance $d_{\mathcal{C}}$, is a geodesic metric space which is simply connected*, CAT(0) *and Gromov-hyperbolic.*

The previous result has important consequences on the behavior of the isometries of the complex, i.e., distance preserving maps. Recall that the translation length, denoted $l(f)$, of an isometry $f : \mathcal{C} \to \mathcal{C}$ is defined by

$$l(f) = \inf_{v \in \mathcal{C}} d_{\mathcal{C}}(v, f(v)).$$

Observe that for any isometry f, the points in the complex where the infimum is reached is invariant by f. We denote by $\mathrm{Min}(f)$ the subset of $\mathcal{C}$ on which the infimum is reached.

Theorem 3.2. *Let $f : \mathcal{C} \to \mathcal{C}$ be an isometry of $\mathcal{C}$ which is also a morphism of complex. Then either $l(f) = 0$ and f fixes a vertex in the complex, either $l(f) > 0$ and one can find f-invariant geodesic line on which f acts by translation by $l(f)$.*

In other words, a tame automorphism f is either *elliptic* (when $l(f) = 0$) or *hyperbolic*.

Proof. Take f an isometry of the complex $\mathcal{C}$. Then $\mathrm{Min}(f)$ is nonempty by [Bridson and Haefliger 1999, II.6.6(2)]. Suppose that $l(f) > 0$, then f satisfies the hypothesis of [loc. cit., II.Theorem 6.8]. More precisely, [loc. cit., II.Theorem 6.8(1)] asserts that an isometry f of a CAT(0) space satisfies $l(f) > 0$ if and only if f translates by $l(f)$ on an invariant geodesic line, as required.

Otherwise $l(f) = 0$, we prove that there exists a vertex which is fixed by f. Define a cell of C to be a vertex, an edge or a square. We note that the intersection of two cells of C is a cell. Since $Min(f) \neq \varnothing$ pick $v \in Min(f)$, then $d_{\mathcal{C}}(v, f(v)) = 0$: So $f(v) = v$. Let S be the minimal cell of $\mathcal{C}$ which contains v. Then v is the intersection of all cells of $\mathcal{C}$ which contains v and we get $f(S) = S$. If S is a vertex, v is a vertex. If S is an edge, since the two vertices of S are of different types, f fixes every vertex of S. Now assume that S is a square. Then f fix the unique type III point of S. □

3B. *Degree growths of elliptic automorphisms.* In this section, we apply the results of the previous section to study the degree growth of particular tame automorphisms. Recall from the previous section that a tame automorphism is *elliptic* or *hyperbolic* if its action on the complex fixes a vertex or preserves a geodesic line of the complex on which it acts by translation respectively.

The following result classifies the degree growth of any elliptic tame automorphisms.

Theorem 3.3. *Let* $f \in \mathrm{Tame(Q)}$ *be any tame automorphism of Q fixing a vertex in the square complex. Then we are in one of the following situations*:

(i) *The sequence* $(\deg(f^n), \deg(f^{-n}))$ *is bounded and* f *is linear or* f^2 *is conjugate via a birational map* $Q \dashrightarrow \mathbb{A}^3$ *to an automorphism of the form* $(x, y, z) \mapsto (ax, by + xR(x), b^{-1}z + xP(x, y))$ *with* $a, b \in k^*$, $P \in k[x, y]$ *and* $R \in k[x]$.

(ii) *There exists a constant* $C > 0$ *such that*

$$\tfrac{1}{C}n \leqslant \deg(f^{\epsilon n}) \leqslant Cn,$$

where $\epsilon \in \{+1, -1\}$ *and* f *is conjugated via a birational map* $Q \dashrightarrow \mathbb{A}^3$ *to an automorphism of the form*

$$(x, y, z) \mapsto (ax, b^{-1}(z + xR(x)), b(y + xP(x)z)),$$

with $a, b \in k^*$, $R \in k[x]$ *and* $P \in k[x] \setminus k$.

(iii) *There exists a constant* $C > 0$ *and an integer* d *such that*

$$\tfrac{1}{C}d^n \leqslant \deg(f^{\epsilon n}) \leqslant Cd^n,$$

where $\epsilon \in \{+1, -1\}$ *and* f *is conjugated via a birational map* $Q \dashrightarrow \mathbb{A}^3$ *to a composition of elements of the form*

$$(x, y, z) \mapsto (ax, b(z + xP(x, y)), b^{-1}(y + xR(x))),$$

where $a, b \in k^*$, $R \in k[x]$ *and* $P \in k[x, y]$ *such that* $\deg_y(P) \geqslant 2$.

Remark 3.4. In case (iii) of the previous theorem, suppose f is a normal form, then $\deg(f^p) = Cd^p + C_0$ where $C > 0$ and $C_0 \in \mathbb{Z}$.

Remark 3.5. The growth of the degree of elliptic automorphisms is summarized in Table 1.

The proof of the theorem relies on the comparison to some reference tame automorphism, for which one computes the degree growth explicitly.

fixed vertex	action on the link	fibration	behavior on the fiber	$\deg(f^n)$
Type III				bounded
Type II		over $\mathbb{P}^2$	flow of a vector field	bounded
Type I	trivial on $\mathcal{T}$	over $\mathbb{P}^1$	flow of a vector field	bounded
Type I	involution on $\mathcal{T}$	over $\mathbb{P}^1$	affine	linear
Type I	hyperbolic on $\mathcal{T}$	over $\mathbb{P}^1$	composition of Henon	$\frac{d^n}{C} \leqslant \deg(f^n) \leqslant Cnd^n$

Table 1. Summary of the degree growth for an elliptic automorphism.

Lemma 3.6. *The following properties hold*:

(i) *If f is belongs to the semidirect product* $\mathrm{Stab}([x,z]) = \tilde{E}_H \rtimes N$ *where*

$$N = \left\{ \begin{pmatrix} ax+bz & a'y+b't \\ cx+dz & c'y+d't \end{pmatrix} \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d' & -b' \\ -c' & a' \end{pmatrix} = I_2 \in M_2(k) \right\},$$

then the sequence $(\deg(f^n), \deg(f^{-n}))$ *is bounded.*

(ii) *If f is of the form*

$$f = \begin{pmatrix} ax & b(y+xP(x)) \\ b^{-1}(z+xS(x)) & a^{-1}(t+zP(x)+yS(x)+xP(x)S(x)) \end{pmatrix}$$

where $P, S \in k[x]\setminus k$, $a, b \in k^*$, then the sequence $(\deg(f^n), \deg(f^{-n}))$ *is bounded.*

(iii) *If f is of the form*

$$f = \begin{pmatrix} ax & b(y+xP(x,z)) \\ b^{-1}(z+xR(x)) & a^{-1}(t+zP(x,z)+yR(x)) \end{pmatrix}$$

with $P \in k[x,z]\setminus k$ and $\deg_z(P) = 1$, *$R \in k[x]$ and $a, b \in k^*$, then the sequence* $(\deg(f^n), \deg(f^{-n}))$ *is bounded.*

(iv) *If f is of the form*

$$f = \begin{pmatrix} ax & b^{-1}(z+xR(x)) \\ b(y+xP(x)z) & a^{-1}(t+z^2P(x)+yR(x)) \end{pmatrix}$$

with $P \in k[x]\setminus k$, $R \in k[x]$, $a, b \in k^*$, then both sequences $(\deg(f^n))$, $(\deg(f^{-n}))$ *grow linearly.*

(v) *If f is of the form*

$$f = \begin{pmatrix} ax & b(y+xP(x,z)) \\ b^{-1}(z+xR(x)) & a^{-1}(t+z^2P(x)+yR(x)) \end{pmatrix}$$

with $P \in k[x,y]$, $R \in k[x]\setminus k$, $a, b \in k^*$, then the sequence $(\deg(f^n), \deg(f^{-n}))$ *is bounded.*

(vi) *If f is a composition of automorphism of the form*

$$\begin{pmatrix} ax & b(z+xP(x,y)) \\ b^{-1}(y+xR(x)) & a^{-1}(t+zR(x)+yP(x,y)+xP(x,y)R(x)) \end{pmatrix},$$

where $R \in k[x]$ and $P \in k[x,y]$ such that $\deg_y(P) \geqslant 2$, *then we have*

$$d^n \leqslant \deg(f^{\pm n}) \leqslant Cd^n, \tag{7}$$

where $C > 0$ and $d \geq 2$ is an integer.

Proof. During the whole proof, we will consider the valuation $\nu : k[Q] \to \mathbb{R}^- \cup \{+\infty\}$ corresponding to $-\deg$. It is defined by the formula

$$\nu(P) = \sup\{-\deg(R) \mid R \in k[x,y,z,t],\, R = P \in k[Q]\}.$$

The fact that such a function gives a valuation will be proved in Proposition 4.2. In each case except the last one, we will also express f^n as $f^n = (x_n, y_n, z_n, t_n)$ where $x_n, y_n, z_n, t_n \in k[Q]$.

Let us prove assertion (i). Denote by N the subgroup

$$N = \left\{ \begin{pmatrix} ax+bz & a'y+b't \\ cx+dz & c'y+d't \end{pmatrix} \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d' & -b' \\ -c' & a' \end{pmatrix} = I_2 \in M_2(k) \right\}.$$

Take $f \in \tilde{E}_H \rtimes N$, then it can be decomposed into

$$f = e \circ g,$$

where $e \in \tilde{E}_H$ and $g \in N$. Since $\tilde{E}_H$ is a normal subgroup, we can consider the elements $e_1 = geg^{-1} \in \tilde{E}_H$, $e_{k+1} = gee_k g^{-1}$ for all $k \geq 1$. We have

$$\begin{aligned} f^n &= (eg)^n = (eg)^{n-2}(egeg) \\ &= (eg)^{n-2} e e_1 g^2 \\ &= (eg)^{n-3} e(gee_1) g^2 \\ &= (eg)^{n-3} e e_2 g^3 \\ &= \cdots \\ f^n &= e e_{n-1} g^n. \end{aligned}$$

Note that since $N \subset O_4$, the degree remain unchanged if we postcompose by elements in this subgroup. In particular, we have $\deg e_{k+1} = \deg(ee_k)$ for all $k \geq 2$. we get

$$\deg(f^n) = \deg(ee_{n-1}).$$

Now we write e as

$$e = \begin{pmatrix} ax & b(y+xP(x,z)) \\ b^{-1}z & a^{-1}(t+zP(x,z)) \end{pmatrix}, \tag{8}$$

where $a, b \in k^*$ and $P \in k[x, z]$, and e_k is of the form

$$e_k = \begin{pmatrix} x'_k & y'_k \\ z'_k & t'_k \end{pmatrix},$$

where $x'_k, y'_k, z'_k, t'_k \in k[Q]$. We claim that the degree of e_k is bounded. We have $\nu(x'_1) = \nu(z'_1) = -1$, $\nu(y'_1) = \nu(t'_1) = -\deg(P) - 1$ and an immediate induction gives

$$\begin{aligned}
\nu(x'_{k+1}) &= \nu(a x'_k) = \nu(x'_k), \\
\nu(y'_{k+1}) &= \nu(b(y'_k + x'_k P(x'_k, z'_k))) = \nu(y'_k + x'_k P(x'_k, z'_k)) \geq -\deg(P) - 1, \\
\nu(z'_{k+1}) &= \nu(b z'_k) = \nu(z'_k), \\
\nu(t'_{k+1}) &= \nu(a^{-1}(t'_k + z'_k P(x'_k, z'_k))) \geq -\deg(P) - 1.
\end{aligned}$$

This shows that $\deg(f^n) = \deg(e e_{n-1})$ is bounded. A similar argument will also show that $\deg(f^{-n})$ is bounded and assertion (i) holds.

Let us prove assertion (ii). Assume f is given by

$$f = \begin{pmatrix} ax & b(y + xP(x)) \\ b^{-1}(z + xS(x)) & a^{-1}(t + zP(x) + yS(x) + xP(x)S(x)) \end{pmatrix}$$

where $P, S \in k[x] \setminus k$ and $a, b \in k^*$. The sequence $\nu(x_n)$ is constant equal to -1 because $\nu(x_{n+1}) = \nu(a x_n) = \nu(x_n) = \nu(x) = -1$. We have $\nu(y_1) = -\deg(P) - 1$, $\deg(z_1) = -\deg(S) - 1$. Observe that $\nu(y_{n+1}) = \nu(b(y_n + x_n P(x_n))) \geq -\deg(P) - 1$ for all $n \geq 1$, and $\nu(z_{n+1}) = \nu(b^{-1}(z_n + x_n S(x_n))) \geq -\deg(S) - 1$ for all $n \geq 1$. Since $\nu(x_n t_n - y_n z_n) = \nu(1) = 0$, and since $\nu(x_n), \nu(y_n), \nu(z_n)$ are all bounded and always negative, we deduce that $\nu(t_n) = \nu(y_n) + \nu(z_n) - \nu(x_n)$ is also bounded. This shows that $\deg(f^n)$ is bounded. Since the inverse f^{-1} can be obtained by replacing P by $-P$, S by $-S$ and a, b by a^{-1}, b^{-1} respectively, we conclude that $\deg(f^{-n})$ is also bounded.

Let us prove assertion (iii). Assume f is of the form

$$f = \begin{pmatrix} ax & b(y + xP(x, z)) \\ b^{-1}(z + xR(x)) & a^{-1}(t + zP(x, z) + yR(x) + xP(x, z)R(x)) \end{pmatrix}$$

with $R \in k[x]$, $a, b \in k^*$, $P \in k[x, z]$ and $\deg_z P = 1$. We first decompose P into

$$P(x, z) = P_0(x) + z P_1(x),$$

where $P_0, P_1 \in k[x]$ and $P_1 \neq 0$.

Observe that $\nu(x_n) = -1$ for all n. Observe that $\nu(z_1) = -\deg(R) - 1$, we obtain by induction

$$\nu(z_{n+1}) = \nu(b^{-1}(z_n + x_n R(x_n))) \geq -\deg(R) - 1.$$

so that $\nu(z_n) \geq -\deg(R) - 1$ for all $n \geq 1$. Observe that $\nu(y_1) = -\deg(P) - 2$, we now estimate $\nu(y_n)$ inductively:

$$\begin{aligned} \nu(y_{n+1}) &= \nu(b(y_n + x_n P_0(x_n) + x_n z_n P_1(x_n))) \\ &= \nu(y_n + x_n P_0(x_n) + x_n z_n P_1(x_n)) \\ &\geq \min(\nu(y_n), -\deg(P_0) - 1, \nu(z_n) - 1 - \deg(P_1)) \\ &\geq \min(\nu(y_n), -\deg(P_0) - 1, -\deg(P_1) - \deg(R) - 2). \end{aligned}$$

We thus obtain that $\nu(y_n) \geq -\deg(P) - \deg(R) - 2$ for all $n \geq 1$. Using the fact that $x_n t_n = 1 + y_n z_n$, we deduce that $\nu(t_n)$ is also bounded since $\nu(x_n)$, $\nu(y_n)$, $\nu(z_n)$ are all bounded. Since f^{-1} has a similar form, we deduce that both $\deg(f^n)$ and $\deg(f^{-n})$ are bounded.

Let us prove assertion (iv). Assume f is of the form

$$f = \begin{pmatrix} ax & b^{-1}(z + xR(x)) \\ b(y + xP(x)z) & a^{-1}(t + z^2 P(x) + yR(x)) \end{pmatrix}$$

with $P \in k[x] \setminus k$, $R \in k[x]$, $a, b \in k^*$. In the case where $R = 0$ one sees easily that the sequence of degrees grows linearly $\deg(f^n) \sim Cn$. Let us now assume $R \neq 0$. We have

$$f^2 = \begin{pmatrix} a^2 x & b^{-1}(b(y + xP(x)z) + axR(ax)) \\ b(b^{-1}(z + xR(x)) + axP(ax)b(y + xP(x)z)) & t_2 \end{pmatrix}. \tag{9}$$

Observe that $\nu(x_n) = -1$ for all $n \geq 1$. We have (using Lemma 4.7 to evaluate the valuation) $\nu(y_1) = \min(-1, -1 - \deg(R))$, $\nu(z_1) = -\deg(P) - 2$, $\nu(y_2) = \min(-1, -2 - \deg(P), -1 - \deg(R))$, $\nu(z_2) = \min(-1, -1 - \deg(R), -2\deg(P) - 3)$ and the inductive relation

$$\begin{cases} \nu(y_{n+1}) = \nu(b^{-1}(z_n + x_n R(x_n))) \geq \min(\nu(z_n), -\deg(R) - 1), \\ \nu(z_{n+1}) = \nu(b(y_n + x_n P(x_n) z_n)) \geq \min(\nu(y_n), -\deg(P) - 1 + \nu(z_n)). \end{cases}$$

Let us show by induction on $n \geq 2$ that:

(a) $\nu(z_n) \leq \nu(y_n)$.

(b) $\nu(z_n) \leq \min(-1, -\deg(R) - 1)$.

The case $n = 2$ was treated above. Let us assume that (a), (b) hold for $n \geq 2$. Since $\nu(z_n x_n P(x_n)) < \nu(y_n)$, we have

$$\nu(z_{n+1}) = \nu(y_n + x_n P(x_n) z_n) = -\deg(P) - 1 + \nu(z_n). \tag{10}$$

Since $\nu(z_n) \leq \min(-1, -\deg(R) - 1)$, we have, by the previous relation,

$$\nu(y_{n+1}) \geq \min(\nu(z_n), -\deg(R) - 1) \geq \nu(z_n) > \nu(z_{n+1}). \tag{11}$$

We have thus showed (a) and (b) for $n + 1$.

Relation (a) shows that $\nu(z_{n+1}) = \nu(z_n) - \deg(P) - 1$ for all $n \geq 2$, so $\nu(z_n)$ grows linearly. Starting from $n \geq 3$, we would have $\nu(z_n) < \min(-1, -\deg(R) - 1)$. And this implies that $\nu(y_n) = \nu(z_{n-1})$ for all $n \geq 4$. Overall, both $\nu(y_n)$, $\nu(z_n)$ grow linearly for $n \geq 4$. Since $\nu(t_n) = \nu(1 + y_n z_n) - \nu(x_n) = \nu(y_n) + \nu(z_n) - \nu(x_n)$

for $n \geqslant 1$, we deduce that $\nu(t_n)$ also grows linearly for large enough n. Overall $\deg(f^n)$ grows linearly. Similarly, f^{-1} is also of the same form, so we also conclude that $\deg(f^{-n})$ also grows linearly.

Let us prove assertion (v). Assume f is of the form

$$f = \begin{pmatrix} ax & b(y + xP(x,z)) \\ b^{-1}(z + xR(x)) & a^{-1}(t + z^2P(x) + yR(x)) \end{pmatrix}$$

with $P \in k[x,y]$, $R \in k[x] \setminus k$, $a, b \in k^*$. We observe that $\nu(x_n) = -1$ and $\nu(z_n) = -\deg(R) - 1$ for all $n \geq 1$. We also have

$$\nu(y_{n+1}) = \nu(y_n + x_n P(x_n, z_n)) \geq \min(\nu(y_n), \nu(x_n) + \nu(P(x_n, z_n))).$$

Write $P = \sum a_{ij} x^i y^j$. Since $\nu(P(x_n, z_n)) \geq \min\{-i + j(-\deg(R) - 1) \mid a_{ij} \neq 0\}$, the above formula also yields that $\nu(y_n)$ is bounded. We then conclude that $\nu(t_n) = \nu(y_n z_n) - \nu(x_n)$ is also bounded. Finally $\deg(f^n)$ is bounded, and so is its inverse which is of a similar form.

Let us prove assertion (vi). Assume $f = g_k \cdots g_1$ is a composition of automorphisms of the form

$$g_i = \begin{pmatrix} a_i x & b_i(z + xP_i(x,y)) \\ b_i^{-1}(y + xR_i(x)) & a_i^{-1}(t + zR_i(x) + yP_i(x,y) + xP_i(x,y)R_i(x)) \end{pmatrix},$$

where $R_i \in k[x]$ and $P_i \in k[x,y]$ such that $\deg_y(P_i) \geqslant 2$. For simplicity, let us reset $x_n, y_n, z_n, t_n \in k[Q]$ defined by

$$g_i \cdots g_2 g_1 f^n = \begin{pmatrix} x_{nk+i} & y_{nk+i} \\ z_{nk+i} & t_{nk+i} \end{pmatrix} \tag{12}$$

for all $i \leqslant k$. We also set $g_{nk+i} := g_i$ for all n, i so that we repeat the same sequence of automorphism periodically.

We will first show that the sequences $\nu(y_n)$, $\nu(z_n)$ are unbounded.

Let us consider the following valuation $\nu_0 : k[Q] \to \mathbb{R}^- \cup \{+\infty\}$ which gives the weight $(0, -1, -1, -2)$ on (x, y, z, t) (see Proposition 4.2 for a precise definition). This valuation gives no weight to x whereas it gives the weight -1 to y, z and the weight -2 to t. By construction, we have $2\nu(P) \leqslant \nu_0(P)$ for all $P \in k[Q]$.

Recall from Section 2D that an automorphism of the form

$$g_n = \begin{pmatrix} a_n x & b_n(z + xP_n(x,y)) \\ b_n^{-1}(y + x_n R(x)) & a_n^{-1}(t + zR(x) + yP(x,y) + xP(x,y)R(x)) \end{pmatrix},$$

induces an element $i(g_n)$ of $\mathrm{Aut}_{k(x)} \mathbb{A}^2$. Namely, the associated element is

$$(y, z) \mapsto (b_n(z + xP_n(x,y)), b_n^{-1}(y + xR_n(x))). \tag{13}$$

One can see that given g_n, g_m of this form, one can find some automorphism $\tilde{g}_{n,m}, \tilde{g}_m$ of the same form determined by some polynomials $\tilde{P}_n = \lambda_n^{-1} P_n(x/\lambda_n, y)$, $\tilde{R}_n = \lambda_n^{-1} R(x/\lambda_n)$ where $\lambda_n \in k^*$ (λ_n depends on g_m for $n \geqslant m$) such that $i(g_n g_m) = i(\tilde{g}_n) i(\tilde{g}_m)$. We thus find them inductively as follows, we first find $\tilde{g}_2$ such that $i(g_2 g_1) = i(\tilde{g}_2) i(g_1)$. Now we find $\tilde{g}_3$ such that $i(g_3(g_2 g_1)) = i(\tilde{g}_3) i(\tilde{g}_2) i(g_1)$ and so on, observe that $\tilde{g}_{n+1}$ depends on all the preceding $g_n, g_{n-1} \dots, g_1$. However the degree in y of those elements remains the same. We thus obtain

$$i(g_n \cdots g_1) = i(\tilde{g}_n) \cdots i(\tilde{g}_2) i(g_1).$$

The elements $i(g_n), i(\tilde{g}_n)$ are Henon-like automorphism of $\mathbb{A}^2_{k[x]}$ and by [Friedland and Milnor 1989, Theorem 2.1] the degree in (y, z) of a composition $i(g_1 \cdots g_n) = i(\tilde{g}_1) \cdots i(\tilde{g}_n)$ of n such elements is a product $d_n = \prod_{i=1}^n \deg_y P_i$ of n integers larger or equal to 2. This implies that $\nu_0(y_n) \sim d_n$, $\nu_0(z_n) \sim d_{n-1}$ grow exponentially fast, and we obtain that $2\nu(y_n) \leqslant \nu_0(y_n)$, $2\nu(z_n) \leqslant \nu_0(z_n)$ are unbounded and that

$$\deg(g_n \cdots g_1) \geq d_n/2 \tag{14}$$

hence $\deg(f^n) \geq d_k^n/2$.

We will now prove the second inequality $\deg(g_n \cdots g_1) \leqslant C' d_n$ for a constant $C' > 0$. Let us first reduce our problem, to any $f \in \operatorname{Stab}([x])$ of the form

$$f = \begin{pmatrix} ax & f_2 \\ f_3 & f_4 \end{pmatrix},$$

where $a \in k^*$, $f_2, f_3, f_4 \in k[Q]$. We set

$$r(f) = \begin{pmatrix} x & f_2 \\ f_3 & af_4 \end{pmatrix}.$$

By construction, $r(f)$ is a tame automorphism which is also in $\operatorname{Stab}([x])$ and one has $\deg(r(f)) = \deg(f)$. Moreover, one can find some automorphism g'_i of the same form as g_i such that

$$r(g_n \cdots g_1) = r(g'_n) \cdots r(g'_1).$$

We can thus replace $g_1, \dots, g_n$ by $r(g'_1), \dots, r(g'_n)$ and assume that the coefficient a_i are all equal to 1.

Let us introduce some particular notations. For any polynomial $P \in k[x, y] \setminus k[x]$ and $b \in k^*$, we write by $h(P)$ the automorphism given by

$$h(P, b) := \begin{pmatrix} x & b(z + xP(x, y)) \\ b^{-1}y & t + yP(x, y) \end{pmatrix}.$$

We also set for all $R, S \in k[x]$ and $b \in k^*$,

$$u(R, S, b) := \begin{pmatrix} x & b(y + xR(x)) \\ b^{-1}(z + xS(x)) & t + zR(x) + yS(x) + xR(x)S(x) \end{pmatrix}. \tag{15}$$

We will use some particular relations via the following lemma:

Lemma 3.7. *For any $P \in k[x, y] \setminus k$ such that $\deg_y P \geq 2$ and any $R, S \in k[x]$, there exist $T \in k[x]$, $\tilde{P} \in k[x, y] \setminus k$ such that the following are satisfied*:

(1) $\deg_y \tilde{P} = \deg_y P - 1$.

