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The wavefront sets of unipotent supercuspidal representations

Dan Ciubotaru, Lucas Mason-Brown and Emile Okada

We prove that the double (or canonical unramified) wavefront set of an irreducible depth-0 supercuspidal representation of a reductive p-adic group is a singleton provided $p > 3(h-1)$, where h is the Coxeter number. We deduce that the geometric wavefront set is also a singleton in this case, proving a conjecture of Mœglin and Waldspurger. When the group is inner to split and the representation belongs to Lusztig's category of unipotent representations, we give an explicit formula for the double and geometric wavefront sets. As a consequence, we show that the nilpotent part of the Deligne–Langlands–Lusztig parameter of a unipotent supercuspidal representation is precisely the image of its geometric wavefront set under Spaltenstein's duality map.

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1. Introduction

Let $\boldsymbol{G}$ be a connected reductive algebraic group defined over a p-adic field k with residue field $\mathbb{F}_q$ and let $\boldsymbol{G}(\mathsf{k})$ be the group of k-rational points. Let (π, X) be an irreducible smooth representation of $\boldsymbol{G}(\mathsf{k})$ with distribution character Θ_X. Let $\mathcal{N}_o^*(\mathsf{k})$ denote the set of nilpotent orbits in $\mathfrak{g}^*(\mathsf{k})$ — the linear dual of the Lie algebra $\mathfrak{g}(\mathsf{k})$ of $\boldsymbol{G}(\mathsf{k})$ — and for each $\mathbb{O}^* \in \mathcal{N}_o^*(\mathsf{k})$ let $\hat{\mu}_{\mathbb{O}^*}$ denote the Fourier transform of the associated orbital integral. In [14], Harish-Chandra proved that there are unique complex numbers $c_{\mathbb{O}^*}(X) \in \mathbb{C}$ such that

$$\Theta_X(\exp(\xi)) = \sum_{\mathbb{O}^* \in \mathcal{N}_o^*(\mathsf{k})} c_{\mathbb{O}^*}(X)\hat{\mu}_{\mathbb{O}^*}(\xi) \tag{1.0.1}$$

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for $\xi \in \mathfrak{g}(\mathsf{k})$ a regular element in a small neighborhood of 0. The formula (1.0.1) is called the *local character expansion* of X.

One of the most important invariants which can be extracted from the local character expansion is the *wavefront set* of X. This is the set of nilpotent coadjoint orbits which appear with nonzero multiplicity in the local character expansion

$$\mathrm{WF}(X) = \max\{\mathbb{O}^* \subset \mathfrak{g}^*(\mathsf{k}) \mid c_{\mathbb{O}^*}(X) \neq 0\}.$$

It is common in the literature to consider a slightly coarser invariant called the *geometric wavefront set*. This is the set of maximal nilpotent coadjoint orbits over an *algebraic closure* $\bar{\mathsf{k}}$ of k which meet some orbit in $\mathrm{WF}(X)$. This set is denoted by ${}^{\bar{\mathsf{k}}}\mathrm{WF}(X)$. A longstanding conjecture of Moeglin and Waldspurger [23, Page 429] asserts that ${}^{\bar{\mathsf{k}}}\mathrm{WF}(X)$ is a singleton for all X. Our first main result is that this conjecture is true when X is a depth-0 supercuspidal representation of any reductive group subject only to the assumption that the residue characteristic of k is sufficiently large. In fact, we prove something stronger. Namely we give formulas for two finer invariants in terms of the unrefined minimal K-type of X. The first of these finer invariants is the *unramified wavefront set* ${}^{K}\mathrm{WF}(X)$. This is the set of maximal nilpotent coadjoint orbits over the maximal unramified extension K of k in $\bar{\mathsf{k}}$ which meet some orbit in $\mathrm{WF}(X)$. The second of these finer invariants is the *double wavefront set* $\mathrm{DWF}(X)$ (also known as the *canonical unramified wavefront set* in earlier work of the authors). This invariant is a subset of $\mathcal{N}_o^* \times \mathcal{N}_o^{*\vee}$ where $\mathcal{N}_o^*$ denotes the set of coadjoint nilpotent orbits of the complex reductive group G with the same absolute root datum as $\boldsymbol{G}$, while $\mathcal{N}_o^{*\vee}$ denotes the analogous set for the Langlands dual group $G^\vee$ of $\boldsymbol{G}$. This set comes equipped with a partial order, defined by the formula

$$(\mathbb{O}_1^*, \mathbb{O}_1^{*\vee}) \leq (\mathbb{O}_2^*, \mathbb{O}_2^{*\vee}) \quad \text{if } \mathbb{O}_1^* \subseteq \overline{\mathbb{O}_2^*},\ \mathbb{O}_2^{*\vee} \subseteq \overline{\mathbb{O}_1^{*\vee}}$$

and receives a natural map from $\mathcal{N}_o^*(\mathsf{k})$

$$i_{\mathsf{k}} : \mathcal{N}_o^*(\mathsf{k}) \to \mathcal{N}_o^* \times \mathcal{N}_o^{*\vee};$$

see Section 2.5 below. The double wavefront set is the set

$$\mathrm{DWF}(X) = \max\{i_{\mathsf{k}}(\mathbb{O}^*) \mid c_{\mathbb{O}^*}(X) \neq 0\} \subseteq \mathcal{N}_o^* \times \mathcal{N}_o^{*\vee}.$$

As this is a relatively new invariant we will say a few words here to motivate its study.

Much like the geometric wavefront set we expect that the double wavefront set of any irreducible depth-0 representation (π, X) is a singleton when the residue characteristic of k is sufficiently large. Under such assumptions the double wavefront set takes on a particularly simple form

$$\mathrm{DWF}(X) = ({}^{\bar{\mathsf{k}}}\mathrm{WF}(X), \mathbb{O}^{*\vee}), \tag{1.0.2}$$

where we view the geometric wavefront set as an element of $\mathcal{N}_o^*$ under a suitable identification of $\bar{\mathsf{k}}$-coadjoint orbits with $\mathcal{N}_o^*$. In this setting, it is natural to ask for a description of the nilpotent orbit $\mathbb{O}^{*\vee}$. The results in the second half of this paper and in [9] strongly suggest that when X is unipotent in the sense of [22], $\mathbb{O}^{*\vee}$ should be the open orbit in the singular support of the perverse sheaf corresponding (in the sense of [11]) to (π, X) under the Langlands correspondence. A theorem to this effect would impose a strict relation between the singular support of the character distribution of (π, X) at the identity and the singular support of its parameterizing sheaf. The relation isn't precise (it may not always be the case that $\mathbb{O}^{*\vee}$ determines ${}^{\bar{\mathsf{k}}}\mathrm{WF}(X)$ as demonstrated in [9, Example 1.4.2]) but the image of i_{k} in $\mathcal{N}_o \times \mathcal{N}_o^\vee$ is small in some sense, and so the possible pairs $(\mathbb{O}, \mathbb{O}^\vee)$ which can arise as double wavefront sets is still very limited.

In the second half of this paper, we specialize our attention to inner to split reductive groups and supercuspidal representations with L-parameters which are trivial on the inertia subgroup of the Weil group. Such representations belong to Lusztig's category of *unipotent representations*, defined in [22, Section 0.3] (see also Definition 4.4.1 below). Irreducible unipotent representations of $\boldsymbol{G}(\mathsf{k})$ are parameterized by so-called Deligne–Langlands–Lusztig (or enhanced Langlands) parameters. The Deligne–Langlands–Lusztig parameter of an irreducible unipotent $\boldsymbol{G}(\mathsf{k})$-representation X is a $G^\vee$-orbit of triples (τ, n, ρ), where τ is a semisimple element in the Langlands dual group $G^\vee$ of $\boldsymbol{G}$, n is a nilpotent element of the Lie algebra $\mathfrak{g}^\vee$ such that $\mathrm{Ad}(\tau)n = qn$, and ρ is an irreducible representation of a certain finite group A^1_φ; see Section 4.5. When $\boldsymbol{G}$ is adjoint, A^1_φ is the component group of the centralizer of τ and n in $G^\vee$.

It is natural to ask how the local character expansion of X is related to its Deligne–Langlands–Lusztig parameter (τ, n, ρ). At one extreme, we have the coefficient $c_0(X)$ of the zero orbit in the local character expansion. It is known that when X is tempered, $c_0(X) \neq 0$ if and only if X is square integrable, and in this case, $c_0(X)$ equals, up to a sign, the ratio between the formal degrees of X and the Steinberg representation. An interpretation of the formal degree in terms of the Langlands parameter was conjectured first by Reeder [32] for unipotent representations and then vastly generalized by Hiraga, Ichino, and Ikeda [15] for all discrete series representations of a semisimple group over a local field. For unipotent representations, this interpretation was verified in the case of split exceptional groups by Reeder [32] and in the remaining cases by Opdam [28]; see also [13].

At the other extreme, we have the various wavefront sets. Our second main result relates the unramified, geometric, and double wavefront sets to the nilpotent element n in the Deligne–Langland–Lusztig parameter (s, n, τ) of X. More precisely, let $\mathbb{O}^\vee_X$ denote the nilpotent $G^\vee$-orbit of n. We prove in Theorem 5.0.2 that

$$d_S({}^{K}\mathrm{WF}(X)) = \mathbb{O}^\vee_X, \quad \mathrm{DWF}(X) = d_A(\mathbb{O}^\vee_X, 1) = (d(\mathbb{O}^\vee_X), \mathbb{O}^\vee_X), \quad {}^{\bar{\mathsf{k}}}\mathrm{WF}(X) = d(\mathbb{O}^\vee_X). \tag{1.0.3}$$

Here we have identified coadjoint nilpotent orbits with adjoint nilpotent orbits using the Killing form, and d, d_S and d_A are the duality maps defined by Spaltenstein, Sommers and Achar, respectively; see Section 2.1. A notable consequence of this result is that the unramified and double wavefront sets (and in fact also the geometric wavefront set since $\mathbb{O}^\vee_X$ is always special for the unipotent supercuspidal representations) determine the nilpotent part of the Deligne–Langlands–Lusztig parameter. We emphasize that the simplicity of the formulas (1.0.3) is due to the fact that X is supercuspidal and therefore equal to its Aubert–Zelevinsky dual [2]. In general, one expects that the wavefront set of X is related by duality to the nilpotent parameter associated to the *AZ dual* of X; see [10]. This expression for the wavefront set is reminiscent of Lusztig's formula for the Kawanaka wavefront set of an irreducible unipotent representation of a finite reductive group [21, Theorem 11.2]. In fact, the finite reductive group results from loc. cit. play an important role in the construction and analysis of test functions in the local character expansion [5; 27]. For a positive-depth analogue of the depth-0 Barbasch–Moy methods, see [8].

1.1. ***Structure of paper.*** In Section 2, we collect some preliminaries on nilpotent orbits, wavefront sets, and depth-0 representations. In Section 3, we prove our main result on depth-0 supercuspidal representations (this result is proved without restrictions on the group). In Section 4, we collect some additional preliminaries on nilpotent orbits, wavefront sets, and Bruhat–Tits theory, necessary for our study of unipotent representations. In Section 5, we state our main result on unipotent supercuspidal representations (this result requires that $\boldsymbol{G}$ is inner to split). The proof of this result, which is completed in Section 8, requires some explicit tabulation of unipotent cuspidal representations of finite groups of Lie type (appearing in Section 6), and unipotent supercuspidal representations of simple adjoint groups (appearing in Section 7).

2. Preliminaries, I

Let k be a nonarchimedean local field of characteristic 0 with residue field $\mathbb{F}_q$ of sufficiently large characteristic, ring of integers $\mathfrak{o} \subset \mathsf{k}$, and valuation $\mathsf{val}_{\mathsf{k}}$. Fix an algebraic closure $\bar{\mathsf{k}}$ of k with Galois group Γ, and let $K \subset \bar{\mathsf{k}}$ be the maximal unramified extension of k in $\bar{\mathsf{k}}$. Let E be a minimal Galois extension of K in $\bar{\mathsf{k}}$ such that $\boldsymbol{G}$ is E-split. Let $\mathfrak{O}$ be the ring of integers of K. Let Frob be the geometric Frobenius element of $\mathrm{Gal}(K/\mathsf{k})$, the topological generator which induces the inverse of the automorphism $x \to x^q$ of $\mathbb{F}_q$. Let $\boldsymbol{G}$ be a connected reductive algebraic group defined over k and write $\boldsymbol{G}(\mathsf{k})$ for the group of k-rational points of $\boldsymbol{G}$. Let G be the complex reductive group defined over $\mathbb{C}$ with the same absolute root datum as $\boldsymbol{G}$, and let $G^\vee$ denote the Langlands dual group. Let $\mathcal{C}(\boldsymbol{G}(\mathsf{k}))$ be the category of smooth complex $\boldsymbol{G}(\mathsf{k})$-representations and let $\Pi(\boldsymbol{G}(\mathsf{k})) \subset \mathcal{C}(\boldsymbol{G}(\mathsf{k}))$ be the set of irreducible objects. Let $R(\boldsymbol{G}(\mathsf{k}))$ denote the Grothendieck group of $\mathcal{C}(\boldsymbol{G}(\mathsf{k}))$. Let $\mathcal{B}(\boldsymbol{G}, \mathsf{k})$ denote the extended Bruhat–Tits building for $\boldsymbol{G}$ over k and $\mathcal{A}(\boldsymbol{T}, \mathsf{k})$ the apartment of $\mathcal{B}(\boldsymbol{G}, \mathsf{k})$ corresponding to a maximal k-split torus $\boldsymbol{T}$ of $\boldsymbol{G}$. For a subset $c \subseteq \mathcal{B}(\boldsymbol{G}, \mathsf{k})$ write $c \subseteq_f \mathcal{B}(\boldsymbol{G}, \mathsf{k})$ to indicate that it is a face of the building. For a face $c \subseteq_f \mathcal{B}(\boldsymbol{G}, \mathsf{k})$ there is a group $\boldsymbol{P}_c^\dagger$ defined over $\mathfrak{o}$ such that $\boldsymbol{P}_c^\dagger(\mathfrak{o})$ identifies with the stabilizer of c in $\boldsymbol{G}(\mathsf{k})$. There is an exact sequence

$$1 \to \boldsymbol{U}_c(\mathfrak{o}) \to \boldsymbol{P}_c^\dagger(\mathfrak{o}) \to \boldsymbol{L}_c^\dagger(\mathbb{F}_q) \to 1, \tag{2.0.1}$$

where $\boldsymbol{U}_c(\mathfrak{o})$ is the pro-unipotent radical of $\boldsymbol{P}_c^\dagger(\mathfrak{o})$ and $\boldsymbol{L}_c^\dagger$ is the reductive quotient of the special fiber of $\boldsymbol{P}_c^\dagger$. Let $\boldsymbol{L}_c$ denote the identity component of $\boldsymbol{L}_c^\dagger$, and let $\boldsymbol{P}_c$ be the subgroup of $\boldsymbol{P}_c^\dagger$ defined over $\mathfrak{o}$ such that $\boldsymbol{P}_c(\mathfrak{o})$ is the inverse image of $\boldsymbol{L}_c(\mathbb{F}_q)$ in $\boldsymbol{P}_c^\dagger(\mathfrak{o})$. The groups $\boldsymbol{P}_c(\mathfrak{o})$ are the so-called parahoric subgroups. When c is a chamber, the group $\boldsymbol{P}_c(\mathfrak{o})$ is called an Iwahori subgroup.

