Terminal orders on arithmetic surfaces

Daniel Chan and Colin Ingalls

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The local structure of terminal Brauer classes on arithmetic surfaces was classified (2021), generalising the classification on geometric surfaces (2005). Part of the interest in these classifications is that it enables the minimal model program to be applied to the noncommutative setting of orders on surfaces. We give étale local structure theorems for terminal orders on arithmetic surfaces, at least when the degree is a prime $p > 5$. This generalises the structure theorem given in the geometric case. They can all be explicitly constructed as algebras of matrices over symbols. From this description one sees that such terminal orders all have global dimension two, thus generalising the fact that terminal (commutative) surfaces are smooth and hence homologically regular.

1. Introduction

Given a smooth point on a complex variety of dimension d, the étale local structure is Spec R where $R = \mathbb{C}\{x_1, \ldots, x_d\}$, the algebra of algebraic power series in d variables. This result elegantly captures, in an algebraic fashion, the idea that manifolds are all locally Euclidean. In [Chan and Ingalls 2005], a noncommutative analogue of this result was given for the case of orders on a complex surface. Here we extend the result to arbitrary surfaces, at least under some mild hypotheses.

To set the ambient framework, we briefly recall here the minimal model program for surfaces enriched by a Brauer class as developed in [Chan and Ingalls 2005; 2021]. The minimal model program enriched by a Brauer class relies on a restricted class of log surfaces described below. Although this program uses log surfaces and log surface contractions, the resulting terminal minimal models have a structure theory more akin to terminal commutative surfaces than log surfaces. In other words, they are noncommutative analogues of regular surfaces rather than quotient singularities. In particular, for a terminal order, the log surface is log smooth, i.e., a regular surface with a normal crossing divisor, and the orders over these surfaces have global dimension two, which is equivalent to regular in the commutative case. To set up the program, let X be a normal surface with function field K and Λ a sheaf of maximal $\mathcal{O}_X$-orders on X in a central simple K-algebra D. Let $\beta \in \operatorname{Br} K$ be the Brauer class corresponding to D whose index we assume is prime to residue characteristics of X. Of fundamental importance in the original commutative minimal model program is the canonical divisor, and the key to the noncommutative version is the canonical divisor $K_{X,\beta}$ of β on X defined in terms of the ramification data of β as follows. If Λ is

not Azumaya at the generic point of an irreducible divisor $C \subset X$, then we say β *ramifies* along C, and it turns out that we can associate to it the *ramification* $a_C(\beta)$ of β along C, which is an element of the torsion étale cohomology group $H^1(K(C), \mathbb{Q}/\mathbb{Z})$. We define the *ramification index* of β along C to be the order e_C of $a_C(\beta)$. Serre duality theory for the order Λ suggests the definition

$$K_{X,\beta} := K_X + \sum_C \left(1 - \frac{1}{e_C}\right)C,$$

where the sum runs over the finitely many ramification curves (alternatively, one can set $e_C = 1$ when β is unramified along C). With this definition, the notions of *discrepancy*, *terminal*, *canonical* etc. naturally follow as in the original commutative minimal model program. Much of classical commutative surface theory goes through such as resolutions of singularities and Castelnuovo's contraction theorem.

We will be concerned with the étale local theory and so let R be a commutative excellent noetherian normal two-dimensional Hensel local domain with fraction field K. In this context, it will be useful to not only consider maximal orders but, more generally, a normal R-order Λ (see Definition 2.1). To this, we attach a localised Brauer class $(\beta, g_\mathfrak{p})$ (in Definition 2.2) where the integers $g_\mathfrak{p}$ measure how much Λ deviates from being maximal at the codimension-one prime $\mathfrak{p}$.

When the residue field κ is algebraically closed, terminal localised Brauer classes $(\beta, g_\mathfrak{p})$ and the corresponding terminal orders were completely classified in [Chan and Ingalls 2005] and shown to always have global dimension two. In that article, it is shown that

i) R is smooth,

ii) β is zero or has ramification along a normal crossing divisor $C_1 \cup C_2$, and

iii) Λ is maximal except possibly along a curve of multiplicity one when $\beta = 0$, or one of the C_i otherwise.

A complete structure theorem was given for terminal normal orders in this case (see [Chan and Ingalls 2005, Section 2]). If Λ is maximal, then it is isomorphic to a full matrix algebra over a symbol $\Delta = R\langle y, z\rangle/(y^m - u, z^m - v, zy - \zeta yz)$ where ζ is some primitive m-th root of unity and $u, v \in R$ is a regular system of parameters. In general, one obtains a triangular modulo z matrix algebra over such symbols as defined in Definition 2.4.

When κ is a finite field of characteristic prime to the order m of β, it turns out there are more possibilities for terminal localised Brauer classes. If m is a prime > 5, the terminal localised Brauer classes were completely classified in [Chan and Ingalls 2021]. The new possibilities are summed up in Definitions 3.1 and 5.1, but, briefly, when R is regular, there is the additional possibility that β is ramified on a single multiplicity-one curve, and, more interestingly, R can also be a type of Hirzebruch–Jung singularity, in which case $\beta \neq 0$, though it is unramified along codimension-one primes of R. Our main theorem is the following result which generalises the aforementioned structure theory of terminal normal orders to this arithmetic situation. Again, symbols feature significantly but in the more general sense of tensor products

of a $\mathbb{Z}/m$-extension of R with a μ_m-extension and graded components skew commute according to the natural pairing $\mathbb{Z}/m \times \mu_m \to \mu_m$ (see Definition 3.2).

Theorem 1.1. *Let R be an excellent noetherian two-dimensional normal Hensel local domain. Let Λ be a terminal R-order whose degree is a prime, say $m > 5$. If the residue field of R contains a primitive m-th root of unity and has trivial Brauer group, then Λ has global dimension two. In fact, all such Λ are explicitly constructed in Propositions 3.3, 3.5 and Theorem 5.7 as various triangular modulo z matrix algebras over symbols.*

To prove the theorem, we construct explicit examples of normal orders with all the possible terminal localised Brauer classes. In the case when R is regular, this is relatively straight forward. In the singular case, we need to show that our Hirzebruch–Jung singularity, defined to have a minimal resolution whose exceptional locus is a string of projective lines defined over the residue field of R, is actually a cyclic quotient singularity. We prove this in Section 4, and give an explicit construction of the regular cyclic cover which is used in the construction of the corresponding terminal orders. The other step is to show that these explicitly constructed orders are sufficiently nice (in particular, have global dimension two) and that any normal order with the same ramification data has to be Morita equivalent to them. We follow the basic framework of [Chan and Ingalls 2005]. Unfortunately, the use of the Cohen structure theorem in that article is unavailable in this setting, so we give a streamlined method avoiding this tool in Section 2.

2. Uniqueness result for regular almost maximal orders

We review basic definitions of orders, their ramification theory and the relationship with the Brauer group. Since the notion of maximal orders is not stable under étale localisation, we review normal orders as introduced in [Chan and Ingalls 2005]. The main result is a uniqueness-type result for certain normal orders which have global dimension two and are maximal everywhere except possibly on a single irreducible divisor. This was proved over an algebraically closed field using a complicated argument in §2.3 of the same reference. We present a streamlined proof here using the classification of normal orders over discrete valuation rings found in the Appendix.

Let R be a noetherian normal domain and K its field of fractions. Given a central simple K-algebra Q, an *order* A in Q is an R-subalgebra such that A is a finitely generated R-module such that $KA = Q$. Then $K \otimes_R A \simeq Q$ so we sometimes dispense with explicitly mentioning Q and say a finite R-algebra A is an R-order if it is a torsion-free R-module such that $K \otimes_R A$ is a central simple K-algebra. We define the *degree* of A to be $\deg A := \deg K \otimes_R A = \sqrt{\dim_K K \otimes_R A}$.

One ought to think of A as a model of the noncommutative "field" Q in this case. The classical noncommutative analogue of the notion of normality is that the order is *maximal*, that is, if A' is another order in Q containing A, then $A = A'$. Unfortunately, this notion is not stable under étale localisation. When R is two dimensional, the following condition was introduced in [Chan and Ingalls 2005] to remedy this defect, taking its cue from Serre's criterion for normality in the commutative case.

