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**Spectral moment formulae for
 $GL(3) \times GL(2)$ L -functions
I: The cuspidal case**

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Spectral moment formulae of various shapes have proven very successful in studying the statistics of central L -values. We establish, in a completely explicit fashion, such formulae for the family of $GL(3) \times GL(2)$ Rankin–Selberg L -functions using the period integral method. Our argument does not rely on either the Kuznetsov or Voronoi formulae. We also prove the essential analytic properties and derive explicit formulae for the integral transform of our moment formulae. We hope that our method will provide deeper insights into moments of L -functions for higher-rank groups.

1. Introduction

1A. Background. The study of L -values at the central point $s = \frac{1}{2}$ has taken center stage in many branches of number theory over the past decades due to their profound arithmetic significance. A variety of perspectives have enriched our understanding of the nature of central L -values. In particular, a statistical perspective can offer valuable insights. Fundamental questions in this direction include the determination of (non)vanishing and sizes of these L -values. An effective way to approach problems of this sort is via *moments of L -functions*. Techniques from analytic number theory have proven fruitful in estimating the sizes of moments of all kinds. Moreover, spectacular results can be obtained when moment estimates join forces with arithmetic geometry and automorphic representations.

This line of investigation is nicely exemplified by the landmark result of [Conrey and Iwaniec 2000]. Let χ be a real primitive Dirichlet character (mod q) with q odd and square-free. The main object of [loc. cit.] is the cubic moment of $GL(2)$ automorphic L -functions of the congruence subgroup $\Gamma_0(q)$ twisted by χ . An *upper bound* of Lindelöf strength in the q -aspect was established therein. When combining this upper bound with the celebrated formula of [Waldspurger 1981], the famous Burgess $\frac{3}{16}$ -bound for Dirichlet L -functions was improved for the first time since the 1960's. In fact, Conrey and Iwaniec [2000] proved the bound

$$L\left(\frac{1}{2}, \chi\right) \ll_{\epsilon} q^{1/6+\epsilon}. \quad (1-1)$$

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Understanding the effects of a sequence of intricate arithmetic and analytic transformations constitutes a significant part of moment calculations as seen in [Conrey and Iwaniec 2000]. Surprisingly, such a sequence of [loc. cit.] ends up in a single elegant *identity* showcasing a duality between the cubic average over a basis of $GL(2)$ automorphic forms (Maass or holomorphic) and the fourth moment of $GL(1)$ L -functions. This remarkable phenomenon was uncovered relatively recently in [Petrow 2015]. His work consists of new elaborate analysis (see also [Young 2017]) building upon the foundation of [Conrey and Iwaniec 2000]. Further contributions to this topic include those in [Frolenkov 2020] and the earlier works [Ivić 2001; 2002], which studied other aspects of the problem. In its basic form, the identity roughly takes the shape

$$\sum_{f:GL(2)\text{-Maass/Holomorphic}} L\left(\frac{1}{2}, f\right)^3 = \int_{-\infty}^{\infty} |\zeta\left(\frac{1}{2} + it\right)|^4 dt + (***) , \quad (1-2)$$

where the weight functions for the moments are suppressed and (***) represents certain polar contributions.

Besides its structural elegance, the identity (1-2) comes with immediate applications. It leads to sharp moment estimates as a consequence of exact evaluation. As an extra benefit, it streamlines the analysis in the traditional, approximate approach. In [Petrow 2015], this identity was referred to as a “*Motohashi-type identity*”. Previously, Motohashi [1993; 1997] discovered a similar identity but with the test function chosen on the fourth moment side, i.e., in the reverse direction of [Conrey and Iwaniec 2000; Petrow 2015; Young 2017; Ivić 2001; 2002]. It greatly enhances our understanding of the fourth moment of the ζ -function. The recent works [Young 2011; Blomer et al. 2020; Topalogullari 2021; Kaneko 2022] have extended Motohashi’s work to Dirichlet L -functions.

Conrey and Iwaniec [2000, Introduction] further envisioned the possibilities and challenges of extending their method to a setting involving a $GL(3)$ automorphic form. This is natural because the cubic moment of $GL(2)$ L -functions can be regarded as the first moment of $GL(3) \times GL(2)$ Rankin–Selberg L -functions, averaged over a basis of $GL(2)$ automorphic forms, where the $GL(3)$ automorphic form is a minimal parabolic Eisenstein series. It is anticipated that advances in harmonic analysis of $GL(3)$ could provide new perspectives towards the Conrey–Iwaniec method. Furthermore, the $GL(3)$ set-up introduces an important new example: the first moment for the $GL(3) \times GL(2)$ family involving a $GL(3)$ cusp form, which necessitates the use of genuine $GL(3)$ techniques.

In the decade following [Conrey and Iwaniec 2000], two key breakthroughs made this extension possible for $GL(3)$. Firstly, the $GL(3)$ *Voronoi formula* was developed in [Miller and Schmid 2006] (see also [Goldfeld and Li 2006; Ichino and Templier 2013]), making it usable for a variety of analytic applications. Notably, the Hecke combinatorics of $GL(3)$ associated to twisting and ramifications are considerably more involved than the classical $GL(2)$ counterpart. Secondly, the $GL(3)$ Voronoi formula was successfully applied in [Li 2011] together with new techniques to obtain strong *upper bounds* for the first moment of $GL(3) \times GL(2)$ Rankin–Selberg L -functions in the $GL(2)$ spectral aspect. As a corollary, she obtained the first instance of subconvexity for $GL(3)$ automorphic L -functions.

1B. Main results. The purpose of this article is to further the investigation of $GL(3) \times GL(2)$ moments of L -functions. However, we will depart from the standard approaches in the existing literature. We are interested in understanding the *intrinsic mechanisms* and examining the *essential ingredients* that lead directly to the complete structure of these moments, including both main terms and off-diagonals. Addressing these aspects carefully is crucial for enabling generalizations to higher-rank groups. We find that the formalism of *period integrals* for $GL(3)$ is particularly effective in achieving these objectives.

We are ready to state the main result of this article, which is the Motohashi type moment identity behind the work of Li [2011].

Theorem 1.1. *Let:*

- Φ be a fixed, Hecke-normalized Maass cusp form of $SL_3(\mathbb{Z})$ with the Langlands parameters $(\alpha_1, \alpha_2, \alpha_3) \in (i\mathbb{R})^3$, and $\tilde{\Phi}$ be the dual form of Φ .
- $(\phi_j)_{j=1}^\infty$ be an orthogonal basis of **even**, Hecke-normalized Maass cusp forms of $SL_2(\mathbb{Z})$ which satisfy

$$\Delta\phi_j = \left(\frac{1}{4} - \mu_j^2\right)\phi_j.$$

- $L(s, \phi_j \otimes \Phi)$ and $L(s, \Phi)$ be the Rankin–Selberg L -function of the pair (ϕ_j, Φ) and the standard L -function of Φ respectively, where Λ denotes the corresponding complete L -functions.
- \mathcal{C}_η ($\eta > 40$) be the class of holomorphic functions H defined on the vertical strip $|\operatorname{Re} \mu| < 2\eta$ such that $H(\mu) = H(-\mu)$ and has rapid decay

$$H(\mu) \ll e^{-2\pi|\mu|} \quad (|\operatorname{Re} \mu| < 2\eta).$$

- For $H \in \mathcal{C}_\eta$, $(\mathcal{F}_\Phi H)(s_0, s)$ is the integral transform defined in (7-6) and it only depends on the Langlands parameters of Φ .

Then on the domain $\frac{1}{4} + \frac{1}{200} < \sigma < \frac{3}{4}$, we have the following moment identity:

$$\begin{aligned} \sum_{j=1}^{\infty} H(\mu_j) \frac{\Lambda(s, \phi_j \otimes \tilde{\Phi})}{\langle \phi_j, \phi_j \rangle} + \int_{(0)} H(\mu) \frac{\Lambda(s + \mu, \tilde{\Phi})\Lambda(1 - s + \mu, \Phi)}{|\Lambda(1 + 2\mu)|^2} \frac{d\mu}{4\pi i} \\ = \frac{\pi^{-3s}}{2} L(2s, \Phi) \int_{(0)} \frac{H(\mu)}{|\Gamma(\mu)|^2} \cdot \prod_{i=1}^3 \Gamma\left(\frac{s + \mu - \alpha_i}{2}\right) \Gamma\left(\frac{s - \mu - \alpha_i}{2}\right) \frac{d\mu}{2\pi i} \\ + \frac{1}{2} L(2s - 1, \Phi) (\mathcal{F}_\Phi H)(2s - 1, s) \\ + \frac{1}{2} \int_{(1/2)} \zeta(2s - s_0) L(s_0, \Phi) (\mathcal{F}_\Phi H)(s_0, s) \frac{ds_0}{2\pi i}. \end{aligned} \tag{1-3}$$

The function $s \mapsto (\mathcal{F}_\Phi H)(2s - 1, s)$ can be computed explicitly, see [Theorem 1.2](#) below.

The temperedness assumption $(\alpha_1, \alpha_2, \alpha_3) \in (i\mathbb{R})^3$ for our fixed Maass cusp form Φ is very mild—it merely serves as a simplification of our exposition (when applying Stirling’s formula in [Section 8](#)) and can be removed with a little more effort. In fact, all Maass cusp forms of $\mathrm{SL}_3(\mathbb{Z})$ are conjectured to be tempered and it was proved in [\[Miller 2001\]](#) that the nontempered forms constitute a density zero set.

We have made no attempt to enlarge the class of test functions for [Theorem 1.1](#) since this is not the focus of this article (but is certainly doable by more refined analysis). The regularity assumptions of \mathcal{C}_η essentially follow from those of the Kontorovich–Lebedev inversion (see [Section 5B](#)). As in [\[Goldfeld and Kontorovich 2013; Goldfeld et al. 2021; 2022; Buttcanne 2020\]](#), the class \mathcal{C}_η already includes good test functions that are useful in a number of applications and allows us to deduce a version of [Theorem 1.1](#) for incomplete L -functions (see [Remark 5.27](#)).

Also, we have obtained the analytic properties and several explicit expressions for the integral transform $(\mathcal{F}_\Phi H)(s_0, s)$. They are written in terms of Mellin–Barnes integrals or hypergeometric functions as in [\[Motohashi 1993; 1997\]](#). We do not record the full formulae here but refer the readers to [Section 10](#) for the detailed discussions. However, we record an interesting identity of special functions as follow:

Theorem 1.2 ([Theorem 10.2](#)). *For $\frac{1}{2} + \frac{1}{100} < \sigma < 1$, we have*

$$(\mathcal{F}_\Phi H)(2s - 1, s) = \pi^{\frac{1}{2}-s} \prod_{i=1}^3 \frac{\Gamma(s - \frac{1}{2} + \frac{\alpha_i}{2})}{\Gamma(1 - s - \frac{\alpha_i}{2})} \cdot \int_{(0)} \frac{H(\mu)}{|\Gamma(\mu)|^2} \cdot \prod_{i=1}^3 \prod_{\pm} \Gamma\left(\frac{1 - s + \alpha_i \pm \mu}{2}\right) \frac{d\mu}{2\pi i}. \quad (1-4)$$

There are actually two additional identities of Barnes type that account for the origins and the combinatorics of six (out of eight) of the off-diagonal main terms for the cubic moment of $\mathrm{GL}(2)$ L -functions. The results align nicely with the predictions of the “*moment conjecture*” (or “*recipe*”) of [\[Conrey et al. 2005\]](#). We refer the interested readers to our papers [\[Kwan 2023; 2024\]](#).

1C. Follow-up works. The current work aims to illustrate the key ideas and address the main analytic issues of our period integral approach. It is the simplest to illustrate all these using the cuspidal case for Φ . However, this is by no means the end of the scope of our method. In our upcoming works [\[Kwan 2023; 2024\]](#), we demonstrate the versatility of our method by:

- (1) Providing a new proof of the cubic moment identity (1-2) (actually for the more general “*shifted moment*”) with a number of technical advantages, as well as a new unified way of extracting the full set of main terms. There are considerable recent interests in understanding the deep works of [\[Motohashi 1993; 1997\]](#) and [\[Conrey and Iwaniec 2000\]](#) from different perspectives, e.g., [\[Nelson 2019; Wu 2022; Balkanova et al. 2021\]](#).
- (2) Establishing a Motohashi’s formula of $\mathrm{GL}(3)$ in the nonarchimedean aspect which dualizes $\mathrm{GL}(2)$ twists of Hecke eigenvalues into $\mathrm{GL}(1)$ twists by Dirichlet characters. This offers insights into the celebrated works [\[Young 2011; Blomer et al. 2020\]](#) on the fourth moment of Dirichlet L -functions. In their works, this kind of change of structures was the result of a long sequence of spectral/harmonic transformations and it was surprising (and useful) to observe such a nice phenomenon.

2. Outline

In [Section 3](#), we discuss the technical features of the method used in this article and draw comparisons with the current literature. In [Section 4](#), we include a sketch of our arguments to demonstrate the essential ideas of our method and sidestep the technical points. In [Section 5](#), we collect the essential notions and results for later parts of the article.

The proof of [Theorem 1.1](#) is divided into four sections. In [Section 6](#), we prove the key identity of this article (see [Corollary 6.2](#)). In [Section 7](#), we develop such an identity into moments of L -functions on the region of absolute convergence. In particular, the intrinsic structure of the problem allows one to easily see the shape of the dual moment (see [Proposition 7.2](#)). In [Section 8](#), we obtain the region of holomorphy and growth of the archimedean transform. In [Section 9](#), a step-by-step analytic continuation argument is performed based on the analytic information obtained in [Section 8](#).

In [Section 10](#), we prove [Theorem 1.2](#). and provide several explicit formulae of the integral transforms.

3. Technical features of our method

3A. Period reciprocity. Our work adds a new instance to the recent banner “*period reciprocity*” which seeks to uncover the underlying structures of moments of L -functions through the lenses of period integrals. The general philosophy of this method is to evaluate a period integral in two distinct manners. Under favorable circumstances, the intrinsic structures of period integrals would lead to interesting, nontrivial moment identities, say connecting two different-looking families of L -functions.

In our case, the generalized Motohashi-type phenomenon of [Theorem 1.1](#) at $s = \frac{1}{2}$ will be shown to be an intrinsic property of a given Maass cusp form Φ of $SL_3(\mathbb{Z})$ via the following trivial identity

$$\int_0^1 \left[\int_0^\infty \Phi \left(\begin{pmatrix} y_0 & & & \\ & y_0 & & \\ & & 1 & u \\ & & & 1 \end{pmatrix} d^\times y_0 \right) e(-u) du \right] = \int_0^\infty \left[\int_0^1 \Phi \left(\begin{pmatrix} 1 & u & & \\ & 1 & y_0 & \\ & & 1 & \\ & & & 1 \end{pmatrix} e(-u) du \right) d^\times y_0. \quad (3-1)$$

Roughly speaking, [Theorem 1.1](#) follows from (1) spectrally expanding the innermost integral on the left in terms of a basis of $GL(2)$ automorphic forms, and (2) computing the innermost integral on the right in terms of the $GL(3)$ Fourier–Whittaker period. A sketch of this will be provided in [Section 4](#). In practice, it turns out to be convenient to work with a more general set-up

$$\int_{SL_2(\mathbb{Z}) \backslash GL_2(\mathbb{R})} P(g; h) \Phi \left(\begin{pmatrix} g & \\ & 1 \end{pmatrix} \right) |\det g|^{s-1/2} dg \quad (3-2)$$

so as to bypass certain technical difficulties, where $P(*; h)$ is a Poincaré series of $SL_2(\mathbb{Z})$.

The current examples for period reciprocity occur rather sporadically and there is currently no systematic method for constructing new examples. Also, techniques differ greatly in each known instance; see [[Michel and Venkatesh 2006; 2010; Nelson 2019; Blomer 2012a; Nunes 2023; Jana and Nunes 2021; Zacharias 2021; 2019](#)]. This stands in stark contrast to the more traditional “Kuznetsov–Voronoi” framework (see [Section 3B](#)). However, period reciprocity seems to address some of the technical complications more softly than the Kuznetsov–Voronoi approach. We shall elaborate more in the upcoming subsections.

Regarding the “classical” Motohashi phenomenon (1-2), a strategy was very recently proposed in [Michel and Venkatesh 2006; 2010] that developed into a fully rigorous method by Nelson [2019] through the use of regularized period integrals, incorporating new insights from automorphic representations. This article provides an alternative approach, which not only includes (1-2) but also generalizes several related instances of this phenomenon. We address the structural and analytic aspects of the formulae rather differently using unipotent integration for $GL(3)$ and method of analytic continuation. (We begin by considering (3-2) for $\text{Re } s \gg 1$.) For further discussions, see Section 4.

We would also like to mention the works [2022; 2021] in which an interesting framework in terms of tempered distributions and relative trace formula of Godement–Jacquet type was developed to address the phenomenon (1-2).

3B. Comparisons with the Conrey–Iwaniec–Li method. The celebrated works of Conrey and Iwaniec [2000] and Li [2009; 2011] are known for their successful analysis based on the Kuznetsov trace formulae and summation formulae of Poisson/Voronoi type. Their accomplishments include a delicate treatment of the arithmetic of exponential sums as well as the stationary phase analysis.

The Kuznetsov trace formula (or more generally the relative trace formula) has been a cornerstone in the analytic theory of L -functions over the past few decades. In the context of Theorem 1.1, which involves summing over a basis of even Maass forms for $SL_2(\mathbb{Z})$ (or equivalently, Maass forms for $PGL_2(\mathbb{Z}) \setminus PGL_2(\mathbb{R})$), it is an equality of the shape

$$\sum_j H(\mu_j) \frac{\lambda_j(n) \overline{\lambda_j(m)}}{L(1, \text{Ad}^2 \phi_j)} + (\text{cts}) = \delta_{m=n} \int_{\mathbb{R}} H(\mu) d_{\text{spec}} \mu + \sum_{\pm} \sum_c \frac{S(\pm m, n; c)}{c} \mathcal{J}^{\pm} \left(\frac{4\pi \sqrt{mn}}{c} \right) \quad (3-3)$$

between the spectral bilinear form of Hecke eigenvalues and the geometric expansion, which consists of Kloosterman sums $S(m, n; c)$ and oscillatory integrals \mathcal{J}^+ and \mathcal{J}^- involving the J -Bessel and K -Bessel function in their kernels respectively. These two pieces have to be treated separately.

As noticed in [Conrey and Iwaniec 2000; Li 2009; 2011; Blomer 2012b] and a number of subsequent works, the J -Bessel piece is particularly interesting due to its striking technical features. These features are crucial for achieving significant cancellations in geometric sums and integrals, a property that appears to be distinctive to higher-rank settings. (In view of this, readers may wish to compare with the analysis in [Liu and Ye 2002] in the $GL(2)$ settings.) More concretely, Li [2011] was able to apply the $GL(3)$ Voronoi formula *twice*, which were surprisingly noninvoluntary, because of a subtle cancellation taking place between the *arithmetic phase* coming from Voronoi and the *analytic phase* coming from the J -Bessel transform.

The treatment of the J -Bessel piece in the Kuznetsov–Voronoi approach is crucial for analyzing more general moments of L -functions, including those involving nonselfdual L -functions or noncentral L -values, as demonstrated in Theorem 1.1.

In our period integral approach, the Kuznetsov formula, the Voronoi formula, and the approximate functional equation, which belong to the standard toolbox in analytic number theory, are completely avoided. This is motivated by several conceptual reasons, which we will now explain:

- Firstly, since the $GL(3) \times GL(2)$ L -functions on the spectral side are interpreted as period integrals, we never need to open up those L -functions into Dirichlet series. As a result, averaging over the Hecke eigenvalues of our basis of $GL(2)$ Maass forms using the Kuznetsov formula is unnecessary.
- Secondly, the dual arithmetic object in our moment identity (1-3) contains the standard L -function of $GL(3)$. The standard L -function is constructed solely from the $GL(3)$ Hecke eigenvalues, whereas the $GL(3)$ Voronoi formula involves *general* Fourier coefficients of $GL(3)$ due to arithmetic twisting. It is thus reasonable to expect a proof of (1-3) that does not rely on the $GL(3)$ Voronoi formula of [Miller and Schmid 2006] nor the full Fourier expansions of [Jacquet et al. 1979a; 1979b]. The set-up (3-1) already suggests that our method meets such an expectation, but see Proposition 6.1 for full details.
- Thirdly, we do not encounter any intermediate exponential sums (e.g., Kloosterman/Ramanujan sums), slow-decaying/very oscillatory special functions, nor shifted convolution sums which are necessary in [Ivić 2001; 2002; Frolenkov 2020] for (1-2). Also, we handle the archimedean component of (1-3) in a unified manner, rather than handling the J - and K -Bessel pieces separately as done in [Conrey and Iwaniec 2000; Li 2009; 2011]. We directly work with the $GL(3)$ Whittaker function associated with the automorphic form Φ .
- Fourthly, we take advantage of the equivariance of the Whittaker functions under unipotent translations which helps to simplify many formulae.

