

Algebra & Number Theory

Volume 18

2024

No. 11

Word measures on $GL_N(q)$ and free group algebras

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Fix a finite field K of order q and a word w in a free group F on r generators. A w -random element in $\mathrm{GL}_N(K)$ is obtained by sampling r independent uniformly random elements $g_1, \dots, g_r \in \mathrm{GL}_N(K)$ and evaluating $w(g_1, \dots, g_r)$. Consider $\mathbb{E}_w[\mathrm{fix}]$, the average number of vectors in K^N fixed by a w -random element. We show that $\mathbb{E}_w[\mathrm{fix}]$ is a rational function in q^N . If $w = u^d$ with u a nonpower, then the limit $\lim_{N \rightarrow \infty} \mathbb{E}_w[\mathrm{fix}]$ depends only on d and not on u . These two phenomena generalize to all stable characters of the groups $\{\mathrm{GL}_N(K)\}_N$.

A main feature of this work is the connection we establish between word measures on $\mathrm{GL}_N(K)$ and the free group algebra $K[F]$. A classical result of Cohn (1964) and Lewin (1969) is that every one-sided ideal of $K[F]$ is a free $K[F]$ -module with a well-defined rank. We show that for w a nonpower, $\mathbb{E}_w[\mathrm{fix}] = 2 + \frac{C}{q^N} + O\left(\frac{1}{q^{2N}}\right)$, where C is the number of rank-2 right ideals $I \leq K[F]$ which contain $w - 1$ but not as a basis element. We describe a full conjectural picture generalizing this result, featuring a new invariant we call the q -primitivity rank of w .

In the process, we prove several new results about free group algebras. For example, we show that if T is any finite subtree of the Cayley graph of F , and $I \leq K[F]$ is a right ideal with a generating set supported on T , then I admits a basis supported on T . We also prove an analog of Kaplansky's unit conjecture for certain $K[F]$ -modules.

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1. Introduction

Fix $r \in \mathbb{Z}_{\geq 1}$. We let F denote the free group on r generators. A word $w \in F$ induces a map on any finite group, $w : G^r \rightarrow G$, by substituting the letters of w with elements of G . This map defines a distribution on the group G : the pushforward of the uniform distribution on G^r . Equivalently, this distribution is

MSC2020: primary 20C07, 20E05; secondary 16S34, 20C33, 20G40, 20H30, 68R15.

Keywords: free group algebra, word measures, q -primitivity rank.

the normalized number of times each element in G is obtained by a substitution in w . We call such a distribution a *word measure* on G , and if w is given, *the w -measure on G* . For example, if $w = abab^{-2}$, a w -random element in G is $ghgh^{-2}$ where g, h are independent, uniformly random elements of G .

The study of word measures on various families of groups revealed structural depth with surprising connections to objects in combinatorial and geometric group theory (see, e.g., [Puder 2014; Puder and Parzanchevski 2015; Magee and Puder 2019; 2021; 2024; Hanany and Puder 2023]). It has proven useful for many questions regarding free groups and their automorphism groups (see, e.g., [Puder and Parzanchevski 2015; Hanany et al. 2020]), as well as for questions about random Schreier graphs and their expansion (see, e.g., [Puder 2015; Hanany and Puder 2023]). Previous works in the subject study word measures on the groups $\text{Sym}(N)$, $U(N)$, $O(N)$, $\text{Sp}(N)$ and generalized symmetric groups. Section 1E explains how some of the results in the current paper relate to the established structure in other families of groups.

We focus on word measures on $\text{GL}_N(K)$, the general linear group over a fixed finite field K of order q . As seen in other families of groups, word measures on this family demonstrate structural depth. Most interestingly, we show that the analysis of word measures on $\text{GL}_N(K)$ is intertwined with the theory of free group algebras.

1A. The average number of fixed vectors. We consider various families of real- or complex-valued functions defined on $\text{GL}_N(K)$, and study their expected value under word measures. Our core example is the function $\text{fix} : \text{GL}_N(K) \rightarrow \mathbb{Z}_{\geq 0}$ counting elements in the vector space $V = K^N$ which are fixed by a given matrix in $\text{GL}_N(K)$. Not only does this special case illustrate our more general results, but is also a case in which our understanding goes deeper. The function fix is, in fact, a family of functions, one for every value of $N \in \mathbb{Z}_{\geq 1}$. We let $\mathbb{E}_w[\text{fix}]$ denote the expected value of fix under the w -measure on $\text{GL}_N(K)$, so $\mathbb{E}_w[\text{fix}]$ is also a sequence of numbers, one for every value of $N \in \mathbb{Z}_{\geq 1}$. Our first result is the following.

Theorem 1.1. *For every $w \in F$ and every large enough N , $\mathbb{E}_w[\text{fix}]$ is given by a rational function in q^N with rational coefficients.*

For example, if $w = [a, b] = aba^{-1}b^{-1}$ is the commutator of two basis elements, then

$$\mathbb{E}_w[\text{fix}] = 2 + \frac{(q-1)^2 q^N - (q-1)^3}{(q^N - 1)(q^N - q)}$$

for every $N \geq 2$ (recall that $q = |K|$ is fixed throughout, so this expression is indeed a rational function in q^N with coefficients in \mathbb{Q}). Consult Table 1 for further examples. For general words, the rational expression is valid for every $N \geq |w|$. See Section 2 for a tighter lower bound on N . Theorem 1.1 is a special case of Theorem 1.11.

Our second result alludes to a result of Nica [1994]. Let $1 \neq w = u^d$ where $d \in \mathbb{N}_{\geq 1}$ and u a nonpower. Nica proved, inter alia, that the distribution of the number of fixed points in a w -random permutation in $\text{Sym}(N)$ has a limit distribution as $N \rightarrow \infty$ which depends solely on d and not on u . A similar phenomenon was later shown to hold in various other families of groups. We add the groups $\{\text{GL}_N(K)\}_N$

w	q	$\mathbb{E}_w[\text{fix}]$	valid for
a	every q	2	$N \geq 1$
a^2	q even	3	$N \geq 2$
	q odd	4	
a^3	$q \equiv 0, 2 \pmod{3}$	4	$N \geq 3$
	$q \equiv 1 \pmod{3}$	8	
$[a, b]$	every q	$2 + \frac{(q-1)^2 q^N - (q-1)^3}{(q^N-1)(q^N-q)}$	$N \geq 2$
$a^2 b^3$	$q = 2$	$2 + \frac{2}{2^N-2}$	$N \geq 3$
	$q = 3$	$2 + \frac{4}{3^N-3}$	
$[a, b]^2$	$q = 2$	$3 + \frac{2(2^{2N}-9 \cdot 2^N+26)}{(2^N-1)(2^N-2)(2^N-8)}$	$N \geq 4$
$a^2 b^2 c^2$	$q = 2$	$2 + \frac{1}{(2^N-2)^2}$	$N \geq 2$
	$q = 3$	$2 + \frac{8(3^{2N}-4 \cdot 3^N+5)}{(3^N-1)^2(3^N-3)^2}$	

Table 1. The rational expressions giving $\mathbb{E}_w[\text{fix}]$ for various words $w \in \mathbf{F}(a, b, c)$ and various values of $q = |K|$. For the first four words, rational expressions are given for all values of q . For the remaining three words, rational expressions are given only for particular values of q .

as such a family. In our illustrative special case, this is captured by the following result, which first appeared in [Ernst-West 2019]. It also appeared independently in [Eberhard and Jezernik 2022, Section 8].

Theorem 1.2. *Let $1 \neq w = u^d$ with $d \geq 1$ and u a nonpower. Then*

$$\lim_{N \rightarrow \infty} \mathbb{E}_w[\text{fix}] = \#\{p \in K[x] : p \mid x^d - 1 \text{ and } p \text{ monic}\}. \tag{1-1}$$

In particular, the limit does not depend on u .

Combined with Theorem 1.1, if c_d is the number of monic divisors of $x^d - 1 \in K[x]$, we get that $\mathbb{E}_w[\text{fix}] = c_d + O(\frac{1}{q^N})$. In particular, for nonpowers, $\mathbb{E}_w[\text{fix}] = 2 + O(\frac{1}{q^N})$, and for proper powers $c_d \geq 3$ (if $d \geq 2$, then $x^d - 1$ admits at least three distinct monic divisors: $1, x - 1$ and $x^d - 1$). Theorem 1.2 is analogous to the result in the symmetric group $\text{Sym}(N)$, where this limit is equal to the number of positive divisors of d in \mathbb{Z} [Nica 1994]. In fact, it is sufficient to prove that the limit in (1-1) depends only on d and not on u , and then the left-hand side of (1-1) is equal to $\lim_{N \rightarrow \infty} \mathbb{E}_{a^d}[\text{fix}]$. This number can then be extracted from the analysis of uniformly random elements in $GL_N(K)$ — see, for example, [Fulman and Stanton 2016]. Theorem 1.2 is a special case of the more general Theorem 1.12 below.

1B. The q -primitivity rank. The analysis of $\mathbb{E}_w[\text{fix}]$, yielding Theorems 1.1 and 1.2, can be performed using elementary linear algebraic arguments. In fact, this is how they were first derived in the [Ernst-West

2019]. However, it turns out to be extremely useful to analyze these quantities using the theory of free group algebras.

Denote by $\mathcal{A} \stackrel{\text{def}}{=} K[\mathbf{F}]$ the free group algebra over K : its elements are finite linear combinations of elements of the free group \mathbf{F} with coefficients from the finite field K . It is a classical result of Cohn [1964] and Lewin¹ [1969] that right ideals of \mathcal{A} are free right \mathcal{A} -modules with a well-defined rank.² An analogous result holds for left ideals, but here we use right ideals only — in fact, from now on, we write “ideals” to mean “right ideals”. In Section 2 below we derive a formula for $\mathbb{E}_w[\text{fix}]$ as a sum over a finite set of finitely generated ideals of \mathcal{A} , and Section 3 shows that the contribution of every such ideal is of order determined by its rank.

In particular, this algebraic perspective allows a further understanding of the deviation of $\mathbb{E}_w[\text{fix}]$ from $\mathbb{E}_a[\text{fix}]$, the analogous expectation under the uniform measure. Namely, as the action $\text{GL}_N(K) \curvearrowright K^N$ admits two orbits (the zero vector and all nonzero vectors), the expected number of vectors in K^N fixed by a uniformly random element of $\text{GL}_N(K)$ is $\mathbb{E}_a[\text{fix}] = 2$, and we consider the difference $\mathbb{E}_w[\text{fix}] - 2$. Theorems 1.1 and 1.2 imply that if w is a proper power, then $\mathbb{E}_w[\text{fix}] - 2$ is of order $\Theta(1)$, and otherwise, it is of order $O\left(\frac{1}{q^N}\right)$. Next, we provide a more refined and accurate description of this difference in the nonpower case. To state our result and conjecture, we first define the notion of primitivity of elements in ideals. Recall that by Cohn and Lewin’s result, every ideal $I \leq \mathcal{A}$ is a free \mathcal{A} -module and so admits a basis. All bases of I have the same cardinality, called the rank of I and denoted $\text{rk } I$.

Definition 1.3. Let $I \leq \mathcal{A}$ be an ideal and let $f \in I$. We say that f is a *primitive* element of I if it is contained in some basis of I (considering I as a free right \mathcal{A} -module). Otherwise, f is *imprimitive* in I .

This is analogous to the notion of a primitive element in a free group: an element belonging to some basis of this group. Our next central result captures the $\frac{1}{q^N}$ -term of the Laurent expansion of $\mathbb{E}_w[\text{fix}]$.

Theorem 1.4. *Let $1 \neq w \in \mathbf{F}$ be a nonpower. Then the expected number of vectors in K^N fixed by a w -random element of $\text{GL}_N(K)$ is*

$$\mathbb{E}_w[\text{fix}] = 2 + \frac{|\text{Crit}_q^2(w)|}{q^N} + O\left(\frac{1}{q^{2N}}\right),$$

where $\text{Crit}_q^2(w)$ is the set of ideals $I \leq \mathcal{A}$ of rank two which contain the element $w - 1$ as an imprimitive element.

As implied by the theorem, the set $\text{Crit}_q^2(w)$ is indeed finite for every nonpower w . We prove this fact directly in Corollary 3.11. To illustrate, consider the commutator word $w = [a, b]$. As mentioned above,

$$\mathbb{E}_{[a,b]}[\text{fix}] = 2 + \frac{(q-1)^2 q^N - (q-1)^3}{(q^N-1)(q^N-q)} = 2 + \frac{(q-1)^2}{q^N} + O\left(\frac{1}{q^{2N}}\right).$$

¹Some claim that the first correct proof of this result (stated formally below as Theorem 3.1) is due to Lewin [1969] — see [Hog-Angeloni 1990, Footnote 5].

²For example, it can be shown that the augmentation ideal $I_{\mathbf{F}} = \{\sum \alpha_w w \mid \sum \alpha_w = 0\} \subseteq \mathcal{A}$ is of rank $r = \text{rk } \mathbf{F}$. For instance, when $\mathbf{F} = \mathbf{F}(a, b, c)$, $I_{\mathbf{F}} = (a-1)\mathcal{A} \oplus (b-1)\mathcal{A} \oplus (c-1)\mathcal{A}$.

In this case there are exactly $(q - 1)^2$ distinct ideals of rank two containing $[a, b] - 1$ as an imprimitive element: these are $(\delta a - 1, \varepsilon b - 1)$ with $\delta, \varepsilon \in K^*$. We conjecture a more general phenomenon, for which we make the following definition.

Definition 1.5. The q -primitivity rank of $w \in \mathbf{F}$, denoted $\pi_q(w)$, is the smallest rank of a proper ideal of \mathcal{A} containing $w - 1$ as an imprimitive element. Namely,

$$\pi_q(w) \stackrel{\text{def}}{=} \min\{\text{rk } I \mid I \subsetneq \mathcal{A}, I \ni w - 1 \text{ and } w - 1 \text{ is imprimitive in } I\}.$$

If this set is empty, we set $\pi_q(w) = \infty$. A critical ideal for w is a proper ideal of rank $\pi_q(w)$ containing $w - 1$ as an imprimitive element. We denote by $\text{Crit}_q(w)$ the set of critical ideals for w .

Corollary 3.17 shows that $\pi_q(w)$ takes values only in $\{0, 1, \dots, r\} \cup \{\infty\}$, where r is the rank of \mathbf{F} . Note that $\pi_q(w) = 0$ if and only if $w = 1$: the only rank-0 ideal is (0) , whose only basis is the empty set. In Section 4A below, we prove that $\pi_q(w) = 1$ if and only if $w \in \mathbf{F}$ is a proper power (Corollary 4.6), and that in this case, if one writes $w = u^d$ with $d \geq 2$ and u a nonpower, the set of critical ideals of w is

$$\text{Crit}_q(u^d) = \{(p(u)) : p \mid x^d - 1 \in K[x], p \text{ monic and } p \neq 1, x^d - 1\}.$$

For example, if $|K| = q = 3$ and $w = u^4$, the critical ideals of w are in one-to-one correspondence with the six nontrivial monic divisors the polynomial $x^4 - 1 \in K[x]$. These rank-1 ideals are $(u - 1)$, $(u + 1)$, $(u^2 - 1)$, $(u^2 + 1)$, $(u^3 - u^2 + u - 1)$ and $(u^3 + u^2 + u + 1)$. The trivial monic divisors of $x^4 - 1$ correspond to the ideal $(1) = \mathcal{A}$ which is not proper, and to the ideal $(u^4 - 1)$ in which $w - 1$ is primitive. By Proposition 3.16, $\pi_q(w) = \infty$ if and only if w is a primitive element of \mathbf{F} .

The following conjecture thus generalizes Theorems 1.2 and 1.4.

Conjecture 1.6. Let $w \in \mathbf{F}$ and let $\pi = \pi_q(w)$. Then the expected number of vectors in K^N fixed by a w -random element of $GL_N(K)$ is

$$\mathbb{E}_w[\text{fix}] = 2 + \frac{|\text{Crit}_q(w)|}{q^{N \cdot (\pi - 1)}} + O\left(\frac{1}{q^{N \cdot \pi}}\right). \tag{1-2}$$

Corollary 3.11 yields that $\text{Crit}_q(w)$ is indeed finite. If $\pi := \pi_q(w) = 0$ (namely, if $w = 1$), then $\text{Crit}_q(w) = \{(0)\}$ and (1-2) is obvious. Theorem 1.2 proves (1-2) when $\pi = 1$, and Theorem 1.4 proves it when $\pi = 2$. As mentioned above, $\pi_q(w) = \infty$ if and only if w is primitive in \mathbf{F} , and in this case a w -random element of $GL_N(K)$ distributes uniformly [Puder and Parzanchevski 2015, Observation 1.2], and so (1-2) holds. In particular, Conjecture 1.6 holds for the free group of rank 2 as the possible values of $\pi_q(w)$ are $\{0, 1, 2, \infty\}$ (Corollary 3.17). We conclude the following analog of a result about S_N [Puder 2014, Theorem 1.5].

Corollary 1.7. Let $w \in \mathbf{F}_2$. Then w induces the uniform measure on $GL_N(K)$ for all N if and only if w is primitive.

Another important background for Conjecture 1.6 is an analogous result in the case of the symmetric group S_N . The primitivity rank of a word $w \in \mathbf{F}$, denoted $\pi(w)$ and introduced in [Puder 2014], is the

smallest rank of a subgroup of F containing w as an imprimitive element. Let $\text{Crit}_F(w)$ denote the set of subgroups of F of rank $\pi(w)$ which contain w as an imprimitive element. Then the S_N -analog of [Conjecture 1.6](#) is [[Puder and Parzanchevski 2015](#), Theorem 1.8]: the expected number of fixed points in a w -random permutation in S_N is

$$1 + \frac{|\text{Crit}_F(w)|}{N^{\pi(w)-1}} + O\left(\frac{1}{N^{\pi(w)}}\right).$$

Alongside its role in word measures on S_N , the original primitivity rank $\pi(w)$ seems to play a universal role in word measures on groups (see [[Hanany and Puder 2023](#), Conjecture 1.13]), it has connections with stable commutator length (see Section 1.6 in the same article) and was recently found relevant to the study of one-relator groups (see, for example, [[Louder and Wilton 2022](#)]). [Definition 1.5](#) seemingly introduces a family of related invariants of words — one for every prime power q . In fact, the same definition can be applied to arbitrary fields — see [Section 7](#). However, it is possible that all these invariants coincide for a given word. We are able to show one inequality and conjecture a full equality.

Proposition 1.8. *For every word $w \in F$ and every prime power q , $\pi_q(w) \leq \pi(w)$.*

Conjecture 1.9. *For every word $w \in F$ and every prime power q , $\pi_q(w) = \pi(w)$.*

[Conjecture 1.9](#), along with [Conjecture 1.6](#), are in line with a universal conjecture — [[Hanany and Puder 2023](#), Conjecture 1.13] — about the role of the primitivity rank $\pi(w)$ in word measures on groups. For more background, see Section 1.6 in the same article.

As part of our study of word measures in $\text{GL}_N(K)$ employing the free group algebras, we also prove some results about these algebras which may be of independent interest. For example, suppose that T is a subtree of the Cayley graph of F with respect to some basis. If $I \leq \mathcal{A}$ is a finitely generated ideal with a generating set supported on T , then I admits a basis which is supported on T ([Theorem 3.8](#)). We also analyze the \mathcal{A} -module $\mathcal{A}/(w-1)$ obtained as the quotient of the right \mathcal{A} -module \mathcal{A} by its submodule $(w-1)$. [Theorem 5.4](#) proves an analog of Kaplansky's unit conjecture for these modules and shows that if w is a nonpower, then the only cyclic generators of $\mathcal{A}/(w-1)$ are the trivial ones. See [Section 7](#) for a further discussion of this line of research.

1C. General stable class functions and characters. As mentioned above, some of the results concerning the function fix and its expectation under word measures are only an illustrative special case of more general results. The variety of functions we consider are those relating to *stable* representations of the family $\text{GL}_\bullet(K)$ (see [[Putman and Sam 2017](#); [Gan and Watterlond 2018](#)]). Below we present the generalizations of [Theorems 1.1](#) and [1.2](#) and of [Conjecture 1.6](#).

First, we must remark on the unconventional definition we make in this paper. Formal words in group theory are usually read from left to right: this is why one usually considers *right* Cayley graphs. As a consequence, we consider here the slightly nonstandard *right* action of $\text{GL}_N(K)$ on $V_N \stackrel{\text{def}}{=} K^N$, namely, we consider V_N as row vectors, and the action of $g \in \text{GL}_N(K)$ on $v \in V_N$ is given by $(v, g) \mapsto vg$. Thus,

the action of $w(g_1, \dots, g_r)$ on a vector $v \in V_N$ can be thought of as the composition of the action, letter by letter, from left to right—the natural direction in which the word is read.

