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Rooted tree maps for multiple L-values from a perspective of harmonic algebras

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We show the image of rooted tree maps forms a subspace of the kernel of the evaluation map of multiple L-values. To prove this, we define the diamond product as a modified harmonic product and describe its properties. We also show that τ-conjugate rooted tree maps are their antipodes.

1. Introduction

In [8] the second author found the Connes–Kreimer's Hopf algebra of rooted trees $\mathcal{H}$ acts on $\mathfrak{H} = \mathbb{Q}\langle x, y\rangle$, the noncommutative polynomial ring in two indeterminates. We refer to the elements in $\mathrm{End}(\mathfrak{H})$ assigned to rooted trees as rooted tree maps. Rooted tree maps possess the rules coming from their coproducts. In particular, to primitive elements in $\mathcal{H}$, derivations in $\mathrm{End}(\mathfrak{H})$ are assigned. Rooted tree maps give rise to a broad class of relations (including duality, derivation relation, and Ohno's relation) among multiple zeta values. It is then shown that the class of relations coming from rooted tree maps is equivalent to the linear part of Kawashima's relation [2]. A pending issue is that the map assigned to the antipode of a rooted tree is nothing but the conjugation of the original map by τ, the antiautomorphism on $\mathfrak{H}$ characterized by interchanging x and y, which is shown in [7] by using additional algebraic properties of rooted tree maps and harmonic algebras.

The second and the third authors generalize the domain of such rooted tree maps so that they must induce a broad class of relations among multiple L-values [9]. Unlike the case of multiple zeta values, they only show that the maps assigned to the antipodes of rooted trees induce relations among multiple L-values. We show in this paper that their first prospect is true owing to further algebraic properties of rooted tree maps and harmonic algebras to establish the basics of rooted tree maps for multiple L-values.

To be more precise, let μ_r be the set of r-th roots of unity. For an index set $(\boldsymbol{k}; \boldsymbol{s}) = (k_1, \dots, k_l; s_1, \dots, s_l)$ with $k_1, \dots, k_l \geq 1$, $s_1, \dots, s_l \in \mu_r$, $(k_1, s_1) \neq (1, 1)$, the multiple L-value of shuffle type (abbreviated as MLV) is defined in [1] by the convergent series

$$L(\boldsymbol{k}; \boldsymbol{s}) = \lim_{m\to\infty} \sum_{m>m_1>\cdots>m_l>0} \frac{s_1^{m_1-m_2} \cdots s_{l-1}^{m_{l-1}-m_l} s_l^{m_l}}{m_1^{k_1} \cdots m_l^{k_l}}.$$

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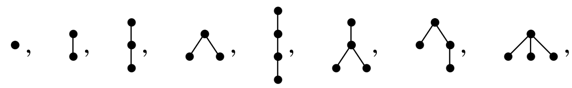
If $r = 1$, this is nothing but the multiple zeta value (abbreviated as MZV). The MZVs and the MLVs have been well-studied in the last three decades.

The index set $(\boldsymbol{k}; \boldsymbol{s})$ is often identified with the word $z_{\boldsymbol{k},\boldsymbol{s}} := z_{k_1,s_1} \cdots z_{k_l,s_l}$, where $z_{k,s}$ stands for $x^{k-1}y_s$, in the noncommutative polynomial algebra $\mathcal{A}_r := \mathbb{Q}\langle x, y_s \mid s \in \mu_r \rangle$. Then MLVs are algebraically discussed via the $\mathbb{Q}$-linear map $\mathcal{L} : \mathcal{A}_r^0 \to \mathbb{C}$ defined by $\mathcal{L}(1) = 1$ and $\mathcal{L}(z_{\boldsymbol{k},\boldsymbol{s}}) = L(\boldsymbol{k}; \boldsymbol{s})$. ($\mathcal{A}_r^0$ is a subalgebra of $\mathcal{A}_r$ generated by admissible words, detailed in the next section.)

On the other hand, (nonplanar) rooted trees are finite and connected graphs with no cycles and a special vertex called the root. For example,



and so on. The topmost vertex of each rooted tree represents the root. The algebra generated by them has a Hopf algebra structure known as the Connes–Kreimer Hopf algebra of rooted trees, which appeared in [4] by Arne Dür. (One can even trace it back to the work by J. Butcher in the 1960s.) In [3], it is used in the study of perturbative quantum field theory and is well-studied in the last quarter century.

Rooted tree maps (abbreviated as RTMs), first defined in [8] based on the Connes–Kreimer Hopf algebra of rooted trees, induce a certain class of relations among MZVs. In other words, a part of $\ker \mathcal{L}$ comes from the RTMs if $r = 1$. Although this phenomenon is expected to be extended naturally to any positive integer r, the only result proved in [9] is for RTMs taken conjugation by a certain involution τ. We study some algebraic properties of RTMs for MLVs using the harmonic algebra as are studied in [7] in the MZVs case. We then show the aforementioned expectation is true and τ-conjugate RTM is nothing but its antipode.

2. Main results

Let $\mathcal{A}_r^1$ and $\mathcal{A}_r^0$ be subalgebras of $\mathcal{A}_r$ given by

$$A_r \supset A_r^1 = \mathbb{Q} \oplus A_{r,+}^1 \supset A_r^0 = \mathbb{Q} \oplus A_{r,+}^0,$$

where

$$A_{r,+}^1 = \bigoplus_{s \in \mu_r} \mathcal{A}_r y_s, \quad A_{r,+}^0 = \bigoplus_{s \in \mu_r} x \mathcal{A}_r y_s \oplus \bigoplus_{\substack{s,t \in \mu_r \\ t \neq 1}} y_t \mathcal{A}_r y_s.$$

Each word $z_{\boldsymbol{k},\boldsymbol{s}} \in \mathcal{A}_{r,+}^0$ is called admissible and corresponds to the index set $(\boldsymbol{k}; \boldsymbol{s})$ with $(k_1, s_1) \neq (1, 1)$. Let $z = x + y_1$, $z_s^\delta = x + \delta(s) y_s \in \mathcal{A}_r$, where $\delta(1) = 0$ and $\delta(s) = 1$ if $s \neq 1$.

Denote by $\mathcal{H}$ the $\mathbb{Q}$-vector space generated by rooted forests, i.e., disjoint unions of rooted trees. This $\mathcal{H}$ has a structure of a connected Hopf algebra, which is briefly described in the next section. We assign to any rooted tree t a linear map $\tilde{t} \in \mathrm{End}_{\mathbb{Q}}(\mathcal{A}_r)$, which we call a RTM, elaborated in Section 4. The assignment $\tilde{\cdot}$ is known to be an algebra homomorphism, and hence we can assign to any $f \in \mathcal{H}$ a linear map $\tilde{f} \in \mathrm{End}_{\mathbb{Q}}(\mathcal{A}_r)$. Using the notation of the diamond product $\diamond_s$ ($s \in \mu_r$), which is described in Section 5, we have the following result.

Theorem 2.1. *For $f \in \mathcal{H}$, there exists a unique $F_f \in \mathcal{A}_1^1$ such that*

$$\tilde{f}(z_s^\delta w) = z_s^\delta (F_f \diamond_s w)$$

for any $s \in \mu_r$ and any $w \in \mathcal{A}_r$.

The product $\diamond_s$ is a variation of the harmonic product. Indeed, Proposition 5.4 below asserts that

$$v \diamond_s w = \psi_s(\varphi(v) * \psi_s^{-1}(w)), \tag{1}$$

where $v \in \mathcal{A}_1$, $w \in \mathcal{A}_r$, and $*$ is the harmonic product. Here, $\psi_s = \varphi \mathcal{I} M_s$, where φ is the automorphism on $\mathcal{A}_r$ determined by $\varphi(x) = z$ and $\varphi(y_s) = z_s^\delta - z (= \delta(s) y_s - y_1)$ for $s \in \mu_r$, and $\mathcal{I}$ and $M_s (s \in \mu_r)$ are linear maps on $\mathcal{A}_r$ defined by

$$\mathcal{I}(z_{\boldsymbol{k},\boldsymbol{s}} x^a) = z_{k_1,s_1} z_{k_2,s_1 s_2} \cdots z_{k_l,s_1\cdots s_l} x^a, \quad M_s(z_{\boldsymbol{k},\boldsymbol{s}} x^a) = z_{k_1,s s_1} z_{k_2,s_2} \cdots z_{k_l,s_l} x^a$$

for $a \geq 0$. Note that φ is an involution. According to [6], we have

$$z_s^\delta \cdot \psi_s(\mathcal{A}_{1,+}^1 * \mathcal{A}_{r,+}^1) \subset \ker \mathcal{L} \tag{2}$$

for any $s \in \mu_r$. Hence, for $s \in \mu_r$, $w \in \mathcal{A}_{r,+}^1$, and $f \in \mathrm{Aug}(\mathcal{H})$, where $\mathrm{Aug}(\mathcal{H})$ denotes the augmentation ideal of $\mathcal{H}$, i.e., $\mathcal{H} = \mathbb{Q} \oplus \mathrm{Aug}(\mathcal{H})$, we have

$$\tilde{f}(z_s^\delta w) = z_s^\delta (F_f \diamond_s w) = z_s^\delta \cdot \psi_s(\varphi(F_f) * \psi_s^{-1}(w)) \in \ker \mathcal{L}.$$

Thus we have the following:

Corollary 2.2. *For $f \in \mathrm{Aug}(\mathcal{H})$, we have $\tilde{f}(\mathcal{A}_{r,+}^0) \subset \ker \mathcal{L}$.*

Remark 2.3. This result was expected but not proved in [9]. Still we do not know the way to prove this directly from the definition of RTM (except that the case of $r = 1$, the MZV case, which is done in [8]).

Let S be the antipode of $\mathcal{H}$. Then, for $f \in \mathcal{H}$, we find that the antipode $\widetilde{S(f)}$ is described similarly by using the diamond product $\diamond_s$.

Theorem 2.4. *For $f \in \mathcal{H}$, there exists a unique $G_f \in \mathcal{A}_1^1$ such that*

$$\widetilde{S(f)}(z_s^\delta w) = z_s^\delta (G_f \diamond_s w)$$

for any $s \in \mu_r$ and any $w \in \mathcal{A}_r$.

As is defined in [9], let τ be the antiautomorphism on $\mathcal{A}_r$ defined by $\tau(x) = y_1$, $\tau(y_1) = x$, and $\tau(y_s) = -y_s$ $(s \neq 1)$. Note that τ is an involution. Then we show the following result, which is a generalization of [7, Theorem 1.5].

Theorem 2.5. *For $f \in \mathcal{H}$, we have $\widetilde{S(f)} = \tau \tilde{f} \tau$.*

Hence, for $s \in \mu_r$, $w \in \mathcal{A}^1_{r,+}$, and $f \in \mathrm{Aug}(\mathcal{H})$, we have

$$\tau \tilde{f} \tau(z_s^\delta w) = \widetilde{S(f)}(z_s^\delta w) = z_s^\delta (G_f \diamond_s w) = z_s^\delta \cdot \psi_s(\varphi(G_f) * \psi_s^{-1}(w)) \in \ker \mathcal{L}$$

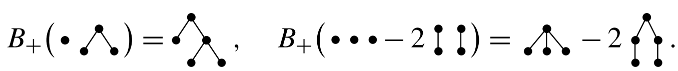
because of (1), (2), and Theorems 2.4 and 2.5. Thus again we have the following, proved first in [9, Theorem 2.4]:

Corollary 2.6. *For* $f \in \mathrm{Aug}(\mathcal{H})$, *we have* $\tau \tilde{f} \tau(\mathcal{A}^0_{r,+}) \subset \ker \mathcal{L}$.