(2) $\deg_x \tilde{P} \leqslant \deg_x P + (\deg_y P - 1)(1 + \deg R)$.

(3) $\deg T \leqslant \max(\deg S, (\deg_y P)(\deg R + 1) + \deg P)$.

(4) $h(P, \beta) \circ u(R, S, b) = u(T, R, \beta b^{-1}) \circ h(y\tilde{P}, 1)$.

Proof. The proof follows easily from the Taylor expansion:

$$P(x, b(y + xR(x))) = P(x, by) + \frac{(bxR(x))^{\deg_y P}}{(\deg_y P)!}\partial_2^{\deg_y P} P(x, by) + \sum_{k=1}^{\deg_y P-1} \frac{(bxR(x))^k}{k!}\partial_2^k P(x, by). \ \square$$

Let us decompose g_i into as $g_i = u(R_i, S_i, b_i) \circ h(yp_i, \beta_i)$, where $p_i \in k[x, y]$ such that $\deg_y p_i = \deg_y P_i - 1$ and $b_i, \beta_i \in k^*$. Note that since the polynomials P_i, R_i are chosen periodically from $\{P_1, \ldots, P_k\}, \{R_1, \ldots, R_k\}$ respectively, the polynomials R_i, S_i, p_i are also chosen among finitely many polynomials. Let us consider

$$M := \max_i(\deg(R_i), \deg S_i),$$
$$M_p := \max(M, \max_i(1 + \deg p_i)).$$

Observe also that $d_n = \prod_{i=1}^n \deg_y P_i = \prod_1^n (1 + \deg_y p_i)$ and that $M \le M_p$ by construction.

We prove by induction on $n \ge 1$ that there exists $T_n, U_n \in k[x]$, $\beta'_n \in k^*$ and $\tilde{P}_1, \ldots, \tilde{P}_n \in k[x, y] \setminus k$ such that

$$g_n \cdots g_1 = u(T_n, U_n, \beta'_n)h(y\tilde{P}_n, 1)\ldots h(y\tilde{P}_1),$$

and satisfying the conditions

$$\deg U_n \leqslant M \quad \text{and} \quad \forall n \ge 2, \deg_x \tilde{P}_n \leqslant \deg_x p_n + \deg_y(p_n)(1 + \deg T_{n-1}), \tag{16}$$

and for all $n \ge 2$

$$\deg T_n \leqslant d_n \deg T_1 + \sum_{i=1}^{n-1} \frac{d_n}{d_i} + M_p \sum_{i=2}^{n} \frac{d_n}{d_i}. \tag{17}$$

The case where $n = 1$ is immediate. Let us assume that $g_n \ldots g_1$ can be decomposed into $u(T_n, U_n, \beta'_n) \circ h(y\tilde{P}_n, 1) \circ \cdots \circ h(y\tilde{P}_1, 1)$. Using the above lemma to $R = T_n, S = U_n$ and $P = yp_{n+1}$, we obtain $\tilde{\beta}_{n+1} \in k^*$, some polynomials $\tilde{T} \in k[x]$ and $\tilde{P}_{n+1} \in k[x, y]$ and get

$$\begin{aligned} g_{n+1}g_n \cdots g_1 &= u(R_{n+1}, S_{n+1}, b_n)h(yp_{n+1}, \beta_{n+1})u(T_n, U_n, \beta'_n)h(y\tilde{P}_n, 1) \cdots h(y\tilde{P}_1), \\ &= u(R_{n+1}, S_{n+1}, \beta_{n+1})u(\tilde{T}, R_{n+1}, \tilde{\beta}_{n+1})h(y\tilde{P}_{n+1}, 1)h(y\tilde{P}_n, 1) \cdots h(y\tilde{P}_1, 1), \end{aligned}$$

and we conclude by setting $T_{n+1} = \tilde{T} + \tilde{\beta}_{n+1}^{-1}R_{n+1}$, $U_{n+1} = R_{n+1} + \tilde{\beta}_{n+1}^{-1}S_{n+1}$, $\beta'_{n+1} = \beta_{n+1}\tilde{\beta}_{n+1}$. We check the bounds on the degree of T_{n+1}, U_{n+1}. The lemma directly implies

$$\deg(U_{n+1}) \leqslant \max(\deg R_{n+1}, \deg S_{n+1}) \leqslant M \quad \text{and} \quad \deg_x \tilde{P}_{n+1} \leqslant \deg_x p_{n+1} + \deg_y(p_{n+1})(1 + \deg T_n).$$

Using the induction hypothesis $\deg U_n \leqslant M$, the inequality (17) and the fact that $\deg R_{n+1} \leqslant M \leqslant M_p$, $(1+\deg p_{n+1}) \leq M_p$, we get

$$\begin{aligned}
\deg(T_{n+1}) &\leqslant \max(\deg R_{n+1}, \deg(\tilde{T})),\\
&\leqslant \max(\deg R_{n+1}, \deg U_n, (1+\deg_y p_{n+1})(\deg T_n + 1) + 1 + \deg p_{n+1}),\\
&\leqslant \max(M, (1+\deg_y p_{n+1})\deg T_n + M_p + (1+\deg_y p_{n+1})),\\
&\leqslant \max\left(M, \frac{d_{n+1}}{d_n}\left(d_n \deg T_1 + \sum_{i=1}^{n-1}\frac{d_n}{d_i} + M_p\sum_{i=2}^{n}\frac{d_n}{d_i}\right) + M_p + \frac{d_{n+1}}{d_n}\right),\\
&\leqslant d_{n+1}\deg T_1 + \sum_{i=1}^{n}\frac{d_{n+1}}{d_i} + M_p\sum_{i=2}^{n+1}\frac{d_{n+1}}{d_i}.
\end{aligned}$$

This finishes the proof by induction.

Using the fact that d_{n+1}/d_n is an integer larger or equal to 2, the inequality (17) implies that for all $n \geq 2$, we have $d_n/d_i \leqslant \left(\frac{1}{2}\right)^{i-1} d_n/d_1$, for all $i \leqslant n$,

$$\deg T_n \leqslant d_n \deg T_1 + (2+M_p)\frac{d_n}{d_1} \leqslant C d_n, \tag{18}$$

where $C = \deg(T_1) + (2+M_p)/d_1$. Now (16) simplifies as

$$\begin{aligned}
\deg_x \tilde{P}_n &\leqslant \deg_x p_n + \deg_y(p_n)(1+\deg T_{n-1}),\\
&\leqslant \deg_x p_n + \deg_y(p_n)(1+Cd_{n-1}),\\
&\leqslant d_n\left(C+\tfrac{1}{2}\right) + \deg_x p_n,
\end{aligned}$$

and we choose an integer N_0 such that for all $n \geq N_0$, $\max(\deg_x p_i) \leqslant d_n$. We then get

$$\deg_x \tilde{P}_n \leqslant C' d_n \tag{19}$$

where $C' = C + \frac{3}{2}$ for all $n \geq N_0$.

Let us now prove that $\deg(h(y\tilde{P}_n, 1)\ldots h(y\tilde{P}_1, 1)) \leqslant C' d_n$ where $C' > 0$. For this particular product, we reproduce the standard arguments for product of Henon transformations.

Let us write

$$h(y\tilde{P}_n, 1)\ldots h(y\tilde{P}_1, 1) = \begin{pmatrix} x & y'_n \\ z'_n & t'_n \end{pmatrix},$$

where $y'_n, z'_n, t'_n \in k[Q]$.

We have the following inductive relation for $n \geqslant 1$:

$$\nu(z'_{n+1}) = \nu(y'_n) \quad \text{and} \quad \nu(y'_{n+1}) = \nu(z'_n + xy'_n\tilde{P}_{n+1}(x, y'_n)).$$

We prove by induction on $n \geq 1$ that $\nu(y'_n) \leqslant \nu(z'_n)$.

Indeed, it is clear for $n=1$ and assume by induction that $\nu(y'_n) \leqslant \nu(z'_n)$ for $n \geq 1$, then since $\tilde{P}_{n+1} \neq 0$, we get

$$\nu(y'_{n+1}) = \nu(z'_n + x y'_n \tilde{P}_{n+1}(x, y'_n)) = \nu(x y'_n \tilde{P}_{n+1}(x, y'_n)) < \nu(y'_n) = \nu(z'_{n+1}),$$

as required. Moreover, by applying (19), we have for all $n \geq N_0$,

$$\begin{aligned} \nu(y'_{n+1}) &\geqslant \min(-\deg_x \tilde{P}_{n+1}, (\deg_y \tilde{P}_n + 1)\nu(y'_n)) \\ &\geqslant \min(-C' d_{n+1}, (\deg_y \tilde{P}_n + 1)\nu(y'_n)) \\ &\geqslant \min\left(-C' d_{n+1}, \frac{d_{n+1}}{d_n}\nu(y'_n)\right). \end{aligned}$$

An immediate induction using the previous inequality shows that

$$\nu(y'_n) \geqslant \min\left(-C' d_n, \frac{d_n}{d_{N_0}}\nu(y'_{N_0})\right), \tag{20}$$

for all $n \geq N_0$ and this shows that $\deg(h(y\tilde{P}_n) \ldots h(y\tilde{P}_1)) \leqslant \tilde{C} d_n$.

Finally, we conclude

$$\begin{aligned} \deg(g_n \ldots g_1) &= \deg(u(T_n, U_n) h(y\tilde{P}_n) \cdots h(y\tilde{P}_1)) \\ &\leqslant \max(\deg T_n, \deg U_n, \deg(h(y\tilde{P}_n) \cdots h(y\tilde{P}_1))) \leqslant C'' d_n, \end{aligned} \tag{21}$$

where $C'' > 0$, as required. $\square$

Proof of Theorem 3.3. Take $f \in \mathrm{Tame}(\mathrm{Q})$ an elliptic automorphism. Since f fixes a vertex on the complex, we will distinguish three cases depending on the type of vertices f fixes. Moreover, recall that the degree growth is an invariant of conjugation and that by Proposition 2.7, the tame group acts transitively on the set of vertices of type I, II and III respectively. We are thus reduced to compute the degree growth for f in the subgroups $\mathrm{Stab}([\mathrm{Id}])$, $\mathrm{Stab}([x, z])$ and $\mathrm{Stab}([x])$ respectively.

First case: If $f \in \mathrm{Stab}([\mathrm{Id}]) = O_4$, the sequence $(\deg(f^n), \deg(f^{-n}))$ is bounded.

Second case: Suppose that $f \in \mathrm{Stab}([x, z])$. By Proposition 2.6, one has

$$\mathrm{Stab}([x, z]) = E_H \rtimes \left\{ \begin{pmatrix} ax + bz & a'y + b't \\ cx + dz & c'y + d't \end{pmatrix} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d' & -b' \\ -c' & a' \end{pmatrix} = I_2 \in M_2(k) \right\}.$$

Denote by $\pi_{xz} : \mathcal{Q} \to \mathbb{A}^2 \setminus \{(0,0)\}$ the map induced by the projection

$$(x, y, z, t) \to (x, z),$$

then f naturally preserves the fibration π_{xz}. Recall that $\pi_{xz}^{-1}(\mathbb{A}^2 \setminus (\{0\} \times \mathbb{A}^1))$ is isomorphic to $\mathbb{A}^2 \setminus (\{0\} \times \mathbb{A}^1) \times \mathbb{A}^1$ and f induces a birational self map. If the induced (linear) action on $\mathbb{A}^2 \setminus \{0\}$ is diagonalizable, then f can be conjugate to an element of the form

$$f : (x, z, y) \mapsto (ax, b^{-1}z, b(y + xP(x, z))).$$

Otherwise, the action on $\mathbb{A}^2$ has Jordan form and f is birationally conjugate to

$$f : (x, z, y) \mapsto (ax, b^{-1}x + az, b(y + xP(x, z))),$$

with $a, b \in k^*$, $P \in k[x, z]$. Moreover, using assertion (i) of Lemma 3.6, the sequence $(\deg(f^n), \deg(f^{-n}))$ is bounded and f satisfies assertion (i).

<u>Third case</u>: Consider $f \in \mathrm{Stab}([x])$ such that $f \notin \mathrm{Stab}([x, y]) \cup \mathrm{Stab}([x, z])$. By definition, there exists a constant $a \in k^*$ such that $x \circ f = ax$. Naturally, f preserves the fibration $\pi_x : \mathcal{Q} \to \mathbb{A}^1$ and since $\pi_x^{-1}(\mathbb{A}^1 \setminus \{0\})$ is isomorphic to $\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2$, the automorphism f is of the form

$$f : (x, y, z) \to (ax, f_1, f_2),$$

where (f_1, f_2) defines an element of $\mathrm{Aut}(\mathbb{A}^2_{k[x,x^{-1}]})$.

By Proposition 2.10, f induces an action on the tree $\mathcal{T}_{\pi_x,k(x)}$. If f induces an action on this tree which fixes the three vertices $[y]$, $[z]$ and $[y, z]$, then f belongs to $A^S_{[x]}$ where S is the 2×2 square containing $[x], [y], [z], [t]$. By Proposition 2.10(iv), f is then of the form

$$\begin{pmatrix} ax & b(y + xP(x)) \\ b^{-1}(z + xS(x)) & a^{-1}(t + zP(x) + yS(x) + xP(x)S(x)) \end{pmatrix}$$

where $P, S \in k[x] \setminus k$. By Lemma 3.6 (ii), the sequences $(\deg(f^n))$ and $\deg(f^{-n}))$ are bounded and f satisfies assertion (i) since in the fixed trivialization, f is of the form

$$(x, y, z) \mapsto (ax, b(y + xP(x)), b^{-1}(z + xS(x))).$$

Recall that the vertices of type II in the Bass–Serre tree $\mathcal{T}_{\pi,k(x)}$ were equivalence classes of components (f_1, f_2) of automorphisms in $\mathrm{Aut}(\mathbb{A}^2_{k(x)})$ where two components $(f_1, f_2) \simeq (g_1, g_2)$ if and only if there exists

$$\begin{pmatrix} a & b & c \\ a' & b' & c' \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(k(x))$$

such that $(g_1, g_2) = (af_1 + bf_2 + c, a'f_1 + b'f_2 + c')$.

Suppose that f, f^2 are not conjugate in $\mathrm{Stab}([x])$ to elements in $A^{(y,z)}_{[x]}$ and the action of f on the subtree $\mathcal{T}_{\pi,k(x)}$ of $\mathcal{T}_{k(x)}$ fixes a vertex. If the fixed vertex in the tree $\mathcal{T}_{\pi,k(x)}$ is of type II, then using Proposition 2.10(i) we can suppose that f fixes the vertex given by $[y, z]$. In particular, this implies that f is conjugated to

$$\begin{pmatrix} ax & b(y{+}xP(x, z)) \\ b^{-1}(z{+}xR(x)) & a^{-1}(t{+}zP(x, z){+}yR(x)) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} ax & b^{-1}(z{+}xR(x)) \\ b(y{+}xP(x, z)) & a^{-1}(t{+}zP(x, z){+}yR(x)) \end{pmatrix}$$

with $P \in k[x, z] \setminus k$ where $\deg_z(P) = 1$ and $R \in k[x]$. Using Lemma 3.6(iii) and (iv), the sequences $\deg(f^n)$ and $\deg(f^{-n})$ are bounded in the first case and grow linearly in the second. In the first case, f satisfies assertion (i) and f satisfies assertion (ii) in the second.

If f, f^2 are not conjugate in Stab($[x]$) to elements in $A^S_{[x]}$ and the action f on $\mathcal{T}_{\pi,k(x)}$ fixes a vertex of type I but no vertices of type II, then using Proposition 2.10(i), f is conjugate (in Stab($[x]$)) to an element which fixes the vertex $[z]$ in the Bass–Serre tree, hence it is an element of the subgroup $\tilde{E}$ defined in assertion (v) of Proposition 2.10. This shows that f is of the form

$$\begin{pmatrix} ax & b(y+xP(x,z)) \\ b^{-1}(z+xR(x)) & a^{-1}(t+z^2P(x)+yR(x)) \end{pmatrix}$$

with $P \in k[x,y]$, $R \in k[x] \setminus k$. By Lemma 3.6(v), the degrees of both f^n and f^{-n} are bounded and f satisfies assertion (i).

The remaining case is when the action on the subtree $\mathcal{T}_{\pi,k(x)}$ is hyperbolic. By Proposition 2.12, f is conjugated to a composition of elements of the form

$$\begin{pmatrix} ax & b(z+xP(x,y)) \\ b^{-1}(y+xR(x)) & a^{-1}(t+zR(x)+yP(x,y)+xP(x,y)R(x)) \end{pmatrix},$$

where $R \in k[x]$ and $P \in k[x,y]$ such that $\deg_y(P) \geqslant 2$. By Lemma 3.6 (vi), the degree sequences $(\deg(f^n))$, $(\deg(f^{-n})$ satisfy

$$d^n \leqslant \deg(f^{\pm n}) \leq Cd^n,$$

where $d \geq 2$ is an integer and f satisfies assertion (iii). □

4. Valuative estimates

This section is devoted to the generalization of the so-called parachute inequalities; see [Bisi et al. 2014, Minoration A.2]. Our proof extends the method of [Lamy and Vénéreau 2013] to more general valuations. The plan of the section is as follows. First we recall some general facts on valuations (Section 4A), then we consider a particular class of valuations in Section 4B. For these particular valuations, we introduce the parachute associated to a pair of regular functions on the quadric allowing us to estimate the degree of a derivative on a given direction (Section 4C). Using this and some elementary facts on key polynomials (Section 4D), we finally deduce our key estimates in Section 4E.

4A. ***Valuations on affine and projective varieties.*** Let X be an affine variety of dimension n over k. By convention for us, a valuation on X is a map $\nu : k[X] \to \mathbb{R} \cup \{+\infty\}$ which satisfies the following properties:

(1) We have $\nu^{-1}(\{+\infty\}) = \{0\}$.

(2) The function ν is not constant on $k[X] \setminus \{0\}$.

(3) For any $a \in k^*$, one has $\nu(a) = 0$.

(4) For any $f_1, f_2 \in k[X]$, one has $\nu(f_1 f_2) = \nu(f_1) + \nu(f_2)$.

(5) For any $f_1, f_2 \in k[X]$, one has $\nu(f_1 + f_2) \geqslant \min(\nu(f_1), \nu(f_2))$.

When the subset $\nu^{-1}(\{+\infty\})$ is not reduced to $\{0\}$, we say that ν is a semivaluation. We endow the space of valuations with the coarsest topology for which all evaluation maps $\nu \mapsto \nu(f)$ are continuous where $f \in k[X]$.

The group $\mathbb{R}_+^*$ naturally acts on the set of valuations by multiplication.

The main examples of valuations are monomial valuations. We recall their definition below. Fix a smooth point p on X, an algebraic system of (local) coordinates $u = (u_0, \dots, u_{n-1})$ at this point and some nonnegative weights $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{R}^+)^n$. We shall denote by $u^I = \prod_{j=0}^n u_j^{i_j}$ when $I = (i_0, \dots, i_{n-1}) \in \mathbb{N}^n$ and by $\langle I, \alpha\rangle = \alpha_0 i_0 + \cdots + \alpha_{n-1} i_{n-1}$ the usual scalar product. The monomial valuation ν with weight α with respect to the system of coordinates u is defined by

$$\nu\Big(\sum_{I\in\mathbb{N}^n} a_I u^I\Big) = \min\{\langle I, \alpha\rangle \mid a_I \neq 0\},$$

where $a_I \in k$.

When $f \in \mathcal{O}_{p,X}$ is a regular function at the point p, then one defines $\nu(f)$ as

$$\nu(f) = \nu\Big(\sum a_I(f) u^I\Big),$$

where $\sum a_I(f) u^I$ is a formal expansion of f near p. The fact that $\nu(f)$ does not depend on the choice of the formal expansion of f near p is proved in [Jonsson and Mustaţă 2012, Proposition 3.1].

Observe that when $\alpha = (1, 0, \dots, 0)$, then the associated valuation coincides with the order of vanishing along $\{u_0 = 0\}$. Furthermore, when $X = \operatorname{Spec}(k[x, y, z, t])$, the valuation $-\deg$ coincides with the monomial valuation on $k[x, y, z, t]$ with weight $(-1, -1, -1, -1)$ with respect to (x, y, z, t) using the same formula.

Consider a regular morphism $f : X \to Y$ where Y is an affine variety and a valuation ν on X. The pushforward of the valuation ν on X by f is denoted $f_*\nu$ is given by the formula

$$f_*\nu = \nu \circ f^\sharp,$$

where $f^\sharp$ denotes the morphism of k-algebra corresponding to f.

We also recall the notion of center of a valuation ν.

For any projective variety $\overline{X}$ containing X as a Zariski open subset, when there exists a regular function $P \in k[X]$ for which $\nu(P) < 0$, the center of ν in $\overline{X}$ is a nonempty Zariski closed irreducible subset which is contained in $\overline{X} \setminus X$. Denote by R_ν the valuation ring and by $\mathcal{M}_\nu$ its maximal ideal, then the center $Z(\nu)$ of ν is a subvariety of $\overline{X}$ defined as follows

$$Z(\nu) = \{p \in \overline{X} \mid \mathcal{O}_{p,\overline{X}} \subset R_\nu, \mathcal{M}_{p,\overline{X}} = \mathcal{M}_\nu \cap \mathcal{O}_{p,\overline{X}}\},$$

where $\mathcal{O}_{p,\overline{X}}$ denotes the local ring of regular functions at the point p and where $\mathcal{M}_{p,\overline{X}}$ is its maximal ideal. The fact that $Z(\nu)$ is nonempty follows from the valuative criterion of properness and we shall refer to [Vaquié 2000] for the general properties of this set.

4B. ***Valuations $\mathcal{V}_0$ on the quadric.*** We denote by $q \in k[x, y, z, t]$ the polynomial $q = xt - yz$ and by $\pi : k[x, y, z, t] \to \mathrm{k}[\mathcal{Q}]$ the canonical projection. Our objective is to define a subset of the set of all valuations on the quadric $\mathcal{Q}$, with different weights on some coordinate axis.

Take a point $p = (x_0, y_0, z_0, t_0) \in k^4$ and a weight $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in (\mathbb{R}^-)^4$, we write by ν_p^α the monomial valuation on $k[x, y, z, t]$ with weight α with respect to the system of coordinates $(x - x_0, y - y_0, z - z_0, t - t_0)$.

We first show that ν_p^α does not depend on p.

Lemma 4.1. *For any weight $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in (\mathbb{R}^-)^4$, we have $\nu_p^\alpha = \nu_{p'}^\alpha$ for any $p, p' \in k^4$.*

Proof. For any multiindices $I = (i_0, i_1, i_2, i_3)$, $J = (j_0, j_1, j_2, j_3) \in \mathbb{Z}_{\geq 0}^4$, denote by $\binom{I}{J} := \Pi_{s=0}^{s} \binom{i_s}{j_s}$.

Set $\nu^\alpha := v_0^\alpha$. We only need to show that for every polynomial $P(x) \in k[x_0, x_1, x_2, x_3]$ and $b = (b_0, b_1, b_2, b_3) \in k^4$, $\nu^\alpha(P(x)) = \nu^\alpha(P(x + b))$. We may assume that $P \neq 0$: Write $P(x) = \sum_I a_I x^I$,

Then $\nu^\alpha(P) = \min\{\langle I, \alpha\rangle a_I \neq 0\}$. Then

$$P(x+b) = \sum_I a_I (x+b)^I = \sum_I \sum_{J \leq I} \binom{I}{J} b^{I-J} a_I x^J = \sum_J \Bigg(\sum_{I \geqslant J} a_I \binom{I}{J} b^{I-J} \Bigg) x^J. \tag{22}$$

Then

$$\nu^\alpha(P(x+b)) = \min\Bigg\{ \langle J, \alpha\rangle \;\Big|\; \sum_{I \geqslant J} \binom{I}{J} a_I b^{I-J} \neq 0 \Bigg\}. \tag{23}$$

If $\sum_{I \geqslant J} \binom{I}{J} a_I b^{I-J} \neq 0$, there is $I \geq J$ such that $a_I \neq 0$. Since $\alpha \leqslant 0$, $\langle I, \alpha\rangle \leqslant \langle J, \alpha\rangle$. This implies that

$$\nu^\alpha(P(x+b)) \geq \nu^\alpha(P(x)). \tag{24}$$

for every $P \in k[x_0, x_1, x_2, x_3] \setminus \{0\}$ and $b \in k^4$. Apply this for $P(x+b) \in k[x_0, x_1, x_2, x_3] \setminus \{0\}$ and $-b \in k^4$, we get $\nu^\alpha(P(x)) = \nu^\alpha(P(x + b - b)) \geq \nu^\alpha(P(x+b))$ which concludes the proof. $\square$

Set $\nu^\alpha := \nu_0^\alpha$.

Proposition 4.2. *For any weight $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in (\mathbb{R}^- \setminus \{0\})^4$ such that $\alpha_0 + \alpha_3 = \alpha_2 + \alpha_1$, the map $\nu : \mathrm{k}[\mathcal{Q}] \to \mathbb{R}^- \cup \{+\infty\}$ given by*

$$\nu(f) := \sup\{\nu^\alpha(R) \mid R \in k[x, y, z, t], \pi(R) = f\},$$

for any $f \in \mathrm{k}[\mathcal{Q}]$ is a valuation on the quadric which is centered at infinity (i.e., whose center is in $\overline{\mathcal{Q}} \setminus \mathcal{Q} \subset \mathbb{P}^4$).

