Moduli of linear slices
of high degree smooth hypersurfaces

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We study the variation of linear sections of hypersurfaces in $\mathbb{P}^n$. We completely classify all plane curves, necessarily singular, whose line sections do not vary maximally in moduli. In higher dimensions, we prove that the family of hyperplane sections of any smooth degree d hypersurface in $\mathbb{P}^n$ varies maximally for $d \geq n+3$. In the process, we generalize the classical Grauert–Mülich theorem about lines in projective space, both to k-planes in projective space and to free rational curves on arbitrary varieties.

1. Introduction

A fundamental technique for studying a degree d complex hypersurface X in projective space $\mathbb{P}^n$ is to intersect it with hyperplanes. The family of varieties thus obtained can be represented by a map to moduli

$$\phi : \mathbb{P}^{n*} \dashrightarrow \mathbb{P}H^0(\mathscr{O}_{\mathbb{P}^{n-1}}(d)) /\!/ \operatorname{SL}_n, \quad [\Lambda] \mapsto [\Lambda \cap X].$$

Basic properties of ϕ are still not understood, even under regularity assumptions on X. Take, for instance, the problem of determining the dimension of its image. If X is assumed to be *general*, then ϕ can directly be shown to have maximal rank, i.e., its image is as large as possible, as done in [van Opstall and Veliche 2007]. However, once we assume X is an *arbitrary* hypersurface, the story becomes more complicated, with several authors studying special cases in the last few decades. Even in the case of a reduced plane curve X, showing maximal variation is not a trivial task. Thirty years ago, while studying PGL_3-orbits of plane curves, Aluffi and Faber [1993, Proposition 4.2] cleverly exploited the classical Plücker formulas to prove that *smooth* plane curves of degree at least 5 always have maximum variation of linear sections. However, if the curve X is singular, then ϕ can fail to have maximal rank, and Aluffi and Faber were not able to completely analyze this case.

Quite generally, if the dimension of the projective automorphism group of X is larger than expected (e.g., if X is a cone), then linear slices must fail to vary maximally in moduli. Outside this class of hypersurfaces, we are unaware of any other examples where ϕ fails to have maximal rank, so we pose the following question:

Question 1.1. If ϕ fails to have maximal rank, must X have a positive-dimensional projective automorphism group?
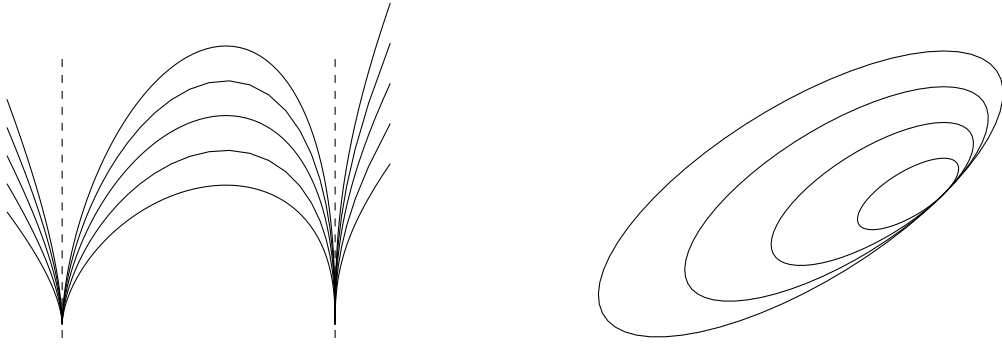
Figure 1. Singular curves for which ϕ fails to have maximal rank. Left: union of orbits under $\mathbb{G}_m$ action. Right: union of orbits under $\mathbb{G}_a$ action (quadritangent conics).

We are concerned exclusively with the case where X is a hypersurface, although one can ask similar questions for other subvarieties of $\mathbb{P}^n$, such as in [Mckernan 1991]. Our first result is to answer Question 1.1 affirmatively when $X \subset \mathbb{P}^2$ is a plane curve:

Theorem 1.2. *If $X \subset \mathbb{P}^2_{\mathbb{C}}$ is an arbitrary plane curve and if ϕ fails to have maximal rank, then X has infinitely many projective automorphisms.*

Given Theorem 1.2, we see that of all curves where ϕ fails to have maximal rank have stabilizer containing $\mathbb{G}_m$ or $\mathbb{G}_a$, where typical examples are depicted in Figure 1.

The map ϕ is even more difficult to understand for larger-dimensional hypersurfaces — we restrict our attention primarily to smooth hypersurfaces. Beauville [1990] investigated the case where ϕ is a *constant map* and classified the smooth hypersurfaces X for which the family of hyperplane sections has constant moduli. This phenomenon happens only for very special hypersurfaces in positive characteristic. In contrast, we prove:

Theorem 1.3. *If $X \subset \mathbb{P}^n_{\mathbb{C}}$ is a smooth hypersurface of degree $d \geq n + 3$, then ϕ has maximal rank.*

We can also intersect a hypersurface X with k-planes for smaller k, obtaining natural analogues

$$\phi_k : \mathbb{G}(k, n) \dashrightarrow \mathbb{P}H^0(\mathscr{O}_{\mathbb{P}^k}(d)) /\!\!/ \, \mathrm{SL}_{k+1}, \quad [\Lambda] \mapsto [\Lambda \cap X],$$

and ask similar questions about ϕ_k. Harris, Mazur, and Pandharipande [Harris et al. 1998], and then later Starr [2006], studied the situation where ϕ_k is expected to be dominant, relating the problem of establishing dominance to the question of unirationality of low degree hypersurfaces. When ϕ_k is expected to be generically finite and dominant, the problem of establishing its degree has also appeared in the literature. In this direction, see [Cadman and Laza 2008; Lee et al. 2020; 2023].

We are able to generalize Theorem 1.3, and prove that ϕ_k has maximal rank under some restrictions on k:

Theorem 1.4. *If $X \subset \mathbb{P}^n_{\mathbb{C}}$ is a smooth hypersurface of degree d, then ϕ_k has maximal rank assuming $d \geq n + 3$ and $k \geq \frac{2}{3}n$.*

For $k < \frac{2}{3}n$, we obtain similar statements, but with d forced to be larger (see Theorems 5.8 and 5.9).

Broadening the topic even further, we can intersect X with other types of varieties, for example, rational curves of degree e. In this way, we obtain a map from the variety of degree e rational curves in $\mathbb{P}^n$ to the moduli space of ed points on $\mathbb{P}^1$. Our methods provide results in this context — see Theorems 4.2 and 5.2.

1A. *Methods*. The log tangent sheaf $T_{\mathbb{P}^n}(-\log X)$ and the Grauert–Mülich theorem play key roles in our approach. We identify the tangent space of the fiber of ϕ at a point $[\Lambda]$ with sections of the log tangent sheaf $T_{\mathbb{P}^n}(-\log X)$ restricted to Λ. Then, we adapt the argument in the usual Grauert–Mülich theorem [Okonek et al. 1980, Theorem 2.1.4] to produce sections or subsheaves of the log tangent sheaf $T_{\mathbb{P}^n}(-\log X)$. In the plane curve case, this forces X to be an integral curve for a vector field on $\mathbb{P}^2$, leading to the classification in Theorem 4.2. In the higher dimensional case, we appeal to a result of Guenancia regarding the semistability of $T_{\mathbb{P}^n}(-\log X)$, when $(\mathbb{P}^n, X)$ is a log-canonical pair and $d \geq n + 2$. In particular, all our results in this case actually hold when $(\mathbb{P}^n, X)$ is a log-canonical pair, not only when X is smooth.

Our methods will produce results in other contexts, for example, if we replace $\mathbb{P}^n$ with a homogeneous space G/P.

2. Preliminaries

We introduce conventions and basic definitions.

2A. *Notation and conventions*. We will work over the complex numbers. We identify vector bundles with locally free sheaves throughout, and all our sheaves are coherent. A *subbundle* of a vector bundle V is a locally free subsheaf $W \subset V$ such that V/W is also locally free. For us, a variety is an integral scheme of finite type. If F is a coherent sheaf on a scheme X, we denote by $\mathrm{ev} : H^0(X, F) \otimes \mathcal{O}_X \to F$ the natural evaluation map.

We denote by $\mathrm{Mor}_e(\mathbb{P}^k, \mathbb{P}^n)$ the variety parameterizing morphisms $f : \mathbb{P}^k \to \mathbb{P}^n$ with $f^*\mathcal{O}(1) = \mathcal{O}(e)$. Explicitly, $\mathrm{Mor}_e(\mathbb{P}^k, \mathbb{P}^n)$ is a Zariski open subset of $\mathbb{P}\big(H^0(\mathcal{O}_{\mathbb{P}^k}(e))^{\oplus n+1}\big)$ parameterizing tuples $(A_0, \ldots, A_n)$ of homogeneous degree e forms on $\mathbb{P}^k$ which do not vanish simultaneously anywhere on $\mathbb{P}^k$. More generally, $\mathrm{Mor}(X, Y)$ denotes the (not finite-type) scheme parameterizing morphisms from the scheme X to the scheme Y.

Given a torsion-free sheaf E on a projective variety X, we let its slope $\mu(E)$ denote the ratio $\frac{\deg(E)}{\mathrm{rank}(E)}$, where $\deg(E) = \int_X c_1(E)\mathcal{O}_X(1)^{\dim(X)-1}$. We call E semistable (respectively stable) if there is no proper subsheaf F with $\mu(F) > \mu(E)$ (respectively $\mu(F) \geq \mu(E)$). In general, the *Harder–Narasimhan* filtration of E is

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_a = E,$$

where the subquotients $E_1/E_0, E_2/E_1, \ldots, E_a/E_{a-1}$ are semistable and have strictly decreasing slopes. Finally, if E, F are two coherent sheaves, then $\mathrm{Hom}(E, F)$ will denote the vector space of global homomorphisms $E \to F$ while $\underline{\mathrm{Hom}}(E, F)$ will denote the sheaf of local homomorphisms.

2B. *The map to moduli* Φ. Suppose $X \in \mathbb{P}^n$ is a degree d hypersurface. After fixing integers $e \geq 1$, $k \leq n - 1$, we get the induced *map to moduli*

$$\Phi : \mathrm{Mor}_e(\mathbb{P}^k, \mathbb{P}^n) \dashrightarrow \mathbb{P}H^0(\mathbb{P}^k, \mathscr{O}_{\mathbb{P}^k}(de)), \quad \iota \mapsto [\iota^{-1}(X)].$$

We say that Φ has maximal rank if the dimension of its image is

$$\max\{\dim(\mathrm{Mor}_e(\mathbb{P}^k, \mathbb{P}^n)), \mathbb{P}H^0(\mathbb{P}^k, \mathscr{O}_{\mathbb{P}^k}(de))\}.$$

Equivalently, since we are working over $\mathbb{C}$, the derivative of Φ at a general point has maximum rank.

Though our methods give results for all e, k, we are primarily interested in the cases where $e = 1$ or $k = 1$. Therefore, we have only stated our results in these cases. In the case $e = 1$, Φ having maximal rank is equivalent to the map

$$\mathbb{G}(k, n) \dashrightarrow \mathbb{P}H^0(\mathbb{P}^k, \mathscr{O}_{\mathbb{P}^k}(d)) /\!\!/ \mathrm{SL}_k, \quad [\Lambda] \mapsto [\Lambda \cap X],$$

having maximal rank, assuming the general k-plane slice of X has no infinitesimal automorphisms. Whenever our results apply, this condition will always be satisfied.