Rather than considering only the number of vectors fixed by g , we consider more generally the number of subspaces of V of a fixed dimension which are invariant under g and on which g acts in a prescribed way. This is formalized as follows:

Definition 1.10. Let $m \in \mathbb{Z}_{\geq 1}$ and $\mathcal{B} \in GL_m(K)$. We define a map $\tilde{\mathcal{B}} : GL_N(K) \rightarrow \mathbb{Z}_{\geq 0}$ (valid for arbitrary N) as follows. For $g \in GL_N(K)$ we let $\tilde{\mathcal{B}}(g)$ be the number of m -tuples of vectors $v_1, \dots, v_m \in V_N = K^N$ on which the (right) action of g can be described by a multiplication from the left by the matrix \mathcal{B} . Namely,

$$\tilde{\mathcal{B}}(g) = \#\{M \in M_{m \times N}(K) \mid Mg = \mathcal{B}M\}.$$

For example, if $\mathcal{B} = (1) \in GL_1(K)$, then $\tilde{\mathcal{B}} = \text{fix}$. For $\mathcal{B} = (\lambda) \in GL_1(K)$, the function $\tilde{\mathcal{B}}$ gives the size of the eigenspace $V_\lambda \leq V_N$ of an element. If $\mathcal{B} = I_m \in GL_m(K)$, then $\tilde{\mathcal{B}}(g) = \text{fix}(g)^m$, and if

$$\mathcal{B} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & & & \end{pmatrix} \in GL_m(K),$$

then $\tilde{\mathcal{B}}(g) = \text{fix}(g^m)$. The following two theorems are the generalization of Theorems 1.1 and 1.2:

Theorem 1.11. Suppose that $w \in F$, $m \in \mathbb{Z}_{\geq 1}$ and $\mathcal{B} \in GL_m(K)$. Then for every large enough N , the expectation $\mathbb{E}_w[\tilde{\mathcal{B}}]$ is given by a rational function in q^N .

Theorem 1.12. Let $1 \neq w = u^d$ with $d \geq 1$ and u a nonpower. For every $m \in \mathbb{Z}_{\geq 1}$ and $\mathcal{B} \in GL_m(K)$, the limit $\lim_{N \rightarrow \infty} \mathbb{E}_w[\tilde{\mathcal{B}}]$ exists and depends only on d and not on u .

In the special case of $\tilde{\mathcal{B}} = I_m \in GL_m(K)$, Theorem 1.11 appeared in [Ernst-West 2019]. The same special case of Theorem 1.12 first appeared in the same thesis, and then, independently, in [Eberhard and Jezernik 2022, Section 8].

In particular, Theorem 1.12 captures all moments of the number of fixed vectors under the w -measure. So if $w = u^d$, all these moments converge, as $N \rightarrow \infty$, to the same limits as for $w = a^d$, namely as for a d -th power of a uniformly random element of $GL_N(K)$. Denote the number of fixed vectors in K^N of a w -random element of $GL_N(K)$ by $\text{fix}_{w,N}$. When $w = a$, a limit distribution as $N \rightarrow \infty$ is known to exist [Fulman and Stanton 2016, Theorem 2.1].³ Although this limit distribution is not determined by its moments, we do prove the following in the Appendix:

Theorem 1.13. Let $1 \neq w \in F$ be a nonpower. Then the random variables $\text{fix}_{w,N}$ have a limit distribution, and this limit distribution is identical to the one of $\text{fix}_{a,N}$ described in [Fulman and Stanton 2016, Theorem 2.1].

³This was originally proved by Rudvalis and Shinoda—see [Fulman and Stanton 2016].

Theorem 1.13 is analogous to the $L = d = 1$ case of Nica’s main Theorem 1.1 [1994], which revolves around the limit distribution of the number of fixed point in w -random permutations. We suspect that **Theorem 1.13** can be generalized to a full analog of Nica’s result (and see Section 7).

Remark 1.14. One can further generalize **Theorem 1.11** to more than one word. For example, for any tuple of words $w_1, \dots, w_\ell \in F$, consider an ℓ -tuple of random elements

$$\overline{w}_1 = w_1(g_1, \dots, g_r), \dots, \overline{w}_\ell = w_\ell(g_1, \dots, g_r) \in \text{GL}_N(K),$$

where g_1, \dots, g_r are independent, uniformly random elements of $\text{GL}_N(K)$, and consider expressions like $\mathbb{E}[\text{fix}(\overline{w}_1) \cdot \text{fix}(\overline{w}_2) \cdots \text{fix}(\overline{w}_\ell)]$. The same argument given in the proof of **Theorem 1.11** shows that this expectation is given by a rational expression in q^N . Also, Corollary 1.3 in [Ernst-West 2019] shows that the difference $\mathbb{E}[\text{fix}(\overline{w}_1) \cdots \text{fix}(\overline{w}_\ell)] - \mathbb{E}_{w_1}[\text{fix}] \cdots \mathbb{E}_{w_\ell}[\text{fix}] = O\left(\frac{1}{q^N}\right)$ if and only if no pair of words is conjugated into the same cyclic subgroup of F .

We further introduce a generalization of **Conjecture 1.6**. Consider

$$\mathcal{R} \stackrel{\text{def}}{=} \mathbb{C}[\{\tilde{\mathcal{B}} \mid \mathcal{B} \in \text{GL}_m(K), m \in \mathbb{Z}_{\geq 0}\}],$$

the \mathbb{C} -algebra generated by all functions $\tilde{\mathcal{B}}$ from **Definition 1.10**. Every element of \mathcal{R} is a (class) function defined on $\text{GL}_N(K)$ for every N . Rather than formal polynomials in the $\tilde{\mathcal{B}}$ ’s, the elements of \mathcal{R} are functions on $\text{GL}_\bullet(K)$, so two elements giving the same function on $\text{GL}_N(K)$ for every N are identified. For example, every conjugate of \mathcal{B} gives rise to the same function as \mathcal{B} . In fact, this is the only case where two elements give the same function: $\tilde{\mathcal{B}}_1 = \tilde{\mathcal{B}}_2$ if and only if \mathcal{B}_1 and \mathcal{B}_2 belong to $\text{GL}_m(K)$ for the same m and are conjugates — see [Ernst-West et al. 2024, Corollary 3.1]. If we also include the constant function 1, thought of as $\tilde{\mathcal{B}}$ where $\mathcal{B} = e \in \text{GL}_0(K) \stackrel{\text{def}}{=} \{e\}$, then \mathcal{R} is the \mathbb{C} -span of the $\tilde{\mathcal{B}}$ ’s: indeed, if $\mathcal{B}_1 \in \text{GL}_{m_1}(K)$ and $\mathcal{B}_2 \in \text{GL}_{m_2}(K)$, then $\tilde{\mathcal{B}}_1 \cdot \tilde{\mathcal{B}}_2 = \overline{\mathcal{B}_1 \oplus \mathcal{B}_2}$ where $\mathcal{B}_1 \oplus \mathcal{B}_2 \in \text{GL}_{m_1+m_2}(K)$ is the suitable block-diagonal matrix. In the same article, it is shown that \mathcal{R} is, in fact, a graded algebra and admits a linear basis consisting of $\{\tilde{\mathcal{B}}\}$, where \mathcal{B} goes over exactly one representative from every conjugacy class in all $\text{GL}_m(K)$ ($m \geq 0$).

Some of the functions in \mathcal{R} coincide, for large enough N , with irreducible characters of $\text{GL}_N(K)$. For example, for $N \geq 2$, the action of $\text{GL}_N(K)$ on the projective space $\mathbb{P}^{N-1}(K)$ decomposes to the trivial representation and an irreducible representation whose character we denote $\chi^{\mathbb{P}}$. Then for every $N \geq 2$, the character $\chi^{\mathbb{P}}$ is equal to an element in \mathcal{R} :

$$\chi^{\mathbb{P}} = \frac{1}{q-1} \sum_{\lambda \in K^*} (\tilde{\lambda} - 1) - 1$$

(here $\tilde{\lambda}$ is the function corresponding to $\lambda \in \text{GL}_1(K)$). In [Ernst-West et al. 2024] it is shown that the set of families of irreducible characters $\{\chi_N \in \text{GL}_N(K)\}_{N \geq N_0}$ which coincide with elements of \mathcal{R} is precisely the set of *stable* irreducible representations of $\text{GL}_\bullet(K)$ as in [Gan and Watterlond 2018]. Our generalization of **Conjecture 1.6** deals with these families of irreducible characters.

Conjecture 1.15. *Let χ be a stable character of $GL_\bullet(K)$, namely, an element of \mathcal{R} which coincides, for every large enough N , with some irreducible character of $GL_N(K)$. Then*

$$\mathbb{E}_w[\chi] = O((\dim \chi)^{1-\pi_q(w)}).$$

By [Theorem 1.11](#) (with $w = 1$), $\dim \chi = \chi(1)$ is a rational function in q^N . [Conjecture 1.15](#), together with a positive answer to [Conjecture 1.9](#), constitute a special case of the more general, albeit not as precise, [[Hanany and Puder 2023](#), [Conjecture 1.13](#)]. (See also [[Ernst-West et al. 2023](#), [Conjecture A.4](#)] for a slightly more ambitious version of [Conjecture 1.15](#).)

For $N \geq 2$, the decomposition of the function fix to irreducible characters is

$$\text{fix} = 2 \cdot \text{triv} + \chi^{\mathbb{P}} + \xi_1 + \dots + \xi_{q-2},$$

where ξ_1, \dots, ξ_{q-2} are distinct irreducible characters, each of dimension $\frac{q^N-1}{q-1}$, all belonging to \mathcal{R} . Thus, they all fall into the framework of [Conjecture 1.15](#), and we get that this conjecture implies, in particular, that $\mathbb{E}_w[\text{fix}] = 2 + O((q^N)^{1-\pi_q(w)})$. In particular, [Conjecture 1.15](#) generalizes (a slightly weaker version of) [Conjecture 1.6](#). Some background for [Conjecture 1.15](#) can be found in [[Hanany and Puder 2023](#), [Section 1](#)].

1D. Reader’s guide.

Notation. The free group F has rank r and a fixed basis $B = \{b_1, \dots, b_r\}$. Recall that all ideals in this paper are one-sided right ideals unless stated otherwise, and we write $I \leq \mathcal{A}$ to mean that I is an ideal of the free group algebra $\mathcal{A} = K[F]$. More generally, we write $M \leq \mathcal{A}^m$ if M is a submodule of the free right \mathcal{A} -module \mathcal{A}^m . For any set $S \subseteq \mathcal{A}^m$, we denote by $\langle S \rangle$ the submodule generated by S , and if $S = \{s_1, \dots, s_t\}$ we may also write $\langle s_1, \dots, s_t \rangle$.

We denote by $E = \{e_1, \dots, e_m\}$ a basis for the free \mathcal{A} -module \mathcal{A}^m . The elements of the form ez with $e \in E$ and $z \in F$ are called *monomials*. For a subset Q of the monomials, we write $M \leq_Q \mathcal{A}^m$ to mean that M has a generating set in \mathcal{A}^m such that each of its elements is supported on Q . Usually, for ideals inside \mathcal{A} , we consider subsets of F corresponding to the vertices in some subtree T of the (right) Cayley graph $\mathcal{C} \stackrel{\text{def}}{=} \text{Cay}(F, B)$ of F with respect to the basis B . In this case, instead of $I \leq_{\text{vert}(T)} \mathcal{A}$ (here, of course, $\text{vert}(T)$ denotes the set of vertices of T), we simply write $I \leq_T \mathcal{A}$. More generally, for submodules of \mathcal{A}^m , we usually consider m disjoint copies $\mathcal{C}_1, \dots, \mathcal{C}_m$ of $\text{Cay}(F, B)$, with origins e_1, \dots, e_m , respectively, and consider a collection of (possibly empty) subtrees $\mathbb{T} = T_1 \cup \dots \cup T_m$, with $T_i \subset \mathcal{C}_i$. We write $M \leq_{\mathbb{T}} \mathcal{A}^m$ to mean that M is generated by elements supported on the vertices of \mathbb{T} .

For a submodule $M \leq \mathcal{A}^m$ and a set S of monomials in \mathcal{A}^m , we let $M|_S$ denote the set of elements of M which are supported on S . This is a vector space over K .

Paper organization. After a very brief survey of related works in [Section 1E](#), [Section 2](#) proves that $\mathbb{E}_w[\text{fix}]$, and likewise $\mathbb{E}_w[\tilde{B}]$, are given by rational functions in q^N ([Theorems 1.1](#) and [1.11](#), respectively). In [Section 3](#) we study the free group algebra and its ideals, show how the computation of $\mathbb{E}_w[\text{fix}]$ is related to “exploration” processes in the Cayley graph of F , and prove some basic properties of the q -primitivity rank including [Proposition 1.8](#). We then study $\lim_{N \rightarrow \infty} \mathbb{E}_w[\text{fix}]$ and $\lim_{N \rightarrow \infty} \mathbb{E}_w[\tilde{B}]$ and prove [Theorems 1.2](#) and

1.12 in Section 4. Section 5 studies the right \mathcal{A} -module $\mathcal{A}/(w-1)$ and specifies its cyclic generators, and also gives a criterion to detect when $w-1$ is primitive in a given rank-2 ideal in \mathcal{A} . Section 6 deals with the coefficient of $\frac{1}{q^N}$ in the Laurent expansion of $\mathbb{E}_w[\text{fix}]$ and proves Theorem 1.4. Section 7 gathers the many open questions that are raised by this work. Finally, the Appendix contains the proof of Theorem 1.13.

1E. Related works. As mentioned above, the two phenomena described in Theorems 1.11 and 1.12 are found in other families of groups. The fact that the expectation under word measures of “natural” class functions over certain families of groups are given by rational functions was first established for the symmetric group [Nica 1994; Linial and Puder 2010]. It was later established for the classical groups $U(N)$ [Magee and Puder 2019] and $O(N)$ and $\text{Sp}(N)$ [Magee and Puder 2024] based on Weingarten calculus (see, for instance, [Collins and Śniady 2006]), and also in the wreath product $G \wr S_N$ for an arbitrary finite group G [Magee and Puder 2021; Shomroni 2023a]. A related phenomenon appears when free groups are replaced by surface groups (fundamental groups of compact closed surfaces). Indeed, there is a natural definition of a measure induced by an element of a surface group on finite groups and certain compact groups, and the expected value of certain characters of the symmetric group $\text{Sym}(N)$ under such measures can be approximated to any degree by a rational function [Magee and Puder 2023]. A similar result holds for measures induced by elements of surface groups on $\text{SU}(N)$ [Magee 2022].

The phenomenon described in Theorems 1.2 and 1.12, that if $w = u^d$ then the limit expectation of natural class functions in the family under the w -measure depends only on d and not on u , is also found in many of the above mentioned cases. It is true in $\text{Sym}(N)$ [Nica 1994; Linial and Puder 2010], in $U(N)$ [Mingo et al. 2007; Rădulescu 2006], as well as in $O(N)$ and in $\text{Sp}(N)$ [Magee and Puder 2024]. It also holds in the characters analyzed in [Magee and Puder 2023] for measures on $\text{Sym}(N)$ induced by elements of surface groups [ibid., Theorem 1.2].

Finally, there are analogs to Theorem 1.4 and Conjectures 1.6 and 1.15, which give an interpretation to the order of $\mathbb{E}_w[f] - \mathbb{E}_x[f]$, an interpretation which lies in invariants of w as an element of the abstract free group F . We mentioned previously that there are very similar results in the case of $\text{Sym}(N)$ [Puder and Parzanchevski 2015; Hanany and Puder 2023]. There are other invariants of w explaining the leading order (and sometimes much more than the leading order) in the expected values of class functions in $U(N)$, $O(N)$, $\text{Sp}(N)$ and $G \wr S_N$ [Magee and Puder 2019; 2021; 2024; Brodsky 2024; Shomroni 2023a; 2023b]. A more detailed summary may be found in [Hanany and Puder 2023, Section 1.3].

2. Rational expressions

We prove Theorems 1.1 and 1.11, which show that the expectations under word measures of the class functions we consider on $\text{GL}_N(K)$ are given by rational functions in q^N . The proof uses only linear algebra and can be written in completely elementary terms. While we start with this approach, we then “translate” the proof to the language of ideals and modules of the free group algebra $\mathcal{A} = K[F]$. Given our additional results and conjectures, the latter language is much more fruitful.

2A. The function fix and Theorem 1.1.

The elementary approach. We first illustrate the proof in the somewhat simpler special case considered in [Theorem 1.1](#): the function fix . Let $V_N = K^N$ be the vector space of row vectors of length N . Given $w \in \mathbf{F}$, one needs to count all $g_1, \dots, g_r \in GL_N(K)$ and $v \in V_N$ such that $v.w(g_1, \dots, g_r) = v$. We consider the entire trajectory of v when the letters of w are applied one by one. Namely, assume that w is written in the basis $B = \{b_1, \dots, b_r\}$ of \mathbf{F} as $w = b_{i_1}^{\varepsilon_1} \dots b_{i_\ell}^{\varepsilon_\ell}$ (where $i_j \in \{1, \dots, r\}$ and $\varepsilon_j \in \{\pm 1\}$). We consider the vectors

$$v^0 \stackrel{\text{def}}{=} v, \quad v^1 \stackrel{\text{def}}{=} v^0 \cdot g_{i_1}^{\varepsilon_1}, \quad v^2 \stackrel{\text{def}}{=} v^1 \cdot g_{i_2}^{\varepsilon_2}, \quad \dots, \quad v^{\ell-1} \stackrel{\text{def}}{=} v^{\ell-2} \cdot g_{i_{\ell-1}}^{\varepsilon_{\ell-1}}, \quad v^\ell \stackrel{\text{def}}{=} v^{\ell-1} \cdot g_{i_\ell}^{\varepsilon_\ell} = v^0. \quad (2-1)$$

We denote this trajectory by $\bar{v} = (v^0, \dots, v^\ell)$. Given that the entire trajectory is determined by g_1, \dots, g_r and $v = v^0$, we do not change our goal by counting $(g_1, \dots, g_r; \bar{v})$ satisfying the equations in (2-1) instead of $(g_1, \dots, g_r; v)$ satisfying $v.w(g_1, \dots, g_r) = v$.

The basic idea behind our counting is grouping together solutions $(g_1, \dots, g_r; \bar{v})$ according to the linear relations over K which v^0, \dots, v^ℓ satisfy. There are finitely many options here (trivially, at most the number of linear subspaces of $K^{\ell+1}$), and, as we show below, for each subspace of $K^{\ell+1}$ the number of solutions $(g_1, \dots, g_r; \bar{v})$ corresponding to it is either identically zero for every N , or its contribution to $\mathbb{E}_w[\text{fix}]$ is given by a rational function in q^N for every large enough N .

Denote by $[1, w]$ the subtree of $\mathcal{C} = \text{Cay}(\mathbf{F}, B)$ corresponding to the path from the origin to the vertex w . For every $b \in B$, denote by $D_b(w)$ the vertices of $[1, w]$ with an outgoing b -edge (within $[1, w]$), and denote by $e_b(w)$ the number of b -edges in $[1, w]$, so $e_b(w) = |D_b(w)|$. Now consider a subspace $\Delta \leq K^{\ell+1}$ thought of as a set of equations on the vectors v^0, \dots, v^ℓ , or, equivalently, on the vertices of $[1, w]$. Below we denote these vertices by the corresponding prefix of w in \mathbf{F} , and write elements of $K^{[1, w]} \stackrel{\text{def}}{=} K^{\text{vert}([1, w])} \cong K^{\ell+1}$ as linear combinations of these vertices. We have

$$\begin{aligned} \mathbb{E}_w[\text{fix}] &= \frac{\#\{g_1, \dots, g_r \in GL_N(K), v \in V_N \mid v.w(g_1, \dots, g_r) = v\}}{|GL_N(K)|^r} \\ &= \frac{\#\{g_1, \dots, g_r \in GL_N(K), \bar{v} \in V_N^{\ell+1} \mid \bar{v} \text{ and } g_1, \dots, g_r \text{ satisfy (2-1)}\}}{|GL_N(K)|^r} \\ &= \sum_{\Delta \leq K^{[1, w]}} \frac{\#\{g_1, \dots, g_r \in GL_N(K), \bar{v} \in V_N^{\ell+1} \mid \bar{v} \text{ satisfies precisely } \Delta, \bar{v}, g_1, \dots, g_r \text{ satisfy (2-1)}\}}{|GL_N(K)|^r} \end{aligned} \quad (2-2)$$

If there are solutions $(g_1, \dots, g_r; \bar{v})$ which satisfy precisely Δ , then the following two conditions hold:

- C1:** $w - 1 \in \Delta$ (here $w - 1$ is the equation $w - 1 = 0$, or, equivalently, $v^\ell - v^0 = 0$).
- C2:** Δ is ‘‘closed under multiplication by $b^{\pm 1}$ ’’. Namely, for every $b \in B$ and every equation $\delta = \sum_{z \in D_b(w)} \lambda_z z$ ($\lambda_z \in K$) supported on $D_b(w)$, denote by $\delta b \stackrel{\text{def}}{=} \sum_{z \in D_b(w)} \lambda_z z b$ the corresponding equation on the vertices on the termini of the corresponding b -edges. Then

$$\delta \in \Delta \iff \delta b \in \Delta.$$

Conversely, if Δ satisfies conditions **C1** and **C2**, then for every large enough N there exist solutions $(g_1, \dots, g_r; \bar{v})$ satisfying precisely Δ , and the contribution of Δ in (2-2) is given by a rational function

in q^N . Indeed, denote by $\dim(\Delta)$ the dimension of the subspace Δ , and by $\dim_b(\Delta)$ the dimension of the subspace of Δ consisting of equations supported on $D_b(w)$. First, we choose a trajectory $\bar{v} \in V_N^{\ell+1}$ satisfying precisely Δ . The number of choices for such \bar{v} is precisely $\text{indep}_{\ell+1-\dim(\Delta)}(V_N)$, where

$$\text{indep}_h(V_N) \stackrel{\text{def}}{=} (q^N - 1)(q^N - q) \cdots (q^N - q^{h-1})$$

is the number of h -tuples of independent vectors in V_N .⁴

Second, given a trajectory \bar{v} satisfying precisely Δ , we choose the tuple $g_1, \dots, g_r \in \text{GL}_N(K)$ so that \bar{v}, g_1, \dots, g_r satisfy (2-1). We choose g_i separately for every $i = 1, \dots, r$. Let $b = b_i$. The vectors of \bar{v} at the starting points of b -edges in $[1, w]$, namely, in $D_b(w)$, span a subspace of V of dimension $e_b(w) - \dim_b(\Delta)$. (Such a trajectory may exist only if $e_b(w) - \dim_b(\Delta) \leq N$). In this case, the element g_i should map a subspace of dimension $e_b(w) - \dim_b(\Delta)$ in a prescribed way, and condition **C2** guarantees this prescribed way is valid and can be realized by a linear transformation. The number of elements in $\text{GL}_N(K)$ satisfying this constraint is

$$(q^N - q^{e_b(w) - \dim_b(\Delta)})(q^N - q^{e_b(w) - \dim_b(\Delta) + 1}) \cdots (q^N - q^{N-1}).$$

If g_1, \dots, g_r satisfy these constraints and as **C1** holds, \bar{v} and g_1, \dots, g_r satisfy (2-1). Overall, if $N \geq e_b(w) - \dim_b(\Delta)$ for every $b \in B$, the term corresponding to Δ in (2-2) is

$$\begin{aligned} \text{indep}_{\ell+1-\dim(\Delta)}(V_N) \cdot \prod_{b \in B} \frac{(q^N - q^{e_b(w) - \dim_b(\Delta)})(q^N - q^{e_b(w) - \dim_b(\Delta) + 1}) \cdots (q^N - q^{N-1})}{(q^N - 1)(q^N - q) \cdots (q^N - q^{N-1})} \\ = \frac{\text{indep}_{\ell+1-\dim(\Delta)}(V_N)}{\prod_{b \in B} \text{indep}_{e_b(w) - \dim_b(\Delta)}(V_N)}, \end{aligned}$$

which is rational in q^N . Overall, we obtain

$$\mathbb{E}_w[\text{fix}] = \sum_{\Delta \leq K^{[1, w]}: \Delta \text{ satisfies } \mathbf{C1}, \mathbf{C2}} \frac{\text{indep}_{\ell+1-\dim(\Delta)}(V_N)}{\prod_{b \in B} \text{indep}_{e_b(w) - \dim_b(\Delta)}(V_N)}, \tag{2-3}$$

which completes the proof of [Theorem 1.1](#).