Our period integral approach offers several technical advantages and is fundamentally distinct from the Kuznetsov–Voronoi approach. Indeed, our approach is *local* and the key result Proposition 6.1 can be easily phrased in terms of adèles (see (4-7)), whereas the Kuznetsov–Voronoi approach is *global* and *nonadelic*. In this article, we focus on the level 1 case and the spectral aspect as a proof of concept and thus we use the classical language of real groups. In our upcoming work, we wish to extend our method in various nonarchimedean aspects.

3C. Prospects for higher-rank. Once we reach $GL(3)$, the geometric expansion for the Kuznetsov formula becomes significantly more intricate and presents a number of obstacles in generalizing the Kuznetsov-based approaches to moments of L -functions of higher-rank:

Remark 3.1 (oscillatory integrals). In $GL(2)$, a couple of coincidences allow us to identify the oscillatory integrals with some well-studied special functions; see [Motohashi 1997; Iwaniec 2002]. However, such phenomena do not occur in $GL(3)$, where unexpected analytic difficulties arise; see [Buttcane 2013; 2016]. The complicated formulae for the oscillatory integrals make the Kuznetsov trace formula for $GL(3)$ challenging to apply; see [Blomer and Buttcane 2020].

Remark 3.2 (Kloosterman sums). The $GL(3)$ Kloosterman sums, e.g.,

$$S(m_1, m_2, n_1, n_2; D_1, D_2) := \sum_{\substack{\dagger \\ B_1(D_1), B_2(D_2) \\ C_1(D_1), C_2(D_2)}} e\left(\frac{m_1 B_1 + n_1(Y_1 D_2 - Z_1 B_2)}{D_1}\right) e\left(\frac{m_2 B_2 + n_2(Y_2 D_1 - Z_2 B_1)}{D_2}\right), \quad (3-4)$$

are clearly much harder to work with than the usual one, where the definitions of Y_i, Z_i 's along with a couple of congruence and coprimality conditions are suppressed. There are two other Kloosterman sums for $GL(3)$; see [Buttcane 2013] for details.

As discussed in Section 3B, further transformations of the exponential sums from the Kuznetsov formulae encode important arithmetic information about the moments of L -functions. In [Blomer and Buttcane 2020] it was demonstrated that this approach for (3-4) after applying a four-fold Poisson summation. However, beyond this specific instance, the general applicability of such transformations to (3-4) remains unclear. On the other hand, applications of Voronoi formulae for $GL(3)$ (see [Conrey and Iwaniec 2000; Li 2009; 2011; Blomer 2012b; Blomer and Khan 2019a; 2019b]) and for $GL(4)$ (see [Blomer et al. 2019; Chandee and Li 2020]) are currently limited to the usual Kloosterman sums of $GL(2)$, with complications arising quickly beyond this familiar context.

Conceptually speaking, the challenges associated with Remarks 3.1–3.2 stem from the *Bruhat decomposition*, which is fundamental to the framework of relative trace formulae in general. However, ideas from period reciprocity offers a way to bypass the Bruhat decomposition and the related geometric sums and integrals, which is a welcoming feature.

Regarding Remark 3.1, the advantages of our method are visible even in the context of Theorem 1.1. Even though we work with the group $GL(3)$ on the dual side, the oscillatory factor in our approach (see (6-8)) is actually simpler than the ones encountered in the “Kuznetsov–Voronoi” approaches (see [Li 2011]). It is more structured in two key ways: (1) It arises naturally from the definition of the archimedean Whittaker function. (2) It serves as an important constituent of the exact Motohashi structure, the exact structures of the main terms predicted by [Conrey et al. 2005], as well as for the analytic continuation past $\text{Re } s = \frac{1}{2}$. Furthermore, our approach is devoid of integrals over noncompact subsets of the unipotent subgroups (or the complements) which are known to result in intricate dual calculations and exponential phases in case of $GL(3)$ Voronoi formula (see Section 4 of [Ichino and Templier 2013]) and Kuznetsov formulae (see Chapter 11 of [Goldfeld 2015]).

It is worth pointing out the crucial archimedean ingredient in our proof generalizes to $GL(n)$ through *Stade's formula* (see [Stade 2001]), which allows us to rewrite the archimedean part completely in terms of integrals Γ -functions. This representation is sufficient for our purposes and possesses remarkable recursive structures beneficial for further analytic manipulations, as detailed in Section 10. Another notable recent application of Stade's formula can be found in [Goldfeld et al. 2021; 2022]. We anticipate that our method will provide insights into the structures of archimedean transforms, pave the way for generalizing to moments of higher-rank L -functions and overcome the technical challenges posed by

the “Kuznetsov–Voronoi” method. We shall return to this subject in our upcoming works, together with treatment of the nonarchimedean places.

4. Informal sketch and discussion

To assist the readers, we first outline the main ideas of this article, before diving into any of the analytic subtleties of our actual argument. In fact, this represents the most intrinsic picture of our method and facilitates comparisons with the strategy of [Michel and Venkatesh 2006]. The style of this section will be largely informal — we shall suppress the constant multiples (say those 2’s and π ’s), assume convergence, and set aside the treatment of main terms.

According to [Michel and Venkatesh 2006], the classical Motohashi formula can be understood as an intrinsic property of the $GL(2)$ Eisenstein series (denoted by E^* below) via the (“regularized”) geodesic period

$$\int_0^\infty |E^*(iy)|^2 d^\times y,$$

which can be evaluated in two ways according to $|E^*|^2$ and $E^* \cdot \overline{E^*}$ respectively:

(1) ($GL(2)$ spectral expansion)

$$\sum_{\phi:GL(2)} \langle |E^*|^2, \phi \rangle \int_0^\infty \phi(iy) d^\times y = \sum_{\phi:GL(2)} \Lambda\left(\frac{1}{2}, \phi\right)^2 \cdot \Lambda\left(\frac{1}{2}, \phi\right) + (\dots). \quad (4-1)$$

(2) ($GL(1) \times GL(1)$ expansion, or the Mellin–Plancherel formula)

$$\int_{(1/2)} |\widetilde{E}^*(s)|^2 \frac{ds}{2\pi i} = \int_{\mathbb{R}} |\Lambda\left(\frac{1}{2} + it\right)|^2 \frac{dt}{2\pi}. \quad (4-2)$$

This seemingly simple sketch turns out to require rather sophisticated regularizations but was skillfully executed very recently in [Nelson 2019].

We now turn to our sketch of the (generalized) Motohashi phenomenon as described in [Theorem 1.1](#). Let Φ be a Maass cusp form of $SL_3(\mathbb{Z})$. As mentioned in the introduction, our starting point is the trivial identity

$$\int_0^1 \left[\int_0^\infty \Phi \left(\begin{pmatrix} y_0 & (1 \ u) \\ & 1 \end{pmatrix} d^\times y_0 \right) e(-u) du \right] = \int_0^\infty \left[\int_0^1 \Phi \left(\begin{pmatrix} (1 \ u) y_0 \\ & 1 \end{pmatrix} e(-u) du \right) \right] d^\times y_0. \quad (4-3)$$

For symmetry, observe that the right side of (4-3) can be written as

$$\int_0^\infty \left[\int_0^1 \widetilde{\Phi} \left[\begin{pmatrix} 1 & & \\ & 1 & u \\ & & 1 \end{pmatrix} \begin{pmatrix} y_0 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right] e(-u) du \right] d^\times y_0 \quad (4-4)$$

with $\widetilde{\Phi}(g) := \Phi({}^t g^{-1})$ being the dual form of Φ .

Remark 4.1. Indeed, the center-invariance of Φ implies that

$$(4-3) = \int_0^\infty \int_0^1 \Phi \left[\begin{pmatrix} 1 & u \\ & 1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & y_0 \end{pmatrix} \right] e(-u) du d^\times y_0.$$

Let $w_\ell := \begin{pmatrix} & -1 \\ 1 & \\ & & 1 \end{pmatrix}$. The observation

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & y_0 \end{pmatrix} = w_\ell^{-1} \begin{pmatrix} y_0 & & \\ & 1 & \\ & & 1 \end{pmatrix} w_\ell \quad \text{and} \quad \begin{pmatrix} 1 & & \\ & 1 & \\ & -u & 1 \end{pmatrix} = w_\ell \begin{pmatrix} 1 & u & \\ & 1 & \\ & & 1 \end{pmatrix} w_\ell^{-1}$$

together with the left and right invariance of Φ by w_ℓ further rewrite (4-3) as

$$\begin{aligned} \int_0^\infty \int_0^1 \Phi \left[\begin{pmatrix} 1 & & \\ & 1 & \\ & -u & 1 \end{pmatrix} \begin{pmatrix} y_0 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right] e(-u) du d^\times y_0 \\ = \int_0^\infty \int_0^1 \tilde{\Phi} \left[\begin{pmatrix} 1 & & \\ & 1 & u \\ & & 1 \end{pmatrix} \begin{pmatrix} y_0 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right] e(-u) du d^\times y_0. \end{aligned}$$

As an overview of our strategy:

- (1) *Similar to Michel–Venkatesh’s strategy*, the integral over $(0, \infty)$ (or the center $Z_{\text{GL}_2}^+(\mathbb{R})$) yields Rankin–Selberg L -functions on the spectral side and a t -integral on the dual side.
- (2) *Different from Michel–Venkatesh’s strategy*, our approach introduces an extra integral over $[0, 1]$ (or the quotient $U_2(\mathbb{Z}) \backslash U_2(\mathbb{R})$ of the unipotent subgroup U_2 of $\text{GL}(2)$). This integral results in Whittaker functions as weight functions on the spectral side, and leads to a product of two distinct L -functions on the dual side.
- (3) The Mellin–Plancherel of (4-2) is replaced by two Fourier expansions over $\mathbb{Z} \backslash \mathbb{R}$ below.

In fact, the *unipotent* nature of our period method is crucial in realizing the spectral duality for the fourth moment of Dirichlet L -functions (see [Kwan 2024]), as well as in ensuring the abundance of admissible test functions on the spectral side, but these features will not be displayed in this section.

4A. The $\text{GL}(2)$ (spectral) side. This side is relatively straight-forward and gives the desired $\text{GL}(3) \times \text{GL}(2)$ moment. Regard Φ as a function of $L^2(\Gamma_2 \backslash \mathfrak{h}^2)$ via

$$(\text{Proj}_2^3 \Phi)(g) := \int_0^\infty \Phi \left(\begin{pmatrix} y_0 g & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) d^\times y_0 \quad (g \in \mathfrak{h}^2),$$

which in turn can be expanded spectrally as

$$(\text{Proj}_2^3 \Phi)(g) = \sum_j \frac{\langle \text{Proj}_2^3 \Phi, \phi_j \rangle}{\|\phi_j\|^2} \phi_j(g) + \frac{\langle \text{Proj}_2^3 \Phi, 1 \rangle}{\|1\|^2} \cdot 1 + (\text{cont}).$$

The spectral coefficients $\langle \text{Proj}_2^3 \Phi, \phi_j \rangle$ are precisely the $GL(3) \times GL(2)$ Rankin–Selberg L -functions. Hence,

$$\text{LHS of (4-3)} = \int_0^1 (\text{Proj}_2^3 \Phi) \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} e(-u) du = \sum_j W_{\mu_j}(1) \cdot \frac{\Lambda(\frac{1}{2}, \phi_j \otimes \Phi)}{\|\phi_j\|^2} + (\text{cont}), \quad (4-5)$$

where $\mu \mapsto W_\mu(1)$ is a weight function.

4B. The $GL(1)$ (dual) side. In view of Point (3) above, we evaluate the innermost integral of (4-4) in terms of the *Fourier–Whittaker periods* for $\tilde{\Phi}$, denoted by $(\tilde{\Phi})_{(\cdot, \cdot)}$ (see Definition 5.12). From Proposition 6.1, (4-4) is given by

$$\int_0^\infty \int_0^1 \int_0^1 \tilde{\Phi} \left[\begin{pmatrix} 1 & u_{1,3} \\ & 1 & u_{2,3} \\ & & 1 \end{pmatrix} \begin{pmatrix} y_0 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right] e(-u_{2,3}) du_{1,3} du_{2,3} d^\times y_0 \\ + \sum_{a_0 \in \mathbb{Z} - \{0\}} \sum_{a_1 \in \mathbb{Z} - \{0\}} \int_0^\infty (\tilde{\Phi})_{(1, a_1)} \left[\begin{pmatrix} 1 & & \\ a_0 & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} y_0 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right] d^\times y_0. \quad (4-6)$$

The first line of (4-6) corresponds to the diagonal term and is precisely the integral representation of the standard L -function of $\tilde{\Phi}$. It is equal to $L(1, \tilde{\Phi}) \cdot Z_\infty(1, \tilde{\Phi})$, where $Z_\infty(\cdot, \tilde{\Phi})$ is the $GL(3)$ local zeta integral at ∞ . The second line of (4-6) is the off-diagonal contribution, denoted by OD_Φ below, and is expressed in terms of the Fourier coefficients of $\tilde{\Phi}$:

$$OD_\Phi = \sum_{a_0 \in \mathbb{Z} - \{0\}} \sum_{a_1 \in \mathbb{Z} - \{0\}} \frac{\mathcal{B}_{\tilde{\Phi}}(1, a_1)}{|a_1|} \int_0^\infty (\tilde{\Phi})_{(1, 1)} \left[\begin{pmatrix} a_1/a_0 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} y_0 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right] d^\times y_0. \quad (4-7)$$

It can be further explicated as

$$OD_\Phi = \sum_{a_0 \in \mathbb{Z} - \{0\}} \sum_{a_1 \in \mathbb{Z} - \{0\}} \frac{\mathcal{B}_{\tilde{\Phi}}(1, a_1)}{|a_1|} \cdot \int_0^\infty W_{\alpha(\Phi)} \left(\left| \frac{a_1}{a_0} \right| \frac{y_0}{1 + y_0^2}, 1 \right) \cdot e \left(\frac{a_1}{a_0} \frac{y_0^2}{1 + y_0^2} \right) d^\times y_0 \quad (4-8)$$

using the $GL(3)$ Whittaker function $W_{\alpha(\Phi)}$, where the oscillatory factor $e(\dots)$ originates from the unipotent translation of Whittaker function.

Roughly speaking, (4-8) suggests some forms of (multiplicative) convolutions between the $GL(3)$ and $GL(1)$ data at both the archimedean and the nonarchimedean places:

- (1) (Archimedean) We apply the Mellin inversion formula for $W_{\alpha(\Phi)}$, a standard result in $GL(3)$ theory, together with the local functional equation for $GL(1)$ in the form

$$e(x) + e(-x) = \int_{-i\infty}^{i\infty} \frac{\Gamma_{\mathbb{R}}(u)}{\Gamma_{\mathbb{R}}(1-u)} |x|^{-u} \frac{du}{2\pi i} \quad (x \neq 0). \quad (4-9)$$

(2) (Nonarchimedean) Observe the following identity of the double Dirichlet series:

$$\sum_{a_0 \neq 0} \sum_{a_1 \neq 0} \frac{\mathcal{B}_\Phi(a_1, 1)}{|a_1|} \left| \frac{a_1}{a_0} \right|^{1-s_0-u} = L(s_0 + u, \tilde{\Phi}) \zeta(1 - s_0 - u). \quad (4-10)$$

We thus arrive at

$$OD_\Phi = \int_{(1/2)} \zeta(1 - s_0) L(s_0, \tilde{\Phi}) \cdot (\dots) \frac{ds_0}{2\pi i}, \quad (4-11)$$

where “ (\dots) ” stands for a certain integral transform that can be described purely in terms of Γ -functions.

Remark 4.2. (1) In (3-2), the test function h of the Poincaré series $P(*; h)$ will be transformed into the Kontorovich–Lebedev transform $h^\#$ on the $GL(2)$ side (see Proposition 5.25) and into the Mellin transform \tilde{h} on the $GL(1)$ side (see (7-6)). This is consistent with the sketch above.

(2) Readers may wish to compare the integral transforms obtained in the sketch with the one described in Section 1.3 of [Balkanova et al. 2021].

Remark 4.3. The choices of unipotent subgroups have been important in the constructions of various L -series for the group $GL(3)$:

- $\left\{ \begin{pmatrix} 1 & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}$ or $\left\{ \begin{pmatrix} 1 & * & * \\ & 1 & \\ & & 1 \end{pmatrix} \right\}$ for the standard L -function.
- $\left\{ \begin{pmatrix} 1 & * \\ & 1 \\ & & 1 \end{pmatrix} \right\}$ for Bump’s double Dirichlet series [Bump 1984].
- $\left\{ \begin{pmatrix} 1 & \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}$ or $\left\{ \begin{pmatrix} 1 & * \\ & 1 \\ & & 1 \end{pmatrix} \right\}$ for the Motohashi phenomenon of this article.

5. Preliminary

The analytic theory of automorphic forms for the group $GL(3)$ has undergone considerable development in the past decade. Readers should beware that the recent articles in the field (e.g., [Buttcane 2013; 2016; 2020; Goldfeld et al. 2021]) have adopted a different set of conventions and normalizations from those in the standard text [Goldfeld 2015]. (Nevertheless, [Goldfeld 2015] remains a useful reference as it thoroughly documents many standard results and their proofs.)

In this article, we follow the more recent conventions (closest to [Buttcane 2020]), which is better aligned with the theory of automorphic representation. We will summarize the essential notions and results below, with extra attention on the archimedean calculations involving Whittaker functions, as they play a key role in our analysis.

5A. Notations and conventions. Throughout this article, we use the following notations: $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2} \Gamma(s/2)$ ($s \in \mathbb{C}$); $e(x) := e^{2\pi i x}$ ($x \in \mathbb{R}$); $\Gamma_n := SL_n(\mathbb{Z})$ ($n \geq 2$). Without otherwise specified, our test function H lies in the class \mathcal{C}_{η} and $H = h^{\#}$. We will often use the same symbol to denote a function (in s) and its analytic continuation.

We will frequently encounter contour integrals of the shape

$$\int_{-i\infty}^{i\infty} \cdots \int_{-i\infty}^{i\infty} (\cdots) \frac{ds_1}{2\pi i} \cdots \frac{ds_k}{2\pi i}$$

where the contours involved should follow Barnes' convention: they pass to the right of all of the poles of the gamma functions in the form $\Gamma(s_i + a)$ and to the left of all of the poles of the gamma functions in the form $\Gamma(a - s_i)$.

We also adopt the following set of conventions:

- (1) All Maass cusp forms will be simultaneous eigenfunctions of the Hecke operators and will be either even or odd. Also, their first Fourier coefficients are equal to 1. In this case, the forms are said to be *Hecke-normalized*. Note that there are no odd form for $SL_3(\mathbb{Z})$; see Proposition 9.2.5 of [Goldfeld 2015].
- (2) Our fixed Maass cusp form Φ of $SL_3(\mathbb{Z})$ is assumed to be *tempered at ∞* , i.e., its Langlands parameters are purely imaginary.
- (3) Denote by θ the best progress towards the Ramanujan conjecture for the Maass cusp forms of $SL_3(\mathbb{Z})$. We have $\theta \leq \frac{1}{2} - \frac{1}{10}$; see Theorem 12.5.1 of [Goldfeld 2015].