2C. *Log tangent sheaves*. We now introduce the main tool of the paper. We suspect Lemma 2.2 is well-known to experts but include a proof for want of a suitable reference. Everything in this section should work for a reduced divisor in an arbitrary smooth ambient variety, but we will focus on the case that the ambient variety is projective space.

Let $D \subset \mathbb{P}^n$ be a reduced hypersurface. Viewing D as a divisor in the smooth ambient variety $\mathbb{P}^n$, we get the log tangent sheaf $T_{\mathbb{P}^n}(-\log D)$, which sits inside the exact sequence

$$0 \to T_{\mathbb{P}^n}(-\log D) \to T_{\mathbb{P}^n} \to \mathscr{O}_D(D) \to \mathscr{O}_{D_{\mathrm{sing}}}(D) \to 0,$$

where D_{sing} is the singular subscheme cut out of $\mathbb{P}^n$ by the equation for D and its partials. In terms of background, we only assume what is covered in [Liao 2013, 2.1.2], but see [Saito 1980] for the original reference. One can check that $T_{\mathbb{P}^n}(-\log D)$ is a vector bundle when D is smooth using local coordinates; in general $T_{\mathbb{P}^n}(-\log D)$ is a reflexive sheaf.

Remark 2.1. Informally, local sections of $T_{\mathbb{P}^n}(-\log D)$ represent local vector fields which are tangent to D. This can be seen explicitly by noting that the map $T_{\mathbb{P}^n} \to \mathscr{O}_D(D)$ in the exact sequence above is given by $\theta \mapsto \theta(f)$, where θ is a vector field and f is the (local) equation for D. If we identify $\mathscr{O}_D(D)$ with $N_{D/\mathbb{P}^n}$, the map $T_{\mathbb{P}^n} \to \mathscr{O}_D(D)$ is also $T_{\mathbb{P}^n} \to T_{\mathbb{P}^n}|_D \to N_{D/\mathbb{P}^n}$.

Let $\mathbb{P}^k \xrightarrow{\iota} \mathbb{P}^n$ be a map defined by degree e homogeneous forms, and suppose $Z \subset \mathbb{P}^n$ is a subscheme. We say an infinitesimal deformation $\iota_\epsilon : \mathbb{P}^k \times \mathrm{Spec}\,\mathbb{C}[\epsilon]/(\epsilon^2) \to \mathbb{P}^n$ *preserves* $\iota^{-1}(Z)$ if $\iota_\epsilon^{-1}(Z) \subset \mathbb{P}^k \times \mathrm{Spec}\,\mathbb{C}[\epsilon]/(\epsilon^2)$ is the trivial deformation $\iota^{-1}(Z) \times \mathrm{Spec}\,\mathbb{C}[\epsilon]/(\epsilon^2)$. The point of this section is to prove the following lemma.

Lemma 2.2. *Let $\mathbb{P}^k \xrightarrow{\iota} \mathbb{P}^n$ be a map defined by degree e homogeneous forms whose image is not contained in D. Global sections of $\iota^* T_{\mathbb{P}^n}(-\log D)$ correspond to deformations of the map ι preserving the hypersurface $\iota^{-1}(D)$.*

Proof. First, sections of $\iota^* T_{\mathbb{P}^n}$ correspond to deformations of ι. More explicitly, ι is defined by an $(n+1)$-tuple of degree e forms in $k+1$ variables $A_0(s_0, \ldots, s_k), \ldots, A_n(s_0, \ldots, s_k)$ sending $[s_0 : \cdots : s_k]$ to $[A_0(s_0, \ldots, s_k) : \cdots : A_n(s_0, \ldots, s_k)]$.

A deformation ι_ϵ of ι is given by another $(n+1)$-tuple of degree e forms in $k+1$ variables $B_0(s_0, \ldots, s_k), \ldots, B_n(s_0, \ldots, s_k)$. Explicitly, as a map from $\mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2)) \times \mathbb{P}^k \to \mathbb{P}^n$ this is given in coordinates by $\epsilon, [s_0, \ldots, s_k]$ mapping to $[A_0(s_0, \ldots, s_k) + \epsilon B_0(s_0, \ldots, s_k) : \cdots : A_n(s_0, \ldots, s_k) + \epsilon B_n(s_0, \ldots, s_k)]$. The vector space of deformations is given by the quotient space of $(n+1)$-tuples of degree e forms $(B_0, \ldots, B_n)$ modulo the 1-dimensional vector space generated by $(A_0, \ldots, A_n)$.

Let D be defined by $F = 0$ where F is a reduced homogeneous form in $n+1$ variables. If we pull back the form F under the deformed map ι_ϵ, we obtain

$$F(A_0(s_0, \ldots, s_k) + \epsilon B_0(s_0, \ldots, s_k), \ldots, A_n(s_0, \ldots, s_k) + \epsilon B_n(s_0, \ldots, s_k))$$
$$= F(A_0(s_0, \ldots, s_k), \ldots, A_n(s_0, \ldots, s_k)) + \epsilon \sum_{i=0}^{n} B_i(s_0, \ldots, s_k) \cdot \partial_i F(A_0(s_0, \ldots, s_k), \ldots, A_n(s_0, \ldots, s_k)).$$

Therefore, deformations ι_ϵ that preserve $\iota^{-1}(D)$ correspond to choices of $B_0, \ldots, B_n$ such that

$$\sum_{i=0}^{n} B_i(s_0, \ldots, s_k) \cdot \partial_i F(A_0(s_0, \ldots, s_k), \ldots, A_n(s_0, \ldots, s_k)) \tag{2-1}$$

is a scalar multiple of F.

Now, we wish to realize this latter condition as producing sections of the pulled back log tangent sheaf. First, the sections of the pulled back tangent sheaf $\iota^* T_{\mathbb{P}^n}$ can be computed via the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^k} \to \mathcal{O}_{\mathbb{P}^k}(e)^{n+1} \to \iota^* T_{\mathbb{P}^n} \to 0$$

to be the quotient space of $(n+1)$-tuples of linear forms $(B_0, \ldots, B_n)$ modulo the 1-dimensional vector space generated by $(A_0, \ldots, A_n)$.

The restricted vector field corresponding to $(B_0, \ldots, B_n)$ is $\sum_{i=0}^{n} B_i \frac{\partial}{\partial x_i}$. Recall that $T_{\mathbb{P}^n}(-\log D)$ is the kernel of the map $T_{\mathbb{P}^n} \to \mathcal{O}_D(D)$ sending a vector field $\theta := \sum_{i=0}^{n} B_i \frac{\partial}{\partial x_i}$ to $\theta(F)$.

In other words, the defining equation is

$$\sum_{i=0}^{n} B_i \frac{\partial}{\partial x_i} F \equiv 0 \pmod{F}.$$

Pulling back this under ι yields exactly (2-1). $\qquad\square$

3. Grauert–Mülich

The goal of this section is to generalize the classical Grauert–Mülich theorem [Okonek et al. 1980, Theorem 2.1.4] in two directions:

Proposition 3.1. *Let $Z \subset \mathbb{P}^N$ be a smooth projective variety and $f : \mathbb{P}^1 \to Z$ be a general map in an open subset $\mathcal{M}$ of $\mathrm{Mor}(\mathbb{P}^1, Z)$ such that $\mathbb{P}^1 \times \mathcal{M} \to Z$ has connected fibers. Suppose $f^* T_Z$ is globally generated.*

Let E be a torsion free sheaf on Z and write $f^* E$ as $\bigoplus_{i=1}^{b} \mathscr{O}(a_i)$ with $a_1 \geq \cdots \geq a_b$. If $a_j > a_{j+1} + 1$ for some j, then E has a subsheaf of rank j and degree $\frac{1}{\deg(f)} \sum_{i=1}^{j} a_i$. In particular, if E is semistable, then the bundle $f^* E$ can be written as $\bigoplus_i \mathscr{O}(a_i)$ with $|a_i - a_{i+1}| \leq 1$.

For applications to slicing by k-planes, we will use Proposition 3.3.

Definition 3.2. Given a torsion free sheaf E on a smooth projective variety, let $\mu^{\max}(E)$ denote the maximum slope of a nontrivial subsheaf of E. It is also the slope of the first subsheaf appearing in its Harder–Narasimhan filtration. Similarly let $\mu^{\min}(E)$ denote the minimum slope of a nontrivial quotient of E. It is also the slope of the quotient of the last two subsheaves appearing in its Harder–Narasimhan filtration.

Proposition 3.3. *Let E be a torsion free sheaf on $\mathbb{P}^n$. Let Λ be a k-plane in $\mathbb{P}^n$, general with respect to E. Let $S \subsetneq E|_\Lambda$ be a sheaf appearing in the Harder–Narasimhan filtration of $E|_\Lambda$ and suppose*

$$\mu^{\min}(S) - \frac{1}{k} > \mu^{\max}(E|_\Lambda/S).$$

Then E is not semistable.

The proofs of Propositions 3.1 and 3.3 are very similar in spirit to standard proofs of Grauert–Mülich, such as the one found in [Okonek et al. 1980]. The argument relies crucially on the following lemma.

Lemma 3.4 (descent lemma from [Okonek et al. 1980, Lemma 2.1.2]). *Let Y and Z be smooth varieties and $\pi : Y \to Z$ be a surjective smooth morphism with connected fibers. Let E be a vector bundle on Z such that $\pi^* E$ has a vector subbundle S with quotient vector bundle Q. If*

$$\mathrm{Hom}(T_{Y/Z}, \underline{\mathrm{Hom}}(S, Q)) = 0,$$

then S is the pullback of a subbundle of E on Z.

The key technical lemma of this section is Lemma 3.6, whose proof will use the following simple fact.

Lemma 3.5. *Let Y be a variety and E and F be two sheaves on Y. Suppose every semistable subquotient in the Harder–Narasimhan filtration of E has greater slope than every semistable subquotient of F, i.e., that $\mu^{\min}(E) > \mu^{\max}(F)$. Then, $\mathrm{Hom}(E, F) = 0$.*

Proof. Let $0 = E_0 \subset E_1 \subset \cdots \subset E_a = E$ be the Harder–Narasimhan filtration for E and $0 = F_0 \subset F_1 \subset \cdots \subset F_b = F$ be the Harder–Narasimhan filtration for F. Consider a map $\phi : E \to F$. We show $\phi = 0$.

First, ϕ induces a map $E_1 \to F_b/F_{b-1}$, which is zero since the source is semistable and has slope greater than the target, which is also semistable. Therefore, ϕ induces a map $E_1 \to F_{b-1}/F_{b-2}$, which again is zero for the same reason. Continuing this, we find the map $E_1 \to F$ is zero.