Moreover, suppose $\nu' : \mathrm{k}[\mathcal{Q}] \to \mathbb{R}^- \cup \{+\infty\}$ is a valuation such that $\nu(\pi(x)) = \nu'(\pi(x))$, $\nu(\pi(y)) = \nu'(\pi(y))$, $\nu(\pi(z)) = \nu'(\pi(z))$ and $\nu(\pi(t)) = \nu'(\pi(t))$, then

$$\nu'(f) \geqslant \nu(f),$$

for any regular function $f \in \mathrm{k}[\mathcal{Q}]$.

Definition 4.3. The set $\mathcal{V}_0$ is set of all valuations $\nu : \mathrm{k}[\mathcal{Q}] \to \mathbb{R}^- \cup \{+\infty\}$ defined by

$$\nu(f) := \sup\{\nu^\alpha(R) \mid \pi(R) = f\},$$

for any $f \in \mathrm{k}[\mathcal{Q}]$ and where $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in (\mathbb{R}^- \setminus \{0\})^4$ is a multiindex for which $\alpha_0+\alpha_3 = \alpha_1+\alpha_2$.

The group $\mathbb{R}^{+,*}$ acts naturally by multiplication on the set of valuations on the quadric and this action descends on an action on $\mathcal{V}_0$.

Remark 4.4. Observe that for $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -1$, the corresponding valuation on the quadric is the order of vanishing along the hyperplane at infinity.

Example 4.5. Consider $\alpha = \left(-\frac{1}{2}, -\frac{3}{5}, -\frac{9}{10}, -1\right)$, then the associated valuation ν is the monomial valuation at the point $[0, 0, 0, 1, 0] \in \bar{\mathcal{Q}}$ with weight $\left(\frac{2}{5}, \frac{1}{10}, 1\right)$ with respect to the coordinate chart $(u, v, w) \mapsto [w^2 + uv, u, v, 1, w] \in \bar{\mathcal{Q}}$. In particular, its center is the point $[0, 0, 0, 1, 0] \in \bar{\mathcal{Q}}$.

Example 4.6. Consider $\alpha = \left(-\frac{1}{2}, -\frac{3}{5}, -\frac{9}{10}, -1\right)$, then the associated valuation ν is the monomial valuation at the point $[6, 2, 3, 1, 0] \in \bar{\mathcal{Q}}$ with weight $\left(\frac{2}{5}, \frac{1}{10}, 1\right)$ with respect to the coordinate chart $(u, v, w) \mapsto [w^2 + (2+u)(3+v), 2+u, 3+v, 1, w] \in \bar{\mathcal{Q}}$. In particular, its center is the point $[6, 2, 3, 1, 0] \in \bar{\mathcal{Q}}$.

To prove the proposition, we shall need the following technical lemma.

Lemma 4.7. *Let $\nu' : k[x, y, z, t] \to \mathbb{R}^- \cup \{+\infty\}$ be a monomial valuation with respect to (x, y, z, t) such that $\nu'_{|k[x,y,z,t]\setminus k} < 0$ and such that $\nu'(xt) = \nu'(yz)$. For any polynomial $R \in k[x, y, z, t]$ given by*

$$R = \sum_{ijmn} a_{ijmn} x^i y^j z^m t^n,$$

with $a_{ijmn} \in k$, the following assertions are equivalent:

(i) *There exists a polynomial $R_1 \in k[x, y, z, t]$ such that $\pi(R_1) = \pi(R) \in \mathrm{k}[\mathcal{Q}]$ and such that $\nu'(R_1) > \nu'(R)$.*

(ii) *The polynomial q divides R^w where R^w is the weighted homogeneous polynomial given by*

$$R^w = \sum_{i\nu'(x)+j\nu'(y)+m\nu'(z)+n\nu'(t)=\nu'(R)} a_{ijmn} x^i y^j z^m t^n.$$

Proof. The implication (ii) $\Rightarrow$ (i) is straightforward. If $q \mid R^w$ then we can decompose R as

$$R = qR_1 + S,$$

where $R_1, S \in k[x, y, z, t]$ such that $\nu'(S) > \nu'(qR_1)$. Hence $\pi(R_1 + S) = \pi(R)$ and $\nu'(R_1 + S) \geqslant \min(\nu'(R_1), \nu'(S)) > \nu'(R)$ as required.

Let us prove the implication (i) $\Rightarrow$ (ii). Take a polynomial R_1 which satisfies (i). Then we can write

$$R_1 = R + (q - 1)S,$$

where $S \in k[x, y, z, t]$. Let us prove that $R^w + qS^w = 0$. As $\nu'(R_1) > \nu'(R)$, the above equality implies that $\nu'(qS) = \nu'(R)$. Since $\alpha_0 + \alpha_3 = \alpha_1 + \alpha_2$, the polynomial q is weighted homogeneous and we have

$(R+(q-1)S)^w = R^w + qS^w$. Let us suppose by contradiction that $R^w + qS^w \neq 0$. This implies that $\nu'(R_1^w) = \nu'(R^w + qS^w) = \nu'(R^w)$ which also contradicts our assumption. Hence $R^w + qS^w = 0$ and $q \mid R^w$ as required. □

The above lemma proves that the supremum $\nu(f)$ in Proposition 4.2 is a maximum which is reached on a value $R \in k[x, y, z, t]$ such that $\pi(R) = f$ and such that q does not divide R^w.

Proof of Proposition 4.2. Fix $\alpha \in (\mathbb{R}^- \setminus \{0\})^4$. Observe that for any $f_1 \in \mathrm{k}[\mathcal{Q}] \setminus \{0\}$, the value $\nu(f_1)$ is smaller or equal than 0. If $a \in k^*$, then by definition $\nu(a) = \nu'(a) = 0$.

Fix $f_1, f_2 \in \mathrm{k}[\mathcal{Q}]$ and let us prove that $\nu(f_1 + f_2) \geqslant \min(\nu(f_1), \nu(f_2))$. Take $R_1, R_2 \in k[x, y, z, t]$ such that $\nu'(R_1) = \nu(\pi(R_1))$ and $\nu'(R_2) = \nu(\pi(R_2))$.

As ν^α is a valuation on $k[x, y, z, t]$, we have by definition

$$\nu'(R_1 + R_2) \geqslant \min(\nu'(R_1), \nu'(R_2)) = \min(\nu(\pi(R_1)), \nu(\pi(R_2))).$$

In particular, the maximal value in the right hand side yields

$$\nu(f_1 + f_2) \geqslant \min(\nu(f_1), \nu(f_2)).$$

We prove that $\nu(\pi(f_1 f_2)) = \nu(\pi(f_1)) + \nu(\pi(f_2))$. Take two polynomials R_1 and $R_2 \in k[x, y, z, t]$ such that $\pi(R_1) = f_1$, $\pi(R_2) = f_2$ and $\nu(f_1) = \nu^\alpha(R_1)$, $\nu(f_2) = \nu^\alpha(R_2)$. Observe that $(R_1 R_2)^w = R_1^w R_2^w$. As the polynomial q does not divide either R_1^w or R_2^w, it does not divide $(R_1 R_2)^w$ since the ideal generated by q is a prime ideal. Hence by Lemma 4.7, one has $\nu(f_1 f_2) = \nu^\alpha(R_1 R_2) = \nu^\alpha(R_1^w) + \nu^\alpha(R_2^w) = \nu(f_1) + \nu(f_2)$ as required.

By construction, the valuation ν is centered at infinity since ν takes negative values on nonzero regular functions on the quadric.

Let us prove that the valuation ν is minimal, take another valuation $\nu' : \mathrm{k}[\mathcal{Q}] \to \mathbb{R}^- \cup \{+\infty\}$ such that $\nu'(\pi(x)) = \nu(x)$, $\nu'(\pi(y)) = \nu(\pi(y))$, $\nu'(\pi(z)) = \nu(\pi(z))$ and $\nu'(\pi(t)) = \nu(\pi(t))$. Then the map $\hat{\nu}' : R \in k[x, y, z, t] \to \nu'(\pi(R))$ defines a semivaluation on $k[x, y, z, t]$. Remark that the monomial valuation ν^α is minimal in $k[x, y, z, t]$, in the sense that for any $R \in k[x, y, z, t]$

$$\hat{\nu}'(R) \geqslant \nu^\alpha(R).$$

Take $f \in \mathrm{k}[\mathcal{Q}]$ and choose a polynomial $R \in k[x, y, z, t]$ such that $\nu_p^\alpha(R) = \nu(f)$, the above inequality implies

$$\nu'(f) \geqslant \nu(f),$$

hence, ν is also minimal. □

4C. *Parachute.* In this subsection, we define the parachute associated to a component of a tame automorphism. For any 4-tuple $(R_1, R_2, R_3, R_4) \in k[x, y, z, t]$ of polynomials, we write

$$dR_1 \wedge dR_2 \wedge dR_3 \wedge dR_4 = \mathrm{Jac}(R_1, R_2, R_3, R_4) dx \wedge dy \wedge dz \wedge dt,$$

with $\mathrm{Jac}(R_1, R_2, R_3, R_4) \in k[x, y, z, t]$.

Definition 4.8. The pseudojacobian of a triple (f_1, f_2, f_3) of regular functions on $\mathcal{Q}$ is defined by

$$j(f_1, f_2, f_3) := \mathrm{Jac}(q, R_1, R_2, R_3))_{|\mathcal{Q}},$$

where $R_i \in k[x, y, z, t]$ are polynomials such that $\pi(R_i) = f_i$ for $i = 1, 2, 3$.

Observe that the pseudojacobian $j(f_1, f_2, f_3)$ is well-defined since any two representatives $R_1, R_2 \in k[x, y, z, t]$ of the same equivalence class in $\mathrm{k}[\mathcal{Q}]$ are equal modulo $(q-1)$.

Lemma 4.9. *Let $\nu \in \mathcal{V}_0$ be a valuation. For any $f_1, f_2, f_3 \in \mathrm{k}[\mathcal{Q}]$, we have*

$$\nu(j(f_1, f_2, f_3)) \geqslant \nu(f_1) + \nu(f_2) + \nu(f_3) - \nu(xt).$$

Proof. Fix $f_1, f_2, f_3 \in \mathrm{k}[\mathcal{Q}]$ and a valuation $\nu \in \mathcal{V}_0$. The valuation $\nu' : k[x, y, z, t] \to \mathbb{R}^- \cup \{+\infty\}$ is monomial for the coordinates (x, y, z, t) with weight $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in (\mathbb{R}^-)^4$ such that $\alpha_0 + \alpha_3 = \alpha_1 + \alpha_2$. We have $\nu(P) = \sup\{\nu'(R) \mid \pi(R) = P\}$ for any $P \in \mathrm{k}[\mathcal{Q}]$ where $\pi : k[x, y, z, t] \to \mathrm{k}[\mathcal{Q}]$ is the canonical projection. Take $R_1, R_2, R_3, R_4 \in k[x, y, z, t]$. We first claim that

$$\nu'(\mathrm{Jac}(R_1, R_2, R_3, R_4)) \geqslant \nu'(R_1) + \nu'(R_2) + \nu'(R_3) + \nu'(R_4) - \nu'(xyzt).$$

Let $a_I^{(k)} \in k$ be the coefficients of R_k for $k = 1, 2, 3, 4$ so that

$$R_k = \sum_{I=(i_1,i_2,i_3,i_4)} a_I^{(k)} x^{i_1} y^{i_2} z^{i_3} t^{i_4}.$$

One obtains by linearity that $\mathrm{Jac}(R_1, R_2, R_3, R_4)$ is a sum of monomials where the valuation of each term is greater or equal to

$$\nu'(R_1) + \nu'(R_2) + \nu'(R_3) + \nu'(R_4) - \nu'(xyzt).$$

Hence

$$\nu'(\mathrm{Jac}(R_1, R_2, R_3, R_4)) \geqslant \nu'(R_1) + \nu'(R_2) + \nu'(R_3) + \nu'(R_4) - \nu'(xyzt).$$

In particular, we apply to $R_4 = q$ and obtain

$$\nu'(\mathrm{Jac}(R_1, R_2, R_3, q)) \geqslant \nu'(R_1) + \nu'(R_2) + \nu'(R_3) - \nu'(xt),$$

since $\nu'(q) = \nu'(xt) = \nu'(yz)$. Take $f_1, f_2, f_3 \in \mathrm{k}[\mathcal{Q}]$, by Lemma 4.7, there exists $R_1, R_2, R_3 \in k[x, y, z, t]$ such that $\pi(R_i) = f_i \in \mathrm{k}[\mathcal{Q}]$ and $\nu(f_i) = \nu'(R_i)$ for all $i = 1, 2, 3$, the above inequality implies

$$\nu(j(f_1, f_2, f_3)) \geqslant \nu'(\mathrm{Jac}(q, R_1, R_2, R_3)) \geqslant \nu'(R_1) + \nu'(R_2) + \nu'(R_3) - \nu'(xt),$$

where the first inequality follows from the definition of ν. Observe that $\nu'(xt) = \nu(xt)$ by Lemma 4.7, hence we have proven that

$$\nu(j(f_1, f_2, f_3)) \geqslant \nu(f_1) + \nu(f_2) + \nu(f_3) - \nu(xt),$$

as required. $\square$

The regular function $j(f_1, f_2, f_3)$ may vanish so that $\nu(j(f_1, f_2, f_3))$ may be equal to $+\infty$, even if $\nu \in \mathcal{V}_0$.

Lemma 4.10. *For any algebraically independent functions $f_1, f_2 \in \mathrm{k}[\mathcal{Q}]$, one of the four regular functions $j(x, f_1, f_2), j(y, f_1, f_2), j(z, f_1, f_2), j(t, f_1, f_2)$ is not identically zero. In particular,*

$$\min(\nu(j(x, f_1, f_2)), \nu(j(y, f_1, f_2)), \nu(j(z, f_1, f_2)), \nu(j(t, f_1, f_2))) < +\infty,$$

for any valuation $\nu \in \mathcal{V}_0$.

Proof. Consider two algebraically independent regular functions $f_1, f_2 \in \mathrm{k}[\mathcal{Q}]$ and suppose by contradiction that $j(x, f_1, f_2) = j(y, f_1, f_2) = j(z, f_1, f_2) = j(t, f_1, f_2) = 0$. If $K \subset L$ are two fields of characteristic zero, then [Lang 2002, Section VIII.5, Proposition 5.5] states that

$$\operatorname{trdeg}_K L = \dim_L \operatorname{Der}_K(L), \tag{25}$$

where $\operatorname{Der}_K(L)$ denotes the vector space of K derivations of L. When $K = k(f_1, f_2)$ and $L = k(\mathcal{Q})$, the above equality implies that any two $k(f_1, f_2)$-derivations are proportional. The conditions $j(x, f_1, f_2) = j(y, f_1, f_2) = j(z, f_1, f_2) = j(t, f_1, f_2) = 0$ imply that $j(x, f_1, \cdot\,), j(y, f_1, \cdot\,), j(z, f_1, \cdot\,)$ and $j(t, f_1, \cdot\,)$ are $k(f_1, f_2)$-derivations, this translates as

$$j(x, f_1, x)j(y, f_1, y) - j(x, f_1, y)j(y, f_1, x) = 0 \in \mathrm{k}[\mathcal{Q}]\,.$$

Hence,

$$j(x, f_1, y) = 0 \in k(\mathcal{Q}).$$

The same argument also yields

$$j(f_1, x, y) = j(f_1, x, z) = j(f_1, x, t) = j(f_1, y, z) = j(f_1, y, t) = j(f_1, z, t) = 0.$$

Hence the maps $j(x, y, \cdot\,), j(x, z, \cdot\,), j(y, z, \cdot\,)$ are also $k(f_1)$-derivations. By (25) applied to $K = k(f_1)$ and to $L = k(\mathcal{Q})$, the space of $k(f_1)$ derivations is 2-dimensional and there exists $a, b, c \in k(\mathcal{Q})$ such that

$$aj(x, y, \cdot\,) + bj(x, z, \cdot\,) + cj(y, z, \cdot\,) = 0,$$

where a, b and c are not all equal to zero. Suppose that $a \neq 0$, we then conclude that

$$aj(x, y, z) = 0 \in k(\mathcal{Q}),$$

which in turn implies that $j(x, y, z) = x = 0 \in \mathrm{k}[\mathcal{Q}]$ and this is impossible. $\square$

Definition 4.11. For any monomial valuation $\nu \in \mathcal{V}_0$ and for any algebraically independent regular functions $f_1, f_2 \in \mathrm{k}[\mathcal{Q}]$, the parachute $\nabla(f_1, f_2)$ with respect to the valuation ν is defined by the following formula

$$\nabla(f_1, f_2) = \min(\nu(j(x, f_1, f_2)), \nu(j(y, f_1, f_2)), \nu(j(z, f_1, f_2)), \nu(j(t, f_1, f_2))) - \nu(f_1) - \nu(f_2).$$

Observe that Lemmas 4.10 and 4.9 imply that $\nabla(f_1, f_2)$ is finite and is strictly greater than zero.

For any polynomial $R \in k[x, y]$, we write by $\partial_2 R \in k[x, y]$ the partial derivative with respect to y. The next identity is similar to [Lamy and Vénéreau 2013, Lemma 5] and is one of the main ingredient to find an upper bound on the value of a valuation.

Lemma 4.12. *Let $\nu \in \mathcal{V}_0$, let $R \in k[x, y] \setminus k$ and let $f_1, f_2 \in \mathrm{k}[Q]$ be two algebraically independent elements. Suppose that there exists an integer $1 \leqslant n \leqslant \deg_y R - 1$ such that $\nu(\partial_2^n R(f_1, f_2))$ is equal to the value on $\partial_2^n R$ of the monomial valuation in two variables having weight $\nu(f_1)$ and $\nu(f_2)$ on x and y respectively. Then*

$$\nu(R(f_1, f_2)) < \deg_y(R)\nu(f_2) + n\nabla(f_1, f_2).$$

Proof. Lemma 4.9 proves that $\nu(j(x, f_1, f_2)) \geqslant \nu(f_1) + \nu(f_2) - \nu(t)$ for any $f_1, f_2 \in \mathrm{k}[Q]$. Using this and the fact that $j(x, f_1, \cdot)$ is a derivation, we obtain

$$\nu(\partial_2 R(f_1, f_2) j(x, f_1, f_2)) = \nu(j(x, f_1, R(f_1, f_2))) \geqslant \nu(f_1) + \nu(R(f_1, f_2)) + \nu(x) - \nu(xt).$$

In particular since $\nu(x) - \nu(xt) = -\nu(t) > 0$, this yields

$$\nu(\partial_2 R(f_1, f_2)) > -(\nu(j(x, f_1, f_2)) - \nu(f_1) - \nu(f_2)) + \nu(R(f_1, f_2)) - \nu(f_2).$$

A similar argument with y, z and t also gives

$$\nu(\partial_2 R(f_1, f_2)) > -\nabla(f_1, f_2) + \nu(R(f_1, f_2)) - \nu(f_2). \tag{26}$$

We apply (26) inductively and obtain the following inequalities:

$$\begin{gathered}\nu(\partial_2^2 R(f_1, f_2)) > -\nabla(f_1, f_2) + \nu(\partial_2 R(f_1, f_2)) - \nu(f_2),\\ \vdots \\ \nu(\partial_2^n R(f_1, f_2)) > -\nabla(f_1, f_2) + \nu(\partial_2^{n-1} R(f_1, f_2)) - \nu(f_2).\end{gathered}$$

This implies that

$$\nu(\partial_2^n R(f_1, f_2)) > -n\nabla(f_1, f_2) - n\nu(f_2) + \nu(R(f_1, f_2)).$$

As $\nu(\partial_2^n R(f_1, f_2))$ is equal to the value of the monomial valuation with weight $(\nu(f_1), \nu(f_2))$ applied to $\partial_2^n(R)$, the last inequality rewrites as

$$(\deg_y R - n)\nu(f_2) \geq \nu(\partial_2^n R(f_1, f_2)) > -n\nabla(f_1, f_2) - n\nu(f_2) + \nu(R(f_1, f_2)).$$

Hence,

$$\nu(R(f_1, f_2)) < \deg_y(R)\nu(f_2) + n\nabla(f_1, f_2),$$

as required. ◻

4D. ***Key polynomials.*** Let us explain how one can find a polynomial which satisfies the hypothesis of Lemma 4.12.

Consider $\mu : k[x, y] \to \mathbb{R}^- \cup \{+\infty\}$ any valuation and $\mu_0 : k[x, y] \to \mathbb{R}^- \cup \{+\infty\}$ the monomial valuation having weight $\mu(x)$ and $\mu(y)$ on x and y respectively. For any polynomial $R \in k[x, y]$, we write by $\bar{R} \in k[x, y]$ the weighted homogeneous polynomial given by

$$\bar{R} = \sum_{i\mu(x)+j\mu(y)=\mu_0(R)} a_{ij} x^i y^j,$$

with $a_{ij} \in k$ such that $R = \sum_{ij} a_{ij} x^i y^j$.

Proposition 4.13. *Consider $\mu : k[x, y] \to \mathbb{R}^- \cup \{+\infty\}$ any valuation and μ_0 the monomial valuation having weights $\mu(x)$ and $\mu(y)$ on x and y respectively. The following properties are satisfied*:

(i) *For any $R \in k[x, y]$, one has $\mu(R) \geqslant \mu_0(R)$.*

(ii) *If $\mu \neq \mu_0$, then there exists two coprime integers s_1, s_2 satisfying $s_1\mu(x) = s_2\mu(y)$ and a unique constant $\lambda \in k$ for which the polynomial $H = x^{s_1} - \lambda y^{s_2}$ satisfies $\mu(H) > \mu_0(H)$. Moreover, for any $R \in k[x, y]$, one has $\mu(R) > \mu_0(R)$ if and only if $H \mid \bar{R}$.*

The polynomial H associated to μ is called a key polynomial associated to μ.

Proof. Let us prove assertion (i). Write $R \in k[x, y]$ as $R = \sum a_{ij} x^i y^j$ where $a_{ij} \in k$. Recall that the fact that μ_0 is monomial implies that

$$\mu_0(R) = \min\{i\mu_0(x) + j\mu_0(y) \mid a_{ij} \neq 0\}.$$

Also, μ is a valuation, hence

$$\mu(R) \geqslant \min\{i\mu_0(x) + j\mu_0(y) \mid a_{ij} \neq 0\} = \mu_0(R).$$

We have thus proved that $\mu(R) \geqslant \mu_0(R)$, as required.

Step 1: Fix s_1, s_2 two coprime integers and $\lambda \in k$. Suppose that $s_1\mu(x) = s_2\mu(y)$ and that the polynomial $H = x^{s_1} - \lambda y^{s_2}$ satisfies $\mu(H) > \mu_0(H)$, we prove that λ is unique. Take $\lambda' \neq \lambda \in k$, then

$$\mu(x^{s_1} - \lambda' y^{s_2}) = \mu(H + (\lambda - \lambda') y^{s_2}) = s_2\mu(y),$$

since $\mu(H) > \mu((\lambda - \lambda') y^{s_2})$. Hence $\mu(x^{s_1} - \lambda' y^{s_2}) = \mu_0(x^{s_1} - \lambda' y^{s_2})$ for any $\lambda' \neq \lambda$.

Step 2: Choose two integers s_1, s_2 such that $s_1\mu(x) = s_2\mu(y)$. We prove that there exists $\lambda \in k^*$ such that $\mu(x^{s_1} - \lambda y^{s_2}) > s_1\mu(x) = s_2\mu(y)$. Suppose by contradiction that for any $\lambda \in k$, one has $\mu(x^{s_1} - \lambda y^{s_2}) = s_1\mu(x)$. We claim that $\mu(R) = \mu_0(R)$ for any polynomial $R \in k[x, y]$. Fix $R \in k[x, y]$. Observe that if R is a homogeneous polynomial with respect to the weight $(\mu(x), \mu(y))$, then R is of the form

$$R = \alpha x^{k_0} \prod_i (x^{s_1} - \lambda_i y^{s_2})$$

where $\alpha, \lambda_i \in k^*$ and $k_0 \in \mathbb{N}$. Our assumption implies that $\mu(R) = \mu_0(R)$ for any homogeneous polynomial R.

If R is a general polynomial, then R can be decomposed into $R = \sum_i R_i$ where each polynomial R_i is homogeneous with respect to the weight $(\mu(x), \mu(y))$. Since $\mu(R_i) = \mu_0(R_i)$ for each i, this proves that $\mu(R) = \mu_0(R)$ for any $R \in k[x, y]$, which contradicts our assumption. We have thus proven the first part of assertion (ii).