3. Connes–Kreimer Hopf algebra of rooted trees

We briefly review the Connes–Kreimer Hopf algebra of rooted trees [3]. A rooted tree is a finite, connected, acyclic, and oriented graph with a special vertex called the root from which every edge directly or indirectly originates. A rooted forest is a product (disjoint union) of rooted trees. The empty forest (with no tree in it) denoted by $\mathbb{I}$ is the neutral element for the product. We denote by $\mathcal{T}$ the $\mathbb{Q}$-vector space freely generated by rooted trees.

As is mentioned in the previous section, we denote by $\mathcal{H}$ the $\mathbb{Q}$-algebra generated by rooted trees. As a vector space, $\mathcal{H}$ is freely generated by rooted forests. The $\mathbb{Q}$-linear map called the grafting operator $B_+ : \mathcal{H} \to \mathcal{T}$ is defined by $B_+(\mathbb{I}) = \bullet$ and, for a rooted forest f of positive degree, all the roots of connected components of f are grafted to a single new vertex, which becomes the new root. For example, we have



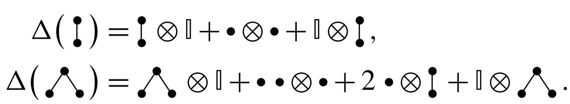
In particular, the map B_+ increases the degree of the graph by 1.

We define the coproduct Δ on $\mathcal{H}$ recursively by multiplicativity and

$$\Delta(t) = t \otimes \mathbb{I} + (\mathbb{I} \otimes B_+)\Delta(f) \tag{3}$$

for $t = B_+(f)$. In terms of Hochschild cohomology of bialgebras, the grafting operator B_+ satisfies the Hochschild 1-cocycle condition. For example, we have

$$\Delta(\mathbb{I}) = \mathbb{I} \otimes \mathbb{I},$$
$$\Delta(\bullet) = \bullet \otimes \mathbb{I} + \mathbb{I} \otimes \bullet,$$
$$\Delta(\bullet\,\bullet) = \bullet\,\bullet \otimes \mathbb{I} + 2\,\bullet \otimes \bullet + \mathbb{I} \otimes \bullet\,\bullet,$$



It is known that the coproduct Δ is coassociative but not cocommutative.

The counit $\hat{\mathbb{I}} : \mathcal{H} \to \mathbb{Q}$ is defined by vanishing on $\mathrm{Aug}(\mathcal{H})$ and $\hat{\mathbb{I}}(\mathbb{I}) = 1$. If we denote the product by $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$, we define the antipode S by the antiautomorphism on $\mathcal{H}$ satisfying

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \mathbb{I} \circ \hat{\mathbb{I}} = m \circ (\mathrm{id} \otimes S) \circ \Delta.$$

Then the tuple $(\mathcal{H}, m, \mathbb{I}, \Delta, \hat{\mathbb{I}}, S)$ forms a Hopf algebra known as the Connes–Kreimer Hopf algebra of rooted trees.

4. Rooted tree maps

We introduce rooted tree maps developed in [9]. Let the identity map on $\mathcal{A}_r$ be assigned to the empty forest $\mathbb{I}$, i.e., $\tilde{\mathbb{I}} = \mathrm{id}$. For any rooted forest f of positive degree, we define the $\mathbb{Q}$-linear map $\tilde{f} : \mathcal{A}_r \to \mathcal{A}_r$ by the following four conditions:

(I) if $f = \bullet$, $\tilde{f}(z_s^\delta) = z_s^\delta(z - z_s^\delta)$ and $\tilde{f}(z) = 0$,

(II) $\widetilde{B_+(f)}(z_s^\delta) = R_{z-z_s^\delta} R_{2z-z_s^\delta} R^{-1}_{z-z_s^\delta} \tilde{f}(z_s^\delta)$ and $\widetilde{B_+(f)}(z) = 0$,

(III) if $f = gh$, $\tilde{f}(v) = \tilde{g}(\tilde{h}(v))$ for $v \in \{z, z_s^\delta \mid s \in \mu_r\}$,

(IV) $\tilde{f}(wv) = M(\widetilde{\Delta(f)}(w \otimes v))$ for $w \in \mathcal{A}_r$, $v \in \{z, z_s^\delta \mid s \in \mu_r\}$,

where $s \in \mu_r$, R_w denotes the right multiplication map by w, that is, $R_w(v) = vw$ for $v, w \in \mathcal{A}_r$, $M : \mathcal{A}_r \otimes \mathcal{A}_r \to \mathcal{A}_r$ denotes the concatenation product, and $\widetilde{\Delta(f)} = \sum_{(f)} \widetilde{f'} \otimes \widetilde{f''}$ when $\Delta(f) = \sum_{(f)} f' \otimes f''$. As a matter of fact, the assignment $\tilde{\cdot} : \mathcal{H} \to \mathrm{End}_{\mathbb{Q}}(\mathcal{A}_r)$ is an algebra homomorphism. We find that $\tilde{f}(z_s^\delta)$ always ends with $z - z_s^\delta$, and hence the condition (II) is well-defined. We also find that the image $\tilde{f}(v)$ in the condition (III) does not depend on how to decompose f into g and h because of the commutativity property of RTMs which is proved by induction on graph order. We can also show the conditions (III) and (IV) hold for any $v \in \mathcal{A}_r$; see [8, Theorem 1.2] or [9, Theorem 2.2]. We call $\tilde{f}$ the RTM assigned to $f \in \mathcal{H}$.

Example 4.1 (calculations of images of RTMs). Since $\tilde{\bullet}(z_s^\delta) = z_s^\delta(z - z_s^\delta)$ and $\Delta(\bullet) = \bullet \otimes \mathbb{I} + \mathbb{I} \otimes \bullet$,

$$\widetilde{\bullet\bullet}(z_s^\delta) = \tilde{\bullet}(z_s^\delta(z - z_s^\delta)) = \tilde{\bullet}(z_s^\delta)(z - z_s^\delta) + z_s^\delta \tilde{\bullet}(z - z_s^\delta) = z_s^\delta(z - z_s^\delta)^2 + (z_s^\delta)^2(z - z_s^\delta).$$

Then we calculate

$$\widetilde{\Lambda}(z_s^\delta) = \widetilde{B_+(\bullet\,\bullet)}(z_s^\delta) = R_{z-z_s^\delta} R_{2z-z_s^\delta} R^{-1}_{z-z_s^\delta} \widetilde{\bullet\bullet}(z_s^\delta) = z_s^\delta(z-z_s^\delta)(2z-z_s^\delta)(z-z_s^\delta) + (z_s^\delta)^2(2z-z_s^\delta)(z-z_s^\delta).$$

5. Harmonic product and diamond product

The harmonic product $* : \mathcal{A}_r \times \mathcal{A}_r \to \mathcal{A}_r$ is defined by $\mathbb{Q}$-bilinearity and

(I) $1 * w = w * 1 = w$,

(II) $vy_s * wy_t = (v * wy_t)y_s + (vy_s * w)y_t + (v * w)xy_{st}$,

(III) $vx * w = v * wx = (v * w)x$

for $v, w \in \mathcal{A}_r$, $s, t \in \mu_r$. It is associative and commutative. The tuples $(\mathcal{A}_r^1, *)$ and $(\mathcal{A}_r^0, *)$ are subalgebras of $(\mathcal{A}_r, *)$. The composition $\mathcal{L}\mathcal{I}$ is known as the evaluation map of MLVs of harmonic type. It is an algebra homomorphism with respect to $*$ (see [1]).

Lemma 5.1. *For $k, l \geq 1$, $s, t \in \mu_r$, and $v, w \in \mathcal{A}_r$, we have*

(i) $vz_{k,s} * wz_{l,t} = (v * wz_{l,t})z_{k,s} + (vz_{k,s} * w)z_{l,t} + (v * w)z_{k+l,st}$,

(ii) $z_{k,s}v * z_{l,t}w = z_{k,s}(v * z_{l,t}w) + z_{l,t}(z_{k,s}v * w) + z_{k+l,st}(v * w)$.

Proof. Because of the condition (III), it is enough to show when $v, w \in \mathcal{A}_r^1$.

To show (i), substitute vx^{k-1} and wx^{l-1} into v and w, respectively, in the condition (II) and then use the condition (III).

We show (ii) by induction on total degree of words. If $v = w = 1$, it follows from (i) for $v = w = 1$.

If $v = v'z_{m,a}$ ($v' \in \mathcal{A}_r^1$, $m \geq 1$, $a \in \mu_r$) and $w = 1$, the left-hand side equals

$$(z_{k,s}v' * z_{l,t})z_{m,a} + z_{k,s}vz_{l,t} + z_{k,s}v'z_{l+m,ta} \tag{4}$$

because of (i). The first term turns into

$$(z_{k,s}(v' * z_{l,t}) + z_{l,t}z_{k,s}v' + z_{k+l,st}v')z_{m,a}$$

by induction, and hence we have

$$(4) = z_{k,s}((v' * z_{l,t})z_{m,a} + vz_{l,t} + v'z_{l+m,ta}) + z_{l,t}z_{k,s}v + z_{k+l,st}v.$$

Again by (i), we see that this coincides with the right-hand side. The proof goes similarly if $v = 1$ and $w = w'z_{n,b}$ ($w' \in \mathcal{A}_r$, $n \geq 1$, $b \in \mu_r$).

If $v = v'z_{m,a}$ and $w = w'z_{n,b}$, the left-hand side equals

$$(z_{k,s}v' * z_{l,t}w)z_{m,a} + (z_{k,s}v * z_{l,t}w')z_{n,b} + (z_{k,s}v' * z_{l,t}w')z_{m+n,ab}$$

because of (i). This turns into

$$\begin{aligned}(z_{k,s}(v' * z_{l,t}w) &+ z_{l,t}(z_{k,s}v' * w) + z_{k+l,st}(v' * w))z_{m,a} \\ &+ (z_{k,s}(v * z_{l,t}w') + z_{l,t}(z_{k,s}v * w') + z_{k+l,st}(v * w'))z_{n,b} \\ &+ (z_{k,s}(v' * z_{l,t}w') + z_{l,t}(z_{k,s}v' * w') + z_{k+l,st}(v' * w'))z_{m+n,ab}\end{aligned}$$

by induction. Again by (i), we see that this coincides with the right-hand side. □

From now on, let $y = y_1$ for simplicity. For $s \in \mu_r$, we define the $\mathbb{Q}$-bilinear map $\diamond_s : \mathcal{A}_1 \times \mathcal{A}_r \to \mathcal{A}_r$ by

$$\begin{aligned}1 \diamond_s w &= w, \\ v \diamond_s 1 &= \psi_s\varphi(v), \\ vx \diamond_s wx &= (v \diamond_s wx)x - (vy \diamond_s w)x, \\ vy \diamond_s wx &= (v \diamond_s wx)y + (vy \diamond_s w)x, \\ vx \diamond_s wy &= (v \diamond_s wy)x + (vx \diamond_s w)y, \\ vy \diamond_s wy &= (v \diamond_s wy)y - (vx \diamond_s w)y, \\ vx \diamond_s wy_t &= (v \diamond_s wy_t)x + (v \diamond_s wz_t)y_t - (vy \diamond_s w)y_t, \\ vy \diamond_s wy_t &= (v \diamond_s wy_t)y - (v \diamond_s wz_t)y_t + (vy \diamond_s w)y_t,\end{aligned} \tag{5}$$

for $v \in \mathcal{A}_1$, $w \in \mathcal{A}_r$ and $1 \neq t \in \mu_r$. When $r = 1$, the product $\diamond_1$ corresponds to the one defined in [5] and is commutative. In general, 1 is the left unit but not the right unit. For example, one checks $y \diamond_s 1 = z - z_s^\delta$.