2.1. ***Nilpotent orbits.*** For rest of the paper we identify coadjoint nilpotent orbits with adjoint nilpotent orbits using the Killing form. Let $\mathcal{N}$ be the functor which takes a field F to the set of nilpotent elements in $\mathfrak{g}(F)$, and let $\mathcal{N}_o$ be the functor which takes F to the set of adjoint $\boldsymbol{G}(F)$-orbits on $\mathcal{N}(F)$. When F is k or K, we view $\mathcal{N}_o(F)$ as a partially ordered set with respect to the closure ordering in the topology induced by the topology on F. When F is algebraically closed, we view $\mathcal{N}_o(F)$ as a partially ordered set with respect to the closure ordering in the Zariski topology. For brevity we will write $\mathcal{N}(F'/F)$ (resp. $\mathcal{N}_o(F'/F)$) for $\mathcal{N}(F \to F')$ (resp. $\mathcal{N}_o(F \to F')$) where $F \to F'$ is a morphism of fields. For $(F, F') = (\mathsf{k}, K)$ (resp. $(\mathsf{k}, \bar{\mathsf{k}})$, $(K, \bar{\mathsf{k}})$), the map $\mathcal{N}_o(F'/F)$ is strictly increasing (resp. strictly increasing, nondecreasing). When we wish to emphasize the group we are working with we include it as a superscript, e.g., $\mathcal{N}_o^{\boldsymbol{G}}$. Define

$$\mathcal{I}_o = \{(c, \mathbb{O}) \mid c \subseteq_f \mathcal{B}(\boldsymbol{G}),\ \mathbb{O} \in \mathcal{N}_o^{\boldsymbol{L}_c}(\overline{\mathbb{F}}_q)\}. \tag{2.1.1}$$

There is a partial order on $\mathcal{I}_o$, defined by

$$(c_1, \mathbb{O}_1) \le (c_2, \mathbb{O}_2) \iff c_1 = c_2 \text{ and } \mathbb{O}_1 \le \mathbb{O}_2.$$

By [27, Section 1.1.2] there is a strictly increasing surjective map

$$\mathcal{L} : (\mathcal{I}_o, \le) \to (\mathcal{N}_o(K), \le). \tag{2.1.2}$$

For a face $c \subseteq_f \mathcal{B}(\boldsymbol{G}, F)$ let $\mathcal{L}_c : \mathcal{N}_o^{\boldsymbol{L}_c}(\bar{F}) \to \mathcal{N}_o^{\boldsymbol{G}}(\mathsf{k})$ denote the map $\mathbb{O} \mapsto \mathcal{L}(c, \mathbb{O})$.

Finally recall the following classical result on nilpotent orbits.

Lemma 2.1.1 [29, Theorem 1.5; 30, Corollary 3.5]. *Let $\boldsymbol{G}$ a connected reductive group defined over a field F with good characteristic for $\boldsymbol{G}$. For any algebraically closed field extension F' of F there is canonical isomorphism of partially ordered sets $\Lambda^{F'} : \mathcal{N}_o^{\boldsymbol{G}}(F') \xrightarrow{\sim} \mathcal{N}_o$.*

2.2. ***Dualities on nilpotent orbits.*** Recall the groups G, $G^\vee$ from Section 2 and the functors $\mathcal{N}$, $\mathcal{N}_0$ from Section 2.1. We will extend the functors $\mathcal{N}$, $\mathcal{N}_o$ to include the field $\mathbb{C}$: Define $\mathcal{N}(\mathbb{C})$ to be the nilpotent elements of the Lie algebra of G, and $\mathcal{N}_o(\mathbb{C})$ to be the G-orbits on $\mathcal{N}$. Since it won't ever cause confusion we will often omit the $\mathbb{C}$ and simply write $\mathcal{N}$, $\mathcal{N}_o$ instead of $\mathcal{N}(\mathbb{C})$, $\mathcal{N}_o(\mathbb{C})$. Define $\mathcal{N}_{o,c}$ (resp. $\mathcal{N}_{o,\bar{c}}$) to be the set of all pairs $(\mathbb{O}, C)$ such that $\mathbb{O} \in \mathcal{N}_o$ and C is a conjugacy class in the fundamental group $A(\mathbb{O})$ of $\mathbb{O}$ (resp. Lusztig's canonical quotient $\bar{A}(\mathbb{O})$ of $A(\mathbb{O})$; see [33, Section 5]). There is a natural map

$$\mathfrak{Q} : \mathcal{N}_{o,c} \to \mathcal{N}_{o,\bar{c}}, \quad (\mathbb{O}, C) \mapsto (\mathbb{O}, \bar{C}), \tag{2.2.1}$$

where $\bar{C}$ is the image of C in $\bar{A}(\mathbb{O})$ under the natural homomorphism $A(\mathbb{O}) \twoheadrightarrow \bar{A}(\mathbb{O})$. There are also projection maps $\mathrm{pr}_1 : \mathcal{N}_{o,c} \to \mathcal{N}_o$, $\mathrm{pr}_1 : \mathcal{N}_{o,\bar{c}} \to \mathcal{N}_o$. We will typically write $\mathcal{N}^\vee$, $\mathcal{N}_o^\vee$, $\mathcal{N}_{o,c}^\vee$, and $\mathcal{N}_{o,\bar{c}}^\vee$ for the sets $\mathcal{N}$, $\mathcal{N}_o$, $\mathcal{N}_{o,c}$, and $\mathcal{N}_{o,\bar{c}}$ associated to the Langlands dual group $G^\vee$.

Let

$$d : \mathcal{N}_0 \to \mathcal{N}_0^\vee, \quad d : \mathcal{N}_0^\vee \to \mathcal{N}_0 \tag{2.2.2}$$

be the duality maps defined by Spaltenstein [34, Proposition 10.3] (see also Lusztig [20, §13.3] and Barbasch and Vogan [6, Appendix A]). Let

$$d_S : \mathcal{N}_{o,c} \twoheadrightarrow \mathcal{N}_o^\vee, \quad d_S : \mathcal{N}_{o,c}^\vee \twoheadrightarrow \mathcal{N}_o \tag{2.2.3}$$

be the duality maps defined by Sommers [33, Section 6] and

$$d_A : \mathcal{N}_{o,\bar{c}} \to \mathcal{N}^\vee_{o,\bar{c}}, \quad d_A : \mathcal{N}^\vee_{o,\bar{c}} \to \mathcal{N}_{o,\bar{c}} \tag{2.2.4}$$

be the duality maps defined by Achar [1, Section 1]. These duality maps are compatible in the following sense. For $\mathbb{O} \in \mathcal{N}_o$,

$$d_S(\mathbb{O}, 1) = d(\mathbb{O})$$

and, for $\xi \in \mathcal{N}_{o,c}$,

$$d_A(\mathfrak{Q}(\xi)) = (d_S(\xi), \bar{C}')$$

for some $\bar{C}'$. In particular

$$d_A(\mathbb{O}, 1) = (d(\mathbb{O}), \bar{C}) \tag{2.2.5}$$

for some $\bar{C}$.

There is a natural map

$$\iota : \mathcal{N}_{o,c} \to \mathcal{N}_o \times \mathcal{N}^\vee_o, \quad \xi \mapsto (\mathrm{pr}_1(\xi), d_S(\xi)).$$

The set $\mathcal{N}_o \times \mathcal{N}^\vee_o$ is equipped with a natural partial order

$$(\mathbb{O}_1, \mathbb{O}^\vee_1) \le (\mathbb{O}_2, \mathbb{O}^\vee_2) \iff \mathbb{O}_1 \le \mathbb{O}_2,\ \mathbb{O}^\vee_2 \le \mathbb{O}^\vee_1.$$

This partial order pulls back to a preorder on $\mathcal{N}_{o,c}$ via ι which coincides with the preorder defined in [1, Introduction]. For $\xi, \xi' \in \mathcal{N}_{o,c}$ define $\xi \sim_A \xi'$ if $\iota(\xi) = \iota(\xi')$. Write $[\xi]$ for the equivalence class of $\xi \in \mathcal{N}_{o,c}$. By [1, Theorem 1], the $\sim_A$-equivalence classes in $\mathcal{N}_{o,c}$ coincide precisely with the fibers of the projection map $\mathfrak{Q} : \mathcal{N}_{o,c} \to \mathcal{N}_{o,\bar{c}}$ and so ι descends to an injection

$$\bar{\iota} : \mathcal{N}_{o,\bar{c}} \to \mathcal{N}_o \times \mathcal{N}^\vee_o.$$

In particular $\le_A$ descends to a partial order on $\mathcal{N}_{o,\bar{c}}$ which we also call $\le_A$. The maps d, d_S, d_A are all order reversing with respect to the relevant pre/partial orders.

Achar duality admits a particularly simple interpretation with this setup. We can view $\mathcal{N}_{o,\bar{c}}$ as a subset of $\mathcal{N}_o \times \mathcal{N}^\vee_o$ via $\bar{\iota}$ and we can similarly view $\mathcal{N}^\vee_{o,\bar{c}}$ as a subset of $\mathcal{N}^\vee_o \times \mathcal{N}_o$. Define

$$D : \mathcal{N}_o \times \mathcal{N}^\vee_o \to \mathcal{N}^\vee_o \times \mathcal{N}_o, \quad (\mathbb{O}, \mathbb{O}^\vee) \mapsto (\mathbb{O}^\vee, \mathbb{O}).$$

Then for the so-called special elements $\bar{\xi} \in \mathcal{N}_{o,\bar{c}}$,

$$\bar{\iota}^\vee \circ d_A(\bar{\xi}) = D \circ \bar{\iota}(\bar{\xi}).$$

In particular the elements of the form $(\mathbb{O}, 1) \in \mathcal{N}_{o,\bar{c}}$ are all special and

$$\bar{\iota}(\mathbb{O}, 1) = (\mathbb{O}, d_{\mathrm{BV}}(\mathbb{O})), \quad \bar{\iota}^\vee(d_A(\mathbb{O}, 1)) = (d_{\mathrm{BV}}(\mathbb{O}), \mathbb{O}). \tag{2.2.6}$$

2.3. ***Nilpotent orbits over E.*** Recall from Section 2 that E denotes a minimal Galois extension of K such that $\boldsymbol{G}$ is E-split. In [27, Section 2] the third author establishes a number of results about the structure of $\mathcal{N}_o^{\boldsymbol{G}}(E)$ which we now briefly summarize.

Let $\boldsymbol{T}$ be a maximal E-split torus of $\boldsymbol{G}$ and let x_0 be a special point in $\mathcal{A}(\boldsymbol{T}, E)$. In [27, Section 2.1.5], we construct a bijection

$$\theta_{x_0,\boldsymbol{T}} : \mathcal{N}_o^{\boldsymbol{G}}(E) \xrightarrow{\sim} \mathcal{N}_{o,c}.$$

Theorem 2.3.1 [27, Theorem 2.20, Theorem 2.27, Proposition 2.29]. *The bijection*

$$\theta_{x_0,\boldsymbol{T}} : \mathcal{N}_o^{\boldsymbol{G}}(E) \xrightarrow{\sim} \mathcal{N}_{o,c}$$

is natural in $\boldsymbol{T}$, equivariant in x_0, and makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{N}_o^{\boldsymbol{G}}(E) & \xrightarrow{\theta_{x_0,\boldsymbol{T}}} & \mathcal{N}_{o,c} \\ {\scriptstyle \mathcal{N}_o(\bar{\mathsf{k}}/E)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_1} \\ \mathcal{N}_o(\bar{\mathsf{k}}) & \xrightarrow{\Lambda^{\bar{\mathsf{k}}}} & \mathcal{N}_o \end{array} \tag{2.3.1}$$

The composition

$$d_{S,\boldsymbol{T}} := d_S \circ \theta_{x_0,\boldsymbol{T}}$$

is independent of the choice of x_0 and natural in $\boldsymbol{T}$ [27, Proposition 2.32] and so we get a map

$$i : \mathcal{N}_o^{\boldsymbol{G}}(E) \to \mathcal{N}_o \times \mathcal{N}_o^{\vee}, \quad \mathbb{O} \mapsto (\Lambda^{\bar{\mathsf{k}}} \circ \mathcal{N}_o(\bar{\mathsf{k}}/E)(\mathbb{O}), d_{S,\boldsymbol{T}}(\mathbb{O})),$$

which only depends naturally on the torus $\boldsymbol{T}$. Let $\le_A$ denote the preorder obtained by pulling back $\le$ along i. So in particular for $\mathbb{O}_1, \mathbb{O}_2 \in \mathcal{N}_o(E)$ we have $\mathbb{O}_1 \le_A \mathbb{O}_2$ if

$$\mathcal{N}_o(\bar{\mathsf{k}}/E)(\mathbb{O}_1) \le \mathcal{N}_o(\bar{\mathsf{k}}/E)(\mathbb{O}_2) \quad \text{and} \quad d_{S,\boldsymbol{T}}(\mathbb{O}_1) \ge d_{S,\boldsymbol{T}_2}(\mathbb{O}_2)$$

and let $\sim_A$ denote the equivalence classes of this preorder. By naturality of $d_{S,\boldsymbol{T}}$, this preorder is independent of the choice of $\boldsymbol{T}$ and the map

$$\theta_{x_0,\boldsymbol{T}} : (\mathcal{N}_o(E), \le_A) \to (\mathcal{N}_{o,c}, \le_A)$$

is an isomorphism of preorders.