The free-group-algebra approach. The key observation that leads to the free-group-algebra approach is that condition **C2** above is a feature of (as always, right) ideals of the free group algebra $\mathcal{A} = K[\mathbf{F}]$: a right ideal $I \leq \mathcal{A}$ is a K -linear subspace of \mathcal{A} satisfying **C2** on the entire Cayley graph \mathcal{C} (rather than on $[1, w]$ alone). To make this formal, let us recall some notation. For a subtree T of the Cayley graph $\mathcal{C} = \text{Cay}(\mathbf{F}, B)$, denote by $D_b(T)$ the set of vertices in the subtree $T \subset \mathcal{C}$ with an outgoing b -edge (inside T), and by $e_b(T) = |D_b(T)|$ the number of such edges. For any ideal $I \leq \mathcal{A}$, its restriction to T , denoted

$$I|_T \stackrel{\text{def}}{=} I \cap K^{\text{vert}(T)},$$

⁴We could not find a conventional notation for the quantity $\text{indep}_h(V_N)$, but it is closely related to existing common notation. For example, $\text{indep}_h(v) = q^{Nh} \cdot (q^{-N}; q)_h$, where $(t; q)_h \stackrel{\text{def}}{=} (1-t)(1-tq) \cdots (1-tq^{h-1})$ is the q -shifted factorial.

is a linear subspace of $K^{\text{vert}(T)}$. We say that a K -linear subspace $\Delta \leq K^{\text{vert}(T)}$ satisfies **C2**(T) if for every $\delta \in K^{\text{vert}(T)}$ supported on $D_b(T)$, we have $\delta \in \Delta$ if and only if $\delta.b \in \Delta$.

Lemma 2.1. *Assume that $\Delta \leq K^{\text{vert}(T)}$ is a K -linear subspace satisfying **C2**(T). Then $(\Delta) \leq \mathcal{A}$, the ideal generated by Δ , does not introduce any new elements supported on T , namely*

$$(\Delta)|_T = \Delta. \tag{2-4}$$

Proof. It is clear that $(\Delta)|_T \supseteq \Delta$, so it is enough to show the converse inclusion. We may assume that T is finite: Every element of \mathcal{A} has finite support, and every element of (Δ) is generated by finitely many elements of Δ . So if T is not finite and (2-4) fails, replace T with the finite subtree $S \subseteq T$ which is the convex hull of the support of an element in $(\Delta)|_T \setminus \Delta$ and its finitely many generators in Δ and replace Δ with $\Delta|_S$.

As in the proof of **Theorem 1.1** above, for large enough N , there are $g_1, \dots, g_r \in GL_N(K)$ and $\bar{v} = \{v_z \in V_N\}_{z \in \text{vert}(T)}$ such that for every b -edge $z_1 \xrightarrow{b} z_2$ in T , we have $v_{z_1}.g_b = v_{z_2}$, and such that the equations over K satisfied by the vectors \bar{v} are *precisely* the elements of Δ . The tuple g_1, \dots, g_r defines a group homomorphism $F \rightarrow GL_N(K)$ by $b_i \mapsto g_i$. This group homomorphism defines, in turn, a homomorphism of K -algebras $\mathcal{A} \rightarrow \text{End}(V_N)$. Equivalently, such a homomorphism of K -algebras defines a structure of an \mathcal{A} -module on V_N . Pick an arbitrary $z_0 \in \text{vert}(T) \subseteq F$. Now \mathcal{A} is itself an \mathcal{A} -module and, moreover, it is a free \mathcal{A} -module with basis $\{z_0\}$. There is a unique \mathcal{A} -module homomorphism $\phi : \mathcal{A} \rightarrow V_N$ such that $\phi(z_0) = v_{z_0}$. Since ϕ is an \mathcal{A} -module homomorphism and T is connected, the choice of the g_i 's guarantees that $\phi(z) = v_z$ for every $z \in \text{vert}(T) \subseteq F$.

Finally, $\ker \phi \leq \mathcal{A}$ is a submodule, or an ideal, and the equations it satisfies on $\text{vert}(T)$ are precisely those satisfied by \bar{v} , namely, Δ . Thus

$$(\Delta)|_T \leq [\ker \phi]|_T = \Delta. \tag{2-5}$$

Returning to the case $T = [1, w]$, recall that we write $I \leq_{[1, w]} \mathcal{A}$ if I is an ideal of \mathcal{A} with generating set supported on $[1, w]$. **Lemma 2.1** yields that there is a one-to-one correspondence

$$\{\Delta \leq K^{[1, w]} \text{ satisfying } \mathbf{C1} \text{ and } \mathbf{C2}\} \iff \{I \leq_{[1, w]} \mathcal{A} \mid w - 1 \in I\}.$$

For an ideal $I \leq \mathcal{A}$ and every finite subtree T of \mathcal{C} , define

$$d^T(I) \stackrel{\text{def}}{=} \dim_K(I|_T).$$

Similarly, for every basis element $b \in B$, denote by $D_b(T)$ the set of vertices in the subtree $T \subset \mathcal{C}$ with an outgoing b -edge (inside T), and let

$$d_b^T(I) \stackrel{\text{def}}{=} \dim_K(I|_{D_b(T)}).$$

With this notation, (2-3) is equivalent to

$$\mathbb{E}_w[\text{fix}] = \sum_{I \leq_{[1, w]} \mathcal{A} : I \ni w-1} \frac{\text{indep}_{|w|+1-d^{[1, w]}(I)}(V_N)}{\prod_{b \in B} \text{indep}_{e_b(w)-d_b^{[1, w]}(I)}(V_N)}. \tag{2-5}$$

The advantage of translating (2-3) to the language of ideals as in (2-5) will soon be apparent. For example, Corollary 3.9 below shows that the summand in (2-5) corresponding to $I \leq_{[1,w]} \mathcal{A}$ is of order $(q^N)^{1-\text{rk} I}$.

2B. The general case: Theorem 1.11. Fix $w \in \mathbf{F}$, $m \in \mathbb{Z}_{\geq 1}$ and $\mathcal{B} \in \text{GL}_m(K)$. Our goal is to prove that for every large enough N , the expectation $\mathbb{E}_w[\tilde{\mathcal{B}}]$ is a rational function in q^N . Now we need to count tuples $v_1, \dots, v_m \in V_N$ and $g_1, \dots, g_r \in \text{GL}_N(K)$ such that, defining $u_i \stackrel{\text{def}}{=} v_i \cdot w(g_1, \dots, g_r)$, we have

$$\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} = \mathcal{B} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}. \tag{2-6}$$

As above, we consider the entire trajectories of v_1, \dots, v_m through the letters of w , namely,

$$\begin{aligned} v_1^0 &= v_1, & v_1^1 &= v_1 \cdot g_{i_1}^{\varepsilon_1}, & \dots, & v_1^\ell &= v_1 \cdot w(g_1, \dots, g_r), \\ & & & \vdots & & & \\ v_m^0 &= v_m, & v_m^1 &= v_m \cdot g_{i_1}^{\varepsilon_1}, & \dots, & v_m^\ell &= v_m \cdot w(g_1, \dots, g_r), \end{aligned}$$

which we denote by \bar{v} . Again we group the solutions $(g_1, \dots, g_r; \bar{v})$ according to the equations over K satisfied by \bar{v} . This time, the equations are not given by ideals in \mathcal{A} , but rather by submodules of the right free \mathcal{A} -module \mathcal{A}^m . Formally, let $E = \{e_1, \dots, e_m\}$ be a basis of the free module \mathcal{A}^m . Every element of \mathcal{A}^m is a finite linear combination, with coefficients from K , of monomials ez with $e \in E$ and $z \in \mathbf{F}$. These monomials are identified with the vertices of m disjoint copies $\mathcal{C}_1, \dots, \mathcal{C}_m$ of $\text{Cay}(\mathbf{F}, B)$, with origins e_1, \dots, e_m , respectively.

Let \mathbb{W} denote the union of the paths $[1, w]$ in $\mathcal{C}_1, \dots, \mathcal{C}_m$, so $\mathbb{W} = \bigcup_{e \in E} [e, ew]$. Recall that $M \leq_{\mathbb{W}} \mathcal{A}^m$ means that M is a submodule of \mathcal{A}^m with a generating set supported on \mathbb{W} . If the equations satisfied by the trajectory \bar{v} are precisely $M|_{\mathbb{W}}$, then M must, in particular, contain the elements dictated by (2-6), which we denote by $\text{EQ}_{\mathcal{B},w} \subseteq \mathcal{A}^m$. For example, if $\mathcal{B} = \begin{pmatrix} 2 & 1 \\ 7 & 3 \end{pmatrix} \in \text{GL}_2(K)$, then $\text{EQ}_{\mathcal{B},w}$ equals $\{e_1w - 2e_1 - e_2, e_2w - 7e_1 - 3e_2\}$.

Generalizing the notation from above, if $\mathbb{T} = T_1 \cup \dots \cup T_m$ is a union of (possibly empty) subtrees $T_i \subseteq \mathcal{C}_i$, and $M \leq \mathcal{A}^m$, define

$$\begin{aligned} d^{\mathbb{T}}(M) &\stackrel{\text{def}}{=} \dim_K(M|_{\mathbb{T}}), \\ d_b^{\mathbb{T}}(M) &\stackrel{\text{def}}{=} \dim_K(M|_{D_b(\mathbb{T})}), \quad b \in B, \\ e_b(\mathbb{T}) &\stackrel{\text{def}}{=} |D_b(\mathbb{T})|. \end{aligned}$$

So $|D_b(\mathbb{W})| = m \cdot e_b(w)$. The same argument as above shows that for every $N \geq \max_{b \in B} e_b(\mathbb{W})$,

$$\mathbb{E}_w[\tilde{\mathcal{B}}] = \sum_{M \leq_{\mathbb{W}} \mathcal{A}^m : M \supseteq \text{EQ}_{\mathcal{B},w}} \frac{\text{indep}_{m(|w|+1)-d^{\mathbb{W}}(M)}(V_N)}{\prod_{b \in B} \text{indep}_{e_b(\mathbb{W})-d_b^{\mathbb{W}}(M)}(V_N)}. \tag{2-7}$$

As there are finitely many submodules $M \leq_{\mathbb{W}} \mathcal{A}^m$, the expression (2-7) is rational in q^N . This completes the proof of Theorem 1.11.

3. The free group algebra and its ideals

We gather some known results and some new results about the free group algebra $\mathcal{A} = K[\mathbf{F}]$ and its (as always in this text, right) ideals, and more generally the free right \mathcal{A} -module \mathcal{A}^m and its submodules. Although we assume throughout this paper that K is a fixed finite field, most results of the current section apply to an arbitrary field (not necessarily finite).

The starting point of the story is a paper of Cohn [1964] and a paper of Lewin [1969] (see note 1) which prove that \mathcal{A} is a *free ideal ring*, in the following sense:

Theorem 3.1 [Cohn 1964; Lewin 1969]. *Every ideal $I \leq \mathcal{A}$ is a free \mathcal{A} -module. More generally, every submodule of a free \mathcal{A} -module is free. Every free \mathcal{A} -module M has a unique rank: all bases of M have the same cardinality.*

See [Hog-Angeloni 1990; Rosenmann and Rosset 1994; Rosenmann 1993] for additional proofs of this result.

There are two main new results in Section 3. In Theorem 3.8 below it is shown that if an ideal $I \leq_T \mathcal{A}$ has a generating set supported on some finite subtree T of $\text{Cay}(\mathbf{F}, B)$, then it also admits a basis supported on T . Our analysis also leads to Corollary 3.9: the order of contribution of every ideal $I \leq_{[1,w]} \mathcal{A}$ with $w - 1 \in I$ to the summation (2-5) of $\mathbb{E}_w[\text{fix}]$ is given by its rank.

Recall that $E = \{e_1, \dots, e_m\}$ is a basis of the free right module \mathcal{A}^m , that the elements of \mathcal{A}^m are K -linear combinations of *monomials* $\{ez\}_{e \in E, z \in F}$, and that we identify these monomials with the vertices of m disjoint copies $\mathcal{C}_1, \dots, \mathcal{C}_m$ of $\text{Cay}(\mathbf{F}, B)$. Let $\mathbb{T} = T_1 \cup \dots \cup T_m$ be a union of m finite, possibly empty, subtrees $T_i \subset \mathcal{C}_i$, and let $M \leq_{\mathbb{T}} \mathcal{A}^m$ be a submodule generated on \mathbb{T} . In order to study M , we expose the vertices of \mathbb{T} one-by-one and with them the elements of M which are supported on the already-exposed vertices. Denote by v_t the vertex exposed in step t , where $t = 1, \dots, \#\text{vert}(\mathbb{T})$, and let M_t denote the submodule generated by $M|_{\{v_1, \dots, v_t\}}$, so

$$(0) = M_0 \leq M_1 \leq \dots \leq M_{\#\text{vert}(\mathbb{T})} = M.$$

The order by which we expose the vertices of \mathbb{T} should have the property that as often as possible, we expose neighbors of already-exposed vertices. Formally, it should be the restriction to \mathbb{T} of a full order on the vertices of $\mathcal{C}_1 \cup \dots \cup \mathcal{C}_m$ which abides to the following assumption.

Definition 3.2 (exploration). We call a full order \leq on the vertices of $\mathcal{C}_1 \cup \dots \cup \mathcal{C}_m$ an *exploration* if it is an enumeration of the vertices (so every vertex has finitely many smaller vertices), and every vertex is either

- (1) a neighbor of a smaller vertex, or
- (2) the smallest vertex in some \mathcal{C}_i .

An order on a collection $\mathbb{T} = T_1 \cup \dots \cup T_m$ of (possibly empty) subtrees $T_i \subseteq \mathcal{C}_i$ is called an *exploration* if it is the restriction of an exploration of $\mathcal{C}_1 \cup \dots \cup \mathcal{C}_m$.

Note that an order on \mathbb{T} is an exploration if and only if it is an enumeration of the vertices of \mathbb{T} which satisfies that every vertex is either a neighbor of a smaller vertex of \mathbb{T} or the first vertex visited in some T_i .

Given a finite \mathbb{T} and $M \leq_{\mathbb{T}} \mathcal{A}^m$ as above, every step is either free, forced or a coincidence, according to the following conventions.⁵ Assume first that v_t is a neighbor of an already-exposed vertex u , and that the edge from u to v_t is labeled by $b \in B \cup B^{-1} = \{b_1^{\pm 1}, \dots, b_r^{\pm 1}\}$:

$$u \xrightarrow{b} v_t. \tag{3-1}$$

Denote by D_b^t the set of already-exposed vertices with an outgoing b -edge leading to another already-exposed vertex. This set should include the vertex u . If $M|_{D_b^t}$ contains an element with u in its support, we say the t -th step is *forced*. If v_t is the first vertex we expose in some T_i , the t -th step is not forced. If a step is not forced, it is a *coincidence* if there is an element of $M|_{\{v_1, \dots, v_t\}}$ with v_t in its support, and otherwise it is *free*.

Lemma 3.3. *Let \mathbb{T} and $M \leq_{\mathbb{T}} \mathcal{A}^m$ be as above and let v_1, v_2, \dots be an exploration of $\text{vert}(\mathbb{T})$. Then step t in the exposure of M along \mathbb{T} is*

$$\begin{aligned} \text{forced} &\iff M_{t-1} = M_t \text{ and } M|_{\{v_1, \dots, v_{t-1}\}} \not\leq M|_{\{v_1, \dots, v_t\}}, \\ \text{free} &\iff M_{t-1} = M_t \text{ and } M|_{\{v_1, \dots, v_{t-1}\}} = M|_{\{v_1, \dots, v_t\}}, \\ \text{a coincidence} &\iff M_{t-1} \leq M_t. \end{aligned}$$

If step t is a coincidence and f is an element of $M|_{\{v_1, \dots, v_t\}}$ with v_t in its support, then M_t is generated by M_{t-1} and f .

Of course, if $M_{t-1} \leq M_t$, then, in particular, $M|_{\{v_1, \dots, v_{t-1}\}} \leq M|_{\{v_1, \dots, v_t\}}$.

Proof. First assume step t is forced. There is some $f \in M|_{D_b^t}$ with u in its support, and then $f.b \in M|_{\{v_1, \dots, v_t\}} \setminus M|_{\{v_1, \dots, v_{t-1}\}}$. Yet $f.b \in M_{t-1}$ and any other element of $M|_{\{v_1, \dots, v_t\}}$, by subtracting a suitable K -multiple of $f.b$, becomes an element of M_{t-1} . Hence $M_{t-1} = M_t$.

If the step is free, then $M|_{\{v_1, \dots, v_{t-1}\}} = M|_{\{v_1, \dots, v_t\}}$ by definition, and so $M_{t-1} = M_t$.

Finally, assume that step t is a coincidence. Fix $N \geq t$, and consider (row) vectors $u_1, \dots, u_{t-1} \in V_N = K^N$ with dependencies corresponding *exactly* to the elements of $M|_{\{v_1, \dots, v_{t-1}\}}$, namely, $\sum_{i=1}^{t-1} \alpha_i u_i = 0$ if and only if $\sum_{i=1}^{t-1} \alpha_i v_i \in M$. Let $u_t \in V_N$ be some vector which is *linearly independent* of u_1, \dots, u_{t-1} . For every $b \in B$, there is an element $g_b \in \text{GL}(V_N)$ with $u.g_b = u'$ for every b -edge (u, u') with $u, u' \in \{u_1, \dots, u_t\}$ (here we rely on that the step is not forced). As in the proof of [Lemma 2.1](#), these g_b 's determine a K -algebra homomorphism $\varphi : \mathcal{A} \rightarrow \text{End}(V_N)$. This φ gives V_N a structure of an \mathcal{A} -module. For every $e \in E$ with T_e already visited, pick an arbitrary $v_e \in T_e \cap \{v_1, \dots, v_t\}$. Then these monomials $\{v_e\}$ form a subbasis of the free \mathcal{A} -module \mathcal{A}^m , and there is a homomorphism of \mathcal{A} -modules $\psi : \mathcal{A}^m \rightarrow V$ mapping v_e to u_e . By design, the linear dependencies among u_1, \dots, u_t correspond precisely to the

⁵This terminology is inspired by [\[Eberhard and Jezernik 2022\]](#), which, in turn, was inspired by earlier works dealing with random Schreier graphs of symmetric groups (see, for example, [\[Broder and Shamir 1987\]](#)). The analog in [\[Eberhard and Jezernik 2022\]](#) of our free step is a free step which is not a coincidence, and the analog in the same article of our coincidence is a free step which is also a coincidence.

elements of $\ker \psi$ supported on $\{v_1, \dots, v_t\}$. As u_t is independent of the rest, we get that

$$M_{t-1} \leq \ker \psi \quad \text{yet} \quad M_t \not\leq \ker \psi,$$

proving that $M_{t-1} \not\cong M_t$.

