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Projective orbifolds of Nikulin type

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We study projective irreducible symplectic orbifolds of dimension four that are deformations of partial resolutions of quotients of hyperkähler manifolds of $K3^{[2]}$-type by symplectic involutions; we call them orbifolds of Nikulin type. We first classify those projective orbifolds that are really quotients, by describing all families of projective fourfolds of $K3^{[2]}$-type with a symplectic involution and the relation with their quotients, and then study their deformations. We compute the Riemann–Roch formula for Weil divisors on orbifolds of Nikulin type and using this we describe the first known locally complete family of singular irreducible symplectic varieties as double covers of special complete intersections $(3, 4)$ in $\mathbb{P}^6$.

1. Introduction

One of the three building blocks [Beauville 1983] of Ricci-flat compact Kähler manifolds, together with Abelian varieties and Calabi–Yau manifolds, are irreducible holomorphic symplectic manifolds, i.e., simply connected manifolds X such that $H^{2,0}(X) = \mathbb{C}\omega_X$ is spanned by a symplectic holomorphic form. This kind of manifolds, also known as irreducible hyperkähler, has been deeply studied ever since the foundational works of Beauville [1983], Bogomolov [1978] and Fujiki [1983].

The first and lower dimensional examples of irreducible holomorphic symplectic manifolds are $K3$ surfaces; a second series of deformation families is given by manifolds of $K3^{[n]}$-type, i.e., deformations of Hilbert schemes $W^{[n]}$ of n points on a $K3$ surface W. Together with generalized Kummer manifolds of dimension $2n$ and two deformation families in dimension six and ten constructed by O'Grady, these are all the infinitely many deformation families of irreducible holomorphic manifolds which are currently known.

A natural attempt at constructing new families, already described by Fujiki [1983], is to study quotients of irreducible holomorphic symplectic manifolds by finite symplectic group actions i.e., those actions which preserve the symplectic form. Symplectic involutions σ on smooth $K3$ surfaces W are nowadays well understood thanks to foundational works of Nikulin [1979a], Morrison [1984] and then of van Geemen and Sarti [2007]. The quotient W/σ admits a resolution of singularities with Picard number ≥ 8 which is again a $K3$ surface, called a Nikulin surface. In higher dimension the quotient of a smooth manifold of $K3^{[n]}$-type by a symplectic action does not admit any desingularization being irreducible holomorphic symplectic.

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More recently, starting from work of Beauville [2000], several authors began to study the question of how to enlarge the class of irreducible holomorphic symplectic manifolds while keeping valid most of their distinguished geometrical properties. One of the main directions has been to admit symplectic varieties with mild singularities. Many definitions can be found in the literature and we refer the interested reader to the nice survey by Perego [2020], and references therein, for more details. Beauville considered the class of *symplectic varieties* which admit a symplectic form ω on the smooth locus and have symplectic singularities i.e., singularities such that the holomorphic 2-form ω extends to any resolution. Nowadays, a class which has attracted great attention is that of *irreducible symplectic orbifolds* [Campana 2004]. They naturally appear as building blocks in the generalization of Beauville–Bogomolov decomposition theorem to compact connected Kähler Ricci-flat orbifolds. A compact Kähler orbifold Y is said irreducible symplectic if $Y \setminus \mathrm{Sing}(Y)$ is simply connected and admits a unique, up to a scalar multiple, nondegenerate holomorphic 2-form.

This paper focuses on a special deformation family of irreducible symplectic orbifolds, which we call *orbifolds of Nikulin type*. Those are constructed as deformations of *Nikulin orbifolds*, whose construction mimics that of Nikulin surfaces (see Definition 3.1). The quotient X/σ of a fourfold X of $K3^{[2]}$-type by a symplectic involution σ is singular along a $K3$ surface and 28 points. As we mentioned above, X/σ does not admit a crepant resolution, but one can partially resolve it blowing up the singular $K3$ surface. This partial resolution Y is an irreducible symplectic orbifold with 28 terminal points, which we call Nikulin orbifold. By [Beauville 2000], orbifolds of Nikulin type are also irreducible symplectic varieties, and the two moduli spaces constructions in [Bakker and Lehn 2022; Menet 2020] agree for this deformation family. Examples were already studied by Markushevich and Tikhomirov [2007] and the main properties of the whole deformation family have been described by Menet [2015] and Menet and Rieß [2020; 2021]. It is worth noticing that not all orbifolds of Nikulin type are Nikulin orbifolds, in fact the latter sit in a family of codimension one. As in the case of $K3$ surfaces, orbifolds of Nikulin type are Kähler and irreducible symplectic but in general not projective. The projective ones correspond to divisors in the period domain of orbifolds of Nikulin type [Menet 2020; Bakker and Lehn 2022].

In many aspects the theory of irreducible symplectic manifolds/varieties/ orbifolds is a generalization to higher dimensions of that of $K3$ surfaces. Most notably, the group $H^2(X,\mathbb{Z})$ can be endowed with an integral quadratic form q_X, so-called Beauville–Bogomolov–Fujiki form (for short, BBF form), and it is a lattice L of signature $(3, b_2(X)-3)$, which is a topological invariant of the deformation family; the existence of this lattice structure allows to study moduli spaces of irreducible symplectic manifolds of a fixed deformation type through periods since a global Torelli theorem, analogous to the one for $K3$ surfaces, also holds. However, a remarkable difference with the theory of $K3$ surfaces is the lack of projective models for general higher dimensional algebraic examples. They are crucial for the understanding of the geometric behavior of these varieties. For this reason, in the early development of the theory of irreducible holomorphic symplectic manifolds, a lot of effort has been put into constructing so called locally complete families of these, i.e., general elements in the family of manifolds with a given degree and type of polarization. Historically, the first known locally complete families of projective irreducible holomorphic symplectic manifolds were the family of Fano varieties of smooth cubic fourfolds, shown to be of $K3^{[2]}$-type by Beauville and Donagi [1985], and the family of double EPW sextics, again of $K3^{[2]}$-type, discovered by O'Grady [2005]. A few more families have been constructed in [Debarre and Voisin 2010; Iliev and Ranestad 2001; Iliev et al. 2019; Lehn et al. 2017; Bayer et al. 2021]: all are algebraic manifolds of $K3^{[n]}$-type for some n and their families have codimension one inside their respective moduli spaces. However, in the case of singular orbifolds no locally complete family has been constructed so far.

The main aim of this paper is to provide tools to study the explicit geometry of orbifolds of Nikulin type. We do it by addressing the following problems that we discuss separately in the remaining part of the introduction:

(1) Classify projective fourfolds of $K3^{[2]}$-type with symplectic involutions and related Nikulin orbifolds.

(2) Provide a Riemann–Roch formula for linear systems on orbifolds of Nikulin type.

(3) Describe a locally complete family of orbifolds of Nikulin type.

1A. ***Classification of polarized Nikulin orbifolds.*** The first aim of this paper is to describe the families of projective Nikulin orbifolds, i.e., the algebraic Noether–Lefschetz locus in the family of Nikulin orbifolds, in analogy with what has been done by van Geemen and Sarti for projective Nikulin surfaces. This is achieved in two steps: first we classify all families (infinitely many of those) of projective fourfolds of $K3^{[2]}$-type X carrying a symplectic involution σ; then we describe the corresponding families of projective Nikulin orbifolds Y.

In Section 2, we look at symplectic involutions σ on projective fourfolds of $K3^{[2]}$-type of degree $2d$. We describe their possible Picard lattices and transcendental groups; as a consequence we identify their families in terms of lattice polarized families of fourfolds of $K3^{[2]}$-type. We prove the following result (see Table 1), which is the analogue of the result [van Geemen and Sarti 2007, Proposition 2.2] for $K3$ surfaces with a symplectic involution.

Theorem 1.1. *Let X be a generic projective fourfold of $K3^{[2]}$-type admitting a symplectic involution. Then the pair* $(\mathrm{T}_X, \mathrm{NS}(X))$ *of the transcendental lattice and the Néron–Severi group of X is one of the following*:

- $(U^{\oplus 2} \oplus E_8(-2) \oplus \langle -2d\rangle \oplus \langle -2\rangle,\ \Lambda_{2d})$.
- $(U^{\oplus 2} \oplus D_4(-1) \oplus \langle -2d\rangle \oplus \langle -2\rangle^{\oplus 5},\ \Lambda_{2d})$, *with* $d \equiv 1 \bmod 2$.
- $(U^{\oplus 2} \oplus E_8(-2) \oplus K_d,\ \Lambda_{2d})$ *with* $d \equiv 3 \bmod 4$.
- $(U^{\oplus 2} \oplus D_4(-1) \oplus \langle -2d\rangle \oplus \langle -2\rangle^{\oplus 5},\ \widetilde{\Lambda}_{2d})$ *with* $d \equiv 0 \bmod 2$.

Where the lattices involved are defined in the notation in Section 2A and d is a positive integer.

Vice versa if X is a projective fourfold of $K3^{[2]}$-type such that $\mathrm{NS}(X)$ *is isometric either to* Λ_{2d} *or to* $\widetilde{\Lambda}_{2d}$*, then it admits a symplectic involution.*

In Section 2B we show that the general member of the above lattice polarized families can be described either as Hilbert scheme of two points on a $K3$ surface or as moduli space of (possibly twisted) sheaves on a $K3$ surface; see Table 2. In both cases the $K3$ surfaces involved lie in 12-dimensional families of lattice polarized $K3$ surfaces and are resolution of singular $K3$ surfaces with 7 nodes.

In Section 3 we consider the quotient X/σ and the corresponding Nikulin orbifold Y. The knowledge of the Néron–Severi group and of the transcendental lattice of X allows one to compute the ones of Y and thus the family of fourfolds of $K3^{[2]}$-type X determines the family of the Nikulin orbifolds Y. In particular we prove the following result (see Table 3), which is the analogue of the result [Garbagnati and Sarti 2008, Corollary 2.2] for Nikulin surfaces.

Theorem 1.2. *Let X be a generic projective fourfold of $K3^{[2]}$-type admitting a symplectic involution σ and Y be the corresponding Nikulin orbifold. Then the pair $(\mathrm{T}_X, \mathrm{NS}(X))$ determines uniquely the transcendental lattice T_Y of Y and vice versa T_Y determines uniquely the pair $(\mathrm{T}_X, \mathrm{NS}(X))$. See Table 3 for the explicit description of T_Y and of its relation with $(\mathrm{T}_X, \mathrm{NS}(X))$.*

In Section 3C we study the $K3$ surface S in the fixed locus of the involution σ on the fourfold of $K3^{[2]}$-type X: we show that there is an isometry between $\mathrm{T}_S \otimes \mathbb{Q}$ and $\mathrm{T}_Y \otimes \mathbb{Q}$, where $\mathrm{T}_\bullet \otimes \mathbb{Q}$ is the transcendental lattice with rational coefficients and Y is the Nikulin orbifold as above (see Proposition 3.16). In particular the Picard number of S is at least 8. Moreover, we conjecture that this isometry holds also with integer coefficients (Conjecture 3.12). We prove the conjecture for many subfamilies of codimension 1, corresponding to Hilbert scheme of points on $K3$ surfaces with natural symplectic involutions, and for two locally complete algebraic families, see Propositions 3.14 and 3.15.

1B. ***Riemann–Roch formula for Nikulin orbifolds and orbifolds of Nikulin type.*** In Section 4 we find the Riemann–Roch formula on the orbifolds of Nikulin type by following step by step the quotient construction of Nikulin orbifolds. Since $H^2(Y, \mathbb{Z})$ is endowed with the BBF quadratic form q_Y, explicitly computed by Menet [2015], the Riemann–Roch formula for a $\mathbb{Q}$-Cartier Weil divisor D can be stated as a relation between $\chi(D)$ and $q_Y(D)$, in the same spirit of [Gross et al. 2003, Example 23.19] and depends also on the number of points where D fails to be Cartier. Using the results from [Buckley et al. 2013; Blache 1996; Camere et al. 2019a] for 2-factorial orbifolds we prove in Corollary 4.4 and in Proposition 4.5 the following result.

Theorem 1.3. *Let Y be an orbifold of Nikulin type and let $D = \frac{m}{2}L$ be a $\mathbb{Q}$-Cartier Weil divisor on Y, with $m \in \mathbb{Z}$ and $L \in \mathrm{NS}(Y)$; let n be the number of points where D fails to be Cartier. Then*

$$\chi(\mathcal{O}(D)) = \tfrac{3}{8}\big(\tfrac{1}{24}m^4 q_Y(L)^2 + m^2 q_Y(L) + 8\big) - \tfrac{1}{16}n,$$

where q_Y denotes the BBF quadratic form on $H^2(Y, \mathbb{Z})$.

In particular, on any orbifold of Nikulin type Y and for any $D \in \mathrm{NS}(Y)$,

$$\chi(\mathcal{O}(D)) = \tfrac{1}{4}(q_Y(D)^2 + 6q_Y(D) + 12).$$

By applying the previous result to some specific divisors on Y, we obtain the dimensions of projective spaces where the quotient X/σ or its partial resolution Y have a natural projective model; see Theorems 4.9, 4.10 and 4.12 and Table 4.

1C. ***A locally complete family of orbifolds of Nikulin type.*** To obtain a locally complete family, we need to understand the projective model of the general elements of a family of irreducible symplectic varieties with a given type of polarization. In Section 5, we describe a locally complete family of polarized orbifolds of Nikulin type of BBF degree 2 (the least possible). As already remarked, this is the first known description of a locally complete family of polarized singular irreducible symplectic varieties; the reader should see this construction as the analogue of O'Grady's double EPW sextics. In this case the analogue of a EPW sextic will be a special complete intersection $(3, 4)$ in $\mathbb{P}^6$.

Theorem 1.4. *The general element Y in a family of orbifolds of Nikulin type with a polarization of BBF degree* 2 *and divisibility* 1 *is a double cover of a special complete intersection* $(3, 4)$ *in* $\mathbb{P}^6$ *branched along a surface of degree* 48.

In Section 5D we discuss the reciprocal of the theorem by describing the possible complete intersections $(3, 4)$ using the Beilinson resolution (see also Problem 5.10). Our strategy to prove Theorem 1.4 is the following. Special examples of orbifolds of Nikulin type of BBF degree 2 are constructed as quotients by a symplectic involution of fourfolds X of $K3^{[2]}$-type with Néron–Severi group isometric to $\widetilde{\Lambda_4}$, which is an extension of index two of $\langle 4\rangle \oplus E_8(-2)$. The polarization of BBF degree 4 on X which is orthogonal to the $E_8(-2)$ summand gives a $2:1$ map (see [Iliev et al. 2017]) to an EPW quartic in the cone $C(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^9$. The symplectic involution on X is then induced by an involution on the linear system of the polarization, i.e., on $\mathbb{P}^9$. After projecting from the $\mathbb{P}^2 \subset \mathbb{P}^9$ which is a component of the fixed locus of the involution on $\mathbb{P}^9$, we obtain a complete intersection $(3, 4)$ in $\mathbb{P}^6$ that is singular in codimension 2 along a surface of degree 52. From the results in Sections 3 and 4 we deduce that the image of the projection is the projective model of the quotient of X by the symplectic involution. By deforming this example and knowing part of the monodromy group of orbifolds of Nikulin type (see [Menet and Rieß 2020]), we prove that a general orbifold of BBF degree 2 as above has a similar description.

2. Fourfolds of $K3^{[2]}$-type with a symplectic involution

We are interested in fourfolds of $K3^{[2]}$-type admitting a symplectic involution and mainly in the projective ones. We will describe the general member of families of fourfolds satisfying these conditions first in a lattice theoretic way and then giving a model as (twisted) moduli space of sheaves on a $K3$ surface. From now on let X be a fourfold of $K3^{[2]}$-type and σ be a symplectic involution on X.

2A. ***Lattice theoretic description of X.*** Let us fix some notation and recall preliminary results on lattices:

- The lattice U is the unique even unimodular lattice of rank 2 and signature $(1, 1)$; we will denote by $\{u_1, u_2\}$ a basis such that $u_1^2 = u_2^2 = 0$ and $u_1 u_2 = 1$.

- The lattice E_8 is the unique even unimodular positive definite lattice of rank 8.

- Given a lattice M and an integer $n \in \mathbb{Z}$, $M(n)$ is the lattice obtained multiplying the bilinear form of M by n.

- We denote by $\{b_1, \dots b_8\}$ the basis of $E_8(-2)$ such that: $b_i^2 = -4$, $i = 1, \dots, 8$; $b_j b_{j+1} = 2$, $j = 1, \dots, 6$; $b_3 b_8 = 2$; the other intersections are zeros.

- The lattice N, called Nikulin lattice, is an even negative definite rank 8 lattice. It is generated by the classes r_i, $i = 1, \dots, 8$ such that $r_i^2 = -2$, $r_i r_j = 0$ and by the class $\frac{1}{2}\big(\sum_{i=1}^{8} r_i\big)$.

- For $n \in \mathbb{Z}$, $u(n)$ is the discriminant form of $U(n)$; for each $m \in \mathbb{Z}$ and $\alpha \in \mathbb{Q}$, $\mathbb{Z}_m(\alpha)$ is the discriminant cyclic group $\mathbb{Z}_m$ endowed with the quadratic form taking value α on a generator. For short, the discriminant quadratic form of $\mathbb{Z}_m\big(\pm\frac{1}{m}\big)$ is denoted by $\big(\pm\frac{1}{m}\big)$.

- The discriminant form of N is $u(2)^{\oplus 3}$ and the discriminant form of $E_8(-2)$ is $u(2)^{\oplus 4}$; see [Nikulin 1983, page 1414].

- The lattice $D_4(-1)$ is the rank 4 negative definite lattice whose bilinear form on the basis $\{d_1, d_2, d_3, d_4\}$ is $d_i^2 = -2$, $d_i d_2 = 1$, $i = 1, 3, 4$, $d_i d_j = 0$ otherwise. Its discriminant group is $(\mathbb{Z}/2\mathbb{Z})^2$ and its discriminant form is called $v(2)$, see e.g., [Nikulin 1979b, Section 8].

- The lattice L_{K3} is the unique even unimodular lattice of rank 22 and signature (3, 19) and is isometric to $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$.

- The lattice $L := L_{K3} \oplus \langle -2 \rangle$ and its discriminant form is $(-\frac{1}{2})$. We will denote by δ the generator of the lattice $\langle -2 \rangle$, orthogonal to L_{K3} in L.

- For every positive integer d, the lattice Λ_{2d} is isometric to $\langle 2d \rangle \oplus E_8(-2)$. We denote by h a generator of the summand $\langle 2d \rangle$.

- For every even positive integer d, the lattice $\widetilde{\Lambda}_{2d}$ is the unique overlattice of index 2 of Λ_{2d} in which $\langle 2d \rangle$ and $E_8(-2)$ are primitively embedded.

- The lattice K_d is the negative definite lattice with the following quadratic form

$$\begin{bmatrix} -\frac{d+1}{2} & 1 \\ 1 & -2 \end{bmatrix}, \quad d \equiv 1 \bmod 2.$$

- The lattice H_d is the indefinite lattice with the following quadratic form

$$\begin{bmatrix} \frac{d-1}{2} & 1 \\ 1 & -2 \end{bmatrix}, \quad d \equiv 1 \bmod 2.$$

- The divisibility $\operatorname{div}(v)$ of $v \in M$ is the generator of the ideal $(vw \mid w \in M) \subset \mathbb{Z}$.

Moreover, since for all the considered varieties there is a canonical isomorphism between the Picard lattice and the Néron–Severi group, we always refer to the Néron–Severi group, even to indicate the Picard lattice.

Proposition 2.1. *A fourfold X of $K3^{[2]}$-type admits a symplectic involution if and only if $E_8(-2)$ is primitively embedded in* $\mathrm{NS}(X)$.

Moreover, if X is projective then there exists a positive $d \in \mathbb{N}$ such that $\Lambda_{2d} \subset \mathrm{NS}(X)$. The Néron–Severi group of the very general element in the family of $\langle 2d\rangle$-polarized fourfolds of $K3^{[2]}$-type admitting a symplectic involution is an overlattice of finite index (possibly equal to 1) of Λ_{2d}, with the property that $E_8(-2)$ is primitively embedded in it.

Proof. The first statement is proved by Mongardi [2012]. If X is projective, then it admits an ample divisor, which has necessarily a positive BBF degree. Since $E_8(-2)$ is negative definite, it follows that $\Lambda_{2d} \subset \mathrm{NS}(X)$ and that if the Picard number of X is the minimal possible, i.e., 9, then $\mathrm{NS}(X)$ is an overlattice of finite index (possibly equal to 1) of Λ_{2d}, with the property that $E_8(-2)$ is primitively embedded in it. □

Lemma 2.2 [van Geemen and Sarti 2007, Proposition 2.2]. *The overlattices of Λ_{2d} containing primitively both $\langle 2d\rangle$ and $E_8(-2)$ are*:

(1) *If $d \equiv 1 \bmod 2$ only Λ_{2d} itself.*

(2) *If $d \equiv 0 \bmod 2$ either Λ_{2d} or the unique overlattice $\widetilde{\Lambda}_{2d}$ of index 2 of Λ_{2d} in which $\langle 2d\rangle$ and $E_8(-2)$ are primitively embedded.*

The discriminant forms of the lattices Λ_{2d} and $\widetilde{\Lambda}_{2d}$ are $\left(\frac{1}{2d}\right) \oplus u(2)^{\oplus 4}$ and $\left(\frac{1}{2d}\right) \oplus u(2)^{\oplus 3}$. The lattices Λ_{2d} and $\widetilde{\Lambda}_{2d}$ admit a unique embedding in L_{K3} (up to isometry).