Step 3: We prove the second part of assertion (ii). The argument in Step 2 showed that when $\mu \neq \mu_0$ and R is homogeneous with respect to the weight $(\mu(x), \mu(y))$, then $\mu(R) > \mu_0(R)$ if $H \mid R$. For a general polynomial $R \in k[x, y]$, write $R = \bar{R} + S$. We have

$$\mu_0(R) = \mu_0(\bar{R}) < \mu_0(S) \leqslant \mu(S).$$

If $H \mid \bar{R}$, then $\mu(\bar{R}) > \mu_0(\bar{R}) = \mu_0(R)$. Then $\mu(R) = \mu(\bar{R} + S) \geqslant \min(\mu(\bar{R}), \mu(S)) > \mu_0(R)$. If $H \nmid \bar{R}$, we have $\mu(\bar{R}) = \mu_0(\bar{R}) < \mu_0(S) \leqslant \mu(S)$. So $\mu(R) = \mu(\bar{R} + S) = \mu(\bar{R}) = \mu_0(\bar{R}) = \mu_0(R)$. □

4E. *Parachute inequalities.* We introduce various notions of resonances of components of a tame automorphism. These notions will play an important role in the theorem below. Consider a valuation $\nu \in \mathcal{V}_0$ and a component (f_1, f_2) of a tame automorphism. We are interested in the value of ν on $R(f_1, f_2)$ where $R \in k[x, y]$. The estimates of the value $\nu(R(f_1, f_2))$ will depend on the possible values of the pair $(\nu(f_1), \nu(f_2))$. We shall distinguish the following three cases:

(1) The family $(\nu(f_1), \nu(f_2))$ is $\mathbb{Q}$-independent and we say that the component (f_1, f_2) is *nonresonant* with respect to ν.

(2) There exists two coprime integers s_1, s_2 such that $s_1 > s_2 \geqslant 2$ or $s_2 > s_1 \geqslant 2$ such that $s_1\nu(f_1) = s_2\nu(f_2)$ and we say in this case that the component (f_1, f_2) is *properly resonant* with respect to ν.

(3) Either $\nu(f_1)$ is a multiple of $\nu(f_2)$ or $\nu(f_2)$ is a multiple of $\nu(f_1)$ and there exists a polynomial $H \in k[x, y]$ of the form $x - \lambda y^k$ where $k \in \mathbb{N}^*$, $\lambda \in k^*$ such that $\nu(H(f_1, f_2)) > \nu(f_1) = k\nu(f_2)$. In this case, the component (f_1, f_2) is called *critically resonant* with respect to ν.

Example 4.14. When $\nu = -\deg : k[Q] \to \mathbb{R}^- \cup \{+\infty\}$, the family (x, y) is not critically resonant, but it is neither properly resonant nor nonresonant (in particular there is no alternative). However, (x, y) is nonresonant for the monomial valuation with weight $(-\sqrt{2}, -\sqrt{3}, -\sqrt{2}, -\sqrt{3})$ on (x, y, z, t).

Example 4.15. Take $f_1 = x$, $f_2 = y + x^2 \in k[Q]$, then (f_1, f_2) is critically resonant with respect to the valuation $\mathrm{ord}_{H_\infty} = -\deg$.

Example 4.16. Take $f_1 = z + x^2$, $f_2 = y + x^3 \in k[Q]$, then (f_1, f_2) is properly resonant with respect the valuation $\mathrm{ord}_{H_\infty} = -\deg$.

For $\nu \in \mathcal{V}_0$ and (f_1, f_2) a component of a tame automorphism, the following theorem allows us to estimate the value of ν on $R(f_1, f_2)$ only when (f_1, f_2) is not critically resonant.

Theorem 4.17. *Let $\nu \in \mathcal{V}_0$ be a valuation and let ν_0 be the monomial valuation on $k[x, y]$ with weight $(\nu(f_1), \nu(f_2))$ with respect to (x, y). The following assertions hold*:

(i) *For any polynomial $R \in k[x, y]$, one has the lower bound $\nu(R(f_1, f_2)) \geqslant \nu_0(R(x, y))$.*

(ii) *If the component (f_1, f_2) is nonresonant with respect to ν, then for any polynomial $R \in k[x, y]$, one has $\nu(R(f_1, f_2)) = \nu_0(R(x, y))$.*

(iii) *Suppose that the component (f_1, f_2) is properly resonant with respect to ν and let s_1, s_2 be two coprime positive integers such that $s_1\nu(f_1) = s_2\nu(f_2)$, then for any polynomial $R \in k[x, y]$, either $\nu(R(f_1, f_2)) = \nu_0(R(x, y))$ or $\nu(R(f_1, f_2)) > \nu_0(R(x, y))$ and we have*

$$\nu(R(f_1, f_2)) < \left(s_1 - 1 - \frac{s_1}{s_2}\right)\nu(f_1) = \left(s_2 - 1 - \frac{s_2}{s_1}\right)\nu(f_2).$$

Remark that the inequalities in Theorem 4.17 are strict and this fact is crucial in our proof. Before giving the proof of Theorem 4.17, we state two consequences of this theorem below.

Corollary 4.18. *Let $\nu \in \mathcal{V}_0$ be a monomial valuation and let $f = (f_1, f_2, f_3, f_4)$ be an element of* Tame(Q). *We suppose that $\nu(f_1) < \nu(f_2)$ and that (f_1, f_2) is not critically resonant with respect to ν. Then for any polynomial $R \in k[x, y] \setminus k[y]$, we have*

$$\nu(f_2 R(f_1, f_2)) < \nu(f_1).$$

Proof. Two cases appear. Either $\nu(R(f_1, f_2)) = \nu_0(R(x, y))$ where ν_0 is the monomial valuation with weight $(\nu(f_1), \nu(f_2))$ with respect to (x, y), and we are finished since $R \in k[x, y] \setminus k[y]$. Or $\nu(R(f_1, f_2)) > \nu_0(R(x, y))$ and there exists some integers s_1, s_2 such that $s_1\nu(f_1) = s_2\nu(f_2)$ where $s_2 > s_1 \geqslant 2$. Using Theorem 4.17(iii) and the fact that $s_1 \geqslant 2$, we have thus

$$\nu(f_2 R(f_1, f_2)) < (s_1 - 1)\nu(f_1) < \nu(f_1),$$

as required. □

We state the second corollary for which the constant $\frac{4}{3}$ appears naturally.

Corollary 4.19. *Let $\nu \in \mathcal{V}_0$ be a valuation and let (f_1, f_2) a properly resonant component with respect to ν such that $\nu(f_1) < \nu(f_2)$. Then for any polynomial $R \in k[x, y] \setminus k[y]$, one has*

$$\nu(f_1 R(f_1, f_2)) < \tfrac{4}{3}\nu(f_1).$$

Similarly,

$$\nu(f_2 R(f_1, f_2)) < \tfrac{3}{2}\nu(f_2) < \tfrac{4}{3}\nu(f_2).$$

Proof. Denote by $\nu_0 : k[x, y] \to \mathbb{R}^- \cup \{+\infty\}$ the monomial valuation with weight $(\nu(f_1), \nu(f_2))$ with respect to (x, y). Two cases appear, either $\nu(R(f_1, f_2)) = \nu_0(R(x, y))$ and we are done since $\nu(f_1 R(f_1, f_2)) \leqslant 2\nu(f_1)$ as $R \in k[x, y] \setminus k[y]$ or $\nu(R(f_1, f_2)) > \nu_0(R(x, y))$. In the latter case, consider two coprime integers s_1, s_2 such that $s_1\nu(f_1) = s_2\nu(f_2)$. Since $\nu(f_1) < \nu(f_2)$ and the component (f_1, f_2) is properly resonant, the inequality $s_2 > s_1 \geqslant 2$ holds. Using Theorem 4.17(iii), we obtain

$$\nu(f_1 R(f_1, f_2)) < \left(s_1 - \frac{s_1}{s_2}\right)\nu(f_1).$$

Since $s_2 \geqslant 3$ and $s_1 \geq 2$, we have $s_1 - s_1/s_2 = s_1(1 - 1/s_2) \geq 2\left(1 - \frac{1}{3}\right) = \frac{4}{3}$. Since $\nu(f_1) < 0$, we get

$$\nu(f_1 R(f_1, f_2)) < \left(s_1 - \frac{s_1}{s_2}\right)\nu(f_1) \leqslant \frac{4}{3}\nu(f_1),$$

as required. The second inequality follows from a similar argument $\nu(f_2 R(f_1, f_2)) < \frac{3}{2}\nu(f_2) < \frac{4}{3}\nu(f_2)$. □

Proof of Theorem 4.17. Let us denote by $R = \sum a_{ij} x^i y^j$. Consider the projection $\pi_{xy} : \mathcal{Q} \to \mathbb{A}^2$ induced by the embedding of $\mathcal{Q}$ into $\mathbb{A}^4$ composed with the projection onto $\mathbb{A}^2$ of the form

$$\pi_{xy} : (x, y, z, t) \in \mathcal{Q}(k) \mapsto (x, y).$$

Choose an automorphism f such that $f = (f_1, f_2, f_3, f_4)$ where $f_3, f_4 \in \mathrm{k}[\mathcal{Q}]$. We denote by μ the valuation on $k[x, y]$ given by $\mu = \pi_{xy*} f_* \nu$.

Observe that for any polynomial $R \in k[x, y]$, we have $\nu(R(f_1, f_2)) = \mu(R(x, y))$ and assertion (i) follows directly from Proposition 4.13(i). Observe also that assertion (ii) follows immediately from the fact that $\nu(f_1)$ and $\nu(f_2)$ are $\mathbb{Q}$-independent.

Let us prove assertion (iii). We can suppose by symmetry that $\nu(f_1) < \nu(f_2)$. Since the component (f_1, f_2) is properly resonant, there exists two coprime integers s_1, s_2 such that $s_1\nu(f_1) = s_2\nu(f_2)$ and such that $s_2 > s_1 \geqslant 2$.

By Proposition 4.13 applied to μ, there exists $\lambda \in k^*$ such that the polynomial $H = x^{s_1} - \lambda y^{s_2}$ satisfies

$$\mu(H(x, y)) = \nu(H(f_1, f_2)) > \nu_0(H) = s_1\nu(f_1).$$

For any polynomial $R \in k[x, y]$, denote by $\bar{R}$ be the polynomial given by

$$\bar{R} = \sum_{i\mu(x)+j\mu(y)=\nu_0(R(x,y))} a_{ij} x^i y^j.$$

By construction, we have that there exists an integer $n \geqslant 1$ such that $\bar{R} \in (H^n) \setminus (H^{n+1})$.

We shall use the following lemma (proved at the end of this section):

Lemma 4.20. *Let $R \in k[x, y]$ such that $H \mid \bar{R}$. Consider the integer $n = \max\{k \mid H^k$* ***divides*** *$\bar{R}\} \geqslant 1$. Then the following properties are satisfied*:

(i) *For any integer $k \leqslant n$, we have $\overline{\partial_2^k(R)} = \partial_2^k \bar{R}$.*

(ii) *For any integer $k \leqslant n$, we have $H^{n-k} \mid \partial_2^k \bar{R}$ but $H^{n-k+1} \nmid \partial_2^k \bar{R}$.*

The above lemma implies that $\overline{\partial_2^k R} = \partial_2^k \bar{R}$ and that $H^{n-k} \mid \partial_2^k \bar{R}$ but $H^{n-k+1} \nmid \partial_2^k \bar{R}$ for any $k \leq n$. In particular, H does not divide $\partial_2^n \bar{R}$ and Proposition 4.13(ii) implies that

$$\mu(\partial_2^n R(x, y)) = \nu_0(\partial_2^n R) = \nu_0(\overline{\partial_2^n R}).$$

The previous equation translates as

$$\nu((\partial_2^n R)(f_1, f_2)) = \nu_0(\partial_2^n R)$$

and R satisfies the conditions of Lemma 4.12 (for the same integer n), which in turn asserts that

$$\nu(R(f_1, f_2)) < \deg_y(R)\nu(f_2) + n\nabla(f_1, f_2).$$

Since $H^n \mid \bar{R}$, one has $\deg_y(R) \geqslant \deg_y(\bar{R}) \geqslant s_2 n$, we get

$$\nu(R(f_1, f_2)) < n(s_2\nu(f_2) + \nabla(f_1, f_2)).$$

As $n \geqslant 1$ and $\nabla(f_1, f_2) \leqslant -\nu(f_1) - \nu(f_2)$ by Lemma 4.10, the above implies that

$$\nu(R(f_1, f_2)) < s_2\nu(f_2) - \nu(f_1) - \nu(f_2).$$

Since $s_1\nu(f_1) = s_2\nu(f_2)$, we finally prove that

$$\nu(R(f_1, f_2)) < \nu(f_1)\left(s_1 - 1 - \frac{s_1}{s_2}\right),$$

as required. □

Proof of Lemma 4.20. Consider a monomial valuation $\nu_0 : k[x, y] \to \mathbb{R}^- \cup \{+\infty\}$ with weight $(\alpha, \beta) \in (\mathbb{R}^{-,*})^2$ with respect to (x, y) and $H = x^{s_1} - \lambda y^{s_2}$ where s_1, s_2 are coprime integers such that $s_1\alpha = s_2\beta$.

Let us prove assertion (i) for $k = 1$. Fix $R \in k[x, y]$ and write R as

$$R = \sum_{ij} a_{ij} x^i y^j,$$

where $a_{ij} \in k$. The partial derivative is given explicitly by

$$\partial_2 R = \sum_{i \geq 0, j \geq 1} j a_{ij} x^i y^{j-1}.$$

Since $H \mid \bar{R}$, one has $\bar{R} \in k[x, y] \backslash k[x]$ and $\nu_0(R) = \nu_0(\bar{R})$. Take (i, j) such that $a_{ij} \neq 0$ and $i\alpha + (j-1)\beta = \nu_0(\partial_2 R)$. Then $i\alpha + j\beta = \nu_0(\partial_2 R) + \nu_0(y) \leq \nu_0(R)$. Conversely, since $H \mid \bar{R}$, there exists (i, j) such that $i\alpha + j\beta = \nu_0(R)$ where $j \geq 1$, hence we have that $i\alpha + (j-1)\beta \geq \nu_0(\partial_2 R)$. Hence, $\nu_0(\partial_2 R) = \nu_0(R) - \beta$ and $\overline{\partial_2 R} = \partial_2 \bar{R}$.

Let us prove assertion (ii) for $k = 1$. We have that $H^n \mid \bar{R}$ but $H^{n+1} \nmid \bar{R}$, then we have

$$\bar{R} = H^n S,$$

where $S \in k[x, y]$ is a homogeneous polynomial such that $H \nmid S$. By definition,

$$\partial_2 \bar{R} = ns_2 H^{n-1} y^{s_2-1} S + H^n \partial_2 S.$$

Hence $H^{n-1} \mid \partial_2 \bar{R}$. Suppose by contradiction that $H^n \mid \partial_2 \bar{R}$, then this implies that $H \mid y^{s_2-1} S$ which is impossible since H does not divide S. We have thus proven that $H^{n-1} \mid \partial_2 \bar{R}$ but $H^n \nmid \partial_2 \bar{R}$, as required.

An immediate induction on $k \leqslant n$ proves assertion (i) and (ii). □

5. Proof of Theorem 1 and Theorem 4

This section is devoted to the proof of Theorems 1 and 4. The proof of these two results are very similar and rely on a lower bound of the degree of an automorphism f by $\left(\frac{4}{3}\right)^p$ where p is an integer that we determine.

Let us explain our general strategy. Take an automorphism $f \notin O_4$.

Step 1: We choose an appropriate valuation ν.

We consider a geodesic line γ in the complex joining [Id] and $[f]$. Recall from Proposition 2.3 that the set of 1×1 squares containing [Id] is in bijection with the points on the hyperplane at infinity $H_\infty \subset \mathcal{Q} \subset \mathbb{P}^4$. Depending on which 1×1 square the geodesic γ near the vertex [Id] is contained, we choose accordingly a valuation ν in $\mathcal{V}_0$ centered on the corresponding point at infinity in $\bar{\mathcal{Q}}$.

Step 2 (see Section 5A): We define an integer p according to the geometry of some geodesics in the complex and according to the choice of the valuation ν.

Recall that a path in the 1-skeleton of $\mathcal{C}$ induces a sequence of numbers obtained by evaluating the valuation ν on the consecutive vertices. The integer p is defined as the distance in a graph denoted $\mathcal{C}_\nu$ and encodes the shortest path in the 1-skeleton with minimal degree sequence.

Step 3: We prove that $\deg(f) \geqslant \left(\frac{4}{3}\right)^p$.

Consider the graph $\mathcal{C}_\nu$ associated to ν and denote by d_ν the distance in this graph. This step is the content of the following theorem. Recall that the standard 2×2 square S_0 is the square whose vertices are $[x]$, $[y]$, $[z]$ and $[t]$.

Theorem 5.1. *Pick any valuation $\nu \in \mathcal{V}_0$ satisfying*

$$\max(2\nu(t), \nu(y) + \nu(t), \nu(z) + \nu(t)) < \nu(x) < \min(\nu(y), \nu(z), \nu(t)). \tag{27}$$

Consider any geodesic segment of $\mathcal{C}$ joining [Id] *to a vertex v of type* I *which intersects an edge of the square S_0, then the following assertions hold*:

(1) *We have*

$$\nu(v) \leqslant \left(\tfrac{4}{3}\right)^{d_\nu([t], v)-1} \max(\nu(x), \nu(y), \nu(z), \nu(t)).$$

(2) *For any valuation $\nu' \in \mathcal{V}_0$ satisfying* (27), *we have*

$$d_\nu([t], v) = d_{\nu'}([t], v).$$

The proof of Theorem 5.1 basically proceeds by induction on the distance between $[t]$ and v in the graph $\mathcal{C}_\nu$. The essential ingredient to bound below the degree inductively are the parachute inequalities stated in Theorem 4.17. We explain in Section 5B how to arrive to the situation where these inequalities can be applied using the local geometry near the vertices of type I (i.e., the geometry of its link). We then use these arguments to compute the degree or estimate the valuation ν when one passes from one square to another in each possible situation, this is done successively in Sections 5C, 5D, 5E and 5F.

Once we conjugate appropriately to arrive to the situation of Theorem 5.1, we then deduce directly both Theorem 1 and Theorem 4.

5A. ***The graph $\mathcal{C}_\nu$ associated to a valuation and the orientation of certain edges of the complex.*** Fix a valuation $\nu \in \mathcal{V}_0$. Given any automorphism $f = (f_1, f_2, f_3, f_4) \in \mathrm{Tame(Q)}$, we remark that $\nu(f_1)$ does not depend on the choice of representative of the class $[f_1]$ so that ν induces a function on the vertices of type I of $\mathcal{C}$.

We say that a vertex $v \in \mathcal{C}$ of type I is *ν-minimal* (resp. *ν-maximal*) in a 2×2 square S if $\nu(v)$ is strictly smaller (resp. greater) than the value of the valuation ν on every other vertices of type I of S. Observe that for some valuations, two vertices of type I can have the same value on ν, hence there can be no ν-minimal or ν-maximal vertices.

We now define a graph $\mathcal{C}_\nu$ associated to a valuation $\nu \in \mathcal{V}_0$ as follows:

(1) The vertices are the vertices of $\mathcal{C}$ type I.

(2) One draws an edge between two vertices v_1 and v_2 of $\mathcal{C}'$ if there exists a 2×2 square S centered at a vertex of type III in $\mathcal{C}$ containing v_1, v_2 such that the vertices v_1, v_2 belong to an edge of S or v_1 and v_2 are the ν-minimal and ν-maximal vertices of S respectively.

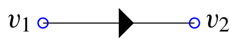
Observe that whenever there is no ν-maximal or minimal vertex in a 2×2 square S centered at a point of type III, then we only draw the four edges of the square S.

The graph $\mathcal{C}_\nu$ is endowed with the distance d_ν such that its the edges have length 1.

Lemma 5.2. *The graph $\mathcal{C}_\nu$ is a connected metric graph.*

Proof. This follows from the fact that the 1-skeleton of $\mathcal{C}$ is connected and the fact that to any path between type I vertices in the 1-skeleton of $\mathcal{C}$, we can find an alternate path in the 1-skeleton of $\mathcal{C}$ with the same endpoints and which takes only edges joining type I and II vertices. If one has a local path in the skeleton passing successively to a type II, then to a type III and then to a type II vertex in the same 1×1 square, then we replace by a path that goes through the vertex of type I within the same square. For path that go through different squares, we use Lemma 2.16 and Lemma 2.15 to modify locally our path so that it takes a corner of a 2×2 square centered at a type III vertex. □

Since we will exploit the properties of this function on the vertices of type I, we introduce the following convention on the figures. Take an edge of length 2 between two type I vertices v_1, v_2, then we put an arrow pointing to v_2 if $\nu(v_2) < \nu(v_1)$ as in the following following:



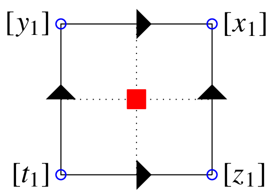
Lemma 5.3. *Let $\nu : \mathrm{k}[Q] \to \mathbb{R} \cup \{+\infty\}$ be a valuation which is trivial over k^* and such that $\nu(x)$, $\nu(y)$, $\nu(z)$, $\nu(t) < 0$. Let S be a 2×2 square of the complex $\mathcal{C}$ centered at a type* III *vertex. Suppose S has a ν-maximal vertex* (*resp. ν-minimal*), *then there exists a ν-minimal* (*resp. ν-maximal*) *vertex and the ν-minimal and ν-maximal vertices are at distance $2\sqrt{2}$ in $\mathcal{C}$.*

Let S be a 2×2 square centered at a vertex of type III which satisfies the conditions of Lemma 5.3. and let ϕ be the associated isometry. Denote by $[x_1]$, $[y_1]$, $[z_1]$ and $[t_1]$ the vertices of type I of the square S where $x_1, y_1, z_1, t_1 \in \mathrm{k}[Q]$ such that the vertex $[x_1]$ is ν-minimal and $[t_1]$ is ν-maximal in S. Then there exists a unique isometry $\phi : S \to [0, 2]^2$ such that

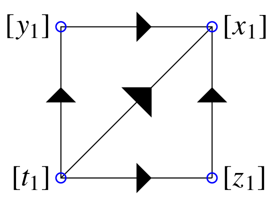
$$\phi([x_1]) = (2, 2) \quad \text{and} \quad \phi([t_1]) = (0, 0),$$

and such that the horizontal edges of S are given the geodesic segments between $[x_1]$ and $[y_1]$, and between $[z_1]$ and $[t_1]$.

Using this convention, Lemma 5.3 implies that we are in the following situation:



In particular, the subgraph of $\mathcal{C}'$ containing the vertices of S looks as follows:



Proof of Lemma 5.3. Let S be a 2×2 square satisfying the hypothesis of the Lemma. Denote $[t_1]$ the ν-maximal vertex of S. Denote also by $[z_1]$, $[y_1]$, $[x_1]$ the type I vertices of S such that the edges between $[t_1]$ and $[z_1]$, between $[t_1]$ and $[y_1]$ are horizontal and vertical respectively.

Observe that $\nu(x_1)$, $\nu(y_1)$, $\nu(z_1)$, $\nu(t_1) < 0$ and that

$$\nu(x_1 t_1 - y_1 z_1) = \nu(1) = 0.$$

This implies that

$$\nu(x_1) + \nu(t_1) = \nu(y_1) + \nu(z_1).$$

In particular, $\nu(t_1) > \nu(y_1)$ implies that

$$\nu(x_1) < \nu(z_1).$$

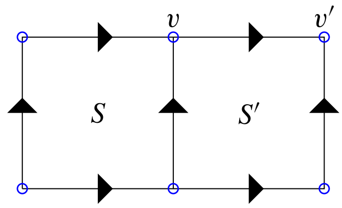
By symmetry, we also prove that $\nu(x_1) < \nu(y_1)$ and this implies that $[x_1]$ is the unique ν-minimal vertex of S, as required. $\square$

Observe that for two distinct valuations $\nu_1, \nu_2 \in \mathcal{V}_0$, the graphs $\mathcal{C}_{\nu_1}$ and $\mathcal{C}_{\nu_2}$ are not in general equal.

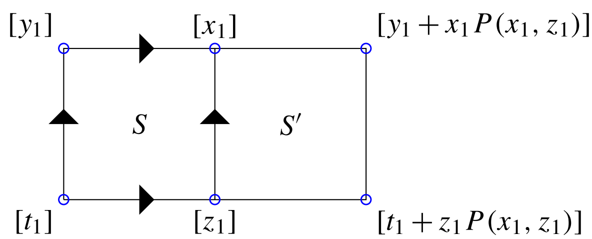
Lemma 5.4. *Fix any valuation $\nu \in \mathcal{V}_0$, and any two adjacent 2×2 squares S, S' centered at a vertex of type III. Suppose that v is a vertex in $S \cap S'$ which is ν-minimal in S.*

Then the unique vertex $v' \in S' \setminus S$ which belongs to an edge containing v is also ν-minimal in S'.

One has the following:



Proof of Lemma 5.4. Take $x_1, y_1, z_1, t_1 \in \mathrm{k}[\mathcal{Q}]$ such that $v = [x_1], [z_1] \in S \cap S'$ and $[y_1], [t_1] \in S$ are the four distinct vertices of S. We claim that we are in the following situation:



where $P \in k[x, y] \setminus k$. Indeed, recall that the tame group acts as $g \cdot [f] = [f \circ g^{-1}]$. In particular, if S_0 is the standard 2×2 square containing $[x], [y], [z], [t]$ and $[\mathrm{Id}]$ and if $f = (x_1, y_1, z_1, t_1)$, then $S = f^{-1} \cdot S_0$. Since S and S' are adjacent along an two edges of type I, there exists an element $e \in E_H$ such that $S' = (f^{-1} \circ e \circ f) \cdot S$. This proves that $S' = (f^{-1} \circ e) \cdot S_0$, and the vertex v' is given by

$$v' = [y \circ e^{-1} \circ f],$$

as required.