Then, we consider the induced map $E_2/E_1 \to F_b/F_{b-1}$ and repeat the argument above to find $E_2/E_1 \to F$ must be the zero map. Continuing this for E_3/E_2 and so on shows that the map ϕ is zero. $\square$

Lemma 3.6. *Let Z be a smooth projective variety and let $\mathcal{U} \to \mathcal{M}$ be a smooth family of projective varieties with a smooth surjective map $\pi : \mathcal{U} \to Z$ having connected fibers. Let E be a torsion free sheaf on Z and let $\mathcal{U}_p$ be a general fiber of $\mathcal{U} \to \mathcal{M}$. Let S be a subsheaf of $\pi^* E|_{\mathcal{U}_p}$ appearing in the Harder–Narasimhan filtration of $\pi^* E|_{\mathcal{U}_p}$ such that*

$$\mu^{\min}(S) + \mu^{\min}(T_{\mathcal{U}/Z}|_{\mathcal{U}_p}) > \mu^{\max}(\pi^* E|_{\mathcal{U}_p}/S).$$

Then there is a subsheaf $\widetilde{S}$ on Z of E such that $\pi^* \widetilde{S}|_{\mathcal{U}_p}$ agrees with S on the locus where S is a vector bundle.

Proof. By [Shatz 1977, Lemmas 5 and 7], we can replace $\mathcal{M}$ by a dense open subset so that the members of the Harder–Narasimham filtration of $\pi^* E|_{\mathcal{U}_p}$ extend to a family over $\mathcal{U}$. Namely, there exists a sequence of subsheaves $0 = S_0 \subset S_1 \subset \cdots \subset S_a = \pi^* E$ that restrict to the Harder–Narasimhan filtration of $E|_{\mathcal{U}_p}$ for all $p \in \mathcal{M}$, so in particular $S = S_i|_{\mathcal{U}_p}$ for some i. If E and all S_j's are locally free, then we can immediately apply Lemma 3.4 to conclude. The next paragraphs deal with the possibility that E or the S_j's are not locally free by passing to a general curve in $\mathcal{U}_p$.

We have an open subset $\mathcal{U}^0 \subset \mathcal{U}$ whose complement has codimension at least 2 and consists of the points over which S_i and $\pi^* E/S_i$ are both vector bundles. The image of $\mathcal{U}^0$ in Z is an open subset Z^0 (by flatness of π) whose complement must also have codimension at least 2.

Now, we can apply Lemma 3.4 in the case $Y = \mathcal{U}^0$ and $Z = Z^0$. In order to do so, we must show that

$$\mathrm{Hom}\big(T_{\mathcal{U}^0/Z^0}, \underline{\mathrm{Hom}}(S_i|_{\mathcal{U}^0}, (\pi^* E/S_i)|_{\mathcal{U}^0})\big) = 0. \tag{3-1}$$

For this, observe that because all sheaves appearing in (3-1) are locally free, it suffices to show the lack of homomorphisms when we restrict to a general fiber $\mathcal{U}_p$. Then, we use the same idea and restrict to a general complete intersection curve $C \subset \mathcal{U}_p$ of sufficiently high degree. Restricting the Harder–Narasimhan filtration of $\pi^* E|_{\mathcal{U}_p}$ to C results in a sequence of vector subbundles because each semistable subquotient on $\mathcal{U}_p$ is in particular torsion-free, so the Harder–Narasimhan filtration is a sequence of vector subbundles away from a set of codimension at least 2 in $\mathcal{U}_p$. Since C can be chosen to avoid this set, restricting a sequence of subbundles yields a sequence of subbundles. By [Mehta and Ramanathan 1982], this sequence of sub vector bundles on C is the Harder–Narasimhan filtration of $\pi^* E|_C$. To show (3-1), it therefore suffices to show

$$\mathrm{Hom}\big(T_{\mathcal{U}/Z}|_C, \underline{\mathrm{Hom}}(S|_C, (\pi^* E|_{\mathcal{U}_p}/S)|_C)\big) = \mathrm{Hom}\big(T_{\mathcal{U}/Z}|_C \otimes S|_C, (\pi^* E|_{\mathcal{U}_p}/S)|_C\big) = 0.$$

We conclude by applying Lemma 3.5, keeping in mind that $S|_C$ is part of the Harder–Narasimhan filtration of $\pi^* E|_C$, and that the following slope equalities hold:

$$\mu^{\max}((\pi^* E|_{\mathcal{U}_p}/S)|_C) = \deg(C)\mu^{\max}(\pi^* E|_{\mathcal{U}_p}/S),$$
$$\mu^{\min}(S|_C) = \deg(C)\mu^{\min}(S),$$
$$\mu^{\min}(T_{\mathcal{U}/Z}|_C) = \deg(C)\mu^{\min}(T_{\mathcal{U}/Z}),$$
$$\mu^{\min}(T_{\mathcal{U}/Z}|_C \otimes S|_C) = \mu^{\min}(S|_C) + \mu^{\min}(T_{\mathcal{U}/Z}|_C),$$

where $\deg(C)$ can be defined using any projective embedding of $\mathcal{U}_p$. $\qquad\square$

In order for us to apply Lemma 3.6, it is necessary to understand the sheaf $T_{\mathcal{U}/Z}|_{\mathcal{U}_p}$. Lemmas 3.8 and 3.9 identify the sheaf in two common situations.

Definition 3.7. Let Y be a variety and V be a globally generated vector bundle on Y. Then, the *Lazarsfeld–Mukai* bundle of V is the kernel of the evaluation map $\mathcal{O}_Y \otimes H^0(V) \to V$.

Lemma 3.8. *Let Z be a smooth projective variety and $\mathcal{M}$ be a smooth open subset of a component of the Hilbert scheme of varieties on Z. Let $\mathcal{U}$ be the universal family, and suppose that the natural map $\pi : \mathcal{U} \to Z$ is smooth. Let $\mathcal{U}_p$ be a general fiber of $\mathcal{U} \to \mathcal{M}$. Then $T_{\mathcal{U}/Z}|_{\mathcal{U}_p}$ is the Lazarsfeld–Mukai bundle for the normal bundle $N_{\mathcal{U}_p/Z}$, defined by the short exact sequence*

$$0 \to T_{\mathcal{U}/Z}|_{\mathcal{U}_p} \to H^0(N_{\mathcal{U}_p/Z}) \otimes \mathcal{O}_{\mathcal{U}_p} \to N_{\mathcal{U}_p/Z} \to 0.$$

Proof. First we compare the normal sheaf of $\mathcal{U}$ in $\mathcal{M} \times Z$ to the normal sheaf $N_{\mathcal{U}_p/Z}$. We have the diagram

In this diagram, we have written $H^0(N_{\mathcal{U}_p/Z}) \otimes \mathcal{O}_{\mathcal{U}_p}$ as $\mathcal{O}_{\mathcal{U}_p}^N$, where $N = h^0(N_{\mathcal{U}_p/Z})$. We see that $N_{\mathcal{U}/\mathcal{M}\times Z}|_{\mathcal{U}_p}$ is isomorphic to $N_{\mathcal{U}_p/Z}$ by the eight lemma. Next we relate $N_{\mathcal{U}/\mathcal{M}\times Z}|_{\mathcal{U}_p}$ to the Lazarsfeld–Mukai bundle. Consider the diagram, where the lower right entry is computed by the eight lemma,

Then since $\pi^* T_{\mathcal{M}}$ is constant on $\mathcal{U}_p$ and $N_{\mathcal{U}/\mathcal{M} \times Z}|_{\mathcal{U}_p} \cong N_{\mathcal{U}_p/Z}$, we see that the last row becomes

$$0 \to T_{\mathcal{U}/Z}|_{\mathcal{U}_p} \to H^0(N_{\mathcal{U}_p/Z}) \otimes \mathcal{O} \to N_{\mathcal{U}_p/Z} \to 0.$$

The result follows. $\qquad\square$

Lemma 3.9. *Let Y and Z be smooth projective schemes and $\mathcal{M}$ be an open subset of $\mathrm{Mor}(Y, Z)$. Let $\pi : Y \times \mathcal{M} \to Z$ be the universal map. For $f : Y \to Z$ in $\mathcal{M}$, suppose $f^* T_Z$ is globally generated. Then, the restriction $T_{Y \times \mathcal{M}/Z}|_{Y \times \{[f]\}}$ is an extension of T_Y by the Lazersfeld–Mukai bundle of $f^* T_Z$.*

Proof. We have the relative tangent sequence

$$0 \to T_{Y \times \mathcal{M}/Z}|_{Y \times \{[f]\}} \to T_{Y \times \mathcal{M}}|_{Y \times \{[f]\}} \to f^* T_Z \to 0.$$

We have the natural decomposition $T_{Y \times \mathcal{M}/Z}|_{Y \times \{[f]\}} \cong H^0(f^* T_Z) \otimes \mathcal{O} \oplus T_Y$, with respect to which the natural map $T_{Y \times \mathcal{M}/Z}|_{Y \times \{[f]\}} \to f^* T_Z$ is $\mathrm{ev} + df$. Consider the following commutative diagram, where K is the Lazersfeld–Mukai bundle of $f^* T_Z$:

The rows and columns are exact and the left column gives $T_{Y \times \mathcal{M}/Z}|_{Y \times \{[f]\}}$ as an extension of K by T_Y. $\quad\square$

Lemma 3.10. *The Lazersfeld–Mukai bundle of any globally generated vector bundle on $\mathbb{P}^1$ is a direct sum of $\mathcal{O}(-1)$'s.*

Proof. Taking Lazersfeld–Mukai bundles behaves well with respect to direct sum, so it remains to show the result for line bundles $\mathcal{O}(a)$ with $a \geq 0$. The Lazersfeld–Mukai bundle M satisfies

$$0 \to M \to \mathcal{O} \otimes H^0(\mathcal{O}(a)) \to \mathcal{O}(a) \to 0.$$

It follows that M has rank a, degree $-a$ and no global sections, so that $M = \mathcal{O}(-1)^a$. The result follows. $\qquad\square$

Proof of Proposition 3.1. We apply Lemma 3.6 to our situation, where $\mathcal{M}$ is an open subset of $\mathrm{Mor}(\mathbb{P}^1, Z)$ containing $[f]$ and $\mathcal{U} = \mathbb{P}^1 \times \mathcal{M}$. Then, applying Lemma 3.9 shows $T_{\mathbb{P}^1 \times \mathcal{M}/Z}|_{\mathbb{P}^1 \times \{[f]\}}$ an extension of $T_{\mathbb{P}^1}$ by the Lazersfeld–Mukai bundle of $f^* T_Z$. By Lemma 3.10, the Lazersfeld–Mukai bundle of $f^* T_Z$

is a sum of $\mathscr{O}(-1)$'s, so $T_{\mathbb{P}^1 \times \mathcal{M}/Z}|_{\mathbb{P}^1 \times \{[f]\}}$ is an extension of $\mathscr{O}(2)$ by a direct sum of $\mathscr{O}(-1)$'s implying $\mu^{\min}(T_{\mathbb{P}^1 \times \mathcal{M}/Z}|_{\mathbb{P}^1 \times \{[f]\}}) \geq -1$.