Proof. The uniqueness of the overlattices is proved in [van Geemen and Sarti 2007, Proposition 2.2], and their discriminant forms are computed in [Camere and Garbagnati 2020, Corollary 3.7]. We briefly recall the proofs here. The lattice Λ_{2d} is described in the list of lattices at the beginning of the section: the discriminant form on $A_{\Lambda_{2d}} = A_{\langle 2d\rangle} \oplus A_{E_8(-2)}$ is $\left(\frac{1}{2d}\right) \oplus u(2)^{\oplus 4}$. We denote $h, u_{i,j}$ for $i = 1, \dots, 4$, $j = 1, 2$ a basis of $A_{\Lambda_{2d}}$ on which the discriminant form is $\left(\frac{1}{2d}\right) \oplus u(2)^{\oplus 4}$. The overlattices $\widetilde{\Lambda}_{2d}$ in which $\langle 2d\rangle$ and $E_8(-2)$ are primitively embedded correspond to isotropic subgroups H of $A_{\Lambda_{2d}}$ which have a nontrivial intersection with both $A_{\langle 2d\rangle}$ and $A_{E_8(-2)}$ in $A_{\Lambda_{2d}}$, by [Nikulin 1979b, Proposition 1.4.1]. So H can be chosen to be generated by $dh + v$, where $v \in A_{E_8(-2)}$ is such that $v^2 = 0$ or 1 respectively when $d \equiv 0 \bmod 4$ or $d \equiv 2 \bmod 4$. We suppose that $d \equiv 0 \bmod 4$ and we can assume that $v = u_{1,1}$. Since $H^{\perp} = \langle h + u_{1,2}, u_{1,1}, u_{i,j} \mid i = 2, 3, 4,\ j = 1, 2\rangle$, $\widetilde{\Lambda}_{2d}$ has discriminant quadratic form $\left(\frac{1}{2d}\right) \oplus u(2)^{\oplus 3}$. The case $d \equiv 2 \bmod 4$ is completely analogous. □

In [van Geemen and Sarti 2007] an explicit basis for the lattice $\widetilde{\Lambda}_{2d}$ is given:

- If $d \equiv 2 \bmod 4$, the lattice $\widetilde{\Lambda}_{2d}$ is generated by the generators of Λ_{2d} and by the class $\frac{1}{2}(h + b_1)$.
- If $d \equiv 0 \bmod 4$, the lattice $\widetilde{\Lambda}_{2d}$ is generated by the generators of Λ_{2d} and by the class $\frac{1}{2}(h + b_1 + b_3)$.

Corollary 2.3. *Let X be a very general element in a family of (possibly not projective) fourfolds of $K3^{[2]}$-type admitting a symplectic involution σ, then $\mathrm{NS}(X) \simeq E_8(-2)$ and vice versa if X is a fourfold of $K3^{[2]}$-type such that $\mathrm{NS}(X) \simeq E_8(-2)$, then X is nonprojective and it admits a symplectic involution.*

Let X be a very general element in a family of projective fourfolds of $K3^{[2]}$-type admitting a symplectic involution σ. Then either $\mathrm{NS}(X) \simeq \Lambda_{2d}$ *for a certain integer $d > 0$ or* $\mathrm{NS}(X) \simeq \widetilde{\Lambda}_{2d}$ *for a certain even integer $d > 0$.*

Vice versa if X is a fourfold of $K3^{[2]}$-type such that $\mathrm{NS}(X)$ *is isometric either to Λ_{2d} for an integer $d > 0$ or to $\widetilde{\Lambda}_{2d}$ for an even integer $d > 0$, then X is projective and admits a symplectic involution.*

We observe that $E_8(-2)$ admits a unique primitive embedding in L, whose orthogonal is $U^{\oplus 3} \oplus E_8(-2) \oplus \langle -2\rangle$.

In order to determine the families of projective fourfolds of $K3^{[2]}$-type admitting a symplectic involution, we consider all possible primitive embeddings of the lattices Λ_{2d} and $\widetilde{\Lambda_{2d}}$ into L.

Proposition 2.4. *For any integer $d > 0$ the lattice Λ_{2d} admits, up to isometry of L, the following primitive embeddings into L*:

(1) *j_1 such that* $j_1(\Lambda_{2d})^\perp \simeq T_{2d,1} := U^{\oplus 2} \oplus E_8(-2) \oplus \langle -2d\rangle \oplus \langle -2\rangle$.

(2) *If $d \equiv 1 \bmod 2$, j_2 such that* $j_2(\Lambda_{2d})^\perp \simeq T_{2d,2} := U^{\oplus 2} \oplus D_4(-1) \oplus \langle -2d\rangle \oplus \langle -2\rangle^{\oplus 5}$.

(3) *If $d \equiv 3 \bmod 4$, j_3 such that* $j_3(\Lambda_{2d})^\perp \simeq T_{2d,3} := U^{\oplus 2} \oplus E_8(-2) \oplus K_d$.

For any $d \equiv 0 \bmod 2$, $\widetilde{\Lambda}_{2d}$ admits a unique primitive embedding $\tilde{j}$ into L, with orthogonal isometric to $\widetilde{T}_{2d} := U^{\oplus 2} \oplus D_4(-1) \oplus \langle -2d\rangle \oplus \langle -2\rangle^{\oplus 5}$.

Proof. First we study possible primitive embeddings of Λ_{2d} inside L. The first embedding j_1 is simply obtained by composing the embedding of Λ_{2d} inside L_{K3} with the embedding of this one inside L. This is unique up to isometry if $d \equiv 0 \bmod 2$.

When $d \equiv 1 \bmod 2$, an application of [Nikulin 1979b, Proposition 1.15.1] shows that there is a second possibility: indeed, in this case $A_{\Lambda_{2d}}$ contains a subgroup H of order two to which the discriminant form restricts as $\left(-\frac{1}{2}\right)$. Standard computations in this case produce the embedding j_2 if $d \equiv 1 \bmod 4$, and the embeddings j_2 and j_3 if $d \equiv 3 \bmod 4$. Up to isometry these are the only possibilities.

Concerning the primitive embeddings of $\widetilde{\Lambda}_{2d}$, $\tilde{j}$ is again obtained by composing the embedding of $\widetilde{\Lambda}_{2d}$ inside L_{K3} with the embedding of this one inside L. The fact that it is the only possible one comes by an application of [loc. cit., Proposition 1.15.1]: we have $A_L \simeq \mathbb{Z}_2\left(-\frac{1}{2}\right)$, whereas the quadratic form on $A_{\widetilde{\Lambda}_{2d}}$ takes values in $\mathbb{Z}/2\mathbb{Z}$ on any subgroup of order two; as a consequence, the only possible choice for two isometric subgroups inside A_L and $A_{\widetilde{\Lambda}_{2d}}$ is $H = \{0\}$, and the discriminant form of the orthogonal R is exactly $(-q_{\widetilde{\Lambda}_{2d}}) \oplus q_{A_L} = u(2)^{\oplus 3} \oplus \left(-\frac{1}{2d}\right) \oplus \left(-\frac{1}{2}\right)$. From [loc. cit., Proposition 1.8.2], we have $u(2)^{\oplus 3} \oplus \left(-\frac{1}{2d}\right) \oplus \left(-\frac{1}{2}\right) \simeq \left(\frac{1}{2}\right)^{\oplus 3} \oplus \left(-\frac{1}{2}\right)^{\oplus 4} \oplus \left(-\frac{1}{2d}\right)$. Moreover, it is easy to show that $\left(\frac{1}{2}\right)^{\oplus 3} \oplus \left(-\frac{1}{2}\right)^{\oplus 4} \simeq v(2) \oplus \left(-\frac{1}{2}\right)^{\oplus 5}$. The signature of R is (2, 12). The genus of the lattices with signature and discriminant form as the ones of R contains a unique class by [loc. cit., Corollary 1.13.3], and so $R \simeq \widetilde{T}_{2d}$. Moreover, by [loc. cit., Theorem 1.14.2], $O(\widetilde{T}_{2d}) \to O(q_{A_{\widetilde{T}_{2d}}})$ is surjective. By [loc. cit., Proposition 1.15.1], we conclude that $\tilde{j}(\widetilde{\Lambda}_{2d})^\perp \simeq \widetilde{T}_{2d}$ ad that the embedding $\tilde{j}$ is unique up to isometries of L. □

To recap, if X is a very general projective fourfold of $K3^{[2]}$-type admitting a symplectic involution, the possibilities for $\mathrm{NS}(X)$ and T_X are found in Table 1.

condition on d	embedding $\mathrm{NS}(X)\subset L$	$\mathrm{NS}(X)$	T_X
$\forall d\in\mathbb{N}$	j_1	Λ_{2d}	$T_{2d,1}:=U^{\oplus 2}\oplus E_8(-2)\oplus\langle -2d\rangle\oplus\langle -2\rangle$
$d\equiv 1 \mod 2$	j_2	Λ_{2d}	$T_{2d,2}:=U^{\oplus 2}\oplus D_4(-1)\oplus\langle -2d\rangle\oplus\langle -2\rangle^{\oplus 5}$
$d\equiv 3 \mod 4$	j_3	Λ_{2d}	$T_{2d,3}:=U^{\oplus 2}\oplus E_8(-2)\oplus K_d$
$d\equiv 0 \mod 2$	$\tilde{j}$	$\widetilde{\Lambda}_{2d}$	$\widetilde{T}_{2d}:=U^{\oplus 2}\oplus D_4(-1)\oplus\langle -2d\rangle\oplus\langle -2\rangle^{\oplus 5}$

Table 1. Possible pairs $(\mathrm{NS}(X),\mathrm{T}_X)$ for general projective fourfolds X of $K3^{[2]}$-type with a symplectic involution.

As observed before, if X is a very general nonprojective fourfold of $K3^{[2]}$-type admitting a symplectic involution, then $\mathrm{NS}(X)=E_8(-2)$ and $\mathrm{T}_X=U^{\oplus 3}\oplus E_8(-2)\oplus\langle -2\rangle$.

As in the case of the $K3$ surfaces, see e.g., [van Geemen and Sarti 2007], to relate the Néron–Severi group of a manifold with an involution to the one of its quotient by the involution, one uses the explicit description of the isometry induced on the second cohomology group by the involution, and the knowledge of a primitive embedding of the Néron–Severi group in the second cohomology group. Therefore here we describe a choice for this embedding, which will be used in Section 3. The uniqueness of the action induced by the involution and of the embeddings up to isometries of the lattice L, will guarantee that the results in Section 3 are independent by the embedding chosen to make the explicit computations.

Hence, we explicitly fix the embeddings j_a, $a=1,2,3$ and $\tilde{j}$ in L which will be used in the following.

Let X be a fourfold of $K3^{[2]}$-type admitting a symplectic involution ι. Fix a basis of $H^2(X,\mathbb{Z})\simeq U^{\oplus 3}\oplus E_8(-1)\oplus E_8(-1)\oplus\langle -2\rangle$: there exists an isometry between $H^2(X,\mathbb{Z})$ and $L=U^{\oplus 3}\oplus E_8(-1)\oplus E_8(-1)\oplus\langle -2\rangle$ such that the involution $\iota^*\in O(H^2(X,\mathbb{Z}))$ switches the two copies of $E_8(-1)$ and acts as the identity on $U\oplus U\oplus U\oplus\langle -2\rangle$. We denote by e_i, (resp. f_i), $i=1,\dots,8$ a basis of the first (resp. second) copy of $E_8(-1)$ in $E_8(-1)\oplus E_8(-1)$, and by b_i a basis of $E_8(-2)$. We fix two different embeddings of the lattice $E_8(-2)$ in $E_8(-1)\oplus E_8(-1)$:

$$\lambda_+(b_i)=e_i+f_i,\ i=1,\dots,8 \quad\text{and}\quad \lambda_-(b_i)=e_i-f_i,\ i=1,\dots,8. \tag{2-1}$$

In particular $H^2(X,\mathbb{Z})^{\iota^*}=U^{\oplus 3}\oplus\lambda_+(E_8(-2))\oplus\langle -2\rangle\simeq U^{\oplus 3}\oplus E_8(-2)\oplus\langle -2\rangle$ and $(H^2(X,\mathbb{Z})^{\iota^*})^{\perp}=\lambda_-(E_8(-2))\simeq E_8(-2)$.

Let $h\in H^2(X,\mathbb{Z})$ be a ι-invariant primitive class with self-intersection $2d>0$. Let us denote by $j(h)$ an embedding of h in $H^2(X,\mathbb{Z})\simeq L$. Since the polarization h is invariant for ι, $j(h)\in H^2(X,\mathbb{Z})^{\iota^*}\simeq U^{\oplus 3}\oplus\lambda_+(E_8(-2))\oplus\langle -2\rangle$ and thus it corresponds to a vector of the form $(\underline{u},\underline{w},\underline{v},\underline{x},\underline{y},k)\in U^{\oplus 3}\oplus E_8(-1)^{\oplus 2}\oplus\langle -2\rangle$ such that $\underline{x}=\underline{y}$.

Proposition 2.5. *Let d be a positive integer and let*

$$j_1(h):=\left(\begin{pmatrix}1\\d\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix},\underline{0},\underline{0},0\right).$$

The embedding $(j_1, \lambda_-) : \langle 2d\rangle \oplus E_8(-2) \to L$ *is a primitive embedding and there exist fourfolds of* $K3^{[2]}$*-type* X_1 *such that* $\mathrm{NS}(X_1) \simeq (j_1, \lambda_-)(\langle 2d\rangle \oplus E_8(-2)) \simeq \Lambda_{2d}$ *and* $\mathrm{T}_{X_1} \simeq T_{2d,1}$.

Proof. The embedding (j_1, λ_-) is clearly primitive, hence there exist fourfolds of $K3^{[2]}$-type admitting this lattice as Néron–Severi group. Since j_1 restricts to an embedding of h in U and λ_- restricts to an embedding of $E_8(-2)$ in $E_8(-1) \oplus E_8(-1)$, one can compute separately the orthogonal in the different direct summands, finding that the orthogonal to $\mathrm{NS}(X_1)$ is $\langle -2d\rangle \oplus U \oplus U \oplus \lambda_+(E_8(-2)) \oplus \langle -2\rangle \simeq T_{2d,1}$. □

Proposition 2.6. *Let d be an odd positive integer. Let*

$$j_2(h) := \left(\begin{pmatrix} 2 \\ 2k+2\end{pmatrix}, \begin{pmatrix} 0 \\ 0\end{pmatrix}, \begin{pmatrix} 0 \\ 0\end{pmatrix}, \underline{e_1}, \underline{e_1}, 1\right) \qquad \text{if } d = 4k+1,$$
$$j_2(h) := \left(\begin{pmatrix} 2 \\ 2k+2\end{pmatrix}, \begin{pmatrix} 0 \\ 0\end{pmatrix}, \begin{pmatrix} 0 \\ 0\end{pmatrix}, \underline{e_1}+\underline{e_3}, \underline{e_1}+\underline{e_3}, 1\right) \quad \text{if } d = 4k-1.$$

The embedding $(j_2, \lambda_-) : \langle 2d\rangle \oplus E_8(-2) \to L$ *is a primitive embedding and there exist fourfolds of* $K3^{[2]}$*-type* X_2 *such that* $\mathrm{NS}(X_2) \simeq (j_2, \lambda_-)(\langle 2d\rangle \oplus E_8(-2)) \simeq \Lambda_{2d}$ *and* $\mathrm{T}_{X_2} \simeq T_{2d,2}$.

Proof. The embedding (j_2, λ_-) is clearly primitive, hence there exist fourfolds of $K3^{[2]}$-type admitting this lattice as Néron–Severi group. By Proposition 2.4 there is an embedding of $\langle 2d\rangle \oplus E_8(-2)$ in $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2\rangle$ which is not equivalent to j_1, computed in Proposition 2.5.

Let

$$\underline{x} = \begin{cases} \underline{e_1} & \text{if } d \equiv 1 \bmod 4, \\ \underline{e_1} + \underline{e_3} & \text{if } d \equiv 3 \bmod 4. \end{cases}$$

By direct computation, the orthogonal lattice $((j_2, \lambda_-)(\Lambda_{2d}))^\perp$ is spanned by the following vectors:

$$(\underline{0}, \underline{a_i}, \underline{0}, \underline{0}, \underline{0}, 0), (\underline{0}, \underline{0}, \underline{a_i}, \underline{0}, \underline{0}, 0), \quad i = 1, 2 \text{ where } a_1, a_2 \text{ is a basis of } U;$$
$$\left(\begin{pmatrix} -1 \\ k+1\end{pmatrix}, \underline{0}, \underline{0}, \underline{0}, \underline{0}, 0\right), \left(\begin{pmatrix} 0 \\ 1\end{pmatrix}, \underline{0}, \underline{0}, \underline{0}, \underline{0}, 1\right), (\underline{0}, \underline{0}, \underline{0}, \underline{w}, \underline{w}, 0), \ \underline{w} \in (\underline{x})^\perp_{E_8(-1)},$$
$$b := (\underline{0}, \underline{0}, \underline{0}, \underline{y}, \underline{y}, 1) \quad \text{with } \underline{y} = \begin{cases} \underline{e_2} & \text{if } d \equiv 1 \bmod 4, \\ \underline{e_4} & \text{if } d \equiv 3 \bmod 4. \end{cases}$$

One can directly compute the form on the previous basis and hence its discriminant form. By [Nikulin 1979b, Corollary 1.13.3] one obtains that there exists a unique, up to isometry, even lattice with signature (2, 12) and the required discriminant form. Such a lattice is isometric to $T_{2d,2}$. □

Proposition 2.7. *Let d be a positive integer such that* $d \equiv 3 \bmod 4$. *Let*

$$j_3(h) := \left(\begin{pmatrix} 2 \\ (d+1)/2\end{pmatrix}, \begin{pmatrix} 0 \\ 0\end{pmatrix}, \begin{pmatrix} 0 \\ 0\end{pmatrix}, \underline{0}, \underline{0}, 1\right).$$

The embedding $(j_3, \lambda_-) : \langle 2d\rangle \oplus E_8(-2) \to L$ *is a primitive embedding and there exist fourfolds of* $K3^{[2]}$*-type* X_3 *such that* $\mathrm{NS}(X_3) \simeq (j_3, \lambda_-)(\langle 2d\rangle \oplus E_8(-2)) \simeq \Lambda_{2d}$ *and* $\mathrm{T}_{X_3} \simeq T_{2d,3}$.

Proof. The embedding (j_3, λ_-) is clearly a primitive embedding of $\langle 2d\rangle \oplus E_8(-2)$ in L and hence there exist fourfolds of $K3^{[2]}$-type admitting this lattice as Néron–Severi group. Since j_3 restricts to an embedding of h in $U \oplus \langle -2\rangle$, one can compute the orthogonal of $j_3(h)$ in $U \oplus \langle -2\rangle$, which is generated by $\left(\binom{0}{1}, 1\right)$ and $\left(\binom{1}{-(d+1)/4}, 0\right)$, with intersection form equal to K_d, so that $\mathrm{T}_{X_3} \simeq T_{2d,3}$. □

Proposition 2.8. *Let d be an even positive integer. Let*

$$\tilde{j}(h) := \left(\begin{pmatrix}2\\2k\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \underline{e_1}, \underline{e_1}, 0\right) \qquad \text{if } d = 4k-2,$$
$$\tilde{j}(h) := \left(\begin{pmatrix}2\\2k\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \underline{e_1}+\underline{e_3}, \underline{e_1}+\underline{e_3}, 0\right) \quad \text{if } d = 4k-4.$$

The embedding $(\tilde{j}, \lambda_-) : \langle 2d\rangle \oplus E_8(-2) \to L$ is not a primitive embedding and the primitive closure of $(\tilde{j}, \lambda_-)(\langle 2d\rangle \oplus E_8(-2))$ is isometric to $\widetilde{\Lambda}_{2d}$. There exist fourfolds of $K3^{[2]}$-type $\widetilde{X}$ such that $\mathrm{NS}(\widetilde{X}) \simeq \widetilde{\Lambda}_{2d}$ and $\mathrm{T}_{\widetilde{X}} \simeq \widetilde{T}_{2d}$.

Proof. Let us consider the case $d = 4k-2$, i.e., $d \equiv 2 \mod 4$. The embedding $(\tilde{j}, \lambda_-)$ is not primitive, since the class $\tilde{j}(h) + \lambda_-(b_1)$ can be divided by 2 in $U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1) \oplus \langle -2\rangle$, whereas $h + b_1$ is primitive inside $\langle 2d\rangle \oplus E_8(-2)$. By adding the class $\frac{1}{2}(\tilde{j}(h) + \lambda_-(b_1))$ to $(\tilde{j}, \lambda_-)(\langle 2d\rangle \oplus E_8(-2))$ one obtains a primitive embedding of $\widetilde{\Lambda}_{2d}$ in L. In particular there exists a fourfold of $K3^{[2]}$-type with $\mathrm{NS}(\widetilde{X}) \simeq \widetilde{\Lambda}_{2d}$ and, by computing its orthogonal complement, one finds $\mathrm{T}_{\widetilde{X}} \simeq U^{\oplus 2} \oplus \langle -2d\rangle \oplus D_4(-1) \oplus \langle -2\rangle^{\oplus 5}$.

The case $d = 4k-4$ is analogous. □

2B. ***Models of X as moduli space of sheaves on a K3 surface.*** In this section we provide at least one model of the very general member of each family of projective fourfolds of $K3^{[2]}$-type admitting a symplectic involution, i.e., of each family described in Table 1. Each of these models will be described either as Hilbert scheme of a certain $K3$ surface or as a moduli space of stable, possibly twisted, sheaves on a $K3$ surface. The main results of this section are summarized in Table 2.

One needs two preliminary definitions in order to list all cases.

Definition 2.9. If $d \equiv 3 \mod 4$, we denote by $(\langle 2d\rangle \oplus \langle -2\rangle^{\oplus 7})'$ the overlattice of $\langle 2d\rangle \oplus \langle -2\rangle^{\oplus 7} = \mathbb{Z}t \bigoplus \oplus_i \mathbb{Z}n_i$ obtained by adding to $\langle 2d\rangle \oplus \langle -2\rangle^{\oplus 7}$ the class $\frac{1}{2}\left(t + \sum_i n_i\right)$.

Lemma 2.10. *The lattice $\langle 2d\rangle \oplus \langle -2\rangle^{\oplus 7}$ admits a unique primitive embedding in L_{K3} and its orthogonal is uniquely determined and isometric to $U^{\oplus 2} \oplus D_4(-1) \oplus \langle -2d\rangle \oplus \langle -2\rangle^{\oplus 5}$.*

If $d \equiv 3 \mod 4$ the lattice $(\langle 2d\rangle \oplus \langle -2\rangle^{\oplus 7})'$ admits a unique primitive embedding in L_{K3} and its orthogonal is uniquely determined and isometric to $U^{\oplus 2} \oplus N \oplus K_d$.

Proof. The discriminant quadratic form of $Q := \langle 2d\rangle \oplus \langle -2\rangle^{\oplus 7}$ is $\left(\frac{1}{2d}\right) \oplus \left(-\frac{1}{2}\right)^{\oplus 7}$. Since L_{K3} is unimodular, the orthogonal $Q^\perp$ needs to have discriminant quadratic form

$$\left(-\tfrac{1}{2d}\right) \oplus \left(\tfrac{1}{2}\right)^{\oplus 7} \simeq \left(-\tfrac{1}{2d}\right) \oplus v(2) \oplus \left(-\tfrac{1}{2}\right)^{\oplus 5}$$

and signature $(2, 12)$; by [Nikulin 1979b, Corollary 1.13.3], there is, up to isometry, a unique lattice with these properties, which is $U^{\oplus 2} \oplus D_4(-1) \oplus \langle -2d\rangle \oplus \langle -2\rangle^{\oplus 5}$, thus the embedding is unique up to isometry of L_{K3}.

embedding $\mathrm{NS}(X) \subset L$	$K3$	model	v	Proposition
$j_1, d \equiv 1 \bmod 2$	W_d	$M_v(W_d, \beta)$	$(0, H', 2)$	Proposition 2.14
$j_1, d \equiv 0 \bmod 2$	W_d	$M_v(W_d, \beta)$	$\left(4, \sum_{i=1}^{7} n_i, 2\right)$	Proposition 2.16
$j_2, d \equiv 1 \bmod 2$	W_d	$W_d^{[2]}$	$-$	Proposition 2.12
$j_3, d \equiv 3 \bmod 4$	W'_d	$M_v(W'_d, \beta)$	$(0, H', 2)$	Proposition 2.15
$\tilde{j}, d \equiv 0 \bmod 2$	W_d	$M_v(W_d)$	$\left(2, \sum_{i=1}^{7} n_i, 4\right)$	Proposition 2.16

Table 2. Birational models of X in the different families.