If step t is a coincidence and $f \in M|_{\{v_1, \dots, v_t\}}$ has v_t in its support, then any other element $g \in M|_{\{v_1, \dots, v_t\}}$ satisfies that $g - \alpha f \in M|_{\{v_1, \dots, v_{t-1}\}}$ for some $\alpha = \alpha(g) \in K$. Hence the final part of the statement of the lemma follows. \square

Lemma 3.4. *Let \mathbb{T} and $M \leq_{\mathbb{T}} \mathcal{A}^m$ be as above. In every exposure process of M along \mathbb{T} as above, the number of coincidences is the same: it does not depend on the order of exposure (as long as it is a valid exploration à la Definition 3.2).*

Proof. Similarly to the definition of $d^{\mathbb{T}}(M)$ and $d_b^{\mathbb{T}}(M)$ from Section 2, let $d^t \stackrel{\text{def}}{=} \dim_K(M|_{\{v_1, \dots, v_t\}})$ and $d_b^t \stackrel{\text{def}}{=} \dim_K(M|_{D_b^t})$. Obviously, $d^0 = d_b^0 = 0$. We now trace how d^t and $\sum_{b \in B} d_b^t$ change with t , depending on the three types of steps defined above. According to the definitions and to Lemma 3.3:

- In a forced step, both d^t and $\sum_b d_b^t$ increase by one (compared to d^{t-1} and $\sum_b d_b^{t-1}$, respectively).
- In a free step, both d^t and $\sum_b d_b^t$ do not change.
- In a coincidence, d^t increases by one, while $\sum_b d_b^t$ does not change.

Therefore, the difference $d^{\mathbb{T}}(M) - \sum_b d_b^{\mathbb{T}}(M)$, which is, of course, independent of the order of exposure, is equal to the number of coincidences. \square

The proof of Lemma 3.4 actually shows that the number of forced and free steps is also independent of the order of exposure, but that is not as useful. The proof also gives the following.

Corollary 3.5. *Consider the expression (2-7) giving $\mathbb{E}_w[\tilde{\mathcal{B}}]$ as a sum over submodules $M \leq_{\mathbb{W}} \mathcal{A}^m$ with $M \supseteq \text{EQ}_{B,w}$. The summand corresponding to such a submodule M is*

$$(q^N)^{m - \#\text{coincidences}} \left(1 + O\left(\frac{1}{q^N}\right) \right),$$

where we count coincidences in an exposure process of M along \mathbb{W} .

Proof. The numerator in the summand corresponding to M in (2-7) is

$$\text{indep}_{m(|w|+1) - d^{\mathbb{W}}(M)}(V_N) = (q^N)^{m(|w|+1) - d^{\mathbb{W}}(M)} \left(1 + O\left(\frac{1}{q^N}\right) \right).$$

The denominator is

$$\prod_b \text{indep}_{e_b(\mathbb{W}) - d_b^{\mathbb{W}}(M)}(V_N) = (q^N)^{\sum_b (e_b(\mathbb{W}) - d_b^{\mathbb{W}}(M))} \left(1 + O\left(\frac{1}{q^N}\right) \right).$$

The result follows as $\sum_b e_b(\mathbb{W}) = m|w|$ and as $d^{\mathbb{W}}(M) - \sum_b d_b^{\mathbb{W}}(M)$ is equal to the number of coincidences. \square

Next, we show that the number of coincidences is identical to the rank of the module M . The proof relies on the main theorem of [Lewin 1969], which makes use of the following notion.

Definition 3.6. A Schreier transversal of a submodule $M \leq \mathcal{A}^m$ is a set ST of monomials of \mathcal{A}^m which satisfies

- (i) ST is closed under prefixes: if $ez \in \text{ST}$ with $e \in E$ and $1 \neq z \in F$, and $b \in B \cup B^{-1}$ is the last letter of z , then $ezb^{-1} \in \text{ST}$, and
- (ii) the linear span $\text{span}_K(\text{ST})$ of ST contains exactly one representative of every coset of \mathcal{A}^m/M .

It is not hard to show that every $M \leq \mathcal{A}^m$ admits Schreier transversals — see [Lewin 1969, pp. 456–457] for an argument as well as for a concrete construction. A Schreier transversal ST consists of the vertices in a collection of (possibly infinite) subtrees, one in \mathcal{C}_i for every $i = 1, \dots, m$. The main theorem of [Lewin 1969] is that one may construct a basis for M which is, roughly, in one-to-one correspondence with the outgoing directed edges from ST to its complement. Although a version of this theorem holds for any submodule of any free \mathcal{A} -module, we only need the case of finitely generated \mathcal{A} -modules.

Theorem 3.7 [Lewin 1969, Theorem 1]. *Let $M \leq \mathcal{A}^m$ be a submodule, and let ST be a Schreier transversal of M . For every $f \in \mathcal{A}^m$, denote by $\phi(f)$ the representative of $f + M$ in $\text{span}_K(\text{ST})$. Then the set*

$$\{ezb - \phi(ezb) \mid ez \in \text{ST}, b \in B, ezb \notin \text{ST}\} \cup \{e - \phi(e) \mid e \in E \setminus \text{ST}\} \tag{3-2}$$

is a basis for M (as a free \mathcal{A} -module).

We stress that in (3-2), b is a proper basis element and not the inverse of one.

Theorem 3.8. *Let $\mathbb{T} = T_1 \cup \dots \cup T_m$ be a collection of finite, possibly empty, subtrees $T_i \subset \mathcal{C}_i$ and assume that $M \leq_{\mathbb{T}} \mathcal{A}^m$. Then the number of coincidences in an exposure of M along \mathbb{T} is **equal** to $\text{rk } M$.*

Moreover, M admits a basis supported on \mathbb{T} . In fact, every set of elements $f_1, \dots, f_{\text{rk } M}$ supported on \mathbb{T} with the leading vertex⁶ of f_i being the monomial exposed in the i -th coincidence is a basis of M .

Proof. Let $s = \text{rk } M$. Let ST be a Schreier transversal for M . Then the basis (3-2) contains s elements. Let \mathbb{S} be the smallest collection of finite subtrees (one in each \mathcal{C}_i) which contain the whole support of these s basis elements. Note that \mathbb{S} contains exactly s vertices (monomials) outside ST, and all these vertices are either leaves or isolated in \mathbb{S} (namely, these are vertices of degree 1 or 0 in \mathbb{S}). Consider an exposure process of M along \mathbb{S} according to some exploration such that the vertices of $\mathbb{S} \cap \text{ST}$ are exposed first and only then the remaining s vertices. Because there is no nonzero element of M supported on ST, the first $|\mathbb{S}| - s$ steps are all free.

We claim that the remaining s steps are all coincidences. Indeed,

$$(0) = M_{|\mathbb{S}|-s} \leq M_{|\mathbb{S}|-s+1} \leq \dots \leq M_{|\mathbb{S}|-1} \leq M_{|\mathbb{S}|} = M.$$

For $i = 1, \dots, s$, let $f_i \in M_{|\mathbb{S}|-s+i}$ be the basis element from (3-2) with the vertex exposed in step $|\mathbb{S}| - s + i$ in its support. Clearly, $f_i \in M_{|\mathbb{S}|-s+i}$. By induction, $M_{|\mathbb{S}|-s+i} = (f_1, \dots, f_i)$. Indeed, $M_{|\mathbb{S}|-s+1} = (f_1)$, and

⁶The leading vertex of $f \in M$ is the last vertex in the support of f to be exposed in the exploration process on \mathbb{T} .

if $M_{|\mathbb{S}|-s+i-1} = (f_1, \dots, f_{i-1})$ then either step i is a coincidence and then $M_{|\mathbb{S}|-s+i} = (M_{|\mathbb{S}|-s+i-1}, f_i)$ by Lemma 3.3, or step i is not a coincidence and then $M_{|\mathbb{S}|-s+i} = M_{|\mathbb{S}|-s+i-1}$. But $\{f_1, \dots, f_s\}$ is a basis by Theorem 3.7, so $f_i \notin (f_1, \dots, f_{i-1}) = M_{|\mathbb{S}|-s+i-1}$. We conclude that $M_{|\mathbb{S}|-s+i-1} \not\supseteq M_{|\mathbb{S}|-s+i}$ so all these s steps are indeed coincidences by Lemma 3.3.

Now consider the collection of finite trees \mathbb{U} , which is the collection of smallest subtrees (one in each \mathcal{C}_i) which contains both \mathbb{S} and the given \mathbb{T} . Expose M along \mathbb{U} by two different explorations. In the first order, expose \mathbb{S} first and then the remaining vertices of \mathbb{U} . There are exactly s coincidences: after we exposed all of \mathbb{S} , we have $M_{|\mathbb{S}|} = M$, so no more coincidences are possible, by Lemma 3.3. By Lemma 3.4, s is also the number of coincidences when we first expose \mathbb{T} and then the remaining vertices of $\mathbb{U} \setminus \mathbb{T}$. But again, because M is generated on \mathbb{T} , we have $M_{|\mathbb{T}|} = M$ and there are no more coincidences after exposing \mathbb{T} . This shows there are exactly $s = \text{rk } M$ coincidences in an exposure of M along \mathbb{T} .

For the second statement, assume that $M \leq_{\mathbb{T}} \mathcal{A}^m$ and consider an exposure of M along \mathbb{T} . If step t is a coincidence, then by Lemma 3.3, $M_t = (M_{t-1}, f_t)$ where $f_t \in M_{|\{v_1, \dots, v_t\}|}$ with v_t in its support. Hence $M = (f_{t_1}, \dots, f_{t_s})$ where t_1, \dots, t_s are the s coincidences. But every set of size $s = \text{rk } M$ which generates M is a basis [Cohn 1964, Proposition 2.2]. \square

From Theorem 3.8 and Corollary 3.5 we immediately obtain that the order of contribution of a given ideal to $\mathbb{E}_w[\text{fix}]$ is given by its rank:

Corollary 3.9. *Consider the expression (2-7) giving $\mathbb{E}_w[\tilde{\mathcal{B}}]$ as a sum over submodules $M \leq_{\mathbb{W}} \mathcal{A}^m$ with $M \supseteq \text{EQ}_{\mathcal{B}, w}$. The summand corresponding to such a submodule M is*

$$(q^N)^{m-\text{rk } M} \left(1 + O\left(\frac{1}{q^N}\right) \right).$$

In Section 4B we show that there are no submodules of rank $< m$ containing $\text{EQ}_{\mathcal{B}, w}$, and so $\lim_{N \rightarrow \infty} \mathbb{E}_w[\tilde{\mathcal{B}}]$, the limit from Theorem 1.12, is equal to the number of rank- m submodules supported on \mathbb{W} and containing $\text{EQ}_{\mathcal{B}, w}$. Using Corollary 3.10, one can show that the restriction to submodules supported on \mathbb{W} is redundant — we elaborate in Section 4.

Recall that Definition 1.5 introduced $\pi_q(w)$ and $\text{Crit}_q(w)$ for every $w \in \mathbf{F}$. Theorem 3.8 can also be used to show that the set $\text{Crit}_q(w)$ is always finite. If N is a free \mathcal{A} -module and $L \leq N$ a submodule (and therefore free as well), we say that L is a free factor of N if some basis (and hence every basis) of L can be extended to a basis of N .

Corollary 3.10. *Let $M \leq N \leq \mathcal{A}^m$ be two finitely generated submodules of \mathcal{A}^m , and assume that there is no intermediate submodule which is a proper free factor of N .⁷ Namely, if $M \leq L \leq N$ and L is a free factor of N then $L = N$. If $M \leq_{\mathbb{T}} \mathcal{A}^m$ with \mathbb{T} a union of subtrees as above, then $N \leq_{\mathbb{T}} \mathcal{A}^m$.*

Proof. Take a collection \mathbb{S} of subtrees which contains \mathbb{T} and such that $N \leq_{\mathbb{S}} \mathcal{A}^m$. Expose N along \mathbb{S} according to some exploration which first exposes \mathbb{T} and then the remaining vertices. Let $N_{|\mathbb{T}|} = (N_{|\mathbb{T}|})$

⁷In analogy with subgroups of the free group \mathbf{F} , one may say that N is an algebraic extension of M — see, e.g., [Puder and Parzanchevski 2015, Definition 2.1].

denote the submodule of N generated by the elements of N supported on \mathbb{T} . Clearly, $M \leq N_{|\mathbb{T}|}$, and using [Theorem 3.8](#) to construct a basis for N from the coincidences of this exposure process, we get that $N_{|\mathbb{T}|}$ is a free factor of N . By assumption we therefore have $N_{|\mathbb{T}|} = N$, so $N \leq_{\mathbb{T}} \mathcal{A}^m$. \square

In the following corollary we use the fact that K is finite. For example, the element $xyx^{-1}y^{-1} - 1 \in \mathcal{A}$ has critical ideals $\{(\alpha x - 1, \beta y - 1) \mid \alpha, \beta \in K^*\}$, which is an infinite set if K is infinite. For a general element $f \in \mathcal{A}$, we say that an ideal $I \leq \mathcal{A}$ is *critical* for f if it contains f as an imprimitive element, and it has minimal rank among all such ideals.

Corollary 3.11. *Let $f \in \mathcal{A}$, and suppose that the subtree $T \subseteq \text{Cay}(\mathbf{F}, B)$ supports f . Then any critical ideal of f is generated on T . In particular, $\text{Crit}_q(w)$ is finite for every word $w \in \mathbf{F}$ and every prime power q .*

Proof. Assume that $I \leq \mathcal{A}$ is critical for f , namely, that it is an ideal of minimal rank which contains f as an imprimitive element. Assume that $f \in J \leq I$ and that J is a free factor of I . In particular, $\text{rk } J \leq \text{rk } I$. If f is primitive in J , it is also primitive in I , which is impossible. So f is imprimitive in J . But I is critical for f , and so $\text{rk } J = \text{rk } I$ and $J = I$. Therefore the assumption of [Corollary 3.10](#) applies to $(f) \leq I$, and for every finite subtree $T \subseteq \text{Cay}(\mathbf{F}, B)$ supporting f , we have $I \leq_T \mathcal{A}$. For every $f \in \mathcal{A}$ we may take T finite, and if K is finite, there are only finitely many ideals supported on T . \square

3A. Properties of the q -primitivity rank. We prove some basic properties of the q -primitivity rank of words. Let H be a subgroup of the free group \mathbf{F} . We associate to H two (right) ideals of interest. The first is its augmentation ideal $I_H \leq K[H]$, defined as the kernel of the augmentation map $\varepsilon_H : K[H] \rightarrow K$ where $\varepsilon_H(\sum_{h \in H} \alpha_h h) = \sum_{h \in H} \alpha_h$. If $\{h_\beta\}_{\beta \in B}$ is a basis for H then $\{h_\beta - 1\}_{\beta \in B}$ is a basis for I_H [[Cohen 1972](#), Proposition 4.8], and in particular $\text{rk } I_H = \text{rk } H$. The second, when considering H as a subgroup of \mathbf{F} , is the (right) ideal J_H of $\mathcal{A} = K[\mathbf{F}]$ generated by $\{h - 1\}_{h \in H}$. The following proposition also follows from [[Cohen 1972](#), Chapter 4], but as it is not stated there explicitly, we add a short proof.

Proposition 3.12. *If $\{h_\beta\}_{\beta \in B}$ is a basis for H then $\{h_\beta - 1\}_{\beta \in B}$ is a basis for J_H . In particular, $\text{rk } J_H = \text{rk } H$.*

Proof. Since $\{h_\beta - 1\}_{\beta \in B}$ already generates I_H in $K[H]$, it generates $h - 1$ for any $h \in H$, and is thus a generating set for J_H . Let T be a right transversal for H in \mathbf{F} (i.e., a set of representatives of the right cosets of H). Then for every $t \in T$ the set $K[H]t$ of elements of \mathcal{A} supported on the coset Ht forms a left $K[H]$ -module, and the group algebra \mathcal{A} admits a left $K[H]$ -module decomposition $\mathcal{A} = \bigoplus_{t \in T} K[H]t$. Let $P_{Ht} : \mathcal{A} \rightarrow K[H]t$ be the projections induced by this decomposition. Suppose now that there is a relation $\sum_{\beta \in B} (h_\beta - 1)a_\beta = 0$ for some coefficients $\{a_\beta\}_{\beta \in B}$ in \mathcal{A} . For every $t \in T$, applying the left $K[H]$ -module map P_{Ht} to both sides yields the relation $\sum_{\beta \in B} (h_\beta - 1)P_{Ht}(a_\beta) = 0$, and multiplying by t^{-1} gives $\sum_{\beta \in B} (h_\beta - 1)(P_{Ht}(a_\beta)t^{-1}) = 0$. Since $P_{Ht}(a_\beta)t^{-1} \in K[H]$ and $\{h_\beta - 1\}_{\beta \in B}$ is a basis (for I_H), we deduce that $P_{Ht}(a_\beta) = 0$ for every $\beta \in B$. Thus, $a_\beta = \sum_{t \in T} P_{Ht}(a_\beta) = 0$ for every $\beta \in B$. \square

Proposition 3.13. *Let $H \leq \mathbf{F}$ be finitely generated and let $w \in \mathbf{F}$. If $w - 1$ is primitive in J_H then w is primitive in H .*

Proof. Assume that $w - 1$ is primitive in J_H . As $w - 1 \in J_H$, by [Cohen 1972, Lemma 4.1], w lies in H . Fix a basis h_1, h_2, \dots, h_k for H . Then $\{h_i - 1\}_{i=1}^k$ is a basis for I_H and $w - 1 \in I_H$, so we can write (uniquely) $w - 1 = \sum_{i=1}^k (h_i - 1)a_i$ for some coefficients $a_i \in K[H]$. By a theorem of Umirbaev⁸ [1994, Corollary on page 184], to deduce that w is primitive in H it is enough to show that the coefficients $\{a_i\}_{i=1}^k$ form a left-invertible column in the sense that there exist $u_1, u_2, \dots, u_k \in K[H]$ such that $\sum_{i=1}^k u_i a_i = 1$. Since $w - 1$ is primitive in J_H , there exist some elements $f_2, \dots, f_k \in J_H$ completing $w - 1$ to a basis of J_H . By Proposition 3.12, $\{h_i - 1\}_{i=1}^k$ is, too, a basis for J_H . Let $C \in M_{kk}(\mathcal{A})$ be a change-of-basis matrix satisfying $(h_1 - 1, h_2 - 1, \dots, h_k - 1)C = (w - 1, f_2, \dots, f_k)$, where by uniqueness of presenting $w - 1$ in the basis $\{h_i - 1\}_{i=1}^k$ the first column of C is

$$\begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix}.$$

As one can also change basis in the other direction, there exists some $D \in M_{kk}(\mathcal{A})$ such that

$$(h_1 - 1, h_2 - 1, \dots, h_k - 1) = (w - 1, f_2, \dots, f_k)D.$$

Thus, $(w - 1, f_2, \dots, f_k)DC = (w - 1, f_2, \dots, f_k)$, which by the uniqueness of presentation implies that DC is the identity matrix. In particular, the first row of D which we denote by (d_1, d_2, \dots, d_k) is a left inverse to the first column of C in the sense that

$$\sum_{i=1}^k d_i a_i = 1. \tag{3-3}$$

We next show that the elements $\{d_i\}_{i=1}^k$ can be replaced by elements $\{u_i\}_{i=1}^k$ lying in $K[H]$. Let T be a left transversal for H in F . Then as a right $K[H]$ -module, \mathcal{A} decomposes as $\mathcal{A} = \bigoplus_{t \in T} tK[H]$. Denote by P_{tH} the projection onto the summand corresponding to t . Then applying the right $K[H]$ -module map P_H to (3-3) gives $\sum_{i=1}^k P_H(d_i)a_i = 1$. We finish by letting $u_i = P_H(d_i)$. \square

Lemma 3.14. *Let $J \leq \mathcal{A}$ be an ideal and $f \in J$ a primitive element. Then f is primitive in every intermediate ideal $I \leq J$.*

Proof. Since f is primitive in J we can write $J = (f) \oplus M$ for some ideal M . We claim that $I = (f) \oplus (M \cap I)$. The directness is obvious (M already intersects (f) trivially). It remains to show that I is indeed the sum of the two summands. Let $a \in I$. Then $a \in J$ and thus can be decomposed as $a = a_1 + m$ where $a_1 \in (f)$ and $m \in M$. But then $m = a - a_1 \in I$ and so $m \in M \cap I$. \square

In the following corollary we do not assume that H is finitely generated.

Corollary 3.15. *Let $H \leq F$ and let $w \in F$. Then w is primitive in H if and only if $w - 1$ is primitive in J_H .*

⁸Umirbaev's result is actually stated for free group rings over the integers. However, the proof uses no specific properties of \mathbb{Z} and hence also applies, mutatis mutandis, to the field K .