Definition 2.1. Let R be a two-dimensional normal domain. An R-order A is said to be *normal* if

(1) A is a reflexive R-module,

(2) for every height-one prime $\mathfrak{p}$, the localisation $A_{\mathfrak{p}}$ is *normal* in the sense that its radical is principal as a left and right ideal.

The second condition is thoroughly analysed in the Appendix. Note that maximal orders are normal, and that normal orders are tame.

Let R be a two-dimensional normal domain and A be a normal R-order. Since $K \otimes_R A$ is a central simple K-algebra, it determines a corresponding Brauer class $\beta_A \in \operatorname{Br} K$. Given any codimension-one prime $\mathfrak{p} \lhd R$, with corresponding residue field $\kappa(\mathfrak{p})$, there is a ramification map

$$a_{\mathfrak{p}} : \operatorname{Br} K \to H^1_{\text{ét}}(\kappa(\mathfrak{p}), \mathbb{Q}/\mathbb{Z}).$$

As noted by Artin and Mumford [1972], this map can be interpreted in terms of orders as follows. First note that $a_{\mathfrak{p}}(\beta_A)$, being an element of $H^1_{\text{ét}}(\kappa(\mathfrak{p}), \mathbb{Q}/\mathbb{Z})$, is given by a cyclic field extension κ' of $\kappa(\mathfrak{p})$ and a choice of generator σ for the Galois group $\operatorname{Gal}(\kappa'/\kappa(\mathfrak{p}))$. Let J be the radical of $A_{\mathfrak{p}}$ which is principal, and so is generated by an element, say π. Then $\kappa' = Z(A_{\mathfrak{p}}/J)$ and σ is the automorphism induced by conjugation by π. Note that $a_{\mathfrak{p}}(\beta_A) = 0$ means that $A_{\mathfrak{p}}$ is Azumaya so the collection of nonzero $a_{\mathfrak{p}}(\beta_A)$ is called the ramification data of A. If $\mathfrak{m} \supset \mathfrak{p}$ is a codimension-two prime, then one can also look at the ramification of the field extension $\kappa'/\kappa(\mathfrak{p})$ at $\mathfrak{m}$, which is referred to as *secondary ramification*. The above results are described in more detail in [Artin and de Jong 2004, §1; Chan 2010, §4; 2011; Grieve and Ingalls 2021, §1] where other phenomena such as the cancellation of secondary ramification data are also explained.

If now B is a normal order contained in the maximal order A above, then from the Appendix, we know that $Z(B_{\mathfrak{p}}/\operatorname{rad} B_{\mathfrak{p}}) \simeq \prod_{i=1}^{d} \kappa'$ for some d. Furthermore, conjugation by a generator t of $\operatorname{rad} B_{\mathfrak{p}}$ permutes the d factors cyclically, and conjugation by t^d reduces to σ. The ramification data of B will thus not only include the $a_{\mathfrak{p}}(\beta_A)$, but also the integers $g_{\mathfrak{p}} := d$. Since B is generically Azumaya and thus maximal, we find on varying $\mathfrak{p}$ that all but finitely many of the $g_{\mathfrak{p}}$ will be one.

Definition 2.2. A *localised Brauer class on R* is a pair $(\beta, g_{\mathfrak{p}})$ consisting of a Brauer class $\beta \in \operatorname{Br} K$ and a function assigning to each codimension-one prime $\mathfrak{p} \lhd R$ a positive integer $g_{\mathfrak{p}}$ which equals one for all but finitely many $\mathfrak{p}$. In particular, the localised Brauer class of the normal order B above is $(\beta_A, g_{\mathfrak{p}})$ in the notation of the previous paragraph.

We now give an instance where the localised Brauer class and R-rank of a normal R-order determines the isomorphism class of the order.

Assumption 2.3. *Suppose now that R is a Hensel local two-dimensional normal domain. Let Δ be a maximal R-order in a division ring and suppose that there exists a normal element $z \in \Delta$ such that*

(1) *the quotient $\Delta/z\Delta$ is supported, as an R-module, on a codimension-one prime $\mathfrak{q}$,*

(2) *the element z generates the radical of $\Delta_{\mathfrak{q}}$,*

(3) *$\Delta/z\Delta$ is hereditary.*

Under these assumptions, we construct the order

$$\Delta_d = \Delta_d(z) := \begin{pmatrix} \Delta & \Delta & \cdots & \Delta \\ z\Delta & \Delta & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ z\Delta & \cdots & z\Delta & \Delta \end{pmatrix} \subseteq M_d(\Delta) \tag{2-1}$$

Definition 2.4. We will refer to the subalgebra Δ_d in (2-1) above as a *triangular modulo z matrix algebra.*

Proposition 2.5. *Under Assumption 2.3, the order Δ_d is normal and has global dimension two. Its localised Brauer class $(\beta, g_{\mathfrak{p}})$ is given by*

(1) *β is the Brauer class of Δ,*

(2) *$g_{\mathfrak{q}} = d$ and all other $g_{\mathfrak{p}} = 1$.*

Proof. Note (1) follows from the fact that Δ_d is an order in $M_n(K\Delta)$. To check Δ_d is a reflexive R-module, note first that the Δ is reflexive being a maximal order. Also, Δ is a domain so z must be a non-zero-divisor. Thus $z\Delta$ and hence also Δ_d are reflexive as well.

We consider now local structure at a codimension-one prime $\mathfrak{p}$. If $\mathfrak{p} \neq \mathfrak{q}$, then from Assumption 2.3(1), we know that $(\Delta_d)_{\mathfrak{p}} \simeq M_d(\Delta_{\mathfrak{p}})$ so is maximal. On the other hand, Assumption 2.3(2) ensures that $(\Delta_d)_{\mathfrak{q}}$ is normal and $g_{\mathfrak{q}} = d$. This completes the verification of (2).

Finally, consider the normal element

$$t := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \vdots & & \ddots & 1 \\ z & 0 & \cdots & \cdots & 0 \end{pmatrix} \in \Delta_d.$$

By Assumption 2.3(3), we see that $\Delta_d / t\Delta_d \simeq \prod_{i=1}^d \Delta/z\Delta$ is hereditary, so Δ_d itself must have global dimension two. $\qquad\square$

In the light of this proposition, we might consider Δ_d to be regular and almost maximal as in the title of this section.

Theorem 2.6. *Suppose that Assumption 2.3 holds and let Δ_d be the normal R-order of (2-1). Let Λ be any normal R-order with the same localised Brauer class as Δ_d. Then $n = \frac{\deg \Lambda}{\deg \Delta_d}$ is an integer and $\Lambda \simeq M_n(\Delta_d)$ so has global dimension two.*

Proof. Suppose $\deg \Delta_d \mid \deg \Lambda$ and let $\Lambda' := M_n(\Delta_d)$. Since the localised Brauer classes of Λ, Λ' coincide, as do their degrees, we may embed them both in a common central simple K-algebra Q. By Corollary A.6, we know that $\Lambda_{\mathfrak{q}} \simeq \Lambda'_{\mathfrak{q}}$ so by altering one of the embeddings, we may suppose that we actually have $\Lambda_{\mathfrak{q}} = \Lambda'_{\mathfrak{q}}$. Consider the (Λ, Λ')-bimodule $B := (\Lambda\Lambda')^{**} \subset Q$, the reflexive hull of the

R-module $\Lambda\Lambda'$. Now Λ' has global dimension two by Proposition 2.5 and B is Cohen–Macaulay as an R-module so is projective as a Λ'-module by [Ramras 1969, Proposition 3.5] (the hypotheses are stated differently there but the proof applies in our case).