Suppose $f^* E$ splits as $\bigoplus_i \mathscr{O}(a_i)$ with $a_1 \geq \cdots \geq a_r$ and $a_j \leq a_{j+1} - 2$. Letting $S = \bigoplus_{i \leq j} \mathscr{O}(a_i)$, we find

$$a_j + (-1) > a_{j+1},$$
$$\mu^{\min}(S) + \mu^{\min}(T_{\mathbb{P}^1 \times \mathcal{M}/Z}|_{\mathbb{P}^1 \times \{[f]\}}) > \mu^{\max}((f^* E)/S),$$

Therefore, we can apply Lemma 3.6 and conclude. $\qquad\square$

Proof of Proposition 3.3. This follows from Lemma 3.6 with $Z = \mathbb{P}^n$, $\mathcal{M} = \mathbb{G}(k, n)$ and $\mathcal{U}$ the universal k-plane. The only thing to check is $\mu^{\min}(T_{\mathcal{U}/\mathbb{P}^n}|_\Lambda) = -\frac{1}{k}$. By Lemma 3.8, $T_{\mathcal{U}/\mathbb{P}^n}|_\Lambda$ lies in the sequence

$$0 \to T_{\mathcal{U}/\mathbb{P}^n}|_\Lambda \to H^0(\mathscr{O}_\Lambda(1)^{n-k}) \otimes \mathscr{O}_\Lambda \to \mathscr{O}_\Lambda(1)^{n-k} \to 0,$$

and so is isomorphic to $\Omega_\Lambda(1)^{n-k}$ by the Euler sequence. Since $\Omega_\Lambda(1)$ is semistable with slope $-\frac{1}{k}$ [Okonek et al. 1980, Theorem 1.3.2], the result follows. $\qquad\square$

4. Plane curves

We now apply the results from the previous section to analyze the map to moduli Φ introduced in Section 2B in the case of plane curves. Throughout this section, C in $\mathbb{P}^2$ denotes a reduced plane curve. (In the nonreduced case, we simply pass to the reduction and apply the results of this section.) Our main results in this section are stated below.

Theorem 4.1. *Let C be a reduced plane curve of degree d. Then, the map*

$$\Phi : \mathrm{Mor}_e(\mathbb{P}^1, \mathbb{P}^2) \dashrightarrow \mathbb{P}(H^0(\mathscr{O}_{\mathbb{P}^1}(ed))), \quad [\iota] \mapsto [\iota^{-1}(C)],$$

has maximal rank if C has finite stabilizer under the action of PGL_3.

In fact, we can classify all cases in Theorem 4.1 where Φ does not have maximal rank.

Theorem 4.2. *We get a complete classification of cases when Φ in Theorem 4.1 does not have maximal rank*:

(1) $d \geq 5$: *C is a union of orbits under an action of $\mathbb{G}_m$ or $\mathbb{G}_a$ on $\mathbb{P}^2$.*

(2) $d = 4$:

    (a) *$e = 1$ and C is the union of four concurrent lines.*

    (b) *$e \geq 2$ and C is a union of orbits under an action of $\mathbb{G}_m$ or $\mathbb{G}_a$ on $\mathbb{P}^2$.*

(3) $d = 3$: *$e \geq 2$ and C is union of concurrent lines.*

Before giving the proofs of these theorems, we need the following two propositions.

Proposition 4.3. *If C is a reduced plane curve and $T_{\mathbb{P}^2}(-\log C)$ admits a nontrivial homomorphism from $\mathscr{O}_{\mathbb{P}^2}(1)$, then C is a union of concurrent lines.*

Proof. First, a nontrivial map from $\mathcal{O}_{\mathbb{P}^2}(1) \to T_{\mathbb{P}^2}(-\log C)$ induces a nontrivial map $\mathcal{O}_{\mathbb{P}^2}(1) \to T_{\mathbb{P}^2}$. Consider the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1)^3 \to T_{\mathbb{P}^2} \to 0.$$

Applying $\underline{\mathrm{Hom}}(\mathcal{O}_{\mathbb{P}^2}(1), \cdot)$ to the Euler sequence, we find

$$\underline{\mathrm{Hom}}(\mathcal{O}_{\mathbb{P}^2}(1), T_{\mathbb{P}^2}) \cong \underline{\mathrm{Hom}}(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(1)^3)$$

and that the composite map $\mathcal{O}_{\mathbb{P}^2}(1) \to T_{\mathbb{P}^2}(-\log C) \to T_{\mathbb{P}^2}$ lifts to a map $\mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{O}_{\mathbb{P}^2}(1)^3$.

After a change of coordinates, we can assume the map $\mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{O}_{\mathbb{P}^2}(1)^3$ is inclusion into the first factor. The map $\mathcal{O}_{\mathbb{P}^2}(1)^3 \to T_{\mathbb{P}^2}$ sends a tuple of linear forms (L_1, L_2, L_3) to $\left(L_1 \frac{\partial}{\partial X}, L_2 \frac{\partial}{\partial Y}, L_3 \frac{\partial}{\partial Z}\right)$, so we conclude that $L \frac{\partial}{\partial X}$ is a section of $T_{\mathbb{P}^2}(-\log C)$ for all linear forms L.

By [Remark 2.1](#), we see that away from the point $[1:0:0]$, the tangent vector $\frac{\partial}{\partial X}$ is in the tangent space of C for every point of C. Restricting to the affine chart $\{Z \neq 0\}$ with coordinates (x, y) and dehomogenizing, this means C restricts to a union of lines parallel to the x-axis. Since these lines and the line at infinity are precisely the lines passing through $[1:0:0]$, we conclude C is a union of concurrent lines. $\qquad\square$

Proposition 4.4. *If C is a reduced plane curve and $T_{\mathbb{P}^2}(-\log C)$ has a section, then C is equivalent to a union of orbits under one of the two actions by $\mathbb{G}_m$ and $\mathbb{G}_a$ as follows*:

$$\mathbb{G}_m \to \mathrm{GL}_3, \quad t \mapsto \begin{pmatrix} t^a & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a, b \in \mathbb{N},$$

$$\mathbb{G}_a \to \mathrm{GL}_3, \quad t \mapsto \exp\left(t \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

Explicitly, there are two cases:

(1) *C is projectively equivalent to a union of curves of the form $X^p Y^q = c Z^{p+q}$, $c \in \mathbb{C}^\times$, and possibly a subset of the three coordinate lines.*

(2) *C is projectively equivalent to a union of members of the family $\{XZ - Y^2 + cZ^2 \mid c \in \mathbb{C}\}$ of conics quadritangent to $\{XZ - Y^2 = 0\}$ at $[0:0:1]$, and possibly the line $\{Z = 0\}$.*

Proof. Let s be a section of $T_{\mathbb{P}^2}(-\log C)$. Then, s is also a section of $T_{\mathbb{P}^2}$ and can be written as $L_X \frac{\partial}{\partial X} + L_Y \frac{\partial}{\partial Y} + L_Z \frac{\partial}{\partial Z}$ where L_X, L_Y, L_Z are homogenous linear forms in X, Y and Z.

Let C_0 be a component of C and let $p \in C$ be a smooth point of C_0. We lift $p \in \mathbb{P}^2$ to a point $\tilde{p} \in \mathbb{C}^3 \setminus \{0\}$. Then, C_0 contains the projection under $\mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2$ of the integral curve $\widetilde{C}$ through $\tilde{p}$ which is the solution to the matrix differential equation

$$\frac{d}{dt} \begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix} = A \begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix}, \quad \begin{pmatrix} X(0) \\ Y(0) \\ Z(0) \end{pmatrix} = \tilde{p}. \tag{4-1}$$

Here A is the 3×3 matrix with complex entries such that

$$A \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} L_X(X, Y, Z) \\ L_Y(X, Y, Z) \\ L_Z(X, Y, Z) \end{pmatrix}.$$

If the projection of $\widetilde{C}$ to $\mathbb{P}^2$ is not a single point, then the image is dense in C_0. Therefore, C must be (the closure of) a finite union of projections of integral curves in $\mathbb{C}^3 \setminus \{0\}$ and 1-dimensional components of the zero locus of s.

After a linear change of coordinates, we can assume that A is in Jordan block form. We keep this choice of coordinates from now on. We let $\tilde{p} = (c_1, c_2, c_3) \in \mathbb{C}^3$ denote a lift of a point on C_0 (to be determined separately in each case) and we let $P(X, Y, Z)$ be a homogenous polynomial defining C_0.

Case 1: A is diagonal. We will show that the first case of Proposition 4.4 happens, so we can assume that $c_1, c_2, c_3 \neq 0$ or else $\tilde{p}$ is contained in a coordinate line. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of A. Then, the solution to (4-1) is $(X(t) \ Y(t) \ Z(t))^T = (e^{\lambda_1 t} c_1 \ e^{\lambda_2 t} c_2 \ e^{\lambda_3 t} c_3)^T$, where $(\cdot)^T$ denotes the transpose.

The defining equation $P(X, Y, Z)$ of C_0 is a homogenous polynomial of minimal degree satisfying

$$P(c_1 e^{\lambda_1 t}, c_2 e^{\lambda_2 t}, c_3 e^{\lambda_3 t}) = 0. \tag{4-2}$$

We can choose a new grading on $\mathbb{C}[X, Y, Z]$ by the complex numbers $\mathbb{C}$ where the monomial $X^a Y^b Z^c$ has the grade $a\lambda_1 + b\lambda_2 + c\lambda_3$. Let P_ω be the homogenous component of P with grade $\omega \in \mathbb{C}$. By linear independence of characters, the elements in $\{e^{\omega t} \mid \omega \in \mathbb{C}\}$ are linearly independent, and hence $P_\omega(c_1 e^{\lambda_1 t}, c_2 e^{\lambda_2 t}, c_3 e^{\lambda_3 t}) = 0$. Therefore, P divides P_ω for all $\omega \in \mathbb{C}$ so P_ω can be nonzero for only one value of ω.

We cannot have $\lambda_1, \lambda_2, \lambda_3$ all equal or else s would be a multiple of $X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + Z\frac{\partial}{\partial Z}$ which induces the zero vector field on $\mathbb{P}^2$. The monomials $X^a Y^b Z^c$ that can appear in P with nonzero coefficients must be the solution to the two linear equations

$$a + b + c = \deg(C_0), \tag{4-3}$$

$$\lambda_1 a + \lambda_2 b + \lambda_3 c = \omega \tag{4-4}$$

for some fixed ω. The solution set to (4-3) is some 1-dimensional complex line ℓ in $\mathbb{C}^3$ and we are interested in the integer solutions $\ell \cap \mathbb{Z}^3$. If $\ell \cap \mathbb{Z}^3$ is empty or a single point, then P is a monomial, hence degree 1 by irreducibility. So the only remaining case is if $\ell \cap \mathbb{Z}^3$ is a 1-dimensional lattice, which can be written in the form $\{(a_0, b_0, c_0) + m(a_1, b_1, c_1) \mid m \in \mathbb{Z}\}$.

Thus, we know that the monomials $X^a Y^b Z^c$ that can appear with nonnegative coefficients in P must be in $S = \{(a_0, b_0, c_0) + m(a_1, b_1, c_1) \mid m \in \mathbb{Z}\} \cap \mathbb{Z}_{\geq 0}^3$. If S contains exactly one element, then P is degree one by irreducibility. If S contains exactly two elements, then P is a binomial and must then be of the form $X^a Y^b + k Z^{a+b}$ for some $k \neq 0$, because P is irreducible. Finally, one can check S cannot contain three or more elements assuming P is irreducible.

