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We construct scattering diagrams for Chekhov–Shapiro generalized cluster algebras where exchange polynomials are factorized into binomials, generalizing the cluster scattering diagrams of Gross, Hacking, Keel and Kontsevich. They turn out to be natural objects arising in Fock and Goncharov's cluster duality. Analogous features and structures (such as positivity and the cluster complex structure) in the ordinary case also appear in the generalized situation. With the help of these scattering diagrams, we show that generalized cluster variables are theta functions and hence have certain positivity property with respect to the coefficients in the binomial factors.

1. Introduction

We study generalized cluster algebras in the sense of [Chekhov and Shapiro 2014]. These algebras are generalizations of the (ordinary) cluster algebras introduced by Fomin and Zelevinsky [2002], allowing more general exchange polynomials (as opposed to only binomials) in mutations.

We will see that generalized cluster algebras cannot only be studied in a similar way as cluster algebras [Fomin and Zelevinsky 2002; 2003; 2007; Berenstein et al. 2005], but that they also naturally appear in the context of the *cluster duality* proposed by Fock and Goncharov [2009]. A modified version of Fock and Goncharov's cluster duality was formulated and proved by Gross, Hacking, Keel and Kontsevich [Gross et al. 2018]. In this paper, we extend the scheme therein to study generalized cluster algebras.

Generalized cluster algebras come in a family containing ordinary cluster algebras. Each algebra in this family can be viewed as (a subalgebra of) the algebra of regular functions of a generalized $\mathcal{A}$-cluster variety. The (generalized version of) cluster duality says this family is in a sense dual to another family of generalized $\mathcal{X}$-cluster varieties. In this paper, we demonstrate this duality by reconstructing a family

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of generalized cluster algebras with principal coefficients $\mathscr{A}^{\mathrm{prin}}$ from a general fiber of the corresponding dual family of $\mathcal{X}$-cluster varieties.

In the ordinary case, the reconstruction is done through a *cluster scattering diagram*, the main technical tool developed in [Gross et al. 2018], which is a mathematical structure associated to the dual $\mathcal{X}$-cluster variety. For our purpose of studying generalized cluster algebras, we construct *generalized cluster scattering diagrams*. This is done by allowing more general wall-crossing functions on the initial incoming walls. It turns out that many features (such as the positivity property of wall-crossings and the cluster complex structure) in the ordinary case still hold in the generalized situation; see Theorems 6.31 and 7.10.

Using the techniques of scattering diagrams (and related objects such as broken lines) transplanted from [Gross et al. 2018], we are able to prove that generalized cluster monomials are theta functions. As a result, they have certain positivity property coming from that of the scattering diagram. We remark that this positivity is with respect to the coefficients appearing in the binomial factors of exchange polynomials, thus weaker than a conjectural positivity of Chekhov and Shapiro (Conjecture 8.13) with respect to the coefficients of exchange polynomials themselves; See Theorem 8.12 and Section 8.5 for the precise statements.

We next describe the contents of the paper in more detail.

1.1. *Generalized cluster algebras.* Our way of generalizing cluster algebras is slightly different from [Chekhov and Shapiro 2014], in the way we deal with coefficients. In a sense, one can go from one formulation to the other, in particular when the coefficients are evaluated in some algebraically closed field; see Sections 3.2, 3.5 and also 8.5. We replace Fomin and Zelevinsky's binomial exchange relation

$$x'_k x_k = p_k^+ \prod_{i=1}^n x_i^{[b_{ik}]_+} + p_k^- \prod_{i=1}^n x_i^{[-b_{ik}]_+}$$

with a more general polynomial exchange relation

$$x'_k x_k = \prod_{j=1}^{r_k} \left(p_{k,j}^+ \prod_{i=1}^n x_i^{[b_{ik}/r_k]_+} + p_{k,j}^- \prod_{i=1}^n x_i^{[-b_{ik}/r_k]_+} \right).$$

We require the coefficients $p_{k,j}^{\pm}$ (in some semifield $(\mathbb{P}, \oplus, \cdot)$) to satisfy the normalized condition $p_{k,j}^+ \oplus p_{k,j}^- = 1$. The normalization makes mutations deterministic and a particular choice of coefficients named *principal coefficients* (as in [Fomin and Zelevinsky 2007]) available in the generalized situation.