2.4. ***Wavefront sets.*** Let (π, X) be an admissible smooth representation of $\boldsymbol{G}(\mathsf{k})$ and let Θ_X be the character of X. Recall that for each nilpotent orbit $\mathbb{O} \in \mathcal{N}_o(\mathsf{k})$ there is an associated distribution $\mu_{\mathbb{O}}$ on $C_c^\infty(\mathfrak{g}^\omega(\mathsf{k}))$ called the *nilpotent orbital integral* of $\mathbb{O}$ [31]. Write $\hat{\mu}_{\mathbb{O}}$ for the Fourier transform of this distribution. Generalizing a result of Howe [16], Harish-Chandra [14] showed that there are complex numbers $c_{\mathbb{O}}(X) \in \mathbb{C}$ such that

$$\Theta_X(\exp(\xi)) = \sum_{\mathbb{O}} c_{\mathbb{O}}(X)\hat{\mu}_{\mathbb{O}}(\xi) \tag{2.4.1}$$

for $\xi \in \mathfrak{g}^{\omega}(\mathsf{k})$ a regular element in a small neighborhood of 0. The formula (2.4.1) is called the *local character expansion* of π. The (p-*adic*) *wavefront set* of X is

$$\mathrm{WF}(X) := \max\{\mathbb{O} \mid c_{\mathbb{O}}(X) \neq 0\} \subseteq \mathcal{N}_o(\mathsf{k}).$$

The *unramified wavefront set* of X is

$${}^K\mathrm{WF}(X) := \max\{\mathcal{N}_o(K/\mathsf{k})(\mathbb{O}) \mid c_{\mathbb{O}}(X) \neq 0\} \subseteq \mathcal{N}_o(K).$$

The *geometric wavefront set* of X is

$${}^{\bar{\mathsf{k}}}\mathrm{WF}(X) := \max\{\mathcal{N}_o(\bar{\mathsf{k}}/\mathsf{k})(\mathbb{O}) \mid c_{\mathbb{O}}(X) \neq 0\} \subseteq \mathcal{N}_o(\bar{\mathsf{k}}).$$

Using the map $\Lambda^{\bar{\mathsf{k}}}$ from Lemma 2.1.1 we will interchangeably view the geometric wavefront set as living in $\mathcal{N}_o(\bar{\mathsf{k}})$ and $\mathcal{N}_o$. The way we will compute the unramified wavefront set is using the tools developed in [27] based on ideas of Barbasch and Moy [5].

Definition 2.4.1. For every face $c \subseteq_f \mathcal{B}(\boldsymbol{G})$, the space of invariants $X^{\boldsymbol{U}_c(\mathfrak{o})}$ is a (finite-dimensional) $\boldsymbol{L}_c(\mathbb{F}_q)$-representation. Let $\mathrm{WF}(X^{\boldsymbol{U}_c(\mathfrak{o})}) \subseteq \mathcal{N}_o^{\boldsymbol{L}_c}(\bar{\mathbb{F}}_q)$ denote the *Kawanaka wavefront set* [18] of $X^{\boldsymbol{U}_c(\mathfrak{o})}$, and define *the local unramified wavefront set of X at c* to be

$${}^K\mathrm{WF}_c(X) := \{\mathcal{L}_c(\mathbb{O}) \mid \mathbb{O} \in \mathrm{WF}(X^{\boldsymbol{U}_c(\mathfrak{o})})\} \subseteq \mathcal{N}_o(K). \tag{2.4.2}$$

Theorem 2.4.2 [27, Theorem 0.1]. *Let (π, X) be a depth-0 representation of $\boldsymbol{G}(\mathsf{k})$ and $\mathcal{C}$ be a collection of faces of $\mathcal{B}(\boldsymbol{G}, \mathsf{k})$ such that every nilpotent orbit lies in the image of $\mathcal{L}_c$ for some $c \in \mathcal{C}$. Then*

$${}^K\mathrm{WF}(X) := \max\{{}^K\mathrm{WF}_c(X) \mid c \in \mathcal{C}\} \subseteq \mathcal{N}_o(K). \tag{2.4.3}$$

2.5. *The double wavefront set.* In [27, Section 2.2.3] the third author introduced a refinement of the geometric wavefront set called the *double wavefront set*.

Remark 2.5.1. (1) The double wavefront set is called the *canonical unramified wavefront set* in [27], but this name is not very descriptive and double wavefront set seems more appropriate due to its cosmetic similarities to the double affine Hecke algebra.

(2) The double wavefront set is only defined for unramified groups in [27], but the definition given here makes sense for any reductive group defined over a nonarchimedean local field with good residue characteristic. The proof of Theorem 2.5.2 in this generality is essentially the same as for unramified groups.

The double wavefront set is defined as follows. Recall the map i and partial order $\leq$ on $\mathcal{N}_o \times \mathcal{N}_o^{\vee}$ from Section 2.3, and the coefficients $c_{\mathbb{O}}(X)$ of the local character expansion from Section 2.4. Let

$$i_{\mathsf{k}} : \mathcal{N}_o(\mathsf{k}) \to \mathcal{N}_o \times \mathcal{N}_o^{\vee}$$

denote the composition $i\circ\mathcal{N}_o(E/\mathsf{k})$. The double wavefront set of a smooth admissible representation (π, X) of $\boldsymbol{G}(\mathsf{k})$ is defined to be

$$\mathrm{DWF}(X) := \max\{i_{\mathsf{k}}(\mathbb{O}) : \mathbb{O}\in\mathcal{N}_o(\mathsf{k}),\ c_{\mathbb{O}}(X)\neq 0\}\subseteq \mathcal{N}_o\times\mathcal{N}_o^\vee.$$

Let

$$\mathrm{pr}_{\mathcal{N}} : \mathcal{N}_o\times\mathcal{N}_o^\vee\to\mathcal{N}_o,\quad \mathrm{pr}_{\mathcal{N}^\vee} : \mathcal{N}_o\times\mathcal{N}_o^\vee\to\mathcal{N}_o^\vee$$

denote the projection maps. It is clear from the definition that the geometric wavefront set is simply the maximal orbit lying in $\mathrm{pr}_{\mathcal{N}}\circ\mathrm{DWF}(X)$. In particular when the double wavefront set is a singleton, then so is the geometric wavefront set, and the double wavefront set must be of the form

$$\mathrm{DWF}(X) = ({}^{\bar{\mathsf{k}}}\mathrm{WF}(X), \mathbb{O}^\vee) \tag{2.5.1}$$

for some orbit $\mathbb{O}^\vee\in\mathcal{N}_o^\vee$.

Let i_K be the composition $i\circ\mathcal{N}_o(E/K)$. Much like the unramified wavefront set, the double wavefront set can be computed building-locally.

Theorem 2.5.2 [27, Lemma 2.36]. *Let (π, X) be a depth-0 representation of $\boldsymbol{G}(\mathsf{k})$ and $\mathcal{C}$ be a collection of faces of $\mathcal{B}(\boldsymbol{G},\mathsf{k})$ such that every nilpotent orbit lies in the image of $\mathcal{L}_c$ for some $c\in\mathcal{C}$. Then*

$$\mathrm{DWF}(X) := \max\{i_K({}^K\mathrm{WF}_c(X)) \mid c\in\mathcal{C}\}\subseteq\mathcal{N}_o\times\mathcal{N}_o^\vee. \tag{2.5.2}$$

2.6. ***Depth-0 representations.*** Let (π, X) be a smooth irreducible representation of $\boldsymbol{G}(\mathsf{k})$. We say that X is depth-0 if $X^{\boldsymbol{U}_c(\mathfrak{o})}\neq 0$ for some face $c\subseteq_f\mathcal{B}(\boldsymbol{G},\mathsf{k})$. Write $\Pi^0(\boldsymbol{G}(\mathsf{k}))$ for the subset of $\Pi(\boldsymbol{G}(\mathsf{k}))$ consisting of depth-0 representations. Let

$$S(\boldsymbol{G}) := \{(c,\sigma) : c\subseteq_f\mathcal{B}(\boldsymbol{G},\mathsf{k}),\ \sigma \text{ a cuspidal representation of } \boldsymbol{L}_c(\mathbb{F}_q)\}$$

and for $(c_1,\sigma_1), (c_2,\sigma_2)\in S(\boldsymbol{G})$ write $(c_1,\sigma_1)\sim(c_2,\sigma_2)$ if they are associate in the sense of [26, Section 5]. By [25, Theorem 5.2] there is a well-defined map

$$\mathrm{type} : \Pi^0(\boldsymbol{G}(\mathsf{k}))\to S(\boldsymbol{G})/\sim,$$

which attaches to X a well-defined and unique association class $(c,\sigma)\in S(\boldsymbol{G})$ such that σ appears as a subrepresentation of $X^{\boldsymbol{U}_c(\mathfrak{o})}$. This association class is called the unrefined minimal K-type of X, and we write type(X) for brevity (for depth-0 representations unrefined minimal K-types are types). For $(c_1,\sigma_1), (c_2,\sigma_2)\in S(\boldsymbol{G})$, if $(c_1,\sigma_1)\sim(c_2,\sigma_2)$ then $(c_1,\mathrm{WF}(\sigma_1))\sim_K(c_2,\mathrm{WF}(\sigma_2))$ where WF denotes the Kawanaka wavefront set and $\sim_K$ is the equivalence relation defined in [27, Section 1.1.1]. In particular, since $\mathcal{L}$ is constant on $\sim_K$-classes we have that

$$\mathcal{L}(c_1,\mathrm{WF}(\sigma_1)) = \mathcal{L}(c_2,\mathrm{WF}(\sigma_2)). \tag{2.6.1}$$

For $X\in\Pi^0(\boldsymbol{G}(\mathsf{k}))$ define ${}^K\mathbb{O}_{\mathrm{type}}(X)$ to be the well-defined orbit $\mathcal{L}_c(\mathrm{WF}(\sigma))$ where $(c,\sigma) = \mathrm{type}(X)$.

3. Wavefront sets of depth-0 supercuspidal representations

Lemma 3.0.1 [24, Proposition 1.4, Proposition 2.1, Corollary 3.5]. *Let $(\pi, X) \in \Pi^0(\boldsymbol{G}(\mathsf{k}))$ be a depth-0 representation and let* $\mathrm{type}(X) = [(c, \sigma)]$. *Then (π, X) is supercuspidal if and only if c is a minimal facet of the building. When X is supercuspidal it is of the form $\mathrm{ind}_{\boldsymbol{P}_c^\dagger(\mathfrak{o})}^{\boldsymbol{G}(\mathsf{k})}(\sigma^\dagger)$ where $\sigma^\dagger$ is some irreducible representation of $\boldsymbol{P}_c^\dagger(\mathfrak{o})$ which contains the inflation of an irreducible Deligne–Lusztig cuspidal unipotent representation σ of $\boldsymbol{L}_c(\mathbb{F}_q)$ upon restriction to $\boldsymbol{P}_c(\mathfrak{o})$.*

Lemma 3.0.2. *For $X \in \Pi^0(\boldsymbol{G}(\mathsf{k}))$ we have that ${}^K\mathbb{O}_{\mathrm{type}}(X) \le \mathbb{O}$ for some $\mathbb{O} \in {}^K\mathrm{WF}(X)$. In particular, when ${}^K\mathrm{WF}(X)$ is a singleton we have*

$$ {}^K\mathbb{O}_{\mathrm{type}}(X) \le {}^K\mathrm{WF}(X). $$

Proof. Let $\mathrm{type}(X) = [(c, \sigma)]$. Then σ is a subrepresentation of $X^{\boldsymbol{U}_c(\mathfrak{o})}$ and so $\mathrm{WF}(\sigma) \le \mathbb{O}$ for some $\mathbb{O} \in \mathrm{WF}(X^{\boldsymbol{U}_c(\mathfrak{o})})$. Thus

$$ \mathcal{L}_c(\mathrm{WF}(\sigma)) \le \mathcal{L}_c(\mathbb{O}) \in {}^K\mathrm{WF}_c(X). $$

The result then follows from the fact that

$$ {}^K\mathrm{WF}(X) = \max\{{}^K\mathrm{WF}_c(X) : c \subseteq_f \mathcal{B}(\boldsymbol{G}, \mathsf{k})\}. \qquad \square $$

For an Iwahori-spherical representation X this inequality says nothing because $\mathrm{type}(X) = [(c_0, \mathrm{triv})]$ where c_0 is a chamber of the building and so ${}^K\mathbb{O}_{\mathrm{type}}(X)$ is the zero orbit. For supercuspidal representations however, the inequality is in fact an equality. We now proceed to prove this.