5B. (Spherical) Whittaker functions and transforms. In the rest of this article, all Whittaker functions will refer to the spherical ones. The Whittaker function of $GL_2(\mathbb{R})$ is more familiar and is given by

$$W_{\mu}(y) := 2\sqrt{y} K_{\mu}(2\pi y) \tag{5-1}$$

for $\mu \in \mathbb{C}$ and $y > 0$. Under this normalization, the following holds:

Proposition 5.1. For $\operatorname{Re}(w + \frac{1}{2} \pm \mu) > 0$, we have

$$\int_0^{\infty} W_{\mu}(y) y^w d^{\times} y = \frac{\pi^{-w-1/2}}{2} \Gamma\left(\frac{w + \frac{1}{2} + \mu}{2}\right) \Gamma\left(\frac{w + \frac{1}{2} - \mu}{2}\right). \tag{5-2}$$

Proof. Standard, see (2.5.2) of [Motohashi 1997] for instance. □

For the group $GL_3(\mathbb{R})$, we first introduce the function

$$I_{\alpha}(y_0, y_1) = I_{\alpha} \begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix} := y_0^{1-\alpha_3} y_1^{1+\alpha_1}$$

for $y_0, y_1 > 0$ and $\alpha \in \mathfrak{a}_{\mathbb{C}}^{(3)} := \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3 : \alpha_1 + \alpha_2 + \alpha_3 = 0\}$. Then the Whittaker function for $\mathrm{GL}_3(\mathbb{R})$, denoted by

$$W_{\alpha}(y_0, y_1) = W_{\alpha} \begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix},$$

is defined in terms of *Jacquet's integral*

$$\prod_{1 \leq j < k \leq 3} \Gamma_{\mathbb{R}}(1 + \alpha_j - \alpha_k) \times \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} I_{\alpha} \left[\begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} \begin{pmatrix} 1 & u_{1,2} & u_{1,3} \\ & 1 & u_{2,3} \\ & & 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix} \right] e^{(-u_{1,2} - u_{2,3})} du_{1,2} du_{1,3} du_{2,3} \quad (5-3)$$

for $y_0, y_1 > 0$ and $\alpha \in \mathfrak{a}_{\mathbb{C}}^{(3)}$; see Chapter 5.5 of [Goldfeld 2015] for details.

Remark 5.2. Notice the differences in the normalizations of I_{α} here compared to that in equation (5.1.1) of [Goldfeld 2015]. Also, the Whittaker functions discussed here are the *complete* Whittaker functions as defined in [loc. cit.].

Moreover, the Whittaker function of $\mathrm{GL}_3(\mathbb{R})$ admits the following useful Mellin–Barnes representation commonly known as the *Vinogradov–Takhtadzhyan formula*:

Proposition 5.3. Assume $\alpha \in \mathfrak{a}_{\mathbb{C}}^{(3)}$ is tempered, i.e., $\mathrm{Re} \alpha_i = 0$ ($i = 1, 2, 3$). Then for any $\sigma_0, \sigma_1 > 0$,

$$W_{-\alpha}(y_0, y_1) = \frac{1}{4} \int_{(\sigma_0)} \int_{(\sigma_1)} G_{\alpha}(s_0, s_1) y_0^{1-s_0} y_1^{1-s_1} \frac{ds_0}{2\pi i} \frac{ds_1}{2\pi i}, \quad y_0, y_1 > 0, \quad (5-4)$$

where

$$G_{\alpha}(s_0, s_1) := \frac{\prod_{i=1}^3 \Gamma_{\mathbb{R}}(s_0 + \alpha_i) \Gamma_{\mathbb{R}}(s_1 - \alpha_i)}{\Gamma_{\mathbb{R}}(s_0 + s_1)}. \quad (5-5)$$

Proof. This can be verified (up to the constant $\frac{1}{4}$) by a brute force yet elementary calculation, i.e., checking the right side of (5-4) satisfies the differential equations of $\mathrm{GL}(3)$; see pages 38–39 of [Bump 1984]. For a cleaner proof starting from (5-3); see Chapter X of [loc. cit.]. \square

Remark 5.4. Notice the sign convention of the α_i in formula (5-4)—it is consistent with [Buttcane 2020] but is opposite to that of (6.1.4)–(6.1.5) in [Goldfeld 2015].

Corollary 5.5. For any $-\infty < A_0, A_1 < 1$, we have

$$|W_{-\alpha}(y_0, y_1)| \ll y_0^{A_0} y_1^{A_1}, \quad y_0, y_1 > 0, \quad (5-6)$$

where the implicit constant depends only on α, A_0, A_1 .

Proof. Follows directly from Proposition 5.3 by contour shifting. \square

We will need the explicit evaluation of the $\mathrm{GL}_3(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$ Rankin–Selberg integral. It is a consequence of the *second Barnes lemma* stated as follows.

Lemma 5.6. For $a, b, c, d, e, f \in \mathbb{C}$ with $f = a + b + c + d + e$, we have

$$\int_{-i\infty}^{i\infty} \frac{\Gamma(w+a)\Gamma(w+b)\Gamma(w+c)\Gamma(d-w)\Gamma(e-w)}{\Gamma(w+f)} \frac{dw}{2\pi i} = \frac{\Gamma(d+a)\Gamma(d+b)\Gamma(d+c)\Gamma(e+a)\Gamma(e+b)\Gamma(e+c)}{\Gamma(f-a)\Gamma(f-b)\Gamma(f-c)}. \quad (5-7)$$

The contours of integration must adhere to Barnes' convention; see [Section 5A](#) for details.

Proof. See [\[Bailey 1935\]](#). □

Proposition 5.7. Let W_μ and $W_{-\alpha}$ be the Whittaker functions of $GL_2(\mathbb{R})$ and $GL_3(\mathbb{R})$ respectively. For $\operatorname{Re} s \gg 0$, we have

$$\begin{aligned} \mathcal{Z}_\infty(s; W_\mu, W_{-\alpha}) &:= \int_0^\infty \int_0^\infty W_\mu(y_1) W_{-\alpha}(y_0, y_1) (y_0^2 y_1)^{s-1/2} \frac{dy_0 dy_1}{y_0 y_1^2} \\ &= \frac{1}{4} \prod_{\pm} \prod_{k=1}^3 \Gamma_{\mathbb{R}}(s \pm \mu - \alpha_k). \end{aligned} \quad (5-8)$$

Proof. See [\[Bump 1988\]](#). □

The following pair of integral transforms plays an important role in the archimedean aspect of this article.

Definition 5.8. Let $h : (0, \infty) \rightarrow \mathbb{C}$ and $H : i\mathbb{R} \rightarrow \mathbb{C}$ be measurable functions with $H(\mu) = H(-\mu)$. Let $W_\mu(y) := 2\sqrt{y}K_\mu(2\pi y)$. Then the Kontorovich–Lebedev transform of h is defined by

$$h^\#(\mu) := \int_0^\infty h(y) W_\mu(y) \frac{dy}{y^2}, \quad (5-9)$$

whereas its inverse transform is defined by

$$H^\flat(y) = \frac{1}{4\pi i} \int_{(0)} H(\mu) W_\mu(y) \frac{d\mu}{|\Gamma(\mu)|^2}, \quad (5-10)$$

provided the integrals converge absolutely. Note: the normalization constant $1/4\pi i$ in (5-10) is consistent with that in [\[Motohashi 1997; Iwaniec 2002\]](#).

Definition 5.9. Let \mathcal{C}_η be the class of holomorphic functions H on the vertical strip $|\operatorname{Re} \mu| < 2\eta$ such that

- (1) $H(\mu) = H(-\mu)$,
- (2) H has rapid decay in the sense that

$$H(\mu) \ll e^{-2\pi|\mu|} \quad (|\operatorname{Re} \mu| < 2\eta). \quad (5-11)$$

In this article, we take $\eta > 40$ without otherwise specifying.

By contour-shifting and Stirling’s formula, we have:

Proposition 5.10. *For any $H \in \mathcal{C}_\eta$, the integral (5-10) defining H^b converges absolutely. Moreover, we have*

$$H^b(y) \ll \min\{y, y^{-1}\}^\eta \quad (y > 0). \tag{5-12}$$

Proof. See Lemma 2.10 of [Motohashi 1997]. □

Proposition 5.11. *Under the same assumptions of Proposition 5.10, we have*

$$(h^\#)^b(g) = h(g) \quad \text{and} \quad (H^b)^\#(\mu) = H(\mu). \tag{5-13}$$

Proof. See Lemma 2.10 of [Motohashi 1997]. It is a consequence of the Rankin–Selberg calculation for $\text{GL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{R})$. □

5C. Automorphic forms of $\text{GL}(2)$ and $\text{GL}(3)$. Let

$$\mathfrak{h}^2 := \left\{ \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} y & \\ & 1 \end{pmatrix} : u \in \mathbb{R}, y > 0 \right\}$$

with its invariant measure given by $y^{-2} du dy$. Let $\Delta := -y^2(\partial_x^2 + \partial_y^2)$. An automorphic form $\phi : \mathfrak{h}^2 \rightarrow \mathbb{C}$ of $\Gamma_2 = \text{SL}_2(\mathbb{Z})$ satisfies $\Delta\phi = (\frac{1}{4} - \mu^2)\phi$ for some $\mu = \mu(\phi) \in \mathbb{C}$. It is often handy to identify μ with the pair $(\mu, -\mu) \in \mathfrak{a}_{\mathbb{C}}^{(2)}$.

For $a \in \mathbb{Z} - \{0\}$, the a -th Fourier coefficient of ϕ , denoted by $\mathcal{B}_\phi(a)$, is defined by

$$(\widehat{\phi})_a(y) := \int_0^1 \phi \left[\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} y & \\ & 1 \end{pmatrix} \right] e(-au) du = \frac{\mathcal{B}_\phi(a)}{\sqrt{|a|}} \cdot W_{\mu(\phi)}(|a|y). \tag{5-14}$$

In the case of the Eisenstein series of Γ_2 , i.e.,

$$\phi = E(z; \mu) := \frac{1}{2} \sum_{\gamma \in U_2(\mathbb{Z}) \backslash \Gamma_2} I_\mu(\text{Im } \gamma z) \quad (z \in \mathfrak{h}^2), \tag{5-15}$$

where $I_\mu(y) := y^{\mu+1/2}$, it is well-known that $\Delta E(*; \mu) = (\frac{1}{4} - \mu^2)E(*; \mu)$ and the Fourier coefficients $\mathcal{B}(a; \mu)$ of $E(*; \mu)$ is given by

$$\mathcal{B}(a; \mu) = \frac{|a|^\mu \sigma_{-2\mu}(|a|)}{\Lambda(1 + 2\mu)}, \tag{5-16}$$

where

$$\Lambda(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s) \quad \text{and} \quad \sigma_{-2\mu}(|a|) := \sum_{d|a} d^{-2\mu}.$$

The series (5-15) converges absolutely for $\text{Re } \mu > \frac{1}{2}$ and it admits a meromorphic continuation to \mathbb{C} .

Next, let

$$\mathfrak{h}^3 := \left\{ \begin{pmatrix} 1 & u_{1,2} & u_{1,3} \\ & 1 & u_{2,3} \\ & & 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix} : u_{i,j} \in \mathbb{R}, y_k > 0 \right\}.$$

Let $\Phi : \mathfrak{h}^3 \rightarrow \mathbb{C}$ be a Maass cusp form of Γ_3 as defined in Definition 5.1.3 of [Goldfeld 2015]. In particular, there exists $\alpha = \alpha(\Phi) \in \mathfrak{a}_{\mathbb{C}}^{(3)}$ such that for any $D \in Z(U\mathfrak{gl}_3(\mathbb{C}))$ (the center of the universal enveloping algebra of the Lie algebra $\mathfrak{gl}_3(\mathbb{C})$), we have

$$D\Phi = \lambda_D \Phi \quad \text{and} \quad DI_\alpha = \lambda_D I_\alpha$$

for some $\lambda_D \in \mathbb{C}$. The triple $\alpha(\Phi)$ is said to be the *Langlands parameters* of Φ .

Definition 5.12. Let $m = (m_1, m_2) \in (\mathbb{Z} - \{0\})^2$ and $\Phi : \mathfrak{h}^3 \rightarrow \mathbb{C}$ be a Maass cusp form of $SL_3(\mathbb{Z})$. For any $y_0, y_1 > 0$, the integral defined by

$$\begin{aligned} & (\widehat{\Phi})_{(m_1, m_2)} \begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix} \\ & := \int_0^1 \int_0^1 \int_0^1 \Phi \left[\begin{pmatrix} 1 & u_{1,2} & u_{1,3} \\ & 1 & u_{2,3} \\ & & 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix} \right] e(-m_1 u_{2,3} - m_2 u_{1,2}) du_{1,2} du_{1,3} du_{2,3}. \end{aligned} \quad (5-17)$$

is said to be the (m_1, m_2) -th *Fourier–Whittaker period* of Φ . Moreover, the (m_1, m_2) -th *Fourier coefficient* of Φ is the complex number $\mathcal{B}_\Phi(m_1, m_2)$ for which

$$(\widehat{\Phi})_{(m_1, m_2)} \begin{pmatrix} y_0 y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix} = \frac{\mathcal{B}_\Phi(m_1, m_2)}{|m_1 m_2|} W_{\alpha(\Phi)}^{\text{sgn}(m_2)} \begin{pmatrix} (|m_1|y_0)(|m_2|y_1) & & \\ & |m_1|y_0 & \\ & & 1 \end{pmatrix} \quad (5-18)$$

holds for any $y_0, y_1 > 0$.

Remark 5.13. (1) The multiplicity-one theorem of Shalika (see Theorem 6.1.6 of [Goldfeld 2015]) guarantees the well-definedness of the Fourier coefficients for Φ .

(2) If Φ is Hecke-normalized (see Section 5A.(1)), then $\mathcal{B}_\Phi(1, n)$ can be shown to be a Hecke eigenvalue of Φ ; see Section 6.4 of [Goldfeld 2015].

5D. Automorphic L -functions. The Maass cusp forms Φ and ϕ below are Hecke-normalized and their Langlands parameters are denoted by $\alpha \in \mathfrak{a}_{\mathbb{C}}^{(3)}$ and $\mu \in \mathfrak{a}_{\mathbb{C}}^{(2)}$ respectively. Let $\widetilde{\Phi}(g) := \Phi({}^t g^{-1})$ be the dual form of Φ . It is not hard to show that the Langlands parameters of $\widetilde{\Phi}$ are given by $-\alpha$.

Definition 5.14. Suppose Φ and ϕ are Maass cusp forms of Γ_3 and Γ_2 respectively. For $\text{Re } s \gg 1$, the Rankin–Selberg L -function of Φ and ϕ is defined by

$$L(s, \phi \otimes \Phi) := \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \frac{\mathcal{B}_\phi(m_2) \mathcal{B}_\Phi(m_1, m_2)}{(m_1^2 m_2)^s}. \quad (5-19)$$

Although we do not make use of the Dirichlet series for $L(s, \phi \otimes \Phi)$ in this article, it is frequently used in the literature, especially in the “Kuznetsov–Voronoi” method. We take this opportunity to indicate our normalization in terms of Dirichlet series to facilitate conversion and comparison, and to correct some minor inaccuracies in Section 12.2 of [Goldfeld 2015].

Proposition 5.15. *Suppose Φ and ϕ are Maass cusp forms of Γ_3 and Γ_2 respectively. In addition, assume that ϕ is even. Then for any $\text{Re } s \gg 1$, we have*

$$\int_{\Gamma_2 \backslash \text{GL}_2(\mathbb{R})} \phi(g) \tilde{\Phi} \begin{pmatrix} g & \\ & 1 \end{pmatrix} |\det g|^{s-1/2} dg = \frac{1}{2} \cdot \Lambda(s, \phi \otimes \tilde{\Phi}), \tag{5-20}$$

where

$$\Lambda(s, \phi \otimes \tilde{\Phi}) := L_\infty(s, \phi \otimes \tilde{\Phi}) \cdot L(s, \phi \otimes \tilde{\Phi}) \tag{5-21}$$

and

$$L_\infty(s, \phi \otimes \tilde{\Phi}) := \prod_{k=1}^3 \Gamma_{\mathbb{R}}(s \pm \mu - \alpha_k). \tag{5-22}$$

Proof. The assumption on the parity of ϕ is missing in [Goldfeld 2015]. Also, the pairing should be taken over the quotient $\Gamma_2 \backslash \text{GL}_2(\mathbb{R})$ instead of $\Gamma_2 \backslash \mathfrak{h}^2$ in [loc. cit.].

As a brief sketch, we replace $\tilde{\Phi} \begin{pmatrix} g & \\ & 1 \end{pmatrix}$ by its Fourier–Whittaker expansion (see Theorem 5.3.2 of [Goldfeld 2015]) on the left side of (5-20) and unfold. Then one may extract the Dirichlet series in (5-19) by using (5-14) and (5-17). The integral of Whittaker functions can be computed by Proposition 5.7. \square

In the rest of this article, we will often make use of the shorthands $(\mathbb{P}_2^3 \Phi)(g) := \Phi \begin{pmatrix} g & \\ & 1 \end{pmatrix}$ and the pairing

$$(\phi, (\mathbb{P}_2^3 \Phi) \cdot |\det *|^{s-1/2})_{\Gamma_2 \backslash \text{GL}_2(\mathbb{R})}$$

for the integral on the left side of (5-20). By the rapid decay of Φ at ∞ , this integral converges absolutely for any $s \in \mathbb{C}$ and uniformly on any compact subset of \mathbb{C} . Thus, the L -function $L(s, \phi \otimes \tilde{\Phi})$ admits an entire continuation.

Remark 5.16. (1) When ϕ is even, the involution $g \mapsto {}^t g^{-1}$ gives the functional equation

$$\Lambda(s, \phi \otimes \tilde{\Phi}) = \Lambda(1 - s, \phi \otimes \Phi).$$

(2) When ϕ is odd, the right side of (5-20) is identical to 0 and hence *does not* provide an integral representation for $\Lambda(s, \phi \otimes \tilde{\Phi})$. One must alter Proposition 5.15 accordingly in this case, say using the raising/lowering operators, or proceed adelicly with an appropriate choice of test vector at ∞ . However, we shall not go into these as our spectral average is taken over even Maass forms of Γ_2 only.

(3) As discussed in Section 3B, the roles of parities and root numbers are rather intricate in the study of moments of L -functions, especially regarding the archimedean integral transforms.

Definition 5.17. Let $\Phi : \mathfrak{h}^3 \rightarrow \mathbb{C}$ be a Maass cusp form of Γ_3 . For $\text{Re } s \gg 1$, the standard L -function of Φ is defined by

$$L(s, \Phi) := \sum_{n=1}^{\infty} \frac{\mathcal{B}_\Phi(1, n)}{n^s}. \tag{5-23}$$

In the rest of this article, we will not make use of the integral representation of $L(s, \Phi)$, i.e., the first line of (4-6) with $\tilde{\Phi}$ replaced by Φ . It suffices to note that $L(s, \Phi)$ admits an entire continuation and satisfies the following functional equation:

Proposition 5.18. *Let $\Phi : \mathfrak{h}^3 \rightarrow \mathbb{C}$ be a Maass cusp form of Γ_3 . For any $s \in \mathbb{C}$, we have*

$$\Lambda(s, \Phi) = \Lambda(1 - s, \tilde{\Phi}), \quad (5-24)$$

where

$$\Lambda(s, \Phi) := L_\infty(s, \Phi) \cdot L(s, \Phi) \quad (5-25)$$

and

$$L_\infty(s, \Phi) := \prod_{k=1}^3 \Gamma_{\mathbb{R}}(s + \alpha_k). \quad (5-26)$$

Proof. See Chapter 6.5 of [Goldfeld 2015] or [Jacquet et al. 1979a; 1979b]. \square

Furthermore, since ϕ and Φ are assumed to be Hecke-normalized, the standard L -functions $L(s, \phi)$ and $L(s, \Phi)$ admit Euler products of the form

$$L(s, \phi) = \prod_p \prod_{j=1}^2 (1 - \beta_{\phi,j}(p) p^{-s})^{-1}, \quad L(s, \Phi) = \prod_p \prod_{k=1}^3 (1 - \alpha_{\Phi,k}(p) p^{-s})^{-1} \quad (5-27)$$

for $\text{Re } s \gg 1$. Then one can show that

$$L(s, \phi \otimes \Phi) = \prod_p \prod_{j=1}^2 \prod_{k=1}^3 (1 - \beta_{\phi,j}(p) \alpha_{\Phi,k}(p) p^{-s})^{-1} \quad (5-28)$$

by Cauchy's identity, see the argument of Proposition 7.4.12 of [Goldfeld 2015].

