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Degree growth for tame automorphisms of an affine quadric threefold

Dang Nguyen-Bac

We consider the degree sequences of the tame automorphisms preserving an affine quadric threefold. Using some valuative estimates derived from the work of Shestakov and Umirbaev and the action of this group on a CAT(0), Gromov-hyperbolic square complex constructed by Bisi, Furter and Lamy, we prove that the dynamical degrees of tame elements avoid any value strictly between 1 and $\frac{4}{3}$. As an application, these methods allow us to characterize when the growth exponent of the degree of a random product of finitely many tame automorphisms is positive.

Introduction

Fix a projective variety X of dimension n defined over an algebraically closed field k of characteristic zero and a rational map f on X. We are interested in the complexity associated to the dynamical system induced by f, more precisely on the growth of the degrees of the p-fold composition $f^p = f \circ \cdots \circ f$. This general problem was addressed in the work of Russakovski and Shiffman [1997] when $X = \mathbb{P}^n$ in which they related the asymptotic behavior of the images by f of the linear subvarieties of $\mathbb{P}^n$ with the degree sequences. The asymptotic ratios of these sequences, denoted $\lambda_i(f)$ for $i \leqslant n$, and referred as dynamical degrees, control the topological entropy of those maps [Dinh and Sibony 2005] and are crucial for the construction of an invariant measure of maximal entropy [Bedford and Smillie 1992; Bedford and Diller 2005; Guedj 2005].

When f is a birational surface map, the situation is completely classified [Blanc and Cantat 2016; Cantat 2011; Diller and Favre 2001; Gizatullin 1980]. For general rational maps on surfaces, the behavior of the degree is known for morphisms of the affine plane [Favre and Jonsson 2011] and when $\lambda_1(f)^2 > \lambda_2(f)$ [Boucksom et al. 2008].

From the dimension three on, only the degree growth of monomial maps [Favre and Wulcan 2012; Lin 2012], regular morphisms, pseudoautomorphisms [Bedford 2015; Oguiso and Truong 2014; 2015; Truong 2016; 2017], birational maps on hyperkähler varieties [Lo Bianco 2019], dominant rational maps satisfying $\lambda_1(f)^2 > \lambda_2(f)$ [Dang and Favre 2021] and sporadic examples [Abarenkova et al. 1999a; 1999b; Anglès d'Auriac et al. 2006; Bedford and Truong 2010; Bedford and Kim 2014] were studied. Recently, new constraints on slow degree growth appeared for polynomial maps of the affine space [Urech

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2018], birational mappings [Cantat and Xie 2020] and nonregularizable birational transformations [Lonjou and Urech 2021], however the general problem of understanding the degree of the iterates of birational transformations of $\mathbb{P}^3$ remains open. The main reason is that we usually rely on the construction of a good birational model (e.g., an algebraically stable model in the sense of Fornaess and Sibony [1995]) to find the degree sequences, but the structure of the set of birational models of threefolds is far more complicated than its analog for surfaces. It is thus natural to ask whether we can find a large class of birational transformations of $\mathbb{P}^3$ for which this sequence is fully understood.

A first natural choice would be the group of polynomial automorphisms of the three dimensional affine space. Even though there has been some recent work on particular subgroups of this group [Lamy 2019; Lamy and Przytycki 2021; Wright 2015], their dynamical degrees were computed explicitly for degree 2 maps [Maegawa 2001], for degree 3 maps [Blanc and van Santen 2019]. Recently, the author proved with C. Favre that the dynamical degrees of polynomial automorphisms of $\mathbb{A}^3$ are all algebraic numbers [Dang and Favre 2021]. However, the problem of classifying all the dynamical degrees and all the possible degree growths remain open. We have thus turned our attention to a simpler situation, namely the subgroup of tame automorphisms of the affine quadric threefold.

We denote by (x, y, z, t) the affine coordinates in $\mathbb{A}^4$ and consider the affine quadric $\mathcal{Q}$ given by

$$\mathcal{Q} = V(xt - yz - 1).$$

Observe that the Picard group of the closure $\overline{\mathcal{Q}}$ of $\mathcal{Q}$ in $\mathbb{P}^4$ is generated by $H = c_1(\mathcal{O}(1)_{|\overline{\mathcal{Q}}})$ so that one can define the algebraic degree of an automorphism by

$$\deg(f) := \deg_1(f) = (\pi_1^* H^2 \cdot \pi_2^* H),$$

where π_1 and π_2 are the projections of the graph of the birational map induced by f in $\overline{\mathcal{Q}} \times \overline{\mathcal{Q}}$ onto the first and the second factor respectively. Observe that by definition $\deg_2(f) = (\pi_1^* H \cdot \pi_2^* H^2) = \deg(f^{-1})$ since f is an automorphism.

The group of automorphism naturally contains the subgroup $O_4 \subset GL_4(k)$ of linear maps of $\mathbb{A}^4$ preserving the quadric Q. The subgroup of tame automorphisms, denoted Tame(Q), is defined as the subgroup generated by O_4 and transformations induced by

$$(x, y, z, t) \mapsto (x, y, z + xP(x, y), t + yP(x, y)),$$

with $P \in k[x, y]$.

Theorem 1. *Let f be a tame automorphism, then one of the following possibilities occur*:

(i) *The sequence $(\deg(f^n), \deg(f^{-n}))$ is bounded. Moreover, f is conjugated to an element of O_4 or f^2 is conjugated to an automorphism of the form*

$$(x, y, z, t) \mapsto (ax, by + xR(x), b^{-1}z + xP(x, y), a^{-1}(t + yP(x, y) + zR(x) + xR(x)P(x, y)))$$

with $a, b \in k^*$, $P \in k[x, y]$ and $R \in k[x]$.

(ii) *There exists a constant* $C > 0$ *such that*

$$\tfrac{1}{C} n \leqslant \deg(f^{\epsilon n}) \leqslant C n,$$

for all $\epsilon \in \{+1, -1\}$ *and* f *is conjugated to an automorphism of the form*

$$(x, y, z, t) \mapsto (ax, b^{-1}(z + xR(x)), b(y + xP(x)z), a^{-1}(t + z^2 P(x) + yR(x) + xzP(x)R(x))),$$

with $a, b \in k^*$, $R \in k[x]$ *and* $P \in k[x] \setminus k$.

(iii) *The sequences* $\deg(f^n)$ *and* $\deg(f^{-n})$ *grow at least exponentially and there exists a constant* $C(f) > 0$ *such that*

$$\min(\deg(f^{-n}), \deg(f^n)) \geqslant C(f)\left(\tfrac{4}{3}\right)^n.$$

Theorem 1 is a first step towards an understanding of the dynamical degrees of these particular automorphisms.

Corollary 2. *The following inclusion is satisfied*:

$$\{\lambda_1(f) \mid f \in \mathrm{Tame}(\mathrm{Q})\} \subset \{1\} \cup \left[\tfrac{4}{3}, +\infty\right[.$$

This result is reminiscent of a theorem of Blanc and Cantat [2016, Corollary 2.7] stating that the set of first dynamical degrees of any birational surface maps is included in $\{1\} \cup [\lambda_L, \infty)$ where $\lambda_L \simeq 1.176280$ denotes the Lehmer number. We conjecture however that the gap should be bigger and that there should be no dynamical degree of Tame(Q) in the interval $]1, 2[$. The verification of such a conjecture would suggest that the dynamical degrees of tame automorphisms of the quadric are always integers.

Another immediate consequence of Theorem 1 is the following corollary.

Corollary 3. *Any tame automorphism* $f \in \mathrm{Tame}(\mathrm{Q})$ *satisfying* $\lambda_1(f) = 1$ *preserves a fibration or belongs to* O_4 *and both sequences* $\deg(f^n)$, $\deg(f^{-n})$ *are either bounded or linear.*

The above result gives a positive answer to a question by Urech [2018, Question 4] in this special situation.

The proof of Theorem 1 exploits extensively the structure of the group of tame automorphisms. We use the natural action of Tame(Q) on a square complex $\mathcal{C}$ which was introduced and studied by Bisi, Furter and Lamy [Bisi et al. 2014]. This action is faithful, transitive on squares, and isometric. The complex $\mathcal{C}$ plays the same role for Tame(Q) as the Bass–Serre tree for Aut$[k^2]$.

One of the main result of [Bisi et al. 2014] is that $\mathcal{C}$ is a geodesic space which is both CAT(0) and Gromov-hyperbolic. As a result, a tame automorphism induces an action on the complex which is rather constrained: either it is elliptic and fixes a vertex in the complex $\mathcal{C}$; or it is hyperbolic and acts by translation on an invariant geodesic line.

Using an explicit description of the stabilizer subgroups of each vertices, we compute the degree sequences of all elliptic tame automorphisms.

The crucial point of the proof is the study in Section 5 of the degree growth of hyperbolic automorphisms. In this case, we show that the sequence of degrees is bounded from below by $C\left(\frac{4}{3}\right)^n$ for some positive constant $C > 0$ and where n depends on the distance of translation on an invariant geodesic line. Let us state a weaker statement which summarizes the overall idea of our proof and which relates the degree with the displacement by f of a vertex v_0 fixed by the linear group.

Theorem 4. *For any tame automorphisms $f \in \mathrm{Tame(Q)}$ for which f is not in O_4, the following inequality holds*:

$$\log(\deg(f)) \geqslant \frac{\log(4/3)}{2\sqrt{2}} d_{\mathcal{C}}(f \cdot v_0, v_0) - 2\log\left(\tfrac{4}{3}\right),$$

where $d_{\mathcal{C}}$ denotes the distance in the complex.

This phenomenon already appears in the case of plane automorphisms since one can bound from below the logarithm of the degree of a plane automorphism by $\log(2)$ multiplied by the distance between two vertices in the Bass–Serre tree associated to the group $\mathrm{Aut}(\mathbb{A}^2)$. Also in the case of $\mathrm{Bir}(\mathbb{P}^2)$, there is a relationship between the degree and the distance on a suitable hyperbolic space. The above result does not imply Theorem 1 and one needs to prove a more refined statement to obtain that the degree of f^n is indeed larger than $\left(\frac{4}{3}\right)^n$. Let us explain how this is done.

Let $f \in \mathrm{Tame(Q)}$ be any hyperbolic automorphism. First we show that by conjugating with an appropriate automorphism, we can suppose that v_0 lies at distance ≤ 2 of an f-invariant geodesic line. Suppose that v_0 is contained in an invariant geodesic of f. Our goal is to prove that

$$\deg(f^n) \geqslant \left(\tfrac{4}{3}\right)^{d_{\mathcal{C}}(v_0, f^n \cdot v_0)} \text{ for all } n \in \mathbb{N}. \tag{1}$$

The sequence of large squares (i.e., isometric to $[0, 2]^2$) cut by the geodesic segment $[v_0, f^n \cdot v_0]$ allows us to write

$$f^n = g_p \circ g_{p-1} \cdots \circ g_1 \tag{2}$$

as a composition of elementary automorphisms and linear transformations (in O_4) preserving the quadric. This decomposition is not unique in general and ideally, one would hope to prove that the degree is multiplicative so that $\deg(f^n) \geq \prod_{i=1}^p \deg(g_i)$. The obstruction to this property is the presence of resonances, which are explained as follows. Two regular functions $P, R \in k[Q]$ are *resonant* if there exists $\lambda \in k^*$ and two integers p, q such that $\deg(P^p - \lambda R^q) < p\deg(P) = q\deg(R)$ and they are called *critical* if $p = 1$ or $q = 1$.

When these resonances are not *critical*, we show that one can apply the so-called parachute inequalities (recalled in Section 4E) to deduce (1). These inequalities are elementary valuative estimates on the values of partial derivatives of suitable polynomials, and are derived from the proof of Nagata's conjecture by Shestakov and Umirbaev; see [Kuroda 2016; Lamy and Vénéreau 2013; Shestakov and Umirbaev 2003].

To get around the appearance of critical resonances, we exploit the structure of the tame group to prove that f^n always admits an appropriate factorization for which the parachute inequalities can be applied inductively. In other words, we write $f^n = g'_p \circ \cdots \circ g'_1$ where g'_i are tame automorphisms such that for each $i \leqslant p$, g'_{i+1} and $(g'_i \circ \cdots \circ g'_1)$ do not have critical resonances.

Using the correspondence between the factorizations of f and the sequences of large squares cut out by the invariant geodesic, we are reduced to proving that one can modify inductively our initial sequence of large squares to avoid critical resonances. The essential point is to choose a valuation ν of monomial type (i.e., with different weights on the coordinate axis x, y, z, t) such that one of the vertex of our initial large square has ν-value strictly less than the three others. The dissymmetry induced by ν will be propagated along any sequence of squares following our geodesic. We then argue that this minimality property on each large square allows us to choose another square with no critical resonances. As a result, the core of our approach relies deeply on the structure of the tame group which is reflected by the geometric properties of the square complex. Our proof is presented using purely combinatorial arguments.

In the last part of this paper, we shall give a random version of Theorem 1. Consider a finitely generated subgroup G of the tame group and an atomic probability measure μ on G such that

$$\int_G \log(\deg(g))\, d\mu(g) < +\infty.$$

The random walk on G with transition law μ is the Markov chain starting at Id with transition law μ. The state of the Markov chain g_n at the time n is equal to the product of n independent, identically distributed random variable on G with distribution law μ. Its distribution law ν_n is the n-fold convolution of μ. Since the degree is submultiplicative, Kingman's subadditivity asserts that the degree exponents given by

$$\lambda_1(\mu) := \limsup_{n\to+\infty} \frac{1}{n} \int_G \log(\deg(g))\, d\nu_n(g) \quad \text{and} \quad \lambda_2(\mu) := \limsup_{n\to+\infty} \frac{1}{n} \int_G \log(\deg(g^{-1}))\, d\nu_n(g)$$

are finite. These numbers measure the complexity of our random walk and one recovers the first and second dynamical degrees of f when μ is equal to the Dirac measure at f.

Since the degree is equal to the norm of the pullback operator induced by f on the Neron–Severi group of the quadric, these quantities play the same role as the Lyapounov exponents of a random products of matrices [Furstenberg 1963; Furstenberg and Kesten 1960] for this group and the existence of these exponents can thus be interpreted as a law of large number [Benoist and Quint 2016, Theorem 0.6].

We now state the following result on the behavior of any symmetric random walks on this particular group.

Theorem 5. *Let G be a finitely generated subgroup of the tame group and let μ be a symmetric atomic measure on G satisfying the condition*

$$\int_G \log(\deg(g))\, d\mu(g) < +\infty.$$

Then the degree exponents $\lambda_1(\mu) = \lambda_2(\mu)$ are positive if and only if G contains two automorphisms with dynamical degree strictly larger than 1 *generating a free group of rank* 2.

Moreover, we also obtain the following classification.

Corollary 6. *When $\lambda_1(\mu) = \lambda_2(\mu) = 0$ then G satisfies one of the following properties*:

(i) *The group G is conjugated to a subgroup of the linear group O_4.*

(ii) *There exists a G-equivariant morphism $\varphi : Q \to \mathbb{A}^2 \setminus \{(0,0)\}$ where G acts on $\mathbb{A}^2 \setminus \{(0,0)\}$ linearly.*

(iii) *The group G contains an automorphism h with $\lambda_1(h) > 1$ and there exists an integer M such that any automorphism $f \in G$ can be decomposed into $g \circ h^p$ where p is an integer and g has a degree bounded by M.*

(iv) *There exists a G-equivariant morphism $\varphi : Q \to \mathbb{A}^1$ where G acts on $\mathbb{A}^1$ by multiplication and any automorphism of G has dynamical degree* 1.

In other words, the degree exponents detect whenever the random walk has a chaotic behavior.

These last two results essentially follow from a classification of the finitely generated subgroups of the tame group and a theorem due to Maher and Tiozzo [2018, Theorem 1.2] which asserts that a random walk on a subgroup G of isometries of a CAT(0) space will drift to the boundary whenever G contains two noncommuting hyperbolic elements. When this happens, we obtain using Theorem 4 that the degree exponent is bounded below by a multiplicative factor of the drift and is thus positive. Otherwise, we prove that G preserves a vertex in the complex or a geodesic line. We then determine the degree sequences explicitly and conclude.

If we pursue the analogy with the random walk on groups, it is natural to ask whether one can obtain a central limit theorem analog to the one for random products of matrices [Benoist and Quint 2016, Theorem 0.7] or for random products of mapping classes [Dahmani and Horbez 2018]. We state it as follows.

Conjecture 7. *Take μ a symmetric atomic measure on the tame group. Then the limit*

$$\sigma^2 := \lim_{n\to+\infty} \frac{1}{n} \int_G (\log \deg(g) - \lambda(\mu)n)^2 \, d\nu_n(g)$$

exists where $\nu_n = \mu^{*n}$ denotes the n-fold convolution of μ and the sequence of random variables

$$\frac{\log \deg(g_n) - \lambda(\mu)n}{\sqrt{n}}$$

converges to the normal distribution law $\mathcal{N}(0, \sigma^2)$.

Structure of the paper. In Section 1, we recall some general facts on the tame group and then review in Section 2 the construction of the associated square complex. In Section 3, we focus on the global properties of the complex and exploit them to describe the degree sequences of particular automorphisms whose action fix a vertex on the complex. We then state in Section 4 the main valuative estimates needed for our proofs of Theorems 1 and 4 which are presented in Section 5. Finally, we apply the previous result to deduce Theorem 5 and Corollary 6 in the last section.

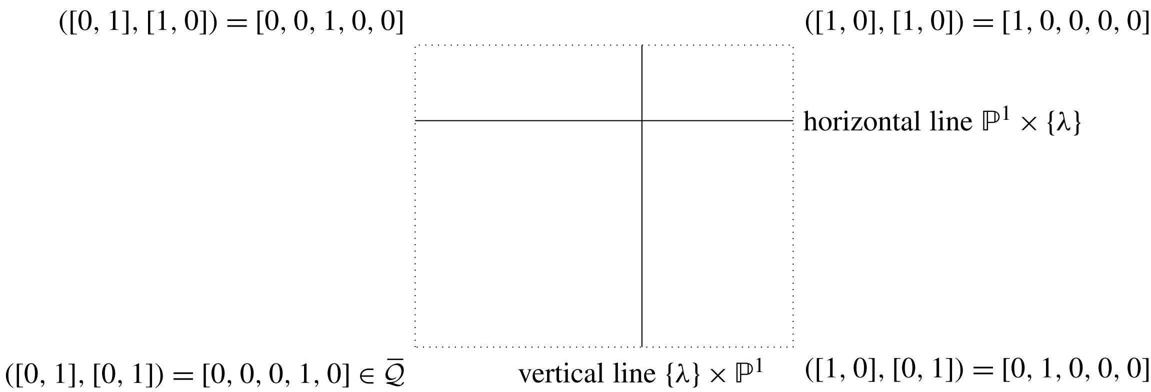


Figure 1. Vertical and horizontal lines in H_∞.

1. General facts on the tame group of the quadric

We work over an algebraically closed field k of characteristic zero. Take some affine coordinates $(x, y, z, t) \in \mathbb{A}^4$ and consider the smooth affine quadric threefold $\mathcal{Q}$ given by

$$\mathcal{Q} := V(xt - yz - 1) \subset \mathbb{A}^4.$$

Let us also fix an open embedding $\mathbb{A}^4 \subset \mathbb{P}^4$ so that $\mathbb{A}^4 = \mathbb{P}^4 \setminus V(w)$ in the homogeneous coordinates $[x, y, z, t, w] \in \mathbb{P}^4$.

In this section, we briefly describe the geometry of the affine quadric and give some preliminary properties of its elementary and orthogonal group of automorphism.

1A. ***The geometry of a quadric threefold and its compactification in*** $\mathbb{P}^4$**.** The affine variety $\mathcal{Q} \subset \mathbb{A}^4$ is a smooth quadric threefold. The Zariski closure $\overline{\mathcal{Q}}$ of the affine quadric is also smooth in $\mathbb{P}^4$ and has Picard rank one by Lefschetz hyperplane theorem. A birational map from $\overline{\mathcal{Q}}$ to $\mathbb{P}^3$ is given by choosing a point $p_0 \in \overline{\mathcal{Q}}$ and sending a point $p \in \overline{\mathcal{Q}}$ to the intersection of the line (pp_0) with a hyperplane in $\mathbb{P}^4$ which does not contain p_0.

We denote by $H_\infty := \overline{\mathcal{Q}} \setminus \mathcal{Q}$ the hyperplane section at infinity. It is a smooth quadric surface given in homogeneous coordinates by

$$H_\infty := V(xt - yz, w) \subset \mathbb{P}^4.$$

We identify H_∞ with $\mathbb{P}^1 \times \mathbb{P}^1$ by the isomorphism induced by the composition of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ with the inclusion $\mathbb{P}^3 = V(w) \hookrightarrow \mathbb{P}^4$. In homogeneous coordinates, it is given by

$$([\xi_0, \xi_1], [\eta_0, \eta_1]) \mapsto [\xi_0\eta_0, \xi_0\eta_1, \xi_1\eta_0, \xi_1\eta_1, 0]. \tag{3}$$

Any line in H_∞ of the form $\{\lambda\} \times \mathbb{P}^1$ (resp. $\mathbb{P}^1 \times \{\lambda\}$) where $\lambda \in \mathbb{P}^1$ is said to be vertical (resp. horizontal); see Figure 1.

The two projection maps $\pi_x : \mathcal{Q} \to \mathbb{A}^1$ and $\pi_y : \mathcal{Q} \to \mathbb{A}^1$ given by

$$\pi_x : (x, y, z, t) \in \mathcal{Q} \mapsto x,$$
$$\pi_y : (x, y, z, t) \in \mathcal{Q} \mapsto y,$$

induce algebraic fibrations which are trivial over $\mathbb{A}^1 \setminus \{0\}$ such that $\pi_x^{-1}(\mathbb{A}^1 \setminus \{0\})$ and $\pi_y^{-1}(\mathbb{A}^1 \setminus \{0\})$ are isomorphic to $\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2$. Observe that the fibers over 0 are both isomorphic to $\mathbb{A}^1 \times \mathbb{A}^1 \setminus \{0\}$ so that the fibrations are not locally trivial over a neighborhood of the origin. Observe that the intersection with H_∞ of the closure of the fiber over 0 in $\overline{\mathcal{Q}}$ is the union of a vertical line and a horizontal line. The projection on the two components

$$\pi_{x,y} : (x, y, z, t) \to (x, y)$$

induces a surjective morphism $\pi_{x,y} : \mathcal{Q} \to \mathbb{A}^2 \setminus (0, 0)$ which is also trivial over $\mathbb{A}^2 \setminus \{x = 0\}$.

The affine quadric Q carries naturally a volume form Ω which is the Poincaré residue of the rational 4-form $dx \wedge dy \wedge dz \wedge dt/f$ along $\mathcal{Q}$. More explicitly, Ω is defined by

$$\Omega = \frac{dx \wedge dy \wedge dz}{x}\Big|_{\mathcal{Q}} = \frac{dy \wedge dz \wedge dt}{t}\Big|_{\mathcal{Q}} = \frac{dx \wedge dz \wedge dt}{z}\Big|_{\mathcal{Q}}.$$

One checks that Ω extends as a rational 3-form $\overline{\Omega}$ on $\overline{\mathcal{Q}}$ such that its divisors of poles and zeros satisfies

$$\mathrm{div}(\overline{\Omega}) = -3[H_\infty].$$

1B. ***The orthogonal group.*** A regular automorphism f of $\mathcal{Q}$ is determined by a morphism $f^\sharp$ of the k-algebra $k[Q]$ and hence by its image on the four regular functions x, y, z, t. If we denote by $f_x, f_y, f_z, f_t \in \mathrm{k}[\mathcal{Q}]$ the image of x, y, z, t by $f^\sharp$, it is convenient to adopt a matrix-like notation for f as follows:

$$f = \begin{pmatrix} f_x & f_y \\ f_z & f_t \end{pmatrix}.$$

Observe that $f_x f_t - f_z f_y = 1$ since $f^\sharp$ is a morphism of the k-algebra $k[Q]$ and that any such automorphism preserves the volume form Ω (up to a constant).

Denote by $q(x, y, z, t) = xt - yz$ the quadratic form defined on the vector space $V = k^4$. The group O_4 is the subgroup of linear automorphisms of k^4 which leave the quadratic form q invariant

$$O_4 = \{f \in \mathrm{GL}_4(k) \mid q \circ f = q\}.$$

An element of O_4 naturally defines an automorphism of the quadric hypersurface Q. As a consequence, we have that for any $f \in O_4$,

$$f^*\Omega = \epsilon(f)\Omega,$$

where $\epsilon : O_4 \to k^*$ is a morphism of groups. Since Ω is the Poincaré residue of the form $dx \wedge dy \wedge dz \wedge dt/(xt - yz - 1)$ to $\mathcal{Q}$, this implies that for any $f \in O_4$, $\epsilon(f)$ is equal to the determinant of the endomorphism of k^4 associated to f, hence $\epsilon(f) \in \{+1, -1\}$. The subgroup SO_4 is the kernel of ϵ and has index 2 in O_4.

Observe that every element of O_4 extends as regular automorphism of $\overline{Q}$ which leaves the hyperplane at infinity invariant. In particular, the restriction map onto H_∞ induces a morphism of groups from O_4 onto $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$.

The main properties of O_4 and SO_4 are summarized in the following proposition.

Proposition 1.1. *The following properties are satisfied*:

(i) *The group* SO_4 *acts transitively on the set of horizontal and vertical lines at infinity respectively*, *and on the set of points at infinity.*

(ii) *Any element of* $f \in O_4$ *which does not belong to* SO_4 *exchanges the horizontal lines at infinity with the vertical lines at infinity.*

(iii) *The following sequence is exact*:

$$1 \to \{+1, -1\} \to O_4 \to \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) \to 1.$$

(iv) *For any element* $f \in O_4$, *we have*

$$f^*\Omega = \epsilon(f)\Omega,$$

where $\epsilon(f) \in \{+1, -1\}$ *and* $\mathrm{Ker}(\epsilon) = \mathrm{SO}_4$.

Proof. Observe that (iii) follows directly from the following exact sequence:

$$1 \to \{+1, -1\} \to O_4 \to \mathrm{PSO}_4 \to 1,$$

and the fact that $\mathrm{PSO}_4 \simeq \mathrm{PGL}_2 \times \mathrm{PGL}_2$ which is given in [Fulton and Harris 1991, Section 23.1].

In particular, (iii) directly implies (i). □

1C. *Elementary transformations.* The group E_V (resp. E_H) of vertical (resp. horizontal) elementary transformations is defined by

$$E_V := \left\{ \begin{pmatrix} ax & by \\ b^{-1}(z + xP(x, y)) & a^{-1}(t + yP(x, y)) \end{pmatrix} \;\middle|\; P \in k[x, y], a, b \in k^* \right\},$$

$$E_H := \left\{ \begin{pmatrix} ax & b(y + xP(x, z)) \\ b^{-1}z & a^{-1}(t + zP(x, z)) \end{pmatrix} \;\middle|\; P \in k[x, y], a, b \in k^* \right\}.$$

The terminology comes from the fact that these transformations are restrictions to the quadric of transformations of $\mathbb{A}^4$ of the form

$$(x, y, z, t) \to (x, y + P(x), z + R(x, y), t + S(x, y, z))$$

where $P \in k[x]$, $R \in k[x, y]$, $S \in k[x, y, z]$, which are elementary in the sense of [Shestakov and Umirbaev 2003].

Any automorphism in E_V fix the two fibrations $\pi_x : (x, y, z, t) \to x$ and $\pi_y : (x, y, z, t) \to y$ and this geometric property characterizes the group E_V; see [Dang 2018, Proposition 3.2.3.1]. An explicit computation proves that any elementary automorphism f preserves the volume form Ω:

$$f^*\Omega = \Omega.$$

We will not focus on the action of these elementary transformations on the compactification $\overline{\mathcal{Q}}$. For more details on the study of the birational transformations induced by these transformations, we refer to [Dang 2018, Chapter 3, Section 3.2.3].

2. The square complex associated to the tame group

The tame group, denoted Tame(Q), is the subgroup of $\mathrm{Aut}(\mathcal{Q})$ generated by E_V and O_4. It is naturally included in $\mathrm{Bir}(\mathbb{P}^3)$ since the variety $\overline{\mathcal{Q}}$ is rational.

Observe that any tame automorphism f fixes the volume form Ω up to a sign, i.e., there exists a group morphism $\epsilon : \mathrm{Tame(Q)} \to \{+1, -1\}$ such that

$$f^*\Omega = \epsilon(f)\Omega.$$

This allows us to identify the kernel $\mathrm{STame}(Q)$ of ϵ as the group generated by SO_4 and E_V. It has index 2 in Tame(Q).

The tame group Tame(Q) is a strict subgroup of $\mathrm{Aut}(\mathcal{Q})$ [Lamy and Vénéreau 2013] and satisfies the Tits alternative; see [Bisi et al. 2014, Theorem C]. The proof of this last fact is due to Bisi, Furter and Lamy and relies on the construction of a square complex on which the group acts by isometry.

The plan of this section is as follows. In Section 2A we detail the construction of the square complex due to Bisi, Furter and Lamy. Then, following the presentation in [Bisi et al. 2014] we shall review in Sections 2B, 2C and 2D the properties of the stabilizer of each vertex of this complex. We will focus particularly on the stabilizer of the vertices which we call of type I in Sections 2C and 2D, for which the analysis is more involving. Finally, we state in Section 2E five technical lemmas on how four squares glue together near each vertices. As before, the situation is also more delicate near the vertices of type I and we need to introduce more terminology to describe the local geometry at those vertices. For a more detailed explanation of the results in this section, we refer to [Bisi et al. 2014, Sections 2, and 3.1] and to [Dang 2018, Chapter 3, Section 3.3].

2A. ***Construction of the square complex.*** The square complex, denoted $\mathcal{C}$, is a 2-dimensional polyhedral complex where the cells of dimension 2 are squares and where the cells of dimension 0 and 1 have some special markings.

We say that a regular function $f_1 \in \mathrm{k}[\mathcal{Q}]$ is a component of an automorphism if there exists $f_2, f_3, f_4 \in \mathrm{k}[\mathcal{Q}]$ such that $f = (f_1, f_2, f_3, f_4)$ defines an automorphism of the quadric. One similarly defines the notion of components for a pair (f_1, f_2) or for a triple (f_1, f_2, f_3) of regular functions on $\mathcal{Q}$ when they can be completed to a 4-tuple defining an automorphism of the affine quadric.

We distinguish three types of vertices for the complex $\mathcal{C}$:

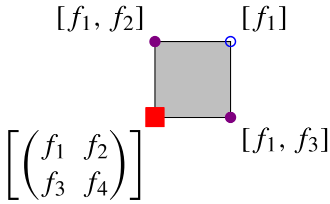


Figure 2. A cell of dimension 2.

- Type I vertices are equivalence classes of components $f_1 \in \mathrm{k}[\mathcal{Q}]$ of a tame automorphism, where two components f_1 and f_2 are identified if there exists an element $a \in k^*$ such that $f_1 = af_2$. A vertex induced by a component $f_1 \in \mathrm{k}[\mathcal{Q}]$ is denoted by $[f_1]$.
- Type II vertices are equivalence class of components (f_1, f_2) of an automorphism where $f_1 = x \circ f$, $f_2 = y \circ f \in \mathrm{k}[\mathcal{Q}]$ for $f \in \mathrm{Tame(Q)}$ and where one identifies two components (f_1, f_2) with (g_1, g_2) if $(g_1, g_2) = (af_1 + bf_2, cf_1 + df_2)$ for some matrix
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2 .$$
A vertex induced by a component (f_1, f_2) is denoted by $[f_1, f_2]$. Denote by $f_3 = z \circ f$ and $f_4 = t \circ f$, the vertices $[f_1, f_2]$, $[f_1, f_3]$, $[f_2, f_4]$, $[f_3, f_4]$ are well-defined since the automorphisms (f_1, f_3, f_2, f_4), $(-f_2, -f_4, f_1, f_3)$ and $(-f_3, f_4, -f_1, f_2)$ are also tame. Moreover, given a component (f_1, f_2) and an invertible matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2$, there exists an automorphism g such that $x \circ g = af_1 + bf_2$ and $y \circ g = cf_1 + df_2$. Let us insist on the fact that on the contrary, there are no vertices of the form $[f_1, f_4]$ or $[f_2, f_3]$.
- Type III vertices are equivalence classes of automorphisms $f \in \mathrm{Tame(Q)}$ where two tame automorphisms f and g are equivalent if there exists $h \in O_4$ such that $f = h \circ g$. An equivalence class of $f \in \mathrm{Tame(Q)}$ is denoted by $[f]$.

The edges of the complex $\mathcal{C}$ are of two types:

- Type I edges join a vertex of type I of the form $[f_1]$ with a vertex of type II of the form $[f_1, f_2]$ where (f_1, f_2) are the components of a tame automorphism.
- Type III edges join a vertex of type II of the form $[f_1, f_2]$ with a vertex of type III $[f]$ where (f_1, f_2) are the components of the automorphism given by f.

The cells of dimension 2 are squares containing two type II vertices of the form $[f_1, f_2]$, $[f_1, f_3]$, one vertex of type I given by $[f_1]$ and one vertex of type III given by $[f]$ where (f_1, f_2, f_3) are the components of the automorphism $f \in \mathrm{Tame(Q)}$. We have the figure of a square given in Figure 2. As in [Bisi et al. 2014], we adopt the following convention for the pictures: the vertices of type I, II and III are represented by the symbol ○, ● and ■ respectively.

The square complex $\mathcal{C}$ is obtained by the quotient of the disjoint union of all cells by the equivalence relation $\sim$ where any two cells C_1, C_2 are identified along $C_1 \cap C_2$.

Each square of the complex is endowed with the euclidean metric d so that each square is isometric to $[0, 1] \times [0, 1]$. For any points p and q in $\mathcal{C}$, define by

$$d_{\mathcal{C}}(p, q) = \inf\left\{\sum_{i=0}^{N} d(p_i, p_{i+1})\right\},$$

where the infimum is taken over all sequence of points $p_0 = p, \dots, p_N = q$ where p_i and p_{i+1} lie on the same square in $\mathcal{C}$. As any cell of the complex $\mathcal{C}$ has only finitely many isometries, we may apply a general result from [Bridson and Haefliger 1999, Section I.7] and conclude that the function $d_{\mathcal{C}}$ induces a metric on the complex and turns $(\mathcal{C}, d_{\mathcal{C}})$ into a complete metric space. We will explain in Section 3 the global properties on the complex induced by this metric.

Let us define the action of the tame group Tame(Q) on the complex $\mathcal{C}$. Pick any two automorphisms $f, g \in \text{Tame(Q)}$. We define the action of g on the each vertices of the complex by setting

$$\begin{aligned} g \cdot [f_1] &:= [f_1 \circ g^{-1}], \\ g \cdot [f_1, f_2] &:= [f_1 \circ g^{-1}, f_2 \circ g^{-1}], \\ g \cdot [f] &:= [f \circ g^{-1}]. \end{aligned}$$

The action on vertices induces a morphism of the square complex which preserves the type of vertices and edges and preserves the distance.

Remark 2.1. Although, this is not clear at this stage, the action of the tame group on this complex will act transitively on the set of squares and the precise study of the stabilizer of type III vertices will result from the geometry of the complex near a vertex of type III done in Proposition 2.3. As a result of the study, a square is determined uniquely once it contains a vertex of type III and a vertex of type I, or by three vertices.

Recall that the subgroup $\text{STame}(Q)$ generated by SO_4 and elementary transformations has index 2 in Tame(Q).

Definition 2.2. An edge E of the complex is called *horizontal* (resp. *vertical*) if there exists an element $f \in \text{STame}(Q)$ such that $f \cdot E$ is equal to the edge joining $[x, y]$ with $[x]$ (resp. $[x, z]$ with $[x]$) or to the edge between $[\text{Id}]$ and $[x, z]$ (resp. $[\text{Id}]$ and $[x, y]$).

We will see that the set of vertical and horizontal edges form a partition of the set of edges (see (iii) and (iv) of Proposition 2.7).

2B. ***Stabilizer of vertices of type III, II and the properties of the action.*** In this section, we shall first review the properties of the stabilizer of type II and III vertices then deduce from these the global properties of the action of the group on this complex. To do so, we shall exploit the relationship between the local geometry near each vertices and their respective stabilizer subgroups. The geometry near a given vertex v

is encoded in its link $\mathcal{L}(v)$ which is constructed as follows. The vertices of $\mathcal{L}(v)$ are in bijection with the vertices v' such that $[v, v']$ is an edge of the complex $\mathcal{C}$. And we draw an edge joining v' and v'' in $\mathcal{L}(v)$ if the vertices v, v', v'' belong to the same square.

Observe that the action of the tame group on the vertices of type III is transitive. As a result, we shall focus on the stabilizer subgroup of the vertex [Id], which is by construction O_4. Its action on the complex induces an action on the link $\mathcal{L}([\mathrm{Id}])$.

Proposition 2.3. *The link $\mathcal{L}([\mathrm{Id}])$ is a complete bipartite graph and there exists an O_4-equivariant bijection between the set of vertices of the link $\mathcal{L}([\mathrm{Id}])$ to the set of lines at infinity such that the vertices which belong to a vertical (resp. horizontal) edge of type* III *are mapped to vertical (resp. horizontal) lines at infinity in H_∞. Moreover, this bijection induces an O_4-equivariant bijection from the edges of $\mathcal{L}([\mathrm{Id}])$ to the set of points at infinity H_∞.*

Remark 2.4. Observe that Proposition 1.1 and Proposition 2.3 imply that the group O_4 acts faithfully and transitively on the link $\mathcal{L}([\mathrm{Id}])$.

Proof. We identify two types of vertices in the link of [Id], the vertices which belong to a horizontal edge containing [Id] or those which are contained in a vertical edge containing [Id].

We define a map φ from the vertices of the link $\mathcal{L}([\mathrm{Id}])$ to the set of lines in H_∞. Take a vertex v in the link $\mathcal{L}([\mathrm{Id}])$ and a component (f_1, f_2) such that $[f_1, f_2] = v$. By definition, there exists an element $f \in O_4$ such that $f_1 = x \circ f$ and $f_2 = y \circ f$ since the stabilizer of [Id] is O_4. The zero locus $V(f_1) \cap V(f_2) \cap H_\infty$ in $\overline{Q}$ is the line at infinity corresponding to the preimage of $\{x = y = 0\} \cap \overline{Q}$ by f. Observe that the line $V(f_1) \cap V(f_2) \cap H_\infty$ does not depend on the choice of representative of the equivalence class v since any two other component in the same class defines the same homogeneous ideal $\langle f_1, f_2, xt - yz - w^2\rangle$. We thus define $\varphi(v)$ to be the line $V(f_1) \cap V(f_2) \cap H_\infty$. Observe that if v is a vertex of type II such that the edge containing v and [Id] is vertical, then $f \in \mathrm{SO}_4$. Hence the line at infinity $V(x \circ f) \cap V(y \circ f) \cap H_\infty$ is vertical. Observe also that φ is naturally O_4-equivariant. The same argument holds for the vertices of type II which belong to horizontal edges containing [Id].

Let us prove that the map φ is surjective. Consider a vertical line $L \subset H_\infty$ at infinity, then there exists by Proposition 1.1(i) an automorphism f in SO_4 such that the image of the vertical line at infinity given by $[0, 1] \times \mathbb{P}^1$ is L. Since $\varphi([x, y])$ corresponds to the line $[0, 1] \times \mathbb{P}^1$, the vertex of type II $[x \circ f, y \circ f]$ defines a component of an automorphism which belongs to the link $\mathcal{L}([\mathrm{Id}])$ such that $\varphi([x \circ f, y \circ f]) = L$. Hence, φ is surjective.

Let us prove that φ is injective. Consider two vertices v_1, v_2 such that their image by φ is equal, we prove that $v_1 = v_2$. Consider two components (f_1, f_2), (g_1, g_2) such that $[f_1, f_2] = v_1$ and $[g_1, g_2] = v_2$. We must prove that (f_1, f_2) and (g_1, g_2) belong to the same equivalence class. By symmetry, we can suppose that the line $\varphi(v_1)$ is vertical. Hence, there exists $f, g \in \mathrm{SO}_4$ such that $f_1 = x \circ f$, $g_1 = x \circ g$, $f_2 = y \circ f$ and $g_2 = y \circ g$. In particular, this implies that $f \circ g^{-1}$ fixes the vertical line at infinity given by $\{[0, 1]\} \times \mathbb{P}^1$.

Using Proposition 1.1(iii), we conclude that $f \circ g^{-1}$ is of the form

$$f \circ g^{-1} = \begin{pmatrix} ax+by & cx+dy \\ a'z+b't & c'z+d't \end{pmatrix},$$

where the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} d' & -b' \\ -c' & a' \end{smallmatrix}\right) \in M_2(k)$ satisfy

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} d' & -b' \\ -c' & a' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In particular, this implies that the components (f_1, f_2) and (g_1, g_2) are equivalent since $f_1 = ag_1 + bg_2$, $f_2 = cg_1 + dg_2$.

One similarly defines a bijection from the edges of the link $\mathcal{L}([\mathrm{Id}])$ to H_∞. The link is complete since a horizontal and a vertical line in H_∞ always intersect at a point in H_∞, hence for any vertices v_1, v_2 in $\mathcal{L}([\mathrm{Id}])$ which are mapped by φ to a vertical and a horizontal line respectively, there exists an edge joining v_1 and v_2. □

Proposition 2.5. *The following properties are satisfied*:

(1) *The stabilizer of a vertex of type* III *in* $\mathrm{STame}(Q)$ *is conjugated in* $\mathrm{STame}(Q)$ *to* SO_4.

(2) *The stabilizer of an edge of type* III *is conjugated in* Tame(Q) *to the subgroup*

$$A \cdot \begin{pmatrix} x & y \\ z & t \end{pmatrix} \cdot B^t,$$

where $A \in \mathrm{SL}_2(k)$ *is a lower triangular matrix and* $B \in \mathrm{SL}_2(k)$.

(3) *The stabilizer of a* 1×1 *square is conjugated in* Tame(Q) *to*

$$\left\{ \begin{pmatrix} ax & b(y+cx) \\ b^{-1}(z+dx) & a^{-1}(t+cz+dy+dcx) \end{pmatrix} \;\middle|\; (a, b, c, d) \in k^* \times k^* \times k \times k \right\} \rtimes \left\{ \begin{pmatrix} x & z \\ y & t \end{pmatrix}, \mathrm{Id} \right\}$$

(4) *The pointwise stabilizer of the union of the four squares containing* [Id] *and* $[x]$, $[y]$, $[z]$ *and* $[t]$ *respectively is equal to*

$$\left\{ \begin{pmatrix} ax & by \\ b^{-1}z & a^{-1}t \end{pmatrix} \;\middle|\; a, b \in k^* \right\}.$$

Proof. Observe that (i) follows directly from the definition of the definition. Moreover, the next assertions (ii), (iii) and (iv) are exactly the content of [Bisi et al. 2014, Lemmas 2.5(2), 2.7 and 2.11]. □

We focus on the stabilizer subgroups of vertices of type II. For that, we also define some special subgroups of E_V, E_H where the constant are all 1. Set $\tilde{E}_H$ the subgroup of E_H of elements of the form

$$\begin{pmatrix} x & y + xP(x, z) \\ z & t + zP(x, z) \end{pmatrix},$$

with $P \in k[x, y]$ and respectively elements in $\tilde{E}_V$ are of the form

$$\begin{pmatrix} x & y \\ z + xP(x, y) & t + yP(x, y) \end{pmatrix}.$$

Proposition 2.6. *The following properties are satisfied*:

(i) *The stabilizer of a vertex of type* II *in* Tame(Q) *is conjugated in* Tame(Q) *to the semidirect product* $\tilde{E}_V \rtimes \mathrm{GL}_2$ *where the group* GL_2 *is identified with the elements of the form*

$$\begin{pmatrix} ax+by & cx+dy \\ a'z+b't & c'z+d't \end{pmatrix}$$

where $a, b, c, d, a', b', c', d' \in k$ *such that* $\det\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \neq 0$ *and*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{a'd'-b'c'} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

(ii) *The stabilizer of a vertical edge of type* I *is conjugated in* $\mathrm{STame}(Q)$ *to the subgroup*

$$\tilde{E}_H \rtimes \left\{ \begin{pmatrix} ax & d^{-1}y \\ dz+cx & a^{-1}t+ca^{-1}d^{-1}y \end{pmatrix} \;\middle|\; (a, c, d) \in k^* \times k \times k^* \right\}.$$

(iii) *The pointwise stabilizer of the geodesic segment of length* 2 *joining the vertices* $[f_1]$, $[f_3]$ *and* $[f_1, f_3]$ *where* $f = (f_1, f_2, f_3, f_4) \in \mathrm{STame}(Q)$ *is conjugated in* $\mathrm{STame}(Q)$ *to*

$$\tilde{E}_H \rtimes \left\{ \begin{pmatrix} ax & by \\ b^{-1}z & a^{-1}t \end{pmatrix}, a, b \in k^* \right\}.$$

Proof. Assertion (i), (ii) and (iii) are given in [Bisi et al. 2014, Lemmas 2.3, 2.5(1) and 2.6(1)] respectively. □

From the description of the previous stabilizer subgroups, we state the following consequences on the action of the group on this complex.

Proposition 2.7. *The tame group* Tame(Q) *acts by isometry on the complex* $\mathcal{C}$ *and this action satisfies the following properties*:

(i) *The action preserves the types of vertices and the types of edges.*

(ii) *The action is faithful and transitive on the set of vertices of type* I , II *and* III *respectively.*

(iii) *The subgroup* $\mathrm{STame}(Q)$ *acts transitively on the set of vertical* (*resp. horizontal*) *edges of type* I *and* III.

(iv) *Any automorphism* $f \in$ Tame(Q) *which does not belong to the subgroup* $\mathrm{STame}(Q)$ *sends a vertical edge to a horizontal edge of the same type.*

(v) *The subgroup* $\mathrm{STame}(Q)$ *acts transitively on the set of* 1×1 *squares.*

(vi) *The group* Tame(Q) *acts transitively on the union of* 4 *squares which is isometric to* $[0, 2] \times [0, 2]$ *and which contains a common vertex of type* III.

Proof. The transitive of the action on the set of vertices of type I, II and III and assertions (i) and (iv) follow from [Bisi et al. 2014, Lemmas 2.1 and 2.4].

To prove (ii), we need to explain why the action is also faithful. Observe that if a tame automorphism fixes every vertices of type III, or type II or type I, then it fixes the whole complex since every vertex of type III (resp. type II or I) is the middle point of a geodesic segment joining type I or type II points. Then the faithfulness follows from the faithfulness of the action on the link $\mathcal{L}([\mathrm{Id}])$.

The assertions (iii), (v) and (vi) are exactly the content of [Bisi et al. 2014, Lemmas 2.4 and 2.7, Corollary 2.10] respectively. □

2C. ***Bass–Serre tree associated to plane automorphisms.*** We consider the field $K = k(x)$. We define the graph $\mathcal{T}_{k(x)}$ which is a bipartite metric graph:

(1) Vertices of type I are equivalence classes of components $f_1 \in k(x)[y, z]$ of plane automorphisms where one identifies two components f_1 and g_1 if there exists $a \in k(x)^*$ and $b \in k(x)$ such that $f_1 = ag_1 + b$. An equivalence class induced by a component f_1 is denoted $[f_1]$.

(2) Vertices of type II are equivalence classes of automorphisms f of $\mathbb{A}^2_{k(x)}$ where one identifies two automorphisms $f = (f_1, f_2)$ and $g = (g_1, g_2)$ if there exists an affine automorphism $h \in \mathbb{A}^2_{k(x)}$ whose coefficients are given by the matrix

$$\begin{pmatrix} a & b & c \\ a' & b' & c' \\ 0 & 0 & 1 \end{pmatrix}$$

such that $(f_1, f_2) = (ag_1 + bg_2 + c, a'g_1 + b'g_2 + c)$. An equivalence class induced by a plane automorphism $f = (f_1, f_2)$ is denoted $[f_1, f_2]$.

(3) Edges link a vertex v_1 of type I with a vertex v_2 of type II if there exists a polynomial automorphism $f = (f_1, f_2)$ such that $[f_1] = v_1$ and $[f_1, f_2] = v_2$.

We endow this graph $\mathcal{T}_{k(x)}$ with the distance such that each edge is of length 1. This graph $\mathcal{T}_{k(x)}$ is thus a complete geodesic metric space.

The action of an automorphism $g \in \mathbb{A}^2_{k(x)}$ on $\mathcal{T}_{k(x)}$ is defined as follows

$$g \cdot [f_1] = [f_1 \circ g^{-1}] \quad \text{and} \quad g \cdot [f_1, f_2] = [f_1 \circ g^{-1}, f_2 \circ g^{-1}]$$

for any automorphism $f = (f_1, f_2) \in \mathrm{Aut}(\mathbb{A}^2_{k(x)})$.

A classical theorem from Jung [1942] proves that the graph $\mathcal{T}_{k(x)}$ is a tree and that the group of plane automorphism acts faithfully, by isometry and transitively on the set of type I and II vertices respectively.

2D. ***Link over a vertex of type I.*** In this subsection, we study the link over the vertex of type I given by $[x]$. Observe that the stabilizer subgroup of the vertex $[x]$ acts naturally in the link of the vertex $[x]$.

Lemma 2.8. *The group* $\mathrm{Stab}([x])$ *acts transitively, faithfully on the set of vertices in the link of* $[x]$ *induced by the edges joining* $[x]$ *and the vertices of type* II *of* $\mathcal{C}$.

Proof. By Proposition 2.7(v), the group $\mathrm{STame}(Q)$ acts transitively on the set of 1×1 squares and since a 1×1 square containing $[x]$ defines an edge in the link $\mathcal{L}([x])$, the induced action of $\mathrm{Stab}([x])$ is transitive on the edges of the link $\mathcal{L}([x])$. Observe also that the involution $\sigma : (x, y, z, t) \mapsto (x, z, y, t)$ induces an action on the link which exchanges the vertices $[x, y]$, $[x, z]$ in the link and fixes the edge between these two vertices. This proves that the action of the stabilizer $\mathrm{Stab}([x])$ is transitive on the link of $[x]$.

Let us prove that the action is faithful. Suppose $f \in \mathrm{Stab}([x])$ acts by the identity map in the link over $[x]$, then in particular, f must fix pointwise the square containing $[\mathrm{Id}]$ and $[x]$. By Proposition 2.5(iii), f is of the form

$$f = \begin{pmatrix} ax & d^{-1}(y+bx) \\ d(z+cx) & a^{-1}(t+cy+bz+bcx) \end{pmatrix},$$

where $a, d \in k^*$ and $b, c \in k$. Since f must also fix the vertices of type II $[x, y+xP(x)]$ and $[x, z+xP(x)]$ where $P \in k[x]$, we have that $a = d = 1$ and $c = b = 0$ as required. $\square$

In the following arguments, we will use the fact that the link $\mathcal{L}([x])$ is connected [Bisi et al. 2014, Lemma 3.2], which is a highly nontrivial argument which relies deeply on the reduction theory inspired by the work of Shestakov and Umirbaev; see [Bisi et al. 2014, Corollary 1.5].

Recall that the general fiber of the projection $\pi_x : \mathcal{Q} \to \mathbb{A}^1$ defined in Section 1C is isomorphic to $\mathbb{A}^2$. We fix an identification of $\pi_x^{-1}(\mathbb{A}^1 \setminus \{0\})$ with $\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2$ given by

$$(x, y, z) \mapsto (x, y, z, (yz+1)/x). \tag{4}$$

The relationship between the stabilizer of the vertex $[x]$ and $\mathrm{Aut}(\mathbb{A}^2_{k(x)})$ is realized explicitly as follows.

Denote by $\mathcal{L}([x])'$ the first barycentric division of $\mathcal{L}([x])$. We shall define a simplicial map $\pi : \mathcal{L}([x])' \to \mathcal{T}_{k(x)}$ as follows.

Let v be a vertex of type II in $\mathcal{C}$ which defines a vertex in the link of $[x]$, then since the action of $\mathrm{Stab}([x])$ on the link $\mathcal{L}([x])$ is transitive by Lemma 2.8, there exists an element $f \in \mathrm{Stab}([x])$ such that $f \cdot [x, z] = v$. Since f naturally fixes the fibration π_x, under the identification $\pi_x^{-1}(\mathbb{A}^1 \setminus \{0\}) \simeq \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2$ given by (4), the regular map f is given by

$$(x, y, z) \mapsto (x \circ f, y \circ f, z \circ f).$$

Under this identification, $(y \circ f, z \circ f)$ induces an element of $\mathbb{A}^2_{k(x)}$. We thus define

$$\pi(v) = [z \circ f^{-1}] \in \mathcal{T}_{k(x)}.$$

Observe that $\pi(v)$ does not depend on the choice of f. Indeed, if $g \in \mathrm{Stab}([x])$ is another automorphism such that $g \cdot [x, z] = f \cdot [x, z]$, there are $a, b \in k^*$ such that $x \circ f^{-1} = ax$, $x \circ g^{-1} = bx$ and $[x, z \circ f^{-1}] = [x, z \circ g^{-1}]$. Then $z \circ g^{-1} = cz \circ f^{-1} + dx$ for some $c \in k^*$, $d \in k$. We obtain that $[z \circ g^{-1}] = [z \circ f^{-1}] \in \mathcal{T}_{k(x)}$.

Let $m \in \mathcal{L}([x])'$ be the middle point of an edge E of $\mathcal{L}([x])$ and let m_0 be the middle point of the geodesic joining $[x, y]$ and $[x, z]$ in $\mathcal{L}([x])'$. Since the action of $\mathrm{Stab}([x])$ in the link $\mathcal{L}([x])$ is transitive by Lemma 2.8, there exists an element $f \in \mathrm{Stab}([x])$ such that $f \circ m_0 = m$. Since f naturally fixes the fibration π_x, it induces an automorphism of $\pi_x^{-1}(\mathbb{A}^1 \setminus \{0\})$ and under the identification given by (4), it is of the form

$$(x, y, z) \mapsto (x \circ f, y \circ f, z \circ f).$$

We thus define

$$\pi(m) = [y \circ f^{-1}, z \circ f^{-1}].$$

Observe also that $\pi(m)$ does not depend on the choice of f. If $g \in \mathrm{Stab}([x])$ such that $g \cdot m_0 = m$, then $g^{-1} \circ f$ belongs to the subgroup

$$\left\{ \begin{pmatrix} ax & b(y+cx) \\ b^{-1}(z+dx) & a^{-1}(t+dy+cz+cdx) \end{pmatrix} \;\middle|\; a, b \in k^*, c, d \in k \right\} \rtimes \left\{ \mathrm{Id}, \begin{pmatrix} x & z \\ y & t \end{pmatrix} \right\},$$

hence $[y \circ g^{-1}, z \circ g^{-1}] = [y \circ f^{-1}, z \circ f^{-1}] \in \mathcal{T}_{k(x)}$ and $\pi(m)$ is well-defined.

If E is an edge of $\mathcal{L}([x])'$ of length 1, then we define the image of E by π as the geodesic joining the image of the endpoints of E by π. As a result,the map π is a simplicial map between $\mathcal{L}([x])'$ and $\mathcal{T}_{k(x)}$ such that the action of $\mathrm{Stab}([x])$ descends into an action on the image $\pi(\mathcal{L}([x])') \subset \mathcal{T}_{k(x)}$ (one can prove that $\pi : \mathcal{L}([x])' \to \pi(\mathcal{L}([x])')$ is the unique $\mathrm{Stab}([x])$-equivariant map for which $\pi([x, y]) = [y]$ and $\pi([x, z]) = [z]$). To simplify the next statement, we denote by $\mathcal{T}_{\pi,k(x)}$ the image in $\mathcal{T}_{k(x)}$ of $\mathcal{L}([x])'$ by π.

Definition 2.9. The subgroup $A^S_{[x]}$ of $\mathrm{Stab}([x])$ is the intersection of $\mathrm{STame}(Q)$ with the stabilizer of the vertices $[y]$, $[z]$, $[y, z]$ in $\mathcal{T}_{\pi,k(x)}$ where S is the standard 2×2 square containing $[x]$, $[y]$, $[z]$, $[t]$. More generally, if v is any vertex of type I contained in a 2×2 square S', the subgroup $A^{S'}_v$ is equal to $g A^S_{[x]} g^{-1}$ where $g \in \mathrm{STame}(Q)$ such that $g \cdot S = S'$ and $g \cdot [x] = v$.

Proposition 2.10. *Denote by $m \in \mathcal{L}([x])'$ the middle point between the point $[x, y]$ and $[x, z]$. The simplicial map $\pi : \mathcal{L}([x])' \to \mathcal{T}_{\pi,k(x)}$ satisfies the following properties*:

(i) *The image of the edge between the point $[x, y]$ and m by π is a fundamental domain of $\mathcal{T}_{\pi,k(x)}$.*

(ii) *The image $\pi(\mathcal{L}([x])') = \mathcal{T}_{\pi,k(x)}$ is a subtree of $\mathcal{T}_{k(x)}$.*

(iii) *The preimage by π of the segment of length 2 joining $[z]$ and $[y]$ is a bipartite graph.*

(iv) *The subgroup $A^S_{[x]} \subset \mathrm{Stab}([x]) \cap \mathrm{STame}(Q)$ is the set of elements of the form*

$$\begin{pmatrix} ax & b(y + xP(x)) \\ b^{-1}(z + xT(x)) & a^{-1}(t + zP(x) + yT(x) + xP(x)T(x)) \end{pmatrix},$$

where $P, T \in k[x]$ and $a, b \in k^*$.

(v) *The group $\mathrm{Stab}([x])$ is the amalgamated product $\tilde{E} * \tilde{A}$ along their intersection where $\tilde{E}$ is the group generated by elements of the form*

$$\begin{pmatrix} ax & b(y + xP(x, z)) \\ b^{-1}(z + xT(x)) & a^{-1}(t + zP(x, z) + yT(x)) \end{pmatrix}, \tag{5}$$

where $a, b \in k^*$, $P \in k[x, z]$, $T \in k[x]$, and $\tilde{A}$ is the group generated by $A^S_{[x]}$ and the involution $(x, y, z, t) \mapsto (x, z, y, t)$.

Remark 2.11. In fact, the action of Stab($[x]$) on $\mathcal{T}_{\pi,k(x)}$ can be extended to an action on the whole tree $\mathcal{T}_{k(x)}$. One can view this since the fundamental domain of the tree $\mathcal{T}_{k(x)}$ is the image of an edge of $\mathcal{L}([x])'$ by π. Another way is to identify the general fiber Q_η using (4) with $\mathbb{A}^2_{k(x)} = \mathrm{Spec}(k(x)[y,z])$. So for every $g, h \in k(x)[y,z]$ such that (g,h) defines an automorphism, we can define $f \cdot [g] = [g \circ f^{-1}]$ and $f \cdot [g,h] = [g \circ f^{-1}, h \circ f^{-1}]$ for any $f \in \mathrm{Stab}([x])$.

Proof. Assertion (i), (ii), (iii) and (v) are the content of [Bisi et al. 2014, Lemmas 3.4(1), 3.5(1) and (2), Proposition 4.11] respectively.

Let us prove statement (iv). Let us denote by $\phi : \mathrm{Stab}([x]) \to \mathrm{Aut}(\mathcal{T}_{\pi,k(x)})$ the morphism of groups induced by the simplicial map $\pi : \mathcal{L}([x])' \to \mathcal{T}_{\pi,k(x)}$ where $\mathrm{Aut}(\mathcal{T}_{\pi,k(x)})$ denotes the induced simplicial map on the tree. It is clear that any element of the form

$$\begin{pmatrix} ax & b(y + xP(x)) \\ b^{-1}(z + xT(x)) & a^{-1}(t + zP(x) + yT(x) + xP(x)T(x)) \end{pmatrix},$$

where $P, T \in k[x]$ and $a, b \in k^*$ induces an action which preserves the vertices $[y]$, $[z]$ and $[y,z]$ on $\mathcal{T}_{\pi,k(x)}$. Conversely, we prove that any element of $A^S_{[x]}$ has this form. Pick $g \in A^S_{[x]}$, since $\phi(g)$ fixes the vertices $[y]$, $[z]$ and $[y,z]$ of $\mathcal{T}_{\pi,k(x)}$, As $\phi(g)$ fixes every vertex of type I and since it belongs to the image of ϕ, the components in x, y, z of the automorphism g must be of the form

$$g = (x, y, z) \to (ax, b(y + xP(x)), c(z + xT(x))),$$

where $P, T \in k[x]$ and where $a, b, c \in k^*$. In particular, as $g \in \mathrm{Tame(Q)}$, $b = c^{-1}$ and g is of the form

$$\begin{pmatrix} ax & b(y + xP(x)) \\ b^{-1}(z + xT(x)) & a^{-1}(t + zP(x) + yT(x) + xP(x)T(x)) \end{pmatrix},$$

proving (iv). □

Proposition 2.12. *Any element of* Stab($[x]$) *whose action on* $\mathcal{T}_{\pi,k(x)}$ *is hyperbolic is conjugated to a composition of automorphisms of the form*

$$\begin{pmatrix} ax & b(z + xP(x,y)) \\ b^{-1}(y + xR(x)) & a^{-1}(t + zR(x) + yP(x,y) + xP(x,y)R(x)) \end{pmatrix},$$

where $R \in k[x]$ *and* $P \in k[x,y]$ *such that* $\deg_y(P) \geqslant 2$.

Proof. Let us fix an element $f \in \mathrm{Stab}([x])$ whose action on $\mathcal{T}_{\pi,k(x)}$ is loxodromic. Using assertion (v) of Proposition 2.10, up to conjugation, we can assume f is decomposed into

$$f = a_1 \circ e_1 \circ \cdots \circ a_n \circ e_n,$$

where $a_i \in \tilde{A} \setminus \tilde{E}$ and $e_i \in \tilde{E} \setminus \tilde{A}$. Let σ be the involution $(x,y,z,t) \mapsto (x,z,y,t)$. Every element of $\tilde{A}$ preserve the set of vertices $[y]$, $[z]$ in $\mathcal{T}_{\pi,k(x)}$. Observe that the elements in $A^S_{[x]}$ fix each vertex $[y]$, $[z]$ in $\mathcal{T}_{\pi,k(x)}$ and belong to $\tilde{E}$. Moreover, an element belongs to $\tilde{A} \cap \tilde{E}$ if it preserves the vertex $[z]$, hence it must be in $A^S_{[x]}$. We deduce that the elements a_i are of the form

$$\begin{pmatrix} ax & b(z+xT(x)) \\ b^{-1}(y+xP(x)) & a^{-1}(t+zP(x)+yT(x)+xP(x)T(x)) \end{pmatrix}, \tag{6}$$

where $a, b \in k^*$, $P, T \in k[x]$. This shows that we can decompose a_i into $a_i = \sigma \circ a'_i$ where $a'_i \in A^S_{[x]} \subset \tilde{A} \cap \tilde{E}$. We can thus absorb the element a'_i in the term in $\tilde{E}$:

$$f = \sigma \circ (a'_1 \circ e_1) \circ \sigma \circ (a'_2 \circ e_2) \circ \cdots \circ \sigma \circ (a'_n \circ e_n).$$

We then conclude since the product $\sigma \circ (a'_i \circ e_i)$ is of the required form. □

2E. ***Five technical consequences on the local geometry at each vertex.*** We say that a subset $S \subset \mathcal{C}$ is a 2×2 square of $\mathcal{C}$ if S is the union of four distinct 1×1 squares such that S isometric to $[0, 2] \times [0, 2]$. Moreover, we say that a 2×2 square is centered on a vertex v if the vertex v corresponds to the image of the point $(1, 1)$ by an isometry from $[0, 2] \times [0, 2]$ to S.

Two 1×1 (resp. 2×2) squares S, S' are said to be *adjacent* if their union $S \cup S'$ is isometric to $[0, 2] \times [0, 1]$ (resp. $[0, 4] \times [0, 2]$). Two 1×1 squares S and S' are adjacent along a vertical (resp. horizontal) edge if they are adjacent and their intersection $S \cap S'$ is a vertical edge (resp. horizontal). Similarly, two 2×2 squares are said to be *adjacent* along a horizontal (resp. vertical) if they intersect along a boundary segment (i.e., $\{0\} \times [0, 2]$, $\{2\} \times [0, 2]$, $[0, 2] \times \{0\}$ or $[0, 2] \times \{2\}$) isometric to $[0, 2]$ which is the union of two horizontal edges (resp. vertical edges).

Two 1×1 (resp. 2×2) squares S_1 and S_2 are said to be *adherent* if they are not adjacent but their intersection is reduced to a vertex which is in a corner of each respective square (i.e., one of the extremal point of each square). If a vertex $v \in \mathcal{C}$ belongs to the intersection of two adherent squares $S_1 \cap S_2$, then S_1 and S_2 are said to be adherent along the vertex v.

We say that two 1×1 (resp. 2×2) squares S, S' are *flat* if there exists two 1×1 (resp. 2×2) squares S_1, S_2 such that the union $S_1 \cup S_2 \cup S \cup S'$ is isometric to $[0, 2] \times [0, 2]$ (resp. $[0, 4] \times [0, 4]$). Similarly, three 1×1 (resp. 2×2) squares are flat if we can find another 1×1 (resp. 2×2) square such that their union is isometric to $[0, 2] \times [0, 2]$ (resp. $[0, 4] \times [0, 4]$). Once Lemma 2.16 and Lemma 2.15 are obtained, they will allow us to work only with 2×2 squares centered around vertices of type III, instead of 1×1 squares.

We will prove that three 1×1 squares S_1, S_2, S_3 such that S_1 and S_2, S_2 and S_3 are adjacent and contain a common vertex of type II or III are necessarily flat; see Lemmas 2.15 and 2.16 below. However, this property does not necessarily hold when the squares contain a common vertex of type I (see Lemma 2.17 below), we prove that they are either flat or contained in a spiral staircase. We explain this terminology below.

A collection (S, S') of 1×1 or 2×2 squares is contained in a *spiral staircase around* v (see 2.14 for an example) if they contain a common vertex v of type I and such that any minimal sequence $S_1 = S, \ldots, S_k = S'$ of squares containing v and connecting S to S' satisfies the following conditions:

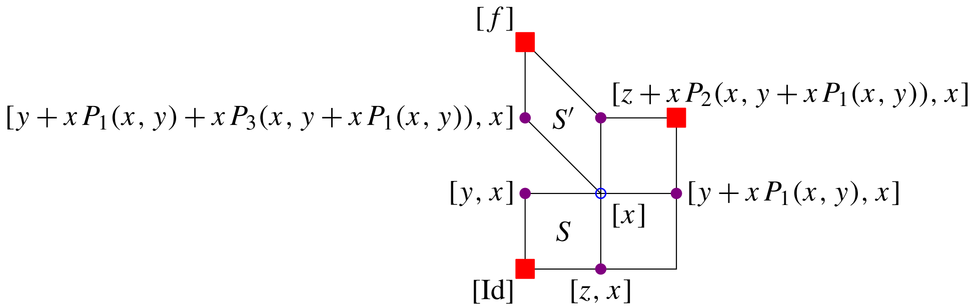


Figure 3. Example of horizontal spiral staircase.

(1) For all integer $i \leqslant k-1$, the squares S_i and S_{i+1} are alternatively adjacent along a vertical or horizontal edge containing v.

(2) Any three consecutive squares (S_i, S_{i+1}, S_{i+2}) for $i \leqslant k-2$ is not flat.

When the first two squares S_1 and S_2 are adjacent along a horizontal edge (containing v), we say that the *spiral staircase around* v is *vertical*. Otherwise the squares S_1 and S_2 are adjacent along a vertical edge and we have a *horizontal spiral staircase around* v.

Remark 2.13. We conjecture that there cannot be any spiral staircase that is both vertical and horizontal but do not need such a statement in our proofs.

When two squares S, S' are flat, then the collection (S, S') is not contained in a spiral staircase.

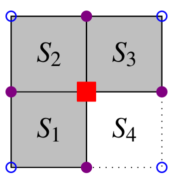
Example 2.14. Consider $P_1, P_2, P_3 \in k[x, y] \setminus k[x]$, denote by S the square containing $[x]$ and [Id] and S' the square containing $[x]$ and $[f]$ where $f \in \mathrm{Tame(Q)}$ is given by

$$f = \begin{pmatrix} x & y + xP_1(x, y) + xP_3(x, z + xP_2(x, y + xP_1(x, y))) \\ z + xP_2(x, y + xP_1(x, y)) & f_4 \end{pmatrix},$$

where $f_4 = t + y(P_1(x, y) + P_3(x, z + xP_2(x, y + xP_1(x, y)))) + yP_2(x, y + xP_1(x, y)) + x(P_1(x, y) + P_3(x, z + xP_2(x, y + xP_1(x, y))))P_2(x, y + xP_1(x, y)))$. Then the pair (S, S') is contained in a horizontal spiral staircase and one has the Figure 3.

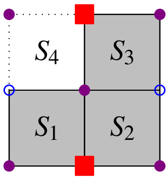
The next lemmas describe when three squares containing a common vertex are flat.

Lemma 2.15. *Let v be a vertex of type* III *and let S_1, S_2, S_3 be three distinct 1×1 squares such that S_1 is adjacent to S_2 along an edge containing v, and S_2 is adjacent to S_3 along an edge containing v. Then the three squares can be completed into a 2×2 square centered along v:*



Proof. Since the group acts transitively on the vertices of type III by Proposition 2.7, we can reduce by conjugating by a tame element to the case where the vertex [Id] is a common point of the three squares. By Proposition 2.3(i) and (ii), the three squares determine 3 distinct points p_1, p_2, p_3 at infinity such that p_1 and p_2 are on a same line at infinity L_{12}, and p_2, p_3 lie on another line L_{23} which is transverse to L_{12}. Denote by p_4 the intersection of the line passing through p_1 transverse to L_{12} with the line passing through p_3 transverse to L_{23}. This point determines a unique square S_4 containing [Id] by Proposition 2.3(ii) and the union $S_1 \cup S_2 \cup S_3 \cup S_4$ is isometric to $[0, 2] \times [0, 2]$ since p_1, p_2, p_3 and p_4 lie on a cycle of four lines at infinity. □

Lemma 2.16. *Let v be a vertex of type* II *and let S_1, S_2, S_3 be three distinct 1×1 squares such that S_1 is adjacent to S_2 along an edge containing v, and S_2 is adjacent to S_3 along an edge containing v. Then the three squares can be completed into a 2×2 square centered along v:*



Proof. Since the tame group and PGL_2 act transitively on the vertices of type III and on the pairs of points on $\mathbb{P}^1$ respectively, we are reduced by conjugating with an appropriate tame automorphism to the situation where the squares S_1 and S_2 contain [Id] and the points $[y]$ and $[x]$ respectively. Take $f \in \mathrm{STame}(Q)$ a tame automorphism such that the vertex $f \cdot S_2 = S_3$. Note that since S_2 and S_3 are adjacent, f preserve the edge $S_2 \cap S_3$. By Proposition 2.6(ii), f belongs to

$$E_V \rtimes \left\{ \begin{pmatrix} ax & dy + cx \\ d^{-1}z & a^{-1}t + ca^{-1}d^{-1}z \end{pmatrix} \,\middle|\, (a, c, d) \in k^* \times k \times k^* \right\}.$$

In particular, f can be decomposed as $f = g \circ e$ where $e \in E_V$ and g belongs to the subgroup

$$\left\{ \begin{pmatrix} ax & dy + cx \\ d^{-1}z & a^{-1}t + ca^{-1}d^{-1}z \end{pmatrix} \,\middle|\, (a, c, d) \in k^* \times k \times k^* \right\}.$$

The vertex of type III in S_3 is determined by $[f] = O_4 f = O_4 g \circ e = O_4 e = [e]$. Set $S_4 = e^{-1} \cdot [S_1]$, by construction, S_4 is adjacent to S_1 and since it also contains the vertices $e^{-1} \cdot [\mathrm{Id}] = [e]$ and $[x, y]$, it is adjacent to S_3. We have thus proved that S_1, S_2, S_3 are flat. □

The important consequence of the above two lemmas is the following: Assume S is a 2×2 square centered around a vertex of type III and that $\tilde{S}'$ is a 1×1 square adjacent to S along an edge of type I. Then using Lemma 2.16 and Lemma 2.15, there exists another 2×2 square S' containing $\tilde{S}'$, centered on a vertex of type III, adjacent to S. This is synthesized in Figure 4.

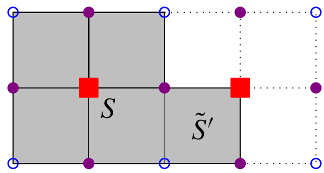
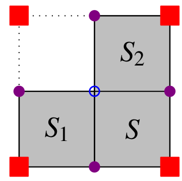
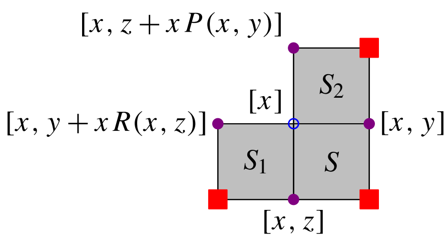


Figure 4. A 2×2 square S centered around a type III vertex with a 1×1 square $\tilde{S}'$ adjacent to S along a type I edge where $\tilde{S}'$ is contained in another 2×2 square S'.

Lemma 2.17. *Let v be a vertex of type* I *and let S, S_1, S_2 be three distinct 1×1 squares such that S is adjacent to S_1 along an edge containing v, and S is adjacent to S_2 along an edge containing v. Let g_1 and $g_2 \in \mathrm{STame}(Q)$ such that $g_1 S = S_1$ and $g_2 S = S_2$. Then the three squares can be completed into a 2×2 square centered along v if and only if g_1 or g_2 belongs to A_v^S:*



Proof. Since the group $\mathrm{STame}(Q)$ is transitive on the set of 1×1 squares, we can suppose that the common vertex v is $[x]$ and that S contains the vertex $[\mathrm{Id}]$. We are thus in the following situation:



where P, $R \in k[x,y]$.

Let us prove the reverse implication ($\Leftarrow$). Assume g_1 or $g_2 \in A_{[x]}^S$. Let us assume that $g_1 \in A_{[x]}^S$, then this implies that $R(x,z) \in k[x,z]$ and $P(x) \in k[x]$ and g_1, g_2 can be taken to be

$$g_1 = \begin{pmatrix} x & y - xR(x) \\ z & t - zR(x) \end{pmatrix} \quad \text{and} \quad g_2 = \begin{pmatrix} x & y \\ z - xP(x,y) & t - yP(x,y) \end{pmatrix}.$$

Set $S' = g_1 g_2 S$, since $g_1 \circ g_2 = g_2 \circ g_1$, we obtain that S' is adjacent to S_2 and S_1, and contains $[x]$ because $g_1 \in A_{[x]}^S \subset \mathrm{Stab}([x])$. In particular, S, S_1, S_2 are flat.

Let us prove the first implication ($\Rightarrow$). Suppose that the squares S_1, S_2, S_3 are flat. Then there exists a component $f_4 \in \mathrm{k}[Q]$ such that the element f given by

$$f = \begin{pmatrix} x & y + xR(x,z) \\ z + xP(x,y) & f_4 \end{pmatrix}$$

belongs to Tame(Q). In particular, it must fix the volume form Ω, this implies that

$$\partial_y P(x, y)\partial_z R(x, z) = 0 \in \mathrm{k}[\mathcal{Q}]\,.$$

This implies that $\partial_y P(x, y) = 0$ or $\partial_z R(x, z) = 0$ hence g_1 or g_2 belongs to $A^S_{[x]}$ as required. □

Lemma 2.18. *Take S and S' two 2×2 squares centered at a vertex of type* III *which are adherent along a vertex of type* I. *Then S and S' satisfy one of the following properties*:

(i) *Either the pair (S, S') is flat.*

(ii) *Either the pair of squares (S, S') is contained in a horizontal or vertical spiral staircase.*

Proof. Consider two squares S, S' such that the pair of square (S, S') is not flat. Up to a conjugation by an element of STame(Q), we can suppose that S and S' are adherent along the vertex $[x]$. Since the group Tame(Q) acts transitively on the set of 2×2 squares centered on type III vertices by Proposition 2.7(vi), there exists an element $g \in$ STame(Q) such that $g \cdot S = S'$. Using the fact that the link of type I vertices is connected, we can choose any minimal sequence S_i of adjacent 2×2 squares centered along a vertex of type III all containing $[x]$ such that $S_1 = S, \dots, S_k = S'$. Observe that we can always construct such a sequence by taking a sequence of $\tilde{S}_1, \dots, \tilde{S}_k$ of 1×1 squares containing the same vertex of type I and such that $\tilde{S}_1 \subset S$, $\tilde{S}_k \subset S'$. We apply Lemmas 2.16 and 2.15 inductively to $S_i, \tilde{S}_{i+1}$ for $i = 1, \dots, k-1$ and we obtain that there exists a square S_{i+1} containing $\tilde{S}_{i+1}$ centered around a vertex of type III, which is adjacent to S_i for each $i = 1, \dots, k-1$.

Since the sequence of square $S_1, \dots, S_k$ is minimal, we claim that the squares S_i and S_{i+2} are adherent along the vertex $[x]$ but the sequence S_i, S_{i+1}, S_{i+2} is not flat. Indeed, if it were the case, there exists a 2×2 square $\tilde{S}_i$ containing $[x]$ such that $\tilde{S}_i$ is adjacent to S_i and S_{i+2} and such that the union $S_i, S_{i+1}, S_{i+2}, \tilde{S}_i$ is isometric to $[0, 4]\times[0, 4]$. Observe that the edge $S_i \cap S_{i-1}$ and $S_i \cap \tilde{S}_i$ are equal so the sequence $S_1, \dots, S_{i-1}, \tilde{S}_i, S_{i+2}, \dots, S_k$ is a sequence of squares connecting S and S' of length $k-1$. This contradicts the fact that the sequence $S_1, \dots, S_k$ is minimal.

Moreover, the squares S_i and S_{i+1} are alternatively adjacent along vertical and horizontal edges. Hence the pair (S, S') is contained in a horizontal or vertical spiral staircase, as required. □

In practice, we will use the following explicit characterization to determine whether two squares adherent along a vertex of type I are flat.

Lemma 2.19. *Consider two 2×2 adjacent squares S_1, S_2 along a horizontal edge containing $[x_1], [y_1]$ and a polynomial $P \in k[x, y]\setminus k$. Denote by $[z_1], [t_1]$ the other vertices of S_1 such that $[x_1], [z_1]$ belong to a vertical edge of S_1 and by $[z_1 + x_1 P(x_1, y_1)], [t_1 + y_1 P(x_1, y_1)]$ the two other vertices of S_2. Let g be the tame automorphism defined by*

$$g = \begin{pmatrix} x & y \\ z + xP(x, y) & t + yP(x, y) \end{pmatrix}$$

so that $g \cdot S_1 = S_2$.

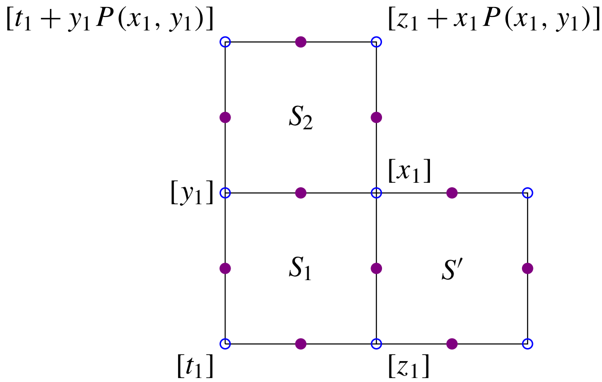


Figure 5. The initial situation of Lemma 2.19.

The following assertions hold:

(i) *We have* $g \in A^{S_1}_{[x_1]}$ *if and only if* $P \in k[x] \setminus k$.

(ii) *For any square* S' *adjacent to* S_1 *along the vertical edge containing* $[x_1]$, $[z_1]$, *the squares* S_1, S', S_2 *are flat if and only if* $P \in k[x] \setminus k$.

Figure 5 summarizes the initial situation in the previous lemma.

Proof. By conjugation, we can suppose that $x_1 = x$, $y_1 = y$, $z_1 = z$ and $t_1 = t$. Assertion (i) follows directly from the definition of $A^{S_1}_{[x]}$.

Let us prove assertion (ii). Choose a square S' such that $g'S_1 = S'$ where $g' \notin A^{S_1}_{[x]}$. Using successively Lemma 2.17, Lemma 2.16 and Lemma 2.15, we obtain that the squares S_1, S_2, S' are flat if and only if g or g' are in $A^{S_1}_{[x]}$. Since $g' \notin A^{S_1}_{[x]}$, then we deduce that S_1, S_2, S' are flat if and only if $g \in A^{S_1}_{[x]}$. Finally the condition $g \in A^{S_1}_{[x]}$ is equivalent to the fact that $P \in k[x] \setminus k$ by assertion (i). $\square$

3. Global geometry of the complex

In this section, we first review the results due to Bisi, Furter and Lamy regarding the global geometric properties of the metric square complex $(\mathcal{C}, d_{\mathcal{C}})$ introduced in Section 2. We then describe the degree of iterates of a tame automorphism fixing a vertex of the complex.

3A. ***Gromov curvature and Gromov-hyperbolicity.*** Recall that a map $\gamma : [0, l] \to (\mathcal{C}, d_{\mathcal{C}})$ defines a geodesic segment of length l if γ induces an isometry from $[0, l]$ to $\gamma([0, l])$. A map $\gamma : \mathbb{R} \to \mathcal{C}$ which is an isometry onto its image is called a geodesic line and a map $\gamma : \mathbb{R}^+ \to \mathcal{C}$ which is an isometry onto its image is called a geodesic half-line. Recall also that $\gamma : I \to \mathcal{C}$ where $I = [0, l]$ or $I = \mathbb{R}, \mathbb{R}^+$ is a quasigeodesic if there exists $\lambda > 0$, $M > 0$ such that for any $s, s' \in I$, the following inequality is satisfied:

$$\tfrac{1}{\lambda}|s - s'| - M \leqslant d_{\mathcal{C}}(\gamma(s), \gamma(s')) \leqslant \lambda|s - s'| + M.$$

As a result, a geodesic line is also a quasigeodesic. When any two points on a metric space can be joined by a geodesic segment, we say that the space is a geodesic metric space.

A geodesic space (X, d) is CAT(0) (see [Bridson and Haefliger 1999, Section II.1]) if its triangles are thinner than euclidean triangles. In other words, (X, d) satisfies the following condition. For any three points p, q, r in X, take a triangle in the euclidean plane $(\mathbb{R}^2, \|\cdot\|)$ with vertices $\bar{p}, \bar{q}, \bar{r} \in \mathbb{R}^2$ such that $d(p, q) = \|\bar{p} - \bar{q}\|$, $d(q, r) = \|\bar{q} - \bar{r}\|$ and $d(r, p) = \|\bar{r} - \bar{p}\|$. Then for any point $m_1 \in X$ and $m_2 \in X$ in the geodesic segment $[p, q]$ and $[q, r]$ respectively, one has

$$d(m_1, m_2) \leqslant \|\bar{m}_1 - \bar{m}_2\|,$$

where $\bar{m}_1$ and $\bar{m}_2$ are the unique points on the segments $[\bar{p}, \bar{q}]$ and $[\bar{q}, \bar{r}]$ respectively such that $d(m_1, p) = \|\bar{p} - \bar{m}_1\|$ and $d(r, m_2) = \|\bar{r} - \bar{m}_2\|$.

Let us recall the notion of Gromov-hyperbolic metric space. Let $\delta > 0$ be a positive real number. A metric space (X, d) is δ-hyperbolic if for any geodesic triangle $T = [p, q] \cup [q, r] \cup [r, p]$ in X and for any point $m \in [p, q]$, we have

$$d(m, [q, r] \cup [r, p]) \leqslant \delta$$

Theorem 3.1 [Bisi et al. 2014, Theorem A]. *The square complex $\mathcal{C}$, endowed with the distance $d_{\mathcal{C}}$, is a geodesic metric space which is simply connected*, CAT(0) *and Gromov-hyperbolic.*

The previous result has important consequences on the behavior of the isometries of the complex, i.e., distance preserving maps. Recall that the translation length, denoted $l(f)$, of an isometry $f : \mathcal{C} \to \mathcal{C}$ is defined by

$$l(f) = \inf_{v \in \mathcal{C}} d_{\mathcal{C}}(v, f(v)).$$

Observe that for any isometry f, the points in the complex where the infimum is reached is invariant by f. We denote by $\mathrm{Min}(f)$ the subset of $\mathcal{C}$ on which the infimum is reached.

Theorem 3.2. *Let $f : \mathcal{C} \to \mathcal{C}$ be an isometry of $\mathcal{C}$ which is also a morphism of complex. Then either $l(f) = 0$ and f fixes a vertex in the complex, either $l(f) > 0$ and one can find f-invariant geodesic line on which f acts by translation by $l(f)$.*

In other words, a tame automorphism f is either *elliptic* (when $l(f) = 0$) or *hyperbolic*.

Proof. Take f an isometry of the complex $\mathcal{C}$. Then $\mathrm{Min}(f)$ is nonempty by [Bridson and Haefliger 1999, II.6.6(2)]. Suppose that $l(f) > 0$, then f satisfies the hypothesis of [loc. cit., II.Theorem 6.8]. More precisely, [loc. cit., II.Theorem 6.8(1)] asserts that an isometry f of a CAT(0) space satisfies $l(f) > 0$ if and only if f translates by $l(f)$ on an invariant geodesic line, as required.

Otherwise $l(f) = 0$, we prove that there exists a vertex which is fixed by f. Define a cell of C to be a vertex, an edge or a square. We note that the intersection of two cells of C is a cell. Since $Min(f) \neq \varnothing$ pick $v \in Min(f)$, then $d_{\mathcal{C}}(v, f(v)) = 0$: So $f(v) = v$. Let S be the minimal cell of $\mathcal{C}$ which contains v. Then v is the intersection of all cells of $\mathcal{C}$ which contains v and we get $f(S) = S$. If S is a vertex, v is a vertex. If S is an edge, since the two vertices of S are of different types, f fixes every vertex of S. Now assume that S is a square. Then f fix the unique type III point of S. □

3B. ***Degree growths of elliptic automorphisms.*** In this section, we apply the results of the previous section to study the degree growth of particular tame automorphisms. Recall from the previous section that a tame automorphism is *elliptic* or *hyperbolic* if its action on the complex fixes a vertex or preserves a geodesic line of the complex on which it acts by translation respectively.

The following result classifies the degree growth of any elliptic tame automorphisms.

Theorem 3.3. *Let $f \in \mathrm{Tame(Q)}$ be any tame automorphism of Q fixing a vertex in the square complex. Then we are in one of the following situations*:

(i) *The sequence $(\deg(f^n), \deg(f^{-n}))$ is bounded and f is linear or f^2 is conjugate via a birational map $Q \dashrightarrow \mathbb{A}^3$ to an automorphism of the form $(x, y, z) \mapsto (ax, by + xR(x), b^{-1}z + xP(x, y))$ with $a, b \in k^*$, $P \in k[x, y]$ and $R \in k[x]$.*

(ii) *There exists a constant $C > 0$ such that*

$$\frac{1}{C}n \leqslant \deg(f^{\epsilon n}) \leqslant Cn,$$

where $\epsilon \in \{+1, -1\}$ and f is conjugated via a birational map $Q \dashrightarrow \mathbb{A}^3$ to an automorphism of the form

$$(x, y, z) \mapsto (ax, b^{-1}(z + xR(x)), b(y + xP(x)z)),$$

with $a, b \in k^*$, $R \in k[x]$ and $P \in k[x] \setminus k$.

(iii) *There exists a constant $C > 0$ and an integer d such that*

$$\frac{1}{C}d^n \leqslant \deg(f^{\epsilon n}) \leqslant Cd^n,$$

where $\epsilon \in \{+1, -1\}$ and f is conjugated via a birational map $Q \dashrightarrow \mathbb{A}^3$ to a composition of elements of the form

$$(x, y, z) \mapsto (ax, b(z + xP(x, y)), b^{-1}(y + xR(x))),$$

where $a, b \in k^*$, $R \in k[x]$ and $P \in k[x, y]$ such that $\deg_y(P) \geqslant 2$.

Remark 3.4. In case (iii) of the previous theorem, suppose f is a normal form, then $\deg(f^p) = Cd^p + C_0$ where $C > 0$ and $C_0 \in \mathbb{Z}$.

Remark 3.5. The growth of the degree of elliptic automorphisms is summarized in Table 1.

The proof of the theorem relies on the comparison to some reference tame automorphism, for which one computes the degree growth explicitly.

fixed vertex	action on the link	fibration	behavior on the fiber	$\deg(f^n)$
Type III				bounded
Type II		over $\mathbb{P}^2$	flow of a vector field	bounded
Type I	trivial on $\mathcal{T}$	over $\mathbb{P}^1$	flow of a vector field	bounded
Type I	involution on $\mathcal{T}$	over $\mathbb{P}^1$	affine	linear
Type I	hyperbolic on $\mathcal{T}$	over $\mathbb{P}^1$	composition of Henon	$\frac{d^n}{C} \leqslant \deg(f^n) \leqslant Cnd^n$

Table 1. Summary of the degree growth for an elliptic automorphism.

Lemma 3.6. *The following properties hold*:

(i) *If f is belongs to the semidirect product* $\mathrm{Stab}([x, z]) = \tilde{E}_H \rtimes N$ *where*

$$N = \left\{ \begin{pmatrix} ax+bz & a'y+b't \\ cx+dz & c'y+d't \end{pmatrix} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d' & -b' \\ -c' & a' \end{pmatrix} = I_2 \in M_2(k) \right\},$$

then the sequence $(\deg(f^n), \deg(f^{-n}))$ *is bounded.*

(ii) *If f is of the form*

$$f = \begin{pmatrix} ax & b(y+xP(x)) \\ b^{-1}(z+xS(x)) & a^{-1}(t+zP(x)+yS(x)+xP(x)S(x)) \end{pmatrix}$$

where $P, S \in k[x] \setminus k$, $a, b \in k^*$, then the sequence $(\deg(f^n), \deg(f^{-n}))$ *is bounded.*

(iii) *If f is of the form*

$$f = \begin{pmatrix} ax & b(y+xP(x,z)) \\ b^{-1}(z+xR(x)) & a^{-1}(t+zP(x,z)+yR(x)) \end{pmatrix}$$

with $P \in k[x, z] \setminus k$ and $\deg_z(P) = 1$, $R \in k[x]$ and $a, b \in k^*$, then the sequence $(\deg(f^n), \deg(f^{-n}))$ *is bounded.*

(iv) *If f is of the form*

$$f = \begin{pmatrix} ax & b^{-1}(z+xR(x)) \\ b(y+xP(x)z) & a^{-1}(t+z^2P(x)+yR(x)) \end{pmatrix}$$

with $P \in k[x] \setminus k$, $R \in k[x]$, $a, b \in k^*$, then both sequences $(\deg(f^n))$, $(\deg(f^{-n}))$ *grow linearly.*

(v) *If f is of the form*

$$f = \begin{pmatrix} ax & b(y+xP(x,z)) \\ b^{-1}(z+xR(x)) & a^{-1}(t+z^2P(x)+yR(x)) \end{pmatrix}$$

with $P \in k[x, y]$, $R \in k[x] \setminus k$, $a, b \in k^*$, then the sequence $(\deg(f^n), \deg(f^{-n}))$ *is bounded.*

(vi) *If f is a composition of automorphism of the form*

$$\begin{pmatrix} ax & b(z+xP(x,y)) \\ b^{-1}(y+xR(x)) & a^{-1}(t+zR(x)+yP(x,y)+xP(x,y)R(x)) \end{pmatrix},$$

where $R \in k[x]$ and $P \in k[x,y]$ such that $\deg_y(P) \geqslant 2$, *then we have*

$$d^n \leqslant \deg(f^{\pm n}) \leqslant Cd^n, \tag{7}$$

where $C > 0$ and $d \geq 2$ is an integer.

Proof. During the whole proof, we will consider the valuation $\nu : k[Q] \to \mathbb{R}^- \cup \{+\infty\}$ corresponding to $-\deg$. It is defined by the formula

$$\nu(P) = \sup\{-\deg(R) \mid R \in k[x,y,z,t],\, R = P \in k[Q]\}.$$

The fact that such a function gives a valuation will be proved in Proposition 4.2. In each case except the last one, we will also express f^n as $f^n = (x_n, y_n, z_n, t_n)$ where $x_n, y_n, z_n, t_n \in k[Q]$.

Let us prove assertion (i). Denote by N the subgroup

$$N = \left\{ \begin{pmatrix} ax+bz & a'y+b't \\ cx+dz & c'y+d't \end{pmatrix} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d' & -b' \\ -c' & a' \end{pmatrix} = I_2 \in M_2(k) \right\}.$$

Take $f \in \tilde{E}_H \rtimes N$, then it can be decomposed into

$$f = e \circ g,$$

where $e \in \tilde{E}_H$ and $g \in N$. Since $\tilde{E}_H$ is a normal subgroup, we can consider the elements $e_1 = geg^{-1} \in \tilde{E}_H$, $e_{k+1} = gee_kg^{-1}$ for all $k \geq 1$. We have

$$\begin{aligned} f^n &= (eg)^n = (eg)^{n-2}(egeg) \\ &= (eg)^{n-2}ee_1g^2 \\ &= (eg)^{n-3}e(gee_1)g^2 \\ &= (eg)^{n-3}ee_2g^3 \\ &= \cdots \\ f^n &= ee_{n-1}g^n. \end{aligned}$$

Note that since $N \subset O_4$, the degree remain unchanged if we postcompose by elements in this subgroup. In particular, we have $\deg e_{k+1} = \deg(ee_k)$ for all $k \geq 2$. we get

$$\deg(f^n) = \deg(ee_{n-1}).$$

Now we write e as

$$e = \begin{pmatrix} ax & b(y+xP(x,z)) \\ b^{-1}z & a^{-1}(t+zP(x,z)) \end{pmatrix}, \tag{8}$$

where $a, b \in k^*$ and $P \in k[x, z]$, and e_k is of the form

$$e_k = \begin{pmatrix} x'_k & y'_k \\ z'_k & t'_k \end{pmatrix},$$

where $x'_k, y'_k, z'_k, t'_k \in k[Q]$. We claim that the degree of e_k is bounded. We have $\nu(x'_1) = \nu(z'_1) = -1$, $\nu(y'_1) = \nu(t'_1) = -\deg(P) - 1$ and an immediate induction gives

$$\begin{aligned}
&\nu(x'_{k+1}) = \nu(ax'_k) = \nu(x'_k), \\
&\nu(y'_{k+1}) = \nu(b(y'_k + x'_k P(x'_k, z'_k))) = \nu(y'_k + x'_k P(x'_k, z'_k)) \geq -\deg(P) - 1, \\
&\nu(z'_{k+1}) = \nu(bz'_k) = \nu(z'_k), \\
&\nu(t'_{k+1}) = \nu(a^{-1}(t'_k + z'_k P(x'_k, z'_k))) \geq -\deg(P) - 1.
\end{aligned}$$

This shows that $\deg(f^n) = \deg(ee_{n-1})$ is bounded. A similar argument will also show that $\deg(f^{-n})$ is bounded and assertion (i) holds.

Let us prove assertion (ii). Assume f is given by

$$f = \begin{pmatrix} ax & b(y + xP(x)) \\ b^{-1}(z + xS(x)) & a^{-1}(t + zP(x) + yS(x) + xP(x)S(x)) \end{pmatrix}$$

where $P, S \in k[x] \setminus k$ and $a, b \in k^*$. The sequence $\nu(x_n)$ is constant equal to -1 because $\nu(x_{n+1}) = \nu(ax_n) = \nu(x_n) = \nu(x) = -1$. We have $\nu(y_1) = -\deg(P) - 1$, $\deg(z_1) = -\deg(S) - 1$. Observe that $\nu(y_{n+1}) = \nu(b(y_n + x_n P(x_n))) \geq -\deg(P) - 1$ for all $n \geq 1$, and $\nu(z_{n+1}) = \nu(b^{-1}(z_n + x_n S(x_n))) \geq -\deg(S) - 1$ for all $n \geq 1$. Since $\nu(x_n t_n - y_n z_n) = \nu(1) = 0$, and since $\nu(x_n)$, $\nu(y_n)$, $\nu(z_n)$ are all bounded and always negative, we deduce that $\nu(t_n) = \nu(y_n) + \nu(z_n) - \nu(x_n)$ is also bounded. This shows that $\deg(f^n)$ is bounded. Since the inverse f^{-1} can be obtained by replacing P by $-P$, S by $-S$ and a, b by a^{-1}, b^{-1} respectively, we conclude that $\deg(f^{-n})$ is also bounded.

Let us prove assertion (iii). Assume f is of the form

$$f = \begin{pmatrix} ax & b(y + xP(x, z)) \\ b^{-1}(z + xR(x)) & a^{-1}(t + zP(x, z) + yR(x) + xP(x, z)R(x)) \end{pmatrix}$$

with $R \in k[x]$, $a, b \in k^*$, $P \in k[x, z]$ and $\deg_z P = 1$. We first decompose P into

$$P(x, z) = P_0(x) + zP_1(x),$$

where $P_0, P_1 \in k[x]$ and $P_1 \neq 0$.

Observe that $\nu(x_n) = -1$ for all n. Observe that $\nu(z_1) = -\deg(R) - 1$, we obtain by induction

$$\nu(z_{n+1}) = \nu(b^{-1}(z_n + x_n R(x_n))) \geq -\deg(R) - 1.$$

so that $\nu(z_n) \geq -\deg(R) - 1$ for all $n \geq 1$. Observe that $\nu(y_1) = -\deg(P) - 2$, we now estimate $\nu(y_n)$ inductively:

$$\begin{aligned}\nu(y_{n+1}) &= \nu(b(y_n + x_n P_0(x_n) + x_n z_n P_1(x_n))) \\ &= \nu(y_n + x_n P_0(x_n) + x_n z_n P_1(x_n)) \\ &\geq \min(\nu(y_n), -\deg(P_0) - 1, \nu(z_n) - 1 - \deg(P_1)) \\ &\geq \min(\nu(y_n), -\deg(P_0) - 1, -\deg(P_1) - \deg(R) - 2).\end{aligned}$$

We thus obtain that $\nu(y_n) \geq -\deg(P) - \deg(R) - 2$ for all $n \geq 1$. Using the fact that $x_n t_n = 1 + y_n z_n$, we deduce that $\nu(t_n)$ is also bounded since $\nu(x_n), \nu(y_n), \nu(z_n)$ are all bounded. Since f^{-1} has a similar form, we deduce that both $\deg(f^n)$ and $\deg(f^{-n})$ are bounded.