Since $\nu(x_1) < \nu(y_1)$ and since $\nu(P(x_1, z_1)) < 0$, this implies that

$$\nu(y_1 + x_1 P(x_1, z_1)) = \nu(x_1 P(x_1, z_1)) < \nu(x_1).$$

Similarly, one has

$$\nu(t_1 + z_1 P(x_1, z_1)) = \nu(z_1 P(x_1, z_1)) < \nu(z_1).$$

Hence since the vertex $[z_1]$ is ν-maximal, we have that $v' = [y_1 + x_1 P(x_1, z_1)]$ is the ν-minimal vertex in S' by Lemma 5.3, as required. $\square$

The following proposition compares the distance d_ν with the distance $d_{\mathcal{C}}$.

Proposition 5.5. *The distance d_ν and the distance $d_{\mathcal{C}}$ are equivalent, i.e., there exists a constant $C > 0$ such that for any vertices $v_1, v_2 \in \mathcal{C}$ of type* I, *one has*

$$\frac{1}{2\sqrt{2}} d_{\mathcal{C}}(v_1, v_2) \leqslant d_\nu(v_1, v_2) \leqslant 2 d_{\mathcal{C}}(v_1, v_2).$$

Proof. For each 2×2 square S centered at a vertex of type III in $\mathcal{C}$, the restriction to $S \cap \mathcal{C}_\nu$ of the distance in $\mathcal{C}_\nu$ and the distance $d_{\mathcal{C}}$ are bi-Lipschitz equivalent. More precisely, for any $v_1, v_2 \in S \cap \mathcal{C}_\nu$, the following inequality holds:

$$\frac{d_{\mathcal{C}}(v_1, v_2)}{2\sqrt{2}} \leqslant d_\nu(v_1, v_2) \leqslant 2 d_{\mathcal{C}}(v_1, v_2).$$

Hence, if we apply the previous inequality to a chain of points which belong successively to the same square, we obtain the distance in $\mathcal{C}$ is equivalent to the distance d_ν and for any vertices v_1, v_2 of type I in $\mathcal{C}$, we have

$$\frac{d_{\mathcal{C}}(v_1, v_2)}{2\sqrt{2}} \leqslant d_\nu(v_1, v_2) \leqslant 2 d_{\mathcal{C}}(v_1, v_2),$$

as required. $\square$

5B. ***Avoiding critical resonances.*** Fix a valuation $\nu \in \mathcal{V}_0$ and fix a 2×2 square S. Consider a vertex $[x_1]$ of type I in S which is ν-minimal in S where $x_1 \in \mathrm{k}[Q]$ and denote by $[z_1]$ another vertex of type I in S such that $[x_1]$ and $[z_1]$ belong to a vertical edge of the square S. For any square S' which is adjacent to S along the edge containing $[x_1]$ and $[z_1]$, Lemma 5.4 implies that the function induced by ν on the vertices is as follows:

$$\begin{tikzcd}
{[y_1]} \arrow[r] & {[x_1]} \arrow[r] & {[y_1 + x_1 P(x_1, z_1)]} \\
{[t_1]} \arrow[u] \arrow[r] \arrow[ur, phantom, "S"] & {[z_1]} \arrow[u] \arrow[r] \arrow[ur, phantom, "S'"] & {[t_1 + z_1 P(x_1, z_1)]} \arrow[u]
\end{tikzcd}$$

where $y_1, t_1, \in \mathrm{k}[Q]$ and $P \in k[x, y] \setminus k$.

If $P \in k[y]$, then using the fact that $\nu(x_1) < \nu(y_1)$ and $\nu(x_1) < \nu(z_1) < \nu(t_1)$, we have $\nu(y_1 + x_1 P(x_1, z_1)) = \nu(x_1) + \deg(P)\nu(z_1) < (\deg(P) + 1)\nu(z_1)$ and $\nu(t_1 + z_1 P(x_1, z_1)) = (\deg(P) + 1)\nu(z_1)$. This degenerate case can be formulated as follows. Take $g \in \mathrm{STame}(Q) \cap \mathrm{Stab}([z_1])$ such that $g \cdot S = S'$, Lemma 2.19 shows that $g \in A^S_{[z_1]}$.

Otherwise, we can suppose that $P \in k[x, y] \setminus k[y]$. The main observation is that when the component (x_1, z_1) is not critically resonant with respect to ν, then by Corollary 4.19, one has

$$\nu(x_1 P(x_1, z_1)) = \max(\nu(y_1 + x_1 P(x_1, z_1), \nu(t_1 + z_1 P(x_1, z_1))) < \tfrac{4}{3}\nu(z_1).$$

Following the discussion when $P \in k[y]$, we deduce that the previous inequality also holds regardless of the condition $P \in k[y]$ or $P \in k[x, y] \setminus k[y]$.

Moreover, when $P \in k[x, y] \setminus k[y]$ the same argument combined with Corollary 4.18 yields

$$\max(\nu(y_1 + x_1 P(x_1, z_1), \nu(t_1 + z_1 P(x_1, z_1))) < \nu(x_1).$$

We have summarized the above argument in the following lemma.

Lemma 5.6. *Fix $\nu \in \mathcal{V}_0$ and S, S' two adjacent 2×2 squares. Consider v_1, v_2 two vertices of the common edge of these squares and suppose that v_1 is ν-minimal in S. Suppose that the edge joining v_1 and v_2 corresponds to a component (f_1, f_2) which is not critically resonant. Then for any vertex $v' \in S'$ distinct from v_1, v_2, we have*

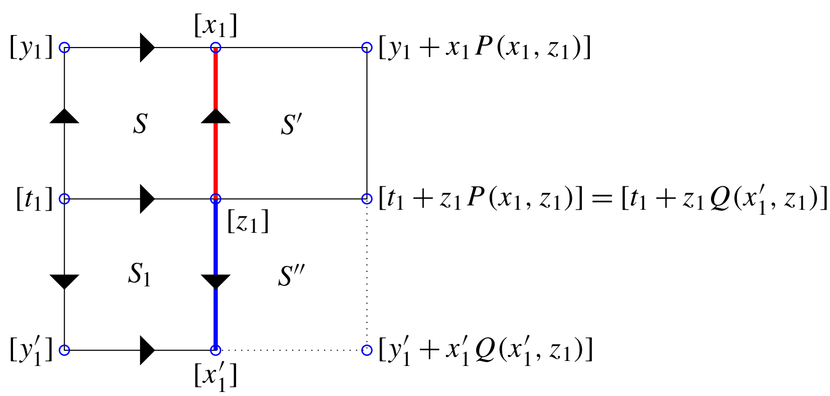
$$\nu(v') < \tfrac{4}{3}\nu(v_2).$$

Moreover, if $g \in \mathrm{Stab}(v_2) \cap \mathrm{STame}(Q)$ such that $g \cdot S = S'$ and $g \notin A^S_{v_2}$, then

$$\nu(v') < \min\left(\tfrac{4}{3}\nu(v_2), \nu(v_1)\right).$$

When the component (x_1, z_1) in the previous figure is critically resonant, then the previous arguments do not necessarily hold since we cannot apply Corollary 4.19.

Our key observation is that the previous inequality remains valid whenever there exists a square S_1 adjacent to S along the edge containing $[t_1]$, $[z_1]$, such that its other edge containing $[z_1]$ is not critically resonant and such that $[t_1]$ is ν-maximal in S_1. If we choose S_1 so that the squares S_1, S, S' are flat, we arrive at the following situation where a *blue edge* means that the corresponding component is not critically resonant and a *red edge* that the component is critically resonant:



where $Q \in k[x, y] \setminus k$ and where S''' is a 2×2 square adjacent to S_1 and S'. We can thus apply the previous argument to the square S_1 and S''', we obtain

$$\nu(y_1' + x_1' Q(x_1', z_1)) < \nu(t_1 + z_1 P(x_1, z_1)) < \nu(z_1).$$

and furthermore

$$\nu(t_1 + z_1 P(x_1, z_1)) = \max(\nu(y_1' + x_1' Q(x_1', z_1)), \nu(t_1 + z_1 P(x_1, z_1))) < \tfrac{4}{3}\nu(z_1).$$

In other words, we obtain that the vertex $[z_1]$ is ν-maximal in S''', which implies that it is also ν-maximal in S'. Overall, the previous inequality with the ν-maximality of $[z_1]$ in S' finally yields

$$\nu(t_1 + z_1 P(x_1, z_1)) = \max(\nu(t_1 + z_1 P(x_1, z_1)), \nu(y_1 + x_1 P(x_1, z_1))) < \tfrac{4}{3}\nu(z_1). \tag{28}$$

In the rest of section, we keep the same convention on the colors of the edges.

The proposition below is the key ingredient in our proof and explains how one can find a square which has an edge which is not critically resonant.

Proposition 5.7. *Fix a valuation $\nu \in \mathcal{V}_0$. Let S be any 2×2 square having a unique ν-minimal vertex, and let $[f_1], [f_2]$ be any horizontal (resp. vertical) edge of S. Suppose that $\nu(f_1) < \nu(f_2)$, that (f_1, f_2) is critically resonant and that for any polynomial $R \in k[x] \setminus k$, one has*

$$\nu(f_1 - f_2 R(f_2)) < \nu(f_2).$$

Then there exists a square S_1 adjacent to S along the vertical (resp. horizontal) edge containing $[f_2]$ which satisfies the following properties:

(i) *For any square S_2 adjacent to S along the edge containing $[f_1], [f_2]$, the squares S_1, S, S_2 are flat.*

(ii) *The horizontal (resp. vertical) edge in S_1 containing $[f_2]$ is not critically resonant.*

(iii) *There exists an element $g \in A^S_{[f_2]}$ such that $g \cdot S = S_1$.*

Proof. Statement (i) and (iii) follow from Lemma 2.19(ii) and (i) respectively. Indeed pick any polynomial $R \in k[x] \setminus k$, and let S_R be the square containing $[f_2], [f_1 - f_2 R(f_2)]$ which is adjacent to S along the vertical edge containing $[f_2]$. Since R depends on a single variable, it follows that for any square S_2 adjacent to S along the edge containing $[f_1], [f_2]$, the squares S_R, S_2, S are flat.

We now prove (ii), and produce a polynomial $R \in k[x] \setminus k$ such that the component $(f_2, f_1 - f_2 R(f_2))$ is not critically resonant. Since the component (f_1, f_2) is critically resonant, there exists a constant $\lambda \in k^*$ and an integer $n \geqslant 1$ such that

$$\nu(f_1 - \lambda f_2^n)) > \nu(f_1) = n\nu(f_2).$$

Since $\nu(f_1) < \nu(f_2)$, we get $n \geqslant 2$ so that $R_1 := \lambda x^{n-1} \in k[x] \setminus k$.

If the component $(f_2, f_1 - f_2 R_1(f_2))$ is not critically resonant, then the square S_1 containing $[f_2]$, $[f_1 - f_2 R_1(f_2)]$ which is adjacent to S along the vertical edge containing $[f_2]$ satisfies assertion (ii) and we are done. Otherwise, $(f_2, f_1 - f_2R_1(f_2))$ is critically resonant. Observe that by assumption, we have

$$\nu(f_1 - f_2R_1(f_2)) < \nu(f_2),$$

so that $\nu(f_1 - f_2R_1(f_2)) = n_2\nu(f_2)$ for some $n_2 \geqslant 2$, and $\nu(f_1 - f_2R_2(f_2)) > n_2\nu(f_2)$ for some polynomial $R_2 \in k[x] \setminus k$ of the form $R_2(x) = R_1(x) + \lambda' x^{n_2-1}$. Repeating this argument we get a sequence of polynomials $R_i \in k[x] \setminus k$, and either $(f_2, f_1 - f_2R_i(f_2))$ is not critically resonant for some index i; or $(f_2, f_1 - f_2R_i(f_2))$ is critically resonant for all i. However in the latter case, the sequence $(\nu(f_1 - f_2R_i(f_2))$ is strictly increasing and $(\nu(f_1 - f_2R_i(f_2))$ are all multiples of $\nu(f_2)$ which yields a contradiction. The proof is complete. □

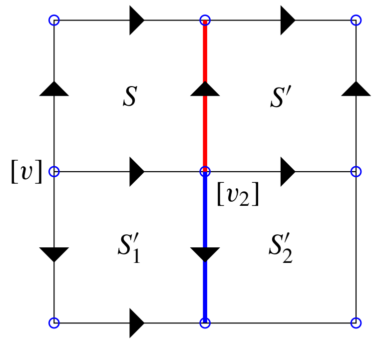
5C. ***Degree estimates between adjacent squares.***

Theorem 5.8. *Take a valuation $\nu \in \mathcal{V}_0$. Let v be any ν-maximal vertex of a 2×2 square S and let S' be an adjacent square which does not contain v and let v_2 be the vertex in $S \cap S'$ which is not ν-minimal in S. Suppose that the vertex v is also ν-maximal in any square $\tilde{S}$ adjacent to S along the edge joining v and v_2. Then S' admits a ν-minimal vertex and for any vertex $v' \in S' \setminus S$, one has*

$$\nu(v') < \tfrac{4}{3}\nu(v_2).$$

Proof. Observe that Lemma 5.4 implies that S' has a unique ν-minimal vertex. If the edge $S \cap S'$ is not critically resonant, then Lemma 5.6 implies the conclusion of the theorem. Otherwise, the edge $S \cap S'$ is critically resonant and we check that the squares S and S' satisfy the hypothesis of Proposition 5.7.

Denote by $[f_1]$ the ν-minimal vertex in S and by $[f_2] = v_2$. For any polynomial $R \in k[x] \setminus k$, take S_R to be the square containing $[f_1 - f_2R(f_2)]$, $[f_2]$ and v. By construction, the square S_R is adjacent to S along an edge containing v, hence the vertex v is ν-maximal in S_R and this implies that the vertex $[f_1 - f_2R(f_2)]$ is ν-minimal in S_R. In particular, we have proved that $\nu(f_1 - f_2R(f_2)) < \nu(f_2)$. By Proposition 5.7, there exists two squares S'_1, S'_2 such that the union $S \cap S'_1 \cup S' \cup S'_2$ forms a 4×4 square centered along the vertex $[v_2]$ and such that the edge $S'_1 \cap S'_2$ is not critically resonant. We thus arrive at the following situation (with the same convention on colors as in the previous section):



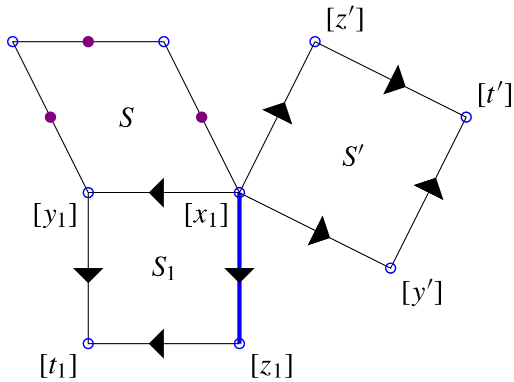


Figure 6. The initial situation of Theorem 5.9.

In particular, by applying Lemma 5.4 to the square S'_1 and S'_2, we find that

$$\nu(v') < \tfrac{4}{3}\nu(v_2),$$

for any vertex $v' \in S' \setminus S$ as required. $\square$

5D. ***Degree estimates at a ν-maximal vertex.*** In this section, we analyze the situation of two 2×2 squares adherent at a vertex of type I.

Recall from Section 2E that a pair of adherent squares (S, S') is contained in a spiral staircase around $v = S \cap S'$ if there exists a sequence of squares $S_0 = S, \ldots, S_p = S'$ connecting S and S', all containing v, which are adjacent alternatively along vertical and horizontal edges and such that any three consecutive squares S_i, S_{i+1}, S_{i+2} are not flat for $i \leqslant p-2$. When the intersection between S_0 and S_1 is a horizontal (resp. vertical) edge, we say that the staircase is vertical (resp. horizontal).

Theorem 5.9. *Fix a valuation $\nu \in \mathcal{V}_0$.*

Consider three 2×2 squares S, S_1 and S' having a vertex $[x_1]$ of type I *in common. We assume that S and S_1 have a common horizontal edge $[x_1]$, $[y_1]$, and that the pair (S, S') is contained in a vertical spiral staircase containing S_1. Denote by $[z_1]$ the vertex in S_1 which forms a vertical edge with $[x_1]$.*

Assume that $[x_1]$ is ν-maximal in S_1, that the component (x_1, z_1) is not critically resonant, that $\nu(z_1) < \nu(y_1)$ and $\nu(z_1) < \left(\frac{4}{3}\right)\nu(x_1)$. Then for any vertex $v \in S'$ distinct from $[x_1]$, one has

$$\nu(v) < \tfrac{4}{3}\nu(x_1).$$

Figure 6 summarizes the situation of Theorem 5.9 (with the convention of Section 5B on the color of the edges).

We shall use repeatedly the following lemma, whose proof is given at the end of this section.

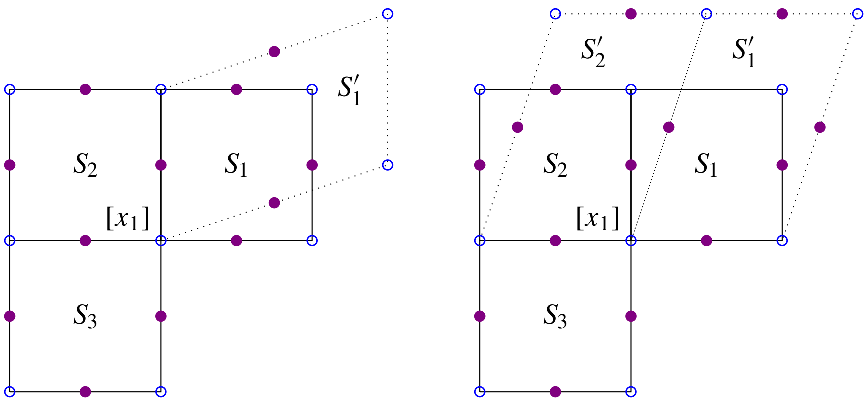
Recall from Section 2D the definition of the subgroup A_v^S of the stabilizer of a vertex v of type I, where S is a 2×2 square containing v.

Lemma 5.10. *Take three 2×2 squares S_1, S_2, S_3 containing $[x_1]$ and which are adjacent alternatively along vertical and horizontal edges. Suppose that S_1, S_2 and S_3 are not flat. Then the following assertions hold*:

(i) *Suppose that S_1' is a 2×2 square which is adjacent to S_2 along $S_1 \cap S_2$ such that there exists an element $g \in A_{[x_1]}^{S_1}$ for which $g \cdot S_1 = S_1'$. Then the squares S_1', S_2, S_3 are not flat.*

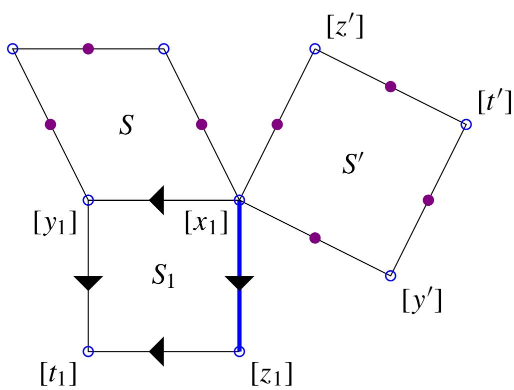
(ii) *For any 2×2 squares S_1', S_2' such that S_1, S_2, S_1', S_2' are flat, the squares S_1', S_2', S_3 are not flat. Moreover, given any $g_1, g_2 \in \mathrm{Stab}([x_1]) \cap \mathrm{STame}(Q)$ such that $g_1 S_1 = S_1'$, and $g_2 S_2 = S_2'$, we have $g_1 \in A_{[x_1]}^{S_1}$ and $g_2 \in A_{[x_1]}^{S_2}$.*

This lemma will allow us to consider alternative spiral staircase around the vertex $[x_1]$. We thus have the following figures in each situation:



Proof of Theorem 5.9. Take a valuation $\nu \in \mathcal{V}_0$ and three squares S, S_1, S' satisfying the conditions of the theorem. By assumption, there exists an integer $p \geqslant 2$ and a sequence of adjacent squares $S_2, \ldots, S_{p-1}$ such that $S_0 = S, S_1, S_2, \ldots, S_p = S'$ forms a vertical staircase.

We denote by $[y_1], [z_1], [t_1], [x_1]$ and $[z'], [y'], [t']$ the vertices of S_1 and S' respectively so that the edges $[x_1], [y_1]$ and $[x_1], [y']$ are horizontal and the edges $[x_1], [z_1]$ and $[x_1], [z']$ are vertical. We are thus in the following situation:



Recall that S and S' are connected by a vertical staircase $S = S_0, S_1, \dots, S_{p-1}, S_p = S'$.

Lemma 5.11. *The theorem holds whenever the edges $S_i \cap S_{i+1}$ are not critically resonant for all $i \geq 1$.*

Lemma 5.12. *For any vertex v such that $[x_1], v$ is an edge of S', there exists a vertical staircase $S = S_0, \tilde{S}_1, \tilde{S}_2, \dots, \tilde{S}_{q-1}, \tilde{S}_q$ such that*

- $\tilde{S}_1 = S_1$;
- $\tilde{S}_q$ *and S' are adjacent along the edge $[x_1], v$;*
- *the edges $\tilde{S}_i \cap \tilde{S}_{i+1}$ are not critically resonant for all $i \geq 1$.*

Take any vertex v of S' such that $[x_1], v$ is an edge of S'. By Lemma 5.12 we get a sequence of squares $\tilde{S}_i$ connecting S to $\tilde{S}_q$ and satisfying the assumptions of Lemma 5.11. This proves $\nu(v) < \frac{4}{3}\nu(x_1)$ as required. ☐

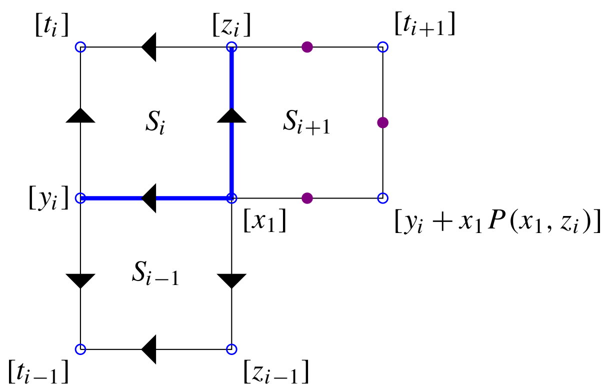
Proof of Lemma 5.11. We prove by induction on i the following two properties:

($\mathcal{P}_1$) For any vertex $v \neq [x_1]$ in $S_i \setminus S_0$, one has $\nu(v) < \frac{4}{3}\nu(x_1)$.

($\mathcal{P}_2$) Let $v_1 \neq [x_1]$ be the unique vertex which is contained in the edge $S_i \cap S_{i-1}$ and let v_2 be the other vertex in S_i which belongs to an edge containing $[x_1]$. Then one has $\nu(v_2) < \nu(v_1)$.

Observe that ($\mathcal{P}_1$) and ($\mathcal{P}_2$) are satisfied when $i = 1$ by our standing assumption on S_1.

Let us prove the induction step. For all i, denote by t_i the unique vertex of S_i which does not lie in $S_{i-1} \cup S_{i+1}$; by y_i the vertex in $S_i \cap S_{i-1}$ distinct from x_1. We also write z_i for the vertex in $S_i \cap S_{i+1}$ distinct from x_1 (so that $y_{i+1} = z_i$). We thus have the following picture:



By our induction hypothesis, we have

$$\nu(z_i) < \nu(y_i) < \nu(x_1).$$

Observe that y_{i+1} is given by

$$y_{i+1} = y_i + x_1 P(x_1, z_i).$$

for some polynomial $P \in k[x, y]$. Since the squares (S_{i-1}, S_i, S_{i+1}) is not flat, Lemma 2.19(i) and Lemma 2.17 imply that $P \notin k[x]$.

Since the component (x_1, z_i) is not critically resonant, Corollary 4.18 applied to $f_1 = z_i$ and $f_2 = x_1$ implies

$$\nu(x_1 P(x_1, z_i)) < \nu(z_i) \leqslant \min\left(\tfrac{4}{3}\nu(x_1), \nu(z_i)\right),$$

where the second inequality follows from the induction hypothesis $\nu(z_i) < \frac{4}{3}\nu(x_1)$, hence

$$\nu(y_{i+1}) = \nu(y_i + x_1 P(x_1, z_i)) = \nu(x_1 P(x_1, z_i)) < \min\left(\tfrac{4}{3}\nu(x_1), \nu(z_i)\right).$$

This proves that $[x_1]$ is ν-maximal in S_{i+1}, hence $[t_{i+1}]$ is ν-minimal in S_{i+1} by Lemma 5.3 and assertion $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ hold for $i+1$, as required. □

Proof of Lemma 5.12. We show that for any vertex v such that $[x_1], v$ is an edge of S', there exists a vertical staircase $S = S_0, \tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_{p-1}, \tilde{S}_p$ around $[x_1]$(of exactly same length) such that:

- $\tilde{S}_1 = S_1$.
- $\tilde{S}_p$ and S' are adjacent along the edge $[x_1], v$, and there exists an element $g \in A^{S'}_{[x_1]}$ for which $g \cdot \tilde{S}_q = S'$.
- The edges $\tilde{S}_i \cap \tilde{S}_{i+1}$ are not critically resonant for all $i \geq 1$.