Proposition 5.19. *For $\text{Re}(s \pm \mu) \gg 1$, we have*

$$(E(*; \mu), (\mathbb{P}_2^3 \Phi) \cdot |\det *|^{\bar{s}-1/2})_{\Gamma_2 \backslash GL_2(\mathbb{R})} = \frac{1}{2} \frac{\Lambda(s + \mu, \tilde{\Phi}) \Lambda(s - \mu, \tilde{\Phi})}{\Lambda(1 + 2\mu)}. \quad (5-29)$$

Proof. Parallel to Proposition 5.15. Meanwhile, we make use of (5-16). \square

Remark 5.20. By analytic continuation, (5-20) and (5-29) hold for $s \in \mathbb{C}$ and away from the poles of $E(*; \mu)$. In fact, the rapid decay of Φ at ∞ guarantees the pairings converge absolutely.

5E. Calculation on the spectral side. As noted before, our approach diverges from the ‘‘Kuznetsov–Voronoi’’ method from the outset. We express the moment of $GL(3) \times GL(2)$ L -functions via the period integral in Proposition 5.15 using a Poincaré series.

Definition 5.21. Let $a \geq 1$ be an integer and $h \in C^\infty(0, \infty)$. The Poincaré series of Γ_2 is defined as

$$P^a(z; h) := \sum_{\gamma \in U_2(\mathbb{Z}) \backslash \Gamma_2} h(a \text{Im } \gamma z) e(a \text{Re } \gamma z) \quad (z \in \mathfrak{h}^2) \quad (5-30)$$

provided the series converges absolutely.

It is not hard to see that if the bounds

$$h(y) \ll y^{1+\epsilon} \quad (\text{as } y \rightarrow 0) \quad \text{and} \quad h(y) \ll y^{1/2-\epsilon} \quad (\text{as } y \rightarrow \infty) \tag{5-31}$$

are satisfied, then the Poincaré series $P^a(z; h)$ converges absolutely and represents an L^2 -function. In this article, we take $h := H^b$ with $H \in \mathcal{C}_\eta$ and $\eta > 40$. By [Proposition 5.10](#), the conditions in (5-31) are clearly met. We will often use the shorthand $P^a := P^a(*; h)$. Also, we denote the Petersson inner product on $\Gamma_2 \backslash \mathfrak{h}^2$ by $\langle \cdot, \cdot \rangle$, defined as

$$\langle \phi_1, \phi_2 \rangle := \int_{\Gamma_2 \backslash \mathfrak{h}^2} \phi_1(g) \cdot \overline{\phi_2(g)} \, dg$$

with dg being the invariant measure on \mathfrak{h}^2 .

Lemma 5.22. *Let ϕ be a Maass cusp form of Γ_2 , $\Delta\phi = (\frac{1}{4} - \mu^2)\phi$, and $\mathcal{B}_\phi(a)$ be the a -th Fourier coefficient of ϕ . Then*

$$\langle P^a, \phi \rangle = |a|^{1/2} \cdot \overline{\mathcal{B}_\phi(a)} \cdot h^\#(\bar{\mu}).$$

Proof. Replace P^a in $\langle P^a, \phi \rangle$ by its definition and unfold, we easily find that

$$\langle P^a, \phi \rangle = \int_0^\infty h(ay) \cdot \overline{(\widehat{\phi})_a(y)} \frac{dy}{y^2}.$$

The result follows at once upon plugging-in (5-14) and making the change of variable $y \rightarrow |a|^{-1}y$. \square

Similarly, the following holds away from the poles of $E(*; \mu)$:

Lemma 5.23. *We have*

$$\langle P^a, E(*; \mu) \rangle = |a|^{1/2} \cdot \frac{|a|^{\bar{\mu}} \sigma_{-2\bar{\mu}}(|a|)}{\zeta^*(1+2\bar{\mu})} \cdot h^\#(\bar{\mu}). \tag{5-32}$$

Proposition 5.24 (spectral expansion). *Suppose $f \in L^2(\Gamma_2 \backslash \mathfrak{h}^2)$ and $\langle f, 1 \rangle = 0$. Then*

$$f(z) = \sum_{j=1}^\infty \frac{\langle f, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle} \cdot \phi_j(z) + \int_{(0)} \langle f, E(*; \mu) \rangle \cdot E(z; \mu) \frac{d\mu}{4\pi i} \quad (z \in \mathfrak{h}^2) \tag{5-33}$$

where $(\phi_j)_{j \geq 1}$ is any orthogonal basis of Maass cusp forms for Γ_2 .

Proof. See Theorem 3.16.1 of [\[Goldfeld 2015\]](#). \square

Proposition 5.25. *Let Φ be a Maass cusp form of Γ_3 and P^a be a Poincaré series of Γ_2 . Then*

$$\begin{aligned} & 2|a|^{-1/2} (P^a, (\mathbb{P}_2^3 \Phi) |\det *|^{\bar{s}-1/2})_{\Gamma_2 \backslash \text{GL}_2(\mathbb{R})} \\ &= \sum'_{j=1}^\infty h^\#(\bar{\mu}_j) \frac{\overline{\mathcal{B}_j(a)} \Lambda(s, \phi_j \otimes \tilde{\Phi})}{\langle \phi_j, \phi_j \rangle} + \int_{(0)} h^\#(\mu) \frac{\sigma_{-2\mu}(|a|) |a|^{-\mu} \Lambda(s+\mu, \tilde{\Phi}) \Lambda(1-s+\mu, \Phi)}{|\Lambda(1+2\mu)|^2} \frac{d\mu}{4\pi i} \end{aligned} \tag{5-34}$$

for any $s \in \mathbb{C}$, where the sum is restricted to an orthogonal basis (ϕ_j) of even Hecke-normalized Maass cusp forms for Γ_2 with $\Delta\phi_j = (\frac{1}{4} - \mu_j^2)\phi_j$ and $\mathcal{B}_j(a) := \mathcal{B}_{\phi_j}(a)$.

Proof. Substitute the spectral expansion of P^a as in (5-33) into the pairing $(P^a, (\mathbb{P}_2^3 \Phi) \cdot |\det *|^{\bar{s}-1/2})_{\Gamma_2 \backslash GL_2(\mathbb{R})}$. The inner products involved have been computed in Lemmas 5.22–5.23 and Definitions 5.15–5.19. \square

Remark 5.26. Good control over spectral aspects and integral transforms, along with flexibility in choosing test functions on the spectral side, are crucial in applications. Also, this helps eliminate extraneous polar contributions (e.g., those not predicted by [Conrey et al. 2005]) for Eisenstein cases. These explain the preference of Kuznetsov-based methods over period-based methods (see the discussions in [Blomer 2012a; Nunes 2023; Zacharias 2019; 2021]).

While our method is period-based, it accommodates a broad class of test functions similar to the Kuznetsov approaches, thanks to the transforms in Definition 5.8. These transforms, generalized to $GL(n)$ as in [Goldfeld and Kontorovich 2012], have significantly contributed to the development of higher-rank Kuznetsov formulae; see [Goldfeld and Kontorovich 2013; Goldfeld et al. 2021; 2022; Buttcane 2020].

Our method effectively combines the strengths of both Kuznetsov and period approaches, balancing precision in the archimedean aspect with structural insights into the nonarchimedean aspect.

Remark 5.27. Within our class \mathcal{C}_η of test functions, a good choice of test function is given by

$$H(\mu) := (e^{((\mu+iM)/R)^2} + e^{((\mu-iM)/R)^2}) \cdot \frac{\Gamma(2\eta + \mu)\Gamma(2\eta - \mu)}{\prod_{i=1}^3 \Gamma(\frac{1/2 + \mu - \alpha_i}{2})\Gamma(\frac{1/2 - \mu - \alpha_i}{2})}, \quad (5-35)$$

where $\eta > 40$, $M \gg 1$, and $R = M^\gamma$ ($0 < \gamma \leq 1$). In (5-35),

- the factor $e^{((\mu+iM)/R)^2} + e^{((\mu-iM)/R)^2}$ serves as a smooth cut-off for $|\mu_j| \in [M - R, M + R]$ and gives the needed decay in Proposition 5.10;
- the factors $\prod_{i=1}^3 \Gamma(\frac{1}{2}(\frac{1}{2} + \mu - \alpha_i))\Gamma(\frac{1}{2}(\frac{1}{2} - \mu - \alpha_i))$ cancel out the archimedean factors of $\Lambda(\frac{1}{2}, \phi_j \otimes \tilde{\Phi})$ on the spectral expansion (5-34) and in the diagonal contribution (6-9);
- the factors $\Gamma(2\eta + \mu)\Gamma(2\eta - \mu)$ balance off the exponential growth from $d\mu/|\Gamma(\mu)|^2$, $\|\phi_j\|^{-2}$ and $|\Lambda(1 + 2i\mu)|^{-2}$. Also, a large enough region of holomorphy of (5-35) is maintained so that $h(y) := H^b(y)$ has sufficient decay at 0 and ∞ .

Remark 5.28. One might consider using an automorphic kernel instead of a Poincaré series for Theorem 1.1. While this offers more structural flexibility, the analysis of the spherical transforms becomes quite complicated; see [Zagier 1981; Buttcane 2013]. The Poincaré series approach appears better suited to the analytic number theory of higher-rank groups.

6. Basic identity for dual moment

6A. Unipotent integration. We are ready to work on the dual side of our moment formula. To simplify our argument, we will only consider $P = P^a(*; h)$ with $a = 1$ in the following. Suppose $\operatorname{Re} s > 1 + \frac{\theta}{2}$, where θ is defined in Section 5A. We start by substituting the definition of P into the pairing in (5-34).

We find upon unfolding

$$(P, \mathbb{P}_2^3 \Phi \cdot |\det *|^{\bar{s}-1/2})_{\Gamma_2 \backslash \text{GL}_2(\mathbb{R})} = \int_0^\infty \int_0^\infty h(y_1) \cdot (y_0^2 y_1)^{s-1/2} \cdot \int_0^1 \tilde{\Phi} \left[\begin{pmatrix} 1 & u_{1,2} \\ & 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & \\ & y_0 \\ & & 1 \end{pmatrix} \right] e(u_{1,2}) du_{1,2} \frac{dy_0 dy_1}{y_0 y_1^2}. \quad (6-1)$$

The main task of this section is to compute the inner, “incomplete” unipotent integral in (6-1) in terms of the Fourier–Whittaker periods of Φ (see Definition 5.12), which are relevant in constructing various L -functions associated with Φ , as discussed in Section 5D.

While this can be achieved using the full Fourier expansion of [Jacquet et al. 1979a; 1979b] (see [Goldfeld 2015, Theorem 5.3.2]) and simplifying, we opt for a self-contained and conceptual treatment, which follows from two one-dimensional Fourier expansions and the automorphy of Φ . Essentially, this is where “summation formulae” come into play in our method, presented in an elementary, clean, and global manner.

Proposition 6.1. *For any automorphic function Φ of Γ_3 , we have, for any $y_0, y_1 > 0$,*

$$\int_0^1 \Phi \left[\begin{pmatrix} 1 & u_{1,2} \\ & 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & \\ & y_0 \\ & & 1 \end{pmatrix} \right] e(-u_{1,2}) du_{1,2} = \sum_{a_0, a_1 = -\infty}^\infty (\widehat{\Phi})_{(a_1, 1)} \left[\begin{pmatrix} 1 & \\ & 1 \\ -a_0 & 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & \\ & y_0 \\ & & 1 \end{pmatrix} \right]. \quad (6-2)$$

Proof. Firstly, we Fourier-expand along the abelian subgroup $\left\{ \begin{pmatrix} 1 & * \\ & 1 \\ & & 1 \end{pmatrix} \right\}$:

$$\int_0^1 \Phi \left[\begin{pmatrix} 1 & u_{1,2} \\ & 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & \\ & y_0 \\ & & 1 \end{pmatrix} \right] e(-u_{1,2}) du_{1,2} = \sum_{a_0 = -\infty}^\infty \int_{\mathbb{Z}^2 \backslash \mathbb{R}^2} \Phi \left[\begin{pmatrix} 1 & u_{1,2} & u_{1,3} \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & \\ & y_0 \\ & & 1 \end{pmatrix} \right] e(-u_{1,2} - a_0 \cdot u_{1,3}) du_{1,2} du_{1,3}. \quad (6-3)$$

Secondly, for each $a_0 \in \mathbb{Z}$, consider a unimodular change of variables of the form $(u_{1,2}, u_{1,3}) = (u'_{1,2}, u'_{1,3}) \cdot \begin{pmatrix} 1 & \\ & 1 \\ -a_0 & 1 \end{pmatrix}$. One can readily observe that

$$\begin{pmatrix} 1 & u_{1,2} & u_{1,3} \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & a_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u'_{1,2} & u'_{1,3} \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & -a_0 & 1 \end{pmatrix}.$$

Together with the automorphy of Φ with respect to Γ_3 , we have

$$\begin{aligned} & \int_0^1 \Phi \left[\begin{pmatrix} 1 & u_{1,2} \\ & 1 \\ & & 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & \\ & y_0 \\ & & 1 \end{pmatrix} \right] e(-a_2 \cdot u_{1,2}) du_{1,2} \\ &= \sum_{a_0=-\infty}^{\infty} \int_{\mathbb{Z}^2 \setminus \mathbb{R}^2} \Phi \left[\begin{pmatrix} 1 & u'_{1,2} & u'_{1,3} \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & 1 \\ & & -a_0 \ 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & \\ & y_0 \\ & & 1 \end{pmatrix} \right] e(-u'_{1,2}) du'_{1,2} du'_{1,3}. \quad (6-4) \end{aligned}$$

The result follows from a third and final Fourier expansion along the abelian subgroup $\left\{ \begin{pmatrix} 1 & & \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}$:

$$\begin{aligned} & \int_0^1 \Phi \left[\begin{pmatrix} 1 & u_{1,2} \\ & 1 \\ & & 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & \\ & y_0 \\ & & 1 \end{pmatrix} \right] e(-u_{1,2}) du_{1,2} \\ &= \sum_{a_0, a_1=-\infty}^{\infty} \int_0^1 \int_0^1 \int_0^1 \Phi \left[\begin{pmatrix} 1 & u_{1,2} & u_{1,3} \\ & 1 & u_{2,3} \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & 1 \\ & & -a_0 \ 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & \\ & y_0 \\ & & 1 \end{pmatrix} \right] \\ & \quad \cdot e(-u_{1,2} - a_1 \cdot u_{2,3}) du_{1,2} du_{1,3} du_{2,3}. \end{aligned}$$

□

We then explicate [Proposition 6.1](#) when Φ is a Maass cusp form of Γ_3 . This constitutes the *basic identity* of the present article. [Theorem 1.1](#) is a natural consequence of this identity and the diagonal/off-diagonal structures on the dual side become apparent (see [Proposition 7.2](#)).

Corollary 6.2. *Suppose Φ is a Maass cusp form of Γ_3 . Then*

$$\begin{aligned} & \int_0^1 \Phi \left[\begin{pmatrix} 1 & u_{1,2} \\ & 1 \\ & & 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & \\ & y_0 \\ & & 1 \end{pmatrix} \right] e(-u_{1,2}) du_{1,2} \\ &= \sum_{a_1 \neq 0} \frac{\mathcal{B}_\Phi(a_1, 1)}{|a_1|} \cdot W_{\alpha(\Phi)}(|a_1| y_0, y_1) + \sum_{a_0 \neq 0} \sum_{a_1 \neq 0} \frac{\mathcal{B}_\Phi(a_1, 1)}{|a_1|} \cdot W_{\alpha(\Phi)} \left(\frac{|a_1| y_0}{1+(a_0 y_0)^2}, y_1 \sqrt{1+(a_0 y_0)^2} \right) \\ & \quad \cdot e \left(-\frac{a_0 a_1 y_0^2}{1+(a_0 y_0)^2} \right). \quad (6-5) \end{aligned}$$

Proof. By cuspidality, $(\widehat{\Phi})_{(0,1)} \equiv 0$. The result follows from a straight-forward linear algebra calculation

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & -a_0 \ 1 \end{pmatrix} \begin{pmatrix} y_0 y_1 & \\ & y_0 \\ & & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & & \\ & 1 & -\frac{a_0 y_0^2}{1+(a_0 y_0)^2} \\ & & 1 \end{pmatrix} \begin{pmatrix} \frac{y_0}{1+(a_0 y_0)^2} \cdot y_1 \sqrt{1+(a_0 y_0)^2} & \\ & \frac{y_0}{1+(a_0 y_0)^2} \\ & & 1 \end{pmatrix} \quad (6-6)$$

under the right quotient by $O_3(\mathbb{R}) \cdot \mathbb{R}^\times$. One can verify Equation (6-6) using the formula stated in Section 2.4 of [Buttcane 2018] or the mathematica command `IwasawaForm[]` in the `GL(n)pack (gln.m)`. The user manual and the package can both be downloaded from Kevin A. Broughan’s website: see <https://www.math.waikato.ac.nz/kab/glnpack.html>. □

6B. Initial simplification and absolute convergence. We temporarily restrict ourselves to the vertical strip $1 + \frac{\theta}{2} < \sigma := \text{Re } s < 4$. As we will see, this guarantees absolute convergence of sums and integrals.

Suppose $H \in \mathcal{C}_\eta$ with $\eta > 40$ (see Proposition 5.10). Then the bound (5-12) for $h := H^b$ implies its Mellin transform $\tilde{h}(w) := \int_0^\infty h(y)y^w d^\times y$ is holomorphic on the strip $|\text{Re } w| < \eta$. Substituting (6-5) into (6-1), and apply the changes of variables $y_0 \rightarrow |a_1|^{-1}y_0, y_1 \rightarrow |a_0|^{-1}y_1$ to the first, second piece of the resultant, we have

$$(P, \mathbb{P}_2^3 \Phi \cdot |\det *|^{\bar{s}-1/2})_{\Gamma_2 \backslash \text{GL}_2(\mathbb{R})} = 2 \cdot L(2s, \Phi) \cdot \int_0^\infty \int_0^\infty h(y_1) \cdot (y_0^2 y_1)^{s-1/2} \cdot W_{-\alpha(\Phi)}(y_0, y_1) \frac{dy_0 dy_1}{y_0 y_1^2} + OD_\Phi(s), \tag{6-7}$$

where $OD_\Phi(s)$ is defined below.

Definition 6.3. Define $OD_\Phi(s)$ as

$$OD_\Phi(s) := \sum_{a_0 \neq 0} \sum_{a_1 \neq 0} \frac{\mathcal{B}_\Phi(1, a_1)}{|a_0|^{2s-1} |a_1|} \cdot \int_0^\infty \int_0^\infty h(y_1) \cdot (y_0^2 y_1)^{s-1/2} \cdot e\left(\frac{a_1}{a_0} \cdot \frac{y_0^2}{1+y_0^2}\right) \cdot W_{-\alpha(\Phi)}\left(\left|\frac{a_1}{a_0}\right| \cdot \frac{y_0}{1+y_0^2}, y_1 \sqrt{1+y_0^2}\right) \frac{dy_0 dy_1}{y_0 y_1^2}. \tag{6-8}$$

Proposition 6.4. When $H \in \mathcal{C}_\eta$ and $4 > \sigma > \frac{1+\theta}{2}$, we have

$$\int_0^\infty \int_0^\infty h(y_1) \cdot (y_0^2 y_1)^{s-1/2} \cdot W_{-\alpha(\Phi)}(y_0, y_1) \frac{dy_0 dy_1}{y_0 y_1^2} = \frac{\pi^{-3s}}{8} \cdot \int_{(0)} \frac{H(\mu)}{|\Gamma(\mu)|^2} \cdot \prod_{i=1}^3 \Gamma\left(\frac{s+\mu-\alpha_i}{2}\right) \Gamma\left(\frac{s-\mu-\alpha_i}{2}\right) \frac{d\mu}{2\pi i}. \tag{6-9}$$

Proof. From Proposition 5.11, we have

$$\int_0^\infty \int_0^\infty h(y_1) \cdot (y_0^2 y_1)^{s-1/2} \cdot W_{-\alpha(\Phi)}(y_0, y_1) \frac{dy_0 dy_1}{y_0 y_1^2} = \frac{1}{2} \cdot \int_{(0)} \frac{H(\mu)}{|\Gamma(\mu)|^2} \cdot \int_0^\infty \int_0^\infty W_\mu(y_1) W_{-\alpha(\Phi)}(y_0, y_1) (y_0^2 y_1)^{s-1/2} \frac{dy_0 dy_1}{y_0 y_1^2} \frac{d\mu}{2\pi i}.$$

The y_0, y_1 -integrals can be evaluated by Proposition 5.7 and (6-9) follows. Moreover, the right side of (6-9) is holomorphic on $\sigma > 0$. □

Proposition 6.5. The off-diagonal $OD_\Phi(s)$ converges absolutely when $4 > \sigma > 1 + \frac{\theta}{2}$ and $H \in \mathcal{C}_\eta$ ($\eta > 40$).