Lemma 5.2. *For $s \in \mu_r$, $v \in \mathcal{A}_1$, and $w \in \mathcal{A}_r$, we have*

$$vz \diamond_s w = v \diamond_s wz = (v \diamond_s w)z.$$

Proof. By definition, we easily see $vz \diamond_s w = (v \diamond_s w)z$.

We prove $v \diamond_s wz = (v \diamond_s w)z$ for words v, w by induction on $d = \deg(v)$. It is obvious if $d = 0$. Assume $d \geq 1$. If $v = v'x$, by definition (in particular, adding the third and fifth identities in (5)), we have

$$\begin{aligned} v'x \diamond_s wz &= (v' \diamond_s wz)x - (v'y \diamond_s w)x + (v'x \diamond_s w)y \\ &= (v' \diamond_s wz)x - (v'z \diamond_s w)x + (v'x \diamond_s w)z. \end{aligned}$$

By the induction hypothesis, the first two terms cancel out, and hence we obtain the assertion. The proof goes similarly when $v = v'y$. □

Lemma 5.3. *For $s, t \in \mu_r$, $v \in \mathcal{A}_1$, and $w \in \mathcal{A}_r$, we have*

(i) $vx \diamond_s wz_t^\delta = (v \diamond_s wz_t^\delta)z_t^\delta - (vy \diamond_s w)z_t^\delta$,

(ii) $vy \diamond_s wz_t^\delta = (v \diamond_s wz_t^\delta)(y \diamond_t 1) + (vy \diamond_s w)z_t^\delta$.

Proof. By the third and seventh identities in (5), we have (i). By (i), Lemma 5.2, and $y \diamond_t 1 = z - z_t^\delta$, we have (ii). □

We put $z_s = x + y_s$ for simplicity (and hence $z_1 = z$). Note that $\varphi(z_s) = z_s^\delta$.

Proposition 5.4. *For $s \in \mu_r$, $v \in \mathcal{A}_1$, and $w \in \mathcal{A}_r$, we have*

$$v \diamond_s w = \psi_s\big(\varphi(v) * \psi_s^{-1}(w)\big).$$

Proof. If $v = 1$ or $w = 1$, it is obvious. Otherwise, the proof goes by induction on $\deg(v) + \deg(w)$.

If $v = v'z$, by definitions, the right-hand side turns into

$$\psi_s(\varphi(v'z) * \psi_s^{-1}(w)) = \psi_s(\varphi(v')x * \psi_s^{-1}(w)) = \psi_s\big((\varphi(v') * \psi_s^{-1}(w))x\big) = \psi_s(\varphi(v') * \psi_s^{-1}(w))z.$$

Then, by the induction hypothesis, this equals $(v' \diamond_s w)z$, which equals the left-hand side because of Lemma 5.2.

Similarly, if $w = w'z$, the right-hand side turns into

$$\psi_s(\varphi(v) * \psi_s^{-1}(w'z)) = \psi_s(\varphi(v) * \psi_s^{-1}(w')x) = \psi_s\big((\varphi(v) * \psi_s^{-1}(w'))x\big) = \psi_s(\varphi(v) * \psi_s^{-1}(w'))z,$$

which equals the left-hand side.

To complete the proof, we show when $v = v'x$ and $w = w'z_t^\delta$. In this case, the right-hand side turns into

$$\psi_s(\varphi(v')z * \psi_s^{-1}(w'z_t^\delta)). \tag{6}$$

Without loss of generality, suppose $\varphi(w') = z_{k_1,t_1} \cdots z_{k_n,t_n}$. Then, by definitions, we find

$$\psi_s^{-1}(w' z_t^{\delta}) = \psi_s^{-1}(w') z_{t/t_n}.$$

Hence, we have

$$(6) = \psi_s\big((\varphi(v')y * \psi_s^{-1}(w'))z_{t/t_n} + (\varphi(v') * \psi_s^{-1}(w')y_{t/t_n})z + (\varphi(v') * \psi_s^{-1}(w'))xz_{t/t_n}\big). \tag{7}$$

Since $\varphi(v') \in \mathcal{A}_1$, $\psi_s^{-1}(w') = z_{k_1,t_1/s} z_{k_2,t_2/t_1} \cdots z_{k_n,t_n/t_{n-1}}$, and the harmonic product has combinatorial meaning of overlapping shuffle, the subscript of 'y' in the last z_{t/t_n} or z changes into

$$s \times \left(\frac{t_1}{s} \times \frac{t_2}{t_1} \times \cdots \times \frac{t_n}{t_{n-1}}\right) \times \frac{t}{t_n} = t \quad \text{or} \quad s \times \left(\frac{t_1}{s} \times \frac{t_2}{t_1} \times \cdots \times \frac{t_n}{t_{n-1}} \times \frac{t}{t_n}\right) = t,$$

respectively, after the map ψ_s applies. Therefore we have

$$\begin{aligned}(7) &= \psi_s\big(\varphi(v')y * \psi_s^{-1}(w') + \varphi(v') * \psi_s^{-1}(w')y_{t/t_n} + (\varphi(v') * \psi_s^{-1}(w'))x\big)z_t^{\delta} \\ &= \psi_s(-\varphi(v'y) * \psi_s^{-1}(w') + \varphi(v') * \psi_s^{-1}(w'z_t^{\delta}))z_t^{\delta} \\ &= (-v'y \diamond_s w' + v' \diamond_s w'z_t^{\delta})z_t^{\delta}.\end{aligned}$$

The last equality is by the induction hypothesis. By Lemma 5.3(i), this coincides with the left-hand side. □

Lemma 5.5. *The product $\diamond_s$ gives a left $\mathcal{A}_1$-module structure to $\mathcal{A}_r$ for any $s \in \mu_r$.*

Proof. For $s \in \mu_r$, $u, v \in \mathcal{A}_1$ and $w \in \mathcal{A}_r$, We have

$$\begin{aligned}(u \diamond_1 v) \diamond_s w &= \big(\varphi(\varphi(u) * \varphi(v))\big) \diamond_s w \\ &= \psi_s\big((\varphi(u) * \varphi(v)) * \psi_s^{-1}(w)\big) \\ &= \psi_s\big(\varphi(u) * (\varphi(v) * \psi_s^{-1}(w))\big) \\ &= \psi_s\big(\varphi(u) * (\psi_s^{-1}(v \diamond_s w))\big) = u \diamond_s (v \diamond_s w)\end{aligned}$$

by Proposition 5.4 and the associativity of $*$. □

Lemma 5.6. *For $s, t \in \mu_r$ and $v, w \in \mathcal{A}_r$, we have*

$$y \diamond_s vz_t^{\delta}w = (y \diamond_s v)z_t^{\delta}w + vz_t^{\delta}(y \diamond_s w) + vz_t^{\delta}(z_s^{\delta} - z_t^{\delta})w.$$

Proof. We prove the lemma by induction on $\deg(w)$. When $w = 1$, we have

$$y \diamond_s vz_t^{\delta} = (y \diamond_s v)z_t^{\delta} + (1 \diamond_s vz_t^{\delta})(y \diamond_t 1) \tag{8}$$

by Lemma 5.3(ii). Since $y \diamond_t 1 = y \diamond_s 1 + (z_s^{\delta} - z_t^{\delta})$, we have

$$(8) = (y \diamond_s v)z_t^{\delta} + vz_t^{\delta}(y \diamond_s 1) + vz_t^{\delta}(z_s^{\delta} - z_t^{\delta})$$

and the assertion. If $w = w'z$ ($w' \in \mathcal{A}_r$), by the induction hypothesis and Lemma 5.2, we have

$$\begin{aligned}\text{L.H.S.} &= (y \diamond_s vz_t^{\delta}w')z \\ &= (y \diamond_s v)z_t^{\delta}w'z + vz_t^{\delta}(y \diamond_s w')z + vz_t^{\delta}(z_s^{\delta} - z_t^{\delta})w'z = \text{R.H.S.}\end{aligned}$$

If $w = w' z_{t'}^{\delta}$ $(w' \in \mathcal{A}_r)$, by Lemma 5.3(ii) and the induction hypothesis, we have

$$\begin{aligned}
\text{L.H.S.} &= (1 \diamond_s v z_t^{\delta} w' z_{t'}^{\delta})(y \diamond_{t'} 1) + (y \diamond_s v z_t^{\delta} w') z_{t'}^{\delta} \\
&= (1 \diamond_s v z_t^{\delta} w' z_{t'}^{\delta})(y \diamond_{t'} 1) + (y \diamond_s v) z_t^{\delta} w + v z_t^{\delta} (y \diamond_s w') z_{t'}^{\delta} + v z_t^{\delta} (z_s^{\delta} - z_t^{\delta}) w \\
&= \text{R.H.S.}
\end{aligned}$$

This finishes the proof. □

Now write $R = R_y R_{x+2y} R_y^{-1}$. For rooted forests f, we define polynomials $F_f \in \mathcal{A}_1^1$ recursively by

- $F_{\mathbb{I}} = 1$,
- $F_{\bullet} = y$,
- $F_t = R(F_f)$ if $t = B_+(f)$ and $f \neq \mathbb{I}$,
- $F_f = F_g \diamond_1 F_h$ if $f = gh$.

The subscript of F is extended linearly.

Proposition 5.7. *For $f \in \mathcal{H}$, put $\Delta(f) = \sum_{(f)} f' \otimes f''$. Then, for $s, s' \in \mu_r$ and $v, w \in \mathcal{A}_r$, we have*

$$F_f \diamond_s v z_{s'}^{\delta} w = \sum_{(f)} (F_{f'} \diamond_s v) z_{s'}^{\delta} (F_{f''} \diamond_{s'} w).$$

Proof. It is enough to consider the case that f is a monomial, i.e., a rooted forest. If $f = \mathbb{I}$, it is obvious. If $f = \bullet$, by Lemma 5.6, we find the proposition holds.