We first show an isomorphism of right Λ' modules, $B_{\Lambda'} \simeq \Lambda'_{\Lambda'}$. Now R is Henselian so Krull–Schmidt holds for Λ'-modules. The indecomposable projective Λ'-modules are isomorphic to summands of Λ' and there are exactly d isomorphism classes of these, say $P_1, \ldots, P_d$, corresponding to the rows of Δ_d. From Proposition A.3 there exists an integer r such that $(P_i)_{\mathfrak{q}}/(P_i)_{\mathfrak{q}}(\operatorname{rad}\Lambda'_{\mathfrak{q}}) \simeq S_i^{\oplus r}$ where S_i is a simple $\Lambda'_{\mathfrak{q}}$ module and that, furthermore, the S_i are all nonisomorphic. In particular, two finitely generated projective Λ'-modules are isomorphic if and only if their localisations at $\mathfrak{q}$ are isomorphic. Now by our choice of embeddings, $B_{\mathfrak{q}} = \Lambda'_{\mathfrak{q}}$ so $B \simeq \Lambda'$ as desired.

To complete the proof of the theorem when $\deg\Delta_d \mid \deg\Lambda$, it suffices to show that the natural map $\Lambda \to \operatorname{End}_{\Lambda'} B = \Lambda'$ is an isomorphism. Since both sides are reflexive, this can be checked on codimension-one primes $\mathfrak{p}$. When $\mathfrak{p} \neq \mathfrak{q}$, it is an isomorphism since Λ is maximal. When $\mathfrak{p} = \mathfrak{q}$, it is an isomorphism since we recalibrated so $\Lambda_{\mathfrak{q}} = \Lambda'_{\mathfrak{q}} = B_{\mathfrak{q}}$.

If $\deg\Delta_d$ does not divide $\deg\Lambda$, we apply the special case proved to $M_{\deg\Delta_d}(\Lambda)$, which shows that at least Λ is Morita equivalent to Δ_d. Hence $\Lambda \simeq \operatorname{End}_{\Delta_d} P$ for some projective Δ_d-module. We can argue as before, looking locally at $\mathfrak{q}$ to see that d indecomposable projective modules occur equally in the decomposition of P, so $P \simeq \Delta_d^{\oplus n}$ for some n as desired. $\qquad\square$

3. Toral terminal orders, regular centre case

We have a series of concepts that are motivated by toric or toroidal geometry, but that does not satisfy either of these definitions. We call these *toral*. Let $(R, \mathfrak{m})$ be a two-dimensional noetherian regular Hensel local domain with field of fractions K. We introduce the notion of a *toral terminal* localised Brauer class on R. These are all terminal. In the special setting of [Chan and Ingalls 2021], i.e., when R is an arithmetic surface with finite residue field and the ramification data are all p-torsion for some prime p, this exactly agrees with terminal. Assuming that R has enough roots of unity and a trivial Brauer group, we then classify the normal R-orders associated to toral terminal localised Brauer classes and show they all are regular in the sense that they have global dimension two.

In the regular centre case, toral terminal localised Brauer classes fall into two types. The first, without secondary ramification is defined below.

Definition 3.1. A localised Brauer class $(\beta, g_{\mathfrak{p}})$ on R is *toral terminal without secondary ramification* if there exists a regular system of parameters $u, v \in R$ such that

(1) β is unramified at every codimension-one prime $\mathfrak{p}$ except possibly $\mathfrak{p} = (u)$ (that is, $a_{\mathfrak{p}}(\beta) = 0$ for $\mathfrak{p} \neq (u)$), and

(2) all $g_{\mathfrak{p}} = 1$ except possibly $\mathfrak{p} = (v)$.

We will construct orders via symbols and so need to assume the existence of enough roots of unity.

Definition 3.2. Suppose $\zeta \in R$ is a primitive n-th root of unity. Given $a, b \in R - 0$ we define the R-symbol $(a, b) := (a, b)_\zeta^R$ to be the R-algebra

$$\Lambda = \frac{R\langle x, y \rangle}{(x^n - a, \, y^n - b, \, yx - \zeta xy)}.$$

Proposition 3.3. *Let R be a two-dimensional regular noetherian Hensel local domain and $(\beta, g_{\mathfrak{p}})$ be a toral terminal localised Brauer class with ramification as given in Definition 3.1. Suppose that $\mathrm{Br}\, R/\mathfrak{m} = 0$ and R possesses a primitive n-th root of unity ζ where n is the order of β in the $\mathrm{Br}\, K$.*

Let $a \in R$ be any element chosen so the ramification of β along (u) is given by adjoining an n-th root of a. Then

(1) *$\Delta = (u, a)_\zeta^R$ is a maximal order in a division ring with the same ramification data as β,*

(2) *any normal order Λ with localised Brauer class $(\beta, g_{\mathfrak{p}})$ is isomorphic to $M_m(\Delta_d(v))$ (see notation in (2-1)) where $d = g_{(v)}$ and m is an arbitrary positive integer.*

In particular, Λ has global dimension two.

Proof. As secondary ramification must cancel, the ramification of β along (u) must be given by an étale cyclic extension of $R/(u)$, say of degree n. Now the cyclic étale extensions of $R/\mathfrak{m}$, $R/(u)$ and R all coincide, so we may find $a \in R$ defining this ramification, by adjoining $\sqrt[n]{a}$. Also, since $\mathrm{Br}\, R = \mathrm{Br}\, R/\mathfrak{m} = 0$, we know n is the order of β.

Note that $\Delta = (u, a)_\zeta^R$ is Azumaya on the open set $u \neq 0$ and is maximal at the generic point $\mathfrak{p}$ of $u = 0$ and has the same ramification as β at $\mathfrak{p}$. Now reflexive orders which are maximal in codimension one are maximal globally by [Auslander and Goldman 1960, Theorem 1.5]. Thus since Δ is also reflexive, it is a maximal order which has the same ramification as that of β. Furthermore, as already observed, $\mathrm{Br}\, R = 0$ so both β and Δ determine the same Brauer class in $\mathrm{Br}\, K$.

We seek now to apply Theorem 2.6. We begin by verifying Assumption 2.3 for $z = v$. First, Δ is a domain since its degree coincides with the period. Clearly $\Delta/v\Delta$ is supported along the prime (v) only, and in fact v generates the radical of the localisation $\Delta_{(v)}$. It remains to verify Assumption 2.3(3). Let $x \in \Delta$ be the n-th root of u as in Definition 3.2. Then x gives a non-zero-divisor in $\Delta/v\Delta$. Furthermore, $\Delta/(x, v)$ is the separable extension $(R/\mathfrak{m})(\sqrt[n]{a})$ so $\Delta/v\Delta$ is indeed hereditary. $\qquad\square$

Definition 3.4. A localised Brauer class $(\beta, g_{\mathfrak{p}})$ on R is *toral terminal with secondary ramification* if there exists a regular system of parameters such that

(1) β is unramified away from (uv),

(2) β is ramified along both (u) and (v) and the ramification at these prime ideals are given by totally ramified field extensions of the residue fields,

(3) all $g_{\mathfrak{p}}$ equal 1 except possibly $g_{(v)}$.

More generally (but still assuming R regular), we say a localised Brauer class $(\beta, g_{\mathfrak{p}})$ is *toral terminal* if it is either toral terminal with secondary ramification as above, or toral terminal without secondary ramification as in Definition 3.1.

Proposition 3.5. *Let R be a two-dimensional regular noetherian Hensel local domain and $(\beta, g_{\mathfrak{p}})$ be a toral terminal localised Brauer class with secondary ramification as given in Definition 3.4. Suppose that $\operatorname{Br} R/\mathfrak{m} = 0$ and R possesses a primitive n-th root of unity where n is the order of β in $\operatorname{Br} K$. Any normal order Λ with localised Brauer class $(\beta, g_{\mathfrak{p}})$ is isomorphic to $M_m(\Delta_d(y))$ (see notation in (2-1)) where*

(1) $d = g_{(v)}$,

(2) $\Delta_d(y)$ *is built from the maximal order*

$$\Delta = (au, bv)_\zeta^R,$$

where $a, b \in R^\times$ are units, ζ is an appropriate n-th root of unity and y is the n-th root of bv used in Definition 3.2.

In particular, Λ has global dimension two.

Proof. We use Theorem 2.6 along the same lines as the proof of Proposition 3.3. It suffices to find u, v, ζ such that $(u, v)_\zeta^R$ has the same ramification as β.