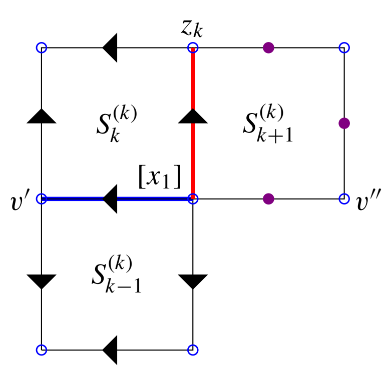
We construct a sequence of spiral staircase $S^{(k)}_0 = S, S^{(k)}_1, \ldots, S^{(k)}_p$ of length p indexed by k as follows. We pick an initial spiral staircase $S^{(1)}_0 = S, S^{(1)}_1 = S_1, \ldots, S^{(1)}_p = S'$ around $[x_1]$ of length p joining S and S'. Observe that the edge $S^{(1)}_1 \cap S^{(1)}_2$ is not critically resonant by our assumption and the sequence $(S^{(1)})_{i \leqslant 2}$ defines a spiral staircase that satisfies the conclusion of the lemma.

Our aim is to construct $(S^{(k)}_i)_{i \leqslant p}$ inductively so that the conclusion of the lemma holds for $(S^{(k)}_i)_{i \leqslant k+1}$ for all $k \leqslant p$.

For $k = 1$, there is nothing to prove since $[x_1], [z_1]$ is not critically resonant by our standing assumption.

For $k \geq 1$, assume that $(S^{(k)}_i)_{i \leqslant p}$ is constructed. If the edge $S^{(k)}_k \cap S^{(k)}_{k+1}$ is not critically resonant, then we set $S^{(k+1)}_i = S^{(k)}_i$ for all i. Otherwise, it is critically resonant and we will replace the two squares $S^{(k)}_k, S^{(k)}_{k+1}$ and keep all the other squares.

Denote by $[z_k]$ and v' the vertices in $S^{(k)}_k$ distinct from x_1 and lying in S' and $\tilde{S}_{p-1}$ respectively. Using the property $\mathcal{P}2$ from the proof of Lemma 5.11, we have $\nu(z_k) < \nu(v') < \nu(x_1)$, and we have the following picture:



We claim that

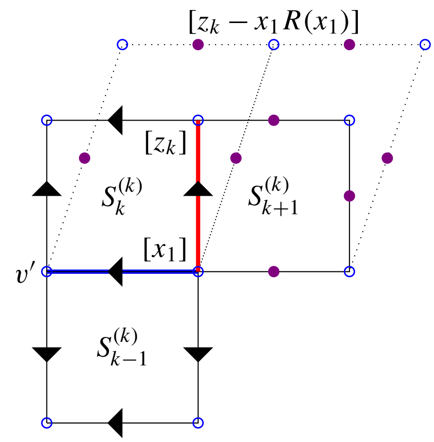
$$\nu(z_k - x_1 R(x_1)) < \nu(x_1),$$

for any polynomial $R \in k[x] \setminus k$. Taking this claim for granted we conclude the proof of the lemma. By Proposition 5.7, we may find a square $S_k^{(k+1)}$ adjacent to $S_k^{(k)}$ along the edge containing $[x_1]$, v' whose edges containing $[x_1]$ are not critically resonant and such that the triple $S_k^{(k)}, S_k^{(k+1)}, S_{k+1}^{(k)}$ is flat. Let $S_{k+1}^{(k+1)}$ be the 2×2 square completing the 4×4 square containing $S_k^{(k)}, S_k^{(k+1)}, S_{k+1}^{(k)}$.

Since the squares $S_{k-1}^{(k)}, S_k^{(k)}$ and $S_{k+1}^{(k)}$ are not flat, Lemma 5.10(ii) implies that the triple $S_{k-1}^{(k)}, S_k^{(k+1)}$ and $S_{k+1}^{(k+1)}$ is also not flat, so that the sequence $(S_1^{(k)}, S_2^{(k)}, \dots, S_{k-1}^{(k)}, S_k^{(k+1)}, S_{k+1}^{(k+1)})$ is a spiral staircase such that any edge lying in two consecutive squares is not critically resonant.

Let us set $S_i^{(k+1)} = S_i^{(k)}$ for all $i \neq k, k+1$. We now show that $(S_1^{(k+1)}, \dots, S_p^{(k+1)})$ defines a spiral staircase. Observe that $S_k^{(k)}, S_{k+1}^{(k)}, S_k^{(k+1)}, S_{k+1}^{(k+1)}$ are flat, since $S_k^{(k)}, S_{k+1}^{(k)}, S_{k+2}^{(k)}$ are not flat, assertion (ii) of Lemma 5.10 implies that $S_k^{(k+1)}, S_{k+1}^{(k+1)}, S_{k+2}^{(k)} = S_{k+2}^{(k+1)}$ are not flat. This shows that $(S_1^{(k+1)}, \dots, S_p^{(k+1)})$ defines a spiral staircase, as required.

We now prove our claim. Fix a polynomial $R \in k[x] \setminus k$, and consider the square S_R containing $[x_1]$, $[z_k - x_1 R(x_1)]$ and v'. Since $x R(x) \in k[x]$, the squares S_R, $S_k^{(k)}$ and $S_{k+1}^{(k)}$ are flat by Lemma 2.19(ii). We thus have the following picture:



By Lemma 2.19, there exists an element $g \in A_{[x_1]}^{S_k^{(k)}}$ such that $g \cdot S_k^{(k)} = S_R$. By Lemma 5.10(i) the triple $S_{k-2}^{(k)}, S_{k-1}^{(k)}, S_R$ are not flat since $S_{k-2}^{(k)}, S_{k-1}^{(k)}, S_k^{(k)}$ are not flat. We have thus proven that the sequence $(S, S_1, S_2^{(k)}, \dots, S_{k-1}^{(k)}, S_R)$ is contained in a spiral staircase for which any edge lying in two consecutive squares is not critically resonant. By Lemma 5.11 the vertex $[x_1]$ is ν-maximal in S_R, hence

$$\nu(z_k - x_1 R(x_1)) < \nu(x_1),$$

as required. □

Proof of Lemma 5.10. By transitivity of the action of $\mathrm{STame}(Q)$ on the 2×2 squares, we can suppose that S_2 is the standard 2×2 square containing $[x]$, $[t]$, $[y]$, $[z]$ and that S_1 and S_3 are adjacent along the vertical and horizontal edge containing $[x]$ respectively. Take $g_1, g_3 \in \operatorname{Stab}([x]) \cap \operatorname{STame}(Q)$ such that $g_1 \cdot S_2 = S_1$ and $g_3 S_2 = S_3$.

Let us prove assertion (i). Since S_1, S_2, S_3 are not flat, Lemma 2.17 implies that $g_1, g_3 \notin A_{[x]}^{S_2}$. Observe that $gg_1 \cdot S_2 = S_1'$ and $g_3 \cdot S_2 = S_3$ where $g \circ g_1 \notin A_{[x]}^{S_2}$, hence the squares S_1', S_2, S_3 are also not flat by Lemma 2.17.

Let us prove assertion (ii). We assume S_1 is the standard square and that $[x] = [x_1]$.

Consider $g \in \operatorname{Stab}([x]) \cap \operatorname{STame}(Q)$ such that $g \cdot S_1 = S_2$. Note that $g_1 \cdot S_1 = S_1'$, $g_2 \cdot S_2 = S_2'$. Assume by contradiction that $g_1 \notin A_{[x]}^{S_1}$. Since the squares S_1', S_1, S_2 are flat, Lemma 2.17 implies that $g \in A_{[x]}^{S_1}$. However, Lemma 2.17 applied to S_1, S_2, S_3 together with the fact that $g^{-1} \in A_{[x]}^{S_2}$ shows that S_1, S_2, S_3 are flat, we have thus obtained a contradiction. We have thus shown that $g_1 \in A_{[x]}^{S_1}$ and a similar argument also gives $g_2 \in A_{[x]}^{S_2}$.

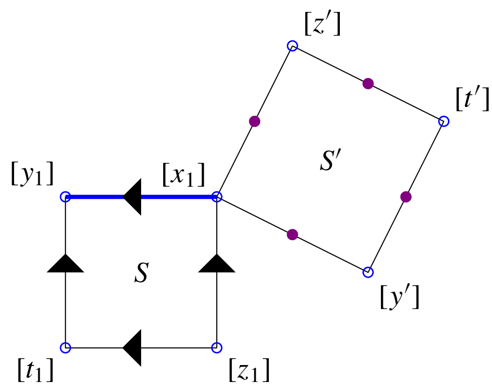
Let us prove that S_1', S_2', S_3 are not flat, consider the element $g_3 \in \operatorname{Stab}([x]) \cap \operatorname{STame}(Q)$ such that $g_3 S_2 = S_3$. Assume by contradiction that S_1', S_2', S_3 are flat. We have $g_1 g^{-1} g_2^{-1} \cdot S_2' = S_1'$ and $g_3 g_2^{-1} \cdot S_2' = S_3$ so Lemma 2.17 shows that one of the element $g_1 g^{-1} g_2^{-1}$, $g_3 g_2^{-1}$ belongs to $A_{[x]}^{S_2'}$. Since $g_3 \notin A_{[x]}^{S_2}$, so $g_3 g_2^{-1} \notin A_{[x]}^{S_2'}$. We deduce that $g_1 g^{-1} g_2^{-1} \in A_{[x]}^{S_2'}$, thus $g_1 g^{-1} \in A_{[x]}^{S_2}$. However $g g_1 g^{-1} \in A_{[x]}^{S_2}$ since $g_1 \in A_{[x]}^{S_1}$ and we get that $g \in A_{[x]}^{S_2}$ which contradicts the fact that S_1, S_2, S_3 are not flat. $\square$

5E. *Degree at a nonextremal vertex.*

Theorem 5.13. *Take a valuation $\nu \in \mathcal{V}_0$. Consider two 2×2 adherent squares S and S' at a vertex of type I given by $[x_1]$ with $x_1 \in \mathrm{k}[Q]$ such that the pair (S, S') is contained in a vertical spiral staircase. Assume $[y_1]$ is the ν-minimal vertex in S distinct from $[x_1]$ which belongs to the horizontal edge containing $[x_1]$ and that the edge containing $[x_1], [y_1]$ is not critically resonant. Then for any vertex v distinct from $[x_1]$ in S' one has*

$$\nu(v) < \tfrac{4}{3}\nu(x_1).$$

One has the following picture:

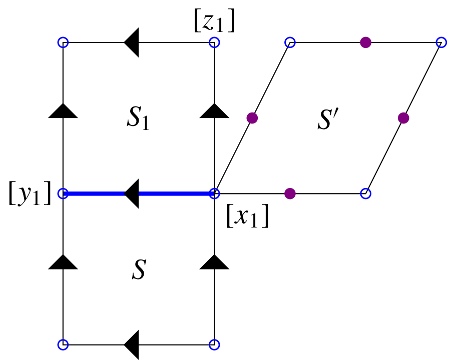


Remark 5.14. By symmetry, observe that the same assertion holds if $[z_1]$ is ν-minimal in S and the pair (S, S') is contained in a horizontal spiral staircase.

Proof. Consider two squares S, S' and the vertices $[x_1], [y_1] \in S$ satisfying the conditions of the Theorem. By definition, there exists an integer p and p adjacent squares $S_0 = S, \dots, S_p = S'$ containing $[x_1]$ connecting S and S'.

Since $S_0 = S$ and S_1 are adjacent, the vertex $[x_1]$ is ν-maximal in S_1 by Lemma 5.4.

Denote by $[z_1]$ the vertex in S_1 such that the vertices $[x_1]$ and $[z_1]$ are contained in the vertical edge of S_1 so that we are in the following situation:



Fix any polynomial $R \in k[x] \setminus k$. Consider S_R the square containing $[x_1]$, $[y_1]$ and $[z_1 - x_1 R(x_1)]$. By Lemma 2.19, the squares S_1, S_R, S_2 are flat. Take $\tilde{S}_R$ the 2×2 square completing the 4×4 square containing S_1, S_R, S_2. Lemma 5.10(ii) implies that S, S_R, $\tilde{S}_R$ are not flat since S, S_1, S_2 are not flat. Moreover, the fact that S, S_1 belong to a spiral staircase shows that any element $g \in \mathrm{Stab}([x_1]) \cap \mathrm{STame}(Q)$ such that $g \cdot S = S_1$ cannot belong to $A^S_{[x_1]}$ by Lemma 2.17. Because $S \cap S_R$ is the noncritically resonant edge containing $[x_1]$, $[y_1]$, Lemma 5.4 applied to S and S_R gives

$$\nu(z_1 - x_1 R(x_1)) < \nu(x_1).$$

By Proposition 5.7, there exists a square S_1' adjacent to S along $[x_1]$, $[y_1]$ such that the squares S_1', S_1, S_2 are flat and such that the vertical edge in S_1' containing $[x_1]$ is not critically resonant. Consider the square S_2' completing the 4×4 square containing S_1', S_1, S_2. By construction, the edge $S_1' \cap S_2'$ is not critically resonant. Observe also that Lemma 5.6 implies that for any vertex $v \in S_1'$ distinct from $[x_1]$ and $[y_1]$, one has

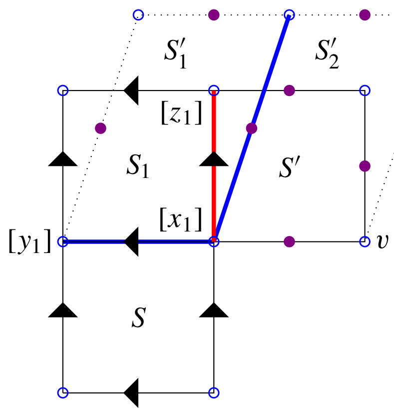
$$\nu(v) < \max\bigl(\nu(y_1), \tfrac{4}{3}\nu(x_1)\bigr).$$

Suppose that $p \geqslant 3$, then the triple (S, S_1', S') satisfies the assumptions of Theorem 5.9 by considering the spiral staircase $(S, S_1', S_2', S_3, \dots, S')$ and we conclude that for any vertex v distinct from $[x_1]$ in S':

$$\nu(v) < \tfrac{4}{3}\nu(x_1).$$

We have thus proven the theorem.

Suppose that $p = 2$ and the squares S' and S_1 are adjacent. We are thus in the following situation:



where v is the unique vertex in S' distinct from $[x_1]$ which belongs to the horizontal edge containing $[x_1]$. By Theorem 5.9, $[x_1]$ is ν-maximal in S'_2, hence it is also ν-maximal in S' and $\nu(v) < \frac{4}{3}\nu(x_1)$. Observe also that Lemma 5.6 implies that

$$\nu(z_1) < \tfrac{4}{3}\nu(x_1).$$

This proves that for any $v \in S'$ distinct from $[x_1]$, one has

$$\nu(v) < \tfrac{4}{3}\nu(x_1),$$

by Lemma 5.4 and the theorem holds. □

5F. *Degree estimates at a ν-minimal vertex.*

Theorem 5.15. *Consider any valuation $\nu \in \mathcal{V}_0$. Let S and S' be two adherent 2×2 squares intersecting at a vertex v which is ν-minimal in S. Then the following holds*:

(i) *The vertex v is the ν-maximal vertex of S'.*

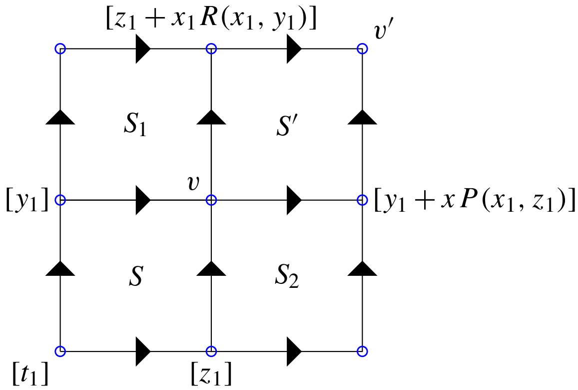
(ii) *If v' is a vertex in S' which does not belong to any square adjacent to S, then we have*

$$\nu(v') < \tfrac{4}{3}\nu(v)$$

Remark 5.16. Suppose that the vertex $v \in S'$ belongs to a square adjacent to S, then we will apply the estimates in Theorem 5.8 instead.

Proof. Let us prove assertions (i) and (ii).

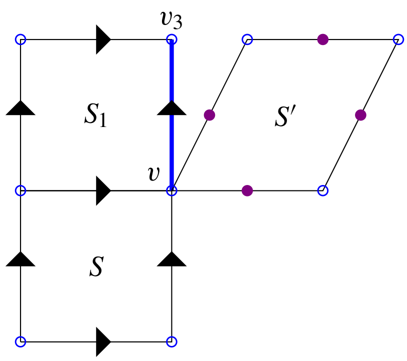
Suppose first that S and S' belong to a 4×4 squares containing S, S', S_1 and S_2 as in the figure below. Since S, S_1 and S, S_2 are adjacent along an edge containing v, Lemma 5.4 implies that we are in the following situation:



where $v = [x_1], [y_1], [z_1], [t_1] \in S$ and $P, R \in k[x, y] \setminus k$. Observe that v is ν-maximal in S' and we have proved assertion (i). Since the squares S, S_1, S_2 are flat, Lemma 2.19 and Lemma 2.17 imply that $P \in k[x] \setminus k$ or $R \in k[x] \setminus k$. Suppose that $P \in k[x] \setminus k$, then we have $\left(\frac{4}{3}\right)\nu(x_1) > \nu(y_1 + x_1 P(x_1)) = (\deg(P) + 1)\nu(x_1) > \nu(v')$ proving (ii) as required.

Suppose next that (S, S') is contained in a spiral staircase. Choose a sequence of squares $S_0 = S, \ldots, S_p = S'$ of squares containing v and connecting S and S' such that each triple of consecutive squares is not flat. By symmetry, we can suppose that S_0 and S_1 are adjacent along a horizontal edge containing v. Observe that Lemma 5.4 applied to S, S_1 implies that the edge $S_1 \cap S_2$ contains the ν-minimal vertex in S_1.

If the edge $S_1 \cap S_2$ is not critically resonant, then the pair (S_1, S') is contained in a horizontal staircase so that one has the following picture:



By Theorem 5.13, the vertex v is ν-maximal in S' and one has $\nu(v') < \left(\frac{4}{3}\right)\nu(v)$ for all $v' \neq v$ in S'. We have thus proved assertion (i) and (ii).

We now suppose that the edge $S_1 \cap S_2$ is critically resonant. Denote by $[f_1]$ the ν-minimal vertex in S_1 and by $v = [f_2]$. Fix any polynomial $R \in k[x] \setminus k$ and take S_R the square containing $[f_1 - f_2 R(f_2)], [f_2]$ and the edge $S_1 \cap S_0$. Lemma 2.19(ii) implies that the squares S_1, S_R, S_2 are flat. Take S'_R the 2×2 square completing the 4×4 square containing S_1, S_R, S_2. Since the squares S, S_1, S_2 are not flat, Lemma 5.10 implies that S, S_R, S'_R are also not flat. In particular, the squares S and S_R intersect along an edge containing v, Lemma 5.4 implies that

$$\nu(f_1 - f_2 R(f_2)) < \nu(f_2).$$

By Proposition 5.7 applied to the edge $[f_1], [f_2]$, we can find a square S'_1 adjacent to S along $S \cap S_1$ and $g \in A_v$ such that $g \cdot S_1 = S'_1$ and such that the vertical edge containing v in S'_1 is not critically resonant. By Lemma 2.17, the squares S_1, S'_1, S_2 are flat. Take S'_2 the 2×2 square completing the 4×4 square containing S_1, S_2, S'_1. As the three squares S, S_1, S_2 are not flat, Lemma 5.10 implies that the squares S, S'_1, S'_2 are also not flat.

If $p \geqslant 3$, then the pair (S'_1, S_p) is contained in a horizontal spiral staircase $(S'_1, S'_2, S_3, \ldots, S_p)$ and the edge $S'_1 \cap S'_2$ is not critically resonant. Hence, by Theorem 5.13, the vertex v is ν-maximal in S' and for any vertex v' distinct from v in S', one has

$$\nu(v') < \tfrac{4}{3}\nu(v),$$

proving (i) and (ii) as required.

Suppose that $p = 2$ so that $S_2 = S'$. Observe that S'_2 and S' are adjacent along a horizontal edge containing v. By Lemma 5.6 applied to S'_1, S'_2, v is ν-maximal in S'_2, it is also ν-maximal on the edge $S'_2 \cap S'$. Since v is ν-maximal on the vertical edge $S_1 \cap S'$, we have thus proven that v is ν-maximal in S' and assertion (i) holds. Take v_2 the vertex contained in $S' \cap S'_2$ distinct from v. Since the edge $S'_1 \cap S'_2$ is not critically resonant, Lemma 5.6 implies that $\nu(v_2) < \frac{4}{3}\nu(v)$. Hence, for any vertex $v' \in S'$ not contained in the same band as S, one has $\nu(v') < \left(\frac{4}{3}\right)\nu(v)$ proving (ii) as required. □

5G. *Proof of Theorem 5.1.* Take S_0 the standard square containing $[x], [y], [z], [t]$. Fix a valuation $\nu \in \mathcal{V}_0$ such that

$$\max(2\nu(t), \nu(y) + \nu(t), \nu(z) + \nu(t)) < \nu(x) < \min(\nu(y), \nu(z), \nu(t)).$$

Pick any vertex v of type I such that the geodesic segment in $\mathcal{C}$ joining [Id] to v intersects an edge of the standard square. Choose any geodesic segment $\gamma : [0, n] \to \mathcal{C}_\nu$ joining $[t]$ to v such that the sequence $(\nu(\gamma(i)))_{0 \le i \le n}$ is maximal for the lexicographic order in $\mathbb{R}^{n+1}$ among all geodesic segments joining $[t]$ to v. Pick any sequence $\tilde{S}_0, \ldots, \tilde{S}_{n-1}$ of 2×2 squares such that $\gamma(i), \gamma(i+1) \in \tilde{S}_i$ for all $i \leqslant n-1$. We claim that the following properties hold:

(A) The vertex $\gamma(i)$ is the unique ν-maximal vertex in $\tilde{S}_i$ for all $0 \le i \le n-1$.

(B) We have $\nu(\gamma(i+1)) < \frac{4}{3}\nu(\gamma(i))$ for all $1 \leqslant i \leqslant n-1$.

(C) For any other valuation $\nu' \in \mathcal{V}_0$ satisfying (27), the vertex $\gamma(i)$ is also ν'-maximal in $\tilde{S}_i$ for all $0 \le i \le n-1$.

Observe first that these properties (A), (B) and (C) imply Theorem 5.1(1) and (2).

Observe the slight discrepancy in the indices between (A), (C) and (B). We do not claim that $\nu(\gamma(1)) < \frac{4}{3}\nu([t])$ in general. This claim is however sufficient to imply Theorem 5.1(1) and (2).

Observe that assertion (C) implies that $d_\nu([t], v) \geq d_{\nu'}([t], v)$ and we conclude by symmetry that $d_\nu([t], v) = d_{\nu'}([t], v)$ for any other valuation $\nu' \in \mathcal{V}_0$ satisfying (27). This proves assertion 2 of the theorem.

We shall prove the claim by induction on $n \geqslant 1$. Fix another valuation $\nu' \in \mathcal{V}_0$ satisfying (27).

Suppose $n = 1$. There is only one square $\tilde{S}_0$ containing $[t]$ and v (it may not be the standard square). Since $n = 1$, we only need to prove assertions (A) and (C).

Lemma 5.17. *Take any* 2×2 *square* S *adjacent to the standard square* S_0 *along an edge containing* $[t]$. *Then the vertex* $[t]$ *is* ν*-maximal in* S.

Moreover, denote by v_1 *the vertex in* $S \cap S_0$ *distinct from* $[t]$ *in* S *and by* v_2 *the vertex distinct from* v_1 *for which the vertices* $[t]$, v_2 *form an edge of* S. *Then one has* $\nu(v_2) < \nu(v_1)$.

Grant this lemma. If $\tilde{S}_0 = S_0$, then (A) and (C) automatically hold. If $\tilde{S}_0$ and S_0 are adjacent along an edge containing $[t]$ Lemma 5.17 implies assertions (A) and (C) immediately. Suppose now that $\tilde{S}_0$ and S_0 are adherent at $[t]$. If the squares $\tilde{S}_0$ and S_0 are flat, then Lemma 5.17 applied to the two squares adjacent to both S_0 and $\tilde{S}_0$ again implies that $[t]$ is also ν-maximal and ν'-maximal in $\tilde{S}_0$.

Otherwise $(S_0, \tilde{S}_0)$ are contained in a spiral staircase. Take an integer $p \geqslant 2$ and a sequence of squares $S_0, S'_1, \ldots, S'_p = \tilde{S}_0$ connecting S_0 to $\tilde{S}_0$ such that each three consecutive squares are not flat. We claim that we can choose some squares S''_1, S''_2 such that $S_0, S''_1, S''_2, S'_3, \ldots, \tilde{S}_0$ is a spiral staircase and such that $S''_1 \cap S''_2$ is not critically resonant. If the edge $S'_2 \cap S'_1$ is not critically resonant, then we set $S''_2 = S'_2$ and $S''_1 = S'_1$.