Let us prove assertion (iv). Assume f is of the form

$$f = \begin{pmatrix} ax & b^{-1}(z + xR(x)) \\ b(y + xP(x)z) & a^{-1}(t + z^2 P(x) + yR(x)) \end{pmatrix}$$

with $P \in k[x] \setminus k$, $R \in k[x], a, b \in k^*$. In the case where $R = 0$ one sees easily that the sequence of degrees grows linearly $\deg(f^n) \sim Cn$. Let us now assume $R \neq 0$. We have

$$f^2 = \begin{pmatrix} a^2 x & b^{-1}(b(y + xP(x)z) + axR(ax)) \\ b(b^{-1}(z + xR(x)) + axP(ax)b(y + xP(x)z)) & t_2 \end{pmatrix}. \tag{9}$$

Observe that $\nu(x_n) = -1$ for all $n \geq 1$. We have (using Lemma 4.7 to evaluate the valuation) $\nu(y_1) = \min(-1, -1 - \deg(R))$, $\nu(z_1) = -\deg(P) - 2$, $\nu(y_2) = \min(-1, -2 - \deg(P), -1 - \deg(R))$, $\nu(z_2) = \min(-1, -1 - \deg(R), -2\deg(P) - 3)$ and the inductive relation

$$\begin{cases} \nu(y_{n+1}) = \nu(b^{-1}(z_n + x_n R(x_n))) \geq \min(\nu(z_n), -\deg(R) - 1), \\ \nu(z_{n+1}) = \nu(b(y_n + x_n P(x_n) z_n)) \geq \min(\nu(y_n), -\deg(P) - 1 + \nu(z_n)). \end{cases}$$

Let us show by induction on $n \geq 2$ that:

(a) $\nu(z_n) \leq \nu(y_n)$.

(b) $\nu(z_n) \leq \min(-1, -\deg(R) - 1)$.

The case $n = 2$ was treated above. Let us assume that (a), (b) hold for $n \geq 2$. Since $\nu(z_n x_n P(x_n)) < \nu(y_n)$, we have

$$\nu(z_{n+1}) = \nu(y_n + x_n P(x_n) z_n) = -\deg(P) - 1 + \nu(z_n). \tag{10}$$

Since $\nu(z_n) \leq \min(-1, -\deg(R) - 1)$, we have, by the previous relation,

$$\nu(y_{n+1}) \geq \min(\nu(z_n), -\deg(R) - 1) \geq \nu(z_n) > \nu(z_{n+1}). \tag{11}$$

We have thus showed (a) and (b) for $n + 1$.

Relation (a) shows that $\nu(z_{n+1}) = \nu(z_n) - \deg(P) - 1$ for all $n \geq 2$, so $\nu(z_n)$ grows linearly. Starting from $n \geq 3$, we would have $\nu(z_n) < \min(-1, -\deg(R) - 1)$. And this implies that $\nu(y_n) = \nu(z_{n-1})$ for all $n \geq 4$. Overall, both $\nu(y_n)$, $\nu(z_n)$ grow linearly for $n \geq 4$. Since $\nu(t_n) = \nu(1 + y_n z_n) - \nu(x_n) = \nu(y_n) + \nu(z_n) - \nu(x_n)$

for $n \geqslant 1$, we deduce that $\nu(t_n)$ also grows linearly for large enough n. Overall $\deg(f^n)$ grows linearly. Similarly, f^{-1} is also of the same form, so we also conclude that $\deg(f^{-n})$ also grows linearly.

Let us prove assertion (v). Assume f is of the form

$$f = \begin{pmatrix} ax & b(y + xP(x,z)) \\ b^{-1}(z + xR(x)) & a^{-1}(t + z^2P(x) + yR(x)) \end{pmatrix}$$

with $P \in k[x, y]$, $R \in k[x] \setminus k$, $a, b \in k^*$. We observe that $\nu(x_n) = -1$ and $\nu(z_n) = -\deg(R) - 1$ for all $n \geq 1$. We also have

$$\nu(y_{n+1}) = \nu(y_n + x_n P(x_n, z_n)) \geq \min(\nu(y_n), \nu(x_n) + \nu(P(x_n, z_n))).$$

Write $P = \sum a_{ij} x^i y^j$. Since $\nu(P(x_n, z_n)) \geq \min\{-i + j(-\deg(R) - 1) \mid a_{ij} \neq 0\}$, the above formula also yields that $\nu(y_n)$ is bounded. We then conclude that $\nu(t_n) = \nu(y_n z_n) - \nu(x_n)$ is also bounded. Finally $\deg(f^n)$ is bounded, and so is its inverse which is of a similar form.

Let us prove assertion (vi). Assume $f = g_k \cdots g_1$ is a composition of automorphisms of the form

$$g_i = \begin{pmatrix} a_i x & b_i(z + xP_i(x,y)) \\ b_i^{-1}(y + xR_i(x)) & a_i^{-1}(t + zR_i(x) + yP_i(x,y) + xP_i(x,y)R_i(x)) \end{pmatrix},$$

where $R_i \in k[x]$ and $P_i \in k[x, y]$ such that $\deg_y(P_i) \geqslant 2$. For simplicity, let us reset $x_n, y_n, z_n, t_n \in k[Q]$ defined by

$$g_i \cdots g_2 g_1 f^n = \begin{pmatrix} x_{nk+i} & y_{nk+i} \\ z_{nk+i} & t_{nk+i} \end{pmatrix} \tag{12}$$

for all $i \leqslant k$. We also set $g_{nk+i} := g_i$ for all n, i so that we repeat the same sequence of automorphism periodically.

We will first show that the sequences $\nu(y_n)$, $\nu(z_n)$ are unbounded.

Let us consider the following valuation $\nu_0 : k[Q] \to \mathbb{R}^- \cup \{+\infty\}$ which gives the weight $(0, -1, -1, -2)$ on (x, y, z, t) (see Proposition 4.2 for a precise definition). This valuation gives no weight to x whereas it gives the weight -1 to y, z and the weight -2 to t. By construction, we have $2\nu(P) \leqslant \nu_0(P)$ for all $P \in k[Q]$.

Recall from Section 2D that an automorphism of the form

$$g_n = \begin{pmatrix} a_n x & b_n(z + xP_n(x,y)) \\ b_n^{-1}(y + x_n R(x)) & a_n^{-1}(t + zR(x) + yP(x,y) + xP(x,y)R(x)) \end{pmatrix},$$

induces an element $i(g_n)$ of $\mathrm{Aut}_{k(x)} \mathbb{A}^2$. Namely, the associated element is

$$(y, z) \mapsto (b_n(z + xP_n(x,y)), b_n^{-1}(y + xR_n(x))). \tag{13}$$

One can see that given g_n, g_m of this form, one can find some automorphism $\tilde{g}_{n,m}, \tilde{g}_m$ of the same form determined by some polynomials $\tilde{P}_n = \lambda_n^{-1} P_n(x/\lambda_n, y)$, $\tilde{R}_n = \lambda_n^{-1} R(x/\lambda_n)$ where $\lambda_n \in k^*$ (λ_n depends on g_m for $n \geqslant m$) such that $i(g_n g_m) = i(\tilde{g}_n) i(\tilde{g}_m)$. We thus find them inductively as follows, we first find $\tilde{g_2}$ such that $i(g_2 g_1) = i(\tilde{g_2}) i(g_1)$. Now we find $\tilde{g_3}$ such that $i(g_3(g_2 g_1)) = i(\tilde{g_3}) i(\tilde{g_2}) i(g_1)$ and so on, observe that $\tilde{g}_{n+1}$ depends on all the preceding $g_n, g_{n-1} \dots, g_1$. However the degree in y of those elements remains the same. We thus obtain

$$i(g_n \cdots g_1) = i(\tilde{g}_n) \cdots i(\tilde{g}_2) i(g_1).$$

The elements $i(g_n), i(\tilde{g}_n)$ are Henon-like automorphism of $\mathbb{A}^2_{k[x]}$ and by [Friedland and Milnor 1989, Theorem 2.1] the degree in (y, z) of a composition $i(g_1 \cdots g_n) = i(\tilde{g}_1) \cdots i(\tilde{g}_n)$ of n such elements is a product $d_n = \prod_{i=1}^n \deg_y P_i$ of n integers larger or equal to 2. This implies that $\nu_0(y_n) \sim d_n$, $\nu_0(z_n) \sim d_{n-1}$ grow exponentially fast, and we obtain that $2\nu(y_n) \leqslant \nu_0(y_n)$, $2\nu(z_n) \leqslant \nu_0(z_n)$ are unbounded and that

$$\deg(g_n \cdots g_1) \geq d_n/2 \tag{14}$$

hence $\deg(f^n) \geq d_k^n/2$.

We will now prove the second inequality $\deg(g_n \cdots g_1) \leqslant C' d_n$ for a constant $C' > 0$. Let us first reduce our problem, to any $f \in \mathrm{Stab}([x])$ of the form

$$f = \begin{pmatrix} ax & f_2 \\ f_3 & f_4 \end{pmatrix},$$

where $a \in k^*$, $f_2, f_3, f_4 \in k[Q]$. We set

$$r(f) = \begin{pmatrix} x & f_2 \\ f_3 & af_4 \end{pmatrix}.$$

By construction, $r(f)$ is a tame automorphism which is also in $\mathrm{Stab}([x])$ and one has $\deg(r(f)) = \deg(f)$. Moreover, one can find some automorphism g_i' of the same form as g_i such that

$$r(g_n \cdots g_1) = r(g_n') \cdots r(g_1').$$

We can thus replace $g_1, \dots, g_n$ by $r(g_1'), \dots, r(g_n')$ and assume that the coefficient a_i are all equal to 1.

Let us introduce some particular notations. For any polynomial $P \in k[x, y] \setminus k[x]$ and $b \in k^*$, we write by $h(P)$ the automorphism given by

$$h(P, b) := \begin{pmatrix} x & b(z + xP(x, y)) \\ b^{-1}y & t + yP(x, y) \end{pmatrix}.$$

We also set for all $R, S \in k[x]$ and $b \in k^*$,

$$u(R, S, b) := \begin{pmatrix} x & b(y + xR(x)) \\ b^{-1}(z + xS(x)) & t + zR(x) + yS(x) + xR(x)S(x) \end{pmatrix}. \tag{15}$$

We will use some particular relations via the following lemma:

Lemma 3.7. *For any $P \in k[x, y] \setminus k$ such that $\deg_y P \geq 2$ and any $R, S \in k[x]$, there exist $T \in k[x]$, $\tilde{P} \in k[x, y] \setminus k$ such that the following are satisfied*:

(1) $\deg_y \tilde{P} = \deg_y P - 1$.

(2) $\deg_x \tilde{P} \leqslant \deg_x P + (\deg_y P - 1)(1 + \deg R)$.

(3) $\deg T \leqslant \max(\deg S, (\deg_y P)(\deg R + 1) + \deg P)$.

(4) $h(P, \beta) \circ u(R, S, b) = u(T, R, \beta b^{-1}) \circ h(y\tilde{P}, 1)$.

Proof. The proof follows easily from the Taylor expansion:

$$P(x, b(y + xR(x))) = P(x, by) + \frac{(bxR(x))^{\deg_y P}}{(\deg_y P)!} \partial_2^{\deg_y P} P(x, by) + \sum_{k=1}^{\deg_y P - 1} \frac{(bxR(x))^k}{k!} \partial_2^k P(x, by). \qquad \square$$

Let us decompose g_i into as $g_i = u(R_i, S_i, b_i) \circ h(yp_i, \beta_i)$, where $p_i \in k[x, y]$ such that $\deg_y p_i = \deg_y P_i - 1$ and $b_i, \beta_i \in k^*$. Note that since the polynomials P_i, R_i are chosen periodically from $\{P_1, \dots, P_k\}, \{R_1, \dots, R_k\}$ respectively, the polynomials R_i, S_i, p_i are also chosen among finitely many polynomials. Let us consider

$$M := \max_i(\deg(R_i), \deg S_i),$$
$$M_p := \max(M, \max_i(1 + \deg p_i)).$$

Observe also that $d_n = \prod_{i=1}^n \deg_y P_i = \prod_1^n (1 + \deg_y p_i)$ and that $M \leq M_p$ by construction.

We prove by induction on $n \geq 1$ that there exists $T_n, U_n \in k[x]$, $\beta'_n \in k^*$ and $\tilde{P}_1, \dots, \tilde{P}_n \in k[x, y] \setminus k$ such that

$$g_n \cdots g_1 = u(T_n, U_n, \beta'_n) h(y\tilde{P}_n, 1) \dots h(y\tilde{P}_1),$$

and satisfying the conditions

$$\deg U_n \leqslant M \quad \text{and} \quad \forall n \geq 2, \deg_x \tilde{P}_n \leqslant \deg_x p_n + \deg_y(p_n)(1 + \deg T_{n-1}), \tag{16}$$

and for all $n \geq 2$

$$\deg T_n \leqslant d_n \deg T_1 + \sum_{i=1}^{n-1} \frac{d_n}{d_i} + M_p \sum_{i=2}^{n} \frac{d_n}{d_i}. \tag{17}$$

The case where $n = 1$ is immediate. Let us assume that $g_n \dots g_1$ can be decomposed into $u(T_n, U_n, \beta'_n) \circ h(y\tilde{P}_n, 1) \circ \dots \circ h(y\tilde{P}_1, 1)$. Using the above lemma to $R = T_n, S = U_n$ and $P = yp_{n+1}$, we obtain $\tilde{\beta}_{n+1} \in k^*$, some polynomials $\tilde{T} \in k[x]$ and $\tilde{P}_{n+1} \in k[x, y]$ and get

$$\begin{aligned} g_{n+1} g_n \cdots g_1 &= u(R_{n+1}, S_{n+1}, b_n) h(yp_{n+1}, \beta_{n+1}) u(T_n, U_n, \beta'_n) h(y\tilde{P}_n, 1) \cdots h(y\tilde{P}_1), \\ &= u(R_{n+1}, S_{n+1}, \beta_{n+1}) u(\tilde{T}, R_{n+1}, \tilde{\beta}_{n+1}) h(y\tilde{P}_{n+1}, 1) h(y\tilde{P}_n, 1) \cdots h(y\tilde{P}_1, 1), \end{aligned}$$

and we conclude by setting $T_{n+1} = \tilde{T} + \tilde{\beta}_{n+1}^{-1} R_{n+1}$, $U_{n+1} = R_{n+1} + \tilde{\beta}_{n+1}^{-1} S_{n+1}$, $\beta'_{n+1} = \beta_{n+1}\tilde{\beta}_{n+1}$. We check the bounds on the degree of T_{n+1}, U_{n+1}. The lemma directly implies

$$\deg(U_{n+1}) \leqslant \max(\deg R_{n+1}, \deg S_{n+1}) \leqslant M \quad \text{and} \quad \deg_x \tilde{P}_{n+1} \leqslant \deg_x p_{n+1} + \deg_y(p_{n+1})(1 + \deg T_n).$$

Using the induction hypothesis $\deg U_n \leqslant M$, the inequality (17) and the fact that $\deg R_{n+1} \leqslant M \leqslant M_p$, $(1+\deg p_{n+1}) \leq M_p$, we get

$$\begin{aligned}
\deg(T_{n+1}) &\leqslant \max(\deg R_{n+1}, \deg(\tilde{T})),\\
&\leqslant \max(\deg R_{n+1}, \deg U_n, (1+\deg_y p_{n+1})(\deg T_n + 1) + 1 + \deg p_{n+1}),\\
&\leqslant \max(M, (1+\deg_y p_{n+1}) \deg T_n + M_p + (1+\deg_y p_{n+1})),\\
&\leqslant \max\left(M, \frac{d_{n+1}}{d_n}\left(d_n \deg T_1 + \sum_{i=1}^{n-1} \frac{d_n}{d_i} + M_p \sum_{i=2}^{n} \frac{d_n}{d_i}\right) + M_p + \frac{d_{n+1}}{d_n}\right),\\
&\leqslant d_{n+1} \deg T_1 + \sum_{i=1}^{n} \frac{d_{n+1}}{d_i} + M_p \sum_{i=2}^{n+1} \frac{d_{n+1}}{d_i}.
\end{aligned}$$

This finishes the proof by induction.

Using the fact that d_{n+1}/d_n is an integer larger or equal to 2, the inequality (17) implies that for all $n \geq 2$, we have $d_n/d_i \leqslant \left(\frac{1}{2}\right)^{i-1} d_n/d_1$, for all $i \leqslant n$,

$$\deg T_n \leqslant d_n \deg T_1 + (2+M_p)\frac{d_n}{d_1} \leqslant C d_n, \tag{18}$$

where $C = \deg(T_1) + (2+M_p)/d_1$. Now (16) simplifies as

$$\begin{aligned}
\deg_x \tilde{P}_n &\leqslant \deg_x p_n + \deg_y(p_n)(1+\deg T_{n-1}),\\
&\leqslant \deg_x p_n + \deg_y(p_n)(1+C d_{n-1}),\\
&\leqslant d_n\left(C+\tfrac{1}{2}\right) + \deg_x p_n,
\end{aligned}$$

and we choose an integer N_0 such that for all $n \geq N_0$, $\max(\deg_x p_i) \leqslant d_n$. We then get

$$\deg_x \tilde{P}_n \leqslant C' d_n \tag{19}$$

where $C' = C + \frac{3}{2}$ for all $n \geq N_0$.

Let us now prove that $\deg(h(y\tilde{P}_n, 1)\dots h(y\tilde{P}_1, 1)) \leqslant C' d_n$ where $C' > 0$. For this particular product, we reproduce the standard arguments for product of Henon transformations.

Let us write

$$h(y\tilde{P}_n, 1)\dots h(y\tilde{P}_1, 1) = \begin{pmatrix} x & y'_n \\ z'_n & t'_n \end{pmatrix},$$

where $y'_n, z'_n, t'_n \in k[Q]$.

We have the following inductive relation for $n \geqslant 1$:

$$\nu(z'_{n+1}) = \nu(y'_n) \quad \text{and} \quad \nu(y'_{n+1}) = \nu(z'_n + x y'_n \tilde{P}_{n+1}(x, y'_n)).$$

We prove by induction on $n \geq 1$ that $\nu(y'_n) \leqslant \nu(z'_n)$.

Indeed, it is clear for $n=1$ and assume by induction that $\nu(y'_n) \leqslant \nu(z'_n)$ for $n \geq 1$, then since $\tilde{P}_{n+1} \neq 0$, we get

$$\nu(y'_{n+1}) = \nu(z'_n + xy'_n \tilde{P}_{n+1}(x, y'_n)) = \nu(xy'_n \tilde{P}_{n+1}(x, y'_n)) < \nu(y'_n) = \nu(z'_{n+1}),$$

as required. Moreover, by applying (19), we have for all $n \geq N_0$,

$$\begin{aligned}\nu(y'_{n+1}) &\geqslant \min(-\deg_x \tilde{P}_{n+1}, (\deg_y \tilde{P}_n + 1)\nu(y'_n)) \\ &\geqslant \min(-C'd_{n+1}, (\deg_y \tilde{P}_n + 1)\nu(y'_n)) \\ &\geqslant \min\left(-C'd_{n+1}, \frac{d_{n+1}}{d_n}\nu(y'_n)\right).\end{aligned}$$

An immediate induction using the previous inequality shows that

$$\nu(y'_n) \geqslant \min\left(-C'd_n, \frac{d_n}{d_{N_0}}\nu(y'_{N_0})\right), \tag{20}$$

for all $n \geq N_0$ and this shows that $\deg(h(y\tilde{P}_n)\dots h(y\tilde{P}_1)) \leqslant \tilde{C}d_n$.

Finally, we conclude

$$\begin{aligned}\deg(g_n \dots g_1) &= \deg(u(T_n, U_n)h(y\tilde{P}_n)\cdots h(y\tilde{P}_1)) \\ &\leqslant \max(\deg T_n, \deg U_n, \deg(h(y\tilde{P}_n)\cdots h(y\tilde{P}_1))) \leqslant C''d_n, \end{aligned} \tag{21}$$

where $C'' > 0$, as required. □

Proof of Theorem 3.3. Take $f \in \mathrm{Tame}(\mathrm{Q})$ an elliptic automorphism. Since f fixes a vertex on the complex, we will distinguish three cases depending on the type of vertices f fixes. Moreover, recall that the degree growth is an invariant of conjugation and that by Proposition 2.7, the tame group acts transitively on the set of vertices of type I, II and III respectively. We are thus reduced to compute the degree growth for f in the subgroups $\mathrm{Stab}([\mathrm{Id}])$, $\mathrm{Stab}([x, z])$ and $\mathrm{Stab}([x])$ respectively.

First case: If $f \in \mathrm{Stab}([\mathrm{Id}]) = O_4$, the sequence $(\deg(f^n), \deg(f^{-n}))$ is bounded.

Second case: Suppose that $f \in \mathrm{Stab}([x, z])$. By Proposition 2.6, one has

$$\mathrm{Stab}([x,z]) = E_H \rtimes \left\{ \begin{pmatrix} ax+bz & a'y+b't \\ cx+dz & c'y+d't \end{pmatrix} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d' & -b' \\ -c' & a' \end{pmatrix} = I_2 \in M_2(k) \right\}.$$

Denote by $\pi_{xz} : \mathcal{Q} \to \mathbb{A}^2 \setminus \{(0,0)\}$ the map induced by the projection

$$(x, y, z, t) \to (x, z),$$

then f naturally preserves the fibration π_{xz}. Recall that $\pi_{xz}^{-1}(\mathbb{A}^2 \setminus (\{0\} \times \mathbb{A}^1))$ is isomorphic to $\mathbb{A}^2 \setminus (\{0\} \times \mathbb{A}^1) \times \mathbb{A}^1$ and f induces a birational self map. If the induced (linear) action on $\mathbb{A}^2 \setminus \{0\}$ is diagonalizable, then f can be conjugate to an element of the form

$$f : (x, z, y) \mapsto (ax, b^{-1}z, b(y + xP(x, z))).$$

Otherwise, the action on $\mathbb{A}^2$ has Jordan form and f is birationally conjugate to

$$f : (x, z, y) \mapsto (ax, b^{-1}x + az, b(y + xP(x, z))),$$

with $a, b \in k^*$, $P \in k[x, z]$. Moreover, using assertion (i) of Lemma 3.6, the sequence $(\deg(f^n), \deg(f^{-n}))$ is bounded and f satisfies assertion (i).

Third case: Consider $f \in \mathrm{Stab}([x])$ such that $f \notin \mathrm{Stab}([x, y]) \cup \mathrm{Stab}([x, z])$. By definition, there exists a constant $a \in k^*$ such that $x \circ f = ax$. Naturally, f preserves the fibration $\pi_x : \mathcal{Q} \to \mathbb{A}^1$ and since $\pi_x^{-1}(\mathbb{A}^1 \setminus \{0\})$ is isomorphic to $\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2$, the automorphism f is of the form

$$f : (x, y, z) \to (ax, f_1, f_2),$$

where (f_1, f_2) defines an element of $\mathrm{Aut}(\mathbb{A}^2_{k[x,x^{-1}]})$.

By Proposition 2.10, f induces an action on the tree $\mathcal{T}_{\pi_x, k(x)}$. If f induces an action on this tree which fixes the three vertices $[y]$, $[z]$ and $[y, z]$, then f belongs to $A^S_{[x]}$ where S is the 2×2 square containing $[x], [y], [z], [t]$. By Proposition 2.10(iv), f is then of the form

$$\begin{pmatrix} ax & b(y + xP(x)) \\ b^{-1}(z + xS(x)) & a^{-1}(t + zP(x) + yS(x) + xP(x)S(x)) \end{pmatrix}$$

where $P, S \in k[x] \setminus k$. By Lemma 3.6 (ii), the sequences $(\deg(f^n))$ and $\deg(f^{-n}))$ are bounded and f satisfies assertion (i) since in the fixed trivialization, f is of the form

$$(x, y, z) \mapsto (ax, b(y + xP(x)), b^{-1}(z + xS(x))).$$

Recall that the vertices of type II in the Bass–Serre tree $\mathcal{T}_{\pi, k(x)}$ were equivalence classes of components (f_1, f_2) of automorphisms in $\mathrm{Aut}(\mathbb{A}^2_{k(x)})$ where two components $(f_1, f_2) \simeq (g_1, g_2)$ if and only if there exists

$$\begin{pmatrix} a & b & c \\ a' & b' & c' \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(k(x))$$

such that $(g_1, g_2) = (af_1 + bf_2 + c, a'f_1 + b'f_2 + c')$.

Suppose that f, f^2 are not conjugate in $\mathrm{Stab}([x])$ to elements in $A^{(y,z)}_{[x]}$ and the action of f on the subtree $\mathcal{T}_{\pi, k(x)}$ of $\mathcal{T}_{k(x)}$ fixes a vertex. If the fixed vertex in the tree $\mathcal{T}_{\pi, k(x)}$ is of type II, then using Proposition 2.10(i) we can suppose that f fixes the vertex given by $[y, z]$. In particular, this implies that f is conjugated to

$$\begin{pmatrix} ax & b(y{+}xP(x,z)) \\ b^{-1}(z{+}xR(x)) & a^{-1}(t{+}zP(x,z){+}yR(x)) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} ax & b^{-1}(z{+}xR(x)) \\ b(y{+}xP(x,z)) & a^{-1}(t{+}zP(x,z){+}yR(x)) \end{pmatrix}$$

with $P \in k[x, z] \setminus k$ where $\deg_z(P) = 1$ and $R \in k[x]$. Using Lemma 3.6(iii) and (iv), the sequences $\deg(f^n)$ and $\deg(f^{-n})$ are bounded in the first case and grow linearly in the second. In the first case, f satisfies assertion (i) and f satisfies assertion (ii) in the second.

If f, f^2 are not conjugate in $\mathrm{Stab}([x])$ to elements in $A^S_{[x]}$ and the action f on $\mathcal{T}_{\pi,k(x)}$ fixes a vertex of type I but no vertices of type II, then using Proposition 2.10(i), f is conjugate (in $\mathrm{Stab}([x])$) to an element which fixes the vertex $[z]$ in the Bass–Serre tree, hence it is an element of the subgroup $\tilde{E}$ defined in assertion (v) of Proposition 2.10. This shows that f is of the form

$$\begin{pmatrix} ax & b(y+xP(x,z)) \\ b^{-1}(z+xR(x)) & a^{-1}(t+z^2P(x)+yR(x)) \end{pmatrix}$$

with $P \in k[x,y]$, $R \in k[x] \setminus k$. By Lemma 3.6(v), the degrees of both f^n and f^{-n} are bounded and f satisfies assertion (i).

The remaining case is when the action on the subtree $\mathcal{T}_{\pi,k(x)}$ is hyperbolic. By Proposition 2.12, f is conjugated to a composition of elements of the form

$$\begin{pmatrix} ax & b(z+xP(x,y)) \\ b^{-1}(y+xR(x)) & a^{-1}(t+zR(x)+yP(x,y)+xP(x,y)R(x)) \end{pmatrix},$$

where $R \in k[x]$ and $P \in k[x,y]$ such that $\deg_y(P) \geqslant 2$. By Lemma 3.6 (vi), the degree sequences $(\deg(f^n))$, $(\deg(f^{-n})$ satisfy

$$d^n \leqslant \deg(f^{\pm n}) \leq Cd^n,$$

where $d \geq 2$ is an integer and f satisfies assertion (iii). □

4. Valuative estimates

This section is devoted to the generalization of the so-called parachute inequalities; see [Bisi et al. 2014, Minoration A.2]. Our proof extends the method of [Lamy and Vénéreau 2013] to more general valuations. The plan of the section is as follows. First we recall some general facts on valuations (Section 4A), then we consider a particular class of valuations in Section 4B. For these particular valuations, we introduce the parachute associated to a pair of regular functions on the quadric allowing us to estimate the degree of a derivative on a given direction (Section 4C). Using this and some elementary facts on key polynomials (Section 4D), we finally deduce our key estimates in Section 4E.

4A. ***Valuations on affine and projective varieties.*** Let X be an affine variety of dimension n over k. By convention for us, a valuation on X is a map $\nu : k[X] \to \mathbb{R} \cup \{+\infty\}$ which satisfies the following properties:

(1) We have $\nu^{-1}(\{+\infty\}) = \{0\}$.

(2) The function ν is not constant on $k[X] \setminus \{0\}$.

(3) For any $a \in k^*$, one has $\nu(a) = 0$.

(4) For any $f_1, f_2 \in k[X]$, one has $\nu(f_1 f_2) = \nu(f_1) + \nu(f_2)$.

(5) For any $f_1, f_2 \in k[X]$, one has $\nu(f_1 + f_2) \geqslant \min(\nu(f_1), \nu(f_2))$.

When the subset $\nu^{-1}(\{+\infty\})$ is not reduced to $\{0\}$, we say that ν is a semivaluation. We endow the space of valuations with the coarsest topology for which all evaluation maps $\nu \mapsto \nu(f)$ are continuous where $f \in k[X]$.

The group $\mathbb{R}_+^*$ naturally acts on the set of valuations by multiplication.

The main examples of valuations are monomial valuations. We recall their definition below. Fix a smooth point p on X, an algebraic system of (local) coordinates $u = (u_0, \dots, u_{n-1})$ at this point and some nonnegative weights $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{R}^+)^n$. We shall denote by $u^I = \prod_{j=0}^n u_j^{i_j}$ when $I = (i_0, \dots, i_{n-1}) \in \mathbb{N}^n$ and by $\langle I, \alpha\rangle = \alpha_0 i_0 + \cdots + \alpha_{n-1} i_{n-1}$ the usual scalar product. The monomial valuation ν with weight α with respect to the system of coordinates u is defined by

$$\nu\Big(\sum_{I \in \mathbb{N}^n} a_I u^I\Big) = \min\{\langle I, \alpha\rangle \mid a_I \neq 0\},$$

where $a_I \in k$.

When $f \in \mathcal{O}_{p,X}$ is a regular function at the point p, then one defines $\nu(f)$ as

$$\nu(f) = \nu\Big(\sum a_I(f) u^I\Big),$$

where $\sum a_I(f) u^I$ is a formal expansion of f near p. The fact that $\nu(f)$ does not depend on the choice of the formal expansion of f near p is proved in [Jonsson and Mustaţă 2012, Proposition 3.1].

Observe that when $\alpha = (1, 0, \dots, 0)$, then the associated valuation coincides with the order of vanishing along $\{u_0 = 0\}$. Furthermore, when $X = \operatorname{Spec}(k[x, y, z, t])$, the valuation $-\deg$ coincides with the monomial valuation on $k[x, y, z, t]$ with weight $(-1, -1, -1, -1)$ with respect to (x, y, z, t) using the same formula.

Consider a regular morphism $f : X \to Y$ where Y is an affine variety and a valuation ν on X. The pushforward of the valuation ν on X by f is denoted $f_*\nu$ is given by the formula

$$f_*\nu = \nu \circ f^\sharp,$$

where $f^\sharp$ denotes the morphism of k-algebra corresponding to f.

We also recall the notion of center of a valuation ν.

For any projective variety $\overline{X}$ containing X as a Zariski open subset, when there exists a regular function $P \in k[X]$ for which $\nu(P) < 0$, the center of ν in $\overline{X}$ is a nonempty Zariski closed irreducible subset which is contained in $\overline{X} \setminus X$. Denote by R_ν the valuation ring and by $\mathcal{M}_\nu$ its maximal ideal, then the center $Z(\nu)$ of ν is a subvariety of $\overline{X}$ defined as follows

$$Z(\nu) = \{p \in \overline{X} \mid \mathcal{O}_{p,\overline{X}} \subset R_\nu, \mathcal{M}_{p,\overline{X}} = \mathcal{M}_\nu \cap \mathcal{O}_{p,\overline{X}}\},$$

where $\mathcal{O}_{p,\overline{X}}$ denotes the local ring of regular functions at the point p and where $\mathcal{M}_{p,\overline{X}}$ is its maximal ideal. The fact that $Z(\nu)$ is nonempty follows from the valuative criterion of properness and we shall refer to [Vaquié 2000] for the general properties of this set.

4B. ***Valuations $\mathcal{V}_0$ on the quadric.*** We denote by $q \in k[x, y, z, t]$ the polynomial $q = xt - yz$ and by $\pi : k[x, y, z, t] \to \mathrm{k}[\mathcal{Q}]$ the canonical projection. Our objective is to define a subset of the set of all valuations on the quadric $\mathcal{Q}$, with different weights on some coordinate axis.

Take a point $p = (x_0, y_0, z_0, t_0) \in k^4$ and a weight $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in (\mathbb{R}^-)^4$, we write by ν_p^α the monomial valuation on $k[x, y, z, t]$ with weight α with respect to the system of coordinates $(x - x_0, y - y_0, z - z_0, t - t_0)$.

We first show that ν_p^α does not depend on p.

Lemma 4.1. *For any weight $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in (\mathbb{R}^-)^4$, we have $\nu_p^\alpha = \nu_{p'}^\alpha$ for any $p, p' \in k^4$.*

Proof. For any multiindices $I = (i_0, i_1, i_2, i_3)$, $J = (j_0, j_1, j_2, j_3) \in \mathbb{Z}^4_{\geq 0}$, denote by $\binom{I}{J} := \Pi_{s=0}^{s} \binom{i_s}{j_s}$.

Set $\nu^\alpha := \upsilon_0^\alpha$. We only need to show that for every polynomial $P(x) \in k[x_0, x_1, x_2, x_3]$ and $b = (b_0, b_1, b_2, b_3) \in k^4$, $\nu^\alpha(P(x)) = \nu^\alpha(P(x + b))$. We may assume that $P \neq 0$: Write $P(x) = \sum_I a_I x^I$,

Then $\nu^\alpha(P) = \min\{\langle I, \alpha\rangle a_I \neq 0\}$. Then

$$P(x+b) = \sum_I a_I (x+b)^I = \sum_I \sum_{J \leq I} \binom{I}{J} b^{I-J} a_I x^J = \sum_J \Bigg(\sum_{I \geqslant J} a_I \binom{I}{J} b^{I-J} \Bigg) x^J. \tag{22}$$

Then

$$\nu^\alpha(P(x+b)) = \min\Bigg\{ \langle J, \alpha\rangle \;\Big|\; \sum_{I \geqslant J} \binom{I}{J} a_I b^{I-J} \neq 0 \Bigg\}. \tag{23}$$

If $\sum_{I \geqslant J} \binom{I}{J} a_I b^{I-J} \neq 0$, there is $I \geq J$ such that $a_I \neq 0$. Since $\alpha \leqslant 0$, $\langle I, \alpha\rangle \leqslant \langle J, \alpha\rangle$. This implies that

$$\nu^\alpha(P(x+b)) \geq \nu^\alpha(P(x)). \tag{24}$$

for every $P \in k[x_0, x_1, x_2, x_3] \setminus \{0\}$ and $b \in k^4$. Apply this for $P(x+b) \in k[x_0, x_1, x_2, x_3] \setminus \{0\}$ and $-b \in k^4$, we get $\nu^\alpha(P(x)) = \nu^\alpha(P(x+b-b)) \geq \nu^\alpha(P(x+b))$ which concludes the proof. □

Set $\nu^\alpha := \nu_0^\alpha$.

Proposition 4.2. *For any weight $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in (\mathbb{R}^- \setminus \{0\})^4$ such that $\alpha_0 + \alpha_3 = \alpha_2 + \alpha_1$, the map $\nu : \mathrm{k}[\mathcal{Q}] \to \mathbb{R}^- \cup \{+\infty\}$ given by*

$$\nu(f) := \sup\{\nu^\alpha(R) \mid R \in k[x, y, z, t], \pi(R) = f\},$$

for any $f \in \mathrm{k}[\mathcal{Q}]$ is a valuation on the quadric which is centered at infinity (i.e., whose center is in $\bar{\mathcal{Q}} \setminus \mathcal{Q} \subset \mathbb{P}^4$).

Moreover, suppose $\nu' : \mathrm{k}[\mathcal{Q}] \to \mathbb{R}^- \cup \{+\infty\}$ is a valuation such that $\nu(\pi(x)) = \nu'(\pi(x))$, $\nu(\pi(y)) = \nu'(\pi(y))$, $\nu(\pi(z)) = \nu'(\pi(z))$ and $\nu(\pi(t)) = \nu'(\pi(t))$, then

$$\nu'(f) \geqslant \nu(f),$$

for any regular function $f \in \mathrm{k}[\mathcal{Q}]$.

Definition 4.3. The set $\mathcal{V}_0$ is set of all valuations $\nu : \mathrm{k}[\mathcal{Q}] \to \mathbb{R}^- \cup \{+\infty\}$ defined by

$$\nu(f) := \sup\{\nu^\alpha(R) \mid \pi(R) = f\},$$

for any $f \in \mathrm{k}[\mathcal{Q}]$ and where $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in (\mathbb{R}^- \setminus \{0\})^4$ is a multiindex for which $\alpha_0+\alpha_3=\alpha_1+\alpha_2$.

The group $\mathbb{R}^{+,*}$ acts naturally by multiplication on the set of valuations on the quadric and this action descends on an action on $\mathcal{V}_0$.

Remark 4.4. Observe that for $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -1$, the corresponding valuation on the quadric is the order of vanishing along the hyperplane at infinity.

Example 4.5. Consider $\alpha = \left(-\frac{1}{2}, -\frac{3}{5}, -\frac{9}{10}, -1\right)$, then the associated valuation ν is the monomial valuation at the point $[0, 0, 0, 1, 0] \in \overline{\mathcal{Q}}$ with weight $\left(\frac{2}{5}, \frac{1}{10}, 1\right)$ with respect to the coordinate chart $(u, v, w) \mapsto [w^2 + uv, u, v, 1, w] \in \overline{\mathcal{Q}}$. In particular, its center is the point $[0, 0, 0, 1, 0] \in \overline{\mathcal{Q}}$.

Example 4.6. Consider $\alpha = \left(-\frac{1}{2}, -\frac{3}{5}, -\frac{9}{10}, -1\right)$, then the associated valuation ν is the monomial valuation at the point $[6, 2, 3, 1, 0] \in \overline{\mathcal{Q}}$ with weight $\left(\frac{2}{5}, \frac{1}{10}, 1\right)$ with respect to the coordinate chart $(u, v, w) \mapsto [w^2 + (2+u)(3+v), 2+u, 3+v, 1, w] \in \overline{\mathcal{Q}}$. In particular, its center is the point $[6, 2, 3, 1, 0] \in \overline{\mathcal{Q}}$.

To prove the proposition, we shall need the following technical lemma.

Lemma 4.7. *Let $\nu' : k[x, y, z, t] \to \mathbb{R}^- \cup \{+\infty\}$ be a monomial valuation with respect to (x, y, z, t) such that $\nu'_{|k[x,y,z,t]\setminus k} < 0$ and such that $\nu'(xt) = \nu'(yz)$. For any polynomial $R \in k[x, y, z, t]$ given by*

$$R = \sum_{ijmn} a_{ijmn} x^i y^j z^m t^n,$$

with $a_{ijmn} \in k$, the following assertions are equivalent:

(i) *There exists a polynomial $R_1 \in k[x, y, z, t]$ such that $\pi(R_1) = \pi(R) \in \mathrm{k}[\mathcal{Q}]$ and such that $\nu'(R_1) > \nu'(R)$.*

(ii) *The polynomial q divides R^w where R^w is the weighted homogeneous polynomial given by*

$$R^w = \sum_{i\nu'(x)+j\nu'(y)+m\nu'(z)+n\nu'(t)=\nu'(R)} a_{ijmn} x^i y^j z^m t^n.$$

Proof. The implication (ii) $\Rightarrow$ (i) is straightforward. If $q \mid R^w$ then we can decompose R as

$$R = qR_1 + S,$$

where $R_1, S \in k[x, y, z, t]$ such that $\nu'(S) > \nu'(qR_1)$. Hence $\pi(R_1 + S) = \pi(R)$ and $\nu'(R_1 + S) \geqslant \min(\nu'(R_1), \nu'(S)) > \nu'(R)$ as required.

Let us prove the implication (i) $\Rightarrow$ (ii). Take a polynomial R_1 which satisfies (i). Then we can write

$$R_1 = R + (q - 1)S,$$

where $S \in k[x, y, z, t]$. Let us prove that $R^w + qS^w = 0$. As $\nu'(R_1) > \nu'(R)$, the above equality implies that $\nu'(qS) = \nu'(R)$. Since $\alpha_0 + \alpha_3 = \alpha_1 + \alpha_2$, the polynomial q is weighted homogeneous and we have

$(R+(q-1)S)^w = R^w + qS^w$. Let us suppose by contradiction that $R^w + qS^w \neq 0$. This implies that $\nu'(R_1^w) = \nu'(R^w + qS^w) = \nu'(R^w)$ which also contradicts our assumption. Hence $R^w + qS^w = 0$ and $q \mid R^w$ as required. □

The above lemma proves that the supremum $\nu(f)$ in Proposition 4.2 is a maximum which is reached on a value $R \in k[x, y, z, t]$ such that $\pi(R) = f$ and such that q does not divide R^w.

Proof of Proposition 4.2. Fix $\alpha \in (\mathbb{R}^- \setminus \{0\})^4$. Observe that for any $f_1 \in \mathrm{k}[\mathcal{Q}] \setminus \{0\}$, the value $\nu(f_1)$ is smaller or equal than 0. If $a \in k^*$, then by definition $\nu(a) = \nu'(a) = 0$.

Fix $f_1, f_2 \in \mathrm{k}[\mathcal{Q}]$ and let us prove that $\nu(f_1 + f_2) \geqslant \min(\nu(f_1), \nu(f_2))$. Take $R_1, R_2 \in k[x, y, z, t]$ such that $\nu'(R_1) = \nu(\pi(R_1))$ and $\nu'(R_2) = \nu(\pi(R_2))$.

As ν^α is a valuation on $k[x, y, z, t]$, we have by definition

$$\nu'(R_1 + R_2) \geqslant \min(\nu'(R_1), \nu'(R_2)) = \min(\nu(\pi(R_1)), \nu(\pi(R_2))).$$

In particular, the maximal value in the right hand side yields

$$\nu(f_1 + f_2) \geqslant \min(\nu(f_1), \nu(f_2)).$$

We prove that $\nu(\pi(f_1 f_2)) = \nu(\pi(f_1)) + \nu(\pi(f_2))$. Take two polynomials R_1 and $R_2 \in k[x, y, z, t]$ such that $\pi(R_1) = f_1$, $\pi(R_2) = f_2$ and $\nu(f_1) = \nu^\alpha(R_1)$, $\nu(f_2) = \nu^\alpha(R_2)$. Observe that $(R_1 R_2)^w = R_1^w R_2^w$. As the polynomial q does not divide either R_1^w or R_2^w, it does not divide $(R_1 R_2)^w$ since the ideal generated by q is a prime ideal. Hence by Lemma 4.7, one has $\nu(f_1 f_2) = \nu^\alpha(R_1 R_2) = \nu^\alpha(R_1^w) + \nu^\alpha(R_2^w) = \nu(f_1) + \nu(f_2)$ as required.

By construction, the valuation ν is centered at infinity since ν takes negative values on nonzero regular functions on the quadric.

Let us prove that the valuation ν is minimal, take another valuation $\nu' : \mathrm{k}[\mathcal{Q}] \to \mathbb{R}^- \cup \{+\infty\}$ such that $\nu'(\pi(x)) = \nu(x)$, $\nu'(\pi(y)) = \nu(\pi(y))$, $\nu'(\pi(z)) = \nu(\pi(z))$ and $\nu'(\pi(t)) = \nu(\pi(t))$. Then the map $\hat{\nu}' : R \in k[x, y, z, t] \to \nu'(\pi(R))$ defines a semivaluation on $k[x, y, z, t]$. Remark that the monomial valuation ν^α is minimal in $k[x, y, z, t]$, in the sense that for any $R \in k[x, y, z, t]$

$$\hat{\nu}'(R) \geqslant \nu^\alpha(R).$$

Take $f \in \mathrm{k}[\mathcal{Q}]$ and choose a polynomial $R \in k[x, y, z, t]$ such that $\nu_p^\alpha(R) = \nu(f)$, the above inequality implies

$$\nu'(f) \geqslant \nu(f),$$

hence, ν is also minimal. □

4C. ***Parachute.*** In this subsection, we define the parachute associated to a component of a tame automorphism. For any 4-tuple $(R_1, R_2, R_3, R_4) \in k[x, y, z, t]$ of polynomials, we write

$$dR_1 \wedge dR_2 \wedge dR_3 \wedge dR_4 = \mathrm{Jac}(R_1, R_2, R_3, R_4) dx \wedge dy \wedge dz \wedge dt,$$

with $\mathrm{Jac}(R_1, R_2, R_3, R_4) \in k[x, y, z, t]$.

Definition 4.8. The pseudojacobian of a triple (f_1, f_2, f_3) of regular functions on $\mathcal{Q}$ is defined by

$$j(f_1, f_2, f_3) := \mathrm{Jac}(q, R_1, R_2, R_3))_{|\mathcal{Q}},$$

where $R_i \in k[x, y, z, t]$ are polynomials such that $\pi(R_i) = f_i$ for $i = 1, 2, 3$.

Observe that the pseudojacobian $j(f_1, f_2, f_3)$ is well-defined since any two representatives $R_1, R_2 \in k[x, y, z, t]$ of the same equivalence class in $\mathrm{k}[\mathcal{Q}]$ are equal modulo $(q - 1)$.

Lemma 4.9. *Let $\nu \in \mathcal{V}_0$ be a valuation. For any $f_1, f_2, f_3 \in \mathrm{k}[\mathcal{Q}]$, we have*

$$\nu(j(f_1, f_2, f_3)) \geqslant \nu(f_1) + \nu(f_2) + \nu(f_3) - \nu(xt).$$

Proof. Fix $f_1, f_2, f_3 \in \mathrm{k}[\mathcal{Q}]$ and a valuation $\nu \in \mathcal{V}_0$. The valuation $\nu' : k[x, y, z, t] \to \mathbb{R}^- \cup \{+\infty\}$ is monomial for the coordinates (x, y, z, t) with weight $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in (\mathbb{R}^-)^4$ such that $\alpha_0 + \alpha_3 = \alpha_1 + \alpha_2$. We have $\nu(P) = \sup\{\nu'(R) \mid \pi(R) = P\}$ for any $P \in \mathrm{k}[\mathcal{Q}]$ where $\pi : k[x, y, z, t] \to \mathrm{k}[\mathcal{Q}]$ is the canonical projection. Take $R_1, R_2, R_3, R_4 \in k[x, y, z, t]$. We first claim that

$$\nu'(\mathrm{Jac}(R_1, R_2, R_3, R_4)) \geqslant \nu'(R_1) + \nu'(R_2) + \nu'(R_3) + \nu'(R_4) - \nu'(xyzt).$$

Let $a_I^{(k)} \in k$ be the coefficients of R_k for $k = 1, 2, 3, 4$ so that

$$R_k = \sum_{I=(i_1,i_2,i_3,i_4)} a_I^{(k)} x^{i_1} y^{i_2} z^{i_3} t^{i_4}.$$

One obtains by linearity that $\mathrm{Jac}(R_1, R_2, R_3, R_4)$ is a sum of monomials where the valuation of each term is greater or equal to

$$\nu'(R_1) + \nu'(R_2) + \nu'(R_3) + \nu'(R_4) - \nu'(xyzt).$$

Hence

$$\nu'(\mathrm{Jac}(R_1, R_2, R_3, R_4)) \geqslant \nu'(R_1) + \nu'(R_2) + \nu'(R_3) + \nu'(R_4) - \nu'(xyzt).$$

In particular, we apply to $R_4 = q$ and obtain

$$\nu'(\mathrm{Jac}(R_1, R_2, R_3, q)) \geqslant \nu'(R_1) + \nu'(R_2) + \nu'(R_3) - \nu'(xt),$$

since $\nu'(q) = \nu'(xt) = \nu'(yz)$. Take $f_1, f_2, f_3 \in \mathrm{k}[\mathcal{Q}]$, by Lemma 4.7, there exists $R_1, R_2, R_3 \in k[x, y, z, t]$ such that $\pi(R_i) = f_i \in \mathrm{k}[\mathcal{Q}]$ and $\nu(f_i) = \nu'(R_i)$ for all $i = 1, 2, 3$, the above inequality implies

$$\nu(j(f_1, f_2, f_3)) \geqslant \nu'(\mathrm{Jac}(q, R_1, R_2, R_3)) \geqslant \nu'(R_1) + \nu'(R_2) + \nu'(R_3) - \nu'(xt),$$

where the first inequality follows from the definition of ν. Observe that $\nu'(xt) = \nu(xt)$ by Lemma 4.7, hence we have proven that

$$\nu(j(f_1, f_2, f_3)) \geqslant \nu(f_1) + \nu(f_2) + \nu(f_3) - \nu(xt),$$

as required. ∎

The regular function $j(f_1, f_2, f_3)$ may vanish so that $\nu(j(f_1, f_2, f_3))$ may be equal to $+\infty$, even if $\nu \in \mathcal{V}_0$.

Lemma 4.10. *For any algebraically independent functions $f_1, f_2 \in \mathrm{k}[\mathcal{Q}]$, one of the four regular functions $j(x, f_1, f_2), j(y, f_1, f_2), j(z, f_1, f_2), j(t, f_1, f_2)$ is not identically zero. In particular,*

$$\min(\nu(j(x, f_1, f_2)), \nu(j(y, f_1, f_2)), \nu(j(z, f_1, f_2)), \nu(j(t, f_1, f_2))) < +\infty,$$

for any valuation $\nu \in \mathcal{V}_0$.

Proof. Consider two algebraically independent regular functions $f_1, f_2 \in \mathrm{k}[\mathcal{Q}]$ and suppose by contradiction that $j(x, f_1, f_2) = j(y, f_1, f_2) = j(z, f_1, f_2) = j(t, f_1, f_2) = 0$. If $K \subset L$ are two fields of characteristic zero, then [Lang 2002, Section VIII.5, Proposition 5.5] states that

$$\operatorname{trdeg}_K L = \dim_L \operatorname{Der}_K(L), \tag{25}$$

where $\operatorname{Der}_K(L)$ denotes the vector space of K derivations of L. When $K = k(f_1, f_2)$ and $L = k(\mathcal{Q})$, the above equality implies that any two $k(f_1, f_2)$-derivations are proportional. The conditions $j(x, f_1, f_2) = j(y, f_1, f_2) = j(z, f_1, f_2) = j(t, f_1, f_2) = 0$ imply that $j(x, f_1, \cdot), j(y, f_1, \cdot), j(z, f_1, \cdot)$ and $j(t, f_1, \cdot)$ are $k(f_1, f_2)$-derivations, this translates as

$$j(x, f_1, x) j(y, f_1, y) - j(x, f_1, y) j(y, f_1, x) = 0 \in \mathrm{k}[\mathcal{Q}].$$

Hence,

$$j(x, f_1, y) = 0 \in k(\mathcal{Q}).$$

The same argument also yields

$$j(f_1, x, y) = j(f_1, x, z) = j(f_1, x, t) = j(f_1, y, z) = j(f_1, y, t) = j(f_1, z, t) = 0.$$

Hence the maps $j(x, y, \cdot), j(x, z, \cdot), j(y, z, \cdot)$ are also $k(f_1)$-derivations. By (25) applied to $K = k(f_1)$ and to $L = k(\mathcal{Q})$, the space of $k(f_1)$ derivations is 2-dimensional and there exists $a, b, c \in k(\mathcal{Q})$ such that

$$a j(x, y, \cdot) + b j(x, z, \cdot) + c j(y, z, \cdot) = 0,$$

where a, b and c are not all equal to zero. Suppose that $a \neq 0$, we then conclude that

$$a j(x, y, z) = 0 \in k(\mathcal{Q}),$$

which in turn implies that $j(x, y, z) = x = 0 \in \mathrm{k}[\mathcal{Q}]$ and this is impossible. □

Definition 4.11. For any monomial valuation $\nu \in \mathcal{V}_0$ and for any algebraically independent regular functions $f_1, f_2 \in \mathrm{k}[\mathcal{Q}]$, the parachute $\nabla(f_1, f_2)$ with respect to the valuation ν is defined by the following formula

$$\nabla(f_1, f_2) = \min(\nu(j(x, f_1, f_2)), \nu(j(y, f_1, f_2)), \nu(j(z, f_1, f_2)), \nu(j(t, f_1, f_2))) - \nu(f_1) - \nu(f_2).$$

Observe that Lemmas 4.10 and 4.9 imply that $\nabla(f_1, f_2)$ is finite and is strictly greater than zero.

For any polynomial $R \in k[x, y]$, we write by $\partial_2 R \in k[x, y]$ the partial derivative with respect to y. The next identity is similar to [Lamy and Vénéreau 2013, Lemma 5] and is one of the main ingredient to find an upper bound on the value of a valuation.

Lemma 4.12. *Let $\nu \in \mathcal{V}_0$, let $R \in k[x, y] \setminus k$ and let $f_1, f_2 \in \mathrm{k}[Q]$ be two algebraically independent elements. Suppose that there exists an integer $1 \leqslant n \leqslant \deg_y R - 1$ such that $\nu(\partial_2^n R(f_1, f_2))$ is equal to the value on $\partial_2^n R$ of the monomial valuation in two variables having weight $\nu(f_1)$ and $\nu(f_2)$ on x and y respectively. Then*

$$\nu(R(f_1, f_2)) < \deg_y(R)\nu(f_2) + n\nabla(f_1, f_2).$$

Proof. Lemma 4.9 proves that $\nu(j(x, f_1, f_2)) \geqslant \nu(f_1) + \nu(f_2) - \nu(t)$ for any $f_1, f_2 \in \mathrm{k}[Q]$. Using this and the fact that $j(x, f_1, \cdot)$ is a derivation, we obtain

$$\nu(\partial_2 R(f_1, f_2) j(x, f_1, f_2)) = \nu(j(x, f_1, R(f_1, f_2))) \geqslant \nu(f_1) + \nu(R(f_1, f_2)) + \nu(x) - \nu(xt).$$

In particular since $\nu(x) - \nu(xt) = -\nu(t) > 0$, this yields

$$\nu(\partial_2 R(f_1, f_2)) > -(\nu(j(x, f_1, f_2)) - \nu(f_1) - \nu(f_2)) + \nu(R(f_1, f_2)) - \nu(f_2).$$

A similar argument with y, z and t also gives

$$\nu(\partial_2 R(f_1, f_2)) > -\nabla(f_1, f_2) + \nu(R(f_1, f_2)) - \nu(f_2). \tag{26}$$

We apply (26) inductively and obtain the following inequalities:

$$\begin{aligned} \nu(\partial_2^2 R(f_1, f_2)) &> -\nabla(f_1, f_2) + \nu(\partial_2 R(f_1, f_2)) - \nu(f_2), \\ &\vdots \\ \nu(\partial_2^n R(f_1, f_2)) &> -\nabla(f_1, f_2) + \nu(\partial_2^{n-1} R(f_1, f_2)) - \nu(f_2). \end{aligned}$$

This implies that

$$\nu(\partial_2^n R(f_1, f_2)) > -n\nabla(f_1, f_2) - n\nu(f_2) + \nu(R(f_1, f_2)).$$

As $\nu(\partial_2^n R(f_1, f_2))$ is equal to the value of the monomial valuation with weight $(\nu(f_1), \nu(f_2))$ applied to $\partial_2^n(R)$, the last inequality rewrites as

$$(\deg_y R - n)\nu(f_2) \geq \nu(\partial_2^n R(f_1, f_2)) > -n\nabla(f_1, f_2) - n\nu(f_2) + \nu(R(f_1, f_2)).$$

Hence,

$$\nu(R(f_1, f_2)) < \deg_y(R)\nu(f_2) + n\nabla(f_1, f_2),$$

as required. □

4D. ***Key polynomials.*** Let us explain how one can find a polynomial which satisfies the hypothesis of Lemma 4.12.

Consider $\mu : k[x, y] \to \mathbb{R}^- \cup \{+\infty\}$ any valuation and $\mu_0 : k[x, y] \to \mathbb{R}^- \cup \{+\infty\}$ the monomial valuation having weight $\mu(x)$ and $\mu(y)$ on x and y respectively. For any polynomial $R \in k[x, y]$, we write by $\bar{R} \in k[x, y]$ the weighted homogeneous polynomial given by

$$\bar{R} = \sum_{i\mu(x)+j\mu(y)=\mu_0(R)} a_{ij} x^i y^j,$$

with $a_{ij} \in k$ such that $R = \sum_{ij} a_{ij} x^i y^j$.

Proposition 4.13. *Consider $\mu : k[x, y] \to \mathbb{R}^- \cup \{+\infty\}$ any valuation and μ_0 the monomial valuation having weights $\mu(x)$ and $\mu(y)$ on x and y respectively. The following properties are satisfied*:

(i) *For any $R \in k[x, y]$, one has $\mu(R) \geqslant \mu_0(R)$.*

(ii) *If $\mu \neq \mu_0$, then there exists two coprime integers s_1, s_2 satisfying $s_1\mu(x) = s_2\mu(y)$ and a unique constant $\lambda \in k$ for which the polynomial $H = x^{s_1} - \lambda y^{s_2}$ satisfies $\mu(H) > \mu_0(H)$. Moreover, for any $R \in k[x, y]$, one has $\mu(R) > \mu_0(R)$ if and only if $H \mid \bar{R}$.*

The polynomial H associated to μ is called a key polynomial associated to μ.

Proof. Let us prove assertion (i). Write $R \in k[x, y]$ as $R = \sum a_{ij} x^i y^j$ where $a_{ij} \in k$. Recall that the fact that μ_0 is monomial implies that

$$\mu_0(R) = \min\{i\mu_0(x) + j\mu_0(y) \mid a_{ij} \neq 0\}.$$

Also, μ is a valuation, hence

$$\mu(R) \geqslant \min\{i\mu_0(x) + j\mu_0(y) \mid a_{ij} \neq 0\} = \mu_0(R).$$

We have thus proved that $\mu(R) \geqslant \mu_0(R)$, as required.

Step 1: Fix s_1, s_2 two coprime integers and $\lambda \in k$. Suppose that $s_1\mu(x) = s_2\mu(y)$ and that the polynomial $H = x^{s_1} - \lambda y^{s_2}$ satisfies $\mu(H) > \mu_0(H)$, we prove that λ is unique. Take $\lambda' \neq \lambda \in k$, then

$$\mu(x^{s_1} - \lambda' y^{s_2}) = \mu(H + (\lambda - \lambda') y^{s_2}) = s_2\mu(y),$$

since $\mu(H) > \mu((\lambda - \lambda') y^{s_2})$. Hence $\mu(x^{s_1} - \lambda' y^{s_2}) = \mu_0(x^{s_1} - \lambda' y^{s_2})$ for any $\lambda' \neq \lambda$.

Step 2: Choose two integers s_1, s_2 such that $s_1\mu(x) = s_2\mu(y)$. We prove that there exists $\lambda \in k^*$ such that $\mu(x^{s_1} - \lambda y^{s_2}) > s_1\mu(x) = s_2\mu(y)$. Suppose by contradiction that for any $\lambda \in k$, one has $\mu(x^{s_1} - \lambda y^{s_2}) = s_1\mu(x)$. We claim that $\mu(R) = \mu_0(R)$ for any polynomial $R \in k[x, y]$. Fix $R \in k[x, y]$. Observe that if R is a homogeneous polynomial with respect to the weight $(\mu(x), \mu(y))$, then R is of the form

$$R = \alpha x^{k_0} \prod_i (x^{s_1} - \lambda_i y^{s_2})$$

where $\alpha, \lambda_i \in k^*$ and $k_0 \in \mathbb{N}$. Our assumption implies that $\mu(R) = \mu_0(R)$ for any homogeneous polynomial R.

If R is a general polynomial, then R can be decomposed into $R = \sum_i R_i$ where each polynomial R_i is homogeneous with respect to the weight $(\mu(x), \mu(y))$. Since $\mu(R_i) = \mu_0(R_i)$ for each i, this proves that $\mu(R) = \mu_0(R)$ for any $R \in k[x, y]$, which contradicts our assumption. We have thus proven the first part of assertion (ii).

Step 3: We prove the second part of assertion (ii). The argument in Step 2 showed that when $\mu \neq \mu_0$ and R is homogeneous with respect to the weight $(\mu(x), \mu(y))$, then $\mu(R) > \mu_0(R)$ if $H \mid R$. For a general polynomial $R \in k[x, y]$, write $R = \bar{R} + S$. We have

$$\mu_0(R) = \mu_0(\bar{R}) < \mu_0(S) \leqslant \mu(S).$$

If $H \mid \bar{R}$, then $\mu(\bar{R}) > \mu_0(\bar{R}) = \mu_0(R)$. Then $\mu(R) = \mu(\bar{R} + S) \geqslant \min(\mu(\bar{R}), \mu(S)) > \mu_0(R)$. If $H \nmid \bar{R}$, we have $\mu(\bar{R}) = \mu_0(\bar{R}) < \mu_0(S) \leqslant \mu(S)$. So $\mu(R) = \mu(\bar{R} + S) = \mu(\bar{R}) = \mu_0(\bar{R}) = \mu_0(R)$. □

4E. *Parachute inequalities.* We introduce various notions of resonances of components of a tame automorphism. These notions will play an important role in the theorem below. Consider a valuation $\nu \in \mathcal{V}_0$ and a component (f_1, f_2) of a tame automorphism. We are interested in the value of ν on $R(f_1, f_2)$ where $R \in k[x, y]$. The estimates of the value $\nu(R(f_1, f_2))$ will depend on the possible values of the pair $(\nu(f_1), \nu(f_2))$. We shall distinguish the following three cases:

(1) The family $(\nu(f_1), \nu(f_2))$ is $\mathbb{Q}$-independent and we say that the component (f_1, f_2) is *nonresonant* with respect to ν.

(2) There exists two coprime integers s_1, s_2 such that $s_1 > s_2 \geqslant 2$ or $s_2 > s_1 \geqslant 2$ such that $s_1\nu(f_1) = s_2\nu(f_2)$ and we say in this case that the component (f_1, f_2) is *properly resonant* with respect to ν.

(3) Either $\nu(f_1)$ is a multiple of $\nu(f_2)$ or $\nu(f_2)$ is a multiple of $\nu(f_1)$ and there exists a polynomial $H \in k[x, y]$ of the form $x - \lambda y^k$ where $k \in \mathbb{N}^*$, $\lambda \in k^*$ such that $\nu(H(f_1, f_2)) > \nu(f_1) = k\nu(f_2)$. In this case, the component (f_1, f_2) is called *critically resonant* with respect to ν.

Example 4.14. When $\nu = -\deg : k[Q] \to \mathbb{R}^- \cup \{+\infty\}$, the family (x, y) is not critically resonant, but it is neither properly resonant nor nonresonant (in particular there is no alternative). However, (x, y) is nonresonant for the monomial valuation with weight $(-\sqrt{2}, -\sqrt{3}, -\sqrt{2}, -\sqrt{3})$ on (x, y, z, t).

Example 4.15. Take $f_1 = x$, $f_2 = y + x^2 \in k[Q]$, then (f_1, f_2) is critically resonant with respect to the valuation $\mathrm{ord}_{H_\infty} = -\deg$.

Example 4.16. Take $f_1 = z + x^2$, $f_2 = y + x^3 \in k[Q]$, then (f_1, f_2) is properly resonant with respect the valuation $\mathrm{ord}_{H_\infty} = -\deg$.

For $\nu \in \mathcal{V}_0$ and (f_1, f_2) a component of a tame automorphism, the following theorem allows us to estimate the value of ν on $R(f_1, f_2)$ only when (f_1, f_2) is not critically resonant.

Theorem 4.17. *Let $\nu \in \mathcal{V}_0$ be a valuation and let ν_0 be the monomial valuation on $k[x, y]$ with weight $(\nu(f_1), \nu(f_2))$ with respect to (x, y). The following assertions hold*:

(i) *For any polynomial $R \in k[x, y]$, one has the lower bound $\nu(R(f_1, f_2)) \geqslant \nu_0(R(x, y))$.*

(ii) *If the component (f_1, f_2) is nonresonant with respect to ν, then for any polynomial $R \in k[x, y]$, one has $\nu(R(f_1, f_2)) = \nu_0(R(x, y))$.*

(iii) *Suppose that the component (f_1, f_2) is properly resonant with respect to ν and let s_1, s_2 be two coprime positive integers such that $s_1\nu(f_1) = s_2\nu(f_2)$, then for any polynomial $R \in k[x, y]$, either $\nu(R(f_1, f_2)) = \nu_0(R(x, y))$ or $\nu(R(f_1, f_2)) > \nu_0(R(x, y))$ and we have*

$$\nu(R(f_1, f_2)) < \left(s_1 - 1 - \frac{s_1}{s_2}\right)\nu(f_1) = \left(s_2 - 1 - \frac{s_2}{s_1}\right)\nu(f_2).$$

Remark that the inequalities in Theorem 4.17 are strict and this fact is crucial in our proof. Before giving the proof of Theorem 4.17, we state two consequences of this theorem below.

Corollary 4.18. *Let $\nu \in \mathcal{V}_0$ be a monomial valuation and let $f = (f_1, f_2, f_3, f_4)$ be an element of* Tame(Q). *We suppose that $\nu(f_1) < \nu(f_2)$ and that (f_1, f_2) is not critically resonant with respect to ν. Then for any polynomial $R \in k[x, y] \setminus k[y]$, we have*

$$\nu(f_2 R(f_1, f_2)) < \nu(f_1).$$

Proof. Two cases appear. Either $\nu(R(f_1, f_2)) = \nu_0(R(x, y))$ where ν_0 is the monomial valuation with weight $(\nu(f_1), \nu(f_2))$ with respect to (x, y), and we are finished since $R \in k[x, y] \setminus k[y]$. Or $\nu(R(f_1, f_2)) > \nu_0(R(x, y))$ and there exists some integers s_1, s_2 such that $s_1\nu(f_1) = s_2\nu(f_2)$ where $s_2 > s_1 \geqslant 2$. Using Theorem 4.17(iii) and the fact that $s_1 \geqslant 2$, we have thus

$$\nu(f_2 R(f_1, f_2)) < (s_1 - 1)\nu(f_1) < \nu(f_1),$$

as required. □

We state the second corollary for which the constant $\frac{4}{3}$ appears naturally.

Corollary 4.19. *Let $\nu \in \mathcal{V}_0$ be a valuation and let (f_1, f_2) a properly resonant component with respect to ν such that $\nu(f_1) < \nu(f_2)$. Then for any polynomial $R \in k[x, y] \setminus k[y]$, one has*

$$\nu(f_1 R(f_1, f_2)) < \tfrac{4}{3}\nu(f_1).$$

Similarly,

$$\nu(f_2 R(f_1, f_2)) < \tfrac{3}{2}\nu(f_2) < \tfrac{4}{3}\nu(f_2).$$

Proof. Denote by $\nu_0 : k[x, y] \to \mathbb{R}^- \cup \{+\infty\}$ the monomial valuation with weight $(\nu(f_1), \nu(f_2))$ with respect to (x, y). Two cases appear, either $\nu(R(f_1, f_2)) = \nu_0(R(x, y))$ and we are done since $\nu(f_1 R(f_1, f_2)) \leqslant 2\nu(f_1)$ as $R \in k[x, y] \setminus k[y]$ or $\nu(R(f_1, f_2)) > \nu_0(R(x, y))$. In the latter case, consider two coprime integers s_1, s_2 such that $s_1\nu(f_1) = s_2\nu(f_2)$. Since $\nu(f_1) < \nu(f_2)$ and the component (f_1, f_2) is properly resonant, the inequality $s_2 > s_1 \geqslant 2$ holds. Using Theorem 4.17(iii), we obtain

$$\nu(f_1 R(f_1, f_2)) < \Big(s_1 - \frac{s_1}{s_2}\Big)\nu(f_1).$$

Since $s_2 \geqslant 3$ and $s_1 \geq 2$, we have $s_1 - s_1/s_2 = s_1(1 - 1/s_2) \geq 2\big(1 - \frac{1}{3}\big) = \frac{4}{3}$. Since $\nu(f_1) < 0$, we get

$$\nu(f_1 R(f_1, f_2)) < \Big(s_1 - \frac{s_1}{s_2}\Big)\nu(f_1) \leqslant \tfrac{4}{3}\nu(f_1),$$

as required. The second inequality follows from a similar argument $\nu(f_2 R(f_1, f_2)) < \frac{3}{2}\nu(f_2) < \frac{4}{3}\nu(f_2)$. $\square$

Proof of Theorem 4.17. Let us denote by $R = \sum a_{ij} x^i y^j$. Consider the projection $\pi_{xy} : \mathcal{Q} \to \mathbb{A}^2$ induced by the embedding of $\mathcal{Q}$ into $\mathbb{A}^4$ composed with the projection onto $\mathbb{A}^2$ of the form

$$\pi_{xy} : (x, y, z, t) \in \mathcal{Q}(k) \mapsto (x, y).$$

Choose an automorphism f such that $f = (f_1, f_2, f_3, f_4)$ where $f_3, f_4 \in \mathrm{k}[\mathcal{Q}]$. We denote by μ the valuation on $k[x, y]$ given by $\mu = \pi_{xy*} f_* \nu$.

Observe that for any polynomial $R \in k[x, y]$, we have $\nu(R(f_1, f_2)) = \mu(R(x, y))$ and assertion (i) follows directly from Proposition 4.13(i). Observe also that assertion (ii) follows immediately from the fact that $\nu(f_1)$ and $\nu(f_2)$ are $\mathbb{Q}$-independent.

Let us prove assertion (iii). We can suppose by symmetry that $\nu(f_1) < \nu(f_2)$. Since the component (f_1, f_2) is properly resonant, there exists two coprime integers s_1, s_2 such that $s_1\nu(f_1) = s_2\nu(f_2)$ and such that $s_2 > s_1 \geqslant 2$.

By Proposition 4.13 applied to μ, there exists $\lambda \in k^*$ such that the polynomial $H = x^{s_1} - \lambda y^{s_2}$ satisfies

$$\mu(H(x, y)) = \nu(H(f_1, f_2)) > \nu_0(H) = s_1\nu(f_1).$$

For any polynomial $R \in k[x, y]$, denote by $\bar{R}$ be the polynomial given by

$$\bar{R} = \sum_{i\mu(x)+j\mu(y)=\nu_0(R(x,y))} a_{ij} x^i y^j.$$

By construction, we have that there exists an integer $n \geqslant 1$ such that $\bar{R} \in (H^n) \setminus (H^{n+1})$.

We shall use the following lemma (proved at the end of this section):

Lemma 4.20. *Let $R \in k[x, y]$ such that $H \mid \bar{R}$. Consider the integer $n = \max\{k \mid H^k$ **divides** $\bar{R}\} \geqslant 1$. Then the following properties are satisfied*:

(i) *For any integer $k \leqslant n$, we have $\overline{\partial_2^k(R)} = \partial_2^k \bar{R}$.*

(ii) *For any integer $k \leqslant n$, we have $H^{n-k} \mid \partial_2^k \bar{R}$ but $H^{n-k+1} \nmid \partial_2^k \bar{R}$.*

The above lemma implies that $\overline{\partial_2^k R} = \partial_2^k \bar{R}$ and that $H^{n-k} \mid \partial_2^k \bar{R}$ but $H^{n-k+1} \nmid \partial_2^k \bar{R}$ for any $k \leq n$. In particular, H does not divide $\partial_2^n \bar{R}$ and Proposition 4.13(ii) implies that

$$\mu(\partial_2^n R(x, y)) = \nu_0(\partial_2^n R) = \nu_0(\overline{\partial_2^n R}).$$

The previous equation translates as

$$\nu((\partial_2^n R)(f_1, f_2)) = \nu_0(\partial_2^n R)$$

and R satisfies the conditions of Lemma 4.12 (for the same integer n), which in turn asserts that

$$\nu(R(f_1, f_2)) < \deg_y(R)\nu(f_2) + n\nabla(f_1, f_2).$$

Since $H^n \mid \bar{R}$, one has $\deg_y(R) \geqslant \deg_y(\bar{R}) \geqslant s_2 n$, we get

$$\nu(R(f_1, f_2)) < n(s_2\nu(f_2) + \nabla(f_1, f_2)).$$

As $n \geqslant 1$ and $\nabla(f_1, f_2) \leqslant -\nu(f_1) - \nu(f_2)$ by Lemma 4.10, the above implies that

$$\nu(R(f_1, f_2)) < s_2\nu(f_2) - \nu(f_1) - \nu(f_2).$$

Since $s_1\nu(f_1) = s_2\nu(f_2)$, we finally prove that

$$\nu(R(f_1, f_2)) < \nu(f_1)\left(s_1 - 1 - \frac{s_1}{s_2}\right),$$

as required. □

Proof of Lemma 4.20. Consider a monomial valuation $\nu_0 : k[x, y] \to \mathbb{R}^- \cup \{+\infty\}$ with weight $(\alpha, \beta) \in (\mathbb{R}^{-,*})^2$ with respect to (x, y) and $H = x^{s_1} - \lambda y^{s_2}$ where s_1, s_2 are coprime integers such that $s_1\alpha = s_2\beta$.

Let us prove assertion (i) for $k = 1$. Fix $R \in k[x, y]$ and write R as

$$R = \sum_{ij} a_{ij} x^i y^j,$$

where $a_{ij} \in k$. The partial derivative is given explicitly by

$$\partial_2 R = \sum_{i \geq 0, j \geq 1} j a_{ij} x^i y^{j-1}.$$

Since $H \mid \bar{R}$, one has $\bar{R} \in k[x, y] \setminus k[x]$ and $\nu_0(R) = \nu_0(\bar{R})$. Take (i, j) such that $a_{ij} \neq 0$ and $i\alpha + (j-1)\beta = \nu_0(\partial_2 R)$. Then $i\alpha + j\beta = \nu_0(\partial_2 R) + \nu_0(y) \leq \nu_0(R)$. Conversely, since $H \mid \bar{R}$, there exists (i, j) such that $i\alpha + j\beta = \nu_0(R)$ where $j \geq 1$, hence we have that $i\alpha + (j-1)\beta \geq \nu_0(\partial_2 R)$. Hence, $\nu_0(\partial_2 R) = \nu_0(R) - \beta$ and $\overline{\partial_2 R} = \partial_2 \bar{R}$.

Let us prove assertion (ii) for $k = 1$. We have that $H^n \mid \bar{R}$ but $H^{n+1} \nmid \bar{R}$, then we have

$$\bar{R} = H^n S,$$

where $S \in k[x, y]$ is a homogeneous polynomial such that $H \nmid S$. By definition,

$$\partial_2 \bar{R} = ns_2 H^{n-1} y^{s_2-1} S + H^n \partial_2 S.$$

Hence $H^{n-1} \mid \partial_2 \bar{R}$. Suppose by contradiction that $H^n \mid \partial_2 \bar{R}$, then this implies that $H \mid y^{s_2-1} S$ which is impossible since H does not divide S. We have thus proven that $H^{n-1} \mid \partial_2 \bar{R}$ but $H^n \nmid \partial_2 \bar{R}$, as required.

An immediate induction on $k \leqslant n$ proves assertion (i) and (ii). □

5. Proof of Theorem 1 and Theorem 4

This section is devoted to the proof of Theorems 1 and 4. The proof of these two results are very similar and rely on a lower bound of the degree of an automorphism f by $\left(\frac{4}{3}\right)^p$ where p is an integer that we determine.

Let us explain our general strategy. Take an automorphism $f \notin O_4$.

Step 1: We choose an appropriate valuation ν.

We consider a geodesic line γ in the complex joining [Id] and $[f]$. Recall from Proposition 2.3 that the set of 1×1 squares containing [Id] is in bijection with the points on the hyperplane at infinity $H_\infty \subset \mathcal{Q} \subset \mathbb{P}^4$. Depending on which 1×1 square the geodesic γ near the vertex [Id] is contained, we choose accordingly a valuation ν in $\mathcal{V}_0$ centered on the corresponding point at infinity in $\bar{\mathcal{Q}}$.

Step 2 (see Section 5A): We define an integer p according to the geometry of some geodesics in the complex and according to the choice of the valuation ν.

Recall that a path in the 1-skeleton of $\mathcal{C}$ induces a sequence of numbers obtained by evaluating the valuation ν on the consecutive vertices. The integer p is defined as the distance in a graph denoted $\mathcal{C}_\nu$ and encodes the shortest path in the 1-skeleton with minimal degree sequence.

Step 3: We prove that $\deg(f) \geqslant \left(\frac{4}{3}\right)^p$.

Consider the graph $\mathcal{C}_\nu$ associated to ν and denote by d_ν the distance in this graph. This step is the content of the following theorem. Recall that the standard 2×2 square S_0 is the square whose vertices are $[x]$, $[y]$, $[z]$ and $[t]$.

Theorem 5.1. *Pick any valuation $\nu \in \mathcal{V}_0$ satisfying*

$$\max(2\nu(t), \nu(y) + \nu(t), \nu(z) + \nu(t)) < \nu(x) < \min(\nu(y), \nu(z), \nu(t)). \tag{27}$$

Consider any geodesic segment of $\mathcal{C}$ joining [Id] *to a vertex v of type* I *which intersects an edge of the square S_0, then the following assertions hold*:

(1) *We have*

$$\nu(v) \leqslant \left(\tfrac{4}{3}\right)^{d_\nu([t], v)-1} \max(\nu(x), \nu(y), \nu(z), \nu(t)).$$

(2) *For any valuation $\nu' \in \mathcal{V}_0$ satisfying (27), we have*

$$d_\nu([t], v) = d_{\nu'}([t], v).$$

The proof of Theorem 5.1 basically proceeds by induction on the distance between $[t]$ and v in the graph $\mathcal{C}_\nu$. The essential ingredient to bound below the degree inductively are the parachute inequalities stated in Theorem 4.17. We explain in Section 5B how to arrive to the situation where these inequalities can be applied using the local geometry near the vertices of type I (i.e., the geometry of its link). We then use these arguments to compute the degree or estimate the valuation ν when one passes from one square to another in each possible situation, this is done successively in Sections 5C, 5D, 5E and 5F.

Once we conjugate appropriately to arrive to the situation of Theorem 5.1, we then deduce directly both Theorem 1 and Theorem 4.

5A. ***The graph $\mathcal{C}_\nu$ associated to a valuation and the orientation of certain edges of the complex.*** Fix a valuation $\nu \in \mathcal{V}_0$. Given any automorphism $f = (f_1, f_2, f_3, f_4) \in \mathrm{Tame(Q)}$, we remark that $\nu(f_1)$ does not depend on the choice of representative of the class $[f_1]$ so that ν induces a function on the vertices of type I of $\mathcal{C}$.

We say that a vertex $v \in \mathcal{C}$ of type I is ν-*minimal* (resp. ν-*maximal*) in a 2×2 square S if $\nu(v)$ is strictly smaller (resp. greater) than the value of the valuation ν on every other vertices of type I of S. Observe that for some valuations, two vertices of type I can have the same value on ν, hence there can be no ν-minimal or ν-maximal vertices.

We now define a graph $\mathcal{C}_\nu$ associated to a valuation $\nu \in \mathcal{V}_0$ as follows:

(1) The vertices are the vertices of $\mathcal{C}$ type I.

(2) One draws an edge between two vertices v_1 and v_2 of $\mathcal{C}'$ if there exists a 2×2 square S centered at a vertex of type III in $\mathcal{C}$ containing v_1, v_2 such that the vertices v_1, v_2 belong to an edge of S or v_1 and v_2 are the ν-minimal and ν-maximal vertices of S respectively.

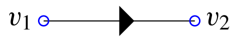
Observe that whenever there is no ν-maximal or minimal vertex in a 2×2 square S centered at a point of type III, then we only draw the four edges of the square S.

The graph $\mathcal{C}_\nu$ is endowed with the distance d_ν such that its the edges have length 1.

Lemma 5.2. *The graph $\mathcal{C}_\nu$ is a connected metric graph.*

Proof. This follows from the fact that the 1-skeleton of $\mathcal{C}$ is connected and the fact that to any path between type I vertices in the 1-skeleton of $\mathcal{C}$, we can find an alternate path in the 1-skeleton of $\mathcal{C}$ with the same endpoints and which takes only edges joining type I and II vertices. If one has a local path in the skeleton passing successively to a type II, then to a type III and then to a type II vertex in the same 1×1 square, then we replace by a path that goes through the vertex of type I within the same square. For path that go through different squares, we use Lemma 2.16 and Lemma 2.15 to modify locally our path so that it takes a corner of a 2×2 square centered at a type III vertex. □

Since we will exploit the properties of this function on the vertices of type I, we introduce the following convention on the figures. Take an edge of length 2 between two type I vertices v_1, v_2, then we put an arrow pointing to v_2 if $\nu(v_2) < \nu(v_1)$ as in the following following:



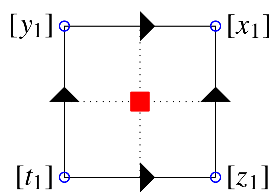
Lemma 5.3. *Let* $\nu : \mathrm{k}[Q] \to \mathbb{R} \cup \{+\infty\}$ *be a valuation which is trivial over* k^* *and such that* $\nu(x)$, $\nu(y)$, $\nu(z)$, $\nu(t) < 0$. *Let* S *be a* 2×2 *square of the complex* $\mathcal{C}$ *centered at a type* III *vertex. Suppose* S *has a* ν*-maximal vertex* (*resp.* ν*-minimal*), *then there exists a* ν*-minimal* (*resp.* ν*-maximal*) *vertex and the* ν*-minimal and* ν*-maximal vertices are at distance* $2\sqrt{2}$ *in* $\mathcal{C}$.

Let S be a 2×2 square centered at a vertex of type III which satisfies the conditions of Lemma 5.3. and let ϕ be the associated isometry. Denote by $[x_1]$, $[y_1]$, $[z_1]$ and $[t_1]$ the vertices of type I of the square S where $x_1, y_1, z_1, t_1 \in \mathrm{k}[Q]$ such that the vertex $[x_1]$ is ν-minimal and $[t_1]$ is ν-maximal in S. Then there exists a unique isometry $\phi : S \to [0, 2]^2$ such that

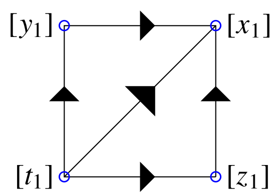
$$\phi([x_1]) = (2, 2) \quad \text{and} \quad \phi([t_1]) = (0, 0),$$

and such that the horizontal edges of S are given the geodesic segments between $[x_1]$ and $[y_1]$, and between $[z_1]$ and $[t_1]$.

Using this convention, Lemma 5.3 implies that we are in the following situation:



In particular, the subgraph of $\mathcal{C}'$ containing the vertices of S looks as follows:



Proof of Lemma 5.3. Let S be a 2×2 square satisfying the hypothesis of the Lemma. Denote $[t_1]$ the ν-maximal vertex of S. Denote also by $[z_1]$, $[y_1]$, $[x_1]$ the type I vertices of S such that the edges between $[t_1]$ and $[z_1]$, between $[t_1]$ and $[y_1]$ are horizontal and vertical respectively.

Observe that $\nu(x_1)$, $\nu(y_1)$, $\nu(z_1)$, $\nu(t_1) < 0$ and that

$$\nu(x_1 t_1 - y_1 z_1) = \nu(1) = 0.$$

This implies that

$$\nu(x_1) + \nu(t_1) = \nu(y_1) + \nu(z_1).$$

In particular, $\nu(t_1) > \nu(y_1)$ implies that

$$\nu(x_1) < \nu(z_1).$$

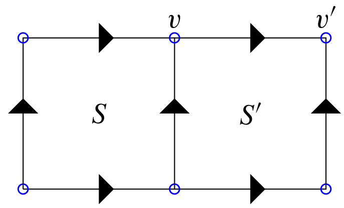
By symmetry, we also prove that $\nu(x_1) < \nu(y_1)$ and this implies that $[x_1]$ is the unique ν-minimal vertex of S, as required. □

Observe that for two distinct valuations $\nu_1, \nu_2 \in \mathcal{V}_0$, the graphs $\mathcal{C}_{\nu_1}$ and $\mathcal{C}_{\nu_2}$ are not in general equal.

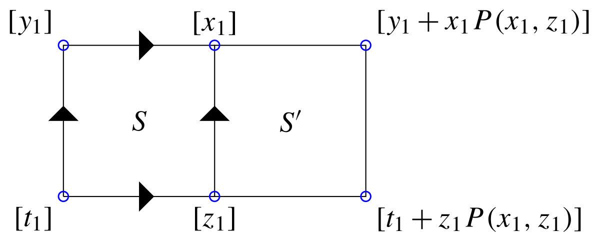
Lemma 5.4. *Fix any valuation $\nu \in \mathcal{V}_0$, and any two adjacent 2×2 squares S, S' centered at a vertex of type III. Suppose that v is a vertex in $S \cap S'$ which is ν-minimal in S.*

Then the unique vertex $v' \in S' \setminus S$ which belongs to an edge containing v is also ν-minimal in S'.

One has the following:



Proof of Lemma 5.4. Take $x_1, y_1, z_1, t_1 \in \mathrm{k}[\mathcal{Q}]$ such that $v = [x_1], [z_1] \in S \cap S'$ and $[y_1], [t_1] \in S$ are the four distinct vertices of S. We claim that we are in the following situation:



where $P \in k[x, y] \setminus k$. Indeed, recall that the tame group acts as $g \cdot [f] = [f \circ g^{-1}]$. In particular, if S_0 is the standard 2×2 square containing $[x], [y], [z], [t]$ and $[\mathrm{Id}]$ and if $f = (x_1, y_1, z_1, t_1)$, then $S = f^{-1} \cdot S_0$. Since S and S' are adjacent along an two edges of type I, there exists an element $e \in E_H$ such that $S' = (f^{-1} \circ e \circ f) \cdot S$. This proves that $S' = (f^{-1} \circ e) \cdot S_0$, and the vertex v' is given by

$$v' = [y \circ e^{-1} \circ f],$$

as required.

Since $\nu(x_1) < \nu(y_1)$ and since $\nu(P(x_1, z_1)) < 0$, this implies that

$$\nu(y_1 + x_1 P(x_1, z_1)) = \nu(x_1 P(x_1, z_1)) < \nu(x_1).$$

Similarly, one has

$$\nu(t_1 + z_1 P(x_1, z_1)) = \nu(z_1 P(x_1, z_1)) < \nu(z_1).$$

Hence since the vertex $[z_1]$ is ν-maximal, we have that $v' = [y_1 + x_1 P(x_1, z_1)]$ is the ν-minimal vertex in S' by Lemma 5.3, as required. $\square$

The following proposition compares the distance d_ν with the distance $d_{\mathcal{C}}$.

Proposition 5.5. *The distance d_ν and the distance $d_{\mathcal{C}}$ are equivalent, i.e., there exists a constant $C > 0$ such that for any vertices $v_1, v_2 \in \mathcal{C}$ of type* I, *one has*

$$\frac{1}{2\sqrt{2}} d_{\mathcal{C}}(v_1, v_2) \leqslant d_\nu(v_1, v_2) \leqslant 2 d_{\mathcal{C}}(v_1, v_2).$$

Proof. For each 2×2 square S centered at a vertex of type III in $\mathcal{C}$, the restriction to $S \cap \mathcal{C}_\nu$ of the distance in $\mathcal{C}_\nu$ and the distance $d_{\mathcal{C}}$ are bi-Lipschitz equivalent. More precisely, for any $v_1, v_2 \in S \cap \mathcal{C}_\nu$, the following inequality holds:

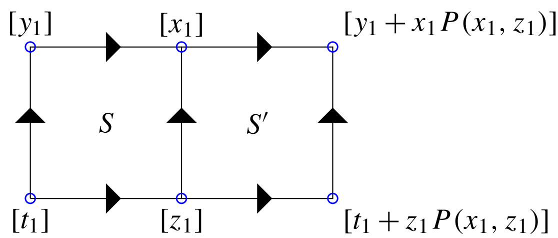
$$\frac{d_{\mathcal{C}}(v_1, v_2)}{2\sqrt{2}} \leqslant d_\nu(v_1, v_2) \leqslant 2 d_{\mathcal{C}}(v_1, v_2).$$

Hence, if we apply the previous inequality to a chain of points which belong successively to the same square, we obtain the distance in $\mathcal{C}$ is equivalent to the distance d_ν and for any vertices v_1, v_2 of type I in $\mathcal{C}$, we have

$$\frac{d_{\mathcal{C}}(v_1, v_2)}{2\sqrt{2}} \leqslant d_\nu(v_1, v_2) \leqslant 2 d_{\mathcal{C}}(v_1, v_2),$$

as required. $\square$

5B. ***Avoiding critical resonances.*** Fix a valuation $\nu \in \mathcal{V}_0$ and fix a 2×2 square S. Consider a vertex $[x_1]$ of type I in S which is ν-minimal in S where $x_1 \in \mathrm{k}[Q]$ and denote by $[z_1]$ another vertex of type I in S such that $[x_1]$ and $[z_1]$ belong to a vertical edge of the square S. For any square S' which is adjacent to S along the edge containing $[x_1]$ and $[z_1]$, Lemma 5.4 implies that the function induced by ν on the vertices is as follows:



where $y_1, t_1, \in \mathrm{k}[Q]$ and $P \in k[x, y] \setminus k$.

If $P \in k[y]$, then using the fact that $\nu(x_1) < \nu(y_1)$ and $\nu(x_1) < \nu(z_1) < \nu(t_1)$, we have $\nu(y_1 + x_1 P(x_1, z_1)) = \nu(x_1) + \deg(P)\nu(z_1) < (\deg(P)+1)\nu(z_1)$ and $\nu(t_1 + z_1 P(x_1, z_1)) = (\deg(P)+1)\nu(z_1)$. This degenerate case can be formulated as follows. Take $g \in \mathrm{STame}(Q) \cap \mathrm{Stab}([z_1])$ such that $g \cdot S = S'$, Lemma 2.19 shows that $g \in A^S_{[z_1]}$.

Otherwise, we can suppose that $P \in k[x,y] \setminus k[y]$. The main observation is that when the component (x_1, z_1) is not critically resonant with respect to ν, then by Corollary 4.19, one has

$$\nu(x_1 P(x_1, z_1)) = \max(\nu(y_1 + x_1 P(x_1, z_1), \nu(t_1 + z_1 P(x_1, z_1))) < \tfrac{4}{3}\nu(z_1).$$

Following the discussion when $P \in k[y]$, we deduce that the previous inequality also holds regardless of the condition $P \in k[y]$ or $P \in k[x,y] \setminus k[y]$.

Moreover, when $P \in k[x,y] \setminus k[y]$ the same argument combined with Corollary 4.18 yields

$$\max(\nu(y_1 + x_1 P(x_1, z_1), \nu(t_1 + z_1 P(x_1, z_1))) < \nu(x_1).$$

We have summarized the above argument in the following lemma.

Lemma 5.6. *Fix $\nu \in \mathcal{V}_0$ and S, S' two adjacent 2×2 squares. Consider v_1, v_2 two vertices of the common edge of these squares and suppose that v_1 is ν-minimal in S. Suppose that the edge joining v_1 and v_2 corresponds to a component (f_1, f_2) which is not critically resonant. Then for any vertex $v' \in S'$ distinct from v_1, v_2, we have*

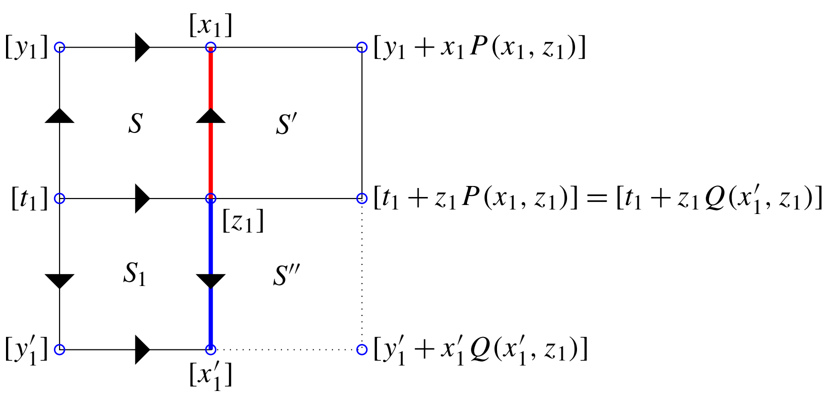
$$\nu(v') < \tfrac{4}{3}\nu(v_2).$$

Moreover, if $g \in \mathrm{Stab}(v_2) \cap \mathrm{STame}(Q)$ such that $g \cdot S = S'$ and $g \notin A^S_{v_2}$, then

$$\nu(v') < \min\big(\tfrac{4}{3}\nu(v_2), \nu(v_1)\big).$$

When the component (x_1, z_1) in the previous figure is critically resonant, then the previous arguments do not necessarily hold since we cannot apply Corollary 4.19.

Our key observation is that the previous inequality remains valid whenever there exists a square S_1 adjacent to S along the edge containing $[t_1], [z_1]$, such that its other edge containing $[z_1]$ is not critically resonant and such that $[t_1]$ is ν-maximal in S_1. If we choose S_1 so that the squares S_1, S, S' are flat, we arrive at the following situation where a *blue edge* means that the corresponding component is not critically resonant and a *red edge* that the component is critically resonant:



where $Q \in k[x, y] \setminus k$ and where S''' is a 2×2 square adjacent to S_1 and S'. We can thus apply the previous argument to the square S_1 and S''', we obtain

$$\nu(y_1' + x_1' Q(x_1', z_1)) < \nu(t_1 + z_1 P(x_1, z_1)) < \nu(z_1).$$

and furthermore

$$\nu(t_1 + z_1 P(x_1, z_1)) = \max(\nu(y_1' + x_1' Q(x_1', z_1)), \nu(t_1 + z_1 P(x_1, z_1))) < \tfrac{4}{3}\nu(z_1).$$

In other words, we obtain that the vertex $[z_1]$ is ν-maximal in S''', which implies that it is also ν-maximal in S'. Overall, the previous inequality with the ν-maximality of $[z_1]$ in S' finally yields

$$\nu(t_1 + z_1 P(x_1, z_1)) = \max(\nu(t_1 + z_1 P(x_1, z_1)), \nu(y_1 + x_1 P(x_1, z_1))) < \tfrac{4}{3}\nu(z_1). \tag{28}$$

In the rest of section, we keep the same convention on the colors of the edges.

The proposition below is the key ingredient in our proof and explains how one can find a square which has an edge which is not critically resonant.

Proposition 5.7. *Fix a valuation $\nu \in \mathcal{V}_0$. Let S be any 2×2 square having a unique ν-minimal vertex, and let $[f_1]$, $[f_2]$ be any horizontal (resp. vertical) edge of S. Suppose that $\nu(f_1) < \nu(f_2)$, that (f_1, f_2) is critically resonant and that for any polynomial $R \in k[x] \setminus k$, one has*

$$\nu(f_1 - f_2 R(f_2)) < \nu(f_2).$$

Then there exists a square S_1 adjacent to S along the vertical (resp. horizontal) edge containing $[f_2]$ which satisfies the following properties:

(i) *For any square S_2 adjacent to S along the edge containing $[f_1]$, $[f_2]$, the squares S_1, S, S_2 are flat.*

(ii) *The horizontal (resp. vertical) edge in S_1 containing $[f_2]$ is not critically resonant.*

(iii) *There exists an element $g \in A^S_{[f_2]}$ such that $g \cdot S = S_1$.*

Proof. Statement (i) and (iii) follow from Lemma 2.19(ii) and (i) respectively. Indeed pick any polynomial $R \in k[x] \setminus k$, and let S_R be the square containing $[f_2]$, $[f_1 - f_2 R(f_2)]$ which is adjacent to S along the vertical edge containing $[f_2]$. Since R depends on a single variable, it follows that for any square S_2 adjacent to S along the edge containing $[f_1]$, $[f_2]$, the squares S_R, S_2, S are flat.

We now prove (ii), and produce a polynomial $R \in k[x] \setminus k$ such that the component $(f_2, f_1 - f_2 R(f_2))$ is not critically resonant. Since the component (f_1, f_2) is critically resonant, there exists a constant $\lambda \in k^*$ and an integer $n \geqslant 1$ such that

$$\nu(f_1 - \lambda f_2^n)) > \nu(f_1) = n\nu(f_2).$$

Since $\nu(f_1) < \nu(f_2)$, we get $n \geqslant 2$ so that $R_1 := \lambda x^{n-1} \in k[x] \setminus k$.

If the component $(f_2, f_1 - f_2 R_1(f_2))$ is not critically resonant, then the square S_1 containing $[f_2]$, $[f_1 - f_2 R_1(f_2)]$ which is adjacent to S along the vertical edge containing $[f_2]$ satisfies assertion (ii) and we are done. Otherwise, $(f_2, f_1 - f_2 R_1(f_2))$ is critically resonant. Observe that by assumption, we have

$$\nu(f_1 - f_2 R_1(f_2)) < \nu(f_2),$$

so that $\nu(f_1 - f_2 R_1(f_2)) = n_2 \nu(f_2)$ for some $n_2 \geqslant 2$, and $\nu(f_1 - f_2 R_2(f_2)) > n_2 \nu(f_2)$ for some polynomial $R_2 \in k[x] \setminus k$ of the form $R_2(x) = R_1(x) + \lambda' x^{n_2 - 1}$. Repeating this argument we get a sequence of polynomials $R_i \in k[x] \setminus k$, and either $(f_2, f_1 - f_2 R_i(f_2))$ is not critically resonant for some index i; or $(f_2, f_1 - f_2 R_i(f_2))$ is critically resonant for all i. However in the latter case, the sequence $(\nu(f_1 - f_2 R_i(f_2))$ is strictly increasing and $(\nu(f_1 - f_2 R_i(f_2))$ are all multiples of $\nu(f_2)$ which yields a contradiction. The proof is complete. □

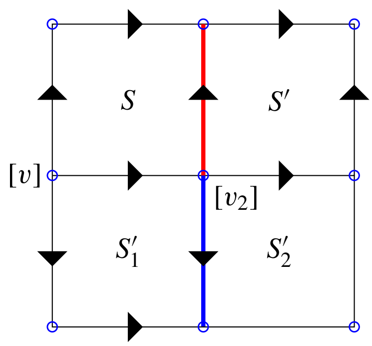
5C. *Degree estimates between adjacent squares.*

Theorem 5.8. *Take a valuation $\nu \in \mathcal{V}_0$. Let v be any ν-maximal vertex of a 2×2 square S and let S' be an adjacent square which does not contain v and let v_2 be the vertex in $S \cap S'$ which is not ν-minimal in S. Suppose that the vertex v is also ν-maximal in any square $\tilde{S}$ adjacent to S along the edge joining v and v_2. Then S' admits a ν-minimal vertex and for any vertex $v' \in S' \setminus S$, one has*

$$\nu(v') < \tfrac{4}{3} \nu(v_2).$$

Proof. Observe that Lemma 5.4 implies that S' has a unique ν-minimal vertex. If the edge $S \cap S'$ is not critically resonant, then Lemma 5.6 implies the conclusion of the theorem. Otherwise, the edge $S \cap S'$ is critically resonant and we check that the squares S and S' satisfy the hypothesis of Proposition 5.7.