Lemma 3.0.3. *Let X be a depth-0 supercuspidal representation. Let c' be a face of $\mathcal{B}(\boldsymbol{G}, \mathsf{k})$ with $X^{\boldsymbol{U}_{c'}(\mathfrak{o})} \ne 0$ and suppose that τ is an irreducible constituent of $X^{\boldsymbol{U}_{c'}(\mathfrak{o})}$. Then τ is a cuspidal representation of $\boldsymbol{L}_{c'}(\mathbb{F}_q)$ and in particular $[(c', \tau)] = \mathrm{type}(X)$.*

Proof. Let $\mathrm{type}(X) = [(c, \sigma)]$ and c', τ be as in the statement of the lemma. Let $[(M, \tau')]$ be the cuspidal data for τ (i.e., a conjugacy class of Levi of $\boldsymbol{L}_{c'}(\mathbb{F}_q)$ and cuspidal representation of said Levi). In particular, if M is included into any parabolic P so that P has Levi decomposition $P = MU$, then τ' is a subrepresentation of τ^U. Now, all of the parabolics of $\boldsymbol{L}_{c'}(\mathfrak{o})$ are conjugate to a parabolic of the form $\boldsymbol{P}_{c''}(\mathfrak{o})/\boldsymbol{U}_{c'}(\mathfrak{o})$ where $c' \subseteq_f \overline{c''}$. Thus (conjugating M appropriately) we can find a c'' such that M is a Levi factor of $\boldsymbol{P}_{c''}(\mathfrak{o})/\boldsymbol{U}_{c'}(\mathfrak{o})$ and so $\boldsymbol{L}_{c''}(\mathbb{F}_q) \simeq M$. We thus have that

$$ \tau' \subseteq \tau^U \subseteq (X^{\boldsymbol{U}_{c'}(\mathfrak{o})})^{\boldsymbol{U}_{c''}(\mathfrak{o})/\boldsymbol{U}_{c'}(\mathfrak{o})} = X^{\boldsymbol{U}_{c''}(\mathfrak{o})}. $$

In particular (c'', τ') is an unrefined minimal K-type for X. Thus by [26, Theorem 5.2], we have that $(c'', \tau') \sim (c, \sigma)$. In particular c'' is also a minimal face and so $c'' = c'$ and $\tau = \tau'$. Thus τ is a cuspidal representation of $\boldsymbol{L}_{c'}(\mathbb{F}_q)$ and $[(c, \sigma)] = [(c', \tau)]$. $\square$

Theorem 3.0.4. *Let (π, X) be a depth-0 supercuspidal representation of $\boldsymbol{G}(\mathsf{k})$. Then*

$$ {}^K\mathrm{WF}(X) = {}^K\mathbb{O}_{\mathrm{type}}(X), \quad \mathrm{DWF}(X) = i_K({}^K\mathbb{O}_{\mathrm{type}}(X)). $$

In particular the unramified, double, and geometric wavefront sets are all singletons.

Proof. By Theorems 2.4.2 and 2.5.2

$$\begin{aligned} {}^K\mathrm{WF}(X) &= \max\{{}^K\mathrm{WF}_c(X) : c \subseteq_f \mathcal{B}(\boldsymbol{G}, \mathsf{k})\} \\ \mathrm{DWF}(X) &= \max\{i_K({}^K\mathrm{WF}_c(X)) : c \subseteq_f \mathcal{B}(\boldsymbol{G}, \mathsf{k})\}. \end{aligned}$$

Suppose $c \subseteq_f \mathcal{B}(\boldsymbol{G}, \mathsf{k})$ is such that $X^{\boldsymbol{U}_c(\mathfrak{o})} \neq 0$. Then for any τ an irreducible constituent of $X^{\boldsymbol{U}_c(\mathfrak{o})}$ we have by Lemma 3.0.3 that $[(c, \tau)] = \mathrm{type}(X)$. Thus we must have that ${}^K\mathrm{WF}_c(X) = {}^K\mathbb{O}_{\mathrm{type}}(X)$. If $c \subseteq_f \mathcal{B}(\boldsymbol{G}, \mathsf{k})$ is such that $X^{\boldsymbol{U}_c(\mathfrak{o})} = 0$ then ${}^K\mathrm{WF}_c(X, \mathbb{C})$ is the zero orbit. Thus

$${}^K\mathrm{WF}(X) = {}^K\mathbb{O}_{\mathrm{type}}(X), \quad \mathrm{DWF}(X) = i_K({}^K\mathbb{O}_{\mathrm{type}}(X)). \qquad \square$$

4. Preliminaries, II

In Section 5 we will use Theorem 3.0.4 to deduce a formula for the wavefront set of a unipotent supercuspidal representation in terms of its Langlands parameters, in the special case when the group is inner to split. For this, we will need some additional notation and preliminaries.

Let $\boldsymbol{G}$ denote a split group defined over $\mathbb{Z}$ and let $\boldsymbol{T} \subset \boldsymbol{G}$ be a maximal torus. For any field F, we write $\boldsymbol{G}(F)$, $\boldsymbol{T}(F)$, etc. for the groups of F-rational points. The $\mathbb{C}$-points are denoted by G, T, etc. Let $\boldsymbol{G}_{\mathrm{ad}} = \boldsymbol{G}/Z(\boldsymbol{G})$ denote the adjoint group of $\boldsymbol{G}$. Write $X^*(\boldsymbol{T}, \bar{\mathsf{k}})$ (resp. $X_*(\boldsymbol{T}, \bar{\mathsf{k}})$) for the lattice of algebraic characters (resp. cocharacters) of $\boldsymbol{T}(\bar{\mathsf{k}})$, and write $\Phi(\boldsymbol{T}, \bar{\mathsf{k}})$ (resp. $\Phi^\vee(\boldsymbol{T}, \bar{\mathsf{k}})$) for the set of roots (resp. coroots). Let

$$\mathcal{R} = (X^*(\boldsymbol{T}, \bar{\mathsf{k}}),\ \Phi(\boldsymbol{T}, \bar{\mathsf{k}}),\ X_*(\boldsymbol{T}, \bar{\mathsf{k}}),\ \Phi^\vee(\boldsymbol{T}, \bar{\mathsf{k}}),\ \langle \cdot, \cdot \rangle)$$

be the root datum corresponding to $\boldsymbol{G}$, and let W be the associated (finite) Weyl group. Let $\boldsymbol{G}^\vee$ be the Langlands dual group of $\boldsymbol{G}$, that is, the connected reductive algebraic group defined over $\mathbb{Z}$ corresponding to the dual root datum

$$\mathcal{R}^\vee = (X_*(\boldsymbol{T}, \bar{\mathsf{k}}),\ \Phi^\vee(\boldsymbol{T}, \bar{\mathsf{k}}),\ X^*(\boldsymbol{T}, \bar{\mathsf{k}}),\ \Phi(\boldsymbol{T}, \bar{\mathsf{k}}),\ \langle \cdot, \cdot \rangle).$$

Set $\Omega = X_*(\boldsymbol{T}, \bar{\mathsf{k}})/\mathbb{Z}\Phi^\vee(\boldsymbol{T}, \bar{\mathsf{k}})$. The center $Z(\boldsymbol{G}^\vee)$ can be naturally identified with the irreducible characters $\mathsf{Irr}\Omega$, and dually, $\Omega \cong X^*(Z(\boldsymbol{G}^\vee))$. For $\omega \in \Omega$, let ζ_ω denote the corresponding irreducible character of $Z(\boldsymbol{G}^\vee)$.

For details regarding the parameterization of inner twists of a group $\boldsymbol{G}(\mathsf{k})$, see [3, §1.3; 13, §1; 17, §2; 19; 36, §2]. We record here only that the set of equivalence classes of inner twists of the split form of $\boldsymbol{G}$ are parameterized by the Galois cohomology group

$$H^1(\Gamma, \boldsymbol{G}_{\mathrm{ad}}) \cong H^1(F, \boldsymbol{G}_{\mathrm{ad}}(K)) \cong \Omega_{\mathrm{ad}} \cong \mathrm{Irr}\, Z(\boldsymbol{G}_{\mathsf{sc}}^\vee),$$

where $\boldsymbol{G}_{\mathsf{sc}}^\vee$ is the Langlands dual group of $\boldsymbol{G}_{\mathrm{ad}}$, i.e., the simply connected cover of $\boldsymbol{G}^\vee$, and F denotes the action of Frob on $\boldsymbol{G}(K)$. We identify Ω_{ad} with the fundamental group of $\boldsymbol{G}_{\mathrm{ad}}$. The isomorphism above is determined as follows: For a cohomology class h in $H^1(F, \boldsymbol{G}_{\mathrm{ad}}(K))$, let z be a representative cocycle. Let $u \in \boldsymbol{G}_{\mathrm{ad}}(K)$ be the image of F under z, and let ω denote the image of u in Ω_{ad}. Set $F_\omega = \mathrm{Ad}(u) \circ F$.

The corresponding rational structure of $\boldsymbol{G}$ is given by F_ω. Let $\boldsymbol{G}^\omega$ be the connected reductive group defined over k such that $\boldsymbol{G}(K)^{F_\omega} = \boldsymbol{G}^\omega(\mathsf{k})$. Note that $\boldsymbol{G}^1 = \boldsymbol{G}$ (where we view $\boldsymbol{G}$ as an algebraic group over k for this equality). The minimal Galois extension of K over which $\boldsymbol{G}^\omega$ splits is K itself. So we may take $E = K$ for the rest of the paper.

If H is a complex reductive group and x is an element of H or $\mathfrak{h}$, we write $H(x)$ for the centralizer of x in H, and $A_H(x)$ for the group of connected components of $H(x)$. If S is a subset of H or $\mathfrak{h}$ (or indeed, of $H \cup \mathfrak{h}$), we can similarly define $H(S)$ and $A_H(S)$. We will sometimes write $A(x)$, $A(S)$ when the group H is implicit. The subgroups of H of the form $H(x)$ where x is a semisimple element of H are called *pseudo-Levi* subgroups of H.

4.1. *The Bruhat–Tits building.* We will recall some standard facts about the Bruhat–Tits building (all of which can be found in [35]).

Fix a $\omega \in \Omega$ and let $\boldsymbol{G}^\omega$ be the inner twist of $\boldsymbol{G}$ corresponding to ω as defined in the previous section. We write $\boldsymbol{T}$ for the split torus scheme over $\mathfrak{o}$ with generic fiber $\boldsymbol{T}$. This scheme $\boldsymbol{T}$ defined over $\mathfrak{o}$ is a subgroup of $\boldsymbol{P}_c$ for any $c \subseteq_f \mathcal{B}(\boldsymbol{G}^\omega, \mathsf{k})$ and the special fiber of $\boldsymbol{T}$, denoted $\bar{\boldsymbol{T}}$, is a maximal torus of $\boldsymbol{L}_c$.

For an apartment $\mathcal{A}$ of $\mathcal{B}(\boldsymbol{G}, K)$ and $\Omega \subseteq \mathcal{A}$ we write $\mathcal{A}(\Omega, \mathcal{A})$ for the smallest affine subspace of $\mathcal{A}$ containing Ω. The inner twist $\boldsymbol{G}^\omega$ of $\boldsymbol{G}$ gives rise to an action of the Galois group $\mathrm{Gal}(K/k)$ on $\mathcal{B}(\boldsymbol{G}, K)$ and we can (and will) identify $\mathcal{B}(\boldsymbol{G}^\omega, \mathsf{k})$ with the fixed points of this action. Write $\Phi(\boldsymbol{T}, K)$ (resp. $\Psi(\boldsymbol{T}, K)$) for the set of roots of $\boldsymbol{G}(K)$ (resp. affine roots) of $\boldsymbol{G}^\omega(K) = \boldsymbol{G}(K)$ relative to $\boldsymbol{T}$. For $\psi \in \Psi(\boldsymbol{T}, \mathsf{k})$ write $\dot{\psi} \in \Phi(\boldsymbol{T}, \mathsf{k})$ for the gradient of ψ, and $W = W(\boldsymbol{T}, \mathsf{k})$ for the Weyl group of $\boldsymbol{G}(\mathsf{k})$ with respect to $\boldsymbol{T}(\mathsf{k})$.

For $c \subseteq_f \mathcal{B}(\boldsymbol{G}, K)$, the stabilizer of c in $\boldsymbol{G}(K)$ identifies with $\boldsymbol{P}_c^\dagger(\mathfrak{O})$. It has pro-unipotent radical $\boldsymbol{U}_c(\mathfrak{O})$ and $\boldsymbol{P}_c^\dagger(\mathfrak{O})/\boldsymbol{U}_c(\mathfrak{O}) = \boldsymbol{L}_c^\dagger(\bar{\mathbb{F}}_q)$. For c a face lying in $\mathcal{B}(\boldsymbol{G}^\omega, \mathsf{k}) \subseteq \mathcal{B}(\boldsymbol{G}, K)$, F_ω stabilizes $\boldsymbol{P}_c(\mathfrak{O})$ and induces a Frobenius on $\boldsymbol{L}_c(\bar{\mathbb{F}}_q)$. The group $\boldsymbol{L}_c(\mathbb{F}_q)$ consists of the fixed points of this Frobenius.

For this paper it will be convenient to fix a maximal k-split torus $\boldsymbol{T}_0$ of $\boldsymbol{G}^\omega$ lying in $\boldsymbol{T}$ of $\boldsymbol{G}^\omega$. We have that $\mathcal{A}(\boldsymbol{T}_0, \mathsf{k}) = \mathcal{A}(\boldsymbol{T}, K)^{\mathrm{Gal}(K/k)}$. We will also fix a $\mathrm{Gal}(K/\mathsf{k})$-stable chamber c_0 of $\mathcal{A}(\boldsymbol{T}, K)$ and a special point $x_0 \in c_0$. Let $\widetilde{W} = W \ltimes X_*(\boldsymbol{T}, K)$ be the (extended) affine Weyl group. The choice of special point x_0 of $\mathcal{B}(\boldsymbol{G}, K)$ fixes an inclusion $\Phi(\boldsymbol{T}, K) \to \Psi(\boldsymbol{T}, K)$ and an isomorphism between $\widetilde{W}$ and $N_{\boldsymbol{G}(K)}(\boldsymbol{T}(K))/\boldsymbol{T}(\mathfrak{O}^\times)$. Write

$$\widetilde{W} \to W, \quad w \mapsto \dot{w}, \tag{4.1.1}$$

for the natural projection map. For a face $c \subseteq_f \mathcal{A}$ let W_c be the subgroup of $\widetilde{W}$ generated by reflections in the hyperplanes through c. The special fiber of $\boldsymbol{T}$ (as a scheme over $\mathfrak{O}$) which we denote by $\overline{\boldsymbol{T}}$, is a split maximal torus of $\boldsymbol{L}_c(\bar{\mathbb{F}}_q)$. Write $\Phi_c(\bar{\boldsymbol{T}}, \bar{\mathbb{F}}_q)$ for the root system of $\boldsymbol{L}_c$ with respect to $\bar{\boldsymbol{T}}$. Then $\Phi_c(\bar{\boldsymbol{T}}, \bar{\mathbb{F}}_q)$ naturally identifies with the set of $\psi \in \Psi(\boldsymbol{T}, K)$ that vanish on c, and the Weyl group of $\bar{\boldsymbol{T}}$ in $\boldsymbol{L}_c$ is isomorphic to W_c.