Let κ_u denote the residue field at the point (u). The ramification $a_{(u)}(\beta) \in H^1(\kappa_u, \mathbb{Q}/\mathbb{Z})$ of β along (u) corresponds to a cyclic field extension $\tilde{\kappa}/\kappa_u$ and a generator of the Galois group. Since we assumed existence of primitive n-th roots of unity, we may use Kummer theory to see that $\tilde{\kappa} = \kappa_u(\sqrt[n]{\bar{v}})$ for some $\bar{v} \in \kappa_u$. Since this is a totally ramified extension, we may change generators and assume that $\bar{v}$ is the restriction of bv for some $b \in R^\times$. If σ is the chosen generator of the Galois group, then $\sigma(\sqrt[n]{bv}) = \zeta \sqrt[n]{bv}$ where ζ is the n-th root of unity required in (2) above. Arguing the same way for ramification of β along (v) and using the fact that secondary ramification cancels, we see that $a_{(v)}(\beta)$ is given by adjoining an n-th root of au for some $a \in R^\times$ and we are done. $\square$

4. Hirzebruch–Jung singularities as cyclic quotient singularities

Over the complex numbers the Hirzebruch–Jung singularities are well understood and all arise as cyclic quotient singularities. We show a similar result for two-dimensional normal singularities which are Hirzebruch–Jung in the sense that their minimal resolution is a string of projective lines defined over the residue field. A related result can be found in [Kollár 2013, Theorem 3.32], the difference being that we do not assume the existence of an underlying ground field.

More precisely, suppose that R is a two-dimensional normal noetherian excellent commutative Hensel local domain with residue field κ. Suppose it is a *κ-rational Hirzebruch–Jung* singularity in the sense that it has a rational minimal resolution $f : Y \to \operatorname{Spec} R =: X$ such that the exceptional locus is a string $E_1, \ldots, E_r$ of exceptional curves isomorphic to the projective line over κ, that is, all E_i are isomorphic to $\mathbb{P}^1_\kappa$, E_i intersects E_{i+1} in a single point which is κ-rational and there are no other intersections. There

are more complicated analogues of Hirzebruch–Jung singularities studied in the literature which we have not studied as they do not arise in the study of the terminal orders considered in this paper.

Since R is Hensel local, we may choose irreducible curves $E_0, E_{r+1} \subset Y$ such that E_0 (respectively, E_{r+1}) intersects E_1 (respectively, E_r) in a single κ-rational point and E_0 and E_2 (respectively, E_{r+1} and E_{r-1}) are disjoint. Let $m_i = -E_i^2$ which, by minimality of the resolution f, must be at least two. We define pairs $v_0 = (0, 1)$, $v_1 = (1, 0)$ and then recursively define

$$v_{i+1} = m_i v_i - v_{i-1} \quad \text{for } i = 1, \ldots, r. \tag{4-1}$$

When R is defined over the complex numbers, these give the exceptional curves in the toric description of the Hirzebruch–Jung singularity. An easy induction shows that the dot product $(1, 1) \cdot v_i$ weakly increases with i, so $(m, -k) := v_{r+1}$ satisfies $0 < k < m$. Similarly, one can show that k, m are relatively prime. The integer m appears in our key theorem below.

Theorem 4.1. *Let R be a κ-rational Hirzebruch–Jung singularity as defined above and $X = \operatorname{Spec} R$. The Weil divisor $f_* E_0$ is m-torsion in the sense that $m f_* E_0$ is Cartier. There is a natural ring structure on*

$$S := \bigoplus_{l=0}^{m-1} \mathcal{O}_X(-l f_* E_0) t^l$$

making it a regular local ring with the same residue field κ as R. In particular, if R contains m-th roots of unity, then R is a cyclic quotient singularity. Furthermore, S/R is étale away from the singular point.

Before launching in to the proof, we set up the appropriate theory first. The idea is to recover as much of the toric theory of Hirzebruch–Jung singularities over the complex numbers as possible. To this end we consider:

Definition 4.2. A divisor D on Y is *toral* if it belongs to $\bigoplus_{i=0}^{r+1} \mathbb{Z}E_i$. We say $w \in R$ is *toral* or is a *toral function* if the associated divisor of $f^* w$ is toral.

Remark 4.3. Definition 4.2 depends on the choice of E_0 and E_{r+1}.

Our first order of business is to classify all toral functions and show their divisors on Y are given by the lattice points in the cone $\mathbb{R}_{\geq 0}(0, 1) + \mathbb{R}_{\geq 0}(m, -k)$ where we recall $(m, -k) = v_{r+1}$. Any function $w \in R$ is determined, up to $H^0(\mathcal{O}_Y^\times) = R^\times$, by its divisor $(f^* w)$ on Y. It thus suffices to classify effective toral divisors D on Y such that $D \sim 0$. Now f is a rational resolution, so by [Lipman 1969, Proposition 11.1 i)], we know $D \sim 0$ if and only if $D.E_i = 0$ for $i = 1, \ldots, r$.

To enumerate all such toral divisors, we work as follows. Let $L = \bigoplus_{i=0}^{r+1} \mathbb{Z}E_i$ and consider $L^* := \operatorname{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) = \bigoplus \mathbb{Z}E_i^*$ where $\{E_i^*\}$ is a dual basis to the E_i. For $i = 1, \ldots, r$, let $E_i^\vee := (C \mapsto E_i.C) \in L^*$ and $\mathbb{E} < L^*$ be the subgroup generated by the $E_i^\vee$. The next result follows from (4-1).

Proposition 4.4. *The homomorphism $v : L^* \to \mathbb{Z}^2$ defined by $E_i^* \mapsto v_i$ is surjective with kernel $\mathbb{E}$.*

The $D \sim 0$ condition ensures that the naturally induced map $L^* \to \mathbb{Z}$ given by $\chi \mapsto \chi(D)$ actually factors through ν to give a homomorphism $\lambda_D : \mathbb{Z}^2 \to \mathbb{Z}$. Note that

$$D = \sum_{i=0}^{r+1} \lambda_D(\nu_i) E_i. \tag{4-2}$$

Suppose now that λ_D is given by dot product with $(i, j) \in \mathbb{Q}^2$. Now $\nu_0 = (0, 1)$, $\nu_1 = (1, 0)$ so $i, j \in \mathbb{Z}$. The divisor D corresponding to (i, j) is effective when $(i, j) \cdot \nu_0 \geq 0$, $(i, j) \cdot \nu_{r+1} \geq 0$, that is, (i, j) lies in the cone $\mathbb{R}_{\geq 0}(1, 0) + \mathbb{R}_{\geq 0}(k, m)$. We now abuse notation and write $x^i y^j$ for any toral function with this D as its divisor in Y. The notation allows us to write $x^i y^j x^{i'} y^{j'} = x^{i+i'} y^{j+j'}$ with the caveat that it holds only modulo $R^\times$. We summarise the results up to this point.

Proposition 4.5. *The toral functions are $x^i y^j$ where $(i, j) \in \mathbb{R}_{\geq 0}(1, 0) + \mathbb{R}_{\geq 0}(k, m)$. Its divisor in Y is $\sum \lambda_l E_l$ where $\lambda_l = (i, j) \cdot \nu_l$.*

Corollary 4.6. *The Weil divisor $f_* E_0$ is m-torsion.*

Proof. The divisor of zeros $\sum \lambda_l E_l$ of the toral function $x^k y^m$ has $\lambda_0 = m$, $\lambda_{r+1} = 0$. $\qquad\square$

Let $\mathfrak{m}$ be the maximal ideal of R and Z be the fundamental cycle so $\mathfrak{m}^n = f_* \mathcal{O}(-nZ)$ for any positive integer n [Liu 2002, Lemma 9.4.14]. Note that $Z = E_1 + \cdots + E_r$. Consider a Weil divisor $C \subset \operatorname{Spec} R$ so $\mathcal{O}(-C)$ is a reflexive ideal. Note that $f^* \mathcal{O}(-C)/\mathcal{T} \simeq \mathcal{O}_Y(-\widetilde{C})$ for some $\mathfrak{m}$-torsion sheaf $\mathcal{T}$ and Cartier divisor $\widetilde{C}$ which is the strict transform of C away from the exceptional locus. Here, by $\mathfrak{m}$-torsion, we mean that $\mathcal{T}$ is annihilated by some positive power of $\mathfrak{m}$.