Proof. Upon inserting absolute values, breaking up the y_0 -integral into $\int_0^1 + \int_1^\infty$, and applying the bounds (5-6) and $|\mathcal{B}_\Phi(1, a_1)| \ll |a_1|^\theta$, observe that

$$OD_\Phi(s) \ll \sum_{a_0=1}^\infty \sum_{a_1=1}^\infty \frac{1}{a_0^{2\sigma-1} a_1^{1-\theta}} \left(\int_{y_0=1}^\infty + \int_{y_0=0}^1 \right) \int_{y_1=0}^\infty |h(y_1)| (y_0^2 y_1)^\sigma \sigma^{-1/2} \left(\frac{a_1 a_0^{-1} y_0}{1+y_0^2} \right)^{A_0} (y_1 \sqrt{1+y_0^2})^{A_1} \frac{dy_0 dy_1}{y_0 y_1^2},$$

where the implicit constant depends only on Φ , A_0 , A_1 with $-\infty < A_0, A_1 < 1$. We are allowed to choose different A_0, A_1 in different ranges of the y_0, y_1 -integrals.

The convergence of both of the series is guaranteed if

$$A_0 < -\theta \quad \text{and} \quad \sigma > 1 - \frac{A_0}{2}. \quad (6-10)$$

We now show that if (6-10) and

$$A_1 < A_0 - 2\sigma + 1 \quad (6-11)$$

both hold, then the y_0 -integrals converge. Indeed, observe that $2\sigma + A_0 - 2 > -1$ (by (6-10)), and

$$\int_{y_0=0}^1 y_0^{2\sigma+A_0-2} (1+y_0^2)^{A_1/2-A_0} dy_0 \asymp_{A_0, A_1} \int_{y_0=0}^1 y_0^{2\sigma+A_0-2} dy_0.$$

So, the last integral converges. Also, (6-10) and (6-11) imply $A_1 < \min\{1, 2A_0\}$ and thus,

$$\int_{y_0=1}^\infty y_0^{2\sigma+A_0-2} (1+y_0^2)^{A_1/2-A_0} dy_0 \leq \int_{y_0=1}^\infty y_0^{2\sigma+A_1-A_0-2} dy_0.$$

The last integral converges because of (6-11).

For the y_1 -integral, the integrals

$$\int_{y_1=1}^\infty |h(y_1)| y_1^{\sigma+A_1-5/2} dy_1 \quad \text{and} \quad \int_{y_1=0}^1 |h(y_1)| y_1^{\sigma+A_1-5/2} dy_1$$

converge whenever $H \in \mathcal{C}_\eta$ (we then have (5-12)) and

$$\eta > \left| \sigma + A_1 - \frac{3}{2} \right|. \quad (6-12)$$

Let $\delta := \sigma - 1 - (\theta/2) (> 0)$. In view of (6-10) and (6-11), we may take $A_0 := -\theta - \delta$ and $A_1 := -2\theta - 1 - 4\delta$. Also, (6-12) trivially holds as $\eta > 40$ and $\sigma < 4$. The result follows. \square

Remark 6.6. Readers may notice the similarity between (3-2) and the inner product construction of the Kuznetsov formula. Indeed, $\mathbb{P}_2^3 \Phi$ is an infinite sum of Poincaré series for $SL_2(\mathbb{Z})$ due to its Fourier expansion, though we never adopt this perspective in this article. This serves as a $GL(3) \times GL(2)$ analog to the Kuznetsov formula. However, there are key differences. Our moment identity equates two unfoldings, rather than comparing spectral and geometric expansions.

The second difference is technical. In the Kuznetsov formula, the oscillatory factors can be eliminated to obtain a “primitive” trace formula, see [Goldfeld and Kontorovich 2013; Zhou 2014; Goldfeld et al.

2021]. However, this does not work here — we have yet to analytically continue into the critical strip in Proposition 6.5. Here, the oscillatory factor in $OD_\Phi(s)$ is crucial, arising naturally from the abstract characterization of Whittaker functions.

7. Structure of the off-diagonal

Fix $\epsilon := \frac{1}{100}$ (say), $0 < \phi < \frac{\pi}{2}$, and consider the domain $1 + \frac{\theta}{2} + \epsilon < \sigma < 4$ in this section to maintain absolute convergence. We will stick with this choice of ϵ for the rest of this article and the number ϕ here should not pose any confusion with the basis of cusp forms (ϕ_j) of Γ_2 . We define a perturbed version of $OD_\Phi(s)$ as follows:

$$OD_\Phi(s; \phi) := \sum_{a_0 \neq 0} \sum_{a_1 \neq 0} \frac{\mathcal{B}_\Phi(1, a_1)}{|a_0|^{2s-1}|a_1|} \int_0^\infty \int_0^\infty h(y_1) \cdot (y_0^2 y_1)^{s-1/2} W_{-\alpha(\Phi)} \left(\left| \frac{a_1}{a_0} \right| \frac{y_0}{1+y_0^2}, y_1 \sqrt{1+y_0^2} \right) \cdot e \left(\frac{a_1}{a_0} \frac{y_0^2}{1+y_0^2}; \phi \right) \frac{dy_0 dy_1}{y_0 y_1^2}, \tag{7-1}$$

where

$$e(x; \phi) := \int_{(\epsilon)} |2\pi x|^{-u} e^{iu\phi \operatorname{sgn}(x)} \Gamma(u) \frac{du}{2\pi i} \quad (x \in \mathbb{R} - \{0\}). \tag{7-2}$$

In Proposition 7.3, we will show that

$$\lim_{\phi \rightarrow \pi/2} OD_\Phi(s; \phi) = OD_\Phi(s) \tag{7-3}$$

on a smaller region of absolute convergence.

Remark 7.1. The goals of this section is to obtain an expression of $OD_\Phi(s; \phi)$ that

- reveals the structure of the dual moment;
- can be analytically continued into the critical strip;
- and will allow us to pass to the limit $\phi \rightarrow \pi/2$ (in the critical strip).

Given these considerations, it is natural to work on the dual side of the Mellin transforms, which also allows for the separation of variables. The main result of this section is as follows:

Proposition 7.2 (dual moment). *Let $H \in \mathcal{C}_\eta$ ($\eta > 40$) and $\phi \in (0, \pi/2)$. On the vertical strip*

$$1 + \frac{\theta}{2} + \epsilon < \sigma < 4, \tag{7-4}$$

we have

$$OD_\Phi(s; \phi) = \frac{1}{4} \int_{(1+\theta+2\epsilon)} \zeta(2s - s_0) L(s_0, \Phi) \cdot \sum_{\delta=\pm} (\mathcal{F}_\Phi^{(\delta)} H)(s_0, s; \phi) \frac{ds_0}{2\pi i}, \tag{7-5}$$

where the transform of H is given by

$$(\mathcal{F}_\Phi^{(\delta)} H)(s_0, s; \phi) := \int_{(15)} \int_{(\epsilon)} \tilde{h}(s - s_1 - \frac{1}{2}) \cdot \mathcal{G}_\Phi^{(\delta)}(s_1, u; s_0, s; \phi) \frac{du}{2\pi i} \frac{ds_1}{2\pi i}, \tag{7-6}$$

with $h := H^\flat$, $G_\Phi := G_{\alpha(\Phi)}$ as defined in (5-5), and

$$\mathcal{G}_\Phi^{(\delta)}(s_1, u; s_0, s; \phi) := G_\Phi(s_0 - u, s_1) \cdot (2\pi)^{-u} e^{i\delta\phi u} \Gamma(u) \cdot \frac{\Gamma\left(\frac{u+1-2s+s_1-s_0}{2}\right) \Gamma\left(\frac{2s-s_0-u}{2}\right)}{\Gamma\left(\frac{1+s_1}{2} - s_0\right)}. \quad (7-7)$$

Proof. Plug-in the expression for $W_{-\alpha(\Phi)}$ from Proposition 5.3 into $OD_\Phi(s; \phi)$ with

$$\sigma_1 := 15 \quad \text{and} \quad 1 + \theta < \sigma_0 < 2\sigma - 1 - \epsilon. \quad (7-8)$$

Insert absolute values to the resulting expression, the sums and integrals are bounded by

$$\begin{aligned} \sum_{\delta := \text{sgn}(a_0 a_1) = \pm} & \left(\sum_{a_0 \neq 0} \frac{1}{|a_0|^{2\sigma - \sigma_0 - \epsilon}} \right) \left(\sum_{a_1 \neq 0} \frac{|\mathcal{B}_\Phi(1, a_1)|}{|a_1|^{\sigma_0 + \epsilon}} \right) \left(\int_{(\sigma_0)} \int_{(\sigma_1)} |G_\Phi(s_0, s_1)| |ds_0| |ds_1| \right) \\ & \cdot \left(\int_{(\epsilon)} |e^{i\delta\phi u} \Gamma(u)| |du| \right) \left(\int_0^\infty y_0^{-\sigma_0 - 2\epsilon + 2\sigma} (1 + y_0^2)^{\sigma_0 + \epsilon - (1 + \sigma_1)/2} d^\times y_0 \right) \\ & \cdot \left(\int_0^\infty |h(y_1)| \cdot y_1^{\sigma - \sigma_1 - 1/2} d^\times y_1 \right). \quad (7-9) \end{aligned}$$

Observe that:

- By Stirling's formula, the s_0, s_1, u -integrals converge as long as

$$\sigma_0, \sigma_1, \epsilon > 0, \quad \phi \in (0, \pi/2). \quad (7-10)$$

- The y_0 -integral converges as long as

$$\sigma_0 + 2\epsilon < 2\sigma < \sigma_1 - \sigma_0 + 1. \quad (7-11)$$

- By the bound $|\mathcal{B}_\Phi(1, a_1)| \ll |a_1|^\theta$, the a_0 -sum and the a_1 -sum converge as long as

$$2\sigma - 1 > \sigma_0 + \epsilon > 1 + \theta. \quad (7-12)$$

Under (7-8), items (7-10), (7-11), (7-12) hold. Moreover, the y_1 -integral converges by (5-12) and $H \in \mathcal{C}_\eta$ ($\eta > 40$). Now, upon rearranging sums and integrals, and noticing that $\mathcal{B}_\Phi(1, a_1) = \mathcal{B}_\Phi(1, -a_1)$, we have

$$\begin{aligned} OD_\Phi(s; \phi) &= 2 \sum_{\delta = \pm} \int_{(\sigma_0)} \int_{(\sigma_1)} \int_{(\epsilon)} \frac{G_\Phi(s_0, s_1)}{4} \cdot (2\pi)^{-u} e^{i\delta\phi u} \Gamma(u) \\ & \cdot \left(\int_0^\infty h(y_1) y_1^{s-s_1-1/2} d^\times y_1 \right) \left(\int_0^\infty y_0^{-s_0-2u+2s} (1 + y_0^2)^{s_0+u-(1+s_1)/2} d^\times y_0 \right) \\ & \cdot \left(\sum_{a_0=1}^\infty \sum_{a_1=1}^\infty \frac{\mathcal{B}_\Phi(1, a_1)}{a_0^{2s-1} a_1} \left(\frac{a_1}{a_0} \right)^{1-s_0-u} \right) \frac{ds_0}{2\pi i} \frac{ds_1}{2\pi i} \frac{du}{2\pi i}. \quad (7-13) \end{aligned}$$

Recall the integral identity

$$\int_0^\infty y_0^v (1 + y_0^2)^A d^\times y_0 = \frac{1}{2} \frac{\Gamma(-A - v/2) \Gamma(v/2)}{\Gamma(-A)} \quad (7-14)$$

for $0 < \operatorname{Re} v < -2 \operatorname{Re} A$. It follows that

$$\begin{aligned}
 OD_\Phi(s; \phi) &= 2 \sum_{\delta=\pm} \int_{(\sigma_0)} \int_{(\sigma_1)} \int_{(\epsilon)} \zeta(2s - s_0 - u) L(s_0 + u; \Phi) \cdot \tilde{h}(s - s_1 - \frac{1}{2}) \\
 &\quad \cdot \frac{G_\Phi(s_0, s_1)}{4} \cdot (2\pi)^{-u} e^{i\delta\phi u} \Gamma(u) \cdot \frac{1}{2} \frac{\Gamma(s - \frac{s_0}{2} - u) \Gamma(\frac{1+s_1-s_0}{2} - s)}{\Gamma(\frac{1+s_1}{2} - s_0 - u)} \frac{ds_0}{2\pi i} \frac{ds_1}{2\pi i} \frac{du}{2\pi i}. \tag{7-15}
 \end{aligned}$$

We pick the contour $(\sigma_0) := (1 + \theta + \epsilon)$, thus imposing (7-4). To isolate the nonarchimedean part of $OD_\Phi(s; \phi)$, we change variables to $s'_0 = s_0 + u$. Substituting the expression for $G_\Phi(s'_0 - u, s_1)$ (see (5-5)), we obtain (7-5)–(7-7). The absolute convergence proven earlier also ensures the holomorphy of the integral transform $(\mathcal{F}_\Phi^{(\delta)} h)(s'_0, s; \phi)$ on the domain

$$\sigma < 4 \quad \text{and} \quad 1 + \theta + \epsilon < \sigma'_0 < 2\sigma - 1. \tag{7-16}$$

This completes the proof. □

Proposition 7.3. *For $4 > \sigma > (3 + \theta)/2$ and $H \in \mathcal{C}_\eta$, we have*

$$\lim_{\phi \rightarrow \pi/2} OD_\Phi(s; \phi) = OD_\Phi(s). \tag{7-17}$$

Proof. Let $\epsilon := \frac{1}{100}$, $\sigma_1 := 15$, and pick any σ_0 satisfying

$$\frac{3}{2} + \theta + \epsilon < \sigma_0 < 2\sigma - 1 - \epsilon. \tag{7-18}$$

Denote by \mathcal{C}_ϵ the indented path consisting of the line segments:

$$-\frac{1}{2} - \epsilon - i\infty \rightarrow -\frac{1}{2} - \epsilon - i \rightarrow \epsilon - i \rightarrow \epsilon + i \rightarrow -\frac{1}{2} - \epsilon + i \rightarrow -\frac{1}{2} - \epsilon + i\infty.$$

Replace $e(x; \phi)$ in (7-13) by the expression:

$$e(x; \phi) = \int_{\mathcal{C}_\epsilon} |2\pi x|^{-u} e^{iu\phi \operatorname{sgn}(x)} \Gamma(u) \frac{du}{2\pi i}. \tag{7-19}$$

Note that $|e^{iu\phi \operatorname{sgn}(x)} \Gamma(u)| \ll_\epsilon (1 + |\operatorname{Im} u|)^{-1-\epsilon}$ for $u \in \mathcal{C}_\epsilon$ and $\phi \in (0, \pi/2]$. Insert absolute values in (7-13). The resulting sums and integrals converge absolutely when $\phi \in (0, \pi/2]$ and (7-18) holds, which can be seen by the same argument following (7-9). Apply dominated convergence and shift the contour of the u -integral to $-\infty$, the residual series obtained is exactly $e((a_1/a_0)(y_0^2/(1 + y_0^2)))$. This completes the proof. □

Now, $OD_\Phi(s; \phi)$ is expressed as Mellin-Barnes integrals. The Γ -factors from Proposition 5.3 and (7-2) alone are not sufficient for our goals (see Remark 7.1 and (7-10), (7-11), (7-12)). The three extra Γ -factors brought by the y_0 -integral, which mix all integration variables, will play an important role in Section 8-9.

8. Analytic properties of the archimedean transform

In (7-5), the factors $\zeta(2s - s_0)$ and $L(s_0, \Phi)$ are known to admit holomorphic continuation and have polynomial growth in vertical strips, except on the line $2s - s_0 = 1$. We also examine the archimedean part of (7-5), i.e., the integral transform

$$(\mathcal{F}_\Phi^{(\delta)} H)(s_0, s; \phi) := \int_{(15)} \int_{(\epsilon)} \tilde{h}(s - s_1 - \frac{1}{2}) \cdot \mathcal{G}_\Phi^{(\delta)}(s_1, u; s_0, s; \phi) \frac{du}{2\pi i} \frac{ds_1}{2\pi i}, \quad (8-1)$$

where $h := H^b$ and $\mathcal{G}_\Phi^{(\delta)}(\dots)$ is defined in (7-7). In Section 7, we have shown that when $\phi \in (0, \pi/2)$, the function $(s_0, s) \mapsto (\mathcal{F}_\Phi^{(\delta)} h)(s_0, s; \phi)$ is holomorphic on the domain (7-16), i.e.,

$$\sigma < 4 \quad \text{and} \quad 1 + \theta + \epsilon < \sigma_0 < 2\sigma - 1.$$

In this section, we establish a larger region of holomorphy for $(s_0, s) \mapsto (\mathcal{F}_\Phi^{(\delta)} H)(s_0, s; \phi)$ that holds for $\phi \in (0, \pi/2]$. We write

$$s = \sigma + it, \quad s_0 = \sigma_0 + it_0, \quad s_1 = \sigma_1 + it_1, \quad \text{and} \quad u = \epsilon + iv,$$

with $\epsilon := \frac{1}{100}$. It is sufficient to consider s inside the rectangular box $\epsilon < \sigma < 4$ and $|t| \leq T$, for any given $T \geq 1000$. Moreover, $\alpha_k := i\gamma_k \in i\mathbb{R}$ ($k = 1, 2, 3$) by our assumptions on Φ . The main result of this section can be stated as follows:

Proposition 8.1. *Suppose $H \in \mathcal{C}_\eta$:*

(1) *For any $\phi \in (0, \pi/2]$, the transform $(\mathcal{F}_\Phi^{(\delta)} H)(s_0, s; \phi)$ is holomorphic on the domain*

$$\sigma_0 > \epsilon, \quad \sigma < 4, \quad \text{and} \quad 2\sigma - \sigma_0 - \epsilon > 0. \quad (8-2)$$

(2) *Whenever $(\sigma_0, \sigma) \in (8-2)$, $|t| < T$, and $\phi \in (0, \pi/2)$, the transform $(\mathcal{F}_\Phi^{(\delta)} H)(s_0, s; \phi)$ has exponential decay as $|t_0| \rightarrow \infty$. Note: The explicit estimate is stated in the proof below and the implicit constant depends only on T and Φ .*

Remark 8.2. The domain (8-2) is chosen in a way that the function $(s_0, s) \mapsto \mathcal{G}_\Phi^{(\delta)}(s_1, u; s_0, s; \phi)$ is holomorphic on (8-2) when $\text{Re } s_1 = \sigma_1 \geq 15$ and $\text{Re } u = \epsilon$. Moreover, if we have $15 \leq \sigma_1 \leq \eta - \frac{1}{2}$ and (8-2), then $s - s_1 - \frac{1}{2}$ lies inside the region of holomorphy of \tilde{h} .

Remark 8.3. As we shall see in Proposition 9.2, the region of holomorphy (8-2) is essentially optimal in terms of σ_0 .