Assume $\deg(f) \geq 2$ and the proposition holds for any elements in $\mathcal{H}$ of degree less than $\deg(f)$. If $f = gh$ $(g, h \neq \mathbb{I})$, we have

$$F_f \diamond_s v z_{s'}^{\delta} w = (F_g \diamond_1 F_h) \diamond_s v z_{s'}^{\delta} w = F_g \diamond_s (F_h \diamond_s v z_{s'}^{\delta} w) \tag{9}$$

because of Lemma 5.5. Since $\deg(g), \deg(h) < \deg(f)$, we have

$$(9) = \sum_{(h)} F_g \diamond_s (F_{h'} \diamond_s v) z_{s'}^{\delta} (F_{h''} \diamond_{s'} w) = \sum_{(g)} \sum_{(h)} (F_{g'} \diamond_s (F_{h'} \diamond_s v)) z_{s'}^{\delta} (F_{g''} \diamond_{s'} (F_{h''} \diamond_{s'} w)) \tag{10}$$

by the induction hypothesis. Again by Lemma 5.5, we have

$$(10) = \sum_{(g)} \sum_{(h)} ((F_{g'} \diamond_1 F_{h'}) \diamond_s v) z_{s'}^{\delta} ((F_{g''} \diamond_1 F_{h''}) \diamond_{s'} w) = \sum_{(f)} (F_{f'} \diamond_s v) z_{s'}^{\delta} (F_{f''} \diamond_{s'} w),$$

and hence the assertion.

If f is a tree and $f = B_+(g)$, we have $F_f = R(F_g)$. In this case, the proof goes inductively on $\deg(w)$. When $w = 1$, we have

$$\begin{aligned}
F_f \diamond_s v z_{s'}^{\delta} &= R(F_g) \diamond_s v z_{s'}^{\delta} \\
&= (R_y^{-1}(F_g) x + 2F_g) y \diamond_s v z_{s'}^{\delta} \\
&= (F_f \diamond_s v) z_{s'}^{\delta} + ((R_y^{-1}(F_g) x + 2F_g) \diamond_s v z_{s'}^{\delta})(z - z_{s'}^{\delta})
\end{aligned} \tag{11}$$

because of Lemma 5.3. Since $\deg(g) < \deg(f)$, we have

$$F_g \diamond_s vz^{\delta}_{s'} = \sum_{(g)} (F_{g'} \diamond_s v) z^{\delta}_{s'} (F_{g''} \diamond_{s'} 1)$$

by the induction hypothesis. Then we have

$$\begin{aligned}(11) &= (F_f \diamond_s v) z^{\delta}_{s'} + (R_y^{-1}(F_g)x \diamond_s vz^{\delta}_{s'})(z - z^{\delta}_{s'}) + 2 \sum_{(g)} (F_{g'} \diamond_s v) z^{\delta}_{s'} (F_{g''} \diamond_{s'} 1)(z - z^{\delta}_{s'}) \\ &= (F_f \diamond_s v) z^{\delta}_{s'} + ((R_y^{-1}(F_g) \diamond_s vz^{\delta}_{s'}) z^{\delta}_{s'} - (F_g \diamond_s v) z^{\delta}_{s'})(z - z^{\delta}_{s'}) \\ &\quad + 2(F_g \diamond_s v) z^{\delta}_{s'} (y \diamond_{s'} 1) + \sum_{\substack{(g) \\ g'' \neq \mathbb{I}}} (F_{g'} \diamond_s v) z^{\delta}_{s'} ((R(F_{g''}) - R_y^{-1}(F_{g''})xy) \diamond_{s'} 1) \qquad (12)\end{aligned}$$

because of Lemma 5.3(i), $y \diamond_{s'} 1 = z - z^{\delta}_{s'}$, and

$$(F_{g''} \diamond_{s'} 1)(z - z^{\delta}_{s'}) = F_{g''} y \diamond_{s'} 1 = \tfrac{1}{2}(R(F_{g''}) - R_y^{-1}(F_{g''})xy) \diamond_{s'} 1.$$

We find

$$\begin{aligned}\sum_{(f)} (F_{f'} \diamond_s v) z^{\delta}_{s'} (F_{f''} \diamond_{s'} 1) &= \sum_{\substack{c(f) \\ f'' \neq \mathbb{I}}} (F_{f'} \diamond_s v) z^{\delta}_{s'} (F_{f''} \diamond_{s'} 1) + (F_f \diamond_s v) z^{\delta}_{s'} \\ &= \sum_{(g)} (F_{g'} \diamond_s v) z^{\delta}_{s'} (F_{B_+(g'')} \diamond_{s'} 1) + (F_f \diamond_s v) z^{\delta}_{s'} \\ &= \sum_{\substack{(g) \\ g'' \neq \mathbb{I}}} (F_{g'} \diamond_s v) z^{\delta}_{s'} (R(F_{g''}) \diamond_{s'} 1) + (F_g \diamond_s v) z^{\delta}_{s'} (y \diamond_{s'} 1) + (F_f \diamond_s v) z^{\delta}_{s'}\end{aligned}$$

because of (3). Therefore we have

$$(12) = \sum_{(f)} (F_{f'} \diamond_s v) z^{\delta}_{s'} (F_{f''} \diamond_{s'} 1) + (R_y^{-1}(F_g) \diamond_s vz^{\delta}_{s'}) z^{\delta}_{s'} (z - z^{\delta}_{s'}) - \sum_{\substack{(g) \\ g'' \neq \mathbb{I}}} (F_{g'} \diamond_s v) z^{\delta}_{s'} (R_y^{-1}(F_{g''})xy \diamond_{s'} 1). \qquad (13)$$

We now see that the second and third terms in (13) cancel out. To see this, we need to show

$$\sum_{\substack{(g) \\ g'' \neq \mathbb{I}}} (F_{g'} \diamond_s v) z^{\delta}_{s'} (R_y^{-1}(F_{g''}) \diamond_{s'} 1) = R_y^{-1}(F_g) \diamond_s vz^{\delta}_{s'}$$

because of $R_y^{-1}(F_{g''})xy \diamond_{s'} 1 = (R_y^{-1}(F_{g''}) \diamond_{s'} 1) z^{\delta}_{s'} (z - z^{\delta}_{s'})$. By $R_y^{-1}(F_{g''}) \diamond_{s'} 1 = R^{-1}_{z - z^{\delta}_{s'}} (F_{g''} \diamond_{s'} 1)$, the induction hypothesis, and Lemma 5.3, we have

$$\begin{aligned}\sum_{\substack{(g) \\ g'' \neq \mathbb{I}}} (F_{g'} \diamond_s v) z^{\delta}_{s'} (R_y^{-1}(F_{g''}) \diamond_{s'} 1) &= R^{-1}_{z - z^{\delta}_{s'}} \left(\sum_{\substack{(g) \\ g'' \neq \mathbb{I}}} (F_{g'} \diamond_s v) z^{\delta}_{s'} (F_{g''} \diamond_{s'} 1) \right) = R^{-1}_{z - z^{\delta}_{s'}} (F_g \diamond_s vz^{\delta}_{s'} - (F_g \diamond_{s'} v) z^{\delta}_{s'}) \\ &= R^{-1}_{z - z^{\delta}_{s'}} (R_y^{-1}(F_g) \diamond_s vz^{\delta}_{s'})(z - z^{\delta}_{s'}) = R_y^{-1}(F_g) \diamond_s vz^{\delta}_{s'}.\end{aligned}$$

Thus we conclude

$$(13)=\sum_{(f)}(F_{f'}\diamond_s v)z_{s'}^{\delta}(F_{f''}\diamond_{s'}1).$$

Now we proceed to the case when $\deg(w)\geq 1$. If $w=w'z$ ($w'\in\mathcal{A}_r$), we have

$$F_f\diamond_s vz_{s'}^{\delta}w=(F_f\diamond_s vz_{s'}^{\delta}w')z=\sum_{(f)}(F_{f'}\diamond_s v)z_{s'}^{\delta}(F_{f''}\diamond_{s'}w')z=\sum_{(f)}(F_{f'}\diamond_s v)z_{s'}^{\delta}(F_{f''}\diamond_{s'}w)$$

by Lemma 5.2 and the induction hypothesis. If $w=w'z_{s''}^{\delta}$ ($w'\in\mathcal{A}_r$, $s''\in\mu_r$), since we have already proved the identity when $w=1$, we have

$$\begin{aligned}F_f\diamond_s vz_{s'}^{\delta}w&=\sum_{(f)}(F_{f'}\diamond_s vz_{s'}^{\delta}w')z_{s''}^{\delta}(F_{f''}\diamond_{s''}1)\\&=\sum_{(f)}\sum_{(f')}(F_{f'_a}\diamond_s v)z_{s'}^{\delta}(F_{f'_b}\diamond_{s'}w)z_{s''}^{\delta}(F_{f''}\diamond_{s''}1),\end{aligned}$$

where we put $\Delta(f')=\sum_{(f')}f'_a\otimes f'_b$. We also have

$$\begin{aligned}\sum_{(f)}(F_{f'}\diamond_s v)z_{s'}^{\delta}(F_{f''}\diamond_{s'}w)&=\sum_{(f)}(F_{f'}\diamond_s v)z_{s'}^{\delta}(F_{f''}\diamond_{s'}w'z_{s''}^{\delta})\\&=\sum_{(f)}\sum_{(f'')}(F_{f'}\diamond_s v)z_{s'}^{\delta}(F_{f''_a}\diamond_{s'}w')z_{s''}^{\delta}(F_{f''_b}\diamond_{s''}1),\end{aligned}$$

where we put $\Delta(f'')=\sum_{(f'')}f''_a\otimes f''_b$. By coassociativity of Δ, these two coincide, and hence we have conclusion. □

The following property plays an important role in our proof of Theorem 2.5 in Section 8.

Proposition 5.8. *For $s,t\in\mu_r$, $v\in\mathcal{A}_1$, and $w\in\mathcal{A}_r$, we have*

$$z_s^{\delta}(vy\diamond_s w(z-z_t^{\delta}))=-\tau(\tau(v)y\diamond_t\tau(w)(z-z_s^{\delta}))(z-z_t^{\delta}).$$

Proof. By Lemmas 5.3(ii) and 5.9(ii), it is equivalent to show the identity

$$vy\diamond_s w-v\diamond_s wz_t^{\delta}=\tau(\tau(v)\diamond_t\tau(w)z_s^{\delta}-\tau(v)y\diamond_t\tau(w)). \tag{14}$$

We prove this by induction on $\deg(v)+\deg(w)$. We consider five cases.

Case 1: $v=1$. If $w=1,z,z-z_u^{\delta}$ ($u\in\mu_r$), we calculate that both sides turn into

$$z-z_s^{\delta}-z_t^{\delta},\quad z(z-z_t^{\delta})-z_s^{\delta}z,\quad (z-z_s^{\delta}-z_u^{\delta})(z-z_u^{\delta})-(z-z_u^{\delta})z_t^{\delta},$$

respectively. If $w=w'z$, we calculate

$$\begin{aligned}\tau(\text{R.H.S.})&=z\tau(w')z_s^{\delta}-(y\diamond_t 1)(1\diamond_t z\tau(w'))-z(y\diamond_t\tau(w'))+z(y\diamond_t 1)(1\diamond_t\tau(w'))\\&=\tau((y\diamond_s w')z)-\tau(w'zz_t^{\delta})=\tau(\text{L.H.S.})\end{aligned}$$

by Lemma 5.9(iii) and the induction hypothesis. If $w=w'(z-z_u^\delta)$, we have

$$\text{L.H.S.}=(y\diamond_s w')(z-z_u^\delta)-w'z_u^\delta(z-z_u^\delta)-w'(z-z_u^\delta)z_t^\delta$$

by Lemma 5.9(ii), and

$$\begin{aligned}\tau(\text{R.H.S.})&=z_u^\delta\tau(w')z_s^\delta-(y\diamond_t 1)z_u^\delta\tau(w')-z_u^\delta(y\diamond_u\tau(w'))\\&=z_u^\delta\tau(y\diamond_s w'-1\diamond_s w'z_u^\delta)-(y\diamond_t 1)z_u^\delta\tau(w')\end{aligned}$$

by Lemma 5.9(iv) and the induction hypothesis. Thus (14) holds.