Proof. One implication is immediate from [Proposition 3.12](#). For the other implication, suppose that $w - 1$ is primitive in J_H . Let S be a basis for H . When writing $w - 1$ in the basis $\{s - 1\}_{s \in S}$ of J_H , only finitely many basis elements appear, so there exist some $h_1, \dots, h_k \in S$ and $a_1, \dots, a_k \in \mathcal{A}$ such that $w - 1 = \sum_{i=1}^k (h_i - 1)a_i$. Let $H' = \langle h_1, h_2, \dots, h_k \rangle$. Then $w - 1$ lies in $J_{H'}$ and by [Lemma 3.14](#) it is primitive in it. Since H' is finitely generated, [Proposition 3.13](#) guarantees that w is primitive in H' . Since the relation of being a free factor is transitive and $\langle w \rangle \leq^* H' \leq^* H$ we are done. \square

We can now prove [Proposition 1.8](#) stating that for every prime power q , the q -primitivity rank is bounded from above by the ordinary primitivity rank, namely, $\pi_q(w) \leq \pi(w)$ for every $w \in \mathbf{F}$.

Proof of Proposition 1.8. Let $w \in \mathbf{F}$. The ordinary primitivity rank of a word is a nonnegative integer or ∞ . We first deal with two trivial cases: if $\pi(w) = \infty$ then there is nothing to prove, and if $\pi(w) = 0$ then $w = 1$ and so $w - 1 = 0$ is contained in the rank-0 trivial ideal of \mathcal{A} as an imprimitive element, so $\pi_q(w) = 0$ as well. Suppose now that $\pi(w) = k \notin \{0, \infty\}$ and let H be a critical subgroup for w in \mathbf{F} , i.e., a subgroup of \mathbf{F} of rank k containing w as an imprimitive element. The ideal $J_H \leq \mathcal{A}$ contains $w - 1$ by its definition as $w \in H$, it contains $w - 1$ as an imprimitive element by [Corollary 3.15](#), it has rank $\text{rk } J_H = k$ by [Proposition 3.12](#), and it is a proper ideal of \mathcal{A} since it is contained in the augmentation ideal $I_{\mathbf{F}} \subsetneq \mathcal{A}$. We conclude that $\pi_q(w) \leq k = \pi(w)$. \square

If $w \in \mathbf{F}$ is primitive, then (analogously to [Lemma 3.14](#)), w is primitive in any subgroup of \mathbf{F} containing it (see, e.g., [[Puder 2014](#), Claim 2.5]). In particular, it has primitivity rank $\pi(w) = \infty$. Furthermore, any imprimitive word $w \in \mathbf{F}$ must have $\pi(w) \leq \text{rk } \mathbf{F}$ since it is already not primitive in \mathbf{F} . Thus, the primitivity rank of words in \mathbf{F} takes values in $\{0, 1, 2, \dots, \text{rk } \mathbf{F}\} \cup \{\infty\}$. We next show that analogous statements hold when π is replaced with π_q .

Proposition 3.16. *For every $w \in \mathbf{F}$ and prime power q , $\pi_q(w) = \infty$ if and only if w is primitive in \mathbf{F} .*

Proof. If $\pi_q(w) = \infty$ then $w - 1$ must be primitive in $J_{\mathbf{F}}$, which implies by [Proposition 3.13](#) that w is primitive in \mathbf{F} . Conversely, let $w \in \mathbf{F}$ be primitive. Then there exists some automorphism $\psi \in \text{Aut } \mathbf{F}$ such that $\psi(w) = b_1$ (recall that $\{b_1, \dots, b_r\}$ is our fixed basis of \mathbf{F}). The automorphism ψ naturally extends (linearly) to an automorphism of the group ring $\psi : \mathcal{A} \rightarrow \mathcal{A}$. Since ψ maps ideals to ideals and bases to bases, it is enough to show that $\psi(w - 1) = b_1 - 1$ is primitive in every ideal containing it. Suppose it is not, and let I be a critical ideal for $b_1 - 1$. By [Corollary 3.11](#), I is generated on $T = \{1, b_1\}$. Let $f \in I|_T$. Then $f = \beta b_1 - \alpha$ for some $\alpha, \beta \in K$. By the definition of a critical ideal, I is a proper ideal and so α, β must be equal because their difference lies in I :

$$\beta - \alpha = f - (b_1 - 1)\beta \in I.$$

Thus, $I|_T = \text{span}_K \{b_1 - 1\}$ and since I is supported on T , I must be the right principal ideal $I = (b_1 - 1)$ in which $b_1 - 1$ is primitive, a contradiction. \square

Corollary 3.17. *For every $w \in \mathbf{F}$ and every prime power q , $\pi_q(w) \in \{0, 1, 2, \dots, \text{rk } \mathbf{F}\} \cup \{\infty\}$.*

Proof. Let $w \in F$. If w is primitive in F then by [Proposition 3.16](#) $\pi_q(w) = \infty$. Otherwise, by [Corollary 3.15](#), $w - 1$ is already imprimitive in J_F and so $\pi_q(w) \leq \text{rk } J_F = \text{rk } F$. \square

4. Powers and the limit of expected values of stable functions

We prove [Theorems 1.2](#) and [1.12](#): if $w \neq 1$, then $\lim_{N \rightarrow \infty} \mathbb{E}_w[\text{fix}]$ and, more generally, $\lim_{N \rightarrow \infty} \mathbb{E}_w[\tilde{\mathcal{B}}]$, exist. If we write $w = u^d$ with u a nonpower and $d \geq 1$, then the limit depends only on d , and, in particular, $\lim_{N \rightarrow \infty} \mathbb{E}_w[\text{fix}]$ is equal to the number of monic divisors of the polynomial $x^d - 1 \in K[x]$.

As mentioned in [Section 1](#), a special case of [Theorem 1.12](#), which includes [Theorem 1.2](#), first appeared in [\[Ernst-West 2019\]](#) and, independently, in [\[Eberhard and Jezernik 2022\]](#). Here, we prove the full version of the theorem while following the strategy from [\[Eberhard and Jezernik 2022\]](#), which is more elegant than the one in [\[Ernst-West 2019\]](#). We use here slightly different notions and give more details than in [\[Eberhard and Jezernik 2022\]](#). As the proof is subtle, and for the sake of readability, we first describe the proof of the special case which is [Theorem 1.2](#).

4A. $\lim_{N \rightarrow \infty} \mathbb{E}_w[\text{fix}]$ and the proof of [Theorem 1.2](#). From [\(2-5\)](#) and [Corollary 3.9](#) it follows that

$$\mathbb{E}_w[\text{fix}] = \sum_{I \leq [1, w], \mathcal{A}: I \ni w-1} (q^N)^{1-\text{rk } I} \left(1 + O\left(\frac{1}{q^N}\right) \right).$$

Clearly, as $w \neq 1$, an ideal containing $w - 1$ has rank at least 1. So

$$\mathbb{E}_w[\text{fix}] = |\{I \leq [1, w], \mathcal{A} \mid I \ni w - 1 \text{ and } \text{rk } I = 1\}| + O\left(\frac{1}{q^N}\right). \tag{4-1}$$

By [Definition 1.5](#), the only noncritical rank-1 ideals containing $w - 1$ are $(w - 1)$ and (1) , which are both generated on $[1, w]$. Any other rank-1 ideal containing $w - 1$ is critical, and [Corollary 3.11](#) guarantees that such ideals are supported on $[1, w]$. We obtain that

$$\mathbb{E}_w[\text{fix}] = |\{I \leq \mathcal{A} \mid I \ni w - 1 \text{ and } \text{rk } I = 1\}| + O\left(\frac{1}{q^N}\right). \tag{4-2}$$

This proves:

Corollary 4.1. *[Conjecture 1.6](#) holds in the case $\pi_q(w) = 1$. Namely, in this case*

$$\mathbb{E}_w[\text{fix}] = 2 + |\text{Crit}_q(w)| + O\left(\frac{1}{q^N}\right).$$

In order to prove [Theorem 1.2](#), it remains to show that if $w = u^d$ with u a nonpower and $d \geq 1$, then the ideals I in [\(4-2\)](#) are precisely $\{(p(u)) : p \mid x^d - 1 \in K[x], p \text{ monic}\}$. First, as any automorphism of F gives rise to an automorphism of \mathcal{A} , we may replace w by any element in its $\text{Aut } F$ -orbit, and, in particular, assume that w is cyclically reduced.

Throughout [Section 4](#), we use the ShortLex order on monomials in \mathcal{A}^m and their finite subsets.

Definition 4.2. Fix an arbitrary full order on the basis E of \mathcal{A}^m , say $e_1 < e_2 < \dots < e_m$. Fix an arbitrary full order on $B \cup B^{-1}$, say $b_1 < b_1^{-1} < b_2 < \dots < b_r^{-1}$. The *ShortLex* order on the monomials $\{ez\}_{e \in E, z \in F}$ is defined by first comparing the length of z (shorter words are smaller) and using lexicographic order to compare ez with $e'z'$ when $|z| = |z'|$. This order induces a full order on finite sets of monomials by comparing the leading monomial in each set, breaking ties by looking at the second monomials, and so on (the empty set is the smallest of all finite sets of monomials). Finally, we get a preorder on the elements of \mathcal{A}^m by comparing their supports. An element $f \in \mathcal{A}^m$ is called *monic* if the K -coefficient of the leading monomial is 1.

For example, $\alpha e_2 b_1^{-1} b_2 < \beta e_2 b_1^{-1} b_2 + e_3 b_1$ and $e_1 b_1 b_2 < e_1 b_3 b_1 < e_2 b_1 b_2$ (here $\alpha, \beta \in K^*$). This ShortLex order is the same one used in [Rosenmann 1993]. (The order used in [Lewin 1969] is not quite the same: it uses length and then *reverse* lexicographic order, and it also fixes a full order on K resulting in a full order on \mathcal{A}^m , rather than a mere preorder.)

Now, let $I \leq \mathcal{A}$ be a rank-1 ideal containing $w - 1$. As noted above, I is generated on $[1, w]$. In the notation of Section 3, consider the exposure of I along the subtree $[1, w]$, starting with the monomial 1 and ending with the monomial w . (This happens to be the restriction of ShortLex to $[1, w]$.) We shift the indices of the vertices by one with respect to Section 3, and define $v_0 = 1, \dots, v_{|w|} = w$. By Theorem 3.8, there is exactly one coincidence in this exposure.⁹ In an exposure along a path, a free step is followed by either another free step or by a coincidence, and in this particular path, the last step is not free. Thus, the first nonfree step must be a coincidence, and the following steps must all be forced. Namely, if v_t is exposed in a coincidence, $t \in \{0, 1, \dots, |w|\}$, then v_0, \dots, v_{t-1} are free steps and $v_{t+1}, \dots, v_{|w|}$ are forced. Denote by $f_t \in I$ the monic element supported on $[1, w]$ with v_t its leading monomial. By Theorem 3.8, $I = (f_t)$. Thus, the map

$$I \mapsto f_I \tag{4-3}$$

is a one-to-one correspondence.

Lemma 4.3. *The ideals I for which f_I is supported on $\langle u \rangle$ are in one-to-one correspondence with monic polynomials in $K[x]$ dividing $x^d - 1$.*

Proof. Consider the subalgebra $K[\langle u \rangle]$ of $\mathcal{A} = K[F]$, the elements of which are linear combinations of the elements in $\langle u \rangle = \{u^i \mid i \in \mathbb{Z}\}$. For every $z \in F$ and $f \in K[\langle u \rangle]$, if $z \notin \langle u \rangle$ then fz is supported on $\langle u \rangle z$, which is disjoint from $\langle u \rangle$. Thus, if $f \in K[\langle u \rangle]$ and $w - 1 = u^d - 1 \in (f) \stackrel{\text{def}}{=} f\mathcal{A}$, then $w - 1$ is also an element of $fK[\langle u \rangle]$, the ideal generated by f inside $K[\langle u \rangle]$. Now

$$K[\langle u \rangle] \cong K[\mathbb{Z}] \cong K[x, x^{-1}]$$

is a commutative ring (a principal ideal domain, in fact). If $p \in K[x, x^{-1}]$ satisfies $(p) \ni x^d - 1$, we may assume, by multiplying p by a unit element if need be, that $p \in K[x]$, p monic, and $p \mid x^d - 1$

⁹We remark that to analyze $\lim_{N \rightarrow \infty} \mathbb{E}_w[\text{fix}]$, one does not really need to go through Theorem 3.8, nor even consider the rank of ideals. Rather, it is enough to rely on Corollary 3.5.

in $K[x]$. Moreover, such $p \in K[x]$ is determined uniquely by the ideal (p) . This completes the proof of the lemma. \square

It remains to show that for every I in (4-2), f_I is supported on $\langle u \rangle$.

Lemma 4.4. *The support of f_I contains $v_0 = 1$.*

Proof. Recall that t denotes the coincidence step in the exposure of I along $[1, w]$. If $t = |w|$, then $f_I = w - 1$ and the lemma holds. Assume that $t < |w|$ and that the support of f_I does not contain 1. For every $s \geq t$, the vertex v_s is (exposed in) a nonfree step, and let $f_s \in I$ be the ShortLex-minimal element among all monic elements in I with v_s their leading monomial and which are supported on $\{v_0, \dots, v_s\}$. (This definition is unambiguous: if $f \neq g$ are two different monic elements with the exact same support, then there is some linear combination $\lambda f + (1 - \lambda)g$ which is strictly smaller). In particular, $f_t = f_I$ and $f_{|w|} = w - 1$. Now fix $s \in \{t + 1, \dots, |w|\}$ to be the smallest index for which $v_0 = 1$ is in the support of f_s . There is no element in $I|_{\{v_0, \dots, v_{s-1}\}}$ with v_0 in its support, because $I|_{\{v_0, \dots, v_{s-1}\}} = \text{span}_K\{f_t, \dots, f_{s-1}\}$. Let $b \in B \cup B^{-1}$ denote the label of the edge from v_{s-1} to v_s . As step s is forced, there is some monic $g \in I|_{\{v_1, \dots, v_{s-1}\}}$ with leading monomial v_{s-1} such that $g.b$ is supported on $\{v_0, \dots, v_s\}$. This $g.b$ must have v_0 in its support, for if not, $f_s - g.b$ does, and the latter is supported on $\{v_0, \dots, v_{s-1}\}$. We conclude that the first edge of w must be b^{-1} . As w is cyclically reduced, $s < |w|$.

Now consider the vertex v_{s+1} , and assume the edge from v_s to v_{s+1} is $c \in B \cup B^{-1}$, $c \neq b^{-1}$:

$$v_{s-1} \xrightarrow{b} v_s \xrightarrow{c} v_{s+1}.$$

As $I|_{\{v_0, \dots, v_s\}} = \text{span}_K\{f_t, \dots, f_s\}$, every element $g \in I|_{\{v_0, \dots, v_s\}}$ with leading monomial v_s must have v_0 in its support. But then, g cannot possibly be supported on starting points of c -edges, contradicting the fact that step $s + 1$ is also forced. Thus f_I contains v_0 in its support. \square

If $t = 0$ then $f_I = 1$ and $I = (1)$. If $t = |w|$, then $f_I = w - 1$ and $I = (w - 1)$. So assume from now on that $0 < t < |w|$. Also, denote by $b \in B \cup B^{-1}$ the first letter of w .

Lemma 4.5. *The letter from v_t to v_{t+1} is b , and f_I must be supported on D_b^{t+1} .*

Proof. Assume the edge from v_t to v_{t+1} is c . The step exposing v_{t+1} is forced, but f_I is the only monic element in $I|_{\{v_0, \dots, v_t\}}$. Thus the corresponding element in D_c^{t+1} must be f_I . As f_I has v_0 in its support, we must have $c = b$. \square

Completing the proof of Theorem 1.2. Recall that we now assume that I is a rank-1 ideal containing $w - 1$ with $I \neq (1)$, $(w - 1)$, and that we need to show that I is supported on $\langle u \rangle$. As I contains $w - 1$ if and only if it contains $wb - b$, the argument above (and Corollary 3.11) show that $I \leq_{[b, wb]} \mathcal{A}$. Expose I along $[b, wb]$ according to the restriction of ShortLex, and denote the vertices by $v_1, \dots, v_{|w|+1}$ (so keeping the same labels as before). As f_I is the only monic element in $I|_{\{v_0, \dots, v_t\}}$, we have that $I|_{\{v_1, \dots, v_t\}} = 0$, that v_{t+1} is the first (and only) coincidence in the exposure along $[b, wb]$, and that the coincidence is given by $f_I.b$. Clearly, $f_I.b$ has v_1 in its support. If b' is the second letter of w , then the same argument

as in [Lemma 4.5](#) shows that $f_I.b$ is supported on vertices with an outgoing b' -edge in $[b, wb]$, and that the edge from v_{t+1} to v_{t+2} is b' .

By iterating the same argument we get that for every prefix w' of w , $f_I.w'$ is supported on $[1, w^2]$. Moreover, the direction in which one can read a prefix of w from some v_j in the support of f_I along $[1, w^2]$ is necessarily forward: if it goes backward, then after $\lfloor \frac{j}{2} \rfloor$ step this path would collide with the path reading w coming from v_0 (a letter in $[1, w]$ cannot be equal to its own inverse or to the inverse of the following letter). We obtain that if f_I has some $v_j = z \in F$ in its support, then $zw = wz$, and so z belongs to the centralizer of w in F , which is $\langle u \rangle$. This completes the proof. \square

Corollary 4.6. *Let $1 \neq w \in F$. Then $\pi_q(w) = 1$ if and only if w is a proper power.*

Proof. Write $w = u^d$ with u a nonpower and $d \geq 1$. The discussion above shows that the rank-1 critical ideals of $w - 1$ are in one-to-one correspondence with the monic divisors of $x^d - 1 \in K[x]$, except for 1 and $x^d - 1$. If $d = 1$, there are no such divisors, and so $\pi_q(w) \geq 2$. If $d \geq 2$, there is at least one such divisor: the polynomial $x - 1$, and so $\pi_q(w) = 1$. \square

Recall that if $\lambda \in \text{GL}_1(K) \cong K^*$, then $\tilde{\lambda} : \text{GL}_N(K) \rightarrow \mathbb{Z}_{\geq 0}$ counts, for every element $g \in \text{GL}_N(K)$, the number of vectors $v \in V = K^N$ satisfying $v.g = \lambda v$. The same argument given above for $\text{fix} = \tilde{1}$ applies to all $\lambda \in K^*$ and gives the following result.

Corollary 4.7. *Let $\lambda \in \text{GL}_1(K) \cong K^*$, let $1 \neq w \in F$ and write $w = u^d$ with u a nonpower and $d \geq 1$. Then,*

$$\lim_{N \rightarrow \infty} \mathbb{E}_w[\tilde{\lambda}] = |\{p \in K[x] : p \mid x^d - \lambda \text{ and } p \text{ monic}\}|.$$

4B. $\lim_{N \rightarrow \infty} \mathbb{E}_w[\tilde{\mathcal{B}}]$ and the proof of [Theorem 1.12](#). Our next goal is proving [Theorem 1.12](#), which states that for any fixed $\mathcal{B} \in \text{GL}_m(K)$, the limit $\lim_{N \rightarrow \infty} \mathbb{E}_w[\tilde{\mathcal{B}}]$ exists and depends only on d , where $w = u^d \neq 1$ as before. As in the proof of [Theorem 1.2](#), we may assume that w is cyclically reduced. From (2-7) and [Corollary 3.9](#) it follows that

$$\mathbb{E}_w[\tilde{\mathcal{B}}] = \sum_{M \leq_{\mathbb{W}} \mathcal{A}^m : M \supseteq \text{EQ}_{\mathcal{B},w}} (q^N)^{m-\text{rk } M} \left(1 + O\left(\frac{1}{q^N}\right) \right), \tag{4-4}$$

where \mathbb{W} is the union of the paths $[e_i, e_i w] \in \mathcal{C}_i$ for $i = 1, \dots, m$. Throughout this [Section 4B](#) we continue using ShortLex from [Definition 4.2](#) and its restriction to collections of subtrees such as \mathbb{W} .

Lemma 4.8. *The smallest rank of a submodule $M \leq_{\mathbb{W}} \mathcal{A}^m$ containing $\text{EQ}_{\mathcal{B},w}$ is m . In particular, $\lim_{N \rightarrow \infty} \mathbb{E}_w[\tilde{\mathcal{B}}]$ exists and*

$$\lim_{N \rightarrow \infty} \mathbb{E}_w[\tilde{\mathcal{B}}] = |\{M \leq_{\mathbb{W}} \mathcal{A}^m \mid M \supseteq \text{EQ}_{\mathcal{B},w} \text{ and } \text{rk } M = m\}|. \tag{4-5}$$

Proof. Let $M \leq_{\mathbb{W}} \mathcal{A}^m$ contain $\text{EQ}_{\mathcal{B},w}$. We expose M along \mathbb{W} in the order induced on \mathbb{W} from ShortLex. So we first expose e_1, \dots, e_m , then, if the first letter of w is $b \in B \cup B^{-1}$, we expose $e_1 b, \dots, e_m b$, and so on. By definition, the last m steps, where $e_1 w, \dots, e_m w$ are exposed, are not free: $\text{EQ}_{\mathcal{B},w}$ contains an element supported on $\{e_i w, e_1, \dots, e_m\}$ with $e_i w$ its leading monomial. We claim that for every $e \in E$,

the first nonfree step in $[e, ew]$ is a coincidence. In particular, there are at least m coincidences and so by [Theorem 3.8](#), $\text{rk } M \geq m$.