Otherwise, $S'_1 \cap S'_2$ is critically resonant. Take $[f_1]$ the vertex distinct from $[t]$ of the edge $S'_2 \cap S'_1$. Denote by $[f_2]$ the vertex in $S_0 \cap S'_1$ distinct from $[t]$. By Lemma 5.17, one has $\nu(f_1) < \nu(t)$ and $\nu(f_1) < \nu(f_2)$. Take any polynomial $R \in k[x] \setminus k$, denote by S_R the square containing $[f_1 - tR(t)], [t], [f_2]$. By construction, S_R is adjacent to S_0 and Lemma 5.17 implies that $\nu(f_1 - tR(t)) < \nu(t)$. By Proposition 5.7, we can find a square $S''_1 = g \cdot S'_1$ with $g \in A^{S'_1}_{[t]}$ such that S'_1, S''_1, S'_2 are flat and the edge containing $[t]$ in S''_1 distinct from $S_0 \cap S'_1$ is not critically resonant. Take S''_2 the 2×2 square completing the 4×4 square containing S'_1, S'_2, S''_1.

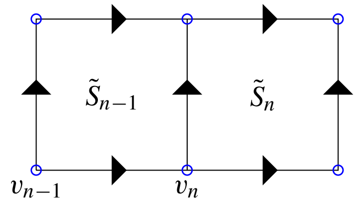
If $p \geqslant 3$, the triple S_0, S'_1, S'_2 is not flat by Lemma 5.10(ii), hence S_0, S''_1, S''_2 are also not flat. The squares $(S_0, S''_1, \tilde{S}_0)$ thus satisfy the conditions of Theorem 5.9, and $[t]$ is ν-maximal in $\tilde{S}_0$. If $p = 2$, then $S'_2 = \tilde{S}_0$ and by Theorem 5.9 applied to (S_0, S''_1, S''_2), the vertex $[t]$ is ν-maximal in S''_2. Since S''_2 and $\tilde{S}_0$ are adjacent along an edge containing $[t]$ and $[t]$ is also ν-maximal in S'_1, it is also ν-maximal in $\tilde{S}_0$, proving assertion (A) as required. Observe that the same argument also applies for $\nu' \in \mathcal{V}_0$, hence assertion (C) also holds.

We have thus proven the claim for $n = 1$.

Let us suppose that the claim is true for $n \geqslant 1$. We shall prove it for $n + 1$. Choose any geodesic $\gamma : [0, n+1] \to \mathcal{C}_\nu$ joining $[t]$ to a vertex v for which the sequence $(\nu(\gamma(i)))_{0 \leq i \leq n}$ is maximal. Denote by $v_i = \gamma(i)$. Take any sequence of squares $\tilde{S}_0, \ldots, \tilde{S}_n$ for which $v_i, v_{i+1} \in \tilde{S}_i$.

By our induction hypothesis applied to the vertex v_n, the sequence $\tilde{S}_0, \ldots, \tilde{S}_{n-1}$ satisfy assertions (A), (B) and (C).

Suppose first that $\tilde{S}_{n-1}$ and $\tilde{S}_n$ are adjacent or equal. Observe that assertion (A) implies that $v = \gamma_{n+1}$ cannot belong to the square $\tilde{S}_{n-1}$, otherwise it would contradict the fact that γ is a geodesic in $\mathcal{C}_\nu$ (recall that in this graph we draw an edge joining the ν-maximal to the ν-minimal vertex). This implies that $\tilde{S}_{n-1}$ and $\tilde{S}_n$ are adjacent along an edge containing the ν-minimal vertex in $\tilde{S}_{n-1}$. Lemma 5.4 shows that the vertex in $\tilde{S}_{n-1} \cap \tilde{S}_n$ which is not ν-minimal in $\tilde{S}_{n-1}$ is ν-maximal in $\tilde{S}_n$. By the maximality of the sequence $(\nu(\gamma(i)))_{0\leq i\leq n}$ the vertex v_n cannot be ν-minimal in $\tilde{S}_{n-1}$, hence is ν-maximal in $\tilde{S}_n$, proving assertion (A). The following figure summarizes the situation:

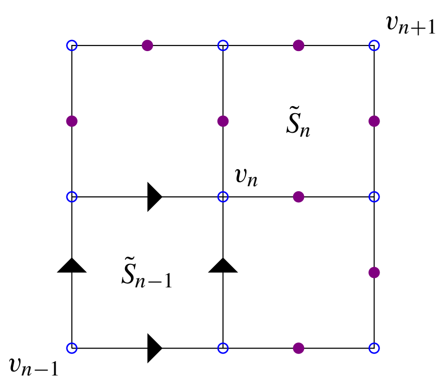


Since v_{n-1} is also ν'-maximal in $\tilde{S}_{n-1}$, the vertex v_n is also ν'-maximal in $\tilde{S}_n$ by Lemma 5.4. We have thus proven assertion (C).

Let us check that $\tilde{S}_{n-1}$ satisfies the condition of Theorem 5.8. Take another square $\tilde{S}$ adjacent to $\tilde{S}_{n-1}$ containing v_{n-1}, v_n. Observe that the sequence $\tilde{S}_0, \ldots, \tilde{S}_{n-2}, \tilde{S}$ satisfies the conditions of the theorem and contains v_n which is at distance n. We apply our induction hypothesis to the vertex v_n and to the sequence of squares $\tilde{S}_0, \ldots, \tilde{S}_{n-2}, \tilde{S}$. Assertion (A) implies that the vertex v_{n-1} is ν-maximal in $\tilde{S}$, as required.

We may thus apply Theorem 5.8 to the band $\tilde{S}_{n-1} \cup \tilde{S}_n$ which yields $\nu(v_{n+1}) < \frac{4}{3}\nu(v_n)$, proving (B), as required.

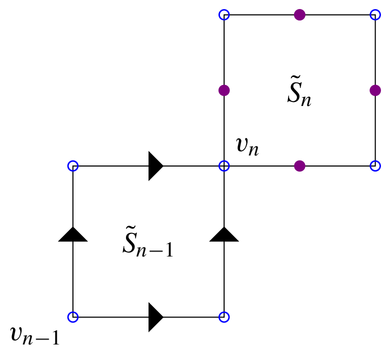
Suppose that the squares $\tilde{S}_{n-1}, \tilde{S}_n$ are adherent and flat. If v_n, v_{n-1} form an edge of $\tilde{S}_{n-1}$, then we can find a band of two squares containing v_{n-1}, v_n, v_{n+1}, which corresponds to the previous situation. Otherwise (v_n, v_{n-1}) is not an edge of $\tilde{S}_{n-1}$, and since v_{n-1} is ν-maximal and ν'-maximal in $\tilde{S}_{n-1}$ by assertions (A) and (C), the vertex v_n is ν-minimal and ν'-minimal in $\tilde{S}_{n-1}$. Observe that the vertex v_{n+1} cannot belong to a band containing v_n, v_{n-1} since we have chosen a geodesic γ for which the sequence $(\nu(\gamma(i)))_{0\leq i\leq n}$ is maximal. We thus arrive at the following situation:



By Theorem 5.15(i) and (ii) applied to $\tilde{S}_{n-1}$ and $\tilde{S}_n$, the vertex v_n is ν-maximal and ν'-maximal in $\tilde{S}_n$ (hence (A), (C) hold), and one has $\nu(v_{n+1}) < \frac{4}{3}\nu(v_n)$, and assertion (B) holds.

Suppose that the squares $\tilde{S}_{n-1}$, $\tilde{S}_n$ are contained in a spiral staircase.

Let us suppose first that the vertices v_{n-1}, v_n do not belong to the same edge of $\tilde{S}_{n-1}$. By assertions (A) and (C) applied to v_{n-1}, the vertex v_{n-1} is ν-maximal and ν'-maximal in $\tilde{S}_{n-1}$, hence v_n is ν-minimal and ν'-minimal in $\tilde{S}_{n-1}$. We thus have the following:



In particular, by Theorem 5.15(i) applied to the squares $\tilde{S}_{n-1}$, $\tilde{S}_n$ implies that v_n is ν-maximal and ν'-maximal in $\tilde{S}$, proving (A) and (C). Observe that v_{n+1} cannot belong to a band containing v_{n-1}, v_n since we have chosen the geodesic such that $\nu(\gamma(i))$ is maximal. In particular, Theorem 5.15(ii) implies that

$$\nu(v_{n+1}) < \tfrac{4}{3}\nu(v_n),$$

proving (B) as required.

Let us suppose that the vertices v_{n-1}, v_n belong to an edge of $\tilde{S}_{n-1}$. Since the argument are similar for horizontal edges, we can suppose that the edge joining v_{n-1}, v_n is vertical, and the pair $(\tilde{S}_{n-1}, \tilde{S}_n)$ belongs to a vertical spiral staircase. Indeed, if $(\tilde{S}_{n-1}, \tilde{S}_n)$ belongs to a horizontal spiral staircase, then we can choose a square $\tilde{S}'_{n-1}$ adjacent to $\tilde{S}_{n-1}$ along the edge containing v_{n-1}, v_n which belongs to the horizontal staircase $(\tilde{S}_{n-1}, \tilde{S}'_{n-1}, \dots, \tilde{S}_n)$. We can thus replace $\tilde{S}_{n-1}$ by $\tilde{S}'_{n-1}$ and the squares $(\tilde{S}'_{n-1}, \tilde{S}_n)$ belong to a vertical spiral staircase or $\tilde{S}'_{n-1}$ and $\tilde{S}_n$ are adjacent. The later case have been treated previously.

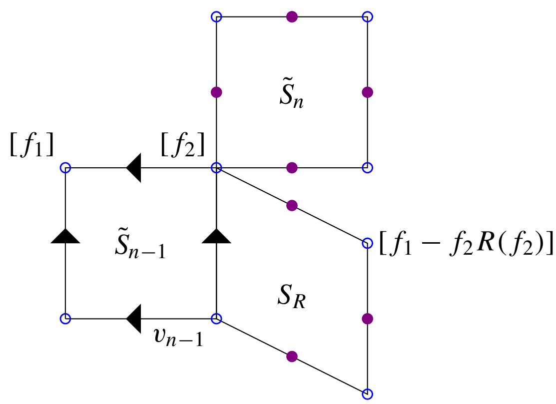
Write by $v_n = [f_2]$ and let $[f_1]$ be the vertex distinct from v_n in $\tilde{S}_{n-1}$ which belongs to the horizontal edge containing v_n. If (f_1, f_2) is not critically resonant, then we can directly apply Theorem 5.13, the vertex v_n is ν-maximal and ν'-maximal in $\tilde{S}_n$ and

$$\nu(v_{n+1}) < \tfrac{4}{3}\nu(v_n),$$

proving (A), (B) and (C) as required.

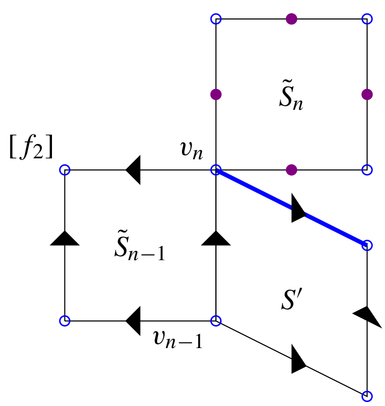
Assume now that (f_1, f_2) is critically resonant. We shall replace the square $\tilde{S}_{n-1}$ by an adjacent square S' along the edge joining v_n and v_{n-1} such that the horizontal edge in S' containing v_n is not critically resonant.

For any polynomial $R \in k[x] \setminus k$, denote by S_R the 2×2 containing $[f_2]$, $[f_1 - f_2 R(f_2)]$, v_{n-1}. We thus have the following:



Using our induction hypothesis for the vertex v_n and to the sequence of squares $\tilde{S}_0, \ldots, \tilde{S}_{n-2}, S_R$, assertions (A) and (C) imply that the vertex v_{n-1} is ν-maximal and ν'-maximal in S_R, hence $\nu(f_1 - f_2 R(f_2)) < \nu(f_2)$ and $\nu'(f_1 - f_2 R(f_2)) < \nu'(f_2)$. By Proposition 5.7, we can find a square S' containing v_{n-1}, v_n for which the horizontal edge containing v_n is not critically resonant and such that there exists $g \in A_{v_n}$ such that $g \cdot S' = \tilde{S}_{n-1}$. By Lemma 5.10, since $(\tilde{S}_{n-1}, \tilde{S}_n)$ is contained in a vertical spiral staircase, this implies that the pair $(S', \tilde{S}_n)$ is also contained in a vertical spiral staircase. Since v_n is neither ν-maximal nor ν-minimal in S', the pair $(S', \tilde{S}_n)$ satisfies the conditions of Theorem 5.13.

One has the following:



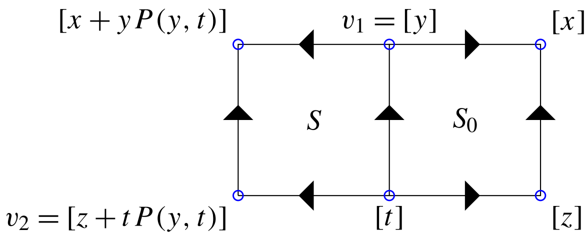
Observe that the same argument applies for ν' and we can find another square S'' adjacent to $\tilde{S}_{n-1}$ along v_n, v_{n-1} such that $S'', \tilde{S}_{n-1}$ is contained in a vertical spiral staircase and such that the horizontal edge in S'' containing v_n is not critically resonant for ν'. By Theorem 5.13, the vertex v_n is ν-maximal and ν'-maximal in $\tilde{S}_n$ and $\nu(v_{n+1}) < \left(\frac{4}{3}\right)\nu(v_n)$, proving (A), (B) and (C) as required.

We have thus proven that our induction step is valid, and the theorem is proved.

Proof of Lemma 5.17 . Fix a valuation $\nu \in \mathcal{V}_0$ satisfying (27) and take a square S adjacent to S_0 along an edge containing $[t]$.

Observe that the edge $S \cap S_0$ is either vertical or horizontal. Since the proof is similar for both cases, we can suppose that $S \cap S_0$ is vertical so that S and S_0 intersect along the edge containing $[y]$, $[t]$. Remark that in this case, we have $v_1 = [y]$ and v_2 is the vertex distinct from $[t]$ which belongs to the horizontal edge in S containing $[t]$.

We are thus in the following situation:



where $P \in k[x, y] \setminus k$.

Since $\nu(P(y, t)) \leqslant \min(\nu(y), \nu(t))$ because P is nonconstant and ν is a quasimonomial valuation, and since (27) implies that $2\nu(t) < \nu(z)$ and $\nu(y) + \nu(t) < \nu(z)$, we get

$$\nu(tP(y, t)) < \nu(z),$$

hence $\nu(z + tP(y, t)) < \nu(z)$ and the vertex $[t]$ is ν-maximal in S. Because ν is monomial satisfying (27), we also have

$$\nu(z + tP(y, t)) < \nu(y),$$

hence $\nu(v_2) < \nu(v_1)$, as required. □

5H. ***Proof of Theorem 1.*** Consider a tame automorphism $f \in \mathrm{Tame}(\mathrm{Q})$. Since the complex $\mathcal{C}$ is $\mathrm{CAT}(0)$ and since the action of f is an isometry and a morphism of complex, the action of f on the complex either fixes a vertex or a geodesic line. In the first case, f is elliptic and by Theorem 3.3, the sequences $(\deg(f^n))$, $(\deg(f^{-n}))$ are either both bounded, both linear or satisfy

$$C^{-1}d^n \leqslant \deg(f^{\pm n}) \leqslant Cd^n,$$

where $C > 0$ and $d \in \mathbb{N}$.

We are thus reduced to prove the theorem in the case where f induces an action which fixes a geodesic line $\gamma : \mathbb{R} \to \mathcal{C}$. Take an hyperbolic automorphism f and a geodesic line $\gamma : \mathbb{R} \to \mathcal{C}$ fixed by f. Denote by S_0 the standard 2×2 square containing $[x], [y], [z]$ and $[t]$. Since for any tame automorphism $h \in \mathrm{Tame}(\mathrm{Q})$, there exists a constant $C > 0$ such that

$$\frac{1}{C} \leqslant \frac{\deg(f^n)}{\deg(h^{-1}f^nh)} \leqslant C,$$

by taking an appropriate conjugate of f, we can suppose that γ starts in S_0 and intersects an edge of S_0. Consider the geodesic segment γ'_n joining [Id] and $[x \circ f^{-n}]$. By construction, γ'_n intersects an edge of the standard square S_0 as γ starts in S_0.

Fix any valuation ν such that (27) is satisfied. There are infinitely many valuations in $\mathcal{V}_0$ satisfying (27) arbitrarily close to $-\deg$. Indeed, consider the sequence of weight $\alpha_i = (-1, -1+2/i, -1+5/i, -1+7/i)$, then by Proposition 4.2, there exists a sequence of valuations ν_i with weight α_i on (x, y, z, t) which converges to $-\deg$.

All assumptions of Theorem 5.1 are then satisfied and we get

$$\nu_i(f^n \cdot [x]) = \nu_i(x \circ f^{-n}) \leqslant \left(\tfrac{4}{3}\right)^{d_{\nu_i}([t],[x \circ f^{-n}])-1} \max(\nu_i(y), \nu_i(z), \nu_i(x), \nu_i(t)).$$

Observe that ν_i tends to $-\deg$, moreover, assertion (2) of Theorem 5.1 implies that the distance $d_{\nu_i}([t], [x \circ f^{-n}])$ are all equal for all i which implies

$$\deg(f^{-n}) \geqslant \left(\tfrac{4}{3}\right)^{d_\nu([t],[x \circ f^{-n}])-1},$$

for a given valuation ν satisfying (27).

We now prove that the sequence $(d_\nu([t], [x \circ f^{-n}]))_n$ grows at least linearly. Indeed since the invariant geodesic γ passes through S_0, then it passes through all the squares $f^i \cdot S_0$ for all $i \leqslant n$. We claim that all the squares $f^i S_0$ are distinct, so there are at least n squares. Indeed each iterate $f^i S_0$ is a 2×2 square containing a piece of the invariant geodesic and the type III vertex in $f^i S_0$ are all distinct otherwise f would be in O_4.

Consider a geodesic segment γ_{1n} in $\mathcal{C}_\nu$ joining $[t]$ and $[x \circ f^{-n}]$ and a shortest path γ_{2n} in $\mathcal{C}_\nu$ contained in a sequence of squares containing the geodesic γ between these two vertices. The path γ_{1n}, γ_{2n} are all in $\mathcal{C}_\nu$ which contains the 1-skeleton of $\mathcal{C}$. The image of those path in $\mathcal{C}$ are both at bounded distance (for the distance $d_{\mathcal{C}}$) from the geodesic γ_n joining $[t]$ and $[x \circ f^n]$ in $\mathcal{C}$. For γ_{2n} this is because γ_{2n} goes through the squares that contain γ_n. For γ_{1n}, this is because γ_{1n} is a geodesic path in $(\mathcal{C}_\nu, d_\nu)$, which is quasiisometric to $(\mathcal{C}, d_{\mathcal{C}})$. In particular, the farthest point in between γ_{1n} and γ_{2n} are apart by a finite number M of squares, which only depends on the distance $d_{\mathcal{C}}(\gamma_{1n}, \gamma_{2n})$. We get

$$l(\gamma_{2n}) \geqslant l(\gamma_{1n}) - M.$$

Since the path γ_{2n} goes through at least the n distinct squares $f^i S_0$ for $i \leqslant n$, the length of γ_{2n} in $\mathcal{C}_\nu$ is larger or equal than n, hence

$$d_\nu([t], [x \circ f^{-n}]) \geqslant n - M.$$

Hence

$$\deg(f^{-n}) \geqslant C\left(\tfrac{4}{3}\right)^{n-1},$$

where $C > 0$. Since the argument is similar for $\deg(f^n)$, we have thus proven that

$$\min(\deg(f^n), \deg(f^{-n})) \geqslant C\left(\tfrac{4}{3}\right)^n$$

where $C > 0$.

5I. ***Proof of Theorem 4.*** Take $f \in \mathrm{Tame(Q)}$. Consider γ the geodesic in $\mathcal{C}$ joining $v_0 = [\mathrm{Id}]$ to $[x \circ f]$. Since the stabilizer of [Id] is the group O_4 by Proposition 2.7 and since the group O_4 acts transitively on the 1×1 squares containing $v_0 = [\mathrm{Id}]$ by Proposition 2.3, we can suppose that the geodesic γ intersects an edge of type I containing $[x]$ of the 1×1 square containing $[x]$, [Id], $[z, x]$ and $[x, y]$. In particular, the geodesic γ intersects an edge of the standard square S_0.

We have proved that the vertex $v = [x \circ f]$ satisfies the conditions of Theorem 5.1, and by considering a sequence of valuations $\nu_p \in \mathcal{V}_0$ converging to $-\deg$ satisfying (27), we have

$$\nu_p(x \circ f) \leqslant \left(\tfrac{4}{3}\right)^{d_{\nu_p}([t],[x \circ f])-1} \max(\nu_p(y), \nu_p(z), \nu_p(x), \nu_p(t)).$$

By Proposition 5.5, we have, for all integer p,

$$\tfrac{1}{2\sqrt{2}} d_{\mathcal{C}}(v_1, v_2) \leqslant d_{\nu_p}(v_1, v_2).$$

for any vertices v_1, v_2 of type I. Since $d_{\mathcal{C}}([t], [x \circ f]) \geqslant d_{\mathcal{C}}([\mathrm{Id}], [f]) - 2\sqrt{2}$, we thus obtain, after taking the limit as $p \to +\infty$,

$$\log \deg(f) \geqslant C d_{\mathcal{C}}([f], [\mathrm{Id}]) - C',$$

where $C' = 2 \log\left(\tfrac{4}{3}\right)$ and $C = \log\left(\tfrac{4}{3}\right)/(2\sqrt{2})$ as required.

6. Application to random walks on the tame group

In this section, we consider a random walk on the tame group and its associated degree sequence. After recalling some general facts on random walks on groups (Section 6A), we then discuss when the degree exponents of a random walk are well-defined and their properties (Section 6B). We then classify in Section 6C the finitely generated subgroup of Tame(Q). Finally we prove Theorem 5, which asserts that the degree exponent of a symmetric random walk on a finitely generated group G is strictly positive if and only if it contains two noncommuting automorphisms with dynamical degree strictly larger than 1 generating a free group of rank 2.

6A. ***General facts on random walks on groups.*** Let G be a finitely generated subgroup of the tame group and let μ be an atomic probability measure on G. The (left) random walk on G with respect to the measure μ is the Markov chain whose initial distribution is the Dirac mass at Id with transition matrix $p(g, g') = \mu(\{g' g^{-1}\})$ for all $g, g' \in G$. We denote by $\Omega = (G^{\mathbb{N}^*}, \mu^{\otimes \mathbb{N}^*})$ the product probability space which encodes the successive increments of the random walk on G with respect to the measure μ. Consider an element $s = (s_1, \dots, s_n, \dots) \in \Omega$, set $g_0(s) = \mathrm{Id}$ and

$$g_n(s) = s_n s_{n-1} \cdots s_1,$$

for all $n \geqslant 1$. The image $\mathcal{P}$ of the map $s \in \Omega \mapsto (\mathrm{Id}, g_1(s), \dots, g_n(s), \dots) \in G^{\mathbb{N}*}$ is called the path space and an element of $\mathcal{P}$ is a path in the group G. We naturally endow $\mathcal{P}$ with the probability measure $\mathbb{P}$ defined on the σ-algebra of cylinders as the pushforward of the product measure on Ω by the map

$s \in \Omega \mapsto (g_i(s))_i$. More explicitly, consider the probability measure ν_n of the projection of $\mathcal{P}$ onto the $(n+1)$-th component g_n, then ν_n is equal to the n-fold convolution $\mu^{*n} * \delta_{\mathrm{Id}}$ so that for all $g \in G$, one has

$$\nu_n(\{g\}) = \mathbb{P}(g_n = g) = \sum_{\substack{s_1,\dots,s_n \\ s_n \cdots s_1 = g}} \prod_{i=1}^{n} \mu(s_i).$$

Fix a reference vertex $v_0 = [\mathrm{Id}]$ in the complex $\mathcal{C}$. Since the tame group acts on the complex, a path in the group $(\mathrm{Id}, g_1, \dots, g_n, \dots)$ induces an element in $\mathcal{C}^{\mathbb{N}*}$ given by $(v_0, g_1 \cdot v_0, \dots, g_n \cdot v_0, \dots)$. The sequence $(v_0, g_1 \cdot v_0, \dots, g_n \cdot v_0, \dots)$ is called a path in the complex.

6B. ***Degree exponents of a random walk.*** Let G be a finitely generated subgroup of the tame group and let μ be an atomic probability measure on G. We shall define in this section the degree exponents of a random walk with respect to the measure μ. To do so, the measure μ must satisfy a finiteness condition on its first moment

$$\int_{g \in G} \log(\deg(g))\, d\mu(g) < +\infty. \tag{29}$$

Let us define the two degree exponents $\lambda_1(\mu), \lambda_2(\mu)$ by

$$\lambda_1(\mu) := \limsup_{n \to +\infty} \frac{1}{n} \int_{g \in G} \log(\deg(g))\, d\nu_n(g), \quad \text{and} \quad \lambda_2(\mu) := \limsup_{n \to +\infty} \frac{1}{n} \int_{g \in G} \log(\deg(g^{-1}))\, d\nu_n(g),$$

where ν_n is the probability measure of g_n.

The following proposition proves that these quantities are finite and give a few basic properties of these numbers.