Denote by $[f_1]$ the ν-minimal vertex in S and by $[f_2] = v_2$. For any polynomial $R \in k[x] \setminus k$, take S_R to be the square containing $[f_1 - f_2 R(f_2)]$, $[f_2]$ and v. By construction, the square S_R is adjacent to S along an edge containing v, hence the vertex v is ν-maximal in S_R and this implies that the vertex $[f_1 - f_2 R(f_2)]$ is ν-minimal in S_R. In particular, we have proved that $\nu(f_1 - f_2 R(f_2)) < \nu(f_2)$. By Proposition 5.7, there exists two squares S'_1, S'_2 such that the union $S \cap S'_1 \cup S' \cup S'_2$ forms a 4×4 square centered along the vertex $[v_2]$ and such that the edge $S'_1 \cap S'_2$ is not critically resonant. We thus arrive at the following situation (with the same convention on colors as in the previous section):



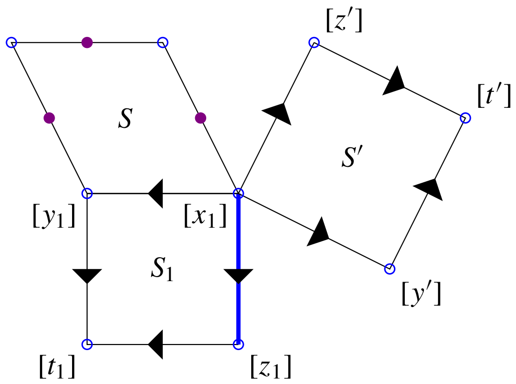


Figure 6. The initial situation of Theorem 5.9.

In particular, by applying Lemma 5.4 to the square S_1' and S_2', we find that

$$\nu(v') < \tfrac{4}{3}\nu(v_2),$$

for any vertex $v' \in S' \setminus S$ as required. □

5D. *Degree estimates at a ν-maximal vertex.* In this section, we analyze the situation of two 2×2 squares adherent at a vertex of type I.

Recall from Section 2E that a pair of adherent squares (S, S') is contained in a spiral staircase around $v = S \cap S'$ if there exists a sequence of squares $S_0 = S, \dots, S_p = S'$ connecting S and S', all containing v, which are adjacent alternatively along vertical and horizontal edges and such that any three consecutive squares S_i, S_{i+1}, S_{i+2} are not flat for $i \leqslant p-2$. When the intersection between S_0 and S_1 is a horizontal (resp. vertical) edge, we say that the staircase is vertical (resp. horizontal).

Theorem 5.9. *Fix a valuation $\nu \in \mathcal{V}_0$.*

Consider three 2×2 squares S, S_1 and S' having a vertex $[x_1]$ of type I *in common. We assume that S and S_1 have a common horizontal edge $[x_1], [y_1]$, and that the pair (S, S') is contained in a vertical spiral staircase containing S_1. Denote by $[z_1]$ the vertex in S_1 which forms a vertical edge with $[x_1]$.*

Assume that $[x_1]$ is ν-maximal in S_1, that the component (x_1, z_1) is not critically resonant, that $\nu(z_1) < \nu(y_1)$ and $\nu(z_1) < \left(\frac{4}{3}\right)\nu(x_1)$. Then for any vertex $v \in S'$ distinct from $[x_1]$, one has

$$\nu(v) < \tfrac{4}{3}\nu(x_1).$$

Figure 6 summarizes the situation of Theorem 5.9 (with the convention of Section 5B on the color of the edges).

We shall use repeatedly the following lemma, whose proof is given at the end of this section.

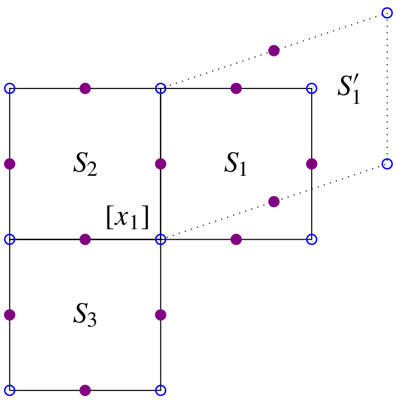
Recall from Section 2D the definition of the subgroup A_v^S of the stabilizer of a vertex v of type I, where S is a 2×2 square containing v.

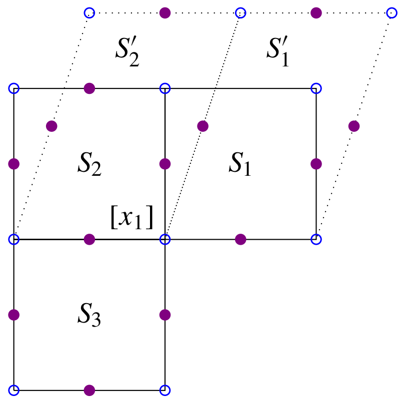
Lemma 5.10. *Take three 2×2 squares S_1, S_2, S_3 containing $[x_1]$ and which are adjacent alternatively along vertical and horizontal edges. Suppose that S_1, S_2 and S_3 are not flat. Then the following assertions hold*:

(i) *Suppose that S_1' is a 2×2 square which is adjacent to S_2 along $S_1 \cap S_2$ such that there exists an element $g \in A_{[x_1]}^{S_1}$ for which $g \cdot S_1 = S_1'$. Then the squares S_1', S_2, S_3 are not flat.*

(ii) *For any 2×2 squares S_1', S_2' such that S_1, S_2, S_1', S_2' are flat, the squares S_1', S_2', S_3 are not flat. Moreover, given any $g_1, g_2 \in \mathrm{Stab}([x_1]) \cap \mathrm{STame}(Q)$ such that $g_1 S_1 = S_1'$, and $g_2 S_2 = S_2'$, we have $g_1 \in A_{[x_1]}^{S_1}$ and $g_2 \in A_{[x_1]}^{S_2}$.*

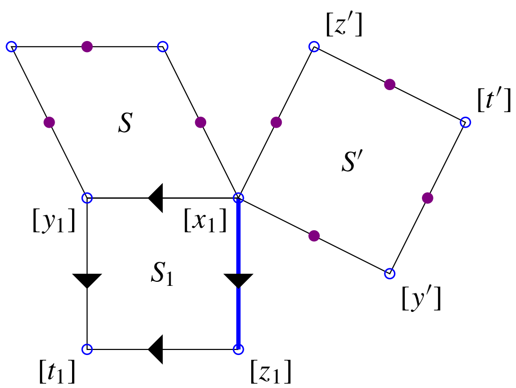
This lemma will allow us to consider alternative spiral staircase around the vertex $[x_1]$. We thus have the following figures in each situation:





Proof of Theorem 5.9. Take a valuation $\nu \in \mathcal{V}_0$ and three squares S, S_1, S' satisfying the conditions of the theorem. By assumption, there exists an integer $p \geqslant 2$ and a sequence of adjacent squares $S_2, \ldots, S_{p-1}$ such that $S_0 = S, S_1, S_2, \ldots, S_p = S'$ forms a vertical staircase.

We denote by $[y_1]$, $[z_1]$, $[t_1]$, $[x_1]$ and $[z']$, $[y']$, $[t']$ the vertices of S_1 and S' respectively so that the edges $[x_1]$, $[y_1]$ and $[x_1]$, $[y']$ are horizontal and the edges $[x_1]$, $[z_1]$ and $[x_1]$, $[z']$ are vertical. We are thus in the following situation:



Recall that S and S' are connected by a vertical staircase $S = S_0, S_1, \dots, S_{p-1}, S_p = S'$.

Lemma 5.11. *The theorem holds whenever the edges $S_i \cap S_{i+1}$ are not critically resonant for all $i \geq 1$.*

Lemma 5.12. *For any vertex v such that $[x_1], v$ is an edge of S', there exists a vertical staircase $S = S_0, \tilde{S}_1, \tilde{S}_2, \dots, \tilde{S}_{q-1}, \tilde{S}_q$ such that*

- $\tilde{S}_1 = S_1$;
- *$\tilde{S}_q$ and S' are adjacent along the edge $[x_1], v$;*
- *the edges $\tilde{S}_i \cap \tilde{S}_{i+1}$ are not critically resonant for all $i \geq 1$.*

Take any vertex v of S' such that $[x_1], v$ is an edge of S'. By Lemma 5.12 we get a sequence of squares $\tilde{S}_i$ connecting S to $\tilde{S}_q$ and satisfying the assumptions of Lemma 5.11. This proves $\nu(v) < \frac{4}{3}\nu(x_1)$ as required. □

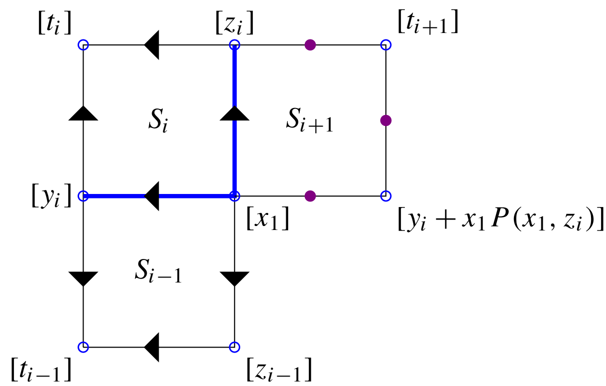
Proof of Lemma 5.11. We prove by induction on i the following two properties:

($\mathcal{P}_1$) For any vertex $v \neq [x_1]$ in $S_i \setminus S_0$, one has $\nu(v) < \frac{4}{3}\nu(x_1)$.

($\mathcal{P}_2$) Let $v_1 \neq [x_1]$ be the unique vertex which is contained in the edge $S_i \cap S_{i-1}$ and let v_2 be the other vertex in S_i which belongs to an edge containing $[x_1]$. Then one has $\nu(v_2) < \nu(v_1)$.

Observe that ($\mathcal{P}_1$) and ($\mathcal{P}_2$) are satisfied when $i = 1$ by our standing assumption on S_1.

Let us prove the induction step. For all i, denote by t_i the unique vertex of S_i which does not lie in $S_{i-1} \cup S_{i+1}$; by y_i the vertex in $S_i \cap S_{i-1}$ distinct from x_1. We also write z_i for the vertex in $S_i \cap S_{i+1}$ distinct from x_1 (so that $y_{i+1} = z_i$). We thus have the following picture:



By our induction hypothesis, we have

$$\nu(z_i) < \nu(y_i) < \nu(x_1).$$

Observe that y_{i+1} is given by

$$y_{i+1} = y_i + x_1 P(x_1, z_i).$$

for some polynomial $P \in k[x, y]$. Since the squares (S_{i-1}, S_i, S_{i+1}) is not flat, Lemma 2.19(i) and Lemma 2.17 imply that $P \notin k[x]$.

Since the component (x_1, z_i) is not critically resonant, Corollary 4.18 applied to $f_1 = z_i$ and $f_2 = x_1$ implies

$$\nu(x_1 P(x_1, z_i)) < \nu(z_i) \leqslant \min\big(\tfrac{4}{3}\nu(x_1), \nu(z_i)\big),$$

where the second inequality follows from the induction hypothesis $\nu(z_i) < \frac{4}{3}\nu(x_1)$, hence

$$\nu(y_{i+1}) = \nu(y_i + x_1 P(x_1, z_i)) = \nu(x_1 P(x_1, z_i)) < \min\big(\tfrac{4}{3}\nu(x_1), \nu(z_i)\big).$$

This proves that $[x_1]$ is ν-maximal in S_{i+1}, hence $[t_{i+1}]$ is ν-minimal in S_{i+1} by Lemma 5.3 and assertion $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ hold for $i+1$, as required. □

Proof of Lemma 5.12. We show that for any vertex v such that $[x_1], v$ is an edge of S', there exists a vertical staircase $S = S_0, \tilde{S}_1, \tilde{S}_2, \dots, \tilde{S}_{p-1}, \tilde{S}_p$ around $[x_1]$(of exactly same length) such that:

- $\tilde{S}_1 = S_1$.
- $\tilde{S}_p$ and S' are adjacent along the edge $[x_1], v$, and there exists an element $g \in A^{S'}_{[x_1]}$ for which $g \cdot \tilde{S}_q = S'$.
- The edges $\tilde{S}_i \cap \tilde{S}_{i+1}$ are not critically resonant for all $i \geq 1$.

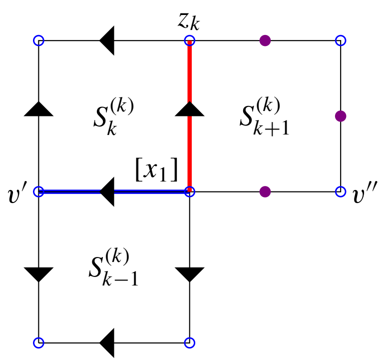
We construct a sequence of spiral staircase $S^{(k)}_0 = S, S^{(k)}_1, \dots, S^{(k)}_p$ of length p indexed by k as follows. We pick an initial spiral staircase $S^{(1)}_0 = S, S^{(1)}_1 = S_1, \dots, S^{(1)}_p = S'$ around $[x_1]$ of length p joining S and S'. Observe that the edge $S^{(1)}_1 \cap S^{(1)}_2$ is not critically resonant by our assumption and the sequence $(S^{(1)})_{i \leqslant 2}$ defines a spiral staircase that satisfies the conclusion of the lemma.

Our aim is to construct $(S^{(k)}_i)_{i \leqslant p}$ inductively so that the conclusion of the lemma holds for $(S^{(k)}_i)_{i \leqslant k+1}$ for all $k \leqslant p$.

For $k = 1$, there is nothing to prove since $[x_1], [z_1]$ is not critically resonant by our standing assumption.

For $k \geq 1$, assume that $(S^{(k)}_i)_{i \leqslant p}$ is constructed. If the edge $S^{(k)}_k \cap S^{(k)}_{k+1}$ is not critically resonant, then we set $S^{(k+1)}_i = S^{(k)}_i$ for all i. Otherwise, it is critically resonant and we will replace the two squares $S^{(k)}_k, S^{(k)}_{k+1}$ and keep all the other squares.

Denote by $[z_k]$ and v' the vertices in $S^{(k)}_k$ distinct from x_1 and lying in S' and $\tilde{S}_{p-1}$ respectively. Using the property $\mathcal{P}2$ from the proof of Lemma 5.11, we have $\nu(z_k) < \nu(v') < \nu(x_1)$, and we have the following picture:



We claim that

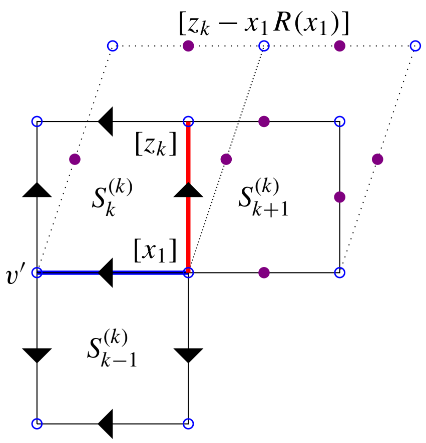
$$\nu(z_k - x_1 R(x_1)) < \nu(x_1),$$

for any polynomial $R \in k[x] \setminus k$. Taking this claim for granted we conclude the proof of the lemma. By Proposition 5.7, we may find a square $S_k^{(k+1)}$ adjacent to $S_k^{(k)}$ along the edge containing $[x_1]$, v' whose edges containing $[x_1]$ are not critically resonant and such that the triple $S_k^{(k)}$, $S_k^{(k+1)}$, $S_{k+1}^{(k)}$ is flat. Let $S_{k+1}^{(k+1)}$ be the 2×2 square completing the 4×4 square containing $S_k^{(k)}$, $S_k^{(k+1)}$, $S_{k+1}^{(k)}$.

Since the squares $S_{k-1}^{(k)}$, $S_k^{(k)}$ and $S_{k+1}^{(k)}$ are not flat, Lemma 5.10(ii) implies that the triple $S_{k-1}^{(k)}$, $S_k^{(k+1)}$ and $S_{k+1}^{(k+1)}$ is also not flat, so that the sequence $(S_1^{(k)}, S_2^{(k)}, \dots, S_{k-1}^{(k)}, S_k^{(k+1)}, S_{k+1}^{(k+1)})$ is a spiral staircase such that any edge lying in two consecutive squares is not critically resonant.

Let us set $S_i^{(k+1)} = S_i^{(k)}$ for all $i \neq k, k+1$. We now show that $(S_1^{(k+1)}, \dots, S_p^{(k+1)})$ defines a spiral staircase. Observe that $S_k^{(k)}$, $S_{k+1}^{(k)}$, $S_k^{(k+1)}$, $S_{k+1}^{(k+1)}$ are flat, since $S_k^{(k)}$, $S_{k+1}^{(k)}$, $S_{k+2}^{(k)}$ are not flat, assertion (ii) of Lemma 5.10 implies that $S_k^{(k+1)}$, $S_{k+1}^{(k+1)}$, $S_{k+2}^{(k)} = S_{k+2}^{(k+1)}$ are not flat. This shows that $(S_1^{(k+1)}, \dots, S_p^{(k+1)})$ defines a spiral staircase, as required.

We now prove our claim. Fix a polynomial $R \in k[x] \setminus k$, and consider the square S_R containing $[x_1]$, $[z_k - x_1 R(x_1)]$ and v'. Since $x R(x) \in k[x]$, the squares S_R, $S_k^{(k)}$ and $S_{k+1}^{(k)}$ are flat by Lemma 2.19(ii). We thus have the following picture:



By Lemma 2.19, there exists an element $g \in A_{[x_1]}^{S_k^{(k)}}$ such that $g \cdot S_k^{(k)} = S_R$. By Lemma 5.10(i) the triple $S_{k-2}^{(k)}$, $S_{k-1}^{(k)}$, S_R are not flat since $S_{k-2}^{(k)}$, $S_{k-1}^{(k)}$, $S_k^{(k)}$ are not flat. We have thus proven that the sequence $(S, S_1, S_2^{(k)}, \dots, S_{k-1}^{(k)}, S_R)$ is contained in a spiral staircase for which any edge lying in two consecutive squares is not critically resonant. By Lemma 5.11 the vertex $[x_1]$ is ν-maximal in S_R, hence

$$\nu(z_k - x_1 R(x_1)) < \nu(x_1),$$

as required. □

Proof of Lemma 5.10. By transitivity of the action of $\mathrm{STame}(Q)$ on the 2×2 squares, we can suppose that S_2 is the standard 2×2 square containing $[x]$, $[t]$, $[y]$, $[z]$ and that S_1 and S_3 are adjacent along the vertical and horizontal edge containing $[x]$ respectively. Take $g_1, g_3 \in \mathrm{Stab}([x]) \cap \mathrm{STame}(Q)$ such that $g_1 \cdot S_2 = S_1$ and $g_3 S_2 = S_3$.

Let us prove assertion (i). Since S_1, S_2, S_3 are not flat, Lemma 2.17 implies that $g_1, g_3 \notin A^{S_2}_{[x]}$. Observe that $gg_1 \cdot S_2 = S'_1$ and $g_3 \cdot S_2 = S_3$ where $g \circ g_1 \notin A^{S_2}_{[x]}$, hence the squares S'_1, S_2, S_3 are also not flat by Lemma 2.17.

Let us prove assertion (ii). We assume S_1 is the standard square and that $[x] = [x_1]$.

Consider $g \in \mathrm{Stab}([x]) \cap \mathrm{STame}(Q)$ such that $g \cdot S_1 = S_2$. Note that $g_1 \cdot S_1 = S'_1$, $g_2 \cdot S_2 = S'_2$. Assume by contradiction that $g_1 \notin A^{S_1}_{[x]}$. Since the squares S'_1, S_1, S_2 are flat, Lemma 2.17 implies that $g \in A^{S_1}_{[x]}$. However, Lemma 2.17 applied to S_1, S_2, S_3 together with the fact that $g^{-1} \in A^{S_2}_{[x]}$ shows that S_1, S_2, S_3 are flat, we have thus obtained a contradiction. We have thus shown that $g_1 \in A^{S_1}_{[x]}$ and a similar argument also gives $g_2 \in A^{S_2}_{[x]}$.

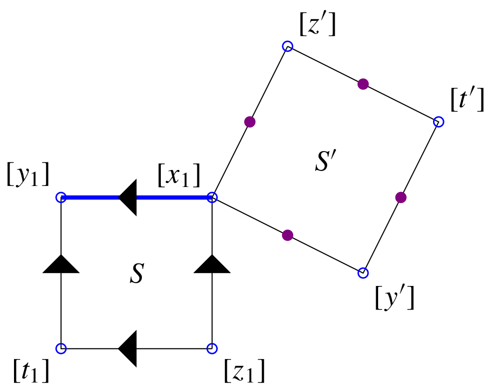
Let us prove that S'_1, S'_2, S_3 are not flat, consider the element $g_3 \in \mathrm{Stab}([x]) \cap \mathrm{STame}(Q)$ such that $g_3 S_2 = S_3$. Assume by contradiction that S'_1, S'_2, S_3 are flat. We have $g_1 g^{-1} g_2^{-1} \cdot S'_2 = S'_1$ and $g_3 g_2^{-1} \cdot S'_2 = S_3$ so Lemma 2.17 shows that one of the element $g_1 g^{-1} g_2^{-1}$, $g_3 g_2^{-1}$ belongs to $A^{S'_2}_{[x]}$. Since $g_3 \notin A^{S_2}_{[x]}$, so $g_3 g_2^{-1} \notin A^{S'_2}_{[x]}$. We deduce that $g_1 g^{-1} g_2^{-1} \in A^{S'_2}_{[x]}$, thus $g_1 g^{-1} \in A^{S_2}_{[x]}$. However $g g_1 g^{-1} \in A^{S_2}_{[x]}$ since $g_1 \in A^{S_1}_{[x]}$ and we get that $g \in A^{S_2}_{[x]}$ which contradicts the fact that S_1, S_2, S_3 are not flat. □

5E. *Degree at a nonextremal vertex.*

Theorem 5.13. *Take a valuation $\nu \in \mathcal{V}_0$. Consider two 2×2 adherent squares S and S' at a vertex of type I given by $[x_1]$ with $x_1 \in \mathrm{k}[Q]$ such that the pair (S, S') is contained in a vertical spiral staircase. Assume $[y_1]$ is the ν-minimal vertex in S distinct from $[x_1]$ which belongs to the horizontal edge containing $[x_1]$ and that the edge containing $[x_1], [y_1]$ is not critically resonant. Then for any vertex v distinct from $[x_1]$ in S' one has*

$$\nu(v) < \tfrac{4}{3}\nu(x_1).$$

One has the following picture:

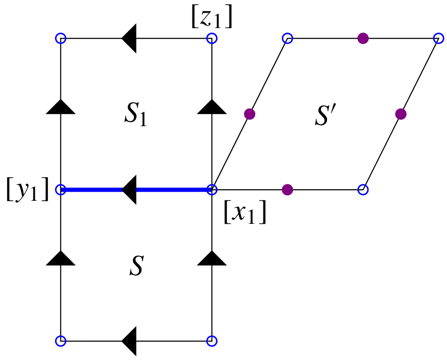


Remark 5.14. By symmetry, observe that the same assertion holds if $[z_1]$ is ν-minimal in S and the pair (S, S') is contained in a horizontal spiral staircase.

Proof. Consider two squares S, S' and the vertices $[x_1]$, $[y_1] \in S$ satisfying the conditions of the Theorem. By definition, there exists an integer p and p adjacent squares $S_0 = S, \dots, S_p = S'$ containing $[x_1]$ connecting S and S'.

Since $S_0 = S$ and S_1 are adjacent, the vertex $[x_1]$ is ν-maximal in S_1 by Lemma 5.4.

Denote by $[z_1]$ the vertex in S_1 such that the vertices $[x_1]$ and $[z_1]$ are contained in the vertical edge of S_1 so that we are in the following situation:



Fix any polynomial $R \in k[x] \setminus k$. Consider S_R the square containing $[x_1]$, $[y_1]$ and $[z_1 - x_1R(x_1)]$. By Lemma 2.19, the squares S_1, S_R, S_2 are flat. Take $\tilde{S}_R$ the 2×2 square completing the 4×4 square containing S_1, S_R, S_2. Lemma 5.10(ii) implies that S, S_R, $\tilde{S}_R$ are not flat since S, S_1, S_2 are not flat. Moreover, the fact that S, S_1 belong to a spiral staircase shows that any element $g \in \mathrm{Stab}([x_1]) \cap \mathrm{STame}(Q)$ such that $g \cdot S = S_1$ cannot belong to $A^S_{[x_1]}$ by Lemma 2.17. Because $S \cap S_R$ is the noncritically resonant edge containing $[x_1]$, $[y_1]$, Lemma 5.4 applied to S and S_R gives

$$\nu(z_1 - x_1R(x_1)) < \nu(x_1).$$

By Proposition 5.7, there exists a square S_1' adjacent to S along $[x_1]$, $[y_1]$ such that the squares S_1', S_1, S_2 are flat and such that the vertical edge in S_1' containing $[x_1]$ is not critically resonant. Consider the square S_2' completing the 4×4 square containing S_1', S_1, S_2. By construction, the edge $S_1' \cap S_2'$ is not critically resonant. Observe also that Lemma 5.6 implies that for any vertex $v \in S_1'$ distinct from $[x_1]$ and $[y_1]$, one has

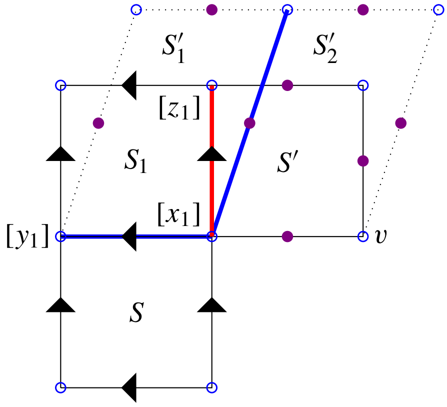
$$\nu(v) < \max\big(\nu(y_1), \tfrac{4}{3}\nu(x_1)\big).$$

Suppose that $p \geqslant 3$, then the triple (S, S_1', S') satisfies the assumptions of Theorem 5.9 by considering the spiral staircase $(S, S_1', S_2', S_3, \dots, S')$ and we conclude that for any vertex v distinct from $[x_1]$ in S':

$$\nu(v) < \tfrac{4}{3}\nu(x_1).$$

We have thus proven the theorem.

Suppose that $p = 2$ and the squares S' and S_1 are adjacent. We are thus in the following situation:



where v is the unique vertex in S' distinct from $[x_1]$ which belongs to the horizontal edge containing $[x_1]$. By Theorem 5.9, $[x_1]$ is ν-maximal in S'_2, hence it is also ν-maximal in S' and $\nu(v) < \frac{4}{3}\nu(x_1)$. Observe also that Lemma 5.6 implies that

$$\nu(z_1) < \frac{4}{3}\nu(x_1).$$

This proves that for any $v \in S'$ distinct from $[x_1]$, one has

$$\nu(v) < \frac{4}{3}\nu(x_1),$$

by Lemma 5.4 and the theorem holds. $\square$

5F. *Degree estimates at a ν-minimal vertex.*

Theorem 5.15. *Consider any valuation $\nu \in \mathcal{V}_0$. Let S and S' be two adherent 2×2 squares intersecting at a vertex v which is ν-minimal in S. Then the following holds*:

(i) *The vertex v is the ν-maximal vertex of S'.*

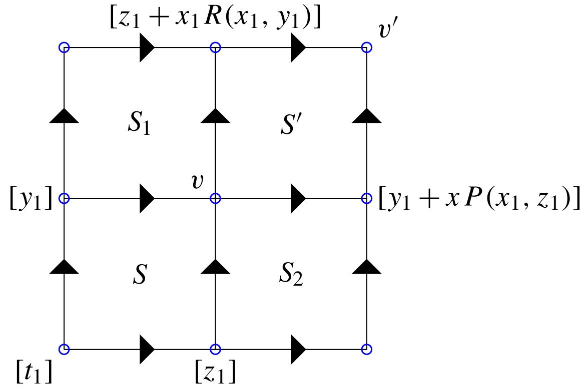
(ii) *If v' is a vertex in S' which does not belong to any square adjacent to S, then we have*

$$\nu(v') < \frac{4}{3}\nu(v)$$

Remark 5.16. Suppose that the vertex $v \in S'$ belongs to a square adjacent to S, then we will apply the estimates in Theorem 5.8 instead.

Proof. Let us prove assertions (i) and (ii).

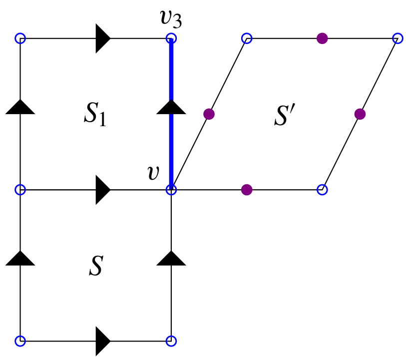
Suppose first that S and S' belong to a 4×4 squares containing S, S', S_1 and S_2 as in the figure below. Since S, S_1 and S, S_2 are adjacent along an edge containing v, Lemma 5.4 implies that we are in the following situation:



where $v = [x_1], [y_1], [z_1], [t_1] \in S$ and $P, R \in k[x, y] \setminus k$. Observe that v is ν-maximal in S' and we have proved assertion (i). Since the squares S, S_1, S_2 are flat, Lemma 2.19 and Lemma 2.17 imply that $P \in k[x] \setminus k$ or $R \in k[x] \setminus k$. Suppose that $P \in k[x] \setminus k$, then we have $\left(\frac{4}{3}\right)\nu(x_1) > \nu(y_1 + x_1 P(x_1)) = (\deg(P) + 1)\nu(x_1) > \nu(v')$ proving (ii) as required.

Suppose next that (S, S') is contained in a spiral staircase. Choose a sequence of squares $S_0 = S, \ldots, S_p = S'$ of squares containing v and connecting S and S' such that each triple of consecutive squares is not flat. By symmetry, we can suppose that S_0 and S_1 are adjacent along a horizontal edge containing v. Observe that Lemma 5.4 applied to S, S_1 implies that the edge $S_1 \cap S_2$ contains the ν-minimal vertex in S_1.

If the edge $S_1 \cap S_2$ is not critically resonant, then the pair (S_1, S') is contained in a horizontal staircase so that one has the following picture:



By Theorem 5.13, the vertex v is ν-maximal in S' and one has $\nu(v') < \left(\frac{4}{3}\right)\nu(v)$ for all $v' \neq v$ in S'. We have thus proved assertion (i) and (ii).

We now suppose that the edge $S_1 \cap S_2$ is critically resonant. Denote by $[f_1]$ the ν-minimal vertex in S_1 and by $v = [f_2]$. Fix any polynomial $R \in k[x] \setminus k$ and take S_R the square containing $[f_1 - f_2 R(f_2)], [f_2]$ and the edge $S_1 \cap S_0$. Lemma 2.19(ii) implies that the squares S_1, S_R, S_2 are flat. Take S'_R the 2×2 square completing the 4×4 square containing S_1, S_R, S_2. Since the squares S, S_1, S_2 are not flat, Lemma 5.10 implies that S, S_R, S'_R are also not flat. In particular, the squares S and S_R intersect along an edge containing v, Lemma 5.4 implies that

$$\nu(f_1 - f_2 R(f_2)) < \nu(f_2).$$

By Proposition 5.7 applied to the edge $[f_1], [f_2]$, we can find a square S'_1 adjacent to S along $S \cap S_1$ and $g \in A_v$ such that $g \cdot S_1 = S'_1$ and such that the vertical edge containing v in S'_1 is not critically resonant. By Lemma 2.17, the squares S_1, S'_1, S_2 are flat. Take S'_2 the 2×2 square completing the 4×4 square containing S_1, S_2, S'_1. As the three squares S, S_1, S_2 are not flat, Lemma 5.10 implies that the squares S, S'_1, S'_2 are also not flat.

If $p \geqslant 3$, then the pair (S'_1, S_p) is contained in a horizontal spiral staircase $(S'_1, S'_2, S_3, \ldots, S_p)$ and the edge $S'_1 \cap S'_2$ is not critically resonant. Hence, by Theorem 5.13, the vertex v is ν-maximal in S' and for any vertex v' distinct from v in S', one has

$$\nu(v') < \tfrac{4}{3}\nu(v),$$

proving (i) and (ii) as required.

Suppose that $p = 2$ so that $S_2 = S'$. Observe that S'_2 and S' are adjacent along a horizontal edge containing v. By Lemma 5.6 applied to S'_1, S'_2, v is ν-maximal in S'_2, it is also ν-maximal on the edge $S'_2 \cap S'$. Since v is ν-maximal on the vertical edge $S_1 \cap S'$, we have thus proven that v is ν-maximal in S' and assertion (i) holds. Take v_2 the vertex contained in $S' \cap S'_2$ distinct from v. Since the edge $S'_1 \cap S'_2$ is not critically resonant, Lemma 5.6 implies that $\nu(v_2) < \frac{4}{3}\nu(v)$. Hence, for any vertex $v' \in S'$ not contained in the same band as S, one has $\nu(v') < \left(\frac{4}{3}\right)\nu(v)$ proving (ii) as required. □

5G. *Proof of Theorem 5.1.* Take S_0 the standard square containing $[x], [y], [z], [t]$. Fix a valuation $\nu \in \mathcal{V}_0$ such that

$$\max(2\nu(t), \nu(y) + \nu(t), \nu(z) + \nu(t)) < \nu(x) < \min(\nu(y), \nu(z), \nu(t)).$$

Pick any vertex v of type I such that the geodesic segment in $\mathcal{C}$ joining [Id] to v intersects an edge of the standard square. Choose any geodesic segment $\gamma : [0, n] \to \mathcal{C}_\nu$ joining $[t]$ to v such that the sequence $(\nu(\gamma(i)))_{0 \le i \le n}$ is maximal for the lexicographic order in $\mathbb{R}^{n+1}$ among all geodesic segments joining $[t]$ to v. Pick any sequence $\tilde{S}_0, \ldots, \tilde{S}_{n-1}$ of 2×2 squares such that $\gamma(i), \gamma(i+1) \in \tilde{S}_i$ for all $i \leqslant n-1$. We claim that the following properties hold:

(A) The vertex $\gamma(i)$ is the unique ν-maximal vertex in $\tilde{S}_i$ for all $0 \le i \le n-1$.

(B) We have $\nu(\gamma(i+1)) < \frac{4}{3}\nu(\gamma(i))$ for all $1 \leqslant i \leqslant n-1$.

(C) For any other valuation $\nu' \in \mathcal{V}_0$ satisfying (27), the vertex $\gamma(i)$ is also ν'-maximal in $\tilde{S}_i$ for all $0 \le i \le n-1$.

Observe first that these properties (A), (B) and (C) imply Theorem 5.1(1) and (2).

Observe the slight discrepancy in the indices between (A), (C) and (B). We do not claim that $\nu(\gamma(1)) < \frac{4}{3}\nu([t])$ in general. This claim is however sufficient to imply Theorem 5.1(1) and (2).

Observe that assertion (C) implies that $d_\nu([t], v) \geq d_{\nu'}([t], v)$ and we conclude by symmetry that $d_\nu([t], v) = d_{\nu'}([t], v)$ for any other valuation $\nu' \in \mathcal{V}_0$ satisfying (27). This proves assertion 2 of the theorem.

We shall prove the claim by induction on $n \geqslant 1$. Fix another valuation $\nu' \in \mathcal{V}_0$ satisfying (27).

Suppose $n = 1$. There is only one square $\tilde{S}_0$ containing $[t]$ and v (it may not be the standard square). Since $n = 1$, we only need to prove assertions (A) and (C).

Lemma 5.17. *Take any* 2×2 *square* S *adjacent to the standard square* S_0 *along an edge containing* $[t]$*. Then the vertex* $[t]$ *is* ν*-maximal in* S*.*

Moreover, denote by v_1 *the vertex in* $S \cap S_0$ *distinct from* $[t]$ *in* S *and by* v_2 *the vertex distinct from* v_1 *for which the vertices* $[t]$, v_2 *form an edge of* S*. Then one has* $\nu(v_2) < \nu(v_1)$*.*

Grant this lemma. If $\tilde{S}_0 = S_0$, then (A) and (C) automatically hold. If $\tilde{S}_0$ and S_0 are adjacent along an edge containing $[t]$ Lemma 5.17 implies assertions (A) and (C) immediately. Suppose now that $\tilde{S}_0$ and S_0 are adherent at $[t]$. If the squares $\tilde{S}_0$ and S_0 are flat, then Lemma 5.17 applied to the two squares adjacent to both S_0 and $\tilde{S}_0$ again implies that $[t]$ is also ν-maximal and ν'-maximal in $\tilde{S}_0$.

Otherwise $(S_0, \tilde{S}_0)$ are contained in a spiral staircase. Take an integer $p \geqslant 2$ and a sequence of squares $S_0, S'_1, \ldots, S'_p = \tilde{S}_0$ connecting S_0 to $\tilde{S}_0$ such that each three consecutive squares are not flat. We claim that we can choose some squares S''_1, S''_2 such that $S_0, S''_1, S''_2, S'_3, \ldots, \tilde{S}_0$ is a spiral staircase and such that $S''_1 \cap S''_2$ is not critically resonant. If the edge $S'_2 \cap S'_1$ is not critically resonant, then we set $S''_2 = S'_2$ and $S''_1 = S'_1$.

Otherwise, $S'_1 \cap S'_2$ is critically resonant. Take $[f_1]$ the vertex distinct from $[t]$ of the edge $S'_2 \cap S'_1$. Denote by $[f_2]$ the vertex in $S_0 \cap S'_1$ distinct from $[t]$. By Lemma 5.17, one has $\nu(f_1) < \nu(t)$ and $\nu(f_1) < \nu(f_2)$. Take any polynomial $R \in k[x] \setminus k$, denote by S_R the square containing $[f_1 - tR(t)], [t], [f_2]$. By construction, S_R is adjacent to S_0 and Lemma 5.17 implies that $\nu(f_1 - tR(t)) < \nu(t)$. By Proposition 5.7, we can find a square $S''_1 = g \cdot S'_1$ with $g \in A^{S'_1}_{[t]}$ such that S'_1, S''_1, S'_2 are flat and the edge containing $[t]$ in S''_1 distinct from $S_0 \cap S'_1$ is not critically resonant. Take S''_2 the 2×2 square completing the 4×4 square containing S'_1, S'_2, S''_1.

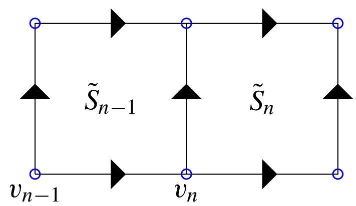
If $p \geqslant 3$, the triple S_0, S'_1, S'_2 is not flat by Lemma 5.10(ii), hence S_0, S''_1, S''_2 are also not flat. The squares $(S_0, S''_1, \tilde{S}_0)$ thus satisfy the conditions of Theorem 5.9, and $[t]$ is ν-maximal in $\tilde{S}_0$. If $p = 2$, then $S'_2 = \tilde{S}_0$ and by Theorem 5.9 applied to (S_0, S''_1, S''_2), the vertex $[t]$ is ν-maximal in S''_2. Since S''_2 and $\tilde{S}_0$ are adjacent along an edge containing $[t]$ and $[t]$ is also ν-maximal in S'_1, it is also ν-maximal in $\tilde{S}_0$, proving assertion (A) as required. Observe that the same argument also applies for $\nu' \in \mathcal{V}_0$, hence assertion (C) also holds.

We have thus proven the claim for $n = 1$.

Let us suppose that the claim is true for $n \geqslant 1$. We shall prove it for $n + 1$. Choose any geodesic $\gamma : [0, n+1] \to \mathcal{C}_\nu$ joining $[t]$ to a vertex v for which the sequence $(\nu(\gamma(i)))_{0 \leq i \leq n}$ is maximal. Denote by $v_i = \gamma(i)$. Take any sequence of squares $\tilde{S}_0, \ldots, \tilde{S}_n$ for which $v_i, v_{i+1} \in \tilde{S}_i$.

By our induction hypothesis applied to the vertex v_n, the sequence $\tilde{S}_0, \ldots, \tilde{S}_{n-1}$ satisfy assertions (A), (B) and (C).

Suppose first that $\tilde{S}_{n-1}$ and $\tilde{S}_n$ are adjacent or equal. Observe that assertion (A) implies that $v = \gamma_{n+1}$ cannot belong to the square $\tilde{S}_{n-1}$, otherwise it would contradict the fact that γ is a geodesic in $\mathcal{C}_\nu$ (recall that in this graph we draw an edge joining the ν-maximal to the ν-minimal vertex). This implies that $\tilde{S}_{n-1}$ and $\tilde{S}_n$ are adjacent along an edge containing the ν-minimal vertex in $\tilde{S}_{n-1}$. Lemma 5.4 shows that the vertex in $\tilde{S}_{n-1} \cap \tilde{S}_n$ which is not ν-minimal in $\tilde{S}_{n-1}$ is ν-maximal in $\tilde{S}_n$. By the maximality of the sequence $(\nu(\gamma(i)))_{0\le i\le n}$ the vertex v_n cannot be ν-minimal in $\tilde{S}_{n-1}$, hence is ν-maximal in $\tilde{S}_n$, proving assertion (A). The following figure summarizes the situation:

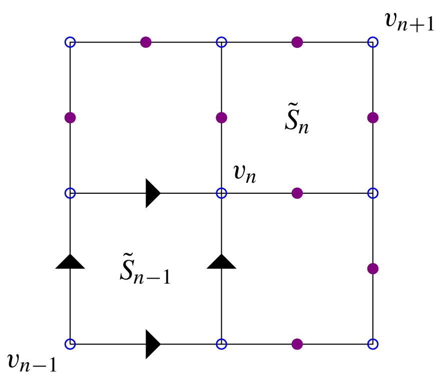


Since v_{n-1} is also ν'-maximal in $\tilde{S}_{n-1}$, the vertex v_n is also ν'-maximal in $\tilde{S}_n$ by Lemma 5.4. We have thus proven assertion (C).

Let us check that $\tilde{S}_{n-1}$ satisfies the condition of Theorem 5.8. Take another square $\tilde{S}$ adjacent to $\tilde{S}_{n-1}$ containing v_{n-1}, v_n. Observe that the sequence $\tilde{S}_0, \ldots, \tilde{S}_{n-2}, \tilde{S}$ satisfies the conditions of the theorem and contains v_n which is at distance n. We apply our induction hypothesis to the vertex v_n and to the sequence of squares $\tilde{S}_0, \ldots, \tilde{S}_{n-2}, \tilde{S}$. Assertion (A) implies that the vertex v_{n-1} is ν-maximal in $\tilde{S}$, as required.

We may thus apply Theorem 5.8 to the band $\tilde{S}_{n-1} \cup \tilde{S}_n$ which yields $\nu(v_{n+1}) < \frac{4}{3}\nu(v_n)$, proving (B), as required.

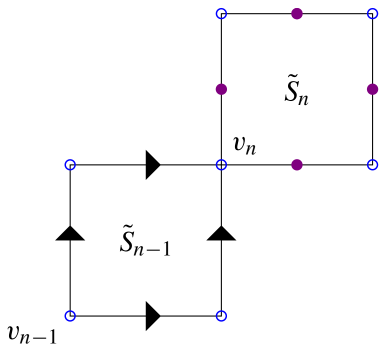
Suppose that the squares $\tilde{S}_{n-1}, \tilde{S}_n$ are adherent and flat. If v_n, v_{n-1} form an edge of $\tilde{S}_{n-1}$, then we can find a band of two squares containing v_{n-1}, v_n, v_{n+1}, which corresponds to the previous situation. Otherwise (v_n, v_{n-1}) is not an edge of $\tilde{S}_{n-1}$, and since v_{n-1} is ν-maximal and ν'-maximal in $\tilde{S}_{n-1}$ by assertions (A) and (C), the vertex v_n is ν-minimal and ν'-minimal in $\tilde{S}_{n-1}$. Observe that the vertex v_{n+1} cannot belong to a band containing v_n, v_{n-1} since we have chosen a geodesic γ for which the sequence $(\nu(\gamma(i)))_{0\le i\le n}$ is maximal. We thus arrive at the following situation:



By Theorem 5.15(i) and (ii) applied to $\tilde{S}_{n-1}$ and $\tilde{S}_n$, the vertex v_n is ν-maximal and ν'-maximal in $\tilde{S}_n$ (hence (A), (C) hold), and one has $\nu(v_{n+1}) < \frac{4}{3}\nu(v_n)$, and assertion (B) holds.

Suppose that the squares $\tilde{S}_{n-1}$, $\tilde{S}_n$ are contained in a spiral staircase.

Let us suppose first that the vertices v_{n-1}, v_n do not belong to the same edge of $\tilde{S}_{n-1}$. By assertions (A) and (C) applied to v_{n-1}, the vertex v_{n-1} is ν-maximal and ν'-maximal in $\tilde{S}_{n-1}$, hence v_n is ν-minimal and ν'-minimal in $\tilde{S}_{n-1}$. We thus have the following:



In particular, by Theorem 5.15(i) applied to the squares $\tilde{S}_{n-1}$, $\tilde{S}_n$ implies that v_n is ν-maximal and ν'-maximal in $\tilde{S}$, proving (A) and (C). Observe that v_{n+1} cannot belong to a band containing v_{n-1}, v_n since we have chosen the geodesic such that $\nu(\gamma(i))$ is maximal. In particular, Theorem 5.15(ii) implies that

$$\nu(v_{n+1}) < \tfrac{4}{3}\nu(v_n),$$

proving (B) as required.

Let us suppose that the vertices v_{n-1}, v_n belong to an edge of $\tilde{S}_{n-1}$. Since the argument are similar for horizontal edges, we can suppose that the edge joining v_{n-1}, v_n is vertical, and the pair $(\tilde{S}_{n-1}, \tilde{S}_n)$ belongs to a vertical spiral staircase. Indeed, if $(\tilde{S}_{n-1}, \tilde{S}_n)$ belongs to a horizontal spiral staircase, then we can choose a square $\tilde{S}'_{n-1}$ adjacent to $\tilde{S}_{n-1}$ along the edge containing v_{n-1}, v_n which belongs to the horizontal staircase $(\tilde{S}_{n-1}, \tilde{S}'_{n-1}, \dots, \tilde{S}_n)$. We can thus replace $\tilde{S}_{n-1}$ by $\tilde{S}'_{n-1}$ and the squares $(\tilde{S}'_{n-1}, \tilde{S}_n)$ belong to a vertical spiral staircase or $\tilde{S}'_{n-1}$ and $\tilde{S}_n$ are adjacent. The later case have been treated previously.

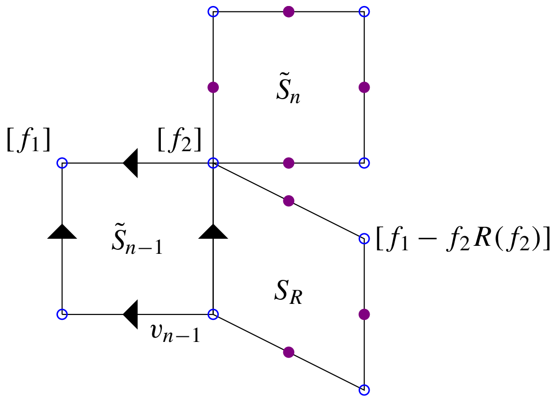
Write by $v_n = [f_2]$ and let $[f_1]$ be the vertex distinct from v_n in $\tilde{S}_{n-1}$ which belongs to the horizontal edge containing v_n. If (f_1, f_2) is not critically resonant, then we can directly apply Theorem 5.13, the vertex v_n is ν-maximal and ν'-maximal in $\tilde{S}_n$ and

$$\nu(v_{n+1}) < \tfrac{4}{3}\nu(v_n),$$

proving (A), (B) and (C) as required.

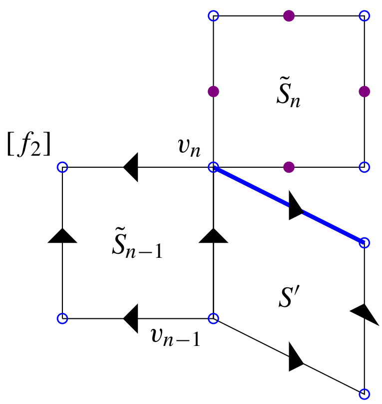
Assume now that (f_1, f_2) is critically resonant. We shall replace the square $\tilde{S}_{n-1}$ by an adjacent square S' along the edge joining v_n and v_{n-1} such that the horizontal edge in S' containing v_n is not critically resonant.

For any polynomial $R \in k[x] \setminus k$, denote by S_R the 2×2 containing $[f_2]$, $[f_1 - f_2R(f_2)]$, v_{n-1}. We thus have the following:



Using our induction hypothesis for the vertex v_n and to the sequence of squares $\tilde{S}_0, \ldots, \tilde{S}_{n-2}, S_R$, assertions (A) and (C) imply that the vertex v_{n-1} is ν-maximal and ν'-maximal in S_R, hence $\nu(f_1 - f_2R(f_2)) < \nu(f_2)$ and $\nu'(f_1 - f_2R(f_2)) < \nu'(f_2)$. By Proposition 5.7, we can find a square S' containing v_{n-1}, v_n for which the horizontal edge containing v_n is not critically resonant and such that there exists $g \in A_{v_n}$ such that $g \cdot S' = \tilde{S}_{n-1}$. By Lemma 5.10, since $(\tilde{S}_{n-1}, \tilde{S}_n)$ is contained in a vertical spiral staircase, this implies that the pair $(S', \tilde{S}_n)$ is also contained in a vertical spiral staircase. Since v_n is neither ν-maximal nor ν-minimal in S', the pair $(S', \tilde{S}_n)$ satisfies the conditions of Theorem 5.13.

One has the following:



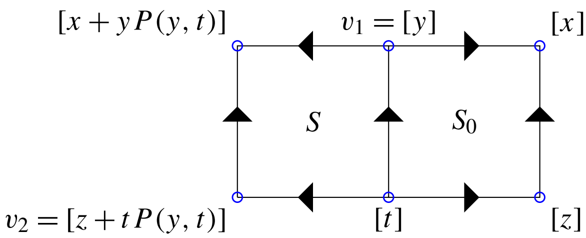
Observe that the same argument applies for ν' and we can find another square S'' adjacent to $\tilde{S}_{n-1}$ along v_n, v_{n-1} such that $S'', \tilde{S}_{n-1}$ is contained in a vertical spiral staircase and such that the horizontal edge in S'' containing v_n is not critically resonant for ν'. By Theorem 5.13, the vertex v_n is ν-maximal and ν'-maximal in $\tilde{S}_n$ and $\nu(v_{n+1}) < \left(\frac{4}{3}\right)\nu(v_n)$, proving (A), (B) and (C) as required.

We have thus proven that our induction step is valid, and the theorem is proved.

Proof of Lemma 5.17 . Fix a valuation $\nu \in \mathcal{V}_0$ satisfying (27) and take a square S adjacent to S_0 along an edge containing $[t]$.

Observe that the edge $S \cap S_0$ is either vertical or horizontal. Since the proof is similar for both cases, we can suppose that $S \cap S_0$ is vertical so that S and S_0 intersect along the edge containing $[y]$, $[t]$. Remark that in this case, we have $v_1 = [y]$ and v_2 is the vertex distinct from $[t]$ which belongs to the horizontal edge in S containing $[t]$.

We are thus in the following situation:



where $P \in k[x, y] \setminus k$.

Since $\nu(P(y,t)) \leqslant \min(\nu(y), \nu(t))$ because P is nonconstant and ν is a quasimonomial valuation, and since (27) implies that $2\nu(t) < \nu(z)$ and $\nu(y) + \nu(t) < \nu(z)$, we get

$$\nu(tP(y,t)) < \nu(z),$$

hence $\nu(z + tP(y,t)) < \nu(z)$ and the vertex $[t]$ is ν-maximal in S. Because ν is monomial satisfying (27), we also have

$$\nu(z + tP(y,t)) < \nu(y),$$

hence $\nu(v_2) < \nu(v_1)$, as required. □

5H. *Proof of Theorem 1.* Consider a tame automorphism $f \in \mathrm{Tame}(\mathrm{Q})$. Since the complex $\mathcal{C}$ is CAT(0) and since the action of f is an isometry and a morphism of complex, the action of f on the complex either fixes a vertex or a geodesic line. In the first case, f is elliptic and by Theorem 3.3, the sequences $(\deg(f^n))$, $(\deg(f^{-n}))$ are either both bounded, both linear or satisfy

$$C^{-1}d^n \leqslant \deg(f^{\pm n}) \leqslant Cd^n,$$

where $C > 0$ and $d \in \mathbb{N}$.

We are thus reduced to prove the theorem in the case where f induces an action which fixes a geodesic line $\gamma : \mathbb{R} \to \mathcal{C}$. Take an hyperbolic automorphism f and a geodesic line $\gamma : \mathbb{R} \to \mathcal{C}$ fixed by f. Denote by S_0 the standard 2×2 square containing $[x]$, $[y]$, $[z]$ and $[t]$. Since for any tame automorphism $h \in \mathrm{Tame}(\mathrm{Q})$, there exists a constant $C > 0$ such that

$$\frac{1}{C} \leqslant \frac{\deg(f^n)}{\deg(h^{-1}f^nh)} \leqslant C,$$

by taking an appropriate conjugate of f, we can suppose that γ starts in S_0 and intersects an edge of S_0. Consider the geodesic segment γ_n' joining [Id] and $[x \circ f^{-n}]$. By construction, γ_n' intersects an edge of the standard square S_0 as γ starts in S_0.

Fix any valuation ν such that (27) is satisfied. There are infinitely many valuations in $\mathcal{V}_0$ satisfying (27) arbitrarily close to $-\deg$. Indeed, consider the sequence of weight $\alpha_i = (-1, -1+2/i, -1+5/i, -1+7/i)$, then by Proposition 4.2, there exists a sequence of valuations ν_i with weight α_i on (x, y, z, t) which converges to $-\deg$.

All assumptions of Theorem 5.1 are then satisfied and we get

$$\nu_i(f^n \cdot [x]) = \nu_i(x \circ f^{-n}) \leqslant \left(\tfrac{4}{3}\right)^{d_{\nu_i}([t],[x\circ f^{-n}])-1} \max(\nu_i(y), \nu_i(z), \nu_i(x), \nu_i(t)).$$

Observe that ν_i tends to $-\deg$, moreover, assertion (2) of Theorem 5.1 implies that the distance $d_{\nu_i}([t], [x \circ f^{-n}])$ are all equal for all i which implies

$$\deg(f^{-n}) \geqslant \left(\tfrac{4}{3}\right)^{d_\nu([t],[x\circ f^{-n}])-1},$$

for a given valuation ν satisfying (27).

We now prove that the sequence $(d_\nu([t], [x \circ f^{-n}]))_n$ grows at least linearly. Indeed since the invariant geodesic γ passes through S_0, then it passes through all the squares $f^i \cdot S_0$ for all $i \leqslant n$. We claim that all the squares $f^i S_0$ are distinct, so there are at least n squares. Indeed each iterate $f^i S_0$ is a 2×2 square containing a piece of the invariant geodesic and the type III vertex in $f^i S_0$ are all distinct otherwise f would be in O_4.

Consider a geodesic segment γ_{1n} in $\mathcal{C}_\nu$ joining $[t]$ and $[x \circ f^{-n}]$ and a shortest path γ_{2n} in $\mathcal{C}_\nu$ contained in a sequence of squares containing the geodesic γ between these two vertices. The path γ_{1n}, γ_{2n} are all in $\mathcal{C}_\nu$ which contains the 1-skeleton of $\mathcal{C}$. The image of those path in $\mathcal{C}$ are both at bounded distance (for the distance $d_{\mathcal{C}}$) from the geodesic γ_n joining $[t]$ and $[x \circ f^n]$ in $\mathcal{C}$. For γ_{2n} this is because γ_{2n} goes through the squares that contain γ_n. For γ_{1n}, this is because γ_{1n} is a geodesic path in $(\mathcal{C}_\nu, d_\nu)$, which is quasiisometric to $(\mathcal{C}, d_{\mathcal{C}})$. In particular, the farthest point in between γ_{1n} and γ_{2n} are apart by a finite number M of squares, which only depends on the distance $d_{\mathcal{C}}(\gamma_{1n}, \gamma_{2n})$. We get

$$l(\gamma_{2n}) \geqslant l(\gamma_{1n}) - M.$$

Since the path γ_{2n} goes through at least the n distinct squares $f^i S_0$ for $i \leqslant n$, the length of γ_{2n} in $\mathcal{C}_\nu$ is larger or equal than n, hence

$$d_\nu([t], [x \circ f^{-n}]) \geqslant n - M.$$

Hence

$$\deg(f^{-n}) \geqslant C\left(\tfrac{4}{3}\right)^{n-1},$$

where $C > 0$. Since the argument is similar for $\deg(f^n)$, we have thus proven that

$$\min(\deg(f^n), \deg(f^{-n})) \geqslant C\left(\tfrac{4}{3}\right)^n$$

where $C > 0$.

5I. ***Proof of Theorem 4.*** Take $f \in \mathrm{Tame}(\mathrm{Q})$. Consider γ the geodesic in $\mathcal{C}$ joining $v_0 = [\mathrm{Id}]$ to $[x \circ f]$. Since the stabilizer of $[\mathrm{Id}]$ is the group O_4 by Proposition 2.7 and since the group O_4 acts transitively on the 1×1 squares containing $v_0 = [\mathrm{Id}]$ by Proposition 2.3, we can suppose that the geodesic γ intersects an edge of type I containing $[x]$ of the 1×1 square containing $[x]$, $[\mathrm{Id}]$, $[z, x]$ and $[x, y]$. In particular, the geodesic γ intersects an edge of the standard square S_0.

We have proved that the vertex $v = [x \circ f]$ satisfies the conditions of Theorem 5.1, and by considering a sequence of valuations $\nu_p \in \mathcal{V}_0$ converging to $-\deg$ satisfying (27), we have

$$\nu_p(x \circ f) \leqslant \left(\tfrac{4}{3}\right)^{d_{\nu_p}([t],[x \circ f])-1} \max(\nu_p(y), \nu_p(z), \nu_p(x), \nu_p(t)).$$

By Proposition 5.5, we have, for all integer p,

$$\frac{1}{2\sqrt{2}} d_{\mathcal{C}}(v_1, v_2) \leqslant d_{\nu_p}(v_1, v_2).$$

for any vertices v_1, v_2 of type I. Since $d_{\mathcal{C}}([t], [x \circ f]) \geqslant d_{\mathcal{C}}([\mathrm{Id}], [f]) - 2\sqrt{2}$, we thus obtain, after taking the limit as $p \to +\infty$,

$$\log \deg(f) \geqslant C d_{\mathcal{C}}([f], [\mathrm{Id}]) - C',$$

where $C' = 2 \log\left(\tfrac{4}{3}\right)$ and $C = \log\left(\tfrac{4}{3}\right)/(2\sqrt{2})$ as required.

6. Application to random walks on the tame group

In this section, we consider a random walk on the tame group and its associated degree sequence. After recalling some general facts on random walks on groups (Section 6A), we then discuss when the degree exponents of a random walk are well-defined and their properties (Section 6B). We then classify in Section 6C the finitely generated subgroup of $\mathrm{Tame}(\mathrm{Q})$. Finally we prove Theorem 5, which asserts that the degree exponent of a symmetric random walk on a finitely generated group G is strictly positive if and only if it contains two noncommuting automorphisms with dynamical degree strictly larger than 1 generating a free group of rank 2.

6A. ***General facts on random walks on groups.*** Let G be a finitely generated subgroup of the tame group and let μ be an atomic probability measure on G. The (left) random walk on G with respect to the measure μ is the Markov chain whose initial distribution is the Dirac mass at Id with transition matrix $p(g, g') = \mu(\{g' g^{-1}\})$ for all $g, g' \in G$. We denote by $\Omega = (G^{\mathbb{N}^*}, \mu^{\otimes \mathbb{N}^*})$ the product probability space which encodes the successive increments of the random walk on G with respect to the measure μ. Consider an element $s = (s_1, \ldots, s_n, \ldots) \in \Omega$, set $g_0(s) = \mathrm{Id}$ and

$$g_n(s) = s_n s_{n-1} \cdots s_1,$$

for all $n \geqslant 1$. The image $\mathcal{P}$ of the map $s \in \Omega \mapsto (\mathrm{Id}, g_1(s), \ldots, g_n(s), \ldots) \in G^{\mathbb{N}*}$ is called the path space and an element of $\mathcal{P}$ is a path in the group G. We naturally endow $\mathcal{P}$ with the probability measure $\mathbb{P}$ defined on the σ-algebra of cylinders as the pushforward of the product measure on Ω by the map

$s \in \Omega \mapsto (g_i(s))_i$. More explicitly, consider the probability measure ν_n of the projection of $\mathcal{P}$ onto the $(n{+}1)$-th component g_n, then ν_n is equal to the n-fold convolution $\mu^{*n} * \delta_{\mathrm{Id}}$ so that for all $g \in G$, one has

$$\nu_n(\{g\}) = \mathbb{P}(g_n = g) = \sum_{\substack{s_1,\dots,s_n \\ s_n \cdots s_1 = g}} \prod_{i=1}^{n} \mu(s_i).$$

Fix a reference vertex $v_0 = [\mathrm{Id}]$ in the complex $\mathcal{C}$. Since the tame group acts on the complex, a path in the group $(\mathrm{Id}, g_1, \dots, g_n, \dots)$ induces an element in $\mathcal{C}^{\mathbb{N}*}$ given by $(v_0, g_1 \cdot v_0, \dots, g_n \cdot v_0, \dots)$. The sequence $(v_0, g_1 \cdot v_0, \dots, g_n \cdot v_0, \dots)$ is called a path in the complex.

6B. *Degree exponents of a random walk.* Let G be a finitely generated subgroup of the tame group and let μ be an atomic probability measure on G. We shall define in this section the degree exponents of a random walk with respect to the measure μ. To do so, the measure μ must satisfy a finiteness condition on its first moment

$$\int_{g \in G} \log(\deg(g))\, d\mu(g) < +\infty. \tag{29}$$

Let us define the two degree exponents $\lambda_1(\mu), \lambda_2(\mu)$ by

$$\lambda_1(\mu) := \limsup_{n\to+\infty} \frac{1}{n} \int_{g\in G} \log(\deg(g))\, d\nu_n(g), \quad \text{and} \quad \lambda_2(\mu) := \limsup_{n\to+\infty} \frac{1}{n} \int_{g\in G} \log(\deg(g^{-1}))\, d\nu_n(g),$$

where ν_n is the probability measure of g_n.

The following proposition proves that these quantities are finite and give a few basic properties of these numbers.

Proposition 6.1. *Take G a countably generated subgroup of the tame group and μ an atomic probability measure on G satisfying condition (29). Then the following properties are satisfied*:

(i) *The degree exponents $\lambda_1(\mu), \lambda_2(\mu)$ are finite and are equal to*

$$\lambda_1(\mu) = \lim_{n\to+\infty} \frac{1}{n} \int_{g\in G} \log(\deg(g))\, d\nu_n(g), \quad \textit{and} \quad \lambda_2(\mu) = \lim_{n\to+\infty} \frac{1}{n} \int_{g\in G} \log(\deg(g^{-1}))\, d\nu_n(g),$$

(ii) *The following inequality holds*

$$\lambda_1(\mu) \geqslant \frac{\lambda_2(\mu)}{2}.$$

(iii) *Consider $\sigma : G \to G$ the inverse map, then $\lambda_2(\mu) = \lambda_1(\sigma_* \mu)$.*

(iv) *The degree exponents are invariant by conjugation, i.e., for any $h \in \mathrm{Tame(Q)}$, we have*

$$\lambda_i(\mathrm{Conj}(h)_* \mu) = \lambda_i(\mu),$$

where $\mathrm{Conj}(h) : \mathrm{Tame(Q)} \to \mathrm{Tame(Q)}$ denotes the conjugation by h in G.

Proof. Let us first prove (i). Since g is an automorphism on a threefold, we have $\deg_2(g) = \deg_1(g^{-1})$ and the Khovanski–Teissier inequalities imply $\deg_1(g)^2 \geqslant \deg_2(g) = \deg_1(g^{-1})$; see e.g., [Dang 2020] for a precise definition of the k-degree. We obtain a finiteness condition on the inverse

$$\int_{g\in G} \log(\deg(g^{-1}))d\mu(g) \leqslant 2\int_{g\in G} \log(\deg(g))\,d\mu(g) < +\infty.$$

Since the function deg is submultiplicative, the random variables

$$s \in \Omega \mapsto \log(\deg(g_n(s)))$$

form a subadditive sequence. The previous equation shows that the average of $\log(\deg(g_1(\cdot)))$ is finite and we can apply Kingman's subadditivity theorem [Kingman 1973, Theorem 1] which implies that the limits

$$\lim_{n\to+\infty} \frac{1}{n}\int_G \log\deg(g)\,d\nu_n(g) \quad \text{and} \quad d\nu_n(g),$$

exist and are finite. This proves assertion (i).

Assertion (ii) follows from the fact that $\deg(g)^2 \geqslant \deg(g^{-1})$ for all $g \in \mathrm{Tame(Q)}$. To prove assertion (iii), observe that for all $(s_1, \dots, s_n, \dots) \in \Omega$, we have

$$s_n^{-1}s_{n-1}^{-1}\cdots s_1^{-1} = (s_1 s_2 \cdots s_n)^{-1}.$$

In particular, we obtain

$$\lim_{n\to+\infty}\frac{1}{n}\int_\Omega \log(\deg((s_1s_2\cdots s_n)^{-1})))\,d\mu^{\otimes n} = \lim_{n\to+\infty}\frac{1}{n}\int_\Omega \log(\deg(s_n s_{n-1}\cdots s_1))\,d\sigma_*\mu^{\otimes n}.$$

Since the right hand side of the equality is equal to $\lambda_1(\sigma_*\mu)$ and the left hand side to $\lambda_2(\mu)$, we have thus proven (iii).

Finally, let us prove assertion (iv). Fix $h \in \mathrm{Tame(Q)}$, recall that there exists a constant $C(h) > 0$ such that for all $g \in \mathrm{Tame(Q)}$, we have

$$\frac{\deg(g)}{C(h)} \leqslant \deg(hgh^{-1}) \leqslant C(h)\deg(g).$$

The last inequality directly implies that $\lambda_i(\mathrm{Conj}(h)_*\mu) = \lambda_i(\mu)$ for all $i = 1, 2$ and all $h \in \mathrm{Tame(Q)}$. □

6C. ***Classification of finitely generated subgroups.*** In this section, we give a classification of the finitely generated subgroups of the tame group. To that end, we recall the terminology due to Gromov [1987] on subgroups of isometries of a hyperbolic space.

Fix a Gromov hyperbolic space X and a group G acting on it by isometry. The action of G on X is called *elementary* if it does not contain two hyperbolic isometries whose action do not fix the same geodesic line. We call the action of G on X *elliptic* if it globally fixes a point in X and we shall say that the action of G is *lineal* if there exists an elliptic subgroup H of G, a geodesic line γ on X invariant by G, pointwise fixed by H on which the quotient G/H acts faithfully by translation.

In our case, any element of the tame group induces an isometry of the complex. We will also need to distinguish among the subgroups which fix a vertex in the complex, more particularly when the fixed vertex is of type I. Remark that a subgroup G of the tame group which fixes a vertex of type I is conjugated to a subgroup of $\mathrm{Stab}([x])$ and recall that we have constructed in Section 2D a natural action from the stabilizer subgroup $\mathrm{Stab}([x])$ on a subtree of the Bass–Serre tree. We have the following classification.

Theorem 6.2. *Let G be a finitely generated subgroup of the tame group. Then one of the following situation occurs*:

(i) *The action of G on the complex is nonelementary.*

(ii) *There exists an automorphism h in G whose action in the complex is hyperbolic and such that any automorphism $f \in G$ can be decomposed into $f = g \circ h^p$ where p is an integer and where g belongs to a subgroup H. Moreover, the subgroup H is conjugated in* $\mathrm{Tame(Q)}$ *to a subgroup of O_4 or to one of*

$$E_H \rtimes \left\{ \begin{pmatrix} ax & by \\ b^{-1}z & a^{-1}t \end{pmatrix} \;\middle|\; a, b \in k^* \right\}.$$

(iii) *The group G is conjugated to a subgroup of the linear group O_4.*

(iv) *There exists a G-equivariant morphism $\varphi : Q \to \mathbb{A}^2 \setminus \{(0, 0)\}$ where G acts on $\mathbb{A}^2 \setminus \{(0, 0)\}$ linearly.*

(v) *The group G contains two noncommuting automorphisms with dynamical degree larger or equal than* 2 *and there exists a G-equivariant morphism $\varphi : Q \to \mathbb{A}^1$ where G acts on $\mathbb{A}^1$ by multiplication.*

(vi) *The group G contains an automorphism h with $\lambda_1(h) \geqslant 2$ and there exists a G-equivariant morphism $\varphi : Q \to \mathbb{A}^1$ on which G acts on $\mathbb{A}^1$ by multiplication and an isomorphism $\varphi^{-1}(\mathbb{A}^1 \setminus \{0\}) \simeq \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2$ such that any automorphisms $f \in G$ can be decomposed into $g \circ h^p$ where p is an integer and g is of the form*

$$g : (x, y, z) \in \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2 \mapsto (ax, by, cz) \in \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2,$$

where $a, b, c \in k^*$.

(vii) *There exists a G-equivariant morphism $\varphi : Q \to \mathbb{A}^1$ where G acts on $\mathbb{A}^1$ by multiplication and any automorphism of G has dynamical degree* 1.

Proof. Figure 7 gives a tree summarizing how our proof proceeds where each end of the tree corresponds to a conclusion of the previous theorem. Denote by $\mathcal{T}$ the associated Bass–Serre tree constructed in Section 2C.

Theorem 6.3. *Let G be a finitely generated subgroup of the tame group whose action on the complex $\mathcal{C}$ is elementary. The following possibilities occur*:

(i) *The action of G on the complex is elliptic*, *i.e.*, *G fixes globally a vertex in the complex.*

(ii) *The action of G is lineal on the complex*, *i.e.*, *there exists an elliptic subgroup H of G*, *a geodesic line γ on $\mathcal{C}$ invariant by G pointwise fixed by H on which the quotient G/H acts faithfully by translation.*

$$\begin{tikzcd}
\text{nonelementary in } \mathcal{C} & \text{elliptic in } \mathcal{T} & \text{nonelementary in } \mathcal{T} & \text{lineal in } \mathcal{T} \\
 & & \text{conjugated to a subgroup of } \operatorname{Stab}([x]) \arrow[ul] \arrow[u] \arrow[ur] & \\
G \arrow[uu] \arrow[r] \arrow[d] & \text{elliptic in } \mathcal{C} \arrow[ur] \arrow[r] \arrow[dr] & \text{conjugated to a subgroup of } \operatorname{Stab}([x,y]) & \\
\text{lineal in } \mathcal{C} & & \text{conjugated to a subgroup of} O_4 &
\end{tikzcd}$$

Figure 7. Outline scheme of the proof of Theorem 6.2.

Moreover, the subgroup H is conjugated in Tame(Q) *to a subgroup of O_4 or to one of*

$$E_H \rtimes \left\{ \begin{pmatrix} ax & by \\ b^{-1}z & a^{-1}t \end{pmatrix} \,\middle|\, a, b \in k^* \right\}.$$

Assume that the above theorem holds, we prove that our classification holds. If G is lineal on the complex, then assertion (ii) in Theorem 6.3 implies that there exists an hyperbolic automorphism $h \in G$ such that any $f \in G$ can be decomposed into $f = g \circ h^p$ where p is an integer and $g \in H$. This falls into situation (ii) of the theorem.

Let us extend the case (i) of Theorem 6.3. Take a group G whose action fixes a vertex of type III, then it is naturally conjugated to a subgroup of O_4 by Proposition 2.5 and assertion (iii) holds. If G fixes a vertex of type II in the complex, then by Proposition 2.6, G satisfies assertion (iv) of the Theorem.

Suppose now that G fixes a vertex of type I then G is conjugated to a subgroup of $\operatorname{Stab}([x])$ by transitivity of the action on the vertices of type I (Proposition 2.7(ii)). In this situation, recall that we have constructed in Section 2D a natural action from the stabilizer subgroup $\operatorname{Stab}([x])$ on a subtree of the Bass–Serre tree, in particular, there exists a G-equivariant morphism $\varphi : Q \to \mathbb{A}^1$ where the action of G on $\mathbb{A}^1$ is multiplicative. In the case where the group G fixes a vertex of type I, its action on the corresponding subtree of the Bass–Serre tree is either nonelementary or elementary. If the action of G is nonelementary on the subtree, then equivalently G contains two noncommuting morphisms with dynamical degree larger or equal to two and G satisfies assertion (v) in our classification.

Suppose that the action of G on the corresponding subtree of the Bass–Serre tree is elementary. Let us fix an isomorphism $\varphi^{-1}(\mathbb{A}^1 \setminus \{0\}) \simeq \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2$. Since G induces an action on $\mathbb{A}^1$ which is multiplicative, the open subset $\varphi^{-1}(\mathbb{A}^1 \setminus \{0\})$ is preserved by each element of G. Assume that G contains only elliptic elements on the subtree. We reproduce some standard arguments for groups acting on trees. Since G is finitely generated, write $G = \langle g_1, \dots, g_n \rangle$ where g_i are the generators of G. Since G is elliptic, any product $g_i g_j$ is elliptic, hence admits a fixed point on $\mathcal{T}_{\pi,k(x)}$. By [Serre 1977, Proposition 26 page 89], the group G has a global fixed point in $\mathcal{T}_{\pi,k(x)}$. Conjugating by an element in $\operatorname{Stab}([x])$, we can suppose the fixed point in $\mathcal{T}_{\pi,k(x)}$ is either $[z]$ or $[y, z]$. In the first case, this implies that $G \subset \tilde{E}$ and $G \subset \tilde{A}$ in the second, where $\tilde{A}$, $\tilde{E}$ are the groups defined in assertion (v) of Proposition 2.10. This shows that assertion (vii) holds.

Otherwise, the action of G on the subtree is lineal. In particular, there exists an automorphism $h \in G$ whose action on the subtree is hyperbolic such that any $f \in G$ can be decomposed into $f = g \circ h^p$ where p is an integer and where g is an automorphism whose action on the subtree is elliptic and fixes pointwise the geodesic line on the subtree fixed by h.

We have thus proved that assertion (vi) holds. □

Proof of Theorem 6.3. Take G a finitely generated subgroup of the tame group. By [Ballmann and Świątkowski 1999, Theorem 2], the subgroup G' must satisfy one of the following three cases:

(a) The group G' is elliptic.

(b) There exists an integer $2 \geqslant k \geqslant 1$, an elliptic subgroup $H \subset G$ and a subspace $E \subset \mathcal{C}$ which is isometric to a k-dimensional euclidean space, is pointwise fixed by H and on which the group G/H acts as a cocompact lattice of translation.

(c) Every automorphism of G is elliptic and there exists a geodesic half-line and a sequence of vertices v_n on this half-line for which the subgroups $G_n = G \cap \operatorname{Stab}(v_n)$ form an increasing filtration which satisfy $G = \cup G_n$.

Since the complex $\mathcal{C}$ is Gromov hyperbolic, it cannot contain any euclidean plane. As a result, the case $k = 2$ in (b) is excluded. The remaining possibility is when $k = 1$ and there exists a geodesic line E globally invariant by G in the complex, a subgroup H of G fixing E pointwise such that G/H acts faithfully transitive by translation on E. Remark also that (c) cannot hold. Indeed, if there was an increasing sequence of subgroups G_n whose union is G, then the each generator would belong to a certain subgroup. Since G is finitely generated, this would mean that $G = G_n$ for a certain n, which contradicts our assumption.

To prove that (ii) holds amounts in proving that in case (b) the elliptic subgroup H is conjugated to a subgroup of O_4 or to a subgroup of

$$E_H \rtimes \left\{ \begin{pmatrix} ax & by \\ b^{-1}z & a^{-1}t \end{pmatrix} \,\middle|\, a, b \in k^* \right\}.$$

Since H fixes, pointwise, a geodesic line E, we can choose a sequence v_n of distinct vertices near E all fixed by H lying on a quasigeodesic line. Consider γ_n a geodesic path in the 1-skeleton of $\mathcal{C}$ joining v_n and v_{n+1}. Since the group G fixes the type of vertices and the endpoint of γ_n, the geodesic γ_n must be fixed pointwise. If one of the geodesic γ_n contains a vertex of type III, then G is conjugated to a subgroup of O_4 and statement (ii) is proved.

Assume now that the geodesics γ_n contain only type I and II vertices. We prove that (ii) also holds. For simplicity, we can assume that v_0, v_1, v_2 pointwise fixed by G are consecutive vertices on a geodesic line of the 1-skeleton of $\mathcal{C}$. Assume also that v_0, v_2 are of type I and that v_1 is of type II. Conjugating with an element of Tame(Q), we can assume that $v_0 = [z]$, $v_1 = [x, z]$ and $v_2 = [x]$. Since G fixes $[x]$, $[x, z]$

and $[z]$, it implies by Proposition 2.6(iii) that G is conjugated to a subgroup of

$$E_H \rtimes \left\{ \begin{pmatrix} ax & by \\ b^{-1}z & a^{-1}t \end{pmatrix} \;\middle|\; a, b \in k^* \right\},$$

proving (ii) as required. □

6D. ***Proof of Theorem 5 and Corollary 6.*** Take G a finitely generated subgroup of the tame group and take μ a symmetric atomic measure on G whose support generates G and such that

$$\int_G \log(\deg(g))\, d\mu(g) < +\infty.$$

We denote by g_n the state of our random walk at the time n. Observe that since μ is symmetric, Proposition 6.1(iii) implies that $\lambda_2(\mu) = \lambda_1(\mu)$.

Let us explain how we proceed to prove our result. By Theorem 6.2, the group G satisfies one of the following conditions:

(i) The action of G on the complex is nonelementary in $\mathcal{C}$.

(ii) There exists an automorphism h in G whose action in the complex is hyperbolic and such that any automorphism $f \in G$ can be decomposed into $f = g \circ h^p$ where p is an integer and where g belongs to a subgroup H.

Moreover, the subgroup H is conjugated in Tame(Q) to a subgroup of O_4 or to one of

$$E_H \rtimes \left\{ \begin{pmatrix} ax & by \\ b^{-1}z & a^{-1}t \end{pmatrix} \;\middle|\; a, b \in k^* \right\}.$$

(iii) The group G is conjugated to a subgroup of the linear group O_4.

(iv) There exists a G-equivariant morphism $\varphi : Q \to \mathbb{A}^2 \setminus \{(0,0)\}$ where G acts on $\mathbb{A}^2 \setminus \{(0,0)\}$ linearly.

(v) The group G contains two noncommuting automorphisms with dynamical degree larger or equal to 2 and there exists a G-equivariant morphism $\varphi : Q \to \mathbb{A}^1$ where G acts on $\mathbb{A}^1$ by multiplication.

(vi) The group G contains an automorphism h with $\lambda_1(h) \geqslant 2$ and there exists a G-equivariant morphism $\varphi : Q \to \mathbb{A}^1$ on which G acts on $\mathbb{A}^1$ by multiplication and an isomorphism $\varphi^{-1}(\mathbb{A}^1 \setminus \{0\}) \simeq \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2$ such that any automorphisms $f \in G$ can be decomposed into $g \circ h^p$ where p is an integer and g is of the form

$$g : (x, y, z) \in \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2 \mapsto (ax, by, cz) \in \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2,$$

where $a, b, c \in k^*$.

(vii) There exists a G-equivariant morphism $\varphi : Q \to \mathbb{A}^1$ where G acts on $\mathbb{A}^1$ by multiplication and any automorphism of G has dynamical degree 1.

Denote by $\lambda = \lambda_1(\mu) = \lambda_2(\mu)$. We shall prove successively the following implications: (i) $\Rightarrow$ ($\lambda > 0$), (v) $\Rightarrow$ ($\lambda > 0$), ((ii), (iv) or (vi)) $\Rightarrow$ ($\lambda = 0$), ((iii) or (vii)) $\Rightarrow$ ($\lambda = 0$). If all the above implications hold, then both Theorem 5 and Corollary 6 hold.

The essential ingredient to compute the degree exponents in situation (i) and (v) is the following result. Suppose that G acts on a Gromov-hyperbolic space (X, d) and fix a reference vertex x_0 in X, then a random path $(\mathrm{Id}, g_1, \dots, g_n, \dots)$ in the group G induces a random path in X given by $(x_0, g_1 \cdot x_0, \dots, g_n \cdot x_0, \dots)$. The following theorem is due to Maher and Tiozzo [2018, Theorem 1.2].

Theorem 6.4. *Let G be a nonelementary countable subgroup of the tame group and let μ be an atomic measure on G whose support generates G and such that the integral*

$$\int_G d(g \cdot v_0, v_0)\, d\mu(g)$$

is finite. Then there exists a constant $L > 0$ such that for almost every sample path in the group, one has

$$\lim_{n \to +\infty} \frac{d(g_n \cdot v_0, v_0)}{n} = L.$$

We will apply this result for $X = \mathcal{C}$ and $X = \mathcal{T}$ in situation (i) and (v) respectively.

Let us prove the implication (i) $\Rightarrow$ $(\lambda > 0)$. Suppose that G is nonelementary in $\mathcal{C}$. By Theorem 4, there exists $C, C' > 0$ such that for any $g \in G$

$$\log(\deg(g)) \geqslant C d_{\mathcal{C}}([\mathrm{Id}], g \cdot [\mathrm{Id}]) \log\left(\tfrac{4}{3}\right) + C'. \tag{30}$$

In particular, the previous inequality and (29) imply that

$$C \log\left(\tfrac{4}{3}\right) \int d_{\mathcal{C}}([\mathrm{Id}], g \cdot [\mathrm{Id}])\, d\mu(g) + C' \leqslant \int_G \log(\deg(g))\, d\mu(g) < +\infty.$$

As G is nonelementary, Theorem 6.4 states that there exists a constant $L > 0$ such that for almost every sample path

$$\liminf_{n \to +\infty} \frac{d_{\mathcal{C}}([\mathrm{Id}], g_n \cdot [\mathrm{Id}])}{n} = L. \tag{31}$$

Moreover, by Theorem 4, the following inequality holds:

$$\log \deg(g_n) \geqslant C d_{\mathcal{C}}([\mathrm{Id}], g_n \cdot [\mathrm{Id}]) \log\left(\tfrac{4}{3}\right) + C', \tag{32}$$

where $C, C' > 0$. As a result, (31) and (32) imply that

$$\frac{\log \deg(g_n)}{n} \geqslant \frac{C}{n} d_{\mathcal{C}}([\mathrm{Id}], g_n \cdot [\mathrm{Id}]) \log\left(\tfrac{4}{3}\right) + \frac{C'}{n},$$

hence taking the limit as $n \to +\infty$ yields

$$\lambda \geqslant C L \log\left(\tfrac{4}{3}\right) > 0,$$

and we have proved that $\lambda > 0$, as required. The implication (i) $\Rightarrow$ $(\lambda > 0)$ holds.

Let us prove the implication (v) $\Rightarrow$ $(\lambda > 0)$. Suppose that G satisfies condition (v). By conjugation, we can suppose that G is a subgroup of $\mathrm{Stab}([x])$. We first relate the distance in the tree $\mathcal{T}_{\pi, k(x)}$ with the degree. Recall from Proposition 2.10(v) that the group $\mathrm{Stab}([x])$ is the amalgamated product $\tilde{E} * \tilde{A}$. Any element g which is neither conjugated to an element of $\tilde{E}$ or $\tilde{A}$ can be decomposed by an alternating product:

$$g = e_1 \circ a_1 \circ e_2 \circ a_2 \cdots \circ a_{p-1} \circ e_p$$

where $e_i \in \tilde{E} \setminus \tilde{A}$, $a_i \notin \tilde{A} \setminus \tilde{E}$. This decomposition reflects the length of the geodesic joining $[\mathrm{Id}] = [y, z]$ and $g[\mathrm{Id}]$ in the sense that

$$d_{\mathcal{T}_{\pi,k(x)}}(g \cdot [\mathrm{Id}], [\mathrm{Id}]) = 2p. \tag{33}$$

Note that when g is elementary or affine, then the above inequality is also true, so we obtain that it holds for any $g \in G$. Using the proof of assertion (vi) of Lemma 3.6 (more precisely (14)), we have

$$\deg\left(\prod i = 1^k e_i \circ a_i\right) \geqslant \prod_{i=1}^{k} d_i \geqslant 2^k,$$

where d_i are the degree in y of the polynomials defining e_i.

In particular, using the fact that G contains two noncommuting hyperbolic isometries on $\mathcal{T}_{\pi,k(x)}$ and Theorem 6.4, we obtain similarly that

$$\frac{1}{n}\int_G \log\deg(g)\, d\nu_n(g) \geqslant \frac{1}{2n}\int_G d_{\mathcal{T}}(g \cdot [\mathrm{Id}], [\mathrm{Id}])\, d\nu_n(g) \to \frac{L}{2},$$

as $n \to +\infty$ where $L > 0$ is the drift of the associated to the random walk on the tree $\mathcal{T}$. We have thus proven that $\lambda > 0$ and the implication (v) $\Rightarrow$ ($\lambda > 0$) holds.

Let us prove that the implication ((ii) or (vi)) $\Rightarrow$ ($\lambda = 0$) holds. Since the proof of the two implications (ii) $\Rightarrow$ ($\lambda = 0$) and (vi) $\Rightarrow$ ($\lambda = 0$) are very similar, we will only give the proof of (ii) $\Rightarrow$ ($\lambda = 0$). Suppose that there exists an hyperbolic automorphism $h \in G$ such that any automorphism $f \in G$ can be decomposed into $f = u(f) \circ h^{p(f)}$ where $p(f)$ is an integer and $u(f)$ belongs to H. We thus have

$$\frac{1}{n}\int_G \log\deg(g)\, d\nu_n(g) = \frac{1}{n}\int_G \log\deg(u(g) \circ h^{p(g)})\, d\nu_n(g).$$

By the submultiplicativity of the degree, we have

$$\deg(u(g) \circ h^{p(g)}) \leqslant C \deg(u(g)) \deg(h^{p(g)})$$

where $C > 0$. In particular, we obtain

$$\frac{1}{n}\int_G \log\deg(g)\, d\nu_n(g) \leqslant \frac{C}{n} + \frac{1}{n}\int_G \log(\deg(u(g)))\, d\nu_n(g) + \frac{1}{n}\int_G \log\deg(h^{p(g)})\, d\nu_n(g). \tag{34}$$

Since the map $p : G \to \mathbb{Z}$ is a morphism of groups, the random walk on G induces a random walk on $\mathbb{Z}$ with transition given by the distribution $p_*\mu$. As the measure $p_*\mu$ is also symmetric, the law of large numbers implies that

$$\frac{1}{n}\int_G \log\deg(h^{p(g)})\, d\nu_n(g) \to 0,$$

as $n \to +\infty$. Observe also that there exists a constant $M > 0$ such that $\deg(u(g)) \leqslant M$ for all $g \in G$. In particular, the integral

$$\frac{1}{n} \int_G \log \deg(u(g))\, d\nu_n(g) \to 0,$$

as $n \to +\infty$. Since each term on the right hand side of (34) tends to zero, we have thus proven that

$$\lambda = \lim_{n \to +\infty} \frac{1}{n} \int_G \log \deg(g)\, d\nu_n(g) = 0$$

and the implication (ii) $\Rightarrow$ ($\lambda = 0$) holds.

Let us prove that the implication ((iii), (iv) or (vii)) $\Rightarrow$ ($\lambda = 0$) holds. Observe that if G satisfies assertion (iii) or (iv) or (vi) then the degree of any element of G is uniformly bounded, hence the degree exponent is zero. We have thus proved the implication ((iii), (iv) or (vii)) $\Rightarrow$ ($\lambda = 0$).

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References

[Abarenkova et al. 1999a] N. Abarenkova, J.-C. Anglès d'Auriac, S. Boukraa, S. Hassani, and J.-M. Maillard, "From Yang–Baxter equations to dynamical zeta functions for birational transformations", pp. 436–490 in *Statistical physics on the eve of the 21st century*, edited by M. T. Batchelor and L. T. Wille, Ser. Adv. Statist. Mech. **14**, World Sci. Publ., River Edge, NJ, 1999. MR

[Abarenkova et al. 1999b] N. Abarenkova, J.-C. Anglès d'Auriac, S. Boukraa, and J.-M. Maillard, "Growth-complexity spectrum of some discrete dynamical systems", *Phys. D* **130**:1-2 (1999), 27–42. MR Zbl

[Anglès d'Auriac et al. 2006] J.-C. Anglès d'Auriac, J.-M. Maillard, and C. M. Viallet, "On the complexity of some birational transformations", *J. Phys. A* **39**:14 (2006), 3641–3654. MR Zbl

[Ballmann and Świątkowski 1999] W. Ballmann and J. Świątkowski, "On groups acting on nonpositively curved cubical complexes", *Enseign. Math.* (2) **45**:1-2 (1999), 51–81. MR Zbl

[Bedford 2015] E. Bedford, "Invertible dynamics on blow-ups of $\mathbb{P}^k$", pp. 67–88 in *Complex analysis and geometry*, edited by F. Bracci et al., Springer Proc. Math. Stat. **144**, Springer, 2015. MR Zbl

[Bedford and Diller 2005] E. Bedford and J. Diller, "Energy and invariant measures for birational surface maps", *Duke Math. J.* **128**:2 (2005), 331–368. MR Zbl

[Bedford and Kim 2014] E. Bedford and K. Kim, "Dynamics of (pseudo) automorphisms of 3-space: periodicity versus positive entropy", *Publ. Mat.* **58**:1 (2014), 65–119. MR Zbl

[Bedford and Smillie 1992] E. Bedford and J. Smillie, "Polynomial diffeomorphisms of $\mathbf{C}^2$, III: Ergodicity, exponents and entropy of the equilibrium measure", *Math. Ann.* **294**:3 (1992), 395–420. MR Zbl

[Bedford and Truong 2010] E. Bedford and T. T. Truong, "Degree complexity of birational maps related to matrix inversion", *Comm. Math. Phys.* **298**:2 (2010), 357–368. MR Zbl

[Benoist and Quint 2016] Y. Benoist and J.-F. Quint, *Random walks on reductive groups*, Ergebnisse der Math. (3) **62**, Springer, 2016. MR

[Bisi et al. 2014] C. Bisi, J.-P. Furter, and S. Lamy, "The tame automorphism group of an affine quadric threefold acting on a square complex", *J. Éc. polytech. Math.* **1** (2014), 161–223. MR

[Blanc and Cantat 2016] J. Blanc and S. Cantat, "Dynamical degrees of birational transformations of projective surfaces", *J. Amer. Math. Soc.* **29**:2 (2016), 415–471. MR Zbl

[Blanc and van Santen 2019] J. Blanc and I. van Santen, "Automorphisms of the affine 3-space of degree 3", preprint, 2019. Zbl arXiv 1912.02144

[Boucksom et al. 2008] S. Boucksom, C. Favre, and M. Jonsson, "Degree growth of meromorphic surface maps", *Duke Math. J.* **141**:3 (2008), 519–538. MR Zbl

[Bridson and Haefliger 1999] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundl. Math. Wissen. **319**, Springer, 1999. MR Zbl

[Cantat 2011] S. Cantat, "Sur les groupes de transformations birationnelles des surfaces", *Ann. of Math.* (2) **174**:1 (2011), 299–340. MR Zbl

[Cantat and Xie 2020] S. Cantat and J. Xie, "On degrees of birational mappings", *Math. Res. Lett.* **27**:2 (2020), 319–337. MR Zbl

[Dahmani and Horbez 2018] F. Dahmani and C. Horbez, "Spectral theorems for random walks on mapping class groups and out (F_N)", *Int. Math. Res. Not.* **2018**:9 (2018), 2693–2744. MR Zbl

[Dang 2018] N.-B. Dang, *Degree growth of rational maps in dimension three*, Ph.D. thesis, Université Paris-Saclay, 2018.

[Dang 2020] N.-B. Dang, "Degrees of iterates of rational maps on normal projective varieties", *Proc. Lond. Math. Soc.* (3) **121**:5 (2020), 1268–1310. MR Zbl

[Dang and Favre 2021] N.-B. Dang and C. Favre, "Spectral interpretations of dynamical degrees and applications", *Ann. of Math.* (2) **194**:1 (2021), 299–359. MR Zbl

[Diller and Favre 2001] J. Diller and C. Favre, "Dynamics of bimeromorphic maps of surfaces", *Amer. J. Math.* **123**:6 (2001), 1135–1169. MR Zbl

[Dinh and Sibony 2005] T.-C. Dinh and N. Sibony, "Une borne supérieure pour l'entropie topologique d'une application rationnelle", *Ann. of Math.* (2) **161**:3 (2005), 1637–1644. MR Zbl

[Favre and Jonsson 2011] C. Favre and M. Jonsson, "Dynamical compactifications of $\mathbf{C}^2$", *Ann. of Math.* (2) **173**:1 (2011), 211–248. MR Zbl

[Favre and Wulcan 2012] C. Favre and E. Wulcan, "Degree growth of monomial maps and McMullen's polytope algebra", *Indiana Univ. Math. J.* **61**:2 (2012), 493–524. MR Zbl

[Fornaess and Sibony 1995] J. E. Fornaess and N. Sibony, "Complex dynamics in higher dimension, II", pp. 135–182 in *Modern methods in complex analysis* (Princeton, NJ, 1992), edited by T. Bloom et al., Ann. of Math. Stud. **137**, Princeton Univ. Press, 1995. MR Zbl

[Friedland and Milnor 1989] S. Friedland and J. Milnor, "Dynamical properties of plane polynomial automorphisms", *Ergodic Theory Dynam. Systems* **9**:1 (1989), 67–99. MR

[Fulton and Harris 1991] W. Fulton and J. Harris, *Representation theory*, Graduate Texts in Mathematics **129**, Springer, 1991. MR Zbl

[Furstenberg 1963] H. Furstenberg, "Noncommuting random products", *Trans. Amer. Math. Soc.* **108** (1963), 377–428. MR Zbl

[Furstenberg and Kesten 1960] H. Furstenberg and H. Kesten, "Products of random matrices", *Ann. Math. Statist.* **31** (1960), 457–469. MR Zbl

[Gizatullin 1980] M. H. Gizatullin, "Rational G-surfaces", *Izv. Akad. Nauk SSSR Ser. Mat.* **44**:1 (1980), 110–144, 239. MR Zbl

[Gromov 1987] M. Gromov, "Hyperbolic groups", pp. 75–263 in *Essays in group theory*, edited by S. M. Gersten, Math. Sci. Res. Inst. Publ. **8**, Springer, 1987. MR Zbl

[Guedj 2005] V. Guedj, "Ergodic properties of rational mappings with large topological degree", *Ann. of Math.* (2) **161**:3 (2005), 1589–1607. MR Zbl

[Jonsson and Mustaţă 2012] M. Jonsson and M. Mustaţă, "Valuations and asymptotic invariants for sequences of ideals", *Ann. Inst. Fourier* (*Grenoble*) **62**:6 (2012), 2145–2209. MR Zbl

[Jung 1942] H. W. E. Jung, "Über ganze birationale Transformationen der Ebene", *J. Reine Angew. Math.* **184** (1942), 161–174. MR Zbl

[Kingman 1973] J. F. C. Kingman, "Subadditive ergodic theory", *Ann. Probability* **1** (1973), 883–909. MR Zbl

[Kuroda 2016] S. Kuroda, "Weighted multidegrees of polynomial automorphisms over a domain", *J. Math. Soc. Japan* **68**:1 (2016), 119–149. MR Zbl

[Lamy 2019] S. Lamy, "Combinatorics of the tame automorphism group", *Ann. Fac. Sci. Toulouse Math.* (6) **28**:1 (2019), 145–207. MR Zbl

[Lamy and Przytycki 2021] S. Lamy and P. Przytycki, "Presqu'un immeuble pour le groupe des automorphismes modérés", *Ann. H. Lebesgue* **4** (2021), 605–651. MR Zbl

[Lamy and Vénéreau 2013] S. Lamy and S. Vénéreau, "The tame and the wild automorphisms of an affine quadric threefold", *J. Math. Soc. Japan* **65**:1 (2013), 299–320. MR

[Lang 2002] S. Lang, *Algebra*, 2nd ed., Graduate Texts in Mathematics **211**, Springer, 2002. MR Zbl

[Lin 2012] J.-L. Lin, "Algebraic stability and degree growth of monomial maps", *Math. Z.* **271**:1-2 (2012), 293–311. MR Zbl

[Lo Bianco 2019] F. Lo Bianco, "On the primitivity of birational transformations of irreducible holomorphic symplectic manifolds", *Int. Math. Res. Not.* **2019**:1 (2019), 1–32. MR Zbl

[Lonjou and Urech 2021] A. Lonjou and C. Urech, "Actions of Cremona groups on CAT(0) cube complexes", *Duke Math. J.* **170**:17 (2021), 3703–3743. MR Zbl

[Maegawa 2001] K. Maegawa, "Classification of quadratic polynomial automorphisms of $\mathbb{C}^3$ from a dynamical point of view", *Indiana Univ. Math. J.* **50**:2 (2001), 935–951. MR Zbl

[Maher and Tiozzo 2018] J. Maher and G. Tiozzo, "Random walks on weakly hyperbolic groups", *J. Reine Angew. Math.* **742** (2018), 187–239. MR Zbl

[Oguiso and Truong 2014] K. Oguiso and T. T. Truong, "Salem numbers in dynamics on Kähler threefolds and complex tori", *Math. Z.* **278**:1-2 (2014), 93–117. MR Zbl

[Oguiso and Truong 2015] K. Oguiso and T. T. Truong, "Explicit examples of rational and Calabi–Yau threefolds with primitive automorphisms of positive entropy", *J. Math. Sci. Univ. Tokyo* **22**:1 (2015), 361–385. MR Zbl

[Russakovskii and Shiffman 1997] A. Russakovskii and B. Shiffman, "Value distribution for sequences of rational mappings and complex dynamics", *Indiana Univ. Math. J.* **46**:3 (1997), 897–932. MR Zbl

[Serre 1977] J.-P. Serre, *Arbres, amalgames*, SL_2, Astérisque **46**, Société Mathématique de France, Paris, 1977. MR Zbl

[Shestakov and Umirbaev 2003] I. P. Shestakov and U. U. Umirbaev, "The Nagata automorphism is wild", *Proc. Natl. Acad. Sci. USA* **100**:22 (2003), 12561–12563. MR Zbl

[Truong 2016] T. T. Truong, "Some dynamical properties of pseudo-automorphisms in dimension 3", *Trans. Amer. Math. Soc.* **368**:1 (2016), 727–753. MR Zbl

[Truong 2017] T. T. Truong, "Automorphisms of blowups of threefolds being Fano or having Picard number 1", *Ergodic Theory Dynam. Systems* **37**:7 (2017), 2255–2275. MR Zbl

[Urech 2018] C. Urech, "Remarks on the degree growth of birational transformations", *Math. Res. Lett.* **25**:1 (2018), 291–308. MR Zbl

[Vaquié 2000] M. Vaquié, "Valuations", pp. 539–590 in *Resolution of singularities* (Obergurgl, 1997), edited by H. Hauser et al., Progr. Math. **181**, Birkhäuser, Basel, 2000. MR Zbl

[Wright 2015] D. Wright, "The generalized amalgamated product structure of the tame automorphism group in dimension three", *Transform. Groups* **20**:1 (2015), 291–304. MR Zbl

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nguyen-bac.dang@universite-paris-saclay.fr
Universite Paris-Saclay, Institut Mathématique d'Orsay, Orsay, France

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A weighted one-level density of families of L-functions

Alessandro Fazzari

This paper is devoted to a weighted version of the one-level density of the nontrivial zeros of L-functions, tilted by a power of the L-function evaluated at the central point. Assuming the Riemann hypothesis and the ratio conjecture, for some specific families of L-functions, we prove that the same structure suggested by the density conjecture also holds in this weighted investigation, if the exponent of the weight is small enough. Moreover, we speculate about the general case, conjecturing explicit formulae for the weighted kernels.