The discriminant quadratic form of $Q' := (\langle 2d \rangle \oplus \langle -2 \rangle^{\oplus 7})'$ is $\left(\frac{2}{d}\right) \oplus u(2)^{\oplus 3}$, hence its orthogonal in L_{K3} has discriminant quadratic form $\left(-\frac{2}{d}\right) \oplus u(2)^{\oplus 3}$ and signature $(2, 12)$: again by [Nikulin 1979b, Corollary 1.13.3], there is, up to isometry, a unique lattice with these properties, which is $U^{\oplus 2} \oplus N \oplus K_d$. □

The previous lemma implies that there exists a well defined family of $K3$ surfaces which is polarized with the lattice $\langle 2d \rangle \oplus \langle -2 \rangle^{\oplus 7}$ (resp. $(\langle 2d \rangle \oplus \langle -2 \rangle^{\oplus 7})'$).

Definition 2.11. For any positive integer d, W_d is a $K3$ surface such that $\mathrm{NS}(W_d) = \langle 2d \rangle \oplus \langle -2 \rangle^{\oplus 7}$.

For the positive integers d such that $d \equiv 3 \bmod 4$, W'_d is a $K3$ surface such that $\mathrm{NS}(W'_d) = (\langle 2d \rangle \oplus \langle -2 \rangle^{\oplus 7})'$

By the previous lemma, the transcendental lattices of the surfaces W_d and W'_d are respectively $\mathrm{T}_{W_d} \simeq U^{\oplus 2} \oplus D_4(-1) \oplus \langle -2d \rangle \oplus \langle -2 \rangle^{\oplus 5}$ and $\mathrm{T}_{W'_d} \simeq U^{\oplus 2} \oplus N \oplus K_d$.

In the following we will denote by H' a primitive vector in

$$\langle 2d \rangle \oplus \langle -2 \rangle^{\oplus 7} \quad \text{or} \quad (\langle 2d \rangle \oplus \langle -2 \rangle^{\oplus 7})'$$

whose square is 2. It surely exists by Lagrange's four squares theorem.

Table 2 summarizes all the birational models given for X: in the first column we identify the family of fourfolds which we are considering (and this is done by exhibiting the embedding $\mathrm{NS}(X) \subset L$, using the results in Table 1); in the second column we declare which $K3$ surface is associated to the model; in the third we describe the model; if the model of X is as moduli space of sheaves determined by a Mukai vector, in the fourth column we write the Mukai vector (we omit the element in the Brauer group giving the twist, when needed); in the last column we give the reference to the proposition where we describe the model and prove that it is the required one.

The easiest description of the fourfold that we obtain is the one associated to the embedding $j_2 : \mathrm{NS}(X) \hookrightarrow L$, indeed in this case X is (birational to) a Hilbert scheme of points on a $K3$ surface, by the following.

Proposition 2.12. *Let d be an odd positive integer. Then $W_d^{[2]}$ is a (Λ_{2d}, j_2)-polarized fourfold.*

Proof. The transcendental lattice T_{W_d} of W_d is isometric to the one of $W_d^{[2]}$. Since $\mathrm{T}_{W_d} \simeq T_{2d,2}$ (see Proposition 2.4), we conclude that $\mathrm{T}_{W_d^{[2]}} \simeq T_{2d,2}$. Moreover, $\mathrm{NS}(W_d^{[2]}) \simeq \mathrm{NS}(W_d) \oplus \langle -2 \rangle$ and, by comparison of the discriminant forms, one has $\mathrm{NS}(W_d) \oplus \langle -2\rangle \simeq \Lambda_{2d}$. So a generic (Λ_{2d}, j_2)-polarized fourfold has the same transcendental lattice and Néron–Severi group of $W_d^{[2]}$, thus their families coincide. □

Remark 2.13. Note that there is no natural symplectic involution on these Hilbert schemes. It is a nice open problem how to construct such involutions on some birational models of those Hilbert schemes, but for $d = 1$ there is the following geometric construction.

If $d = 1$, then the generic $(\langle 2\rangle \oplus \langle -2\rangle^{\oplus 7})$-polarized $K3$ surface W_1 is a double cover of a del Pezzo surface of degree 2, denoted by dP_2, thus it admits a nonsymplectic involution, which is the cover involution. We denote it as ι_{W_1} and we observe that it acts as the identity on $\mathrm{NS}(W_1)$. Moreover, since the anticanonical model of dP_2 exhibits dP_2 as double cover of $\mathbb{P}^2$ branched on a quartic curve, the surface W_1 admits a model (induced by the anticanonical one of dP_2) as a quartic hypersurface in $\mathbb{P}^3$ which does not contain lines. Therefore the fourfold $X = W_1^{[2]}$ admits two nonsymplectic involutions: one is $\iota_{W_1}^{[2]}$, the natural involution induced by ι_{W_1}, and the other is Beauville's involution β; see [Beauville 1983, Proposition 11] for the definition. The isometry $(\iota_{W_1}^{[2]})^*$ acts as the identity on $\mathrm{NS}(X)$ and as minus the identity on T_X, hence it commutes with every isometry induced by an involution on X (since they commute both on the transcendental lattice and on the Néron–Severi group). In particular $\iota_{W_1}^{[2]}$ and β are two commuting nonsymplectic involutions, whose composition is necessarily a symplectic involution on X. Such an involution can be constructed also on a birational model, as done in [Markushevich and Tikhomirov 2007].

In the case $d = 3$, by Proposition 2.12 examples of (Λ_6, j_2)-polarized fourfolds are given by Hilbert squares of $K3$ surfaces W_3 which are $(\langle 6\rangle \oplus \langle -2\rangle^{\oplus 7})$-polarized. In this case, one can show that such Hilbert squares are in fact birational to the Fano varieties of cubic fourfolds with 8 nodes [Lehn 2018, Theorem 1.1]. It is an open question whether it is possible to describe geometrically a symplectic involution on these manifolds.

Proposition 2.14. *Let d be an odd positive integer. There exist a Brauer class $\beta \in H^2(\mathcal{O}^*_{W_d})_2$ and a Mukai vector $v \in H^*(W_d, \mathbb{Z})$ such that the moduli space $X = M_v(W_d, \beta)$ is a (Λ_{2d}, j_1)-polarized fourfold of $K3^{[2]}$-type.*

Proof. The transcendental lattice T_{W_d} of W_d is of the form $U \oplus \Xi$ for Ξ an even hyperbolic lattice; we denote by f_1, f_2 a basis of the hyperbolic plane U. Then we consider $B = \frac{f_1}{2} \in \mathrm{T}_{W_d} \otimes \mathbb{Q}$ and $\beta \in H^2(\mathcal{O}^*_{W_d})_2$ the Brauer class of order two corresponding to $(\_, B) : \mathrm{T}_{W_d} \to \mathbb{Z}_2$. The twisted Néron–Severi group $\mathrm{NS}(W_d, \beta)$ is thus the sublattice of $H^*(W_d, \mathbb{Z})$ generated by $\mathrm{NS}(W_d)$, $(0, 0, 1)$ and $(2, f_1, 0)$, hence it is isomorphic to $U(2) \oplus \mathrm{NS}(W_d)$, and its orthogonal in the Mukai lattice is isometric to $U(2) \oplus \Xi$. It follows from work of Yoshioka [2006, Section 3] that $\mathrm{NS}(M_v(W_d, \beta)) \simeq v_B^{\perp} \cap \mathrm{NS}(W_d, \beta)$ and that the transcendental lattice of $M_v(W_d, \beta)$ is isometric to $U(2) \oplus \Xi$.

We conclude by choosing as Mukai vector $v = (0, H', 2)$ where $H' \in \mathrm{NS}(W_d)$ is a primitive effective class of square two. The orthogonal P of H' in $\mathrm{NS}(W_d)$ is a negative definite lattice with rank and length 7 and discriminant group $\mathbb{Z}_{2d} \oplus (\mathbb{Z}_2)^{\oplus 6}$ with discriminant quadratic form $q = \left(\frac{1}{2d}\right) \oplus v(2) \oplus \left(-\frac{1}{2}\right)^{\oplus 4}$. For such a choice we have $v_B = v$ primitive of square two and the orthogonal to v in $U(2)\oplus \mathrm{NS}(W_d)$ is a hyperbolic lattice of rank 9 and discriminant group $\mathbb{Z}_{2d}\oplus(\mathbb{Z}_2)^{\oplus 8}$. Its 2-adic component is isometric to the one of $\langle 2\rangle\oplus\langle -2\rangle^8\simeq\langle 2\rangle\oplus E_8(-2)$ and there is only one even indefinite lattice in this genus by [Nikulin 1979b, Theorem 1.13.2]. Thus the orthogonal to v is isometric to Λ_{2d}. □

Proposition 2.15. *Let d be a positive integer such that $d\equiv 3 \bmod 4$. There exist a Brauer class $\beta\in H^2(\mathcal{O}^*_{W'_d})_2$ and a Mukai vector $v\in H^*(W'_d,\mathbb{Z})$ such that the moduli space $X = M_v(W'_d,\beta)$ is a (Λ_{2d}, j_3)-polarized fourfold of $K3^{[2]}$-type.*

Proof. Denote by Ξ the lattice $U\oplus N\oplus K_d$, it holds

$$T_{2d,3}\simeq U^{\oplus 2}\oplus E_8(-2)\oplus K_d\simeq U(2)\oplus U\oplus N\oplus K_d\simeq U(2)\oplus\Xi \tag{2-2}$$

and $\mathrm{T}_{W'_d}\simeq U^{\oplus 2}\oplus N\oplus K_d\simeq U\oplus\Xi$. Now reasoning as in Proposition 2.14, one chooses $B=\frac{f_1}{2}\in \mathrm{T}_{W'_d}\otimes\mathbb{Q}$ and $\beta\in H^2(\mathcal{O}^*_{W'_d})_2$ the Brauer class of order two corresponding to $(\_, B):\mathrm{T}_{W'_d}\to\mathbb{Z}_2$. So $\mathrm{NS}(M_v(W'_d,\beta))\simeq v_B^{\perp}\cap\mathrm{NS}(W'_d,\beta)$ and the transcendental lattice of $M_v(W'_d,\beta)$ is isometric to $U(2)\oplus\Xi\simeq T_{2d,3}$.

We conclude by choosing as Mukai vector $v=(0,H',2)$ where $H'\in\mathrm{NS}(W'_d)\simeq(\langle 2d\rangle\oplus\langle -2\rangle^{\oplus 7})'$ is a primitive effective class of square two. □

Proposition 2.16. *If $d\equiv 0 \bmod 2$, then*:

- *A general fourfold of $K3^{[2]}$-type (Λ_{2d}, j_1)-polarized is birational to $M_v(W_d,\beta)$ where $v=\big(4,\sum_i n_i,2\big)$ and β are as above.*
- *A general fourfold of $K3^{[2]}$-type $(\widetilde{\Lambda}_{2d},\tilde{j})$-polarized is birational to $M_w(W_d)$ with $w=\big(2,\sum n_i,4\big)\in H^*(W_d,\mathbb{Z})$.*

Proof. Let us fix β as in Proposition 2.14. Then, since $\mathrm{T}_{W_d}\simeq U^{\oplus 2}\oplus D_4(-1)\oplus\langle -2\rangle^{\oplus 5}\oplus\langle -2d\rangle$, $\mathrm{T}_{M_v(W_d,\beta)}\simeq U\oplus U(2)\oplus D_4(-1)\oplus\langle -2\rangle^{\oplus 5}\oplus\langle -2d\rangle$ for every possible choice of the Mukai vector v. Moreover, the twisted Néron–Severi group $\mathrm{NS}(W_d,\beta)$ is $U(2)\oplus\mathrm{NS}(W_d)$ (as in the proof of Proposition 2.14) and it is generated by $(0,0,1)$, $(2,f_1,0)$, $(0,n_i,0)$, $i=1,\dots 7$, $(0,t,0)$ (where t, n_i are the generators of $\mathrm{NS}(W_d)$, $t^2=2d$, and f_1 is as in Proposition 2.14). We now fix $v=(4,\sum_i n_i,2)$, then $v_B=(4,\sum_i n_i+2f_1,2)\in H^*(W_d,\mathbb{Z})$ and we compute $v_B^{\perp}\cap\mathrm{NS}(W_d,\beta)$. It is generated by $(2,f_1,-1)$, $(0,2n_1,1)$, $(0,n_i-n_{i+1},0)$, $i=1,\dots 6$, $(0,t,0)$. One can directly check that $(0,t,0)$ is orthogonal to all the other generators and the form computed on all the other generators is $R(2)$ where R is an even negative definite unimodular lattice of rank 8. It follows that $R\simeq E_8(-1)$ and so the orthogonal to v_B in $\mathrm{NS}(W_d,\beta)$ is isometric to $E_8(-2)\oplus\langle 2d\rangle\simeq\Lambda_{2d}$. Hence $M_v(W_d,\beta)$ is (Λ_{2d},j_1)-polarized and gives a birational model of the general (Λ_{2d},j_1)-polarized fourfold of $K3^{[2]}$-type.

To prove the similar result for a general fourfold of $K3^{[2]}$-type $(\widetilde{\Lambda}_{2d},\tilde{j})$-polarized we observe that $\mathrm{T}_{W_d}\simeq\widetilde{T}_{2d}$. Moreover, the $(1,1)$-part in $H^*(W_d,\mathbb{Z})$ is $U\oplus\mathrm{NS}(W_d)$. Next, we observe that $\widetilde{\Lambda}_{2d}\simeq\langle 2d\rangle\oplus N$, where N is the Nikulin lattice, obtained by $\langle -2\rangle^{\oplus 8}$ by gluing the class $n:=\sum r_i/2$ and it is generated by the first seven roots $r_1,\dots,r_7$ and by n such that $n^2=-4$ and $nr_i=-1$.

Let $g_1, g_2, t, n_1, \dots, n_7$ be a basis of $U \oplus \mathrm{NS}(W_d)$, i.e., of the $(1,1)$ part of $H^*(W_d, \mathbb{Z})$. Consider now the explicit primitive embedding $\langle 2d \rangle \oplus N \subset U \oplus \mathrm{NS}(W_d)$ which sends the $\langle 2d \rangle$ summand in the lattice spanned by t and which sends $r_i \mapsto n_i + g_1$ for $i = 1, \dots, 7$, $n \mapsto 2g_1 - g_2$. The Mukai vector $w = (2, \sum_i n_i, 4)$ is $4g_1 + 2g_2 + n_1 + \cdots + n_7$ and its orthogonal is spanned by t, n and r_i with $i = 1, \dots 7$. So the orthogonal to the Mukai vector w in $U \oplus \mathrm{NS}(W_d)$ is isometric to $\langle 2d \rangle \oplus N \simeq \widetilde{\Lambda}_{2d}$ and this ends the proof. □

Remark 2.17 (induced automorphisms from autoequivalences). The symplectic automorphism considered in Proposition 2.16 is induced by a symplectic autoequivalence on $D^b(W_d)$ that is not induced by a symplectic action on W_d. The result [Beckmann and Oberdieck 2022, Proposition 1.4] gives a way to further investigate these symplectic involutions. If [Beckmann and Oberdieck 2022, Proposition 1.4] is generalized for twisted sheaves then this would give a way to study also the other involutions considered here.

3. Nikulin orbifolds

After having described the moduli spaces of projective fourfolds X of $K3^{[2]}$-type admitting a symplectic involution σ, we now turn to the study of their quotients. It is well-known, since work of Fujiki [1983], that the quotient does not admit a crepant resolution of singularities. Nevertheless, there is a partial resolution $Y \to X/\sigma$ which is a so-called *irreducible symplectic orbifold*.

Definition 3.1. Let X be a fourfold of $K3^{[2]}$-type and let σ be a symplectic involution on X. The partial resolution Y of X/σ obtained by blowing up the $K3$ surface contained in $\mathrm{Sing}(X/\sigma)$ is called the *Nikulin orbifold* corresponding to (X, σ).

Deformations in the sense of [Bakker and Lehn 2022; Menet 2020] of Nikulin orbifolds are said to be *orbifolds of Nikulin type*.

We recall the following result by Menet.

Theorem 3.2 [Menet 2015]. *The second cohomology group $H^2(Y, \mathbb{Z})$ of an orbifold Y of Nikulin type is endowed with a symmetric bilinear form, which is the Beauville–Bogomolov–Fujiki form B_Y and thus it is a lattice. Let q_Y denote the corresponding quadratic form. Let Σ be the exceptional divisor of $Y \to X/\sigma$ and let Δ be the divisor induced by δ; then*

$$q_Y(\Sigma) = q_Y(\Delta) = -4, \quad \tfrac{1}{2}(\Sigma \pm \Delta) \in H^2(Y, \mathbb{Z}).$$

The lattice $(H^2(Y, \mathbb{Z}), q_Y)$ is isometric to $U(2)^{\oplus 3} \oplus E_8(-1) \oplus \langle -2 \rangle \oplus \langle -2 \rangle$, where the last two summands are generated by $\frac{1}{2}(\Delta \pm \Sigma)$.

It follows that Σ is a class with self-intersection -4 and divisibility 2 in $H^2(Y, \mathbb{Z})$.

As a consequence of the previous theorem we get the following

Corollary 3.3. *Let X be fourfold of $K3^{[2]}$-type with a symplectic involution σ and such that $\mathrm{NS}(X) \simeq E_8(-2)$; then the corresponding Nikulin orbifold Y has $\mathrm{NS}(Y) \simeq \langle -4 \rangle$.*

embedding $\mathrm{NS}(X) \subset L$	$\mathrm{NS}(Y)$	T_Y	Proposition
j_1	$\langle 4d\rangle \oplus \langle -4\rangle$	$U(2)^{\oplus 2} \oplus E_8(-1) \oplus \langle -4d\rangle \oplus \langle -4\rangle$	Proposition 3.5
$j_2, d \equiv 1 \bmod 2$	$\begin{bmatrix} d-1 & 2 \\ 2 & -4 \end{bmatrix}$	$U(2)^{\oplus 2} \oplus E_7(-1) \oplus K_d(2) \oplus \langle -2\rangle$	Proposition 3.6
$j_3, d \equiv 3 \bmod 4$	$\begin{bmatrix} d-1 & 2 \\ 2 & -4 \end{bmatrix}$	$U(2)^{\oplus 2} \oplus K_d(2) \oplus E_8(-1)$	Proposition 3.7
$\tilde{j}, d \equiv 0 \bmod 2$	$\langle d\rangle \oplus \langle -4\rangle$	$U^{\oplus 2} \oplus \langle -d\rangle \oplus N \oplus \langle -4\rangle$	Proposition 3.8

Table 3. Pairs $(\mathrm{NS}(Y), \mathrm{T}_Y)$ for the Nikulin orbifold Y associated to X.

Hence, deformations of Y are not necessarily Nikulin orbifolds, since it follows from Corollary 3.3 that Nikulin orbifolds are contained in a family of codimension 1.

3A. ***Families of projective Nikulin orbifolds and the map π_*.*** In Corollary 3.3 we describe the explicit relations between $\mathrm{NS}(X)$ and $\mathrm{NS}(Y)$ in the generic case. In the following we will consider the same problem for special subfamilies, those of the projective fourfolds X.

If one specializes to the projective case one has four different families of fourfolds of $K3^{[2]}$-type X admitting a symplectic involution σ, which depend on the chosen embedding of $\mathrm{NS}(X)$ in L and are those listed in Table 1. The aim of this section is to associate to each of these families the family of Nikulin orbifolds Y which are partial resolution of X/σ. The results of this section are summarized in Table 3: in the first column we identify the family by choosing the embedding $\mathrm{NS}(X) \subset L$; in the second column we describe the Néron–Severi group of Y, in the third its transcendental lattice and in the last we give the reference to the propositions where the results are proved.

To prove these results we will use the explicit embeddings described in Section 2A and also the following explicit description of the map π_* induced by the quotient map $\pi : X \to X/\sigma$.

The map

$$\pi_* : H^2(X, \mathbb{Z}) \to H^2(X/\sigma, \mathbb{Z}) \subset H^2(Y, \mathbb{Z}) \tag{3-1}$$

is compatible (as explained below) with the lattice structure induced by the Beauville–Bogomolov–Fujiki form both on $H^2(X, \mathbb{Z})$ and on $H^2(Y, \mathbb{Z})$. Hence we can interpret π_* as a map between the lattices $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2\rangle$ and $U(2)^{\oplus 3} \oplus E_8(-1) \oplus \langle -2\rangle \oplus \langle -2\rangle$. To describe this map we consider, as in Section 2A, a basis of $H^2(X, \mathbb{Z})$ such that $\sigma^* \in O(H^2(X, \mathbb{Z}))$ switches the two copies of $E_8(-1)$ and acts as the identity on $U \oplus U \oplus U \oplus \langle -2\rangle$. We consider again the embeddings of the lattice $E_8(-2)$ in $E_8(-1) \oplus E_8(-1)$:

$$\lambda_+(b_i) = e_i + f_i \quad i = 1, \dots, 8,$$

$$\lambda_-(b_i) = e_i - f_i \quad i = 1, \dots, 8.$$

In particular $H^2(X, \mathbb{Z})^{\sigma^*} = U^{\oplus 3} \oplus \lambda_+(E_8(-2)) \oplus \langle -2\rangle \simeq U^{\oplus 3} \oplus E_8(-2) \oplus \langle -2\rangle$ and $(H^2(X, \mathbb{Z})^{\sigma^*})^\perp = \lambda_-(E_8(-2)) \simeq E_8(-2)$.