Case 2: $v=z$. If $w=1,z,z-z_u^\delta$, we calculate that both sides turn into

$$z^2-zz_s^\delta-z_t^\delta z,\quad z(z-z_s^\delta-z_t^\delta)z,\quad (z^2-zz_s^\delta-z_u^\delta z)(z-z_u^\delta)-(z-z_u^\delta)z_t^\delta z,$$

respectively. If $w=w'z$, we calculate

$$\begin{aligned}\tau(\text{R.H.S.})&=z(z\diamond_t\tau(w')z_s^\delta)-z\big(y\diamond_t\tau(w)+zy\diamond_t\tau(w')-z(y\diamond_t\tau(w'))\big)\\&=z\tau(zy\diamond_s w'-z\diamond_s w'z_t^\delta)-z\big(y\diamond_t\tau(w)-z(y\diamond_t\tau(w'))\big)\end{aligned}$$

by Lemma 5.9(v) and the induction hypothesis. Applying Lemma 5.9(iii) to the third term, we find that this is $\tau(\text{L.H.S.})$. Note that

$$xv\diamond_s z_t^\delta w=z_s^\delta(v\diamond_s z_t^\delta w)+z_t^\delta(xv\diamond_t w)-zz_t^\delta(v\diamond_t w)\tag{15}$$

by Lemma 5.9(iv) and (vi). If $w=w'(z-z_u^\delta)$, we have

$$\begin{aligned}\tau(\text{R.H.S.})&=z_u^\delta(z\diamond_u\tau(w')z_s^\delta)-z(y\diamond_t z_u^\delta\tau(w'))-z_u^\delta(zy\diamond_u\tau(w'))+zz_u^\delta(y\diamond_u\tau(w'))\\&=z_u^\delta\tau(zy\diamond_s w'-z\diamond_s w'z_u^\delta)-z(y\diamond_t 1)(1\diamond_t z_u^\delta\tau(w'))\end{aligned}$$

by (15) and the induction hypothesis. This is $\tau(\text{L.H.S.})$ because of Lemma 5.9(ii). Thus (14) holds.

Case 3: $v=y$. If $w=1,z,z-z_u^\delta$, we calculate that both sides turn into

$$\begin{aligned}&(z-z_s^\delta-z_t^\delta)(z-z_t^\delta)-(z-z_s^\delta)z_s^\delta,\quad (z-z_s^\delta)^2z-(z-z_s^\delta)zz_t^\delta-zz_t^\delta(z-z_t^\delta),\\&(z-z_s^\delta-z_u^\delta)(z-z_u^\delta)(z-z_t^\delta-z_u^\delta)-(z-z_s^\delta)z_s^\delta(z-z_u^\delta)-(z-z_u^\delta)z_t^\delta(z-z_t^\delta),\end{aligned}$$

respectively. Note that

$$xv\diamond_s zw=z_s^\delta(v\diamond_s zw)+z(xv\diamond_s w)-zz_s^\delta(v\diamond_s w)\tag{16}$$

by Lemma 5.9(iii) and (v). If $w=w'z$, we calculate

$$\begin{aligned}\tau(\text{R.H.S.})&=z_t^\delta(1\diamond_t z\tau(w')z_s^\delta)+z(x\diamond_t\tau(w')z_s^\delta)-zz_t^\delta(1\diamond_t\tau(w')z_s^\delta)\\&\qquad-\big(z_t^\delta(y\diamond_t z\tau(w'))+z(xy\diamond_t\tau(w'))-zz_t^\delta(y\diamond_t\tau(w'))\big)\\&=z_t^\delta\tau(y\diamond_s w-1\diamond_s wz_t^\delta)+z\tau(y^2\diamond_s w'-y\diamond_s w'z_t^\delta)-zz_t^\delta(y\diamond_s w'-1\diamond_s w'z_t^\delta)\end{aligned}$$

by (16) and the induction hypothesis. Hence, applying τ and using Lemma 5.3(ii), we find (14) also holds in this case. If $w=w'(z-z_u^\delta)$, we have

$$\begin{aligned}\tau(\text{R.H.S.}) &= z_t^\delta(1\diamond_t \tau(w)z_s^\delta)+z_u^\delta(x\diamond_u \tau(w')z_s^\delta)-zz_u^\delta(1\diamond_u \tau(w')z_s^\delta)\\ &\qquad -\big(z_t^\delta(y\diamond_t \tau(w))+z_u^\delta(xy\diamond_u \tau(w'))-zz_u^\delta(y\diamond_u \tau(w'))\big)\\ &= z_t^\delta\tau(y\diamond_s w-1\diamond_s wz_t^\delta)+z_u^\delta\tau(y^2\diamond_s w'-y\diamond_s w'z_u^\delta)-zz_u^\delta\tau(y\diamond_s w'-1\diamond_s w'z_u^\delta)\end{aligned}$$

by (15) and the induction hypothesis. This is τ(L.H.S.) because of Lemmas 5.9(ii) and 5.3(ii). Thus (14) holds.

Case 4: $v=v'z$. If $w=1$,

$$\begin{aligned}\tau(\text{R.H.S.}) &= z(\tau(v')\diamond_t z_s^\delta)+(z_s^\delta z-zz_s^\delta)(\tau(v')\diamond_s 1)-z(\tau(v')y\diamond_t 1)\\ &= z_s^\delta z\tau(v'\diamond_s 1)-\tau(v\diamond_s z_t^\delta)\end{aligned}$$

by Lemma 5.9(i), (vi), and (vii) and the induction hypothesis. This is τ(L.H.S.) because of Lemmas 5.2 and 5.9(i). If $w=z$,

$$\begin{aligned}\tau(\text{R.H.S.}) &= z\big(\tau(v')\diamond_t zz_s^\delta+\tau(v)\diamond_t z_s^\delta-z(\tau(v')\diamond_t z_s^\delta)\big)-z\big(\tau(v')y\diamond_t z+\tau(v)y\diamond_t 1-z(\tau(v')y\diamond_t 1)\big)\\ &= z\tau(v'y\diamond_s z-v'\diamond_s zz_t^\delta)+z\tau(vy\diamond_s 1-v\diamond_s z_t^\delta)-z\tau(v'y\diamond_s 1-v'\diamond_s z_t^\delta)\end{aligned}$$

by Lemma 5.9(v) and the induction hypothesis. This is τ(L.H.S.) because of Lemma 5.2. If $w=z-z_u^\delta$,

$$\begin{aligned}\tau(\text{R.H.S.}) &= z(\tau(v')\diamond_t z_u^\delta z_s^\delta)+z_u^\delta(\tau(v)\diamond_u z_s^\delta)-zz_u^\delta(\tau(v')\diamond_u z_s^\delta)\\ &\qquad -\big(z(\tau(v')y\diamond_t z_u^\delta)+z_u^\delta(\tau(v)y\diamond_u 1)-zz_u^\delta(\tau(v')y\diamond_u 1)\big)\\ &= z\tau(v'y\diamond_s (z-z_u^\delta)-v'\diamond_s (z-z_u^\delta)z_t^\delta)+z_u^\delta\tau(vy\diamond_s 1-v\diamond_s z_u^\delta)-zz_u^\delta\tau(v'y\diamond_s 1-v'\diamond_s z_u^\delta)\end{aligned}$$

by Lemma 5.9(vi) and the induction hypothesis. This is τ(L.H.S.) because of Lemmas 5.2 and 5.3(ii). If $w=w'z$,

$$\begin{aligned}\tau(\text{R.H.S.}) &= z\big(\tau(v')\diamond_t \tau(w)z_s^\delta+\tau(v)\diamond_t \tau(w')z_s^\delta-z(\tau(v')\diamond_t \tau(w')z_s^\delta)\big)\\ &\qquad -z\big(\tau(v')y\diamond_t \tau(w)+\tau(v)y\diamond_t \tau(w')-z(\tau(v')y\diamond_t \tau(w'))\big)\\ &= z\tau(v'y\diamond_s w-v'\diamond_s wz_t^\delta)+z\tau(vy\diamond_s w'-v\diamond_s w'z_t^\delta)-z\tau(v'y\diamond_s w'-v'\diamond_s w'z_t^\delta)\end{aligned}$$

by Lemma 5.9(v) and the induction hypothesis. This is τ(L.H.S.) because of Lemma 5.2. If $w=w'(z-z_u^\delta)$,

$$\begin{aligned}\tau(\text{R.H.S.}) &= z(\tau(v')\diamond_t \tau(w)z_s^\delta)+z_u^\delta(\tau(v)\diamond_u \tau(w')z_s^\delta)-zz_u^\delta(\tau(v')\diamond_u \tau(w')z_s^\delta)\\ &\qquad -\big(z(\tau(v')y\diamond_t \tau(w))+z_u^\delta(\tau(v)y\diamond_u \tau(w'))-zz_u^\delta(\tau(v')y\diamond_u \tau(w'))\big)\\ &= z\tau(v'y\diamond_s w-v'\diamond_s wz_t^\delta)+z_u^\delta\tau(vy\diamond_s w'-v\diamond_s w'z_u^\delta)-zz_u^\delta\tau(v'y\diamond_s w'-v'\diamond_s w'z_u^\delta)\end{aligned}$$

by Lemma 5.9(vi) and the induction hypothesis. This is τ(L.H.S.) because of Lemmas 5.2 and 5.3(ii). Thus (14) holds.