1. A weighted version of the one-level density

Let us assume the Riemann hypothesis for all the L-functions that arise. The classical one-level density considers a smooth localization at the central point of the counting function of the nontrivial zeros of an L-function, averaged over a "natural" family of L-functions in the Selberg class[1]. More specifically, given an even and real-valued function f in the Schwartz space[2] and an L-function $L(s)$ in a family $\mathcal{F}$, we consider the quantity

$$\sum_{\gamma_L} f(c(L)\gamma_L), \tag{1-1}$$

where γ_L denotes the imaginary part of a generic nontrivial zero of L and $c(L)$ denotes the log-conductor of $L(s)$ at the central point. We recall that $1/c(L)$ is the mean spacing of the nontrivial zeros of $L(s)$ around $s=\frac{1}{2}$. The one-level density for the family $\mathcal{F}$ is the average of the above quantity over the family, i.e.,

$$\lim_{X\to\infty} \frac{1}{\sum_{L\in\mathcal{F}_X} 1} \sum_{L\in\mathcal{F}_X} \sum_{\gamma_L} f(c(L)\gamma_L), \tag{1-2}$$

with

$$\mathcal{F}_X := \{L \in \mathcal{F} : c(L) \le \log X\}.$$

In the literature, this is also referred to as the "low-lying zeros" density, as the sum (1-1) gives information on the distribution of the zeros of L which are close to the central point. Indeed, if a zero is substantially more than $1/c(L)$ away from the central point, then it does not contribute significantly to the sum (see, e.g., [Iwaniec et al. 2000] for a complete overview).

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[1] We refer, e.g., to [Kaczorowski and Perelli 1999] for the definitions and the basic properties of the Selberg class.

[2] In practice we will see that this condition can be weakened and a decay like $f(x) \ll 1/(1+x^2)$ at infinity will suffice.

Katz and Sarnak [1999] studied a wide variety of families and attached to each of these families of L-functions is a symmetry type (i.e., unitary, symplectic, or orthogonal, hereafter identified by a group G), which should govern the one-level density of the considered family. Namely, the density conjecture predicts that

$$\lim_{X\to\infty} \frac{1}{\sum_{L\in\mathcal{F}_X} 1} \sum_{L\in\mathcal{F}_X} \sum_{\gamma_L} f(c(L)\gamma_L) = \int_{-\infty}^{+\infty} f(x) W_{\mathcal{F}}(x)\,dx, \tag{1-3}$$

where $W_{\mathcal{F}}$ equals the one-level density function W_G for the (scaled) limit of

$$G \in \{\mathrm{U}(N), \mathrm{USp}(2N), \mathrm{O}(N), \mathrm{SO}(2N), \mathrm{SO}(2N+1)\}^3,$$

i.e., the kernel appearing in the analogous average in the corresponding random matrix theory setting. In particular, the kernel $W_{\mathcal{F}}$ is predicted to depend on G only. We recall that the function W_G is known for all of the classical compact groups, being

$$\begin{aligned}
W_{\mathrm{U}}(x) &= 1,\\
W_{\mathrm{USp}}(x) &= 1 - \frac{\sin(2\pi x)}{2\pi x},\\
W_{\mathrm{O}}(x) &= 1 + \frac{1}{2}\delta_0(x),\\
W_{\mathrm{SO}^+}(x) &= 1 + \frac{\sin(2\pi x)}{2\pi x},\\
W_{\mathrm{SO}^-}(x) &= \delta_0(x) + 1 - \frac{\sin(2\pi x)}{2\pi x},
\end{aligned}$$

where δ_0 is the Dirac δ-function centered at 0. Examples of one-level density theorems which prove (1-3) in specific cases can be found, e.g., in [Iwaniec et al. 2000; Hughes and Rudnick 2002; 2003; Özlük and Snyder 1999; Conrey 2005; Conrey and Snaith 2007].

In this paper, we investigate a weighted analogue of the one-level density. In particular, we consider a tilted average over the family $\mathcal{F}$ of the quantity (1-1), multiplied by a power of L evaluated at the central point. The philosophy of this tilted average is similar to that of Fazzari [2021a; 2021b]; the weight has the effect of giving more relevance to the L-functions which are large at the central point, near which zeros are expected to be rarer.

More specifically, given $k \in \mathbb{N}$, we are interested in

$$\mathcal{D}_k^{\mathcal{F}}(f) = \mathcal{D}_k^{\mathcal{F}}(f, X) := \frac{1}{\sum\limits_{L\in\mathcal{F}_X} V(L(1/2))^k} \sum_{L\in\mathcal{F}_X} \sum_{\gamma_L} f(c(L)\gamma_L) V\big(L\big(\tfrac{1}{2}\big)\big)^k \tag{1-4}$$

in the limit $X \to \infty$, where V depends on the symmetry type of the family; in particular, $V(z) = |z|^2$ in the unitary case and $V(z) = z$ for the symplectic and orthogonal cases. The quantity $\mathcal{D}_k^{\mathcal{F}}(f)$ links the

[3]Note that the scaled limit of $\mathrm{SO}(2N)$ (respectively, $\mathrm{SO}(2N+1)$) is commonly denoted by SO^+ (respectively, SO^-) in the literature and also in the rest of this paper.

moments to the one-level density, making the connection between nontrivial zeros and the size of $L\big(\frac{1}{2}\big)$ explicit. Indeed, $\mathcal{D}_k^{\mathcal{F}}(f)$ can be seen as a special case of

$$\frac{1}{\sum_{L\in\mathcal{F}_X} V(L(1/2))^k}\sum_{L\in\mathcal{F}_X} g(L)V\big(L\big(\tfrac{1}{2}\big)\big)^k \tag{1-5}$$

where $g(L)$ is a function over the L-functions of a given family $\mathcal{F}$. In the unitary case, for example, we know from Soundararajan's work [2009] that the dominant contribution to the $2k$-th moment comes from those L-functions such that the size of $\big|L\big(\frac{1}{2}\big)\big|$ is about $(\log X)^{k+o(1)}$, which form a thin subset of size about $\#\mathcal{F}_X/(\log T)^{k^2+o(1)}$. Thus, if the function g has size 1, then only these L-functions contribute to the main term of the sum in (1-5). With the choice we made in (1-4), we have $g(L)=\sum_{\gamma_L} f(c(L)\gamma_L)$, which is not bounded but only $\ll c(L)$, by the Riemann–von Mangoldt formula. However, the standard n-th level density [Rudnick and Sarnak 1996] implies that $g(L)\ll c(L)^{\varepsilon}$ for all but $\#\mathcal{F}_X/(\log X)^A$ L-functions in the family, for every $A>0$. Therefore, also in (1-4), we have that only the L-functions such that $\big|L\big(\frac{1}{2}\big)\big|\asymp(\log X)^{k\pm\varepsilon}$ contribute significantly to the main term of the sum. For this reason, for unitary families, $\mathcal{D}_k^{\mathcal{F}}$ can be interpreted as a (weighted) one-level density for the thin subset

$$\big\{L\in\mathcal{F}:(\log X)^{k-\varepsilon}\ll\big|L\big(\tfrac{1}{2}\big)\big|\ll(\log X)^{k+\epsilon}\big\}.$$

Similarly, in the symplectic and orthogonal cases, $\mathcal{D}_k^{\mathcal{F}}$ is a weighted one-level density, focused on the L-functions in the family which are responsible to the k-th moment.

From the computations we perform throughout this paper in some specific cases, we speculate that the structure suggested by the density conjecture (1-3) holds also in the weighted case. Namely, we expect that

$$\mathcal{D}_k^{\mathcal{F}}(f)=\int_{-\infty}^{+\infty} f(x)W_G^k(x)\,dx+O\Big(\frac{1}{\log X}\Big), \tag{1-6}$$

where the weighted one-level density function W_G^k only depends on k and on the symmetry type of the family $\mathcal{F}$. Note that the superscript k is an index, indicating that we are weighting with the k-th power of $V\big(L\big(\frac{1}{2}\big)\big)$; in particular W_G^k is not the k-th power of W_G.

This kind of weighting naturally appears also in other contexts, such as Kowalski, Saha and Tsimerman's paper [Kowalski et al. 2012]. Given a Siegel modular form F of genus 2, the authors compute the one-level density of the spinor L-functions of F, with a weight ω^F which is essentially the modulus square of the first Fourier coefficient[4] of F. This family is expected to be orthogonal, but with this weight one does not obtain the usual kernel W_{O}. This discrepancy can be explained by Böcherer's conjecture [1986] and [Dickson et al. 2020] (now proved by Furusawa and Morimoto [2021]), which claims that ω^F is proportional to the central value $L\big(\frac{1}{2},F\big)$. To be more precise, it says that $\omega^F\approx L\big(\frac{1}{2},F\big)L\big(\frac{1}{2},F\times\chi_4\big)$. Since $L\big(\frac{1}{2},F\times\chi_4\big)$ is "uncorrelated" with $L(s,F)$ and with its zeros, then the kernel they obtained is indeed $W^1_{\mathrm{SO}^+}$ (see, e.g., (5-17)[5] and note that weighting with $L\big(\frac{1}{2},F\big)^k$, the odd part of the family does not contribute, if $k>0$). Moreover, they notice that this kernel is the one that arises from symplectic symmetry types.

[4] I.e., the Fourier coefficient corresponding to the identity matrix.

[5] In [Kowalski et al. 2012], the kernel is written as $1-\delta_0/2$, which is equivalent to $W^1_{\mathrm{SO}^+}$ for test functions whose Fourier transforms are supported in $[-1,1]$, which is an assumption in [Kowalski et al. 2012].

Thus, the symmetry of the family jumps from O to USp, after weighting with ω^F (see also [Knightly and Reno 2019; Sugiyama 2021] for other examples where this phenomenon of change of symmetry type is observed). This transition can be seen as a particular case of (2-4) below, which conjecturally predicts a relation between the weighted one-level density functions of different symmetry types.

2. Statement of main results

In the following, we focus on three specific families of L-functions, each with a different symmetry type; first we consider the unitary family $\boldsymbol{\zeta} := \{\zeta(\frac{1}{2}+it) : t \in \mathbb{R}\}$, i.e., the continuous family of the Riemann zeta function parametrized by a vertical shift. Then we study the symplectic family $\boldsymbol{L}_\chi$ of quadratic Dirichlet L-functions. Finally, we look at the orthogonal family $\boldsymbol{L}_{\Delta,\chi}$ of the quadratic twists of the L-function associated with the discriminant modular form Δ. For these families, under the assumption of the relevant Riemann hypothesis and ratio conjecture, we perform an asymptotic analysis of $\mathcal{D}_k^{\mathcal{F}}(f)$. Our results confirm our prediction (1-6), for small values of k. We recall that the case $k=0$ is already known in the literature for all of these families, both assuming the ratio conjecture (see [Conrey and Snaith 2007]) and without (for restricted ranges for f, see, e.g., [Hughes and Rudnick 2002; Conrey 2005; Özlük and Snyder 1999]).

We start with the unitary family. Note that, since this is a continuous family, the average over the family in the definition of $\mathcal{D}_k^{\boldsymbol{\zeta}}(f)$ is given by an integration over $t \in [T, 2T]$ instead of the sum in (1-4). In this case, setting

$$\begin{aligned}
W_{\mathrm{U}}^0(x) &:= W_{\mathrm{U}}(x) = 1,\\
W_{\mathrm{U}}^1(x) &:= 1 - \frac{\sin^2(\pi x)}{(\pi x)^2}\\
W_{\mathrm{U}}^2(x) &:= 1 - \frac{2+\cos(2\pi x)}{(\pi x)^2} + \frac{3\sin(2\pi x)}{(\pi x)^3} + \frac{3(\cos(2\pi x)-1)}{2(\pi x)^4},
\end{aligned}$$

we prove the following theorem:

Theorem 2.1. *Let us assume the Riemann hypothesis and the ratio conjecture* (*see Conjecture 3.1*). *Let us consider a test function f, which is holomorphic throughout the strip $|\Im(z)| < 2$, real on the real line, even and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$. Then, for $k \in \mathbb{N}$ and $k \le 2$, we have*

$$\mathcal{D}_k^{\boldsymbol{\zeta}}(f) = \int_{-\infty}^{+\infty} f(x) W_U^k(x)\,dx + \mathrm{O}\Big(\frac{1}{\log T}\Big).$$

For this unitary family, in [Bettin and Fazzari 2022] we also develop an alternative method built on Hughes and Rudnick's technique [2002], which allows us to show (1-6) unconditionally[6]. Moreover, the analogue of Theorem 2.1 can be proved in the random matrix theory setting without any assumptions, since the formula for the ratios of characteristic polynomials averaged over the unitary group is known

[6]Neither the Riemann Hypothesis nor the ratio conjecture is required. However, this unconditional strategy works only for test functions whose Fourier transform's support is small enough.

unconditionally (see [Conrey et al. 2008, Theorem 4.1] and also [Conrey et al. 2005b; Huckleberry et al. 2016]). Therefore, denoting

$$Z = Z(A,\theta) = \det(I - Ae^{i\theta}) \tag{2-1}$$

for the characteristic polynomial of $N \times N$ matrices A and $\theta_1, \dots, \theta_N$ for the phases of the eigenvalues of A, we prove a result that is the random matrix analogy of Theorem 2.1, essentially with the same proof. When we work on the random matrix theory side, to ensure that the one-level density is well defined, we need the test function to be 2π-periodic. Hence, given $f : \mathbb{R} \to \mathbb{R}$ an even Schwartz function, we define

$$F_N(x) = \sum_{h\in\mathbb{Z}} f\Big(\frac{N}{2\pi}(x - 2\pi h)\Big), \tag{2-2}$$

and we prove the following:

Theorem 2.2. *Let us consider $f : \mathbb{R} \to \mathbb{R}$ an even Schwartz function and F_N as in (2-2). Then, for $k \in \mathbb{N}$ and $k \le 2$, we have*

$$\frac{1}{\int_{U(N)} |Z|^{2k}\, d_{\mathrm{Haar}}} \int_{U(N)} \sum_{j=1}^{N} F_N(\theta_j)|Z|^{2k}\, d_{\mathrm{Haar}} \xrightarrow{N\to\infty} \int_{-\infty}^{+\infty} f(x) W_{\mathrm{U}}^{k}(x)\, dx.$$

In the symplectic case, we compute the weighted one-level density functions for any nonnegative integer $k \le 4$. We set

$$\begin{aligned}
W_{\mathrm{USp}}^{0}(x) &:= W_{\mathrm{USp}}(x) = 1 - \frac{\sin(2\pi x)}{2\pi x},\\
W_{\mathrm{USp}}^{1}(x) &:= 1 + \frac{\sin(2\pi x)}{2\pi x} - \frac{2\sin^2(\pi x)}{(\pi x)^2},\\
W_{\mathrm{USp}}^{2}(x) &:= 1 - \frac{\sin(2\pi x)}{2\pi x} - \frac{24(1-\sin^2(\pi x))}{(2\pi x)^2} + \frac{48\sin(2\pi x)}{(2\pi x)^3} - \frac{96\sin^2(\pi x)}{(2\pi x)^4},\\
W_{\mathrm{USp}}^{3}(x) &:= 1 + \frac{\sin(2\pi x)}{2\pi x} - \frac{12\sin^2(\pi x)}{(\pi x)^2} - \frac{240\sin(2\pi x)}{(2\pi x)^3}\\
&\qquad - \frac{15(6-10\sin^2(\pi x))}{(\pi x)^4} + \frac{2880\sin(2\pi x)}{(2\pi x)^5} - \frac{90\sin^2(\pi x)}{(\pi x)^6},\\
W_{\mathrm{USp}}^{4}(x) &:= 1 - \frac{\sin(2\pi x)}{2\pi x} - \frac{10(1+\cos(2\pi x))}{(\pi x)^2} + \frac{90\sin(2\pi x)}{(\pi x)^3} - \frac{15(3-31\cos(2\pi x))}{(\pi x)^4}\\
&\qquad - \frac{1470\sin(2\pi x)}{(\pi x)^5} - \frac{315(1+9\cos(2\pi x))}{(\pi x)^6} + \frac{3150\sin(2\pi x)}{(\pi x)^7} - \frac{1575(1-\cos(2\pi x))}{(\pi x)^8},
\end{aligned}$$

and we prove the following result:

Theorem 2.3. *Let us assume the Riemann hypothesis and the ratio conjecture for the L-functions in the family $\boldsymbol{L}_\chi$ (see Conjecture 4.1). Let us consider a test function f, which is holomorphic throughout the strip $|\Im(z)| < 2$, real on the real line, even and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$. Then, for $k \in \mathbb{N}$ and $k \le 4$, we have*

$$\mathcal{D}_k^{\boldsymbol{L}_\chi}(f) = \int_{-\infty}^{+\infty} f(x) W_{\mathrm{USp}}^{k}(x)\, dx + \mathrm{O}\Big(\frac{1}{\log X}\Big).$$

Also in the symplectic case, with the same proof we also get the corresponding result in the random matrix theory setting unconditionally, as [Conrey et al. 2008, Theorem 4.2] provides the analogue of Conjecture 4.1. Note that, in the symplectic (respectively, orthogonal) case, Z defined as in (2-1) is the characteristic polynomial of $2N \times 2N$ symplectic (respectively, orthogonal) matrices.

Theorem 2.4. *Let us consider $f : \mathbb{R} \to \mathbb{R}$ an even Schwartz function and F_N as in (2-2). Then, for $k \in \mathbb{N}$ and $k \leq 4$, we have*

$$\frac{1}{\int_{\mathrm{USp}(2N)} Z^k d_{\mathrm{Haar}}} \int_{\mathrm{USp}(2N)} \sum_{j=1}^{N} F_{2N}(\theta_j)\, Z^k\, d_{\mathrm{Haar}} \xrightarrow{N\to\infty} \int_{-\infty}^{+\infty} f(x) W_{\mathrm{USp}}^k(x)\, dx.$$

Finally, for the (even) orthogonal family $\boldsymbol{L}_{\Delta,\chi}$, we let

$$\begin{aligned}
W_{\mathrm{SO}^+}^0(x) &:= W_{\mathrm{SO}^+}(x) = 1 + \frac{\sin(2\pi x)}{2\pi x},\\
W_{\mathrm{SO}^+}^1(x) &:= 1 - \frac{\sin(2\pi x)}{2\pi x},\\
W_{\mathrm{SO}^+}^2(x) &:= 1 + \frac{\sin(2\pi x)}{2\pi x} - \frac{2\sin^2(\pi x)}{(\pi x)^2},\\
W_{\mathrm{SO}^+}^3(x) &:= 1 - \frac{\sin(2\pi x)}{2\pi x} - \frac{24(1-\sin^2(\pi x))}{(2\pi x)^2} + \frac{48\sin(2\pi x)}{(2\pi x)^3} - \frac{96\sin^2(\pi x)}{(2\pi x)^4},\\
W_{\mathrm{SO}^+}^4(x) &:= 1 + \frac{\sin(2\pi x)}{2\pi x} - \frac{12\sin^2(\pi x)}{(\pi x)^2} - \frac{240\sin(2\pi x)}{(2\pi x)^3}\\
&\qquad - \frac{15(6-10\sin^2(\pi x))}{(\pi x)^4} + \frac{2880\sin(2\pi x)}{(2\pi x)^5} - \frac{90\sin^2(\pi x)}{(\pi x)^6}.
\end{aligned}$$

Notice that there are strong similarities with the symplectic kernels; we will discuss these analogies below. With these notations, we prove the following theorem:

Theorem 2.5. *Let us assume the Riemann hypothesis and the ratio conjecture for the L-functions in the family $\boldsymbol{L}_{\Delta,\chi}$ (see Conjecture 5.1). Let us consider a test function f, which is holomorphic throughout the strip $|\Im(z)| < 2$, real on the real line, even and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$. Then, for $k \in \mathbb{N}$ and $k \leq 4$, we have*

$$\mathcal{D}_k^{\boldsymbol{L}_{\Delta,\chi}}(f) = \int_{-\infty}^{+\infty} f(x) W_{\mathrm{SO}^+}^k(x)\, dx + \mathrm{O}\Big(\frac{1}{\log X}\Big).$$

Again the analogous result in random matrix theory is instead unconditional (relying on [Conrey et al. 2008, Theorem 4.3] in place of Conjecture 5.1).

Theorem 2.6. *Let us consider $f : \mathbb{R} \to \mathbb{R}$ an even Schwartz function and F_N as in (2-2). Then, for $k \in \mathbb{N}$ and $k \leq 4$, we have*

$$\frac{1}{\int_{\mathrm{SO}(2N)} Z^k\, d_{\mathrm{Haar}}} \int_{\mathrm{SO}(2N)} \sum_{j=1}^{N} F_{2N}(\theta_j) Z^k\, d_{\mathrm{Haar}} \xrightarrow{N\to\infty} \int_{-\infty}^{+\infty} f(x) W_{\mathrm{SO}^+}^k(x)\, dx.$$

P_G^k	$G=\mathrm{U}$	$G=\mathrm{USp}$	$G=\mathrm{SO}^+$
$k=0$	0	$-\frac{1}{2}$	$\frac{1}{2}$
$k=1$	$y-1$	$2y-\frac{3}{2}$	$-\frac{1}{2}$
$k=2$	$-2y^3+4y-2$	$-4y^3+6y-\frac{5}{2}$	$2y-\frac{3}{2}$
$k=3$		$12y^5-20y^3+12y-\frac{7}{2}$	$-4y^3+6y-\frac{5}{2}$
$k=4$		$-40y^7+84y^5-60y^3+20y-\frac{9}{2}$	$12y^5-20y^3+12y-\frac{7}{2}$

Table 1. Values of P_G^k obtained for k small.

2A. ***A general conjecture for W_G^k.*** Thanks to the explicit expressions we get for the kernels W_G^k in the range $k \le 4$, we can speculate about what happens for any $k \in \mathbb{N}$. First of all, we notice that the Fourier transform of the kernels W_G^k exhibits a structure. From the explicit formulae for W_G^k we get in the range $k \le 4$, $\widehat{W}_G^k$ turns out to be an even function, supported on $[-1, 1]$, uniquely determined by a polynomial on $[0, 1]$. More precisely, we conjecture that

$$\widehat{W}_G^k(y) = \delta_0(y) + P_G^k(|y|)\chi_{[-1,1]}(y), \tag{2-3}$$

where P_G^k is a polynomial depending on k and G only. In particular, in the unitary case and with $k \ge 1$, we expect the degree of P_{U}^k to be $2k-1$ and $P_{\mathrm{U}}^k(0) = -k$, $P_{\mathrm{U}}^k(1) = 0$. For the symplectic family, if $k \ge 1$, we predict P_{USp}^k with degree $2k-1$ and $P_{\mathrm{USp}}^k(0) = -(2k+1)/2$, $P_{\mathrm{USp}}^k(1) = (-1)^{k+1}/2$. Finally for the orthogonal symmetry type, we conjecture the degree of $P_{\mathrm{SO}^+}^k$ to be $2k-3$ and $P_{\mathrm{SO}^+}^k(0) = -(2k-1)/2$, $P_{\mathrm{SO}^+}^k(1) = (-1)^k/2$ for any $k \ge 2$ $\big($the case $k=1$ yields $\widehat{W}_{\mathrm{SO}^+}^1(y) = \delta_0(y) - \frac{1}{2}\big)$. We collect in Table 1 all the values of P_G^k we obtained for k small, which support our speculations. Note that the case $k=0$, corresponding to the first row in the table, was already known in the literature, while all other results are new.

Looking at Table 1, we can detect relations between the weighted one-level density functions with different symmetry types. In particular, from the above discussion, it seems natural to expect that

$$W_{\mathrm{SO}^+}^k(x) = W_{\mathrm{USp}}^{k-1}(x) \tag{2-4}$$

for any $k \in \mathbb{Z}_+$. Moreover, the Fourier transforms of W_G^k suggest that the weighted one-level density function in the unitary case is the average of the symplectic and orthogonal cases; namely, we conjecture that

$$W_{\mathrm{U}}^k(x) = \frac{W_{\mathrm{USp}}^k(x) + W_{\mathrm{SO}^+}^k(x)}{2}. \tag{2-5}$$

We note that also the leading order moment coefficients $f_G(k)$ for the three compact groups U, USp and SO^+ satisfy relations linking them with each other (see [Keating 2005, Equations (6.10) and (6.11)]):

$$f_{\mathrm{SO}^+}(k) = 2^k f_{\mathrm{USp}}(k-1) \quad \text{and} \quad 2^{k^2} f_U(k) = f_{\mathrm{USp}}(k) f_{\mathrm{SO}^+}(k).$$

Equations (2-4) and (2-5) can be seen as the analogue of the above formulae, in the context of the weighted one-level density.

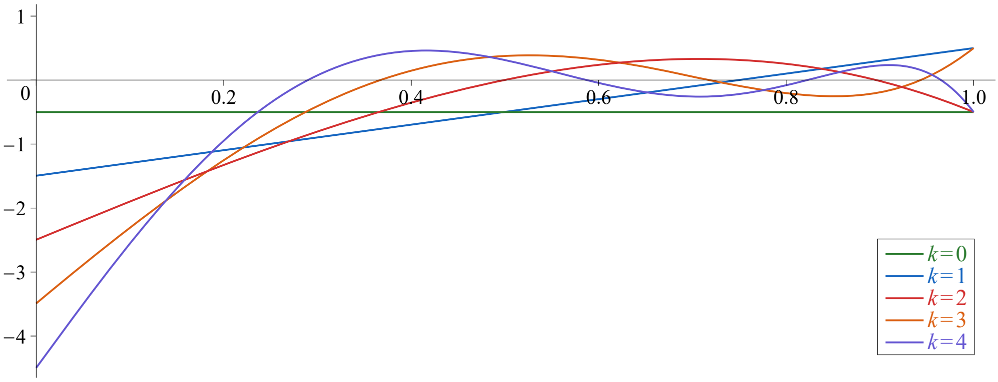

Figure 1. $P^k_{\mathrm{USp}}(y)$, for $y \in [0, 1]$.

Finally, we conjecture an explicit formula for the polynomials P^k_G, which together with (2-3) provides a precise conjecture for the weighted kernels W^k_G. In view of (2-4) and (2-5), it suffices to focus on the symplectic case only. Looking at what happens for $k \le 4$ (see Figure 1), we speculate that for every positive integer k,

$$P^k_{\mathrm{USp}}(y) = -\frac{2k+1}{2} - k(k+1)\sum_{j=1}^{k}(-1)^j c_{j,k}\frac{y^{2j-1}}{2j-1}, \tag{2-6}$$

where the coefficient $c_{j,k}$ is defined by

$$c_{j,k} = \frac{1}{j}\binom{k-1}{j-1}\binom{k+j}{j-1}.$$

We note that the sequence of the $c_{j,k}$ appears in OEIS[7], as the number of diagonal dissections of a convex $(k+2)$-gon into j regions. By Fourier inversion, from (2-3) and (2-6), we get an explicit conjectural formula for W^k_{USp}, namely,

$$W^k_{\mathrm{USp}}(x) = 1 - (2k+1)\frac{\sin(2\pi x)}{2\pi x} + \sum_{j=1}^{k}\frac{k(k+1)}{2^{2j-2}\pi^{2j-1}}\frac{c_{j,k}}{2j-1}\frac{d^{2j-1}}{dx^{2j-1}}\left[\frac{1-\cos(2\pi x)}{2\pi x}\right],$$

see Figure 2.

[7] https://oeis.org/A033282.

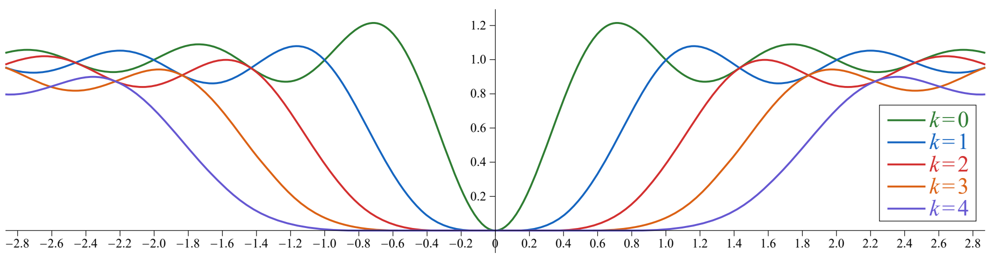

Figure 2. $W^k_{\mathrm{USp}}(x)$ for $k = 0, \ldots, 4$.

From all these discussions, we can formulate the following conjecture:

Conjecture 2.1. *Let us consider a test function f, holomorphic in the strip $|\Im(z)| < 2$, even, real on the real line and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$. For any $k \in \mathbb{N}$, given a family $\mathcal{F}$ of L-functions with symmetry type $G \in \{U, \mathrm{USp}, \mathrm{SO}^+\}$, we have*

$$\mathcal{D}_k^{\mathcal{F}}(f) = \int_{-\infty}^{+\infty} f(x) W_G^k(x)\, dx + \mathrm{O}\Big(\frac{1}{\log X}\Big)$$

as $X \to \infty$, where the weighted one-level density function W_G^k depends on k and G only. In addition, the following relations hold:

$$W_{\mathrm{SO}^+}^k(x) = W_{\mathrm{USp}}^{k-1}(x) \quad \textit{and} \quad W_{\mathrm{U}}^k(x) = \frac{W_{\mathrm{USp}}^k(x) + W_{\mathrm{SO}^+}^k(x)}{2}$$

for any $k \in \mathbb{Z}_+$ and $k \in \mathbb{N}$, respectively. Moreover, for every $k \in \mathbb{Z}_+$, in the symplectic case (*the others can be recovered by the above relations*), *we have that*

$$\widehat{W}_{\mathrm{USp}}^k(y) = \delta_0(y) + P_{\mathrm{USp}}^k(|y|)\chi_{[-1,1]}(y),$$

where P_{USp}^k is a polynomial of degree $2k-1$, given by

$$P_{\mathrm{USp}}^k(y) = -\frac{2k+1}{2} - k(k+1)\sum_{j=1}^{k}(-1)^j c_{j,k}\frac{y^{2j-1}}{2j-1},$$

with

$$c_{j,k} = \frac{1}{j}\binom{k-1}{j-1}\binom{k+j}{j-1}.$$

2B. ***An expression for $W_G^k(x)$ in terms of hypergeometric functions and its vanishing at $x = 0$.*** We now focus on the behavior of the weighted kernels $W_G^k(x)$ at $x = 0$. For all symmetry types, it seems clear that the order of vanishing of $W_G^k(x)$ for $x \to 0$ increases as k grows. This phenomenon reflects the effect of the weight $V\big(L\big(\frac{1}{2}\big)\big)^k$ in the average over the family, which gives more and more relevance to those L-functions that are large at the central point, as k increases. More precisely, for the unitary family we conjecture that

$$W_U^k(x) \sim \frac{\pi^{2k}x^{2k}}{(2k-1)!!(2k+1)!!}, \tag{2-7}$$

as $x \to 0$ and $k \in \mathbb{N}$. In particular, together with (1-6), this suggests that, on weighted average over the considered family, the number of normalized zeros which are less than ε away from the central point is typically $\asymp_k \varepsilon^{2k+1}$. Analogously, the asymptotic behavior of the symplectic and orthogonal kernels can be deduced from (2-7) by (2-4) and (2-5). For small values of k, the behavior of $W_G^k(x)$ at $x = 0$ is outlined in Table 2; the first row was already known in the literature, all the others are new.

In the following conjecture, we condense all the speculations about the behavior of the weighted kernels $W_G^k(x)$ as $x \to 0$:

$W_G^k,\ x\to 0$	$G=U$	$G=\mathrm{USp}$	$G=SO^+$
$k=0$	1	$2\pi^2x^2/3$	2
$k=1$	$\pi^2x^2/3$	$2\pi^4x^4/45$	$2\pi^2x^2/3$
$k=2$	$\pi^4x^4/45$	$2\pi^6x^6/1575$	$2\pi^4x^4/45$
$k=3$		$2\pi^8x^8/99225$	$2\pi^6x^6/1575$
$k=4$		$2\pi^{10}x^{10}/9823275$	$2\pi^8x^8/99225$

Table 2. The behavior of $W_G^k(x)$ at $x=0$ for small values of k.

Conjecture 2.2. *For $G\in\{\mathrm{U},\mathrm{USp},\mathrm{SO}^+\}$ and $k\in\mathbb{N}$, the weighted kernels W_G^k defined in Conjecture 2.1 satisfy the following asymptotic relations as $x\to 0$:*

$$W_U^k(x)\sim\frac{\pi^{2k}x^{2k}}{(2k-1)!!(2k+1)!!},\qquad W_{\mathrm{USp}}^k(x)\sim\frac{2\pi^{2(k+1)}x^{2(k+1)}}{(2k+1)!!(2k+3)!!},\qquad W_{\mathrm{SO}^+}^k(x)\sim\frac{2\pi^{2k}x^{2k}}{(2k-1)!!(2k+1)!!}.$$

Finally, assuming Conjecture 2.1, we obtain the expansion of $W_G^k(x)$ at $x=0$. In particular, we show that the asymptotic behavior of $W_G^k(x)$ can be deduced from the explicit formulae that we conjectured in Section 2A. In view of (2-4) and (2-5), it suffices to consider the symplectic case only.

Theorem 2.7. *Let us assume Conjecture 2.1. Then for any $k\in\mathbb{N}$, we have*

$$W_{\mathrm{USp}}^k(x)=\sum_{m=1}^{\infty}\beta_{m,k}x^{2m}$$

with

$$\beta_{m,k}=(-1)^{m+1}\frac{(2\pi)^{2m}}{(2m+1)!}\left((-1)^k+\frac{k(k+1)}{m+1}{}_3F_2\left[\begin{matrix}1-k,\ k+2,\ m+1\\ m+2,\ 2\end{matrix};1\right]\right),$$

where ${}_3F_2$ denotes the generalized hypergeometric function. Moreover, we have

$${}_3F_2\left[\begin{matrix}1-k,\ k+2,\ m+1\\ m+2,\ 2\end{matrix};1\right]=\begin{cases}\dfrac{(m+1)(-1)^{k+1}}{k(k+1)}, & \text{if } 1\le m\le k,\\[2ex] \dfrac{2(-1)^{k+1}(k-1)!(k+2)!}{(2k+2)!}\left(\binom{2k+1}{k+1}-1\right), & \text{if } m=k+1.\end{cases}$$

In particular, Conjecture 2.2 follows.

3. Proof of Theorems 2.1 and 2.2

We first tilt the Lebesgue measure multiplying by $\left|\zeta\left(\frac{1}{2}+it\right)\right|^2$ and denote it by

$$\langle g\rangle_{|\zeta|^2}:=\frac{1}{T\log T}\int_T^{2T}g(t)\left|\zeta\left(\frac{1}{2}+it\right)\right|^2dt. \tag{3-1}$$

Then we consider f, an even test function, and its Fourier transform

$$\hat f(y):=\int_{-\infty}^{+\infty}f(x)e^{-2\pi ixy}\,dx.$$

We recall that Conrey, Farmer and Zirnbauer [Conrey et al. 2008] applied a modification of the recipe for integral moments to the case of ratios getting the following statement, called the ratio conjecture (here, we state this conjecture in a slightly weaker form than in [Conrey et al. 2008], as far as the shifts are concerned):

Conjecture 3.1 [Conrey et al. 2008, Conjecture 5.1]. *Let us denote by $\chi(s)$ the explicit factor in the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$. For any positive integers K, L, Q and R and for any $\alpha_1, \dots, \alpha_{K+L}, \gamma_1, \dots, \gamma_Q, \delta_1, \dots, \delta_R$ complex shifts with real part $\asymp (\log T)^{-1}$ and imaginary part $\ll_\varepsilon T^{1-\varepsilon}$ for every $\varepsilon > 0$, we have*

$$\frac{1}{T}\int_T^{2T} \frac{\prod_{k=1}^{K}\zeta(s+\alpha_k)\prod_{l=K+1}^{K+L}\zeta(1-s-\alpha_l)}{\prod_{q=1}^{Q}\zeta(s+\gamma_q)\prod_{r=1}^{R}\zeta(1-s+\delta_r)}\,dt = \frac{1}{T}\int_T^{2T}\sum_{\sigma\in\Xi_{K,L}}\prod_{k=1}^{K}\frac{\chi(s+\alpha_k)}{\chi(s-\alpha_{\sigma(k)})}Y_U A_\zeta(\cdots)\,dt + O(T^{1/2+\varepsilon}),$$

with $(\cdots) = (\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(K)}; -\alpha_{\sigma(K+1)}, \dots, -\alpha_{\sigma(K+L)}; \gamma; \delta)$, where

$$Y_U(\alpha;\beta;\gamma;\delta) := \frac{\prod_{k=1}^{K}\prod_{l=1}^{L}\zeta(1+\alpha_k+\beta_l)\prod_{q=1}^{Q}\prod_{r=1}^{R}\zeta(1+\gamma_q+\delta_r)}{\prod_{k=1}^{K}\prod_{r=1}^{R}\zeta(1+\alpha_k+\delta_r)\prod_{l=1}^{L}\prod_{q=1}^{Q}\zeta(1+\beta_l+\gamma_q)}$$

and A_ζ is an Euler product, absolutely convergent for all of the variables in small disks around 0, which is given by

$$A_\zeta(\alpha;\beta;\gamma;\delta) := \prod_p \frac{\prod_{k=1}^{K}\prod_{l=1}^{L}(1-1/p^{1+\alpha_k+\beta_l})\prod_{q=1}^{Q}\prod_{r=1}^{R}(1-1/p^{1+\gamma_q+\delta_r})}{\prod_{k=1}^{K}\prod_{r=1}^{R}(1-1/p^{1+\alpha_k+\delta_r})\prod_{l=1}^{L}\prod_{q=1}^{Q}(1-1/p^{1+\beta_l+\gamma_q})} \times \sum_{\sum a_k+\sum c_q=\sum b_l+\sum d_r}\frac{\prod\mu(p^{c_q})\prod\mu(p^{d_r})}{p^{\sum(1/2+\alpha_k)a_k+\sum(1/2+\beta_l)b_l+\sum(1/2+\gamma_q)c_q+\sum(1/2+\delta_r)d_r}},$$

while $\Xi_{K,L}$ denotes the subset of permutations $\sigma \in S_{K+L}$ of $\{1, 2, \dots, K+L\}$ for which we have $\sigma(1) < \sigma(2) < \cdots < \sigma(K)$ and $\sigma(K+1) < \sigma(K+2) < \cdots < \sigma(K+L)$.

By assuming this conjecture about the moments of zeta and denoting

$$N_f(t) = \sum_\gamma f\left(\frac{\log T}{2\pi}(\gamma - t)\right),$$

we can prove the following result:

Proposition 3.1. *Let us assume Conjecture 3.1 and the Riemann hypothesis. We consider a test function $f(z)$ which is holomorphic throughout the strip $|\Im(z)| < 2$, real on the real line, even and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$. Then*

$$\mathcal{D}_1^\zeta(f) := \langle N_f\rangle_{|\zeta|^2} = \int_{-\infty}^{+\infty} f(x)W_U^1(x)\,dx + O\left(\frac{1}{\log T}\right),$$

with

$$W_U^1(x) := 1 - \operatorname{sinc}^2(x) = 1 - \frac{\sin^2(\pi x)}{(\pi x)^2}.$$

In addition, with the same strategy as in the proof of Proposition 3.1 (but much longer computations, which can be done by using Sage[8]), we can also study the "fourth moment" case. Namely, we let

$$\langle g\rangle_{|\zeta|^4} := \frac{1}{(T(\log T)^4)/(2\pi^2)} \int_T^{2T} g(t)\Big|\zeta\Big(\frac{1}{2}+it\Big)\Big|^4\,dt, \tag{3-2}$$

and we prove the following result:

Proposition 3.2. *Let us assume Conjecture 3.1 and the Riemann hypothesis. We consider a test function $f(z)$ which is holomorphic throughout the strip $|\Im(z)| < 2$, real on the real line, even and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$. Then*

$$\mathcal{D}_2^{\zeta}(f) := \langle N_f\rangle_{|\zeta|^4} = \int_{-\infty}^{+\infty} f(x)W_U^2(x)\,dx + O\Big(\frac{1}{\log T}\Big) \tag{3-3}$$

with

$$W_U^2(x) := 1 - \frac{2+\cos(2\pi x)}{(\pi x)^2} + \frac{3\sin(2\pi x)}{(\pi x)^3} + \frac{3(\cos(2\pi x)-1)}{2(\pi x)^4}.$$

3A. *Proof of Proposition 3.1.* To prove Proposition 3.1, we strongly rely on Conjecture 3.1, which allows us to perform a similar computation as in [Conrey and Snaith 2007, Section 3]. We introduce two parameters $\alpha, \beta \in \mathbb{R}$ of size $\asymp 1/\log T$, we let

$$\boldsymbol{\zeta}_{\alpha,\beta}(t) := \zeta\big(\tfrac{1}{2}+\alpha+it\big)\zeta\big(\tfrac{1}{2}+\beta-it\big) \tag{3-4}$$

and we look at

$$\langle N_f\rangle_{|\zeta|^2}^{\alpha,\beta} := \frac{1}{T\log T}\int_T^{2T}\sum_{\gamma} f\Big(\frac{\log T}{2\pi}(\gamma - t)\Big)\boldsymbol{\zeta}_{\alpha,\beta}(t)\,dt, \tag{3-5}$$

with $\gamma \in \mathbb{R}$ since we are assuming the Riemann hypothesis (we recall that $\rho = \frac{1}{2}+i\gamma$ are the nontrivial zeros of ζ). By the residue theorem, we have that

$$\langle N_f\rangle_{|\zeta|^2}^{\alpha,\beta} = \frac{1}{T\log T}\int_T^{2T}\frac{1}{2\pi i}\Big(\int_{(c)} - \int_{(1-c)}\Big)\frac{\zeta'}{\zeta}(s+it)\cdot f\Big(\frac{-i\log T}{2\pi}\Big(s-\frac{1}{2}\Big)\Big)\,ds\,\boldsymbol{\zeta}_{\alpha,\beta}(t)\,dt, \tag{3-6}$$

where $c \in \big(\frac{1}{2}, 1\big)$ and $\int_{(c)}$ denotes the integral over the vertical line of those s such that $\Re(s) = c$. We select $c = \frac{1}{2}+\delta$ with $\delta \asymp (\log T)^{-1}$, and we first consider the integral over the c-line

$$\begin{aligned}\mathcal{I} &:= \frac{1}{T\log T}\int_T^{2T}\frac{1}{2\pi i}\int_{(c)}\frac{\zeta'}{\zeta}(s+it)f\Big(\frac{-i\log T}{2\pi}\Big(s-\frac{1}{2}\Big)\Big)\,ds\,\boldsymbol{\zeta}_{\alpha,\beta}(t)\,dt\\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f\Big(\frac{\log T}{2\pi}(y-i\delta)\Big)\frac{d}{d\gamma}\Big[\frac{I(\alpha;\beta;\delta+iy+\gamma;\delta+iy)}{T\log T}\Big]_{\gamma=0}\,dy,\end{aligned}$$

where

$$I(A;B;C;D) := \int_T^{2T}\frac{\zeta(1/2+A+it)\zeta(1/2+B-it)\zeta(1/2+C+it)}{\zeta(1/2+D+it)}\,dt. \tag{3-7}$$

[8] SageMath, the Sage Mathematics Software System (Version 0.6.3), The Sage Developers, 2021, https://www.sagemath.org.

Moments like (3-7) can be computed thanks to Conjecture 3.1, and it turns out to be

$$I(A;B;C;D)=\int_T^{2T}\left\{\frac{\zeta(1+A+B)\zeta(1+B+C)}{\zeta(1+B+D)}+\left(\frac{t}{2\pi}\right)^{-A-B}\frac{\zeta(1-A-B)\zeta(1-A+C)}{\zeta(1-A+D)}\right.$$
$$\left.+\left(\frac{t}{2\pi}\right)^{-B-C}\frac{\zeta(1+A-C)\zeta(1-B-C)}{\zeta(1-C+D)}\right\}dt+O\bigl(T^{1/2+\varepsilon}\bigr) \quad (3\text{-}8)$$

for suitable shifts A, B, C and D, i.e., with real part $\asymp(\log T)^{-1}$ and imaginary part $\ll_\varepsilon T^{1-\varepsilon}$, for every $\varepsilon>0$ (see, e.g., [Conrey and Snaith 2007, Section 2.1]). Notice that the arithmetical factor $A_\zeta(\alpha;\beta;\gamma;\delta)$ from Conjecture 3.1 equals 1 in our case, with $K=2$, $L=1$, $Q=1$ and $R=0$ (this can be easily proven by direct computation or deduced by [Conrey et al. 2005a, Corollary 2.6.2]). We now want to apply (3-8) with $A=\alpha$, $B=\beta$, $C=\delta+iy+\gamma$ and $D=\delta+iy$ and to do so, we need that the imaginary parts of all the shifts are $\ll_\varepsilon T^{1-\varepsilon}$. A standard technique to avoid this issue is splitting the integral over y in two pieces; the contribution to $\mathcal{I}$ coming from $|y|>T^{1-\varepsilon}$ is $\ll T^{-1+\varepsilon}$, thanks to the good decaying of f and to RH, since

$$\frac{1}{T\log T}\int_T^{2T}|\boldsymbol{\zeta}_{\alpha,\beta}(t)|\int_{|y|>T^{1-\varepsilon}}|f(y\log T)|\left|\frac{\zeta'}{\zeta}\left(\frac12+\delta+iy+it\right)\right|dy\,dt$$
$$\ll\frac{T^{\varepsilon/100}}{T}\int_T^{2T}\int_{|y|>T^{1-\varepsilon}}\frac{\log(|y|t)}{|y|^2}\,dy\,dt\ll T^{-1+\varepsilon}.$$

Therefore, we can truncate the integral over y at height $T^{1-\varepsilon}$, apply (3-8) and then re-extend the integration over y to infinity with a small error term. Thus, differentiating with respect to γ at $\gamma=0$, moving the path of integration to $\delta=0$ (we are allowed to since now the integral is regular at $\delta=0$), we get

$$\mathcal{I}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}f\left(\frac{\log T}{2\pi}y\right)\frac{1}{T\log T}\int_T^{2T}g_{\alpha,\beta}(y;t)\,dt\,dy+O(T^{-1/2+\varepsilon}), \quad (3\text{-}9)$$

with

$$g_{\alpha,\beta}(y;t):=\frac{\zeta(1+\alpha+\beta)\zeta'(1+\beta+iy)}{\zeta(1+\beta+iy)}+\left(\frac{t}{2\pi}\right)^{-\alpha-\beta}\frac{\zeta(1-\alpha-\beta)\zeta'(1-\alpha+iy)}{\zeta(1-\alpha+iy)}$$
$$-\left(\frac{t}{2\pi}\right)^{-\beta-iy}\zeta(1+\alpha-iy)\zeta(1-\beta-iy). \quad (3\text{-}10)$$

We notice that, when computing this derivative, it is useful to observe that if $f(z)$ is analytic at $z=0$, then (see [Conrey and Snaith 2007, Equation (2.13)])

$$\frac{d}{d\gamma}\left[\frac{f(\gamma)}{\zeta(1-\gamma)}\right]_{\gamma=0}=-f(0).$$

Similarly we deal with the integral over the $(1-c)$-line in (3-6)

$$\mathcal{J}:=\frac{1}{T\log T}\int_T^{2T}\frac{1}{2\pi i}\int_{(1-c)}\frac{\zeta'}{\zeta}(s+it)f\left(\frac{-i\log T}{2\pi}\left(s-\frac12\right)\right)ds\,\boldsymbol{\zeta}_{\alpha,\beta}(t)\,dt$$
$$=\frac{1}{2\pi i}\int_{(c)}f\left(\frac{-i\log T}{2\pi}\left(s-\frac12\right)\right)\frac{1}{T\log T}\int_T^{2T}\frac{\zeta'}{\zeta}(1-s+it)\boldsymbol{\zeta}_{\alpha,\beta}(t)\,dt\,ds.$$

Using the functional equation
$$\frac{\zeta'}{\zeta}(1-z)=\frac{X'}{X}(z)-\frac{\zeta'}{\zeta}(z),$$
where
$$\frac{X'}{X}(z):=\log\pi-\frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{z}{2}\Big)-\frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{1-z}{2}\Big),$$
we express $\mathcal{J}$ as a sum of two terms
$$\mathcal{J}=\mathcal{J}_1-\mathcal{J}_2, \tag{3-11}$$
with
$$\mathcal{J}_1:=\frac{1}{2\pi i}\int_{(c)}f\Big(\frac{-i\log T}{2\pi}\Big(s-\frac{1}{2}\Big)\Big)\frac{1}{T\log T}\int_T^{2T}\frac{X'}{X}(s-it)\boldsymbol{\zeta}_{\alpha,\beta}(t)\,dt\,ds$$
and
$$\mathcal{J}_2:=\frac{1}{2\pi i}\int_{(c)}f\Big(\frac{-i\log T}{2\pi}\Big(s-\frac{1}{2}\Big)\Big)\frac{1}{T\log T}\int_T^{2T}\frac{\zeta'}{\zeta}(s-it)\boldsymbol{\zeta}_{\alpha,\beta}(t)\,dt\,ds.$$
With $c=\frac{1}{2}+\delta$, where $\delta\to 0$, it is easy to see that
$$\begin{aligned}\mathcal{J}_1&=\frac{1}{2\pi}\int_{-\infty}^{+\infty}f\Big(\frac{\log T}{2\pi}y\Big)\frac{1}{T\log T}\int_T^{2T}\frac{X'}{X}\Big(\frac{1}{2}+iy-it\Big)\boldsymbol{\zeta}_{\alpha,\beta}(t)\,dt\,dy\\&=-\frac{1}{2\pi}\int_{-\infty}^{+\infty}f\Big(\frac{\log T}{2\pi}y\Big)\frac{\log T+O(1)}{T\log T}\int_T^{2T}\boldsymbol{\zeta}_{\alpha,\beta}(t)\,dt\,dy,\end{aligned}\tag{3-12}$$
since, using Stirling's approximation to estimate the gamma-factors, we have (again, we can assume $y\ll T^{1-\varepsilon}$ because of the great decaying of f)
$$\begin{aligned}\frac{X'}{X}\Big(\frac{1}{2}+iy-it\Big)&=-\frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{iy}{2}-\frac{it}{2}\Big)-\frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}-\frac{iy}{2}+\frac{it}{2}\Big)+O(1)\\&=-\frac{1}{2}\log\Big(-\frac{it}{2}\Big)-\frac{1}{2}\log\Big(\frac{it}{2}\Big)+O(1)=-\log T+O(1).\end{aligned}$$
Moreover, with the same choice of c as before, if we set $\alpha=\beta$, we get
$$\mathcal{J}_2=\mathcal{I}. \tag{3-13}$$
Then (3-6), (3-11) and (3-13) imply that
$$\langle N_f\rangle_{|\zeta|^2}^{\alpha,\alpha}=\mathcal{I}+\mathcal{J}=-\mathcal{J}_1+2\mathcal{I} \tag{3-14}$$
and the function $\mathcal{J}_1=\mathcal{J}_1(\alpha)$ is regular at $\alpha=0$. We can then take the limit in (3-12), getting
$$\begin{aligned}\lim_{\alpha\to 0}\mathcal{J}_1&=-\frac{1}{2\pi}\int_{-\infty}^{+\infty}f\Big(\frac{\log T}{2\pi}y\Big)\frac{\log T+O(1)}{T\log T}\int_T^{2T}\Big|\zeta\Big(\frac{1}{2}+it\Big)\Big|^2\,dt\,dy\\&=-\frac{\log T+O(1)}{2\pi}\int_{-\infty}^{+\infty}f\Big(\frac{\log T}{2\pi}y\Big)\,dy\\&=-\int_{-\infty}^{+\infty}f(x)\,dx+O\Big(\frac{1}{\log T}\Big)\end{aligned}\tag{3-15}$$

with the change of variable $\log T/(2\pi)y = x$. Lastly, we study the remaining term $\mathcal{I}$, from (3-9) and (3-10). We set $\alpha = \beta = a/\log T$, with $0 < a < 1$. Then we perform the same change of variable $\log T/(2\pi)y = x$ as before, and we get

$$\mathcal{I} = \int_{-\infty}^{+\infty} f(x) \frac{1}{T(\log T)^2} \int_T^{2T} g_{a/\log T, a/\log T}\Big(\frac{2\pi x}{\log T}; t\Big)\, dt\, dx + O(T^{1/2+\varepsilon}).$$

Since $\log(t/(2\pi)) = \log T + O(1)$ as $t \in [T, 2T]$, we have

$$\begin{aligned}\mathcal{I} = \Big(\frac{1}{(\log T)^2} + O\Big(\frac{1}{(\log T)^3}\Big)\Big) \int_{-\infty}^{+\infty} f(x) \Big(\zeta\Big(1 + \frac{2a}{\log T}\Big)\frac{\zeta'}{\zeta}\Big(1 + \frac{a+2\pi i x}{\log T}\Big) \\ + e^{-2a}\zeta\Big(1 - \frac{2a}{\log T}\Big)\frac{\zeta'}{\zeta}\Big(1 - \frac{a-2\pi i x}{\log T}\Big) - e^{-a-2\pi i x}\zeta\Big(1 + \frac{a-2\pi i x}{\log T}\Big)\zeta\Big(1 - \frac{a+2\pi i x}{\log T}\Big)\Big)\, dx,\end{aligned} \tag{3-16}$$

where the error term is uniform in a. Now, we will prove that the above expression is regular at $a = 0$, showing that

$$\lim_{a\to 0} \mathcal{I} = \int_{-\infty}^{+\infty} f(x)\mathcal{P}(x)\, dx + O\Big(\frac{1}{\log T}\Big), \tag{3-17}$$

as $T \to \infty$, where

$$\mathcal{P}(x) := \frac{-1 + 2\pi i x + e^{-2\pi i x}}{4\pi^2 x^2}.$$

Intuitively, if we replace each zeta function with its leading term in the expansion at the point 1 given by $\zeta(1+z) \sim 1/z$, we have

$$\begin{aligned}\mathcal{I} &\approx \frac{1}{(\log T)^2} \int_{-\infty}^{+\infty} f(x)\Big(\frac{-(\log T)^2}{2a(a+2\pi i x)} + e^{-2a}\frac{-(\log T)^2}{2a(a-2\pi i x)} + e^{-a-2\pi i x}\frac{(\log T)^2}{(a-2\pi i x)(a+2\pi i x)}\Big)\, dx \\ &= \int_{-\infty}^{+\infty} f(x)\Big(-\frac{1}{2a(a+2\pi i x)} - \frac{e^{-2a}}{2a(a-2\pi i x)} + \frac{e^{-a-2\pi i x}}{(a-2\pi i x)(a+2\pi i x)}\Big)\, dx.\end{aligned}$$

The function inside the parentheses above equals

$$\frac{-a(1+e^{-2a}) + 2\pi i x(1-e^{-2a}) + 2ae^{-a-2\pi i x}}{2a(a^2 + 4\pi^2 x^2)} = \frac{-1 + 2\pi i x + e^{-2\pi i x} + O(a)}{4\pi^2 x^2 + O(a^2)},$$

and then tends to $\mathcal{P}(x)$ as $a \to 0$.

To show (3-17) rigorously, we split the integral over x into two parts. We start with the case $x \ll \log T$; from Taylor approximation $f(1+s\pm y) = f(1+s) \pm yf'(1+s) + O_s(y^2)$, we get

$$\begin{aligned}\frac{\zeta'}{\zeta}\Big(1 + \frac{2\pi i x}{\log T} \pm \frac{a}{\log T}\Big) &= \frac{\zeta'}{\zeta}\Big(1 + \frac{2\pi i x}{\log T}\Big) \pm \frac{a}{\log T}\Big(\frac{\zeta'}{\zeta}\Big)'\Big(1 + \frac{2\pi i x}{\log T}\Big) + O_T(a^2) \\ &=: c_1(x) \pm \frac{a}{\log T}c_2(x) + O_T(a^2)\end{aligned}$$

and

$$\zeta\Big(1 - \frac{2\pi i x}{\log T} \pm \frac{a}{\log T}\Big) = \zeta\Big(1 - \frac{2\pi i x}{\log T}\Big) + O_T(a) =: k(x) + O_T(a)$$

as $a \to 0$, with the notations

$$\begin{aligned} c_1(x) = c_1(x,T) &:= \frac{\zeta'}{\zeta}\Big(1+\frac{2\pi i x}{\log T}\Big), \\ c_2(x) = c_1(x,T) &:= \Big(\frac{\zeta'}{\zeta}\Big)'\Big(1+\frac{2\pi i x}{\log T}\Big), \\ k(x) = k(x,T) &:= \zeta\Big(1-\frac{2\pi i x}{\log T}\Big). \end{aligned}$$

Moreover, we use the asymptotic expansion

$$\zeta(1+z) = \frac{1}{z}+\gamma+O(z), \quad z\to 0, \tag{3-18}$$

and we get

$$\begin{aligned} &\int_{x\ll \log T} f(x)\bigg(\zeta\Big(1+\frac{2a}{\log T}\Big)\frac{\zeta'}{\zeta}\Big(1+\frac{a+2\pi i x}{\log T}\Big)+e^{-2a}\zeta\Big(1-\frac{2a}{\log T}\Big)\frac{\zeta'}{\zeta}\Big(1-\frac{a-2\pi i x}{\log T}\Big) \\ &\qquad\qquad\qquad -e^{-a-2\pi i x}\zeta\Big(1+\frac{a-2\pi i x}{\log T}\Big)\zeta\Big(1-\frac{a+2\pi i x}{\log T}\Big)\bigg)\,dx \\ &= \int_{x\ll \log T} f(x)\bigg(\Big[\frac{\log T}{2a}+\gamma+O_T(a)\Big]\Big[c_1(x)+\frac{a}{\log T}c_2(x)+O_T(a^2)\Big] \\ &\qquad +e^{-2a}\Big[\frac{-\log T}{2a}+\gamma+O_T(a)\Big]\Big[c_1(x)-\frac{a}{\log T}c_2(x)+O_T(a^2)\Big]-e^{-a-2\pi i x}\big[k(x)+O_T(a)\big]^2\bigg)\,dx, \end{aligned}$$

whose limit as $a\to 0$ is

$$\int_{x\ll \log T} f(x)\big\{c_1(x)\log T + c_2(x) + 2\gamma c_1(x) - e^{-2\pi i x}k(x)^2\big\}\,dx.$$

By definition of $c_1(x)$, $c_2(x)$ and $k(x)$, the asymptotic expansion (3-18) yields

$$\begin{aligned} c_1(x) &= -\frac{\log T}{2\pi i x}+O(1), \\ c_2(x) &= \frac{(\log T)^2}{(2\pi i x)^2}+O(1), \\ k(x) &= \frac{(\log T)^2}{(2\pi i x)^2}-\frac{2\gamma\log T}{2\pi i x}+O(1), \end{aligned}$$

uniformly for $x \ll \log T$. Then the above is equal to

$$\int_{x\ll \log T} f(x)\bigg\{-\frac{(\log T)^2}{2\pi i x}+\frac{(\log T)^2}{(2\pi i x)^2}-e^{-2\pi i x}\frac{(\log T)^2}{(2\pi i x)^2}+O(\log T)\bigg\}\,dx$$

(note that the sum $2\gamma c_1(x) - e^{-2\pi i x}k(x)^2$ gives the third term in the parentheses with an error $O(\log T)$, a possible pole at $x=0$ cancels out), which is

$$= (\log T)^2\int_{x\ll \log T} f(x)\frac{-1+2\pi i x+e^{-2\pi i x}}{4\pi^2x^2}\,dx + O(\log T).$$

Finally, we can re-extend the range of integration with a small error term (being $f(x) \ll 1/(1+x^2)$ and $\mathcal{P}(x)$ bounded), getting that the contribution of $x \ll \log T$ in the integral over x in (3-16), in the limit as $a \to 0$, equals

$$(\log T)^2 \int_{-\infty}^{+\infty} f(x)\mathcal{P}(x)\,dx + O(\log T).$$

To prove (3-17), we finally have to bound the contribution of $x \gg \log T$ in the integral on the right-hand side of (3-16), as $a \to 0$; to do so, we use the bounds $\zeta(1+iy) \ll \log y$ (see [Titchmarsh 1986, Theorem 3.5]) and $(\zeta'/\zeta)(1+iy) \ll \log y$ (see [Titchmarsh 1986, Equation (3.11.9)]) for $y \gg 1$, thus the contribution coming from $x \gg \log T$ is equal to

$$\begin{aligned}
\lim_{a\to 0} \int_{x\gg \log T} f(x)\biggl(\Bigl[\frac{\log T}{2a} + O(1)\Bigr]\Bigl[\frac{\zeta'}{\zeta}\Bigl(1+\frac{2\pi i x}{\log T}\Bigr) + O\Bigl(\frac{a}{\log T}\Bigr)\Bigr] \\
+ e^{-2a}\Bigl[-\frac{\log T}{2a} + O(1)\Bigr]\Bigl[\frac{\zeta'}{\zeta}\Bigl(1+\frac{2\pi i x}{\log T}\Bigr) + O\Bigl(\frac{a}{\log T}\Bigr)\Bigr] + O\bigl((\log x)^2\bigr)\biggr)\,dx \\
= \int_{x\gg\log T} f(x)\biggl(\log T\,\frac{\zeta'}{\zeta}\Bigl(1+\frac{2\pi i x}{\log T}\Bigr)\lim_{a\to 0}\Bigl[\frac{1-e^{-2a}}{2a}\Bigr] + O\bigl((\log x)^2\bigr)\biggr)\,dx \\
\ll \int_{x\gg \log T} |f(x)|\bigl(\log T\log x + (\log x)^2\bigr)\,dx \ll \log T \int_{x\gg\log T} \frac{\log x}{x^2}\,dx,
\end{aligned}$$

then (3-17) follows, being $\int_{x\gg\log T}(\log x/x^2)\,dx \ll 1$. Finally, if we decompose $\mathcal{P}(x)$ in even and odd parts

$$\mathcal{P}(x) = -\frac{1}{2}\frac{\sin^2(\pi x)}{(\pi x)^2} - \frac{i\bigl(\sin(2\pi x) - 2\pi x\bigr)}{4\pi^2 x^2}, \tag{3-19}$$

since f is even and $\mathcal{P}(x)$ bounded, we have

$$\lim_{a\to 0}\mathcal{I} = -\frac{1}{2}\int_{-\infty}^{+\infty} f(x)\frac{\sin^2(\pi x)}{(\pi x)^2}\,dy + O\Bigl(\frac{1}{\log T}\Bigr). \tag{3-20}$$

Putting together (3-14), (3-15) and (3-20), we finally get

$$\langle N_f\rangle_{|\zeta|^2} = \int_{-\infty}^{+\infty} f(x)\Bigl(1 - \frac{\sin^2(\pi x)}{(\pi x)^2}\Bigr)\,dx + O\Bigl(\frac{1}{\log T}\Bigr),$$

as $T \to \infty$, and the theorem has been proved.

3B. ***Proof of Proposition 3.2.*** This proof builds on the same ideas as that of Proposition 3.1, even though we have to handle longer computations; to begin with, we introduce four parameters $\alpha, \beta, \nu, \eta \in \mathbb{R}$ of size $1/\log T$, we let

$$\boldsymbol{\zeta}_{\alpha,\beta,\nu,\eta}(t) := \zeta\bigl(\tfrac{1}{2}+\alpha+it\bigr)\zeta\bigl(\tfrac{1}{2}+\beta+it\bigr)\zeta\bigl(\tfrac{1}{2}+\nu-it\bigr)\zeta\bigl(\tfrac{1}{2}+\eta-it\bigr),$$

and we look at

$$\langle N_f\rangle^{\alpha,\beta,\nu,\eta}_{|\zeta|^4} := \frac{1}{(T(\log T)^4)/(2\pi^2)}\int_T^{2T}\sum_{\gamma} f\Bigl(\frac{\log T}{2\pi}(\gamma - t)\Bigr)\boldsymbol{\zeta}_{\alpha,\beta,\nu,\eta}(t)\,dt, \tag{3-21}$$

with $\gamma \in \mathbb{R}$ since we are assuming the Riemann hypothesis. Analogous to (3-14), the residue theorem yields

$$\langle N_f \rangle_{|\zeta|^4}^{\alpha,\beta,\alpha,\beta} = -\mathcal{J}_1 + 2\mathcal{I}, \tag{3-22}$$

with

$$\mathcal{J}_1 = \mathcal{J}_1(\alpha,\beta) = -\int_{-\infty}^{+\infty} f(x) \frac{\log T + O(1)}{(T(\log T)^5)/(2\pi^2)} \int_T^{2T} \zeta_{\alpha,\beta,\alpha,\beta}(t)\, dt\, dx$$
$$\xrightarrow{\alpha,\beta\to 0} -\int_{-\infty}^{+\infty} f(x)\, dx + O\Big(\frac{1}{\log T}\Big) \tag{3-23}$$

and

$$\begin{aligned}\mathcal{I} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f\Big(\frac{\log T}{2\pi}(y - i\delta)\Big) \frac{d}{d\gamma}\bigg[\frac{I(\alpha;\beta;\gamma;\delta+iy;\alpha;\beta)}{T(\log T)^4/(2\pi^2)}\bigg]_{\gamma=\delta+iy} dy \\ &= \int_{-\infty}^{+\infty} f\Big(x - \frac{i\delta \log T}{2\pi}\Big) \\ &\qquad \times \frac{d}{d\gamma}\bigg[\frac{I(a/\log T; b/\log T; \gamma; \delta + 2\pi i x/\log T; a/\log T; b/\log T)}{T(\log T)^5/(2\pi^2)}\bigg]_{\gamma=\delta+2\pi i x/\log T} dx,\end{aligned}$$

where $a, b \asymp 1$, $\delta \asymp 1/\log T$ and $I(A;B;C;D;F;G)$ is defined by

$$\int_T^{2T} \frac{\zeta(1/2+A+it)\zeta(1/2+B+it)\zeta(1/2+C+it)\zeta(1/2+F-it)\zeta(1/2+G-it)}{\zeta(1/2+D+it)}\, dt.$$

If the shifts satisfy the conditions prescribed by Conjecture 3.1 then such an integral can be evaluated by using the ratio conjecture. According to the recipe, up to an error $O(T^{1/2+\varepsilon})$, the above moment is a sum of ten pieces, the first being

$$\int_T^{2T} \frac{\zeta(1+A+F)\zeta(1+A+G)\zeta(1+B+F)\zeta(1+B+G)\zeta(1+C+F)\zeta(1+C+G)}{\zeta(1+D+F)\zeta(1+D+G)} \mathcal{A}_{A,B,C,D,F,G}\, dt,$$

where

$$\begin{aligned}\mathcal{A}_{A,B,C,D,F,G} = \prod_p &\Big(1 - \frac{1}{p^{1+A+F}}\Big)\Big(1 - \frac{1}{p^{1+A+G}}\Big)\Big(1 - \frac{1}{p^{1+B+F}}\Big)\Big(1 - \frac{1}{p^{1+B+G}}\Big) \\ &\times \Big(1 - \frac{1}{p^{1+C+F}}\Big)\Big(1 - \frac{1}{p^{1+C+G}}\Big)\Big(1 - \frac{1}{p^{1+D+F}}\Big)^{-1}\Big(1 - \frac{1}{p^{1+D+G}}\Big)^{-1} \\ &\times \sum_{a+b+c+d=f+g} \frac{\mu(p^d)}{p^{(1/2+A)a+(1/2+B)b+(1/2+C)c+(1/2+D)d+(1/2+F)f+(1/2+G)g}}.\end{aligned}$$

It will be useful to notice that if all the shifts equal zero, then

$$\mathcal{A} := \mathcal{A}_{0,0,0,0,0,0} = \frac{1}{\zeta(2)};$$

again this can be proven by direct computation or deduced by [Conrey et al. 2005a, Corollary 2.6.2]. All the other nine terms can be recovered from the first one just by swapping the shifts as prescribed by the recipe;

doing so yields a formula for $I(A; B; C; D; F; G)$ and differentiating with respect to C at $C = D$, we get

$$\begin{aligned}\frac{d}{dC}\big[I(A; B; C; D; F; G)\big]_{C=D}\\ = \int_T^{2T}\bigg(R_1+\Big(\frac{t}{2\pi}\Big)^{-A-F}R_2+\Big(\frac{t}{2\pi}\Big)^{-A-G}R_3+\Big(\frac{t}{2\pi}\Big)^{-B-F}R_4+\Big(\frac{t}{2\pi}\Big)^{-B-G}R_5\\ +\Big(\frac{t}{2\pi}\Big)^{-D-F}R_6+\Big(\frac{t}{2\pi}\Big)^{-D-G}R_7+\Big(\frac{t}{2\pi}\Big)^{-A-B-F-G}R_8\\ +\Big(\frac{t}{2\pi}\Big)^{-A-D-F-G}R_9+\Big(\frac{t}{2\pi}\Big)^{-B-D-F-G}R_{10}\bigg)\,dt+O\big(T^{1/2+\varepsilon}\big), \quad (3\text{-}24)\end{aligned}$$

with

$$\begin{aligned}R_1 &= R_1(A, B, D, F, G)\\ &= \frac{\mathcal{A}_{A,B,D,D,F,G}}{\zeta(1+A+F)\zeta(1+A+G)\zeta(1+B+F)\zeta(1+B+G)\zeta(1+D+F)\zeta(1+D+G)}\\ &\qquad\times\bigg[\frac{\zeta'}{\zeta}(1+D+F)+\frac{\zeta'}{\zeta}(1+D+G)+\frac{\mathcal{A}'_{A,B,D,D,F,G}}{A_{A,B,D,D,F,G}}\bigg],\\ R_2 &= R_1(-F, B, D, -A, G),\\ R_3 &= R_1(-G, B, D, F, -A),\\ R_4 &= R_1(A, -F, D, -B, G),\\ R_5 &= R_1(A, -G, D, F, -B),\\ R_6 &= R_6(A, B, D, F, G)\\ &= -\frac{\zeta(1+A-D)\zeta(1+A+G)\zeta(1+B-D)\zeta(1+B+G)\zeta(1-F-D)\zeta(1-F+G)}{\zeta(1+D+G)}\mathcal{A}_{A,B,D,D,F,G},\\ R_7 &= R_6(A, B, D, G, F),\\ R_8 &= R_1(-F, -G, D, -A, -B),\\ R_9 &= R_6(-F, B, D, G, -A),\\ R_{10} &= R_6(A, -F, D, G, -B).\end{aligned}$$

If the shifts A, B, D, F, G are $\ll 1/\log T$, the above formula simplifies a lot, since we have

$$R_1 = \frac{(-2D-F-G)\mathcal{A}}{(A+F)(A+G)(B+F)(B+G)(D+F)(D+G)} + O\bigg(\frac{(\log T)^5}{\log T}\bigg)$$

and

$$R_6 = \frac{-(D+G)\mathcal{A}}{(A-D)(A+G)(B-D)(B+G)(-F-D)(-F+G)} + O\bigg(\frac{(\log T)^5}{\log T}\bigg).$$

As in the proof of Proposition 3.1, by a truncation of the integral over x and Taylor approximations, we can use (3-24) to evaluate $\mathcal{I}$; one can use Sage to carry out this massive computation, getting

$$\lim_{\substack{a\to 0\\ b\to 0}} \mathcal{I} = \int_{-\infty}^{+\infty} f(x)\frac{(\log T)^5\mathcal{A}}{(\log T)^5/(2\pi^2)}h(2\pi i x)\,dx + O\Big(\frac{1}{\log T}\Big) \tag{3-25}$$

with
$$h(y) := -\frac{y^3-2y^2+6-e^{-y}(y^2+6y+6)}{6y^4}.$$
Note that, as in the last section, we moved the path of integration over x to $\delta=0$, being the integral regular at $\delta=0$. Therefore, putting together (3-22), (3-23) and (3-25), we get that
$$\begin{aligned}\langle N_f\rangle_{|\zeta|^4} &= \int_{-\infty}^{+\infty} f(x)\Bigl(1+2\frac{2\pi^2}{\zeta(2)}h(2\pi i x)\Bigr)\,dx + O\Bigl(\frac{1}{\log T}\Bigr)\\ &= \int_{-\infty}^{+\infty} f(x)\bigl(1+24h(2\pi i x)\bigr)\,dx + O\Bigl(\frac{1}{\log T}\Bigr) = \int_{-\infty}^{+\infty} f(x)W_U^2(x)\,dx + O\Bigl(\frac{1}{\log T}\Bigr),\end{aligned}$$
since f is even.

3C. ***Proof of Theorem 2.2.*** As mentioned in the introduction, the analogue of the ratio conjecture is a theorem in random matrix theory. Therefore, the same machinery described above proves Theorem 2.2, using [Conrey et al. 2008, Theorem 4.1] in place of Conjecture 3.1.

4. Proof of Theorems 2.3 and 2.4

The family $\{L(\frac{1}{2},\chi_d) : d>0,\ d \text{ fundamental discriminant}\}$ is a symplectic family, in the sense that it can be modeled by characteristic polynomials of symplectic matrices in the group $\mathrm{USp}(2N)$, if we identify $2N \approx \log(d/\pi)$. Indeed d/π is the analytic conductor of $L(s,\chi_d)$, thus $\log(d/\pi)$ (i.e., the density of zeros) plays the role of $2N$ in the random matrix theory setting[9].

We consider the moments of quadratic Dirichlet L-functions at the critical point $s=\frac{1}{2}$, i.e., the mean value
$$\sum_{d\le X} L\bigl(\tfrac{1}{2},\chi_d\bigr)^k \tag{4-1}$$
in the limit $X\to\infty$, where the summation over d has to be interpreted as the sum over all the positive fundamental discriminants d below X, here and in the following. Also, we will denote by $X^*\sim 1/(2\zeta(2))X$ the number of fundamental discriminants below X. We recall that Jutila [1981] proved asymptotic formulae for the first moment, showing that
$$\sum_{d\le X} L\Bigl(\frac{1}{2},\chi_d\Bigr) \sim \frac{\mathcal{A}}{2}\frac{1}{2\zeta(2)}X\log X, \tag{4-2}$$
where
$$\mathcal{A} = \prod_p \Bigl(1-\frac{1}{p(p+1)}\Bigr) \tag{4-3}$$
and also for the second moment, proving
$$\sum_{d\le X} L\Bigl(\frac{1}{2},\chi_d\Bigr)^2 \sim \frac{\mathcal{B}}{24}\frac{1}{2\zeta(2)}X(\log X)^3, \tag{4-4}$$

[9] See [Conrey et al. 2005a, Conjecture 1.5.3] and the comments below for some clarification concerning the “conductor”.

with

$$\mathcal{B} = \prod_p \left(1 - \frac{4p^2 - 3p + 1}{p^3(p+1)}\right). \tag{4-5}$$

It is believed that

$$\sum_{d \le X} L\left(\tfrac{1}{2}, \chi_d\right)^k \sim C_k X (\log X)^{k(k+1)/2}, \tag{4-6}$$

and using analogies with random matrix theory, Keating and Snaith [2000] also conjectured a precise value for the constant C_k. Moreover, the recipe produces a conjectural asymptotic formula with all the main terms for the moments (4-1) with k integer and also for ratios of products of quadratic Dirichlet L-functions (see [Conrey et al. 2008]), which is a symplectic analogue of Conjecture 3.1.

Conjecture 4.1 [Conrey et al. 2008, Conjecture 5.2]. *Let K, Q be two positive integers, $\alpha_1, \dots, \alpha_K$ and $\gamma_1, \dots, \gamma_Q$ be complex shifts with real part $\asymp (\log T)^{-1}$ and imaginary part $\ll_\varepsilon T^{1-\varepsilon}$ for every $\varepsilon > 0$, then*

$$\sum_{d \le X} \frac{\prod_{k=1}^K L(1/2 + \alpha_k, \chi_d)}{\prod_{q=1}^Q L(1/2 + \gamma_q, \chi_d)} = \sum_{d \le X} \sum_{\epsilon \in \{-1,1\}^K} \left(\frac{d}{\pi}\right)^{(1/2)\sum_k (\epsilon_k \alpha_k - \alpha_k)} \prod_{k=1}^K g_S\left(\frac{1}{2} + \frac{\alpha_k - \epsilon_k \alpha_k}{2}\right) Y_S \mathcal{A}_S(\cdots) + O(X^{1/2+\varepsilon}),$$

with $(\cdots) = (\epsilon_1 \alpha_1, \dots, \epsilon_K \alpha_K; \gamma)$, where

$$Y_S(\alpha; \gamma) := \frac{\prod_{j \le k \le K} \zeta(1 + \alpha_j + \alpha_k) \prod_{q < r \le Q} \zeta(1 + \gamma_q + \gamma_r)}{\prod_{k=1}^K \prod_{q=1}^Q \zeta(1 + \alpha_k + \gamma_q)}$$

and $\mathcal{A}_S$ is an Euler product, absolutely convergent for all of the variables in small disks around 0, which is given by

$$\mathcal{A}_S(\alpha; \gamma) := \prod_p \frac{\prod_{j \le k \le K} (1 - 1/p^{1+\alpha_j+\alpha_k}) \prod_{q < r \le Q} (1 - 1/p^{1+\gamma_q+\gamma_r})}{\prod_{k=1}^K \prod_{q=1}^Q (1 - 1/p^{1+\alpha_k+\gamma_q})} \times \left(1 + (1 + 1/p)^{-1} \sum_{0 < \sum_k a_k + \sum_q c_q \text{ is even}} \frac{\prod_q \mu(p^{c_q})}{p^{\sum_k a_k(1/2+\alpha_k) + \sum_q c_q(1/2+\gamma_q)}}\right),$$

while

$$g_S(z) := \frac{\Gamma((1-z)/2)}{\Gamma(z/2)}.$$

In particular, for our applications to the weighted one-level density, we are interested in the case where $Q = 1$ and $2 \le K \le 5$.

4A. *Conjecture 4.1 in the case $K = 2$, $Q = 1$.* We start with

$$\sum_{d \le X} \frac{L(1/2 + A, \chi_d) L(1/2 + C, \chi_d)}{L(1/2 + D, \chi_d)}, \tag{4-7}$$

with A, C, D shifts which satisfy the hypotheses prescribed by Conjecture 4.1; by the ratio conjecture, up to a negligible error $O(X^{1/2+\varepsilon})$, this is a sum of four terms and the first is

$$\sum_{d\le X}\frac{\zeta(1+2A)\zeta(1+2C)\zeta(1+A+C)}{\zeta(1+A+D)\zeta(1+C+D)}\mathcal{A}(A,C;D),$$

where

$$\mathcal{A}(A,C;D)=\prod_p\Big(1-\frac{1}{p^{1+2A}}\Big)\Big(1-\frac{1}{p^{1+2C}}\Big)\Big(1-\frac{1}{p^{1+A+C}}\Big)\Big(1-\frac{1}{p^{1+A+D}}\Big)^{-1}\Big(1-\frac{1}{p^{1+C+D}}\Big)^{-1}\\ \times\bigg(1+\frac{p}{p+1}\sum_{0<a+c+d\text{ even}}\frac{\mu(p^d)}{p^{a(1/2+A)+c(1/2+C)+d(1/2+D)}}\bigg).$$

In the following, it will be relevant to notice that for small values of the shifts, then the arithmetical coefficient $\mathcal{A}(A,C;D)$ tends to $\mathcal{A}$, defined in (4-3); this essentially follows from [Conrey et al. 2008, Corollary 6.4]. All the other terms can be easily recovered from the first one, just by changes of sign of the shifts, as the recipe suggests. This yields a formula for (4-7), written as a sum of four pieces; by computing the derivative $\frac{d}{dC}[\cdots]_{C=D}$, we get

$$\sum_{d\le X}\frac{L'}{L}\Big(\frac12+D,\chi_d\Big)L\Big(\frac12+A,\chi_d\Big)=\sum_{d\le X}\bigg(Q_1+\Big(\frac{d}{\pi}\Big)^{-A}g_S\Big(\frac12+A\Big)Q_2+\Big(\frac{d}{\pi}\Big)^{-D}g_S\Big(\frac12+D\Big)Q_3\\ +\Big(\frac{d}{\pi}\Big)^{-A-D}g_S\Big(\frac12+A+D\Big)Q_4\bigg)+O\big(X^{1/2+\varepsilon}\big),\quad (4\text{-}8)$$

with

$$Q_1=\mathcal{A}(A,D;D)\frac{\zeta(1+2A)}{\zeta(1+A+D)}\bigg(\frac{2\zeta'(1+2D)\zeta(1+A+D)}{\zeta(1+2D)}+\frac{\zeta'(1+A+D)\zeta(1+2D)}{\zeta(1+2D)}\\ -\frac{\zeta'(1+2D)\zeta(1+A+D)}{\zeta(1+2D)}\bigg)+\mathcal{A}'(A,D;D)\zeta(1+2A)\\ =\mathcal{A}(A,D;D)\frac{\zeta(1+2A)}{\zeta(1+A+D)}\bigg(\frac{\zeta'}{\zeta}(1+2D)\zeta(1+A+D)+\zeta'(1+A+D)\bigg)+\mathcal{A}'(A,D;D)\zeta(1+2A),$$

$$Q_2=\mathcal{A}(-A,D;D)\frac{\zeta(1-2A)}{\zeta(1-A+D)}\bigg(\frac{\zeta'}{\zeta}(1+2D)\zeta(1-A+D)+\zeta'(1-A+D)\bigg)+\mathcal{A}'(-A,D;D)\zeta(1-2A),$$

$$Q_3=-\mathcal{A}(A,-D;D)\frac{\zeta(1+2A)\zeta(1-2D)\zeta(1+A-D)}{\zeta(1+A+D)},$$

$$Q_4=-\mathcal{A}(-A,-D;D)\frac{\zeta(1-2A)\zeta(1-2D)\zeta(1-A-D)}{\zeta(1-A+D)}.$$

Moreover, we notice that if the shifts are $\ll(\log X)^{-1}$, then we can approximate (4-8), getting

$$\sum_{d\le X}\frac{L'}{L}\Big(\frac12+D,\chi_d\Big)L\Big(\frac12+A,\chi_d\Big)\\ =\mathcal{A}X^*\bigg(\frac{-A-3D}{(2A)(2D)(A+D)}+X^{-A}\frac{A-3D}{(-2A)(2D)(-A+D)}\\ +X^{-D}\frac{A+D}{(2A)(2D)(A-D)}+X^{-A-D}\frac{-A+D}{(-2A)(2D)(-A-D)}\bigg)+O(\log X),\quad (4\text{-}9)$$

being that $\mathcal{A}(\pm A,\pm D,D)=\mathcal{A}+O(1/\log X)$ and $\zeta(1+z)=1/z+O(1)$ as $z\to 0$.

4B. *Conjecture 4.1* in the case $K = 3$, $Q = 1$. Now, we study in detail

$$\sum_{d\le X}\frac{L(1/2+A,\chi_d)L(1/2+B,\chi_d)L(1/2+C,\chi_d)}{L(1/2+D,\chi_d)},\tag{4-10}$$

with A, B, C and D as prescribed by Conjecture 4.1. This time, the asymptotic formula suggested by recipe is a sum of eight terms; the first is

$$\sum_{d\le X}\frac{\zeta(1+2A)\zeta(1+2B)\zeta(1+2C)\zeta(1+A+B)\zeta(1+A+C)\zeta(1+B+C)}{\zeta(1+A+D)\zeta(1+B+D)\zeta(1+C+D)}\mathcal{A}(A,B,C;D),$$

where the (rather horrible) arithmetical coefficient is given by

$$\begin{aligned}
&\mathcal{A}(A,B,C;D)\\
&=\prod_p\Big(1-\frac{1}{p^{1+2A}}\Big)\Big(1-\frac{1}{p^{1+2B}}\Big)\Big(1-\frac{1}{p^{1+2C}}\Big)\Big(1-\frac{1}{p^{1+A+B}}\Big)\Big(1-\frac{1}{p^{1+A+C}}\Big)\\
&\qquad\times\Big(1-\frac{1}{p^{1+B+C}}\Big)\Big(1-\frac{1}{p^{1+A+D}}\Big)^{-1}\Big(1-\frac{1}{p^{1+B+D}}\Big)^{-1}\Big(1-\frac{1}{p^{1+C+D}}\Big)^{-1}\\
&\qquad\times\Bigg(1+\frac{p}{p+1}\bigg[\Big(1+\frac{1}{p^{1+A+B}}+\frac{1}{p^{1+A+C}}+\frac{1}{p^{1+B+C}}\Big)\Big(1-\frac{1}{p^{1+2A}}\Big)^{-1}\Big(1-\frac{1}{p^{1+2B}}\Big)^{-1}\Big(1-\frac{1}{p^{1+2C}}\Big)^{-1}\\
&\qquad\quad-1-\Big(\frac{1}{p^{1+A+D}}+\frac{1}{p^{1+B+D}}+\frac{1}{p^{1+C+D}}+\frac{1}{p^{2+A+B+C+D}}\Big)\Big(1-\frac{1}{p^{1+2A}}\Big)^{-1}\Big(1-\frac{1}{p^{1+2B}}\Big)^{-1}\Big(1-\frac{1}{p^{1+2C}}\Big)^{-1}\bigg]\Bigg).
\end{aligned}$$

We notice that, as in the proof of [Conrey et al. 2008, Corollary 6.4], we can prove that the arithmetical coefficient is convergent if all the variables are in small disk around 0, being $\mathcal{A}(\underline{0})=\mathcal{B}$, with $\mathcal{B}$ defined in (4-5). As in the previous example, this gives a formula for (4-10) with all the main terms and error $O(X^{1/2+\varepsilon})$. Differentiating this formula with respect to C at $C=D$, we get

$$\begin{aligned}
&\sum_{d\le X}\frac{L'}{L}\Big(\frac12+D,\chi_d\Big)L\Big(\frac12+A,\chi_d\Big)L\Big(\frac12+B,\chi_d\Big)\\
&\quad=\sum_{d\le X}\Bigg(R_1+\Big(\frac{d}{\pi}\Big)^{-A}g_S\Big(\frac12+A\Big)R_2+\Big(\frac{d}{\pi}\Big)^{-B}g_S\Big(\frac12+B\Big)R_3+\Big(\frac{d}{\pi}\Big)^{-D}g_S\Big(\frac12+D\Big)R_4\\
&\qquad\qquad+\Big(\frac{d}{\pi}\Big)^{-A-B}g_S\Big(\frac12+A+B\Big)R_5+\Big(\frac{d}{\pi}\Big)^{-A-D}g_S\Big(\frac12+A+D\Big)R_6\\
&\qquad\qquad+\Big(\frac{d}{\pi}\Big)^{-B-D}g_S\Big(\frac12+B+D\Big)R_7+\Big(\frac{d}{\pi}\Big)^{-A-B-D}g_S\Big(\frac12+A+B+D\Big)R_8\Bigg)+O(X^{1/2+\varepsilon})
\end{aligned}\tag{4-11}$$

with

$$\begin{aligned}
R_1&=R_1(A,B,D)\\
&=\mathcal{A}(A,B,D;D)\frac{\zeta(1+2A)\zeta(1+2B)\zeta(1+A+B)}{\zeta(1+A+D)\zeta(1+B+D)}\\
&\qquad\times\bigg(\frac{2\zeta'(1+2D)\zeta(1+A+D)\zeta(1+B+D)+\zeta(1+2D)\zeta'(1+A+D)\zeta(1+B+D)}{\zeta(1+2D)}\\
&\qquad\qquad+\frac{\zeta(1+2D)\zeta(1+A+D)\zeta'(1+B+D)-\zeta(1+A+D)\zeta(1+B+D)\zeta'(1+2D)}{\zeta(1+2D)}\bigg)\\
&\qquad\qquad+\zeta(1+2A)\zeta(1+2B)\zeta(1+A+B)\mathcal{A}'(A,B,D;D),
\end{aligned}$$

and

$$
\begin{aligned}
R_2 &= R_1(-A, B, D),\\
R_3 &= R_1(A, -B, D),\\
R_4 &= R_4(A, B, D)\\
&= -\frac{\zeta(1+2A)\zeta(1+2B)\zeta(1+A+B)\zeta(1-2D)\zeta(1+A-D)\zeta(1+B-D)}{\zeta(1+A+D)\zeta(1+B+D)}\mathcal{A}(A, B, -D; D),\\
R_5 &= R_1(-A, -B, D),\\
R_6 &= R_4(-A, B, D),\\
R_7 &= R_4(A, -B, D),\\
R_8 &= R_4(-A, -B, D).
\end{aligned}
$$

If $A, B, D \ll (\log X)^{-1}$ the above formula simplifies a lot, since in this case

$$
R_1 = \frac{-AB-3AD-3BD-5D^2}{(2A)(2B)(2D)(A+B)(A+D)(B+D)}\mathcal{B} + O\biggl(\frac{(\log X)^6}{(\log X)^3}\biggr) =: f(A, B, D)\mathcal{B} + O\bigl((\log X)^3\bigr)
$$

and

$$
R_4 = \frac{-(A+D)(B+D)}{(2A)(2B)(-2D)(A+B)(A-D)(B-D)}\mathcal{B} + O\biggl(\frac{(\log X)^6}{(\log X)^3}\biggr) =: g(A, B, D)\mathcal{B} + O\bigl((\log X)^3\bigr),
$$

giving

$$
\begin{aligned}
\sum_{d\le X} \frac{L'}{L}\Bigl(\frac{1}{2}+D, \chi_d\Bigr)L\Bigl(\frac{1}{2}+A, \chi_d\Bigr)L\Bigl(\frac{1}{2}+B, \chi_d\Bigr)
&= \mathcal{B}X^*\bigl(f(A, B, D)+X^{-A}f(-A, B, D)+X^{-B}f(A, -B, D)+X^{-D}g(A, B, D)\\
&\quad +X^{-A-B}f(-A, -B, D)+X^{-A-D}g(-A, B, D)\\
&\quad +X^{-B-D}g(A, -B, D)+X^{-A-B-D}g(-A, -B, D)\bigr)+O\bigl((\log X)^3\bigr). \qquad (4\text{-}12)
\end{aligned}
$$

Analogous (but longer) formulae can be obtained also in the cases $K = 4$, $Q = 1$ and $K = 5$, $Q = 1$. With exactly the same ideas (but much longer computations) also the case $K > 5$, $Q = 1$ can be dealt.

4C. *The weighted one-level density for $\{L(\frac{1}{2}, \chi_d)\}_d$.* We recall that the one-level density for the symplectic family of quadratic Dirichlet L-functions has been studied originally by Özlük and Snyder [1999] and independently by Katz and Sarnak [1999][10], who proved that

$$
\lim_{X\to\infty} \frac{1}{X^*}\sum_{d\le X}\sum_{\gamma_d} f\Bigl(\frac{\log X}{2\pi}\gamma_d\Bigr) = \int_{-\infty}^{+\infty} f(x)\Bigl(1-\frac{\sin(2\pi x)}{2\pi x}\Bigr)\,dx \qquad (4\text{-}13)
$$

under GRH, for any f such that supp $\hat{f} \subset (-2, 2)$. Moreover, Conrey and Snaith [2007] showed (4-13) (also with lower order terms) with no constraint on the support of $\hat{f}$, under the assumption of the

[10] See also *"Zeroes of Zeta Functions, their Spaces and their Spectral Nature"* by Katz and Sarnak, the 1997 preprint version of [Katz and Sarnak 1999].

ratio conjecture; namely, they consider f a test function, holomorphic throughout the strip $|\Im(z)| < 2$, even, real on the real line and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$, and they study

$$\mathcal{D}_0^{L_\chi}(f) := \frac{1}{X^*} \sum_{d \le X} \sum_{\gamma_d} f\left(\frac{\log X}{2\pi}\gamma_d\right). \tag{4-14}$$

As $X \to \infty$, they show that the above is asymptotic to the right-hand side of (4-14), which matches with the one-level density for the eigenvalues of the matrices from the symplectic group $\mathrm{USp}(2N)$. In particular, we notice that the one-level density function $1 - \sin(2\pi x)/(2\pi x)$ vanishes of order 2 at $x = 0$, being $\sim (2\pi^2/3)x^2$ as $x \to 0$.