Definition 4.7. We call $\widetilde{C}$ the *pullback* of C. This agrees with the usual pullback in the case that C is Cartier.

Reflexivity ensures that $f_* \mathcal{O}_Y(-\widetilde{C}) = \mathcal{O}(-C)$ so $\mathcal{O}(-C)$ is *contracted* in the language of [Lipman 1969, Definition 6.1]. Suppose more generally that $D \in \operatorname{Div} Y$ is such that $\mathcal{O}_Y(-D)$ is generated by global sections so [Lipman 1969, Corollary to 7.3] ensures that $\mathfrak{m}^n f_* \mathcal{O}_Y(-D) = f_* \mathcal{O}_Y(-D - nZ)$. Applying f_* to the exact sequence

$$0 \to \mathcal{O}_Y(-D - Z) \to \mathcal{O}_Y(-D) \to \mathcal{O}_Z(-D) \to 0$$

gives the exact sequence

$$0 \to f_* \mathcal{O}(-D) \otimes_R R/\mathfrak{m} \to H^0(\mathcal{O}_Z(-D)) \to R^1 f_* \mathcal{O}_Y(-D - Z).$$

Now $\mathcal{O}_Y(-D - Z)$ is generated by global sections since the same is true of $\mathcal{O}_Y(-D)$ and $\mathcal{O}_Y(-Z)$, so $R^1 f_* \mathcal{O}_Y(-D - Z)) = 0$ from which follows the next result.

Lemma 4.8. *Let D be a divisor on Y such that $\mathcal{O}_Y(-D)$ is generated by global sections. Write $I = f_* \mathcal{O}_Y(-D)$ which will be an ideal in R if D is effective. Then we have $I \otimes_R R/\mathfrak{m} \simeq H^0(\mathcal{O}_Z(-D))$. In particular, a set of generators for I can be found by giving a set of global sections of $\mathcal{O}_Y(-D)$ whose*

restriction to Z gives a spanning set for $H^0(\mathcal{O}_Z(-D))$. This applies in particular to $I = \mathcal{O}(-C)$ where C is an effective Weil divisor on X and $D = \widetilde{C}$ is the pullback.

We will use this lemma to find toral generators for $\mathcal{O}(-if_*E_0)$. First, we need a "toral" basis for $H^0(Z, \mathcal{L})$ where $\mathcal{L}$ is a line bundle on Z with all $d_i := \deg_{E_i} \mathcal{L}$ greater than or equal to 0.

Definition 4.9. The intersections $E_i \cap E_{i+1}$, $i = 0, \ldots, r$, are said to be the *toral points* of Z. A nonzero section $s \in H^0(Z, \mathcal{L})$ is *toral* if its zero set is a union of exceptional curves and toral points. We say s is *basic toral* if it also satisfies the following condition: whenever $s|_{E_i} \neq 0$ but has a zero at $E_i \cap E_{i+1}$ (respectively, $E_i \cap E_{i-1}$), then $s|_{E_j} = 0$ for $j > i$ (respectively, $j < i$).

We can construct basic toral sections as follows. Start with some nonzero section $s_i \in H^0(E_i, \mathcal{L}|_{E_i}) \simeq H^0(\mathbb{P}^1, \mathcal{O}(d_i))$ which is "toral" in the sense that its zeros are confined to $E_{i-1} \cup E_{i+1}$. Up to a scalar in κ, there are $d_i + 1$ of these. We show it can be extended uniquely to a basic toral section s of $\mathcal{L}$. Now if s_i has a zero at $E_{i-1} \cap E_i$, then we simply extend by setting $s|_{E_j} = 0$ for $j < i$. If on the other hand s_i is nonzero at $E_{i-1} \cap E_i$, then there is a unique way to extend it to a toral section on E_{i-1} and we can continue by induction. A similar argument determines s on E_j for $j > i$. This gives the following:

Lemma 4.10. *Any basic toral section s is uniquely determined by any nonzero restriction $s|_{E_l}$ and has the form constructed in the preceding paragraph.*

Proposition 4.11. *Given a line bundle $\mathcal{L}$ on Z with nonnegative degrees $d_i := \deg_{E_i} \mathcal{L} \geq 0$, there exists a basis for $H^0(Z, \mathcal{L})$ consisting of basic toral sections. This basis is unique up to scaling the basis elements.*

Proof. First, $H^0(Z, \mathcal{L})$ is naturally isomorphic to the kernel of the natural map $\bigoplus_{l=1}^r H^0(E_l, \mathcal{O}_{E_i}(d_i)) \to H^0(T, \mathcal{O}_T)$ where T is the set of nodes in Z. This has dimension $d = \sum(d_i + 1) - (r - 1)$. The only linear relations between basic toral sections are those which are scalar multiples of each other so it suffices to find d basic toral sections, no two of which are multiples of each other. The above construction provides these once we note that the basic toral section constructed from some toral section $s_i \in H^0(E_i, \mathcal{L}|_{E_i})$ coincides with one constructed from $s_{i-1} \in H^0(E_{i-1}, \mathcal{L}_{E_{i-1}})$ if and only if s_i, s_{i-1} take on the same nonzero value at $E_{i-1} \cap E_i$. $\square$

We can now construct toral generators for reflexive ideals in R.

Proposition 4.12. *Let D be an effective toral divisor on Y such that $\mathcal{O}_Y(-D)$ is generated by global sections and $I = f_*\mathcal{O}_Y(-D)$ be the associated ideal of R. Then I is generated by toral functions.*

Proof. Combining Lemma 4.8 with Proposition 4.11, it suffices to lift every basic toral section of $\mathcal{O}_Z(-D)$ to a toral section of $\mathcal{O}_Y(-D)$. Let $d_i := -D.E_i$ and consider a basic toral section whose restriction to E_i has a zero of order e at E_{i-1} and order $d_i - e$ at E_{i+1}. Lifting this to a toral function amounts to finding an effective toral divisor $\Delta = \sum_{j=0}^{r+1} \delta_j E_j$ such that

a) $-D \sim \Delta$, and

b) $\delta_{i-1} = e$, $\delta_i = 0$, $\delta_{i+1} = d_i - e$.

Condition a) amounts to checking that all the intersection numbers $(D + \Delta).E_j$ equal 0. We solve these equations for δ_j by induction on $|j - i|$ and simultaneously prove effectivity of Δ by proving δ_j is nondecreasing for $j \geq i$ and nonincreasing for $j \leq i$. The base case is satisfied since

$$(D + \Delta).E_i = \delta_{i-1} + D.E_i + \delta_{i+1} = e - d_i + (d_i - e) = 0.$$

Suppose now that nondecreasing integers $\delta_i, \ldots, \delta_j$ have now been defined satisfying $(D + \Delta).E_l = 0$ for $l = i, \ldots, j$. We examine the equation

$$0 = (D + \Delta).E_j = -d_j + \delta_{j-1} - m_j \delta_j + \delta_{j+1}.$$

We may thus solve for the integer δ_{j+1} which further satisfies

$$\delta_{j+1} - \delta_j = d_j + ((m_j - 1)\delta_j - \delta_{j-1}) \geq 0$$

by the inductive hypothesis. A similar argument works for nonincreasing δ_j when $j \leq i$. $\qquad\square$

Proof. We can now complete the proof of Theorem 4.1. Inspired by Corollary 4.6, or rather its proof, we define a ring structure on

$$S = \bigoplus_{l=0}^{m-1} \mathcal{O}(-lf_*E_0)t^l$$

by defining $t^{-m} = x^k y^m$. Given a toral function $x^i y^j \in \mathcal{O}(-lf_*E_0)$, we say

$$x^i y^j t^l = x^{i-kl/m} y^{j-l}$$

is a toral function in S. It suffices to find two toral functions in $f_1, f_2 \in S$ which generate the maximal ideal

$$\mathfrak{n} := \mathfrak{m} \oplus \mathcal{O}(-f_*E_0)t \oplus \cdots \oplus \mathcal{O}(-(m-1)f_*E_0)t^{m-1}.$$

By Proposition 4.12, it suffices to show f_1, f_2 will generate all the toral elements in the summands $\mathfrak{m}, \ldots, \mathcal{O}(-(m-1)f_*E_0)t^{m-1}$. Our sloppiness in notation for $x^i y^j$ is warranted since we only care about the ideal generated by f_1, f_2.