Recall that a choice of x_0 fixes an embedding $\Phi(\boldsymbol{T}, K) \to \Psi(\boldsymbol{T}, L)$. If we fix a set of simple roots $\Delta \subset \Phi(\boldsymbol{T}, K)$, this embedding determines a set of extended simple roots $\tilde{\Delta} \subseteq \Psi(\boldsymbol{T}, K)$. When $\Phi(\boldsymbol{T}, K)$ is irreducible, $\tilde{\Delta}$ is just the set $\Delta \cup \{1 - \alpha_0\}$ where α_0 is the highest root of $\Phi(\boldsymbol{T}, K)$ with respect to Δ.

When $\Phi(\boldsymbol{T}, K)$ is reducible, say $\Phi(\boldsymbol{T}, K) = \cup_i \Phi_i$ where each Φ_i is irreducible, then $\tilde{\Delta} = \cup_i \tilde{\Delta}_i$ where $\Delta_i = \Phi_i \cap \Delta$. Fix Δ so that the chamber cut out by $\tilde{\Delta}$ is c_0. Let

$$\boldsymbol{P}(\tilde{\Delta}) := \{J \subsetneq \tilde{\Delta} : J \cap \tilde{\Delta}_i \subsetneq \tilde{\Delta}_i, \forall i\}.$$

Each $J \in \boldsymbol{P}(\tilde{\Delta})$ cuts out a face of c_0 which we denote by $c(J)$. In particular $c(\Delta) = x_0$. Since $\Omega \simeq \widetilde{W}/W \ltimes \mathbb{Z}\Phi(\boldsymbol{T}, K)$, and $W \ltimes \mathbb{Z}\Phi(\boldsymbol{T}, K)$ acts simply transitively on the chambers of $\mathcal{A}(\boldsymbol{T}, K)$, the action of $\widetilde{W}$ on $\mathcal{A}(\boldsymbol{T}, K)$ induces an action of Ω on the faces of c_0 and hence on $\tilde{\Delta}$ (and $\boldsymbol{P}(\tilde{\Delta})$). For $\omega \in \Omega$ let σ_ω denote the corresponding permutation of $\tilde{\Delta}$. Let

$$\boldsymbol{P}^\omega(\tilde{\Delta}) := \{J \in \boldsymbol{P}(\tilde{\Delta}) \mid \sigma_\omega(J) = J\}$$

and let c_0^ω be the chamber of $\mathcal{B}(\boldsymbol{G}^\omega, \mathsf{k})$ lying in c_0. The set $\boldsymbol{P}^\omega(\tilde{\Delta})$ is an indexing set for the faces of c_0^ω. For $J \in \boldsymbol{P}^\omega(\tilde{\Delta})$ write $c^\omega(J)$ for the face of c_0^ω corresponding to J. The face $c^\omega(J)$ lies in $c(J)$. For $J, J' \in \boldsymbol{P}^\omega(\tilde{\Delta})$ (resp. $\boldsymbol{P}(\tilde{\Delta})$) we have $J \subseteq J'$ if and only if $\overline{c^\omega(J)} \supseteq c^\omega(J')$ (resp. $\overline{c(J)} \supseteq c(J')$).

4.2. ***Lifting nilpotent orbits.*** Recall the group G from Section 2 and the definition of psuedo-Levi subgroups from Section 4. Fix a maximal torus T of G. Call a pseudo-Levi subgroup L of G *standard* if it contains T and write Z_L for its center. Let $\mathcal{A} = \mathcal{A}(\boldsymbol{T}, K)$.

Lemma 4.2.1 [27, Section 2.14, Corollary 2.19]. *There is a W-equivariant map*

$$\mathfrak{L}_{x_0} : \{\textit{faces of } \mathcal{A}\} \to \{(L, tZ_L^\circ) \mid L \textit{ a standard pseudo-Levi}, \ Z_G^\circ(tZ_L^\circ) = L\}, \tag{4.2.1}$$

where c_1, c_2 lie in the same fiber if and only if

$$\mathcal{A}(c_1, \mathcal{A}) + X_*(\boldsymbol{T}, K) = \mathcal{A}(c_2, \mathcal{A}) + X_*(\boldsymbol{T}, K).$$

If $\mathfrak{L}_{x_0}(c) = (L, tZ_L^\circ)$ then L is the complex reductive group with the same root datum as $\boldsymbol{L}_c(\bar{F}_q)$ and thus there is an isomorphism $\Lambda_c^{\bar{F}_q} : \mathcal{N}_o^{\boldsymbol{L}_c}(\bar{\mathbb{F}}_q) \xrightarrow{\sim} \mathcal{N}_o^L$.

Recall from the end of Section 4.1 the definitions of Δ, $\tilde{\Delta}$, c_0, $\boldsymbol{P}(\tilde{\Delta})$ and $c(J)$. The definitions of c_0 and $c(J)$ depend on a choice of x_0 and Δ. Let L_J denote the pseudo-Levi subgroup of G generated by T and the root groups corresponding to $\dot{\alpha}$ for $\alpha \in J$. Then $\mathrm{pr}_1 \circ \mathfrak{L}_{x_0}(c(J)) = L_J$ (by [27, Lemma 2.21], this group should not depend on x_0). Define

$$\begin{aligned} \mathcal{I}_{x_0, \tilde{\Delta}} &= \{(J, \mathbb{O}) \mid J \in \boldsymbol{P}(\tilde{\Delta}), \ \mathbb{O} \in \mathcal{N}_o^{\boldsymbol{L}_{c(J)}}(\bar{\mathbb{F}}_q)\}, \\ \mathcal{K}_{\tilde{\Delta}} &= \{(J, \mathbb{O}) \mid J \in \boldsymbol{P}(\tilde{\Delta}), \ \mathbb{O} \in \mathcal{N}_o^{L_J}(\mathbb{C})\}. \end{aligned} \tag{4.2.2}$$

The map

$$\iota_{x_0} : \mathcal{I}_{x_0, \tilde{\Delta}} \to \mathcal{K}_{\tilde{\Delta}}, \quad (J, \mathbb{O}) \mapsto (J, \Lambda_{c(J)}^{\bar{F}_q}(\mathbb{O})),$$

is an isomorphism. Let

$$\mathbb{L} : \mathcal{K}_{\tilde{\Delta}} \to \mathcal{N}_{c,o} \tag{4.2.3}$$

be the map that sends $(J, \mathbb{O})$ to $(Gx, tZ_G^\circ(x))$ where $x \in \mathbb{O}$ and $(L, tZ_L^\circ) = \mathfrak{L}(c(J))$ where $\mathfrak{L}$ is the map from [27, Corollary 2.19]. By [10, Theorem 2.1.7] the diagram

$$\begin{tikzcd} \mathcal{I}_{x_0,\tilde{\Delta}} \arrow[r, "\sim", "\iota_{x_0}"'] \arrow[d, two heads, "\mathcal{L}"] & \mathcal{K}_{\tilde{\Delta}} \arrow[d, two heads, "\mathbb{L}"] \\ \mathcal{N}_o(K) \arrow[r, "\sim", "\theta_{x_0,\boldsymbol{T}}"'] & \mathcal{N}_{o,c} \end{tikzcd} \tag{4.2.4}$$

commutes. Define

$$\bar{\mathbb{L}} = \mathfrak{Q} \circ \mathbb{L}.$$

This map can be computed using Achar's algorithms in [1, Section 3.4].

4.3. ***Isogenies.*** Let $f : \boldsymbol{H}' \to \boldsymbol{H}$ be an isogeny of connected reductive groups defined over k. Let $f_k : \boldsymbol{H}'(\mathsf{k}) \to \boldsymbol{H}(\mathsf{k})$ denote the corresponding homomorphism of k-points. We note that $\mathcal{N}_{o,\bar{c}}^{\boldsymbol{H}} \simeq \mathcal{N}_{o,\bar{c}}^{\boldsymbol{H}'}$ and so we can compare the double wavefront sets of representations of the two groups.

Lemma 4.3.1. *Let X be an irreducible smooth depth-0 representation of $\boldsymbol{H}(\mathsf{k})$ and write X' for the representation of $\boldsymbol{H}'(\mathsf{k})$ obtained by pulling back along f_k. Then X' decomposes as a finite sum of irreducible smooth representations $X' = \bigoplus_i X_i'$ and $\mathrm{DWF}(X) = \mathrm{DWF}(X_i')$ for all i.*

Proof. This is an easy consequence of Clifford theory since the image of G under the isogeny is normal and of finite index. □

4.4. ***Unipotent supercuspidal representations.*** Recall Lusztig's notion of a unipotent representation of a finite group of Lie type [20, Section 6.5]. Let

$$S_{\mathrm{unip}}(\boldsymbol{G}) := \{(c, \sigma) \in S(\boldsymbol{G}) : \sigma \text{ is } \textit{unipotent}\}.$$

Definition 4.4.1. Let X be a depth-0 irreducible $\boldsymbol{G}(\mathsf{k})$-representation. We say that X has *unipotent cuspidal support* if $\mathrm{type}(X) \in S_{\mathrm{unip}}(\boldsymbol{G})$. Write $\Pi^{\mathsf{Lus}}(\boldsymbol{G}(\mathsf{k}))$ for the subset of $\Pi^0(\boldsymbol{G}(\mathsf{k}))$ consisting of all such representations. We call a supercuspidal representation 'unipotent' if it is depth-0 and has unipotent cuspidal support.

4.5. ***Langlands classification of unipotent supercuspidal representations.*** Let W_{k} be the Weil group of k with inertia subgroup I_{k} and set $W_{\mathsf{k}}' = W_{\mathsf{k}} \times \mathrm{SL}(2, \mathbb{C})$. We will think of a *Langlands parameter* for $\boldsymbol{G}$ as a continuous morphism $\varphi : W_{\mathsf{k}}' \to G^\vee$ such that $\varphi(w)$ is semisimple for each $w \in W_{\mathsf{k}}$ and the restriction of φ to $\mathrm{SL}(2, \mathbb{C})$ is algebraic. A Langlands parameter φ is called *unramified* if $\varphi(I_{\mathsf{k}}) = \{1\}$. Let $G^\vee(\varphi)$ denote the centralizer of $\varphi(W_{\mathsf{k}}')$ in $G^\vee$. Define

$$Z^1_{G^\vee_{\mathsf{sc}}}(\varphi) = \text{ preimage of } G^\vee(\varphi)/Z(G^\vee) \text{ under the projection } G^\vee_{\mathsf{sc}} \to G^\vee_{\mathrm{ad}},$$

and let A^1_φ denote the component group of $Z^1_{G^\vee_{\mathsf{sc}}}(\varphi)$. An *enhanced Langlands parameter* is a pair (φ, ρ), where $\rho \in \mathrm{Irr}(A^1_\varphi)$. A parameter (φ, ρ) is called $\boldsymbol{G}^\omega$*-relevant* (recall that $\boldsymbol{G}^\omega$ is an inner twist of the split form, $\omega \in \Omega_{\mathrm{ad}}$) if ρ acts on $Z(G^\vee_{\mathsf{sc}})$ by a multiple of the character ζ_ω.

Define the elements

$$s_\varphi = \varphi(\mathrm{Frob}, 1), \quad u_\varphi = \varphi\left(1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right).$$

Following [4], consider the possibly disconnected reductive group

$$\mathcal{G}_\varphi = Z^1_{G^\vee_{\mathrm{sc}}}(\varphi(W_{\mathsf{k}})),$$

which is defined analogously to $Z^1_{G^\vee_{\mathrm{sc}}}(\varphi)$. Then $u_\varphi \in \mathcal{G}^\circ_\varphi$ and by [4, (92)]

$$A^1_\varphi \cong \mathcal{G}_\varphi(u_\varphi)/\mathcal{G}_\varphi(u_\varphi)^\circ.$$

An enhanced Langlands parameter (φ, ρ) is called *discrete* if $G^\vee(\varphi)$ does not contain a nontrivial torus (this notion is independent of ρ). A discrete parameter is called *cuspidal* if (u_φ, ρ) is a cuspidal pair. This means that every ρ° which occurs in the restriction of ρ to $A_{\mathcal{G}^\circ_\varphi}(u_\varphi)$ defines a $\mathcal{G}^\circ_\varphi$-equivariant local system on the $\mathcal{G}^\circ_\varphi$-conjugacy class of u_φ which is cuspidal in the sense of Lusztig.

A Langlands correspondence for unipotent supercuspidal representations has been obtained by [24] when $\boldsymbol{G}$ is simple and adjoint; see also [22]. For arbitrary reductive K-split groups, this correspondence is available by [12; 13]. Let $\mathrm{Irr}(\boldsymbol{G}^\omega(\mathsf{k}))_{\mathrm{cusp,unip}}$ denote the set of equivalence classes of irreducible unipotent supercuspidal $\boldsymbol{G}^\omega(\mathsf{k})$-representations. Let $\Phi(G^\vee)^\omega_{\mathrm{cusp,nr}}$ denote the set of $G^\vee$-equivalence classes of unramified cuspidal enhanced Langlands parameters (φ, ρ) which are $\boldsymbol{G}^\omega$-relevant.