Similarly to what we did in Section 3, we now want to compute the weighted one-level density in the symplectic case, tilted by $L\left(\frac{1}{2}, \chi_d\right)$. We note that, differently from what happens in the Riemann zeta function case, here we are allowed to consider the first power as well, as $L\left(\frac{1}{2}, \chi_d\right)$ is real. The analogue of (3-1) in this context is

$$\mathcal{D}_1^{L_\chi}(f) := \frac{1}{\sum_{d \le X} L(1/2, \chi_d)} \sum_{d \le X} \sum_{\gamma_d} f\left(\frac{\log X}{2\pi}\gamma_d\right) L\left(\frac{1}{2}, \chi_d\right), \tag{4-15}$$

and via ratio conjecture in the form of (4-8) this can be studied asymptotically, as shown in the following result:

Proposition 4.1. *Assume GRH and Conjecture 4.1 for $K = 2$, $Q = 1$. For any test function f, holomorphic in the strip $\Im(z) < 2$, even, real on the real line and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$, we have*

$$\mathcal{D}_1^{L_\chi}(f) = \int_{-\infty}^{+\infty} f(x) W_{\mathrm{USp}}^1(x)\, dx + O\left(\frac{1}{\log X}\right)$$

as $X \to \infty$, where

$$W_{\mathrm{USp}}^1(x) := 1 + \frac{\sin(2\pi x)}{2\pi x} - \frac{2\sin^2(\pi x)}{(\pi x)^2}.$$

Proof. We start looking at

$$\frac{1}{(\mathcal{A}/2) X^* \log X} \sum_{d \le X} \sum_{\gamma_d} f\left(\frac{\log X}{2\pi}\gamma_d\right) L\left(\frac{1}{2} + \alpha, \chi_d\right), \tag{4-16}$$

with $\alpha \asymp (\log X)^{-1}$; note that, as $\alpha \to 0$, $\sum_{d \le X} L\left(\frac{1}{2} + \alpha, \chi_d\right)$ tends to $1/(2\zeta(2))(\mathcal{A}/2) X \log X$, which is the normalization $(\mathcal{A}/2) X^* \log X$ we have in (4-16). As usual, we use the Cauchy theorem and the functional equation for $(L'/L)(s, \chi_d)$ to write

$$\frac{1}{(\mathcal{A}/2) X^* \log X} \sum_{d \le X} \sum_{\gamma_d} f\left(\frac{\log X}{2\pi}\gamma_d\right) L\left(\frac{1}{2} + \alpha, \chi_d\right) = -\mathcal{J}(\alpha) + 2\mathcal{I}(\alpha) + O\left(\frac{1}{\log X}\right) \tag{4-17}$$

where

$$\begin{aligned}
\mathcal{J}(\alpha) &:= \frac{2}{\mathcal{A} X^* \log X} \frac{1}{2\pi} \int_{-\infty}^{+\infty} (-\log X) f\left(\frac{y \log X}{2\pi}\right) \sum_{d \le X} L\left(\frac{1}{2} + \alpha, \chi_d\right) dy \\
&= \frac{-2}{\mathcal{A} X^* \log X} \int_{-\infty}^{+\infty} f(x) \sum_{d \le X} \left(L\left(\frac{1}{2}, \chi_d\right) + O\left(\frac{1}{\log X}\right)\right) dx = -\int_{-\infty}^{+\infty} f(x)\, dx + O\left(\frac{1}{\log X}\right)
\end{aligned}$$

and

$$\mathcal{I}(\alpha) := \frac{2}{\mathcal{A}X^*(\log X)^2}\int_{-\infty}^{+\infty} f(x)\sum_{d\le X}\frac{L'}{L}\Big(\frac{1}{2}+\delta+\frac{2\pi i x}{\log X},\chi_d\Big)L\Big(\frac{1}{2}+\alpha,\chi_d\Big)\,dx,$$

with $\delta \asymp (\log X)^{-1}$. Now we rely on the assumption of the ratio conjecture (in particular, (4-8) and (4-9)) to compute the sum over d; in particular, in the same way as in the proof of Proposition 3.1, by a truncation of the integral over x and Taylor approximations, we get

$$\mathcal{I}(\alpha) = \frac{2}{(\log X)^2}\int_{-\infty}^{+\infty} f(x)g_X\Big(\alpha,\delta+\frac{2\pi i x}{\log X}\Big)\,dx + O\Big(\frac{1}{\log X}\Big),$$

where

$$g_X(\alpha,w) := \frac{-\alpha-3w}{(2\alpha)(2w)(\alpha+w)} + X^{-\alpha}\frac{\alpha-3w}{(-2\alpha)(2w)(-\alpha+w)} + X^{-w}\frac{\alpha+w}{(2\alpha)(2w)(\alpha-w)} + X^{-\alpha-w}\frac{-\alpha+w}{(-2\alpha)(2w)(-\alpha-w)}.$$

The integral is regular at $\delta = 0$ then, if we denote $\alpha = a/\log X$, we get

$$\mathcal{I}\Big(\frac{a}{\log X}\Big) = \frac{2}{(\log X)^2}\int_{-\infty}^{+\infty} f(x)g_X\Big(\frac{a}{\log X},\frac{2\pi i x}{\log X}\Big)\,dx + O\Big(\frac{1}{\log X}\Big),$$

which is regular at $a = 0$; indeed, if we take the limit as $a \to 0$, we get

$$\mathcal{I}(0) = \lim_{a\to 0}\mathcal{I}\Big(\frac{a}{\log X}\Big) = 2\int_{-\infty}^{+\infty} f(x)g(2\pi i x)\,dx + O\Big(\frac{1}{\log X}\Big)$$

where

$$\begin{aligned} g(w) &:= \lim_{a\to 0}\bigg(\frac{-a-3w}{(2a)(2w)(a+w)} + e^{-a}\frac{a-3w}{(-2a)(2w)(-a+w)} + e^{-w}\frac{a+w}{(2a)(2w)(a-w)} + e^{-a-w}\frac{-a+w}{(-2a)(2w)(-a-w)}\bigg) \\ &= \frac{-we^{-w}-3w-4e^{-w}+4}{4w^2}. \end{aligned}$$

Then

$$\begin{aligned} \mathcal{D}_1^{\boldsymbol{L}_\chi}(f) &= \lim_{\alpha\to 0}\frac{1}{(\mathcal{A}/2)X^*\log X}\sum_{d\le X}\sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big)L\Big(\frac{1}{2}+\alpha,\chi_d\Big) \\ &= -\mathcal{J}(0) + 2\mathcal{I}(0) + O\Big(\frac{1}{\log X}\Big) = \int_{-\infty}^{+\infty} f(x)\big(1+4g(2\pi i x)\big)\,dx + O\Big(\frac{1}{\log X}\Big), \end{aligned}$$

and since f is even, the main term above equals

$$\int_{-\infty}^{+\infty} f(x)\Big(1+\frac{\sin(2\pi x)}{2\pi x}-\frac{2\sin^2(\pi x)}{(\pi x)^2}\Big)\,dx. \qquad \square$$

Analogously, we can compute the weighted one-level density, tilted by the second power of $L(\frac{1}{2},\chi_d)$, i.e.,

$$\mathcal{D}_2^{\boldsymbol{L}_\chi}(f) := \frac{1}{\sum_{d\le X} L(1/2,\chi_d)^2}\sum_{d\le X}\sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big)L\Big(\frac{1}{2},\chi_d\Big)^2 \tag{4-18}$$

under the assumption of Conjecture 4.1, in the case $K = 3$, $Q = 1$.

Proposition 4.2. *Assume GRH and Conjecture 4.1 for $K = 3$, $Q = 1$. For any function f holomorphic in the strip $\Im(z) < 2$, even, real on the real line and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$, we have*

$$\mathcal{D}_2^{\boldsymbol{L}_X}(f) = \int_{-\infty}^{+\infty} f(x) W_{\mathrm{USp}}^2(x)\,dx + O\Big(\frac{1}{\log X}\Big)$$

as $X \to \infty$, where

$$W_{\mathrm{USp}}^2(x) := 1 - \frac{\sin(2\pi x)}{2\pi x} - \frac{24(1-\sin^2(\pi x))}{(2\pi x)^2} + \frac{48\sin(2\pi x)}{(2\pi x)^3} - \frac{96\sin^2(\pi x)}{(2\pi x)^4}.$$

Proof. The proof works like that of Proposition 4.1; first, for $\alpha = a/\log X \asymp (\log X)^{-1}$ and for $\beta = b/\log X \asymp (\log X)^{-1}$, we analyze

$$\frac{1}{(\mathcal{B}/24)X^*(\log X)^3} \sum_{d\le X} \sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big) L\Big(\frac{1}{2}+\alpha, \chi_d\Big) L\Big(\frac{1}{2}+\beta, \chi_d\Big), \tag{4-19}$$

which can be written as

$$-\mathcal{J}(\alpha,\beta) + 2\mathcal{I}(\alpha,\beta) + O\Big(\frac{1}{\log X}\Big), \tag{4-20}$$

where

$$\begin{aligned}\mathcal{J}(\alpha,\beta) &:= \frac{-24\log X}{\mathcal{B}X^*(\log X)^3}\frac{1}{2\pi}\int_{-\infty}^{+\infty} f\Big(\frac{y\log X}{2\pi}\Big) \sum_{d\le X} L\Big(\frac{1}{2}+\alpha, \chi_d\Big) L\Big(\frac{1}{2}+\beta, \chi_d\Big)\\ &= -\int_{-\infty}^{+\infty} f(x)\,dx + O\Big(\frac{1}{\log X}\Big)\end{aligned} \tag{4-21}$$

and

$$\mathcal{I}(\alpha,\beta) := \frac{24}{\mathcal{B}X^*(\log X)^4}\int_{-\infty}^{+\infty} f(x) \sum_{d\le X} \frac{L'}{L}\Big(\frac{1}{2}+\delta+\frac{2\pi i x}{\log X}, \chi_d\Big) L\Big(\frac{1}{2}+\alpha, \chi_d\Big) L\Big(\frac{1}{2}+\beta, \chi_d\Big)\,dx,$$

where $\delta \asymp (\log T)^{-1}$, as usual. With the usual machinery, the ratio conjecture (see (4-11) and (4-12)) allows us to evaluate the sum over d; the resulting quantity is regular at $\delta = 0$ and at $\alpha = a/\log X = 0$, $\beta = b/\log X = 0$, thus taking the limit, we get

$$\mathcal{I}(0,0) = 24\int_{-\infty}^{+\infty} f(x) h(2\pi i x)\,dx, \tag{4-22}$$

with

$$h(y) := \frac{y^3e^{-y} - 5y^3 + 12y^2e^{-y} + 12y^2 + 48ye^{-y} + 48e^{-y} - 48}{48y^4}.$$

Putting all together, from (4-19), (4-20), (4-21) and (4-22), we finally get

$$\mathcal{D}_2^{\boldsymbol{L}_X} = -\mathcal{J}(0,0) + 2\mathcal{I}(0,0) + O\Big(\frac{1}{\log X}\Big) = \int_{-\infty}^{+\infty} f(x)\big(1 + 48h(2\pi i x)\big)\,dx + O\Big(\frac{1}{\log X}\Big).$$

Moreover, since f is even, the main term equals

$$\int_{-\infty}^{+\infty} f(x)\Big(1 - \frac{\sin(2\pi x)}{2\pi x} - \frac{24(1-\sin^2(\pi x))}{(2\pi x)^2} + \frac{48\sin(2\pi x)}{(2\pi x)^3} - \frac{96\sin^2(\pi x)}{(2\pi x)^4}\Big)\,dx. \qquad \square$$

In the same way, we study

$$\mathcal{D}_3^{\boldsymbol{L}_\chi}(f) := \frac{1}{\sum_{d\le X} L(1/2,\chi_d)^3} \sum_{d\le X}\sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big) L\Big(\frac{1}{2},\chi_d\Big)^3, \tag{4-23}$$

assuming Conjecture 4.1 in the case $K=4$, $Q=1$.

Proposition 4.3. *Assume GRH and Conjecture 4.1 for $K=4$, $Q=1$. For any function f holomorphic in the strip $\Im(z)<2$, even, real on the real line and such that $f(x)\ll 1/(1+x^2)$ as $x\to\infty$, we have*

$$\mathcal{D}_3^{\boldsymbol{L}_\chi}(f) = \int_{-\infty}^{+\infty} f(x) W^3_{\mathrm{USp}}(x)\,dx + O\Big(\frac{1}{\log X}\Big)$$

as $X\to\infty$, where

$$W^3_{\mathrm{USp}}(x) := 1 + \frac{\sin(2\pi x)}{2\pi x} - \frac{12\sin^2(\pi x)}{(\pi x)^2} - \frac{240\sin(2\pi x)}{(2\pi x)^3} - \frac{15(6-10\sin^2(\pi x))}{(\pi x)^4} + \frac{2880\sin(2\pi x)}{(2\pi x)^5} - \frac{90\sin^2(\pi x)}{(\pi x)^6}.$$

Proof. We consider $\alpha,\beta,\nu\in\mathbb{R}$ of size $\asymp 1/\log X$. We denote

$$\boldsymbol{L}_{\alpha,\beta,\nu}\big(\tfrac12,\chi_d\big) := L\big(\tfrac12+\alpha,\chi_d\big)L\big(\tfrac12+\beta,\chi_d\big)L\big(\tfrac12+\nu,\chi_d\big),$$

and we look at

$$\frac{1}{\sum_{d\le X} L(1/2,\chi_d)^3} \sum_{d\le X}\sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big) \boldsymbol{L}_{\alpha,\beta,\nu}\Big(\frac{1}{2},\chi_d\Big).$$

With the usual machinery we get that the above equals

$$\int_{-\infty}^{+\infty} f(x)\Big(1 + 2\frac{1}{\sum_{d\le X} L(1/2,\chi_d)^3} \sum_{d\le X} \frac{L'}{L}\Big(\frac12+\delta+\frac{2\pi i x}{\log X},\chi_d\Big)\boldsymbol{L}_{\alpha,\beta,\nu}\Big(\frac12,\chi_d\Big)\Big)\,dx$$

up to an error $O(1/\log X)$, with $\delta\asymp 1/\log X$. The remaining sum over d can be evaluated asymptotically by using the ratio conjecture (i.e., Conjecture 4.1 for $K=4$, $Q=1$). This can be done by using Sage to carry out the easy but very long computations. Doing so, letting $\alpha,\beta,\nu\to 0$, we obtain

$$\mathcal{D}_3^{\boldsymbol{L}_\chi}(f) = \int_{-\infty}^{+\infty} f(x)\big(1+2\cdot 2880\cdot h(2\pi i x)\big)\,dx + O\Big(\frac{1}{\log X}\Big),$$

with

$$h(y) := \frac{-7y^5+24y^4-240y^2+2880}{5760y^6} + \frac{e^{-y}(-y^5-24y^4-240y^3-1200y^2-2880y-2880)}{5760y^6}.$$

The claim follows, since f is even. □

Finally, we look at

$$\mathcal{D}_4^{\boldsymbol{L}_\chi}(f) := \frac{1}{\sum_{d\le X} L(1/2,\chi_d)^4} \sum_{d\le X}\sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big) L\Big(\frac{1}{2},\chi_d\Big)^4 \tag{4-24}$$

assuming Conjecture 4.1, in the case $K=5$, $Q=1$.

Proposition 4.4. *Assume GRH and Conjecture 4.1 for $K=5$, $Q=1$. For any function f holomorphic in the strip $\Im(z)<2$, even, real on the real line and such that $f(x)\ll 1/(1+x^2)$ as $x\to\infty$, we have*

$$\mathcal{D}_4^{\boldsymbol{L}_\chi}(f)=\int_{-\infty}^{+\infty} f(x)W_{\mathrm{USp}}^4(x)\,dx+O\Big(\frac{1}{\log X}\Big)$$

as $X\to\infty$, where

$$\begin{aligned}W_{\mathrm{USp}}^4(x):=1&-\frac{\sin(2\pi x)}{2\pi x}-\frac{10(1+\cos(2\pi x))}{(\pi x)^2}+\frac{90\sin(2\pi x)}{(\pi x)^3}-\frac{15(3-31\cos(2\pi x))}{(\pi x)^4}\\&-\frac{1470\sin(2\pi x)}{(\pi x)^5}-\frac{315(1+9\cos(2\pi x))}{(\pi x)^6}+\frac{3150\sin(2\pi x)}{(\pi x)^7}-\frac{1575(1-\cos(2\pi x))}{(\pi x)^8}.\end{aligned}$$

Proof. The proof works in the same way as the previous ones. We consider $\alpha,\beta,\nu,\eta\in\mathbb{R}$ of size $\asymp 1/\log X$, we let

$$\boldsymbol{L}_{\alpha,\beta,\nu,\eta}\big(\tfrac12,\chi_d\big):=L\big(\tfrac12+\alpha,\chi_d\big)L\big(\tfrac12+\beta,\chi_d\big)L\big(\tfrac12+\nu,\chi_d\big)L\big(\tfrac12+\eta,\chi_d\big),$$

and we look at

$$\frac{1}{\sum_{d\le X}L(1/2,\chi_d)^4}\sum_{d\le X}\sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big)\boldsymbol{L}_{\alpha,\beta,\nu,\eta}\Big(\frac12,\chi_d\Big).$$

By the usual manipulations, the above equals

$$\int_{-\infty}^{+\infty} f(x)\bigg(1+2\frac{1}{\sum_{d\le X}L(1/2,\chi_d)^4}\sum_{d\le X}\frac{L'}{L}\Big(\frac12+\delta+\frac{2\pi i x}{\log X},\chi_d\Big)\boldsymbol{L}_{\alpha,\beta,\nu,\eta}\Big(\frac12,\chi_d\Big)\bigg)\,dx,$$

up to an error $O(1/\log X)$, with $\delta\asymp 1/\log X$. Thanks to Conjecture 4.1 with $K=5$, $Q=1$, the above can be computed asymptotically. As $\alpha,\beta,\nu,\eta\to 0$, with the help of Sage, we then obtain

$$\mathcal{D}_4^{\boldsymbol{L}_\chi}(f)=\int_{-\infty}^{+\infty} f(x)\big(1+2\cdot 4838400\cdot h(2\pi i x)\big)\,dx+O\Big(\frac{1}{\log X}\Big),$$

with

$$\begin{aligned}h(y):=\frac{-9y^7+40y^6-720y^4+20160y^2-403200}{9676800y^8}&+\frac{e^{-y}(y^7+40y^6+720y^5+7440y^4)}{9676800y^8}\\&+\frac{e^{-y}(47040y^3+181440y^2+403200y+403200)}{9676800y^8}.\end{aligned}$$

Again, being f even, the claim follows. □

4D. ***Proof of Theorem 2.4.*** The same remark as in Section 3C applies here, relying on [Conrey et al. 2008, Theorem 4.2] instead of Conjecture 4.1.

5. Proof of Theorems 2.5 and 2.6

As a last example, we analyze the orthogonal case of the family of quadratic twists of the L-functions associated with the discriminant modular form Δ. which is the unique normalized cusp form of weight 12.

Its Fourier coefficients define the Ramanujan tau function $\tau(n)$, being

$$\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n,$$

with $q = e^{2\pi iz}$. Thus, the L-function associated with Δ is defined by

$$L_\Delta(s) := \sum_{n=1}^{\infty}\frac{\tau^*(n)}{n^s},$$

where $\tau^*(n) = \tau(n)/n^{11/2}$. The family we want to describe is the collection of the quadratic twists of L_Δ, that are

$$L_\Delta(s,\chi_d) = \sum_{n=1}^{\infty}\frac{\chi_d(n)\tau^*(n)}{n^s} = \prod_p\left(1-\frac{\tau^*(p)\chi_d(p)}{p^s}+\frac{\chi_d(p^2)}{p^{2s}}\right)^{-1},$$

and, for $d > 0$, they satisfy the functional equation

$$\left(\frac{d^2}{4\pi^2}\right)^{s/2}\Gamma\left(s+\frac{11}{2}\right)L_\Delta(s,\chi_d) = \left(\frac{d^2}{4\pi^2}\right)^{(1-s)/2}\Gamma\left(1-s+\frac{11}{2}\right)L_\Delta(1-s,\chi_d).$$

Finally, we also record that

$$\frac{1}{L_\Delta(s,\chi_d)} = \prod_p\left(1-\frac{\tau^*(p)\chi_d(p)}{p^s}+\frac{\chi_d(p^2)}{p^{2s}}\right) =: \sum_{n=1}^{\infty}\frac{\chi_d(n)\mu_\Delta(n)}{n^s},$$

where μ_Δ is the multiplicative function defined by $\mu_\Delta(p) = -\tau^*(p)$, $\mu_\Delta(p^2) = 1$ and $\mu_\Delta(p^\alpha) = 0$ if $\alpha \geq 3$.

The family $\{L_\Delta(1/2,\chi_d) : d > 0,\ f.d.\}$ is an even orthogonal family, modeled by the group $\mathrm{SO}(2N)$ with the identification $2N \approx \log(d^2/(4\pi^2))$.

The moments at the central value of L-functions associated with quadratic twists of a modular form have been studied extensively in recent years, but only the first moment [Bump et al. 1990; Iwaniec 1990; Murty and Murty 1991] and partially the second [Soundararajan and Young 2010; Radziwiłł and Soundararajan 2015] have been obtained. It is known that such a family can be either symplectic or orthogonal, depending on the specific L-function we twist; in particular, if we start with the L-function associated with the discriminant modular form Δ, then we are in the latter case. For an orthogonal family $\mathcal{F}$, ordered by the conductor $C(f)$, Conrey and Farmer [2000] and Keating and Snaith [2000] predict that

$$\frac{1}{X^*}\sum_{\substack{f\in\mathcal{F}\\ C(f)\leq X}} L_f\left(\frac{1}{2}\right)^k \sim \frac{f_O(k)}{2}a(k)(\log X^A)^{k(k-1)/2}, \tag{5-1}$$

where the above sum is over the X^* elements of the family $\mathcal{F}$ such that $C(f) \leq X$; $f_O(k)$ is the leading order coefficient of the moments of characteristic polynomials of matrices in $SO(2N)$; $a(k)$ is a constant depending on the particular family involved; A is a constant depending on the functional equation satisfied by the L-functions in the family, in particular on the degree of the relevant parameter in the functional equation for $L_f(s)$ (see [Conrey and Farmer 2000, Equation (1.3)] for further details and examples). Moreover, in this case the recipe [Conrey et al. 2005a] provides a precise formula with all the main terms for any integral moment, extended by [Conrey et al. 2008] to ratios. The ratio conjecture for the orthogonal family of quadratic twists of the discriminant modular form can be stated as follows:

Conjecture 5.1 [Conrey et al. 2008, Conjecture 5.3]. *Let K, Q be two positive integers, $\alpha_1, \dots, \alpha_K$ and $\gamma_1, \dots, \gamma_Q$ be complex shifts with real part $\asymp (\log X)^{-1}$ and imaginary part $\ll_\varepsilon X^{1-\varepsilon}$ for every $\varepsilon > 0$, then*

$$\sum_{d \le X} \frac{\prod_{k=1}^K L_\Delta(1/2+\alpha_k, \chi_d)}{\prod_{q=1}^Q L_\Delta(1/2+\gamma_q, \chi_d)} = \sum_{d\le X} \sum_{\epsilon \in \{-1,1\}^K} \left(\frac{d^2}{4\pi^2}\right)^{(1/2)\sum_k(\epsilon_k\alpha_k - \alpha_k)} \prod_{k=1}^K g_O\left(\frac{1}{2} + \frac{\alpha_k - \epsilon_k\alpha_k}{2}\right) Y_O \mathcal{A}_O(\cdots) + O(X^{1/2+\varepsilon}),$$

with $(\cdots) = (\epsilon_1\alpha_1, \dots, \epsilon_K\alpha_K; \gamma)$, where

$$Y_O(\alpha;\gamma) := \frac{\prod_{j<k\le K} \zeta(1+\alpha_j+\alpha_k) \prod_{q<r\le Q} \zeta(1+\gamma_q+\gamma_r) \prod_{q \le Q} \zeta(1+2\gamma_q)}{\prod_{k=1}^K \prod_{q=1}^Q \zeta(1+\alpha_k+\gamma_q)}$$

and $\mathcal{A}_O$ is an Euler product, absolutely convergent for all of the variables in small disks around 0, which is given by

$$\mathcal{A}_O(\alpha;\gamma) := \prod_p \frac{\prod_{j<k\le K}(1-1/p^{1+\alpha_j+\alpha_k}) \prod_{q<r\le Q}(1-1/p^{1+\gamma_q+\gamma_r}) \prod_{q\le Q}(1-1/p^{1+2\gamma_q})}{\prod_{k=1}^K\prod_{q=1}^Q (1-1/p^{1+\alpha_k+\gamma_q})} \times \left(1 + \left(1+\frac{1}{p}\right)^{-1} \sum_{0<\sum_k a_k + \sum_q c_q \text{ is even}} \frac{\prod_k \tau^*(p^{a_k}) \prod_q \mu_\Delta(p^{c_q})}{p^{\sum_k a_k(1/2+\alpha_k) + \sum_q c_q(1/2+\gamma_q)}}\right)$$

while

$$g_O(s) := \frac{\Gamma(1/2-s+6)}{\Gamma(s-1/2+6)}.$$

In the following, we will analyze the applications of this conjecture to the weighted one-level density, as we did in Section 4C for a symplectic family. To do so, we first look at what Conjecture 5.1 gives in a few specific examples.

5A. *Conjecture 5.1 in the case $K = 1$, $Q = 0$.* This is the easiest situation possible, corresponding to the first moment of $L_\Delta(\frac{1}{2}, \chi_d)$; for A a complex number which satisfies the hypotheses prescribed by Conjecture 5.1, the ratio conjecture yields

$$\frac{1}{X^*} \sum_{d \le X} L_\Delta\left(\frac{1}{2}+A, \chi_d\right) = \mathcal{A}(A) + \left(\frac{d}{2\pi}\right)^{-2A} \frac{\Gamma(6-A)}{\Gamma(6+A)} \mathcal{A}(-A) + O(X^{-1/2+\varepsilon}),$$

with

$$\mathcal{A}(A) := \prod_p \left(1 + \frac{p}{p+1}\left[-1 + \sum_{m=0}^\infty \frac{\tau^*(p^{2m})}{p^{(1/2+A)2m}}\right]\right).$$

We note that $\mathcal{A}(A)$ is regular at $A = 0$; indeed the $m = 0$ and $m = 1$ terms give 1 and $\tau^*(p^2)p^{-1-2A}$ respectively, therefore an approximation for $\mathcal{A}(A)$ would be $\prod_p(1 + \tau^*(p^2)/p^{1+2A} + \cdots)$. Differently from the unitary and symplectic cases, where the first term in the corresponding Euler products gives the polar factor $\zeta(1+2A)$, here we would have $L_\Delta(\mathrm{sym}^2, 1+2A)$ the symmetric square of L_Δ, which is well known to be regular and nonzero at 1 (see [Iwaniec 1997, Chapter 13] for a complete overview about the symmetric square and its properties). However, for the sake of brevity, we prefer not to factor out $L_\Delta(\mathrm{sym}^2, 1+2A)$, and we leave the contribution of the symmetric square encoded in the arithmetical factor $\mathcal{A}(A)$, which converges in a small disk around 0. Thus, for $A=0$, we immediately get

$$\frac{1}{X^*}\sum_{d\le X} L_\Delta\Big(\frac{1}{2},\chi_d\Big) = 2\mathcal{A} + O(X^{-1/2+\varepsilon}), \tag{5-2}$$

where

$$\mathcal{A} := \mathcal{A}(0) = \prod_p\bigg(1+\frac{p}{p+1}\bigg[-1+\sum_{m=0}^{\infty}\frac{\tau^*(p^{2m})}{(\sqrt{p})^{2m}}\bigg]\bigg). \tag{5-3}$$

5B. *Conjecture 5.1 in the case K = 2, Q = 1.* We consider

$$\sum_{d\le X}\frac{L_\Delta(1/2+A,\chi_d)L_\Delta(1/2+C,\chi_D)}{L_\Delta(1/2+D,\chi_d)}, \tag{5-4}$$

with A, C, D shifts satisfying the usual hypotheses prescribed by the ratio conjecture; by Conjecture 5.1, up to a negligible error, this is a sum of four terms and the first is

$$\sum_{d\le X}\frac{\zeta(1+A+C)\zeta(1+2D)}{\zeta(1+A+D)\zeta(1+C+D)}\mathcal{A}(A,C;D),$$

where

$$A\Big(A,C;D\Big) = \prod_p\Big(1-\frac{1}{p^{1+A+C}}\Big)\Big(1-\frac{1}{p^{1+2D}}\Big)\Big(1-\frac{1}{p^{1+A+D}}\Big)^{-1}\Big(1-\frac{1}{p^{1+C+D}}\Big)^{-1} \times\bigg(1+\frac{p}{p+1}\sum_{0<a+c+d \text{ even}}\frac{\tau^*(p^a)\tau^*(p^c)\mu(p^d)}{p^{a(1/2+A)+c(1/2+C)+d(1/2+D)}}\bigg).$$

As usual, we note that $\mathcal{A}(A,C;D)\sim\mathcal{A}(0,0;0)=\mathcal{A}$ defined in (5-3) as $A, C, D\to 0$; this can be proved by a modification of the proof of [Conrey et al. 2008, Corollary 6.4] or by direct computation. All the other terms can be easily recovered from the first one, then we get a formula for (5-4), written as a sum of four pieces; by computing the derivative $\frac{d}{dC}[\cdots]_{C=D}$, we get

$$\sum_{d\le X}\frac{L'_\Delta}{L_\Delta}\Big(\frac{1}{2}+D,\chi_d\Big)L\Big(\frac{1}{2}+A,\chi_d\Big) = \sum_{d\le X}\bigg(Q_1+\Big(\frac{d}{2\pi}\Big)^{-2A}g_O\Big(\frac{1}{2}+A\Big)Q_2+\Big(\frac{d}{2\pi}\Big)^{-2D}g_O\Big(\frac{1}{2}+D\Big)Q_3 + \Big(\frac{d}{2\pi}\Big)^{-2A-2D}g_O\Big(\frac{1}{2}+A+D\Big)Q_4\bigg)+O(X^{1/2+\varepsilon}), \tag{5-5}$$

with

$$Q_1 = \mathcal{A}(A,D;D)\Big(\frac{\zeta'}{\zeta}(1+A+D)-\frac{\zeta'}{\zeta}(1+2D)\Big)+\mathcal{A}'(A,D;D),$$

$$Q_2 = \mathcal{A}(-A,D;D)\Big(\frac{\zeta'}{\zeta}(1-A+D)-\frac{\zeta'}{\zeta}(1+2D)\Big)+\mathcal{A}'(-A,D;D),$$

$$Q_3 = -\mathcal{A}(A,-D;D)\frac{\zeta(1+A-D)\zeta(1+2D)}{\zeta(1+A+D)},$$

$$Q_4 = -\mathcal{A}(-A,-D;D)\frac{\zeta(1-A-D)\zeta(1+2D)}{\zeta(1-A+D)}.$$

Moreover, we notice that if the shifts are of order $\asymp (\log X)^{-1}$, then we can approximate (5-5), getting

$$\sum_{d\le X}\frac{L'_\Delta}{L_\Delta}\Big(\frac{1}{2}+D,\,\chi_d\Big)L\Big(\frac{1}{2}+A,\,\chi_d\Big)$$
$$=\mathcal{A}X^*\bigg(\frac{A-D}{(A+D)2D}+X^{-2A}\frac{-A-D}{(-A+D)2D}+X^{-2D}\frac{-A-D}{(A-D)2D}+X^{-2A-2D}\frac{A-D}{(-A-D)2D}\bigg)+O(1),\tag{5-6}$$

since $\mathcal{A}(\pm A,\pm D,D)=\mathcal{A}+O(1/\log X)$ and $\zeta(1+z)=1/z+O(1)$ as $z\to 0$.

5C. *Conjecture 5.1 in the case $K=2$, $Q=0$.* We now analyze closely the second moment of $L_\Delta\big(\frac{1}{2},\chi_d\big)$; we take two complex shifts A, B such that $A, B\asymp(\log X)^{-1}$, and we look at

$$\frac{1}{X^*}\sum_{d\le X}L_\Delta\Big(\frac{1}{2}+A,\,\chi_d\Big)L_\Delta\Big(\frac{1}{2}+B,\,\chi_d\Big).$$

By Conjecture 5.1, ignoring the negligible error term $O(X^{1/2+\varepsilon})$, the above is

$$f(A,B)+X^{-2A}f(-A,B)+X^{-2B}f(A,-B)+X^{-2A-2B}f(-A-B),\tag{5-7}$$

with

$$f(A,B):=\zeta(1+A+B)\prod_p\Big(1-\frac{1}{p^{1+A+B}}\Big)\bigg(1+\frac{p}{p+1}\sum_{\substack{m+n>0\\ \text{even}}}\frac{\tau^*(p^m)\tau^*(p^n)}{p^{(1/2+A)m+(1/2+B)n}}\bigg).$$

Since $A, B\asymp(\log X)^{-1}$, we set $a=A\log X\asymp 1$ and $b=B\log X\asymp 1$, so that (5-7) becomes

$$\mathcal{B}\log X\bigg(\frac{1}{a+b}+\frac{e^{-2a}}{-a+b}+\frac{e^{-2b}}{a-b}+\frac{e^{-2a-2b}}{-a-b}\bigg)\bigg(1+O\Big(\frac{1}{\log X}\Big)\bigg),\tag{5-8}$$

where

$$\mathcal{B}:=\prod_p\Big(1-\frac{1}{p}\Big)\bigg(1+\frac{p}{p+1}\sum_{\substack{m+n>0\\ \text{even}}}\frac{\tau^*(p^m)\tau^*(p^n)}{p^{(m+n)/2}}\bigg).\tag{5-9}$$

The expression in (5-8) is regular at $a=0$ and $b=0$, since the limit of the first parentheses as $a,b\to 0$ equals 4. Therefore, we finally get

$$\frac{1}{X^*}\sum_{d\le X}L_\Delta\Big(\frac{1}{2},\,\chi_d\Big)^2\sim 4\mathcal{B}\log X.\tag{5-10}$$

5D. *Conjecture 5.1 in the case $K=3$, $Q=1$.* Finally, we look at

$$\sum_{d\le X}\frac{L_\Delta(1/2+A,\chi_d)L_\Delta(1/2+B,\chi_d)L_\Delta(1/2+C,\chi_d)}{L_\Delta(1/2+D,\chi_d)},\tag{5-11}$$

with A, B, C and D as Conjecture 5.1 prescribes. The first of the eight terms given by the recipe is

$$\sum_{d\le X}\frac{\zeta(1+2D)\zeta(1+A+B)\zeta(1+A+C)\zeta(1+B+C)}{\zeta(1+A+D)\zeta(1+B+D)\zeta(1+C+D)}\mathcal{A}(A,B,C;D),$$

where

$$\mathcal{A}(A,B,C;D)=\prod_p \frac{(1-1/p^{1+2D})(1-1/p^{1+A+B})(1-1/p^{1+A+C})(1-1/p^{1+B+C})}{(1-1/p^{1+A+D})(1-1/p^{1+B+D})(1-1/p^{1+C+D})} \times\biggl(1+\frac{p}{p+1}\sum_{\substack{0<a+b+c+d\\ \text{even}}}\frac{\tau^*(p^a)\tau^*(p^b)\tau^*(p^c)\mu_\Delta(p^d)}{p^{(1/2+A)a+(1/2+B)b+(1/2+C)c+(1/2+D)d}}\biggr)$$

is the arithmetical coefficient, absolutely convergent in small disks around 0, such that $\mathcal{A}(0,0,0;0)=\mathcal{B}$. As in all the previous examples, this gives a formula for (5-11) with all the main terms and error $O(X^{1/2+\varepsilon})$ and differentiating this formula with respect to C at $C=D$, we get

$$\begin{aligned}\sum_{d\le X}\frac{L'_\Delta}{L_\Delta}\Bigl(\frac12+D,\chi_d\Bigr)&L_\Delta\Bigl(\frac12+A,\chi_d\Bigr)L_\Delta\Bigl(\frac12+B,\chi_d\Bigr)\\ =\sum_{d\le X}\biggl(&R_1+\Bigl(\frac{d}{2\pi}\Bigr)^{-2A}g_O\Bigl(\frac12+A\Bigr)R_2+\Bigl(\frac{d}{2\pi}\Bigr)^{-2B}g_O\Bigl(\frac12+B\Bigr)R_3\\ &+\Bigl(\frac{d}{2\pi}\Bigr)^{-2D}g_O\Bigl(\frac12+D\Bigr)R_4+\Bigl(\frac{d}{2\pi}\Bigr)^{-2A-2B}g_O\Bigl(\frac12+A+B\Bigr)R_5\\ &+\Bigl(\frac{d}{2\pi}\Bigr)^{-2A-2D}g_O\Bigl(\frac12+A+D\Bigr)R_6+\Bigl(\frac{d}{2\pi}\Bigr)^{-2B-2D}g_O\Bigl(\frac12+B+D\Bigr)R_7\\ &+\Bigl(\frac{d}{2\pi}\Bigr)^{-2A-2B-2D}g_O\Bigl(\frac12+A+B+D\Bigr)R_8\biggr)+O(X^{1/2+\varepsilon}),\end{aligned}\tag{5-12}$$

with

$$\begin{aligned}R_1&=R_1(A,B,D)\\ &=\mathcal{A}(A,B,D;D)\zeta(1+A+B)\\ &\qquad\times\Bigl(\frac{\zeta'}{\zeta}(1+A+D)+\frac{\zeta'}{\zeta}(1+B+D)-\frac{\zeta'}{\zeta}(1+2D)\Bigr)+\zeta(1+A+B)\mathcal{A}'(A,B,D;D),\\ R_2&=R_1(-A,B,D),\\ R_3&=R_1(A,-B,D),\\ R_4&=R_4(A,B,D)=-\frac{\zeta(1+2D)\zeta(1+A+B)\zeta(1+A-D)\zeta(1+B-D)}{\zeta(1+A+D)\zeta(1+B+D)}\mathcal{A}(A,B,-D;D),\\ R_5&=R_1(-A,-B,D),\\ R_6&=R_4(-A,B,D),\\ R_7&=R_4(A,-B,D),\\ R_8&=R_4(-A,-B,D).\end{aligned}\tag{5-13}$$

If $A,B,D\asymp(\log X)^{-1}$, the above formula simplifies a lot, since

$$R_1=\frac{AB-AD-BD-3D^2}{2D(A+B)(A+D)(B+D)}\mathcal{B}+O\biggl(\frac{(\log X)^4}{(\log X)^3}\biggr)=: f(A,B,D)+O(\log X)$$

and

$$R_4=\frac{(A+D)(B+D)}{(-2D)(A+B)(A-D)(B-D)}\mathcal{B}+\biggl(\frac{(\log X)^4}{(\log X)^3}\biggr)=: g(A,B,D)+O(\log X),$$

giving

$$\sum_{d\le X}\frac{L'_\Delta}{L_\Delta}\Big(\frac{1}{2}+D,\,\chi_d\Big)L_\Delta\Big(\frac{1}{2}+A,\,\chi_d\Big)L_\Delta\Big(\frac{1}{2}+B,\,\chi_d\Big)$$
$$\begin{aligned}=\mathcal{B}X^*\big(f(A,B,D)&+X^{-2A}f(-A,B,D)+X^{-2B}f(A,-B,D)\\&+X^{-2D}g(A,B,D)+X^{-2A-2B}f(-A,-B,D)+X^{-2A-2D}g(-A,B,D)\\&+X^{-2B-2D}g(A,-B,D)+X^{-2A-2B-2D}g(-A,-B,D)\big)+O(\log X).\end{aligned}\tag{5-14}$$

Analogous formulae can be obtained in the cases $K=4$, $Q=1$ and $K=5$, $Q=1$. Again, with the same technique, one can get formulae also in the case $K>5$, $Q=1$.

5E. *The weighted one-level density for $\{L_\Delta(\frac{1}{2},\chi_d)\}_d$.* In analogy to what we did in Section 4C, we now compute the weighted one-level density for the orthogonal family of quadratic twists of L_Δ. We assume the Riemann hypothesis for the L-functions we are considering, and we denote with $\gamma_{\Delta,d}$ the imaginary part of a generic zero of $L_\Delta(s,\chi_d)$. In the classical case, assuming the ratio conjecture, Conrey and Snaith [2007] proved that

$$\lim_{X\to\infty}\frac{1}{X^*}\sum_{d\le X}\sum_{\gamma_{\Delta,d}}f\Big(\frac{\log X}{\pi}\gamma_{\Delta,d}\Big)=\int_{-\infty}^{+\infty}f(x)\Big(1+\frac{\sin(2\pi x)}{2\pi x}\Big)\,dx\tag{5-15}$$

for any test function f, satisfying the usual properties as in Theorem 2.5. We now use the formulae of the previous section to derive the weighted one-level density; we let

$$\mathcal{D}_1^{L_{\Delta,\chi}}(f):=\frac{1}{\sum_{d\le X}L_\Delta(1/2,\chi_d)}\sum_{d\le X}\sum_{\gamma_{\Delta,d}}f\Big(\frac{\log X}{\pi}\gamma_{\Delta,d}\Big)L_\Delta\Big(\frac{1}{2},\chi_d\Big),\tag{5-16}$$

and we prove the following result:

Proposition 5.1. *Assume GRH and Conjecture 5.1 for $K=2$, $Q=1$. For any function f holomorphic in the strip $\Im(z)<2$, even, real on the real line and such that $f(x)\ll 1/(1+x^2)$ as $x\to\infty$, we have*

$$\mathcal{D}_1^{L_{\Delta,\chi}}(f)=\int_{-\infty}^{+\infty}f(x)W^1_{\mathrm{SO}^+}(x)\,dx+O\Big(\frac{1}{\log X}\Big)$$

as $X\to\infty$, where

$$W^1_{\mathrm{SO}^+}(x):=1-\frac{\sin(2\pi x)}{2\pi x}.\tag{5-17}$$

Proof. The strategy of the proof is the same as in the unitary and symplectic cases, thus we will just sketch how the proof works, highlighting the differences with the other cases. For $a\asymp 1/\log X$ a real parameter, we consider the quantity

$$\frac{1}{2\mathcal{A}X^*}\sum_{d\le X}\sum_{\gamma_{\Delta,d}}f\Big(\frac{\log X}{\pi}\gamma_{\Delta,d}\Big)L_\Delta\Big(\frac{1}{2}+\frac{a}{\log X},\,\chi_d\Big),$$

which can be written as ($\delta\asymp(\log X)^{-1}$)

$$\frac{\log(X^2)}{2\pi}\int_{-\infty}^{+\infty}f\Big(\frac{\log X}{\pi}y\Big)\,dy+2\frac{1}{2\pi}\int_{-\infty}^{+\infty}f\Big(\frac{\log X}{\pi}y\Big)\frac{1}{2\mathcal{A}X^*}\sum_{d\le X}\frac{L'_\Delta}{L_\Delta}\Big(\frac{1}{2}+\delta+iy\Big)L_\Delta\Big(\frac{1}{2}+\frac{a}{\log X},\,\chi_d\Big)\,dy,$$

with an error $O((\log X)^{-1})$, by using the Cauchy's theorem and the functional equation

$$\frac{L'_\Delta}{L_\Delta}(1-s,\chi_d)=\frac{Y'_\Delta}{Y_\Delta}(s,\chi_d)-\frac{L'_\Delta}{L_\Delta}(s,\chi_d),$$

with $(Y'_\Delta/Y_\Delta)(s,\chi_d)=-\log d^2+O(1)$ (note that the square here is due to the conductor of $L_\Delta(s,\chi_d)$, which is $d^2/(4\pi^2)$). With the change of variable $(\log X/\pi)y=x$ the above equals

$$\int_{-\infty}^{+\infty} f(x)\biggl(1+\frac{1}{2\mathcal{A}X^*\log X}\sum_{d\le X}\frac{L'_\Delta}{L_\Delta}\Bigl(\frac{1}{2}+\delta+\frac{\pi i x}{\log X}\Bigr)L_\Delta(\tfrac{1}{2}+\tfrac{a}{\log X},\chi_d)\biggr)\,dx.$$

Now we use the assumption of the ratio conjecture in the form of (5-5) and (5-6) to evaluate the sum over d, getting

$$\int_{-\infty}^{+\infty} f(x)\biggl(1+\frac{1}{2\log X}h_X\Bigl(\frac{a}{\log X},\frac{\pi i x}{\log X}\Bigr)\biggr)\,dx+O\Bigl(\frac{1}{\log X}\Bigr),$$

with

$$h_X(\alpha,w):=\frac{\alpha-w}{(\alpha+w)2w}+X^{-2\alpha}\frac{-\alpha-w}{(-\alpha+w)2w}+X^{-2w}\frac{-\alpha-w}{(\alpha-w)2w}+X^{-2\alpha-2w}\frac{\alpha-w}{(-\alpha-w)2w}.$$

Letting $a\to 0$, and since $h_X(0,w)=(X^{-2w}-1)/w$, we get

$$\int_{-\infty}^{+\infty} f(x)\biggl(1+\frac{e^{-2\pi i x}-1}{2\pi i x}\biggr)\,dx+O\Bigl(\frac{1}{\log X}\Bigr).$$

Putting all together, since f is even, we finally have

$$\mathcal{D}_1^{L_{\Delta,\chi}}(f)=\int_{-\infty}^{+\infty} f(x)\biggl(1-\frac{\sin(2\pi x)}{2\pi x}\biggr)\,dx+O\Bigl(\frac{1}{\log X}\Bigr).$$ □

Similarly we compute the analogue of (5-15), tilting by the second power of $L_\Delta\bigl(\frac{1}{2},\chi_d\bigr)$, i.e.,

$$\mathcal{D}_2^{L_{\Delta,\chi}}(f):=\frac{1}{\sum_{d\le X}L_\Delta(1/2,\chi_d)^2}\sum_{d\le X}\sum_{\gamma_{\Delta,d}} f\Bigl(\frac{\log X}{\pi}\gamma_{\Delta,d}\Bigr)L_\Delta\Bigl(\frac{1}{2},\chi_d\Bigr)^2 \tag{5-18}$$

under the assumption of Conjecture 5.1, in the case $K=3$, $Q=1$. This is achieved in the following proposition:

Proposition 5.2. *Assume GRH and Conjecture 5.1 for $K=3$, $Q=1$. For any function f holomorphic in the strip $\Im(z)<2$, even, real on the real line and such that $f(x)\ll 1/(1+x^2)$ as $x\to\infty$, we have*

$$\mathcal{D}_2^{L_{\Delta,\chi}}(f)=\int_{-\infty}^{+\infty} f(x)W^2_{\mathrm{SO}^+}(x)\,dx+O\Bigl(\frac{1}{\log X}\Bigr)$$

as $X\to\infty$, where

$$W^2_{\mathrm{SO}^+}(x):=1+\frac{\sin(2\pi x)}{\pi x}-\frac{2\sin^2(\pi x)}{(\pi x)^2}.$$

Proof. Again we start with

$$\frac{1}{4\mathcal{B}X^*\log X}\sum_{d\le X}\sum_{\gamma_{\Delta,d}} f\Bigl(\frac{\log X}{\pi}\gamma_{\Delta,d}\Bigr)L_\Delta\Bigl(\frac{1}{2}+\frac{a}{\log X},\chi_d\Bigr)L_\Delta\Bigl(\frac{1}{2}+\frac{b}{\log X},\chi_d\Bigr),$$

and with the usual machinery, we write it as

$$\int_{-\infty}^{+\infty} f(x)\Big(1+\frac{1}{4\mathcal{B}X^*(\log X)^2}\mathcal{I}\Big(\frac{a}{\log X},\frac{b}{\log X},\delta+\frac{\pi i x}{\log X}\Big)\Big)\,dx+O\Big(\frac{1}{\log X}\Big),$$

with

$$\mathcal{I}(\alpha,\beta,w):=\sum_{d\le X}\frac{L'_\Delta}{L_\Delta}\Big(\frac12+w,\chi_d\Big)L_\Delta\Big(\frac12+\alpha,\chi_d\Big)L_\Delta\Big(\frac12+\beta,\chi_d\Big).$$

Thanks to the assumption of the ratio conjecture (see (5-12) and (5-14)) we are able to evaluate asymptotically the above sum, which is regular at $\alpha,\beta,\delta=0$. More specifically, we have that

$$\lim_{\substack{a\to0\\ b\to0\\ \delta\to0}}\mathcal{I}\Big(\frac{a}{\log X},\frac{b}{\log X},\delta+\frac{y}{\log X}\Big)=\mathcal{B}X^*(\log X)^2h(y)+O(\log X),$$

with

$$h(y):=\frac{-2ye^{-2y}-6y-4e^{-2y}+4}{y^2}.$$

Then we get

$$\begin{aligned}\mathcal{D}_2^{\boldsymbol{L}_{\Delta,\chi}}(f)&=\int_{-\infty}^{+\infty} f(x)\Big(1+\frac14 h(\pi i x)\Big)\,dx+O\Big(\frac{1}{\log X}\Big)\\&=\int_{-\infty}^{+\infty} f(x)\Big(1+\frac{\sin(2\pi x)}{2\pi x}-\frac{\sin^2(\pi x)}{(\pi x)^2}\Big)\,dx+O\Big(\frac{1}{\log X}\Big),\end{aligned}$$

since f is even. □

We go on and define

$$\mathcal{D}_3^{\boldsymbol{L}_{\Delta,\chi}}(f):=\frac{1}{\sum_{d\le X}L_\Delta(1/2,\chi_d)^3}\sum_{d\le X}\sum_{\gamma_{\Delta,d}} f\Big(\frac{\log X}{\pi}\gamma_{\Delta,d}\Big)L_\Delta\Big(\frac12,\chi_d\Big)^3,$$

analyzing the third-moment case.

Proposition 5.3. *Assume GRH and Conjecture 5.1 for $K=4$, $Q=1$. For any function f holomorphic in the strip $\Im(z)<2$, even, real on the real line and such that $f(x)\ll 1/(1+x^2)$ as $x\to\infty$, we have*

$$\mathcal{D}_3^{\boldsymbol{L}_{\Delta,\chi}}(f)=\int_{-\infty}^{+\infty} f(x)W^3_{\mathrm{SO}^+}(x)\,dx+O\Big(\frac{1}{\log X}\Big)$$

as $X\to\infty$, where

$$W^3_{\mathrm{SO}^+}(x):=1-\frac{\sin(2\pi x)}{2\pi x}-\frac{24(1-\sin^2(\pi x))}{(2\pi x)^2}+\frac{48\sin(2\pi x)}{(2\pi x)^3}-\frac{96\sin^2(\pi x)}{(2\pi x)^4}.$$

Proof. We introduce the usual real parameters α,β,ν of size $\asymp 1/\log X$, we let

$$\boldsymbol{L}_\Delta^{\alpha,\beta,\nu}\big(\tfrac12,\chi_d\big):=L_\Delta\big(\tfrac12+\alpha,\chi_d\big)L_\Delta\big(\tfrac12+\beta,\chi_d\big)L_\Delta\big(\tfrac12+\nu,\chi_d\big),$$

and we consider

$$\frac{1}{\sum_{d\le X}L_\Delta(1/2,\chi_d)^3}\sum_{d\le X}\sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big)\boldsymbol{L}_\Delta^{\alpha,\beta,\nu}\Big(\frac12,\chi_d\Big).$$

With the usual strategy we get that the above equals

$$\int_{-\infty}^{+\infty} f(x)\Big(1+\frac{1}{\sum_{d\le X} L(1/2,\,\chi_d)^3}\sum_{d\le X}\frac{L_\Delta'}{L_\Delta}\Big(\frac12+\delta+\frac{2\pi i x}{\log X},\,\chi_d\Big)\boldsymbol{L}_\Delta^{\alpha,\beta,\nu}\big(\tfrac12,\,\chi_d\big)\Big)\,dx$$

up to an error $O(1/\log X)$, with $\delta \asymp 1/\log X$. We evaluate asymptotically the remaining sum over d thanks to Conjecture 5.1 for $K=4$, $Q=1$), using Sage to carry out the computations. Doing so, letting $\alpha,\,\beta,\,\nu\to 0$, we obtain

$$\mathcal{D}_3^{\boldsymbol{L}_{\Delta,X}}(f)=\int_{-\infty}^{+\infty} f(x)\Big(1+\frac12 h(\pi i x)\Big)\,dx+O\Big(\frac{1}{\log X}\Big),$$

with

$$h(y):=\frac{-5y^3+6y^2-6+e^{-2y}(y^3+6y^2+12y+6)}{y^4}.$$

The claim follows, since f is even. □

Finally, in the following result, we study the case $k=4$, given by

$$\mathcal{D}_4^{\boldsymbol{L}_{\Delta,X}}(f):=\frac{1}{\sum_{d\le X} L_\Delta(1/2,\,\chi_d)^4}\sum_{d\le X}\sum_{\gamma_{\Delta,d}} f\Big(\frac{\log X}{\pi}\gamma_{\Delta,d}\Big)L_\Delta\Big(\frac12,\,\chi_d\Big)^4:$$

Proposition 5.4. *Assume GRH and Conjecture 5.1 for $K=5$, $Q=1$. For any function f holomorphic in the strip $\Im(z)<2$, even, real on the real line and such that $f(x)\ll 1/(1+x^2)$ as $x\to\infty$, we have*

$$\mathcal{D}_4^{\boldsymbol{L}_{\Delta,X}}(f)=\int_{-\infty}^{+\infty} f(x)W_{\mathrm{SO}^+}^4(x)\,dx+O\Big(\frac{1}{\log X}\Big)$$

as $X\to\infty$, where

$$W_{\mathrm{SO}^+}^4(x)$$
$$:=1+\frac{\sin(2\pi x)}{2\pi x}-\frac{12\sin^2(\pi x)}{(\pi x)^2}-\frac{240\sin(2\pi x)}{(2\pi x)^3}-\frac{15(6-10\sin^2(\pi x))}{(\pi x)^4}+\frac{2880\sin(2\pi x)}{(2\pi x)^5}-\frac{90\sin^2(\pi x)}{(\pi x)^6}.$$

Proof. As usual, if we set

$$\boldsymbol{L}_\Delta^{\alpha,\beta,\nu,\eta}\big(\tfrac12,\,\chi_d\big):=L_\Delta\big(\tfrac12+\alpha,\,\chi_d\big)L_\Delta\big(\tfrac12+\beta,\,\chi_d\big)L_\Delta\big(\tfrac12+\nu,\,\chi_d\big)L_\Delta\big(\tfrac12+\eta,\,\chi_d\big),$$

then we express $\mathcal{D}_4^{\boldsymbol{L}_{\Delta,X}}(f)$ as the limit for $\alpha,\,\beta,\,\nu,\,\eta\to 0$ of

$$\int_{-\infty}^{+\infty} f(x)\Big(1+\frac{1}{\sum_{d\le X} L(1/2,\,\chi_d)^4}\sum_{d\le X}\frac{L_\Delta'}{L_\Delta}\Big(\frac12+\delta+\frac{2\pi i x}{\log X},\,\chi_d\Big)\boldsymbol{L}_\Delta^{\alpha,\beta,\nu,\eta}\Big(\frac12,\,\chi_d\Big)\Big)\,dx$$

up to an error $O(1/\log X)$, with $\delta\asymp 1/\log X$. The above can be evaluated asymptotically (again Sage is of help in carrying out the computation), and we get

$$\mathcal{D}_4^{\boldsymbol{L}_{\Delta,X}}(f)=\int_{-\infty}^{+\infty} f(x)\Big(1+\frac12 h(\pi i x)\Big)\,dx+O\Big(\frac{1}{\log X}\Big),$$

with

$$h(y):=\frac{-7y^5+12y^4-30y^2+90}{y^6}-\frac{e^{-2y}(y^5+12y^4+60y^3+150y^2+180y+90)}{y^6}.$$

Since f is even, the claim follows. □

Theorem 2.5 follows by Propositions 5.1–5.4.

5F. ***Proof of Theorem 2.6.*** Citing [Conrey et al. 2008, Theorem 4.3] instead of assuming Conjecture 5.1, the same strategy gives an unconditional proof in random matrix theory.

6. Proof of Theorem 2.7

By Fourier inversion, we have that

$$\begin{aligned} W^k_{\mathrm{USp}}(x) &= \int_{-\infty}^{+\infty} \widehat{W}^k_{\mathrm{USp}}(y) e^{2\pi i x y} dy \\ &= \sum_{n=0}^{\infty} \biggl(\frac{(2\pi i)^n}{n!} \int_{-\infty}^{+\infty} \widehat{W}^k_{\mathrm{USp}}(y) y^n \, dy \biggr) x^n. \end{aligned}$$

Moreover, since $\widehat{W}^k_{\mathrm{USp}}$ is even, then $\int_{-\infty}^{+\infty} \widehat{W}^k_{\mathrm{USp}}(y) y^n \, dy = 0$ if n is odd. Hence, by definition of $\widehat{W}^k_{\mathrm{USp}}$,

$$W^k_{\mathrm{USp}}(x) = \sum_{m=0}^{\infty} \beta_{m,k} x^{2m},$$

with

$$\beta_{m,k} = \frac{(2\pi i)^{2m}}{(2m)!} \int_{-\infty}^{+\infty} \biggl(\delta_0(y) + \chi_{[-1,1]}(y) \biggl(-\frac{2k+1}{2} - k(k+1) \sum_{j=1}^{k} (-1)^j c_{j,k} \frac{|y|^{2j-1}}{2j-1} \biggr) \biggr) y^{2m} \, dy,$$

where $\chi_{[-1,1]}$ denotes the indicator function of the interval $[-1, 1]$. By computing the integral, being $\int_{-1}^{1} y^{2m} \, dy = 2/(2m+1)$ and $\int_{-1}^{1} y^{2m} |y|^{2j-1} \, dy = 1/(m+j)$, the above yields

$$\beta_{m,k} := \delta_0(m) - \frac{(2\pi i)^{2m}}{(2m)!} \biggl[\frac{2k+1}{2m+1} + k(k+1) \sum_{j=1}^{k} \frac{(-1)^j}{(2j-1)(j+m)} c_{j,k} \biggr].$$

Since

$$c_{j,k} = \frac{1}{j} \binom{k-1}{j-1} \binom{k+j}{j-1} = \frac{j}{k(k+1)} \binom{k+j}{j} \binom{k}{j},$$

we get

$$\beta_{m,k} = \delta_0(m) - \frac{(2\pi i)^{2m}}{(2m)!} \biggl[\frac{2k+1}{2m+1} + S_m(1) \biggr], \tag{6-1}$$

where

$$S_{m,k}(x) := \sum_{j=1}^{\infty} \frac{(-1)^j j}{(2j-1)(j+m)} \binom{k+j}{j} \binom{k}{j} x^j.$$

Now we write the factors in the above sum in terms of the Pochhammer symbol, defined as

$$(a)_0 := 1 \quad \text{and} \quad (a)_n := a(a+1)(a+2) \cdots (a+n-1) \quad \text{for } n \geq 1;$$

namely,

$$\binom{k+j}{j} = \frac{(k+1)_j}{j!},$$
$$(-1)^j\binom{k}{j} = \frac{(-k)_j}{(1)j},$$
$$\frac{j}{(2j-1)(j+m)} = \frac{1}{2m+1}\left(\frac{1}{2j-1}+\frac{m}{j+m}\right) = \frac{1}{2m+1}\left(-\frac{(-1/2)_j}{(1/2)_j}+\frac{(m)_j}{(m+1)_j}\right),$$

so that we have

$$S_{m,k}(x) = \frac{1}{2m+1}\sum_{j=1}^{\infty}\left(-\frac{(-1/2)_j}{(1/2)_j}+\frac{(m)_j}{(m+1)_j}\right)\frac{(-k)_j(k+1)_j}{(1)_j}\frac{x^j}{j!}. \tag{6-2}$$

Reparametrizing the sum and using $(a)_{j+1} = a(a+1)_j$, this gives

$$S_{m,k}(x) = S^1_{m,k}(x)+S^2_{m,k}(x),$$

where

$$S^1_{m,k}(x) := \frac{-k(k+1)x}{2m+1}\sum_{j=0}^{\infty}\frac{(1/2)_j(-k+1)_j(k+2)_j}{(3/2)_j(2)_j}\frac{x^j}{j!}\frac{1}{j+1}$$

and

$$S^2_{m,k}(x) := \frac{-mk(k+1)x}{(2m+1)(m+1)}\sum_{j=0}^{\infty}\frac{(m+1)_j(-k+1)_j(k+2)_j}{(m+2)_j(2)_j}\frac{x^j}{j!}\frac{1}{j+1}.$$

By writing

$$\frac{1}{j+1} = 2\left(1-\frac{2j+1}{2j+2}\right) \quad\text{and}\quad \frac{(1/2)_j}{(3/2)_j} = \frac{1}{2j+1},$$

we get

$$S^1_{m,k}(x) = -\frac{2xk(k+1)}{m+1}\,{}_3F_2\left[\begin{matrix}1-k,\ k+2,\ 1/2\\ 3/2,\ 2\end{matrix};x\right]-\frac{{}_2F_1\left[\begin{smallmatrix}-k,\,k+1\\ 1\end{smallmatrix};x\right]-1}{2m+1},$$

where ${}_pF_q$ denotes the generalized hypergeometric function, defined as

$${}_pF_q\left[\begin{matrix}a_1, & \dots, & a_p\\ b_1, & \dots, & b_q\end{matrix};x\right] := \sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{x^n}{n!}.$$

Similarly, since

$$-\frac{1}{j+1} = \frac{1}{m}-\frac{j+m+1}{(j+1)m},$$

we have

$$S^2_{m,k}(x) = \frac{xk(k+1)}{(2m+1)(m+1)}\,{}_3F_2\left[\begin{matrix}1-k,\ k+2,\ m+1\\ m+2,\ 2\end{matrix};x\right]+\frac{{}_2F_1\left[\begin{smallmatrix}-k,\,k+1\\ 1\end{smallmatrix};x\right]-1}{2m+1}.$$

Therefore, substituting in (6-2) yields

$$S_{m,k}(x) = -\frac{2xk(k+1)}{m+1}\,{}_3F_2\left[\begin{matrix}1-k,\ k+2,\ 1/2\\ 3/2,\ 2\end{matrix};x\right]+\frac{xk(k+1)}{(2m+1)(m+1)}\,{}_3F_2\left[\begin{matrix}1-k,\ k+2,\ m+1\\ m+2,\ 2\end{matrix};x\right]. \tag{6-3}$$

Plugging (6-3) into (6-1), we obtain

$$\beta_{m,k} = \delta_0(m) - \frac{(-1)^m(2\pi)^{2m}}{(2m+1)!}\Bigg(2k+1-2k(k+1)\,{}_3F_2\left[\begin{matrix}1-k,\ k+2,\ 1/2\\ 3/2,\ 2\end{matrix};1\right] + \frac{k(k+1)}{(m+1)}\,{}_3F_2\left[\begin{matrix}1-k,\ k+2,\ m+1\\ m+2,\ 2\end{matrix};1\right]\Bigg). \tag{6-4}$$

Now we need a few lemmas, in order to be able to compute the remaining hypergeometric functions.

Lemma 6.1. *For any $k \in \mathbb{N}$, we have*

$${}_3F_2\left[\begin{matrix}1-k,\ k+2,\ 1/2\\ 3/2,\ 2\end{matrix};1\right] = \begin{cases}\dfrac{1}{k+1}, & \text{if } k \text{ even},\\ \dfrac{1}{k}, & \text{if } k \text{ odd}.\end{cases}$$

Proof. We recall the reduction formula for the generalized hypergeometric function (see, e.g., [Gottschalk and Maslen 1988, Equation (17), when $n = 1$],), being

$${}_{A+1}F_{B+1}\left[\begin{matrix}a_1,\ \ldots,\ a_A,\ c+n\\ b_1,\ \ldots,\ b_B,\ c\end{matrix};x\right] = \sum_{j=0}^{n}\binom{n}{j}\frac{1}{(c)_j}\frac{\prod_{i=1}^{A}(a_i)_j}{\prod_{i=1}^{B}(b_i)_j}\,{}_AF_B\left[\begin{matrix}a_1+j,\ \ldots,\ a_A+j\\ b_1+j,\ \ldots,\ b_B+j\end{matrix};x\right]$$

for any A, B positive integers, $n \in \mathbb{N}$. The left hand side can be then written as

$$\begin{aligned}{}_3F_2\left[\begin{matrix}1-k,\ 1/2,\ k+2\\ 3/2,\ 2\end{matrix};1\right] &= \sum_{j=0}^{k}\binom{k}{j}\frac{1}{(2)_j}\frac{(1-k)_j(1/2)_j}{(3/2)_j}\,{}_2F_1\left[\begin{matrix}1-k+j,\ 1/2+j\\ 3/2+j\end{matrix};1\right]\\ &= \sum_{j=0}^{k-1}\binom{k}{j}\frac{1}{(2)_j}\frac{(1-k)_j(1/2)_j}{(3/2)_j}\,{}_2F_1\left[\begin{matrix}1-k+j,\ 1/2+j\\ 3/2+j\end{matrix};1\right],\end{aligned} \tag{6-5}$$

as $(1-k)_k = 0$. The remaining hypergeometric function can be computed by applying Gauss' summation theorem (see, e.g., [Koepf 2014, Equation (3.1)]), i.e., the formula

$${}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix};1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad \Re(c) > \Re(a+b).$$

We recall that if $a = -n$, $n \in \mathbb{N}$, this is the Chu–Vandermonde identity (see, again, [Koepf 2014, immediately below Equation (3.1)])

$${}_2F_1\left[\begin{matrix}-n,\ b\\ c\end{matrix};1\right] = \frac{(c-b)_n}{(c)_n}.$$

This yields

$${}_2F_1\left[\begin{matrix}1-k+j,\ 1/2+j\\ 3/2+j\end{matrix};1\right] = \frac{(k-j-1)!(j+1/2)!}{(k-1/2)!} \tag{6-6}$$

for $k > j$. Plugging (6-6) into (6-5), we get

$$\begin{aligned}{}_3F_2\left[\begin{matrix}1-k,\ 1/2,\ k+2\\ 3/2,\ 2\end{matrix};1\right] &= \frac{(k-1)!}{2(k-1/2)!}\sum_{j=0}^{k-1}\binom{k}{j}(-1)^j\frac{(j-1/2)!}{(j+1)!}\\ &= \frac{(k-1)!}{2(k-1/2)!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{(j-1/2)!}{(j+1)!}-\frac{(-1)^k}{2k(k+1)}.\end{aligned}\tag{6-7}$$

Moreover, since $\left(\frac{1}{2}\right)_j = (1/\sqrt{\pi})(j-\frac{1}{2})!$, $(-1)^j\binom{k}{j} = (-k)_j/j!$ and $(2)_j = (j+1)!$, we have

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{(j-1/2)!}{(j+1)!} = \sqrt{\pi}\,{}_2F_1\left[\begin{matrix}-k,\ 1/2\\ 2\end{matrix};1\right] = 2\frac{(k+1/2)!}{(k+1)!}$$

by applying the Chu–Vandermonde identity. Putting this into (6-7), we finally get

$${}_3F_2\left[\begin{matrix}1-k,\ 1/2,\ k+2\\ 3/2,\ 2\end{matrix};1\right] = \frac{k+1/2}{k(k+1)}-\frac{(-1)^k}{2k(k+1)},$$

and the claim follows. □

By using Lemma 6.1, (6-4) becomes

$$\beta_{m,k} = \delta_0(m)-\frac{(-1)^m(2\pi)^{2m}}{(2m+1)!}\left((-1)^k+\frac{k(k+1)}{(m+1)}{}_3F_2\left[\begin{matrix}1-k,\ k+2,\ m+1\\ m+2,\ 2\end{matrix};1\right]\right).\tag{6-8}$$

The coefficient $\beta_{0,k}$ can be then computed, thanks to the following lemma:

Lemma 6.2. *For any $k \in \mathbb{N}$ we have*

$${}_3F_2\left[\begin{matrix}1-k,\ k+2,\ 1\\ 2,\ 2\end{matrix};1\right] = \begin{cases}0, & \text{if } k \text{ even},\\ \dfrac{2}{k(k+1)}, & \text{if } k \text{ odd}.\end{cases}$$

Proof. By definition, we have

$${}_3F_2\left[\begin{matrix}1-k,\ k+2,\ 1\\ 2,\ 2\end{matrix};1\right] = \sum_{j=0}^{\infty}\frac{(1-k)_j(k+2)_j(1)_j}{(2)_j(2)_j}\frac{1}{j!} = -\frac{1}{k(k+1)}\sum_{j=0}^{\infty}\frac{(-k)_{j+1}(k+1)_{j+1}}{(1)_{j+1}}\frac{1}{j!},$$

since $(1)_j/(2)_j = 1/(j+1)$, $(1-k)_j = (-k)_{j+1}/(-k)$ and $(2)_j = (1)_{j+1}$. Reparametrizing the series with $l = j+1$, the above yields

$$-\frac{1}{k(k+1)}\left(\sum_{l=0}^{\infty}\frac{(-k)_l(k+1)_l}{(1)_l}\frac{1}{l!}-1\right) = \frac{1-{}_2F_1\left[\begin{matrix}-k,\ k+1\\ 1\end{matrix};1\right]}{k(k+1)}.$$

The claim is then proven, by noticing that ${}_2F_1\left[\begin{smallmatrix}-k,k+1\\ 1\end{smallmatrix};1\right] = (-1)^k$, thanks to the Chu–Vandermonde identity. □

This implies that $\beta_{0,k}=0$ for any $k\in\mathbb{N}$, proving the first part of Theorem 2.7. To complete our proof, we need to show that

$$\beta_{k+1,k}=\frac{2\pi^{2(k+1)}}{(2k+1)!!(2k+1)!!} \tag{6-9}$$

is the first nonzero coefficient. As a first step, the following lemma shows that $\beta_{i,k}=0$ for all $1\le i\le k$:

Lemma 6.3. *For any $k\in\mathbb{N}$ and for any $1\le m\le k$, we have*

$$ {}_3F_2\left[\begin{matrix}1-k,\ k+2,\ m+1\\ m+2,\ 2\end{matrix};1\right]=\frac{(m+1)(-1)^{k+1}}{k(k+1)}.$$

Proof. We begin by applying the reduction formula, which yields

$$ {}_3F_2\left[\begin{matrix}1-k,\ k+2,\ m+1\\ m+2,\ 2\end{matrix};1\right]=\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{1}{(2)_j}\frac{(1-k)_j(k+2)_j}{(m+2)_j}\,{}_2F_1\left[\begin{matrix}1-k+j,\ k+2+j\\ m+2+j\end{matrix};1\right]. \tag{6-10}$$

Moreover, the Chu–Vandermonde identity gives

$$ {}_2F_1\left[\begin{matrix}1-k+j,\ k+2+j\\ m+2+j\end{matrix};1\right]=\frac{(m-k)_{k-j-1}}{(m+j+2)_{k-j-1}}.$$

Since $(m-k)_{k-j-1}=0$ for all $j<m-1$, only the term $j=m-1$ survives in the sum in (6-10). Hence, we get

$$\begin{aligned} {}_3F_2\left[\begin{matrix}1-k,\ k+2,\ m+1\\ m+2,\ 2\end{matrix};1\right]&=\frac{(1-k)_{m-1}}{(2)_{m-1}}\frac{(k+2)_{m-1}}{(m+2)_{m-1}}\frac{(m-k)_{k-m}}{(2m+1)_{k-m}}\\ &=\frac{(-1)^{m-1}(k-1)!}{(k-m)!\,m!}\frac{(k+m)!(m+1)!}{(k+1)!(2m)!}\frac{(-1)^{k-m}(k-m)!(2m)!}{(m+k)!}\\ &=\frac{(-1)^{k-1}(k-1)!}{m!}\frac{(m+1)!}{(k+1)!}=\frac{(-1)^{k-1}(m+1)}{k(k+1)},\end{aligned} \tag{6-11}$$

where in the first line we applied the equalities $(m+2)_{m-1}=(2m)!/(m+1)!$, $(k+2)_{m-1}=(k+m)!/(k+1)!$ and $(1-k)_{m-1}=(-1)^{m-1}(k-1)!/(k-m)!$. Similarly also $(m-k)_{k-m}=(-1)^{k-m}(k-m)!$ and $(2m+1)_{k-m}=(k+m)!/(2m)!$. □

Finally, with the following lemma, we can also compute $\beta_{k+1,k}$:

Lemma 6.4. *For any $k\in\mathbb{N}$, we have*

$$ {}_3F_2\left[\begin{matrix}1-k,\ k+2,\ k+2\\ k+3,\ 2\end{matrix};1\right]=\frac{2(-1)^{k+1}(k-1)!(k+2)!}{(2k+2)!}\left(\binom{2k+1}{k+1}-1\right).$$

Proof. The idea of the proof is similar the one of Lemma 6.3. First we apply the reduction formula in order to write ${}_3F_2\left[\begin{smallmatrix}1-k,k+2,k+2\\ k+3,2\end{smallmatrix};1\right]$ as a finite sum of terms involving ${}_2F_1$, namely,

$$ {}_3F_2\left[\begin{matrix}1-k,\ k+2,\ k+2\\ k+3,\ 2\end{matrix};1\right]=\sum_{j=0}^{k-1}\binom{k}{j}\frac{1}{(2)_j}\frac{(1-k)_j(k+2)_j}{(k+3)_j}\,{}_2F_1\left[\begin{matrix}1-k+j,\ k+2+j\\ k+3+j\end{matrix};1\right]. \tag{6-12}$$

Note that the term $j=k$ vanishes, as $(1-k)_k=0$. Now we use Gauss' summation theorem and compute the remaining hypergeometric function, i.e.,

$$ {}_2F_1\left[\begin{matrix}1-k+j,\ k+2+j\\ k+3+j\end{matrix};1\right]=\frac{(k+j+2)!}{(2k+1)!}. $$

Plugging this into (6-12), since $(-1)^j\binom{k}{j}=(-k)_j/j!$ and $(k+2)_j=(k+j+1)!/(k+1)!$, we have

$$ \begin{aligned} {}_3F_2\left[\begin{matrix}1-k,\ k+2,\ k+2\\ k+3,\ 2\end{matrix};1\right]&=\frac{(k-1)!(k+2)!}{(2k+1)!}\sum_{j=0}^{k-1}\frac{(-k)_j(k+2)_j}{(2)_j}\frac{1}{j!}\\ &=\frac{(k-1)!(k+2)!}{(2k+1)!}\left({}_2F_1\left[\begin{matrix}-k,\ k+2\\ 2\end{matrix};1\right]-\frac{(-k)_k(k+2)_k}{(2)_k}\frac{1}{k!}\right). \end{aligned} \tag{6-13} $$

Therefore, since ${}_2F_1\left[\begin{smallmatrix}-k,k+2\\ 2\end{smallmatrix};1\right]=(-1)^k/(k+1)$, we get

$$ \begin{aligned} {}_3F_2\left[\begin{matrix}1-k,\ k+2,\ k+2\\ k+3,\ 2\end{matrix};1\right]&=\frac{(k-1)!(k+2)!}{(2k+1)!}\left(\frac{(-1)^k}{k+1}-\frac{(-1)^k(2k+1)!}{((k+1)!)^2}\right)\\ &=\frac{2(k-1)!(k+2)!(-1)^k}{(2k+2)!}\left(1-\frac{(2k+1)!}{(k+1)!\,k!}\right), \end{aligned} $$

and the claim follows. □

To conclude the proof of Theorem 2.7, we just combine (6-8) with Lemma 6.4, getting

$$ \begin{aligned} \beta_{k+1,k}&=\frac{(2\pi)^{2k+2}}{(2k+3)!}\left(1-\frac{k(k+1)}{k+2}\frac{2(k-1)!(k+2)!}{(2k+2)!}\left(\binom{2k+1}{k+1}-1\right)\right)\\ &=\frac{(2\pi)^{2k+2}}{(2k+3)!}\left(1-\frac{2[(k+1)!]^2}{(2k+2)!}\left(\frac{(2k+1)!}{(k+1)!\,k!}-1\right)\right)\\ &=\frac{(2\pi)^{2k+2}}{(2k+3)!}\left(1-\frac{2(k+1)}{2k+2}+\frac{2[(k+1)!]^2}{(2k+2)!}\right)=\frac{(2\pi)^{2k+2}}{(2k+3)!}\frac{(k+1)!\,k!}{(2k+1)!}. \end{aligned} $$

Equation (6-9) follows by the identities $(2k+1)!=2^kk!(2k+1)!!$ and $(2k+3)!=2^{k+1}(k+1)!(2k+3)!!$.

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References

[Bettin and Fazzari 2022] S. Bettin and A. Fazzari, "A weighted one-level density of the non-trivial zeros of the Riemann zeta-function", preprint, 2022. arXiv 2208.08421

[Böcherer 1986] S. Böcherer, "Bemerkungen über die Dirichletreihen von Koecher und Maaß", *Preprint Math. Gottingensis Heft* **69** (1986), 36. Zbl

[Bump et al. 1990] D. Bump, S. Friedberg, and J. Hoffstein, "Nonvanishing theorems for L-functions of modular forms and their derivatives", *Invent. Math.* **102**:3 (1990), 543–618. MR Zbl

[Conrey 2005] B. Conrey, "Families of L-functions and 1-level densities", pp. 225–249 in *Recent perspectives in random matrix theory and number theory*, edited by F. Mezzadri and N. C. Snaith, London Math. Soc. Lecture Note Ser. **322**, Cambridge Univ. Press, 2005. MR Zbl

[Conrey and Farmer 2000] J. B. Conrey and D. W. Farmer, "Mean values of L-functions and symmetry", *Internat. Math. Res. Notices* **17** (2000), 883–908. MR Zbl

[Conrey and Snaith 2007] J. B. Conrey and N. C. Snaith, "Applications of the L-functions ratios conjectures", *Proc. Lond. Math. Soc.* (3) **94**:3 (2007), 594–646. MR Zbl

[Conrey et al. 2005a] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, and N. C. Snaith, "Integral moments of L-functions", *Proc. London Math. Soc.* (3) **91**:1 (2005), 33–104. MR Zbl

[Conrey et al. 2005b] J. B. Conrey, D. W. Farmer, and M. R. Zirnbauer, "Howe pairs, supersymmetry, and ratios of random characteristic polynomials for the unitary groups U(N)", preprint, 2005. arXiv math-ph/0511024

[Conrey et al. 2008] B. Conrey, D. W. Farmer, and M. R. Zirnbauer, "Autocorrelation of ratios of L-functions", *Commun. Number Theory Phys.* **2**:3 (2008), 593–636. MR Zbl

[Dickson et al. 2020] M. Dickson, A. Pitale, A. Saha, and R. Schmidt, "Explicit refinements of Böcherer's conjecture for Siegel modular forms of squarefree level", *J. Math. Soc. Japan* **72**:1 (2020), 251–301. MR Zbl

[Fazzari 2021a] A. Fazzari, "A weighted central limit theorem for $\log\left|\zeta\left(\frac{1}{2}+it\right)\right|$", *Mathematika* **67**:2 (2021), 324–341. MR

[Fazzari 2021b] A. Fazzari, "Weighted value distributions of the Riemann zeta function on the critical line", *Forum Math.* **33**:3 (2021), 579–592. MR Zbl

[Furusawa and Morimoto 2021] M. Furusawa and K. Morimoto, "Refined global Gross–Prasad conjecture on special Bessel periods and Böcherer's conjecture", *J. Eur. Math. Soc.* (*JEMS*) **23**:4 (2021), 1295–1331. MR Zbl

[Gottschalk and Maslen 1988] J. E. Gottschalk and E. N. Maslen, "Reduction formulae for generalised hypergeometric functions of one variable", *J. Phys. A* **21**:9 (1988), 1983–1998. MR Zbl

[Huckleberry et al. 2016] A. Huckleberry, A. Püttmann, and M. R. Zirnbauer, "Haar expectations of ratios of random characteristic polynomials", *Complex Anal. Synerg.* **2**:1 (2016), art. id. 1. MR Zbl

[Hughes and Rudnick 2002] C. Hughes and Z. Rudnick, "Linear statistics for zeros of Riemann's zeta function", *C. R. Math. Acad. Sci. Paris* **335**:8 (2002), 667–670. MR Zbl

[Hughes and Rudnick 2003] C. P. Hughes and Z. Rudnick, "Linear statistics of low-lying zeros of L-functions", *Q. J. Math.* **54**:3 (2003), 309–333. MR Zbl

[Iwaniec 1990] H. Iwaniec, "On the order of vanishing of modular L-functions at the critical point", *Sém. Théor. Nombres Bordeaux* (2) **2**:2 (1990), 365–376. MR Zbl

[Iwaniec 1997] H. Iwaniec, *Topics in classical automorphic forms*, Graduate Studies in Mathematics **17**, American Math. Society, Providence, RI, 1997. MR Zbl

[Iwaniec et al. 2000] H. Iwaniec, W. Luo, and P. Sarnak, "Low lying zeros of families of L-functions", *Inst. Hautes Études Sci. Publ. Math.* **91** (2000), 55–131. MR Zbl

[Jutila 1981] M. Jutila, "On the mean value of $L\left(\frac{1}{2},\ \chi\right)$ for real characters", *Analysis* **1**:2 (1981), 149–161. MR Zbl

[Kaczorowski and Perelli 1999] J. Kaczorowski and A. Perelli, "The Selberg class: A survey", pp. 953–992 in *Number theory in progress, vol.* 2 (Zakopane–Kościelisko, 1997)), edited by K. Győry et al., de Gruyter, Berlin, 1999. MR Zbl

[Katz and Sarnak 1999] N. M. Katz and P. Sarnak, "Zeroes of zeta functions and symmetry", *Bull. Amer. Math. Soc.* (*N.S.*) **36**:1 (1999), 1–26. MR Zbl

[Keating 2005] J. P. Keating, "L-functions and the characteristic polynomials of random matrices", pp. 251–277 in *Recent perspectives in random matrix theory and number theory*, edited by F. Mezzadri and N. C. Snaith, London Math. Soc. Lecture Note Ser. **322**, Cambridge Univ. Press, 2005. MR Zbl

[Keating and Snaith 2000] J. P. Keating and N. C. Snaith, "Random matrix theory and L-functions at $s = \frac{1}{2}$", *Comm. Math. Phys.* **214**:1 (2000), 91–110. MR

[Knightly and Reno 2019] A. Knightly and C. Reno, "Weighted distribution of low-lying zeros of GL(2) L-functions", *Canad. J. Math.* **71**:1 (2019), 153–182. MR Zbl

[Koepf 2014] W. Koepf, *Hypergeometric summation: An algorithmic approach to summation and special function identities*, 2nd ed., Springer, London, 2014. MR Zbl

[Kowalski et al. 2012] E. Kowalski, A. Saha, and J. Tsimerman, "Local spectral equidistribution for Siegel modular forms and applications", *Compos. Math.* **148**:2 (2012), 335–384. MR Zbl

[Murty and Murty 1991] M. R. Murty and V. K. Murty, "Mean values of derivatives of modular L-series", *Ann. of Math.* (2) **133**:3 (1991), 447–475. MR Zbl

[Özlük and Snyder 1999] A. E. Özlük and C. Snyder, "On the distribution of the nontrivial zeros of quadratic L-functions close to the real axis", *Acta Arith.* **91**:3 (1999), 209–228. MR Zbl

[Radziwiłł and Soundararajan 2015] M. Radziwiłł and K. Soundararajan, "Moments and distribution of central L-values of quadratic twists of elliptic curves", *Invent. Math.* **202**:3 (2015), 1029–1068. MR

[Rudnick and Sarnak 1996] Z. Rudnick and P. Sarnak, "Zeros of principal L-functions and random matrix theory", *Duke Math. J.* **81**:2 (1996), 269–322. MR Zbl

[Soundararajan 2009] K. Soundararajan, "Moments of the Riemann zeta function", *Ann. of Math.* (2) **170**:2 (2009), 981–993. MR Zbl

[Soundararajan and Young 2010] K. Soundararajan and M. P. Young, "The second moment of quadratic twists of modular L-functions", *J. Eur. Math. Soc.* (*JEMS*) **12**:5 (2010), 1097–1116. MR Zbl

[Sugiyama 2021] S. Sugiyama, "Low-lying zeros of symmetric power L-functions weighted by symmetric square L-values", preprint, 2021. arXiv 2101.06705

[Titchmarsh 1986] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd ed., Clarendon, New York, 1986. MR Zbl

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fazzari@dima.unige.it *Dipartimento di Matematica, Università di Genova, Genova, Italy*
Department of Algebra, Charles University, Praha, Czech Republic

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Semisimple algebras and PI-invariants of finite dimensional algebras

Eli Aljadeff and Yakov Karasik

Let Γ be the T-ideal of identities of an affine PI-algebra over an algebraically closed field F of characteristic zero. Consider the family $\mathcal{M}_\Gamma$ of finite dimensional algebras Σ with $\mathrm{Id}(\Sigma)=\Gamma$. By Kemer's theory $\mathcal{M}_\Gamma$ is not empty. We show there exists $A\in\mathcal{M}_\Gamma$ with Wedderburn–Malcev decomposition $A\cong A_{ss}\oplus J_A$, where J_A is the Jacobson's radical and A_{ss} is a semisimple supplement with the property that if $B\cong B_{ss}\oplus J_B\in\mathcal{M}_\Gamma$ then A_{ss} is a direct summand of B_{ss}. In particular A_{ss} is unique minimal, thus an invariant of Γ. More generally, let Γ be the T-ideal of identities of a PI algebra and let $\mathcal{M}_{\mathbb{Z}_2,\Gamma}$ be the family of finite dimensional superalgebras Σ with $\mathrm{Id}(E(\Sigma))=\Gamma$. Here E is the unital infinite dimensional Grassmann algebra and $E(\Sigma)$ is the Grassmann envelope of Σ. Again, by Kemer's theory $\mathcal{M}_{\mathbb{Z}_2,\Gamma}$ is not empty. We prove there exists a superalgebra $A\cong A_{ss}\oplus J_A\in\mathcal{M}_{\mathbb{Z}_2,\Gamma}$ such that if $B\in\mathcal{M}_{\mathbb{Z}_2,\Gamma}$, then A_{ss} is a direct summand of B_{ss} as superalgebras. Finally, we fully extend these results to the G-graded setting where G is a finite group. In particular we show that if A and B are finite dimensional $G_2:=\mathbb{Z}_2\times G$-graded simple algebras then they are G_2-graded isomorphic if and only if $E(A)$ and $E(B)$ are G-graded PI-equivalent.

1. Introduction

Let F be an algebraically closed field of characteristic zero and $F\langle X\rangle$ the free associative algebra over F on a countable set of variables X. Let Γ be a T-ideal of $F\langle X\rangle$ (i.e., invariant under all algebra endomorphisms of $F\langle X\rangle$). It is easy to see that Γ is in fact the ideal of polynomial identities of a suitable associative algebra (e.g., $\Gamma=\mathrm{Id}(F\langle X\rangle/\Gamma)$). Kemer's representability theorem says that if $\Gamma\neq 0$, then it is the T-ideal of identities of an algebra of the form $E(B)$, the Grassmann envelope of some finite dimensional $\mathbb{Z}_2$-graded algebra $B=B_0\oplus B_1$ over F. Here $E=E_0\oplus E_1$ is the infinite dimensional unital Grassmann algebra over F with the natural $\mathbb{Z}_2$-grading and $E(B)=E_0\otimes B_0\oplus E_1\otimes B_1$ viewed as an ungraded algebra. In case Γ is the T-ideal of identities of an affine PI algebra, or equivalently, in case Γ contains a nontrivial Capelli polynomial, Kemer's representability theorem says that $\Gamma=\mathrm{Id}(A)$ where A is a finite dimensional algebra over F. Kemer's representability theorem is the key step towards the positive solution of the Specht problem which claims that every T-ideal is finitely based.

The purpose of this paper is to prove, roughly speaking, that if A is a finite dimensional algebra over an algebraically closed field of characteristic zero F, then the maximal semisimple subalgebra of A, namely a supplement A_{ss} of the Jacobson's radical J_A in A, is "basically uniquely determined" by $\mathrm{Id}(A)$. We

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show also that a similar result holds for the algebra $E(B)$, that is, a $\mathbb{Z}_2$-graded semisimple supplement of J_B in a finite dimensional superalgebra B is basically uniquely determined by $\Gamma = \mathrm{Id}(E(B))$. Finally, we extend our results to the G-graded setting where G is a finite group. Before we state the results precisely, we should remark right away that strictly speaking the semisimple part of a finite dimensional algebra cannot be determined by its T-ideal of identities for the simple reason that e.g., $\mathrm{Id}(A) = \mathrm{Id}(A \oplus A)$. So, by "basically uniquely determined" we mean the following.

Theorem 1.1. *Let Γ be a T-ideal of identities and suppose Γ contains a Capelli polynomial c_n for some n. Then there exists a finite dimensional semisimple F-algebra U that satisfies the following conditions*:

(1) *There exists a finite dimensional algebra A over F with* $\mathrm{Id}(A) = \Gamma$ *and such that $A \cong U \oplus J_A$ is its Wedderburn–Malcev decomposition.*

(2) *If B is any finite dimensional algebra over F with* $\mathrm{Id}(B) = \Gamma$ *and B_{ss} is its maximal semisimple subalgebra*, *then U is a direct summand of B_{ss}.*

Clearly, up to an algebra isomorphism, the semisimple algebra U is unique minimal and hence it is an invariant of Γ.

Let A be a finite dimensional algebra over F and let $A \cong A_1 \times \cdots \times A_q \oplus J$ be its Wedderburn–Malcev decomposition where A_i is simple, $i = 1, \ldots, q$, and $J = J_A$ is the Jacobson radical.

Definition 1.2. We say A is *full* if up to a permutation of the simple components $A_1 \cdot J \cdot A_2 \cdots J \cdot A_q \neq 0$.

The following theorem plays a key role in the proof of Theorem 1.1.

Theorem 1.3. *If two full algebras A and B are PI-equivalent then their maximal semisimple subalgebras are isomorphic. In particular this holds in case A and B are fundamental algebras.*

Remark 1.4. Fundamental algebras are special type of full algebras. They are important in Kemer's theory but will not play a role in this paper; see [Aljadeff et al. 2020].

Let us show how Theorem 1.3 follows from Theorem 1.1. Let A_0 be a finite dimensional algebra PI equivalent to A and with minimal semisimple subalgebra U. We show $U \cong A_{ss}$. Recall that for a finite dimensional algebra W, $\exp(W) \leq \dim_F(W_{ss})$ and equality holds if (and only if) W is full. Here $\exp(W)$ is the exponent of the algebra W, an asymptotic PI invariant attached to the T-ideal $\mathrm{Id}(W)$ and so $\exp(A_0) = \exp(A)$; see [Giambruno and Zaicev 1998, Corollary 1]. Furthermore, by Theorem 1.1 we have that U is a direct summand of A_{ss} and the result follows.

For *fundamental algebras* the result of Theorem 1.3 was proved by Procesi [2016, Corollary 3.15]. Procesi's result is based on a geometric construction which corresponds to a T-ideal Γ containing a Capelli polynomial, or equivalently, a T-ideal of identities of a finite dimensional algebra A. Let us comment briefly on Procesi's approach. He considers the coordinate ring $\mathcal{T}_t(Y)$ of the variety of the semisimple representations of the free algebra $F\langle X\rangle$ into the algebra of $t \times t$-matrices over F, where X is a set of cardinality m and t is the *exponent* of Γ; see [Aljadeff et al. 2020, Chapter 21]. The commutative algebra $\mathcal{T}_t(Y)$ acts on the T-ideal K generated by Kemer polynomials of Γ via a quotient algebra $\mathcal{T}_D$, an algebra which is generated by traces. It turns out, and this is a key idea of Kemer [Aljadeff et al. 2016, Section 10], that replacing suitable variables x_i which alternate in a Kemer polynomial f by zx_i (z is an auxiliary variable), it gives rise to the multiplication of f by a trace function. This determines the action of $\mathcal{T}_D$ and hence of $\mathcal{T}_t(Y)$ on K. Finally, it is shown that the support variety W for the $\mathcal{T}_t(Y)$-module K carries the information we need. Indeed, it turns out that if A is any fundamental algebra with $\mathrm{Id}(A) = \Gamma$ and with semisimple part $A_{ss} = A_1 \times \cdots \times A_q$, then the tuple $(t_1, \ldots, t_q)$, where $A_i \cong M_{t_i}(F)$, is an invariant of W.

Our approach instead is mostly combinatorial. It uses a *refined version* of the so called "Kemer's lemma 1" [Aljadeff et al. 2016, Section 6; Kanel-Belov and Rowen 2005, Proposition 4.44; Kemer 1987, Section 2] which deals with full algebras (an important ingredient in Kemer's solution of the Specht problem). We do not use however the more subtle result of Kemer, namely "Kemer's lemma 2" [Aljadeff et al. 2016, Section 7; Kanel-Belov and Rowen 2005, Proposition 4.54; Kemer 1987, Section 2] which concerns with fundamental algebras. The advantage of full algebras comparing to fundamental algebras (beside being a much larger class) is that they are easier to define and in particular they can be characterized without using polynomial identities. This allows us to extend Theorem 1.3 to (1) nonaffine algebras (2) group graded algebras.

Let us turn now to the case where Γ contains no Capelli polynomials. In that case we have the following result.

Theorem 1.5. *Let $\Gamma \leq F\langle X\rangle$ be a nonzero T-ideal and suppose $c_n \notin \Gamma$ for every n. Then there exists a finite dimensional semisimple superalgebra U over F which satisfies the following conditions*:

(1) *There exists a finite dimensional superalgebra A over F with $\mathrm{Id}(E(A)) = \Gamma$ and such that $A \cong U \oplus J_A$ is its Wedderburn–Malcev decomposition.*

(2) *If B is any finite dimensional superalgebra over F with $\mathrm{Id}(E(B)) = \Gamma$ and B_{ss} is its maximal semisimple subalgebra, then U is a direct summand of B_{ss} as superalgebras.*

The proof of Theorem 1.1 is given in the next section (Section 2). In Section 3 we treat the nonaffine case, Theorem 1.5.

In the last two sections of this article we extend the main results to the setting of G-graded T-ideals and G-graded algebras where G is a finite group. The main obstacle here is due to the fact that a G-graded simple algebra A is not determined up to a G-graded isomorphism by the dimensions of the homogeneous components A_g, $g \in G$. The proof uses the extension of Kemer's theory to G-graded algebras where G is a finite group; see [Aljadeff and Kanel-Belov 2010].

Remark 1.6. The extension of the results above to algebras over fields of finite characteristic and in particular over finite fields does not seem to be straightforward. One of the reasons is that *alternation* and *symmetrization*, operations which appear in the proofs, may result as zero multiplication. We refer to the work of Belov, Rowen and Vishne on full quivers of representations of algebras over fields of arbitrary characteristic and more generally over commutative Noetherian domains; see [Belov-Kanel et al. 2010;

2011; 2012]. The notion of full quiver is useful for studying the interactions between the radical and the semisimple component of Zariski closed algebras, a notion that appears in Belov's remarkable solution of the Specht problem for affine algebras over fields of finite characteristic; see [Belov 2010]. We emphasize that such interactions for Zariski closed algebras are considerably more subtle than for finite dimensional algebras over a field of characteristic zero.

2. Preliminaries and proof of the affine case

We start by introducing some combinatorial terminology.

Let $\alpha = (a_1, \dots, a_q)$ be a q-tuple, $q \geq 0$, (or multiset rather, since the order of the a_i will not play a role) of positive integers. For any sub-tuple γ of α we let $\sigma(\gamma) = \sum_{a \in \gamma} a$ be the weight of γ. We set $\sigma(\gamma) = 0$ if γ is the empty tuple.

In what follows the tuple α will correspond to the dimensions of the simple components of a finite dimensional semisimple algebra. More precisely, if A is a finite dimensional algebra over F, we let $A \cong A_1 \times \cdots \times A_q \oplus J_A$ be its Wedderburn–Malcev decomposition. Then $\mathfrak{m}_A = (\dim_F(A_1), \dots, \dim_F(A_q))$ is the tuple corresponding to A. With this notation $\mathfrak{m}_A$ is empty if and only if A is nilpotent.

Definition 2.1. Let $\alpha = (a_1, \dots, a_r)$ and $\beta = (b_1, \dots, b_s)$ be tuples of positive integers. We say β *covers* α if the tuple α may be decomposed into s disjoint, possibly empty, sub-tuples $T_1, \dots, T_s$ such that $\sigma(T_i) \leq b_i$, $i = 1, \dots, s$.

Example 2.2. The tuple (16, 12) covers the tuple (10, 9, 3, 3) but it does not cover the tuple (15, 8, 5).

Note 2.3. (1) The covering relation is antisymmetric.

(2) The covering relation is strictly stronger than majorization.

(3) The covering relation is in fact a partial order relation, if one considers multisets rather than tuples.

Next we recall some definitions and a result from Kemer's theory.

Let A be a finite dimensional algebra over F. Let $A \cong A_{ss} \oplus J_A$ be its Wedderburn–Malcev decomposition where J_A is the Jacobson radical and A_{ss} is a semisimple subalgebra supplementing J_A. The algebra A_{ss} decomposes uniquely (up to permutation) into a direct product of simple algebras $A_1 \times \cdots \times A_q$, where $A_i \cong M_{n_i}(F)$ is the algebra of $n_i \times n_i$-matrices over F. Furthermore, it is well known that all semisimple supplements of J_A in A are isomorphic.

It is clear that in order to test whether a multilinear polynomial p is an identity of A it is sufficient to evaluate the polynomial on a basis of A and so we fix from now on a basis $\mathcal{B} = \{e^i_{k,l}, u_1, \dots, u_d\}$. Here, the elements $\{e^i_{k,l}\}$, $1 \leq k, l \leq n_i$ are the elementary matrices of $M_{n_i}(F)$, $i = 1, \dots, q$, and $\{u_1, \dots, u_d\}$ is a basis of J_A.

Definition 2.4. Let $p = p(x_1, \dots, x_n)$ be a multilinear polynomial. We say an evaluation of p on A is *admissible* if the variables of p assume values only from the basis $\mathcal{B}$. We refer to an evaluation of a variable as *semisimple* (resp. *radical*) if the value is an elementary matrix $e^i_{k,l}$ (resp. an element $u_i \in J_A$).

For the rest of the paper we will consider only admissible evaluations.

Definition 2.5. Let A be a full algebra (Definition 1.2). We say a multilinear polynomial $p(x_1, \dots, x_n)$ is:

(1) A-*weakly full* (or *weakly full* of A or *weakly full* when the algebra in question is clear) if it has a nonzero admissible evaluation on A where elements from all simple components are represented in the evaluation.

(2) A-*full* if every simple component of A_{ss} is represented in *every* admissible nonzero evaluation on A. Also here we may use the terminology *full* of A or just *full*.

(3) A-*strongly full* if every basis element of A_{ss} appears in every admissible nonzero evaluation of p.

Remark 2.6. In this paper we make use of polynomials that are *weakly full* or *strongly full*. We mention *full* polynomials here just for completeness. They appear in Kemer's theory; see [Aljadeff et al. 2016, Definition 5.10].

It is clear that if p is A-strongly full then it is full. Also, every full polynomial is weakly full. We are interested in the opposite direction. We start with:

Lemma 2.7. *If A is a full algebra then it admits a weakly full polynomial.*

Proof. Let A be as above. Then the multilinear monomial of degree $2q-1$ is weakly full. Indeed, we get a nonzero evaluation where we put q semisimple (resp. $q-1$ radical) values in the odd (resp. even) positions. □

The following theorem is basically *Kemer's lemma* 1; see [Aljadeff et al. 2016].

Theorem 2.8. *The following hold*:

(1) *Every full algebra admits a multilinear strongly full polynomial and therefore admits a full polynomial.*

(2) *Let A be a full algebra and f_0 be a multilinear weakly full polynomial of A. Then there exists a full polynomial f of A in $\langle f_0\rangle_T$, the T-ideal generated by f_0.*

(3) *Let A be a full algebra and f_0 a multilinear weakly full polynomial of A. Then there exists a strongly full polynomial $f \in \langle f_0\rangle_T$ of A.*

Proof. Clearly, the third statement implies the second and together with Lemma 2.7 it implies the first statement. Statement (3) follows from the construction in the proof of Kemer's lemma 1; see [Aljadeff et al. 2016]. □

As we shall need to refer to the precise construction of strongly full polynomials starting from a weakly full polynomial f_0, let us recall their construction here. It is convenient to illustrate first the construction on the weakly full polynomial mentioned above.