From Proposition 4.5, we know that $x^i y^j$ is a toral function in $\mathcal{O}(-lf_*E_0)$ if and only if $(i, j) \in \mathbb{R}_{\geq 0}(1, 0) + \mathbb{R}_{\geq 0}(k, m)$ and furthermore $l \leq (i, j).v_0 = j$. We may thus let

$$f_1 = x^k y^m t^{m-1} = x^{k/m} y. \tag{4-3}$$

To find f_2 we first find $l \in \{1, \ldots, m-1\}$ which solves $kl \equiv -1 \mod m$, which is possible since k and m are relatively prime. Let $i = \frac{kl+1}{m}$ and

$$f_2 = x^i y^l t^l = x^{1/m}. \tag{4-4}$$

An elementary calculation shows that the toral elements of S all have the form $x^i y^j$ where $i \in \frac{1}{m}\mathbb{Z}$, $j \in \mathbb{Z}$ and $0 \leq j \leq \frac{m}{k}i$. It follows that all the toral elements in $\mathfrak{n}$ are generated by f_1 and f_2.

Finally, the construction of S here is the cyclic covering trick, see, for example, [Lazarsfeld 2004, 4.1.B], which away from the singularity uses an m-torsion line bundle so S/R is étale. □

For use in the next section, we record the following fact which follows from Proposition 4.5.

Lemma 4.13. *The toral function $f_1 \in S$ defined in (4-3) is such that $f_1^m \in R$ and its divisor is $m f_* E_0$.*

5. Toral terminal orders, singular centre case

Unlike in the geometric case where the residue fields are algebraically closed, there are now terminal orders with singular centre [Chan and Ingalls 2021]. Their ramification data were classified in the case where the "index" [Chan and Ingalls 2021, §3, p. 6] was a prime $m > 5$. Here we classify the corresponding orders, giving explicit constructions of them.

Throughout this section, we let $(R, \mathfrak{m})$ be an excellent normal two-dimensional noetherian Hensel local domain with residue field κ. The classification of terminal ramification data on R is best encapsulated via the following definition.

Definition 5.1. A localised Brauer class $(\beta, g_{\mathfrak{p}})$ on R is *toral terminal* if either R is regular and we are in the case of Definitions 3.1 or 3.4, or if the following hold:

(1) R is a κ-rational Hirzebruch–Jung singularity whose residue field has trivial Brauer group. Let $E_1, \ldots, E_r$ be the string of exceptional curves in the minimal resolution (indexed naturally so E_i intersects E_{i+1}).

(2) β is unramified along codimension-one primes in R.

(3) The order m of β equals the *determinant* of R which is defined to be $\det R := \det(M_R)$ where

$$
M_R := - \begin{pmatrix}
E_1^2 & 1 & 0 & \cdots & 0 \\
1 & E_2^2 & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 1 \\
0 & \cdots & 0 & 1 & E_r^2
\end{pmatrix}.
$$

(4) At most one $g_{\mathfrak{p}}$ does not equal 1, in which case $\mathfrak{p}$ corresponds to an irreducible curve C on the minimal resolution which (scheme-theoretically) intersects the exceptional curve in a single κ-rational point $y \notin E_2 \cup \cdots \cup E_{r-1}$.

Remark 5.2. (1) Given a toral terminal localised Brauer class $(\beta, g_{\mathfrak{p}})$ on R as above, [Chan and Ingalls 2021, Theorem 7.1] shows that it is terminal whenever m is a prime > 2. If m is a prime > 5 and κ is finite, then these are the only terminal localised Brauer classes on singular R.

(2) In the same article, the residue fields are all assumed to be finite and so have trivial Brauer group.

Let $(R, \mathfrak{m})$ be a κ-rational Hirzebruch–Jung singularity of determinant m (see Definition 5.1(3)). Suppose that κ contains a primitive m-th root of unity. From Theorem 4.1, there exists a regular local

ring $(S, \mathfrak{n})$ such that $S/\mathfrak{n} = \kappa$ and S/R is a cyclic extension which is étale away from the singular point. In Theorem 4.1, it is presented as a $\mathbb{Z}/m$-graded algebra $S = \bigoplus_{i \in \mathbb{Z}/m} S_i$ so $(\mathbb{Z}/m)^\vee = \mu_m$ acts on it naturally. Let $\alpha \in H^1(\kappa, \mathbb{Z}/p)$ correspond to a cyclic degree-m field extension $\tilde{\kappa}/\kappa$ with some chosen action of $\mathbb{Z}/m$. Since R is Hensel local, there is a corresponding cyclic étale extension $\tilde{R}/R$ and a corresponding μ_m-graded decomposition $\tilde{R} = \bigoplus_{\omega \in \mu_m} \tilde{R}_\omega$.

Definition 5.3. We define the *symbol* (S, α) to be the R-algebra whose underlying R-module structure is given by

$$\Delta := S \otimes_R \tilde{R}$$

and multiplication given by the skew-commutation relations

$$r s = \omega^i s r \quad \text{for all } s \in S_i, \ r \in \tilde{R}_\omega.$$

Proposition 5.4. *The symbol $\Delta = (S, \alpha)$ defined above is a maximal order in a division ring, and Δ is Azumaya in codimension one.*

Proof. The commutative algebra $S \otimes_R \tilde{R}$ is an étale extension of S and hence regular and thus Cohen–Macaulay. It follows that Δ is a reflexive R-module. In codimension one, both S/R and $\tilde{R}/R$ are étale so Δ is defined using the usual symbol construction of Azumaya algebras. We see thus that Δ is a maximal order and is Azumaya in codimension one.

It only remains to show that $\Delta_K := \Delta \otimes_R K(R)$ is a division ring, which we do by showing that it cannot have period $< m$. Suppose that $\tilde{R}$ is obtained by adjoining an m-th root of $\alpha \in R^\times$ to R. Suppose that the order of Δ_K is $n \mid m$. If σ denotes the action of some fixed primitive m-th root of unity on S, then the cyclic algebra $A := K(S)[z; \sigma]/(z^m - \alpha^n)$ is a full matrix algebra over $K(R)$. We see thus from [Gille and Szamuely 2006, Corollary 4.7.5] that $\alpha^n \in K(R)$ is a norm from $K(S)$, say $\alpha^n = N(\beta)$, where $\beta = \beta_1 \beta_2^{-1}$ for $\beta_1, \beta_2 \in S$. Now S is a UFD so we may prime factorise both β_1 and β_2.

We first show that by modifying β by a 1-coboundary $\gamma^{-1} \sigma(\gamma)$, $\gamma \in K(S)$, we may assume that $\beta \in S^\times$. Note that $N(\beta_1), N(\beta_2)$ differ by the unit $\alpha^n \in R^\times$. Thus if there is any prime factor $p_2 \mid \beta_2$, there is some prime factor $p_1 \mid \beta_1$ and i such that $p_2 \mid \sigma^i(p_1)$. We may thus multiply by some 1-coboundary so that these factors now cancel. Having reduced the number of prime factors of β_2, we are done by induction.