Proof. The proof is based on a careful application of the Stirling estimate

$$|\Gamma(a + ib)| \asymp_a (1 + |b|)^{a-1/2} e^{-(\pi/2)|b|} \quad (a \neq 0, -1, -2, \dots, b \in \mathbb{R}) \quad (8-3)$$

to the kernel function $\mathcal{G}_\Phi^{(\delta)}(s_1, u; s_0, s; \phi)$. The following set of conditions will be repeated throughout the proof:

$$\begin{aligned} 0 < \phi &\leq \pi/2, \\ \sigma_0 > \epsilon, \quad \sigma < 4, \quad 2\sigma - \sigma_0 - \epsilon > 0, \\ \operatorname{Re} s_1 = \sigma_1 &\geq 15, \quad \operatorname{Re} u = \epsilon. \end{aligned} \tag{8-4}$$

Assuming (8-4), we apply (8-3) to the kernel function (7-7). It follows that

$$\begin{aligned} &|\mathcal{G}_\Phi^{(\delta)}(s_1, u; s_0, s; \phi)| \\ &\asymp (1+|v|)^{\epsilon-1/2} e^{-((\pi/2)-\phi)|v|} \cdot \prod_{k=1}^3 (1+|t_1-\gamma_k|)^{(\sigma_1-1)/2} e^{-(\pi/4)|t_1-\gamma_k|} \\ &\quad \cdot \prod_{k=1}^3 (1+|t_0-v+\gamma_k|)^{(\sigma_0-\epsilon-1)/2} e^{-(\pi/4)|t_0-v+\gamma_k|} \cdot (1+|2t-t_0-v|)^{(2\sigma-1-\sigma_0-\epsilon)/2} e^{-(\pi/4)|2t-t_0-v|} \\ &\quad \cdot (1+|v-2t+t_1-t_0|)^{(\epsilon-2\sigma+\sigma_1-\sigma_0)/2} e^{-(\pi/4)|v-2t+t_1-t_0|} \\ &\quad \cdot (1+|t_1-2t_0|)^{-(\sigma_1/2-\sigma_0)} e^{\frac{\pi}{4}|t_1-2t_0|} \cdot (1+|t_0+t_1-v|)^{-(\sigma_0+\sigma_1-\epsilon-1)/2} e^{(\pi/4)|t_0+t_1-v|}, \end{aligned} \tag{8-5}$$

where the implicit constant depends at most on σ_1 . Note that the domain (8-2) for (σ, σ_0) is bounded and thus the estimate is uniform in $\sigma, \sigma_0, \epsilon$. This will be assumed for all estimates in the rest of this section.

Let $\mathcal{P}_s^\Phi(t_0, t_1, v)$ be the ‘‘polynomial part’’ of (8-5) and the ‘‘exponential phase’’ of (8-5) be

$$\mathcal{E}_s^\Phi(t_0, t_1, v) := \sum_{k=1}^3 \{|t_1-\gamma_k| + |t_0-v+\gamma_k|\} + |2t-t_0-v| + |v-2t+t_1-t_0| - |t_1-2t_0| - |t_0+t_1-v|.$$

We first examine $\mathcal{E}_s^\Phi(t_0, t_1, v)$, which determines the effective support of $(\mathcal{F}_\Phi^{(\delta)} H)(s_0, s; \phi)$. By the triangle inequality and the fact $\gamma_1 + \gamma_2 + \gamma_3 = 0$, we have

$$|\mathcal{G}_\Phi^{(\delta)}(s_1, u; s_0, s; \phi)| \ll_{\sigma_1} e^{\pi T} \cdot \mathcal{P}_s^\Phi(t_0, t_1, v) \cdot \exp\left(-\frac{\pi}{4}\mathcal{E}(t_0, t_1, v)\right) \cdot e^{-(\pi/2-\phi)|v|} \tag{8-6}$$

with

$$\mathcal{E}(t_0, t_1, v) := 3|t_1| + 3|t_0-v| - |t_1-2t_0| + |v+t_1-t_0| + |t_0+v| - |t_0+t_1-v|, \tag{8-7}$$

whenever we have (8-4) and $|t| \leq T$.

Claim 8.4. *For any $t_0, t_1, v \in \mathbb{R}$, we have $\mathcal{E}(t_0, t_1, v) \geq 0$. Equality holds if and only if*

$$t_1 = 0 \quad \text{and} \quad t_0 - v = 0. \tag{8-8}$$

Proof. Adding up the inequalities $|t_1| + |t_0-v| \geq |t_0+t_1-v|$ and $|v+t_1-t_0| + |t_0+v| \geq |t_1-2t_0|$, we have

$$\mathcal{E}(t_0, t_1, v) \geq 2(|t_1| + |t_0-v|) \geq 0. \tag{8-9}$$

The equality case is apparent. □

Claim 8.5. When (8-4) and $|t| \leq T$ hold, the integral

$$\iint_{\substack{(\operatorname{Re} s_1, \operatorname{Re} u) = (\sigma_1, \epsilon), \\ (t_1, v): (8-11) \text{ holds}}} \tilde{h}\left(s - s_1 - \frac{1}{2}\right) \cdot \mathcal{G}_\Phi^{(\delta)}(s_1, u; s_0, s; \phi) \frac{du}{2\pi i} \frac{ds_1}{2\pi i} \quad (8-10)$$

has exponential decay as $|t_0| \rightarrow \infty$, where

$$|t_1| > \log^2(3 + |t_0|) \quad \text{or} \quad |v - t_0| > \log^2(3 + |t_0|). \quad (8-11)$$

Proof. In the case of (8-11), we have

$$\mathcal{E}(t_0, t_1, v) > \log^2(3 + |t_0|) + |t_1| + |t_0 - v| \quad (8-12)$$

from (8-9). The polynomial part $\mathcal{P}_s^\Phi(t_0, t_1, v)$ can be crudely bounded by

$$\mathcal{P}_s^\Phi(t_0, t_1, v) \ll_{\Phi, \sigma_1, T} [(1 + |t_1|)(1 + |v - t_0|)(1 + |t_0|)]^{A(\sigma_1)}, \quad (8-13)$$

where $A(\sigma_1) > 0$ is some constant.

Putting (8-12), (8-13), and the bound $e^{-(\pi/2 - \phi)|v|} \leq 1$ ($\phi \in (0, \pi/2]$) into (8-6), we obtain

$$\begin{aligned} & |\mathcal{G}_\Phi^{(\delta)}(s_1, u; s_0, s; \phi)| \\ & \ll_{\Phi, \sigma_1, T} (1 + |t_0|)^{A(\sigma_1)} e^{-(\pi/4) \log^2(3 + |t_0|)} \cdot [(1 + |t_1|)(1 + |v - t_0|)]^{A(\sigma_1)} e^{-(\pi/4)[|t_1| + |t_0 - v|]} \end{aligned} \quad (8-14)$$

whenever (8-11), (8-4), and $|t| \leq T$ hold. The boundedness of \tilde{h} on vertical strips implies that (8-10) is

$$\ll_{\sigma_1, \Phi, T} (1 + |t_0|)^{A(\sigma_1)} e^{-(\pi/4) \log^2(3 + |t_0|)}. \quad (8-15)$$

This proves Claim 8.5. \square

Now, let $\phi \in (0, \pi/2]$ and consider $(\mathcal{F}_\Phi^{(\delta)} H)(s_0, s; \phi)$ as a function on the bounded domain

$$(\sigma_0, \sigma) \in (8-2), \quad |t|, |t_0| \leq T. \quad (8-16)$$

When $|t_1| > \log^2(3 + T)$ or $|v| > T + \log^2(3 + T)$, observe that (8-11) is satisfied and from (8-14),

$$|\mathcal{G}_\Phi^{(\delta)}(s_1, u; s_0, s; \phi)| \ll_{\Phi, T} [(1 + |t_1|)(1 + |v|)]^{A(15)} \cdot e^{-(\pi/4)[|t_1| + |v|]}. \quad (8-17)$$

The last function is clearly jointly integrable with respect to t_1, v , and by Remark 8.2, $(\mathcal{F}_\Phi^{(\delta)} H)(s_0, s; \phi)$ is a holomorphic function on (8-16). Since the choice of T is arbitrary, we arrive at the first conclusion of Proposition 8.1.

In the remaining part of this section, we prove the second assertion of Proposition 8.1. We estimate the contribution from

$$|t_1| \leq \log^2(3 + |t_0|) \quad \text{and} \quad |v - t_0| \leq \log^2(3 + |t_0|), \quad (8-18)$$

where the complementary part has been treated in Claim 8.5.

It suffices to restrict to the effective support (8-8). The polynomial part can be essentially computed by substituting $t_1 := 0$ and $v := t_0$. More precisely, when (8-18) and $|t_0| \gg_T 1$ hold, there are only two

possible scenarios for the factors $1 + |(\dots)|$ in (8-5): either $1 + |(\dots)| \asymp |t_0|$, or $\log^{-C}(3 + |t_0|) \ll 1 + |(\dots)| \ll \log^C(3 + |t_0|)$ for some absolute constant $C > 0$.

In the case of (8-18), we apply the bounds $e^{-(\pi/4)\mathcal{E}(t_0, t_1, v)} \leq 1$ and $e^{-(\pi/2-\phi)|v|} \leq e^{-(1/2)(\pi/2-\phi)|t_0|}$ for $|t_0| \gg 1$ to (8-6). As a result, if we also have (8-4), $|t| < T$, and $|t_0| > 8T$, then

$$|\mathcal{G}_\Phi^{(\delta)}(s_1, u; s_0, s; \phi)| \ll_{\sigma_1, \Phi, T} |t_0|^{7-\sigma_1/2} e^{-(1/2)(\pi/2-\phi)|t_0|} \log^{B(\sigma_1)} |t_0| \tag{8-19}$$

and

$$\iint_{\substack{(\operatorname{Re} s_1, \operatorname{Re} u) = (\sigma_1, \epsilon) \\ (t_1, v): (8-18) \text{ holds}}} \tilde{h}(s - s_1 - \frac{1}{2}) \cdot \mathcal{G}_\Phi^{(\delta)}(s_1, u; s_0, s; \phi) \frac{du}{2\pi i} \frac{ds_1}{2\pi i} \\ \ll_{\sigma_1, \Phi, T} |t_0|^{7-\sigma_1/2} e^{-(1/2)(\pi/2-\phi)|t_0|} \log^{4+B(\sigma_1)} |t_0|, \tag{8-20}$$

where $B(\sigma_1) > 0$ is some constant. If $\phi < \pi/2$, then there is exponential decay in (8-20) as $|t_0| \rightarrow \infty$. Therefore, the second conclusion of the proposition follows from (8-20) and (8-15) (putting $\sigma_1 = 15$). \square

9. Analytic continuation of the off-diagonal (proof of Theorem 1.1)

Recall that

$$OD_\Phi(s; \phi) = \frac{1}{4} \int_{(1+\theta+2\epsilon)} \zeta(2s - s_0) L(s_0, \Phi) \cdot \sum_{\delta=\pm} (\mathcal{F}_\Phi^{(\delta)} H)(s_0, s; \phi) \frac{ds_0}{2\pi i} \tag{9-1}$$

for $1 + \frac{\theta}{2} + \epsilon < \sigma < 4$ and $\phi \in (0, \pi/2)$, see Proposition 7.2.

9A. Step 1: We first obtain a holomorphic continuation of $OD_\Phi(s; \phi)$ up to $\operatorname{Re} s > \frac{1}{2} + \epsilon$ by shifting the s_0 -integral to the left.

Fix any $\phi \in (0, \pi/2)$ and $T \geq 1000$. We first restrict ourselves to

$$1 + \frac{\theta}{2} + 2\epsilon < \sigma < 4, \quad |t| < T. \tag{9-2}$$

Clearly, the pole $s_0 = 2s - 1$ of $\zeta(2s - s_0)$ is on the right of the contour $\operatorname{Re} s_0 = 1 + \theta + 2\epsilon$ of the integral (7-5).

Let $T_0 \gg 1$. The rectangle with vertices $2\epsilon \pm iT_0$ and $(1 + \theta + 2\epsilon) \pm iT_0$ in the s_0 -plane lies inside the region of holomorphy (8-2) of $(\mathcal{F}_\Phi^{(\delta)} H)(s_0, s; \phi)$. The contribution from the horizontal segments $[2\epsilon \pm iT_0, (1 + \theta + 2\epsilon) \pm iT_0]$ tends to 0 as $T_0 \rightarrow \infty$ by the exponential decay of $(\mathcal{F}_\Phi^{(\delta)} H)(s_0, s; \phi)$ (see Proposition 8.1), which surely counteracts the polynomial growth from $L(s_0, \Phi)$ and $\zeta(2s - s_0)$. As a result, we may shift the line of integration to $\operatorname{Re} s_0 = 2\epsilon$ and no pole is crossed. Hence,

$$OD_\Phi(s; \phi) = \frac{1}{4} \int_{(2\epsilon)} \zeta(2s - s_0) L(s_0, \Phi) \cdot \sum_{\delta=\pm} (\mathcal{F}_\Phi^{(\delta)} H)(s_0, s; \phi) \frac{ds_0}{2\pi i} \tag{9-3}$$

on (9-2). The right side of (9-3) is holomorphic on

$$\frac{1}{2} + \epsilon < \sigma < 4, \quad |t| < T \tag{9-4}$$

and serves as an analytic continuation of $OD_{\Phi}(s; \phi)$ to (9-4) by using Proposition 8.1. Note that $\sigma > \frac{1}{2} + \epsilon$ implies the holomorphy of $\zeta(2s - s_0)$.

9B. Step 2: Crossing the polar line (shifting the s_0 -integral again). Consider a subdomain of (9-4):

$$\frac{1}{2} + \epsilon < \sigma < \frac{3}{4}, \quad |t| < T. \quad (9-5)$$

Different from step 1, the pole $s_0 = 2s - 1$ is now inside the rectangle with vertices $2\epsilon \pm iT_0$ and $\frac{1}{2} \pm iT_0$ provided $T_0 > 4T$. Such a rectangle lies in the region of holomorphy (8-2) of $(\mathcal{F}_{\Phi}^{(\delta)} H)(s_0, s; \phi)$. When $\phi < \pi/2$, the exponential decay of $(\mathcal{F}_{\Phi}^{(\delta)} H)(s_0, s; \phi)$ once again allows us to shift the line of integration from $\text{Re } s_0 = 2\epsilon$ to $\text{Re } s_0 = \frac{1}{2}$, crossing the pole of $\zeta(2s - s_0)$ which has residue -1 . In other words,

$$OD_{\Phi}(s; \phi) = \frac{1}{4} L(2s - 1, \Phi) \sum_{\delta=\pm} (\mathcal{F}_{\Phi}^{(\delta)} H)(2s - 1, s; \phi) + \frac{1}{4} \int_{(1/2)} \zeta(2s - s_0) L(s_0, \Phi) \cdot \sum_{\delta=\pm} (\mathcal{F}_{\Phi}^{(\delta)} H)(s_0, s; \phi) \frac{ds_0}{2\pi i}. \quad (9-6)$$

On the line $\text{Re } s_0 = \frac{1}{2}$, observe that $s \mapsto (\mathcal{F}_{\Phi}^{(\delta)} H)(s_0, s; \phi)$ is holomorphic on $\sigma > \frac{1}{4} + \frac{\epsilon}{2}$ by (8-2); whereas $s \mapsto \zeta(2s - s_0)$ is holomorphic on $\sigma < \frac{3}{4}$ as $2\sigma - s_0 < 1$. As a result, the function $s \mapsto \int_{(1/2)} (\cdots) \frac{ds_0}{2\pi i}$ in (9-6) is holomorphic on the vertical strip

$$\frac{1}{4} + \frac{\epsilon}{2} < \sigma < \frac{3}{4}, \quad (9-7)$$

which is sufficient for our purpose.

Proposition 8.1 only asserts that the function $s \mapsto (\mathcal{F}_{\Phi}^{(\delta)} H)(2s - 1, s; \phi)$ is holomorphic on $\frac{1}{2} + \epsilon < \sigma < 4$. However, it actually admits a continuation to the domain $\epsilon < \sigma < 4$ as we will see in Proposition 9.2.

9C. Step 3: Putting back $\phi \rightarrow \pi/2$ —shifting the s_1 -integral and refining Steps 1–2. By using estimate (8-14) and dominated convergence,

$$\lim_{\phi \rightarrow \pi/2} (\mathcal{F}_{\Phi}^{(\delta)} H)(2s - 1, s; \phi) = (\mathcal{F}_{\Phi}^{(\delta)} H)(2s - 1, s; \pi/2) \quad (9-8)$$

for $\frac{1}{2} + \epsilon < \sigma < 4$ and $|t| < T$. For the continuous part of (9-6), we need to extend Proposition 8.1 to pass to the limit $\phi \rightarrow \pi/2$. Using the Γ -factors from Proposition 5.3 and the analytic properties of \tilde{h} , we shift the line of integration for the s_1 -integral to achieve the necessary polynomial decay.

Proposition 9.1. *Let $H \in \mathcal{C}_{\eta}$. There exists a constant $B = B_{\eta}$ such that whenever $(\sigma_0, \sigma) \in (8-2)$, $|t| < T$, and $|t_0| \gg_T 1$, we have the estimate*

$$|(\mathcal{F}_{\Phi}^{(\delta)} H)(s_0, s; \pi/2)| \ll |t_0|^{8-\eta/2} \log^B |t_0|, \quad (9-9)$$

where the implicit constant depends only on η, T, Φ .

Proof. On domain (8-2), observe that the vertical strip $\text{Re } s_1 \in [15, \eta - \frac{1}{2}]$ contains no pole of the function $s_1 \mapsto \mathcal{G}_{\Phi}^{(\delta)}(s_1, u; s_0, s; \phi)$, and it lies within the region of holomorphy of \tilde{h} (see Remark 8.2). The estimate

(8-14) allows us to shift the line of integration from $\text{Re } s_1 = 15$ to $\text{Re } s_1 = \eta - \frac{1}{2}$ in (7-6). Notice that the estimates done in Proposition 8.1 works for $\phi = \pi/2$ too. In particular, from (8-20) and (8-15), the bound (9-9) follows by taking $\sigma_1 := \eta - \frac{1}{2}$ therein (after the contour shift). This completes the proof. \square

Suppose $(3 + \theta)/2 < \sigma < 4$. By Proposition 7.3, (7-5) and (9-3), we have

$$\begin{aligned} OD_\Phi(s) &= \lim_{\phi \rightarrow \pi/2} OD_\Phi(s; \phi) \\ &= \lim_{\phi \rightarrow \pi/2} \frac{1}{4} \int_{(1+\theta+2\epsilon)} \zeta(2s - s_0)L(s_0, \Phi) \cdot \sum_{\delta=\pm} (\mathcal{F}_\Phi^{(\delta)} H)(s_0, s; \phi) \frac{ds_0}{2\pi i} \\ &= \lim_{\phi \rightarrow \pi/2} \frac{1}{4} \int_{(2\epsilon)} \zeta(2s - s_0)L(s_0, \Phi) \cdot \sum_{\delta=\pm} (\mathcal{F}_\Phi^{(\delta)} H)(s_0, s; \phi) \frac{ds_0}{2\pi i}. \end{aligned} \tag{9-10}$$

Proposition 9.1 ensures enough polynomial decay and hence the absolute convergence of (9-11) at $\phi = \pi/2$:

$$OD_\Phi(s) = \frac{1}{4} \int_{(2\epsilon)} \zeta(2s - s_0)L(s_0, \Phi) \cdot \sum_{\delta=\pm} (\mathcal{F}_\Phi^{(\delta)} H)(s_0, s; \pi/2) \frac{ds_0}{2\pi i}. \tag{9-11}$$

Now, (9-11) serves as an analytic continuation of $OD_\Phi(s)$ to the domain $\frac{1}{2} + \epsilon < \sigma < 4$.

On the smaller domain $\frac{1}{2} + \epsilon < \sigma < \frac{3}{4}$, the expressions (9-10) and (9-6) are equal. Then

$$\begin{aligned} OD_\Phi(s) &= (9-10) \\ &= \frac{1}{4} L(2s - 1, \Phi) \sum_{\delta=\pm} (\mathcal{F}_\Phi^{(\delta)} H)(2s - 1, s; \pi/2) \\ &\quad + \frac{1}{4} \int_{(1/2)} \zeta(2s - s_0)L(s_0, \Phi) \cdot \sum_{\delta=\pm} (\mathcal{F}_\Phi^{(\delta)} H)(s_0, s; \pi/2) \frac{ds_0}{2\pi i} \end{aligned} \tag{9-12}$$

by dominated convergence and Proposition 8.1. The last integral is holomorphic on $\frac{1}{4} + \frac{\epsilon}{2} < \sigma < \frac{3}{4}$.