Let $A \cong A_1 \times \cdots \times A_q \oplus J_A$, where $A_i \cong M_{n_i}(F)$, $i = 1, \dots, q$ (as above) and suppose that after reordering the simple components we have $A_1 J A_2 \dots J A_q \neq 0$. Let $f_0 = X_1 \cdot w_1 \dots w_{q-1} \cdot X_q$ be a monomial of $2q-1$ variables which is clearly weakly full by the obvious evaluation. Let

$$Z_n = Z_n(x_1, \dots, x_{n^2}; y_1, \dots, y_{n^2+1}) = y_1 \cdot x_1 \cdot y_2 \cdots y_{n^2} \cdot x_{n^2} \cdot y_{n^2+1}$$

be a multilinear monomial on $2n^2+1$ variables. For $i = 1, \dots, q$, we consider k monomials Z_{n_i} in disjoint variables, denoted by $Z_{n_i,l}$, $l = 1, \dots, k$, where the integer k is sufficiently large and will be determined later. We set $\Delta_i = Z_{n_i,1} \cdots Z_{n_i,k}$, the product of k copies of the monomial Z_{n_i} with disjoint sets of variables. Finally, in view of the inequality $A_1 J A_2 \cdots J A_q \neq 0$ we apply the T-operation and replace the variable X_i by $X_i \cdot \Delta_i$ in the polynomial f_0 (here it is just a monomial) and obtain the monomial

$$\Omega = X_1 \cdot \Delta_1 \cdot w_1 \cdot X_2 \cdot \Delta_2 \cdot w_2 \cdots w_{q-1} \cdot X_q \cdot \Delta_q.$$

We refer to the x's (lower case) in Ω as *designated* variables, the y's as *frame* variables and w's as *bridge* variables. Now, it is not difficult to see that the monomial Ω admits a nonzero evaluation where the x's from $Z_{n_i,l}$ get values consisting of the full basis of the i-th simple component, that is the elementary matrices $\{e^i_{t,s}\}$, the y's from $Z_{n_i,l}$ get values of the form $e^i_{t,t}$ and the w's get radical values which bridge the different simple components. Fixing $r = 1, \dots, k$, we alternate all x's from the monomials $Z_{n_i,r}$, $i = 1, \dots, q$, so we obtain k alternating sets of cardinality $\dim_F(A_{ss})$. We denote the polynomial obtained by f_A. We adopt the terminology used in Kemer's theory and refer to each alternating set of designated variables as a *small set*. Moreover, we shall refer to the set of variables x in a small set together with the corresponding frames, that is the y variables that border the x variables, as an *augmented small set*.

Remark 2.9. In Kemer's theory there is also a notion of a *big set*. These are sets which, roughly speaking, involve the alternation of semisimple and bridge variables. We will not make use of big sets here.

Suppose the integer k, namely the number of small sets in f_A, exceeds the nilpotency index of A. Let us show that f_A is a strongly full polynomial of A. We will show that if δ is any admissible *nonzero* evaluation of f_A, then there is at least one small set which assumes precisely a full basis of A_{ss}. Indeed, by the alternation of designated variables we are forced to evaluate each small set on different basis elements and if this is not a full basis of A_{ss}, we have that at least one of the designated variables assumes a radical value. Since k is larger than the nilpotency index of A, we cannot have a radical evaluation in every small set. This shows f_A is strongly full. In fact this proves the last statement of Theorem 2.8 for the weakly full polynomial $f_0 = X_1 \cdot w_1 \cdots w_{q-1} \cdot X_q$.

Let us proceed now to the general case, namely where f_0 is assumed to be an arbitrary multilinear weakly full polynomial of A. Denote by Φ a nonzero evaluation of f_0 which visits every simple component of A. Let us denote the variables of f_0 which assume values from the simple components $A_1, \dots, A_q$ by $X_1, \dots, X_q$ respectively. Since the evaluation $\Phi(f_0)$ is nonzero, it is nonzero on one of the monomials of f_0 which we fix from now on and denote it by R_e. We have then that $f_0 = \sum_{\sigma \in S_m} \lambda_\sigma R_\sigma$ where $\lambda_\sigma \in F$ and $\lambda_e = 1$. Here m is the number of variables in f_0. We proceed now as in the previous case, namely replace the variables X_i by $X_i \cdot \Delta_i$ and obtain a polynomial which we denote by Ω. We have that if $f_0 = f_0(X_1, \dots, X_q; M)$ then $\Omega = f_0(X_1\Delta_1, \dots, X_q\Delta_q, M) \in \langle f_0 \rangle_T$ where M is a suitable set of variables. By an appropriate evaluation of the monomials Δ_i, $i = 1, \ldots, q$, we see that Ω is a nonidentity of A and is clearly weakly full. Finally we alternate the designated variables as above and obtain a polynomial which we denote by f_A. It is not difficult to see that f_A satisfies the third condition of Theorem 2.8 with respect the given weakly full polynomial f_0.

Lemma 2.10 (main lemma-affine). *Notation as above. Suppose A and B are full algebras. Suppose $\mathfrak{m}_B$ does not cover $\mathfrak{m}_A$. Then there exists a strongly full polynomial f_A of A which vanishes on B. In fact, if f_0 is any weakly full polynomial of A then there exists a strongly full polynomial $f_A \in \langle f_0 \rangle_T$ of A which vanishes on B.*

Proof. Let f_A be the strongly full polynomial of A as constructed above in case $f_0 = X_1 \cdot w_1 \cdots w_{q-1} \cdot X_q$. We take a large number of small sets k, exceeding the nilpotency index of B. We claim f_A is an identity of B. We will show that if this is not the case then necessarily B covers A. Let us fix a nonzero evaluation Φ of f_A on B and consider one monomial, which we assume as we may is the monomial Ω of f_A (see the construction above), whose value is nonzero. Note that by the condition on k, there exists an augmented small set, say the j-th set where $j \in \{1, \ldots, k\}$, which is free of radical values. It follows that the Φ-values of each segment in $\{Z_{n_1,j}, \ldots, Z_{n_q,j}\}$ consist only of semisimple elements in B, and moreover semisimple elements from the same simple component. But because the evaluation of Φ on f_A is nonzero and the variables in the j-th small set alternate, the semisimple values of B must be linearly independent. This implies that B covers A as desired.

In the general case we may argue as follows. Let f_0 be an arbitrary weakly full polynomial of A and let $R_\sigma = R_\sigma(X_1, \ldots, X_q; M)$ be any monomial of f_0. Applying the T-operation on R_σ we obtain $\Omega_\sigma = R_\sigma(X_1\Delta_1, \ldots, X_q\Delta_q, M) \in \langle R_\sigma \rangle_T$. Next we alternate the designated variables as above and obtain a polynomial which we denote by $(R_\sigma)_A$. As in the first case considered, that is in case where $f_0 = X_1 \cdot w_1 \cdots w_{q-1} \cdot X_q$, we see that if $(R_\sigma)_A$ admits a nonzero evaluation on B, then B covers A. It follows that if f_A admits a nonzero evaluation on B, this is true also for the polynomial $(R_\sigma)_A$, some σ, and so B covers A. □

Corollary 2.11. *Let A and B full algebras. If they are PI-equivalent, then their semisimple parts, A_{ss} and B_{ss} are isomorphic.*

Proof. Indeed, A and B must cover each other. It follows that the tuple of dimensions of the simple components of A and B coincide up to a permutation (see Note 2.3) and hence A_{ss} and B_{ss} are isomorphic. □

In what follows we will need a somewhat stronger statement.

Corollary 2.12. *Let A be a full algebra and $B_1, \ldots, B_t$ be a finite family of full algebras, each not covering A. If f_0 is a weakly full polynomial of A then there is a strongly full polynomial $f_A \in \langle f_0 \rangle_T$ of A that vanishes on B_i, $i = 1, \ldots, t$. In particular if B is a direct sum of full algebras, each not covering A, then there exists a strongly full polynomial $f_A \in \langle f_0 \rangle_T$ of A which vanishes on B.*

Proof. We only need to pay attention to the number of small sets k in f_A, namely it should exceed the nilpotency index of each J_{B_i}, $i = 1, \ldots, t$. □

Recall that any affine PI-algebra A and in particular any finite dimensional algebra is PI-equivalent to a direct sum of full algebras; see for instance [Aljadeff et al. 2016; 2020]. Here we will need a more precise statement.

Definition 2.13. Let A be finite dimensional algebra. We say

$$P(A) = T_1 \oplus \cdots \oplus T_n$$

is a *presentation of A by full algebras* if the following hold:

(1) T_i is full for $i = 1, \ldots, n$.

(2) $P(A)$ is PI equivalent to A.

Remark 2.14. Note that an algebra may have two different presentations which are isomorphic as algebras (e.g., a radical direct summand may be attached to different full subalgebras). Thus, when referring to a presentation $P(A)$, we are fixing the set of full algebras $\{T_1, \ldots, T_n\}$ up to permutation. Note that if Γ is a T-ideal containing Capelli polynomials we may view $T_1 \oplus \cdots \oplus T_n$ as a presentation of Γ so we may denote it by $P(\Gamma)$.

Proposition 2.15. *Let A be a finite dimensional algebra. Then there exists a presentation $T_1 \oplus \cdots \oplus T_n$ of A. Moreover, there exists such presentation where the semisimple subalgebra $(T_i)_{ss}$ of T_i is a direct summand of A_{ss}, for $i = 1, \ldots, n$.*

Proof. In fact the stronger statement follows from the construction in [Aljadeff et al. 2020, Subsection 17.2.4]. Let $A \cong A_1 \times \cdots \times A_q \oplus J_A$ be the Wedderburn–Malcev decomposition. Clearly we may assume A is not full. Consider the subalgebra

$$\mathcal{A}_i = \langle A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_q;\, J_A \rangle.$$

We claim A and $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_q$ are PI-equivalent. Clearly $\mathrm{Id}(A) \subseteq \mathrm{Id}(\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_q)$. For the converse, if f is a nonidentity of A, it must be a nonidentity of at least one $\mathcal{A}_i$ for otherwise it is a full polynomial of A which implies A is full, contrary to our assumption. The proposition is then proved by induction. □

For any presentation $P(A)$ of A we let $P(A)_{\dim(ss)}$ be the set of *tuples* consisting of the dimensions of the simple components that appear in the different full algebras of $P(A)$ and denote by $P(A)_{\dim(ss),max}$ the set of maximal tuples in $P(A)_{\dim(ss)}$ with respect to covering.

Corollary 2.16. *The set $P(A)_{\dim(ss),\max}$ depends on A but not on the presentation $P(A)$. Hence we can denote the set $P(A)_{\dim(ss),\max}$ by $\mathcal{A}_{\dim(ss),\max}$.*

Proof. Suppose the contrary holds. Let P_1 and P_2 be presentations of A as above. Then without loss of generality there exists a full subalgebra M of P_1 whose tuple is maximal and does not appear as a maximal tuple in P_2. We may assume M is not covered by tuples of P_2 for otherwise M is strictly covered by a tuple of P_2 and in that case we may exchange the roles of P_1 and P_2. Now, by the lemma, there exists a nonidentity polynomial of M which is an identity of every full subalgebra of P_2 and the claim is proved. □

In the next lemma we show we can *fuse* finite dimensional algebras A and B with isomorphic semisimple subalgebras. More generally, suppose the semisimple subalgebra of A is a direct summand of B_{ss}, the semisimple subalgebra of B. We claim $A \times B$ is PI-equivalent to an algebra of the form $B_{ss} \oplus \hat{J}$. Yet more generally, suppose A and B have a common semisimple component U (up to isomorphism), then there exists an algebra C, PI equivalent to $A \times B$, in which the semisimple algebra isomorphic to U appears in C only once. Here is the precise statement.

Lemma 2.17. *Let $A_1 \times \cdots \times A_q \oplus J(A)$ and $B_1 \times \cdots \times B_r \oplus J(B)$ be the Wedderburn–Malcev decompositions of A and B respectively. Suppose $A_1 \times \cdots \times A_k \cong B_1 \times \cdots \times B_k \cong U$. Then $A \times B$ is PI-equivalent to $C = U \times A_{k+1} \times \cdots \times A_q \times B_{k+1} \times \cdots \times B_r \oplus J(A) \oplus J(B)$.*

Proof. We consider the vector space embedding

$$\begin{aligned} C = U \times A_{k+1} \times \cdots \times A_q \times B_{k+1} \times \cdots \times B_r \oplus J(A) \oplus J(B) \\ \hookrightarrow [U \times A_{k+1} \times \cdots \times A_q \oplus J(A)] \times [U \times B_{k+1} \times \cdots \times B_r \oplus J(B)] \end{aligned}$$

where the elements of U are mapped diagonally. It is easy to see that the image is closed under multiplication, yielding an algebra structure on C. As for the polynomial identities the above embedding (now, as algebras) yields $\mathrm{Id}(C) \supseteq \mathrm{Id}(A \times B) = \mathrm{Id}(A) \cap \mathrm{Id}(B)$. On the other hand the algebras A and B are embedded in C and the result follows. □

Definition 2.18. Notation as in the lemma above. We say the algebra C is the fusion of the algebras A and B along U.

Proposition 2.19. *Let $P_1 = P_1(A)$ and $P_2 = P_2(A)$ be presentations of A and let $T_1 \cong (T_1)_{ss} \oplus J_{T_1}$ and $T_2 \cong (T_2)_{ss} \oplus J_{T_2}$ be full subalgebras summands of P_1 and P_2 respectively. Suppose $(T_1)_{ss}$ and $(T_2)_{ss}$, the semisimple parts of T_1 and T_2, are isomorphic and let $U \cong (T_1)_{ss} \cong (T_2)_{ss}$. Let $T_1' = U \oplus J_{T_1} \oplus J_{T_2}$. Then T_1' is full. Furthermore, if we replace $T_1 \cong U \oplus J_{T_1}$ by $T_1' = U \oplus J_{T_1} \oplus J_{T_2}$ in the presentation P_1 we obtain a presentation P_1' of A.*

Proof. From the embedding $U \oplus J_{T_1} \hookrightarrow U \oplus J_{T_1} \oplus J_{T_2}$ we see that every weakly full polynomial of T_1 is weakly full of T_1', so T_1' is full. Furthermore, because $\mathrm{Id}(T_1), \mathrm{Id}(T_2) \supseteq \mathrm{Id}(A)$ we have that $\mathrm{Id}(P_1') \supseteq \mathrm{Id}(A)$. On the other hand $\mathrm{Id}(P_1') \subseteq \mathrm{Id}(P_1)$ $(= \mathrm{Id}(A))$ and the result follows. □

Remark 2.20. Note that fusion of fundamental algebras A and B with isomorphic semisimple subalgebras yields a fundamental algebra; see [Aljadeff et al. 2020] for the definition of fundamental algebras.

Let Γ be the T-ideal of identities of a finite dimensional algebra. Denote by $\mathcal{M}_\Gamma$ the family of presentations $A = T_1 \oplus \cdots \oplus T_n$ of Γ (we simplify the notation slightly and write A rather than $P(A)$ for a presentation of Γ).

In what follows we shall present a procedure in which we iterate 4 steps (numbered 0–3). In each step we replace an algebra $A \in \mathcal{M}_\Gamma$ by an algebra $A' \in \mathcal{M}_\Gamma$ (in particular PI equivalent to A) that is "better" behaved. Then, in one final step (step 4), we construct the algebra A of Theorem 1.1.

Step 0 (deletion): Let $A \in \mathcal{M}_\Gamma$. We delete from A full subalgebras that do not alter $\mathrm{Id}(A)$. Let A_i be a full subalgebra of A. Denote by $\widehat{A_i}$ the summand of A consisting the direct sum of full algebra A_j, $j \neq i$. Then, we delete A_i from the direct sum if $\mathrm{Id}(A_i) \supseteq \mathrm{Id}(\widehat{A_i}) = \cap_{j\neq i} \mathrm{Id}(A_j)$. We abuse notation and simply write the outcome by $F_0(A)$, an operation of type 0 on A, although the operation depends on the choice of the full algebra A_i. Clearly $F_0(A)$ and A are PI equivalent. We write $A = A_{\mathrm{red}_0}$ if *every* operation of type 0 on A is the identity.

Step 1 (fusion): $A \in \mathcal{M}_\Gamma$ and suppose $A = A_{\mathrm{red}_0}$. We fuse full subalgebras with isomorphic semisimple subalgebras. More generally, if A_i and A_j $i \neq j$, are full subalgebras of A and $(A_i)_{ss}$ is a direct summand of $(A_j)_{ss}$, then the operation $F_1 = (F_1)_{A_i,A_j}$ on A is the fusion of A_i and A_j. We abuse notation and simply write the outcome by $F_1(A)$, an operation of type 1 on A, although the operation depends on the choice of the full algebras A_i and A_j. Note that by Proposition 2.19 the algebras $F_1(A)$ and A are PI equivalent. We write $A = A_{\mathrm{red}_{0,1}}$ if every operation of type 0 or 1 on A is the identity.

We come now to a step where we *decompose* full algebras.

Step 2 (decomposition): Let $A \in \mathcal{M}_\Gamma$ and suppose that $A = A_{\mathrm{red}_{0,1}}$. We define an operation of type 2 on A, denoted by F_2, as follows. Choose a full algebra Q appearing in the decomposition of A into full algebras and let $A_{\mathrm{supp}(Q)} = (\hat{Q})_1 \oplus \cdots \oplus (\hat{Q})_n$ be the supplement of Q in A. Note that since $A = A_{\mathrm{red}_{0,1}}$ there is no full algebra component of $A_{\mathrm{supp}(Q)}$ with semisimple part $\cong Q_{ss}$. Suppose there exists a weakly full polynomial p of Q which vanishes on $A_{\mathrm{supp}(Q)}$. In that case we leave the algebra A unchanged, that is $F_2(A) = A$. Otherwise we proceed as follows.

Clearly Q is not nilpotent because A is not nilpotent and $A = A_{\mathrm{red}_{0,1}}$. Let us treat the case where Q_{ss} is simple separately. If Q_{ss} is simple and every weakly full polynomial of Q is a nonidentity of $A_{\mathrm{supp}(Q)}$ we claim $\mathrm{Id}(A) = \mathrm{Id}(A_{\mathrm{supp}(Q)} \oplus J_Q)$ where J_Q is the radical of Q. It is clear that $\mathrm{Id}(A) \subseteq \mathrm{Id}(A_{\mathrm{supp}(Q)} \oplus J_Q)$. Conversely, suppose p is a nonidentity of A. If p is a nonidentity of $A_{\mathrm{supp}(Q)}$ it is also a nonidentity of $\mathrm{Id}(A_{\mathrm{supp}(Q)} \oplus J_Q)$ as needed, so let us assume p is an identity of $A_{\mathrm{supp}(Q)}$. In that case p must be a nonidentity of Q. However, by assumption, p is not weakly full of Q which means here that no indeterminate of p gets a semisimple value in any nonzero evaluation of p. It follows that p is a nonidentity of J_Q and we are done. Suppose now $q > 1$ and let $Q \cong \Delta_1 \times \cdots \times \Delta_q \oplus J_Q$ be the Wedderburn–Malcev decomposition of Q. We are assuming every weakly full polynomial of Q is a nonidentity of $A_{\mathrm{supp}(Q)}$. In that case we claim the following.

Claim 2.21. *We can replace the full subalgebra Q of A by a direct sum of full subalgebras $\mathcal{Q}_1 \oplus \cdots \oplus \mathcal{Q}_q$, where for each $i = 1, \ldots, q$, the semisimple algebra $(\mathcal{Q}_i)_{ss}$ is a proper summand of Q_{ss} (in particular strictly covered by Q) and if $\bar{A}$ denotes the algebra obtained, we have* $\mathrm{Id}(\bar{A}) = \mathrm{Id}(A) = \Gamma$.

Proof. Consider the algebras $\mathcal{Q}_i$, $i = 1, \dots, q$, obtained from Q by deleting one simple component Δ_i and keeping the radical unchanged. We claim A is PI-equivalent to $A_{\mathrm{supp}(Q)} \oplus \mathcal{Q}_1 \oplus \cdots \oplus \mathcal{Q}_q$. Indeed, it is clear that every identity of the former algebra vanishes on the latter one. Conversely, let p be a nonidentity of the former one. We show it does not vanish on the latter. Clearly, we may assume p vanishes on $A_{\mathrm{supp}(Q)}$ and so, by our assumption above, p is not a weakly full polynomial of Q. This means that p has no nonzero evaluation on Q which visits all simple components of Q and so, being a nonidentity of Q, it must be a nonidentity of $\mathcal{Q}_i$ for some i and hence a nonidentity of the latter. $\square$

We write $A = A_{\mathrm{red}_{0,1,2}}$ if any operation of type 0, 1 or 2 on A is the identity.

Similarly to our notation for the operations F_0 and F_1 above we abuse notation here and simply write $F_2(A) = F_{2,Q}(A)$. It follows from the claim that $F_2(A)$ and A are PI equivalent.

Step 3 (absorption): Fix a presentation $A \in \mathcal{M}_\Gamma$ and suppose $A = A_{\mathrm{red}_{0,1,2}}$. Let $B \in \mathcal{M}_\Gamma$. We denote by F_3^{cond} an operation which replaces, roughly speaking, a full subalgebra Q of A with the fusion of Q with certain full subalgebras of B. More precisely, choose a full subalgebra Q of A and a full subalgebra V of B such that V_{ss} is a direct summand of (possibly isomorphic to) Q_{ss}. Then replace the full subalgebra Q in A by the fusion of Q and V. We denote the outcome by $(F_3^{\mathrm{cond}})_{B,Q,V}(A)$ or simply by $(F_3^{\mathrm{cond}})(A)$. The superscript cond means that this operation is conditional. We define $(F_3)_{B,Q,V}(A)$ as follows. Let $A^{\mathrm{cond}} = (F_3^{\mathrm{cond}})_{B,Q,V}(A)$. If $A^{\mathrm{cond}} = (A^{\mathrm{cond}})_{\mathrm{red}_{0,1,2}}$, we set $(F_3)_{B,Q,V}(A) = A$, otherwise we set $(F_3)_{B,Q,V}(A) = A^{\mathrm{cond}}$. As above we write $F_3(A) = (F_3)_{B,Q,V}(A)$ and have, by Proposition 2.19, that the algebras $F_3(A)$ and A are PI equivalent.

Remark 2.22. The point for introducing the conditional operation is that we want an operation of type 3 to be nontrivial only if an operation of type 0, 1 or 2 has a real effect on A^{cond}. This is to prevent the radical from growing indefinitely.

We write $A = A_{\mathrm{red}_{0,1,2,3}}$ if every operation of type 0, 1, 2 or 3 on A is the identity.

Let us describe now the procedure applied to $A \in \mathcal{M}_\Gamma$:

(1) Apply operations of type 0 on A until any additional operation of type 0 acts as an identity. Denote the outcome by A'.

(2) If there exists an operation of type 1 with $F_1(A') \neq A'$, we apply F_1 on A' and return to step 0 with $A := F_1(A')$. We continue until we get an algebra A'' such that $F_\epsilon(A'') = A''$, $\epsilon = 0, 1$.

(3) If there exists an operation of type 2 with $F_2(A'') \neq A''$, we apply F_2 on A'' and return to step 0. We continue until we get an algebra A''' such that $F_\epsilon(A''') = A'''$, $\epsilon = 0, 1, 2$.

(4) If there exists an operation of type 3 with $F_3(A''') \neq A'''$, we apply F_3 on A''' and return to step 0. We continue until we get an algebra A'''' such that $F_\epsilon(A'''') = A''''$, $\epsilon = 0, 1, 2, 3$.

Theorem 2.23. *For every presentation $A \in \mathcal{M}_\Gamma$ the process above stops. In particular, given a presentation A, applying operations of type $0-3$ we obtain a presentation $\mathcal{A} \in \mathcal{M}_\Gamma$ such that $\mathcal{A} = \mathcal{A}_{\mathrm{red}_{0,1,2,3}}$.*

Before giving the proof let us introduce some notation.

Definition 2.24. (1) We let A_{part} be the multiset (i.e., repetitions are allowed) of unordered tuples whose entries are the dimensions of the simple components of semisimple subalgebras of the full algebras appearing in the decomposition of A. Alternatively, we may think of A_{part} as the multiset of semisimple algebras appearing in the full algebras, summands of A.

(2) Let $A \in \mathcal{M}_\Gamma$. We denote by r_A the number of full subalgebras in the presentation of A.

(3) Let $A \in \mathcal{M}_\Gamma$ with A_{part} as above. If $\sigma = (\sigma_1, \ldots, \sigma_m) \in A_{\text{part}}$, i.e., a tuple corresponding to a full algebra, summand of A, we let $n_\sigma = 2^{m^2} \sum_i \sigma_i$ be the weight of σ. Note that the function $f(m) = 2^{m^2}$ satisfies the condition $(m-1)f(m-1) < f(m)$, a condition that will be used later. We let $n_A = n_{A_{\text{part}}} = \sum_{\sigma \in A_{\text{part}}} n_\sigma$ be the *weight* of A.

Proof. We claim:

(1) Let $A \in \mathcal{M}_\Gamma$ and let $\bar{A} = F_\epsilon(A)$, $\epsilon = 0, 1$. If $\bar{A} \neq A$ then $r_{\bar{A}} < r_A$ and $n_{\bar{A}} \leq n_A$.

(2) Let $A \in \mathcal{M}_\Gamma$ and suppose $A = A_{\text{red}_{0,1}}$. Let $\bar{A} = F_2(A)$. If $\bar{A} \neq A$ then $n_{\bar{A}} < n_A$.

The first claim is clear since in these cases we are suppressing a full subalgebra of the presentation of A. Note that if we are suppressing a nilpotent algebra $n_{\bar{A}} = n_A$. For the proof of (2) let $A = A_{\text{red}_{0,1}} \in \mathcal{M}_\Gamma$. This implies no full subalgebras of A are nilpotent unless A is nilpotent, a case we have already addressed (see paragraph above Proposition 2.19). Suppose $F_2(A) \neq A$. This means that one tuple $\sigma = (\sigma_1, \ldots, \sigma_m)$, $m \geq 1$ is replaced by m tuples each of which has length $m-1$ and is obtained from σ by deleting σ_i, $i = 1, \ldots, m$. It follows that the quantity $2^{(m^2)} \sum_i \sigma_i$, the contribution of σ to n_A, is replaced by $(m-1)2^{((m-1)^2)} \sum_i \sigma_i$. As $(m-1)2^{((m-1)^2)} < 2^{(m^2)}$, the result follows. This proves the second claim.

Consider the pairs $\Theta_A = (n_A, r_A)$, $A \in \mathcal{M}_\Gamma$ with the lexicographic order $\preceq$ (and $\prec$ if the inequality is strict). Let $\bar{A} = F_\epsilon(A)$, $\epsilon = 0, 1, 2$. It follows that if $\bar{A} \neq A$, invoking the claims above, we have $\Theta_{\bar{A}} \prec \Theta_A$. In order to complete the proof of the Theorem we need to treat the operation F_3. We note first that F_3 does not change (and in particular does not increase) Θ_A. Recall that F_3 is effective on A, i.e., $F_3(A) \neq A$, only if $F_\epsilon(F_3(A)) \neq F_3(A)$, $\epsilon = 0, 1, 2$, and also that two operations of type 3 are always separated by an *effective* operation of type 0, 1, 2. Finally, since the nontrivial operations of type 0, 1, 2 lower Θ_A the result follows. □

Corollary 2.25. *Given a presentation $A \in \mathcal{M}_\Gamma$, the application of steps $0-3$ to A yields a presentation $\bar{A} \in \mathcal{M}_\Gamma$ with the following properties*:

(1) *If Q is any full subalgebra of $\bar{A}$ then there exists a full subalgebra V of A such that Q_{ss} is a direct summand of V_{ss}.*

(2) *If Q is a full subalgebra of $\bar{A}$, then there is a strongly full polynomial of Q which vanishes on the supplement of Q in $\bar{A}$.*

(3) *If Q is a full subalgebra of $\bar{A}$, and $B \in \mathcal{M}_\Gamma$, then there is a strongly full polynomial of Q which vanishes on every full algebra V of B whose semisimple subalgebra V_{ss} strictly covers Q_{ss} and appears as a summand of the semisimple subalgebra of a full subalgebra of $\bar{A}$.*

Proof. By Theorem 2.23 we may assume $\bar{A} = \bar{A}_{\mathrm{red}_{0,1,2,3}}$ The operations of type 0 and 1 suppress full algebras of A whereas in operation 2 we decompose the semisimple part of a full algebra Q into the direct sum of full algebras whose semisimple part is a direct summand of Q_{ss}. This proves the first statement. Also the second statement follows easily from the construction. Indeed, if this is not the case there is an operation of type 2 which is not the identity on $\bar{A}$ contradicting $\bar{A} = \bar{A}_{\mathrm{red}_{0,1,2,3}}$.

Let us prove the last statement. By the claim we have that if such polynomial does not exist for a suitable full subalgebras V of an algebra $B \in \mathcal{M}_\Gamma$, fusion of V with the corresponding full algebras of $\bar{A}$ generates a decomposition of Q into full algebras whose semisimple algebra is a strict summand of Q_{ss}. This contradicts $\bar{A} = \bar{A}_{\mathrm{red}_{0,1,2,3}}$ and the result follows. □

Remark 2.26. Note that it is possible that a presentation $B \in \mathcal{M}_\Gamma$ contains a full algebra V whose semisimple part V_{ss} does not appear as a direct summand of a full algebra of $\bar{A}$. This does not contradict the last statement of Corollary 2.25.

Let $\bar{A}$ be the algebra obtained from A as in the theorem above and let $\bar{A} = T_1 \oplus \cdots \oplus T_n$ be its decomposition into the direct sum of full algebras. Let $\bar{A}_{\mathrm{part}}$ be the multiset of semsimple algebras appearing in $\bar{A}$, that is $\bar{A}_{\mathrm{part}} = \{(T_i)_{ss}\}_{i=1,\dots,n}$ (see Definition 2.24). Note that here we may replace "multiset" by "set" since at this stage repetitions do not occur.

Our goal is to show $\bar{A}_{\mathrm{part}}$ is uniquely determined by Γ. More precisely

Theorem 2.27. *If $A, B \in \mathcal{M}_\Gamma$ then $\bar{A}_{\mathrm{part}} = \bar{B}_{\mathrm{part}}$.*

Remark 2.28. Note that we know the result for maximal points where $A, B \in \mathcal{M}_\Gamma$ are arbitrary (see Corollary 2.16).

Proof. Suppose the theorem is false and consider the family Ω of all full subalgebras of $\bar{A}$ (resp. $\bar{B}$) whose semisimple part does not appear in $\bar{B}$ (resp. $\bar{A}$). Let $Q \in \Omega$ be maximal with respect to covering and assume without loss of generality that $Q = Q_{\bar{A}}$ is a full subalgebra of $\bar{A}$. Now, by the maximality of $Q_{\bar{A}}$ the semisimple part of every full subalgebra of $\bar{B}$ that strictly covers $Q_{\bar{A}}$ appears in $\bar{A}$. It follows, by Corollary 2.25(3), there exists a full polynomial p which vanishes on every full subalgebra of $\bar{B}$ that strictly covers $Q_{\bar{A}}$. Furthermore, by our construction of strongly full polynomials there exists such p that vanishes on every full subalgebra of $\bar{B}$ that does not cover $Q_{\bar{A}}$ and so p vanishes on $\bar{B}$. This contradicts $\bar{A}$ and $\bar{B}$ are PI equivalent and the theorem is proved. □

Step 4 (merging): In this final step we merge full subalgebras. Let $\bar{A}$ be an algebra as in the theorem. For each isomorphism type of a simple algebra $M_n(F)$ we let d_n be the maximal appearance of $M_n(F)$ in a full subalgebra of $\bar{A}$. Then we let $\mathcal{A}_{\Gamma,ss} = \Lambda_{n_1} \oplus \cdots \oplus \Lambda_{n_t}$ where Λ_{n_i} is the direct sum of d_{n_i} copies $M_{n_i}(F)$. Finally we let $\mathcal{A} \cong \mathcal{A}_{\Gamma,ss} \oplus J_A$, where the direct sum is of vector spaces.

Theorem 2.29. *There is exists an algebra structure on $\mathcal{A}_\Gamma$ so that:*

(1) $\mathrm{Id}(\mathcal{A}_\Gamma) = \Gamma$.

(2) *If B is finite dimensional and* $\mathrm{Id}(B) = \Gamma$ *then $\mathcal{A}_{\Gamma,ss}$ is isomorphic to a direct summand of B_{ss}.*

Proof. For the algebra structure on $\mathcal{A}$ we set the product as follows. The product on $\mathcal{A}_{\Gamma,ss}$ is already determined. Products of radical elements which belong to different full algebras is set to be zero. Let us determine the multiplication of semisimple elements with radicals. Using distributivity we let $z \in J_{\bar{A}_i}$ where $\bar{A}_i$ is a full summand of $\bar{A}$. Choose a summand of $(U_i)_{ss}$ of $\mathcal{A}_{\Gamma,ss}$ isomorphic to $(\bar{A}_i)_{ss}$. Let K be the semisimple supplement of $(U_i)_{ss}$ in $\mathcal{A}_{\Gamma,ss}$, that is

$$(U_i)_{ss} \oplus K \cong \mathcal{A}_{\Gamma,ss}.$$

Then we set the product of z with semsimple elements of $(U_i)_{ss}$ as in $\bar{A}_i$ whereas the multiplication of z with elements of K is set to be zero. Let us show $\mathrm{Id}(\mathcal{A}) = \Gamma$. Each $\bar{A}_i$ is isomorphic to a summand of $\mathcal{A}$ and so $\mathrm{Id}(\bar{A}) \supseteq \mathrm{Id}(\mathcal{A})$. For the opposite inclusion let p be a multilinear nonidentity of $\mathcal{A}$ and fix a nonzero evaluation on $\mathcal{A}$. Since the multiplication of radical elements of different summands $J_{\bar{A}_i}$ and $J_{\bar{A}_j}$ is zero the evaluation may involve at most radicals from $J_{\bar{A}_i}$, for a unique i. For that i, semisimple elements that appear in the evaluation must belong to the summand $(U_i)_{ss}$. We see the polynomial p is a nonidentity of $\bar{A}_i$ and so a nonidentity of $\bar{A}$. For the proof of the second statement, by the construction of $\mathcal{A}_\Gamma$ from $\bar{A}$ we see $\mathcal{A}_{\Gamma,ss}$ is a direct summand of $\bar{A}_{ss}$ and hence, by Theorem 2.27, also of $\bar{B}_{ss}$. Furthermore, we see from step 4 that every Λ_{n_i} is a direct summand of the semisimple part of a full summand of $\bar{A}$ and hence of $\bar{B}$. We complete the proof of the theorem invoking Corollary 2.25(1). $\square$

3. Nonaffine algebras

In this section we prove Theorem 1.5.

We note that the key point in the construction of strongly full polynomials of a finite dimensional full algebra A was the fact that in any nonzero evaluation we were forced to evaluate the designated variables in at least one small set by a *complete* basis of semisimple elements. Then, for such polynomial we showed it is an identity of any full algebra B that does not cover A. Now, if A is a finite dimensional full superalgebra (see [Aljadeff and Kanel-Belov 2010] or Definition 3.3 below), it is not difficult to construct a *super* strongly full polynomial with a similar property, that is, a polynomial p that visits a full basis of the semisimple part of A in every nonzero evaluation. However, this is not what we need. For the proof, we need an *ungraded* polynomial $f_{E(A)}$, nonidentity of $E(A)$, which visits the different supersimple components of A in any nonzero evaluations of the form $\epsilon \otimes u$. Here, $\epsilon = 1 \in E$ or $= \epsilon_i \in E$, where ϵ_i is a generator, and $u \in A$. Furthermore, as in the affine case, we shall need a full basis $\{u\} \subseteq A_{ss}$ to appear in every nonzero evaluation of $f_{E(A)}$. In fact, as in the affine case, we will need to construct such polynomials for $E(A)$ that belong to the T-ideal generated by an arbitrary weakly full polynomial of $E(A)$.

Once we have constructed such polynomials for $E(A)$ where A is a finite dimensional full superalgebra, we will be able to show the analogue of the Main Lemma in the nonaffine setting. The proof of Theorem 1.5 will then follow the same lines of the proof of the affine case.

We start by defining a partial ordering on finite dimensional semisimple $\mathbb{Z}_2$-graded algebras.

Let $A = A_1 \oplus \cdots \oplus A_q$ and $B = B_1 \oplus \cdots \oplus B_s$ be the decompositions of semisimple algebras A and B into direct sum of finite dimensional $\mathbb{Z}_2$-graded simple algebras A_i and B_j respectively. Consider the pair $\mathfrak{m}_A = (\mathfrak{m}_{A,0}, \mathfrak{m}_{A,1})$ where $\mathfrak{m}_{A,0} = (a_{0,1}, \ldots, a_{0,q})$ and $\mathfrak{m}_{A,1} = (a_{1,1}, \ldots, a_{1,q})$ are q-tuples consisting the dimensions of the 0-components and the 1-components of the $\mathbb{Z}_2$-graded simple summands of A. Similarly we have the pair $\mathfrak{m}_B = (\mathfrak{m}_{B,0}, \mathfrak{m}_{B,1})$ and t-tuples $\mathfrak{m}_{B,0} = (b_{0,1}, \ldots, b_{0,t})$ and $\mathfrak{m}_{B,1} = (b_{1,1}, \ldots, b_{1,q})$ for the algebra B.

Definition 3.1. We say B covers A (or $\mathfrak{m}_B$ covers $\mathfrak{m}_A$) if there exists a decomposition of the tuple $(1, \ldots, q)$ into t subsets (possibly empty) such that the sum of the elements of $\mathfrak{m}_{A,0} = (a_{0,1}, \ldots, a_{0,q})$ corresponding to the i-th subset is bounded from above by $b_{0,i}$ and the corresponding sum of odd elements in $\mathfrak{m}_{A,1} = (a_{1,1}, \ldots, a_{1,q})$ is bounded from above by $b_{1,i}$ (same i), $i = 1, \ldots, t$.

Example 3.2. Consider the pair of tuples $\mathfrak{m}_B = (\mathfrak{m}_{B_0}, \mathfrak{m}_{B_1})$ where $\mathfrak{m}_{B_0} = (17, 13)$ and $\mathfrak{m}_{B_1} = (8, 12)$. It covers the pair $\mathfrak{m}_A = (\mathfrak{m}_{A_0}, \mathfrak{m}_{A_1})$ where $\mathfrak{m}_{A_0} = (16, 10, 2)$ and $\mathfrak{m}_{A_1} = (0, 4, 2)$. On the other hand the pair $\mathfrak{m}_B = (\mathfrak{m}_{B_0}, \mathfrak{m}_{B_1})$ where $\mathfrak{m}_{B_0} = (17, 13)$ and $\mathfrak{m}_{B_1} = (8, 12)$ does not cover the pair $\mathfrak{m}_A = (\mathfrak{m}_{A_0}, \mathfrak{m}_{A_1})$ where $\mathfrak{m}_{A_0} = (10, 10, 4)$ and $\mathfrak{m}_{A_1} = (6, 6, 4)$. Note, however, that the tuple $(17, 13)$ (resp. $(8, 12)$) does cover $(10, 10, 4)$ (resp. $(6, 6, 4)$).

Let A be a finite dimensional superalgebra over an algebraically closed field F of characteristic zero. Let $A \cong A_{ss} \oplus J$ be the Wedderburn–Malcev decomposition of A. Let $A_{ss} \cong A_1 \times \cdots \times A_q$ where A_i are supersimple algebras.

Definition 3.3. We say A is full if up to ordering of the supersimple components we have $A_1 \cdot J \cdot A_2 \cdots J \cdot A_q \neq 0$.

Before stating the Main Lemma, let us make precise definitions of admissible evaluations of polynomials as well as weakly full, full and strongly full polynomials of $E(A)$ where A is a finite dimensional full superalgebra.

Let U be a finite dimensional $\mathbb{Z}_2$-simple algebra. It is well known that U is isomorphic to a superalgebra of the form (1) $M_{l,f}(F)$ where the grading is elementary and is determined by an $(l+f)$-tuple with l e's and f σ's, where an elementary matrix $e_{i,j}$ has degree e if $1 \leq i, j \leq l$ or $l+1 \leq i, j \leq l+f$ and degree σ otherwise (2) $FC_2 \otimes M_n(F)$, where FC_2 is the group (super)algebra of $C_2 = \{e, \sigma\}$, and where elements of the form $u_e \otimes e_{i,j}$ have degree e and elements of the form $u_\sigma \otimes e_{i,j}$ have degree σ. Note that the set $\{e_{i,j}\}$ (resp. $\{u_g \otimes e_{i,j}; g \in \{e, \sigma\}\}$) is a basis of $M_{l,f}(F)$ (resp. of $FC_2 \otimes M_n(F)$). We denote by β_{ss} a basis of A_{ss} consisting of all elements of that form. Note that the basis elements in β_{ss} are homogeneous. If U is any simple component of A_{ss}, and z denotes a basis element of U as above, we consider a basis Σ_{ss} of $E(A_{ss})$ consisting of all elements of the form $\epsilon_{i_1} \ldots \epsilon_{i_n} \otimes z$, n is even and $z \in \beta_{ss}$ has degree e (in case $n = 0$, we set $\epsilon_{i_1} \ldots \epsilon_{i_n} = 1$) or $\epsilon_{i_1} \ldots \epsilon_{i_n} \otimes z$, n is odd and $z \in \beta_{ss}$ has degree σ. Here $\epsilon_{i_1}, \ldots, \epsilon_{i_n}$ are different generators of the Grassmann algebra E. Finally, we choose an *homogeneous* basis β_J of the Jacobson radical J of A and consider a basis Σ_J of $E(J)$ consisting of all elements of the form $\epsilon_{i_1} \cdots \epsilon_{i_n} \otimes w$ where (as above) n is even and $w \in \beta_J$ is of degree e or n is odd and $w \in \beta_J$ is of degree σ.

Definition 3.4. Let p be a multilinear polynomial. We say an evaluation of p on $E(A)$ is admissible if all values are taken from Σ_{ss} or Σ_J.

Definition 3.5. Let A be a finite dimensional full superalgebra as above:

(1) We say a multilinear polynomial p is *weakly full* of $E(A)$ if there is an admissible nonzero evaluation of p on $E(A)$ where among the elements $\epsilon_{i_1}\cdots\epsilon_{i_n}\otimes z$, $z\in A_{ss}$ that appear in the evaluation, we have at least one elements z from each $\mathbb{Z}_2$-simple component of A_{ss}.

(2) We say a multilinear polynomial p is *full* of $E(A)$ if all $\mathbb{Z}_2$-simple subalgebras of A_{ss} are represented in every nonzero admissible evaluation of p on $E(A)$. That is, given a nonzero evaluation of p, for every $\mathbb{Z}_2$-simple component A_i, $i=1,\ldots,q$, there is a variable of p whose value is of the form $\epsilon_{i_1}\cdots\epsilon_{i_n}\otimes z$ for some $z\in A_i$.

(3) We say a multilinear polynomial p is *strongly full* of $E(A)$ if for every nonzero admissible evaluation of p on $E(A)$ and every $z\in A_{ss}$, there is variable of p whose value is of the form $\epsilon_{i_1}\cdots\epsilon_{i_n}\otimes z$.

The following statement is the main lemma in the nonaffine case.

Lemma 3.6 (main lemma-nonaffine). *Suppose A and B are finite dimensional $\mathbb{Z}_2$-graded full algebras. Suppose $\mathfrak{m}_B$ does not cover $\mathfrak{m}_A$. Then there exists a strongly full polynomial $f_{E(A)}$ of $E(A)$ which is an identity of $E(B)$. Furthermore, if f_0 is an arbitrary weakly full polynomial of $E(A)$, then there exists a strongly full polynomial $f_{E(A)}\in\langle f_0\rangle_T$ of $E(A)$ which is an identity of $E(B)$.*

The proof of the main lemma will be presented in 4 propositions: (1) Construction of a strongly full polynomial $f_{E(A)}$ of $E(A)$ (Propositions 3.8 and 3.9) (2) Construction of a strongly full polynomial $f_{E(A)}\in\langle f_0\rangle_T$ of $E(A)$ where f_0 is an arbitrary weakly full polynomial of $E(A)$ (Proposition 3.10) (3) The polynomial $f_{E(A)}$ is an identity of $E(B)$ (Proposition 3.11).

We start with the construction of a strongly full polynomial of $E(A)$.

Consider the monomial

$$f_0 = X_1\cdot w_1\cdots w_{q-1}\cdot X_q$$

of degree $2q-1$ where the variables are *ungraded*. Note that f_0 is weakly full of $E(A)$ (that is, there is an admissible nonzero evaluation of f_0 which visits every $\mathbb{Z}_2$-simple component of A). We proceed with the construction of a strongly full polynomial $f_{E(A)}$ in $\langle f_0\rangle_T$.

Let d_0 (resp. d_1) be the dimension of the even (resp. odd) homogeneous component of A_{ss}. We consider a diagram composed of two strips of semisimple elements, denoted by $\alpha_{i,j}$ and similarly two strips of variables $x_{i,j}$, horizontal and vertical, where the horizontal strip has d_0 rows and k columns and the vertical strip has d_1 columns and k rows (k to be determined).

$$\boxed{\begin{array}{cccc}\alpha_{1,d_1+1} & \alpha_{1,d_1+2} & \cdots & \alpha_{1,d_1+k} \\ \alpha_{2,d_1+1} & \alpha_{2,d_1+2} & \cdots & \alpha_{2,d_1+k} \\ \vdots & \vdots & & \vdots \\ \alpha_{d_0,d_1+1} & \alpha_{d_0,d_1+2} & \cdots & \alpha_{d_0,d_1+k}\end{array}}$$

$$\boxed{\begin{array}{ccc}\alpha_{d_0+1,1} & \cdots & \alpha_{d_0+1,d_1} \\ \alpha_{d_0+2,1} & \cdots & \alpha_{d_0+2,d_1} \\ \alpha_{d_0+3,1} & \cdots & \alpha_{d_0+3,d_1} \\ \vdots & & \vdots \\ \alpha_{d_0+k,1} & \cdots & \alpha_{d_0+k,d_1}\end{array}}$$

$$\boxed{\begin{array}{cccc}x_{1,d_1+1} & x_{1,d_1+2} & \cdots & x_{1,d_1+k} \\ x_{2,d_1+1} & x_{2,d_1+2} & \cdots & x_{2,d_1+k} \\ \vdots & \vdots & & \vdots \\ x_{d_0,d_1+1} & x_{d_0,d_1+2} & \cdots & x_{d_0,d_1+k}\end{array}}$$

$$\boxed{\begin{array}{ccc}x_{d_0+1,1} & \cdots & x_{d_0+1,d_1} \\ x_{d_0+2,1} & \cdots & x_{d_0+2,d_1} \\ x_{d_0+3,1} & \cdots & x_{d_0+3,d_1} \\ \vdots & & \vdots \\ x_{d_0+k,1} & \cdots & x_{d_0+k,d_1}\end{array}}$$

Remark 3.7. The variables $x_{i,j}$ in the last two strips will appear in the polynomial $f_{E(A)}$ we are about to construct. The role of these strips is to indicate which sets of variables will alternate in $f_{E(A)}$ and which sets of variables will symmetrize. The elements $\alpha_{i,j}$ appearing in the first two strips are the evaluations of the variables $x_{i,j}$.

We construct a *long* monomial consisting of elements of A as follows.

For each $\mathbb{Z}_2$-graded simple component we write a *nonzero* product of the standard basis, namely elements of the form $e_{i,j} \in M_{l,f}(F)$ or $u_g \otimes e_{i,j} \in FC_2 \otimes M_n$ where $g = e, \sigma$. It is known that such a product exists. We refer to these elements as *designated* elements. In order to keep a unified notation we shall replace $e_{i,j} \in M_{l,f}(F)$ by $u_e \otimes e_{i,j}$. Furthermore, we may assume for simplicity that the nonzero product starts (resp. ends) with an element of the form $u_e \otimes e_{1,y}$ (resp. $u_g \otimes e_{x,1}$). Next we border each basis element $u_g \otimes e_{i,j}$ from left (resp. right) with the element $u_e \otimes e_{i,i}$ (resp. $u_e \otimes e_{j,j}$) which we call *frame*, so that the product of the monomial remains nonzero. Let us denote the product above, namely the product corresponding to the $\mathbb{Z}_2$-graded simple algebra A_i by Z_i. We take now the product of k copies of this monomial $Z_{i,1} \cdots Z_{i,k}$. This is clearly nonzero. Next, we bridge the $\mathbb{Z}_2$-graded simple components with appropriate radical values $w_{s,s+1}$ and get a nonzero product as dictated by the expression $A_1 J A_2 \cdots J A_q \neq 0$.

Finally, we tensor the basis elements with Grassmann elements, where even elements of A are tensored with 1 and odd elements are tensored with *different* generators ϵ_i (odd degree). We shall always view these tensors as ungraded elements of $E(A)$ although, abusing language, we will refer to them as even and odd elements respectively.

We obtained a nonzero expression of the form

$$Z_{1,1}\cdots Z_{1,k}\cdot w_{1,2}\cdot Z_{2,1}\cdots Z_{2,k}\cdot w_{2,3}\cdots w_{q-1,q}\cdot Z_{q,1}\cdots Z_{q,k}.$$

Consider the set $U_{\text{even},1}$ of designated even elements in the tuple

$$(Z_{1,1}, Z_{2,1}, \dots, Z_{q,1}).$$

Similarly, we let $U_{\text{even},i}$ be the designated even elements in the tuple $(Z_{1,i}, Z_{2,i}, \dots, Z_{q,i})$, $i = 1, \dots, k$. Observe that the cardinality of $U_{\text{even},i}$ is $d_o = \dim_F A_{ss,0}$. We denote the elements of $U_{\text{even},i}$ by $\alpha_{1,d_1+i}, \dots, \alpha_{d_0,d_1+i}$, that is, as the i-th column of the horizontal strip above. Furthermore, it will be convenient to denote the designated even elements in $(Z_{1,i}, Z_{2,i}, \dots, Z_{q,i})$ in the same order as they appear in the i-th column.

Similar to the even elements above, $U_{\text{odd},j}$ consists of all designated odd elements in the tuple

$$(Z_{1,j}, Z_{2,j}, \dots, Z_{q,j})$$

and we denote them by $\alpha_{d_0+j,1}, \dots, \alpha_{d_0+j,d_1}$, i.e., the elements in the j-th row of the vertical strip.

For each $t = 1, \dots, k$, we alternate the designated (even) elements

$$\alpha_{1,d_1+t}, \dots, \alpha_{d_0,d_1+t}$$

and symmetrize the designated (odd) elements $\alpha_{d_0+t,1}, \dots, \alpha_{d_0+t,d_1}$. We claim the expression obtained is nonzero. Indeed, any *nontrivial* permutation (independently of its sign) of designated even elements will be surrounded by frames where not all match and hence vanishes. Similarly with the odd elements of A. In particular alternating the even elements and symmetrizing the odd elements yields a nonzero value.

We now *symmetrize* the sets of k elements corresponding to the rows of the horizontal strip and *alternate* the sets of k elements corresponding to the columns of the vertical strip. We claim we get a nonzero value. For the proof we may assume each tuple of k even elements are equal and are of the form $u_e \otimes e_{i,j}$ whereas for the odd elements we assume as we may, the elements of each k tuple have the form $\epsilon_{i,j,g} \otimes u_g \otimes e_{i,j}$, $g \in \{e, \sigma\}$, $\epsilon_{i,j,g}$ are generators of the Grassmann algebra and the elements $u_g \otimes e_{i,j}$ of A are equal. It follows that symmetrization of the rows in the horizontal strip and alternation of the columns in the vertical strip yield the multiplication of each monomial by a factor of $(k!)^{d_0}$. In particular, if the corresponding operation is performed on a vanishing product it remains zero whereas, since $\mathrm{char}(F) = 0$, it is nonzero if the operation were performed on a nonvanishing product.

We now replace the elements of $E(A)$ appearing in the monomial

$$Z_{1,1}\cdots Z_{1,k}\cdot w_{1,2}\cdot Z_{2,1}\cdots Z_{2,k}\cdot w_{2,3}\cdots w_{q-1,q}\cdot Z_{q,1}\cdots Z_{q,k}$$

by variables which we call designated variables, frames and bridges. Note that the monomial obtained is in $\langle f_0\rangle_T$ where $f_0 = X_1\cdot w_1\cdots w_{q-1}\cdot X_q$. It is convenient to arrange the designated variables $x_{r,s}$ in the two strips in $1-1$ correspondence with the designated elements $\alpha_{r,s} \in E(A)$. Finally, we perform the alternations and symmetrizations on these variables and obtain (by construction) a multilinear nonidentity of $E(A)$ which we denote by $f_{E(A)}$. We summarize the above paragraph in the following proposition.

Proposition 3.8. *Let A be a finite dimensional $\mathbb{Z}_2$-graded algebra over F. Suppose A is full and let $f_{E(A)}$ be as above. Then $f_{E(A)}$ is a nonidentity of $E(A)$. Furthermore, $f_{E(A)} \in \langle f_0 \rangle_T$ where $f_0 = X_1 \cdot w_1 \cdots w_{q-1} \cdot X_q$.*

Proposition 3.9. *For k large enough, the polynomial $f_{E(A)}$ is strongly full of $E(A)$.*

Proof. Suppose this is not the case. We claim that only in a bounded number of columns in the horizontal strip of the diagram we can put either radical elements or odd semisimple elements. Indeed, it is clear that the number of radical values is bounded. If we put arbitrary many odd semisimple values, by the pigeonhole principle, there will be variables in the same row which will get values of the form $\epsilon_{i_1} \cdots \epsilon_{i_n} \otimes a$ and $\epsilon_{j_1} \cdots \epsilon_{j_m} \otimes a$, same a, where n and m are odd. Then the symmetrization of the corresponding variables yields zero. Similarly, in any nonzero evaluation, the number of rows in the vertical strip of the diagram in which we can put radical or even elements is bounded. It follows then that for k large enough there exists a column in the horizontal strip, say the i-th column, which assumes only even elements and there is a j-th row in the vertical strip which assumes only odd elements. But more than that, taking k large enough we may assume $i = j$. It follows that by the alternation of the columns in the horizontal strip (resp. symmetrization of the rows in the vertical strip), in any nonzero evaluation, we are forced to evaluate these on basis elements of the form $\epsilon_{i_1} \cdots \epsilon_{i_n} \otimes a$ where a runs over a full basis of $A_{ss,0}$ (resp. $A_{ss,1}$). This proves the proposition. □

We extend the proposition, namely starting with an arbitrary weakly full polynomial f_0 of $E(A)$.

Proposition 3.10. *Let A be a finite dimensional $\mathbb{Z}_2$-graded algebra over F. Suppose A is full. Let f_0 be a multilinear weakly full polynomial of $E(A)$. Then there exists a polynomial $f_{E(A)} \in \langle f_0 \rangle_T$ which is strongly full of $E(A)$.*

Proof. Let us fix a nonzero admissible evaluation Φ of f_0 in $E(A)$ which visits all $\mathbb{Z}_2$-graded simple components of A_{ss}. Denote by $X_1, \dots, X_q$ the variables of f_0 which assume values from the q different $\mathbb{Z}_2$-graded simple components of A. Applying the T-operation we replace the variables $X_1, \dots, X_q$ with $X_1\Delta_1, \dots, X_q\Delta_q$ where $\Delta_t = Z_{t,1} \cdots Z_{t,k}$. Finally we alternate and symmetrize the designated variables as above. The polynomial obtained $f_{E(A)} \in \langle f_0 \rangle_T$ is strongly full for the algebra $E(A)$. The proof is similar to the proof above when f_0 is a monomial. Details are omitted. □

Proposition 3.11. *Let A and B be finite dimensional full superalgebras. Suppose B does not cover A. Let $f_{E(A)}$ be the polynomial constructed above. Then for k sufficiently large, the polynomial $f_{E(A)}$ is an identity of $E(B)$. More generally, suppose B is a direct sum of full superalgebras, each not covering A. Then for k sufficiently large, the polynomial $f_{E(A)}$ is an identity of $E(B)$.*

Proof. The proof is similar to the proof in the affine case. In any nonzero evaluation on $E(B)$ we must have an index i which obtains linearly independent semisimple elements of B. If the evaluation is nonzero, we must have a monomial with nonzero value and hence the semisimple elements appearing in each segment must come from the same $\mathbb{Z}_2$-graded simple component of B. We have then that B covers A. Contradiction. $\square$

Corollary 3.12. *Let A and B full superalgebras. If $E(A)$ and $E(B)$ are PI-equivalent then their semisimple parts A_{ss} and B_{ss} are isomorphic.*

Proof. Indeed, B and A cover each other. It follows that the tuple of pairs of *dimensions* of the simple components of A and B coincide (up to a permutation). Finally we note (see below) that the superstructure of a supersimple algebra A is determined by the dimensions of A_0 and A_1 and hence if these coincide, A_{ss} and B_{ss} must be isomorphic as superalgebras. $\square$

For the rest of the proof we follow the proof in the affine case step by step. Along the proof two basic propositions are needed.

Proposition 3.13. *Let A be a finite dimensional superalgebra over F. Then $E(A)$ is PI-equivalent to the direct sum of algebras $E(A_i)$ where A_i is a finite dimensional full superalgebra.*

Proof. Recall that a finite dimensional superalgebra A is PI-equivalent to the direct sum of full superalgebras $\mathfrak{A} = A_1 \oplus \cdots \oplus A_n$. We claim firstly: $E(A)$ and $E(\mathfrak{A})$ are PI-equivalent: Indeed, a superpolynomial f is an identity of A if and only if the superpolynomial f^* is a superidentity of $E(A)$ as a superalgebra where the 0 component is spanned by elements of the form $\epsilon_{i_1} \cdots \epsilon_{i_{2r}} \otimes a_0$ and the 1-component is spanned by elements of the form $\epsilon_{i_1} \cdots \epsilon_{i_{2r+1}} \otimes a_1$; see [Aljadeff et al. 2020, Subsection 19.4.1]. Here ϵ_j is a generator of E, $a_0 \in A^0$, $a_1 \in A^1$, the even and odd elements of A respectively. Then, if $E(A)$ and $E(\mathfrak{A})$ are PI-equivalent as superalgebras, they are PI-equivalent as ungraded algebras. Next we argue that $E(\mathfrak{A}) \cong E(A_1) \oplus \cdots \oplus E(A_n)$ and the proposition is proved. $\square$

The second statement we need is

Proposition 3.14. *Let A and B be finite dimensional supersimple algebras over F. If $\dim_F(A^0) = \dim_F(B^0)$ and $\dim_F(A^1) = \dim_F(B^1)$ then A and B are $\mathbb{Z}_2$-graded isomorphic.*

Proof. Recall that a $\mathbb{Z}_2$-graded simple algebra over an algebraically closed field F of characteristic 0 is isomorphic to $M_{l,f}(F)$, where $l \geq 1$, $f \geq 0$ or $FC_2 \otimes_F M_n(F)$, $n \geq 1$. In the case of $M_{l,f}(F)$ the dimension of the 0-component (resp. 1-component) is $l^2 + f^2$ (resp. $2lf$) and in particular the total dimension is a square number whereas in the case of $FC_2 \otimes_F M_n(F)$ the dimensions of the homogeneous components are each equal to n^2 and hence not a square number. This proves the proposition. $\square$

4. G-graded algebras

In this section we extend the main theorem to the setting of affine G-graded algebras where G is a finite group. The case of nonaffine G-graded algebras is treated in the next section. Here is the precise statement.

Theorem 4.1. *Let G be a finite group and let Γ be a G-graded T-ideal over F. Suppose Γ contains an ungraded Capelli polynomial c_n, some n. Then there exists a finite dimensional semisimple G-graded algebra U over F which satisfies the following conditions*:

(1) *There exists a finite dimensional G-graded algebra A over F with $\mathrm{Id}_G(A) = \Gamma$ and such that $A \cong U \oplus J_A$ is its Wedderburn–Malcev decomposition as G-graded algebras.*

(2) *If B is any finite dimensional G-graded algebra over F with $\mathrm{Id}_G(B) = \Gamma$ and B_{ss} is its maximal semisimple G-graded subalgebra*, *then U is a direct summand of B_{ss} as G-graded algebras.*

The proof basically follows the main lines of the proof of the ungraded case yet there is a substantial obstacle here due to the fact that G-graded simple algebras are *not determined* up to isomorphism by the dimensions of the corresponding homogeneous components. In the following examples, as usual, F is an algebraically closed field of characteristic zero.

Example 4.2. (1) If G is a finite group, $F^{\alpha}G$ and $F^{\beta}G$, $\alpha, \beta \in H^2(G, F^*)$, are twisted group algebras, then they are G-graded isomorphic if and only if $\alpha = \beta$. Clearly, the dimensions of the homogeneous components equal 1 independently of the cohomology class.

(2) Let $G = \{e, \sigma, \tau, \sigma\tau\}$ be the Klein 4-group. Consider the crossed product grading on $A \cong M_4(F)$, that is the elementary grading determined by the tuple $(e, \sigma, \tau, \sigma\tau)$, and the algebras $B_i \cong F^{\beta_i}G \otimes M_2(F)$, $\beta_1, \beta_2 \in H^2(G, F^*)$. Here β_1 (resp. β_2) is the trivial (resp. nontrivial) cohomology class on G with values on F^*. The dimension of each homogeneous component is 4. The algebras A and B_2 are isomorphic as ungraded algebras ($\cong M_4(F)$) but not isomorphic to $B_1 \cong M_2(F) \oplus M_2(F) \oplus M_2(F) \oplus M_2(F)$. It is easy to see that A and B_2 are nonisomorphic as G-graded algebras; see [Aljadeff and Haile 2014].

Let G be a finite group and let A be a finite dimensional G-graded algebra over F. We decompose A into $A_{ss} \oplus J$ where A_{ss} is a maximal G-graded semisimple algebra which supplements J, the Jacobson radical. The algebra A_{ss} decomposes into a direct product of G-graded simple components $A_1 \times \cdots \times A_q$. As in the ungraded case, the G-graded simple components are uniquely determined up to a G-graded isomorphism.

We start with the definition of the covering relation.

Definition 4.3. Let Q and V be finite dimensional G-graded semisimple algebras over F. We say V covers Q if the G-graded simple components of Q can be decomposed into subsets such that the sum of the dimensions of the corresponding homogeneous components are bounded by the dimensions of the homogeneous components of V. Explicitly, if $Q \cong Q_1 \times \cdots \times Q_q$ and $V \cong V_1 \times \cdots \times V_r$ are the decompositions of Q and V into their G-graded simple components. Let $u_{i,g} = \dim_F(Q_i)_g$ (resp. $v_{j,g} = \dim_F(V_j)_g$) be the dimension of the g-homogeneous component of Q_i (resp. of V_j). Then V covers Q if and only if the indices $1, \dots, q$ can be decomposed into r subsets $\Lambda_1, \dots, \Lambda_r$ such that $\sum_{i \in \Lambda_j} u_{i,g} \le v_{j,g}$.

Definition 4.4. Let A be a finite dimensional G-graded algebra over F. Let $A \cong A_1 \times \cdots \times A_q \oplus J_A$ be its Wedderburn–Malcev decomposition, where A_i are G-graded simple. We say A is full if up to a permutation of the indices we have $A_1 \cdot J \cdots J \cdot A_q \neq 0$.

Next, we introduce G-graded *weakly full*, *full* and *strongly full* for a given full G-graded algebra A.

Definition 4.5. Let A be a finite dimensional full G-graded algebra. A G-graded polynomial p, nonidentity of A, is A-strongly full if it is homogeneous, multilinear and vanishes when evaluated on A unless every basis element of A_{ss} appears as a value of one of its variables. A G-graded homogeneous polynomial p is A-weakly full if there exists an admissible nonzero evaluation on A that visits each G-graded simple subalgebra of A_{ss}. Finally, p is full if this is so for *every* admissible nonzero evaluation.

Strongly full polynomials were constructed in [Aljadeff and Kanel-Belov 2010]. Nevertheless, we shall need their precise structure so let us recall here their construction.

For each G-graded simple component A_i of A consider a *nonzero* product of all basis elements of A_i. These are elements of the form $u_h \otimes e_{r,s}$, $h \in H$ and $1 \leq r, s \leq m$, whose homogeneous degree is $g_r^{-1} h g_s$. Here, the G-grading on A_i is determined by a triple $(H, \alpha, (g_1, \ldots, g_m))$ where H is a subgroup of G, α is a 2-cocycle representing a class in $H^2(H, F^*)$ and $(p_1, \ldots, p_m) \in G^{(m)}$; see [Bakhturin et al. 2008] and [Aljadeff and Haile 2014, Theorem 1.1]for more on this notation. It is known that such a product exists; see [Aljadeff and Kanel-Belov 2010]. As above we border from right and left each basis element with frames of the form $u_e \otimes e_{i,i}$. We denote such product of basis elements, namely the designated and frame elements, by Z_i. We refer to Z_i as the monomial of basis elements of A_i. We may assume the product starts with an element of the form $u_e \otimes e_{1,1}$ and ends with an element of the form $u_h \otimes e_{r,1}$ and so if $Z_{i,l} = Z_i$, $l = 1, \ldots, k$, we have that the product $Z_{i,1} \cdots Z_{i,k}$ is nonzero. Next we bridge products corresponding to different G-graded simple components by radical (homogeneous) elements w_i. We obtain a nonzero product

$$Z_{1,1} \cdots Z_{1,k} \cdot w_1 Z_{2,1} \cdots Z_{2,k} \cdot w_2 \cdots w_{q-1} \cdot Z_{q,1} \cdots Z_{q,k}.$$

As in the ungraded case we consider the i-th set Λ_i, $i = 1, \ldots, k$ consisting of the designated (semisimple) elements in $Z_{1,i}, \ldots, Z_{q,i}$. We denote by $\Lambda_{i,g}$, $g \in G$, the subset of Λ_i consisting of elements of homogeneous degree g. We claim any nontrivial permutation of designated elements in $\Lambda_{i,g}$ yields a zero product. Clearly, it suffices to consider transpositions T. The claim is clear if T exchanges basis elements which belong to G-graded simple components A_i and A_j with $i \neq j$. Suppose T exchanges basis elements $u_{h_1} \otimes e_{r_1,s_1} \neq u_{h_2} \otimes e_{r_2,s_2}$ of the same G-graded simple component. Since they are of *equal* homogeneous degree, we have that $g_{r_1}^{-1} h_1 g_{s_1} = g_{r_2}^{-1} h_2 g_{s_2}$ and so we must have $(r_1, s_1) \neq (r_2, s_2)$. This implies that frames bordering different designated elements of the same homogeneous degree are different and the claim is proved.

We proceed as in the ungraded case where the monomials consisting of elements of A are replaced by monomials of different graded variables with the corresponding homogeneous degree. The *small* sets here are alternating sets of variables of degree $g \in G$ of cardinality equal the dimension of the g-homogeneous component of A_{ss}. The polynomial obtained is denoted by p. This completes the construction of a G-graded strongly full polynomial of A. As in previous cases we shall need a more general statement.

Proposition 4.6. *Let A be a full G-graded algebra and f_0 a G-graded multilinear polynomial which is weakly full of A. Then there exists a multilinear G-graded strongly full polynomial f_A such that $f_A \in \langle f_0 \rangle_T$.*

Proof. The proof is similar to the proof of Theorem 2.8(3). □

Lemma 4.7. *Let A be a G-graded full algebra and f_A a G-graded strongly full polynomial of A with sufficiently many small sets. If B does not cover A, then f_A is an identity of B.*

Proof. The proof is similar to the proof of Lemma 2.10. □

Note that in the ungraded case this was sufficient in order to deduce that the semisimple subalgebras of A and B are isomorphic.

Theorem 4.8. *Let A and B be finite dimensional G-graded full algebras. Suppose they are G-graded PI-equivalent. Then the maximal semisimple subalgebras A_{ss} and B_{ss} are G-graded isomorphic.*

Proof. By the preceding lemma we know that A and B cover each other and hence the tuples of the dimensions of the homogeneous components of the G-graded simple algebras appearing in the decomposition of A_{ss} and B_{ss} are equal. Our goal is to show the corresponding G-graded simple components are G-graded isomorphic.

For the proof we shall need to insert *suitable* e-central polynomials in the full G-graded polynomial f_A of A constructed above. We recall from [Karasik 2019] that every finite dimensional G-graded simple admits an e-central multilinear polynomial c_A, that is a nonidentity of A, central and G-homogeneous of degree e. Furthermore, it follows from its construction, that the polynomial c_A alternates on certain sets of variables of equal homogeneous degree of cardinality equal $\dim_F(A_g)$, for every $g \in G$. For the proof of Theorem 4.8 we shall need e-central polynomials with some additional properties.

Theorem 4.9. *Let A_i, $i = 1, \dots, q$, be the simple components of A_{ss}. Then there exists a polynomial $m_i(X_G)$ with the following properties*:

(1) *$m_i(X_G)$ is e-central of A_i.*

(2) *$m_i(X_G)$ is an identity of every algebra Σ which satisfies the following conditions*:

  (a) *Σ is finite dimensional G-graded simple.*

  (b) $\dim_F(\Sigma_g) = \dim_F((A_i)_g)$ *for every $g \in G$.*

  (c) $\mathrm{Id}_G(A_i) \not\supseteq \mathrm{Id}_G(\Sigma)$.

Proof. By condition (2c) there is a G-graded homogeneous nonidentity $f_{i,\Sigma}$ of A_i, of homogeneous degree $g \in G$ say, which vanishes on Σ. Then replacing a variable of degree g in an alternating set of c_{A_i} by $f_{i,\Sigma}$ we obtain a nonidentity e-central polynomial $m_{i,\Sigma}(X_G)$ of A_i which vanishes on Σ. Now recall from [Aljadeff and Karasik 2022] that the number of G-graded simple algebras Σ satisfying conditions

(2a) and (2b) above is finite and so, because the nonzero values of $m_{i,\Sigma}(X_G)$ are invertible in F^*, we have that $m_i(X_G) = \Pi_\Sigma m_{i,\Sigma}(X_G)$ is an e-central polynomial of A with the desired properties. $\square$

Finally we insert in f_A polynomials with disjoint sets of variables $m_i(X_G)$ adjacent to each monomial $Z_{i,l}$. This completes the construction of a special strongly full polynomial which we denote by $\mathfrak{f}_A$.

We can complete now the proof of the Theorem 4.8. We are assuming the algebras A and B are PI-equivalent and so by Lemma 4.7, the algebras A and B cover each other. It follows that A_{ss} and B_{ss} have the same number of G-graded simple components. Furthermore, if $A_{ss} \cong A_1 \times \cdots \times A_q$ and $B_{ss} \cong B_1 \times \cdots \times B_q$ then there is a permutation $\sigma \in \mathrm{Sym}(q)$ such that $\dim_F((A_i)_g) = \dim_F((B_{\sigma(i)})_g)$, $i = 1, \ldots, q$ and every $g \in G$.

We claim there is a permutation of the G-graded simple components of B_{ss} such that in addition to the condition above we have that $\mathrm{Id}_G(A_i) \supseteq \mathrm{Id}_G(B_{\sigma(i)})$, $i = 1, \ldots, q$. Suppose not. Then for every permutation σ satisfying the condition above, there is a $j = j(\sigma)$ such that $\mathrm{Id}_G(A_j) \not\supseteq \mathrm{Id}_G(B_{\sigma(j)})$. We will show that the strongly full polynomial $\mathfrak{f}_A$ is an identity of B, in contradiction to the PI-equivalence of A and B. Indeed, evaluating $\mathfrak{f}_A$ on B, the value will be zero unless there is a monomial Z_i, together with the inserted central polynomials, whose value is nonzero. This implies there is a permutation σ of the components of B_{ss} such that the i-th segment of p is evaluated on $B_{\sigma(i)}$. This already implies the condition above on the dimensions. But by assumption there is j such that $\mathrm{Id}_G(A_j) \not\supseteq \mathrm{Id}_G(B_{\sigma(j)})$ and so the central polynomial $m_j(X_G)$ vanishes on $B_{\sigma(j)}$.

We conclude there is a permutation $\sigma \in \mathrm{Sym}(q)$ of the simple components of B_{ss} such that:

(1) $\dim_F((A_i)_g) = \dim_F((B_{\sigma(i)})_g)$, $i = 1, \ldots, q$, and every $g \in G$

(2) $\mathrm{Id}_G(A_i) \supseteq \mathrm{Id}_G(B_{\sigma(i)})$, $i = 1, \ldots, q$.

Our goal is to show that in fact $\mathrm{Id}_G(A_i) = \mathrm{Id}_G(B_{\sigma(i)})$, $i = 1, \ldots, q$. Indeed, this would imply what we need, that is $A_i \cong B_{\sigma(i)}$, $i = 1, \ldots, q$, as G-graded algebras; see [Aljadeff and Haile 2014].

Suppose that G is abelian. In that case let us recall the following general result of O. David [2012].

Theorem 4.10. *Let G be a finite abelian group and let A and B finite dimensional G-graded simple algebras over an algebraically closed field F. Then there is an embedding $A \hookrightarrow B$ as G-graded algebras if and only if* $\mathrm{Id}_G(A) \supseteq \mathrm{Id}_G(B)$.

Clearly, it follows at once from the theorem that G-graded algebras satisfying conditions (1) and (2) above must be G-graded isomorphic. David's result is not known in case G is an arbitrary finite group.

Here, instead, we argue as follows. By symmetry there is a permutation $\tau \in \mathrm{Sym}(q)$ such that:

(1) $\dim_F((B_i)_g) = \dim_F((A_{\tau(i)})_g)$, $i = 1, \ldots, q$ and every $g \in G$.

(2) $\mathrm{Id}_G(B_i) \supseteq \mathrm{Id}_G(A_{\tau(i)})$, $i = 1, \ldots, q$.

Consequently there is a permutation $\rho \in \mathrm{Sym}(q)$ such that A_i and $A_{\rho(i)}$ have equal dimensions of the homogeneous components and $\mathrm{Id}_G(A_i) \supseteq \mathrm{Id}_G(A_{\rho(i)})$. We need to show equality holds. Indeed, we see that $\mathrm{Id}_G(A_i) = \mathrm{Id}_G(A_j)$ for i and j which belong to the same orbit determined by ρ and so, in particular $\mathrm{Id}_G(A_i) = \mathrm{Id}_G(A_{\rho(i)})$, $i = 1, \ldots, q$. $\square$

The remaining steps in the proof of Theorem 4.1 are similar to those in the proof of Theorem 1.1. Details are omitted.

5. PI-equivalence of Grassmann envelopes of finite dimensional G_2-graded algebras

In this section we treat the case where the algebra A is finite dimensional $\mathbb{Z}_2 \times G$-graded and $E(A)$ is the Grassmann envelope of A viewed as a G-graded algebra.

The main result in this case is the following.

Theorem 5.1. *Let G be a finite group. Let Γ be a G-graded T-ideal. Suppose Γ contains a nonzero ungraded polynomial but contains no ungraded Capelli c_n, any n. Then there exists a finite dimensional semisimple $\mathbb{Z}_2 \times G$-graded algebra U over F which satisfies the following conditions*:

(1) *There exists a finite dimensional $\mathbb{Z}_2 \times G$-graded algebra A over F with $\mathrm{Id}_G(E(A)) = \Gamma$ and such that $A \cong U \oplus J_A$ is its Wedderburn–Malcev decomposition as $\mathbb{Z}_2 \times G$-graded algebras.*

(2) *If B is any finite dimensional $\mathbb{Z}_2 \times G$-graded algebra over F with $\mathrm{Id}_G(E(B)) = \Gamma$ and B_{ss} is its maximal semisimple $\mathbb{Z}_2 \times G$-graded subalgebra, then U is a direct summand of B_{ss} as $\mathbb{Z}_2 \times G$-graded algebras.*

The general approach is based on cases that were treated earlier, namely the cases where (1) Γ is a T-ideal of identities of a G-graded affine algebra (2) Γ is a T-ideal of identities of an ungraded nonaffine algebra. It turns out however, that also here there is a substantial difficulty, and this is in the very first step of the general approach (see Theorem 5.2 below). In fact, nearly the entire section will be devoted to the proof of Theorem 5.2.

Before we state the theorem let us set some notation.

Let G be a finite group and denote $G_2 := \mathbb{Z}_2 \times G$. We denote $G_{\text{even}} := 0 \times G$; $G_{\text{odd}} = 1 \times G$ and similarly for a G_2 algebra A we write $A_{\text{even}} = A_{G_{\text{even}}}$; $A_{\text{odd}} = A_{G_{\text{odd}}}$.

Theorem 5.2. *Suppose that A and B are two finite dimensional G_2-graded simple algebras. Then A and B are G_2-graded isomorphic if and only if $E(A)$ and $E(B)$ have the same G-graded identities.*

It is worth noting that the Grassmann $*$ operation allows one to pass from a superidentity of A to a superidentity of $E(A)$ (resp. from a G_2-identity of A to a G_2-identity of $E(A)$). The challenge here lies in transforming a superidentity of A into an ordinary identity of $E(A)$ (resp. from a G_2-identity of A into a G-identity of $E(A)$). The main part of the proof of the above Theorem is to find such a transformation.

We start with the construction of the transformation and in Proposition 5.5 we show the key property that makes it work. We emphasize that the construction and also the Theorem are guaranteed to work only in the case where the algebras in question are finite dimensional G_2-graded simple. In general it is not true that if $E(A)$ and $E(B)$ have the same G-graded identities then A and B have the same G_2-graded identities. An example can be found in [Giambruno and Zaicev 2005, Section 8.2].

The construction we are about to present is a generalization to the G-graded setting of the one in Section 3. Its main property appears in Proposition 5.5. In fact, the previous construction could be applied also here. And if we did, it would enable us to show as above that if $E(A)$ and $E(B)$ have the same G-graded identities then $\dim A_{\bar{g}} = \dim B_{\bar{g}}$ for all $\bar{g}$. However this would not be sufficient here since, as pointed out in the previous section, for general groups G one can easily find examples of nonisomorphic G_2-graded simple algebras having this property.

Let $f = f(X_0; Y_0)$ be a multilinear G_2-graded polynomial, where

$$X_0 = \coprod_{\bar{g} \in G_2} \coprod_{i=1}^{T} X_{\bar{g},i}$$

is a union of T *small sets* of degree $\bar{g}$-variables $X_{\bar{g},i} = \{x_{\bar{g},i}^{(1)}, \dots, x_{\bar{g},i}^{(\dim A_{\bar{g}})}\}$ (here $\bar{g}$ runs over all of G_2), and $Y_0 = \coprod_{\bar{g} \in G_2} Y_{\bar{g},0}$ are some additional variables. Assume that f has a G_2-graded evaluation $\phi : F\langle X_0; Y_0\rangle \to A$ with the following properties:

(1) For every nontrivial permutation $\sigma \in \prod_{\bar{g} \in G_2} \prod_{i=1}^{T} S_{X_{\bar{g},i}}$ (here S_W is the symmetric group on the set W) the value of $f(\sigma(X_0); Y_0)$ under the evaluation ϕ is 0.

(2) For all $\bar{g} \in G_2$ the value $\phi(x_{\bar{g},i}^{(j)}) =: a_{\bar{g}}^{(j)}$ is independent of $i = 1, \dots, T$. Furthermore, all $a_{\bar{g}}^{(j)}$, $j = 1, \dots, \dim A_{\bar{g}}$, are distinct.

We will see later that in the case which is relevant to the proof of Theorem 5.2 it is indeed possible to construct such a polynomial.

Let $k > 0$ be a natural number and consider the polynomial

$$f_k := f(X_1; Y_1) \cdots f(X_k; Y_k),$$

where all X_t and Y_t are disjoint copies of X_0 and Y_0 respectively. Notice that

$$X_t = \coprod_{\bar{g} \in G_2} \coprod_{i=1}^{T} X_{\bar{g},(t-1)T+i}.$$

We extend ϕ to $F\langle X; Y\rangle$, where $X = \coprod_{t=1}^{k} X_t$ and $Y = \coprod_{t=1}^{k} Y_t$, by duplicating the evaluation on X_0 and Y_0 to X_t and Y_t respectively (for all $t = 1, \dots, k$). As a result, we have in particular for all $\bar{g}$, i and j that $\phi(x_{\bar{g},i}^{(j)}) = a_{\bar{g}}^{(j)}$ (we rely here on property (2)).

For $a \in A$ we set $X_\phi(a) \subset X$ to be all the variables from X which ϕ assigns to them the value a. In other words, $X_\phi(a) = (\phi|_X)^{-1}(a)$. In particular, $X_\phi(a_{\bar{g}}^{(j)}) = \{x_{\bar{g},1}^{(j)}, \dots, x_{\bar{g},kT}^{(j)}\}$.

Remark 5.3. For every $\bar{g} \in G_2$ we have

$$\coprod_{i=1}^{kT} X_{\bar{g},i} = \coprod_{j=1}^{\dim A_{\bar{g}}} X_\phi(a_{\bar{g}}^{(j)}).$$

One should visualize this equality as "union of columns" (the $X_{\bar{g},i}$'s) = "union of rows" (the $X_\phi(a_{\bar{g}}^{(j)})$) in the matrix

$$\begin{bmatrix} x_{\bar{g},1}^{(1)} & \cdots & x_{\bar{g},kT}^{(1)} \\ \vdots & x_{\bar{g},i}^{(j)} & \vdots \\ x_{\bar{g},1}^{(\dim A_{\bar{g}})} & \cdots & x_{\bar{g},kT}^{(\dim A_{\bar{g}})} \end{bmatrix}.$$

Next, we alternate and symmetrize different subsets of X in the following fashion to obtain a new graded polynomial $s_{k;A}$. For each even (odd) element $\bar{g} \in G_2$ we apply alternation (symmetrization) on all variables $X_{\bar{g},i}$; afterwards we apply symmetrization (alternation) for every set of variables of the form $X_\phi(a)$. All in all, we have

$$s_{k;\phi;A}(f) = \prod_{\bar{g}\in G_{\mathrm{odd}}} \prod_{j=1}^{\dim A_{\bar{g}}} \mathrm{Alt}_{X_\phi(a_{\bar{g}}^{(j)})} \circ \prod_{\bar{g}\in G_{\mathrm{even}}} \prod_{j=1}^{\dim A_{\bar{g}}} \mathrm{Sym}_{X_\phi(a_{\bar{g}}^{(j)})} \circ \prod_{\bar{g}\in G_{\mathrm{odd}}} \prod_{i=1}^{kT} \mathrm{Sym}_{X_{\bar{g},i}} \circ \prod_{\bar{g}\in G_{\mathrm{even}}} \prod_{i=1}^{kT} \mathrm{Alt}_{X_{\bar{g},i}}(f_k).$$

We also consider a "forgetful" operator $F_G^{G_2}$ which transforms G_2-graded polynomials into G-graded polynomials by changing the degree of every variable from $(\epsilon, g) \in G_2$ to $g \in G$. We finally have the G-graded polynomial

$$F_G^{G_2}(s_{k;\phi;A}(f)).$$

We remark that for $g \in G$ the variables $F_G^{G_2}(x_{(0,g),t})$ and $F_G^{G_2}(x_{(1,g),t})$ are two different variables of degree $g \in G$.

Definition 5.4. Let B be a G_2-graded algebra. An evaluation of a G-graded polynomial f on B is called *almost* G_2 if every variable x of f of degree g is evaluated in some $B_{(\epsilon,g)}$.

Furthermore, if B_0 is a subset of B, we say that an evaluation ψ of f on B is a B_0*-evaluation* if every variable of f is evaluated in B_0.

Suppose we have a G_2-graded polynomial f and consider the G-polynomial $F_G^{G_2}(f)$. Note that if ψ is an almost G_2-evaluation of $F_G^{G_2}(f)$, then typically there is no reason that $\deg \psi(F_G^{G_2}(x_{\bar{g}})) = \bar{g}$ (i.e., the parities might not agree). The next Proposition shows that our construction of $F_G^{G_2}(s_{k;\phi;A})(f)$ will ensure that "almost always" the above equality occurs *given that* ψ gives a nonzero value to the polynomial.

Proposition 5.5. *Let B be a finite dimensional G_2-graded algebra. If*

$$\psi : F\langle X_0; Y_0\rangle \to E(B)$$

is a nonzero almost G_2-evaluation, then for every $\bar{g} \in G_2$ we have

$$\deg \psi(F_G^{G_2}(x_{\bar{g},i})) = \bar{g},$$

except possibly for $\dim A \cdot \dim B$ *of the* i.

Furthermore, if there is some $\bar{g}_0 = (\epsilon_0, g_0) \in G_2$ *such that the dimension of* $B_{\bar{g}_0}$ *is strictly smaller than that of* $A_{\bar{g}_0}$*, then for* $k > \dim A \cdot \dim B$*, the polynomial* $F_G^{G_2}(s_{k;\phi;A}(f))$ *is an identity of* $E(B)$*.*

Proof. We focus on proving the "furthermore" part and along the way we get a proof for the main claim. In order to show that $F_G^{G_2}(s_{k;\phi;A}(f))$ is an identity of $E(B)$, it is enough to show that it is 0 under any almost G_2-evaluation of $F_G^{G_2}(s_{k;\phi;A}(f))$, since this polynomial is multilinear. Let $\psi : F\langle X_0; Y_0\rangle \to E(B)$ be an almost G_2-evaluation of $F_G^{G_2}(s_{k;\phi;A}(f))$.

Suppose that $\psi(F_G^{G_2}(s_{k;\phi;A}(f))) \neq 0$. Then, there is some

$$\sigma \in \prod_{\bar{g}\in G_{\text{odd}}} \prod_{i=1}^{kT} S_{X_{\bar{g},i}} \cdot \prod_{\bar{g}\in G_{\text{even}}} \prod_{i=1}^{kT} S_{X_{\bar{g},i}}$$

such that, under ψ, the polynomial

$$F_G^{G_2}\Bigg(\prod_{\bar{g}\in G_{\text{odd}}} \prod_{j=1}^{\dim A_{\bar{g}}} \mathrm{Alt}_{X_\phi(a_{\bar{g}}^{(j)})} \circ \prod_{\bar{g}\in G_{\text{even}}} \prod_{j=1}^{\dim A_{\bar{g}}} \mathrm{Sym}_{X_\phi(a_{\bar{g}}^{(j)})}(f_k(\sigma(X)); Y)\Bigg) \neq 0.$$

Notice that for all i, the set $X_{\bar{g},i}$ stays the same even after applying σ.

We claim that all *small sets* $F_G^{G_2}(X_{\bar{g}_0,i})$, except possibly $\dim A \cdot \dim B$ of them, have all of their variables assigned to elements of degree $\bar{g}_0$. Indeed, we only need to show that the parity is ϵ_0. Assume that $\epsilon_0 = 0$ (the proof for $\epsilon_0 = 1$ is similar).

If on the contrary there are more than $\dim A \cdot \dim B$ small sets $F_G^{G_2}(X_{\bar{g}_0,i})$ having at least one variable which has an odd evaluation, as $k > \dim A \cdot \dim B \geq \dim A_{\bar{g}_0} \cdot \dim B_{(1,g_0)}$, and in view of Remark 5.3, there is some $l_0 \in \{1, \dots, \dim A_{\bar{g}_0}\}$ such that at least $\dim B_{(1,g_0)}$ distinct variables from $F_G^{G_2}(X_\phi(a_{\bar{g}_0}^{(l_0)}))$ are assigned by ψ values from $B_{(1,g_0)} \otimes E_1$. However as we symmetrize that set, we must get 0 — a contradiction. Notice that we have also proved here the main claim.

Denote by $F_G^{G_2}(X_{\bar{g}_0,i_0})$ a small set with the property from the previous paragraph. Since $\dim B_{\bar{g}_0} < \dim A_{\bar{g}_0}$, the alternation (symmetrization) of size $\dim A_{\bar{g}_0}$ must nullify the polynomial. □

We are now ready to prove Theorem 5.2:

Proof of Theorem 5.2. By [Aljadeff and Haile 2014], A and B are G_2-isomorphic if and only if A and B share the same G_2-identities; see also [Bahturin and Yasumura 2019] for a far reaching generalization of the statement in [Aljadeff and Haile 2014]. As a result, it is enough to show that if A and B are not G_2-PI-equivalent, then $E(A)$ and $E(B)$ are not G-PI-equivalent.

Assume, without loss of generality, that there is a multilinear G_2-polynomial $p(x_{\bar{g}_1,1}, \dots, x_{\bar{g}_n,n})$ which is an identity of B but not of A. We consider the G_2-graded basis $\mathcal{B}_A = \{a_{\bar{g}}^{(j)} : \bar{g} \in G_2,\ j = 1, \dots, \dim_F A_{\bar{g}}\}$ of A as in [Aljadeff and Haile 2014] Theorem 1.1. Let ϕ be a nonzero $\mathcal{B}_A$-evaluation of p. We may also assume that $\phi(p) = \delta$, where δ is a nonzero idempotent of A. In the next few paragraphs we are going to construct a G_2-graded polynomial f from p on which we will perform the construction from the beginning of the section to obtain a polynomial $F_G^{G_2}(s_{k;\phi;A}(f))$ which will be an identity of $E(B)$ and a nonidentity of $E(A)$.

For $i=1,\dots,n$ let $X(i)=\{x_{\bar{g},i}^{(j)} : \bar{g}\in G_2,\ j=1,\dots,\dim A_{\bar{g}}\}$ be disjoint variables from the ones of p and set $\phi(x_{\bar{g},i}^{(j)})=a_{\bar{g}}^{(j)}$. For every i let $j(i)$ be such that $\phi(x_{\bar{g}_i,i})=a_{\bar{g}_i}^{(j(i))}$. We identify $x_{\bar{g}_i,i}$ with $x_{\bar{g}_i,i}^{(j(i))}$ for every i. We set $X_0=\coprod_{i=1}^{n} X(i)$.

Similarly to the construction in the proof of Theorem 4.8, one can construct a multilinear G_2-monomial $M=M(X_0;Y)$ with the property that there is an evaluation ϕ_Y of the Y-variables such that the only extension of ϕ_Y to a nonzero $\mathcal{B}_A$-evaluation ϕ_Z of $M(X_0;Y)$ must satisfy $\phi_M|_{X_0}=\phi|_{X_0}$(i.e., ϕ_M also extends ϕ) and if ϕ_M satisfies $\phi_M|_{X_0}=\phi|_{X_0}$ then $\phi_M(M)=\delta$. In what follows we shall denote the unique nonzero evaluation ϕ_M of M by ϕ. Furthermore, one can also arrange that $\phi(M)=\delta$.

Clearly, $\phi(M\cdot p)=\delta$. However, $M\cdot p$ is not multilinear, and so we make some small changes to solve this issue. Consider a new set of variables $z_{\bar{g}_1,1},\dots,z_{\bar{g}_n,n}$ and replace in Z (only) the variables $z_{\bar{g}_i,i}$ by $x_{\bar{g}_i,i}$ for every i and let M' be the new polynomial. Clearly $M'\cdot p$ is multilinear. We extend ϕ to include all the z-variables by declaring $\phi(z_{\bar{g}_i,i})=\phi(x_{\bar{g}_i,i})$ so that $\phi(Mp)=\phi(M'p)=\delta$.

Finally let

$$f=M'\cdot p^*,$$

where $*$ is the Grassmann star operation.

We claim that f satisfies properties (1)–(2): By construction property (2) holds. Hence we are left with verifying property (1). Indeed, any nontrivial permutation of any of the variables in some $X(i)$ induces a new evaluation of f, which we call ϕ', that differs from ϕ only on the set $X(i)$. By the construction of M (and M') we get that $\phi'(M')=0$; showing property (1).

We now consider our final polynomial $F_G^{G_2}(s_{k;\phi;A}(f))$, where $k=n\cdot\dim A\cdot\dim B+1$. Notice that it is a G-polynomial and that the construction also extends ϕ to an evaluation of all of $s_{k;\phi;A}(f)$ (a G_2-graded evaluation!). We claim that it is an identity of $E(B)$ but not of $E(A)$. It is not an identity of $E(A)$ since we can consider the following G-evaluation ψ in $E(A)$: for every variable v appearing in $s_{k;\phi;A}(f)$ we set

$$\psi(F_G^{G_2}(v))=\phi(v)\otimes w_v,$$

where $w_v\in E_{\deg v}$ and all the w_v are chosen so that the product of all of them is nonzero. By the definition of $*$, we have that $\psi(F_G^{G_2}(p^*))=\delta\otimes\prod_{v\in p} w_v$ and so $\psi(F_G^{G_2}(f_k))=\delta\otimes\prod_{v\in f_k} w_v$. By property (1) of f we conclude that

$$\psi\left(F_G^{G_2}\left(\prod_{\bar{g}\in G_{\mathrm{odd}}}\prod_{i=1}^{kn}\mathrm{Sym}_{X_{g,i}}\circ\prod_{\bar{g}\in G_{\mathrm{even}}}\prod_{i=1}^{kn}\mathrm{Alt}_{X_{\bar{g},i}}(f_k)\right)\right)=\psi(F_G^{G_2}(f_k))=\delta\otimes\prod_{v\in f_k} w_v.$$

Finally, since ϕ gives the same value $a_{\bar{g}}^{(i)}$ for every variable in $X_\phi(a_{\bar{g}}^{(i)})$, we have that

$$\psi(F_G^{G_2}(s_{k;\phi;A}(f)))=C\cdot\delta\otimes\prod_{v\in f_k} w_v\neq 0,$$

where $C=\prod_{\bar{g}\in G_2}\prod_{i=1}^{\dim A_{\bar{g}}}|X_\phi(a_{\bar{g}}^{(i)})|!=((kn)!)^{\dim A}$.

We are left with showing that $F_G^{G_2}(s_{k;\phi;A}(f))$ is an identity of $E(B)$. Suppose that $F_G^{G_2}(s_{k;\phi;A}(f))$ is a nonidentity of $E(B)$. Hence there is a nonzero almost G_2-evaluation ψ on $E(B)$. As $\psi(F_G^{G_2}(s_{k;\phi;A}(f)))\neq 0$, there are two permutations

$$\sigma \in \prod_{\bar{g}\in G_2}\prod_{i=1}^{\dim A_{\bar{g}}} S_{X_\phi(a_{\bar{g}}^{(i)})},\ \tau \in \prod_{\bar{g}\in G_2}\prod_{i=1}^{kn} S_{X_{\bar{g},i}}$$

such that

$$\psi(F_G^{G_2}(f_k(\sigma\tau(X),Y,Z)))\neq 0.$$

Clearly, $\sigma\tau(X_i)=\sigma(X_i)$ and σ preserves the G_2-degree. By Proposition 5.5 and the choice of k, there is some $i_0\in\{1,\dots,k\}$ such that for every $x_{\bar{g}}\in\sigma(X_{i_0})$ we have that $\deg\psi(x_{\bar{g}})=\bar{g}$. As a result, as p is an identity of B, we can deduce that $\psi(p^*(\sigma(X_{i_0})))=0$ and so also $\psi(f(\sigma(X_{i_0}),Y,Z))=0$. This clearly forces that $\psi(F_G^{G_2}(f_k(\sigma\tau(X),Y,Z)))=0$, hence reaching a contradiction. □

We may extend Theorem 5.2 to full $G_2=\mathbb{Z}_2\times G$-graded algebras.

Theorem 5.6. *Let A and B be finite dimensional G_2-graded algebras over F. Suppose A and B are full. If $E(A)$ and $E(B)$ are G-graded PI-equivalent then the semisimple parts A_{ss} and B_{ss} are isomorphic as G_2-algebras.*

Proof. For the proof we shall combine the constructions in Section 3 and Section 4, that is for nonaffine ungraded algebras and for affine G-graded algebras, together with the Theorem 5.2. For each G_2-graded simple algebra A_i we let $\mathcal{B}_{A_i}$ be a basis of A_i whose elements are G_2-homogeneous of the form $\{u_h\otimes e_{r,s}\}$. Let K_i denote a nonzero product of the elements in $\mathcal{B}_{A_i}$. We refer to these elements as *designated elements*. Each basis element is bordered by basis elements where for convenience we may assume all but possibly one are of the form $1\otimes e_{i,j}$. As usual we refer to these as *frame elements*. We may use one of the frame elements so the value of the monomial is an idempotent δ of A. We denote this product by Z_i. We let $Z_{i,j}$, $j=1,\dots,k$ be a duplicate of the monomial Z_i and let $\bar{Z}_{i,k}=Z_{i,1}\cdot Z_{i,2}\cdots Z_{i,k}$. Here, k is a large integer which needs to be determined. We let $\Theta_l=(a,\dots,a)$ be the k-tuple where a is the l-th element appearing in the monomial K_i. Since the algebra A is full, we have up to ordering of the G_2-graded simple components of A a nonvanishing product $\bar{Z}_{1,k}\cdot w_1\cdot\bar{Z}_{2,k}\cdots w_{q-1}\cdot\bar{Z}_{q,k}\neq 0$. For every $\bar{g}\in G_2$ we consider k *small sets* , each consisting of $\dim_F(A_{ss})_{\bar{g}}$ designated elements where the j-th small set consists of the designated elements in $K_{1,j},\dots,K_{q,j}$. We have as in previous cases that any nontrivial permutation on a small set leads to a zero product. Our next step is to tensor even elements with the identity of E (the Grassmann algebra), and odd elements with different generators of E. Note that the product remains nonzero. As in previous cases we will view the elements obtained as G-graded elements but for convenience we will still refer to them using the adjective even or odd. Moreover we shall refer as small sets, a set of the form $(1\otimes a_1,\dots,1\otimes a_m)$ where $(a_1,\dots,a_m)$ is a small set of even homogeneous elements of degree $(0,g)$, $g\in G$ or a set the form $(\epsilon_1\otimes b_1,\dots,\epsilon_m\otimes b_m)$ where $(b_1,\dots,b_m)$ is a small set of odd homogeneous elements of degree $(1,g)$, $g\in G$. By abuse of notation we keep the notation Θ_l after multiplying the basis elements with Grassmann generators.

Next we alternate and symmetrize small sets of even and odd elements respectively. Then we symmetrize sets $\Theta_l = (a, \dots, a)$ where a is even and alternate sets $\Theta_t = (b, \dots, b)$ where b is odd. One shows the product is nonzero.

Next we replace the designated elements by X variables, the frames by Y's and the bridges by W's where we forget the $\mathbb{Z}_2$-degree, that is X, Y, W are G-graded variables. Clearly by construction we have a nonidentity f of A. Let us denote the nonzero evaluation above by ϕ. As in previous cases with such polynomial one shows that if B_{ss} does not cover A_{ss} as G_2-algebras then f is a nonidentity of $E(A)$ and an identity of $E(B)$ as a G-graded algebras. Thus, since we are assuming $E(A)$ and $E(B)$ are G-graded PI-equivalent we have that A and B cover each other as G_2-graded algebras. We conclude that up to permutation of the simple components of B we have $A \cong A_1 \times \cdots \times A_q \oplus J_A$ and $B \cong B_1 \times \cdots \times B_q \oplus J_B$ where $\dim_F(A_j)_g = \dim_F(B_j)_g$, $g \in G_2$. We want to prove there is a permutation on the G_2-graded simple components of B such that $A_j \cong B_j$ as G_2-graded algebras.