Take $\underline{u}$, $\underline{v}$, $\underline{w}$ vectors in U and $\underline{x}$, $\underline{y}$ vectors in $E_8(-1)$; for ease of notation, we will denote by $k \in \mathbb{Z}$ an element of $\langle -2\rangle$, referring to the k-th multiple of its generator (depending on the lattice this will be either δ, $\frac{1}{2}(\Delta+\Sigma)$ or $\frac{1}{2}(\Delta-\Sigma)$). Thus $(\underline{u}, \underline{w}, \underline{v}, \underline{x}, \underline{y}, k)$ is a vector in $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2\rangle$. Then

$$\pi_*(\underline{u}, \underline{w}, \underline{v}, \underline{x}, \underline{y}, k) = (\underline{u}, \underline{w}, \underline{v}, \underline{x}+\underline{y}, k, k) \in U(2)^{\oplus 3} \oplus E_8(-1) \oplus \langle -2\rangle \oplus \langle -2\rangle. \tag{3-2}$$

Hence the restriction of π_* to $U^{\oplus 3}$ acts as the identity on the vector space, but the form is multiplied by 2; the restriction of π_* to $E_8(-1)^{\oplus 2}$ acts as the sum of the two components on the vector space and divides the form by 2 in the quotient.

Lemma 3.4. *One has $\pi_*(\lambda_-(E_8(-2)))$ is trivial*; $\pi_*(\lambda_+(E_8(-2))) = E_8(-1)$;

$$\pi_*(H^2(X,\mathbb{Z})^{\sigma^*}) = U(2)^{\oplus 3} \oplus E_8(-1) \oplus \langle -4\rangle.$$

Proof. It suffices to choose a basis of the sublattices $\lambda_-(E_8(-2))$, $\lambda_+(E_8(-2))$, $H^2(X,\mathbb{Z})^{\sigma^*}$ of $H^2(X,\mathbb{Z})$ and then to apply the map π_* as given in (3-2). □

Proposition 3.5. *Let d be a positive integer and X_1 be a (Λ_{2d}, j_1)-polarized fourfold of $K3^{[2]}$-type. The fourfold X_1 admits a symplectic involution σ and*, *denoted by Y_1 the corresponding Nikulin orbifold*, *one has* $\mathrm{NS}(Y_1) \simeq \langle 4d\rangle \oplus \langle -4\rangle$ *and* $\mathrm{T}_{Y_1} \simeq \langle -4d\rangle \oplus U(2)^{\oplus 2} \oplus E_8(-1) \oplus \langle -4\rangle$.

Proof. By Proposition 2.5 one can choose the embedding j_1 such that $j_{1|E_8(-2)} = \lambda_-$ and $j_1(h) := \left(\binom{1}{d}, \binom{0}{0}, \binom{0}{0}, \underline{0}, \underline{0}, 0\right)$. Since $\pi_*(\mathrm{NS}(X_1)) \subset \mathrm{NS}(Y_1)$, one first considers $\pi_*(\mathrm{NS}(X_1)) = \pi_*((j_1, \lambda_-)(\langle 2d\rangle \oplus E_8(-2))) = \pi_*(j_1(h))$ (where the last identity is due to Lemma 3.4). By (3-2),

$$\pi_*(j_1(h)) = \left(\begin{pmatrix}1\\d\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \underline{0}, 0, 0\right) \in U(2)^3 \oplus E_8(-1) \oplus \langle -2\rangle^{\oplus 2},$$

so $q_Y(\pi_*(j_1(h))) = 4d$. Moreover, the class

$$\Sigma = \left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \underline{0}, 1, -1\right)$$

is contained in $\mathrm{NS}(Y_1)$. Hence $\mathrm{NS}(Y_1)$ is spanned by $\pi_*(j_1(h))$ and Σ and there are no linear combinations with rational noninteger coefficients of these classes which are also contained in $H^2(Y_1,\mathbb{Z})$. So $\mathrm{NS}(Y_1) = \langle \pi_*(j_1(h)), \Sigma\rangle \simeq \langle 4d\rangle \oplus \langle -4\rangle$. By definition T_{Y_1} is the orthogonal of $\mathrm{NS}(Y_1)$ in $H^2(Y_1,\mathbb{Z})$. So

$$\mathrm{T}_{Y_1} \simeq \langle -4d\rangle \oplus U(2)^{\oplus 2} \oplus E_8(-1) \oplus \langle -4\rangle. \qquad \square$$

Proposition 3.6. *Let d be an odd positive integer and X_2 be a (Λ_{2d}, j_2)-polarized fourfold of $K3^{[2]}$-type. The fourfold X_2 admits a symplectic involution σ and*, *denoted by Y_2 the corresponding Nikulin orbifold*, *one has* $\mathrm{NS}(Y_2) \simeq H_d(2) := \left[\begin{smallmatrix} d-1 & 2\\ 2 & -4\end{smallmatrix}\right]$ *and* $\mathrm{T}_{Y_2} \simeq U(2)^{\oplus 2} \oplus E_7(-1) \oplus K_d(2) \oplus \langle -2\rangle$.

Proof. By Proposition 2.6 one can choose the embedding j_2 such that

$$j_{2|E_8(-2)} = \lambda_- \quad \text{and} \quad j_2(h) := \begin{cases} \left(\binom{2}{2k+2}, \binom{0}{0}, \binom{0}{0}, \underline{e_1}, \underline{e_1}, 1\right) & \text{if } d = 4k+1,\\ \left(\binom{2}{2k+2}, \binom{0}{0}, \binom{0}{0}, \underline{e_1}+\underline{e_3}, \underline{e_1}+\underline{e_3}, 1\right) & \text{if } d = 4k-1.\end{cases}$$

As above, to compute $\mathrm{NS}(Y_2)$ one observes that a $\mathbb{Q}$-basis is given by $\pi_*(j_2(h))$ and Σ. By (3-2), $\pi_*(j_2(h)) = \big(\binom{2}{2k+1}, \binom{0}{0}, \binom{0}{0}, 2\underline{x}, 1, 1\big)$ and $q_Y(\pi_*(j_2(h))) = 4d$. The class $\pi_*(j_2(h)) - \Sigma = \big(\binom{2}{2k+1}, \binom{0}{0}, \binom{0}{0}, 2\underline{x}, 0, 2\big)$ is divisible by 2 in $H^2(Y_2, \mathbb{Z})$, thus $\frac{1}{2}(\pi_*(j_2(h)) - \Sigma) \in \mathrm{NS}(Y_2)$. Finally, we get

$$\mathrm{NS}(Y_2) = \big\langle \tfrac{1}{2}(\pi_*(j_2(h)) - \Sigma), \Sigma \big\rangle = \begin{bmatrix} d-1 & 2 \\ 2 & -4 \end{bmatrix}.$$

The transcendental lattice is the orthogonal to Σ and $\pi_*(j_2(h))$ in $H^2(Y_2, \mathbb{Z})$. A $\mathbb{Q}$-basis is obtained by computing the image via π^* of the generators of T_{X_2} listed above; then one observes that the only elements which are two-divisible are those of the form $(\underline{0}, \underline{0}, \underline{0}, 2\underline{w}, 0, 0)$, and this allows to deduce a $\mathbb{Z}$-basis of the lattice T_{Y_2}, which is of discriminant $2^8 d$. Direct computation now shows that

$$\mathrm{T}_{Y_2} \simeq U(2)^{\oplus 2} \oplus E_7(-1) \oplus K_d(2) \oplus \langle -2 \rangle. \qquad \square$$

Proposition 3.7. *Let d be a positive integer such that $d \equiv 3 \bmod 4$ and X_3 be a (Λ_{2d}, j_3)-polarized fourfold of $K3^{[2]}$-type. The fourfold X_3 admits a symplectic involution σ and, denoted by Y_3 the corresponding Nikulin orbifold, one has $\mathrm{NS}(Y_3) \simeq H_d(2)$ and $\mathrm{T}_{Y_3} \simeq U(2)^{\oplus 2} \oplus K_d(2) \oplus E_8(-1)$.*

Proof. By Proposition 2.7 one can choose the embedding j_3 such that

$$j_{3|E_8(-2)} = \lambda_- \quad \text{and} \quad j_3(h) := \left(\begin{pmatrix} 2 \\ (d+1)/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \underline{0}, \underline{0}, 1\right).$$

Since both

$$\pi_*(j_3(h)) = \left(\begin{pmatrix} 2 \\ (d+1)/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \underline{0}, 1, 1\right) \quad \text{and} \quad \tfrac{1}{2}(\pi_*(j_3(h)) - \Sigma)$$

are contained in $\mathrm{NS}(Y_3)$,

$$\mathrm{NS}(Y_3) = \big\langle \tfrac{1}{2}(\pi_*(j_3(h)) - \Sigma), \Sigma \big\rangle \simeq \begin{bmatrix} d-1 & 2 \\ 2 & -4 \end{bmatrix}$$

and T_{Y_3} is its orthogonal complement inside $U(2)^{\oplus 3} \oplus E_8(-1) \oplus \langle -2 \rangle^{\oplus 2}$. Hence

$$\mathrm{T}_{Y_3} \simeq U(2)^{\oplus 2} \oplus K_d(2) \oplus E_8(-1). \qquad \square$$

Proposition 3.8. *Let d be an even positive integer and $\widetilde{X}$ be a $(\widetilde{\Lambda}_{2d}, \tilde{j})$-polarized fourfold of $K3^{[2]}$-type. The fourfold $\widetilde{X}$ admits a symplectic involution σ and, denoted by $\widetilde{Y}$ the corresponding Nikulin orbifold, one has $\mathrm{NS}(\widetilde{Y}) \simeq \langle d \rangle \oplus \langle -4 \rangle$ and $\mathrm{T}_{\widetilde{Y}} \simeq U^{\oplus 2} \oplus \langle -d \rangle \oplus N \oplus \langle -4 \rangle$.*

Proof. By Proposition 2.8 one can choose the embedding $\tilde{j}$ such that

$$\tilde{j}_{|E_8(-2)} = \lambda_- \quad \text{and} \quad \tilde{j}(h) := \begin{cases} \big(\binom{2}{2k}, \binom{0}{0}, \binom{0}{0}, \underline{e_1}, \underline{e_1}, 0\big) & \text{if } d = 4k-2 \text{ and,} \\ \big(\binom{2}{2k}, \binom{0}{0}, \binom{0}{0}, \underline{e_1} + \underline{e_3}, \underline{e_1} + \underline{e_3}, 0\big) & \text{if } d = 4k-4. \end{cases}$$

Let us consider the case $d = 4k-2$. Since $\pi_*(\tilde{j}(h)) = \big(\binom{2}{2k}, \binom{0}{0}, \binom{0}{0}, 2\underline{e_1}, 0, 0\big)$, $\frac{1}{2}(\pi_*(\tilde{j}(h))) \in \mathrm{NS}(\widetilde{Y})$ and a basis of $\mathrm{NS}(\widetilde{Y})$ is given by $\frac{1}{2}(\pi_*(\tilde{j}(h)))$ and Σ. So $\mathrm{NS}(\widetilde{Y}) = \langle d \rangle \oplus \langle -4 \rangle$ and $\mathrm{T}_{\widetilde{Y}}$ is the orthogonal complement in $U(2)^{\oplus 3} \oplus E_8(-1) \oplus \langle -2 \rangle^{\oplus 2}$ to

$$\left\langle \left(\binom{1}{k}, \binom{0}{0}, \binom{0}{0}, \underline{e_1}, 0, 0 \right), \left(\binom{0}{0}, \binom{0}{0}, \binom{0}{0}, \underline{0}, 1, -1 \right) \right\rangle.$$

One obtains $\mathrm{T}_{\widetilde{Y}} \simeq U^{\oplus 2} \oplus \langle -d \rangle \oplus N \oplus \langle -4 \rangle$. The case $d = 4k - 4$ is analogous. □

Remark 3.9. The classes of divisors considered in Propositions 3.5, 3.6, 3.7, 3.8 have a geometric meaning: the class Σ is the effective class of the exceptional divisor; the class $\pi_*(j(h))$ is a pseudoample polarization induced on Y by the ample polarization $j(h)$ on X, and it is orthogonal to Σ. Its pullback via π^* is $2j(h)$; the class $(j(h) - \Sigma)$ corresponds to a divisor which has positive intersection with the exceptional divisor Σ and its pullback via π^* is $2j(h)$.

3B. ***Nikulin orbifolds related with natural involutions on Hilbert squares of K3 surfaces.*** We described, in Corollary 3.3, the relations between $\mathrm{NS}(X)$ and $\mathrm{NS}(Y)$ for a very general X of $K3^{[2]}$-type admitting a symplectic involution σ. In Section 3A we specialize X by requiring that it is projective. In this section we specialize X by requiring that it is the Hilbert scheme of two points of a $K3$ surface W and that the involution σ is natural, i.e., it is induced by a symplectic involution on W because of the equivariance of the construction of the Hilbert scheme $W^{[2]}$.

Proposition 3.10. *Let W be a generic nonprojective $K3$ surface admitting a symplectic involution σ_W, i.e., $\mathrm{NS}(W) = E_8(-2)$. Let $X := W^{[2]}$ be its Hilbert square and $\sigma := \sigma_W^{[2]}$ be the natural involution induced by σ_W. Then $\mathrm{NS}(X) = E_8(-2) \oplus \langle -2 \rangle$, $\mathrm{T}_X \simeq U^{\oplus 3} \oplus E_8(-2)$ and $\mathrm{NS}(Y) \simeq \langle -2 \rangle^{\oplus 2}$, $\mathrm{T}_Y \simeq U(2)^{\oplus 3} \oplus E_8(-1)$.*

Proof. By construction, the embedding of $\mathrm{NS}(X)$ in $H^2(X, \mathbb{Z})$ is given by $\lambda_-(E_8(-2)) \oplus \delta \simeq E_8(-2) \oplus \langle -2 \rangle$. By Lemma 3.4, $\pi_*(\lambda_-(E_8(-2)) \oplus \delta) = \pi_*(\delta)$. Since π_* maps $\mathrm{NS}(X)$ to $\mathrm{NS}(Y)$, one deduces that $\Delta = \pi_*(\delta) = (\underline{0}, \underline{0}, \underline{0}, \underline{0}, 1, 1) \in U(2)^{\oplus 3} \oplus E_8(-1) \oplus \langle -2 \rangle \oplus \langle -2 \rangle$ is a class in $\mathrm{NS}(Y)$. Moreover, $\mathrm{NS}(Y)$ always contains the class $\Sigma = (\underline{0}, \underline{0}, \underline{0}, \underline{0}, 1, -1)$. Since $\mathrm{NS}(Y)$ contains both Δ and Σ, it contains all their linear combinations which belong to $H^2(Y, \mathbb{Z})$. In particular $\mathrm{NS}(Y) = \left\langle \frac{1}{2}(\Delta + \Sigma), \frac{1}{2}(\Delta - \Sigma) \right\rangle \simeq \langle -2 \rangle \oplus \langle -2 \rangle$. The transcendental lattices are directly computed respectively as orthogonal to the Néron–Severi groups inside $H^2(X, \mathbb{Z})$ and $H^2(Y, \mathbb{Z})$. □

Proposition 3.11. *Let W be a projective $K3$ surface admitting a symplectic involution σ_W such that $\rho(W) = 9$. Then either $\mathrm{NS}(W) \simeq \Lambda_{2d}$ or $\mathrm{NS}(W) \simeq \widetilde{\Lambda_{2d}}$.*

Let $X = W^{[2]}$ be the Hilbert square on W, σ be the natural symplectic involution induced by σ_W and Y be the corresponding Nikulin orbifold.

If $\mathrm{NS}(W) \simeq \Lambda_{2d}$, then $\mathrm{NS}(W^{[2]}) = \Lambda_{2d} \oplus \langle -2 \rangle$, $\mathrm{T}_{W^{[2]}} \simeq \langle -2d \rangle \oplus U^{\oplus 2} \oplus E_8(-2)$, $\mathrm{NS}(Y) \simeq \langle 4d \rangle \oplus \langle -2 \rangle \oplus \langle -2 \rangle$ and $\mathrm{T}_Y \simeq \langle -4d \rangle \oplus U(2)^{\oplus 2} \oplus E_8(-1)$.

If $\mathrm{NS}(W) \simeq \widetilde{\Lambda_{2d}}$, then $\mathrm{NS}(W^{[2]}) = \widetilde{\Lambda_{2d}} \oplus \langle -2 \rangle$, $\mathrm{T}_{W^{[2]}} \simeq \langle -2d \rangle \oplus U \oplus U \oplus N$, $\mathrm{NS}(Y) \simeq \langle d \rangle \oplus \langle -2 \rangle \oplus \langle -2 \rangle$ and $\mathrm{T}_Y \simeq \langle -d \rangle \oplus U^{\oplus 2} \oplus N$.

Proof. The Néron–Severi group of W is given in [van Geemen and Sarti 2007]. The rest of the proof is analogous to the previous ones and we sketch it. If $\mathrm{NS}(W) \simeq \Lambda_{2d}$ the embedding of $\mathrm{NS}(W^{[2]})$ in

$H^2(X,\mathbb{Z})$ is $(j_1, \lambda_-, \mathrm{id})(h, E_8(-2), \delta)$, where $j_1(h)$ is defined in Proposition 3.5, $\lambda_-(E_8(-2))$ is as above, and

$$\mathrm{id}(\delta) = (\underline{0}, \underline{0}, \underline{0}, \underline{0}, \underline{0}, 1) \in U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2 \rangle.$$

Then one applies π_* as in (3-2): if both Δ and Σ are contained in $\mathrm{NS}(Y)$, then also $\frac{1}{2}(\Delta \pm \Sigma)$ is contained in $\mathrm{NS}(Y)$.

If $\mathrm{NS}(W) \simeq \widetilde{\Lambda_{2d}}$, the embedding of $\mathrm{NS}(W^{[2]})$ in $H^2(X,\mathbb{Z})$ is $(\tilde{j}, \lambda_-, \mathrm{id})(h, E_8(-2), \delta)$, where $\tilde{j}(h)$ is defined in Proposition 3.8 and $\lambda_-(E_8(-2))$ is as above. Then one applies π_* as in (3-2) and concludes. □

3C. ***A conjecture: the transcendental lattices of Y and of the fixed K3 surface.*** In Section 3A, we computed T_Y for every possible embedding j_i. We observe that for all the computed T_Y one can embed T_Y not only in $H^2(Y,\mathbb{Z})$ as we did, but also in L_{K3}. The orthogonal of $\mathrm{T}_Y \hookrightarrow L_{K3}$ is the Néron–Severi group of a $K3$ surface whose transcendental lattice is isometric to T_Y. In this section we discuss the following conjecture, which relates this $K3$ surface with the one in the fixed locus of the symplectic involution σ on X.

Conjecture 3.12. *Let X be a fourfold of $K3^{[2]}$-type admitting a symplectic involution σ, let Y be the partial resolution of X/σ as above, let S be the $K3$ surface contained in* $\mathrm{Fix}_\sigma(X)$. *Then* $\mathrm{T}_Y \simeq \mathrm{T}_S$.

As a first evidence to the conjecture we observe the following.

Proposition 3.13. *Let W be a $K3$ surface (projective or not) admitting a symplectic involution σ_W, such that* $\mathrm{NS}(W)$ *is one of the following lattices $E_8(-2)$, Λ_{2d} or $\widetilde{\Lambda}_{2d}$. Let X be $W^{[2]}$ and σ be the natural symplectic involution induced by σ_W. Then Conjecture 3.12 holds for X.*

Proof. Let us denote by $\hat{W}$ the minimal resolution of W/σ_W. It is a $K3$ surface and its Néron–Severi group and transcendental lattice are determined by those of W by Garbagnati and Sarti [2008, Corollary 2.2]. We will denote by $\widetilde{\Gamma_{2e}}$ the unique even overlattice of index 2 of $\Gamma_{2e} := \langle 2e \rangle \oplus N$ where both N and $\langle 2e \rangle$ are primitively embedded. One has the following relations between the Néron–Severi groups:

$$\begin{aligned}
&\mathrm{NS}(W) = E_8(-2) && \text{if and only if } \mathrm{NS}(\hat{W}) = N.\\
&\mathrm{NS}(W) = \Lambda_{2d} && \text{if and only if } \mathrm{NS}(\hat{W}) = \widetilde{\Gamma}_{4d}.\\
&\mathrm{NS}(W) = \widetilde{\Lambda}_{2d},\, d \equiv 0 \bmod 2, && \text{if and only if } \mathrm{NS}(\hat{W}) = \Gamma_d.
\end{aligned} \tag{3-3}$$

Which correspond to the following relations between the transcendental lattices:

$$\begin{aligned}
&\mathrm{T}_W = U^{\oplus 3} \oplus E_8(-2) && \text{if and only if } \mathrm{T}_{\hat{W}} = U^{\oplus 3} \oplus N.\\
&\mathrm{T}_W = \langle -2d \rangle \oplus U^{\oplus 2} \oplus E_8(-2) && \text{if and only if } \mathrm{T}_{\hat{W}} = \langle -4d \rangle \oplus U(2)^{\oplus 2} \oplus E_8(-1).\\
&\mathrm{T}_W = \langle -2d \rangle \oplus U^{\oplus 2} \oplus N,\, d \equiv 0 \bmod 2, && \text{if and only if } \mathrm{T}_{\hat{W}} = \langle -d \rangle \oplus U^{\oplus 2} \oplus N.
\end{aligned} \tag{3-4}$$

For every fourfold of $K3^{[2]}$-type X with a symplectic involution σ the fixed locus of σ consists of 28 isolated fixed points and a $K3$ surface S. If $X = W^{[2]}$ and $\sigma = \sigma_W^{[2]}$, then the surface S is the Nikulin surface constructed as minimal resolution of W/σ_W, i.e., the surface $\hat{W}$. Hence, to conclude the proof it suffices to show that, for every W (and thus every X), one has $\mathrm{T}_Y \simeq \mathrm{T}_{\hat{W}}$.

If $\mathrm{NS}(W) = E_8(-2)$, then $\mathrm{T}_W = U^{\oplus 3} \oplus E_8(-2)$. By Proposition 3.10, $\mathrm{T}_Y \simeq U^{\oplus 3} \oplus N$ and by (3-4) also $\mathrm{T}_{\hat{W}} \simeq U^{\oplus 3} \oplus N$.

If $\mathrm{NS}(W) = \langle 2d\rangle \oplus E_8(-2)$, then $\mathrm{T}_W = U^{\oplus 2} \oplus \langle -2d\rangle \oplus E_8(-2)$. By Proposition 3.11,

$$\mathrm{T}_Y \simeq \langle -4d\rangle \oplus U(2)^{\oplus 2} \oplus E_8(-1)$$

and by (3-4) also $\mathrm{T}_{\hat{W}} \simeq \langle -4d\rangle \oplus U(2)^{\oplus 2} \oplus E_8(-1)$.

If $\mathrm{NS}(W) = \widetilde{\Lambda}_{2d}$, with $d \equiv 0 \bmod 2$, then $\mathrm{T}_W = \langle -2d\rangle \oplus U^{\oplus 2} \oplus N$. By Proposition 3.11,

$$\mathrm{T}_Y \simeq \langle -d\rangle \oplus U^{\oplus 2} \oplus N$$

and by (3-4) also $\mathrm{T}_{\hat{W}} \simeq \langle -d\rangle \oplus U^{\oplus 2} \oplus N$. □

We can also show that Conjecture 3.12 holds for two locally complete families when $d = 1, 3$ and the embeddings of Λ_{2d} are respectively j_2 and j_3.