Theorem 4.5.1. *For every $\omega \in \Omega_{\mathrm{ad}}$, there is a bijection*

$$\Phi(G^\vee)^\omega_{\mathrm{cusp,nr}} \longleftrightarrow \mathrm{Irr}(\boldsymbol{G}^\omega(\mathsf{k}))_{\mathrm{cusp,unip}}.$$

This bijection satisfies several natural desiderata (*including formal degrees*, *equivariance with respect to tensoring by weakly unramified characters*); *see* [13, Theorem 2].

For X a unipotent supercuspidal representation of $\boldsymbol{G}^\omega(\mathsf{k})$ let φ denote the corresponding Langlands parameter. We will write $\mathbb{O}^\vee_X \in \mathcal{N}^\vee_o$ for the $G^\vee$-orbit of

$$n_\varphi = d\varphi\left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right).$$

Lemma 4.5.2. *Let $X, X' \in \mathrm{Irr}(\boldsymbol{G}^\omega(\mathsf{k}))_{\mathrm{cusp,unip}}$. If* $\mathrm{type}(X) = \mathrm{type}(X')$ *then* $\mathbb{O}^\vee_X = \mathbb{O}^\vee_{X'}$.

Proof. This follows by inspecting the explicit classification in [13]. □

For $[(c, \sigma)] \in S_{\mathrm{unip}}(\boldsymbol{G})$ with c a minimal face we write $\mathbb{O}^\vee(c, \sigma)$ for the common nilpotent parameter of all $X \in \mathrm{Irr}(\boldsymbol{G}^\omega(\mathsf{k}))_{\mathrm{cusp,unip}}$ with $\mathrm{type}(X) = [(c, \sigma)]$.

We will recall the explicit classification in Section 7.

5. Wavefront sets of unipotent supercuspidal representations

Proposition 5.0.1. *Suppose $\boldsymbol{G}$ is simple and adjoint and let $[(c,\sigma)] \in S_{\text{unip}}(\boldsymbol{G}^\omega)$ be such that c is a minimal face. Then*

$$\bar{\mathbb{L}}(c, \mathrm{WF}(\sigma)) = d_A(\mathbb{O}^\vee(c,\sigma), 1).$$

This proposition will be proved in Section 8.

Theorem 5.0.2. *Let $\boldsymbol{G}$ be a split reductive group defined over k. Let $\omega \in \Omega$ and let $\boldsymbol{G}^\omega$ denote the corresponding inner twist of $\boldsymbol{G}$. Let (π, X) be an irreducible supercuspidal $\boldsymbol{G}^\omega(\mathsf{k})$-representation with unipotent cuspidal support. Then ${}^K\mathrm{WF}(X)$, $\mathrm{DWF}(X)$, ${}^{\bar{\mathsf{k}}}\mathrm{WF}(X)$ are singletons, and*

$$d_S({}^K\mathrm{WF}(X)) = \mathbb{O}_X^\vee, \quad \mathrm{DWF}(X) = (d(\mathbb{O}_X^\vee), \mathbb{O}_X^\vee), \quad {}^{\bar{\mathsf{k}}}\mathrm{WF}(X) = d(\mathbb{O}_X^\vee).$$

Proof. By Theorem 3.0.4, the second component of $\mathrm{DWF}(X)$ is exactly $d_S({}^K\mathrm{WF}(X))$. By (2.5.1) the first component of $\mathrm{DWF}(X)$ is ${}^{\bar{\mathsf{k}}}\mathrm{WF}(X)$. Thus it suffices to prove the second equality only. Suppose first that $\boldsymbol{G}$ is simple and adjoint.

Let X be a unipotent supercuspidal representation of $\boldsymbol{G}^\omega(\mathsf{k})$. By Theorem 3.0.4 we have that

$$\mathrm{DWF}(X) = i_K({}^K\mathbb{O}_{\text{type}}(X)).$$

Write $\text{type}(X) = [(c,\sigma)]$. By definition

$${}^K\mathbb{O}_{\text{type}}(X) = \mathbb{L}(c, \mathrm{WF}(\sigma)).$$

By Proposition 5.0.1 we have that

$$\bar{\mathbb{L}}(c, \mathrm{WF}(\sigma)) = d_A(\mathbb{O}^\vee(c,\sigma), 1).$$

Since $\mathbb{O}_X^\vee = \mathbb{O}^\vee(c,\sigma)$ we get that

$$\mathrm{DWF}(X) = d_A(\mathbb{O}_X^\vee, 1)$$

as required.

Applying Lemma 4.3.1 we get that the theorem holds for all simply connected simple groups. Since wavefront sets behave as expected with respect to products, the theorem holds for all simply connected reductive groups. Finally, applying Lemma 4.3.1 again we get that the theorem holds for all split reductive groups $\boldsymbol{G}$. $\square$

6. Unipotent cuspidal representations of finite reductive groups

Let $\boldsymbol{G}$ be a reductive group defined over $\mathbb{F}_q$. We list the unipotent cuspidal representations of $\boldsymbol{G}(\mathbb{F}_q)$ and their Kawanaka wavefront sets. Since the classification of unipotent representations is independent of the isogeny, we may assume without loss of generality that the group G is simple and adjoint. For the explicit results about the parameterization of unipotent representations of finite groups of Lie type, we refer to [20, §4, §8.1] and [7, §13.8, §13.9]. The relevant results for the Kawanaka wavefront sets and unipotent support are in [21, §10, §11].

6.1. *Classical groups.*

6.1.1. $A_{n-1}(q)$. The group $\boldsymbol{G} = \mathrm{PGL}(n)$ does not have unipotent cuspidal representations.

6.1.2. ${}^2A_n(q^2)$. The group $\boldsymbol{G} = \mathrm{PU}(n+1)$ has unipotent representations if and only if $n = \frac{r(r+1)}{2} - 1$, for some integer $r \geq 2$. The unipotent ${}^2A_n(q^2)$-representations are in one-to-one correspondence with partitions of $n+1$, and so are the geometric nilpotent orbits of $\boldsymbol{G}$. When $n = \frac{r(r+1)}{2} - 1$, the cuspidal unipotent representation σ is unique and it is parameterized by the partition

$$(1, 2, 3, \ldots, r).$$

Its Kawanaka wavefront set is $\mathrm{WF}(\sigma) = (1, 2, 3, \ldots, r)$.

6.1.3. $B_n(q)$, $C_n(q)$. Suppose $\boldsymbol{G}$ is $\mathrm{SO}(2n+1)$ or $\mathrm{PSp}(2n)$ over $\mathbb{F}_q$. The group $\boldsymbol{G}(\mathbb{F}_q)$ has a unipotent cuspidal representation (and in this case the cuspidal representation is unique) if and only if $n = r^2 + r$ for a positive integer r. The unipotent representations of $\boldsymbol{G}(\mathbb{F}_q)$ are parameterized by symbols

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_a \\ & \mu_1 & \cdots & \mu_b \end{pmatrix},$$

$0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_a$, $0 \leq \mu_1 < \mu_2 < \cdots < \mu_b$, $a - b$ odd and positive, and λ_1, μ_1 are not both zero, such that

$$n = \sum \lambda_i + \sum \mu_j - \left(\frac{a+b-1}{2}\right)^2.$$

Let $d = a - b$ be the defect of the symbol. Two unipotent representations belong to the same family if their symbols have the same entries with the same multiplicities. For the unipotent cuspidal representation σ, the corresponding symbol has defect $d = 2r+1$ and it is

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & 2r \\ & & - & & \end{pmatrix}.$$

The geometric nilpotent orbits of $\mathrm{SO}(2n+1)$ (resp. $\mathrm{PSp}(2n)$) are parameterized by partitions of $2n+1$ (resp. $2n$), where the even (resp. odd) parts occur with even multiplicity. The Kawanaka wavefront set of the unipotent cuspidal representation σ is

$$\mathrm{WF}(\sigma) = \begin{cases} (1, 1, 3, 3, \ldots, 2r-1, 2r-1, 2r+1) & \text{if } G = \mathrm{SO}(2n+1), \\ (2, 2, 4, 4, \ldots, 2r, 2r) & \text{if } G = \mathrm{PSp}(2n). \end{cases} \tag{6.1.1}$$

6.1.4. $D_n(q)$. Suppose $\boldsymbol{G}$ is the split orthogonal group $\mathrm{PSO}(2n)$ over $\mathbb{F}_q$. There exists a unipotent cuspidal representation (and in this case it is unique) if and only if $n = r^2$ for a positive even integer r. The type D_n-symbols are

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_a \\ & \mu_1 & \cdots & \mu_b \end{pmatrix},$$

$0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_a$, $0 \leq \mu_1 < \mu_2 < \cdots < \mu_b$, $a-b$ is divisible by 4, and λ_1, μ_1 are not both zero, such that

$$n = \sum \lambda_i + \sum \mu_j - \frac{(a+b)(a+b-2)}{4}.$$

One symbol and the symbol if the row swapped are regarded the same. The irreducible unipotent $\boldsymbol{G}(\mathbb{F}_q)$-representations are in one-to-one correspondence with the type D_n-symbols, except if the symbol has identical rows; in that case there are two nonisomorphic irreducible unipotent representations attached to it. The defect $d = a - b$ is even.

For the unipotent cuspidal representation σ, the corresponding symbol has defect $d = 2r$ and it is

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & 2r-1 \\ & & - & & \end{pmatrix}.$$

The geometric nilpotent orbits of $\mathrm{PSO}(2n)$ are parameterized by partitions of $2n$ with the even parts occurring with even multiplicity. The Kawanaka wavefront set of the unipotent cuspidal representation is

$$\mathrm{WF}(\sigma) = (1, 1, 3, 3, \ldots, 2r-1, 2r-1). \tag{6.1.2}$$

6.1.5. ${}^2D_n(q^2)$. The group ${}^2D_n(q^2)$ admits unipotent cuspidal representations if and only if $n = r^2$, for some odd positive integer r, and in this case the unipotent cuspidal representation is unique. The type ${}^2D_n$-symbols are

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_a \\ & \mu_1 & \cdots & \mu_b \end{pmatrix},$$

$0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_a$, $0 \leq \mu_1 < \mu_2 < \cdots < \mu_b$, $a - b \equiv 2 \pmod 4$, and λ_1, μ_1 are not both zero, such that

$$n = \sum \lambda_i + \sum \mu_j - \frac{(a+b)(a+b-2)}{4}.$$

One symbol and the symbol if the row swapped are regarded the same. The irreducible unipotent ${}^2D_n(q^2)$-representations are in one-to-one correspondence with the type ${}^2D_n$ symbols.

For the unipotent cuspidal representation σ, the corresponding symbol and Kawanaka wavefront set are the same as in the split case $D_n(q)$ (except r is now odd).

6.1.6. ${}^3D_4(q^3)$. The group ${}^3D_4(q^3)$ has eight unipotent representations: six are in the principal series, in one-to-one correspondence with the irreducible representations of the Weyl group of type G_2, and two unipotent cuspidal representations, denoted ${}^3D_4[1]$ and ${}^3D_4[-1]$.

The geometric nilpotent orbits of ${}^3D_4(q^3)$ are parameterized by partitions of 8 with even parts occurring with even multiplicity. The unipotent cuspidal representations have Kawanaka wavefront set

$$\mathrm{WF}({}^3D_4[1]) = \mathrm{WF}({}^3D_4[1]) = (1, 1, 3, 3). \tag{6.1.3}$$

6.2. ***Exceptional groups.*** Suppose $\boldsymbol{G}(\mathbb{F}_q)$ is $\mathbb{F}_q$-split. In Table 1, we list all unipotent cuspidal $\boldsymbol{G}(\mathbb{F}_q)$-representations. The irreducible unipotent representations of $\boldsymbol{G}(\mathbb{F}_q)$ are partitioned into families, each

$\boldsymbol{G}(\mathbb{F}_q)$	cuspidal σ	$\mathrm{WF}(\sigma)$	$\mathbb{O}^\vee_\sigma$	$\bar{A}(\mathbb{O}^\vee)$	(x,τ)
G_2	$G_2[1]$ $G_2[-1]$ $G_2[\theta^l]$, $l=1,2$	$G_2(a_1)$	$G_2(a_1)$	S_3	$(1,\epsilon)$ (g_2,ϵ) (g_3,θ^l)
F_4	$F_4^{II}[1]$ $F_4[-1]$ $F_4^I[1]$ $F_4[\theta^l]$, $l=1,2$ $F_4[\pm i]$	$F_4(a_3)$	$F_4(a_3)$	S_4	$(1,\lambda^3)$ (g_2,ϵ) (g_2',ϵ) (g_3,θ^l) $(g_4,\pm i)$
E_6	$E_6[\theta^l]$, $l=1,2$	$D_4(a_1)$	$D_4(a_1)$	S_3	(g_3,θ^l)
E_7	$E_7[\zeta]$ $E_7[\zeta]$	A_4+A_1	A_4+A_1	$\mathbb{Z}/2$	$(g_2,1)$ (g_2,ϵ)
E_8	$E_8^{II}[1]$ $E_8[-1]$ $E_8^I[1]$ $E_8[\theta^l]$, $l=1,2$ $E_8[-\theta^l]$, $l=1,2$ $E_8[\pm i]$ $E_8[\zeta^j]$, $1\le j\le 4$	$E_8(a_7)$	$E_8(a_7)$	S_5	$(1,\lambda^4)$ $(g_2,-\epsilon)$ (g_2',ϵ) $(g_3,\epsilon\theta^l)$ $(g_6,-\theta^l)$ $(g_4,\pm i)$ (g_5,ζ^j)

Table 1. Unipotent cuspidal representations of exceptional groups $\boldsymbol{G}(\mathbb{F}_q)$.

family being in one-to-one correspondence with the set

$$M(\Gamma)=\Gamma\text{-orbits in }\{(x,\tau)\mid x\in\Gamma,\ \tau\in\widehat{Z_\Gamma(x)}\},$$

for a finite group Γ. Each group Γ is uniquely attached to a special nilpotent orbit $\mathbb{O}^\vee$ in the dual Lie algebra, such that $\Gamma=\bar{A}(\mathbb{O}^\vee)$, where $\bar{A}(\mathbb{O}^\vee)$ is Lusztig's canonical quotient.