It turns out many algebraic and combinatorial features of cluster algebras are also inherited by generalized cluster algebras. The same finite type classification as for cluster algebras [Fomin and Zelevinsky 2003] and the generalized Laurent phenomenon have already been obtained in [Chekhov and Shapiro 2014]. We show that the dependence on coefficients in the generalized case behaves very much like the ordinary case [Fomin and Zelevinsky 2007]. In particular, a generalized version of the separation formula, Theorem 3.20, is made available through an analogous notion of principal coefficients. The well-known sign coherence of c-vectors (see Section 3.3) is also extended to the generalized case in Proposition 3.17. We note that there is a rather different version of normalized generalized cluster algebras with a certain reciprocal restriction in [Nakanishi 2015] where some results on the structures of seeds parallel to [Fomin and Zelevinsky 2007] were also established.

Another remarkable feature of an ordinary cluster algebra is the positivity of its cluster variables, i.e., they are all Laurent polynomials in the initial variables x_i and coefficients $p_i^{\pm}$ with nonnegative integer coefficients. This was proved by Lee and Schiffler [2015] for skew-symmetric types and by Gross, Hacking, Keel, and Kontsevich [Gross et al. 2018] for the more general skew-symmetrizable types. We extend the positivity to our generalized case (see Theorem 3.8), showing that the Laurent expression of any cluster variable in x_i and $p_{k,j}^{\pm}$ has nonnegative integer coefficients. We note that the positivity obtained here is (in the reciprocal case) a weak form of a positivity conjecture of Chekhov and Shapiro (which we reformulated in Conjecture 8.13); see Remark 3.9 and Section 8.5.

1.2. Generalized cluster varieties. Let L be a lattice of finite rank. Fix an algebraically closed field $\Bbbk$ of characteristic zero. The (ordinary) cluster varieties of [Fock and Goncharov 2009] are schemes of the form

$$V = \bigcup_s T_{L,s}$$

where each $T_{L,s}$ is a copy of the torus $L \otimes \Bbbk^*$ and they are glued together via birational maps called *cluster mutations*. Here s runs over a set of *seeds* (a seed roughly being a labeled basis of L) iteratively generated by mutations. A cluster mutation is give by the birational map

$$\mu_{(m,n)} : T_L \dashrightarrow T_L, \quad \mu_{(m,n)}^*(z^\ell) = z^\ell (1 + z^m)^{\langle \ell, n \rangle}, \quad \ell \in L^*,$$

for a pair of vectors $(n, m) \in L \times L^*$, where $\langle \cdot, \cdot \rangle$ denotes the natural paring between L^* and L. It has a natural dual by switching the roles of m and n, $\mu_{(-n,m)} : T_{L^*} \to T_{L^*}$. Gluing T_{L^*} via these maps gives the *dual cluster variety* $V^\vee := \bigcup_s T_{L^*,s}$.

Depending on the types of seeds and mutations chosen, one obtains either Fock–Goncharov $\mathcal{A}$-cluster varieties or $\mathcal{X}$-cluster varieties, which are dual constructions as above. A cluster algebra $\mathscr{A}$ can be embedded into the upper cluster algebra $\overline{\mathscr{A}}$, defined to be the algebra of regular functions on the corresponding $\mathcal{A}$-variety, while the dual $\mathcal{X}$-variety encodes the so-called Y-variables; see Section 4.

One can actually encode coefficients in each cluster mutation, the above construction thus leading to families of cluster varieties. They mutate along with seeds under certain rules. In the $\mathcal{A}$-case, they mutate as Y-variables (see [Fomin and Zelevinsky 2007; Fock and Goncharov 2009]). In the $\mathcal{X}$-case, the mutation rule of the coefficients has been worked out by Bossinger, Frías-Medina, Magee and Nájera Chávez [Bossinger et al. 2020].

We extend these dynamics of coefficients to the generalized situation for both the $\mathcal{A}$- and $\mathcal{X}$-cases. We define a *generalized cluster mutation* as

$$\mu^*(z^\ell) = z^\ell \prod_{j=1}^r (t_j^- + t_j^+ z^m)^{\langle \ell, n \rangle},$$

which depends on some coefficients $t_j^{\pm}$ in $\Bbbk^*$; see Section 4. Thus an ordinary cluster mutation can be viewed as a specialization of a generalized one. Generalized cluster varieties are then defined by gluing tori via the generalized mutations. We obtain two families of generalized cluster varieties

$$\pi_{\mathcal{A}} : \mathcal{A} \to \mathrm{Spec}(R), \quad \pi_{\mathcal{X}} : \mathcal{X} \to \mathrm{Spec}(R),$$

where the coefficients vary in some torus $\mathrm{Spec}(R) = (\mathbb{G}_m)^d$.