Proposition 6.1. *Take G a countably generated subgroup of the tame group and μ an atomic probability measure on G satisfying condition* (29). *Then the following properties are satisfied*:

(i) *The degree exponents $\lambda_1(\mu), \lambda_2(\mu)$ are finite and are equal to*

$$\lambda_1(\mu) = \lim_{n \to +\infty} \frac{1}{n} \int_{g \in G} \log(\deg(g))\, d\nu_n(g), \quad \textit{and} \quad \lambda_2(\mu) = \lim_{n \to +\infty} \frac{1}{n} \int_{g \in G} \log(\deg(g^{-1}))\, d\nu_n(g),$$

(ii) *The following inequality holds*

$$\lambda_1(\mu) \geqslant \frac{\lambda_2(\mu)}{2}.$$

(iii) *Consider $\sigma : G \to G$ the inverse map, then $\lambda_2(\mu) = \lambda_1(\sigma_* \mu)$.*

(iv) *The degree exponents are invariant by conjugation, i.e., for any $h \in$* Tame(Q), *we have*

$$\lambda_i(\mathrm{Conj}(h)_* \mu) = \lambda_i(\mu),$$

where $\mathrm{Conj}(h) : \mathrm{Tame(Q)} \to \mathrm{Tame(Q)}$ *denotes the conjugation by h in G.*

Proof. Let us first prove (i). Since g is an automorphism on a threefold, we have $\deg_2(g) = \deg_1(g^{-1})$ and the Khovanski–Teissier inequalities imply $\deg_1(g)^2 \geqslant \deg_2(g) = \deg_1(g^{-1})$; see e.g., [Dang 2020] for a precise definition of the k-degree. We obtain a finiteness condition on the inverse

$$\int_{g\in G} \log(\deg(g^{-1}))d\mu(g) \leqslant 2\int_{g\in G} \log(\deg(g))\,d\mu(g) < +\infty.$$

Since the function deg is submultiplicative, the random variables

$$s \in \Omega \mapsto \log(\deg(g_n(s)))$$

form a subadditive sequence. The previous equation shows that the average of $\log(\deg(g_1(\cdot)))$ is finite and we can apply Kingman's subadditivity theorem [Kingman 1973, Theorem 1] which implies that the limits

$$\lim_{n\to+\infty} \frac{1}{n}\int_G \log\deg(g)\,d\nu_n(g) \quad \text{and} \quad d\nu_n(g),$$

exist and are finite. This proves assertion (i).

Assertion (ii) follows from the fact that $\deg(g)^2 \geqslant \deg(g^{-1})$ for all $g \in \mathrm{Tame}(\mathrm{Q})$. To prove assertion (iii), observe that for all $(s_1, \dots, s_n, \dots) \in \Omega$, we have

$$s_n^{-1}s_{n-1}^{-1}\cdots s_1^{-1} = (s_1s_2\cdots s_n)^{-1}.$$

In particular, we obtain

$$\lim_{n\to+\infty} \frac{1}{n}\int_\Omega \log(\deg((s_1s_2\cdots s_n)^{-1})))\,d\mu^{\otimes n} = \lim_{n\to+\infty} \frac{1}{n}\int_\Omega \log(\deg(s_ns_{n-1}\cdots s_1))\,d\sigma_*\mu^{\otimes n}.$$

Since the right hand side of the equality is equal to $\lambda_1(\sigma_*\mu)$ and the left hand side to $\lambda_2(\mu)$, we have thus proven (iii).

Finally, let us prove assertion (iv). Fix $h \in \mathrm{Tame}(\mathrm{Q})$, recall that there exists a constant $C(h) > 0$ such that for all $g \in \mathrm{Tame}(\mathrm{Q})$, we have

$$\frac{\deg(g)}{C(h)} \leqslant \deg(hgh^{-1}) \leqslant C(h)\deg(g).$$

The last inequality directly implies that $\lambda_i(\mathrm{Conj}(h)_*\mu) = \lambda_i(\mu)$ for all $i = 1, 2$ and all $h \in \mathrm{Tame}(\mathrm{Q})$. □

6C. ***Classification of finitely generated subgroups.*** In this section, we give a classification of the finitely generated subgroups of the tame group. To that end, we recall the terminology due to Gromov [1987] on subgroups of isometries of a hyperbolic space.

Fix a Gromov hyperbolic space X and a group G acting on it by isometry. The action of G on X is called *elementary* if it does not contain two hyperbolic isometries whose action do not fix the same geodesic line. We call the action of G on X *elliptic* if it globally fixes a point in X and we shall say that the action of G is *lineal* if there exists an elliptic subgroup H of G, a geodesic line γ on X invariant by G, pointwise fixed by H on which the quotient G/H acts faithfully by translation.

In our case, any element of the tame group induces an isometry of the complex. We will also need to distinguish among the subgroups which fix a vertex in the complex, more particularly when the fixed vertex is of type I. Remark that a subgroup G of the tame group which fixes a vertex of type I is conjugated to a subgroup of $\mathrm{Stab}([x])$ and recall that we have constructed in Section 2D a natural action from the stabilizer subgroup $\mathrm{Stab}([x])$ on a subtree of the Bass–Serre tree. We have the following classification.

Theorem 6.2. *Let G be a finitely generated subgroup of the tame group. Then one of the following situation occurs*:

(i) *The action of G on the complex is nonelementary.*

(ii) *There exists an automorphism h in G whose action in the complex is hyperbolic and such that any automorphism $f \in G$ can be decomposed into $f = g \circ h^p$ where p is an integer and where g belongs to a subgroup H. Moreover, the subgroup H is conjugated in* Tame(Q) *to a subgroup of O_4 or to one of*

$$E_H \rtimes \left\{ \begin{pmatrix} ax & by \\ b^{-1}z & a^{-1}t \end{pmatrix} \,\middle|\, a, b \in k^* \right\}.$$

(iii) *The group G is conjugated to a subgroup of the linear group O_4.*

(iv) *There exists a G-equivariant morphism $\varphi : Q \to \mathbb{A}^2 \setminus \{(0,0)\}$ where G acts on $\mathbb{A}^2 \setminus \{(0,0)\}$ linearly.*

(v) *The group G contains two noncommuting automorphisms with dynamical degree larger or equal than* 2 *and there exists a G-equivariant morphism $\varphi : Q \to \mathbb{A}^1$ where G acts on $\mathbb{A}^1$ by multiplication.*

(vi) *The group G contains an automorphism h with $\lambda_1(h) \geqslant 2$ and there exists a G-equivariant morphism $\varphi : Q \to \mathbb{A}^1$ on which G acts on $\mathbb{A}^1$ by multiplication and an isomorphism $\varphi^{-1}(\mathbb{A}^1 \setminus \{0\}) \simeq \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2$ such that any automorphisms $f \in G$ can be decomposed into $g \circ h^p$ where p is an integer and g is of the form*

$$g : (x, y, z) \in \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2 \mapsto (ax, by, cz) \in \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2,$$

where $a, b, c \in k^*$.

(vii) *There exists a G-equivariant morphism $\varphi : Q \to \mathbb{A}^1$ where G acts on $\mathbb{A}^1$ by multiplication and any automorphism of G has dynamical degree* 1.

Proof. Figure 7 gives a tree summarizing how our proof proceeds where each end of the tree corresponds to a conclusion of the previous theorem. Denote by $\mathcal{T}$ the associated Bass–Serre tree constructed in Section 2C.

Theorem 6.3. *Let G be a finitely generated subgroup of the tame group whose action on the complex $\mathcal{C}$ is elementary. The following possibilities occur*:

(i) *The action of G on the complex is elliptic*, *i.e.*, *G fixes globally a vertex in the complex.*

(ii) *The action of G is lineal on the complex*, *i.e.*, *there exists an elliptic subgroup H of G*, *a geodesic line γ on $\mathcal{C}$ invariant by G pointwise fixed by H on which the quotient G/H acts faithfully by translation.*

$$\begin{tikzcd}
\text{nonelementary in } \mathcal{C} & \text{elliptic in } \mathcal{T} & \text{nonelementary in } \mathcal{T} & \text{lineal in } \mathcal{T} \\
 & & \text{conjugated to a subgroup of } \operatorname{Stab}([x]) \arrow[ul] \arrow[u] \arrow[ur] & \\
G \arrow[uu] \arrow[d] \arrow[r] & \text{elliptic in } \mathcal{C} \arrow[ur] \arrow[r] \arrow[dr] & \text{conjugated to a subgroup of } \operatorname{Stab}([x,y]) & \\
\text{lineal in } \mathcal{C} & & \text{conjugated to a subgroup of} O_4 &
\end{tikzcd}$$

Figure 7. Outline scheme of the proof of Theorem 6.2.

Moreover, the subgroup H is conjugated in Tame(Q) *to a subgroup of O_4 or to one of*

$$E_H \rtimes \left\{ \begin{pmatrix} ax & by \\ b^{-1}z & a^{-1}t \end{pmatrix} \,\middle|\, a, b \in k^* \right\}.$$

Assume that the above theorem holds, we prove that our classification holds. If G is lineal on the complex, then assertion (ii) in Theorem 6.3 implies that there exists an hyperbolic automorphism $h \in G$ such that any $f \in G$ can be decomposed into $f = g \circ h^p$ where p is an integer and $g \in H$. This falls into situation (ii) of the theorem.

Let us extend the case (i) of Theorem 6.3. Take a group G whose action fixes a vertex of type III, then it is naturally conjugated to a subgroup of O_4 by Proposition 2.5 and assertion (iii) holds. If G fixes a vertex of type II in the complex, then by Proposition 2.6, G satisfies assertion (iv) of the Theorem.

Suppose now that G fixes a vertex of type I then G is conjugated to a subgroup of $\operatorname{Stab}([x])$ by transitivity of the action on the vertices of type I (Proposition 2.7(ii)). In this situation, recall that we have constructed in Section 2D a natural action from the stabilizer subgroup $\operatorname{Stab}([x])$ on a subtree of the Bass–Serre tree, in particular, there exists a G-equivariant morphism $\varphi : Q \to \mathbb{A}^1$ where the action of G on $\mathbb{A}^1$ is multiplicative. In the case where the group G fixes a vertex of type I, its action on the corresponding subtree of the Bass–Serre tree is either nonelementary or elementary. If the action of G is nonelementary on the subtree, then equivalently G contains two noncommuting morphisms with dynamical degree larger or equal to two and G satisfies assertion (v) in our classification.

Suppose that the action of G on the corresponding subtree of the Bass–Serre tree is elementary. Let us fix an isomorphism $\varphi^{-1}(\mathbb{A}^1 \setminus \{0\}) \simeq \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2$. Since G induces an action on $\mathbb{A}^1$ which is multiplicative, the open subset $\varphi^{-1}(\mathbb{A}^1 \setminus \{0\})$ is preserved by each element of G. Assume that G contains only elliptic elements on the subtree. We reproduce some standard arguments for groups acting on trees. Since G is finitely generated, write $G = \langle g_1, \dots, g_n \rangle$ where g_i are the generators of G. Since G is elliptic, any product $g_i g_j$ is elliptic, hence admits a fixed point on $\mathcal{T}_{\pi,k(x)}$. By [Serre 1977, Proposition 26 page 89], the group G has a global fixed point in $\mathcal{T}_{\pi,k(x)}$. Conjugating by an element in $\operatorname{Stab}([x])$, we can suppose the fixed point in $\mathcal{T}_{\pi,k(x)}$ is either $[z]$ or $[y, z]$. In the first case, this implies that $G \subset \tilde{E}$ and $G \subset \tilde{A}$ in the second, where $\tilde{A}$, $\tilde{E}$ are the groups defined in assertion (v) of Proposition 2.10. This shows that assertion (vii) holds.

Otherwise, the action of G on the subtree is lineal. In particular, there exists an automorphism $h \in G$ whose action on the subtree is hyperbolic such that any $f \in G$ can be decomposed into $f = g \circ h^p$ where p is an integer and where g is an automorphism whose action on the subtree is elliptic and fixes pointwise the geodesic line on the subtree fixed by h.

We have thus proved that assertion (vi) holds. □

Proof of Theorem 6.3. Take G a finitely generated subgroup of the tame group. By [Ballmann and Świątkowski 1999, Theorem 2], the subgroup G' must satisfy one of the following three cases:

(a) The group G' is elliptic.

(b) There exists an integer $2 \geqslant k \geqslant 1$, an elliptic subgroup $H \subset G$ and a subspace $E \subset \mathcal{C}$ which is isometric to a k-dimensional euclidean space, is pointwise fixed by H and on which the group G/H acts as a cocompact lattice of translation.

(c) Every automorphism of G is elliptic and there exists a geodesic half-line and a sequence of vertices v_n on this half-line for which the subgroups $G_n = G \cap \mathrm{Stab}(v_n)$ form an increasing filtration which satisfy $G = \cup G_n$.

Since the complex $\mathcal{C}$ is Gromov hyperbolic, it cannot contain any euclidean plane. As a result, the case $k = 2$ in (b) is excluded. The remaining possibility is when $k = 1$ and there exists a geodesic line E globally invariant by G in the complex, a subgroup H of G fixing E pointwise such that G/H acts faithfully transitive by translation on E. Remark also that (c) cannot hold. Indeed, if there was an increasing sequence of subgroups G_n whose union is G, then the each generator would belong to a certain subgroup. Since G is finitely generated, this would mean that $G = G_n$ for a certain n, which contradicts our assumption.

To prove that (ii) holds amounts in proving that in case (b) the elliptic subgroup H is conjugated to a subgroup of O_4 or to a subgroup of

$$E_H \rtimes \left\{ \begin{pmatrix} ax & by \\ b^{-1}z & a^{-1}t \end{pmatrix} \,\middle|\, a, b \in k^* \right\}.$$

Since H fixes, pointwise, a geodesic line E, we can choose a sequence v_n of distinct vertices near E all fixed by H lying on a quasigeodesic line. Consider γ_n a geodesic path in the 1-skeleton of $\mathcal{C}$ joining v_n and v_{n+1}. Since the group G fixes the type of vertices and the endpoint of γ_n, the geodesic γ_n must be fixed pointwise. If one of the geodesic γ_n contains a vertex of type III, then G is conjugated to a subgroup of O_4 and statement (ii) is proved.

Assume now that the geodesics γ_n contain only type I and II vertices. We prove that (ii) also holds. For simplicity, we can assume that v_0, v_1, v_2 pointwise fixed by G are consecutive vertices on a geodesic line of the 1-skeleton of $\mathcal{C}$. Assume also that v_0, v_2 are of type I and that v_1 is of type II. Conjugating with an element of Tame(Q), we can assume that $v_0 = [z]$, $v_1 = [x, z]$ and $v_2 = [x]$. Since G fixes $[x]$, $[x, z]$

and $[z]$, it implies by Proposition 2.6(iii) that G is conjugated to a subgroup of

$$E_H \rtimes \left\{ \begin{pmatrix} ax & by \\ b^{-1}z & a^{-1}t \end{pmatrix} \,\middle|\, a, b \in k^* \right\},$$

proving (ii) as required. □

6D. ***Proof of Theorem 5 and Corollary 6.*** Take G a finitely generated subgroup of the tame group and take μ a symmetric atomic measure on G whose support generates G and such that

$$\int_G \log(\deg(g))\, d\mu(g) < +\infty.$$

We denote by g_n the state of our random walk at the time n. Observe that since μ is symmetric, Proposition 6.1(iii) implies that $\lambda_2(\mu) = \lambda_1(\mu)$.

Let us explain how we proceed to prove our result. By Theorem 6.2, the group G satisfies one of the following conditions:

(i) The action of G on the complex is nonelementary in $\mathcal{C}$.

(ii) There exists an automorphism h in G whose action in the complex is hyperbolic and such that any automorphism $f \in G$ can be decomposed into $f = g \circ h^p$ where p is an integer and where g belongs to a subgroup H.

Moreover, the subgroup H is conjugated in Tame(Q) to a subgroup of O_4 or to one of

$$E_H \rtimes \left\{ \begin{pmatrix} ax & by \\ b^{-1}z & a^{-1}t \end{pmatrix} \,\middle|\, a, b \in k^* \right\}.$$

(iii) The group G is conjugated to a subgroup of the linear group O_4.

(iv) There exists a G-equivariant morphism $\varphi : Q \to \mathbb{A}^2 \setminus \{(0,0)\}$ where G acts on $\mathbb{A}^2 \setminus \{(0,0)\}$ linearly.

(v) The group G contains two noncommuting automorphisms with dynamical degree larger or equal to 2 and there exists a G-equivariant morphism $\varphi : Q \to \mathbb{A}^1$ where G acts on $\mathbb{A}^1$ by multiplication.

(vi) The group G contains an automorphism h with $\lambda_1(h) \geqslant 2$ and there exists a G-equivariant morphism $\varphi : Q \to \mathbb{A}^1$ on which G acts on $\mathbb{A}^1$ by multiplication and an isomorphism $\varphi^{-1}(\mathbb{A}^1 \setminus \{0\}) \simeq \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2$ such that any automorphisms $f \in G$ can be decomposed into $g \circ h^p$ where p is an integer and g is of the form

$$g : (x, y, z) \in \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2 \mapsto (ax, by, cz) \in \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2,$$

where $a, b, c \in k^*$.

(vii) There exists a G-equivariant morphism $\varphi : Q \to \mathbb{A}^1$ where G acts on $\mathbb{A}^1$ by multiplication and any automorphism of G has dynamical degree 1.

Denote by $\lambda = \lambda_1(\mu) = \lambda_2(\mu)$. We shall prove successively the following implications: (i) $\Rightarrow (\lambda > 0)$, (v) $\Rightarrow (\lambda > 0)$, ((ii), (iv) or (vi)) $\Rightarrow (\lambda = 0)$, ((iii) or (vii)) $\Rightarrow (\lambda = 0)$. If all the above implications hold, then both Theorem 5 and Corollary 6 hold.

The essential ingredient to compute the degree exponents in situation (i) and (v) is the following result. Suppose that G acts on a Gromov-hyperbolic space (X, d) and fix a reference vertex x_0 in X, then a random path $(\mathrm{Id}, g_1, \dots, g_n, \dots)$ in the group G induces a random path in X given by $(x_0, g_1 \cdot x_0, \dots, g_n \cdot x_0, \dots)$. The following theorem is due to Maher and Tiozzo [2018, Theorem 1.2].

Theorem 6.4. *Let G be a nonelementary countable subgroup of the tame group and let μ be an atomic measure on G whose support generates G and such that the integral*

$$\int_G d(g \cdot v_0, v_0)\, d\mu(g)$$

is finite. Then there exists a constant $L > 0$ such that for almost every sample path in the group, one has

$$\lim_{n \to +\infty} \frac{d(g_n \cdot v_0, v_0)}{n} = L.$$

We will apply this result for $X = \mathcal{C}$ and $X = \mathcal{T}$ in situation (i) and (v) respectively.

Let us prove the implication (i) $\Rightarrow$ ($\lambda > 0$). Suppose that G is nonelementary in $\mathcal{C}$. By Theorem 4, there exists $C, C' > 0$ such that for any $g \in G$

$$\log(\deg(g)) \geqslant C d_{\mathcal{C}}([\mathrm{Id}], g \cdot [\mathrm{Id}]) \log\left(\tfrac{4}{3}\right) + C'. \tag{30}$$

In particular, the previous inequality and (29) imply that

$$C \log\left(\tfrac{4}{3}\right) \int d_{\mathcal{C}}([\mathrm{Id}], g \cdot [\mathrm{Id}])\, d\mu(g) + C' \leqslant \int_G \log(\deg(g))\, d\mu(g) < +\infty.$$

As G is nonelementary, Theorem 6.4 states that there exists a constant $L > 0$ such that for almost every sample path

$$\liminf_{n \to +\infty} \frac{d_{\mathcal{C}}([\mathrm{Id}], g_n \cdot [\mathrm{Id}])}{n} = L. \tag{31}$$

Moreover, by Theorem 4, the following inequality holds:

$$\log \deg(g_n) \geqslant C d_{\mathcal{C}}([\mathrm{Id}], g_n \cdot [\mathrm{Id}]) \log\left(\tfrac{4}{3}\right) + C', \tag{32}$$

where $C, C' > 0$. As a result, (31) and (32) imply that

$$\frac{\log \deg(g_n)}{n} \geqslant \frac{C}{n} d_{\mathcal{C}}([\mathrm{Id}], g_n \cdot [\mathrm{Id}]) \log\left(\tfrac{4}{3}\right) + \frac{C'}{n},$$

hence taking the limit as $n \to +\infty$ yields

$$\lambda \geqslant CL \log\left(\tfrac{4}{3}\right) > 0,$$

and we have proved that $\lambda > 0$, as required. The implication (i) $\Rightarrow$ ($\lambda > 0$) holds.

Let us prove the implication (v) $\Rightarrow$ ($\lambda > 0$). Suppose that G satisfies condition (v). By conjugation, we can suppose that G is a subgroup of $\mathrm{Stab}([x])$. We first relate the distance in the tree $\mathcal{T}_{\pi,k(x)}$ with the degree. Recall from Proposition 2.10(v) that the group $\mathrm{Stab}([x])$ is the amalgamated product $\tilde{E} * \tilde{A}$. Any element g which is neither conjugated to an element of $\tilde{E}$ or $\tilde{A}$ can be decomposed by an alternating product:

$$g = e_1 \circ a_1 \circ e_2 \circ a_2 \cdots \circ a_{p-1} \circ e_p$$

where $e_i \in \tilde{E} \setminus \tilde{A}$, $a_i \notin \tilde{A} \setminus \tilde{E}$. This decomposition reflects the length of the geodesic joining $[\mathrm{Id}] = [y, z]$ and $g[\mathrm{Id}]$ in the sense that

$$d_{\mathcal{T}_{\pi,k(x)}}(g \cdot [\mathrm{Id}], [\mathrm{Id}]) = 2p. \tag{33}$$

Note that when g is elementary or affine, then the above inequality is also true, so we obtain that it holds for any $g \in G$. Using the proof of assertion (vi) of Lemma 3.6 (more precisely (14)), we have

$$\deg\left(\prod i = 1^k e_i \circ a_i\right) \geqslant \prod_{i=1}^{k} d_i \geqslant 2^k,$$

where d_i are the degree in y of the polynomials defining e_i.

In particular, using the fact that G contains two noncommuting hyperbolic isometries on $\mathcal{T}_{\pi,k(x)}$ and Theorem 6.4, we obtain similarly that

$$\frac{1}{n}\int_G \log\deg(g)\, d\nu_n(g) \geqslant \frac{1}{2n}\int_G d_{\mathcal{T}}(g \cdot [\mathrm{Id}], [\mathrm{Id}])\, d\nu_n(g) \to \frac{L}{2},$$

as $n \to +\infty$ where $L > 0$ is the drift of the associated to the random walk on the tree $\mathcal{T}$. We have thus proven that $\lambda > 0$ and the implication (v) $\Rightarrow$ ($\lambda > 0$) holds.

Let us prove that the implication ((ii) or (vi)) $\Rightarrow$ ($\lambda = 0$) holds. Since the proof of the two implications (ii) $\Rightarrow$ ($\lambda = 0$) and (vi) $\Rightarrow$ ($\lambda = 0$) are very similar, we will only give the proof of (ii) $\Rightarrow$ ($\lambda = 0$). Suppose that there exists an hyperbolic automorphism $h \in G$ such that any automorphism $f \in G$ can be decomposed into $f = u(f) \circ h^{p(f)}$ where $p(f)$ is an integer and $u(f)$ belongs to H. We thus have

$$\frac{1}{n}\int_G \log\deg(g)\, d\nu_n(g) = \frac{1}{n}\int_G \log\deg(u(g) \circ h^{p(g)})\, d\nu_n(g).$$

By the submultiplicativity of the degree, we have

$$\deg(u(g) \circ h^{p(g)}) \leqslant C \deg(u(g)) \deg(h^{p(g)})$$

where $C > 0$. In particular, we obtain

$$\frac{1}{n}\int_G \log\deg(g)\, d\nu_n(g) \leqslant \frac{C}{n} + \frac{1}{n}\int_G \log(\deg(u(g)))\, d\nu_n(g) + \frac{1}{n}\int_G \log\deg(h^{p(g)})\, d\nu_n(g). \tag{34}$$

Since the map $p : G \to \mathbb{Z}$ is a morphism of groups, the random walk on G induces a random walk on $\mathbb{Z}$ with transition given by the distribution $p_*\mu$. As the measure $p_*\mu$ is also symmetric, the law of large numbers implies that

$$\frac{1}{n}\int_G \log\deg(h^{p(g)})\, d\nu_n(g) \to 0,$$

as $n \to +\infty$. Observe also that there exists a constant $M > 0$ such that $\deg(u(g)) \leqslant M$ for all $g \in G$. In particular, the integral

$$\frac{1}{n}\int_G \log \deg(u(g))\, d\nu_n(g) \to 0,$$

as $n \to +\infty$. Since each term on the right hand side of (34) tends to zero, we have thus proven that

$$\lambda = \lim_{n\to+\infty} \frac{1}{n}\int_G \log \deg(g)\, d\nu_n(g) = 0$$

and the implication (ii) $\Rightarrow$ $(\lambda = 0)$ holds.

Let us prove that the implication ((iii), (iv) or (vii)) $\Rightarrow$ $(\lambda = 0)$ holds. Observe that if G satisfies assertion (iii) or (iv) or (vi) then the degree of any element of G is uniformly bounded, hence the degree exponent is zero. We have thus proved the implication ((iii), (iv) or (vii)) $\Rightarrow$ $(\lambda = 0)$.

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