In Table 1, for each unipotent cuspidal representation σ, we will record the corresponding Kawanaka wavefront set, the nilpotent orbit $\mathbb{O}^\vee$ corresponding to σ and its canonical quotient $\bar{A}(\mathbb{O}^\vee)$, the pair $(x,\tau)\in M(\bar{A}(\mathbb{O}^\vee))$ that parameterizes σ. The geometric nilpotent orbits are given in the Bala–Carter notation.

Finally, for the twisted group ${}^2E_6(q^2)$, there are three unipotent cuspidal representations, denoted ${}^2E_6[1]$, ${}^2E_6[\theta]$, ${}^2E_6[\theta^2]$. All three of them have Kawanaka wavefront set $D_4(a_1)$ in E_6.

7. Langlands parameters for unipotent supercuspidal representations

Recall from Section 4.5 the notation $\mathbb{O}^\vee_X$ and $\mathbb{O}^\vee(c,\sigma)$. We record $\mathbb{O}^\vee_X$ for each unipotent supercuspidal representations (π,X) of an inner to split simple adjoint algebraic group. To only list the information we need we make the following observations. By Lemma 4.5.2, $\mathbb{O}^\vee_X$ only depends on $\mathrm{type}(X)=[(c,\sigma)]$ and by conjugating appropriately we may assume that c is of the form $c^\omega(J)$ for some $J\in\boldsymbol{P}^\omega(\tilde{\Delta})$. Since $c^\omega(J)$

is a minimal face if and only if J is maximal in $\boldsymbol{P}^{\omega}(\tilde{\Delta})$, the possible values for type(X) can be indexed by pairs (J,σ) where J is maximal in $\boldsymbol{P}^{\omega}(\tilde{\Delta})$ and σ is a unipotent cuspidal representation of $\boldsymbol{L}_{c^{\omega}(J)}(\mathbb{F}_q)$. So in this section we list the possible ω along with the set of possible pairs (J,σ) (up to $\sim$) and their corresponding $\mathbb{O}^{\vee}(c^{\omega}(J),\sigma)$. We use the conventions of [22, Section 6.10] to specify the set $J\subseteq\tilde{\Delta}$. When $\boldsymbol{G}$ is of classical type, the group $\boldsymbol{L}_{c^{\omega}(J)}(\mathbb{F}_q)$ is also of classical type and so if it admits a unipotent cuspidal representation, then it has *exactly one* unipotent cuspidal representation. Thus for the classical types we will only record the J and $\mathbb{O}^{\vee}(c^{\omega}(J),\sigma)$. The explicit parameters can be found in [12, §4.7; 13; 22; 32].

7.1. ***Classical groups.***

7.1.1. PGL(n). If $\boldsymbol{G}=\mathrm{PGL}(n)$, then $G^{\vee}=\mathrm{SL}(n,\mathbb{C})$ and $Z(G^{\vee})=\mathbb{Z}/n\mathbb{Z}$. Hence $\Omega=\mathrm{Irr}(Z(G^{\vee})$ can be identified with C_n. For $\omega\in\Omega$, the inner form $\boldsymbol{G}^{\omega}$ admits unipotent supercuspidal representations if and only if ω has order n and $J=\varnothing$. In this case $\mathbb{O}^{\vee}(c^{\omega}(J),\sigma)$ is the principal nilpotent orbit.

7.1.2. SO($2n+1$). If $\boldsymbol{G}=\mathrm{SO}(2n+1)$, $G^{\vee}=\mathrm{Sp}(2n,\mathbb{C})$ and $Z(G^{\vee})=\mathbb{Z}/2\mathbb{Z}$. The inner forms are parameterized by $\widehat{Z(G^{\vee})}\cong C_2=\{1,-1\}$.

(1) If $\omega=1$, then J is of the form $D_\ell\times B_t$, where $\ell+t=n$, $\ell=a^2$, $t=b(b+1)$, a, b nonnegative integers, a even. Let

$$\delta=\begin{cases} b-a & \text{if } b\geq a,\\ a-b-1 & \text{if } a>b,\end{cases} \tag{7.1.1}$$

and $\Sigma=a+b$. The nilpotent orbit $\mathbb{O}^{\vee}(c^{\omega}(J),\sigma)$ is parameterized by the partition

$$\lambda=(2,4,\dots,2\delta)\cup(2,4,\dots,2\Sigma). \tag{7.1.2}$$

(2) If $\omega=-1$, then J is of the form $D_\ell\times B_t$, where $\ell+t=n$, $\ell=a^2$, $t=b(b+1)$, a, b nonnegative integers, where a is now odd. The nilpotent orbit $\mathbb{O}^{\vee}(c^{\omega}(J),\sigma)$ is defined analogously to the $\omega=1$ case.

7.1.3. PSp($2n$). If $\boldsymbol{G}=\mathrm{PSp}(2n)$, then $G^{\vee}=\mathrm{Spin}(2n+1,\mathbb{C})$, and $Z(G^{\vee})=\mathbb{Z}/2\mathbb{Z}$. The inner forms are parameterized by $\widehat{Z(G^{\vee})}\cong C_2=\{1,-1\}$.

(1) If $\omega=1$, then J is of the form $C_\ell\times C_t$, where $\ell+t=n$, $\ell=a(a+1)$, $t=b(b+1)$, a, b nonnegative integers and $a\geq b$. Let $\delta=a-b$ and $\Sigma=a+b$. The nilpotent orbit $\mathbb{O}^{\vee}(c^{\omega}(J),\sigma)$ is parameterized by the partition

$$\lambda=(1,3,\dots,2\delta-1)\cup(1,3,\dots,2\Sigma+1), \tag{7.1.3}$$

where $\cup$ means union of partitions.

(2) If $\omega=-1$, then J is of the form $J=C_\ell\,{}^2A_t\,C_\ell$, where $2\ell+t=n-1$ and $t=\frac{a(a+1)}{2}-1$, $\ell=b(b+1)$, a, b are nonnegative integers. If $a=0,1$, we interpret J as being $J=C_\ell\times C_\ell$. Let a' be such that $a=2a'$ if a is even and $a=2a'+1$ if a is odd. Let $\Sigma=b+a'$ and

$$\delta=\begin{cases} b-a' & \text{if } 2b\geq a,\\ a'-b & \text{if } 2b<a.\end{cases}$$

The nilpotent orbit $\mathbb{O}^\vee(c^\omega(J),\sigma)$ is parameterized by the partition

$$\lambda=\begin{cases}(1,5,\dots,4\Sigma+1)\cup(3,7,\dots,4\delta-1) & \text{if } a \text{ is even and } 2b\geq a,\\ (1,5,\dots,4\Sigma+1)\cup(1,5,\dots,4\delta-3) & \text{if } a \text{ is even and } 2b<a,\\ (3,7,\dots,4\Sigma+3)\cup(1,5,\dots,4\delta-3) & \text{if } a \text{ is odd and } 2b\geq a,\\ (3,7,\dots,4\Sigma+3)\cup(3,7,\dots,4\delta-1) & \text{if } a \text{ is odd and } 2b<a.\end{cases}$$

7.1.4. PSO$(2n)$. If $\boldsymbol{G}=\mathrm{PSO}(2n)$, then $G^\vee=\mathrm{Spin}(2n,\mathbb{C})$, and

$$Z(G^\vee)=\begin{cases}(\mathbb{Z}/2\mathbb{Z})^2 & \text{if } n \text{ is even},\\ \mathbb{Z}/4\mathbb{Z} & \text{if } n \text{ is odd}.\end{cases}$$

Let τ be the standard diagram automorphism of type D_n. Let $\{1,-1\}$ be the kernel of the isogeny $\mathrm{Spin}(2n,\mathbb{C})\to\mathrm{SO}(2n,\mathbb{C})$. Write the four characters of $Z(G^\vee)$ as $\Omega=\{1,\eta,\rho,\eta\rho\}$, where $\tau(\eta)=\eta$ and $\eta(-1)=1$.

(1) If $\omega=1$, then J is of the form $D_\ell\times D_t$, where $\ell+t=n$, $\ell=a^2$, $t=b^2$, a,b even nonnegative integers, $a\geq b$. Let $\delta=a-b$, $\Sigma=a+b$. The nilpotent orbit $\mathbb{O}^\vee(c^\omega(J),\sigma)$ is parameterized by the partition

$$\lambda=(1,3,\dots,2\delta-1)\cup(1,3,\dots,2\Sigma-1). \tag{7.1.4}$$

(2) If $\omega=\eta$, then J is of the form ${}^2D_\ell\times{}^2D_t$, where $\ell+t=n$, $\ell=a^2$, $t=b^2$, and a,b are odd positive integers, $a\geq b$. The nilpotent orbit $\mathbb{O}^\vee(c^\omega(J),\sigma)$ is defined analogously to the $\omega=1$ case.

(3) If $\omega=\rho,\eta\rho$, then J can take one of the following two forms:

(i) J is of the form ${}^2A_t$, where $t=n-1$ is even, $t=\frac{a(a+1)}{2}-1$, a is a nonnegative integer. This means that $a\equiv 0,3 \pmod 4$. There are four ways to embed J into the affine Dynkin diagram $\widetilde{D}_n$, two of them are ρ-stable, and the other two $\eta\rho$-stable. In all cases the nilpotent orbit $\mathbb{O}^\vee(c^\omega(J),\sigma)$ is parameterized by the partition

$$\lambda=\begin{cases}(3,3,7,7,\dots,2a-1,2a-1) & \text{if } a\equiv 0 \pmod 4,\\ (1,1,5,5,\dots,2a-1,2a-1) & \text{if } a\equiv 3 \pmod 4.\end{cases} \tag{7.1.5}$$

(ii) J is of the form $D_\ell\,{}^2A_t D_\ell$, where $2\ell+t=n-1$, $t=\frac{a(a+1)}{2}-1$ and $\ell=b^2$, a,b are nonnegative integers. Let a' be such that $a=2a'$ if a is even and $a=2a'+1$ if a is odd. Let $\Sigma=b+a'$ and

$$\delta=\begin{cases}b-a' & \text{if } 2b>a,\\ a'-b & \text{if } 2b\leq a.\end{cases}$$

The nilpotent orbit $\mathbb{O}^\vee(c^\omega(J),\sigma)$ is parameterized by the partition

$$\lambda=\begin{cases}(3,7,\dots,4\Sigma-1)\cup(1,5,\dots,4\delta-3) & \text{if } a \text{ is even and } 2b>a,\\ (3,7,\dots,4\Sigma-1)\cup(3,7,\dots,4\delta-1) & \text{if } a \text{ is even and } 2b\leq a,\\ (1,5,\dots,4\Sigma+1)\cup(3,7,\dots,4\delta-5) & \text{if } a \text{ is odd and } 2b>a,\\ (1,5,\dots,4\Sigma+1)\cup(1,5,\dots,4\delta+1) & \text{if } a \text{ is odd and } 2b\leq a.\end{cases}$$

7.2. ***Exceptional groups.***

7.2.1. G_2. If $\boldsymbol{G} = G_2$, then $G^\vee = G_2(\mathbb{C})$, and $Z(G^\vee) = \{1\}$. If $\omega = 1$ then J is of the form G_2 and there are four choices for σ as enumerated in Table 1. In all cases

$$\mathbb{O}^\vee(c^\omega(J), \sigma) = G_2(a_1).$$

7.2.2. F_4. If $\boldsymbol{G} = F_4$, then $G^\vee = F_4(\mathbb{C})$, and $Z(G^\vee) = \{1\}$. If $\omega = 1$ then J is of the form F_4 and there are seven choices for σ as enumerated in Table 1. In all cases

$$\mathbb{O}^\vee(c^\omega(J), \sigma) = F_4(a_3).$$

7.2.3. E_6. If $\boldsymbol{G} = E_6$, then $G^\vee = E_6(\mathbb{C})$, and $Z(G^\vee) = \{1, \zeta, \zeta^2\}$.

(1) If $\omega = 1$ then J is of the form E_6 and there are two choices for σ as enumerated in Table 1. In both cases

$$\mathbb{O}^\vee(c^\omega(J), \sigma) = D_4(a_1).$$

(2) If $\omega \in \{\zeta, \zeta^2\}$ then J is of the form ${}^3D_4$ and $\sigma = D_4[1]$ or $D_4[-1]$. In both cases

$$\mathbb{O}^\vee(c^\omega(J), \sigma) = E_6(a_3).$$

7.2.4. E_7. If $\boldsymbol{G} = E_7$, then $G^\vee = E_7(\mathbb{C})$, and $Z(G^\vee) = \{1, -1\}$.

(1) If $\omega = 1$ then J is of the form E_7 and there are two choices for σ as enumerated in Table 1. In both cases

$$\mathbb{O}^\vee(c^\omega(J), \sigma) = A_4 + A_1.$$

(2) If $\omega = -1$, then J is of the form ${}^2E_6$. There are three cuspidal unipotent representations afforded by J: ${}^2E_6[1]$, ${}^2E_6[\theta]$, ${}^2E_6[\theta^2]$. In all cases

$$\mathbb{O}^\vee(c^\omega(J), \sigma) = E_7(a_5).$$

7.2.5. E_8. If $\boldsymbol{G} = E_8$, then $G^\vee = E_8(\mathbb{C})$, and $Z(G^\vee) = \{1\}$. If $\omega = 1$ then J is of the form E_8 and there are thirteen choices for σ as enumerated in Table 1. In all cases

$$\mathbb{O}^\vee(c^\omega(J), \sigma) = E_8(a_7).$$

8. Proof of Proposition 5.0.1

8.1. ***Classical groups.*** In each case we show that

$$\bar{\mathbb{L}}(J, \mathrm{WF}(\sigma)) = d_A(\lambda, 1),$$

where λ is the partition parameterizing $\mathbb{O}^\vee(c^\omega(J), \sigma)$. We will use the machinery of [1, Section 3.4] to prove this equality.