Proposition 3.14. *Let X be a (Λ_2, j_2)-polarized fourfold of $K3^{[2]}$-type and σ the symplectic involution described in Remark 2.13. Conjecture 3.12 holds in this case.*

Proof. By Remark 2.13, $X = W_1^{[2]}$. We must describe the fixed locus of the symplectic involution $\sigma = \iota_{W_1}^{[2]} \circ \beta$ on X; see also [Markushevich and Tikhomirov 2007, Lemmma 5.3]. The surface W_1 has a model as quartic in $\mathbb{P}^3$ and its nonsymplectic involution ι_{W_1} is the restriction of an automorphism of $\mathbb{P}^3$, still denoted by ι_{W_1}. For any point $P \in W_1$ we consider the line $r_P := \langle P, \iota_{W_1}(P)\rangle$. The line r_P is invariant for ι_{W_1} and thus the set of intersection points $r_P \cap W_1$ is invariant for ι_{W_1}, hence there exists a point $Q \in W_1$ such that

$$r_P \cap W_1 = \{P, \iota_{W_1}(P), Q, \iota_{W_1}(Q)\}.$$

We consider the pair of points (P, Q), which corresponds to a point in $W_1^{[2]}$. This point is a fixed point of σ, indeed $\beta(P, Q) = (\iota_{W_1}(P), \iota_{W_1}(Q))$ and $\iota_{W_1}^{[2]}(\iota_{W_1}(P), \iota_{W_1}(Q)) = (P, Q)$, so $\sigma(P, Q) = (P, Q)$. We get a fixed point of σ for each point $P \in W_1$. Vice versa each fixed point of σ in $W_1^{[2]}$ necessarily corresponds to a pair of points in W_1 which lie on a ι_{W_1}-invariant line. So the fixed surface S of σ is parametrized by points in W_1 and thus it is birational to W_1 (birational because in order to construct $W_1^{[2]}$ we blow up a surface and it is possible, a priori, that this introduces some exceptional divisors in the fixed locus). Nevertheless the surface S contained in the fixed locus of σ is a $K3$ surface as W_1 and thus if they are birational, they are isomorphic. So S is a surface isomorphic to W_1 and in particular its transcendental lattice is $\mathrm{T}_S \simeq \mathrm{T}_{W_1} \simeq U^{\oplus 2} \oplus D_4(-1) \oplus \langle -2\rangle^{\oplus 6}$. This lattice is a 2-elementary lattice with signature $(2, 12)$ and $\delta = 1$, so it is isometric to any other 2-elementary lattice with these properties, in particular to

$$U(2)^{\oplus 2} \oplus E_7(-1) \oplus K_1(2) \oplus \langle -2\rangle$$

and the conjecture holds. □

In the case of (Λ_6, j_3)-polarized fourfolds, the orthogonal of Λ_6 is $T_{6,3} = U^{\oplus 2} \oplus E_8(-2) \oplus K_3$ and $j_3(h)$ is a polarization on X of degree 6 and divisibility 2, hence X is birational to the Fano variety of a smooth cubic fourfold. In fact, this is the family of Fano varieties $F(Z)$ of smooth symmetric cubic fourfolds Z carrying a symplectic involution, as discussed in [Camere 2012, Section 7]. In this case, the ample polarization h of degree 6 is of nonsplit type and its orthogonal complement is $h^{\perp} \simeq U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus A_2(-1)$; since $E_8(-2)$ has to be orthogonal to h, we obtain that the orthogonal complement of Λ_6 into L is the sublattice

$$T_{6,3} \simeq U^{\oplus 2} \oplus E_8(-2) \oplus A_2(-1).$$

In this case the equation of the cubic fourfold can be chosen to be

$$X_0^2 L_0(X_2:X_3:X_4:X_5) + X_1^2 L_1(X_2:X_3:X_4:X_5) + X_0 X_1 L_2(X_2:X_3:X_4:X_5) + G(X_2:X_3:X_4:X_5) = 0$$

where $L_i(X_2 : X_3 : X_4 : X_5)$ and $G(X_2 : X_3 : X_4 : X_5)$ are homogeneous polynomials, $\deg(L_i) = 1$, $\deg(G) = 3$. The symplectic involution is induced on the Fano variety by the projective transformation

$$(X_0 : X_1 : X_2 : X_3 : X_4 : X_5) \to (-X_0 : -X_1 : X_2 : X_3 : X_4 : X_5).$$

The fixed locus consists of 28 points, in the $(+1)$-eigenspace, and of a $K3$ surface S, in the (-1)-eigenspace, which has bidegree (2, 1) in $\mathbb{P}^1 \times V(G)$.

Proposition 3.15. *Let Z, $F(Z)$, S be as above. Then $\mathrm{T}_S \simeq \mathrm{T}_Y \simeq U(2)^{\oplus 2} \oplus K_3(2) \oplus E_8(-1)$ and Conjecture 3.12 holds for $F(Z)$.*

Proof. Since $V(G)$ is a cubic in the projective space $\mathbb{P}^3_{(X_2:X_3:X_4:X_5)}$ the $K3$ surface S in the fixed locus is a complete intersection of two hypersurfaces of bidegree (2, 1) and (0, 3) in $\mathbb{P}^1 \times \mathbb{P}^3$. We denote by dP_3 the del Pezzo cubic surface defined by $V(G)$. We recall that dP_3 is obtained as blow up of $\mathbb{P}^2$ in six points and, denoted by m the class of a line in $\mathbb{P}^2$ and by E_i the exceptional divisors of the blow up, $\mathrm{NS}(dP_3)$ is generated (over $\mathbb{Z}$) by $m, E_1, \dots, E_6$. The surface dP_3 is embedded in $\mathbb{P}^3$ by the anticanonical linear system $H := 3m - \sum_i E_i$. So

$$m = \tfrac{1}{3}\Big(H + \sum_i E_i\Big) \in \mathrm{NS}(dP_3).$$

To compute $\mathrm{NS}(S)$ we first observe that it is generated, at least over $\mathbb{Q}$, by the classes $h_1, h_2, \ell_i, i = 1, \dots, 6$ where h_1 (resp. h_2) is the restriction to the surface of the pullback in $\mathbb{P}^1 \times \mathbb{P}^3$ of the hyperplane section of $\mathbb{P}^1$ (resp. $\mathbb{P}^3$) and ℓ_i is the pullback of the class $E_i \in \mathrm{NS}(dP_3)$. The intersection properties of these classes are the following: $h_1^2 = 0$, $h_1 h_2 = 3$, $h_1 \ell_i = 1$, $i = 1, \dots, 6$, $h_2^2 = 6$, $h_2 \ell_i = 2$, $(\ell_i)^2 = -2$ and $\ell_i \ell_j = 0$ if $i \neq j$. In particular, we observe that h_2 is the pullback of the divisor $H \in \mathrm{NS}(dP_3)$ and since

$$\tfrac{1}{3}\Big(H + \sum_i E_i\Big) \in \mathrm{NS}(dP_3),$$

we obtain that $\frac{1}{3}(h_2+\sum_i \ell_i) \in \mathrm{NS}(S)$ (this divisor exhibits S as double cover of $\mathbb{P}^2$ and contracts the rational curves ℓ_i to nodes of the branch locus of the double cover). So $\left\{h_1, \frac{1}{3}(h_2+\sum_i \ell_i), \ell_i\right\}$ is a set of generators of $\mathrm{NS}(S)$. The discriminant group of this lattice is $\mathbb{Z}_6 \oplus (\mathbb{Z}_2)^{\oplus 5}$ and the discriminant form is the opposite of the one of $U(2)^{\oplus 2} \oplus A_2(-2)$. We deduce that the transcendental lattice of S is

$$\mathrm{T}_S \simeq U(2)^{\oplus 2} \oplus A_2(-2) \oplus E_8(-1).$$

Recalling that $A_2(-1) \simeq K_3$, we obtain that

$$\mathrm{T}_S \simeq U(2)^{\oplus 2} \oplus K_3(2) \oplus E_8(-1) \simeq \mathrm{T}_Y;$$

see Table 3. So Conjecture 3.12 holds in this case. □

The conjecture is true at least with rational coefficients, or, in other words, the transcendental lattice of the symplectic orbifold Y is the same of a (possibly twisted) Fourier–Mukai partner of the fixed $K3$ surface.

Proposition 3.16. *Let X be a fourfold of $K3^{[2]}$-type admitting a symplectic involution σ, let Y be the corresponding Nikulin orbifold and let S be the $K3$ surface contained in* $\mathrm{Fix}_\sigma(X)$. *Then* $\mathrm{T}_Y \otimes \mathbb{Q} \simeq \mathrm{T}_S \otimes \mathbb{Q}$. *In particular*, $\rho(Y) = \rho(S) - 6$.

Proof. Let $\nu : S \to X$ be the embedding of the $K3$ surface, we consider the restriction of forms $\nu^* : H^2(X, \mathbb{C}) \to H^2(S, \mathbb{C})$, which gives a morphism of Hodge structures of weight two.

Let $\omega_S \in H^{2,0}(S)$ be the restriction of a symplectic form $\omega_X \in H^{2,0}(X)$, i.e., $\omega_S = \nu^*\omega_X$; since S is the fixed $K3$ surface, this restriction is again a symplectic form on S, hence $\omega_X \notin \ker \nu^*$. Moreover, the rational transcendental lattice $\mathrm{T}_X \otimes \mathbb{Q}$ can be defined as the smallest rational Hodge substructure of $H^2(X, \mathbb{Q})$ such that $\mathrm{T}_X \otimes \mathbb{C}$ contains ω_X. This implies that the restriction $\nu^*_{|\mathrm{T}_X \otimes \mathbb{Q}}$ is injective; indeed, both the transcendental lattice and the kernel of a morphism of Hodge structures are irreducible Hodge substructures, thus either their intersection is trivial or they coincide, which is not the case here. In the same way one observes that the image of $\nu^*_{|\mathrm{T}_X \otimes \mathbb{Q}}$ is exactly $\mathrm{T}_S \otimes \mathbb{Q}$; both these Hodge substructures of $H^2(S, \mathbb{Q})$ are irreducible, and their intersection contains at least $\omega_S \neq 0$, thus they coincide. In the rest of the proof we denote $\nu^* : \mathrm{T}_X \otimes \mathbb{Q} \to \mathrm{T}_S \otimes \mathbb{Q}$: it is an isomorphism of irreducible Hodge structures of weight two.

Let now $\tilde\rho : \widetilde{X} \to X$ be the blow-up of the fixed $K3$ surface S, $\widetilde{\Sigma}$ be the exceptional divisor of ρ and let $\tilde\pi : \widetilde{X} \to Y$ be the quotient by the involution induced on $\widetilde{X}$ by σ. We use the following diagram:

$$\begin{tikzcd} & \widetilde{X} \arrow[r, "\tilde\rho"] \arrow[d, "\tilde\pi"] & X \arrow[d, "\pi"] \\ \widetilde{Y} \arrow[r] & Y \arrow[r, "\rho"] & X/\sigma \end{tikzcd}$$

We know from [Shioda 1986, Proposition 5] that the transcendental lattice of a smooth resolution $\widetilde{Y}$ of a quotient X/Γ, where X is smooth and Γ is a finite group, is a Hodge structure isomorphic to the

Γ-invariant part of T_X. In our case, a smooth resolution $\widetilde{Y}$ of singularities of X/σ is also a resolution of singularities for the orbifold Y, hence $\mathrm{T}_Y \otimes \mathbb{Q}$ is isomorphic to $\mathrm{T}_{\widetilde{Y}} \otimes \mathbb{Q}$ as Hodge structures. Finally we obtained an isomorphism of rational Hodge structures of weight two

$$\mathrm{T}_Y \otimes \mathbb{Q} \cong (\mathrm{T}_X \otimes \mathbb{Q})^\sigma = \mathrm{T}_X \otimes \mathbb{Q} \cong \mathrm{T}_S \otimes \mathbb{Q},$$

where the first and the last isomorphisms are respectively given by $\tilde{\rho}_* \circ \tilde{\pi}^*$ and ν^*.

We now show that this isomorphism is in fact an isometry over $\mathbb{Q}$. Let $\mu_{[S]} : H^2(X, \mathbb{Q}) \to H^6(X, \mathbb{Q})$ be the cup-product with $[S]$, where $[S]$ is the cohomology class of S; Voisin [2022, Proposition B.2] shows that $\ker \mu_{[S]} = \ker \nu^*$ and that, as a consequence, on $\operatorname{im} \nu^*$ the cup-product on S is induced by cup-product on X via the following equality:

$$\langle \nu^* x, \nu^* y\rangle_S = \langle \mu_{[S]}(x), y\rangle_X = x.y.[S].$$

In our particular case, this equality holds for all $x, y \in \mathrm{T}_X \otimes \mathbb{Q}$.

Denote by $\widetilde{\Sigma}$ and Σ respectively the exceptional divisors of $\tilde{\rho}$ and of ρ. Let $\alpha, \beta \in \mathrm{T}_Y \otimes \mathbb{Q}$; by [Menet 2015, Proposition 2.11] we have $B_Y(\alpha, \beta) = -\frac{1}{8}\alpha.\beta.\Sigma^2$. Moreover, observing that $\tilde{\pi}^*\Sigma = 2\widetilde{\Sigma}$, a standard computation in intersection theory yields

$$\alpha.\beta.\Sigma^2 = 2\tilde{\pi}^*\alpha.\tilde{\pi}^*\beta.\widetilde{\Sigma}^2 = -2\tilde{\rho}_*\tilde{\pi}^*\alpha.\tilde{\rho}_*\tilde{\pi}^*\beta.[S] = -2\langle \nu^*\tilde{\rho}_*\tilde{\pi}^*\alpha, \nu^*\tilde{\rho}_*\tilde{\pi}^*\beta\rangle_S,$$

where the second equality follows from projection formula (see [Fulton 1998, Proposition 8.3(c)]) and the equality $\widetilde{\Sigma}^2 = -\rho^*[S]$, which is proven in [Menet 2015, Lemma 2.12].

This shows that $B_Y(\alpha, \beta) = \frac{1}{4}\langle \nu^*\tilde{\rho}_*\tilde{\pi}^*\alpha, \nu^*\tilde{\rho}_*\tilde{\pi}^*\beta\rangle_S$ for all $\alpha, \beta \in \mathrm{T}_Y \otimes \mathbb{Q}$, thus $\mathrm{T}_S \otimes \mathbb{Q} \simeq \mathrm{T}_Y(4) \otimes \mathbb{Q} \simeq \mathrm{T}_Y \otimes \mathbb{Q}$. $\square$

Remark 3.17. The $K3$ surfaces in the fixed locus of a symplectic involution can be seen as a generalization of Nikulin surfaces as their moduli space is densely covered by families of Nikulin surfaces. It would be interesting to study the rationality of such moduli spaces as in [Farkas and Verra 2016].

4. Orbifold Riemann–Roch formula

4A. ***Orbifold Riemann–Roch.*** In order to study projective models of Nikulin orbifolds, we need to apply the theory of orbifold Riemann–Roch, as developed in [Blache 1996] and in [Buckley et al. 2013]. We first treat the case of Nikulin orbifolds, and then we generalize it to orbifolds of Nikulin type.

We consider again the following diagram:

$$\begin{tikzcd}
V \arrow[r, "\widetilde{\beta}"] \arrow[rr, bend left, "\tilde{r}"] \arrow[d, "q"] & \widetilde{X} \arrow[r, "\tilde{\rho}"] \arrow[d, "\tilde{\pi}"] & X \arrow[d, "\pi"] \\
\widetilde{Y} \arrow[r, "\beta"] \arrow[rr, bend right] & Y \arrow[r, "\rho"] & X/\sigma
\end{tikzcd}$$

Where:

- X is a fourfold of $K3^{[2]}$-type, $\sigma \in \mathrm{Aut}(X)$ is a symplectic involution and $S \subset \mathrm{Fix}_\sigma(X)$ is the fixed surface; we will denote by $\mathcal{N}_{S|X}$ the normal sheaf.
- Y is the Nikulin orbifold corresponding to (X, σ); Σ is the exceptional divisor of $\rho : Y \to X/\sigma$ and $\widetilde{X}$ is the blow-up of X along S.
- $\widetilde{Y}$ is the total smooth resolution of X/σ, and hence of Y, and V is the blow-up of $\widetilde{X}$ in the inverse image via $\tilde{\rho}$ of the 28 isolated fixed points of σ. Denote respectively by $E_1, \dots, E_{28}$ and $\widetilde{E_1}, \dots, \widetilde{E_{28}}$ the exceptional divisors on V and on $\widetilde{Y}$. Moreover, let E_S and $\widetilde{E_S}$ be the exceptional divisors on V and $\widetilde{Y}$ over S and over its image in X/σ respectively. Finally, let E and $\widetilde{E}$ be respectively $\sum_{i=1}^{28} E_i + E_S$ and $\sum_{i=1}^{28} \widetilde{E}_i + \widetilde{E}_S$.

Lemma 4.1. *Let X, Y, $\widetilde{Y}$ be as described above, and let $\nu : S \hookrightarrow X$ be the embedding of the fixed $K3$ surface. Then*

$$c_1(\widetilde{Y}) = \tfrac{1}{2}(q_* c_1(V) + \widetilde{E}) = -\sum_{i=1}^{28} \widetilde{E}_i,$$

$$c_2(\widetilde{Y}) = \tfrac{1}{2} q_* \tilde{r}^*(c_2(X) + \nu_*[S]) + q_*\left(-8\sum_{i=1}^{28} E_i^2 - E_S^2\right) + \tfrac{3}{2} K_{\widetilde{Y}} \widetilde{E} + 2K_{\widetilde{Y}}^2.$$

Proof. The proof follows from an application of Grothendieck–Riemann–Roch formula [Fulton 1998, Theorem 15.2] combined with well-known properties of smooth blow-ups; see [loc. cit., Example 15.4.3]:

$$K_V = 3\sum_{i=1}^{28} E_i + E_S,\ c_2(V) = \tilde{r}^*(c_2(X) + \nu_*[S]) + 2\sum_{i=1}^{28} E_i^2.$$

It is a generalization of the proof of [Camere et al. 2019a, Proof of Proposition 7.2]. □

Theorem 4.2 (orbifold Riemann–Roch formula). *Let D be a $\mathbb{Q}$-Cartier Weil divisor on Y, then $q^*\beta^* D$ is equivalent to $\tilde{r}^* H + kE_S$, with $H \in \mathrm{NS}(X)$, $k \in \mathbb{Z}$; let n be the number of points in which the divisor D fails to be Cartier. Then*

$$\chi(Y, D) = \tfrac{1}{48} H^4 + \tfrac{1}{48} H^2.c_2(X) + \left(\tfrac{1}{16} - \tfrac{1}{8}k^2\right)(H_{|S})^2 + 3 - \tfrac{1}{16} n + \tfrac{1}{4} k^4 - \tfrac{1}{2} 3k^2.$$

Proof. Since D is a $\mathbb{Q}$-Cartier Weil divisor on Y, then there exists an effective divisor $\widetilde{D} \in \mathrm{NS}(\widetilde{Y})$ such that $\beta^* D = \widetilde{D} + \sum_{i=1}^{28} \lambda_i \widetilde{E}_i$ with $\lambda_i \in \mathbb{Q}$: $\lambda_i = \frac{1}{2}$ if D fails to be Cartier in $p_i \in \mathrm{Sing}(Y)$ for $i = 1, \dots, 28$, it is zero otherwise. We have $\beta^* D.\widetilde{E}_i = 0$ for all i. Then the orbifold Riemann–Roch formula ([Buckley et al. 2013, Theorem 3.3]) is

$$\begin{aligned}\chi(Y, D) &= \chi(\widetilde{Y}, \widetilde{D}) \\ &= \tfrac{1}{24}(\beta^* D)^4 + \tfrac{1}{12}(\beta^* D)^3.c_1(\widetilde{Y}) + \tfrac{1}{24}(\beta^* D)^2.(c_1(\widetilde{Y})^2 + c_2(\widetilde{Y})) \\ &\qquad + \tfrac{1}{24}(\beta^* D).c_1(\widetilde{Y}).c_2(\widetilde{Y}) + \chi(\mathcal{O}_{\widetilde{Y}}) + \sum_{i=1}^{28} \gamma_i(D),\end{aligned}$$

where for each singular point $p_i \in Y$ we define $\gamma_i(D) = -\frac{1}{16}$ if D is not Cartier in p_i, $\gamma_i(D) = 0$ otherwise.

It was proven in [Fu and Menet 2021] that $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_{\widetilde{Y}}) = 3$. Moreover, it follows from $K_{\widetilde{Y}} = \sum_i \widetilde{E}_i$, as shown in Lemma 4.1, that $\beta^* D.c_1(\widetilde{Y}) = 0$, hence the formula above reduces to computing $(\beta^* D)^4$ and $(\beta^* D)^2.c_2(\widetilde{Y})$. Our aim is now to reduce the intersection theory on $\widetilde{Y}$ to the intersection theory on X.

In our situation, we have $q^*\beta^* D = \tilde{r}^* H + kE_S$ (indeed, if there were components in the E_i's, we would have $\beta^* D.\widetilde{E}_i \neq 0$). Moreover, $q^*\widetilde{E_S} = 2E_S$ and $q_* E_S = \widetilde{E_S}$; hence $E_S^4 = 12$, since Fujiki's relation on Y implies $\widetilde{E_S}^4 = 6 \cdot 16$.