Case 5: $v = v'y$. If $w = 1$,

$$\begin{aligned}\tau(\text{R.H.S.}) &= z_t^\delta(\tau(v') \diamond_t z_s^\delta) + z_s^\delta(\tau(v) \diamond_s 1) - zz_s^\delta(\tau(v') \diamond_s 1) - z_t^\delta(\tau(v')y \diamond_t 1)\\ &= \tau(v'y \diamond_s 1 - v' \diamond_s z_t^\delta) + z_s^\delta(\tau(v) \diamond_s 1) - zz_s^\delta(\tau(v') \diamond_s 1)\end{aligned}$$

by (15), Lemma 5.9(i), $x \diamond_t 1 = z_t^\delta$, and the induction hypothesis. This is τ(L.H.S.) because of Lemmas 5.9(i) and 5.3(ii). If $w = z$,

$$\begin{aligned}\tau(\text{R.H.S.}) &= z_t^\delta(\tau(v') \diamond_t zz_s^\delta) + z(\tau(v) \diamond_t z_s^\delta) - zz_t^\delta(\tau(v') \diamond_t z_s^\delta)\\ &\qquad - z_t^\delta(\tau(v')y \diamond_t z) + z(\tau(v)y \diamond_t 1) - zz_t^\delta(\tau(v')y \diamond_t 1)\\ &= z_t^\delta\tau(v'y \diamond_s z - v' \diamond_s zz_t^\delta) + z\tau(vy \diamond_s 1 - v \diamond_s z_t^\delta) - zz_t^\delta\tau(v'y \diamond_s 1 - v' \diamond_s z_t^\delta)\end{aligned}$$

by (16) and the induction hypothesis. This is τ(L.H.S.) because of Lemmas 5.2 and 5.3(ii). If $w = z - z_u^\delta$,

$$\begin{aligned}\tau(\text{R.H.S.}) &= z_t^\delta(\tau(v') \diamond_t z_u^\delta z_s^\delta) + z_u^\delta(\tau(v) \diamond_u z_s^\delta) - zz_u^\delta(\tau(v') \diamond_u z_s^\delta)\\ &\qquad - \big(z_t^\delta(\tau(v')y \diamond_t z_u^\delta) + z_u^\delta(\tau(v)y \diamond_u 1) - zz_u^\delta(\tau(v')y \diamond_u 1)\big)\\ &= z_t^\delta\tau(v'y \diamond_s (z - z_u^\delta) - v' \diamond_s (z - z_u^\delta)z_t^\delta) + z_u^\delta\tau(vy \diamond_s 1 - v \diamond_s z_u^\delta) - zz_u^\delta\tau(v'y \diamond_s 1 - v' \diamond_s z_u^\delta)\end{aligned}$$

by (15) and the induction hypothesis. This is τ(L.H.S.) because of Lemmas 5.2 and 5.3(ii). If $w = w'z$,

$$\begin{aligned}\tau(\text{R.H.S.}) &= z_t^\delta(\tau(v') \diamond_t \tau(w)z_s^\delta) + z(\tau(v) \diamond_t \tau(w')z_s^\delta) - zz_t^\delta(\tau(v') \diamond_t \tau(w')z_s^\delta)\\ &\qquad - \big(z_t^\delta(\tau(v')y \diamond_t \tau(w)) + z(\tau(v)y \diamond_t \tau(w')) - zz_t^\delta(\tau(v')y \diamond_t \tau(w'))\big)\\ &= z_t^\delta\tau(v \diamond_s w - v' \diamond_s wz_t^\delta) + z\tau(vy \diamond_s w' - v \diamond_s w'z_t^\delta) - zz_t^\delta\tau(v \diamond_s w' - v' \diamond_s w'z_t^\delta)\end{aligned}$$

by (16) and the induction hypothesis. This is τ(L.H.S.) because of Lemmas 5.2 and 5.3(ii). If $w = w'(z - z_u^\delta)$,

$$\begin{aligned}\tau(\text{R.H.S.}) &= z_t^\delta(\tau(v') \diamond_t \tau(w)z_s^\delta) + z_u^\delta(\tau(v) \diamond_u \tau(w')z_s^\delta) - zz_u^\delta(\tau(v') \diamond_u \tau(w')z_s^\delta)\\ &\qquad - \big(z_t^\delta(\tau(v')y \diamond_t \tau(w)) + z_u^\delta(\tau(v)y \diamond_u \tau(w')) - zz_u^\delta(\tau(v')y \diamond_u \tau(w'))\big)\\ &= z_t^\delta\tau(v'y \diamond_s w - v' \diamond_s wz_t^\delta) + z_u^\delta\tau(vy \diamond_s w' - v \diamond_s w'z_u^\delta) - zz_u^\delta\tau(v'y \diamond_s w' - v' \diamond_s w'z_u^\delta)\end{aligned}$$

by (15) and the induction hypothesis. This is τ(L.H.S.) because of Lemmas 5.2 and 5.3(ii). Thus (14) holds and we complete the proof. □

Lemma 5.9. *For $s, t \in \mu_r$, $v, v' \in \mathcal{A}_1$, and $w \in \mathcal{A}_r$, the following equalities hold*:

(i) $vv' \diamond_s 1 = (v \diamond_s 1)(v' \diamond_s 1)$.

(ii) $vy \diamond_s w(z - z_t^\delta) = (vy \diamond_s w - v \diamond_s wz_t^\delta)(y \diamond_t 1)$.

(iii) $yv \diamond_s zw = (y \diamond_s 1)(v \diamond_s zw) + z(yv \diamond_s w) - z(y \diamond_s 1)(v \diamond_s w)$.

(iv) $yv \diamond_s z_t^\delta w = (y \diamond_s 1)(v \diamond_s z_t^\delta w) + z_t^\delta(yv \diamond_t w)$.

(v) $zv \diamond_s zw = z(v \diamond_s zw + zv \diamond_s w - z(v \diamond_s w))$.

(vi) $zv \diamond_s z_t^\delta w = z(v \diamond_s z_t^\delta w) + z_t^\delta(zv \diamond_t w) - zz_t^\delta(v \diamond_t w)$.

(vii) $\tau(v \diamond_s 1) = \tau(v) \diamond_s 1$.

Proof. (i): If $v=1$ or $v'=1$, it is obvious. Otherwise, it is enough to show when $v=z^{k_1-1}y\cdots z^{k_m-1}y$ and $v'=z^{l_1-1}y\cdots z^{l_n-1}y$. One calculates

$$vv'\diamond_s 1=z^{k_1-1}(z-z_s^\delta)\cdots z^{k_m-1}(z-z_s^\delta)z^{l_1-1}(z-z_s^\delta)\cdots z^{l_n-1}(z-z_s^\delta),$$

which is clearly equal to $(v\diamond_s 1)(v'\diamond_s 1)$.

(ii): This is a direct consequence of Lemmas 5.2 and 5.3(ii).

(iii): We first consider the case $v=1$. If $w=1$, it is obvious because of Lemma 5.2. If $w=w'z$ ($w'\in\mathcal{A}_r$), the left-hand side turns into

$$(y\diamond_s zw')z=((y\diamond_s 1)zw'+z(y\diamond_s w')-z(y\diamond_s 1)w')z$$

by Lemma 5.2 and the induction hypothesis on degree of words. This is equal to the right-hand side again by Lemma 5.2. If $w=w'z_t^\delta$ ($w'\in\mathcal{A}_r$, $t\in\mu_r$), the left-hand side turns into

$$(y\diamond_s zw')z_t^\delta+zw(y\diamond_t 1)=(y\diamond_s 1)zw+z(y\diamond_s w')z_t^\delta-z(y\diamond_s 1)w+zw(y\diamond_t 1)$$

by Lemma 5.3(ii) and the induction hypothesis. This is equal to the right-hand side again by Lemma 5.3(ii). If $v=z$, by Lemma 5.2, we have

$$\text{L.H.S.}=(y\diamond_s zw)z$$

and

$$\text{R.H.S.}=((y\diamond_s 1)(1\diamond_s zw)+z(y\diamond_s w)-z(y\diamond_s 1)(1\diamond_s w))z,$$

which are equal as shown just before. If $v=y$, we need to show when $w=1$, $w'z$, $w'z_t^\delta$ ($w'\in\mathcal{A}_r$, $t\in\mu_r$). If $w=1$,

$$\text{L.H.S.}=(y^2\diamond_s 1)z=(y\diamond_s 1)(y\diamond_s z)$$

and

$$\text{R.H.S.}=(y\diamond_s 1)(y\diamond_s z)+z(y^2\diamond_s 1)-z(y\diamond_s 1)^2,$$

which coincide. If $w=w'z$, by induction on degree of words, the left-hand side turns into

$$(y^2\diamond_s zw')z=(y\diamond_s 1)(y\diamond_s zw')z+z(y^2\diamond_s w')z-z(y\diamond_s 1)(y\diamond_s w')z,$$

which is equal to the right-hand side due to Lemma 5.2. If $w=w'z_t^\delta$, by using Lemma 5.3(ii) and the induction hypothesis, one calculates

$$\begin{aligned}\text{L.H.S.}&=(y\diamond_s zw)(y\diamond_t 1)+(y^2\diamond_s zw')z_t^\delta\\&=((y\diamond_s 1)zw+z(y\diamond_s w)-z(y\diamond_s 1)w)(y\diamond_t 1)\\&\qquad+((y\diamond_s 1)(y\diamond_s zw')+z(y^2\diamond_s w')-z(y\diamond_s 1)(y\diamond_s w'))z_t^\delta\end{aligned}$$

and

$$\begin{aligned}\text{R.H.S.}=(y\diamond_s 1)(zw(y\diamond_t 1)+(y\diamond_s zw')z_t^\delta)+z((y\diamond_s w)(y\diamond_t 1)+(y^2\diamond_s w')z_t^\delta)\\-z(y\diamond_s 1)(w(y\diamond_t 1)+(y\diamond_s w')z_t^\delta),\end{aligned}$$

which are equal. If $v = v'z$, it is obvious by Lemma 5.2 and the induction hypothesis. If $v = v'y$, we need to show when $w = 1$, $w'z$, $w'z_t^\delta$ ($w' \in \mathcal{A}_r$, $t \in \mu_r$). If $w = 1$,

$$\text{L.H.S.} = yv \diamond_s z = (yv \diamond_s 1)z$$

and

$$\text{R.H.S.} = (y \diamond_s 1)(v \diamond_s z) + z(yv \diamond_s 1) - z(y \diamond_s 1)(v \diamond_s 1),$$

which are equal. If $w = w'z$, it is obvious by Lemma 5.2 and the induction hypothesis. If $w = w'z_t^\delta$, by using Lemma 5.3(ii) and the induction hypothesis, one calculates

$$\begin{aligned}\text{L.H.S.} &= (yv' \diamond_s zw)(y \diamond_t 1) + (yv \diamond_s zw')z_t^\delta \\ &= ((y \diamond_s 1)(v' \diamond_s zw) + z(yv' \diamond_s w) - z(y \diamond_s 1)(v' \diamond_s w))(y \diamond_t 1) \\ &\qquad + ((y \diamond_s 1)(v \diamond_s zw') + z(yv \diamond_s w') - z(y \diamond_s 1)(v \diamond_s w'))z_t^\delta\end{aligned}$$

and

$$\begin{aligned}\text{R.H.S.} = (y \diamond_s 1)((v' \diamond_s zw)(y \diamond_t 1) + (v \diamond_s zw')z_t^\delta) + z((yv' \diamond_s w)(y \diamond_t 1) + (yv \diamond_s w')z_t^\delta) \\ - z(y \diamond_s 1)((v' \diamond_s w)(y \diamond_t 1) + (v \diamond_s w')z_t^\delta),\end{aligned}$$

which coincide.