We now use the fact that $S/\mathfrak{n} = R/\mathfrak{m}$ to see that modulo $\mathfrak{m}$, α^n is a m-th power. Since $\tilde{\kappa}$ is a degree-m extension of κ obtained by adjoining an m-th root of α, we must have $n = m$. $\square$

Proposition 5.5. *Let $(\beta, g_\mathfrak{p})$ be a toral terminal localised Brauer class on a κ-rational Hirzebruch–Jung singularity R. Suppose that R has a primitive m-th root of unity where m is the order of β. Then there is an $\alpha \in H^1(\kappa, \mathbb{Z}/m)$ such that class of the symbol (S, α) in $\operatorname{Br} K(R)$ is β.*

Proof. We use the notation in Definition 5.1. Let $X \to \operatorname{Spec} R$ be the minimal resolution of the κ-rational Hirzebruch–Jung singularity R. Note that β is ramified only along the exceptional curve, so by the Artin–Mumford–Saltman sequence [1972; 2008, Theorem 6.12], the ramification covers of the E_i are all étale. Now $\operatorname{Br} \kappa = 0$ so the ramification along E_i is given by cyclic cover of the form $\mathbb{P}^1_{\kappa_i} \to \mathbb{P}^1_\kappa \simeq E_i$

where κ_i is a cyclic extension of κ of degree $n \mid m$. The ramification is thus given by an element of $H^1(\kappa, \mathbb{Z}/p)$.

We now use the theory developed in [Chan and Ingalls 2021, Setion 4]. There was an assumption there that the residue field was finite, but the theory goes through in this case, as long as one realises that the absolute Galois group G of κ is now not necessarily $\hat{\mathbb{Z}}$. In particular, we have the following version of [Chan and Ingalls 2021, Proposition 9.8].

Lemma 5.6. *There exists a homomorphism* $z : \mathbb{Z}^2 \to H^1(\kappa, \mathbb{Z}/m)$ *such that the ramification of β along* E_i *is given by $z(\nu_i)$ where ν_i is as defined in* (4-1).

In particular, we see that the ramification of β, and hence β itself is completely determined by $z(0, 1) = z(\nu_0) = 0$ and $z(1, 0) = z(\nu_1)$, the ramification along E_1. It is now clear that we can pick $\alpha \in H^1(\kappa, \mathbb{Z}/m)$ so that the symbol (S, α) has the same ramification as β along E_1, and hence belongs to the same Brauer class over $K(R)$. $\qquad\square$

Theorem 5.7. *Let Λ be a normal order over an excellent two-dimensional Hensel local noetherian domain $(R, \mathfrak{m})$ which is not regular. Suppose its localised Brauer class $(\beta, g_{\mathfrak{p}})$ is toral terminal. If R has a primitive m-th root of unity where m is the order of β, then $\Lambda \simeq M_n(\Delta_d(z))$ where*

(1) Δ *is the symbol (S, α) where S is the regular cyclic cover of R constructed in* Theorem 4.1 *and* $\alpha \in H^1(\kappa, \mathbb{Z}/m)$,

(2) $n \in \mathbb{N}$, d *is either 1 or the unique $g_{\mathfrak{p}}$ not equal to 1, and $z \in S \subset \Delta$ is the normal element denoted f_1 in* (4-3).

In particular, Λ has global dimension two.

Proof. From Proposition 5.5, we may choose α so that $\Delta = (S, \alpha)$ represents the Brauer class β. We also know from Proposition 2.5 that Δ is a maximal order in a division ring. The result will thus follow from Theorem 2.6 once we verify Assumption 2.3. Let $f : X \to \operatorname{Spec} R$ be the minimal resolution. Using the notation in Section 4, toral terminal implies that we may pick $E_0 \subset X$ to be such that $C := f_* E_0$ corresponds to the codimension-one prime $\mathfrak{q}$ with $g_{\mathfrak{q}} \neq 1$ if such a prime exists (and is otherwise an arbitrary prime divisor intersecting $E_1 \setminus E_2$ in a κ-rational point).

Note that z is a toral function and hence gives a normal element of Δ. We also know from Lemma 4.13 that $z^m \in R$ and that its associated divisor is $m f_* E_0$. It follows that $(z) \lhd S$ is the unique prime lying over $\mathfrak{q}$. Thus $\Delta/z\Delta$ is supported on C as an R-module and Assumption 2.3(1) is verified. It also follows that z lies in the radical of $\Delta_{\mathfrak{q}}$. Consider now

$$\bar{\Delta} := \Delta/z\Delta \simeq S/(z) \otimes_{R/\mathfrak{q}} \tilde{R},$$

where $\tilde{R}$ is the cyclic étale extension of R determined by α. To see that z generates the radical of $\Delta_{\mathfrak{q}}$ it suffices to observe that $\bar{\Delta}_{\mathfrak{q}}$ is a central simple $K(R/\mathfrak{q})$-algebra since it is readily identified with a symbol. This completes the verification of Assumption 2.3(2) so it remains only to show that $\bar{\Delta}$ is hereditary. To this end, let y be the other generator of $\operatorname{rad} S$ denoted f_2 in (4-4). It is normal in Δ and thus $\bar{\Delta}$. Then

$\bar{\Delta}/y\bar{\Delta} \simeq \kappa \otimes_R \tilde{R}$ which is a field and hence has global dimension zero. This completes the proof of the theorem. $\square$

Corollary 5.8. *Let Λ be a normal order over an excellent two-dimensional normal noetherian Hensel local domain R with finite residue field. Suppose that its localised Brauer class $(\beta, g_{\mathfrak{p}})$ is terminal,*

(1) *the order m of β is prime > 5, and*

(2) *R has primitive m-th roots of unity.*

Then Λ has global dimension two.

Appendix

The theory of normal and more generally hereditary orders over a complete discrete valuation ring is well known and can be found in standard texts such as Reiner's classic text [1975]. We extend some results to arbitrary discrete valuation rings R which are not necessarily complete.

Let $\mathfrak{m}$ be the maximal ideal of R and K be its field of fractions. Let Δ be a maximal order in some K-central division ring $K\Delta$. Let $\Lambda \subseteq M_n(\Delta)$ be a hereditary order in $M_n(K\Delta)$ with say Jacobson radical J. Note that $\Delta/\mathrm{rad}\,\Delta$ is central simple, say isomorphic to $M_r(D)$ where D is a division ring. The case when R is complete is simpler because we always have $r = 1$ then.

Proposition A.1. *Consider a right projective Λ-module P such that $\mathrm{End}\,P \simeq \Delta$. Then $P/PJ \simeq S^{\oplus r}$ where S is a simple Λ-module and r is the integer such that $\Delta/\mathrm{rad}\,\Delta \simeq M_r(D)$ for some division ring D. This result holds in particular for $P = \Delta^n$.*

Proof. Consider the natural ring homomorphism

$$\Delta = \mathrm{End}_\Lambda P \to \mathrm{End}_\Lambda P/PJ.$$

This map is surjective since P is projective. From the ideal theory of maximal orders [Auslander and Goldman 1960, Theorem 2.3], the only semisimple quotient of Δ is $\Delta/\mathrm{rad}\,\Delta \simeq M_r(D)$ so P/PJ must be the direct sum of r copies of a single simple.

Finally, $\mathrm{End}_\Lambda \Delta^n = \mathrm{End}_{M_n(\Delta)} \Delta^n$ since Λ is an order in $M_n(K\Delta)$. The final statement follows thus from $\Delta = \mathrm{End}_{M_n(\Delta)} \Delta^n$ which is a consequence of Morita theory. $\square$

Definition A.2. We say that Λ is *normal* if its Jacobson radical J is free of rank 1 as a left and right module.

Suppose from now on that Λ is normal, so that one can choose a *uniformiser* $t \in J$ such that $J = \Lambda t = t\Lambda$ so the inner automorphism $r \mapsto trt^{-1}$ induces an automorphism of the semisimple ring Λ/J. We refer to this as the *t-action* on the Wedderburn components, which induces an analogous t-action on the simples (it maps a simple S to $S \otimes_\Lambda t\Lambda$). Suppose there are d simples $S_1, \ldots, S_d$. The following show that the t-action permutes the Wedderburn components cyclically.