Indeed, assume that the first nonfree vertex in $[e_i, e_i w]$ is $e_i z$ for some prefix z of w . If $z = 1$, then $e_i z$ is a coincidence by definition. Now assume that $z \neq 1$ and that $b \in B \cup B^{-1}$ is the last letter of z . As $e_i z$ is the first nonfree step in $[e_i, e_i w]$, we have that $e_i z b^{-1}$ was free, so there is no element of $M|_{\mathbb{W}}$ with leading monomial $e_i z b^{-1}$. Between the exposure of $e_i z b^{-1}$ and that of $e_i z$, the vertices exposed do not admit outgoing b -edges (in the already-exposed part of \mathbb{W}): these vertices are either $e_j z b^{-1}$ for $j > i$, where the only outgoing edge is headed backwards and cannot be b as w is reduced; or $e_j z$ for $j < i$, where the only outgoing edge is b^{-1} . Thus, when exposing $e_i z$ at step t , the largest monomial in D_b^t is $e_i z b^{-1}$, but as $e_i z b^{-1}$ is free, there are no elements of $M|_{D_b^t} \subseteq M|_{\mathbb{W}}$ with leading monomial $e_i z b^{-1}$. So step t cannot be forced and must be a coincidence. \square

The proof of [Lemma 4.8](#) actually shows that a free vertex in $[e, ew]$ cannot be followed by a forced vertex in the same path. As the last vertex in $[e, ew]$ is nonfree, we get the following.

Corollary 4.9. *If $M \leq_{\mathbb{W}} \mathcal{A}^m$ has rank m and contains $\text{EQ}_{\mathcal{B},w}$, then for every $e \in E$, the first nonfree step in $[e, ew]$ is a coincidence, and all later steps in $[e, ew]$ are forced.*

Remark 4.10. It is possible to extend [Corollary 3.11](#) from elements and ideals in \mathcal{A} to subsets and submodules in \mathcal{A}^m , and conclude that every rank- m submodule of \mathcal{A}^m containing $\text{EQ}_{\mathcal{B},w}$ is supported on \mathbb{W} .

Lemma 4.11. *Assume that $1 \neq w = u^d$ with $d \geq 1$ and u a nonpower. To prove [Theorem 1.12](#), it is enough to show that every submodule $M \leq_{\mathbb{W}} \mathcal{A}^m$ of rank m with $M \supseteq \text{EQ}_{\mathcal{B},w}$ is generated on $\{eu^j\}_{e \in E, j \in \{0, \dots, d\}}$.*

Proof. Assume that every submodule M from (4-5) is generated on $\{eu^j\}_{e \in E, j \in \{0, \dots, d\}}$. Then, as in the proof of [Lemma 4.3](#), these submodules are in one-to-one correspondence with rank- m submodules of $K[\langle u \rangle]^m$ containing $\text{EQ}_{\mathcal{B},w}$ (and generated on $\{eu^j\}_{e \in E, j \in \{0, \dots, d\}}$), where $K[\langle u \rangle]^m$ is the rank- m free module over $K[\langle u \rangle]$. As before, $K[\langle u \rangle] \cong K[\mathbb{Z}] \cong K[x, x^{-1}]$, and the image of $\text{EQ}_{\mathcal{B},w} \subseteq K[\langle u \rangle]^m$ in $K[\mathbb{Z}]^m$ through the corresponding isomorphism does not depend on u but only on d . Hence, the number of submodules in (4-5) does not depend on u , proving [Theorem 1.12](#). \square

Remark 4.12. It is quite straightforward to show that every submodule of $K[\langle u \rangle]^m$ containing $\text{EQ}_{\mathcal{B},w}$ must be of rank exactly m : after the first coincidence in each of the m paths, all remaining steps are clearly forced.

Now fix $M \leq_{\mathbb{W}} \mathcal{A}^m$ of rank m containing $\text{EQ}_{\mathcal{B},w}$. For every $f \in M|_{\mathbb{W}}$, denote by $\theta(f)$ the projection of f to the monomials e_1, \dots, e_m , so $\theta(f)$ is a K -linear combination of e_1, \dots, e_m . For $t = 0, \dots, |\mathbb{W}|$, let

$$\Theta_t \stackrel{\text{def}}{=} \text{span}_K \{ \theta(f) \mid f \in M|_{D^t(\mathbb{W})} \} \leq \text{span}_K \{ e_1, \dots, e_m \}$$

(recall that $D^t(\mathbb{W})$ is the set of first t monomials exposed in \mathbb{W} through ShortLex). So we have

$$\{0\} = \Theta_0 \leq \Theta_1 \leq \dots \leq \Theta_{|\mathbb{W}|} = \text{span}_K \{ e_1, \dots, e_m \},$$

where the last equality is due to the fact that $M \supseteq \text{EQ}_{\mathcal{B},w}$, the equations in $\text{EQ}_{\mathcal{B},w}$ are supported on \mathbb{W} , the linear combinations of e_1, \dots, e_m given by the m equations in $\text{EQ}_{\mathcal{B},w}$ are precisely the rows of \mathcal{B} , and \mathcal{B} is regular by definition. Recall (Corollary 4.9) that there is a sole coincidence in $[e_i, e_i w]$ for every $i = 1, \dots, m$, and let z_i denote the prefix of w so that $e_i z_i$ is the step in which the coincidence of $[e_i, e_i w]$ takes place.

Lemma 4.13. *We have $\Theta_{t-1} \not\subseteq \Theta_t$ if and only if step t is a coincidence. In particular, if $g_i \in M|_{\mathbb{W}}$ is a (monic) element with leading monomial $e_i z_i$, then the vectors $\theta(g_1), \dots, \theta(g_m)$ are linearly independent.*

Proof. We already explained why $\dim(\Theta|_{\mathbb{W}}) = m$. Note that $\dim \Theta_t - \dim \Theta_{t-1} \in \{0, 1\}$, because every two monic elements $g_1, g_2 \in M|_{\mathbb{W}}$ with leading monomial v_t satisfy $\theta(g_1) - \theta(g_2) = \theta(g_1 - g_2) \in \Theta_{t-1}$. As there are exactly m coincidences, it is enough to prove that $\Theta_{t-1} = \Theta_t$ whenever step t is forced or free. If step t is free, then $M|_{D^{t-1}(\mathbb{W})} = M|_{D^t(\mathbb{W})}$ and obviously $\Theta_{t-1} = \Theta_t$. It thus remains to show that this is the case also if step t is forced.

Let ez be the monomial exposed in step t which is forced, and let $b \in B \cup B^{-1}$ be the edge leading to ez . There exists some $g \in M|_{D_b^t(\mathbb{W})}$ with ezb^{-1} in its support (in fact, its leading monomial), such that the coefficient of ezb^{-1} in g is $1 \in K$. If $\theta(g.b) \in \Theta_{t-1}$, then every other monic $f \in M|_{\mathbb{W}}$ with leading monomial ez satisfies $\theta(f) = \theta(f - g.b) + \theta(g.b) \in \Theta_{t-1}$ and we are done. So assume that $\theta(g.b) \notin \Theta_{t-1}$. In particular, $\theta(g.b) \neq 0$, so $g.b$ has some $e' \in E$ in its support, and so b^{-1} is the first letter of w . As w is assumed to be cyclically reduced, z is a proper prefix of w .

Now consider the monomial following ez in $[e, ew]$. Say it is ezc for some $b^{-1} \neq c \in B \cup B^{-1}$, and it is exposed at time s (so $s = t + m$). Because step t is forced, so is step s (by Corollary 4.9). As in the proof of Lemma 4.8, the monomials exposed between ez and ezc do not belong to $D_c^s(\mathbb{W})$, so $D_c^s(\mathbb{W}) \subseteq D^t(\mathbb{W})$. As step s is forced, there exists some monic $f \in M|_{D_c^s(\mathbb{W})} \subseteq M|_{D^t(\mathbb{W})}$ with ez its leading monomial. As before, as $\theta(f - g.b) \in \Theta_{t-1}$ but $\theta(g.b) \notin \Theta_{t-1}$, we get $\theta(f) = \theta(f - g.b) + \theta(g.b) \notin \Theta_{t-1}$. In particular, $\theta(f) \neq 0$. But $c \neq b^{-1}$ is not the first letter of w , so f cannot have any $e \in E$ in its support — a contradiction. This completes the proof of the first statement of the lemma. This also shows there exist $g_i \in M|_{\mathbb{W}}$ with leading monomial $e_i z_i$, for $i = 1, \dots, m$, such that $\theta(g_1), \dots, \theta(g_m)$ are linearly independent. The second statement of the lemma now follows from the fact that if $f, g \in M|_{\mathbb{W}}$ are both monic with leading monomial the t -th vertex, then $\theta(f) - \theta(g) \in \Theta_{t-1}$. □

Define $\mathbb{W}^2 \stackrel{\text{def}}{=} [e_1, e_1 w^2] \cup \dots \cup [e_m, e_m w^2]$, and let $b \in B \cup B^{-1}$ be the first letter of w . For every $i = 1, \dots, m$, let $f_{e_i z_i} \in M|_{\mathbb{W}}$ be the minimal monic element with leading monomial $e_i z_i$.

Lemma 4.14. *For every $i = 1, \dots, m$, $f_{e_i z_i}$ is supported on $D_b(\mathbb{W}^2)$, and the outgoing b -edge at $e_i z_i$ is headed forward (towards $e_i w^2$).*

Proof. We proceed by induction on the order induced by ShortLex on $\{e_i z_i\}_{i=1, \dots, m}$. The argument that follows works for both the base case and the induction step. If $z_i = w$, then there is an element in $\text{EQ}_{\mathcal{B},w}$ with leading monomial $e_i w$ which is supported on $E \cup \{e_i w\}$, so $f_{e_i z_i}$ is also supported on $E \cup \{e_i w\}$ and the claim is clear. So assume that $|z_i| < |w|$, and that $e_i z_i$ is exposed at time t and admits

an outgoing c -edge towards $e_i w$. Then step $t + m$, in which $e_i z_i c$ is exposed, is forced, and there exists some $g \in M|_{D_c^{t+m}(\mathbb{W})} \subseteq M|_{D^t(\mathbb{W})}$ with leading monomial $e_i z_i$. By Lemma 4.13, $\theta(g) \notin \Theta_{t-1}$ so g has some $e \in E$ in its support, and therefore $c = b$.

Moreover, we may assume that g is supported on free steps and coincidences only. Indeed, the submodule M_{t+m-1} is generated on the free steps and coincidences exposed up to step $t + m - 1$ (this is always the case in every valid exposure process), but by Corollary 4.9, in our case these vertices form a valid collection of subtrees \mathbb{T} ($\mathbb{T} = T_1 \cup \dots \cup T_m$, where $T_j = [e_j, e_j z_j] \cap [e_j, e_j z_i]$ for $j \geq i$ and $T_j = [e_j, e_j z_j] \cap [e_j, e_j z_i b]$ for $j < i$). But $e_i z_i b$ is forced, so every element with leading monomial $e_i z_i b$ belongs to M_{t+m-1} , and if we extend \mathbb{T} to $e_i z_i b$ it is still a forced step (by Lemma 3.3). Thus there is some $g \in M|_{D_b(\mathbb{T} \cup \{e_i z_i b\})}$ with leading monomial $e_i z_i$.

If g has some coincidence $e_j z_j$ in its support other than $e_i z_i$, then as $e_j z_j < e_i z_i$, our induction hypothesis applies and $f_{e_j z_j} \in D_b(\mathbb{W}^2)$. Hence we may subtract $\alpha f_{e_j z_j}$ from g for some $\alpha \in K^*$ to decrease g , and $g - \alpha f_{e_j z_j} \in D_b(\mathbb{W}^2)$. If we repeat such subtractions as long as we can, we end up with a monic element f which is supported entirely on free vertices inside $D_b(\mathbb{W}^2)$ along with its leading monomial $e_i z_i$. Because all its nonleading monomials are free, this f is exactly $f_{e_i z_i}$ (otherwise $f - f_{e_i z_i} \neq 0$ is supported on free vertices, which is impossible), and we are done. \square

Completing the proof of Theorem 1.12. Recall that $M \leq_{\mathbb{W}} \mathcal{A}^m$ is a fixed submodule satisfying $\text{rk } M = m$ and $M \supseteq \text{EQ}_{\mathcal{B}, w}$. By Lemma 4.11, it is enough to show that M is generated by elements supported on $\{eu^j\}_{e \in E, j \in \{0, \dots, d\}}$. By Theorem 3.8, $M = (f_{e_1 z_1}, \dots, f_{e_m z_m})$, so it is enough to show that $f_{e_i z_i}$ is supported on $\{eu^j\}_{e \in E, j \in \mathbb{Z}}$ for all i .

Recall that b is the first letter of w . The submodule M contains $\text{EQ}_{\mathcal{B}, w}$ if and only if it contains $\text{EQ}_{\mathcal{B}, w} \cdot b \stackrel{\text{def}}{=} \{f \cdot b \mid f \in \text{EQ}_{\mathcal{B}, w}\}$. Define

$$\mathbb{W}^b \stackrel{\text{def}}{=} \bigcup_{e \in E} [b, wb],$$

and consider the exposure of M along \mathbb{W}^b in the order induced from ShortLex. Clearly, the monomials that were free in the exposure along \mathbb{W} are free now as well. We claim that the former coincidences $e_i z_i$ are now also free: as above, if $f \in M|_{\mathbb{W}^b}$ is monic with leading monomial $e_i z_i$, then $f_{e_i z_i} - f \in M$ is an element with $\theta(f_{e_i z_i} - f) = \theta(f_{e_i z_i})$ but with leading monomial smaller than $e_i z_i$, which contradicts Lemma 4.13. On the other hand, by Lemma 4.14, $f_{e_i z_i} \cdot b \in M|_{\mathbb{W}^b}$ has leading monomial $e_i z_i b$, and so $e_i z_i b$ is a coincidence in the exposure of M along \mathbb{W}^b . Moreover, the nonleading monomials of $f_{e_i z_i} \cdot b$ are all free in the exposure along \mathbb{W}^b , so $f_{e_i z_i} \cdot b$ is the minimal monic element in $M|_{\mathbb{W}^b}$ with leading monomial $e_i z_i b$. The same argument as in Lemma 4.14 shows that $f_{e_i z_i} \cdot b$ is supported on $D_c(\mathbb{W}^2)$ and the outgoing c -edge at $e_i z_i b$ is headed towards $e_i w^2$, where $c \in B \cup B^{-1}$ is the second letter of w .

This argument can now go on to the exposure of M along \mathbb{W}^{bc} and so on, and shows that for every prefix w' of w and every i , $f_{e_i z_i} \cdot w'$ is supported on $[1, w^2]$. This completes the proof exactly as in the proof of Theorem 1.2 in Section 4A. \square

5. The quotient module $K[F]/(w - 1)$

Fix $w \in F$, and consider the right \mathcal{A} -module obtained as a quotient of the \mathcal{A} -module \mathcal{A} by its submodule $(w - 1)$. We denote this quotient by

$$\mathcal{A}_w \stackrel{\text{def}}{=} K[F]/(w - 1) = \mathcal{A}/(w - 1).$$

We study this module and prove two main results about it. First, we show that if w is a nonpower, then the only cyclic generators of \mathcal{A}_w are the “obvious ones” (Theorem 5.4). Second, we prove that whenever a subtree $T \subseteq \text{Cay}(F, B)$ supports both $w - 1$ and a rank-2 ideal $I \leq_T \mathcal{A}$ in which $w - 1$ is primitive, there is an element $f \in \mathcal{A}$ supported on T so that $\{f, w - 1\}$ is a basis of I (Corollary 5.2). In particular, the latter result yields an algorithm to test whether $w - 1$ is primitive in a given rank-2 ideal (Corollary 5.3). We need these two results for our proof of Theorem 1.4 in Section 6, but we also find them interesting for their own right. See Section 7 for a discussion on potential generalizations of these results.

Consider the Schreier graph

$$\mathcal{S}_w \stackrel{\text{def}}{=} \text{Sch}(F \curvearrowright \langle w \rangle \backslash F, B) = \langle w \rangle \backslash \text{Cay}(F, B).$$

This is a graph whose vertices correspond to the right cosets of the subgroup $\langle w \rangle$ in F . For every vertex $\langle w \rangle z$ and every $b \in B$, there is a directed b -edge from the vertex $\langle w \rangle z$ to the vertex $\langle w \rangle zb$. In other words, this is the quotient of $\text{Cay}(F, B)$ by the action of $\langle w \rangle$ from the left. Note that \mathcal{S}_w is made of a cycle (reading the cyclic reduction of w) with infinite trees hanging from it (unless $\text{rk } F = 1$, in which case \mathcal{S}_w is a mere cycle). This is illustrated in Figure 1.

An element $f \in \mathcal{A}$ belongs to the ideal $(w - 1)$ if and only if for every $z \in F$, the coefficients in f of the elements in the right coset $\langle w \rangle z$ sum up to zero. Therefore, the elements of \mathcal{A}_w are given by K -linear combinations of right cosets of $\langle w \rangle$, namely, K -linear combinations of the vertices of \mathcal{S}_w . This can also

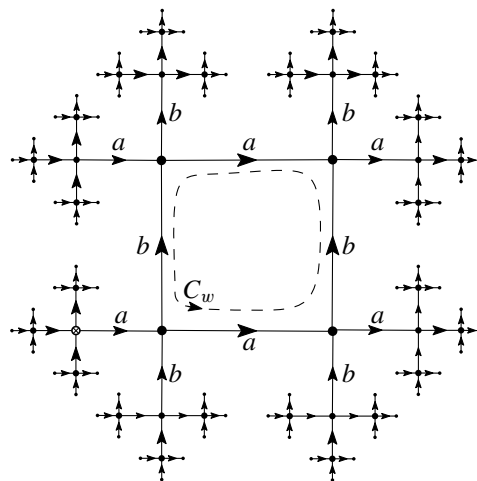


Figure 1. The Schreier graph \mathcal{S}_w for $w = a[a, b]a^{-1}$. The unique simple cycle is marked by C_w .

be seen by the fact that a possible Schreier transversal of the ideal $(w - 1)$ is obtained by considering $\text{Cay}(\mathbf{F}, B)$, cutting the axis¹⁰ of w on both sides of one period of the cyclic reduction of w , and taking the connected component of this period.

Now consider the quotient map

$$\rho : \mathcal{A} \twoheadrightarrow \mathcal{A}_w,$$

which, by abuse of notation, we also regard as the graph morphism

$$\rho : \text{Cay}(\mathbf{F}, B) \rightarrow \mathcal{S}_w.$$

Whenever a subtree $T \subseteq \text{Cay}(\mathbf{F}, B)$ contains $[1, w]$, its image $\rho(T) \subseteq \mathcal{S}_w$ contains the cycle in \mathcal{S}_w . In fact, it suffices that T contains any interval in the axis of w of length at least the length of the cyclic reduction of w .

Lemma 5.1. *Let $G \subseteq \mathcal{S}_w$ be a connected subgraph which contains the cycle of \mathcal{S}_w . Let $f \in \mathcal{A}_w$ satisfy that none of $\{f.z \mid z \in \mathbf{F}\}$ is supported on G . Then the submodule $f\mathcal{A} \leq \mathcal{A}_w$ does not contain any nonzero element supported on G .*

Proof. On the vertices of $\mathcal{S}_w \setminus G$ define an “exploration” as in Definition 3.2: this is an enumeration of these vertices such that every vertex is a neighbor of some vertex in G or of a smaller vertex. This exploration induces a preorder on the orbit $\{f.z \mid z \in \mathbf{F}\}$ obtained by comparing the largest vertex in their support with respect to this exploration order (by assumption, every element $f.z$ in this orbit has at least one vertex outside G in its support). Assume without loss of generality that f is an element of the orbit with the smallest possible maximal vertex in its support. Denote this vertex v_{\max} . Denote by \bar{G} the (connected) subgraph of \mathcal{S}_w consisting of G together with the prefix $\{v \in \text{vert}(\mathcal{S}_w \setminus G) \mid v \leq v_{\max}\}$ of the exploration order on $\mathcal{S}_w \setminus G$.