(iv): We first consider the case $v = 1$. If $w = 1$, it is obvious because of Lemma 5.3(ii). If $w = w'z$ ($w' \in \mathcal{A}_r$), the left-hand side turns into

$$(y \diamond_s z_t^\delta w')z = ((y \diamond_s 1)z_t^\delta w' + z_t^\delta (y \diamond_t w'))z$$

by Lemma 5.2 and the induction hypothesis on degree of words. This is equal to the right-hand side again by Lemma 5.2. If $w = w'z_u^\delta$ ($w' \in \mathcal{A}_r$, $u \in \mu_r$), the left-hand side turns into

$$(y \diamond_s z_t^\delta w')z_u^\delta + z_t^\delta w(y \diamond_u 1) = (y \diamond_s 1)z_t^\delta w + z_t^\delta (y \diamond_t w')z_u^\delta + z_t^\delta w(y \diamond_u 1)$$

by Lemma 5.3(ii) and the induction hypothesis. This is equal to the right-hand side again by Lemma 5.3(ii). If $v = z$, by Lemma 5.2, we have

$$\text{L.H.S.} = (y \diamond_s z_t^\delta w)z$$

and

$$\text{R.H.S.} = ((y \diamond_s 1)(1 \diamond_s z_t^\delta w) + z_t^\delta (y \diamond_t w))z,$$

which are equal as shown just before. If $v = y$, we need to show when $w = 1$, $w'z$, $w'z_u^\delta$ ($w' \in \mathcal{A}_r$, $u \in \mu_r$). If $w = 1$,

$$\text{L.H.S.} = (y \diamond_s z_t^\delta)(y \diamond_t 1) + (y^2 \diamond_s 1)z_t^\delta$$

and

$$\text{R.H.S.} = (y \diamond_s 1)(y \diamond_s z_t^\delta) + z_t^\delta (y^2 \diamond_t 1),$$

which are equal because of Lemma 5.3(ii). If $w = w'z$, it is obvious by Lemma 5.2 and the induction hypothesis. If $w = w'z_u^\delta$, by the induction hypothesis, one calculates

$$\begin{aligned}\text{L.H.S.} &= (y \diamond_s z_t^\delta w)(y \diamond_u 1) + (y^2 \diamond_s z_t^\delta w')z_u^\delta \\ &= ((y \diamond_s 1)z_t^\delta w + z_t^\delta(y \diamond_t w))(y \diamond_u 1) + ((y \diamond_s 1)(y \diamond_s z_t^\delta w') + z_t^\delta(y^2 \diamond_t w'))z_u^\delta\end{aligned}$$

and

$$\text{R.H.S.} = (y \diamond_s 1)(z_t^\delta w(y \diamond_u 1) + (y \diamond_s z_t^\delta w')z_u^\delta) + z_t^\delta((y \diamond_t w)(y \diamond_u 1) + (y^2 \diamond_t w')z_u^\delta),$$

which coincide. If $v = v'z$, it is obvious by Lemma 5.2 and the induction hypothesis. If $v = v'y$, we need to show when $w = 1$, $w'z$, $w'z_u^\delta$ ($w' \in \mathcal{A}_r$, $u \in \mu_r$). If $w = 1$,

$$\text{L.H.S.} = (yv' \diamond_s z_t^\delta)(y \diamond_t 1) + (yv \diamond_s 1)z_t^\delta$$

and

$$\text{R.H.S.} = (y \diamond_s 1)(v \diamond_s z_t^\delta) + z_t^\delta(yv \diamond_t 1),$$

which are equal by Lemma 5.3(ii) and the induction hypothesis. If $w = w'z$, it is obvious by Lemma 5.2 and the induction hypothesis. If $w = w'z_u^\delta$, by using Lemma 5.3(ii) and the induction hypothesis, one calculates

$$\begin{aligned}\text{L.H.S.} &= (yv' \diamond_s z_t^\delta w)(y \diamond_u 1) + (yv \diamond_s z_t^\delta w')z_u^\delta \\ &= ((y \diamond_s 1)(v' \diamond_s z_t^\delta w) + z_t^\delta(yv' \diamond_t w))(y \diamond_u 1) + ((y \diamond_s 1)(v \diamond_s z_t^\delta w') + z_t^\delta(yv \diamond_t w'))z_u^\delta\end{aligned}$$

and

$$\text{R.H.S.} = (y \diamond_s 1)((v' \diamond_s z_t^\delta w)(y \diamond_u 1) + (v \diamond_s z_t^\delta w')z_u^\delta) + z_t^\delta((yv' \diamond_t w)(y \diamond_u 1) + (yv \diamond_t w')z_u^\delta),$$

which coincide.

(v): If $v = 1$, by using Lemma 5.2, the right-hand side turns into

$$z^2w + zwz - z^2w = zwz,$$

which is equal to the left-hand side. If $v = z$, we have

$$\text{L.H.S.} = (z \diamond_s zw)z = zwz^2$$

and

$$\text{R.H.S.} = z(zwz + wz^2 - zwz) = zwz^2,$$

which coincide. If $v = y$, we need to show when $w = 1$, $w'z$, $w'z_t^\delta$ ($w' \in \mathcal{A}_r$, $t \in \mu_r$). If $w = 1$,

$$\text{L.H.S.} = zy \diamond_s z = zyz$$

and

$$\text{R.H.S.} = z(y \diamond_s z + zy \diamond_s 1 - z(y \diamond_s 1)) = zyz,$$

which coincide. If $w = w'z$, by induction on degree of words, the left-hand side turns into

$$(zy \diamond_s zw')z = z(y \diamond_s zw' + zy \diamond_s w' - z(y \diamond_s w'))z,$$

which is equal to the right-hand side due to Lemma 5.2. If $w = w'z_t^\delta$, by using Lemma 5.3(ii) and the induction hypothesis, one calculates

$$\text{L.H.S.} = (z \diamond_s zw)(y \diamond_t 1) + (zy \diamond_s zw')z_t^\delta = zwz(y \diamond_t 1) + z(y \diamond_s zw' + zy \diamond_s w' - z(y \diamond_s w'))z_t^\delta$$

and

$$\text{R.H.S.} = z\big(zw(y \diamond_t 1) + (y \diamond_s zw')z_t^\delta + wz(y \diamond_t 1) + (zy \diamond_s w')z_t^\delta - z(w(y \diamond_t 1) + (y \diamond_s w')z_t^\delta)\big),$$

which are equal.

If $v = v'z$, it is obvious by Lemma 5.2 and the induction hypothesis. If $v = v'y$, we need to show when $w = 1$, $w'z$, $w'z_t^\delta$ ($w' \in \mathcal{A}_r$, $t \in \mu_r$). If $w = 1$,

$$\text{L.H.S.} = zv \diamond_s z = zvz$$

and

$$\text{R.H.S.} = z(v \diamond_s z + zv \diamond_s 1 - z(v \diamond_s 1)) = zvz,$$

which coincide. If $w = w'z$, it is obvious by Lemma 5.2 and the induction hypothesis. If $w = w'z_t^\delta$, by using Lemma 5.3(ii) and the induction hypothesis, one calculates

$$\begin{aligned}\text{L.H.S.} &= (zv' \diamond_s zw)(y \diamond_t 1) + (zv \diamond_s zw')z_t^\delta \\ &= z(v' \diamond_s zw + zv' \diamond_s w - z(v' \diamond_s w))(y \diamond_t 1) + z(v \diamond_s zw' + zv \diamond_s w' - z(v \diamond_s w'))z_t^\delta\end{aligned}$$

and

$$\begin{aligned}\text{R.H.S.} = z\big(&(v' \diamond_s zw)(y \diamond_t 1) + (v \diamond_s zw')z_t^\delta + (zv' \diamond_s w)(y \diamond_t 1) \\ &+ (zv \diamond_s w')z_t^\delta - z((v' \diamond_s w)(y \diamond_t 1) + (v \diamond_s w')z_t^\delta)\big),\end{aligned}$$

which are equal.

(vi): If $v = 1$, by using Lemma 5.2, the right-hand side turns into

$$zz_t^\delta w + z_t^\delta(z \diamond_t w) - zz_t^\delta w = z_t^\delta wz,$$

which is equal to the left-hand side. If $v = z$, we have

$$\text{L.H.S.} = (z \diamond_s z_t^\delta w)z = z_t^\delta wz^2$$

and

$$\text{R.H.S.} = zz_t^\delta wz + z_t^\delta wz^2 - zz_t^\delta wz = z_t^\delta wz^2,$$

which coincide. If $v = y$, we need to show when $w = 1$, $w'z$, $w'z_u^\delta$ ($w' \in \mathcal{A}_r$, $u \in \mu_r$). If $w = 1$,

$$\text{L.H.S.} = zy \diamond_s z_t^\delta = z_t^\delta z(t \diamond_t 1) + (zy \diamond_s 1)z_t^\delta$$

and

$$\begin{aligned}\text{R.H.S.} &= z(y \diamond_s z_t^\delta) + z_t^\delta(zy \diamond_t 1) - zz_t^\delta(y \diamond_t 1) \\ &= z(z_t^\delta(y \diamond_t 1) + (y \diamond_s 1)z_t^\delta) + z_t^\delta z(y \diamond_t 1) - zz_t^\delta(y \diamond_t 1),\end{aligned}$$

which coincide. If $w = w'z$, it is obvious by Lemma 5.2 and the induction hypothesis. If $w = w'z_u^\delta$, by using Lemma 5.3(ii) and the induction hypothesis, one calculates

$$\begin{aligned}\text{L.H.S.} &= (z \diamond_s z_t^\delta w)(y \diamond_u 1) + (zy \diamond_s z_t^\delta w')z_u^\delta \\ &= z_t^\delta wz(y \diamond_u 1) + z(y \diamond_s z_t^\delta w')z_u^\delta + z_t^\delta(zy \diamond_t w')z_u^\delta - zz_t^\delta(y \diamond_s w')z_u^\delta\end{aligned}$$

and

$$\text{R.H.S.} = z(z_t^\delta w(y \diamond_u 1) + (y \diamond_s z_t^\delta w')z_u^\delta) + z_t^\delta(wz(y \diamond_u 1) + (zy \diamond_t w')z_u^\delta) - zz_t^\delta(w(y \diamond_u 1) + (y \diamond_t w')z_u^\delta),$$

which are equal. If $v = v'z$, it is obvious by Lemma 5.2 and the induction hypothesis. If $v = v'y$, we need to show when $w = 1, w'z, w'z_u^\delta$ ($w' \in \mathcal{A}_r$, $u \in \mu_r$). If $w = 1$, by Lemma 5.3(ii) and the induction hypothesis, one calculates

$$\begin{aligned}\text{L.H.S.} &= (zv' \diamond_s z_t^\delta)(y \diamond_t 1) + (zv \diamond_s 1)z_t^\delta \\ &= (z(v' \diamond_s z_t^\delta) + z_t^\delta(zv' \diamond_t 1) - zz_t^\delta(v' \diamond_t 1))(y \diamond_t 1) + z(v \diamond_s 1)z_t^\delta\end{aligned}$$

and

$$\begin{aligned}\text{R.H.S.} &= z(v \diamond_s z_t^\delta) + z_t^\delta(zv \diamond_t 1) - zz_t^\delta(v \diamond_t 1) \\ &= z((v' \diamond_s z_t^\delta)(y \diamond_t 1) + (v \diamond_s 1)z_t^\delta) + z_t^\delta z(v' \diamond_t 1)(y \diamond_t 1) - zz_t^\delta(v' \diamond_t 1)(y \diamond_t 1),\end{aligned}$$

which are equal. If $w = w'z$, it is obvious by Lemma 5.2 and the induction hypothesis. If $w = w'z_u^\delta$, by using Lemma 5.3(ii) and the induction hypothesis, one calculates

$$\begin{aligned}\text{L.H.S.} &= (zv' \diamond_s z_t^\delta w)(y \diamond_u 1) + (zv \diamond_s z_t^\delta w')z_u^\delta \\ &= (z(v' \diamond_s z_t^\delta w) + z_t^\delta(zv' \diamond_t w) - zz_t^\delta(v' \diamond_t w))(y \diamond_u 1) + (z(v \diamond_s z_t^\delta w') + z_t^\delta(zv \diamond_t w') - zz_t^\delta(v \diamond_t w'))z_u^\delta\end{aligned}$$

and

$$\begin{aligned}\text{R.H.S.} = z((v' \diamond_s z_t^\delta w)(y \diamond_u 1) + (v \diamond_s z_t^\delta w')z_u^\delta) + z_t^\delta((zv' \diamond_t w)(y \diamond_u 1) + (zv \diamond_t w')z_u^\delta) \\ - zz_t^\delta((v' \diamond_t w)(y \diamond_u 1) + (v \diamond_t w')z_u^\delta),\end{aligned}$$

which are equal.