Hence we obtain the following equalities of intersection numbers in $\mathbb{Q}$, by using Lemma 4.1 and the projection formula [Fulton 1998, Proposition 8.3(c)] (see also [Camere et al. 2019a] for further details):

$$\begin{aligned}
(\beta^* D)^4 = \tfrac{1}{2}(q^*\beta^* D)^4 = \tfrac{1}{2}((\tilde{r}^* H)^4 + k^4 E_S^4 + 6k^2(\tilde{r}^* H)^2.E_S^2) = \tfrac{1}{2}H^4 + 6k^4 - 3k^2(H_{|S})^2,\\
(\beta^* D)^2.q_*\tilde{r}^* c_2(X) = \tilde{r}^*(H^2.c_2(X)) + k^2 E_S^2.\tilde{r}^* c_2(X) = H^2.c_2(X) - k^2 c_2(X).\nu_*[S],\\
(\beta^* D)^2.q_*\tilde{r}^*\nu_*[S] = \tilde{r}^*((H_{|S})^2) + k^2 E_S^2.\tilde{r}^*\nu_*[S] = (H_{|S})^2 - k^2 c_2(\mathcal{N}_{S|X}),\\
(\beta^* D)^2.q_*(E_S^2) = -\tilde{r}^*((H_{|S})^2) + k^2 E_S^4 = -(H_{|S})^2 + 12k^2.
\end{aligned}$$

Many equalities and vanishings of some terms in the formulas above use the following equality for $\alpha \in A_{4-i}(X)$ (easy generalization of [Bădescu and Beltrametti 2013, Lemma 1.1]):

$$E_S^i.\tilde{r}^*\alpha = (-1)^{i-1} s_{i-2}(\mathcal{N}_{S|X}).\nu^*\alpha,$$

combined with $\nu^*\nu_*[S] = c_2(\mathcal{N}_{S|X})$ (see [Fulton 1998, Corollary 6.3]) and with the results contained in [Camere 2012, Proof of Theorem 5], which give $s_1(\mathcal{N}_{S|X}) = 0$, $c_2(X).[S] = 36$, $s_2(\mathcal{N}_{S|X}) = -c_2(\mathcal{N}_{S|X}) = -c_2(X).[S] + c_2(S) = -12$. □

Lemma 4.3. *Let $H \in \mathrm{NS}(X)$ as in Theorem 4.2; then $(H_{|S})^2 = 2q_X(H)$, where q_X is the BBF quadratic form on $H^2(X, \mathbb{Z})$.*

Proof. This is proven in [Menet 2015, Proposition 2.24(4)], once recalled that $(H_{|S})^2 = -E_S^2.\tilde{r}^* H^2$. □

Corollary 4.4 (Riemann–Roch formula for Cartier divisors on Y). *If $D \in \mathrm{NS}(Y)$ then $\chi(Y, D) = \frac{1}{4}(q_Y(D)^2 + 6q_Y(D) + 12)$.*

Proof. In this particular case, Theorem 4.2 simplifies into

$$\chi(D) = \tfrac{1}{48}H^4 + \tfrac{1}{48}H^2.c_2(X) + \left(\tfrac{1}{16} - \tfrac{1}{8}k^2\right)(H_{|S})^2 + 3 + \tfrac{1}{4}k^4 - \tfrac{1}{2}3k^2.$$

Since $q^*\beta^* D = \tilde{r}^* H + kE_S$, by push-pull formula [Fulton 1998, proof of Proposition 2.3(c)] and the commutativity of the diagram above, we have $D = \frac{1}{2}\tilde{\pi}_*\widetilde{\rho}^* H + \frac{k}{2}\Sigma$. The statement then follows from $q_Y(\Sigma) = -4$, $q_Y(\tilde{\pi}_*\widetilde{\rho}^* H) = 2q_X(H)$ [Menet 2015, Proposition 2.9], Riemann–Roch formula on X [Gross et al. 2003, Example 23.19] and Lemma 4.3. □

Corollary 4.4 holds for all orbifolds of Nikulin type, since it is topological in nature. Indeed, we can deform any orbifold of Nikulin type with a Cartier divisor to a Nikulin orbifold while keeping the class of the divisor algebraic (one just needs to require an additional (-4)-class of divisibility 2 in the same monodromy orbit of Σ in Theorem 3.2). We deduce the following general result.

Proposition 4.5. *Let Y be an orbifold of Nikulin type and let D and $\frac{m}{2}L$ be equivalent $\mathbb{Q}$-Cartier Weil divisors on Y, with $m \in \mathbb{Z}$ and L a Cartier divisor. Let n be the number of points where D fails to be Cartier. Then*

$$\chi(D) = \tfrac{3}{8}\big(\tfrac{1}{24}m^4 q_Y(L)^2 + m^2 q_Y(L) + 8\big) - \tfrac{1}{16}n.$$

Proof. Since Y is an orbifold of Nikulin type it is singular in 28 points. Let $\beta : \widetilde{Y} \to Y$ be a smooth resolution of singularities. By [Buckley et al. 2013, Theorem 3.3], $\chi(D) = \chi(\beta^* D) - \frac{n}{16}$ as integers. Our assumptions imply that $\beta^* D = \frac{m}{2}\beta^* L$, hence $(\beta^* D)^4 = \frac{m^4}{16}L^4 = \frac{3m^4}{8}q_Y(L)^2$.

Moreover, it follows from Corollary 4.4 that

$$\tfrac{1}{24}(\beta^* D)^2.c_2(\widetilde{Y}) = \tfrac{1}{96}m^2(\beta^* L)^2.c_2(\widetilde{Y}) = \tfrac{1}{4}m^2(\chi(L) - 3 - \tfrac{1}{4}q_Y(L)^2) = \tfrac{3}{8}m^2 q_Y(L),$$

since $L^4 = 6q_Y(L)^2$ and

$$\tfrac{1}{24}(\beta^* L)^4 + \tfrac{1}{24}(\beta^* L)^2.c_2(\widetilde{Y}) + 3 = \chi(\beta^* L) = \chi(L) = \tfrac{1}{4}(q_Y(L)^2 + 6q_Y(L) + 12).$$

Hence, $\chi(D) = \chi(\beta^* D) - \frac{n}{16} = \frac{3}{8}(\frac{m^4}{24}q_Y(L)^2 + m^2 q_Y(L) + 8) - \frac{n}{16}$. □

4B. ***Projective models of quotients.*** Let X be as above, with $\rho(X) = 9$. Let us denote by A the ample generator of the orthogonal to $E_8(-2)$ in $\mathrm{NS}(X)$. In particular A is preserved by σ. Then the map $\varphi_{|A|} : X \to \mathbb{P}(H^0(X, A)^\vee)$ is such that the automorphism σ on X is induced by a projective transformation on $\mathbb{P}(H^0(X, A)^\vee)$, still denoted by σ. Hence σ acts on the vector space $U := H^0(X, A)^\vee$, splitting it in the direct sum $U_+ \oplus U_-$ where U_+ and U_- are the eigenspaces of the eigenvalues $+1$ and -1 respectively.

The fourfold X projects to $\mathbb{P}(U_+)$ and $\mathbb{P}(U_-)$; since we are considering projective spaces which are invariant for σ, these two projections induce maps on the quotient, i.e., they induce the maps $X/\sigma \dashrightarrow \mathbb{P}(U_+)$ and $X/\sigma \dashrightarrow \mathbb{P}(U_-)$. These rational maps extend to the partial resolution Y, so we obtained two maps $Y \dashrightarrow \mathbb{P}(U_+)$ and $Y \dashrightarrow \mathbb{P}(U_-)$. We are interested in these maps, which essentially give the projective models of the quotient orbifold keeping trace of the construction of this orbifold as quotient of X.

The maps $Y \dashrightarrow \mathbb{P}(U_+)$ and $Y \dashrightarrow \mathbb{P}(U_-)$ are of course induced by some linear systems on Y and in order to find them we are looking for divisors D on Y such that $\tilde{\rho}_* \tilde{\pi}^* D = \pi^* \rho_* D = A$ (because the maps to $\mathbb{P}(U_\pm)$ are induced by the projections from $\mathbb{P}(H^0(X, A)^\vee)$).

If a connected component Z of the fixed locus $\mathrm{Fix}_\sigma(X)$ of σ on X is contained in one of the two eigenspaces, then the generic member of the linear system giving the projection to the other eigenspace has to pass through Z. Thus the corresponding divisor on X/σ is Weil but not necessarily Cartier and passes through n of the 28 singular points of X/σ and possibly through the singular surface of X/σ. Nevertheless, since the map is just $2:1$, we can assume that generically the divisor on X/σ passes simply through the singularities. Let us now consider the partial resolution $\rho : Y \to X/\sigma$. The divisor which we are considering on X/σ induces a divisor D_1 on Y. Since ρ is an isomorphism outside Σ (which is the exceptional divisor of ρ mapped to the singular surface), the Weil divisor D_1 passes simply through n of the 28 isolated singular points of Y and then fails to be Cartier on these points. Moreover, if the divisor on X/σ passes through the singular surface, then D_1 has a component on the exceptional divisor Σ, with multiplicity 1; otherwise it has none.

embedding $\mathrm{NS}(X) \subset L$	(n_1, n_2)	m_1	m_2	Proposition
$j_1, d \equiv 1 \bmod 2$	(12, 16)	$\frac{d^2}{4}+\frac{3d}{2}+\frac{5}{4}$	$\frac{d^2}{4}+d-\frac{1}{4}$	Theorem 4.9
$j_1, d \equiv 0 \bmod 2$	(16, 12)	$\frac{d^2}{4}+\frac{3d}{2}+1$	$\frac{d^2}{4}+d$	Theorem 4.9
$j_2, d \equiv 1 \bmod 2$	(28, 0)	$\frac{d^2}{4}+\frac{3d}{2}+\frac{1}{4}$	$\frac{d^2}{4}+d+\frac{3}{4}$	Theorem 4.10
$j_3, d \equiv 3 \bmod 4$	(28, 0)	$\frac{d^2}{4}+\frac{3d}{2}+\frac{1}{4}$	$\frac{d^2}{4}+d+\frac{3}{4}$	Theorem 4.10
$\tilde{j}, d \equiv 0 \bmod 2$	(0, 28)	$\frac{d^2}{4}+\frac{3d}{2}+2$	$\frac{d^2}{4}+d-1$	Theorem 4.12

Table 4. Summary of the properties of D_1 and D_2 and of the dimensions m_i of the projective spaces target of the map $\varphi_{|D_i|}$.

We observe that the linear system on X which corresponds to one of the projections and which is not a complete linear system (since its members have to pass through a part of $\mathrm{Fix}_\sigma(X)$) induces a complete linear system on V (where all the fixed locus is blown up).

By the previous discussion we deduce that the divisors that we are looking for on Y are two divisors D_1 and D_2 (each associated to one of the two projections on the two eigenspaces) such that

$$q^*\beta^*D_i = \tilde{r}^*A + k_i E_S, \quad \text{with } k_i = 0, -1 \text{ and thus } \tilde{\rho}_*\tilde{\pi}^*D_i = A. \tag{4-1}$$

Exactly one between D_1 and D_2 fails to be Cartier in a specific point (indeed a specific isolated fixed point is contained in exactly one eigenspace). The same holds true for the fixed surface (it is contained in exactly one of the eigenspaces), hence $k_i = -1$ for exactly one value among 1 and 2 and $k_i = 0$ for the other one. Indeed, if D_i is orthogonal to Σ, then β^*D_i is orthogonal to $\widetilde{E_S}$ and $q^*\beta^*D_i$ is orthogonal to E_S, i.e., $k_i = 0$. Similarly if the intersection of D_i with Σ is nontrivial, then $k_i = -1$.

So, given X a generic member of a family of fourfolds of $K3^{[2]}$-type with a symplectic involution, we determine two $\mathbb{Q}$-Cartier Weil divisors D_1 and D_2 which give two maps $\varphi_{|D_i|} : Y_i \to \mathbb{P}^{m_i}$. In Table 4 we summarize the properties of D_1 and D_2 and the dimensions m_i of the projective spaces target of the map $\varphi_{|D_i|}$. We choose D_1 to be always orthogonal to the exceptional divisor Σ and hence D_2 is always the divisor meeting Σ. Hence we have also to declare the number of points where D_i fails to be Cartier (and this is always denoted by n_i). As in the other tables, in the first column we identify the family of X (and hence of Y) by giving the explicit embedding of $\mathrm{NS}(X)$ in L and in the last we give the reference to the propositions were the results are proved.

Proposition 4.6. *Let $\rho(X) = 9$, A, D_1 and D_2 be as above and $q(A) = 2d$. Then both $\chi(Y, D_1)$ and $\chi(Y, D_2)$ are integer if and only if n_i and k_i are as in the following (up to a possible switch between D_1 and D_2):*

- *If d is even then*
  - $(n_1, k_1) = (0, 0)$ *and* $(n_2, k_2) = (28, -1)$ *or*
  - $(n_1, k_1) = (16, 0)$ *and* $(n_2, k_2) = (12, -1)$.

- *If d is odd then*
  - $(n_1, k_1) = (28, 0)$ *and* $(n_2, k_2) = (0, -1)$ *or*
  - $(n_1, k_1) = (12, 0)$ *and* $(n_2, k_2) = (16, -1)$.

Proof. We recall that for a divisor A on a fourfold of $K3^{[2]}$-type X it holds

$$\tfrac{1}{48}A^4 + \tfrac{1}{4}A^2.c_2(X) + \tfrac{3}{2} = \tfrac{1}{2}\chi(A) = \tfrac{1}{16}(q_X(A)+4)(q_X(A)+6),$$

which, combined with Theorem 4.2, gives

$$\chi(Y, D_i) = \tfrac{1}{16}(q_X(A)+4)(q_X(A)+6) - \tfrac{3}{2} + \big(\tfrac{1}{16} - \tfrac{1}{8}k_i^2\big)(A_{|S})^2 + 3 - \tfrac{1}{16}n_i + \tfrac{1}{4}k_i^4 - 3\tfrac{1}{2}k_i^2.$$

Now we recall that $q_X(A) = 2d$ and, by Lemma 4.3, $A^2_{|S} = 2q_X(A) = 4d$, so

$$\chi(Y, D_i) = \tfrac{1}{4}d^2 + \tfrac{5}{4}d + \tfrac{1}{4}d - \tfrac{1}{2}k_i^2 d + 3 - \tfrac{1}{16}n_i + \tfrac{1}{4}k_i^4 - 3\tfrac{1}{2}k_i^2.$$

We observe that if $k_i = 0, -1$, then $|k_i| = k_i^2 = k_i^4$, hence we obtain the following formula:

$$\chi(Y, D_i) = \tfrac{1}{4}d^2 + \tfrac{1}{2}3d - \tfrac{1}{2}|k_i|d - \tfrac{1}{16}n_i - \tfrac{1}{4}5|k_i| + 3.$$

Let us assume $k_1 = 0$ and then $k_2 = -1$. If d is even, then $\chi(Y, D_2) \in \mathbb{Z}$ forces $n_2 \equiv 12 \bmod 16$, which implies $n_2 = 12$ or $n_2 = 28$. If d is odd then $\chi(Y, D_2) \in \mathbb{Z}$ forces $n_2 \equiv 0 \bmod 16$, which implies $n_2 = 0$ or $n_2 = 16$. □

We observe that if $n_i = 0$ for a certain divisor D_i, then it is a Cartier divisor on Y. In this case D_i is $\pi_*(A)$ and it is orthogonal to the exceptional divisor if $k_i = 0$, it has a positive intersection with the divisor Σ if $k_i = -1$.

Lemma 4.7. *The variety Y is normal with terminal singularities. In particular Y is a klt variety.*

Proof. The variety Y is smooth outside 28 points where its singularities are locally the quotient of $\mathbb{C}^4$ by an involution g. In particular it is an orbifold. Hence it is normal. Moreover, the local action of the automorphism g is given by the diagonal matrix $\mathrm{diag}(-1, -1, -1, -1) \in \mathrm{SL}(4)$. The age of g is 2, hence the singularities of Y are terminal singularities (see [Joyce 2000, Theorem 6.4.3]), and in particular the pair $(Y, 0)$ is a klt pair. □

Proposition 4.8. *Let X, A, D_1 and D_2 be as in Proposition 4.6. Then*

$$\chi(Y, D_i) = h^0(Y, D_i).$$

Proof. The Kawamata–Vieweg vanishing theorem holds, see [Kollár and Mori 1998, Theorem 2.70], for the variety Y. With respect to the notation in [loc. cit.] one can assume $\Delta = 0$, and $N \equiv D_i$, $i = 1, 2$. It remains to prove that the D_i's are nef and big divisors. Since $\rho(X) = 9$ and A is the generator of $E_8(-2)^{\perp}$ in $\mathrm{NS}(X)$, it can be assumed to be an ample divisor. In particular it is nef, so $\pi_*(A)$ is a nef divisors. Since the sign of the top self-intersection of A is the same as the sign of the self-intersection of $\pi_*(A)$, we deduce that $\pi_*(A)$ is a nef and big divisor, by [Kollár and Mori 1998, Proposition 2.61].

Moreover, $\rho^*(D_i) = \pi_*(A)$ and since the properties of being big and nef are birational invariants, we deduce that D_i is nef and big. □

In Section 2 we associated the divisor A to a certain embedding of $\mathrm{NS}(X)$ in $H^2(X, \mathbb{Z})$, i.e., we considered $A = j(h)$ where h is vector in $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2\rangle$. In Propositions 3.5, 3.6, 3.7, 3.8, we studied the image of this divisor under the map π_* and we determined the generators of $\mathrm{NS}(Y)$. So, by comparing the conditions on D_i with the Néron–Severi group of Y computed in Section 3, one obtains the following theorems.

Theorem 4.9. *Let* $\mathrm{NS}(X) = (j_1, \lambda_-)(\langle 2d\rangle \oplus E_8(-2))$ *and A the generator of* $j_1(\langle 2d\rangle)$.

Let D_1 and D_2 be $\mathbb{Q}$-Cartier Weil divisors such that $2D_1 = \rho^*(\pi_*(j_1(h))) \in \mathrm{NS}(Y)$ *and* $2D_2 = \rho^*\pi_*(j_1(h)) - \Sigma \in \mathrm{NS}(Y)$. *Then if d is even* (*resp. odd*), *D_1 fails to be Cartier in* 16 (*resp.* 12) *points and D_2 in the other* 12 (*resp. the other* 16) *points. These divisors are such that* $\tilde{\rho}_*\tilde{\pi}^*(D_1) = \tilde{\rho}_*\tilde{\pi}^*(D_2) = A$ *and*

$$H^0(X, A) = (\rho^{-1} \circ \pi)^* H^0(Y, D_1) \oplus (\rho^{-1} \circ \pi)^* H^0(Y, D_2).$$

Proof. Let us consider the case d even. The other one is similar. One first considers $\rho^*(\pi_*(j_1(h))) \in \mathrm{NS}(Y)$, $\rho^*(\pi_*(j_1(h))) - \Sigma \in \mathrm{NS}(Y)$. Then there exist D_1 and D_2 $\mathbb{Q}$-Cartier Weil divisors such that a multiple of D_i, denoted by $h_i D_i$ is one prescribed element in $\mathrm{NS}(Y)$. We choose h_i to be the minimum among positive integers such that $h_i D_i \in \mathrm{NS}(Y)$. In particular, due to the singularities of Y, h_i is either 1 or 2. If $h_i = 1$, then D_i is Cartier, otherwise it is a $\mathbb{Q}$-Cartier Weil divisor on Y and it fails to be Cartier in n_i points. The possibilities for the divisors D_1 and D_2 are given in Proposition 4.6: the divisor D_1 is orthogonal to Σ, hence it is characterized by $k_1 = 0$; then there are two possibilities for n_1: either $n_1 = 0$ or $n_1 = 16$. If $n_1 = 0$, then D_1 is Cartier and $h_1 = 1$, otherwise D_1 is not Cartier and $h_1 = 2$. The choice of one of these two possibilities determines also the properties of D_2, which is necessarily $\mathbb{Q}$-Cartier Weil and not Cartier, hence h_2 is necessarily 2.

If $h_1 = 1$, then the divisors D_1 would be Cartier, but this is not the case, since the divisor $\frac{1}{2}\rho^*(\pi_*(j_1(h)))$ is not Cartier ($\mathrm{NS}(Y)$ is described in Proposition 3.5). We deduce that $h_1 = 2$, so $n_1 = 16$, $n_2 = 12$.

The map ρ is the contraction of Σ so, if $B \in \mathrm{NS}(Y)$ and $B \neq r\Sigma$, then $\rho_*(B)$ is a multiple of the unique generator of $\mathrm{NS}(X/\sigma)$. Since π is a $2:1$ map and A is invariant for σ, we have $\pi^*(\rho_*(h_i D_i)) = 2A$ for each $h_i D_i \in \mathrm{NS}(Y)$ as above. In particular we have $\pi^*(\rho_*(D_2)) = \tilde{\rho}_*\tilde{\pi}^*(D_2) = A$ (since $h_2 = 2$) and thus the sections of D_2 correspond to sections of A which are either all invariant or all antiinvariant for the action of the involution σ. So the sections of D_2 span a subspace of $H^0(X, A)$ which is contained (possibly coincides) either in U_+ or in U_- where $U_\pm$ are the eigenspaces of $H^0(X, A)$ for the action of σ^*. Similarly, the span of the sections of $2(D_1/2) = D_1$ is contained in the other eigenspace. In order to conclude that each one of $\varphi_{|D_1|}$ and $\varphi_{|D_2|}$ is associated to one of the two projections of X to $\mathbb{P}(U_+)$ and to $\mathbb{P}(U_-)$, it suffices to prove that the space spanned by the sections of D_2 (resp. D_1) is not just contained, but coincides with one of the eigenspaces. So it suffices to prove that $\dim(H^0(Y, D_2) \oplus H^0(Y, D_1)) = \dim(H^0(X, A))$.