Recall from the Theorem 5.2 above that if $\dim_F(A_j)_g = \dim_F(B_{j'})_g$, all $g \in G_2$, for some j and j', there exists a G-polynomial $p_{j,j'}$ which is a G-graded nonidentity of $E(A_j)$ and an identity of $E(B_{j'})$ unless A_j and $B_{j'}$ are G_2-graded isomorphic. Moreover, we may assume the value of the polynomial $p_{j,j'}$ is the idempotent δ of A we fixed above. Denote by $p_i = \prod_{j'} p_{i,j'}$. We note that p_i is a G-polynomial nonidentity of $E(A_i)$ and an identity of $E(B_{j'})$ for every G_2-graded simple algebra whose dimension of the homogeneous G_2-components are equal to the corresponding dimensions of the homogeneous components of A_j but is not isomorphic to A_j. Finally, we insert to the right of every monomial $Z_{i,l}$ a copy of the polynomial p_i with disjoint variables. The polynomial obtained m_A is a G-graded nonidentity of $E(A)$. By assumption it is a nonidentity of $E(B)$, which forces the existence of a permutation on the G_2-graded simple components of B such that $A_j \cong B_j$ as G_2-graded algebras. This completes the proof. $\square$

We can now complete the proof of Theorem 5.1 as in the proof of Theorem 1.1, that is by performing Steps $0-4$ on the set of finite dimensional G_2-graded algebras A with $\mathrm{Id}_G(E(A)) = \Gamma$ (see Section 2). Details are omitted.

References

[Aljadeff and Haile 2014] E. Aljadeff and D. Haile, "Simple G-graded algebras and their polynomial identities", *Trans. Amer. Math. Soc.* **366**:4 (2014), 1749–1771. MR Zbl

[Aljadeff and Kanel-Belov 2010] E. Aljadeff and A. Kanel-Belov, "Representability and Specht problem for G-graded algebras", *Adv. Math.* **225**:5 (2010), 2391–2428. MR Zbl

[Aljadeff and Karasik 2022] E. Aljadeff and Y. Karasik, "On generic G-graded Azumaya algebras", *Adv. Math.* **399** (2022), Paper No. 108292, 42. MR Zbl

[Aljadeff et al. 2016] E. Aljadeff, A. Kanel-Belov, and Y. Karasik, "Kemer's theorem for affine PI algebras over a field of characteristic zero", *J. Pure Appl. Algebra* **220**:8 (2016), 2771–2808. MR Zbl

[Aljadeff et al. 2020] E. Aljadeff, A. Giambruno, C. Procesi, and A. Regev, *Rings with polynomial identities and finite dimensional representations of algebras*, American Mathematical Society Colloquium Publications **66**, American Mathematical Society, Providence, RI, 2020. MR Zbl

[Bahturin and Yasumura 2019] Y. Bahturin and F. Yasumura, "Graded polynomial identities as identities of universal algebras", *Linear Algebra Appl.* **562** (2019), 1–14. MR Zbl

[Bakhturin et al. 2008] Y. A. Bakhturin, M. V. Zaĭtsev, and S. K. Segal, "Finite-dimensional simple graded algebras", *Mat. Sb.* **199**:7 (2008), 21–40. In Russian; translated in *Sb. Math.* **199**:7–8 (2008), 965–983. MR

[Belov 2010] A. Y. Belov, "Local finite basis property and local representability of varieties of associative rings", *Izv. Ross. Akad. Nauk Ser. Mat.* **74**:1 (2010), 3–134. In Russian; translated in *Izv. Math.* **74**:01 (2010), 1–126. MR Zbl

[Belov-Kanel et al. 2010] A. Belov-Kanel, L. Rowen, and U. Vishne, "Structure of Zariski-closed algebras", *Trans. Amer. Math. Soc.* **362**:9 (2010), 4695–4734. MR Zbl

[Belov-Kanel et al. 2011] A. Belov-Kanel, L. Rowen, and U. Vishne, "Application of full quivers of representations of algebras, to polynomial identities", *Comm. Algebra* **39**:12 (2011), 4536–4551. MR Zbl

[Belov-Kanel et al. 2012] A. Belov-Kanel, L. H. Rowen, and U. Vishne, "Full quivers of representations of algebras", *Trans. Amer. Math. Soc.* **364**:10 (2012), 5525–5569. MR Zbl

[David 2012] O. David, "Graded embeddings of finite dimensional simple graded algebras", *J. Algebra* **367** (2012), 120–141. MR Zbl

[Giambruno and Zaicev 1998] A. Giambruno and M. Zaicev, "On codimension growth of finitely generated associative algebras", *Adv. Math.* **140**:2 (1998), 145–155. MR Zbl

[Giambruno and Zaicev 2005] A. Giambruno and M. Zaicev, *Polynomial identities and asymptotic methods*, Mathematical Surveys and Monographs **122**, American Mathematical Society, Providence, RI, 2005. MR Zbl

[Kanel-Belov and Rowen 2005] A. Kanel-Belov and L. H. Rowen, *Computational aspects of polynomial identities*, Research Notes in Mathematics **9**, A K Peters, Ltd., Wellesley, MA, 2005. MR Zbl

[Karasik 2019] Y. Karasik, "G-graded central polynomials and G-graded Posner's theorem", *Trans. Amer. Math. Soc.* **372**:8 (2019), 5531–5546. MR Zbl

[Kemer 1987] A. R. Kemer, "Finite basability of identities of associative algebras", *Algebra i Logika* **26**:5 (1987), 597–641, 650. MR Zbl

[Prochezi 2016] K. Prochezi, "The geometry of polynomial identities", *Izv. Ross. Akad. Nauk Ser. Mat.* **80**:5 (2016), 103–152. In Russian; translated in *Izv. Math.* **80**:05 (2016), 910–953. MR

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aljadeff@technion.ac.il *Department of Mathematics, Technion, Haifa, Israel*

theyakov@gmail.com *Department of Mathematics, Technion, Haifa, Israel*

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Projective orbifolds of Nikulin type

Chiara Camere, Alice Garbagnati, Grzegorz Kapustka and Michał Kapustka

We study projective irreducible symplectic orbifolds of dimension four that are deformations of partial resolutions of quotients of hyperkähler manifolds of $K3^{[2]}$-type by symplectic involutions; we call them orbifolds of Nikulin type. We first classify those projective orbifolds that are really quotients, by describing all families of projective fourfolds of $K3^{[2]}$-type with a symplectic involution and the relation with their quotients, and then study their deformations. We compute the Riemann–Roch formula for Weil divisors on orbifolds of Nikulin type and using this we describe the first known locally complete family of singular irreducible symplectic varieties as double covers of special complete intersections $(3, 4)$ in $\mathbb{P}^6$.

1. Introduction

One of the three building blocks [Beauville 1983] of Ricci-flat compact Kähler manifolds, together with Abelian varieties and Calabi–Yau manifolds, are irreducible holomorphic symplectic manifolds, i.e., simply connected manifolds X such that $H^{2,0}(X) = \mathbb{C}\omega_X$ is spanned by a symplectic holomorphic form. This kind of manifolds, also known as irreducible hyperkähler, has been deeply studied ever since the foundational works of Beauville [1983], Bogomolov [1978] and Fujiki [1983].

The first and lower dimensional examples of irreducible holomorphic symplectic manifolds are $K3$ surfaces; a second series of deformation families is given by manifolds of $K3^{[n]}$-type, i.e., deformations of Hilbert schemes $W^{[n]}$ of n points on a $K3$ surface W. Together with generalized Kummer manifolds of dimension $2n$ and two deformation families in dimension six and ten constructed by O'Grady, these are all the infinitely many deformation families of irreducible holomorphic manifolds which are currently known.

A natural attempt at constructing new families, already described by Fujiki [1983], is to study quotients of irreducible holomorphic symplectic manifolds by finite symplectic group actions i.e., those actions which preserve the symplectic form. Symplectic involutions σ on smooth $K3$ surfaces W are nowadays well understood thanks to foundational works of Nikulin [1979a], Morrison [1984] and then of van Geemen and Sarti [2007]. The quotient W/σ admits a resolution of singularities with Picard number ≥ 8 which is again a $K3$ surface, called a Nikulin surface. In higher dimension the quotient of a smooth manifold of $K3^{[n]}$-type by a symplectic action does not admit any desingularization being irreducible holomorphic symplectic.

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More recently, starting from work of Beauville [2000], several authors began to study the question of how to enlarge the class of irreducible holomorphic symplectic manifolds while keeping valid most of their distinguished geometrical properties. One of the main directions has been to admit symplectic varieties with mild singularities. Many definitions can be found in the literature and we refer the interested reader to the nice survey by Perego [2020], and references therein, for more details. Beauville considered the class of *symplectic varieties* which admit a symplectic form ω on the smooth locus and have symplectic singularities i.e., singularities such that the holomorphic 2-form ω extends to any resolution. Nowadays, a class which has attracted great attention is that of *irreducible symplectic orbifolds* [Campana 2004]. They naturally appear as building blocks in the generalization of Beauville–Bogomolov decomposition theorem to compact connected Kähler Ricci-flat orbifolds. A compact Kähler orbifold Y is said irreducible symplectic if $Y \setminus \mathrm{Sing}(Y)$ is simply connected and admits a unique, up to a scalar multiple, nondegenerate holomorphic 2-form.

This paper focuses on a special deformation family of irreducible symplectic orbifolds, which we call *orbifolds of Nikulin type*. Those are constructed as deformations of *Nikulin orbifolds*, whose construction mimics that of Nikulin surfaces (see Definition 3.1). The quotient X/σ of a fourfold X of $K3^{[2]}$-type by a symplectic involution σ is singular along a $K3$ surface and 28 points. As we mentioned above, X/σ does not admit a crepant resolution, but one can partially resolve it blowing up the singular $K3$ surface. This partial resolution Y is an irreducible symplectic orbifold with 28 terminal points, which we call Nikulin orbifold. By [Beauville 2000], orbifolds of Nikulin type are also irreducible symplectic varieties, and the two moduli spaces constructions in [Bakker and Lehn 2022; Menet 2020] agree for this deformation family. Examples were already studied by Markushevich and Tikhomirov [2007] and the main properties of the whole deformation family have been described by Menet [2015] and Menet and Rieß [2020; 2021]. It is worth noticing that not all orbifolds of Nikulin type are Nikulin orbifolds, in fact the latter sit in a family of codimension one. As in the case of $K3$ surfaces, orbifolds of Nikulin type are Kähler and irreducible symplectic but in general not projective. The projective ones correspond to divisors in the period domain of orbifolds of Nikulin type [Menet 2020; Bakker and Lehn 2022].

In many aspects the theory of irreducible symplectic manifolds/varieties/ orbifolds is a generalization to higher dimensions of that of $K3$ surfaces. Most notably, the group $H^2(X, \mathbb{Z})$ can be endowed with an integral quadratic form q_X, so-called Beauville–Bogomolov–Fujiki form (for short, BBF form), and it is a lattice L of signature $(3, b_2(X) - 3)$, which is a topological invariant of the deformation family; the existence of this lattice structure allows to study moduli spaces of irreducible symplectic manifolds of a fixed deformation type through periods since a global Torelli theorem, analogous to the one for $K3$ surfaces, also holds. However, a remarkable difference with the theory of $K3$ surfaces is the lack of projective models for general higher dimensional algebraic examples. They are crucial for the understanding of the geometric behavior of these varieties. For this reason, in the early development of the theory of irreducible holomorphic symplectic manifolds, a lot of effort has been put into constructing so called locally complete families of these, i.e., general elements in the family of manifolds with a given degree and type of polarization. Historically, the first known locally complete families of projective irreducible holomorphic symplectic manifolds were the family of Fano varieties of smooth cubic fourfolds, shown to be of $K3^{[2]}$-type by Beauville and Donagi [1985], and the family of double EPW sextics, again of $K3^{[2]}$-type, discovered by O'Grady [2005]. A few more families have been constructed in [Debarre and Voisin 2010; Iliev and Ranestad 2001; Iliev et al. 2019; Lehn et al. 2017; Bayer et al. 2021]: all are algebraic manifolds of $K3^{[n]}$-type for some n and their families have codimension one inside their respective moduli spaces. However, in the case of singular orbifolds no locally complete family has been constructed so far.

The main aim of this paper is to provide tools to study the explicit geometry of orbifolds of Nikulin type. We do it by addressing the following problems that we discuss separately in the remaining part of the introduction:

(1) Classify projective fourfolds of $K3^{[2]}$-type with symplectic involutions and related Nikulin orbifolds.

(2) Provide a Riemann–Roch formula for linear systems on orbifolds of Nikulin type.

(3) Describe a locally complete family of orbifolds of Nikulin type.

1A. ***Classification of polarized Nikulin orbifolds.*** The first aim of this paper is to describe the families of projective Nikulin orbifolds, i.e., the algebraic Noether–Lefschetz locus in the family of Nikulin orbifolds, in analogy with what has been done by van Geemen and Sarti for projective Nikulin surfaces. This is achieved in two steps: first we classify all families (infinitely many of those) of projective fourfolds of $K3^{[2]}$-type X carrying a symplectic involution σ; then we describe the corresponding families of projective Nikulin orbifolds Y.

In Section 2, we look at symplectic involutions σ on projective fourfolds of $K3^{[2]}$-type of degree $2d$. We describe their possible Picard lattices and transcendental groups; as a consequence we identify their families in terms of lattice polarized families of fourfolds of $K3^{[2]}$-type. We prove the following result (see Table 1), which is the analogue of the result [van Geemen and Sarti 2007, Proposition 2.2] for $K3$ surfaces with a symplectic involution.

Theorem 1.1. *Let X be a generic projective fourfold of $K3^{[2]}$-type admitting a symplectic involution. Then the pair $(\mathrm{T}_X, \mathrm{NS}(X))$ of the transcendental lattice and the Néron–Severi group of X is one of the following*:

- $(U^{\oplus 2} \oplus E_8(-2) \oplus \langle -2d\rangle \oplus \langle -2\rangle,\ \Lambda_{2d})$.
- $(U^{\oplus 2} \oplus D_4(-1) \oplus \langle -2d\rangle \oplus \langle -2\rangle^{\oplus 5},\ \Lambda_{2d})$, *with* $d \equiv 1 \bmod 2$.
- $(U^{\oplus 2} \oplus E_8(-2) \oplus K_d,\ \Lambda_{2d})$ *with* $d \equiv 3 \bmod 4$.
- $(U^{\oplus 2} \oplus D_4(-1) \oplus \langle -2d\rangle \oplus \langle -2\rangle^{\oplus 5},\ \widetilde{\Lambda}_{2d})$ *with* $d \equiv 0 \bmod 2$.

Where the lattices involved are defined in the notation in Section 2A and d is a positive integer.

Vice versa if X is a projective fourfold of $K3^{[2]}$-type such that $\mathrm{NS}(X)$ *is isometric either to Λ_{2d} or to $\widetilde{\Lambda}_{2d}$, then it admits a symplectic involution.*

In Section 2B we show that the general member of the above lattice polarized families can be described either as Hilbert scheme of two points on a $K3$ surface or as moduli space of (possibly twisted) sheaves on a $K3$ surface; see Table 2. In both cases the $K3$ surfaces involved lie in 12-dimensional families of lattice polarized $K3$ surfaces and are resolution of singular $K3$ surfaces with 7 nodes.

In Section 3 we consider the quotient X/σ and the corresponding Nikulin orbifold Y. The knowledge of the Néron–Severi group and of the transcendental lattice of X allows one to compute the ones of Y and thus the family of fourfolds of $K3^{[2]}$-type X determines the family of the Nikulin orbifolds Y. In particular we prove the following result (see Table 3), which is the analogue of the result [Garbagnati and Sarti 2008, Corollary 2.2] for Nikulin surfaces.

Theorem 1.2. *Let X be a generic projective fourfold of $K3^{[2]}$-type admitting a symplectic involution σ and Y be the corresponding Nikulin orbifold. Then the pair $(\mathrm{T}_X, \mathrm{NS}(X))$ determines uniquely the transcendental lattice T_Y of Y and vice versa T_Y determines uniquely the pair $(\mathrm{T}_X, \mathrm{NS}(X))$. See Table 3 for the explicit description of T_Y and of its relation with $(\mathrm{T}_X, \mathrm{NS}(X))$.*

In Section 3C we study the $K3$ surface S in the fixed locus of the involution σ on the fourfold of $K3^{[2]}$-type X: we show that there is an isometry between $\mathrm{T}_S \otimes \mathbb{Q}$ and $\mathrm{T}_Y \otimes \mathbb{Q}$, where $\mathrm{T}_\bullet \otimes \mathbb{Q}$ is the transcendental lattice with rational coefficients and Y is the Nikulin orbifold as above (see Proposition 3.16). In particular the Picard number of S is at least 8. Moreover, we conjecture that this isometry holds also with integer coefficients (Conjecture 3.12). We prove the conjecture for many subfamilies of codimension 1, corresponding to Hilbert scheme of points on $K3$ surfaces with natural symplectic involutions, and for two locally complete algebraic families, see Propositions 3.14 and 3.15.

1B. ***Riemann–Roch formula for Nikulin orbifolds and orbifolds of Nikulin type.*** In Section 4 we find the Riemann–Roch formula on the orbifolds of Nikulin type by following step by step the quotient construction of Nikulin orbifolds. Since $H^2(Y, \mathbb{Z})$ is endowed with the BBF quadratic form q_Y, explicitly computed by Menet [2015], the Riemann–Roch formula for a $\mathbb{Q}$-Cartier Weil divisor D can be stated as a relation between $\chi(D)$ and $q_Y(D)$, in the same spirit of [Gross et al. 2003, Example 23.19] and depends also on the number of points where D fails to be Cartier. Using the results from [Buckley et al. 2013; Blache 1996; Camere et al. 2019a] for 2-factorial orbifolds we prove in Corollary 4.4 and in Proposition 4.5 the following result.

Theorem 1.3. *Let Y be an orbifold of Nikulin type and let $D = \frac{m}{2}L$ be a $\mathbb{Q}$-Cartier Weil divisor on Y, with $m \in \mathbb{Z}$ and $L \in \mathrm{NS}(Y)$; let n be the number of points where D fails to be Cartier. Then*

$$\chi(\mathcal{O}(D)) = \tfrac{3}{8}\big(\tfrac{1}{24}m^4 q_Y(L)^2 + m^2 q_Y(L) + 8\big) - \tfrac{1}{16}n,$$

where q_Y denotes the BBF quadratic form on $H^2(Y, \mathbb{Z})$.

In particular, on any orbifold of Nikulin type Y and for any $D \in \mathrm{NS}(Y)$,

$$\chi(\mathcal{O}(D)) = \tfrac{1}{4}(q_Y(D)^2 + 6q_Y(D) + 12).$$

By applying the previous result to some specific divisors on Y, we obtain the dimensions of projective spaces where the quotient X/σ or its partial resolution Y have a natural projective model; see Theorems 4.9, 4.10 and 4.12 and Table 4.

1C. *A locally complete family of orbifolds of Nikulin type.* To obtain a locally complete family, we need to understand the projective model of the general elements of a family of irreducible symplectic varieties with a given type of polarization. In Section 5, we describe a locally complete family of polarized orbifolds of Nikulin type of BBF degree 2 (the least possible). As already remarked, this is the first known description of a locally complete family of polarized singular irreducible symplectic varieties; the reader should see this construction as the analogue of O'Grady's double EPW sextics. In this case the analogue of a EPW sextic will be a special complete intersection $(3, 4)$ in $\mathbb{P}^6$.

Theorem 1.4. *The general element Y in a family of orbifolds of Nikulin type with a polarization of BBF degree* 2 *and divisibility* 1 *is a double cover of a special complete intersection* $(3, 4)$ *in* $\mathbb{P}^6$ *branched along a surface of degree* 48.

In Section 5D we discuss the reciprocal of the theorem by describing the possible complete intersections $(3, 4)$ using the Beilinson resolution (see also Problem 5.10). Our strategy to prove Theorem 1.4 is the following. Special examples of orbifolds of Nikulin type of BBF degree 2 are constructed as quotients by a symplectic involution of fourfolds X of $K3^{[2]}$-type with Néron–Severi group isometric to $\widetilde{\Lambda}_4$, which is an extension of index two of $\langle 4\rangle \oplus E_8(-2)$. The polarization of BBF degree 4 on X which is orthogonal to the $E_8(-2)$ summand gives a $2:1$ map (see [Iliev et al. 2017]) to an EPW quartic in the cone $C(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^9$. The symplectic involution on X is then induced by an involution on the linear system of the polarization, i.e., on $\mathbb{P}^9$. After projecting from the $\mathbb{P}^2 \subset \mathbb{P}^9$ which is a component of the fixed locus of the involution on $\mathbb{P}^9$, we obtain a complete intersection $(3, 4)$ in $\mathbb{P}^6$ that is singular in codimension 2 along a surface of degree 52. From the results in Sections 3 and 4 we deduce that the image of the projection is the projective model of the quotient of X by the symplectic involution. By deforming this example and knowing part of the monodromy group of orbifolds of Nikulin type (see [Menet and Rieß 2020]), we prove that a general orbifold of BBF degree 2 as above has a similar description.

2. Fourfolds of $K3^{[2]}$-type with a symplectic involution

We are interested in fourfolds of $K3^{[2]}$-type admitting a symplectic involution and mainly in the projective ones. We will describe the general member of families of fourfolds satisfying these conditions first in a lattice theoretic way and then giving a model as (twisted) moduli space of sheaves on a $K3$ surface. From now on let X be a fourfold of $K3^{[2]}$-type and σ be a symplectic involution on X.

2A. *Lattice theoretic description of X.* Let us fix some notation and recall preliminary results on lattices:

- The lattice U is the unique even unimodular lattice of rank 2 and signature $(1, 1)$; we will denote by $\{u_1, u_2\}$ a basis such that $u_1^2 = u_2^2 = 0$ and $u_1 u_2 = 1$.

- The lattice E_8 is the unique even unimodular positive definite lattice of rank 8.

- Given a lattice M and an integer $n \in \mathbb{Z}$, $M(n)$ is the lattice obtained multiplying the bilinear form of M by n.

- We denote by $\{b_1, \dots b_8\}$ the basis of $E_8(-2)$ such that: $b_i^2 = -4$, $i = 1, \dots, 8$; $b_j b_{j+1} = 2$, $j = 1, \dots, 6$; $b_3 b_8 = 2$; the other intersections are zeros.

- The lattice N, called Nikulin lattice, is an even negative definite rank 8 lattice. It is generated by the classes r_i, $i = 1, \dots, 8$ such that $r_i^2 = -2$, $r_i r_j = 0$ and by the class $\frac{1}{2}\left(\sum_{i=1}^{8} r_i\right)$.

- For $n \in \mathbb{Z}$, $u(n)$ is the discriminant form of $U(n)$; for each $m \in \mathbb{Z}$ and $\alpha \in \mathbb{Q}$, $\mathbb{Z}_m(\alpha)$ is the discriminant cyclic group $\mathbb{Z}_m$ endowed with the quadratic form taking value α on a generator. For short, the discriminant quadratic form of $\mathbb{Z}_m\left(\pm\frac{1}{m}\right)$ is denoted by $\left(\pm\frac{1}{m}\right)$.

- The discriminant form of N is $u(2)^{\oplus 3}$ and the discriminant form of $E_8(-2)$ is $u(2)^{\oplus 4}$; see [Nikulin 1983, page 1414].

- The lattice $D_4(-1)$ is the rank 4 negative definite lattice whose bilinear form on the basis $\{d_1, d_2, d_3, d_4\}$ is $d_i^2 = -2$, $d_i d_2 = 1$, $i = 1, 3, 4$, $d_i d_j = 0$ otherwise. Its discriminant group is $(\mathbb{Z}/2\mathbb{Z})^2$ and its discriminant form is called $v(2)$, see e.g., [Nikulin 1979b, Section 8].

- The lattice L_{K3} is the unique even unimodular lattice of rank 22 and signature $(3, 19)$ and is isometric to $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$.

- The lattice $L := L_{K3} \oplus \langle -2 \rangle$ and its discriminant form is $(-\frac{1}{2})$. We will denote by δ the generator of the lattice $\langle -2 \rangle$, orthogonal to L_{K3} in L.

- For every positive integer d, the lattice Λ_{2d} is isometric to $\langle 2d \rangle \oplus E_8(-2)$. We denote by h a generator of the summand $\langle 2d \rangle$.

- For every even positive integer d, the lattice $\widetilde{\Lambda}_{2d}$ is the unique overlattice of index 2 of Λ_{2d} in which $\langle 2d \rangle$ and $E_8(-2)$ are primitively embedded.

- The lattice K_d is the negative definite lattice with the following quadratic form

$$\begin{bmatrix} -\frac{d+1}{2} & 1 \\ 1 & -2 \end{bmatrix}, \qquad d \equiv 1 \bmod 2.$$

- The lattice H_d is the indefinite lattice with the following quadratic form

$$\begin{bmatrix} \frac{d-1}{2} & 1 \\ 1 & -2 \end{bmatrix}, \qquad d \equiv 1 \bmod 2.$$

- The divisibility $\operatorname{div}(v)$ of $v \in M$ is the generator of the ideal $(vw \mid w \in M) \subset \mathbb{Z}$.

Moreover, since for all the considered varieties there is a canonical isomorphism between the Picard lattice and the Néron–Severi group, we always refer to the Néron–Severi group, even to indicate the Picard lattice.

Proposition 2.1. *A fourfold X of $K3^{[2]}$-type admits a symplectic involution if and only if $E_8(-2)$ is primitively embedded in* $\mathrm{NS}(X)$.

Moreover, if X is projective then there exists a positive $d \in \mathbb{N}$ such that $\Lambda_{2d} \subset \mathrm{NS}(X)$. The Néron–Severi group of the very general element in the family of $\langle 2d \rangle$-polarized fourfolds of $K3^{[2]}$-type admitting a symplectic involution is an overlattice of finite index (*possibly equal to* 1) *of Λ_{2d}, with the property that $E_8(-2)$ is primitively embedded in it.*

Proof. The first statement is proved by Mongardi [2012]. If X is projective, then it admits an ample divisor, which has necessarily a positive BBF degree. Since $E_8(-2)$ is negative definite, it follows that $\Lambda_{2d} \subset \mathrm{NS}(X)$ and that if the Picard number of X is the minimal possible, i.e., 9, then $\mathrm{NS}(X)$ is an overlattice of finite index (possibly equal to 1) of Λ_{2d}, with the property that $E_8(-2)$ is primitively embedded in it. □

Lemma 2.2 [van Geemen and Sarti 2007, Proposition 2.2]. *The overlattices of Λ_{2d} containing primitively both $\langle 2d \rangle$ and $E_8(-2)$ are*:

(1) *If $d \equiv 1 \bmod 2$ only Λ_{2d} itself.*

(2) *If $d \equiv 0 \bmod 2$ either Λ_{2d} or the unique overlattice $\widetilde{\Lambda}_{2d}$ of index 2 of Λ_{2d} in which $\langle 2d \rangle$ and $E_8(-2)$ are primitively embedded.*

The discriminant forms of the lattices Λ_{2d} and $\widetilde{\Lambda}_{2d}$ are $\left(\frac{1}{2d}\right) \oplus u(2)^{\oplus 4}$ and $\left(\frac{1}{2d}\right) \oplus u(2)^{\oplus 3}$. The lattices Λ_{2d} and $\widetilde{\Lambda}_{2d}$ admit a unique embedding in L_{K3} (*up to isometry*).

Proof. The uniqueness of the overlattices is proved in [van Geemen and Sarti 2007, Proposition 2.2], and their discriminant forms are computed in [Camere and Garbagnati 2020, Corollary 3.7]. We briefly recall the proofs here. The lattice Λ_{2d} is described in the list of lattices at the beginning of the section: the discriminant form on $A_{\Lambda_{2d}} = A_{\langle 2d \rangle} \oplus A_{E_8(-2)}$ is $\left(\frac{1}{2d}\right) \oplus u(2)^{\oplus 4}$. We denote $h, u_{i,j}$ for $i = 1, \dots, 4$, $j = 1, 2$ a basis of $A_{\Lambda_{2d}}$ on which the discriminant form is $\left(\frac{1}{2d}\right) \oplus u(2)^{\oplus 4}$. The overlattices $\widetilde{\Lambda}_{2d}$ in which $\langle 2d \rangle$ and $E_8(-2)$ are primitively embedded correspond to isotropic subgroups H of $A_{\Lambda_{2d}}$ which have a nontrivial intersection with both $A_{\langle 2d \rangle}$ and $A_{E_8(-2)}$ in $A_{\Lambda_{2d}}$, by [Nikulin 1979b, Proposition 1.4.1]. So H can be chosen to be generated by $dh + v$, where $v \in A_{E_8(-2)}$ is such that $v^2 = 0$ or 1 respectively when $d \equiv 0 \bmod 4$ or $d \equiv 2 \bmod 4$. We suppose that $d \equiv 0 \bmod 4$ and we can assume that $v = u_{1,1}$. Since $H^{\perp} = \langle h + u_{1,2}, u_{1,1}, u_{i,j} \mid i = 2, 3, 4,\ j = 1, 2 \rangle$, $\widetilde{\Lambda}_{2d}$ has discriminant quadratic form $\left(\frac{1}{2d}\right) \oplus u(2)^{\oplus 3}$. The case $d \equiv 2 \bmod 4$ is completely analogous. □

In [van Geemen and Sarti 2007] an explicit basis for the lattice $\widetilde{\Lambda}_{2d}$ is given:

- If $d \equiv 2 \bmod 4$, the lattice $\widetilde{\Lambda}_{2d}$ is generated by the generators of Λ_{2d} and by the class $\frac{1}{2}(h + b_1)$.
- If $d \equiv 0 \bmod 4$, the lattice $\widetilde{\Lambda}_{2d}$ is generated by the generators of Λ_{2d} and by the class $\frac{1}{2}(h + b_1 + b_3)$.

Corollary 2.3. *Let X be a very general element in a family of* (*possibly not projective*) *fourfolds of $K3^{[2]}$-type admitting a symplectic involution σ, then $\mathrm{NS}(X) \simeq E_8(-2)$ and vice versa if X is a fourfold of $K3^{[2]}$-type such that $\mathrm{NS}(X) \simeq E_8(-2)$, then X is nonprojective and it admits a symplectic involution.*

Let X be a very general element in a family of projective fourfolds of $K3^{[2]}$-type admitting a symplectic involution σ. Then either $\mathrm{NS}(X) \simeq \Lambda_{2d}$ for a certain integer $d > 0$ or $\mathrm{NS}(X) \simeq \widetilde{\Lambda}_{2d}$ for a certain even integer $d > 0$.

Vice versa if X is a fourfold of $K3^{[2]}$-type such that $\mathrm{NS}(X)$ is isometric either to Λ_{2d} for an integer $d > 0$ or to $\widetilde{\Lambda}_{2d}$ for an even integer $d > 0$, then X is projective and admits a symplectic involution.

We observe that $E_8(-2)$ admits a unique primitive embedding in L, whose orthogonal is $U^{\oplus 3} \oplus E_8(-2) \oplus \langle -2\rangle$.

In order to determine the families of projective fourfolds of $K3^{[2]}$-type admitting a symplectic involution, we consider all possible primitive embeddings of the lattices Λ_{2d} and $\widetilde{\Lambda_{2d}}$ into L.

Proposition 2.4. *For any integer $d > 0$ the lattice Λ_{2d} admits, up to isometry of L, the following primitive embeddings into L:*

(1) *j_1 such that $j_1(\Lambda_{2d})^\perp \simeq T_{2d,1} := U^{\oplus 2} \oplus E_8(-2) \oplus \langle -2d\rangle \oplus \langle -2\rangle$.*

(2) *If $d \equiv 1 \bmod 2$, j_2 such that $j_2(\Lambda_{2d})^\perp \simeq T_{2d,2} := U^{\oplus 2} \oplus D_4(-1) \oplus \langle -2d\rangle \oplus \langle -2\rangle^{\oplus 5}$.*

(3) *If $d \equiv 3 \bmod 4$, j_3 such that $j_3(\Lambda_{2d})^\perp \simeq T_{2d,3} := U^{\oplus 2} \oplus E_8(-2) \oplus K_d$.*

For any $d \equiv 0 \bmod 2$, $\widetilde{\Lambda}_{2d}$ admits a unique primitive embedding $\tilde{j}$ into L, with orthogonal isometric to $\widetilde{T}_{2d} := U^{\oplus 2} \oplus D_4(-1) \oplus \langle -2d\rangle \oplus \langle -2\rangle^{\oplus 5}$.

Proof. First we study possible primitive embeddings of Λ_{2d} inside L. The first embedding j_1 is simply obtained by composing the embedding of Λ_{2d} inside L_{K3} with the embedding of this one inside L. This is unique up to isometry if $d \equiv 0 \bmod 2$.

When $d \equiv 1 \bmod 2$, an application of [Nikulin 1979b, Proposition 1.15.1] shows that there is a second possibility: indeed, in this case $A_{\Lambda_{2d}}$ contains a subgroup H of order two to which the discriminant form restricts as $\left(-\frac{1}{2}\right)$. Standard computations in this case produce the embedding j_2 if $d \equiv 1 \bmod 4$, and the embeddings j_2 and j_3 if $d \equiv 3 \bmod 4$. Up to isometry these are the only possibilities.

Concerning the primitive embeddings of $\widetilde{\Lambda}_{2d}$, $\tilde{j}$ is again obtained by composing the embedding of $\widetilde{\Lambda}_{2d}$ inside L_{K3} with the embedding of this one inside L. The fact that it is the only possible one comes by an application of [loc. cit., Proposition 1.15.1]: we have $A_L \simeq \mathbb{Z}_2\left(-\frac{1}{2}\right)$, whereas the quadratic form on $A_{\widetilde{\Lambda}_{2d}}$ takes values in $\mathbb{Z}/2\mathbb{Z}$ on any subgroup of order two; as a consequence, the only possible choice for two isometric subgroups inside A_L and $A_{\widetilde{\Lambda}_{2d}}$ is $H = \{0\}$, and the discriminant form of the orthogonal R is exactly $(-q_{\widetilde{\Lambda}_{2d}}) \oplus q_{A_L} = u(2)^{\oplus 3} \oplus \left(-\frac{1}{2d}\right) \oplus \left(-\frac{1}{2}\right)$. From [loc. cit., Proposition 1.8.2], we have $u(2)^{\oplus 3} \oplus \left(-\frac{1}{2d}\right) \oplus \left(-\frac{1}{2}\right) \simeq \left(\frac{1}{2}\right)^{\oplus 3} \oplus \left(-\frac{1}{2}\right)^{\oplus 4} \oplus \left(-\frac{1}{2d}\right)$. Moreover, it is easy to show that $\left(\frac{1}{2}\right)^{\oplus 3} \oplus \left(-\frac{1}{2}\right)^{\oplus 4} \simeq v(2) \oplus \left(-\frac{1}{2}\right)^{\oplus 5}$. The signature of R is $(2, 12)$. The genus of the lattices with signature and discriminant form as the ones of R contains a unique class by [loc. cit., Corollary 1.13.3], and so $R \simeq \widetilde{T}_{2d}$. Moreover, by [loc. cit., Theorem 1.14.2], $O(\widetilde{T}_{2d}) \to O(q_{A_{\widetilde{T}_{2d}}})$ is surjective. By [loc. cit., Proposition 1.15.1], we conclude that $\tilde{j}(\widetilde{\Lambda}_{2d})^\perp \simeq \widetilde{T}_{2d}$ ad that the embedding $\tilde{j}$ is unique up to isometries of L. □

To recap, if X is a very general projective fourfold of $K3^{[2]}$-type admitting a symplectic involution, the possibilities for $\mathrm{NS}(X)$ and T_X are found in Table 1.

condition on d	embedding $\mathrm{NS}(X) \subset L$	$\mathrm{NS}(X)$	T_X
$\forall d \in \mathbb{N}$	j_1	Λ_{2d}	$T_{2d,1} := U^{\oplus 2} \oplus E_8(-2) \oplus \langle -2d \rangle \oplus \langle -2 \rangle$
$d \equiv 1 \bmod 2$	j_2	Λ_{2d}	$T_{2d,2} := U^{\oplus 2} \oplus D_4(-1) \oplus \langle -2d \rangle \oplus \langle -2 \rangle^{\oplus 5}$
$d \equiv 3 \bmod 4$	j_3	Λ_{2d}	$T_{2d,3} := U^{\oplus 2} \oplus E_8(-2) \oplus K_d$
$d \equiv 0 \bmod 2$	$\tilde{j}$	$\widetilde{\Lambda}_{2d}$	$\widetilde{T}_{2d} := U^{\oplus 2} \oplus D_4(-1) \oplus \langle -2d \rangle \oplus \langle -2 \rangle^{\oplus 5}$

Table 1. Possible pairs $(\mathrm{NS}(X), \mathrm{T}_X)$ for general projective fourfolds X of $K3^{[2]}$-type with a symplectic involution.

As observed before, if X is a very general nonprojective fourfold of $K3^{[2]}$-type admitting a symplectic involution, then $\mathrm{NS}(X) = E_8(-2)$ and $\mathrm{T}_X = U^{\oplus 3} \oplus E_8(-2) \oplus \langle -2 \rangle$.

As in the case of the $K3$ surfaces, see e.g., [van Geemen and Sarti 2007], to relate the Néron–Severi group of a manifold with an involution to the one of its quotient by the involution, one uses the explicit description of the isometry induced on the second cohomology group by the involution, and the knowledge of a primitive embedding of the Néron–Severi group in the second cohomology group. Therefore here we describe a choice for this embedding, which will be used in Section 3. The uniqueness of the action induced by the involution and of the embeddings up to isometries of the lattice L, will guarantee that the results in Section 3 are independent by the embedding chosen to make the explicit computations.

Hence, we explicitly fix the embeddings j_a, $a = 1, 2, 3$ and $\tilde{j}$ in L which will be used in the following.

Let X be a fourfold of $K3^{[2]}$-type admitting a symplectic involution ι. Fix a basis of $H^2(X, \mathbb{Z}) \simeq U^{\oplus 3} \oplus E_8(-1) \oplus E_8(-1) \oplus \langle -2 \rangle$: there exists an isometry between $H^2(X, \mathbb{Z})$ and $L = U^{\oplus 3} \oplus E_8(-1) \oplus E_8(-1) \oplus \langle -2 \rangle$ such that the involution $\iota^* \in O(H^2(X, \mathbb{Z}))$ switches the two copies of $E_8(-1)$ and acts as the identity on $U \oplus U \oplus U \oplus \langle -2 \rangle$. We denote by e_i, (resp. f_i), $i = 1, \dots, 8$ a basis of the first (resp. second) copy of $E_8(-1)$ in $E_8(-1) \oplus E_8(-1)$, and by b_i a basis of $E_8(-2)$. We fix two different embeddings of the lattice $E_8(-2)$ in $E_8(-1) \oplus E_8(-1)$:

$$\lambda_+(b_i) = e_i + f_i, \ i = 1, \dots, 8 \quad \text{and} \quad \lambda_-(b_i) = e_i - f_i, \ i = 1, \dots, 8. \tag{2-1}$$

In particular $H^2(X, \mathbb{Z})^{\iota^*} = U^{\oplus 3} \oplus \lambda_+(E_8(-2)) \oplus \langle -2 \rangle \simeq U^{\oplus 3} \oplus E_8(-2) \oplus \langle -2 \rangle$ and $(H^2(X, \mathbb{Z})^{\iota^*})^\perp = \lambda_-(E_8(-2)) \simeq E_8(-2)$.

Let $h \in H^2(X, \mathbb{Z})$ be a ι-invariant primitive class with self-intersection $2d > 0$. Let us denote by $j(h)$ an embedding of h in $H^2(X, \mathbb{Z}) \simeq L$. Since the polarization h is invariant for ι, $j(h) \in H^2(X, \mathbb{Z})^{\iota^*} \simeq U^{\oplus 3} \oplus \lambda_+(E_8(-2)) \oplus \langle -2 \rangle$ and thus it corresponds to a vector of the form $(\underline{u}, \underline{w}, \underline{v}, \underline{x}, \underline{y}, k) \in U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2 \rangle$ such that $\underline{x} = \underline{y}$.

Proposition 2.5. *Let d be a positive integer and let*

$$j_1(h) := \left(\begin{pmatrix} 1 \\ d \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \underline{0}, \underline{0}, 0 \right).$$

The embedding $(j_1, \lambda_-) : \langle 2d\rangle \oplus E_8(-2) \to L$ is a primitive embedding and there exist fourfolds of $K3^{[2]}$-type X_1 such that $\mathrm{NS}(X_1) \simeq (j_1, \lambda_-)(\langle 2d\rangle \oplus E_8(-2)) \simeq \Lambda_{2d}$ and $\mathrm{T}_{X_1} \simeq T_{2d,1}$.

Proof. The embedding (j_1, λ_-) is clearly primitive, hence there exist fourfolds of $K3^{[2]}$-type admitting this lattice as Néron–Severi group. Since j_1 restricts to an embedding of h in U and λ_- restricts to an embedding of $E_8(-2)$ in $E_8(-1) \oplus E_8(-1)$, one can compute separately the orthogonal in the different direct summands, finding that the orthogonal to $\mathrm{NS}(X_1)$ is $\langle -2d\rangle \oplus U \oplus U \oplus \lambda_+(E_8(-2)) \oplus \langle -2\rangle \simeq T_{2d,1}$. □

Proposition 2.6. *Let d be an odd positive integer. Let*

$$j_2(h) := \left(\begin{pmatrix} 2 \\ 2k+2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \underline{e_1}, \underline{e_1}, 1\right) \qquad \text{if } d = 4k+1,$$
$$j_2(h) := \left(\begin{pmatrix} 2 \\ 2k+2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \underline{e_1} + \underline{e_3}, \underline{e_1} + \underline{e_3}, 1\right) \quad \text{if } d = 4k-1.$$

The embedding $(j_2, \lambda_-) : \langle 2d\rangle \oplus E_8(-2) \to L$ is a primitive embedding and there exist fourfolds of $K3^{[2]}$-type X_2 such that $\mathrm{NS}(X_2) \simeq (j_2, \lambda_-)(\langle 2d\rangle \oplus E_8(-2)) \simeq \Lambda_{2d}$ and $\mathrm{T}_{X_2} \simeq T_{2d,2}$.

Proof. The embedding (j_2, λ_-) is clearly primitive, hence there exist fourfolds of $K3^{[2]}$-type admitting this lattice as Néron–Severi group. By Proposition 2.4 there is an embedding of $\langle 2d\rangle \oplus E_8(-2)$ in $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2\rangle$ which is not equivalent to j_1, computed in Proposition 2.5.

Let
$$\underline{x} = \begin{cases} \underline{e_1} & \text{if } d \equiv 1 \mod 4, \\ \underline{e_1} + \underline{e_3} & \text{if } d \equiv 3 \mod 4. \end{cases}$$

By direct computation, the orthogonal lattice $((j_2, \lambda_-)(\Lambda_{2d}))^{\perp}$ is spanned by the following vectors:

$$(\underline{0}, \underline{a_i}, \underline{0}, \underline{0}, \underline{0}, 0), (\underline{0}, \underline{0}, \underline{a_i}, \underline{0}, \underline{0}, 0), \quad i = 1, 2 \text{ where } a_1, a_2 \text{ is a basis of } U;$$
$$\left(\begin{pmatrix} -1 \\ k+1 \end{pmatrix}, \underline{0}, \underline{0}, \underline{0}, \underline{0}, 0\right), \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \underline{0}, \underline{0}, \underline{0}, \underline{0}, 1\right), (\underline{0}, \underline{0}, \underline{0}, \underline{w}, \underline{w}, 0),\ \underline{w} \in (\underline{x})^{\perp}_{E_8(-1)},$$
$$b := (\underline{0}, \underline{0}, \underline{0}, \underline{y}, \underline{y}, 1) \quad \text{with } \underline{y} = \begin{cases} \underline{e_2} & \text{if } d \equiv 1 \mod 4, \\ \underline{e_4} & \text{if } d \equiv 3 \mod 4. \end{cases}$$

One can directly compute the form on the previous basis and hence its discriminant form. By [Nikulin 1979b, Corollary 1.13.3] one obtains that there exists a unique, up to isometry, even lattice with signature (2, 12) and the required discriminant form. Such a lattice is isometric to $T_{2d,2}$. □

Proposition 2.7. *Let d be a positive integer such that $d \equiv 3 \mod 4$. Let*

$$j_3(h) := \left(\begin{pmatrix} 2 \\ (d+1)/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \underline{0}, \underline{0}, 1\right).$$

The embedding $(j_3, \lambda_-) : \langle 2d\rangle \oplus E_8(-2) \to L$ is a primitive embedding and there exist fourfolds of $K3^{[2]}$-type X_3 such that $\mathrm{NS}(X_3) \simeq (j_3, \lambda_-)(\langle 2d\rangle \oplus E_8(-2)) \simeq \Lambda_{2d}$ and $\mathrm{T}_{X_3} \simeq T_{2d,3}$.

Proof. The embedding (j_3, λ_-) is clearly a primitive embedding of $\langle 2d\rangle \oplus E_8(-2)$ in L and hence there exist fourfolds of $K3^{[2]}$-type admitting this lattice as Néron–Severi group. Since j_3 restricts to an embedding of h in $U\oplus\langle -2\rangle$, one can compute the orthogonal of $j_3(h)$ in $U\oplus\langle -2\rangle$, which is generated by $\left(\binom{0}{1},1\right)$ and $\left(\binom{1}{-(d+1)/4},0\right)$, with intersection form equal to K_d, so that $\mathrm{T}_{X_3}\simeq T_{2d,3}$. $\square$

Proposition 2.8. *Let d be an even positive integer. Let*

$$\tilde{j}(h) := \left(\begin{pmatrix}2\\2k\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix},\underline{e_1},\underline{e_1},0\right) \qquad \text{if } d=4k-2,$$
$$\tilde{j}(h) := \left(\begin{pmatrix}2\\2k\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix},\underline{e_1}+\underline{e_3},\underline{e_1}+\underline{e_3},0\right) \quad \text{if } d=4k-4.$$

The embedding $(\tilde{j},\lambda_-):\langle 2d\rangle\oplus E_8(-2)\to L$ is not a primitive embedding and the primitive closure of $(\tilde{j},\lambda_-)(\langle 2d\rangle\oplus E_8(-2))$ is isometric to $\widetilde{\Lambda}_{2d}$. There exist fourfolds of $K3^{[2]}$-type $\widetilde{X}$ such that $\mathrm{NS}(\widetilde{X})\simeq\widetilde{\Lambda}_{2d}$ and $\mathrm{T}_{\widetilde{X}}\simeq\widetilde{T}_{2d}$.

Proof. Let us consider the case $d=4k-2$, i.e., $d\equiv 2 \bmod 4$. The embedding $(\tilde{j},\lambda_-)$ is not primitive, since the class $\tilde{j}(h)+\lambda_-(b_1)$ can be divided by 2 in $U\oplus U\oplus U\oplus E_8(-1)\oplus E_8(-1)\oplus\langle -2\rangle$, whereas $h+b_1$ is primitive inside $\langle 2d\rangle\oplus E_8(-2)$. By adding the class $\frac{1}{2}(\tilde{j}(h)+\lambda_-(b_1))$ to $(\tilde{j},\lambda_-)(\langle 2d\rangle\oplus E_8(-2))$ one obtains a primitive embedding of $\widetilde{\Lambda}_{2d}$ in L. In particular there exists a fourfold of $K3^{[2]}$-type with $\mathrm{NS}(\widetilde{X})\simeq\widetilde{\Lambda}_{2d}$ and, by computing its orthogonal complement, one finds $\mathrm{T}_{\widetilde{X}}\simeq U^{\oplus 2}\oplus\langle -2d\rangle\oplus D_4(-1)\oplus\langle -2\rangle^{\oplus 5}$.

The case $d=4k-4$ is analogous. $\square$

2B. *Models of X as moduli space of sheaves on a K3 surface.* In this section we provide at least one model of the very general member of each family of projective fourfolds of $K3^{[2]}$-type admitting a symplectic involution, i.e., of each family described in Table 1. Each of these models will be described either as Hilbert scheme of a certain $K3$ surface or as a moduli space of stable, possibly twisted, sheaves on a $K3$ surface. The main results of this section are summarized in Table 2.

One needs two preliminary definitions in order to list all cases.

Definition 2.9. If $d\equiv 3 \bmod 4$, we denote by $(\langle 2d\rangle\oplus\langle -2\rangle^{\oplus 7})'$ the overlattice of $\langle 2d\rangle\oplus\langle -2\rangle^{\oplus 7} = \mathbb{Z}t\bigoplus\oplus_i\mathbb{Z}n_i$ obtained by adding to $\langle 2d\rangle\oplus\langle -2\rangle^{\oplus 7}$ the class $\frac{1}{2}\left(t+\sum_i n_i\right)$.

Lemma 2.10. *The lattice $\langle 2d\rangle\oplus\langle -2\rangle^{\oplus 7}$ admits a unique primitive embedding in L_{K3} and its orthogonal is uniquely determined and isometric to $U^{\oplus 2}\oplus D_4(-1)\oplus\langle -2d\rangle\oplus\langle -2\rangle^{\oplus 5}$.*

If $d\equiv 3 \bmod 4$ the lattice $(\langle 2d\rangle\oplus\langle -2\rangle^{\oplus 7})'$ admits a unique primitive embedding in L_{K3} and its orthogonal is uniquely determined and isometric to $U^{\oplus 2}\oplus N\oplus K_d$.

Proof. The discriminant quadratic form of $Q:=\langle 2d\rangle\oplus\langle -2\rangle^{\oplus 7}$ is $\left(\frac{1}{2d}\right)\oplus\left(-\frac{1}{2}\right)^{\oplus 7}$. Since L_{K3} is unimodular, the orthogonal $Q^{\perp}$ needs to have discriminant quadratic form

$$\left(-\tfrac{1}{2d}\right)\oplus\left(\tfrac{1}{2}\right)^{\oplus 7}\simeq\left(-\tfrac{1}{2d}\right)\oplus v(2)\oplus\left(-\tfrac{1}{2}\right)^{\oplus 5}$$

and signature $(2,12)$; by [Nikulin 1979b, Corollary 1.13.3], there is, up to isometry, a unique lattice with these properties, which is $U^{\oplus 2}\oplus D_4(-1)\oplus\langle -2d\rangle\oplus\langle -2\rangle^{\oplus 5}$, thus the embedding is unique up to isometry of L_{K3}.

embedding $\mathrm{NS}(X) \subset L$	$K3$	model	v	Proposition
$j_1, d \equiv 1 \bmod 2$	W_d	$M_v(W_d, \beta)$	$(0, H', 2)$	Proposition 2.14
$j_1, d \equiv 0 \bmod 2$	W_d	$M_v(W_d, \beta)$	$\left(4, \sum_{i=1}^{7} n_i, 2\right)$	Proposition 2.16
$j_2, d \equiv 1 \bmod 2$	W_d	$W_d^{[2]}$	$-$	Proposition 2.12
$j_3, d \equiv 3 \bmod 4$	W'_d	$M_v(W'_d, \beta)$	$(0, H', 2)$	Proposition 2.15
$\tilde{j}, d \equiv 0 \bmod 2$	W_d	$M_v(W_d)$	$\left(2, \sum_{i=1}^{7} n_i, 4\right)$	Proposition 2.16

Table 2. Birational models of X in the different families.

The discriminant quadratic form of $Q' := (\langle 2d\rangle \oplus \langle -2\rangle^{\oplus 7})'$ is $\left(\frac{2}{d}\right) \oplus u(2)^{\oplus 3}$, hence its orthogonal in L_{K3} has discriminant quadratic form $\left(-\frac{2}{d}\right) \oplus u(2)^{\oplus 3}$ and signature $(2, 12)$: again by [Nikulin 1979b, Corollary 1.13.3], there is, up to isometry, a unique lattice with these properties, which is $U^{\oplus 2} \oplus N \oplus K_d$. □

The previous lemma implies that there exists a well defined family of $K3$ surfaces which is polarized with the lattice $\langle 2d\rangle \oplus \langle -2\rangle^{\oplus 7}$ (resp. $(\langle 2d\rangle \oplus \langle -2\rangle^{\oplus 7})'$).

Definition 2.11. For any positive integer d, W_d is a $K3$ surface such that $\mathrm{NS}(W_d) = \langle 2d\rangle \oplus \langle -2\rangle^{\oplus 7}$.

For the positive integers d such that $d \equiv 3 \bmod 4$, W'_d is a $K3$ surface such that $\mathrm{NS}(W'_d) = (\langle 2d\rangle \oplus \langle -2\rangle^{\oplus 7})'$

By the previous lemma, the transcendental lattices of the surfaces W_d and W'_d are respectively $\mathrm{T}_{W_d} \simeq U^{\oplus 2} \oplus D_4(-1) \oplus \langle -2d\rangle \oplus \langle -2\rangle^{\oplus 5}$ and $\mathrm{T}_{W'_d} \simeq U^{\oplus 2} \oplus N \oplus K_d$.

In the following we will denote by H' a primitive vector in

$$\langle 2d\rangle \oplus \langle -2\rangle^{\oplus 7} \quad \text{or} \quad (\langle 2d\rangle \oplus \langle -2\rangle^{\oplus 7})'$$

whose square is 2. It surely exists by Lagrange's four squares theorem.

Table 2 summarizes all the birational models given for X: in the first column we identify the family of fourfolds which we are considering (and this is done by exhibiting the embedding $\mathrm{NS}(X) \subset L$, using the results in Table 1); in the second column we declare which $K3$ surface is associated to the model; in the third we describe the model; if the model of X is as moduli space of sheaves determined by a Mukai vector, in the fourth column we write the Mukai vector (we omit the element in the Brauer group giving the twist, when needed); in the last column we give the reference to the proposition where we describe the model and prove that it is the required one.

The easiest description of the fourfold that we obtain is the one associated to the embedding $j_2 : \mathrm{NS}(X) \hookrightarrow L$, indeed in this case X is (birational to) a Hilbert scheme of points on a $K3$ surface, by the following.

Proposition 2.12. *Let d be an odd positive integer. Then $W_d^{[2]}$ is a (Λ_{2d}, j_2)-polarized fourfold.*

Proof. The transcendental lattice T_{W_d} of W_d is isometric to the one of $W_d^{[2]}$. Since $\mathrm{T}_{W_d} \simeq T_{2d,2}$ (see Proposition 2.4), we conclude that $\mathrm{T}_{W_d^{[2]}} \simeq T_{2d,2}$. Moreover, $\mathrm{NS}(W_d^{[2]}) \simeq \mathrm{NS}(W_d) \oplus \langle -2\rangle$ and, by comparison of the discriminant forms, one has $\mathrm{NS}(W_d) \oplus \langle -2 \rangle \simeq \Lambda_{2d}$. So a generic (Λ_{2d}, j_2)-polarized fourfold has the same transcendental lattice and Néron–Severi group of $W_d^{[2]}$, thus their families coincide. □

Remark 2.13. Note that there is no natural symplectic involution on these Hilbert schemes. It is a nice open problem how to construct such involutions on some birational models of those Hilbert schemes, but for $d = 1$ there is the following geometric construction.

If $d = 1$, then the generic $(\langle 2 \rangle \oplus \langle -2 \rangle^{\oplus 7})$-polarized $K3$ surface W_1 is a double cover of a del Pezzo surface of degree 2, denoted by dP_2, thus it admits a nonsymplectic involution, which is the cover involution. We denote it as ι_{W_1} and we observe that it acts as the identity on $\mathrm{NS}(W_1)$. Moreover, since the anticanonical model of dP_2 exhibits dP_2 as double cover of $\mathbb{P}^2$ branched on a quartic curve, the surface W_1 admits a model (induced by the anticanonical one of dP_2) as a quartic hypersurface in $\mathbb{P}^3$ which does not contain lines. Therefore the fourfold $X = W_1^{[2]}$ admits two nonsymplectic involutions: one is $\iota_{W_1}^{[2]}$, the natural involution induced by ι_{W_1}, and the other is Beauville's involution β; see [Beauville 1983, Proposition 11] for the definition. The isometry $(\iota_{W_1}^{[2]})^*$ acts as the identity on $\mathrm{NS}(X)$ and as minus the identity on T_X, hence it commutes with every isometry induced by an involution on X (since they commute both on the transcendental lattice and on the Néron–Severi group). In particular $\iota_{W_1}^{[2]}$ and β are two commuting nonsymplectic involutions, whose composition is necessarily a symplectic involution on X. Such an involution can be constructed also on a birational model, as done in [Markushevich and Tikhomirov 2007].

In the case $d = 3$, by Proposition 2.12 examples of (Λ_6, j_2)-polarized fourfolds are given by Hilbert squares of $K3$ surfaces W_3 which are $(\langle 6 \rangle \oplus \langle -2 \rangle^{\oplus 7})$-polarized. In this case, one can show that such Hilbert squares are in fact birational to the Fano varieties of cubic fourfolds with 8 nodes [Lehn 2018, Theorem 1.1]. It is an open question whether it is possible to describe geometrically a symplectic involution on these manifolds.

Proposition 2.14. *Let d be an odd positive integer. There exist a Brauer class $\beta \in H^2(\mathcal{O}_{W_d}^*)_2$ and a Mukai vector $v \in H^*(W_d, \mathbb{Z})$ such that the moduli space $X = M_v(W_d, \beta)$ is a (Λ_{2d}, j_1)-polarized fourfold of $K3^{[2]}$-type.*

Proof. The transcendental lattice T_{W_d} of W_d is of the form $U \oplus \Xi$ for Ξ an even hyperbolic lattice; we denote by f_1, f_2 a basis of the hyperbolic plane U. Then we consider $B = \frac{f_1}{2} \in \mathrm{T}_{W_d} \otimes \mathbb{Q}$ and $\beta \in H^2(\mathcal{O}_{W_d}^*)_2$ the Brauer class of order two corresponding to $(\_, B) : \mathrm{T}_{W_d} \to \mathbb{Z}_2$. The twisted Néron–Severi group $\mathrm{NS}(W_d, \beta)$ is thus the sublattice of $H^*(W_d, \mathbb{Z})$ generated by $\mathrm{NS}(W_d)$, $(0, 0, 1)$ and $(2, f_1, 0)$, hence it is isomorphic to $U(2) \oplus \mathrm{NS}(W_d)$, and its orthogonal in the Mukai lattice is isometric to $U(2) \oplus \Xi$. It follows from work of Yoshioka [2006, Section 3] that $\mathrm{NS}(M_v(W_d, \beta)) \simeq v_B^\perp \cap \mathrm{NS}(W_d, \beta)$ and that the transcendental lattice of $M_v(W_d, \beta)$ is isometric to $U(2) \oplus \Xi$.

We conclude by choosing as Mukai vector $v = (0, H', 2)$ where $H' \in \mathrm{NS}(W_d)$ is a primitive effective class of square two. The orthogonal P of H' in $\mathrm{NS}(W_d)$ is a negative definite lattice with rank and length 7 and discriminant group $\mathbb{Z}_{2d} \oplus (\mathbb{Z}_2)^{\oplus 6}$ with discriminant quadratic form $q = \left(\frac{1}{2d}\right) \oplus v(2) \oplus \left(-\frac{1}{2}\right)^{\oplus 4}$. For such a choice we have $v_B = v$ primitive of square two and the orthogonal to v in $U(2) \oplus \mathrm{NS}(W_d)$ is a hyperbolic lattice of rank 9 and discriminant group $\mathbb{Z}_{2d} \oplus (\mathbb{Z}_2)^{\oplus 8}$. Its 2-adic component is isometric to the one of $\langle 2\rangle \oplus \langle -2\rangle^8 \simeq \langle 2\rangle \oplus E_8(-2)$ and there is only one even indefinite lattice in this genus by [Nikulin 1979b, Theorem 1.13.2]. Thus the orthogonal to v is isometric to Λ_{2d}. □

Proposition 2.15. *Let d be a positive integer such that $d \equiv 3 \bmod 4$. There exist a Brauer class $\beta \in H^2(\mathcal{O}^*_{W'_d})_2$ and a Mukai vector $v \in H^*(W'_d, \mathbb{Z})$ such that the moduli space $X = M_v(W'_d, \beta)$ is a (Λ_{2d}, j_3)-polarized fourfold of $K3^{[2]}$-type.*

Proof. Denote by Ξ the lattice $U \oplus N \oplus K_d$, it holds

$$T_{2d,3} \simeq U^{\oplus 2} \oplus E_8(-2) \oplus K_d \simeq U(2) \oplus U \oplus N \oplus K_d \simeq U(2) \oplus \Xi \tag{2-2}$$

and $\mathrm{T}_{W'_d} \simeq U^{\oplus 2} \oplus N \oplus K_d \simeq U \oplus \Xi$. Now reasoning as in Proposition 2.14, one chooses $B = \frac{f_1}{2} \in \mathrm{T}_{W'_d} \otimes \mathbb{Q}$ and $\beta \in H^2(\mathcal{O}^*_{W'_d})_2$ the Brauer class of order two corresponding to $(\_, B) : \mathrm{T}_{W'_d} \to \mathbb{Z}_2$. So $\mathrm{NS}(M_v(W'_d, \beta)) \simeq v_B^{\perp} \cap \mathrm{NS}(W'_d, \beta)$ and the transcendental lattice of $M_v(W'_d, \beta)$ is isometric to $U(2) \oplus \Xi \simeq T_{2d,3}$.

We conclude by choosing as Mukai vector $v = (0, H', 2)$ where $H' \in \mathrm{NS}(W'_d) \simeq (\langle 2d\rangle \oplus \langle -2\rangle^{\oplus 7})'$ is a primitive effective class of square two. □

Proposition 2.16. *If $d \equiv 0 \bmod 2$, then*:

- *A general fourfold of $K3^{[2]}$-type (Λ_{2d}, j_1)-polarized is birational to $M_v(W_d, \beta)$ where $v = \left(4, \sum_i n_i, 2\right)$ and β are as above.*
- *A general fourfold of $K3^{[2]}$-type $(\widetilde{\Lambda}_{2d}, \tilde{j})$-polarized is birational to $M_w(W_d)$ with $w = \left(2, \sum n_i, 4\right) \in H^*(W_d, \mathbb{Z})$.*

Proof. Let us fix β as in Proposition 2.14. Then, since $\mathrm{T}_{W_d} \simeq U^{\oplus 2} \oplus D_4(-1) \oplus \langle -2\rangle^{\oplus 5} \oplus \langle -2d\rangle$, $\mathrm{T}_{M_v(W_d,\beta)} \simeq U \oplus U(2) \oplus D_4(-1) \oplus \langle -2\rangle^{\oplus 5} \oplus \langle -2d\rangle$ for every possible choice of the Mukai vector v. Moreover, the twisted Néron–Severi group $\mathrm{NS}(W_d, \beta)$ is $U(2) \oplus \mathrm{NS}(W_d)$ (as in the proof of Proposition 2.14) and it is generated by $(0, 0, 1)$, $(2, f_1, 0)$, $(0, n_i, 0)$, $i = 1, \dots 7$, $(0, t, 0)$ (where t, n_i are the generators of $\mathrm{NS}(W_d)$, $t^2 = 2d$, and f_1 is as in Proposition 2.14). We now fix $v = (4, \sum_i n_i, 2)$, then $v_B = (4, \sum_i n_i + 2f_1, 2) \in H^*(W_d, \mathbb{Z})$ and we compute $v_B^{\perp} \cap \mathrm{NS}(W_d, \beta)$. It is generated by $(2, f_1, -1)$, $(0, 2n_1, 1)$, $(0, n_i - n_{i+1}, 0)$, $i = 1, \dots 6$, $(0, t, 0)$. One can directly check that $(0, t, 0)$ is orthogonal to all the other generators and the form computed on all the other generators is $R(2)$ where R is an even negative definite unimodular lattice of rank 8. It follows that $R \simeq E_8(-1)$ and so the orthogonal to v_B in $\mathrm{NS}(W_d, \beta)$ is isometric to $E_8(-2) \oplus \langle 2d\rangle \simeq \Lambda_{2d}$. Hence $M_v(W_d, \beta)$ is (Λ_{2d}, j_1)-polarized and gives a birational model of the general (Λ_{2d}, j_1)-polarized fourfold of $K3^{[2]}$-type.

To prove the similar result for a general fourfold of $K3^{[2]}$-type $(\widetilde{\Lambda}_{2d}, \tilde{j})$-polarized we observe that $\mathrm{T}_{W_d} \simeq \widetilde{T}_{2d}$. Moreover, the $(1, 1)$-part in $H^*(W_d, \mathbb{Z})$ is $U \oplus \mathrm{NS}(W_d)$. Next, we observe that $\widetilde{\Lambda}_{2d} \simeq \langle 2d\rangle \oplus N$, where N is the Nikulin lattice, obtained by $\langle -2\rangle^{\oplus 8}$ by gluing the class $n := \sum r_i/2$ and it is generated by the first seven roots $r_1, \dots, r_7$ and by n such that $n^2 = -4$ and $nr_i = -1$.

Let $g_1, g_2, t, n_1, \dots, n_7$ be a basis of $U \oplus \mathrm{NS}(W_d)$, i.e., of the $(1,1)$ part of $H^*(W_d, \mathbb{Z})$. Consider now the explicit primitive embedding $\langle 2d\rangle \oplus N \subset U \oplus \mathrm{NS}(W_d)$ which sends the $\langle 2d\rangle$ summand in the lattice spanned by t and which sends $r_i \mapsto n_i + g_1$ for $i = 1, \dots, 7$, $n \mapsto 2g_1 - g_2$. The Mukai vector $w = (2, \sum_i n_i, 4)$ is $4g_1 + 2g_2 + n_1 + \dots + n_7$ and its orthogonal is spanned by t, n and r_i with $i = 1, \dots 7$. So the orthogonal to the Mukai vector w in $U \oplus \mathrm{NS}(W_d)$ is isometric to $\langle 2d\rangle \oplus N \simeq \widetilde{\Lambda}_{2d}$ and this ends the proof. $\square$

Remark 2.17 (induced automorphisms from autoequivalences). The symplectic automorphism considered in Proposition 2.16 is induced by a symplectic autoequivalence on $D^b(W_d)$ that is not induced by a symplectic action on W_d. The result [Beckmann and Oberdieck 2022, Proposition 1.4] gives a way to further investigate these symplectic involutions. If [Beckmann and Oberdieck 2022, Proposition 1.4] is generalized for twisted sheaves then this would give a way to study also the other involutions considered here.

3. Nikulin orbifolds

After having described the moduli spaces of projective fourfolds X of $K3^{[2]}$-type admitting a symplectic involution σ, we now turn to the study of their quotients. It is well-known, since work of Fujiki [1983], that the quotient does not admit a crepant resolution of singularities. Nevertheless, there is a partial resolution $Y \to X/\sigma$ which is a so-called *irreducible symplectic orbifold*.

Definition 3.1. Let X be a fourfold of $K3^{[2]}$-type and let σ be a symplectic involution on X. The partial resolution Y of X/σ obtained by blowing up the $K3$ surface contained in $\mathrm{Sing}(X/\sigma)$ is called the *Nikulin orbifold* corresponding to (X, σ).

Deformations in the sense of [Bakker and Lehn 2022; Menet 2020] of Nikulin orbifolds are said to be *orbifolds of Nikulin type*.

We recall the following result by Menet.

Theorem 3.2 [Menet 2015]. *The second cohomology group $H^2(Y, \mathbb{Z})$ of an orbifold Y of Nikulin type is endowed with a symmetric bilinear form, which is the Beauville–Bogomolov–Fujiki form B_Y and thus it is a lattice. Let q_Y denote the corresponding quadratic form. Let Σ be the exceptional divisor of $Y \to X/\sigma$ and let Δ be the divisor induced by δ; then*

$$q_Y(\Sigma) = q_Y(\Delta) = -4, \qquad \tfrac{1}{2}(\Sigma \pm \Delta) \in H^2(Y, \mathbb{Z}).$$

The lattice $(H^2(Y, \mathbb{Z}), q_Y)$ is isometric to $U(2)^{\oplus 3} \oplus E_8(-1) \oplus \langle -2\rangle \oplus \langle -2\rangle$, where the last two summands are generated by $\frac{1}{2}(\Delta \pm \Sigma)$.

It follows that Σ is a class with self-intersection -4 and divisibility 2 in $H^2(Y, \mathbb{Z})$.

As a consequence of the previous theorem we get the following

Corollary 3.3. *Let X be fourfold of $K3^{[2]}$-type with a symplectic involution σ and such that* $\mathrm{NS}(X) \simeq E_8(-2)$; *then the corresponding Nikulin orbifold Y has* $\mathrm{NS}(Y) \simeq \langle -4\rangle$.

embedding $\mathrm{NS}(X) \subset L$	$\mathrm{NS}(Y)$	T_Y	Proposition
j_1	$\langle 4d\rangle \oplus \langle -4\rangle$	$U(2)^{\oplus 2} \oplus E_8(-1) \oplus \langle -4d\rangle \oplus \langle -4\rangle$	Proposition 3.5
$j_2, d \equiv 1 \bmod 2$	$\begin{bmatrix} d-1 & 2 \\ 2 & -4 \end{bmatrix}$	$U(2)^{\oplus 2} \oplus E_7(-1) \oplus K_d(2) \oplus \langle -2\rangle$	Proposition 3.6
$j_3, d \equiv 3 \bmod 4$	$\begin{bmatrix} d-1 & 2 \\ 2 & -4 \end{bmatrix}$	$U(2)^{\oplus 2} \oplus K_d(2) \oplus E_8(-1)$	Proposition 3.7
$\tilde{j}, d \equiv 0 \bmod 2$	$\langle d\rangle \oplus \langle -4\rangle$	$U^{\oplus 2} \oplus \langle -d\rangle \oplus N \oplus \langle -4\rangle$	Proposition 3.8

Table 3. Pairs $(\mathrm{NS}(Y), \mathrm{T}_Y)$ for the Nikulin orbifold Y associated to X.

Hence, deformations of Y are not necessarily Nikulin orbifolds, since it follows from Corollary 3.3 that Nikulin orbifolds are contained in a family of codimension 1.

3A. ***Families of projective Nikulin orbifolds and the map*** $\boldsymbol{\pi_*}$**.** In Corollary 3.3 we describe the explicit relations between $\mathrm{NS}(X)$ and $\mathrm{NS}(Y)$ in the generic case. In the following we will consider the same problem for special subfamilies, those of the projective fourfolds X.

If one specializes to the projective case one has four different families of fourfolds of $K3^{[2]}$-type X admitting a symplectic involution σ, which depend on the chosen embedding of $\mathrm{NS}(X)$ in L and are those listed in Table 1. The aim of this section is to associate to each of these families the family of Nikulin orbifolds Y which are partial resolution of X/σ. The results of this section are summarized in Table 3: in the first column we identify the family by choosing the embedding $\mathrm{NS}(X) \subset L$; in the second column we describe the Néron–Severi group of Y, in the third its transcendental lattice and in the last we give the reference to the propositions where the results are proved.

To prove these results we will use the explicit embeddings described in Section 2A and also the following explicit description of the map π_* induced by the quotient map $\pi : X \to X/\sigma$.

The map

$$\pi_* : H^2(X, \mathbb{Z}) \to H^2(X/\sigma, \mathbb{Z}) \subset H^2(Y, \mathbb{Z}) \tag{3-1}$$

is compatible (as explained below) with the lattice structure induced by the Beauville–Bogomolov–Fujiki form both on $H^2(X, \mathbb{Z})$ and on $H^2(Y, \mathbb{Z})$. Hence we can interpret π_* as a map between the lattices $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2\rangle$ and $U(2)^{\oplus 3} \oplus E_8(-1) \oplus \langle -2\rangle \oplus \langle -2\rangle$. To describe this map we consider, as in Section 2A, a basis of $H^2(X, \mathbb{Z})$ such that $\sigma^* \in O(H^2(X, \mathbb{Z}))$ switches the two copies of $E_8(-1)$ and acts as the identity on $U \oplus U \oplus U \oplus \langle -2\rangle$. We consider again the embeddings of the lattice $E_8(-2)$ in $E_8(-1) \oplus E_8(-1)$:

$$\lambda_+(b_i) = e_i + f_i \quad i = 1, \dots, 8,$$
$$\lambda_-(b_i) = e_i - f_i \quad i = 1, \dots, 8.$$

In particular $H^2(X, \mathbb{Z})^{\sigma^*} = U^{\oplus 3} \oplus \lambda_+(E_8(-2)) \oplus \langle -2\rangle \simeq U^{\oplus 3} \oplus E_8(-2) \oplus \langle -2\rangle$ and $(H^2(X, \mathbb{Z})^{\sigma^*})^\perp = \lambda_-(E_8(-2)) \simeq E_8(-2)$.

Take $\underline{u}$, $\underline{v}$, $\underline{w}$ vectors in U and $\underline{x}$, $\underline{y}$ vectors in $E_8(-1)$; for ease of notation, we will denote by $k \in \mathbb{Z}$ an element of $\langle -2\rangle$, referring to the k-th multiple of its generator (depending on the lattice this will be either δ, $\frac{1}{2}(\Delta+\Sigma)$ or $\frac{1}{2}(\Delta-\Sigma)$). Thus $(\underline{u}, \underline{w}, \underline{v}, \underline{x}, \underline{y}, k)$ is a vector in $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2\rangle$. Then

$$\pi_*(\underline{u}, \underline{w}, \underline{v}, \underline{x}, \underline{y}, k) = (\underline{u}, \underline{w}, \underline{v}, \underline{x}+\underline{y}, k, k) \in U(2)^{\oplus 3} \oplus E_8(-1) \oplus \langle -2\rangle \oplus \langle -2\rangle. \tag{3-2}$$

Hence the restriction of π_* to $U^{\oplus 3}$ acts as the identity on the vector space, but the form is multiplied by 2; the restriction of π_* to $E_8(-1)^{\oplus 2}$ acts as the sum of the two components on the vector space and divides the form by 2 in the quotient.

Lemma 3.4. *One has $\pi_*(\lambda_-(E_8(-2)))$ is trivial; $\pi_*(\lambda_+(E_8(-2))) = E_8(-1)$;*

$$\pi_*(H^2(X, \mathbb{Z})^{\sigma^*}) = U(2)^{\oplus 3} \oplus E_8(-1) \oplus \langle -4\rangle.$$

Proof. It suffices to choose a basis of the sublattices $\lambda_-(E_8(-2))$, $\lambda_+(E_8(-2))$, $H^2(X, \mathbb{Z})^{\sigma^*}$ of $H^2(X, \mathbb{Z})$ and then to apply the map π_* as given in (3-2). □

Proposition 3.5. *Let d be a positive integer and X_1 be a (Λ_{2d}, j_1)-polarized fourfold of $K3^{[2]}$-type. The fourfold X_1 admits a symplectic involution σ and, denoted by Y_1 the corresponding Nikulin orbifold, one has* $\mathrm{NS}(Y_1) \simeq \langle 4d\rangle \oplus \langle -4\rangle$ *and* $\mathrm{T}_{Y_1} \simeq \langle -4d\rangle \oplus U(2)^{\oplus 2} \oplus E_8(-1) \oplus \langle -4\rangle$.

Proof. By Proposition 2.5 one can choose the embedding j_1 such that $j_{1|E_8(-2)} = \lambda_-$ and $j_1(h) := \left(\binom{1}{d}, \binom{0}{0}, \binom{0}{0}, \underline{0}, \underline{0}, 0\right)$. Since $\pi_*(\mathrm{NS}(X_1)) \subset \mathrm{NS}(Y_1)$, one first considers $\pi_*(\mathrm{NS}(X_1)) = \pi_*((j_1, \lambda_-)(\langle 2d\rangle \oplus E_8(-2))) = \pi_*(j_1(h))$ (where the last identity is due to Lemma 3.4). By (3-2),

$$\pi_*(j_1(h)) = \left(\begin{pmatrix}1\\d\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \underline{0}, 0, 0\right) \in U(2)^3 \oplus E_8(-1) \oplus \langle -2\rangle^{\oplus 2},$$

so $q_Y(\pi_*(j_1(h))) = 4d$. Moreover, the class

$$\Sigma = \left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \underline{0}, 1, -1\right)$$

is contained in $\mathrm{NS}(Y_1)$. Hence $\mathrm{NS}(Y_1)$ is spanned by $\pi_*(j_1(h))$ and Σ and there are no linear combinations with rational noninteger coefficients of these classes which are also contained in $H^2(Y_1, \mathbb{Z})$. So $\mathrm{NS}(Y_1) = \langle \pi_*(j_1(h)), \Sigma\rangle \simeq \langle 4d\rangle \oplus \langle -4\rangle$. By definition T_{Y_1} is the orthogonal of $\mathrm{NS}(Y_1)$ in $H^2(Y_1, \mathbb{Z})$. So

$$\mathrm{T}_{Y_1} \simeq \langle -4d\rangle \oplus U(2)^{\oplus 2} \oplus E_8(-1) \oplus \langle -4\rangle. \qquad \square$$

Proposition 3.6. *Let d be an odd positive integer and X_2 be a (Λ_{2d}, j_2)-polarized fourfold of $K3^{[2]}$-type. The fourfold X_2 admits a symplectic involution σ and, denoted by Y_2 the corresponding Nikulin orbifold, one has* $\mathrm{NS}(Y_2) \simeq H_d(2) := \begin{bmatrix} d-1 & 2 \\ 2 & -4\end{bmatrix}$ *and* $\mathrm{T}_{Y_2} \simeq U(2)^{\oplus 2} \oplus E_7(-1) \oplus K_d(2) \oplus \langle -2\rangle$.

Proof. By Proposition 2.6 one can choose the embedding j_2 such that

$$j_{2|E_8(-2)} = \lambda_- \quad \text{and} \quad j_2(h) := \begin{cases} \left(\binom{2}{2k+2}, \binom{0}{0}, \binom{0}{0}, \underline{e_1}, \underline{e_1}, 1\right) & \text{if } d = 4k+1, \\ \left(\binom{2}{2k+2}, \binom{0}{0}, \binom{0}{0}, \underline{e_1}+\underline{e_3}, \underline{e_1}+\underline{e_3}, 1\right) & \text{if } d = 4k-1. \end{cases}$$

As above, to compute $\mathrm{NS}(Y_2)$ one observes that a $\mathbb{Q}$-basis is given by $\pi_*(j_2(h))$ and Σ. By (3-2), $\pi_*(j_2(h)) = \left(\binom{2}{2k+1}, \binom{0}{0}, \binom{0}{0}, 2\underline{x}, 1, 1\right)$ and $q_Y(\pi_*(j_2(h))) = 4d$. The class $\pi_*(j_2(h)) - \Sigma = \left(\binom{2}{2k+1}, \binom{0}{0}, \binom{0}{0}, 2\underline{x}, 0, 2\right)$ is divisible by 2 in $H^2(Y_2, \mathbb{Z})$, thus $\frac{1}{2}(\pi_*(j_2(h)) - \Sigma) \in \mathrm{NS}(Y_2)$. Finally, we get

$$\mathrm{NS}(Y_2) = \left\langle \tfrac{1}{2}(\pi_*(j_2(h)) - \Sigma), \Sigma \right\rangle = \begin{bmatrix} d-1 & 2 \\ 2 & -4 \end{bmatrix}.$$

The transcendental lattice is the orthogonal to Σ and $\pi_*(j_2(h))$ in $H^2(Y_2, \mathbb{Z})$. A $\mathbb{Q}$-basis is obtained by computing the image via π^* of the generators of T_{X_2} listed above; then one observes that the only elements which are two-divisible are those of the form $(\underline{0}, \underline{0}, \underline{0}, 2\underline{w}, 0, 0)$, and this allows to deduce a $\mathbb{Z}$-basis of the lattice T_{Y_2}, which is of discriminant $2^8 d$. Direct computation now shows that

$$\mathrm{T}_{Y_2} \simeq U(2)^{\oplus 2} \oplus E_7(-1) \oplus K_d(2) \oplus \langle -2 \rangle. \qquad \square$$

Proposition 3.7. *Let d be a positive integer such that $d \equiv 3 \bmod 4$ and X_3 be a (Λ_{2d}, j_3)-polarized fourfold of $K3^{[2]}$-type. The fourfold X_3 admits a symplectic involution σ and, denoted by Y_3 the corresponding Nikulin orbifold, one has $\mathrm{NS}(Y_3) \simeq H_d(2)$ and $\mathrm{T}_{Y_3} \simeq U(2)^{\oplus 2} \oplus K_d(2) \oplus E_8(-1)$.*

Proof. By Proposition 2.7 one can choose the embedding j_3 such that

$$j_{3|E_8(-2)} = \lambda_- \quad \text{and} \quad j_3(h) := \left(\begin{pmatrix} 2 \\ (d+1)/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \underline{0}, \underline{0}, 1\right).$$

Since both

$$\pi_*(j_3(h)) = \left(\begin{pmatrix} 2 \\ (d+1)/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \underline{0}, 1, 1\right) \quad \text{and} \quad \tfrac{1}{2}(\pi_*(j_3(h)) - \Sigma)$$

are contained in $\mathrm{NS}(Y_3)$,

$$\mathrm{NS}(Y_3) = \left\langle \tfrac{1}{2}(\pi_*(j_3(h)) - \Sigma), \Sigma \right\rangle \simeq \begin{bmatrix} d-1 & 2 \\ 2 & -4 \end{bmatrix}$$

and T_{Y_3} is its orthogonal complement inside $U(2)^{\oplus 3} \oplus E_8(-1) \oplus \langle -2 \rangle^{\oplus 2}$. Hence

$$\mathrm{T}_{Y_3} \simeq U(2)^{\oplus 2} \oplus K_d(2) \oplus E_8(-1). \qquad \square$$

Proposition 3.8. *Let d be an even positive integer and $\widetilde{X}$ be a $(\widetilde{\Lambda}_{2d}, \tilde{j})$-polarized fourfold of $K3^{[2]}$-type. The fourfold $\widetilde{X}$ admits a symplectic involution σ and, denoted by $\widetilde{Y}$ the corresponding Nikulin orbifold, one has $\mathrm{NS}(\widetilde{Y}) \simeq \langle d \rangle \oplus \langle -4 \rangle$ and $\mathrm{T}_{\widetilde{Y}} \simeq U^{\oplus 2} \oplus \langle -d \rangle \oplus N \oplus \langle -4 \rangle$.*

Proof. By Proposition 2.8 one can choose the embedding $\tilde{j}$ such that

$$\tilde{j}_{|E_8(-2)} = \lambda_- \quad \text{and} \quad \tilde{j}(h) := \begin{cases} \left(\binom{2}{2k}, \binom{0}{0}, \binom{0}{0}, \underline{e_1}, \underline{e_1}, 0\right) & \text{if } d = 4k-2 \text{ and,} \\ \left(\binom{2}{2k}, \binom{0}{0}, \binom{0}{0}, \underline{e_1} + \underline{e_3}, \underline{e_1} + \underline{e_3}, 0\right) & \text{if } d = 4k-4. \end{cases}$$

Let us consider the case $d = 4k-2$. Since $\pi_*(\tilde{j}(h)) = \left(\binom{2}{2k}, \binom{0}{0}, \binom{0}{0}, 2\underline{e_1}, 0, 0\right)$, $\frac{1}{2}(\pi_*(\tilde{j}(h))) \in \mathrm{NS}(\widetilde{Y})$ and a basis of $\mathrm{NS}(\widetilde{Y})$ is given by $\frac{1}{2}(\pi_*(\tilde{j}(h)))$ and Σ. So $\mathrm{NS}(\widetilde{Y}) = \langle d \rangle \oplus \langle -4 \rangle$ and $\mathrm{T}_{\widetilde{Y}}$ is the orthogonal complement in $U(2)^{\oplus 3} \oplus E_8(-1) \oplus \langle -2\rangle^{\oplus 2}$ to

$$\left\langle \left(\begin{pmatrix}1\\k\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \underline{e_1}, 0, 0 \right), \left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \underline{0}, 1, -1 \right) \right\rangle.$$

One obtains $\mathrm{T}_{\widetilde{Y}} \simeq U^{\oplus 2} \oplus \langle -d\rangle \oplus N \oplus \langle -4\rangle$. The case $d = 4k-4$ is analogous. □

Remark 3.9. The classes of divisors considered in Propositions 3.5, 3.6, 3.7, 3.8 have a geometric meaning: the class Σ is the effective class of the exceptional divisor; the class $\pi_*(j(h))$ is a pseudoample polarization induced on Y by the ample polarization $j(h)$ on X, and it is orthogonal to Σ. Its pullback via π^* is $2j(h)$; the class $(j(h)-\Sigma)$ corresponds to a divisor which has positive intersection with the exceptional divisor Σ and its pullback via π^* is $2j(h)$.

3B. *Nikulin orbifolds related with natural involutions on Hilbert squares of K3 surfaces.* We described, in Corollary 3.3, the relations between $\mathrm{NS}(X)$ and $\mathrm{NS}(Y)$ for a very general X of $K3^{[2]}$-type admitting a symplectic involution σ. In Section 3A we specialize X by requiring that it is projective. In this section we specialize X by requiring that it is the Hilbert scheme of two points of a $K3$ surface W and that the involution σ is natural, i.e., it is induced by a symplectic involution on W because of the equivariance of the construction of the Hilbert scheme $W^{[2]}$.

Proposition 3.10. *Let W be a generic nonprojective $K3$ surface admitting a symplectic involution σ_W, i.e., $\mathrm{NS}(W) = E_8(-2)$. Let $X := W^{[2]}$ be its Hilbert square and $\sigma := \sigma_W^{[2]}$ be the natural involution induced by σ_W. Then $\mathrm{NS}(X) = E_8(-2) \oplus \langle -2\rangle$, $\mathrm{T}_X \simeq U^{\oplus 3} \oplus E_8(-2)$ and $\mathrm{NS}(Y) \simeq \langle -2\rangle^{\oplus 2}$, $\mathrm{T}_Y \simeq U(2)^{\oplus 3} \oplus E_8(-1)$.*

Proof. By construction, the embedding of $\mathrm{NS}(X)$ in $H^2(X,\mathbb{Z})$ is given by $\lambda_-(E_8(-2)) \oplus \delta \simeq E_8(-2) \oplus \langle -2\rangle$. By Lemma 3.4, $\pi_*(\lambda_-(E_8(-2)) \oplus \delta) = \pi_*(\delta)$. Since π_* maps $\mathrm{NS}(X)$ to $\mathrm{NS}(Y)$, one deduces that $\Delta = \pi_*(\delta) = (\underline{0}, \underline{0}, \underline{0}, \underline{0}, 1, 1) \in U(2)^{\oplus 3} \oplus E_8(-1) \oplus \langle -2\rangle \oplus \langle -2\rangle$ is a class in $\mathrm{NS}(Y)$. Moreover, $\mathrm{NS}(Y)$ always contains the class $\Sigma = (\underline{0}, \underline{0}, \underline{0}, \underline{0}, 1, -1)$. Since $\mathrm{NS}(Y)$ contains both Δ and Σ, it contains all their linear combinations which belong to $H^2(Y,\mathbb{Z})$. In particular $\mathrm{NS}(Y) = \left\langle \frac{1}{2}(\Delta+\Sigma), \frac{1}{2}(\Delta-\Sigma)\right\rangle \simeq \langle -2\rangle \oplus \langle -2\rangle$. The transcendental lattices are directly computed respectively as orthogonal to the Néron–Severi groups inside $H^2(X,\mathbb{Z})$ and $H^2(Y,\mathbb{Z})$. □

Proposition 3.11. *Let W be a projective $K3$ surface admitting a symplectic involution σ_W such that $\rho(W) = 9$. Then either $\mathrm{NS}(W) \simeq \Lambda_{2d}$ or $\mathrm{NS}(W) \simeq \widetilde{\Lambda_{2d}}$.*

Let $X = W^{[2]}$ be the Hilbert square on W, σ be the natural symplectic involution induced by σ_W and Y be the corresponding Nikulin orbifold.

If $\mathrm{NS}(W) \simeq \Lambda_{2d}$, then $\mathrm{NS}(W^{[2]}) = \Lambda_{2d} \oplus \langle -2\rangle$, $\mathrm{T}_{W^{[2]}} \simeq \langle -2d\rangle \oplus U^{\oplus 2} \oplus E_8(-2)$, $\mathrm{NS}(Y) \simeq \langle 4d\rangle \oplus \langle -2\rangle \oplus \langle -2\rangle$ and $\mathrm{T}_Y \simeq \langle -4d\rangle \oplus U(2)^{\oplus 2} \oplus E_8(-1)$.

If $\mathrm{NS}(W) \simeq \widetilde{\Lambda_{2d}}$, then $\mathrm{NS}(W^{[2]}) = \widetilde{\Lambda_{2d}} \oplus \langle -2\rangle$, $\mathrm{T}_{W^{[2]}} \simeq \langle -2d\rangle \oplus U \oplus U \oplus N$, $\mathrm{NS}(Y) \simeq \langle d\rangle \oplus \langle -2\rangle \oplus \langle -2\rangle$ and $\mathrm{T}_Y \simeq \langle -d\rangle \oplus U^{\oplus 2} \oplus N$.

Proof. The Néron–Severi group of W is given in [van Geemen and Sarti 2007]. The rest of the proof is analogous to the previous ones and we sketch it. If $\mathrm{NS}(W) \simeq \Lambda_{2d}$ the embedding of $\mathrm{NS}(W^{[2]})$ in

$H^2(X,\mathbb{Z})$ is $(j_1, \lambda_-, \mathrm{id})(h, E_8(-2), \delta)$, where $j_1(h)$ is defined in Proposition 3.5, $\lambda_-(E_8(-2))$ is as above, and

$$\mathrm{id}(\delta) = (\underline{0}, \underline{0}, \underline{0}, \underline{0}, \underline{0}, 1) \in U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2 \rangle.$$

Then one applies π_* as in (3-2): if both Δ and Σ are contained in $\mathrm{NS}(Y)$, then also $\frac{1}{2}(\Delta \pm \Sigma)$ is contained in $\mathrm{NS}(Y)$.

If $\mathrm{NS}(W) \simeq \widetilde{\Lambda_{2d}}$, the embedding of $\mathrm{NS}(W^{[2]})$ in $H^2(X,\mathbb{Z})$ is $(\tilde{j}, \lambda_-, \mathrm{id})(h, E_8(-2), \delta)$, where $\tilde{j}(h)$ is defined in Proposition 3.8 and $\lambda_-(E_8(-2))$ is as above. Then one applies π_* as in (3-2) and concludes. $\square$

3C. ***A conjecture: the transcendental lattices of Y and of the fixed K3 surface.*** In Section 3A, we computed T_Y for every possible embedding j_i. We observe that for all the computed T_Y one can embed T_Y not only in $H^2(Y,\mathbb{Z})$ as we did, but also in L_{K3}. The orthogonal of $\mathrm{T}_Y \hookrightarrow L_{K3}$ is the Néron–Severi group of a $K3$ surface whose transcendental lattice is isometric to T_Y. In this section we discuss the following conjecture, which relates this $K3$ surface with the one in the fixed locus of the symplectic involution σ on X.

Conjecture 3.12. *Let X be a fourfold of $K3^{[2]}$-type admitting a symplectic involution σ, let Y be the partial resolution of X/σ as above, let S be the $K3$ surface contained in $\mathrm{Fix}_\sigma(X)$. Then $\mathrm{T}_Y \simeq \mathrm{T}_S$.*

As a first evidence to the conjecture we observe the following.

Proposition 3.13. *Let W be a $K3$ surface (projective or not) admitting a symplectic involution σ_W, such that $\mathrm{NS}(W)$ is one of the following lattices $E_8(-2)$, Λ_{2d} or $\widetilde{\Lambda}_{2d}$. Let X be $W^{[2]}$ and σ be the natural symplectic involution induced by σ_W. Then Conjecture 3.12 holds for X.*

Proof. Let us denote by $\hat{W}$ the minimal resolution of W/σ_W. It is a $K3$ surface and its Néron–Severi group and transcendental lattice are determined by those of W by Garbagnati and Sarti [2008, Corollary 2.2]. We will denote by $\widetilde{\Gamma_{2e}}$ the unique even overlattice of index 2 of $\Gamma_{2e} := \langle 2e \rangle \oplus N$ where both N and $\langle 2e \rangle$ are primitively embedded. One has the following relations between the Néron–Severi groups:

$$\begin{aligned} &\mathrm{NS}(W) = E_8(-2) && \text{if and only if } \mathrm{NS}(\hat{W}) = N.\\ &\mathrm{NS}(W) = \Lambda_{2d} && \text{if and only if } \mathrm{NS}(\hat{W}) = \widetilde{\Gamma}_{4d}.\\ &\mathrm{NS}(W) = \widetilde{\Lambda}_{2d}, d \equiv 0 \bmod 2, && \text{if and only if } \mathrm{NS}(\hat{W}) = \Gamma_d. \end{aligned} \tag{3-3}$$

Which correspond to the following relations between the transcendental lattices:

$$\begin{aligned} &\mathrm{T}_W = U^{\oplus 3} \oplus E_8(-2) && \text{if and only if } \mathrm{T}_{\hat{W}} = U^{\oplus 3} \oplus N.\\ &\mathrm{T}_W = \langle -2d \rangle \oplus U^{\oplus 2} \oplus E_8(-2) && \text{if and only if } \mathrm{T}_{\hat{W}} = \langle -4d \rangle \oplus U(2)^{\oplus 2} \oplus E_8(-1).\\ &\mathrm{T}_W = \langle -2d \rangle \oplus U^{\oplus 2} \oplus N, d \equiv 0 \bmod 2, && \text{if and only if } \mathrm{T}_{\hat{W}} = \langle -d \rangle \oplus U^{\oplus 2} \oplus N. \end{aligned} \tag{3-4}$$

For every fourfold of $K3^{[2]}$-type X with a symplectic involution σ the fixed locus of σ consists of 28 isolated fixed points and a $K3$ surface S. If $X = W^{[2]}$ and $\sigma = \sigma_W^{[2]}$, then the surface S is the Nikulin surface constructed as minimal resolution of W/σ_W, i.e., the surface $\hat{W}$. Hence, to conclude the proof it suffices to show that, for every W (and thus every X), one has $\mathrm{T}_Y \simeq \mathrm{T}_{\hat{W}}$.

If $\mathrm{NS}(W) = E_8(-2)$, then $\mathrm{T}_W = U^{\oplus 3} \oplus E_8(-2)$. By Proposition 3.10, $\mathrm{T}_Y \simeq U^{\oplus 3} \oplus N$ and by (3-4) also $\mathrm{T}_{\hat{W}} \simeq U^{\oplus 3} \oplus N$.

If $\mathrm{NS}(W) = \langle 2d\rangle \oplus E_8(-2)$, then $\mathrm{T}_W = U^{\oplus 2} \oplus \langle -2d\rangle \oplus E_8(-2)$. By Proposition 3.11,

$$\mathrm{T}_Y \simeq \langle -4d\rangle \oplus U(2)^{\oplus 2} \oplus E_8(-1)$$

and by (3-4) also $\mathrm{T}_{\hat{W}} \simeq \langle -4d\rangle \oplus U(2)^{\oplus 2} \oplus E_8(-1)$.

If $\mathrm{NS}(W) = \widetilde{\Lambda}_{2d}$, with $d \equiv 0 \bmod 2$, then $\mathrm{T}_W = \langle -2d\rangle \oplus U^{\oplus 2} \oplus N$. By Proposition 3.11,

$$\mathrm{T}_Y \simeq \langle -d\rangle \oplus U^{\oplus 2} \oplus N$$

and by (3-4) also $\mathrm{T}_{\hat{W}} \simeq \langle -d\rangle \oplus U^{\oplus 2} \oplus N$. □

We can also show that Conjecture 3.12 holds for two locally complete families when $d = 1, 3$ and the embeddings of Λ_{2d} are respectively j_2 and j_3.

Proposition 3.14. *Let X be a (Λ_2, j_2)-polarized fourfold of $K3^{[2]}$-type and σ the symplectic involution described in Remark 2.13. Conjecture 3.12 holds in this case.*

Proof. By Remark 2.13, $X = W_1^{[2]}$. We must describe the fixed locus of the symplectic involution $\sigma = \iota_{W_1}^{[2]} \circ \beta$ on X; see also [Markushevich and Tikhomirov 2007, Lemmma 5.3]. The surface W_1 has a model as quartic in $\mathbb{P}^3$ and its nonsymplectic involution ι_{W_1} is the restriction of an automorphism of $\mathbb{P}^3$, still denoted by ι_{W_1}. For any point $P \in W_1$ we consider the line $r_P := \langle P, \iota_{W_1}(P)\rangle$. The line r_P is invariant for ι_{W_1} and thus the set of intersection points $r_P \cap W_1$ is invariant for ι_{W_1}, hence there exists a point $Q \in W_1$ such that

$$r_P \cap W_1 = \{P, \iota_{W_1}(P), Q, \iota_{W_1}(Q)\}.$$

We consider the pair of points (P, Q), which corresponds to a point in $W_1^{[2]}$. This point is a fixed point of σ, indeed $\beta(P, Q) = (\iota_{W_1}(P), \iota_{W_1}(Q))$ and $\iota_{W_1}^{[2]}(\iota_{W_1}(P), \iota_{W_1}(Q)) = (P, Q)$, so $\sigma(P, Q) = (P, Q)$. We get a fixed point of σ for each point $P \in W_1$. Vice versa each fixed point of σ in $W_1^{[2]}$ necessarily corresponds to a pair of points in W_1 which lie on a ι_{W_1}-invariant line. So the fixed surface S of σ is parametrized by points in W_1 and thus it is birational to W_1 (birational because in order to construct $W_1^{[2]}$ we blow up a surface and it is possible, a priori, that this introduces some exceptional divisors in the fixed locus). Nevertheless the surface S contained in the fixed locus of σ is a $K3$ surface as W_1 and thus if they are birational, they are isomorphic. So S is a surface isomorphic to W_1 and in particular its transcendental lattice is $\mathrm{T}_S \simeq \mathrm{T}_{W_1} \simeq U^{\oplus 2} \oplus D_4(-1) \oplus \langle -2\rangle^{\oplus 6}$. This lattice is a 2-elementary lattice with signature $(2, 12)$ and $\delta = 1$, so it is isometric to any other 2-elementary lattice with these properties, in particular to

$$U(2)^{\oplus 2} \oplus E_7(-1) \oplus K_1(2) \oplus \langle -2\rangle$$

and the conjecture holds. □

In the case of (Λ_6, j_3)-polarized fourfolds, the orthogonal of Λ_6 is $T_{6,3} = U^{\oplus 2} \oplus E_8(-2) \oplus K_3$ and $j_3(h)$ is a polarization on X of degree 6 and divisibility 2, hence X is birational to the Fano variety of a smooth cubic fourfold. In fact, this is the family of Fano varieties $F(Z)$ of smooth symmetric cubic fourfolds Z carrying a symplectic involution, as discussed in [Camere 2012, Section 7]. In this case, the ample polarization h of degree 6 is of nonsplit type and its orthogonal complement is $h^\perp \simeq U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus A_2(-1)$; since $E_8(-2)$ has to be orthogonal to h, we obtain that the orthogonal complement of Λ_6 into L is the sublattice

$$T_{6,3} \simeq U^{\oplus 2} \oplus E_8(-2) \oplus A_2(-1).$$

In this case the equation of the cubic fourfold can be chosen to be

$$X_0^2 L_0(X_2:X_3:X_4:X_5) + X_1^2 L_1(X_2:X_3:X_4:X_5) + X_0 X_1 L_2(X_2:X_3:X_4:X_5) + G(X_2:X_3:X_4:X_5) = 0$$

where $L_i(X_2 : X_3 : X_4 : X_5)$ and $G(X_2 : X_3 : X_4 : X_5)$ are homogeneous polynomials, $\deg(L_i) = 1$, $\deg(G) = 3$. The symplectic involution is induced on the Fano variety by the projective transformation

$$(X_0 : X_1 : X_2 : X_3 : X_4 : X_5) \to (-X_0 : -X_1 : X_2 : X_3 : X_4 : X_5).$$

The fixed locus consists of 28 points, in the $(+1)$-eigenspace, and of a $K3$ surface S, in the (-1)-eigenspace, which has bidegree (2, 1) in $\mathbb{P}^1 \times V(G)$.