Proposition A.3. *The action of t permutes all the simples cyclically. In other words, by reindexing if necessary, we may assume that $S_i^{\oplus r} \simeq Pt^{i-1}/Pt^i$ where $P = \Delta^n$.*

Proof. From Proposition A.1, we may assume S_1 is the simple such that $P/Pt \simeq S_1^{\oplus r}$. The composition factors of any finite-length quotient of P all lie in the t-orbit of S_1. For any simple S_i, we may choose a finitely generated projective Λ-module Q which surjects onto S_i. If the $M_n(K\Delta)$-module $Q \otimes_R K$ is isomorphic to $(K\Delta^n)^{\oplus a}$, then by clearing denominators, we can find an embedding $Q \hookrightarrow P^{\oplus a}$ such that the cokernel of $Qt \hookrightarrow P^{\oplus a}$ has finite length. It follows that Q/Qt has composition factors in the t-orbit of S_1 so, in particular, S_i lies in the t-orbit. $\qquad\square$

For our structure theory, we will need the order

$$\Delta_d := \begin{pmatrix} \Delta & \Delta & \cdots & \Delta \\ \operatorname{rad}\Delta & \Delta & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \operatorname{rad}\Delta & \cdots & \operatorname{rad}\Delta & \Delta \end{pmatrix} \subseteq M_d(\Delta),$$

whose radical is

$$\operatorname{rad}\Delta_d := \begin{pmatrix} \operatorname{rad}\Delta & \Delta & \cdots & \Delta \\ \operatorname{rad}\Delta & \operatorname{rad}\Delta & & \vdots \\ \vdots & \ddots & \ddots & \Delta \\ \operatorname{rad}\Delta & \cdots & \operatorname{rad}\Delta & \operatorname{rad}\Delta \end{pmatrix}.$$

If $\pi \in \Delta$ is a generator for $\operatorname{rad}\Delta$, then one readily shows that

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ \pi & 0 & \cdots & \cdots & 0 \end{pmatrix} \in \Delta_d$$

generates the radical on the left and right so Δ_d is normal.

The order Δ_d arises naturally:

Proposition A.4. *Let P be the projective Λ-module Δ^n.*

(1) *As subsets of $\operatorname{Hom}_{M_n(K\Delta)}(P \otimes_R K, P \otimes_R K) = K\Delta$, we have*

$$\operatorname{Hom}_\Lambda(Pt^i, Pt^j) = \begin{cases} \Delta & \text{if } 0 \leq i - j < d, \\ \operatorname{rad}\Delta & \text{if } 0 < j - i \leq d. \end{cases}$$

(2) *In particular*, $\operatorname{End}_\Lambda(P \oplus Pt \oplus \cdots \oplus Pt^{d-1}) = \Delta_d$.

Proof. Part (2) follows from (1) which we now prove. We may as well assume that $i = 0$. First, $\operatorname{Hom}_\Lambda(P, Pt)$ is the kernel of the natural surjection $\operatorname{End}_\Lambda P \to \operatorname{End}_\Lambda P/Pt$ which in turn is $\operatorname{rad}\Delta$. For

$-d < j \leq 0$, the composition factors of Pt^j/P are all nonisomorphic to the simple summands of P/Pt. Hence $\mathrm{Hom}_\Lambda(P, Pt^j) = \mathrm{Hom}_\Lambda(P, P) = \Delta$ in this case. A similar argument shows that

$$\mathrm{Hom}_\Lambda(P, Pt) = \mathrm{Hom}_\Lambda(P, Pt^2) = \cdots = \mathrm{Hom}_\Lambda(P, Pt^d). \qquad \square$$

Theorem A.5. *Let R be a normal order over a discrete valuation ring R, and r be the integer in Proposition A.1. Then there exists $b \mid r$ such that $M_b(\Lambda) \simeq M_c(\Delta_d)$ for some c.*

Proof. We know from Proposition A.3 that conjugation by t permutes the Wedderburn components of $\Lambda/\mathrm{rad}\,\Lambda$ so Λ_Λ must be the projective cover of a semisimple module of the form $(S_1 \oplus \cdots \oplus S_d)^{\oplus a}$ for some a. We let $c = l/r$, $b = l/a$ where l is the lowest common multiple of a and r. It follows that $\Lambda^{\oplus b} \simeq (P \oplus Pt \oplus \cdots \oplus Pt^{d-1})^{\oplus c}$. Using Proposition A.4 to compute endomorphism rings of both sides gives the theorem. $\qquad \square$

Corollary A.6. *Up to isomorphism, a normal R-order Λ is uniquely determined by the Brauer class of $K\Delta$ (in $\mathrm{Br}\,K$), the number of simples of Λ and the degree (or R-rank) of Λ.*

Proof. Suppose the Morita equivalence between Λ and Δ_d is given by the Morita bimodule ${}_\Lambda Q_{\Delta_d}$. If S is the direct sum of one copy of each simple Δ_d-module, then Q_{Δ_d} is the projective cover of a semisimple module $S^{\oplus a}$ for some a. The degree of Λ determines a uniquely. $\qquad \square$

To determine Λ itself, it suffices to classify all possible indecomposable projective Δ_d-modules Q and their tops $Q/Q(\mathrm{rad}\,\Delta_d)$, a task which we address now.

Let $\bar\Delta = \Delta/\mathrm{rad}\,\Delta$. We define a $\bar\Delta$-*flag* to be a sequence of $\bar\Delta$-submodules

$$0 \leq \bar I_1 \leq \bar I_2 \leq \cdots \leq \bar I_d = \bar\Delta.$$

Their inverse images in Δ gives the sequence of Δ-modules

$$\mathrm{rad}\,\Delta \leq I_1 \leq I_2 \leq \cdots \leq I_d = \Delta.$$

The module of row vectors $Q = (I_1\ I_2\ \cdots\ I_d)$ defines a Δ_d-submodule of Δ^d. It is projective since Δ_d is normal.

Proposition A.7. *The order Δ_d is normal.*

(1) *The projective Δ_d-module Q constructed from a $\bar\Delta$-flag as above is indecomposable.*

(2) $Q/Q(\mathrm{rad}\,\Delta_d) \simeq (\bar I_1\ I_2/I_1 \cdots I_d/I_{d-1})$.

(3) *Every indecomposable projective Δ_d-module has this form.*

(4) *In particular, the indecomposable projectives are precisely the projective covers of any direct sum of r simple Δ_d-modules.*

Proof. Note that $Q \otimes_R K \simeq (K\Delta)^d$ is an indecomposable $M_d(K\Delta)$-module so Q is also indecomposable.

If S denotes a simple Δ-module, then the simple Δ_d-modules are

$$(S\ 0 \cdots 0),\ (0\ S\ 0 \cdots 0),\ \ldots,\ (0 \cdots 0\ S).$$

Thus Q is the projective cover of the semisimple module

$$Q/Q(\operatorname{rad} \Delta_d) \simeq (\bar{I}_1 \; I_2/I_1 \cdots I_d/I_{d-1}),$$

which is a direct sum of exactly r simples. By varying the $\bar{\Delta}$-flag, we can construct the projective cover of any direct sum of r simples we like. It thus remains to show there are no other indecomposable projective modules. Let L be one such and suppose $L/L(\operatorname{rad} \Delta_d)$ is a direct sum of more than r simples. Then we can find a direct summand T consisting of precisely r simples and use an appropriate $\bar{\Delta}$-flag to construct the projective cover Q of T. Then the natural surjection $L \to T$ lifts to a surjection $L \to Q$ which must split, a contradiction. If on the other hand, $L/L(\operatorname{rad} \Delta_d)$ had fewer than r simples, then we could apply the same argument to show that some projective Q constructed using a $\bar{\Delta}$-flag decomposes, another contradiction. $\qquad\square$

Example A.8. In the case where $d = r$, we can get examples of normal orders which differ most significantly from $\Delta_d = \Delta_r$. Let Q be the indecomposable projective Δ_r-module corresponding to the "complete" $\bar{\Delta}$-flag where all I_{i+1}/I_i are simple. The top $Q/Q(\operatorname{rad} \Delta_r)$ of Q contains exactly one copy of every simple Δ_r-module so the resulting order $\Lambda := \operatorname{End}_{\Delta_r} Q$ will indeed be normal. Interestingly, Λ is an order in $K\Delta$ so is much smaller than Δ_r.

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danielc@unsw.edu.au   School of Mathematics and Statistics, University of New South Wales Sydney, Sydney, Australia

cingalls@math.carleton.ca   School of Mathematics and Statistics, Carleton University, Ottawa, ON, Canada

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