Case 2: A has exactly two Jordan blocks Let λ_1 be the eigenvalue of the 2×2 block and λ_2 be the eigenvalue of the 1×1 block. We will show that the first case of Proposition 4.4 happens. We can assume C_0 is not contained in a coordinate line, and therefore assume $\tilde{p}$ is such that $c_2, c_3 \neq 0$. Then, a solution to (4-1) is $(X(t) \ Y(t) \ Z(t))^T = (e^{\lambda_1 t} c_1 + c_2 t e^{\lambda_1 t} \ e^{\lambda_1 t} c_2 \ e^{\lambda_2 t} c_3)^T$ for $\tilde{p} = (c_1, c_2, c_3)$.

This means $P(e^{\lambda_1 t} c_1 + c_2 t e^{\lambda_1 t}, e^{\lambda_1 t} c_2, e^{\lambda_2 t} c_3) = 0$. Dividing by $e^{\deg(P) \lambda_1 t}$ and letting $\lambda = \lambda_2 - \lambda_1$, we find

$$P(c_1 + c_2 t, c_2, e^{\lambda t} c_3) = 0.$$

Reparameterizing t by $t - \frac{c_1}{c_2}$, we can assume $c_1 = 0$. We claim now that the map $\mathbb{C}[X, Y, Z] \to \mathbb{C}[\![t]\!]$ sending $P(X, Y, Z)$ to $P(c_2 t, c_2, e^{\lambda t} c_3)$ is an injection because $c_2 t, c_2, e^{\lambda t} c_3$ are algebraically independent. The latter claim follows from the fact that the functions $\{t^m e^{\omega t} \mid m \in \mathbb{Z}_{\geq 0}, \ \omega \in \mathbb{C}\}$ are linearly independent. Therefore, $P = 0$, i.e., C_0 must be contained in either the $\{Y = 0\}$ or $\{Z = 0\}$ coordinate lines, establishing this case.

Case 3: A has exactly one Jordan block Let λ be the unique eigenvalue of A. Subtracting a diagonal matrix from A is equivalent to subtracting the Euler vector field $X \frac{\partial}{\partial X} + Y \frac{\partial}{\partial Y} + Z \frac{\partial}{\partial Z}$ from the vector field s, so we can assume $\lambda = 0$. Then, a solution to (4-1) is $(X(t) \ Y(t) \ Z(t))^T = (c_1 + c_2 t + c_3 \frac{1}{2} t^2 \ c_2 + c_3 t \ c_3)^T$ for $\tilde{p} = (c_1, c_2, c_3)$. We will show that the second case of Proposition 4.4 happens, so we can assume that $c_3 \neq 0$ or else C_0 is contained in $\{Z = 0\}$.

We know that $P(c_1 + c_2 t + c_3 \frac{1}{2} t^2, c_2 + c_3 t, c_3) = 0$. We change coordinates on t. Letting $t \mapsto t - \frac{c_2}{c_3}$ yields

$$P\left(c_1 + \frac{1}{2} \frac{c_2^2}{c_3} + c_3 \frac{1}{2} t^2, c_3 t, c_3\right) = 0.$$

Dividing out by a power of c_3 and replacing c_1 with another constant c_1', we find

$$P\left(c_1' + \frac{1}{2} t^2, t, 1\right) = 0.$$

As t varies, the curve $(c_1' + \frac{1}{2} t^2, t, 1)$ parameterizes the conic $XZ - \frac{1}{2} Y^2 - c_1' Z^2$ in $\mathbb{P}^2$, settling this case. $\qquad \square$

Proofs of Theorems 4.1 and 4.2. We will prove Theorem 4.2 which implies Theorem 4.1. Let $f : \mathbb{P}^1 \to \mathbb{P}^2$ be a general map of degree e. The log tangent sheaf $T_{\mathbb{P}^2}(-\log C)$ is a vector bundle since it is a reflexive sheaf on a surface. Pulling back $T_{\mathbb{P}^2}(-\log C)$ to $\mathbb{P}^1$ yields a rank-2 vector bundle E of degree $(3 - d)e$. We split our analysis into cases.

Case: $d \geq 5$ or $d = 4$ and $e \geq 2$. If Φ is not of maximal rank, then $E \cong \mathcal{O}(a) \oplus \mathcal{O}(b)$ where $a \geq 0$. Since the total degree of E is at most -2, we get $a - b \geq 2$ and we can apply Proposition 3.1 to find a line subbundle of $T_{\mathbb{P}^2}(-\log C)$ of nonnegative degree. This means $T_{\mathbb{P}^2}(-\log C)$ has a section and we conclude by Proposition 4.4.

Case: $d = 4$ and $e = 1$. If Φ is not of maximal rank, then $h^0(E) \geq 2$. This means $E \cong \mathcal{O}(a) \oplus \mathcal{O}(b)$ where $a \geq 1$. In this case $a - b \geq 3$, so we can apply Proposition 3.1 to find a line subbundle $\mathcal{O}(a)$ of $T_{\mathbb{P}^2}(-\log C)$. Applying Proposition 4.3, we are done.

Case: $d = 3$ **and** $e \geq 2$. In this case, $\deg(E) = 0$ and a dimension count shows that Φ is not of maximal rank whenever $h^0(E) \geq 3$. Hence, we can apply Proposition 3.1 to find a line subbundle of $T_{\mathbb{P}^2}(-\log C)$ of positive degree, so we again conclude using Proposition 4.3.

Case: $d = 3$ **and** $e = 1$. We can find a line ℓ meeting C in three distinct points. This means Φ is automatically surjective, so it is of maximal rank.

Case: $d = 2$. In this case, $\deg(E) = e$ and Φ is not of maximal rank if and only if $E \cong O(a) \oplus O(b)$ where $a \geq e+2$ and $b \leq -2$. Applying Proposition 3.1, we find a line subbundle of $T_{\mathbb{P}^2}(-\log C)$ of degree at least $\left\lceil \frac{e+2}{e} \right\rceil = 2$. However, there are no nontrivial maps $\mathcal{O}(2) \to T_{\mathbb{P}^2}$, showing Φ must have maximal rank. $\square$

5. Hyperplane sections

We let X be a smooth degree d hypersurface in $\mathbb{P}^n$. Using the notation from Section 2B, our objective is to prove that Φ has maximal rank when $k = n - 1$ and $e = 1$. Unlike the plane curve case, we are unable to obtain a complete classification statement like Theorem 4.2. However, we are able to prove that if d is larger than $n + 1$, the hyperplane sections of X vary maximally in moduli. We prove Theorem 1.4 and some generalizations, captured below in Theorems 5.8 and 5.9.

Our results all rely on a stability result from Guenancia [2016]. The following version comes from Guenancia's Theorem A by observing that the canonical bundle of a degree d hypersurface in $\mathbb{P}^n$ is ample when $d \geq n + 2$.

Theorem 5.1 [Guenancia 2016, Theorem A]. *If X is a smooth hypersurface of degree $d \geq n + 2$, then $T_{\mathbb{P}^n}(-\log X)$ is semistable.*

Using Theorem 5.1, the basic strategy is to understand how large the degree d can be such that the restriction of $T_{\mathbb{P}^n}(-\log X)$ to the curve or k-plane can have a section. We use results from Section 3 to do this.

Theorem 5.2. *If X in $\mathbb{P}^n$ is a smooth hypersurface of degree d, then the space of degree e rational curve sections of X vary maximally in modulus when $d > \frac{n(n-1)}{2e} + n + 1$.*

Proof. Consider the bundle $T_{\mathbb{P}^n}(-\log X)$. By Theorem 5.1, this bundle is semistable. For d larger than $n + 1$, we see that a section of this bundle would give a destabilizing subsheaf, so we know that $T_{\mathbb{P}^n}(-\log X)$ has no sections.

Let $\mathcal{M} = \mathrm{Mor}_e(\mathbb{P}^1, \mathbb{P}^n)$ be the space of parameterized degree e rational curves in $\mathbb{P}^n$. Given a choice of F with $X = V(F)$, there is a natural map $\Phi : \mathcal{M} \to H^0(\mathcal{O}_{\mathbb{P}^1}(ed))$ sending a map $f : \mathbb{P}^1 \to \mathbb{P}^n$ to the pullback $f^*F \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(de))$. We know by Lemma 2.2 that the tangent space to the fiber of Φ at a given map $f : \mathbb{P}^1 \to \mathbb{P}^n$ is simply $H^0(f^*T_{\mathbb{P}^n}(-\log X))$. To show that Φ is generically finite, we need only show that $h^0(f^*T_{\mathbb{P}^n}(-\log X)) = 0$.

By Proposition 3.1, we see that $f^*T_{\mathbb{P}^n}(-\log X)$ is a direct sum of line bundles $\bigoplus \mathcal{O}(a_i)$ with consecutive a_i differing by at most 1. Thus, any such bundle on $\mathbb{P}^1$ that has a section will have degree larger than that of the bundle $\mathcal{O} \oplus \mathcal{O}(-1) \oplus \cdots \oplus \mathcal{O}(-n+1)$. From this it follows that any semistable bundle E on $\mathbb{P}^n$ such that f^*E has a section for a general map $f : \mathbb{P}^1 \to \mathbb{P}^n$ will have degree at least $-\frac{n(n-1)}{2}$. Thus, if

$\deg f^* T_{\mathbb{P}^n}(-\log X) < -\frac{n(n-1)}{2}$ then $h^0(f^* T_{\mathbb{P}^n}(-\log X)) = 0$. Since $\deg f^* T_{\mathbb{P}^n}(-\log X) = e(n+1-d)$, the result follows. $\qquad\square$

We now consider k-plane sections of smooth hypersurfaces. By Proposition 3.3, we need to understand torsion free sheaves on $\mathbb{P}^k$ whose Harder–Narasimhan filtration has subquotients whose slopes do not decrease too quickly, namely $\mu_1 > \mu_2 > \cdots > \mu_a$ with $\mu_i - \mu_{i+1} \le \frac{1}{k}$ for all i. Understanding the possible slopes that may appear in the Harder–Narasimhan filtration is an interesting combinatorial problem, which we describe below.

Definition 5.3. Let a sequence $(d_1, r_1), (d_2, r_2), \ldots, (d_a, r_a)$ in $\mathbb{Z}_{\ge 0} \times \mathbb{Z}_{>0}$ be *k-admissible* if $d_1 \le 0$ and $0 \le \frac{d_{i+1}}{r_{i+1}} - \frac{d_i}{r_i} \le \frac{1}{k}$ for each i. Let $A_{k,n}$ denote the set of k-admissible sequences with $\sum_i r_i = n$ (where a is arbitrary).

Definition 5.4. Define $B_k(n)$ to be $\max\left\{ \sum_{i=1}^a d_i \mid (d_1, r_1), \ldots, (d_a, r_a) \text{ in } A_{k,n} \right\}$.