Now consider the element $fg \in \mathcal{A}_w$ for an arbitrary $g \in \mathcal{A}$ not supported on the identity $e \in \mathbf{F}$. It suffices to show that fg is not supported on \bar{G} (let alone on G). Write $f = \alpha_1 f_1 + \dots + \alpha_m f_m$ with $\alpha_1, \dots, \alpha_m \in K^*$ and distinct $f_1, \dots, f_m \in \text{vert}(\mathcal{S}_w)$, and write $g = \beta_1 g_1 + \dots + \beta_\ell g_\ell$ with $\beta_1, \dots, \beta_\ell \in K^*$ and distinct $g_1, \dots, g_\ell \in \mathbf{F}$ and so that $|g_1| \geq |g_2| \geq \dots \geq |g_\ell|$. Denote by $b \in B \cup B^{-1}$ the first letter in g_1 . Then $f.b$ cannot be supported on \bar{G} : otherwise, $f.b$ would be supported on G together with vertices strictly smaller than v_{\max} in $\mathcal{S}_w \setminus G$ (we use here the fact that v_{\max} is a leaf in \bar{G}), contradicting our assumption about f . So there is a monomial f_i in the support of \bar{G} such that $f_i.b$ is a monomial outside \bar{G} . But then $f_i g_1$ is at distance $|g_1|$ from \bar{G} , with the closest vertex of \bar{G} being f_i . Clearly, $f_i g_1 \neq f_j g_k$ for every $(j, k) \neq (i, 1)$, because the only path of length $|g_1|$ from \bar{G} to $f_i g_1$ in \mathcal{S}_w , is the path starting at f_i and reading g_1 . Thus $f_i g_1$ belongs to the support of fg , and fg is not supported on \bar{G} . \square

Corollary 5.2. *Let $1 \neq w \in \mathbf{F}$ and $T \subseteq \text{Cay}(\mathbf{F}, B)$ be a subtree which contains $[1, w]$. Assume that $I \leq_T \mathcal{A}$ is a rank-2 ideal supported on T which contains $w - 1$ as a primitive element. Then there is an element $f \in \mathcal{A}$ supported on T so that $\{f, w - 1\}$ is a basis for I .*

¹⁰The axis of w is composed of the points in $\text{Cay}(\mathbf{F}, B)$ moved by left multiplication by w by the least distance.

Proof. As $w - 1$ is primitive in I , there is some $f \in \mathcal{A}$ which completes it to a basis of I . Consider $\bar{T} = \rho(T)$, the image of T in \mathcal{S}_w and let $\bar{f} = \rho(f) \in \mathcal{A}_w$. If $\{f, w - 1\}$ is a basis for I , then so is $\{g, w - 1\}$ for every $g \in \rho^{-1}(\bar{f})$, because in this case $f - g \in (w - 1)$. So if $\bar{f}.z$ is supported on \bar{T} for some $z \in \mathbf{F}$, we are done: if $\{f, w - 1\}$ is a basis then so is $\{f.z, w - 1\}$. Otherwise, we are in the situation of [Lemma 5.1](#), and $\bar{f}\mathcal{A}$ does not contain any element supported on \bar{T} . But $\bar{f}\mathcal{A}$ contains $\rho(I)$ (in fact $\bar{f}\mathcal{A} = \rho(I)$), and as I is generated on T , I contains an element $h \in I \setminus (w - 1)$ which is supported on T . Then $\bar{f}\mathcal{A} \ni \rho(h)$, which is a contradiction as $\rho(h) \neq 0$ and is supported on \bar{T} . \square

Corollary 5.3. *If the field K is finite,¹¹ there is an algorithm to test, given a (generating set of a) rank-2 ideal $I \leq \mathcal{A}^m$ and a word $w \in \mathbf{F}$, whether $w - 1$ is primitive in I .*

Proof. By [[Cohn 1964](#), Proposition 2.2], every pair of generators of I is a basis. So $\{f, w - 1\}$ is a basis for I if and only if $f, w - 1 \in I$ and $(f, w - 1)$ contains the given generating set of I . By [Corollary 5.2](#), $w - 1$ is primitive in I if and only if there exists an element f supported on T such that $\{f, w - 1\}$ is a basis of I . As K is finite, there are finitely many elements supported on T . Finally, Rosenmann [[1993](#)] describes an algorithm to test whether a given element belongs to a given ideal in \mathcal{A} (where the ideal is given by a finite generating set). \square

Corollaries [5.2](#) and [5.3](#) naturally raise the question to what extent they can be generalized for ideals of rank larger than two and for elements of \mathcal{A} which are not of the form $w - 1$ — see [Section 7](#) for a discussion around it.

5A. Cyclic generators of $K[\mathbf{F}]/(w - 1)$. The group algebra $\mathcal{A} = K[\mathbf{F}]$ has only trivial units — a scalar times an element of the group¹² (this property was conjectured by Kaplansky to hold in all group algebras of torsion-free groups over fields but a counterexample has recently been found [[Gardam 2021](#)]). The goal of this subsection is to prove a similar result for $\mathcal{A}_w = \mathcal{A}/(w - 1)$. While \mathcal{A}_w is not a ring and therefore does not admit units, it does admit cyclic generators as an \mathcal{A} -module: elements $f \in \mathcal{A}_w$ such that $f\mathcal{A} = \mathcal{A}_w$. Clearly, for every unit of \mathcal{A} , its image in \mathcal{A}_w is a cyclic generator. Here we prove that provided that w is not a power, all cyclic generators of \mathcal{A}_w are of this sort.

Theorem 5.4. *Assume that $1 \neq w \in \mathbf{F}$ is a nonpower. Then every cyclic generator of the right \mathcal{A} -module $\mathcal{A}_w = \mathcal{A}/(w - 1)$ is an image of a unit of \mathcal{A} .*

Namely, every cyclic generator of \mathcal{A}_w is a coset of the form $\alpha z + (w - 1)$ for some $\alpha \in K^*$ and $z \in \mathbf{F}$.

Remark 5.5. [Theorem 5.4](#) is false for proper powers. For example, if $|K| = 3$ and $w = a^3$, then $\rho(a + 1) \in \mathcal{A}_w$ is not a ρ -image of a unit of \mathcal{A} : its support in \mathcal{S}_w is of size two. Yet $a^3 + 1 \in (a + 1)$ and so $\rho(2) = \rho(a^3 + 1) \in \rho(a + 1)\mathcal{A}$. Thus $\rho(a + 1)$ is a cyclic generator of \mathcal{A}_w .

¹¹We assume throughout the paper that K is finite, but some of the results about free group algebras, such as [Corollary 5.2](#), hold for infinite fields just as well. In contrast, [Corollary 5.3](#) relies on K being finite.

¹²This is well known. It can also be seen, for example, by an argument similar to the one in the proof of [Lemma 5.1](#): for any $0 \neq f \in \mathcal{A}$ with support of size at least 2, take a minimal subtree T of $\mathcal{C} = \text{Cay}(\mathbf{F}, B)$ which supports an element in the orbit $\{f.z \mid z \in \mathbf{F}\}$. Then the argument in the proof of [Lemma 5.1](#) shows that $f\mathcal{A}$ does not contain elements supported on T except for scalar multiples of f .

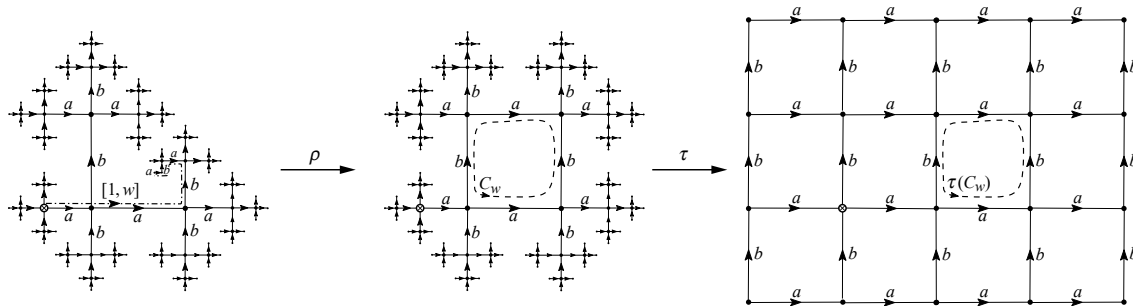


Figure 2. Let $w = a[a, b]a^{-1}$. The Cayley graph of $F = F(a, b)$ is on the left with $[1, w]$ marked. The middle graph is S_w , and the graph at the right side is a piece of the Cayley graph of $F/\langle\langle w \rangle\rangle$. In all graphs, the vertex corresponding to the identity element or its ρ -image is marked with \otimes .

First we show that cyclic generators in \mathcal{A}_w may be assumed to be supported on the cycle of S_w .

Lemma 5.6. *If $f \in \mathcal{A}_w$ is a cyclic generator of \mathcal{A}_w , then there is some $z \in F$ such that fz is supported on the cycle in S_w .*

Proof. This follows immediately from Lemma 5.1 applied to G being the cycle in S_w . □

Let $w \in F$ be a nonpower. If w' is the cyclic reduction of w then the automorphism of F mapping w to w' extends to an automorphism of \mathcal{A} and induces isomorphisms $\mathcal{A}_w \cong \mathcal{A}_{w'}$ and $S_w \cong S_{w'}$. Thus we may assume without loss of generality that w is cyclically reduced. Denote by $\langle\langle w \rangle\rangle$ the normal closure of w in F , and denote by $((w - 1))$ the two-sided ideal of \mathcal{A} generated by $w - 1$. Since we have $\{0\} \subseteq (w - 1) \subseteq ((w - 1))$, we get canonical epimorphisms of right \mathcal{A} -modules

$$\mathcal{A} \xrightarrow{\rho} \mathcal{A}_w \xrightarrow{\tau} \mathcal{A}/((w - 1)).$$

See Figure 2.

Lemma 5.7. *Let $p : F \rightarrow F/\langle\langle w \rangle\rangle$ be the canonical projection. The map $\varphi : \mathcal{A}/((w - 1)) \rightarrow K[F/\langle\langle w \rangle\rangle]$ defined by $\sum_{z \in F} \alpha_z z + ((w - 1)) \mapsto \sum_{z \in F} \alpha_z p(z)$ is an isomorphism of K -algebras.*

Proof. The proof is a standard argument in algebra, but we include it for completeness. By the universal property of group rings, the group homomorphism $p : F \rightarrow F/\langle\langle w \rangle\rangle \subseteq K[F/\langle\langle w \rangle\rangle]$ extends to a unique K -algebra epimorphism $\psi : \mathcal{A} \rightarrow K[F/\langle\langle w \rangle\rangle]$. The ideal $((w - 1))$ lies in the kernel of ψ : it is enough to show that $u(w - 1)v \in \ker \psi$ for $u, v \in F$, and

$$\psi(u(w - 1)v) = \psi(uwv - uv) = \psi(uwv) - \psi(uv) = p(uwv) - p(uv) = 0,$$

where the last equality is because uwv and uv lie in the same coset of $\langle\langle w \rangle\rangle$. Thus, the homomorphism ψ induces an epimorphism $\psi' : \mathcal{A}/((w - 1)) \rightarrow K[F/\langle\langle w \rangle\rangle]$. For every $z \in F$, ψ' satisfies $\psi'(z + ((w - 1))) = \psi(z) = p(z)$, and so by linear extension ψ' agrees with φ from the statement of the lemma (in particular, φ is a well-defined epimorphism of K -algebras).

It is left to show that φ is injective. Suppose that

$$\varphi\left(\sum_{z \in F} \alpha_z z + ((w - 1))\right) = \sum_{z \in F} \alpha_z p(z) = 0.$$

For every coset C of $\langle\langle w \rangle\rangle$ we have $\sum_{z \in C} \alpha_z = 0$. We complete the proof by showing that this implies that $\sum_{z \in C} \alpha_z z \in ((w - 1))$ — and then the sum over all cosets would also lie in $((w - 1))$. Such a finite sum can always be decomposed as a sum over elements of the form $\alpha(z_2 - z_1)$ where $\alpha \in K$ and $z_1, z_2 \in C$. In every such element, z_2 can be obtained from z_1 by a finite sequence of multiplications from the right by conjugates of w or w^{-1} , and so it is enough to show that $z_2 - z_1 \in ((w - 1))$ for $z_2 = z_1 \cdot uw^\varepsilon u^{-1}$ where $u \in F$ and $\varepsilon \in \{\pm 1\}$. And indeed we have $z_2 - z_1 = z_1 u (w^\varepsilon - 1) u^{-1} \in ((w - 1))$. \square

In our proof of [Theorem 5.4](#) we use the following well-known concept.

Definition 5.8. A *right order* on a group Γ is a linear order on Γ such that for every $r, s, t \in \Gamma$ with $r < s$ we have $rt < st$. A group is called *right-orderable* if it admits a right order.

It is well-known that Kaplansky’s unit conjecture, mentioned above, is true for right orderable groups — see, e.g., [\[Clay and Rolfsen 2016, Theorem 1.58\]](#). We add a proof for completeness.

Lemma 5.9. *Let K be a field and Γ a right-orderable group. If $ts = 1$ for $t, s \in K[\Gamma]$ then $t = \lambda g$ for some $\lambda \in K^*$ and $g \in \Gamma$.*

Proof. Write $t = \sum_{i=1}^n \lambda_i g_i$ for $\lambda_i \in K^*$ and $g_1, \dots, g_n \in \Gamma$ distinct. Since $ts = 1$ we know that $n \neq 0$. Now assume towards contradiction that $n \geq 2$. Let $<$ be a right order for Γ . Assume without loss of generality that $g_1 < g_2 < \dots < g_n$. Write similarly $s = \sum_{j=1}^m \mu_j h_j$ for $h_1 < h_2 < \dots < h_m$ and $\mu_j \in K^*$. Then we have $1 = ts = \sum_{i,j} \lambda_i \mu_j g_i h_j$.

We now find two elements of Γ such that their coefficients in ts are nonzero. Let j_{\min} be the index such that $g_1 h_{j_{\min}} = \min\{g_1 h_1, g_1 h_2, \dots, g_1 h_m\}$. In particular, $g_1 h_{j_{\min}}$ is strictly smaller than any other $g_1 h_j$ for $j \neq j_{\min}$. In addition, if $i \neq 1$ then $g_1 h_{j_{\min}} \leq g_i h_j < g_i h_j$. Thus, the coefficient of $g_1 h_{j_{\min}}$ in ts is $\lambda_1 \mu_{j_{\min}} \neq 0$. Similarly, let j_{\max} be the index such that $g_n h_{j_{\max}} = \max\{g_n h_1, g_n h_2, \dots, g_n h_m\}$. A similar argument shows that the coefficient of $g_n h_{j_{\max}}$ in ts is $\lambda_n \mu_{j_{\max}} \neq 0$. Finally, since $n \geq 2$, we have $g_1 h_{j_{\min}} < g_n h_{j_{\min}} \leq g_n h_{j_{\max}}$ and so $g_1 h_{j_{\min}}$ and $g_n h_{j_{\max}}$ are distinct elements of Γ with nonzero coefficients in $ts = 1$ — a contradiction. \square

The following theorem is a well-known result in the theory of one-relator groups.

Theorem 5.10. *If $1 \neq w \in F$ is a nonpower then the one-relator group $F/\langle\langle w \rangle\rangle$ is right-orderable.*

Proof. As w is a nonpower, we deduce that $F/\langle\langle w \rangle\rangle$ is torsion-free by a theorem of Karass, Magnus and Solitar [\[Karrass et al. 1960, Theorem 1\]](#). By a theorem proven independently by Brodskii [\[1984, Corollary 2.3\]](#) and Howie [\[1982, Corollary 4.3\]](#), every torsion-free one-relator group has the property of being *locally indicable*, which means that each of its nontrivial finitely generated subgroups admits a nontrivial homomorphism to \mathbb{Z} . Finally, the Burns–Hale theorem [\[1972, Theorem 2\]](#) states that a group H is right-orderable if and only if any nontrivial finitely generated subgroup of H admits a nontrivial

homomorphism to some right-orderable group. Since \mathbb{Z} is right-orderable (the usual order on \mathbb{Z} is a right order), the Burns–Hale theorem implies that every locally indicable group is right-orderable. The combination of the theorems above gives that $F/\langle\langle w \rangle\rangle$ is right-orderable. \square

Proof of Theorem 5.4. Let $1 \neq w \in F$ be a nonpower and suppose that $\bar{f} \in \mathcal{A}_w$ generates \mathcal{A}_w . Since ρ is surjective, there exists some $f \in \mathcal{A}$ such that $\rho(f) = \bar{f}$. As \bar{f} generates \mathcal{A}_w , there exists some $s \in \mathcal{A}$ such that $\bar{f}s = \rho(1)$ or, equivalently, $\rho(fs) = \rho(1)$. Applying τ to both sides of the equation and using the fact that $\tau \circ \rho$ is a homomorphism of K -algebras we obtain $\tau\rho(f) \cdot \tau\rho(s) = \tau\rho(1)$, and in particular $\tau\rho(f)$ has a right inverse in the quotient K -algebra $\mathcal{A}/((w - 1))$. Now, since $\tau\rho(f)$ has a right inverse in $\mathcal{A}/((w - 1))$, its image under the isomorphism φ from Lemma 5.7 has a right inverse in $K[F/\langle\langle w \rangle\rangle]$. By Theorem 5.10, as w is not a power, $F/\langle\langle w \rangle\rangle$ is right-orderable. Lemma 5.9 applied for $\Gamma = F/\langle\langle w \rangle\rangle$, implies that $\varphi(\tau\rho(f)) = \lambda g$ for some $\lambda \in K^*$ and $g \in F/\langle\langle w \rangle\rangle$.

Without loss of generality, by Lemma 5.6, we may assume that $\bar{f} = \rho(f)$ is supported on cosets of $\langle w \rangle$ belonging to the unique simple cycle of S_w . The Weinbaum subword theorem [1972, Theorem 2] asserts that none of the nontrivial proper subwords of the cyclic reduction of w lies in its normal closure $\langle\langle w \rangle\rangle$. This implies that two distinct vertices of the cycle of S_w have distinct images through τ , namely, their images belong to different elements of $F/\langle\langle w \rangle\rangle$. But the τ -image of \bar{f} is λg , which is supported on a single element $g \in K[F/\langle\langle w \rangle\rangle]$. Thus \bar{f} itself is supported on a single element of the cycle of S_w and can be lifted to an element $f \in \mathcal{A}$ supported on a single element of F . \square

6. Critical ideals of rank 2

Throughout this section fix a nonpower $1 \neq w \in F$ and assume without loss of generality that it is cyclically reduced. Theorems 1.1 and 1.2 yield that $\mathbb{E}_w[\text{fix}] = 2 + \frac{c}{q^N} + O\left(\frac{1}{q^{2N}}\right)$ for some constant c . Our goal is to prove Theorem 1.4:

$$c = |\text{Crit}_q^2(w)|,$$

where $\text{Crit}_q^2(w)$ is the set of rank-2 ideals $I \leq \mathcal{A}$ containing $w - 1$ as an imprimitive element.

Recall our formula (2-5) for $\mathbb{E}_w[\text{fix}]$ and Corollary 3.9. It follows that the $\frac{1}{q^N}$ -coefficient of $\mathbb{E}_w[\text{fix}]$ consists of the contributions of the rank-1 and rank-2 ideals in the set

$$\mathcal{I} \stackrel{\text{def}}{=} \{I \leq_{[1,w]} \mathcal{A} \mid I \ni w - 1\}.$$

As $w \neq 1$ and is a nonpower, by Corollary 4.6 the rank-1 ideals in \mathcal{I} are precisely (1) and $(w - 1)$. The contribution of (1) to (2-5) is precisely 1, so it does not affect c . Denote by β_w the coefficient of $\frac{1}{q^N}$ in the contribution of $(w - 1)$, namely, this contribution is $1 + \frac{\beta_w}{q^N} + O\left(\frac{1}{q^{2N}}\right)$. The summand in (2-5) corresponding to a rank-2 ideal is $\frac{1}{q^N} + O\left(\frac{1}{q^{2N}}\right)$, so such an ideal contributes exactly 1 to c . Recall that all the ideals in $\text{Crit}_q^2(w)$ are in \mathcal{I} , by Corollary 3.11. Denote by $\text{Prim}^2(w)$ the set of rank-2 ideals in \mathcal{I} in which $w - 1$ is primitive. With this notation, the coefficient c of $\frac{1}{q^N}$ in $\mathbb{E}_w[\text{fix}]$ is $c = \beta_w + |\text{Prim}^2(w)| + |\text{Crit}_q^2(w)|$. Our goal is, thus, to prove that

$$\beta_w + |\text{Prim}^2(w)| = 0.$$

Recall from Section 5 the quotient \mathcal{A} -module $\mathcal{A}_w \stackrel{\text{def}}{=} \mathcal{A}/(w-1)$, the projection $\rho : \mathcal{A} \rightarrow \mathcal{A}_w$ and the Schreier graph $\mathcal{S}_w = \langle w \rangle \backslash \text{Cay}(\mathbf{F}, B)$. The elements of \mathcal{A}_w are K -linear combinations of the vertices of \mathcal{S}_w , and we use ρ to denote also the quotient in the graph level $\rho : \text{Cay}(\mathbf{F}, B) \rightarrow \mathcal{S}_w$. Let $C_w = \rho([1, w])$ denote the unique simple cycle in \mathcal{S}_w (here we use the fact that w is assumed to be cyclically reduced).