We are now able to compute $\chi(D_i)$, $i=1,2$, by Theorem 4.2 and we know that $\chi(D_i)=h^0(D_i)$, by Proposition 4.8. Since $q_X(A)=2d$, one checks

$$\begin{aligned}\tfrac{1}{8}(q_X(A)+6)(q_X(A)+4) &= \dim(H^0(X,A))\\ &= \dim(H^0(Y,D_1))+\dim(H^0(Y,D_2))\\ &= \left(\tfrac{1}{4}d^2+\tfrac{1}{2}3d-1+3\right)+\left(\tfrac{1}{4}d^2+\tfrac{1}{2}3d-\tfrac{1}{2}d-\tfrac{12}{16}-\tfrac{5}{4}+3\right)\\ &= \tfrac{1}{2}d^2+\tfrac{1}{2}5d+3.\end{aligned}$$

Since $\tilde{\rho}_*\tilde{\pi}^*(D_i)=A$ we conclude that

$$H^0(X,A)=(\rho^{-1}\circ\pi)^*H^0(Y,D_1)\oplus(\rho^{-1}\circ\pi)^*H^0(Y,D_2). \qquad \square$$

Theorem 4.10. *Let* $d\equiv 1 \bmod 2$, $s=2,3$ *and* $\mathrm{NS}(X)\simeq(j_s,\lambda_-)(\langle 2d\rangle\oplus E_8(-2))$. *Let A be the generator of* $j_s(\langle 2d\rangle)$. *Let D_1 be the $\mathbb{Q}$-Cartier Weil divisor such that* $2D_1=\rho^*(\pi_*(j_2(h)))\in\mathrm{NS}(Y)$ *and D_2 the Cartier divisor* $D_2:=\frac{1}{2}(\rho^*\pi_*(j_1(h))-\Sigma)\in\mathrm{NS}(Y)$. *Then D_1 fails to be Cartier in 28 points,* $\tilde{\rho}_*\tilde{\pi}^*(D_1)=\tilde{\rho}_*\tilde{\pi}^*(D_2)=A$ *and*

$$H^0(X,A)=(\rho^{-1}\circ\pi)^*H^0(Y,D_1)\oplus(\rho^{-1}\circ\pi)^*H^0(Y,D_2).$$

Proof. The proof is similar to the previous one. One first observes that $\rho^*(\pi_*(j_2(h)))\in\mathrm{NS}(Y)$ and $\frac{1}{2}(\rho^*(\pi_*(j_2(h)))-\Sigma)\in\mathrm{NS}(Y)$ by the Propositions 3.6 and 3.7. There exist D_1 and D_2 $\mathbb{Q}$-Cartier Weil divisors such that a multiple of D_i, denoted by h_iD_i is one prescribed element in $\mathrm{NS}(Y)$. We choose h_i to be the minimum among positive integers such that $h_iD_i\in\mathrm{NS}(Y)$. In particular, due to the singularities of Y, h_i is either 1 or 2. If $h_i=1$, then D_i is Cartier, otherwise it is a $\mathbb{Q}$-Cartier Weil divisor on Y and it fails to be Cartier in n_i points. The possibilities for the divisors D_1 and D_2 are given in Proposition 4.6: the divisor D_1 is orthogonal to Σ, hence it is characterized by $k_1=0$; then there are two possibilities for n_1, which in turn determine uniquely the values of n_2: either $n_1=28$ and $n_2=0$ or $n_1=12$ and $n_2=16$. If $n_1=28$, then $n_2=0$ and so D_2 is Cartier, otherwise (if $n_1=12$), neither D_1 nor D_2 are Cartier. As in the previous proof, we are looking for divisors D_i, $i=1,2$, such that $\pi^*(\rho_*(D_i))=A$. Since

$$\pi^*\left(\rho_*\left(\frac{\rho^*(\pi_*(j_2(h)))-\Sigma}{2}\right)\right)=A,$$

we obtain

$$D_2=\tfrac{1}{2}(\rho^*(\pi_*(j_2(h)))-\Sigma),$$

$h_2=1$ and D_2 is Cartier. This implies that $n_1=28$ and D_1 fails to be Cartier in all the 28 singular points of Y. As in the previous proposition one is able to compute $\chi(D_i)$, $i=1,2$, by Theorem 4.2 and we know that $\chi(D_i) = h^0(D_i)$, by Proposition 4.8. So, recalling that $q_X(A) = 2d$, one can check that

$$\begin{aligned}\tfrac{1}{8}(q_X(A)+6)(q_X(A)+4) &= \dim(H^0(X, A))\\ &= \dim(H^0(Y, D_1)) + \dim(H^0(Y, D_2))\\ &= \left(\tfrac{1}{4}d^2 + \tfrac{1}{2}3d - \tfrac{28}{16} + 3\right) + \left(\tfrac{1}{4}d^2 + \tfrac{1}{2}3d - \tfrac{1}{2}d - \tfrac{5}{4} + 3\right)\\ &= \tfrac{1}{2}d^2 + \tfrac{1}{2}5d + 3.\end{aligned}$$

Since $\tilde{\rho}_*\tilde{\pi}^*(D_i) = A$ we conclude that

$$H^0(X, A) = (\rho^{-1} \circ \pi)^* H^0(Y, D_1) \oplus (\rho^{-1} \circ \pi)^* H^0(Y, D_2). \qquad \square$$

Remark 4.11. When $d = 1$ and $j_s = j_2$, in the case discussed in Proposition 3.14, we obtain $h^0(D_1) = h^0(D_2) = 3$, respectively with $(n_1, k_1) = (28, 0)$ and $(n_2, k_2) = (0, -1)$. When $d = 3$ and $j_s = j_3$, in the case of the Fano variety of a symmetric cubic discussed before Proposition 3.15, we obtain $h^0(D_1) = 8$ and $h^0(D_2) = 7$, respectively with $(n_1, k_1) = (28, 0)$ and $(n_2, k_2) = (0, -1)$.

Theorem 4.12. *Let $d \equiv 0 \bmod 2$ and let $\mathrm{NS}(X) \simeq \widetilde{\Lambda}_{2d}$ be the primitive closure of the embedding $(\tilde{j}, \lambda_-)(\langle 2d\rangle \oplus E_8(-2))$ where A is the generator of $\tilde{j}(\langle 2d\rangle)$.*

Let D_1 be the Cartier divisor $D_1 \simeq \rho^* \frac{1}{2}(\pi_*(\tilde{j}(h))) \in \mathrm{NS}(Y)$ and D_2 be a $\mathbb{Q}$-Cartier Weil divisor such that $2D_2 \simeq \left(\frac{1}{2}(\pi_*(\tilde{j}(h))) - \Sigma\right) \in \mathrm{NS}(Y)$. Then D_2 fails to be Cartier in 28 points, $\tilde{\rho}_*\tilde{\pi}^*(D_1) = \tilde{\rho}_*\tilde{\pi}^*(D_2) = A$ and

$$H^0(X, A) = (\rho^{-1} \circ \pi)^* H^0(Y, D_1) \oplus (\rho^{-1} \circ \pi)^* H^0(Y, D_2).$$

Proof. The proof is completely analogous to the previous ones. We omit it. $\square$

We now give an example of application of the previous theorems, in particular of Theorem 4.9 with $d = 1$.

Proposition 2.4 shows that, when $d = 1$, the lattice $\Lambda_2 \simeq \langle 2\rangle \oplus \langle -2\rangle^{\oplus 8}$ admits two nonisometric embeddings inside $L = L_{K3} \oplus \langle -2\rangle$, and in particular j_1 with orthogonal isometric to $T_{2,1} := U^{\oplus 2} \oplus E_8(-2) \oplus \langle -2\rangle^{\oplus 2}$.

An explicit construction of this family is given in [Camere 2012, Section 8]: it is the family of smooth double EPW sextics which carry a symplectic involution, as it is observed in [Mongardi and Wandel 2015, Example 6.8].

Indeed, the very general element of this family is $X = X_{\mathbb{A}}$ a double EPW sextic, as defined in [O'Grady 2005], associated with a Lagrangian subspace $\mathbb{A} \in \mathbb{LG}(\bigwedge^3 V)$ invariant for the action on $\bigwedge^3 V$ induced by the involution i of the six-dimensional vector space V which has exactly four eigenvalues $+1$. The fourfold $X_{\mathbb{A}}$ is defined as a double cover of a so-called EPW sextic $Z_{\mathbb{A}} \subset \mathbb{P}(V) \simeq \mathbb{P}^5$, which in this case is invariant for i, and it carries an ample invariant class $A \in \mathrm{NS}(X_{\mathbb{A}})$ of degree two; the map $\varphi_{|A|} : X_{\mathbb{A}} \to \mathbb{P}^5$ associated to A factors through the double cover $f : X_{\mathbb{A}} \to Z_{\mathbb{A}}$.

As a consequence, we get two involutions induced by i on $X_{\mathbb{A}}$ and we call σ the symplectic one among the two lifts. It is proven in [Camere 2012, Proposition 19] that the fixed locus $\mathrm{Fix}_\sigma(X_{\mathbb{A}})$ is the union of

28 isolated fixed points and one $K3$ surfaces. In fact, 12 points are the preimages in the double cover of six points $q_1, \dots, q_6 \in \mathbb{P}(V_-)$, whereas the other 16 points lie in the intersection of the ramification of f with $\mathbb{P}(V_+)$.

Finally, the fixed $K3$ surface S is the $K3$ surface obtained as double cover of a quadric surface $Q \subset Z_{\mathbb{A}} \cap \mathbb{P}(V_+)$ ramified along its intersection with a quartic surface. The double cover endows S with a nonsymplectic involution and a copy of $U(2)$ is primitively embedded in $\mathrm{NS}(S)$. By Proposition 3.5, if the conjecture holds we should have $\mathrm{T}_S \simeq U(2)^{\oplus 2} \oplus E_8(-1) \oplus \langle -4 \rangle^{\oplus 2}$.

Next we look at the Nikulin fourfold Y obtained as partial resolution of $X_{\mathbb{A}}/\sigma$. Using the notation of Proposition 4.6 and of Theorem 4.9, we obtain on Y two divisors D_1 and D_2 with $(n_1, k_1) = (12, 0)$ and $(n_2, k_2) = (16, -1)$. The orbifold Riemann–Roch formula in Theorem 4.2 implies $h^0(D_1) = 4$ and $h^0(D_2) = 2$; compare also with Table 4.

The quotient of $\mathbb{P}^5$ by the involution is the join in $\mathbb{P}^{12}$ of a conic $C \subset \mathbb{P}^2_1$ and a second Veronese $v_2(\mathbb{P}^3) \subset \mathbb{P}^9_2$ where $\mathbb{P}^2_1$ an $\mathbb{P}^9_2$ are general linear subspaces of $\mathbb{P}^{12}$. With the notation as in Theorem 4.9 we have a polarization $2D_1$ on Y such that $q_Y(2D_1) = 4$ (see the proof of Theorem 4.9).

Lemma 4.13. *The image $\varphi_{|2D_1|}(Y) = \bar{Z} \subset \mathbb{P}^{12}$ is the intersection of $J(C, v_2(\mathbb{P}^3))$ with a special cubic I. The map $\varphi_{|2D_1|}$ is generically $2:1$ ramified along a surface.*

Proof. The image $\varphi_{|2D_1|}(Y) = \bar{Z} \subset \mathbb{P}^{12}$ can be seen as the image of a symmetric EPW sextic through the involution described above. The equation of the sextic can be written as a cubic in term of invariant quadric polynomials. Such polynomials can be seen as coordinates of $\mathbb{P}^{12}$ so the image is defined as the intersection of $J(C, v_2(\mathbb{P}^3))$ with a cubic.

The image $\bar{Z}$ is singular along 6 points $C \cap I$ and three surfaces $I \cap v_2(\mathbb{P}^3) \subset \mathbb{P}^9_2$ (two of the components are quadric surfaces, one is a Kummer quartic) and the image of the singular surface of degree 40 on $Z_{\mathbb{A}}$. Only the Kummer quartic is in the ramification since the quadric component is in the ramification of the symplectic involution. □

Note that $J(C, v_2(\mathbb{P}^3)) \subset \mathbb{P}^2$ can be seen as the intersection of the cone $C(\mathbb{P}^2_1, v_2(\mathbb{P}^3))$ with a quadric cone with vertex $\mathbb{P}^9_2$. For general projective models of a general deformation we expect the above quadric cone to be more general.

5. Orbifolds of Nikulin type of BBF degree 2

The aim of this section is to study the first locally complete family of projective orbifolds of Nikulin type. This will be a family polarized by a class of BBF degree 2. It follows from the Riemann-Roch Theorem 1.3 that their projective models are fourfolds in $\mathbb{P}^6$.

Note that there are two types of classes of BBF degree 2 in the second cohomology group $U(2)^{\oplus 3} \oplus E_8(-1) \oplus \langle -2 \rangle \oplus \langle -2 \rangle$ of an orbifold of Nikulin type, respectively with divisibility 1 and 2.

An example of a Nikulin orbifold with the class of the polarization of divisibility 2 is given by the quotient of the Fano variety of lines on a symmetric cubic fourfold by the involution with signature $(2, 4)$. Indeed, by Remark 4.11 the model of X in $\mathbb{P}^{14}$ is symmetric with respect to an involution with invariant space $\mathbb{P}^7$. After projecting from it we obtain a fourfold in $\mathbb{P}^6$ being a special projective model of a Nikulin orbifold of BBF degree 2 with divisibility 2.

In this section, we are interested in the case of a polarization of BBF degree 2 divisibility 1. We first describe some special elements of the locally complete family, given by the Nikulin orbifolds. We show that they correspond to double EPW quartics, see Lemma 5.1, and that they are double covers of complete intersections of type $(3, 4)$ in $\mathbb{P}^6$; see Proposition 5.5. Then in Section 5B we generalize the previous results, showing that all the projective deformations of these Nikulin orbifolds are double covers of special complete intersections of a cubic and a quartic in $\mathbb{P}^6$.

5A. ***Geometry of*** **$(\widetilde{\Lambda}_4, \tilde{j})$*-polarized*** **$K3^{[2]}$*-fourfolds.*** (Compare with [van Geemen and Sarti 2007, Section 3.5].) We consider a fourfold of $K3^{[2]}$-type with Néron–Severi group $\mathrm{NS}(X) \simeq \widetilde{\Lambda}_4$. Then X admits a symplectic involution σ such that the corresponding Nikulin orbifold Y has a polarization of BBF degree 2 orthogonal to the exceptional divisor Σ, see Proposition 3.8. We denote by D_1 this divisor.

We now describe the image $\varphi_{|D_1|}(Y) \subset \mathbb{P}^6$. As in Proposition 2.8, we can assume that $\mathrm{NS}(X)$ is generated by A, $E_8(-2)$ and F_1 where $q_X(A) = 4$ and $F_1 = \frac{1}{2}(A + v)$ and $v \in E_8(-2)$ with $q_X(v) = -4$. Let $F_2 = \frac{1}{2}(A - v)$ then $F_i^2 = 0$ and $A = F_1 + F_2$. After monodromy operations we can assume that A is big and nef. We denote $C(\mathbb{P}^2 \times \mathbb{P}^2)$ the cone in $\mathbb{P}^9$ over the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$.

Lemma 5.1. *The linear system $|A|$ defines a $2:1$ map to $C(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^9$. The image is symmetric with respect to a linear involution σ with signature $(3, 7)$ on $\mathbb{P}^9$ that exchanges the factors in the Segre product. Moreover, the image is isomorphic to an EPW quartic corresponding to a Verra threefold that is symmetric with respect to the involution exchanging the factors in $\mathbb{P}^2 \times \mathbb{P}^2$.*

Proof. By the construction of $\mathrm{NS}(X)$ given in Proposition 2.8, one obtains that F_i are primitive in $\mathrm{NS}(X)$ and that the linear system of $A = F_1 + F_2$ defines a $2:1$ map to $C(\mathbb{P}^2 \times \mathbb{P}^2)$; see [Iliev et al. 2017, Theorem 1.1]. The symplectic involution σ acts as -1 on $E_8(-2)$, hence $\sigma^* F_1 = F_2$. So σ switches the two copies of $\mathbb{P}^2$ in $C(\mathbb{P}^2 \times \mathbb{P}^2)$ and $\varphi_{|A|}(X)$ is symmetric with respect to the linear involution which induces σ and which has signature $(3, 7)$ on $\mathbb{P}^9$.

Moreover, $U(2) \simeq \langle F_1, F_2 \rangle$ is primitive in $\mathrm{NS}(X)$. It follows that X is in the moduli space of lattice polarized fourfolds of $K3^{[2]}$-type with $U(2)$ contained in the Néron–Severi lattice. It is thus a deformation of double EPW quartics described in [Iliev et al. 2017].

It follows as in [Camere et al. 2019b, Section 6.5] that X is related to a threefold $V \subset \mathbb{P}^2 \times \mathbb{P}^2$ symmetric with respect to the involution interchanging the factors. $\square$

Lemma 5.2. *The quotient of $C(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^9$ by σ is isomorphic to the projection of this cone from the invariant $\mathbb{P}^2_- \subset \mathbb{P}^9$. This quotient is a cubic hypersurface Z_3 that is isomorphic to a cone in $\mathbb{P}^6$ over a symmetric determinantal cubic fourfold in $\mathbb{P}^5$. In particular its singular locus is a cone over the Veronese surface in $\mathbb{P}^5$.*

Proof. We can assume $C(\mathbb{P}^2 \times \mathbb{P}^2)$ is defined by 2×2 minors of a 3×3 matrix with entries being a basis of the hyperplane in $\mathbb{P}^{9\vee}$ orthogonal to the vertex of the cone. So elements of $\mathbb{P}^9$ can be thought as classes of pairs (x, M) such that $x \in \mathbb{C}$ and M is a 3×3 matrix of rank 1. The involution σ is then just the map transposing M i.e., $(x, M) \mapsto (x, M^T)$ and

$$\mathbb{P}^2_- = \{(0, M) \mid M \neq 0, M + M^T = 0\}.$$

The corresponding projection is then

$$\mathbb{P}^9 \ni (x, M) \mapsto (x, M + M^T) \in \mathbb{P}^6_+$$

where $\mathbb{P}^6_+ = \{(x, M) \mid x \in \mathbb{C}, M = M^T\}$. Since for a rank 1 matrix M, we have $M + M^T$ is a matrix of rank at most 2, the image of the projection is a cone over the space of symmetric matrices with trivial determinant. The latter is singular in the cone over the locus of rank 1 symmetric matrices, which is a cone over a Veronese surface. □

We denote by $p : \mathbb{P}^9 \to \mathbb{P}^6$ the projection from the σ-invariant $\mathbb{P}^2_- \subset \mathbb{P}^9$ described in the previous lemma and we observe that p restricts to a $2:1$ map

$$C(\mathbb{P}^2 \times \mathbb{P}^2) \to p(C(\mathbb{P}^2 \times \mathbb{P}^2))$$

with branch locus isomorphic to the cone over the diagonal of $\mathbb{P}^2 \times \mathbb{P}^2$.

Proposition 5.3. *Let $J := p(\varphi_{|A|}(X)) \subset \mathbb{P}^6$, then J is a complete intersection $Z_3 \cap Z_4 \subset \mathbb{P}^6$ of two hypersurfaces Z_3 and Z_4 of degrees* 3 *and* 4 *respectively. Moreover*, *J is singular along a surface which is the disjoint union of two* (*possibly reducible*) *surfaces*: *S_{16} of degree* 16 *and S_{36} of degree* 36.

Proof. This proof is supported by a calculation using Macaulay2 whose script is presented in the online supplement. Using the script we find an explicit example, defined in positive characteristic, of fourfold J satisfying the assertion of the theorem. We need only to argue that the invariants (degree, dimensions of the variety and decomposition of its singular locus) of the constructed variety are as expected to conclude by semicontinuity. Note that, once we get the expected invariants it is not important if the computation is made in positive or 0 characteristic as semicontinuity permits us to pass to characteristic zero in any case.

Observe first that, by the definition and properties of the map p, the variety J is contained in the hypersurface Z_3 which is the cone over the symmetric determinantal cubic hypersurface in $\mathbb{P}^5$ described in Lemma 5.2. In particular Z_3 is singular along a threefold cone over the Veronese surface and has generically A_1 singularities. Moreover, by construction, J is the image of a symmetric quartic hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$, being a symmetric EPW quartic, via the quotient by the symmetry. It follows that J is a fourfold of degree 12 in Z_3. Using our script in Macaualy2 we find an explicit case where J is complete intersection of Z_3 with a quartic and this must hence be the generic behavior.

Moreover the intersection of the singular locus of Z_3 with J is a quartic section of a cone over the Veronese surface and is part of the singular locus of J. Using our Macaulay 2 script we check on an example that this quartic section of the cone over the Veronese surface is a surface of degree 16 as expected hence this is also the generic case for J.

Recall now, that a very general EPW quartic is singular along a surface of degree 72 in $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^9$. It follows that a general symmetric EPW quartic has also at least a surface of degree 72 as singularities. Our Macaulay2 computation shows that there are symmetric EPW quartics for which the singular surface is indeed of degree 72. This surface is mapped via the map p to a surface in J which is necessarily part of the singular locus and has degree at least 36. Our Macaulay 2 script produces an example where this surface of degree 72 is mapped to a surface of degree 36, hence this must be the general behavior for symmetric EPW quartics.

Summing up, the variety J in general must contain in its singular locus the following surfaces:

(1) The intersection of the singular locus of Z_3 with Z_4. Since $\mathrm{Sing}(Z_3)$ is a cone over the Veronese surface, $\mathrm{Sing}(Z_3) \cap Z_4$ has degree 16.

(2) The quotient of the singular locus of the symmetric EPW quartic by the involution, which is a variety of degree $72 : 2 = 36$.

Since in our explicit example the singular locus of J consists of two disjoint surfaces one of degree 16 and one of degree 36 and both need to appear in the very general case this concludes the proof. □

Remark 5.4. In the above proof we can avoid computer calculation with some additional effort. First, we prove that J is in general smooth in codimension 1. Indeed, if J was singular in codimension 1, then the corresponding symmetric EPW quartic would either be singular in codimension 1 or would need to contain the whole ramification locus including the vertex of the cone. Both these cases cannot occur; see [Iliev et al. 2017, Section 3].

Next, from the shape of the Néron–Severi lattice of a very general symmetric double EPW quartic we can deduce that there are no divisors contracted by the map from the double EPW quartic to the cone over $\mathbb{P}^2 \times \mathbb{P}^2$ hence the very general symmetric EPW quartic is singular in a surface of degree 72 and has no additional singularities as in the very general nonsymmetric case.

Finally, following the construction in [Iliev et al. 2017, Proposition 2.14], we can describe the symmetric EPW quartic via the varieties of (1, 1) conics on their corresponding symmetric Verra fourfolds (see Lemma 5.1) and deduce that the singular surface of degree 72 has no component contained in the cone over the diagonal. Hence the image of this singular surface of degree 72, which is necessarily symmetric, via the projection p is a surface of degree 36 and by the fact that p is a local isomorphism outside its branch locus is necessarily a component of the singular locus of J. For the same reason J is smooth outside the union of the branch locus and the surface of degree 36.

Proposition 5.5. *The map $\varphi_{|D_1|} : Y \to \mathbb{P}^6$ is $2:1$ onto $\varphi_{|D_1|}(Y) \subset \mathbb{P}^6$ and its image is isomorphic to J, the complete intersection $Z_3 \cap Z_4 \subset \mathbb{P}^6$. The exceptional divisor $\Sigma \subset Y$ is mapped to a component of degree 4 of the surface S_{16}. Moreover, the (-2)-class $D_1 - \Sigma$ is effective on Y and contracted to a surface via $2D_1 - \Sigma$. There are no more contractible classes on any birational model of Y.*

Proof. By Section 4B, $\varphi_{|D_1|}(Y)$ is the image of the projection of $\varphi_{|A|}(X)$ from a σ-invariant subspace in $H^0(X, A)^\vee$. In our context this implies that $\varphi_{|D_1|}(Y) = p(\varphi_{|A|}(X))$ and hence Lemma 5.1 shows that

$\varphi_{|D_1|}$ is $2:1$ and Proposition 5.3 describes $\varphi_{|D_1|}(Y)$. In particular we have the following diagram

$$\begin{tikzcd}
X \arrow[r, "2:1"] \arrow[d, dashed, "2:1"] \arrow[rrr, bend left, "\varphi_{|A|}"] & \varphi_{|A|}(X) \arrow[r, hook] \arrow[d, dashed, "2:1"] & C(\mathbb{P}^2 \times \mathbb{P}^2) \arrow[r, hook] \arrow[d, dashed, "2:1"] & \mathbb{P}^9 \arrow[d, "p"] \\
Y \arrow[r, "2:1"] \arrow[rrr, bend right, "\varphi_{|D_1|}"'] & J = Z_4 \cap Z_3 \arrow[r, hook] & Z_3 = p(C(\mathbb{P}^2 \times \mathbb{P}^2)) \arrow[r, hook] & \mathbb{P}^6.
\end{tikzcd}$$

where $J := p(\varphi_{|A|}(X))$ is $Z_4 \cap Z_3$ by Proposition 5.3. The exceptional divisor Σ resolves the singularity of X/σ in the $K3$ surface image of the σ-fixed surface S on X. The latter surface is in $C(\mathbb{P}^2 \times \mathbb{P}^2)$. The symplectic involution on X is induced by the symmetry on $\mathbb{P}^9$ that interchanges the factors of $\mathbb{P}^2 \times \mathbb{P}^2$. So the $K3$ surface S in $C(\mathbb{P}^2 \times \mathbb{P}^2)$, being fixed by the involution, is contained in the cone over the diagonal in $\mathbb{P}^2 \times \mathbb{P}^2$. It follows that its image is a component of S_{16}. By Lemma 4.3 it is a surface S_4 of degree 4 which is necessarily projectively isomorphic to the Veronese surface.