(vii): If $v = 1$, it is obvious. Otherwise, putting $v = z^{k_1-1}y \cdots z^{k_m-1}y$, one calculates

$$\tau(v \diamond_s 1) = \tau(z^{k_1-1}(z - z_s^\delta) \cdots z^{k_m-1}(z - z_s^\delta)) = z_s^\delta z^{k_m-1} \cdots z_s^\delta z^{k_1-1}$$

and

$$\tau(v) \diamond_s 1 = \psi_s(zx^{k_m-1} \cdots zx^{k_1-1}) = \varphi(z_s x^{k_m-1} \cdots z_s x^{k_1-1}),$$

which are equal. □

6. Proof of Theorem 2.1

We prove that the polynomial F_f defined just before Proposition 5.7 satisfies the theorem. The proof goes by induction on $\deg(f)$ for rooted forests f and $\deg(w)$ for words w. First, we prove the theorem when $f = \bullet$. If $w = 1$, we have

$$\tilde{f}(z_s^\delta) = z_s^\delta(z - z_s^\delta)$$

and

$$z_s^\delta(F_f \diamond_s 1) = z_s^\delta(y \diamond_s 1) = z_s^\delta(z - z_s^\delta),$$

which are equal. Suppose $\deg(w) \geq 1$. If $w = w'z$ ($w' \in \mathcal{A}_r$), by [9, Theorem 2.2(d)], which asserts that R_z and any RTM commute, the induction hypothesis, and Lemma 5.2, we have

$$\tilde{f}(z_s^\delta w'z) = \tilde{f}(z_s^\delta w')z = z_s^\delta(F_f \diamond_s w')z = z_s^\delta(F_f \diamond_s w). \tag{17}$$

If $w = w'z_t^\delta$ ($w' \in \mathcal{A}_r$), we have

$$\tilde{f}(z_s^\delta w'z_t^\delta) = \tilde{f}(z_s^\delta w')z_t^\delta + z_s^\delta w'z_t^\delta(z - z_t^\delta)$$

and, by Lemma 5.3,

$$z_s^\delta(y \diamond_s w'z_t^\delta) = z_s^\delta(y \diamond_s w')z_t^\delta + z_s^\delta w'z_t^\delta(z - z_t^\delta),$$

which are equal by the induction hypothesis.

Next, suppose $\deg(f) \geq 2$. If $f = gh$ ($g, h \neq \mathbb{I}$), we have

$$\tilde{f}(z_s^\delta w) = \tilde{g}\tilde{h}(z_s^\delta w) = \tilde{g}(z_s^\delta(F_h \diamond_s w)) = z_s^\delta(F_g \diamond_s (F_h \diamond_s w)) = z_s^\delta((F_g \diamond_1 F_h) \diamond_s w) = z_s^\delta(F_f \diamond_s w)$$

since $\deg(g), \deg(h) < \deg(f)$ and Lemma 5.5. Let f be a rooted tree and put $f = B_+(g)$. When $w = 1$, we have

$$\tilde{f}(z_s^\delta) = R_{z-z_s^\delta} R_{2z-z_s^\delta} R_{z-z_s^\delta}^{-1} \tilde{g}(z_s^\delta) = R_{z-z_s^\delta} R_{2z-z_s^\delta} R_{z-z_s^\delta}^{-1} z_s^\delta(F_g \diamond_s 1) \tag{18}$$

by the induction hypothesis. Since $\psi_s\varphi R_x = R_{z_s^\delta}\psi_s\varphi$ and $\psi_s\varphi R_y = R_{z-z_s^\delta}\psi_s\varphi$ on $\mathcal{A}_1^1$, we have

$$(18) = z_s^\delta(\psi_s\varphi(R_y R_{x+2y} R_y^{-1}(F_g)) = z_s^\delta(F_f \diamond_s 1). \tag{19}$$

Suppose $\deg(w) \geq 1$. If $w = w'z$ ($w' \in \mathcal{A}_r$), we have (17) again (but this time we consider $\deg(f) \geq 2$). If $w = w'z_t^\delta$ ($w' \in \mathcal{A}$) and $\Delta(f) = \sum_{(f)} f' \otimes f''$, we have

$$\tilde{f}(z_s^\delta w'z_t^\delta) = \sum_{(f)} \widetilde{f'}(z_s^\delta w')\widetilde{f''}(z_t^\delta) = \sum_{(f)} z_s^\delta(F_{f'} \diamond_s w')z_t^\delta(F_{f''} \diamond_t 1)$$

by the induction hypothesis on degree of words and (19). This is equal to $z_s^\delta(F_f \diamond_s w)$ by Proposition 5.7.

Uniqueness of F_f is shown as follows. If $F_f' \in \mathcal{A}_1^1$ also satisfies the theorem, we have

$$(F_f - F_f') \diamond_s w = 0$$

for any $s \in \mu_r$ and any $w \in \mathcal{A}_r$. In particular, putting $w = 1$ we have

$$(F_f - F'_f) \diamond_s 1 = 0,$$

and hence

$$F_f - F'_f = \varphi\psi_s^{-1}(0) = 0. \qquad \square$$

7. Proof of Theorem 2.4

For rooted forests f, we define polynomials $G_f \in \mathcal{A}_1^1$ recursively by

- $G_{\mathbb{I}} = 1$,
- $G_{\bullet} = -y$,
- $G_t = L_{2x+y}(G_f)$ if $t = B_+(f)$ and $f \neq \mathbb{I}$,
- $G_f = G_g \diamond_1 G_h$ if $f = gh$,

where L_v denotes the left multiplication map by v, i.e., $L_v(w) = vw$ $(v, w \in \mathcal{A}_r)$. The subscript of G is extended linearly. In [7], we find that $G_f = F_{S(f)}$.

Lemma 7.1. *For $f \in \mathrm{Aug}(\mathcal{H})$, put $\Delta(f) = \sum_{(f)} f' \otimes f''$. Then we have*

$$\sum_{(f)} F_{f'} \diamond_1 G_{f''} = 0.$$

Proof. See [7, Proposition 4.5]. $\square$

Proof of Theorem 2.4. If $f = \bullet$, the theorem holds since $S(f) = -\bullet$, $G_f = -y$, and Theorem 2.1 for $f = \bullet$. Assume $\deg(f) \geq 2$. If $f = gh$ $(g, h \neq \mathbb{I})$, we have

$$\widetilde{S(f)} = \widetilde{S(gh)} = \widetilde{S(h)S(g)} = \widetilde{S(h)}\widetilde{S(g)} = \widetilde{S(g)}\widetilde{S(h)}$$

because the antipode S is an antiautomorphism, $\widetilde{\cdot}$ is an algebra homomorphism, and RTMs commute with each other. Then, since $\deg(g), \deg(h) < \deg(f)$ and Lemma 5.5, we have

$$\widetilde{S(f)}(z_s^\delta w) = \widetilde{S(g)}(\widetilde{S(h)}(z_s^\delta w)) = z_s^\delta(G_g \diamond_s (G_h \diamond_s w)) = z_s^\delta((G_g \diamond_1 G_h) \diamond_s w) = z_s^\delta(G_f \diamond_s w)).$$

If f is a tree, by letting $\Delta(f) = \sum_{(f)} f' \otimes f''$ and Lemma 7.1, we have

$$z_s^\delta(G_f \diamond_s w)) = -z_s^\delta \sum_{\substack{(f)\\ f' \neq \mathbb{I}}} (F_{f'} \diamond_1 G_{f''}) \diamond_s w. \tag{20}$$

By Lemma 5.5, Theorem 2.1, and the induction hypothesis, we have

$$(20) = -z_s^\delta \sum_{\substack{(f)\\ f' \neq \mathbb{I}}} F_{f'} \diamond_s (G_{f''} \diamond_s w) = -\sum_{\substack{(f)\\ f' \neq \mathbb{I}}} \widetilde{f}'(z_s^\delta(G_{f''} \diamond_s w)) = -\sum_{\substack{(f)\\ f' \neq \mathbb{I}}} \widetilde{f}'(\widetilde{S(f'')}(z_s^\delta w)).$$

Since $\sum_{(f)} f'S(f'') = 0$, we get the theorem. $\square$

8. Proof of Theorem 2.5

Lemma 8.1. *For* $f \in \mathrm{Aug}(\mathcal{H})$, *we have*

$$F_f = -R_y \tau R_y^{-1}(F_{S(f)}).$$

Proof. See [7, Proposition 5.1]. □

Proof of Theorem 2.5. First, we prove the theorem when $w = z_s^\delta w'(z - z_t^\delta) \in z_s^\delta \mathcal{A}_r(z - z_t^\delta)$. By Theorem 2.4, we have

$$\widetilde{S(f)}(w) = z_s^\delta(F_{S(f)} \diamond_s w'(z - z_t^\delta)).$$

We also have

$$\tau \tilde{f} \tau(w) = \tau \tilde{f}(z_t^\delta \tau(w')(z - z_s^\delta)) = \tau\big(z_t^\delta(F_f \diamond_t \tau(w')(z - z_s^\delta))\big) \tag{21}$$

by Theorem 2.1. Then, by Lemma 8.1, we have

$$(21) = -\tau\big(z_t^\delta(R_y \tau R_y^{-1}(F_{S(f)}) \diamond_t \tau(w')(z - z_s^\delta))\big),$$

which is equal to $z_s^\delta(F_{S(f)} \diamond_s w'(z - z_t^\delta))$ because of Proposition 5.8.

Next, we consider when $w = w'z \in \mathcal{A}_r z$. Since R_z and RTMs commute, we have

$$\widetilde{S(f)}(w) = \widetilde{S(f)}(xw')z$$

and

$$\tau \tilde{f} \tau(w) = \tau \tilde{f} \tau(xw'z) = \tau \tilde{f} \tau(xw')z,$$

which are equal by the induction hypothesis. Similarly, since L_z and RTMs commute, we have the same consequence when $w = zw' \in z\mathcal{A}_r$. □

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