Proposition 3.15. *Let Z, $F(Z)$, S be as above. Then $\mathrm{T}_S \simeq \mathrm{T}_Y \simeq U(2)^{\oplus 2} \oplus K_3(2) \oplus E_8(-1)$ and Conjecture 3.12 holds for $F(Z)$.*

Proof. Since $V(G)$ is a cubic in the projective space $\mathbb{P}^3_{(X_2:X_3:X_4:X_5)}$ the $K3$ surface S in the fixed locus is a complete intersection of two hypersurfaces of bidegree (2, 1) and (0, 3) in $\mathbb{P}^1 \times \mathbb{P}^3$. We denote by dP_3 the del Pezzo cubic surface defined by $V(G)$. We recall that dP_3 is obtained as blow up of $\mathbb{P}^2$ in six points and, denoted by m the class of a line in $\mathbb{P}^2$ and by E_i the exceptional divisors of the blow up, $\mathrm{NS}(dP_3)$ is generated (over $\mathbb{Z}$) by $m, E_1, \dots, E_6$. The surface dP_3 is embedded in $\mathbb{P}^3$ by the anticanonical linear system $H := 3m - \sum_i E_i$. So

$$m = \tfrac{1}{3}\left(H + \sum_i E_i\right) \in \mathrm{NS}(dP_3).$$

To compute $\mathrm{NS}(S)$ we first observe that it is generated, at least over $\mathbb{Q}$, by the classes $h_1, h_2, \ell_i, i = 1, \dots, 6$ where h_1 (resp. h_2) is the restriction to the surface of the pullback in $\mathbb{P}^1 \times \mathbb{P}^3$ of the hyperplane section of $\mathbb{P}^1$ (resp. $\mathbb{P}^3$) and ℓ_i is the pullback of the class $E_i \in \mathrm{NS}(dP_3)$. The intersection properties of these classes are the following: $h_1^2 = 0$, $h_1 h_2 = 3$, $h_1 \ell_i = 1$, $i = 1, \dots, 6$, $h_2^2 = 6$, $h_2 \ell_i = 2$, $(\ell_i)^2 = -2$ and $\ell_i \ell_j = 0$ if $i \neq j$. In particular, we observe that h_2 is the pullback of the divisor $H \in \mathrm{NS}(dP_3)$ and since

$$\tfrac{1}{3}\left(H + \sum_i E_i\right) \in \mathrm{NS}(dP_3),$$

we obtain that $\frac{1}{3}(h_2+\sum_i \ell_i) \in \mathrm{NS}(S)$ (this divisor exhibits S as double cover of $\mathbb{P}^2$ and contracts the rational curves ℓ_i to nodes of the branch locus of the double cover). So $\left\{h_1, \frac{1}{3}(h_2+\sum_i \ell_i), \ell_i\right\}$ is a set of generators of $\mathrm{NS}(S)$. The discriminant group of this lattice is $\mathbb{Z}_6 \oplus (\mathbb{Z}_2)^{\oplus 5}$ and the discriminant form is the opposite of the one of $U(2)^{\oplus 2} \oplus A_2(-2)$. We deduce that the transcendental lattice of S is

$$\mathrm{T}_S \simeq U(2)^{\oplus 2} \oplus A_2(-2) \oplus E_8(-1).$$

Recalling that $A_2(-1) \simeq K_3$, we obtain that

$$\mathrm{T}_S \simeq U(2)^{\oplus 2} \oplus K_3(2) \oplus E_8(-1) \simeq \mathrm{T}_Y;$$

see Table 3. So Conjecture 3.12 holds in this case. □

The conjecture is true at least with rational coefficients, or, in other words, the transcendental lattice of the symplectic orbifold Y is the same of a (possibly twisted) Fourier–Mukai partner of the fixed $K3$ surface.

Proposition 3.16. *Let X be a fourfold of $K3^{[2]}$-type admitting a symplectic involution σ, let Y be the corresponding Nikulin orbifold and let S be the $K3$ surface contained in* $\mathrm{Fix}_\sigma(X)$. *Then* $\mathrm{T}_Y \otimes \mathbb{Q} \simeq \mathrm{T}_S \otimes \mathbb{Q}$. *In particular*, $\rho(Y) = \rho(S) - 6$.

Proof. Let $\nu : S \to X$ be the embedding of the $K3$ surface, we consider the restriction of forms $\nu^* : H^2(X, \mathbb{C}) \to H^2(S, \mathbb{C})$, which gives a morphism of Hodge structures of weight two.

Let $\omega_S \in H^{2,0}(S)$ be the restriction of a symplectic form $\omega_X \in H^{2,0}(X)$, i.e., $\omega_S = \nu^*\omega_X$; since S is the fixed $K3$ surface, this restriction is again a symplectic form on S, hence $\omega_X \notin \ker \nu^*$. Moreover, the rational transcendental lattice $\mathrm{T}_X \otimes \mathbb{Q}$ can be defined as the smallest rational Hodge substructure of $H^2(X, \mathbb{Q})$ such that $\mathrm{T}_X \otimes \mathbb{C}$ contains ω_X. This implies that the restriction $\nu^*_{|\mathrm{T}_X \otimes \mathbb{Q}}$ is injective; indeed, both the transcendental lattice and the kernel of a morphism of Hodge structures are irreducible Hodge substructures, thus either their intersection is trivial or they coincide, which is not the case here. In the same way one observes that the image of $\nu^*_{|\mathrm{T}_X \otimes \mathbb{Q}}$ is exactly $\mathrm{T}_S \otimes \mathbb{Q}$; both these Hodge substructures of $H^2(S, \mathbb{Q})$ are irreducible, and their intersection contains at least $\omega_S \neq 0$, thus they coincide. In the rest of the proof we denote $\nu^* : \mathrm{T}_X \otimes \mathbb{Q} \to \mathrm{T}_S \otimes \mathbb{Q}$: it is an isomorphism of irreducible Hodge structures of weight two.

Let now $\tilde{\rho} : \widetilde{X} \to X$ be the blow-up of the fixed $K3$ surface S, $\widetilde{\Sigma}$ be the exceptional divisor of ρ and let $\tilde{\pi} : \widetilde{X} \to Y$ be the quotient by the involution induced on $\widetilde{X}$ by σ. We use the following diagram:

$$\begin{tikzcd} & \widetilde{X} \arrow[r, "\tilde{\rho}"] \arrow[d, "\tilde{\pi}"] & X \arrow[d, "\pi"] \\ \widetilde{Y} \arrow[r] & Y \arrow[r, "\rho"] & X/\sigma \end{tikzcd}$$

We know from [Shioda 1986, Proposition 5] that the transcendental lattice of a smooth resolution $\widetilde{Y}$ of a quotient X/Γ, where X is smooth and Γ is a finite group, is a Hodge structure isomorphic to the

Γ-invariant part of T_X. In our case, a smooth resolution $\widetilde{Y}$ of singularities of X/σ is also a resolution of singularities for the orbifold Y, hence $\mathrm{T}_Y \otimes \mathbb{Q}$ is isomorphic to $\mathrm{T}_{\widetilde{Y}} \otimes \mathbb{Q}$ as Hodge structures. Finally we obtained an isomorphism of rational Hodge structures of weight two

$$\mathrm{T}_Y \otimes \mathbb{Q} \cong (\mathrm{T}_X \otimes \mathbb{Q})^{\sigma} = \mathrm{T}_X \otimes \mathbb{Q} \cong \mathrm{T}_S \otimes \mathbb{Q},$$

where the first and the last isomorphisms are respectively given by $\tilde{\rho}_* \circ \tilde{\pi}^*$ and ν^*.

We now show that this isomorphism is in fact an isometry over $\mathbb{Q}$. Let $\mu_{[S]} : H^2(X, \mathbb{Q}) \to H^6(X, \mathbb{Q})$ be the cup-product with $[S]$, where $[S]$ is the cohomology class of S; Voisin [2022, Proposition B.2] shows that $\ker \mu_{[S]} = \ker \nu^*$ and that, as a consequence, on $\operatorname{im} \nu^*$ the cup-product on S is induced by cup-product on X via the following equality:

$$\langle \nu^* x, \nu^* y \rangle_S = \langle \mu_{[S]}(x), y \rangle_X = x.y.[S].$$

In our particular case, this equality holds for all $x, y \in \mathrm{T}_X \otimes \mathbb{Q}$.

Denote by $\widetilde{\Sigma}$ and Σ respectively the exceptional divisors of $\tilde{\rho}$ and of ρ. Let $\alpha, \beta \in \mathrm{T}_Y \otimes \mathbb{Q}$; by [Menet 2015, Proposition 2.11] we have $B_Y(\alpha, \beta) = -\frac{1}{8}\alpha.\beta.\Sigma^2$. Moreover, observing that $\tilde{\pi}^*\Sigma = 2\widetilde{\Sigma}$, a standard computation in intersection theory yields

$$\alpha.\beta.\Sigma^2 = 2\tilde{\pi}^*\alpha.\tilde{\pi}^*\beta.\widetilde{\Sigma}^2 = -2\tilde{\rho}_*\tilde{\pi}^*\alpha.\tilde{\rho}_*\tilde{\pi}^*\beta.[S] = -2\langle \nu^*\tilde{\rho}_*\tilde{\pi}^*\alpha, \nu^*\tilde{\rho}_*\tilde{\pi}^*\beta \rangle_S,$$

where the second equality follows from projection formula (see [Fulton 1998, Proposition 8.3(c)]) and the equality $\widetilde{\Sigma}^2 = -\rho^*[S]$, which is proven in [Menet 2015, Lemma 2.12].

This shows that $B_Y(\alpha, \beta) = \frac{1}{4}\langle \nu^*\tilde{\rho}_*\tilde{\pi}^*\alpha, \nu^*\tilde{\rho}_*\tilde{\pi}^*\beta \rangle_S$ for all $\alpha, \beta \in \mathrm{T}_Y \otimes \mathbb{Q}$, thus $\mathrm{T}_S \otimes \mathbb{Q} \simeq \mathrm{T}_Y(4) \otimes \mathbb{Q} \simeq \mathrm{T}_Y \otimes \mathbb{Q}$. □

Remark 3.17. The $K3$ surfaces in the fixed locus of a symplectic involution can be seen as a generalization of Nikulin surfaces as their moduli space is densely covered by families of Nikulin surfaces. It would be interesting to study the rationality of such moduli spaces as in [Farkas and Verra 2016].

4. Orbifold Riemann–Roch formula

4A. ***Orbifold Riemann–Roch.*** In order to study projective models of Nikulin orbifolds, we need to apply the theory of orbifold Riemann–Roch, as developed in [Blache 1996] and in [Buckley et al. 2013]. We first treat the case of Nikulin orbifolds, and then we generalize it to orbifolds of Nikulin type.

We consider again the following diagram:

$$\begin{tikzcd}
V \arrow[r, "\widetilde{\beta}"] \arrow[d, "q"] \arrow[rr, bend left, "\tilde{r}"] & \widetilde{X} \arrow[r, "\tilde{\rho}"] \arrow[d, "\tilde{\pi}"] & X \arrow[d, "\pi"] \\
\widetilde{Y} \arrow[r, "\beta"] \arrow[rr, bend right] & Y \arrow[r, "\rho"] & X/\sigma
\end{tikzcd}$$

Where:

- X is a fourfold of $K3^{[2]}$-type, $\sigma \in \mathrm{Aut}(X)$ is a symplectic involution and $S \subset \mathrm{Fix}_\sigma(X)$ is the fixed surface; we will denote by $\mathcal{N}_{S|X}$ the normal sheaf.
- Y is the Nikulin orbifold corresponding to (X, σ); Σ is the exceptional divisor of $\rho : Y \to X/\sigma$ and $\widetilde{X}$ is the blow-up of X along S.
- $\widetilde{Y}$ is the total smooth resolution of X/σ, and hence of Y, and V is the blow-up of $\widetilde{X}$ in the inverse image via $\tilde{\rho}$ of the 28 isolated fixed points of σ. Denote respectively by $E_1, \dots, E_{28}$ and $\widetilde{E_1}, \dots, \widetilde{E_{28}}$ the exceptional divisors on V and on $\widetilde{Y}$. Moreover, let E_S and $\widetilde{E_S}$ be the exceptional divisors on V and $\widetilde{Y}$ over S and over its image in X/σ respectively. Finally, let E and $\widetilde{E}$ be respectively $\sum_{i=1}^{28} E_i + E_S$ and $\sum_{i=1}^{28} \widetilde{E}_i + \widetilde{E}_S$.

Lemma 4.1. *Let X, Y, $\widetilde{Y}$ be as described above, and let $\nu : S \hookrightarrow X$ be the embedding of the fixed K3 surface. Then*

$$c_1(\widetilde{Y}) = \tfrac{1}{2}(q_* c_1(V) + \widetilde{E}) = -\sum_{i=1}^{28} \widetilde{E}_i,$$

$$c_2(\widetilde{Y}) = \tfrac{1}{2} q_* \tilde{r}^*(c_2(X) + \nu_*[S]) + q_*\left(-8\sum_{i=1}^{28} E_i^2 - E_S^2\right) + \tfrac{3}{2} K_{\widetilde{Y}} \widetilde{E} + 2K_{\widetilde{Y}}^2.$$

Proof. The proof follows from an application of Grothendieck–Riemann–Roch formula [Fulton 1998, Theorem 15.2] combined with well-known properties of smooth blow-ups; see [loc. cit., Example 15.4.3]:

$$K_V = 3\sum_{i=1}^{28} E_i + E_S,\ c_2(V) = \tilde{r}^*(c_2(X) + \nu_*[S]) + 2\sum_{i=1}^{28} E_i^2.$$

It is a generalization of the proof of [Camere et al. 2019a, Proof of Proposition 7.2]. □

Theorem 4.2 (orbifold Riemann–Roch formula). *Let D be a $\mathbb{Q}$-Cartier Weil divisor on Y, then $q^*\beta^* D$ is equivalent to $\tilde{r}^* H + kE_S$, with $H \in \mathrm{NS}(X)$, $k \in \mathbb{Z}$; let n be the number of points in which the divisor D fails to be Cartier. Then*

$$\chi(Y, D) = \tfrac{1}{48} H^4 + \tfrac{1}{48} H^2.c_2(X) + \left(\tfrac{1}{16} - \tfrac{1}{8}k^2\right)(H_{|S})^2 + 3 - \tfrac{1}{16} n + \tfrac{1}{4}k^4 - \tfrac{1}{2}3k^2.$$

Proof. Since D is a $\mathbb{Q}$-Cartier Weil divisor on Y, then there exists an effective divisor $\widetilde{D} \in \mathrm{NS}(\widetilde{Y})$ such that $\beta^* D = \widetilde{D} + \sum_{i=1}^{28} \lambda_i \widetilde{E}_i$ with $\lambda_i \in \mathbb{Q}$: $\lambda_i = \frac{1}{2}$ if D fails to be Cartier in $p_i \in \mathrm{Sing}(Y)$ for $i = 1, \dots, 28$, it is zero otherwise. We have $\beta^* D.\widetilde{E}_i = 0$ for all i. Then the orbifold Riemann–Roch formula ([Buckley et al. 2013, Theorem 3.3]) is

$$\begin{aligned}\chi(Y, D) &= \chi(\widetilde{Y}, \widetilde{D})\\ &= \tfrac{1}{24}(\beta^* D)^4 + \tfrac{1}{12}(\beta^* D)^3.c_1(\widetilde{Y}) + \tfrac{1}{24}(\beta^* D)^2.(c_1(\widetilde{Y})^2 + c_2(\widetilde{Y}))\\ &\qquad + \tfrac{1}{24}(\beta^* D).c_1(\widetilde{Y}).c_2(\widetilde{Y}) + \chi(\mathcal{O}_{\widetilde{Y}}) + \sum_{i=1}^{28} \gamma_i(D),\end{aligned}$$

where for each singular point $p_i \in Y$ we define $\gamma_i(D) = -\frac{1}{16}$ if D is not Cartier in p_i, $\gamma_i(D) = 0$ otherwise.

It was proven in [Fu and Menet 2021] that $\chi(\mathcal{O}_Y)=\chi(\mathcal{O}_{\widetilde{Y}})=3$. Moreover, it follows from $K_{\widetilde{Y}}=\sum_i \widetilde{E}_i$, as shown in Lemma 4.1, that $\beta^*D.c_1(\widetilde{Y})=0$, hence the formula above reduces to computing $(\beta^*D)^4$ and $(\beta^*D)^2.c_2(\widetilde{Y})$. Our aim is now to reduce the intersection theory on $\widetilde{Y}$ to the intersection theory on X.

In our situation, we have $q^*\beta^*D=\tilde{r}^*H+kE_S$ (indeed, if there were components in the E_i's, we would have $\beta^*D.\widetilde{E}_i\neq 0$). Moreover, $q^*\widetilde{E_S}=2E_S$ and $q_*E_S=\widetilde{E_S}$; hence $E_S^4=12$, since Fujiki's relation on Y implies $\widetilde{E_S}^4=6\cdot 16$.

Hence we obtain the following equalities of intersection numbers in $\mathbb{Q}$, by using Lemma 4.1 and the projection formula [Fulton 1998, Proposition 8.3(c)] (see also [Camere et al. 2019a] for further details):

$$(\beta^*D)^4=\tfrac{1}{2}(q^*\beta^*D)^4=\tfrac{1}{2}((\tilde{r}^*H)^4+k^4E_S^4+6k^2(\tilde{r}^*H)^2.E_S^2)=\tfrac{1}{2}H^4+6k^4-3k^2(H_{|S})^2,$$
$$(\beta^*D)^2.q_*\tilde{r}^*c_2(X)=\tilde{r}^*(H^2.c_2(X))+k^2E_S^2.\tilde{r}^*c_2(X)=H^2.c_2(X)-k^2c_2(X).\nu_*[S],$$
$$(\beta^*D)^2.q_*\tilde{r}^*\nu_*[S]=\tilde{r}^*((H_{|S})^2)+k^2E_S^2.\tilde{r}^*\nu_*[S]=(H_{|S})^2-k^2c_2(\mathcal{N}_{S|X}),$$
$$(\beta^*D)^2.q_*(E_S^2)=-\tilde{r}^*((H_{|S})^2)+k^2E_S^4=-(H_{|S})^2+12k^2.$$

Many equalities and vanishings of some terms in the formulas above use the following equality for $\alpha\in A_{4-i}(X)$ (easy generalization of [Bădescu and Beltrametti 2013, Lemma 1.1]):

$$E_S^i.\tilde{r}^*\alpha=(-1)^{i-1}s_{i-2}(\mathcal{N}_{S|X}).\nu^*\alpha,$$

combined with $\nu^*\nu_*[S]=c_2(\mathcal{N}_{S|X})$ (see [Fulton 1998, Corollary 6.3]) and with the results contained in [Camere 2012, Proof of Theorem 5], which give $s_1(\mathcal{N}_{S|X})=0$, $c_2(X).[S]=36$, $s_2(\mathcal{N}_{S|X})=-c_2(\mathcal{N}_{S|X})=-c_2(X).[S]+c_2(S)=-12$. □

Lemma 4.3. *Let $H\in \mathrm{NS}(X)$ as in Theorem 4.2; then $(H_{|S})^2=2q_X(H)$, where q_X is the BBF quadratic form on $H^2(X,\mathbb{Z})$.*

Proof. This is proven in [Menet 2015, Proposition 2.24(4)], once recalled that $(H_{|S})^2=-E_S^2.\tilde{r}^*H^2$. □

Corollary 4.4 (Riemann–Roch formula for Cartier divisors on Y). *If $D\in\mathrm{NS}(Y)$ then $\chi(Y,D)=\frac{1}{4}(q_Y(D)^2+6q_Y(D)+12)$.*

Proof. In this particular case, Theorem 4.2 simplifies into

$$\chi(D)=\tfrac{1}{48}H^4+\tfrac{1}{48}H^2.c_2(X)+\left(\tfrac{1}{16}-\tfrac{1}{8}k^2\right)(H_{|S})^2+3+\tfrac{1}{4}k^4-\tfrac{1}{2}3k^2.$$

Since $q^*\beta^*D=\tilde{r}^*H+kE_S$, by push-pull formula [Fulton 1998, proof of Proposition 2.3(c)] and the commutativity of the diagram above, we have $D=\frac{1}{2}\tilde{\pi}_*\widetilde{\rho}^*H+\frac{k}{2}\Sigma$. The statement then follows from $q_Y(\Sigma)=-4$, $q_Y(\tilde{\pi}_*\widetilde{\rho}^*H)=2q_X(H)$ [Menet 2015, Proposition 2.9], Riemann–Roch formula on X [Gross et al. 2003, Example 23.19] and Lemma 4.3. □

Corollary 4.4 holds for all orbifolds of Nikulin type, since it is topological in nature. Indeed, we can deform any orbifold of Nikulin type with a Cartier divisor to a Nikulin orbifold while keeping the class of the divisor algebraic (one just needs to require an additional (-4)-class of divisibility 2 in the same monodromy orbit of Σ in Theorem 3.2). We deduce the following general result.

Proposition 4.5. *Let Y be an orbifold of Nikulin type and let D and $\frac{m}{2}L$ be equivalent $\mathbb{Q}$-Cartier Weil divisors on Y, with $m \in \mathbb{Z}$ and L a Cartier divisor. Let n be the number of points where D fails to be Cartier. Then*

$$\chi(D) = \tfrac{3}{8}\big(\tfrac{1}{24}m^4 q_Y(L)^2 + m^2 q_Y(L) + 8\big) - \tfrac{1}{16}n.$$

Proof. Since Y is an orbifold of Nikulin type it is singular in 28 points. Let $\beta : \widetilde{Y} \to Y$ be a smooth resolution of singularities. By [Buckley et al. 2013, Theorem 3.3], $\chi(D) = \chi(\beta^* D) - \frac{n}{16}$ as integers. Our assumptions imply that $\beta^* D = \frac{m}{2}\beta^* L$, hence $(\beta^* D)^4 = \frac{m^4}{16}L^4 = \frac{3m^4}{8}q_Y(L)^2$.

Moreover, it follows from Corollary 4.4 that

$$\tfrac{1}{24}(\beta^* D)^2.c_2(\widetilde{Y}) = \tfrac{1}{96}m^2(\beta^* L)^2.c_2(\widetilde{Y}) = \tfrac{1}{4}m^2(\chi(L) - 3 - \tfrac{1}{4}q_Y(L)^2) = \tfrac{3}{8}m^2 q_Y(L),$$

since $L^4 = 6q_Y(L)^2$ and

$$\tfrac{1}{24}(\beta^* L)^4 + \tfrac{1}{24}(\beta^* L)^2.c_2(\widetilde{Y}) + 3 = \chi(\beta^* L) = \chi(L) = \tfrac{1}{4}(q_Y(L)^2 + 6q_Y(L) + 12).$$

Hence, $\chi(D) = \chi(\beta^* D) - \frac{n}{16} = \frac{3}{8}(\frac{m^4}{24}q_Y(L)^2 + m^2 q_Y(L) + 8) - \frac{n}{16}$. □

4B. ***Projective models of quotients.*** Let X be as above, with $\rho(X) = 9$. Let us denote by A the ample generator of the orthogonal to $E_8(-2)$ in $\mathrm{NS}(X)$. In particular A is preserved by σ. Then the map $\varphi_{|A|} : X \to \mathbb{P}(H^0(X, A)^\vee)$ is such that the automorphism σ on X is induced by a projective transformation on $\mathbb{P}(H^0(X, A)^\vee)$, still denoted by σ. Hence σ acts on the vector space $U := H^0(X, A)^\vee$, splitting it in the direct sum $U_+ \oplus U_-$ where U_+ and U_- are the eigenspaces of the eigenvalues $+1$ and -1 respectively.

The fourfold X projects to $\mathbb{P}(U_+)$ and $\mathbb{P}(U_-)$; since we are considering projective spaces which are invariant for σ, these two projections induce maps on the quotient, i.e., they induce the maps $X/\sigma \dashrightarrow \mathbb{P}(U_+)$ and $X/\sigma \dashrightarrow \mathbb{P}(U_-)$. These rational maps extend to the partial resolution Y, so we obtained two maps $Y \dashrightarrow \mathbb{P}(U_+)$ and $Y \dashrightarrow \mathbb{P}(U_-)$. We are interested in these maps, which essentially give the projective models of the quotient orbifold keeping trace of the construction of this orbifold as quotient of X.

The maps $Y \dashrightarrow \mathbb{P}(U_+)$ and $Y \dashrightarrow \mathbb{P}(U_-)$ are of course induced by some linear systems on Y and in order to find them we are looking for divisors D on Y such that $\tilde{\rho}_*\tilde{\pi}^* D = \pi^*\rho_* D = A$ (because the maps to $\mathbb{P}(U_\pm)$ are induced by the projections from $\mathbb{P}(H^0(X, A)^\vee)$).

If a connected component Z of the fixed locus $\mathrm{Fix}_\sigma(X)$ of σ on X is contained in one of the two eigenspaces, then the generic member of the linear system giving the projection to the other eigenspace has to pass through Z. Thus the corresponding divisor on X/σ is Weil but not necessarily Cartier and passes through n of the 28 singular points of X/σ and possibly through the singular surface of X/σ. Nevertheless, since the map is just $2:1$, we can assume that generically the divisor on X/σ passes simply through the singularities. Let us now consider the partial resolution $\rho : Y \to X/\sigma$. The divisor which we are considering on X/σ induces a divisor D_1 on Y. Since ρ is an isomorphism outside Σ (which is the exceptional divisor of ρ mapped to the singular surface), the Weil divisor D_1 passes simply through n of the 28 isolated singular points of Y and then fails to be Cartier on these points. Moreover, if the divisor on X/σ passes through the singular surface, then D_1 has a component on the exceptional divisor Σ, with multiplicity 1; otherwise it has none.

embedding $\mathrm{NS}(X) \subset L$	(n_1, n_2)	m_1	m_2	Proposition
$j_1, d \equiv 1 \bmod 2$	$(12, 16)$	$\frac{d^2}{4}+\frac{3d}{2}+\frac{5}{4}$	$\frac{d^2}{4}+d-\frac{1}{4}$	Theorem 4.9
$j_1, d \equiv 0 \bmod 2$	$(16, 12)$	$\frac{d^2}{4}+\frac{3d}{2}+1$	$\frac{d^2}{4}+d$	Theorem 4.9
$j_2, d \equiv 1 \bmod 2$	$(28, 0)$	$\frac{d^2}{4}+\frac{3d}{2}+\frac{1}{4}$	$\frac{d^2}{4}+d+\frac{3}{4}$	Theorem 4.10
$j_3, d \equiv 3 \bmod 4$	$(28, 0)$	$\frac{d^2}{4}+\frac{3d}{2}+\frac{1}{4}$	$\frac{d^2}{4}+d+\frac{3}{4}$	Theorem 4.10
$\tilde{j}, d \equiv 0 \bmod 2$	$(0, 28)$	$\frac{d^2}{4}+\frac{3d}{2}+2$	$\frac{d^2}{4}+d-1$	Theorem 4.12

Table 4. Summary of the properties of D_1 and D_2 and of the dimensions m_i of the projective spaces target of the map $\varphi_{|D_i|}$.

We observe that the linear system on X which corresponds to one of the projections and which is not a complete linear system (since its members have to pass through a part of $\mathrm{Fix}_\sigma(X)$) induces a complete linear system on V (where all the fixed locus is blown up).

By the previous discussion we deduce that the divisors that we are looking for on Y are two divisors D_1 and D_2 (each associated to one of the two projections on the two eigenspaces) such that

$$q^*\beta^*D_i = \tilde{r}^*A + k_iE_S, \quad \text{with } k_i = 0, -1 \text{ and thus } \tilde{\rho}_*\tilde{\pi}^*D_i = A. \tag{4-1}$$

Exactly one between D_1 and D_2 fails to be Cartier in a specific point (indeed a specific isolated fixed point is contained in exactly one eigenspace). The same holds true for the fixed surface (it is contained in exactly one of the eigenspaces), hence $k_i = -1$ for exactly one value among 1 and 2 and $k_i = 0$ for the other one. Indeed, if D_i is orthogonal to Σ, then β^*D_i is orthogonal to $\widetilde{E_S}$ and $q^*\beta^*D_i$ is orthogonal to E_S, i.e., $k_i = 0$. Similarly if the intersection of D_i with Σ is nontrivial, then $k_i = -1$.

So, given X a generic member of a family of fourfolds of $K3^{[2]}$-type with a symplectic involution, we determine two $\mathbb{Q}$-Cartier Weil divisors D_1 and D_2 which give two maps $\varphi_{|D_i|} : Y_i \to \mathbb{P}^{m_i}$. In Table 4 we summarize the properties of D_1 and D_2 and the dimensions m_i of the projective spaces target of the map $\varphi_{|D_i|}$. We choose D_1 to be always orthogonal to the exceptional divisor Σ and hence D_2 is always the divisor meeting Σ. Hence we have also to declare the number of points where D_i fails to be Cartier (and this is always denoted by n_i). As in the other tables, in the first column we identify the family of X (and hence of Y) by giving the explicit embedding of $\mathrm{NS}(X)$ in L and in the last we give the reference to the propositions were the results are proved.

Proposition 4.6. *Let $\rho(X) = 9$, A, D_1 and D_2 be as above and $q(A) = 2d$. Then both $\chi(Y, D_1)$ and $\chi(Y, D_2)$ are integer if and only if n_i and k_i are as in the following (up to a possible switch between D_1 and D_2):*

- *If d is even then*
  - $(n_1, k_1) = (0, 0)$ *and* $(n_2, k_2) = (28, -1)$ *or*
  - $(n_1, k_1) = (16, 0)$ *and* $(n_2, k_2) = (12, -1)$.

- *If d is odd then*
  - $(n_1, k_1) = (28, 0)$ *and* $(n_2, k_2) = (0, -1)$ *or*
  - $(n_1, k_1) = (12, 0)$ *and* $(n_2, k_2) = (16, -1)$.

Proof. We recall that for a divisor A on a fourfold of $K3^{[2]}$-type X it holds

$$\tfrac{1}{48}A^4 + \tfrac{1}{4}A^2.c_2(X) + \tfrac{3}{2} = \tfrac{1}{2}\chi(A) = \tfrac{1}{16}(q_X(A)+4)(q_X(A)+6),$$

which, combined with Theorem 4.2, gives

$$\chi(Y, D_i) = \tfrac{1}{16}(q_X(A)+4)(q_X(A)+6) - \tfrac{3}{2} + \left(\tfrac{1}{16} - \tfrac{1}{8}k_i^2\right)(A_{|S})^2 + 3 - \tfrac{1}{16}n_i + \tfrac{1}{4}k_i^4 - 3\tfrac{1}{2}k_i^2.$$

Now we recall that $q_X(A) = 2d$ and, by Lemma 4.3, $A_{|S}^2 = 2q_X(A) = 4d$, so

$$\chi(Y, D_i) = \tfrac{1}{4}d^2 + \tfrac{5}{4}d + \tfrac{1}{4}d - \tfrac{1}{2}k_i^2 d + 3 - \tfrac{1}{16}n_i + \tfrac{1}{4}k_i^4 - 3\tfrac{1}{2}k_i^2.$$

We observe that if $k_i = 0, -1$, then $|k_i| = k_i^2 = k_i^4$, hence we obtain the following formula:

$$\chi(Y, D_i) = \tfrac{1}{4}d^2 + \tfrac{1}{2}3d - \tfrac{1}{2}|k_i|d - \tfrac{1}{16}n_i - \tfrac{1}{4}5|k_i| + 3.$$

Let us assume $k_1 = 0$ and then $k_2 = -1$. If d is even, then $\chi(Y, D_2) \in \mathbb{Z}$ forces $n_2 \equiv 12 \bmod 16$, which implies $n_2 = 12$ or $n_2 = 28$. If d is odd then $\chi(Y, D_2) \in \mathbb{Z}$ forces $n_2 \equiv 0 \bmod 16$, which implies $n_2 = 0$ or $n_2 = 16$. □

We observe that if $n_i = 0$ for a certain divisor D_i, then it is a Cartier divisor on Y. In this case D_i is $\pi_*(A)$ and it is orthogonal to the exceptional divisor if $k_i = 0$, it has a positive intersection with the divisor Σ if $k_i = -1$.

Lemma 4.7. *The variety Y is normal with terminal singularities. In particular Y is a klt variety.*

Proof. The variety Y is smooth outside 28 points where its singularities are locally the quotient of $\mathbb{C}^4$ by an involution g. In particular it is an orbifold. Hence it is normal. Moreover, the local action of the automorphism g is given by the diagonal matrix $\mathrm{diag}(-1, -1, -1, -1) \in \mathrm{SL}(4)$. The age of g is 2, hence the singularities of Y are terminal singularities (see [Joyce 2000, Theorem 6.4.3]), and in particular the pair $(Y, 0)$ is a klt pair. □

Proposition 4.8. *Let X, A, D_1 and D_2 be as in Proposition 4.6. Then*

$$\chi(Y, D_i) = h^0(Y, D_i).$$

Proof. The Kawamata–Vieweg vanishing theorem holds, see [Kollár and Mori 1998, Theorem 2.70], for the variety Y. With respect to the notation in [loc. cit.] one can assume $\Delta = 0$, and $N \equiv D_i$, $i = 1, 2$. It remains to prove that the D_i's are nef and big divisors. Since $\rho(X) = 9$ and A is the generator of $E_8(-2)^{\perp}$ in $\mathrm{NS}(X)$, it can be assumed to be an ample divisor. In particular it is nef, so $\pi_*(A)$ is a nef divisors. Since the sign of the top self-intersection of A is the same as the sign of the self-intersection of $\pi_*(A)$, we deduce that $\pi_*(A)$ is a nef and big divisor, by [Kollár and Mori 1998, Proposition 2.61].

Moreover, $\rho^*(D_i) = \pi_*(A)$ and since the properties of being big and nef are birational invariants, we deduce that D_i is nef and big. □

In Section 2 we associated the divisor A to a certain embedding of $\mathrm{NS}(X)$ in $H^2(X, \mathbb{Z})$, i.e., we considered $A = j(h)$ where h is vector in $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2\rangle$. In Propositions 3.5, 3.6, 3.7, 3.8, we studied the image of this divisor under the map π_* and we determined the generators of $\mathrm{NS}(Y)$. So, by comparing the conditions on D_i with the Néron–Severi group of Y computed in Section 3, one obtains the following theorems.

Theorem 4.9. *Let* $\mathrm{NS}(X) = (j_1, \lambda_-)(\langle 2d\rangle \oplus E_8(-2))$ *and A the generator of* $j_1(\langle 2d\rangle)$.

Let D_1 and D_2 be $\mathbb{Q}$-Cartier Weil divisors such that $2D_1 = \rho^*(\pi_*(j_1(h))) \in \mathrm{NS}(Y)$ *and* $2D_2 = \rho^*\pi_*(j_1(h)) - \Sigma \in \mathrm{NS}(Y)$. *Then if d is even* (*resp. odd*), *D_1 fails to be Cartier in* 16 (*resp.* 12) *points and D_2 in the other* 12 (*resp. the other* 16) *points. These divisors are such that* $\tilde{\rho}_*\tilde{\pi}^*(D_1) = \tilde{\rho}_*\tilde{\pi}^*(D_2) = A$ *and*

$$H^0(X, A) = (\rho^{-1} \circ \pi)^* H^0(Y, D_1) \oplus (\rho^{-1} \circ \pi)^* H^0(Y, D_2).$$

Proof. Let us consider the case d even. The other one is similar. One first considers $\rho^*(\pi_*(j_1(h))) \in \mathrm{NS}(Y)$, $\rho^*(\pi_*(j_1(h))) - \Sigma \in \mathrm{NS}(Y)$. Then there exist D_1 and D_2 $\mathbb{Q}$-Cartier Weil divisors such that a multiple of D_i, denoted by $h_i D_i$ is one prescribed element in $\mathrm{NS}(Y)$. We choose h_i to be the minimum among positive integers such that $h_i D_i \in \mathrm{NS}(Y)$. In particular, due to the singularities of Y, h_i is either 1 or 2. If $h_i = 1$, then D_i is Cartier, otherwise it is a $\mathbb{Q}$-Cartier Weil divisor on Y and it fails to be Cartier in n_i points. The possibilities for the divisors D_1 and D_2 are given in Proposition 4.6: the divisor D_1 is orthogonal to Σ, hence it is characterized by $k_1 = 0$; then there are two possibilities for n_1: either $n_1 = 0$ or $n_1 = 16$. If $n_1 = 0$, then D_1 is Cartier and $h_1 = 1$, otherwise D_1 is not Cartier and $h_1 = 2$. The choice of one of these two possibilities determines also the properties of D_2, which is necessarily $\mathbb{Q}$-Cartier Weil and not Cartier, hence h_2 is necessarily 2.

If $h_1 = 1$, then the divisors D_1 would be Cartier, but this is not the case, since the divisor $\frac{1}{2}\rho^*(\pi_*(j_1(h)))$ is not Cartier ($\mathrm{NS}(Y)$ is described in Proposition 3.5). We deduce that $h_1 = 2$, so $n_1 = 16$, $n_2 = 12$.

The map ρ is the contraction of Σ so, if $B \in \mathrm{NS}(Y)$ and $B \neq r\Sigma$, then $\rho_*(B)$ is a multiple of the unique generator of $\mathrm{NS}(X/\sigma)$. Since π is a $2:1$ map and A is invariant for σ, we have $\pi^*(\rho_*(h_i D_i)) = 2A$ for each $h_i D_i \in \mathrm{NS}(Y)$ as above. In particular we have $\pi^*(\rho_*(D_2)) = \tilde{\rho}_*\tilde{\pi}^*(D_2) = A$ (since $h_2 = 2$) and thus the sections of D_2 correspond to sections of A which are either all invariant or all antiinvariant for the action of the involution σ. So the sections of D_2 span a subspace of $H^0(X, A)$ which is contained (possibly coincides) either in U_+ or in U_- where $U_\pm$ are the eigenspaces of $H^0(X, A)$ for the action of σ^*. Similarly, the span of the sections of $2(D_1/2) = D_1$ is contained in the other eigenspace. In order to conclude that each one of $\varphi_{|D_1|}$ and $\varphi_{|D_2|}$ is associated to one of the two projections of X to $\mathbb{P}(U_+)$ and to $\mathbb{P}(U_-)$, it suffices to prove that the space spanned by the sections of D_2 (resp. D_1) is not just contained, but coincides with one of the eigenspaces. So it suffices to prove that $\dim(H^0(Y, D_2) \oplus H^0(Y, D_1)) = \dim(H^0(X, A))$.

We are now able to compute $\chi(D_i)$, $i=1,2$, by Theorem 4.2 and we know that $\chi(D_i)=h^0(D_i)$, by Proposition 4.8. Since $q_X(A)=2d$, one checks

$$\begin{aligned}\tfrac{1}{8}(q_X(A)+6)(q_X(A)+4) &= \dim(H^0(X,A))\\ &= \dim(H^0(Y,D_1))+\dim(H^0(Y,D_2))\\ &= \left(\tfrac{1}{4}d^2+\tfrac{1}{2}3d-1+3\right)+\left(\tfrac{1}{4}d^2+\tfrac{1}{2}3d-\tfrac{1}{2}d-\tfrac{12}{16}-\tfrac{5}{4}+3\right)\\ &= \tfrac{1}{2}d^2+\tfrac{1}{2}5d+3.\end{aligned}$$

Since $\tilde{\rho}_*\tilde{\pi}^*(D_i)=A$ we conclude that

$$H^0(X,A)=(\rho^{-1}\circ\pi)^*H^0(Y,D_1)\oplus(\rho^{-1}\circ\pi)^*H^0(Y,D_2). \qquad \square$$

Theorem 4.10. *Let* $d\equiv 1 \bmod 2$, $s=2,3$ *and* $\mathrm{NS}(X)\simeq(j_s,\lambda_-)(\langle 2d\rangle\oplus E_8(-2))$. *Let A be the generator of* $j_s(\langle 2d\rangle)$. *Let D_1 be the $\mathbb{Q}$-Cartier Weil divisor such that* $2D_1=\rho^*(\pi_*(j_2(h)))\in\mathrm{NS}(Y)$ *and D_2 the Cartier divisor* $D_2:=\frac{1}{2}(\rho^*\pi_*(j_1(h))-\Sigma)\in\mathrm{NS}(Y)$. *Then D_1 fails to be Cartier in 28 points,* $\tilde{\rho}_*\tilde{\pi}^*(D_1)=\tilde{\rho}_*\tilde{\pi}^*(D_2)=A$ *and*

$$H^0(X,A)=(\rho^{-1}\circ\pi)^*H^0(Y,D_1)\oplus(\rho^{-1}\circ\pi)^*H^0(Y,D_2).$$

Proof. The proof is similar to the previous one. One first observes that $\rho^*(\pi_*(j_2(h)))\in\mathrm{NS}(Y)$ and $\frac{1}{2}(\rho^*(\pi_*(j_2(h)))-\Sigma)\in\mathrm{NS}(Y)$ by the Propositions 3.6 and 3.7. There exist D_1 and D_2 $\mathbb{Q}$-Cartier Weil divisors such that a multiple of D_i, denoted by h_iD_i is one prescribed element in $\mathrm{NS}(Y)$. We choose h_i to be the minimum among positive integers such that $h_iD_i\in\mathrm{NS}(Y)$. In particular, due to the singularities of Y, h_i is either 1 or 2. If $h_i=1$, then D_i is Cartier, otherwise it is a $\mathbb{Q}$-Cartier Weil divisor on Y and it fails to be Cartier in n_i points. The possibilities for the divisors D_1 and D_2 are given in Proposition 4.6: the divisor D_1 is orthogonal to Σ, hence it is characterized by $k_1=0$; then there are two possibilities for n_1, which in turn determine uniquely the values of n_2: either $n_1=28$ and $n_2=0$ or $n_1=12$ and $n_2=16$. If $n_1=28$, then $n_2=0$ and so D_2 is Cartier, otherwise (if $n_1=12$), neither D_1 nor D_2 are Cartier. As in the previous proof, we are looking for divisors D_i, $i=1,2$, such that $\pi^*(\rho_*(D_i))=A$. Since

$$\pi^*\left(\rho_*\left(\frac{\rho^*(\pi_*(j_2(h)))-\Sigma}{2}\right)\right)=A,$$

we obtain

$$D_2=\tfrac{1}{2}(\rho^*(\pi_*(j_2(h)))-\Sigma),$$

$h_2=1$ and D_2 is Cartier. This implies that $n_1=28$ and D_1 fails to be Cartier in all the 28 singular points of Y. As in the previous proposition one is able to compute $\chi(D_i)$, $i=1,2$, by Theorem 4.2 and we know that $\chi(D_i)=h^0(D_i)$, by Proposition 4.8. So, recalling that $q_X(A)=2d$, one can check that

$$\begin{aligned}\tfrac{1}{8}(q_X(A)+6)(q_X(A)+4) &= \dim(H^0(X,A))\\ &= \dim(H^0(Y,D_1))+\dim(H^0(Y,D_2))\\ &= \left(\tfrac{1}{4}d^2+\tfrac{1}{2}3d-\tfrac{28}{16}+3\right)+\left(\tfrac{1}{4}d^2+\tfrac{1}{2}3d-\tfrac{1}{2}d-\tfrac{5}{4}+3\right)\\ &= \tfrac{1}{2}d^2+\tfrac{1}{2}5d+3.\end{aligned}$$

Since $\tilde{\rho}_*\tilde{\pi}^*(D_i)=A$ we conclude that

$$H^0(X,A)=(\rho^{-1}\circ\pi)^*H^0(Y,D_1)\oplus(\rho^{-1}\circ\pi)^*H^0(Y,D_2).$$ □

Remark 4.11. When $d=1$ and $j_s=j_2$, in the case discussed in Proposition 3.14, we obtain $h^0(D_1)=h^0(D_2)=3$, respectively with $(n_1,k_1)=(28,0)$ and $(n_2,k_2)=(0,-1)$. When $d=3$ and $j_s=j_3$, in the case of the Fano variety of a symmetric cubic discussed before Proposition 3.15, we obtain $h^0(D_1)=8$ and $h^0(D_2)=7$, respectively with $(n_1,k_1)=(28,0)$ and $(n_2,k_2)=(0,-1)$.

Theorem 4.12. *Let $d\equiv 0 \bmod 2$ and let $\mathrm{NS}(X)\simeq\widetilde{\Lambda}_{2d}$ be the primitive closure of the embedding $(\tilde{j},\lambda_-)(\langle 2d\rangle\oplus E_8(-2))$ where A is the generator of $\tilde{j}(\langle 2d\rangle)$.*

Let D_1 be the Cartier divisor $D_1\simeq\rho^*\frac{1}{2}(\pi_*(\tilde{j}(h)))\in\mathrm{NS}(Y)$ and D_2 be a $\mathbb{Q}$-Cartier Weil divisor such that $2D_2\simeq\left(\frac{1}{2}(\pi_*(\tilde{j}(h)))-\Sigma\right)\in\mathrm{NS}(Y)$. Then D_2 fails to be Cartier in 28 points, $\tilde{\rho}_*\tilde{\pi}^*(D_1)=\tilde{\rho}_*\tilde{\pi}^*(D_2)=A$ and

$$H^0(X,A)=(\rho^{-1}\circ\pi)^*H^0(Y,D_1)\oplus(\rho^{-1}\circ\pi)^*H^0(Y,D_2).$$

Proof. The proof is completely analogous to the previous ones. We omit it. □

We now give an example of application of the previous theorems, in particular of Theorem 4.9 with $d=1$.

Proposition 2.4 shows that, when $d=1$, the lattice $\Lambda_2\simeq\langle 2\rangle\oplus\langle -2\rangle^{\oplus 8}$ admits two nonisometric embeddings inside $L=L_{K3}\oplus\langle -2\rangle$, and in particular j_1 with orthogonal isometric to $T_{2,1}:=U^{\oplus 2}\oplus E_8(-2)\oplus\langle -2\rangle^{\oplus 2}$.

An explicit construction of this family is given in [Camere 2012, Section 8]: it is the family of smooth double EPW sextics which carry a symplectic involution, as it is observed in [Mongardi and Wandel 2015, Example 6.8].

Indeed, the very general element of this family is $X=X_{\mathbb{A}}$ a double EPW sextic, as defined in [O'Grady 2005], associated with a Lagrangian subspace $\mathbb{A}\in\mathbb{LG}(\bigwedge^3 V)$ invariant for the action on $\bigwedge^3 V$ induced by the involution i of the six-dimensional vector space V which has exactly four eigenvalues $+1$. The fourfold $X_{\mathbb{A}}$ is defined as a double cover of a so-called EPW sextic $Z_{\mathbb{A}}\subset\mathbb{P}(V)\simeq\mathbb{P}^5$, which in this case is invariant for i, and it carries an ample invariant class $A\in\mathrm{NS}(X_{\mathbb{A}})$ of degree two; the map $\varphi_{|A|}:X_{\mathbb{A}}\to\mathbb{P}^5$ associated to A factors through the double cover $f:X_{\mathbb{A}}\to Z_{\mathbb{A}}$.

As a consequence, we get two involutions induced by i on $X_{\mathbb{A}}$ and we call σ the symplectic one among the two lifts. It is proven in [Camere 2012, Proposition 19] that the fixed locus $\mathrm{Fix}_\sigma(X_{\mathbb{A}})$ is the union of

28 isolated fixed points and one $K3$ surfaces. In fact, 12 points are the preimages in the double cover of six points $q_1, \dots, q_6 \in \mathbb{P}(V_-)$, whereas the other 16 points lie in the intersection of the ramification of f with $\mathbb{P}(V_+)$.

Finally, the fixed $K3$ surface S is the $K3$ surface obtained as double cover of a quadric surface $Q \subset Z_{\mathbb{A}} \cap \mathbb{P}(V_+)$ ramified along its intersection with a quartic surface. The double cover endows S with a nonsymplectic involution and a copy of $U(2)$ is primitively embedded in $\mathrm{NS}(S)$. By Proposition 3.5, if the conjecture holds we should have $\mathrm{T}_S \simeq U(2)^{\oplus 2} \oplus E_8(-1) \oplus \langle -4 \rangle^{\oplus 2}$.

Next we look at the Nikulin fourfold Y obtained as partial resolution of $X_{\mathbb{A}}/\sigma$. Using the notation of Proposition 4.6 and of Theorem 4.9, we obtain on Y two divisors D_1 and D_2 with $(n_1, k_1) = (12, 0)$ and $(n_2, k_2) = (16, -1)$. The orbifold Riemann–Roch formula in Theorem 4.2 implies $h^0(D_1) = 4$ and $h^0(D_2) = 2$; compare also with Table 4.

The quotient of $\mathbb{P}^5$ by the involution is the join in $\mathbb{P}^{12}$ of a conic $C \subset \mathbb{P}^2_1$ and a second Veronese $v_2(\mathbb{P}^3) \subset \mathbb{P}^9_2$ where $\mathbb{P}^2_1$ an $\mathbb{P}^9_2$ are general linear subspaces of $\mathbb{P}^{12}$. With the notation as in Theorem 4.9 we have a polarization $2D_1$ on Y such that $q_Y(2D_1) = 4$ (see the proof of Theorem 4.9).

Lemma 4.13. *The image $\varphi_{|2D_1|}(Y) = \bar{Z} \subset \mathbb{P}^{12}$ is the intersection of $J(C, v_2(\mathbb{P}^3))$ with a special cubic I. The map $\varphi_{|2D_1|}$ is generically $2:1$ ramified along a surface.*

Proof. The image $\varphi_{|2D_1|}(Y) = \bar{Z} \subset \mathbb{P}^{12}$ can be seen as the image of a symmetric EPW sextic through the involution described above. The equation of the sextic can be written as a cubic in term of invariant quadric polynomials. Such polynomials can be seen as coordinates of $\mathbb{P}^{12}$ so the image is defined as the intersection of $J(C, v_2(\mathbb{P}^3))$ with a cubic.

The image $\bar{Z}$ is singular along 6 points $C \cap I$ and three surfaces $I \cap v_2(\mathbb{P}^3) \subset \mathbb{P}^9_2$ (two of the components are quadric surfaces, one is a Kummer quartic) and the image of the singular surface of degree 40 on $Z_{\mathbb{A}}$. Only the Kummer quartic is in the ramification since the quadric component is in the ramification of the symplectic involution. □

Note that $J(C, v_2(\mathbb{P}^3)) \subset \mathbb{P}^2$ can be seen as the intersection of the cone $C(\mathbb{P}^2_1, v_2(\mathbb{P}^3))$ with a quadric cone with vertex $\mathbb{P}^9_2$. For general projective models of a general deformation we expect the above quadric cone to be more general.

5. Orbifolds of Nikulin type of BBF degree 2

The aim of this section is to study the first locally complete family of projective orbifolds of Nikulin type. This will be a family polarized by a class of BBF degree 2. It follows from the Riemann-Roch Theorem 1.3 that their projective models are fourfolds in $\mathbb{P}^6$.

Note that there are two types of classes of BBF degree 2 in the second cohomology group $U(2)^{\oplus 3} \oplus E_8(-1) \oplus \langle -2 \rangle \oplus \langle -2 \rangle$ of an orbifold of Nikulin type, respectively with divisibility 1 and 2.

An example of a Nikulin orbifold with the class of the polarization of divisibility 2 is given by the quotient of the Fano variety of lines on a symmetric cubic fourfold by the involution with signature $(2, 4)$. Indeed, by Remark 4.11 the model of X in $\mathbb{P}^{14}$ is symmetric with respect to an involution with invariant space $\mathbb{P}^7$. After projecting from it we obtain a fourfold in $\mathbb{P}^6$ being a special projective model of a Nikulin orbifold of BBF degree 2 with divisibility 2.

In this section, we are interested in the case of a polarization of BBF degree 2 divisibility 1. We first describe some special elements of the locally complete family, given by the Nikulin orbifolds. We show that they correspond to double EPW quartics, see Lemma 5.1, and that they are double covers of complete intersections of type $(3, 4)$ in $\mathbb{P}^6$; see Proposition 5.5. Then in Section 5B we generalize the previous results, showing that all the projective deformations of these Nikulin orbifolds are double covers of special complete intersections of a cubic and a quartic in $\mathbb{P}^6$.

5A. *Geometry of $(\widetilde{\Lambda}_4, \tilde{j})$-polarized $K3^{[2]}$-fourfolds.* (Compare with [van Geemen and Sarti 2007, Section 3.5].) We consider a fourfold of $K3^{[2]}$-type with Néron–Severi group $\mathrm{NS}(X) \simeq \widetilde{\Lambda}_4$. Then X admits a symplectic involution σ such that the corresponding Nikulin orbifold Y has a polarization of BBF degree 2 orthogonal to the exceptional divisor Σ, see Proposition 3.8. We denote by D_1 this divisor.

We now describe the image $\varphi_{|D_1|}(Y) \subset \mathbb{P}^6$. As in Proposition 2.8, we can assume that $\mathrm{NS}(X)$ is generated by A, $E_8(-2)$ and F_1 where $q_X(A) = 4$ and $F_1 = \frac{1}{2}(A + v)$ and $v \in E_8(-2)$ with $q_X(v) = -4$. Let $F_2 = \frac{1}{2}(A - v)$ then $F_i^2 = 0$ and $A = F_1 + F_2$. After monodromy operations we can assume that A is big and nef. We denote $C(\mathbb{P}^2 \times \mathbb{P}^2)$ the cone in $\mathbb{P}^9$ over the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$.

Lemma 5.1. *The linear system $|A|$ defines a $2:1$ map to $C(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^9$. The image is symmetric with respect to a linear involution σ with signature $(3, 7)$ on $\mathbb{P}^9$ that exchanges the factors in the Segre product. Moreover, the image is isomorphic to an EPW quartic corresponding to a Verra threefold that is symmetric with respect to the involution exchanging the factors in $\mathbb{P}^2 \times \mathbb{P}^2$.*

Proof. By the construction of $\mathrm{NS}(X)$ given in Proposition 2.8, one obtains that F_i are primitive in $\mathrm{NS}(X)$ and that the linear system of $A = F_1 + F_2$ defines a $2:1$ map to $C(\mathbb{P}^2 \times \mathbb{P}^2)$; see [Iliev et al. 2017, Theorem 1.1]. The symplectic involution σ acts as -1 on $E_8(-2)$, hence $\sigma^* F_1 = F_2$. So σ switches the two copies of $\mathbb{P}^2$ in $C(\mathbb{P}^2 \times \mathbb{P}^2)$ and $\varphi_{|A|}(X)$ is symmetric with respect to the linear involution which induces σ and which has signature $(3, 7)$ on $\mathbb{P}^9$.

Moreover, $U(2) \simeq \langle F_1, F_2 \rangle$ is primitive in $\mathrm{NS}(X)$. It follows that X is in the moduli space of lattice polarized fourfolds of $K3^{[2]}$-type with $U(2)$ contained in the Néron–Severi lattice. It is thus a deformation of double EPW quartics described in [Iliev et al. 2017].

It follows as in [Camere et al. 2019b, Section 6.5] that X is related to a threefold $V \subset \mathbb{P}^2 \times \mathbb{P}^2$ symmetric with respect to the involution interchanging the factors. □

Lemma 5.2. *The quotient of $C(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^9$ by σ is isomorphic to the projection of this cone from the invariant $\mathbb{P}^2_- \subset \mathbb{P}^9$. This quotient is a cubic hypersurface Z_3 that is isomorphic to a cone in $\mathbb{P}^6$ over a symmetric determinantal cubic fourfold in $\mathbb{P}^5$. In particular its singular locus is a cone over the Veronese surface in $\mathbb{P}^5$.*

Proof. We can assume $C(\mathbb{P}^2 \times \mathbb{P}^2)$ is defined by 2×2 minors of a 3×3 matrix with entries being a basis of the hyperplane in $\mathbb{P}^{9\vee}$ orthogonal to the vertex of the cone. So elements of $\mathbb{P}^9$ can be thought as classes of pairs (x, M) such that $x \in \mathbb{C}$ and M is a 3×3 matrix of rank 1. The involution σ is then just the map transposing M i.e., $(x, M) \mapsto (x, M^T)$ and

$$\mathbb{P}^2_- = \{(0, M) \mid M \neq 0, M + M^T = 0\}.$$

The corresponding projection is then

$$\mathbb{P}^9 \ni (x, M) \mapsto (x, M + M^T) \in \mathbb{P}^6_+$$

where $\mathbb{P}^6_+ = \{(x, M) \mid x \in \mathbb{C}, M = M^T\}$. Since for a rank 1 matrix M, we have $M + M^T$ is a matrix of rank at most 2, the image of the projection is a cone over the space of symmetric matrices with trivial determinant. The latter is singular in the cone over the locus of rank 1 symmetric matrices, which is a cone over a Veronese surface. □

We denote by $p : \mathbb{P}^9 \to \mathbb{P}^6$ the projection from the σ-invariant $\mathbb{P}^2_- \subset \mathbb{P}^9$ described in the previous lemma and we observe that p restricts to a $2:1$ map

$$C(\mathbb{P}^2 \times \mathbb{P}^2) \to p(C(\mathbb{P}^2 \times \mathbb{P}^2))$$

with branch locus isomorphic to the cone over the diagonal of $\mathbb{P}^2 \times \mathbb{P}^2$.

Proposition 5.3. *Let $J := p(\varphi_{|A|}(X)) \subset \mathbb{P}^6$, then J is a complete intersection $Z_3 \cap Z_4 \subset \mathbb{P}^6$ of two hypersurfaces Z_3 and Z_4 of degrees 3 and 4 respectively. Moreover, J is singular along a surface which is the disjoint union of two (possibly reducible) surfaces: S_{16} of degree 16 and S_{36} of degree 36.*

Proof. This proof is supported by a calculation using Macaulay2 whose script is presented in the online supplement. Using the script we find an explicit example, defined in positive characteristic, of fourfold J satisfying the assertion of the theorem. We need only to argue that the invariants (degree, dimensions of the variety and decomposition of its singular locus) of the constructed variety are as expected to conclude by semicontinuity. Note that, once we get the expected invariants it is not important if the computation is made in positive or 0 characteristic as semicontinuity permits us to pass to characteristic zero in any case.

Observe first that, by the definition and properties of the map p, the variety J is contained in the hypersurface Z_3 which is the cone over the symmetric determinantal cubic hypersurface in $\mathbb{P}^5$ described in Lemma 5.2. In particular Z_3 is singular along a threefold cone over the Veronese surface and has generically A_1 singularities. Moreover, by construction, J is the image of a symmetric quartic hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$, being a symmetric EPW quartic, via the quotient by the symmetry. It follows that J is a fourfold of degree 12 in Z_3. Using our script in Macaualy2 we find an explicit case where J is complete intersection of Z_3 with a quartic and this must hence be the generic behavior.

Moreover the intersection of the singular locus of Z_3 with J is a quartic section of a cone over the Veronese surface and is part of the singular locus of J. Using our Macaulay 2 script we check on an example that this quartic section of the cone over the Veronese surface is a surface of degree 16 as expected hence this is also the generic case for J.

Recall now, that a very general EPW quartic is singular along a surface of degree 72 in $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^9$. It follows that a general symmetric EPW quartic has also at least a surface of degree 72 as singularities. Our Macaulay2 computation shows that there are symmetric EPW quartics for which the singular surface is indeed of degree 72. This surface is mapped via the map p to a surface in J which is necessarily part of the singular locus and has degree at least 36. Our Macaulay 2 script produces an example where this surface of degree 72 is mapped to a surface of degree 36, hence this must be the general behavior for symmetric EPW quartics.

Summing up, the variety J in general must contain in its singular locus the following surfaces:

(1) The intersection of the singular locus of Z_3 with Z_4. Since $\mathrm{Sing}(Z_3)$ is a cone over the Veronese surface, $\mathrm{Sing}(Z_3) \cap Z_4$ has degree 16.

(2) The quotient of the singular locus of the symmetric EPW quartic by the involution, which is a variety of degree $72 : 2 = 36$.

Since in our explicit example the singular locus of J consists of two disjoint surfaces one of degree 16 and one of degree 36 and both need to appear in the very general case this concludes the proof. □

Remark 5.4. In the above proof we can avoid computer calculation with some additional effort. First, we prove that J is in general smooth in codimension 1. Indeed, if J was singular in codimension 1, then the corresponding symmetric EPW quartic would either be singular in codimension 1 or would need to contain the whole ramification locus including the vertex of the cone. Both these cases cannot occur; see [Iliev et al. 2017, Section 3].

Next, from the shape of the Néron–Severi lattice of a very general symmetric double EPW quartic we can deduce that there are no divisors contracted by the map from the double EPW quartic to the cone over $\mathbb{P}^2 \times \mathbb{P}^2$ hence the very general symmetric EPW quartic is singular in a surface of degree 72 and has no additional singularities as in the very general nonsymmetric case.

Finally, following the construction in [Iliev et al. 2017, Proposition 2.14], we can describe the symmetric EPW quartic via the varieties of (1, 1) conics on their corresponding symmetric Verra fourfolds (see Lemma 5.1) and deduce that the singular surface of degree 72 has no component contained in the cone over the diagonal. Hence the image of this singular surface of degree 72, which is necessarily symmetric, via the projection p is a surface of degree 36 and by the fact that p is a local isomorphism outside its branch locus is necessarily a component of the singular locus of J. For the same reason J is smooth outside the union of the branch locus and the surface of degree 36.

Proposition 5.5. *The map $\varphi_{|D_1|} : Y \to \mathbb{P}^6$ is $2:1$ onto $\varphi_{|D_1|}(Y) \subset \mathbb{P}^6$ and its image is isomorphic to J, the complete intersection $Z_3 \cap Z_4 \subset \mathbb{P}^6$. The exceptional divisor $\Sigma \subset Y$ is mapped to a component of degree 4 of the surface S_{16}. Moreover, the (-2)-class $D_1 - \Sigma$ is effective on Y and contracted to a surface via $2D_1 - \Sigma$. There are no more contractible classes on any birational model of Y.*

Proof. By Section 4B, $\varphi_{|D_1|}(Y)$ is the image of the projection of $\varphi_{|A|}(X)$ from a σ-invariant subspace in $H^0(X, A)^\vee$. In our context this implies that $\varphi_{|D_1|}(Y) = p(\varphi_{|A|}(X))$ and hence Lemma 5.1 shows that

$\varphi_{|D_1|}$ is $2:1$ and Proposition 5.3 describes $\varphi_{|D_1|}(Y)$. In particular we have the following diagram

$$\begin{tikzcd}
X \arrow[r, "2:1"] \arrow[d, dashed, "2:1"] \arrow[rrr, bend left, "\varphi_{|A|}"] & \varphi_{|A|}(X) \arrow[r, hook] \arrow[d, dashed, "2:1"] & C(\mathbb{P}^2 \times \mathbb{P}^2) \arrow[r, hook] \arrow[d, dashed, "2:1"] & \mathbb{P}^9 \arrow[d, "p"] \\
Y \arrow[r, "2:1"] \arrow[rrr, bend right, "\varphi_{|D_1|}"'] & J = Z_4 \cap Z_3 \arrow[r, hook] & Z_3 = p(C(\mathbb{P}^2 \times \mathbb{P}^2)) \arrow[r, hook] & \mathbb{P}^6.
\end{tikzcd}$$

where $J := p(\varphi_{|A|}(X))$ is $Z_4 \cap Z_3$ by Proposition 5.3. The exceptional divisor Σ resolves the singularity of X/σ in the $K3$ surface image of the σ-fixed surface S on X. The latter surface is in $C(\mathbb{P}^2 \times \mathbb{P}^2)$. The symplectic involution on X is induced by the symmetry on $\mathbb{P}^9$ that interchanges the factors of $\mathbb{P}^2 \times \mathbb{P}^2$. So the $K3$ surface S in $C(\mathbb{P}^2 \times \mathbb{P}^2)$, being fixed by the involution, is contained in the cone over the diagonal in $\mathbb{P}^2 \times \mathbb{P}^2$. It follows that its image is a component of S_{16}. By Lemma 4.3 it is a surface S_4 of degree 4 which is necessarily projectively isomorphic to the Veronese surface.

For the second part we observe that the proper transform on Y of the intersection of $\varphi_{|D_1|}(Y)$ with the span of S_4 in $\mathbb{P}^6$ is the (-2)-class $D_1 - \Sigma$. The system $2D_1 - \Sigma$ is big and induces its contraction since on $\varphi_{|D_1|}(Y)$ it can be seen as a system of quadrics containing the Veronese surface S_4 i.e., it contracts the planes spanned by conics on S_4 which fill the cubic Z_3 intersected with the span of S_4. The locus contracted by $2D_1 - \Sigma$ is hence exactly the (-2)-class $D_1 - \Sigma$. Observe that there can be no more contractible divisorial classes on any birational model of Y. For that, we work in codimension 1 knowing [Menet and Rieß 2020, Lemma 3.2] that any birational map is regular in codimension 1. Now, since the Picard number of Y is 2, among three big divisor classes one of them is a positive combination of the two other ones. In particular, if we have three divisor classes each contracted by some map associated to a big divisor then one of these big divisors is a positive linear combination of the two remaining ones. But a positive combination of two big divisors can only contract subvarieties which are contracted by both divisors, so all three contracted divisors would need to have a common component. However, both Σ and $D_1 - \Sigma$ are represented by distinct irreducible effective divisors so have no common component. $\square$

In the next section we will study the locally complete projective family of orbifolds of Nikulin type to which Y as in Proposition 5.5 belongs and we will show that they can all be realized as certain double covers, in complete analogy with what happens in the case of double EPW sextics. Since the full monodromy group of orbifolds of Nikulin type of dimension 4 is not known yet, we will first use the nonsymplectic involution on Y given by the double cover to produce an involution of $H^2(Y, \mathbb{Z})$ which is a monodromy operator and has the span of the divisor D_1 as the only invariant classes. We recall the following notation: given an element $e \in H^2(Y, \mathbb{Z})$, the reflection r_e in e is the isometry defined by $r_e(x) = x - (2B_Y(x, e)/q_Y(e))e$ (it is integral only for special values of $q_Y(e)$ and $\operatorname{div}(e)$).

Lemma 5.6. *Let D_1 be the class with $q_Y(D_1) = 2$ and divisibility 1 considered above. The isometry $-r_{D_1}$, such that $x \mapsto -x + B_Y(x, D_1)D_1$, in $H^2(Y, \mathbb{Z})$ is a monodromy operator of $H^2(Y, \mathbb{Z})$.*

Proof. The map $\varphi_{|D_1|}$ is a generically $2:1$ map onto its image and so there exists the involution Θ which is the cover involution of $Y \to \varphi_{|D_1|}(Y)$. First $\varphi_{|D_1|}$ contracts the exceptional (-4)-class Σ and then it identifies points switched by Θ. So Θ^* acts as -1 on the transcendental lattice T_Y and acts trivially on $\mathrm{NS}(Y)$, generated by D_1 and Σ, see Proposition 3.8. Moreover Θ^* is a monodromy operator of $H^2(Y, \mathbb{Z})$, since it is induced by an automorphism of Y.

Let r_Σ be the reflection given by $r_\Sigma(x) = x + \frac{1}{2}B_Y(x, \Sigma)\Sigma$ for $x \in H^2(Y, \mathbb{Z})$. It is a monodromy operator by [Menet and Rieß 2021, Proposition 1.5]. We observe that $-r_{D_1} = \Theta^* \circ r_\Sigma$ and so $-r_{D_1}$ is a monodromy operator. □

5B. ***The family of complete intersections* (3, 4).** Let Y be an orbifold of Nikulin type such that

$$(H^2(Y, \mathbb{Z}), q_Y) \simeq U(2)^{\oplus 3} \oplus E_8(-1) \oplus \langle -2 \rangle \oplus \langle -2 \rangle,$$

and there exists an ample Cartier divisor H on Y with degree $q_Y(H) = 2$ and divisibility 1. Such an orbifold exists by surjectivity of the period map. Since the Fujiki constant for Y is 6 we have $H^4 = 24$.

Theorem 5.7. *The map $\varphi_{|H|} : Y \to \mathbb{P}^6$ is $2:1$ and its image is a special fourfold of codimension 2 in $\mathbb{P}^6$ being the complete intersection of a cubic and a quartic. The map is branched along a surface of degree 48.*

Proof. By Corollary 4.4 and the Kawamata–Viehweg vanishing theorem we have

$$h^0(Y, \mathcal{O}(H)) = 7.$$

Hence the target space of $\varphi_{|H|}$ is $\mathbb{P}^6$.

Let Y_0 be the special Nikulin orbifold considered in Section 5A and H_0 be the divisor $D_1 \in \mathrm{NS}(Y_0)$ considered in Proposition 5.5.

From Proposition 5.5, $\varphi_{|H_0|}$ is $2:1$ and hence there exists an involution Θ_0 on Y_0, which is the cover involution and it is nonsymplectic. Moreover the image $\varphi_{|H_0|}(Y_0)$ is a normal complete intersection of type $(3, 4)$. The idea of the proof is to show that a general deformation of (Y_0, H_0) is of the same shape.

Let $(\pi : \mathcal{Y} \to B, \mathcal{H})$ be a family of polarized orbifolds of degree 2 and divisibility 1 with central fiber (Y_0, H_0) over a small disc $B \ni 0$. From Lemma 5.6, $-r_{H_0}$ is a monodromy operator of $H^2(Y_0, \mathbb{Z})$.

Let $t_n \in B$, Y_{t_n} be the fiber of π over t_n and let H_{t_n} be the restriction of $\mathcal{H}$ to Y_{t_n}. We fix a sequence $t_n \to 0$ such that $\mathrm{NS}(Y_{t_n}) = \mathbb{Z}H_{t_n}$. By parallel transport $-r_{H_{t_n}}$ is a monodromy operator of $H^2(Y_{t_n}, \mathbb{Z})$ and, by a standard argument using $\rho(Y_{t_n}) = 1$ and the global Torelli theorem (see for example [Menet and Rieß 2020, Theorem 1.1]), $-r_{H_{t_n}}$ lifts to an involution $\Theta_{t_n} : Y_{t_n} \to Y_{t_n}$.

Arguing as in [O'Grady 2005, Section 2], the limit of Θ_{t_n} is an involution on Y_0 and we show that it is Θ_0. We denote the graph of Θ_{t_n} by Γ_{t_n}. The analytic cycles Γ_{t_n} converge (see the proof of [Huybrechts 1999, Theorem 4.3]) to Γ_0 with a decomposition $\Gamma + n_i\Omega_i$ where Γ is the graph of a birational map $Y_0 \dashrightarrow Y_0$ and Ω_i are irreducible in $D_i \times E_i$ with $D_i, E_i \subset Y_0$ proper subsets. As in [O'Grady 2005, Section 2], Γ_0 induces on $H^2(Y_0, \mathbb{Z})$ exactly the monodromy operator $-r_{H_0}$ via parallel transport.

Again as in [loc. cit.], the invariance of Γ_{t_n} with respect to the exchange of the two factors in $Y_0 \times Y_0$ implies that Γ_0 is invariant as well and, due to the different nature of the two parts in the decomposition above, that Γ is the graph of a birational involution.

If D_i has codimension > 1, the action on $H^2(Y_0, \mathbb{Z})$ of $[\Omega_i]_*$ is zero, thus we assume that D_i is an effective divisor in $\mathrm{NS}(Y_0) = \langle H_0, \Sigma \rangle$, but this implies that the action of Γ on T_{Y_0} coincides with the action of Γ_0, i.e., it acts as $-\mathrm{id}$ on the transcendental lattice. It follows from Proposition 5.5 that there are exactly 2 contractible classes on Y_0: the (-4)-class Σ and the (-2)-class $H_0 - \Sigma$. Hence Σ and $H_0 - \Sigma$ are preserved by any birational map, and thus also by $[\Gamma]_*$ i.e.,

$$[\Gamma]_*(H_0) = H_0, \quad [\Gamma]_*(\Sigma) = \Sigma.$$

We conclude that, since $[\Gamma]_*$ acts on $H^2(Y_0, \mathbb{Z})$ preserving both H_0 and Σ and acts as minus the identity on their orthogonal in $H^2(Y_0, \mathbb{Z})$, it coincides with Θ_0^*. In the case of orbifolds of Nikulin type, the only automorphism acting trivially in cohomology is the identity, as shown in [Menet and Rieß 2021, Proposition 8.1], hence the birational involution associated to Γ is exactly the nonsymplectic involution $\Theta_0 \in \mathrm{Aut}(Y_0)$.

We thus have a sequence $(Y_{t_n}, H_{t_n}, \Theta_{t_n})$ of polarized orbifolds of Nikulin type each equipped with an involution Θ_{t_n} preserving H_{t_n} and such that (Y_0, H_0, Θ_0) is its limit in the sense above. The involutions Θ_{t_n} induce a sequence of involutions on $H^0(Y_{t_n}, H_{t_n}) = H^0(Y_0, H_0)$ whose limit is the map induced by Θ_0 on $H^0(Y_0, H_0)$. The latter is the identity map because Θ_0 is the cover involution of $\varphi_{|H_0|}$. It follows that for $n \gg 1$ the action of Θ_{t_n} on $H^0(Y_{t_n}, H_{t_n})$ is also trivial and hence $\varphi_{|H_{t_n}|}$ is $2:1$ for $n \gg 1$. We conclude that for general (Y_t, H_t) in a neighborhood of (Y_0, H_0) the map $\varphi_{|H_t|}$ is $2:1$ onto the image contained in $\mathbb{P}^6$.

We saw that $J_0 := \varphi_{|H_0|}(Y_0)$ is normal, hence by the openness of normality the image J_t of Y_t through $|H_t|$ is also normal of codimension 2 in $\mathbb{P}^6$. Thus J_t is necessarily the quotient of Y_t through the involution Θ_t. In particular, J_t has ODP points along a surface that is smooth outside the 28 orbifold points. Let us show that the general J_t is also a complete intersection. We consider the family $\{G_t\}_{t\in\Delta}$, with Δ a small disc, with $G_t = \varphi_{H_t}^{-1}(\Pi_1 \cap \Pi_2)$ and Π_i two chosen general hyperplanes in $\mathbb{P}^6$. Note that G_0 is smooth and maps via φ_{H_0} to $J_0 \cap \Pi_1 \cap \Pi_2$ which is a complete intersection $(3, 4)$ in $\mathbb{P}^4$ and which must admit only nodes as singularities. It follows that the general G_t maps via φ_{H_t} to a nodal surface in $R_t = J_t \cap \Pi_1 \cap \Pi_2 \subset \mathbb{P}^4 = \Pi_1 \cap \Pi_2$ being the quotient of G_t through an involution. Such surface is of degree 12 and half-canonical i.e., $K_{R_t} = 2H$ (where H is the hyperplane from $\mathbb{P}^4$).

We can now mimic [Decker et al. 1990, Proposition 1.2] to prove that R_t is a complete intersection. Indeed, R_t is a half canonical surface and since R_t has complete intersection singularities it is the zero locus of a rank 2 vector bundle E on $\mathbb{P}^4$ hence the methods of [loc. cit., Proposition 1.2] apply also in this case. More precisely, the case $c_1(E)^2 - 4c_2(E) \leq 0$ from [loc. cit., Proposition 1.2] cannot occur by a generalization of the double point formula for nodal hypersurfaces proved in [Catanese and Oguiso 2020, Theorem 5.1] ($\delta = 0$ in our case). Thus $c_1(E)^2 - 4c_2(E) > 0$ and we conclude as in Case 2 of [Decker et al. 1990, Proposition 1.2] that $R_t \subset \mathbb{P}^4$ is a complete intersection $(3, 4)$.

We thus know that a general codimension two linear section R_t of J_t is a complete intersection $(3,4)$. To conclude that J_t must also be such a complete intersection let us consider $U_t \supset R_t$ a general hyperplane section of J_t containing R_t and the exact sequences

$$0 \to \mathcal{I}_{J_t} \to \mathcal{I}_{J_t}(1) \to \mathcal{I}_{U_t|\mathbb{P}^5}(1) \to 0 \quad \text{and} \quad 0 \to \mathcal{I}_{U_t} \to \mathcal{I}_{U_t}(1) \to \mathcal{I}_{R_t|\mathbb{P}^4}(1) \to 0.$$

To conclude it is enough to show that $h^1(\mathcal{I}_{J_t}(2)) = h^1(\mathcal{I}_{U_t}(2)) = 0$ and $h^1(\mathcal{I}_{J_t}(3)) = h^1(\mathcal{I}_{U_t}(3)) = 0$ (then the cubic and a quartic defining R_t extend to the ideal of U_t and then J_t). But applying again the long exact sequence of cohomology the vanishing of $h^1(\mathcal{I}_{U_t}(k))$ will follow from the vanishing of $h^2(\mathcal{I}_{J_t}(k-1))$ and $h^1(\mathcal{I}_{J_t}(k))$. It is hence enough to prove

$$h^2(\mathcal{I}_{J_t}(2)) = h^2(\mathcal{I}_{J_t}(1)) = h^1(\mathcal{I}_{J_t}(3)) = h^1(\mathcal{I}_{J_t}(2)) = 0. \tag{5-1}$$

We compute these dimensions using the finite map $f : Y_t \to J_t$: there exists a sheaf $\mathcal{F}$ on J_t such that

$$f_*\mathcal{O}_{Y_t}(k) = \mathcal{O}_{J_t}(k) \oplus \mathcal{F}(k).$$

We get our vanishings (5-1) from the fact that $h^i(\mathcal{O}_{Y_t}(k)) = 0$ for $i = 1, 2$ and $k = 2, 3$. We conclude that $\mathcal{I}_{J_t}$ admits a cubic and a quartic generator which, after restriction to a codimension 2 linear space, define a complete intersection. Since J_t is of codimension 2 and degree 12, J_t is a complete intersection.

Let us compute the degree of the singular surface of J_t (in fact we can deduce this from the singular locus of J_0 finding $52 - 4 = 48$). If we denote by $F \subset Y_t$ a general intersection of two divisors in the system H_t in Y_t then we find that $K_F = 2H_{t|F}$ and $\chi(\mathcal{O}_F(nH_t)) = 12n^2 - 24n + 20$. Denote by $G \subset \mathbb{P}^4$ the image of F being a complete intersection $(3,4)$. The involution given by $|H_t|$ cannot fix varieties of odd codimension (since the smooth locus of the orbifold Y_t has a symplectic form and the singular locus consists of isolated points). Moreover, it cannot fix smooth points, since it is a nonsymplectic involution. The orbifold points are in the fixed locus otherwise they would map to noncomplete intersection singularities. So the ramification of the map is a surface. We find that G is nodal and $F \to G$ is branched at the nodes. Let μ be the number of nodes. We shall compute μ by comparing the Euler characteristics of appropriate sheaves on F and G. First observe that $\chi(\mathcal{O}_G) = 16$ since G is a complete intersection. Next we consider the minimal resolution $\overline{G}$ of G and the blow up $\overline{F}$ of F at the preimages of the nodes together with the induced map $f : \overline{F} \to \overline{G}$. We find $f_*\mathcal{O}_{\overline{F}} = \mathcal{O}_{\overline{G}} \oplus \mathcal{O}_{\overline{G}}(L)$ where $2L$ is the sum of the exceptional divisors on $\overline{G}$. We compare the Riemann–Roch formulas for $\overline{F}$ and $\overline{G}$ and conclude from $2\chi(\mathcal{O}_G) - \frac{\mu}{4} = \chi(\mathcal{O}_{\overline{F}}) = \chi(\mathcal{O}_F) = 20$ that $\mu = 48$. □

5C. ***A special subfamily of BBF degree 2.*** We consider Nikulin orbifolds with a Cartier divisor of BBF degree 2 and divisibility 1 that form a subfamily of codimension 2 of the locally complete family described in Theorem 5.7.

These orbifolds are constructed as quotients of $W^{[2]}$ by a natural symplectic involution $\sigma^{[2]}$, where W is a $K3$ surface with $\mathrm{NS}(W) \simeq \widetilde{\Lambda_4}$ and σ is a symplectic involution on it. The surfaces W are double covers of a quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$ branched along a $(2,2)$ curve C that is symmetric with respect to the involution ι_Q exchanging the two factors of Q; see [van Geemen and Sarti 2007]. We denote by j the cover involution of $W \to Q$ and we observe that ι_Q lifts to two involutions on W: a nonsymplectic involution ι and a symplectic involution σ. We observe that $\iota = j \circ \sigma$.

Then σ induces a natural involution $\sigma^{[2]}$ on $W^{[2]}$ fixing 28 points and a $K3$ surface S. The ample divisor of degree 4 on W invariant for σ^* induces a divisor A on $W^{[2]}$ which is orthogonal to the exceptional divisor of $W^{[2]} \to \mathrm{Sym}^2(W)$.

The map given by $|A|$ can be described as follows:

$$\varphi_{|A|}\colon (\mathbb{P}^3)^{[2]} \supset W^{[2]} \to \mathrm{Sym}^2(Q) \subset \mathrm{Sym}^2(\mathbb{P}^3) \subset \mathbb{P}^9.$$

The involution ι induces on $\mathbb{P}^9$ a linear involution of the form $(-,-,-,+,+,+,+,+,+,+)$ so we have two invariant linear spaces $\mathbb{P}^6_+$ and $\mathbb{P}^2_-$. By Proposition 3.11 the Nikulin orbifold $W^{[2]}/\sigma^{[2]}$ admits a polarization H of BBF degree 2 and divisibility 1 induced by A.

Lemma 5.8. *The image of the Nikulin orbifold $W^{[2]}/\sigma^{[2]}$ through the $4:1$ map given by H is a special complete intersection $(2,3)$ in $\mathbb{P}^6$. This fourfold is a degeneration of the family of $(3,4)$ intersections described in Theorem 5.7.*

Proof. The image of the map φ is a fourfold of degree 12 in $\mathbb{P}^9$ that can be seen as the secant variety of the second Veronese embedding of a quadric surface. The projection from $\mathbb{P}^2_-$ is no longer $2:1$, as it can be checked on a special fiber. The image is contained in a quadric since we find that $\varphi_{|A|}(W^{[2]})$ is contained in a quadric being a cone over $\mathbb{P}^2_-$. We conclude knowing the degree of the fourfold in $\mathbb{P}^6$. □

Remark 5.9. One can show using computer calculations that the intersection $(2,3)$ above is singular along a surface of degree 12.

Note that the involution $\iota^{(2)}$ on $Q^{(2)}$ has two fixed surfaces: B_1 consisting of the pairs $(x, j(x))$ for $x \in Q$ and B_2 consisting of the pairs $(c_1, c_2) \subset C^{(2)} \subset Q^{(2)}$. We see that the fixed $K3$ surface S is mapped to B_1 and the isolated points are mapped to B_2.

5D. ***Lagrangian type description.*** Let us describe an object analogous to the Lagrangian subspace of dimension 10 of $\bigwedge^3 \mathbb{C}^6$ for double EPW sextics. Suppose (Y, H) is a polarized orbifold of Nikulin type with degree $q_Y(H) = 2$ such that $|H|$ induces a finite $2:1$ morphism. Then $|H|$ defines a $2:1$ map f with image J being a 4-dimensional variety of degree 12 in $\mathbb{P}^6$ singular along a surface. Since f is finite, there exists a sheaf $\mathcal{F}$ on J such that

$$f_*\mathcal{O}_Y = \mathcal{O}_J \oplus \mathcal{F}.$$

We infer from the Riemann–Roch theorem the following table:

$$\begin{array}{ccccccc}
H^4(\mathcal{F}(-3)) & H^4(\mathcal{F}(-2)) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbb{C} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & H^0(\mathcal{F}(2)) & H^0(\mathcal{F}(3))
\end{array}$$

So we have the following symmetric Beilinson resolution:

$$0 \to 28\Omega^6(6) \xrightarrow{M^*} 3\Omega^5(5) \oplus \Omega^3(3) \oplus 3\Omega^1(1) \xrightarrow{M} 28\mathcal{O} \to \mathcal{F}(3) \to 0$$

The matrix corresponding to M from the Beilinson resolution is a matrix with three rows of 1 forms, one row of 3-forms and three rows of 5-forms. Moreover, it has the property that $MM^* = 0$ as matrices of 4-forms (the product is induced by the exterior product of forms).

The choice of M is thus the choice of a 28 dimensional linear subspace in

$$3 \bigwedge^5 V_7 \oplus \bigwedge^3 V_7 \oplus 3V_7,$$

isotropic for the product b that can be seen as a kind of symplectic form

$$b : \left(3V_7 \oplus \bigwedge^3 V_7 \oplus 3 \bigwedge^5 V_7\right)^2 \to \bigwedge^6 V_7$$

given by the formula

$$b((l_1, l_2, l_3, \alpha, w_1, w_2, w_3), (L_1, L_2, L_3, \beta, W_1, W_2, W_3)) \\ = L_1 \wedge w_1 + L_2 \wedge w_2 + L_3 \wedge w_3 + \alpha \wedge \beta + l_1 \wedge W_1 + l_2 \wedge W_2 + l_3 \wedge W_3.$$

Note that the variety J, being the support of $\mathcal{F}(3)$, appears as a degeneracy locus of such a map M.

Problem 5.10. (1) *Describe the "Lagrangian"* 28 *space corresponding to Nikulin orbifolds i.e., quotients of fourfolds of* $K3^{[2]}$*-type as described in Section 5A.*

(2) *How to describe the cubic in the complete intersection* $(3, 4)$?

(3) *Is the moduli space of polarized orbifolds of Nikulin type of dimension* 4 *and BBF degree* 2 *unirational*?

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References

[Bakker and Lehn 2022] B. Bakker and C. Lehn, "The global moduli theory of symplectic varieties", *J. Reine Angew. Math.* **790** (2022), 223–265. MR Zbl

[Bayer et al. 2021] A. Bayer, M. Lahoz, E. Macrì, H. Nuer, A. Perry, and P. Stellari, "Stability conditions in families", *Publ. Math. Inst. Hautes Études Sci.* **133** (2021), 157–325. MR

[Beauville 1983] A. Beauville, "Some remarks on Kähler manifolds with $c_1 = 0$", pp. 1–26 in *Classification of algebraic and analytic manifolds* (Katata, 1982), edited by K. Ueno, Progr. Math. **39**, Birkhäuser, Boston, 1983. MR Zbl

[Beauville 2000] A. Beauville, "Symplectic singularities", *Invent. Math.* **139**:3 (2000), 541–549. MR Zbl

[Beauville and Donagi 1985] A. Beauville and R. Donagi, "La variété des droites d'une hypersurface cubique de dimension 4", *C. R. Acad. Sci. Paris Sér. I Math.* **301**:14 (1985), 703–706. MR Zbl

[Beckmann and Oberdieck 2022] T. Beckmann and G. Oberdieck, "Equivariant categories of symplectic surfaces and fixed loci of Bridgeland moduli spaces", *Algebr. Geom.* **9**:4 (2022), 400–442. MR Zbl

[Blache 1996] R. Blache, "Chern classes and Hirzebruch–Riemann–Roch theorem for coherent sheaves on complex-projective orbifolds with isolated singularities", *Math. Z.* **222**:1 (1996), 7–57. MR Zbl

[Bogomolov 1978] F. A. Bogomolov, "Hamiltonian Kählerian manifolds", *Dokl. Akad. Nauk SSSR* **243**:5 (1978), 1101–1104. In Russian; translated in *Soviet Math. Dokl.* **19**:6 (1978), 1462–1465. MR

[Bădescu and Beltrametti 2013] L. Bădescu and M. C. Beltrametti, "Seshadri positive submanifolds of polarized manifolds", *Cent. Eur. J. Math.* **11**:3 (2013), 447–476. MR Zbl

[Buckley et al. 2013] A. Buckley, M. Reid, and S. Zhou, "Ice cream and orbifold Riemann–Roch", *Izv. Ross. Akad. Nauk Ser. Mat.* **77**:3 (2013), 29–54. In Russian; translated in *Izv. Math.* **77**:3 (2013), 461–486. MR Zbl

[Camere 2012] C. Camere, "Symplectic involutions of holomorphic symplectic four-folds", *Bull. Lond. Math. Soc.* **44**:4 (2012), 687–702. MR Zbl

[Camere and Garbagnati 2020] C. Camere and A. Garbagnati, "On certain isogenies between K3 surfaces", *Trans. Amer. Math. Soc.* **373**:4 (2020), 2913–2931. MR Zbl

[Camere et al. 2019a] C. Camere, A. Garbagnati, and G. Mongardi, "Calabi–Yau quotients of hyperkähler four-folds", *Canad. J. Math.* **71**:1 (2019), 45–92. MR Zbl

[Camere et al. 2019b] C. Camere, G. Kapustka, M. Kapustka, and G. Mongardi, "Verra four-folds, twisted sheaves, and the last involution", *Int. Math. Res. Not.* **2019**:21 (2019), 6661–6710. MR Zbl

[Campana 2004] F. Campana, "Orbifoldes à première classe de Chern nulle", pp. 339–351 in *The Fano Conference*, edited by A. Collino et al., Univ. Torino, Turin, 2004. MR Zbl

[Catanese and Oguiso 2020] F. Catanese and K. Oguiso, "The double point formula with isolated singularities and canonical embeddings", *J. Lond. Math. Soc.* (2) **102**:3 (2020), 1337–1356. MR Zbl

[Debarre and Voisin 2010] O. Debarre and C. Voisin, "Hyper-Kähler fourfolds and Grassmann geometry", *J. Reine Angew. Math.* **649** (2010), 63–87. MR Zbl

[Decker et al. 1990] W. Decker, J. Le Potier, T. Peternell, and M. Schneider, "Half-canonical surfaces in $\mathbf{P}_4$", pp. 91–110 in *Algebraic geometry* (L'Aquila, 1988), edited by A. J. Sommese et al., Lecture Notes in Math. **1417**, Springer, 1990. MR Zbl

[Farkas and Verra 2016] G. Farkas and A. Verra, "Prym varieties and moduli of polarized Nikulin surfaces", *Adv. Math.* **290** (2016), 314–328. MR

[Fu and Menet 2021] L. Fu and G. Menet, "On the Betti numbers of compact holomorphic symplectic orbifolds of dimension four", *Math. Z.* **299**:1-2 (2021), 203–231. MR

[Fujiki 1983] A. Fujiki, "On primitively symplectic compact Kähler V-manifolds of dimension four", pp. 71–250 in *Classification of algebraic and analytic manifolds* (Katata, 1982), edited by K. Ueno, Progr. Math. **39**, Birkhäuser, Boston, 1983. MR Zbl

[Fulton 1998] W. Fulton, *Intersection theory*, 2nd ed., Ergeb. Math. Grenzgeb. **2**, Springer, 1998. MR Zbl

[Garbagnati and Sarti 2008] A. Garbagnati and A. Sarti, "Projective models of $K3$ surfaces with an even set", *Adv. Geom.* **8**:3 (2008), 413–440. MR

[van Geemen and Sarti 2007] B. van Geemen and A. Sarti, "Nikulin involutions on $K3$ surfaces", *Math. Z.* **255**:4 (2007), 731–753. MR

[Gross et al. 2003] M. Gross, D. Huybrechts, and D. Joyce, *Calabi–Yau manifolds and related geometries*, Springer, 2003. MR Zbl

[Huybrechts 1999] D. Huybrechts, "Compact hyper-Kähler manifolds: basic results", *Invent. Math.* **135**:1 (1999), 63–113. MR Zbl

[Iliev and Ranestad 2001] A. Iliev and K. Ranestad, "$K3$ surfaces of genus 8 and varieties of sums of powers of cubic fourfolds", *Trans. Amer. Math. Soc.* **353**:4 (2001), 1455–1468. MR Zbl

[Iliev et al. 2017] A. Iliev, G. Kapustka, M. Kapustka, and K. Ranestad, "Hyper-Kähler fourfolds and Kummer surfaces", *Proc. Lond. Math. Soc.* (3) **115**:6 (2017), 1276–1316. MR Zbl

[Iliev et al. 2019] A. Iliev, G. Kapustka, M. Kapustka, and K. Ranestad, "EPW cubes", *J. Reine Angew. Math.* **748** (2019), 241–268. MR Zbl

[Joyce 2000] D. D. Joyce, *Compact manifolds with special holonomy*, Oxford University Press, 2000. MR Zbl

[Kollár and Mori 1998] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics **134**, Cambridge University Press, 1998. MR Zbl

[Lehn 2018] C. Lehn, "Twisted cubics on singular cubic fourfolds—on Starr's fibration", *Math. Z.* **290**:1-2 (2018), 379–388. MR Zbl

[Lehn et al. 2017] C. Lehn, M. Lehn, C. Sorger, and D. van Straten, "Twisted cubics on cubic fourfolds", *J. Reine Angew. Math.* **731** (2017), 87–128. MR Zbl

[Markushevich and Tikhomirov 2007] D. Markushevich and A. S. Tikhomirov, "New symplectic V-manifolds of dimension four via the relative compactified Prymian", *Internat. J. Math.* **18**:10 (2007), 1187–1224. MR Zbl

[Menet 2015] G. Menet, "Beauville–Bogomolov lattice for a singular symplectic variety of dimension 4", *J. Pure Appl. Algebra* **219**:5 (2015), 1455–1495. MR Zbl

[Menet 2020] G. Menet, "Global Torelli theorem for irreducible symplectic orbifolds", *J. Math. Pures Appl.* (9) **137** (2020), 213–237. MR Zbl

[Menet and Rieß 2020] G. Menet and U. Rieß, "On the Kähler cone of irreducible symplectic orbifolds", preprint, 2020. arXiv 2009.04873

[Menet and Rieß 2021] G. Menet and U. Rieß, "Wall divisors on irreducible symplectic orbifolds of M'-type", preprint, 2021, available at https://tinyurl.com/MenetRiess-WallDivisors.

[Mongardi 2012] G. Mongardi, "Symplectic involutions on deformations of $K3^{[2]}$", *Cent. Eur. J. Math.* **10**:4 (2012), 1472–1485. MR Zbl

[Mongardi and Wandel 2015] G. Mongardi and M. Wandel, "Induced automorphisms on irreducible symplectic manifolds", *J. Lond. Math. Soc.* (2) **92**:1 (2015), 123–143. MR Zbl

[Morrison 1984] D. R. Morrison, "On $K3$ surfaces with large Picard number", *Invent. Math.* **75**:1 (1984), 105–121. MR Zbl

[Nikulin 1979a] V. V. Nikulin, "Finite groups of automorphisms of Kählerian $K3$ surfaces", *Trudy Moskov. Mat. Obshch.* **38** (1979), 75–137. In Russian; translated in *Moscow Math. Soc* **38**(1980), 71–135. MR Zbl

[Nikulin 1979b] V. V. Nikulin, "Integer symmetric bilinear forms and some of their geometric applications", *Izv. Akad. Nauk SSSR Ser. Mat.* **43**:1 (1979), 111–177, 238. In Russian; translated in *Math USSR-Izv.* **14**:1 (1979), 103–167. MR

[Nikulin 1983] V. V. Nikulin, "Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections. Algebrogeometric applications", *Journal of Soviet Mathematics* **22** (1983), 1401–1475. Zbl

[O'Grady 2005] K. G. O'Grady, "Involutions and linear systems on holomorphic symplectic manifolds", *Geom. Funct. Anal.* **15**:6 (2005), 1223–1274. MR Zbl

[Perego 2020] A. Perego, "Examples of irreducible symplectic varieties", pp. 151–172 in *Birational geometry and moduli spaces*, edited by E. Colombo et al., Springer INdAM Ser. **39**, Springer, 2020. MR Zbl

[Shioda 1986] T. Shioda, "An explicit algorithm for computing the Picard number of certain algebraic surfaces", *Amer. J. Math.* **108**:2 (1986), 415–432. MR Zbl

[Voisin 2022] C. Voisin, "A note on the topology of fibrations", (2022). Appendix to J. Shen and Q. Yin, "Topology of Lagrangian fibrations and Hodge theory of hyper-Kähler manifolds", *Duke Math. J.* **171**:1 (2022), 209–241. MR Zbl

[Yoshioka 2006] K. Yoshioka, "Moduli spaces of twisted sheaves on a projective variety", pp. 1–30 in *Moduli spaces and arithmetic geometry*, edited by S. Mukai et al., Adv. Stud. Pure Math. **45**, Math. Soc. Japan, Tokyo, 2006. MR Zbl

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chiara.camere@unimi.it *Dipartimento di Matematica, Università degli Studi di Milano, Milano, Italy*

alice.garbagnati@unimi.it *Dipartimento di Matematica, Università degli Studi di Milano, Milano, Italy*

grzegorz.kapustka@uj.edu.pl *Department of Mathematics and Informatics, Jagiellonian University, Kraków, Poland*

michal.kapustka@impan.pl *Institute of Mathematics of the Polish Academy of Sciences, Warsaw, Poland*

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