In the following, we write $(\mathcal{F}_\Phi H)(s_0, s) := (\mathcal{F}_\Phi^+ H)(s_0, s; \pi/2) + (\mathcal{F}_\Phi^- H)(s_0, s; \pi/2)$. Duplication and reflection formulae of Γ -functions in the form

$$2^{-u} \Gamma(u) = \frac{1}{2\sqrt{\pi}} \cdot \Gamma\left(\frac{u}{2}\right) \Gamma\left(\frac{u+1}{2}\right) \quad \text{and} \quad \Gamma\left(\frac{1+u}{2}\right) \Gamma\left(\frac{1-u}{2}\right) = \pi \sec \frac{\pi u}{2},$$

lead to

$$\begin{aligned} (\mathcal{F}_\Phi H)(s_0, s) &= \sqrt{\pi} \int_{(\eta-1/2)} \tilde{h}\left(s - s_1 - \frac{1}{2}\right) \pi^{-s_1} \frac{\prod_{i=1}^3 \Gamma\left(\frac{s_1 - \alpha_i}{2}\right)}{\Gamma\left(\frac{1+s_1}{2} - s_0\right)} \\ &\quad \cdot \int_{(\epsilon)} \frac{\Gamma\left(\frac{u}{2}\right) \Gamma\left(\frac{s_1 - (s_0 - u)}{2} + \frac{1}{2} - s\right) \cdot \prod_{i=1}^3 \Gamma\left(\frac{(s_0 - u) + \alpha_i}{2}\right) \Gamma\left(s - \frac{s_0 + u}{2}\right)}{\Gamma\left(\frac{1-u}{2}\right) \Gamma\left(\frac{(s_0 - u) + s_1}{2}\right)} \frac{du}{2\pi i} \frac{ds_1}{2\pi i}. \end{aligned} \tag{9-13}$$

In Section 10, we will work with this expression further.

9D. Step 4: Continuation of the residual term — shifting the u -integral.

Proposition 9.2. *Let $H \in \mathcal{C}_\eta$. The function $s \mapsto (\mathcal{F}_\Phi H)(2s - 1, s)$ can be holomorphically continued to the vertical strip $\epsilon < \sigma < 4$ except at the three simple poles: $s = (1 - \alpha_i)/2$ ($i = 1, 2, 3$), where $(\alpha_1, \alpha_2, \alpha_3)$ are the Langlands parameters of the Maass cusp form Φ .*

Proof. We will prove a stronger result in [Theorem 10.2](#). However, a simpler argument suffices for the time being. Suppose $\frac{1}{2} + \epsilon < \sigma < 4$ and $s_0 = 2s - 1$. In [\(9-13\)](#), we shift the line of integration from $\operatorname{Re} u = \epsilon$ to $\operatorname{Re} u = -1.9$:

$$\begin{aligned} (\mathcal{F}_\Phi H)(2s - 1, s) &= 2\sqrt{\pi} \prod_{i=1}^3 \Gamma\left(s - \frac{1}{2} + \frac{\alpha_i}{2}\right) \int_{(\eta-1/2)} \tilde{h}\left(s - s_1 - \frac{1}{2}\right) \frac{\pi^{-s_1} \prod_{i=1}^3 \Gamma\left(\frac{s_1 - \alpha_i}{2}\right) \Gamma\left(\frac{s_1}{2} + 1 - 2s\right)}{\Gamma\left(\frac{1+s_1}{2} + 1 - 2s\right) \Gamma\left(s - \frac{1}{2} + \frac{s_1}{2}\right)} \frac{ds_1}{2\pi i} \\ &\quad + \sqrt{\pi} \int_{(\eta-1/2)} \int_{(-1.9)} \text{(same as the integrand of (9-13))} \frac{du}{2\pi i} \frac{ds_1}{2\pi i}. \end{aligned}$$

By Stirling's formula and the same argument following [\(8-17\)](#), the integrals above represent holomorphic functions on $\epsilon < \sigma < 4$. \square

9E. Step 5: Conclusion. Apply [Proposition 9.2](#) to [\(9-12\)](#) and observe that the poles of

$$s \mapsto (\mathcal{F}_\Phi H)(2s - 1, s)$$

are exactly the trivial zeros of the arithmetic factor $L(2s - 1, \Phi)$ in [\(9-6\)](#). We conclude that the product of functions

$$s \mapsto L(2s - 1, \Phi) \cdot (\mathcal{F}_\Phi H)(2s - 1, s)$$

is holomorphic on $\epsilon < \sigma < 4$ and thus [\(9-12\)](#) provides a holomorphic continuation of $OD_\Phi(s)$ to the vertical strip $\frac{1}{4} + \frac{\epsilon}{2} < \sigma < \frac{3}{4}$. By the rapid decay of Φ at ∞ , the pairing $s \mapsto (P, \mathbb{P}_2^3 \Phi \cdot |\det *|^{\bar{s}-1/2})_{\Gamma_2 \backslash GL_2(\mathbb{R})}$ represents an entire function. Putting [Proposition 5.25](#), [\(6-9\)](#) and [\(9-12\)](#) together, we arrive at [Theorem 1.1](#).

Remark 9.3. Analytic continuation for moments of degree 6 automorphic L -functions (i.e., our case) is significantly more complicated than those of degree 4, as seen in the second moment formula for $GL(2)$ by Iwaniec–Sarnak and Motohashi. This complication is partly due to the off-diagonal main terms when Φ is an Eisenstein series (see [\[Kwan 2023\]](#)), which are absent in degree 4 cases (see [\[Conrey et al. 2005, page 35\]](#)).

The key distinction lies in the off-diagonal arithmetic. For Iwaniec–Sarnak and Motohashi, the arithmetic is captured by the shifted Dirichlet series of divisor functions, with holomorphy for the dual side depending on the absolute convergence of this series. In our case, the absolute convergence provided by [Proposition 7.2](#) is insufficient. We must carefully move the contour to ensure the L -functions in the off-diagonal evaluate on $\operatorname{Re} s_0 = \frac{1}{2}$ when $s = \frac{1}{2}$.

10. Explication of the off-diagonal — main terms and integral transform

The power of spectral summation formulae (including [Theorem 1.1](#)) is encoded in the archimedean transforms. It is important to express these transforms explicitly, often in terms of *special functions*. While the special functions for $GL(2)$ exhibit numerous symmetries and identities, this is less true for higher-rank groups, leaving much to explore.

Nevertheless, there has been success in higher-rank cases. For example, Stade [[2001](#); [2002](#)] computed the Mellin transforms and certain Rankin–Selberg integrals of Whittaker functions for $GL_n(\mathbb{R})$; Goldfeld et al. [[Goldfeld and Kontorovich 2013](#); [Goldfeld et al. 2021](#); [2022](#)] obtained (harmonic-weighted) spherical Weyl laws of $GL_3(\mathbb{R})$, $GL_4(\mathbb{R})$ and $GL_n(\mathbb{R})$ with strong power-saving error terms; Buttcane [[2013](#); [2016](#)] developed the Kuznetsov formulae for $GL(3)$. These works heavily rely on *Mellin–Barnes integrals*, suggesting this approach effectively handles the archimedean aspects of higher-rank problems.

In this final section, we continue such investigation and record several formulae for the archimedean transform $(\mathcal{F}_\Phi H)(s_0, s)$.

Lemma 10.1. *Suppose $H \in \mathcal{C}_\eta$ and $h := H^b$. On the vertical strip $-\frac{1}{2} < \operatorname{Re} w < \eta$, we have*

$$\tilde{h}(w) := \int_0^\infty h(y)y^w d^\times y = \frac{\pi^{-w-1/2}}{4} \int_{(0)} H(\mu) \cdot \frac{\Gamma(\frac{w+1/2+\mu}{2})\Gamma(\frac{w+1/2-\mu}{2})}{|\Gamma(\mu)|^2} \frac{d\mu}{2\pi i}, \tag{10-1}$$

Proof. Since $H \in \mathcal{C}_\eta$, both sides of (10-1) converge absolutely on the strip $-\frac{1}{2} < \operatorname{Re} w < \eta$ by Stirling’s formula and [Proposition 5.11](#). Substituting the definition of h as in (5-10) into $\tilde{h}(w)$, the result follows from (5-2). □

10A. The off-diagonal main term in [Theorem 1.1](#). In this subsection, we show that the off-diagonal main term of [Theorem 1.1](#) (i.e., $L(2s - 1, \Phi) \cdot (\mathcal{F}_\Phi H)(2s - 1, s)/2$) aligns with the prediction of [[Conrey et al. 2005](#)]. This follows immediately from proving a Mellin–Barnes integral identity, after which the matching follows from the functional equation (5-24).

The proof is more involved than that of [Proposition 5.7](#), as the u -integral (see [Section 7](#)) adds intricacies. However, the introduction of new Γ -factors reveals symmetries in the u -integral, leading to several cancellations and reductions.

Theorem 10.2. *Suppose $\frac{1}{2} + \epsilon < \sigma < 1$. Then*

$$(\mathcal{F}_\Phi H)(2s - 1, s) = \pi^{1/2-s} \cdot \prod_{i=1}^3 \frac{\Gamma(s - \frac{1}{2} + \frac{\alpha_i}{2})}{\Gamma(1 - s - \frac{\alpha_i}{2})} \cdot \int_{(0)} \frac{H(\mu)}{|\Gamma(\mu)|^2} \cdot \prod_{i=1}^3 \prod_{\pm} \Gamma\left(\frac{1 - s + \alpha_i \pm \mu}{2}\right) \frac{d\mu}{2\pi i}. \tag{10-2}$$

Proof. Suppose $\frac{1}{2} + \epsilon < \sigma < 4$. When $s_0 = 2s - 1$, observe that the factor $\Gamma((1-u)/2)$ in the denominator of (9-13) cancels with the factor $\Gamma(s - (s_0 + u)/2)$ in the numerator of (9-13). This gives

$$(\mathcal{F}_\Phi H)(2s - 1, s) = \sqrt{\pi} \int_{(\eta-1/2)} \tilde{h}\left(s - s_1 - \frac{1}{2}\right) \frac{\pi^{-s_1} \prod_{i=1}^3 \Gamma\left(\frac{s_1 - \alpha_i}{2}\right)}{\Gamma\left(\frac{1+s_1}{2} + 1 - 2s\right)} \cdot \int_{(\epsilon)} \frac{\Gamma\left(\frac{u}{2}\right) \Gamma\left(\frac{u+s_1}{2} + 1 - 2s\right) \cdot \prod_{i=1}^3 \Gamma\left(s - \frac{1}{2} + \frac{\alpha_i - u}{2}\right)}{\Gamma\left(s - \frac{1}{2} + \frac{s_1 - u}{2}\right)} \frac{du}{2\pi i} \frac{ds_1}{2\pi i}. \quad (10-3)$$

We make the change of variable $u \rightarrow -2u$ and take

$$(a, b, c; d, e) = \left(s - \frac{1}{2} + \frac{\alpha_1}{2}, s - \frac{1}{2} + \frac{\alpha_2}{2}, s - \frac{1}{2} + \frac{\alpha_3}{2}; 0, \frac{s_1}{2} + 1 - 2s\right)$$

in (5-7). Notice that

$$(a + b + c) + d + e = 3\left(s - \frac{1}{2}\right) + \frac{s_1}{2} + 1 - 2s = s - \frac{1}{2} + \frac{s_1}{2} (:= f)$$

because of $\alpha_1 + \alpha_2 + \alpha_3 = 0$. We find the u -integral is equal to

$$2 \cdot \prod_{i=1}^3 \frac{\Gamma\left(s - \frac{1}{2} + \frac{\alpha_i}{2}\right) \Gamma\left(\frac{1}{2} - s + \frac{s_1 + \alpha_i}{2}\right)}{\Gamma\left(\frac{s_1 - \alpha_i}{2}\right)}. \quad (10-4)$$

Notice that the three Γ -factors in denominator of the last expression cancel with the three in the numerator of the first line of (10-3). Hence, we have

$$(\mathcal{F}_\Phi H)(2s - 1, s) = 2\sqrt{\pi} \cdot \prod_{i=1}^3 \Gamma\left(s - \frac{1}{2} + \frac{\alpha_i}{2}\right) \int_{(\eta-1/2)} \tilde{h}\left(s - s_1 - \frac{1}{2}\right) \frac{\pi^{-s_1} \prod_{i=1}^3 \Gamma\left(\frac{1}{2} - s + \frac{s_1 + \alpha_i}{2}\right)}{\Gamma\left(\frac{1+s_1}{2} + 1 - 2s\right)} \frac{ds_1}{2\pi i}. \quad (10-5)$$

We must now further restrict to $\frac{1}{2} + \epsilon < \sigma < 1$. We shift the line of integration to the left from $\text{Re } s_1 = \eta - \frac{1}{2}$ to $\text{Re } s_1 = \sigma_1$ satisfying

$$2\sigma - 1 < \sigma_1 < \sigma.$$

It is easy to see no pole is crossed and we may now apply Lemma 10.1:

$$(\mathcal{F}_\Phi H)(2s - 1, s) = \frac{\pi^{1/2-s}}{2} \cdot \prod_{i=1}^3 \Gamma\left(s - \frac{1}{2} + \frac{\alpha_i}{2}\right) \cdot \int_{(0)} \frac{H(\mu)}{|\Gamma(\mu)|^2} \cdot \int_{(\sigma_1)} \frac{\prod_{i=1}^3 \Gamma\left(\frac{1}{2} - s + \frac{s_1 + \alpha_i}{2}\right) \cdot \Gamma\left(\frac{s - s_1 + \mu}{2}\right) \Gamma\left(\frac{s - s_1 - \mu}{2}\right)}{\Gamma\left(\frac{1+s_1}{2} + 1 - 2s\right)} \frac{ds_1}{2\pi i} \frac{d\mu}{2\pi i}. \quad (10-6)$$

For the s_1 -integral, applying the change of variable $s_1 \rightarrow 2s_1$ and (5-7) the second time but with

$$(a, b, c; d, e) = \left(\frac{1}{2} - s + \frac{\alpha_1}{2}, \frac{1}{2} - s + \frac{\alpha_2}{2}, \frac{1}{2} - s + \frac{\alpha_3}{2}; \frac{s+\mu}{2}, \frac{s-\mu}{2}\right). \quad (10-7)$$

Observe that

$$(a + b + c) + (d + e) = 3\left(\frac{1}{2} - s\right) + s := \frac{3}{2} - 2s(=: f).$$

The s_1 -integral is thus equal to

$$\prod_{i=1}^3 \frac{\prod_{\pm} \Gamma\left(\frac{1-s+\alpha_i \pm \mu}{2}\right)}{\Gamma\left(1-s-\frac{\alpha_i}{2}\right)}$$

and the result follows. □

10B. Integral transform. Based on the experience of Stade [2001; 2002], we *do not* expect the Mellin–Barnes integrals of $(\mathcal{F}_{\Phi}H)(s_0, s)$ (see (10-12) below) to be completely reducible as in Theorem 10.2 if (s_0, s) is in a *general position*. However, reductions can occur if the integrals take certain special forms, most clearly seen when expressed as *hypergeometric functions*.

We define

$$\begin{aligned} & {}_4\widehat{F}_3\left(\begin{matrix} A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & \end{matrix} \middle| z\right) \\ & := \frac{\Gamma(A_1)\Gamma(A_2)\Gamma(A_3)\Gamma(A_4)}{\Gamma(B_1)\Gamma(B_2)\Gamma(B_3)} \cdot {}_4F_3\left(\begin{matrix} A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & \end{matrix} \middle| z\right) \\ & := \sum_{n=0}^{\infty} \frac{\Gamma(A_1+n)\Gamma(A_2+n)\Gamma(A_3+n)\Gamma(A_4+n)}{\Gamma(B_1+n)\Gamma(B_2+n)\Gamma(B_3+n)} \frac{z^n}{n!}. \end{aligned} \tag{10-8}$$

The series converges absolutely when $|z| < 1$ and $A_1, A_2, A_3, A_4 \notin \mathbb{Z}_{\leq 0}$; and on $|z| = 1$ if

$$\operatorname{Re}(B_1 + B_2 + B_3 - A_1 - A_2 - A_3 - A_4) > 0.$$

In fact, our hypergeometric functions are of *Saalschütz* type, i.e., $B_1 + B_2 + B_3 - A_1 - A_2 - A_3 - A_4 = 1$. Only such special type of hypergeometric functions at $z = 1$ possess many functional relations and integral representations; see [Mishev 2012].

Proposition 10.3. *Suppose $H \in \mathcal{C}_{\eta}$ and $h := H^{\flat}$. On the region $\sigma_0 > \epsilon$, $\sigma < 4$, and $2\sigma - \sigma_0 - \epsilon > 0$, we have $(\mathcal{F}_{\Phi}H)(s_0, s)$ equal to $2\pi^{3/2}$ times*

$$\begin{aligned} & \int_{(\eta-1/2)} \widetilde{h}\left(s - s_1 - \frac{1}{2}\right) \cdot \frac{\prod_{i=1}^3 \Gamma\left(\frac{s_1 - \alpha_i}{2}\right)}{\Gamma\left(\frac{1+s_1}{2} - s_0\right)} \cdot \pi^{-s_1} \sec \frac{\pi}{2} (2s + s_0 - s_1) \\ & \quad {}_4\widehat{F}_3\left(\begin{matrix} s - \frac{s_0}{2} & \frac{s_0 + \alpha_1}{2} & \frac{s_0 + \alpha_2}{2} & \frac{s_0 + \alpha_3}{2} \\ \frac{1}{2} & \frac{s_0 + s_1}{2} & s + \frac{1}{2} + \frac{s_0 - s_1}{2} & \end{matrix} \middle| 1\right) \frac{ds_1}{2\pi i} \\ & - \int_{(\eta-1/2)} \widetilde{h}\left(s - s_1 - \frac{1}{2}\right) \cdot \frac{\prod_{i=1}^3 \Gamma\left(\frac{s_1 - \alpha_i}{2}\right)}{\Gamma\left(\frac{1+s_1}{2} - s_0\right)} \cdot \pi^{-s_1} \sec \frac{\pi}{2} (2s + s_0 - s_1) \\ & \quad {}_4\widehat{F}_3\left(\begin{matrix} \frac{1}{2} - s_0 + \frac{s_1}{2} & \frac{1}{2} - s + \frac{s_1 + \alpha_1}{2} & \frac{1}{2} - s + \frac{s_1 + \alpha_2}{2} & \frac{1}{2} - s + \frac{s_1 + \alpha_3}{2} \\ \frac{1}{2} - s + s_1 & 1 - s - \frac{s_0 - s_1}{2} & \frac{3}{2} - s - \frac{s_0 - s_1}{2} & \end{matrix} \middle| 1\right) \frac{ds_1}{2\pi i}. \end{aligned} \tag{10-9}$$

Proof. By Stirling's formula, we can shift the line of integration of the u -integral in (9-13) to $-\infty$. The residual series obtained can then be identified in terms of hypergeometric series as asserted in the present proposition. This can also be verified by `InverseMellinTransform[]` command in Mathematica. More systematically, one rewrites the u -integral in the form of a Meijer's G -function. The conversion between Meijer's G -functions and generalized hypergeometric functions is known as *Slater's theorem*; see Chapter 8 of [Prudnikov et al. 1990]. \square

Recently, the articles [Balkanova et al. 2020; Balkanova et al. 2021] have brought in powerful asymptotic analysis of hypergeometric functions to study moments, yielding sharp spectral estimates. Our class of admissible test functions in Theorem 1.1 is broad enough for such prospects, see Remark 5.27.