Lemma 5.5. *If E is a semistable sheaf on $\mathbb{P}^n$ of rank n such that its restriction to a general k-plane has a section, then $\deg E \ge -B_k(n)$.*

Proof. Let Λ be a general k-plane and $0 = E_0 \subset E_1 \subset \cdots \subset E_a = E|_\Lambda$ be the Harder–Narasimhan filtration of $E|_\Lambda$. Let $-d_i$ be the degree of E_i/E_{i-1} and r_i be the rank of E_i/E_{i-1}. Since $E|_\Lambda$ has a section, we see that $d_1 \le 0$. Since E is semistable, by Proposition 3.3 it follows that the sequence $(-d_1, r_1), \ldots, (-d_a, r_a)$ will be k-admissible. The result follows. $\qquad\square$

We can compute a bound for when k-plane sections of a degree d hypersurface in $\mathbb{P}^n$ will vary maximally in moduli in terms of $B_k(n)$.

Theorem 5.6. *Let X be a smooth, degree d hypersurface in $\mathbb{P}^n$ with $d > B_k(n) + n + 1$. Then,*

$$\Phi : \operatorname{Mor}_1(\mathbb{P}^k, \mathbb{P}^n) \dashrightarrow \mathbb{P}H^0(\mathbb{P}^k, \mathscr{O}_{\mathbb{P}^k}(d)), \quad \iota \mapsto [\iota^{-1}(X)],$$

is of maximal rank.

Proof. By Theorem 5.1, $T_{\mathbb{P}^n}(-\log X)$ will be semistable. By Proposition 3.3, $T_{\mathbb{P}^n}(-\log X)|_\Lambda$ will have Harder–Narasimhan filtration as described in the statement of the theorem. Given a hypersurface X together with a choice of defining equation f, we get a map $\phi : \operatorname{Mor}_1(\mathbb{P}^k, \mathbb{P}^n) \to H^0(\mathscr{O}_{\mathbb{P}^k}(d))$ sending a k-plane to the pull-back of f by the k-plane. We wish to show that ϕ is generically finite.

To get a contradiction, suppose ϕ has only positive-dimensional fibers. By Lemma 2.2, the tangent space to a fiber of ϕ at a general point Λ is $H^0(T_{\mathbb{P}^n}(-\log X)|_\Lambda)$, so we know that $T_{\mathbb{P}^n}(-\log X)|_\Lambda$ has a global section. Thus, by Theorem 5.1 and Lemma 5.5, the degree of $T_{\mathbb{P}^n}(-\log X)$ will be at least $-B_k(n)$. It follows that

$$n + 1 - d \ge -B_k(n).$$

This is impossible given the assumptions in the statement of the theorem. $\qquad\square$

Then, Theorem 1.4 follows from the following result on $B_k(n)$.

Proposition 5.7. *If $k \ge \frac{2n}{3}$, then $B_k(n) = 1$.*

Proof. Let $(d_1, r_1), \ldots, (d_a, r_a)$ be an admissible sequence of total degree $B_k(n)$. Without loss of generality, we may assume $d_1 = 0$, $d_i > 0$ for $i > 1$. Then it follows that $r_2 \geq k$, since $\frac{d_2}{r_2} \leq \frac{1}{k}$. Since $\frac{d_3}{r_3} \leq \frac{2}{k}$, we see that $r_3 \geq \frac{k}{2}$, provided that there are at least three terms in the sequence. However, in this case, $r_1 + r_2 + r_3 \geq 1 + k + \frac{k}{2} = 1 + \frac{3k}{2} > n$, which is impossible. Thus, $a \leq 2$.

Next, we observe that $d_2 \leq 1$, since if $d_2 \geq 2$, then $r_2 \geq d_2 k \geq 2k > n$, a contradiction. It follows that the sum of the d_i is at most 1, and since we know that 1 is achievable with the admissible sequence $(0, n - k), (1, k)$, the result follows. $\qquad\square$

We defer more detailed analysis of $B_k(n)$ to the Appendix. From the results in the Appendix and Theorem 5.6 we get the following results.

Theorem 5.8. *If $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree d with $d > 4\left(\frac{n^2}{k^{3/2}} + k^{3/2}\right)$, then the map*

$$\Phi : \mathrm{Mor}_1(\mathbb{P}^k, \mathbb{P}^n) \dashrightarrow \mathbb{P}H^0(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(d)), \quad \iota \mapsto [\iota^{-1}(X)],$$

is of maximal rank.

Proof. This follows from Proposition A.1, where it is shown $B_k(n) \leq 3\left(\frac{n^2}{k^{3/2}} + k^{3/2}\right)$. To finish, one checks that $\frac{n^2}{k^{3/2}} + k^{3/2} \geq 2n \geq n + 2$. This follows from the AM-GM equality and the fact $n \geq 2$. $\qquad\square$

In Theorem 5.8, we prioritized giving a clean statement and proof over giving an optimal constant. Still, one can wonder what the optimal constant by computing $B_k(n)$ for small k and all n. In this case, Corollary A.5 gives the following result:

Theorem 5.9. *If $k \leq 5$, then there is a linear function $\ell(n)$ and an integer C_k such that $|B_k(n) - \frac{n^2}{C_k}| \leq \ell(n)$. Here, $C_2 = 3$, $C_3 = 7$, $C_4 = 11$, $C_5 = 19$. In particular, the map*

$$\Phi : \mathrm{Mor}_1(\mathbb{P}^k, \mathbb{P}^n) \dashrightarrow \mathbb{P}H^0(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(d)), \quad \iota \mapsto [\iota^{-1}(X)],$$

is of maximal rank if $X \subset \mathbb{P}^n$ is smooth and has degree $d \geq C_k n^2 + \ell(n) + n + 2$.

We expect Theorem 5.9 to hold for all values of k, but we can only check a finite number of cases with a computer. Roughly up to $k = 100$ is what is reasonable with our methods.

Given Theorem 5.9, one can ask how fast C_k grows with k. We trivially know $C_k = O(k^2)$ by relaxing the condition that the d_i are integers in the definition of an admissible sequence to compute $B_k(n)$ (in which case we let all the r_i be equal to 1). We also get $C_k = \Omega(k^{3/2})$ from Proposition A.1. From experimental evidence, we think that the actual answer is strictly between $k^{3/2}$ and k^2 but closer to $k^{3/2}$.

Appendix: Bounds and computations for $B_k(n)$

We will bound $B_k(n)$ for all k, n in Proposition A.1. We also compute $B_k(n)$ for k small and arbitrary n, and give some conjectures about $B_k(n)$ in general.

To give an idea of how the function $B_k(n)$ behaves, we note the results in the Appendix can show $B_5(39) = 39$, corresponding to the admissible sequence

$$(0, 1), (1, 5), (1, 3), (1, 2), (2, 3), (4, 5), (1, 1), (6, 5), (4, 3), (3, 2), (5, 3), (9, 5), (2, 1).$$

There are a couple of features of this admissible sequence we believe hold in general that we will only prove in special cases. First, this admissible sequence can be generated greedily, where we use greed to maximize the ratio $\frac{d}{r}$ of the last piece of the sequence. Second, the admissible sequence is essentially periodic in that the (1, 1), (6, 5), (4, 3), (3, 2), (5, 3), (9, 5) is obtained from (0, 1), (1, 5), (1, 3), (1, 2), (2, 3), (4, 5) by replacing each (d, r) with $(d + r, r)$. We give a finite criterion that can be applied to show both the greedy property and the periodicity in Lemma A.4.

We expect there are many other interesting patterns that can be found. For example, the segment (0, 1), (1, 5), (1, 3), (1, 2), (2, 3), (4, 5), (1, 1) of the admissible sequence above is preserved under reversing the order and replacing each (d, r) with $(r - d, r)$. This pattern continues to hold for larger k and suggests that these optimal admissible sequences can also be generated greedily backwards as well as forwards.

Proposition A.1. *We have $B_k(n) \leq 3\left(\frac{n^2}{k^{3/2}} + k^{3/2}\right)$*

Proof. Let $(d_1, r_1), \ldots, (d_a, r_a)$ be an admissible sequence with $\sum_i d_i = B_k(n)$. Let $\mu_i = \frac{d_i}{r_i}$. Let $n(j)$ be the sum of the r_i such that $\mu_i \in [j - 1, j)$.

Since the μ_i contributing to $n(j)$ are all less than j, we observe that

$$B_k(n) \leq \sum_{j=1}^{\infty} jn(j).$$

Thus, understanding the $n(j)$ allows us to bound $B_k(n)$. The sum of all of the $n(j)$ is n. Let J be the last nonzero $n(j)$, so $B_k(n) \leq \sum_{j=1}^{J} jn(j)$.

Let $n_k^{\min}$ be a positive number that is at most $n(j)$ for any $j < J$. Then we obtain an upper bound for $B_k(n)$

$$B_k(n) \leq \sum_{i=1}^{J} jn(j) \leq n_k^{\min} + 2n_k^{\min} + \cdots + \left\lceil \frac{n}{n_k^{\min}} \right\rceil n_k^{\min}$$

$$= n_k^{\min} \frac{\left\lceil \frac{n}{n_k^{\min}} \right\rceil \left(1 + \left\lceil \frac{n}{n_k^{\min}} \right\rceil\right)}{2}$$

$$\leq n_k^{\min} \frac{\left(\frac{n}{n_k^{\min}} + 1\right)\left(\frac{n}{n_k^{\min}} + 2\right)}{2}.$$

Thus, it remains to give a bound for $n_k^{\min}$. Fix $j < J$ and let $(d_{i_j+1}, r_{i_j+1}), \ldots, (d_{i_j+c(j)}, r_{i_j+c(j)})$ be the part of the admissible sequence with slopes $\frac{d_{i_j+1}}{r_{i_j+1}}, \ldots, \frac{d_{i_j+c(j)}}{r_{i_j+c(j)}}$ in $[j - 1, j)$. By definition,

$$r_{i_j+1} + \cdots + r_{i_j+c(j)} = n(j).$$

First, we show $c(j) \geq k$: Note that $\frac{d_{i_j+1}}{r_{i_j+1}} < (j - 1) + \frac{1}{k}$. If $j = 1$, then this is true because $\frac{d_{i_j+1}}{r_{i_j+1}} \leq 0$ by definition. If $j > 1$, then this is true because $\frac{d_{i_j+1}}{r_{i_j+1}} \leq \frac{d_{i_j}}{r_{i_j}} + \frac{1}{k} < (j - 1) + \frac{1}{k}$.

Since

$$\frac{d_{i_j+1}}{r_{i_j+1}} < (j - 1) + \frac{1}{k},$$

we then see that

$$\frac{d_{i_j+2}}{r_{i_j+2}} \le \frac{d_{i_j+1}}{r_{i_j+1}} + \frac{1}{k} < (j-1) + \frac{2}{k}$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{d_{i_j+k}}{r_{i_j+k}} \le \frac{d_{i_j+k-1}}{r_{i_j+k-1}} + \frac{1}{k} < j,$$

so $c(j) \ge k$.