Lemma 6.1. *The $\frac{1}{q^N}$ -coefficient of the summand corresponding to $I = (w-1)$ in (2-5) is*

$$\beta_w = -\frac{q^{v(C_w)} - 1}{q - 1} + \sum_{b \in B} \frac{q^{e_b(C_w)} - 1}{q - 1}. \tag{6-1}$$

Proof. Recall that β_w is the $\frac{1}{q^N}$ -coefficient of the Laurent expansion of

$$\frac{\text{indep}_{|w|+1-d^{[1,w]}(I)}(V_N)}{\prod_{b \in B} \text{indep}_{e_b(w)-d_b^{[1,w]}(I)}(V_N)}, \tag{6-2}$$

for $I = (w-1)$. Because $f \in \mathcal{A}_{[1,w]}$ belongs to $I = (w-1)$ if and only if its coefficients in every fiber over C_w sum up to zero, the dimension over K of $I|_{[1,w]}$ is precisely $d^{[1,w]}(I) = v([1, w]) - v(C_w) = 1$. Similarly, the dimension over K of $I|_{D_b([1,w])}$ is precisely $d_b^{[1,w]}(I) = e_b([1, w]) - e_b(C_w) = 0$. Hence, (6-2) is equal to

$$\frac{\text{indep}_{v(C_w)}(V_N)}{\prod_{b \in B} \text{indep}_{e_b(C_w)}(V_N)} = \frac{(q^N - 1)(q^N - q) \cdots (q^N - q^{v(C_w)-1})}{\prod_{b \in B} (q^N - 1)(q^N - q) \cdots (q^N - q^{e_b(C_w)-1})}. \tag{6-3}$$

Because C_w is a cycle, the number of vertices is identical to the total number of edges. Hence (6-3) is equal to

$$\frac{(1 - \frac{1}{q^N})(1 - \frac{q}{q^N}) \cdots (1 - \frac{q^{v(C_w)-1}}{q^N})}{\prod_{b \in B} (1 - \frac{1}{q^N})(1 - \frac{q}{q^N}) \cdots (1 - \frac{q^{e_b(C_w)-1}}{q^N})},$$

and the $\frac{1}{q^N}$ -coefficient of the Laurent expansion of this expression is

$$(-1 - q - \cdots - q^{v(C_w)-1}) - \sum_{b \in B} (-1 - q - \cdots - q^{e_b(C_w)-1}),$$

which is equal to (6-1). □

Denote by D_w the set of proper nontrivial cyclic submodules¹³ of \mathcal{A}_w generated by some element supported on the cycle C_w :

$$D_w \stackrel{\text{def}}{=} \{g\mathcal{A} \not\subseteq \mathcal{A}_w \mid 0 \neq g \in \mathcal{A}_w \text{ and } g \text{ supported on } C_w\}.$$

Lemma 6.2. *There is a one-to-one correspondence between D_w and $\text{Prim}^2(w)$.*

¹³Recall that a cyclic submodule is a submodule generated by a single element.

Proof. By Corollary 5.2, the rank-2 ideals $I \leq_{[1,w]} \mathcal{A}$ containing $w - 1$ as a primitive element are exactly the rank-2 ideals of the form $(w - 1, f)$ with f supported on $[1, w]$ (here we use again the fact that every pair of generators of a rank-2 ideal is a basis — [Cohn 1964, Proposition 2.2]). Now $g \in \mathcal{A}_w$ is supported on C_w if and only if there is some $f \in \mathcal{A}$ supported on $[1, w]$ with $\rho(f) = g$. Note that

$$(w - 1, f) = \rho^{-1}(\rho(f)\mathcal{A}),$$

where $\rho(f)\mathcal{A}$ is the submodule of \mathcal{A}_w generated by the image of f in \mathcal{A}_w . Because the only rank-1 ideals containing $w - 1$ are (1) and $(w - 1)$, we have that $(w - 1, f)$ is of rank 2 if and only if $\rho(f)\mathcal{A}$ is a nonzero proper submodule of \mathcal{A}_w . \square

Next, we study the different elements $g \in \mathcal{A}_w$ supported on C_w . In order to understand when two different elements g, g' generate the same submodule, we construct a graph Υ . The vertices of Υ are the 1-dimensional linear subspaces of $K^{\text{vert}(C_w)}$, so their number is $v(\Upsilon) = \frac{q^{v(C_w)} - 1}{q - 1}$. For every $b \in B$ and every 1-dimensional subspace $U \leq K^{b\text{-edges}(C_w)}$ (here $K^{b\text{-edges}}$ is the space of K -linear combinations of the b -edges in C_w), the subspace U corresponds to a 1-dimensional subspace $o(U)$ of the vertices supported on the origins of the b -edges, as well as a 1-dimensional subspace $t(U)$ supported on the termini of the b -edges. For every such U we draw a directed b -edge from the vertex $o(U)$ to the vertex $t(U)$ in Υ . Note that $e(\Upsilon) = \sum_{b \in B} \frac{q^{e_b(C_w)} - 1}{q - 1}$, so overall

$$\chi(\Upsilon) \stackrel{\text{def}}{=} v(\Upsilon) - e(\Upsilon) = -\beta_w, \tag{6-4}$$

where the second equality is by Lemma 6.1. Denote by $\mathcal{C}(\Upsilon)$ the connected components of Υ . Because $g\mathcal{A} = g.b\mathcal{A}$ for every $g \in \mathcal{A}_w$ and $b \in B$, there is a well-defined surjective map

$$\Phi : \mathcal{C}(\Upsilon) \twoheadrightarrow \{g\mathcal{A} \mid 0 \neq g \in \mathcal{A}_w \text{ supported on } C_w\} = D_w \cup \{\mathcal{A}_w\}.$$

One of the connected components of Υ is isomorphic to C_w : this is the component consisting of vertices and edges of Υ corresponding to 1-dimensional subspaces supported on a single vertex or on a single edge. Denote this component by C_0 . Clearly, $\Phi(C_0) = \mathcal{A}_w$.

Lemma 6.3. *All connected components of Υ except for C_0 are paths.*

Proof. The degree of every vertex in Υ is at most 2, so every connected component is a path or a cycle. Assume that some component $C_0 \neq C \in \mathcal{C}(\Upsilon)$ is a cycle. Let U be a vertex in C , and assume that this cycle reads the (cyclically reduced) word $z \in \mathbf{F}$ starting (and ending) at U . Recall that U is a 1-dimensional subspace of $K^{\text{vert}(C_w)}$ supported on at least two vertices of C_w , and denote the support of U by $\text{supp}(U)$, so $|\text{supp}(U)| \geq 2$. In particular, for every $s \in \text{supp}(U)$, there is a path in C_w reading z leaving s and reaching some $s' \in \text{supp}(U)$. Hence, some power z^k of z is a path from s to itself for every $s \in \text{supp}(U)$. Because w is not conjugate to w^{-1} , every such copy of z^k has the same orientation along C_w . We get that there is some $y \in \mathbf{F} \setminus \langle w \rangle$ so that $ywy^{-1} = w$. This is not possible unless w is a proper power, which is not the case. \square

Lemma 6.4. *The map $\Phi : \mathcal{C}(\Upsilon) \rightarrow D_w \cup \{\mathcal{A}_w\}$ is one-to-one.*

Proof. [Theorem 5.4](#) states the only cyclic generators of \mathcal{A}_w are elements supported on a single vertex of \mathcal{S}_w , and so C_0 is the only connected component in Υ mapped to \mathcal{A}_w . It remains to show that every element of D_w has a single preimage in $\mathcal{C}(\Upsilon)$. Suppose that $g, g' \in \mathcal{A}_w$, both supported on C_w , so that $g'\mathcal{A} = g\mathcal{A} \in D_w$, namely, $\{0\} \neq g\mathcal{A} = g'\mathcal{A} \not\subseteq \mathcal{A}_w$. Let $f, f' \in \mathcal{A}$ be preimages of g, g' , respectively, through ρ^{-1} , which are supported on $[1, w]$. Then $(f, w-1) = (f', w-1)$ is a rank-2 ideal by [Lemma 6.2](#). Thus there are $p_1, p_2, q_1, q_2 \in \mathcal{A}$ such that

$$f' = fp_1 + (w-1)p_2, \quad f = f'q_1 + (w-1)q_2,$$

so

$$f = (fp_1 + (w-1)p_2)q_1 + (w-1)q_2 = f \cdot p_1q_1 + (w-1)(p_2q_1 + q_2).$$

But $\{f, w-1\}$ is a basis, so by uniqueness we get $p_1q_1 = 1$ (and $p_2q_1 + q_2 = 0$). The only units of \mathcal{A} are scalar product of monomials of the form αz with $\alpha \in K^*$ and $z \in \mathbf{F}$ (this was mentioned and explained in [Section 5A](#)). By multiplying g' by a scalar if necessary, we may thus assume that $p_1 \in \mathbf{F}$ is a word, and we get that

$$g' = \rho(f') = \rho(fp_1 + (w-1)p_2) = \rho(fp_1) = \rho(f)p_1 = gp_1.$$

But C_w contains every reduced path between every two of its vertices, so inside Υ there is a path (reading p_1) from the vertex corresponding to g to the one corresponding to g' . In particular, they both belong to the same connected component. □

Completing the proof of [Theorem 1.4](#). Recall that we need to show that $\beta_w + |\text{Prim}^2(w)| = 0$. Consider the above mentioned map $\Phi : \mathcal{C}(\Upsilon) \rightarrow D_w \cup \{\mathcal{A}_w\}$. As C_0 is a cycle isomorphic to C_w , we have $\chi(C_0) = 0$. By [Lemma 6.3](#), $\chi(C) = 1$ for any $C_0 \neq C \in \mathcal{C}(\Upsilon)$, so $|\mathcal{C}(\Upsilon) \setminus \{C_0\}| = \chi(\Upsilon)$. Thus

$$|\text{Prim}^2(w)| = |D_w| = |\mathcal{C}(\Upsilon) \setminus \{C_0\}| = \chi(\Upsilon) = -\beta_w,$$

where the first equality is by [Lemma 6.2](#), the second equality is by [Lemma 6.4](#), and the fourth equality is by (6-4). □

7. Open questions

This paper raises quite a few questions and directions for future research, and we gather the main ones here. As above, $\mathcal{A} = K[\mathbf{F}]$ and $\pi_q(w)$ is the q -primitivity rank of $w \in \mathbf{F}$ (see [Definition 1.5](#)).

Expected number of fixed vectors. As stated in [Conjecture 1.6](#), is it true that for every $w \in \mathbf{F}$, we have $\mathbb{E}_w[\text{fix}] = 2 + \frac{|\text{Crit}_q(w)|}{q^{N(\pi-1)}} + O\left(\frac{1}{q^{N\pi}}\right)$ where $\pi = \pi_q(w)$? If true, this would generalize [Corollary 1.7](#) and yield that in free groups of arbitrary finite rank the words inducing the uniform measure on $\text{GL}_N(K)$ for every N are precisely the primitive words — a result analogous to [[Puder and Parzanchevski 2015](#), [Theorem 1.1](#)] dealing with S_N .

The q -primitivity rank. Recall [Conjecture 1.9](#): is it true that $\pi_q(w) = \pi(w)$ for every $w \in F$ and every prime power q ? What is the value for a generic word (compare with [\[Puder 2015, Corollary 8.3\]](#) and [\[Kapovich 2022\]](#))? The Cohn–Lewin theorem applies to the free group algebra over an arbitrary field, not necessarily finite, and one can analogously define the K -primitivity rank of w for an arbitrary field K (and even for certain rings). Is it true that the K -primitivity rank is equal to $\pi(w)$ for every field K ?

What about general elements of \mathcal{A} ? One can define the primitivity rank $\pi_{\mathcal{A}}(f)$ of arbitrary $f \in \mathcal{A}$ as the rank of critical ideals, so $\pi_q(w) = \pi_{\mathcal{A}}(w - 1)$ (and see the paragraph preceding [Corollary 3.11](#)). What are the possible values of $\pi_{\mathcal{A}}(f)$ for $f \in \mathcal{A}$? Does this number have any combinatorial meaning (à la [Conjecture 1.6](#))?

The expected value of stable irreducible characters. Recall [Conjecture 1.15](#) which says that for every stable irreducible character χ of $\mathrm{GL}_{\bullet}(K)$, $\mathbb{E}_w[\chi] = O((\dim \chi)^{1-\pi_q(w)})$. This conjecture should be quite difficult to tackle, as it is not even known in the somewhat simpler case of the symmetric group. It is more conceivable that one may be able to prove the weaker result that $\mathbb{E}_w[\chi] = O(q^{-N \cdot \pi_q(w)})$ for every nonpower w and every stable irreducible character of dimension $\Omega(q^{2N})$. This kind of result was proved for stable irreducible characters of $\{S_N\}_N$ [\[Hanany and Puder 2023, Corollary 1.7\]](#), for $\{U(N)\}_N$ [\[Brodsky 2024\]](#) and for $\{G \wr S_N\}_N$ for any finite group G [\[Shomroni 2023b\]](#). (See also [\[Ernst-West et al. 2023, Appendix A\]](#) for further discussion and a more refined conjecture.)

Spectral gap in random Schreier graphs of $\mathrm{GL}_N(K)$. Part of the original motivation for studying word measures on $\mathrm{GL}_N(K)$ lies in questions regarding expansion and spectral gaps in random Schreier graphs of the groups $\mathrm{GL}_N(K)$ when K is fixed and $N \rightarrow \infty$. A recent milestone here is [\[Eberhard and Jezernik 2022\]](#). Still, the following question is still open: Consider a random Schreier graph depicting the linear action of $\mathrm{GL}_N(K)$ on $K^N \setminus \{0\}$ with respect to two random generators. Do these graphs admit a uniform spectral gap with probability $\rightarrow 1$ as $N \rightarrow \infty$? If so, is the spectral gap optimal? It is plausible that the results and conjectures in this paper may contribute to obtaining such results, in a fashion similar to analogous proofs for Schreier graphs of S_N [\[Linial and Puder 2010; Puder 2015; Friedman and Puder 2023; Hanany and Puder 2023\]](#).

Limit distributions. [Theorem 1.13](#) states that for w a nonpower, the distribution of the number of fixed vectors in a w -random element of $\mathrm{GL}_N(K)$ converges in distribution, as $N \rightarrow \infty$, to a limit distribution which is independent of w . Is this true for powers too? Is this true for an arbitrary stable class function in the ring \mathcal{R} from page 2054? (This is known for S_N — see [\[Nica 1994, Theorem 1.1\]](#) and [\[Puder and Zimhoni 2024, Theorem 1.14\]](#) for a more general result about cycles of bounded length.)

Free group algebras. This paper gives rise to quite a few questions about the free group algebra \mathcal{A} . First, it is natural to guess that [Corollary 5.2](#) can be generalized as follows: if $T \subseteq \mathrm{Cay}(F, B)$ is a subtree and f , supported on T , is a primitive element of $I \leq_T \mathcal{A}$, can $\{f\}$ be extended to a basis of I which is supported on T ?

Recall [Theorem 5.4](#) that when w is a nonpower, the only cyclic generators of the right \mathcal{A} -module $\mathcal{A}/(w - 1)$ are images of unit elements of \mathcal{A} . Is this true for general subgroups of F ? Namely, let $H \leq F$ be a finitely generated subgroup which is not contained in any other subgroup of equal or smaller rank (in the language of [\[Puder 2014\]](#), this is $\pi(H) > \text{rk } H$). Let $J_H \stackrel{\text{def}}{=} I_H \mathcal{A} = (\{h - 1 \mid h \in H\})$ (see [\[Cohen 1972, Chapter 4\]](#)). Is it true that the only cyclic generators of the quotient \mathcal{A} -module \mathcal{A}/J_H are images of unit elements of \mathcal{A} ? This would be a Kaplansky-type result for such modules.

There are many other famous theorems and algorithms about free groups and their subgroups and we wonder if they have versions that apply to the free group algebra and its ideals. For example, is there an analog of Whitehead’s cut vertex criterion which may detect efficiently whether a given element belongs to a free factor of a given ideal? See the recent survey [\[Delgado and Ventura 2022\]](#) giving a list of results about free groups and their subgroups using Stallings core graphs.

Appendix: The limit distribution of fix

Fix a nonpower $1 \neq w \in F$. Recall that $\text{fix}_{w,N}$ denotes the number of fixed vectors of a w -random element in $\text{GL}_N(K)$. We explain why the method of moments is applicable for proving convergence in distribution for $\text{fix}_{w,N}$, thus proving [Theorem 1.13](#). We begin by recalling some basic definitions for the moment problem.

Given a sequence of real numbers $(m_n)_{n \geq 0}$ and an interval $I \subseteq \mathbb{R}$, a solution to the associated moment problem is a positive Borel measure θ supported on I with moments $\int_I x^n d\theta(x) = m_n$. When $I = \mathbb{R}$ (respectively, $I = [0, \infty)$), the problem is called a *Hamburger* (respectively, *Stieltjes*) moment problem. If a solution exists, the moment problem is said to be *solvable*. A solvable moment problem is further categorized by the number of solutions: if a unique solution exists, the moment problem is said to be *determinate* and otherwise it is called *indeterminate*, in which case there are infinitely many solutions since the set of solutions is convex.

The limiting measure of $\text{fix}_{w,N}$ is a special case of a well-studied family of measures in the field of orthogonal polynomials. We next recall this family of measures, and then explain how previous analysis of the determinacy of its associated moment problems allows us to deduce the desired convergence in distribution.

Let $p \in (0, 1)$ and $a > 0$. The *Al-Salam Carlitz* polynomials of the second kind $V_n^{(a)}(x; p)$ (see [\[Chihara 1978, pp. 195–198; Koekoek et al. 2010, Section 14.24; Christiansen 2004, pp. 30-33\]](#)) are orthogonal with respect to the probability measure supported on the sequence $\{p^{-k}\}_{k \geq 0}$ with masses

$$w_{AC}(p^{-k}; a; p) = (ap; p)_\infty \frac{a^k p^{k^2}}{(p; p)_k (ap; p)_k}, \tag{A-1}$$

where $(x; y)_n = \prod_{j=0}^{n-1} (1 - xy^j)$ is the q -shifted factorial, or q -Pochhammer symbol, and $(x; y)_\infty = \prod_{j=0}^\infty (1 - xy^j)$.

Let q be a prime power. The limiting measure ν of $\text{fix}_{w,N}$ is a special case of the family of measures [\(A-1\)](#) with parameters $p = q^{-1}$ and $a = 1$. Explicitly, $\nu = \sum_{k=0}^\infty w_{AC}(q^k; 1; q^{-1}) \delta_{q^k}$. The n -th moment of ν is

equal to the number of linear subspaces of an n -dimensional vector space over a field with q elements (see [Fulman and Stanton 2016, Proposition 5.7] or [Chihara 1978, Equation 10.10]).

Let ν' be the pushforward of ν under the translation map $x \mapsto x - 1$, i.e.,

$$\nu' = \sum_{k=0}^{\infty} w_{AC}(q^k; 1; q^{-1})\delta_{q^k-1}.$$

The measure ν' exhibits the interesting phenomenon of having its Hamburger moment problem be indeterminate while its Stieltjes moment problem is determinate (see [Berg and Valent 1994, Section 4]). Since the moments of a random variable Z determine the moments of $Z - 1$ and vice versa, the pushforward map induced by $x \mapsto x - 1$ forms a bijection between solutions to the moment problem associated to ν on $I = [1, \infty)$ and solutions to the Stieltjes moment problem associated to ν' . In particular, any measure supported on $[1, \infty)$ with the same moments as ν must be equal to ν .

Let ν_n be a sequence of Borel probability measures on \mathbb{R} supported on $[1, \infty)$, and suppose that for every $k \in \mathbb{N}$ the k -th moment of ν_n converges as $k \rightarrow \infty$ to the k -th moment of ν , as Theorem 1.12 applied with $\mathcal{B} = I_k \in GL_k(K)$ yields for $\text{fix}_{w,N}$. We are now ready to deduce that ν_n converges weakly¹⁴ to ν . The set of Borel probability measures on \mathbb{R} equipped with the topology of weak convergence is metrizable (the Lévy metric, for example; see [Durrett 2019, Exercise 3.2.6]), and so it is enough to show that every subsequence of ν_n has a further subsequence converging weakly to ν . Let ν_{n_k} be such a subsequence. The convergence of the second moments implies that the sequence $(\nu_n)_{n \in \mathbb{N}}$ is tight [Durrett 2019, Theorem 3.2.14], and by Prokhorov's theorem [Billingsley 1999, Theorem 5.1], ν_{n_k} has a further subsequence $\nu_{n_{k_l}}$ converging weakly to some probability measure $\tilde{\nu}$. The convergence of moments of ν_n to the moments of ν implies that $\tilde{\nu}$ has the same sequence of moments as ν [Durrett 2019, Exercise 3.2.5]. Furthermore, using the Portmanteau theorem [Durrett 2019, Theorem 3.2.11] on the closed set $[1, \infty) \subseteq \mathbb{R}$, we get

$$\tilde{\nu}([1, \infty)) \geq \limsup_{l \rightarrow \infty} \nu_{n_{k_l}}([1, \infty)) = \limsup_{l \rightarrow \infty} 1 = 1,$$

and so $\tilde{\nu}$ must also be supported on $[1, \infty)$. The determinacy of the moment problem associated to ν on $I = [1, \infty)$ implies that $\tilde{\nu} = \nu$, finishing the argument.

Acknowledgments

We thank Khalid Bou-Rabee for some prehistorical discussions on word measures on $GL_2(K)$. We corresponded with Christian Berg on subjects around Theorem 1.13 and its proof, and we thank him for some very helpful pointers he gave us. We also thank George Bergman, Michael Magee and Yotam Shomroni for helpful discussions, and an anonymous referee for a thorough reading of the paper and many suggestions for small improvements. This research has received funding from the Israel Science Foundation: ISF grant 1071/16 and from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement no. 850956).

¹⁴The convergence in distribution of random variables is equivalent to the weak convergence of their measures.

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Communicated by Michael J. Larsen

Received 2022-12-06 Revised 2023-04-21 Accepted 2023-11-27

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

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