8.1.1. $\mathrm{PGL}(n)$. Let $\omega \in \Omega \simeq C_n$ be of order n. Let $J = \varnothing$. Then $\sigma = \mathrm{triv}$, $\mathrm{WF}(\sigma) = \{0\}$, and $\mathbb{O}^\vee(c^\omega(\varnothing), \sigma)$ is the principal orbit $\mathbb{O}^\vee_{\mathrm{prin}}$. We need to show that

$$\bar{\mathbb{L}}(\varnothing, \{0\}) = d_A(\mathbb{O}^\vee_{\mathrm{prin}}, 1).$$

But both sides are equal to $(\{0\}, 1)$ and so we have equality.

8.1.2. $\mathrm{SO}(2n+1)$. Consider the cases $\omega = 1, -1$ simultaneously. Fix integers a, b as in Section 7.1.2 to fix J and hence σ. By Section 6

$$\mathrm{WF}(\sigma) = (1, 1, 3, 3, \dots, 2a-1, 2a-1) \times (1, 1, 3, 3, \dots, 2b-1, 2b-1, 2b+1).$$

Let δ, Σ, λ be as in Section 7.1.2. We have that

$$\begin{aligned}\lambda^t &= (\delta, \delta, \delta-1, \delta-1, \dots, 1, 1) \vee (\Sigma, \Sigma, \Sigma-1, \Sigma-1, \dots, 1, 1)\\ &= \begin{cases} (2b, 2b, 2b-2, 2b-2, \dots, 2a, 2a, 2a-1, 2a-1, \dots, 1, 1) & \text{if } b \geq a, \\ (2a-1, 2a-1, 2a-3, 2a-3, \dots, 2b+1, 2b+1, 2b, 2b, \dots, 1, 1) & \text{if } a > b, \end{cases}\end{aligned}$$

so $\pi(\lambda) = \varnothing$. We also have

$$d(\lambda) = (2b+1, 2b-1, 2b-1, \dots, 1, 1) \cup (2a-1, 2a-1, \dots, 1, 1).$$

Since $(1, 1, 3, 3, \dots, 2a-1, 2a-1)$ only has parts with even multiplicity,

$$\bar{\mathbb{L}}(J, \mathrm{WF}(\sigma)) = {}^{\langle (1,1,3,3,\dots,2a-1,2a-1) \rangle} d(\lambda) = {}^{\langle \varnothing \rangle} d(\lambda) = {}^{\langle \pi(\lambda) \rangle} d(\lambda) = d_A(\lambda, 1),$$

where $\pi(\lambda)$ is the subpartition of λ^t defined by Achar in [1, Equation 8].

8.1.3. $\mathrm{PSp}(2n)$.

(1) Let $\omega = 1$. Fix integers a, b as in Section 7.1.3(1) to fix J and hence σ. By Section 6

$$\mathrm{WF}(\sigma) = (2, 2, 4, 4, \dots, 2a, 2a) \times (2, 2, 4, 4, \dots, 2b, 2b).$$

Let δ, Σ, λ be as in Section 7.1.3(1). We have that

$$\begin{aligned}\lambda^t &= (\delta, \delta-1, \delta-1, \dots, 1, 1) \vee (\Sigma+1, \Sigma, \Sigma, \dots, 1, 1)\\ &= (2a+1, 2a-1, 2a-1, \dots, 2b+1, 2b+1, 2b, 2b, \dots, 1, 1)\end{aligned}$$

so $\pi(\lambda) = \varnothing$. We also have

$$d(\lambda) = (2a, 2a, \dots, 2b+2, 2b+2, 2b, 2b, 2b, 2b, \dots, 2, 2, 2, 2).$$

Since $(2, 2, 4, 4, \dots, 2a, 2a)$ only has parts with even multiplicity,

$$\bar{\mathbb{L}}(J, \mathrm{WF}(\sigma)) = {}^{\langle (2,2,4,4,\dots,2a,2a) \rangle} d(\lambda) = {}^{\langle \varnothing \rangle} d(\lambda) = {}^{\langle \pi(\lambda) \rangle} d(\lambda) = d_A(\lambda, 1).$$

(2) Let $\omega = -1$. Fix integers a, b as in Section 7.1.3(2) to fix J and hence σ. Then

$$\mathrm{WF}(\sigma) = (2, 2, 4, 4, \dots, 2b, 2b) \times (1, 2, \dots, a) \times (2, 2, 4, 4, \dots, 2b, 2b).$$

Let δ, Σ, λ be as in Section 7.1.3(2). We have that

$$\lambda^t = \begin{cases} (\Sigma+1, \Sigma^4, \dots, 1^4) \vee (\delta^3, (\delta-1)^4, \dots, 1^4) & \text{if } a \text{ is even and } 2b \geq a, \\ (\Sigma+1, \Sigma^4, \dots, 1^4) \vee (\delta, (\delta-1)^4, \dots, 1^4) & \text{if } a \text{ is even and } 2b < a, \\ ((\Sigma+1)^3, \Sigma^4, \dots, 1^4) \vee (\delta, (\delta-1)^4, \dots, 1^4) & \text{if } a \text{ is odd and } 2b \geq a, \\ ((\Sigma+1)^3, \Sigma^4, \dots, 1^4) \vee (\delta^3, (\delta-1)^4, \dots, 1^4) & \text{if } a \text{ is odd and } 2b < a, \end{cases} \tag{8.1.1}$$

$$= \begin{cases} (2b+1, (2b)^2, \dots, (a+1)^2, a^4, \dots, 1^4) & \text{if } a \text{ is even and } 2b \geq a, \\ (a+1, (a-1)^4, \dots, (2b+1)^4, (2b)^4, \dots, 1^4) & \text{if } a \text{ is even and } 2b < a, \\ (2b+1, (2b)^2, \dots, (a+1)^2, a^4, \dots, 1^4) & \text{if } a \text{ is odd and } 2b \geq a, \\ (a^3, (a-2)^4, \dots, (2b+1)^4, (2b)^4, \dots, 1^4) & \text{if } a \text{ is odd and } 2b < a. \end{cases} \tag{8.1.2}$$

Thus $\pi(\lambda) = \varnothing$ since all even parts of λ^t have even multiplicity. Moreover

$$d(\lambda) = (2, 2, 4, 4, \dots, 2b, 2b) \cup (1, 1, 2, 2, \dots, a, a) \cup (2, 2, 4, 4, \dots, 2b, 2b)$$

in all cases. Thus

$$\begin{aligned} \bar{\mathbb{L}}(J, \mathrm{WF}(\sigma)) &= \bar{\mathbb{L}}(\tilde{J}, (2, 2, \dots, 2b, 2b) \times (1, 1, \dots, a, a) \cup (2, 2, \dots, 2b, 2b)) \\ &= {}^{\langle (2,2,4,4,\dots,2b,2b) \rangle} d(\lambda) = {}^{\langle \varnothing \rangle} d(\lambda) = {}^{\langle \pi(\lambda) \rangle} d(\lambda) = d_A(\lambda, 1), \end{aligned}$$

where $\tilde{J} = C_l \times C_{t+1+l}$.

8.1.4. PSO$(2n)$.

(1) Let $\omega \in \{1, \eta\}$. Fix integers a, b as in Section 7.1.4(1) and (2) to fix J and hence σ. By Section 6

$$\mathrm{WF}(\sigma) = (1, 1, 3, 3, \dots, 2a-1, 2a-1) \times (1, 1, 3, 3, \dots, 2b-1, 2b-1).$$

Let δ, Σ, λ be as in Section 7.1.4(1). We have that

$$\begin{aligned} \lambda^t &= (\delta, \delta-1, \delta-1, \dots, 1, 1) \vee (\Sigma, \Sigma-1, \Sigma-1, \dots, 1, 1) \\ &= (2a, 2a-2, 2a-2, \dots, 2b, 2b, 2b-1, 2b-1, \dots, 1, 1) \end{aligned}$$

so $\pi(\lambda) = \varnothing$ since all odd parts have even multiplicity. We also have

$$d(\lambda) = (2a-1, 2a-1, \dots, 2b+1, 2b+1, 2b-1, 2b-1, 2b-1, 2b-1, \dots, 1, 1, 1, 1).$$

Since $(1, 1, 3, 3, \dots, 2a-1, 2a-1)$ only has parts with even multiplicity,

$$\bar{\mathbb{L}}(J, \mathrm{WF}(\sigma)) = {}^{\langle (1,1,3,3,\dots,2a-1,2a-1) \rangle} d(\lambda) = {}^{\langle \varnothing \rangle} d(\lambda) = {}^{\pi(\lambda)} d(\lambda) = d_A(\lambda, 1).$$

(2) Let $\omega \in \{\rho, \eta\rho\}$. We will treat the cases (i) and (ii) simultaneously. Fix integers a, b as in Section 7.1.4(3)(ii) to fix J and hence σ (we treat (i) as the case with $b = 0$). By Section 6

$$\mathrm{WF}(\sigma) = (1, 1, 3, 3, \dots, 2b-1, 2b-1) \times (1, 2, \dots, a) \times (1, 1, 3, 3, \dots, 2b-1, 2b-1).$$

Let δ, Σ, λ be as in Section 7.1.4(3)(ii). We have that

$$\lambda^t = \begin{cases} (\Sigma^3, (\Sigma-1)^4, \dots, 1^4) \vee (\delta, (\delta-1)^4, \dots, 1^4) & \text{if } a \text{ is even and } 2b \le a, \\ (\Sigma^3, (\Sigma-1)^4, \dots, 1^4) \vee (\delta^3, (\delta-1)^4, \dots, 1^4) & \text{if } a \text{ is even and } 2b \le a, \\ (\Sigma+1, \Sigma^4, \dots, 1^4) \vee (\delta, (\delta-1)^4, \dots, 1^4) & \text{if } a \text{ is odd and } 2b < a, \\ ((\Sigma+1)^3, \Sigma^4, \dots, 1^4) \vee (\delta^3, (\delta-1)^4, \dots, 1^4) & \text{if } a \text{ is odd and } 2b < a, \end{cases} \tag{8.1.3}$$

$$= \begin{cases} (2b+1, (2b)^2, \dots, (a+1)^2, a^4, \dots, 1^4) & \text{if } a \text{ is even and } 2b \ge a, \\ (a+1, (a-1)^4, \dots, (2b+1)^4, (2b)^4, \dots, 1^4) & \text{if } a \text{ is even and } 2b < a, \\ (2b+1, (2b)^2, \dots, (a+1)^2, a^4, \dots, 1^4) & \text{if } a \text{ is odd and } 2b \ge a, \\ (a^3, (a-2)^4, \dots, (2b+1)^4, (2b)^4, \dots, 1^4) & \text{if } a \text{ is odd and } 2b < a. \end{cases} \tag{8.1.4}$$

Thus $\pi(\lambda) = \varnothing$ since all even parts of λ^t have even multiplicity. Moreover

$$d(\lambda) = (2, 2, 4, 4, \dots, 2b, 2b) \cup (1, 1, 2, 2, \dots, a, a) \cup (2, 2, 4, 4, \dots, 2b, 2b)$$

in all cases. Thus

$$\begin{aligned} \bar{\mathbb{L}}(J, \mathrm{WF}(\sigma)) &= \bar{\mathbb{L}}(\tilde{J}, (1, 1, \dots, 2b-1, 2b-1) \times (1, 1, \dots, a, a) \cup (1, 1, \dots, 2b-1, 2b-1)) \\ &= {}^{\langle (2,2,4,4,\dots,2b,2b) \rangle} d(\lambda) = {}^{\langle \varnothing \rangle} d(\lambda) = {}^{\langle \pi(\lambda) \rangle} d(\lambda) = d_A(\lambda, 1), \end{aligned}$$

where $\tilde{J} = D_l \times D_{t+1+l}$.

8.2. *Exceptional groups.*

8.2.1. *Split forms.* Suppose that $\boldsymbol{G}$ is split, of exceptional type, and that $\omega = 1$. As can be seen in Section 7.2, J is always equal to Δ. Thus,

$$\bar{\mathbb{L}}(J, \mathrm{WF}(\sigma)) = (\mathrm{WF}(\sigma), 1).$$

On the other hand, the nilpotent orbit $\mathbb{O}^\vee := \mathbb{O}^\vee(c^\omega(J), \sigma)$ is always special. Thus,

$$d_A(\mathbb{O}^\vee, 1) = (d(\mathbb{O}^\vee), 1)$$

by the general properties of d_A; see [1, Section 3]. So for Proposition 5.0.1 it suffices to show that

$$\mathrm{WF}(\sigma) = d(\mathbb{O}^\vee)$$

for all σ. This follows by comparing the orbits in Table 1 and in Section 7.2.

8.2.2. *Nonsplit forms of E_6.* Suppose $\boldsymbol{G}$ is of type E_6 and $\omega \in \{\zeta, \zeta^2\}$. Then J is of the form ${}^3D_4$, and $\mathrm{WF}(\sigma) = (1, 1, 3, 3)$ for both $\sigma = D_4[1]$ and $\sigma = D_4[-1]$. The orbit $(1, 1, 3, 3)$ is the orbit A_2 in Bala–Carter notation. Thus we need to show that

$$\bar{\mathbb{L}}(J, (1, 1, 3, 3)) = d_A(E_6(a_3), 1).$$

We note that $E_6(a_3)$ is special and $d(E_6(a_3)) = A_2$ so we must show that

$$\bar{\mathbb{L}}(J, A_2) = (A_2, 1).$$

Since $J \subseteq \Delta$ this follows from [27, Proposition 2.30].

8.2.3. *Nonsplit forms of E_7.* Suppose $\boldsymbol{G}$ is of type E_7 and $\omega = -1$. Then J is of the form ${}^2E_6$, and $\mathrm{WF}(\sigma) = D_4(a_1)$ for all possible σ. Thus we need to show that

$$\bar{\mathbb{L}}(J, D_4(a_1)) = d_A(E_7(a_5), 1).$$

We note that $E_7(a_5)$ is special and $d(E_7(a_5)) = D_4(a_1)$ so we must show that

$$\bar{\mathbb{L}}(J, D_4(a_1)) = (D_4(a_1), 1).$$

Since $J \subseteq \Delta$ this follows from [27, Proposition 2.30].

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