For the second part we observe that the proper transform on Y of the intersection of $\varphi_{|D_1|}(Y)$ with the span of S_4 in $\mathbb{P}^6$ is the (-2)-class $D_1 - \Sigma$. The system $2D_1 - \Sigma$ is big and induces its contraction since on $\varphi_{|D_1|}(Y)$ it can be seen as a system of quadrics containing the Veronese surface S_4 i.e., it contracts the planes spanned by conics on S_4 which fill the cubic Z_3 intersected with the span of S_4. The locus contracted by $2D_1 - \Sigma$ is hence exactly the (-2)-class $D_1 - \Sigma$. Observe that there can be no more contractible divisorial classes on any birational model of Y. For that, we work in codimension 1 knowing [Menet and Rieß 2020, Lemma 3.2] that any birational map is regular in codimension 1. Now, since the Picard number of Y is 2, among three big divisor classes one of them is a positive combination of the two other ones. In particular, if we have three divisor classes each contracted by some map associated to a big divisor then one of these big divisors is a positive linear combination of the two remaining ones. But a positive combination of two big divisors can only contract subvarieties which are contracted by both divisors, so all three contracted divisors would need to have a common component. However, both Σ and $D_1 - \Sigma$ are represented by distinct irreducible effective divisors so have no common component. $\square$

In the next section we will study the locally complete projective family of orbifolds of Nikulin type to which Y as in Proposition 5.5 belongs and we will show that they can all be realized as certain double covers, in complete analogy with what happens in the case of double EPW sextics. Since the full monodromy group of orbifolds of Nikulin type of dimension 4 is not known yet, we will first use the nonsymplectic involution on Y given by the double cover to produce an involution of $H^2(Y, \mathbb{Z})$ which is a monodromy operator and has the span of the divisor D_1 as the only invariant classes. We recall the following notation: given an element $e \in H^2(Y, \mathbb{Z})$, the reflection r_e in e is the isometry defined by $r_e(x) = x - (2B_Y(x, e)/q_Y(e))e$ (it is integral only for special values of $q_Y(e)$ and $\mathrm{div}(e)$).

Lemma 5.6. *Let D_1 be the class with $q_Y(D_1) = 2$ and divisibility 1 considered above. The isometry $-r_{D_1}$, such that $x \mapsto -x + B_Y(x, D_1)D_1$, in $H^2(Y, \mathbb{Z})$ is a monodromy operator of $H^2(Y, \mathbb{Z})$.*

Proof. The map $\varphi_{|D_1|}$ is a generically $2:1$ map onto its image and so there exists the involution Θ which is the cover involution of $Y \to \varphi_{|D_1|}(Y)$. First $\varphi_{|D_1|}$ contracts the exceptional (-4)-class Σ and then it identifies points switched by Θ. So Θ^* acts as -1 on the transcendental lattice T_Y and acts trivially on $\mathrm{NS}(Y)$, generated by D_1 and Σ, see Proposition 3.8. Moreover Θ^* is a monodromy operator of $H^2(Y,\mathbb{Z})$, since it is induced by an automorphism of Y.

Let r_Σ be the reflection given by $r_\Sigma(x) = x + \frac{1}{2}B_Y(x,\Sigma)\Sigma$ for $x \in H^2(Y,\mathbb{Z})$. It is a monodromy operator by [Menet and Rieß 2021, Proposition 1.5]. We observe that $-r_{D_1} = \Theta^* \circ r_\Sigma$ and so $-r_{D_1}$ is a monodromy operator. □

5B. ***The family of complete intersections* (3, 4).** Let Y be an orbifold of Nikulin type such that

$$(H^2(Y,\mathbb{Z}), q_Y) \simeq U(2)^{\oplus 3} \oplus E_8(-1) \oplus \langle -2\rangle \oplus \langle -2\rangle,$$

and there exists an ample Cartier divisor H on Y with degree $q_Y(H) = 2$ and divisibility 1. Such an orbifold exists by surjectivity of the period map. Since the Fujiki constant for Y is 6 we have $H^4 = 24$.

Theorem 5.7. *The map $\varphi_{|H|} : Y \to \mathbb{P}^6$ is $2:1$ and its image is a special fourfold of codimension 2 in $\mathbb{P}^6$ being the complete intersection of a cubic and a quartic. The map is branched along a surface of degree* 48.

Proof. By Corollary 4.4 and the Kawamata–Viehweg vanishing theorem we have

$$h^0(Y, \mathcal{O}(H)) = 7.$$

Hence the target space of $\varphi_{|H|}$ is $\mathbb{P}^6$.

Let Y_0 be the special Nikulin orbifold considered in Section 5A and H_0 be the divisor $D_1 \in \mathrm{NS}(Y_0)$ considered in Proposition 5.5.

From Proposition 5.5, $\varphi_{|H_0|}$ is $2:1$ and hence there exists an involution Θ_0 on Y_0, which is the cover involution and it is nonsymplectic. Moreover the image $\varphi_{|H_0|}(Y_0)$ is a normal complete intersection of type $(3,4)$. The idea of the proof is to show that a general deformation of (Y_0, H_0) is of the same shape.

Let $(\pi : \mathcal{Y} \to B, \mathcal{H})$ be a family of polarized orbifolds of degree 2 and divisibility 1 with central fiber (Y_0, H_0) over a small disc $B \ni 0$. From Lemma 5.6, $-r_{H_0}$ is a monodromy operator of $H^2(Y_0,\mathbb{Z})$.

Let $t_n \in B$, Y_{t_n} be the fiber of π over t_n and let H_{t_n} be the restriction of $\mathcal{H}$ to Y_{t_n}. We fix a sequence $t_n \to 0$ such that $\mathrm{NS}(Y_{t_n}) = \mathbb{Z}H_{t_n}$. By parallel transport $-r_{H_{t_n}}$ is a monodromy operator of $H^2(Y_{t_n},\mathbb{Z})$ and, by a standard argument using $\rho(Y_{t_n}) = 1$ and the global Torelli theorem (see for example [Menet and Rieß 2020, Theorem 1.1]), $-r_{H_{t_n}}$ lifts to an involution $\Theta_{t_n} : Y_{t_n} \to Y_{t_n}$.

Arguing as in [O'Grady 2005, Section 2], the limit of Θ_{t_n} is an involution on Y_0 and we show that it is Θ_0. We denote the graph of Θ_{t_n} by Γ_{t_n}. The analytic cycles Γ_{t_n} converge (see the proof of [Huybrechts 1999, Theorem 4.3]) to Γ_0 with a decomposition $\Gamma + n_i\Omega_i$ where Γ is the graph of a birational map $Y_0 \dashrightarrow Y_0$ and Ω_i are irreducible in $D_i \times E_i$ with $D_i, E_i \subset Y_0$ proper subsets. As in [O'Grady 2005, Section 2], Γ_0 induces on $H^2(Y_0,\mathbb{Z})$ exactly the monodromy operator $-r_{H_0}$ via parallel transport.

Again as in [loc. cit.], the invariance of Γ_{t_n} with respect to the exchange of the two factors in $Y_0 \times Y_0$ implies that Γ_0 is invariant as well and, due to the different nature of the two parts in the decomposition above, that Γ is the graph of a birational involution.

If D_i has codimension > 1, the action on $H^2(Y_0, \mathbb{Z})$ of $[\Omega_i]_*$ is zero, thus we assume that D_i is an effective divisor in $\mathrm{NS}(Y_0) = \langle H_0, \Sigma\rangle$, but this implies that the action of Γ on T_{Y_0} coincides with the action of Γ_0, i.e., it acts as $-\,\mathrm{id}$ on the transcendental lattice. It follows from Proposition 5.5 that there are exactly 2 contractible classes on Y_0: the (-4)-class Σ and the (-2)-class $H_0 - \Sigma$. Hence Σ and $H_0 - \Sigma$ are preserved by any birational map, and thus also by $[\Gamma]_*$ i.e.,

$$[\Gamma]_*(H_0) = H_0, \quad [\Gamma]_*(\Sigma) = \Sigma.$$

We conclude that, since $[\Gamma]_*$ acts on $H^2(Y_0, \mathbb{Z})$ preserving both H_0 and Σ and acts as minus the identity on their orthogonal in $H^2(Y_0, \mathbb{Z})$, it coincides with Θ_0^*. In the case of orbifolds of Nikulin type, the only automorphism acting trivially in cohomology is the identity, as shown in [Menet and Rieß 2021, Proposition 8.1], hence the birational involution associated to Γ is exactly the nonsymplectic involution $\Theta_0 \in \mathrm{Aut}(Y_0)$.

We thus have a sequence $(Y_{t_n}, H_{t_n}, \Theta_{t_n})$ of polarized orbifolds of Nikulin type each equipped with an involution Θ_{t_n} preserving H_{t_n} and such that (Y_0, H_0, Θ_0) is its limit in the sense above. The involutions Θ_{t_n} induce a sequence of involutions on $H^0(Y_{t_n}, H_{t_n}) = H^0(Y_0, H_0)$ whose limit is the map induced by Θ_0 on $H^0(Y_0, H_0)$. The latter is the identity map because Θ_0 is the cover involution of $\varphi_{|H_0|}$. It follows that for $n \gg 1$ the action of Θ_{t_n} on $H^0(Y_{t_n}, H_{t_n})$ is also trivial and hence $\varphi_{|H_{t_n}|}$ is $2:1$ for $n \gg 1$. We conclude that for general (Y_t, H_t) in a neighborhood of (Y_0, H_0) the map $\varphi_{|H_t|}$ is $2:1$ onto the image contained in $\mathbb{P}^6$.

We saw that $J_0 := \varphi_{|H_0|}(Y_0)$ is normal, hence by the openness of normality the image J_t of Y_t through $|H_t|$ is also normal of codimension 2 in $\mathbb{P}^6$. Thus J_t is necessarily the quotient of Y_t through the involution Θ_t. In particular, J_t has ODP points along a surface that is smooth outside the 28 orbifold points. Let us show that the general J_t is also a complete intersection. We consider the family $\{G_t\}_{t\in\Delta}$, with Δ a small disc, with $G_t = \varphi_{H_t}^{-1}(\Pi_1 \cap \Pi_2)$ and Π_i two chosen general hyperplanes in $\mathbb{P}^6$. Note that G_0 is smooth and maps via φ_{H_0} to $J_0 \cap \Pi_1 \cap \Pi_2$ which is a complete intersection $(3, 4)$ in $\mathbb{P}^4$ and which must admit only nodes as singularities. It follows that the general G_t maps via φ_{H_t} to a nodal surface in $R_t = J_t \cap \Pi_1 \cap \Pi_2 \subset \mathbb{P}^4 = \Pi_1 \cap \Pi_2$ being the quotient of G_t through an involution. Such surface is of degree 12 and half-canonical i.e., $K_{R_t} = 2H$ (where H is the hyperplane from $\mathbb{P}^4$).

We can now mimic [Decker et al. 1990, Proposition 1.2] to prove that R_t is a complete intersection. Indeed, R_t is a half canonical surface and since R_t has complete intersection singularities it is the zero locus of a rank 2 vector bundle E on $\mathbb{P}^4$ hence the methods of [loc. cit., Proposition 1.2] apply also in this case. More precisely, the case $c_1(E)^2 - 4c_2(E) \le 0$ from [loc. cit., Proposition 1.2] cannot occur by a generalization of the double point formula for nodal hypersurfaces proved in [Catanese and Oguiso 2020, Theorem 5.1] ($\delta = 0$ in our case). Thus $c_1(E)^2 - 4c_2(E) > 0$ and we conclude as in Case 2 of [Decker et al. 1990, Proposition 1.2] that $R_t \subset \mathbb{P}^4$ is a complete intersection $(3, 4)$.

We thus know that a general codimension two linear section R_t of J_t is a complete intersection $(3,4)$. To conclude that J_t must also be such a complete intersection let us consider $U_t \supset R_t$ a general hyperplane section of J_t containing R_t and the exact sequences

$$0 \to \mathcal{I}_{J_t} \to \mathcal{I}_{J_t}(1) \to \mathcal{I}_{U_t|\mathbb{P}^5}(1) \to 0 \quad \text{and} \quad 0 \to \mathcal{I}_{U_t} \to \mathcal{I}_{U_t}(1) \to \mathcal{I}_{R_t|\mathbb{P}^4}(1) \to 0.$$

To conclude it is enough to show that $h^1(\mathcal{I}_{J_t}(2)) = h^1(\mathcal{I}_{U_t}(2)) = 0$ and $h^1(\mathcal{I}_{J_t}(3)) = h^1(\mathcal{I}_{U_t}(3)) = 0$ (then the cubic and a quartic defining R_t extend to the ideal of U_t and then J_t). But applying again the long exact sequence of cohomology the vanishing of $h^1(\mathcal{I}_{U_t}(k))$ will follow from the vanishing of $h^2(\mathcal{I}_{J_t}(k-1))$ and $h^1(\mathcal{I}_{J_t}(k))$. It is hence enough to prove

$$h^2(\mathcal{I}_{J_t}(2)) = h^2(\mathcal{I}_{J_t}(1)) = h^1(\mathcal{I}_{J_t}(3)) = h^1(\mathcal{I}_{J_t}(2)) = 0. \tag{5-1}$$

We compute these dimensions using the finite map $f\colon Y_t \to J_t$: there exists a sheaf $\mathcal{F}$ on J_t such that

$$f_*\mathcal{O}_{Y_t}(k) = \mathcal{O}_{J_t}(k) \oplus \mathcal{F}(k).$$

We get our vanishings (5-1) from the fact that $h^i(\mathcal{O}_{Y_t}(k)) = 0$ for $i = 1, 2$ and $k = 2, 3$. We conclude that $\mathcal{I}_{J_t}$ admits a cubic and a quartic generator which, after restriction to a codimension 2 linear space, define a complete intersection. Since J_t is of codimension 2 and degree 12, J_t is a complete intersection.

Let us compute the degree of the singular surface of J_t (in fact we can deduce this from the singular locus of J_0 finding $52-4=48$). If we denote by $F \subset Y_t$ a general intersection of two divisors in the system H_t in Y_t then we find that $K_F = 2H_{t|_F}$ and $\chi(\mathcal{O}_F(nH_t)) = 12n^2 - 24n + 20$. Denote by $G \subset \mathbb{P}^4$ the image of F being a complete intersection $(3,4)$. The involution given by $|H_t|$ cannot fix varieties of odd codimension (since the smooth locus of the orbifold Y_t has a symplectic form and the singular locus consists of isolated points). Moreover, it cannot fix smooth points, since it is a nonsymplectic involution. The orbifold points are in the fixed locus otherwise they would map to noncomplete intersection singularities. So the ramification of the map is a surface. We find that G is nodal and $F \to G$ is branched at the nodes. Let μ be the number of nodes. We shall compute μ by comparing the Euler characteristics of appropriate sheaves on F and G. First observe that $\chi(\mathcal{O}_G) = 16$ since G is a complete intersection. Next we consider the minimal resolution $\overline{G}$ of G and the blow up $\overline{F}$ of F at the preimages of the nodes together with the induced map $f\colon \overline{F} \to \overline{G}$. We find $f_*\mathcal{O}_{\overline{F}} = \mathcal{O}_{\overline{G}} \oplus \mathcal{O}_{\overline{G}}(L)$ where $2L$ is the sum of the exceptional divisors on $\overline{G}$. We compare the Riemann–Roch formulas for $\overline{F}$ and $\overline{G}$ and conclude from $2\chi(\mathcal{O}_G) - \frac{\mu}{4} = \chi(\mathcal{O}_{\overline{F}}) = \chi(\mathcal{O}_F) = 20$ that $\mu = 48$. □

5C. ***A special subfamily of BBF degree 2.*** We consider Nikulin orbifolds with a Cartier divisor of BBF degree 2 and divisibility 1 that form a subfamily of codimension 2 of the locally complete family described in Theorem 5.7.

These orbifolds are constructed as quotients of $W^{[2]}$ by a natural symplectic involution $\sigma^{[2]}$, where W is a $K3$ surface with $\mathrm{NS}(W) \simeq \widetilde{\Lambda_4}$ and σ is a symplectic involution on it. The surfaces W are double covers of a quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$ branched along a $(2,2)$ curve C that is symmetric with respect to the involution ι_Q exchanging the two factors of Q; see [van Geemen and Sarti 2007]. We denote by j the cover involution of $W \to Q$ and we observe that ι_Q lifts to two involutions on W: a nonsymplectic involution ι and a symplectic involution σ. We observe that $\iota = j \circ \sigma$.

Then σ induces a natural involution $\sigma^{[2]}$ on $W^{[2]}$ fixing 28 points and a $K3$ surface S. The ample divisor of degree 4 on W invariant for σ^* induces a divisor A on $W^{[2]}$ which is orthogonal to the exceptional divisor of $W^{[2]} \to \mathrm{Sym}^2(W)$.

The map given by $|A|$ can be described as follows:

$$\varphi_{|A|} : (\mathbb{P}^3)^{[2]} \supset W^{[2]} \to \mathrm{Sym}^2(Q) \subset \mathrm{Sym}^2(\mathbb{P}^3) \subset \mathbb{P}^9.$$

The involution ι induces on $\mathbb{P}^9$ a linear involution of the form $(-,-,-,+,+,+,+,+,+,+)$ so we have two invariant linear spaces $\mathbb{P}^6_+$ and $\mathbb{P}^2_-$. By Proposition 3.11 the Nikulin orbifold $W^{[2]}/\sigma^{[2]}$ admits a polarization H of BBF degree 2 and divisibility 1 induced by A.

Lemma 5.8. *The image of the Nikulin orbifold $W^{[2]}/\sigma^{[2]}$ through the $4:1$ map given by H is a special complete intersection $(2,3)$ in $\mathbb{P}^6$. This fourfold is a degeneration of the family of $(3,4)$ intersections described in Theorem 5.7.*

Proof. The image of the map φ is a fourfold of degree 12 in $\mathbb{P}^9$ that can be seen as the secant variety of the second Veronese embedding of a quadric surface. The projection from $\mathbb{P}^2_-$ is no longer $2:1$, as it can be checked on a special fiber. The image is contained in a quadric since we find that $\varphi_{|A|}(W^{[2]})$ is contained in a quadric being a cone over $\mathbb{P}^2_-$. We conclude knowing the degree of the fourfold in $\mathbb{P}^6$. $\square$

Remark 5.9. One can show using computer calculations that the intersection $(2,3)$ above is singular along a surface of degree 12.

Note that the involution $\iota^{(2)}$ on $Q^{(2)}$ has two fixed surfaces: B_1 consisting of the pairs $(x, j(x))$ for $x \in Q$ and B_2 consisting of the pairs $(c_1, c_2) \subset C^{(2)} \subset Q^{(2)}$. We see that the fixed $K3$ surface S is mapped to B_1 and the isolated points are mapped to B_2.

5D. *Lagrangian type description.* Let us describe an object analogous to the Lagrangian subspace of dimension 10 of $\bigwedge^3 \mathbb{C}^6$ for double EPW sextics. Suppose (Y, H) is a polarized orbifold of Nikulin type with degree $q_Y(H) = 2$ such that $|H|$ induces a finite $2:1$ morphism. Then $|H|$ defines a $2:1$ map f with image J being a 4-dimensional variety of degree 12 in $\mathbb{P}^6$ singular along a surface. Since f is finite, there exists a sheaf $\mathcal{F}$ on J such that

$$f_*\mathcal{O}_Y = \mathcal{O}_J \oplus \mathcal{F}.$$

We infer from the Riemann–Roch theorem the following table:

$$\begin{array}{ccccccc}
H^4(\mathcal{F}(-3)) & H^4(\mathcal{F}(-2)) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbb{C} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & H^0(\mathcal{F}(2)) & H^0(\mathcal{F}(3))
\end{array}$$

So we have the following symmetric Beilinson resolution:

$$0 \to 28\Omega^6(6) \xrightarrow{M^*} 3\Omega^5(5) \oplus \Omega^3(3) \oplus 3\Omega^1(1) \xrightarrow{M} 28\mathcal{O} \to \mathcal{F}(3) \to 0$$

The matrix corresponding to M from the Beilinson resolution is a matrix with three rows of 1 forms, one row of 3-forms and three rows of 5-forms. Moreover, it has the property that $MM^* = 0$ as matrices of 4-forms (the product is induced by the exterior product of forms).

The choice of M is thus the choice of a 28 dimensional linear subspace in

$$3\bigwedge^5 V_7 \oplus \bigwedge^3 V_7 \oplus 3V_7,$$

isotropic for the product b that can be seen as a kind of symplectic form

$$b : \Bigl(3V_7 \oplus \bigwedge^3 V_7 \oplus 3\bigwedge^5 V_7\Bigr)^2 \to \bigwedge^6 V_7$$

given by the formula

$$\begin{aligned} &b((l_1, l_2, l_3, \alpha, w_1, w_2, w_3), (L_1, L_2, L_3, \beta, W_1, W_2, W_3)) \\ &\qquad = L_1 \wedge w_1 + L_2 \wedge w_2 + L_3 \wedge w_3 + \alpha \wedge \beta + l_1 \wedge W_1 + l_2 \wedge W_2 + l_3 \wedge W_3. \end{aligned}$$

Note that the variety J, being the support of $\mathcal{F}(3)$, appears as a degeneracy locus of such a map M.

Problem 5.10. (1) *Describe the "Lagrangian" 28 space corresponding to Nikulin orbifolds i.e., quotients of fourfolds of $K3^{[2]}$-type as described in Section 5A.*

(2) *How to describe the cubic in the complete intersection* (3, 4)?

(3) *Is the moduli space of polarized orbifolds of Nikulin type of dimension* 4 *and BBF degree* 2 *unirational*?

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