Next, we establish the existence of a kernel function for the integral transform $(\mathcal{F}_\Phi H)(s_0, s)$ when integrating against a chosen test function $H(\mu)$ on the spectral side. This formula serves as a step toward a more practical result for $(\mathcal{F}_\Phi H)(s_0, s)$. While the proof requires care, it is relatively manageable for our case. However, this is not always true; for example, in the spectral Kuznetsov formulae for $GL(2)$ and $GL(3)$, kernel existence can be more challenging, as noted by [Buttcane 2016; Motohashi 1997].

Proposition 10.4. *Suppose $H \in \mathcal{C}_\eta$. On the domain*

$$\sigma_0 > \epsilon := \frac{1}{100}, \quad \sigma < 4, \quad 2\sigma - \sigma_0 - \epsilon > 0, \quad \sigma_0 + 2\sigma - 1 - \epsilon > 0, \quad 1 + \epsilon - \sigma_0 - \sigma > 0, \quad (10-10)$$

we have

$$(\mathcal{F}_\Phi H)(s_0, s) = \frac{\pi^{1/2-s}}{4} \int_{(0)} \frac{H(\mu)}{|\Gamma(\mu)|^2} \cdot \mathcal{K}(s_0, s; \alpha, \mu) \frac{d\mu}{2\pi i}, \quad (10-11)$$

where the kernel function $\mathcal{K}(s_0, s; \alpha, \mu)$ is given explicitly by the double Barnes integrals

$$\begin{aligned} \mathcal{K}(s_0, s; \alpha, \mu) := & \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \frac{\Gamma\left(\frac{s-s_1+\mu}{2}\right) \Gamma\left(\frac{s-s_1-\mu}{2}\right) \prod_{i=1}^3 \Gamma\left(\frac{s_1-\alpha_i}{2}\right)}{\Gamma\left(\frac{1+s_1}{2}-s_0\right)} \\ & \cdot \frac{\Gamma\left(\frac{u}{2}\right) \Gamma\left(\frac{s_1-s_0+u}{2} + \frac{1}{2}-s\right) \prod_{i=1}^3 \Gamma\left(\frac{s_0-u+\alpha_i}{2}\right) \Gamma\left(s-\frac{s_0+u}{2}\right)}{\Gamma\left(\frac{1-u}{2}\right) \Gamma\left(\frac{s_0-u+s_1}{2}\right)} \frac{du}{2\pi i} \frac{ds_1}{2\pi i}, \end{aligned} \quad (10-12)$$

and the contours follow the Barnes convention.

Remark 10.5. (1) The domain (10-10) is certainly nonempty as it includes our point of interest

$$(\sigma_0, \sigma) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

(2) The contours of (10-12) may be taken explicitly as the vertical lines $\operatorname{Re} u = \epsilon$ and $\operatorname{Re} s_1 = \sigma_1$ with

$$\sigma_0 + 2\sigma - 1 - \epsilon < \sigma_1 < \sigma. \quad (10-13)$$

Proof. Suppose

$$\sigma_0 > \epsilon, \quad \sigma < 4, \quad \text{and} \quad 2\sigma - \sigma_0 - \epsilon > 0 \tag{10-14}$$

as in [Proposition 8.1](#). Recall the expression [\(9-13\)](#) for $(\mathcal{F}_\Phi H)(s_0, s)$. This time, we shift the line of integration of the s_1 -integral to $\text{Re } s_1 = \sigma_1$ satisfying

$$\sigma_1 < \sigma \tag{10-15}$$

and no pole is crossed during this shift as long as

$$\sigma_1 > 0 \quad \text{and} \quad \sigma_1 > \sigma_0 + 2\sigma - 1 - \epsilon. \tag{10-16}$$

Now, assume [\(10-10\)](#). The restrictions [\(10-14\)](#), [\(10-15\)](#), [\(10-16\)](#) hold and such a line of integration for the s_1 -integral exists. Upon shifting the line of integration to such a position, substituting [\(10-1\)](#) into [\(9-13\)](#) and the result follows. □

The second step is to apply a very useful rearrangement of the Γ -factors in the $(n - 1)$ -fold Mellin transform of the $\text{GL}(n)$ spherical Whittaker function as discovered in [\[Ishii and Stade 2007\]](#). We shall only need the case of $n = 3$ which we describe as follows. Recall

$$G_\alpha(s_1, s_2) := \pi^{-s_1-s_2} \cdot \frac{\prod_{i=1}^3 \Gamma\left(\frac{s_1+\alpha_i}{2}\right)\Gamma\left(\frac{s_2-\alpha_i}{2}\right)}{\Gamma\left(\frac{s_1+s_2}{2}\right)} \tag{10-17}$$

from [Proposition 5.3](#). The first Barnes lemma, i.e.,

$$\int_{-i\infty}^{i\infty} \Gamma(w + \alpha)\Gamma(w + \mu)\Gamma(\gamma - w)\Gamma(\delta - w) \frac{dw}{2\pi i} = \frac{\Gamma(\alpha + \gamma)\Gamma(\alpha + \delta)\Gamma(\mu + \gamma)\Gamma(\gamma + \delta)}{\Gamma(\alpha + \mu + \gamma + \delta)}, \tag{10-18}$$

can be applied *in reverse* such that [\(10-17\)](#) can be rewritten as

$$G_\alpha(s_1, s_2) = \pi^{-s_1-s_2} \cdot \Gamma\left(\frac{s_1+\alpha_1}{2}\right)\Gamma\left(\frac{s_2-\alpha_1}{2}\right) \cdot \int_{-i\infty}^{i\infty} \Gamma\left(z + \frac{s_1}{2} - \frac{\alpha_1}{4}\right)\Gamma\left(z + \frac{s_2}{2} + \frac{\alpha_1}{4}\right)\Gamma\left(\frac{\alpha_2}{2} + \frac{\alpha_1}{4} - z\right)\Gamma\left(\frac{\alpha_3}{2} + \frac{\alpha_1}{4} - z\right) \frac{dz}{2\pi i}, \tag{10-19}$$

see Section 2 of [\[Ishii and Stade 2007\]](#). Although [\(10-19\)](#) is less symmetric than [\(10-17\)](#), it more clearly displays the recursive structure of the $\text{GL}(3)$ Whittaker function in terms of the K -Bessel function.

Theorem 10.6. *Suppose $\text{Re } s_0 = \text{Re } s = \frac{1}{2}$ and $\text{Re } \alpha_i = \text{Re } \mu = 0$. Then $\mathcal{K}(s_0, s; \alpha, \mu)$ is equal to*

$$4 \cdot \gamma\left(-\frac{s_0+\alpha_1}{2}\right) \cdot \prod_{\pm} \Gamma\left(\frac{s \pm \mu - \alpha_1}{2}\right) \cdot \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \Gamma(s+t)\Gamma\left(\frac{1-\alpha_1}{2}+t\right) \cdot \Gamma\left(\frac{\alpha_2}{2} + \frac{\alpha_1}{4} - z\right) \cdot \Gamma\left(\frac{\alpha_3}{2} + \frac{\alpha_1}{4} - z\right) \cdot \prod_{\pm} \Gamma\left(\frac{-s \pm \mu}{2} + \frac{\alpha_1}{4} + z - t\right) \cdot \frac{\gamma(t+s_0/2)\gamma(\alpha_1/4-z-s_0/2)}{\gamma(\alpha_1/4+t-z)} \frac{dz}{2\pi i} \frac{dt}{2\pi i}, \tag{10-20}$$

where the contours may be explicitly taken as the vertical lines $\operatorname{Re} t = a$ and $\operatorname{Re} z = b$ satisfying

$$-\frac{1}{2} < a < -\frac{1}{4}, \quad -\frac{1}{4} < b < 0, \quad \text{and} \quad b - a > \frac{1}{4} \quad (10-21)$$

and

$$\gamma(x) := \frac{\Gamma(-x)}{\Gamma(1/2 + x)}. \quad (10-22)$$

Remark 10.7. (1) The assumptions in [Theorem 10.6](#) cover the most interesting cases of [Theorem 1.1](#), particularly on the critical line and for tempered forms, though they are not strictly necessary. These were chosen for a clean description of the contours [\(10-21\)](#).

(2) Furthermore, if either of the following holds:

(a) The cusp form Φ is fixed, allowing implicit constants to depend on $\alpha(\Phi)$.

(b) $\Phi = E_{\min}^{(3)}(*; \alpha)$, where the “shifts” α_i are small as in [\[Conrey et al. 2005\]](#) (i.e., $\ll 1/\log R$, per [Remark 5.27](#)).

Then by continuity, it suffices to assume $\alpha_1 = \alpha_2 = \alpha_3 = 0$. With $s = \frac{1}{2}$, this leads to a simpler formula for [\(10-20\)](#):

$$4 \cdot \gamma\left(-\frac{s_0}{2}\right) \prod_{\pm} \Gamma\left(\frac{\frac{1}{2} \pm \mu}{2}\right) \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \Gamma\left(\frac{1}{2} + t\right)^2 \Gamma(-z)^2 \prod_{\pm} \Gamma\left(\frac{-\frac{1}{2} \pm \mu}{2} + z - t\right) \cdot \frac{\gamma\left(t + \frac{s_0}{2}\right) \gamma\left(-z - \frac{s_0}{2}\right)}{\gamma(t - z)} \frac{dz}{2\pi i} \frac{dt}{2\pi i}.$$

(3) For analytic applications involving Whittaker functions for $GL(n)$, the formula from [\[Ishii and Stade 2007\]](#) has proven more effective than the ones obtained previously. For example:

(a) Buttane [\[2020\]](#) used the formula [\(10-19\)](#) to significantly simplify the archimedean Rankin–Selberg calculation for $GL(3)$, earlier done in [\[Stade 1993\]](#).

(b) In [\[Goldfeld et al. 2022\]](#), it was crucial for deriving strong bounds for Whittaker functions and their inverse transforms, and the Weyl law.

(This was pointed out to the author by Prof. Eric Stade and Prof. Dorian Goldfeld. The author would like to thank them for their comments here.)

(4) Finally, Stirling’s formula shows that the integrand in the Mellin–Barnes representation [\(10-20\)](#) decays exponentially as $|\operatorname{Im} z|, |\operatorname{Im} t| \rightarrow \infty$, independent of $|\operatorname{Im} s_0|$. This advantage is not shared by the integrand in [\(8-1\)](#).

Proof of Theorem 10.6. Substitute (10-19) into (10-12) rearrange the integrals, we find that

$$\begin{aligned} &\mathcal{K}(s_0, s; \alpha, \mu) \\ &:= \int_{-i\infty}^{i\infty} \frac{\Gamma\left(\frac{s-s_1+\mu}{2}\right)\Gamma\left(\frac{s-s_1-\mu}{2}\right)\Gamma\left(\frac{s_1-\alpha_1}{2}\right)}{\Gamma\left(\frac{1+s_1}{2}-s_0\right)} \\ &\quad \cdot \int_{-i\infty}^{i\infty} \Gamma\left(\frac{\alpha_2}{2} + \frac{\alpha_1}{4} - z\right)\Gamma\left(\frac{\alpha_3}{2} + \frac{\alpha_1}{4} - z\right)\Gamma\left(z + \frac{s_1}{2} + \frac{\alpha_1}{4}\right) \\ &\quad \cdot \int_{-i\infty}^{i\infty} \frac{\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{s_1-(s_0-u)}{2} + \frac{1}{2} - s\right)\Gamma\left(s - \frac{s_0+u}{2}\right)\Gamma\left(\frac{s_0-u+\alpha_1}{2}\right)\Gamma\left(z + \frac{s_0-u}{2} - \frac{\alpha_1}{4}\right)}{\Gamma\left(\frac{1-u}{2}\right)} \frac{du}{2\pi i} \frac{dz}{2\pi i} \frac{ds_1}{2\pi i}. \end{aligned} \tag{10-23}$$

The innermost u -integral, originally of ${}_4F_3(1)$ -type (Saalschütz), reduces to a ${}_3F_2(1)$ -type (non-Saalschütz), allowing further transformations. We apply the following Barnes integral identity for ${}_3F_2(1)$ -type (see [Bailey 1935]):

$$\begin{aligned} &\int_{-i\infty}^{i\infty} \frac{\Gamma(a+u)\Gamma(b+u)\Gamma(c+u)\Gamma(f-u)\Gamma(-u)}{\Gamma(e+u)} \frac{du}{2\pi i} \\ &= \frac{\Gamma(b)\Gamma(c)\Gamma(f+a)}{\Gamma(f+a+b+c-e)\Gamma(e-b)\Gamma(e-c)} \\ &\quad \cdot \int_{-i\infty}^{i\infty} \frac{\Gamma(a+t)\Gamma(e-c+t)\Gamma(e-b+t)\Gamma(f+b+c-e-t)\Gamma(-t)}{\Gamma(e+t)} \frac{dt}{2\pi i}. \end{aligned} \tag{10-24}$$

Make a change of variable $u \rightarrow -2u$ and take

$$a = s - \frac{1}{2}s_0, \quad b = \frac{1}{2}(s_0 + \alpha_1), \quad c = z + \frac{1}{2}s_0 - \frac{1}{4}\alpha_1, \quad f = \frac{1}{2}(s_1 - s_0) + \frac{1}{2} - s, \quad e = \frac{1}{2} \tag{10-25}$$

in (10-24), the u -integral of (10-23) can be written as

$$\begin{aligned} &2 \cdot \frac{\Gamma\left(\frac{s_0+\alpha_1}{2}\right)\Gamma\left(z + \frac{s_0}{2} - \frac{\alpha_1}{4}\right)\Gamma\left(\frac{1+s_1}{2}-s_0\right)}{\Gamma\left(\frac{s_1}{2}+z+\frac{\alpha_1}{4}\right)\Gamma\left(\frac{1-s_0-\alpha_1}{2}\right)\Gamma\left(\frac{1}{2}-z-\frac{s_0}{2}+\frac{\alpha_1}{4}\right)} \\ &\quad \cdot \int_{-i\infty}^{i\infty} \frac{\Gamma\left(t+s-\frac{s_0}{2}\right)\Gamma\left(t+\frac{1}{2}-z-\frac{s_0}{2}+\frac{\alpha_1}{4}\right)\Gamma\left(t+\frac{1}{2}-\frac{s_0+\alpha_1}{2}\right)\Gamma\left(\frac{s_0+s_1}{2}+z-s+\frac{\alpha_1}{4}-t\right)\Gamma(-t)}{\Gamma\left(\frac{1}{2}+t\right)} \frac{dt}{2\pi i}. \end{aligned} \tag{10-26}$$

Putting this back into (10-23). Observe that two pairs of Γ -factors involving s_1 will be canceled and we can then execute the s_1 -integral. More precisely,

$$\begin{aligned} &\frac{1}{2} \cdot \mathcal{K}(s_0, s; \alpha, \mu) \\ &= \frac{\Gamma\left(\frac{s_0+\alpha_1}{2}\right)}{\Gamma\left(\frac{1-s_0-\alpha_1}{2}\right)} \cdot \int_{-i\infty}^{i\infty} \frac{dt}{2\pi i} \frac{\Gamma\left(t+s-\frac{s_0}{2}\right)\Gamma\left(t+\frac{1}{2}-\frac{s_0+\alpha_1}{2}\right)\Gamma(-t)}{\Gamma\left(\frac{1}{2}+t\right)} \\ &\quad \cdot \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \frac{\Gamma\left(\frac{\alpha_2}{2} + \frac{\alpha_1}{4} - z\right)\Gamma\left(\frac{\alpha_3}{2} + \frac{\alpha_1}{4} - z\right)\Gamma\left(z + \frac{s_0}{2} - \frac{\alpha_1}{4}\right)}{\Gamma\left(\frac{1}{2} - z - \frac{s_0}{2} + \frac{\alpha_1}{4}\right)} \Gamma\left(t + \frac{1}{2} - z - \frac{s_0}{2} + \frac{\alpha_1}{4}\right) \\ &\quad \cdot \int_{-i\infty}^{i\infty} \frac{ds_1}{2\pi i} \Gamma\left(\frac{s_0+s_1}{2} + z - s + \frac{\alpha_1}{4} - t\right)\Gamma\left(\frac{s-s_1+\mu}{2}\right)\Gamma\left(\frac{s-s_1-\mu}{2}\right)\Gamma\left(\frac{s_1-\alpha_1}{2}\right). \end{aligned} \tag{10-27}$$

Applying (10-18) once again, we obtain

$$\begin{aligned}
 \frac{1}{4} \cdot \mathcal{K}(s_0, s; \alpha, \mu) &= \frac{\Gamma\left(\frac{s_0+\alpha_1}{2}\right)}{\Gamma\left(\frac{1-s_0-\alpha_1}{2}\right)} \Gamma\left(\frac{s+\mu-\alpha_1}{2}\right) \Gamma\left(\frac{s-\mu-\alpha_1}{2}\right) \\
 &\cdot \int_{-i\infty}^{i\infty} \frac{dt}{2\pi i} \Gamma\left(s+t-\frac{s_0}{2}\right) \Gamma\left(\frac{1-\alpha_1}{2}+t-\frac{s_0}{2}\right) \frac{\Gamma(-t)}{\Gamma\left(\frac{1}{2}+t\right)} \\
 &\cdot \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \Gamma\left(\frac{\alpha_2}{2}+\frac{\alpha_1}{4}-z\right) \Gamma\left(\frac{\alpha_3}{2}+\frac{\alpha_1}{4}-z\right) \frac{\Gamma\left(z+\frac{s_0}{2}-\frac{\alpha_1}{4}\right)}{\Gamma\left(\frac{1}{2}-z-\frac{s_0}{2}+\frac{\alpha_1}{4}\right)} \\
 &\cdot \Gamma\left(\frac{-s+\mu}{2}+\frac{\alpha_1}{4}+\frac{s_0}{2}+z-t\right) \Gamma\left(\frac{-s-\mu}{2}+\frac{\alpha_1}{4}+\frac{s_0}{2}+z-t\right) \\
 &\cdot \frac{\Gamma\left(\frac{1}{2}+\frac{\alpha_1}{4}-\frac{s_0}{2}-z+t\right)}{\Gamma\left(-\frac{\alpha_1}{4}+\frac{s_0}{2}+z-t\right)}. \tag{10-28}
 \end{aligned}$$

A final cleaning can be performed via the change of variables $t \rightarrow t + \frac{s_0}{2}$. This leads to (10-20) and completes the proof. \square

11. Notes

Remark 11.1 (Note added in December 2021). The first version of our preprint appeared on Arxiv in December 2021. Peter Humphries has kindly informed the author that the moment of [Theorem 1.1](#) arises naturally from the context of the L^4 -norm problem of $GL(2)$ Maass forms and can also be investigated under another set of ‘‘Kuznetsov–Voronoi’’ method (see [\[Blomer and Khan 2019a; Blomer and Khan 2019b; Blomer et al. 2019\]](#)) that is distinct from [\[Li 2009; 2011\]](#). This is his on-going work with Rizwanur Khan.

Remark 11.2 (Note added in October 2022/April 2023). The preprint of Humphries–Khan has now appeared; see [\[Humphries and Khan 2022\]](#). The spectral moments considered in [\[loc. cit.\]](#) and the present paper are distinct in a number of ways. In one case, our spectral moments coincide when both $\Phi = \tilde{\Phi}$ and $s = \frac{1}{2}$ hold true, but otherwise extra twistings by root numbers are present in the one considered by [\[loc. cit.\]](#). This would then lead to different conclusions in view of the moment conjecture of [\[Conrey et al. 2005\]](#) (see the discussions in [Section 3B](#)). In the other case, our spectral moments differ by a full holomorphic spectrum and thus give rise to distinct conclusions in applications toward nonvanishing (say). All these result in different ways of making choices of test functions, as well as different shapes of the dual sides. The self-duality assumption was used in [\[Humphries and Khan 2022\]](#) to annihilate two of the terms in their proof, but no such treatment is necessary for our method.

There is also the recent preprint [\[Biro 2022\]](#) which studies another instance of reciprocity closely related to ours, but with the decomposition ‘‘ $4 = 2 \times 2$ ’’ on the dual side instead. His integral construction consists of a product of an automorphic kernel with a copy of θ -function and Maass cusp form of $SL_2(\mathbb{Z})$ attached to each variable. The integration is taken over both variables and over the quotient $\Gamma_0(4) \backslash \mathfrak{h}^2$; see equation (3.15) therein.

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