Now, we want to bound $r_{i_j+1} + \cdots + r_{i_j+c(j)} = n(j)$. In the multiset $\{r_{i_j+1}, \ldots, r_{i_j+c(j)}\}$, we know that there is at most element that is equal to 1, fewer than two elements that are equal to 2, fewer than three elements that are equal to 3 and so on. Therefore, if m is the largest integer such that $1 + (1 + \cdots + m - 1) = 1 + \frac{m(m-1)}{2}$ is at most k, then $\frac{(m-1)^2}{2} < \frac{m(m_1)}{2} + 1 \le k$, so $m \le \sqrt{2k} + 1$. Thus,

$$n(j) = r_{i_j+1} + \cdots + r_{i_j+c(j)} \ge 1 + (2 \cdot 1 + 3 \cdot 2 + \cdots m \cdot (m-1))$$

$$= 1 + 2\left(\binom{2}{2} + \binom{3}{2} + \cdots + \binom{m}{2}\right)$$

$$= 1 + \frac{(m+1)m(m-1)}{3}$$

$$\ge \frac{(\sqrt{2k}+2)(\sqrt{2k}+1)\sqrt{2k}}{3} > \frac{2\sqrt{2}}{3} k^{3/2}.$$

Thus, choosing $n_k^{\min}$ to be $\frac{2\sqrt{2}}{3} k^{3/2}$ suffices. Plugging into our earlier bound, we get an upper bound for $B_k(n)$ as

$$n_k^{\min} \frac{\left(\frac{n}{n_k^{\min}} + 1\right)\left(\frac{n}{n_k^{\min}} + 2\right)}{2} = \frac{n^2}{2n_k^{\min}} + \frac{3n}{2} + n_k^{\min} = \frac{n^2}{k^{3/2}} \frac{9}{4\sqrt{2}} + \frac{3n}{2} + \frac{2\sqrt{2}}{3} k^{3/2},$$

which is at most $2\left(\frac{n^2}{k^{3/2}} + n + k^{3/2}\right)$. Applying the AM-GM inequality yields

$$\frac{n^2}{k^{3/2}} + k^{3/2} \ge 2n,$$

yielding the claimed bound. $\qquad\square$

We now move on to computing exact values of $B_k(n)$ for small k. Our strategy is a recursive algorithm that requires some conditions to be met, and we suspect that these conditions are always met. In the course of our proof, we will use the three quantities $\mu^{\max}(n)$, $B_k^{\text{upper}}(n)$ and $B_k^{\text{lower}}(n)$. We define $\mu^{\max}(n)$ by

$$\mu^{\max}(n) := \max\left\{\frac{r_a}{d_a} \,\Big|\, (d_1, r_1), \ldots, (d_a, r_a) \text{ in } A_n\right\}.$$

Lemma A.2. *We can compute $\mu^{\max}(n)$ inductively by $\mu^{\max}(1) = 0$ and*

$$\mu^{\max}(n) = \max\left\{\frac{\lfloor(\mu^{\max}(i) + \frac{1}{k})(n-i)\rfloor}{n-i} \,\Big|\, 0 < i < n\right\}.$$

Proof. Let

$$\mu_n^{\text{max}'} = \max\left\{\frac{\lfloor(\mu^{\text{max}}(i) + \frac{1}{k})(n - i)\rfloor}{n - i} \,\middle|\, 0 < i < n\right\}.$$

We use induction. The base case $n = 1$ is vacuous, so assume $n > 1$. First, we show that $\mu_n^{\text{max}'} \le \mu_n^{\text{max}}$. Given any $0 < i < n$, $\frac{\lfloor(\mu_i^{\text{max}'} + \frac{1}{k})(n - i)\rfloor}{n - i}$ is a slope achieved by taking an admissible sequence $(d_1, r_1), \ldots, (d_a, r_a)$ in A_i and appending $(\lfloor(\mu_i^{\text{max}'} + \frac{1}{k})(n - i)\rfloor, n - i)$. So by definition $\mu_n^{\text{max}'} \le \mu_n^{\text{max}}$.

Now we show $\mu_n^{\text{max}'} \ge \mu_n^{\text{max}}$. Let $(d_1, r_1), \ldots, (d_a, r_a)$ be an admissible sequence in A_n achieving $\frac{d_a}{r_a} = \mu_n^{\text{max}}$. If $a = 1$, then $d_a = 0$ so $\mu_n^{\text{max}} = 0$ while $\mu_n^{\text{max}'}$ is by definition nonnegative. Otherwise, let $i = r_1 + \cdots + r_{a-1}$, so $\frac{d_{a-1}}{r_{a-1}} \le \mu_i^{\text{max}} = \mu_i^{\text{max}'}$ by definition and the assumption hypothesis. Then, $r_a = n - i$ and

$$\frac{d_a}{r_a} \le \mu_i^{\text{max}'} + \frac{1}{k},$$

so $d_a \le \lfloor(\mu_i^{\text{max}'} + \frac{1}{k})(n - i)\rfloor$. Then, by definition $\mu_n^{\text{max}'} \ge \mu_n^{\text{max}}$. Therefore, we are done and $\mu_n^{\text{max}'} = \mu_n^{\text{max}}$ for all n. $\square$

We define $B_k^{\text{upper}}(n)$ recursively by $B_k^{\text{upper}}(1) = 0$ and

$$B_k^{\text{upper}}(n) = \max\left\{B_k^{\text{upper}}(i) + \left\lfloor\left(\mu_i^{\text{max}} + \frac{1}{k}\right)(n - i)\right\rfloor \,\middle|\, 0 < i < n\right\}.$$

For B_k^{lower}, we let $B_k^{\text{lower}}(1) = 0$ and let $i(n)$ be the smallest i that maximizes $\frac{\lfloor(\mu^{\text{max}}(i) + \frac{1}{k})(n - i)\rfloor}{n - i}$. Then define B_k^{lower} inductively by

$$B_k^{\text{lower}}(n) = B_k^{\text{lower}}(i(n)) + \left\lfloor\left(\mu^{\text{max}}(i(n)) + \frac{1}{k}\right)(n - i(n))\right\rfloor.$$

We now show that $B_k(n)$ is bounded by $B_k^{\text{upper}}(n)$ and $B_k^{\text{lower}}(n)$.

Lemma A.3. *We have $B_k^{\text{lower}}(n) \le B_k(n) \le B_k^{\text{upper}}(n)$.*

Proof. First we show $B_k^{\text{lower}}(n) \le B_k(n)$ by induction. To do this, we show by induction that $B_k^{\text{lower}}(n)$ is always achieved by an admissible sequence $(d_1, r_1), \ldots, (d_a, r_a)$ with $\frac{d_a}{r_a} = \mu^{\text{max}}(n)$ and $d_1 + \cdots + d_a = B_k^{\text{lower}}(n)$. The base case $n = 1$ vacuous, so we assume $n > 1$. Let i be the minimal index maximizing

$$\frac{\lfloor(\mu_i^{\text{max}} + \frac{1}{k})(n - i)\rfloor}{n - i}.$$

By the induction assumption, there is an admissible sequence $(d_1, r_1), \ldots, (d_{a-1}, r_{a-1})$ achieving $\frac{d_{a-1}}{r_{a-1}} = \mu_i^{\text{max}}$ and $d_1 + \cdots + d_{a-1} = B_k^{\text{lower}}(i)$. By appending $(\lfloor(\mu_i^{\text{max}} + \frac{1}{k})(n - i)\rfloor, n - i)$ to the sequence we get an admissible sequence $(d_1, r_1), \ldots, (d_a, r_a)$ with $\frac{d_a}{r_a} = \mu_n^{\text{max}}$ and $d_1 + \cdots + d_a = B_k^{\text{lower}}(n)$.

Finally, we show $B_k^{\text{upper}}(n) \ge B_k(n)$ by induction. The base case $n = 1$ is vacuous, so assume $n > 1$. Let $(d_1, r_1), \ldots, (d_a, r_a)$ be an admissible sequence in A_n achieving $d_1 + \cdots + d_a = B_k(n)$. If $a = 1$, then $B_k(n) = 0$ and $B_k^{\text{upper}}(n)$ is always nonnegative by definition. If $a > 1$, then let $i = r_1 + \cdots + r_{a-1}$ so

$(d_1, r_1), \ldots, (d_{a-1}, r_{a-1})$ is an admissible sequence in A_i. By the inductive hypothesis, $d_1 + \cdots + d_{a-1} \leq B_k^{\text{upper}}(i)$. We have $r_a = n - i$ and the maximum d_a can be is $\lfloor (\mu^{\text{max}}(i) + \frac{1}{k})(n - i) \rfloor$. Therefore,

$$d_1 + \cdots + d_a \leq B_k^{\text{upper}}(i) + \left\lfloor \left(\mu^{\text{max}}(i) + \frac{1}{k} \right)(n - i) \right\rfloor \leq B_k^{\text{upper}}(n),$$

finishing the proof.   □

From experimental evidence, we suspect $B_k^{\text{lower}}(n)$ and $B_k^{\text{upper}}(n)$ always coincide, which would give a recursive algorithm for $B_k(n)$. However, to give results for small values of k and all n, we want to have a finite criterion that can be verified by a computer. We believe admissible sequences achieving $B_k(n)$ will always following a periodic structure in n with k fixed reflected in Lemma A.4 below.

Lemma A.4. *Suppose $\mu^{\text{max}}(i_0) = \frac{k-1}{k}$ for some i_0. Then $\mu^{\text{max}}(n) = \mu^{\text{max}}(n - i_0) + 1$ all $n \geq i_0$. If in addition $B_k^{\text{upper}}(i) = B_k^{\text{lower}}(i)$ for each $i \leq 3i_0$, then $B_k^{\text{lower}}(n) = B_k(n) = B_k^{\text{upper}}(n)$ for all n.*

Using Lemma A.4, one can show $B_k(n + i_0) = B_k(n) + n + B_k(i_0)$. Iterating this shows

$$B_k(n + Ni_0) = B_k(n) + nN + NB_k(i_0) + \frac{N(N-1)i_0}{2}$$

for $1 < n \leq i_0$ and $N \geq 0$. In particular, $B_k(n) = \Theta\left(\frac{1}{i_0} N^2 \right)$.

Proof. First note that if $\mu^{\text{max}}(i) = m + \frac{k-1}{k}$ for m an integer, then

$$\mu^{\text{max}}(i + 1) = \max_j \left\{ \frac{\lfloor (\mu^{\text{max}}(j) + \frac{1}{k})(i + 1 - j) \rfloor}{i + 1 - j} \,\Big|\, 0 < j \leq i \right\} \tag{A-1}$$

$$= \left\lfloor m + \frac{k-1}{k} + \frac{1}{k} \right\rfloor = m + 1. \tag{A-2}$$

In particular, there is a unique i_0 for which $\mu^{\text{max}}(i_0) = \frac{k-1}{k}$ and $\mu^{\text{max}}(i_0 + 1) = 1 = 1 + \mu^{\text{max}}(1)$.

We will now show $\mu^{\text{max}}(n) = \mu^{\text{max}}(n - i_0) + 1$ for all $n \geq i_0$ using induction on n. For the case $n = i_0 + 1$, $\mu^{\text{max}}(i_0 + 1) = 1$ from above.

Now suppose $n > i_0 + 1$. We first note that

$$\mu^{\text{max}}\left(i_0 \left\lfloor \frac{n-1}{i_0} \right\rfloor \right) = \left(\left\lfloor \frac{n-1}{i_0} \right\rfloor - 1 \right) + \frac{k-1}{k}$$

is $\frac{k-1}{k}$ by induction.

Now, we claim that $\mu^{\text{max}}(i) < \mu^{\text{max}}\left(i_0 \left\lfloor \frac{n-1}{i_0} \right\rfloor \right)$ for all $i < i_0 \left\lfloor \frac{n-1}{i_0} \right\rfloor$. Since $\mu^{\text{max}}(j)$ is weakly increasing in j, the point is to prove they are not equal. If $\mu^{\text{max}}(i)$ was equal to $\mu^{\text{max}}\left(i_0 \left\lfloor \frac{n-1}{i_0} \right\rfloor \right)$, then $\mu^{\text{max}}(i + 1) = \mu^{\text{max}}(i)$, which contradicts (A-1).

By Lemma A.2, $\mu^{\text{max}}(n)$ will be determined by the i between 0 and n such that $\frac{\lfloor (\mu^{\text{max}}(i) + \frac{1}{k})(n-i) \rfloor}{n-i}$ is maximized. Let $i(n)$ be this i. We claim $i(n) \geq i_0 \left\lfloor \frac{n-1}{i_0} \right\rfloor$. To get a contradiction, suppose that $i(n) < i_0 \left\lfloor \frac{n-1}{i_0} \right\rfloor$.

Since we have shown above that $\mu^{\max}(i(n)) < \mu^{\max}\left(i_0 \left\lfloor \frac{n-1}{i_0} \right\rfloor\right)$,

$$\mu^{\max}(i(n)) + \frac{1}{k} < \mu^{\max}\left(i_0 \left\lfloor \frac{n-1}{i_0} \right\rfloor\right) + \frac{1}{k} = \left\lfloor \frac{n-1}{i_0} \right\rfloor,$$

contradicting $i(n) < i_0 \left\lfloor \frac{n-1}{i_0} \right\rfloor$.

Since $i(n) \geq i_0 \left\lfloor \frac{n}{i_0} \right\rfloor$,

$$
\begin{aligned}
\mu^{\max}(n) &= \frac{\left\lfloor \left(\mu^{\max}(i(n)) + \frac{1}{k}\right)(n - i(n)) \right\rfloor}{n - i(n)} \\
&= \frac{\left\lfloor \left(\mu^{\max}(i(n) - i_0) + 1 + \frac{1}{k}\right)(n - i(n)) \right\rfloor}{n - i(n)} \\
&= \frac{\left\lfloor \left(\mu^{\max}(i(n) - i_0) + \frac{1}{k}\right)((n - i_0) - (i(n) - i_0)) \right\rfloor}{(n - i_0) - (i(n) - i_0)} + 1 \\
&= \max_j \left\{ \frac{\left\lfloor \left(\mu^{\max}(j) + \frac{1}{k}\right)(n - i_0 - j) \right\rfloor}{n - i_0 - j} \,\middle|\, 0 < j \leq n - i_0 \right\} + 1 \\
&= \mu^{\max}(n - i_0) + 1,
\end{aligned}
$$

where the second and fourth line are by induction and the fifth line is by definition. This concludes our induction for $\mu^{\max}$. From our proof, we also see that

$$i(n) - i_0 = i(n - i_0). \tag{A-3}$$

Next, we want to show the statement regarding $B_k(n)$. It suffices to show that $B_k^{\text{lower}}(n) = B_k^{\text{upper}}(n)$ for all n. We will show this by induction and can assume $n > 3i_0$ and $B_k^{\text{lower}}(i) = B_k^{\text{upper}}(i)$ for all $0 < i < n$. As before, let $i(n)$ be the minimum i that maximizes $\frac{\left\lfloor (\mu^{\max}(i) + \frac{1}{k})(n - i) \right\rfloor}{n - i}$. By definition, we want to show

$$B_k(i(n)) + \left\lfloor \left(\mu^{\max}(i(n)) + \frac{1}{k}\right)(n - i(n)) \right\rfloor = \max\left\{ B_k(i) + \left\lfloor \left(\mu^{\max}(i) + \frac{1}{k}\right)(n - i) \right\rfloor \,\middle|\, 0 < i < n \right\}. \tag{A-4}$$

The inequality $\leq$ is clear as the left side is one of the terms on the right side. Let i' be an index maximizing the right side. We want to show that $i' > i_0$. If $i' \leq i_0$, then $B_k^{\text{upper}}(i') + \left\lfloor \left(\mu^{\max}(i') + \frac{1}{k}\right)(n - i') \right\rfloor$ is less than n as $\mu_{i'}^{\max} + \frac{1}{k} \leq 1$ and $B^{\text{upper}}(i') \leq \mu^{\max}(i')i' < i'$. We also crudely bound $B_k^{\text{lower}}(n)$ from below.

To do so, we first bound $B_k^{\text{lower}}(3i_0)$ by $3i_0$. From the statement of Lemma A.4 regarding $\mu^{\max}$, we know $\mu^{\max}(j) \geq m$ for all $i > m \cdot i_0$. Then,

$$B_k^{\text{lower}}(n) \geq 0 \cdot i_0 + 1 \cdot i_0 + 2 \cdot i_0 + 3(n - 3i_0).$$

Since $n > 3i_0$,

$$3i_0 + 3(n - 3i_0) = 2(n - 3i_0) + n > n,$$

yielding a contradiction.

Since $i' > i_0$, the right side of (A-4) is

$$\max\left\{ B_k(i) + \left\lfloor \left(\mu^{\max}(i) + \frac{1}{k} \right)(n-i) \right\rfloor \,\Big|\, 0 < i < n \right\}$$

$$= \max\left\{ B_k(i) + \left\lfloor \left(\mu^{\max}(i) + \frac{1}{k} \right)(n-i) \right\rfloor \,\Big|\, i_0 < i < n \right\}$$

$$= \max\left\{ B_k(i+i_0) + \left\lfloor \left(\mu^{\max}(i+i_0) + \frac{1}{k} \right)(n-i-i_0) \right\rfloor \,\Big|\, 0 < i < n - i_0 \right\}$$

$$= \max\left\{ B_k(i) + i + B_k(i_0) + \left\lfloor \left(\mu^{\max}(i) + 1 + \frac{1}{k} \right)(n-i-i_0) \right\rfloor \,\Big|\, 0 < i < n - i_0 \right\}$$

$$= \max\left\{ B_k(i) + \left\lfloor \left(\mu^{\max}(i) + \frac{1}{k} \right)(n-i-i_0) \right\rfloor \,\Big|\, 0 < i < n - i_0 \right\} + n - i_0 + B_k(i_0).$$

But

$$\max\left\{ B_k(i) + \left\lfloor \left(\mu^{\max}(i) + \frac{1}{k} \right)(n-i-i_0) \right\rfloor \,\Big|\, 0 < i < n - i_0 \right\} = B(n - i_0)$$

by induction. Looking at the left side of (A-4), we get

$$B_k(i(n)) + \left\lfloor \left(\mu^{\max}(i(n)) + \frac{1}{k} \right)(n-i(n)) \right\rfloor$$

$$= B_k(i(n) - i_0) + (i(n) - i_0) + B_k(i_0) + \left\lfloor \left(\mu^{\max}(i(n) - i_0 + i_0) + \frac{1}{k} \right)(n-i(n)) \right\rfloor$$

$$= B_k(i(n) - i_0) + (i(n) - i_0) + B_k(i_0) + \left\lfloor \left(\mu^{\max}(i(n) - i_0) + 1 + \frac{1}{k} \right)(n-i(n)) \right\rfloor$$

$$= B_k(i(n) - i_0) + \left\lfloor \left(\mu^{\max}(i(n) - i_0) + \frac{1}{k} \right)(n - i_0 - (i(n) - i_0) \right\rfloor + n - i_0 + B_k(i_0)$$

$$= B_k(n - i_0) + B_k(i_0) + n - i_0,$$

where the last line is by (A-3). Therefore, both sides of (A-4) are equal, which is what we wanted. □

We can verify the conditions of Lemma A.4 using a Python program for small k. For example, the answer for $k = 2, 3, 4, 5$ are given below.

Corollary A.5. *We have the following closed-form expressions for $B_k(n)$ for $k = 2, 3, 4, 5$. For $k = 2$ and $n \geq 0$,*

$$B_2(3n+1) = \frac{3n^2+n}{2}, \quad B_2(3n+2) = \frac{3n^2+3n}{2}, \quad B_2(3n+3) = \frac{3n^2+5n+2}{2}.$$

For $k = 3$,

$$B_3(7n+1) = \frac{7n^2+n}{2}, \qquad B_3(7n+2) = \frac{7n^2+3n}{2}, \qquad B_3(7n+3) = \frac{7n^2+5n}{2},$$

$$B_3(7n+4) = \frac{7n^2+7n+2}{2}, \quad B_3(7n+5) = \frac{7n^2+9n+2}{2}, \quad B_3(7n+6) = \frac{7n^2+11n+4}{2},$$

$$B_3(7n+7) = \frac{7n^2+13n+6}{2}.$$

For $k = 4$,

$$B_4(11n+1) = \frac{11n^2+n}{2}, \qquad B_4(11n+2) = \frac{11n^2+3n}{2}, \qquad B_4(11n+3) = \frac{11n^2+5n}{2},$$

$$B_4(11n+4) = \frac{11n^2+7n}{2}, \qquad B_4(11n+5) = \frac{11n^2+9n+2}{2}, \qquad B_4(11n+6) = \frac{11n^2+11n+2}{2},$$

$$B_4(11n+7) = \frac{11n^2+13n+4}{2}, \qquad B_4(11n+8) = \frac{11n^2+15n+4}{2}, \qquad B_4(11n+9) = \frac{11n^2+17n+6}{2},$$

$$B_4(11n+10) = \frac{11n^2+19n+8}{2}, \qquad B_4(11n+11) = \frac{11n^2+21n+10}{2}.$$

For $k = 5$,

$$B_5(19n+1) = \frac{19n^2+n}{2}, \qquad B_5(19n+2) = \frac{19n^2+3n}{2}, \qquad B_5(19n+3) = \frac{19n^2+5n}{2},$$

$$B_5(19n+4) = \frac{19n^2+7n}{2}, \qquad B_5(19n+5) = \frac{19n^2+9n}{2}, \qquad B_5(19n+6) = \frac{19n^2+11n+2}{2},$$

$$B_5(19n+7) = \frac{19n^2+13n+2}{2}, \qquad B_5(19n+8) = \frac{19n^2+15n+2}{2}, \qquad B_5(19n+9) = \frac{19n^2+17n+4}{2},$$

$$B_5(19n+10) = \frac{19n^2+19n+4}{2}, \qquad B_5(19n+11) = \frac{19n^2+21n+6}{2}, \qquad B_5(19n+12) = \frac{19n^2+23n+6}{2},$$

$$B_5(19n+13) = \frac{19n^2+25n+8}{2}, \qquad B_5(19n+14) = \frac{19n^2+27n+10}{2}, \qquad B_5(19n+15) = \frac{19n^2+29n+10}{2},$$

$$B_5(19n+16) = \frac{19n^2+31n+12}{2}, \qquad B_5(19n+17) = \frac{19n^2+33n+14}{2}, \qquad B_5(19n+18) = \frac{19n^2+35n+16}{2},$$

$$B_5(19n+19) = \frac{19n^2+37n+18}{2}.$$

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