One key observation of Gross, Hacking and Keel [Gross et al. 2015] is that a cluster variety can be investigated through its toric models, and mutations between seeds are basically switching between neighboring toric models. A toric model is a construction of a log Calabi–Yau variety by blowing up a hypersurface on the toric boundary of some toric variety. In the cluster situation, the toric variety depends on the choice of a seed s which also tells us where on the toric boundary to blow up. The resulting log Calabi–Yau variety is shown in [Gross et al. 2015] (under certain nice conditions) to be isomorphic to the corresponding cluster variety up to codimension two subsets. We extend this description to the generalized case, for both $\mathcal{A}$- and $\mathcal{X}$-type varieties; see Theorem 5.4 and Section 5.3.

1.3. Scattering diagrams. Cluster scattering diagrams are the main technical tool in [Gross et al. 2018]. They have their origin in [Kontsevich and Soibelman 2006; Gross and Siebert 2011] in mirror symmetry. Roughly speaking, in the cluster case, a scattering diagram is a collection of walls in a real vector space with attached *wall-crossing functions* (some of them giving information on mutations). Similar to a cluster algebra which starts with one cluster with information to perform mutations in n directions iteratively, its scattering diagram can be constructed by initially setting up n incoming walls and letting them propagate, generating only outgoing walls.

To get a generalized cluster scattering diagram, we replace ordinary wall-crossings (which correspond to ordinary cluster mutations) on the initial incoming walls by the generalized ones of the form

$$f = \prod_{j=1}^{r} (1 + t_j z^m),$$

where the t_j are treated as formal parameters. Given a seed s (in the generalized sense), the associated data of incoming walls uniquely determines a consistent scattering diagram $\mathfrak{D}_s$, which we call *the generalized cluster scattering diagram*.

We show that the behavior of $\mathfrak{D}_s$ under mutations is analogous to that of the ordinary case, in a way it is canonically associated to a mutation equivalence class of seeds. This is called the *mutation invariance* in [Gross et al. 2018, Theorem 1.24]. See Theorem 6.27 for the precise description of the following theorem.

Theorem 1.1 (Theorem 6.27). *There is a piecewise linear operation T_k such that $T_k(\mathfrak{D}_s)$ is equivalent to $\mathfrak{D}_{\mu_k(s)}$, where $\mu_k(s)$ denotes the mutation in direction k of the seed s.*

In analogy with the ordinary case, the mutation invariance leads to the *cluster complex structure* of $\mathfrak{D}_s$.

Theorem 1.2 (Theorem 7.10). *There is the cluster cone complex Δ_s^+ such that $\mathfrak{D}_s$ is a union of codimension one cones of Δ_s^+ (with explicit attached wall-crossing functions) and walls outside Δ_s^+.*

We observe in Lemma 6.19 that $\mathfrak{D}_s$ is equivalent to the *tropical vertex scattering diagram* $\mathfrak{D}_{(X_\Sigma, H)}$ of [Argüz and Gross 2022] associated to the corresponding $\mathcal{X}$-type toric model associated to s. It is shown in [Argüz and Gross 2022, Theorem 6.1] that $\mathfrak{D}_{(X_\Sigma, H)}$ is further equivalent (after a certain piecewise linear operation) to the *canonical scattering diagram* $\mathfrak{D}_{(X,D)}$ (see [Gross and Siebert 2022; Argüz and Gross 2022, Section 2]) of the log Calabi–Yau pair (X, D) from the toric model. We thus see that $\mathfrak{D}_s$

is canonically associated to the corresponding $\mathcal{X}$-cluster variety, with a different seed s' simply giving another representative $\mathfrak{D}_{s'}$.

1.4. Cluster dualities. The cluster duality of Fock and Goncharov predicts that, in the ordinary case, the varieties $\mathcal{A}$ and $\mathcal{X}$ (see Section 4 for our convention as we do not need the Langlands dual data) are dual in the sense that the upper cluster algebra $\overline{\mathscr{A}}$ has a basis parametrized by the tropical set $\mathcal{X}^{\text{trop}}(\mathbb{Z})$ (see [Gross et al. 2018, Section 2] for a definition) and vice versa. A modified version of this duality (and when it is true) is the main subject of study of [Gross et al. 2018].

The strategy of [Gross et al. 2018] to get the desired basis is as follows. First the tropical set $\mathcal{X}^{\text{trop}}(\mathbb{Z})$ (resp. $\mathcal{X}^{\text{trop}}(\mathbb{R})$) can be identified with the cocharacter lattice M (resp. $M_{\mathbb{R}} := M \otimes \mathbb{R}$) of a chosen seed torus $T_{M,s} = M \otimes \mathbb{k}^*$ contained in the variety $\mathcal{X}$. By the mutation invariance, the ordinary cluster scattering diagram $\mathfrak{D}_s^{\text{ord}}$ (see Section 6.3) naturally lives in $\mathcal{X}^{\text{trop}}(\mathbb{R})$. Denote by Δ^+ the set of integral points inside the cluster complex (which is again a canonical subset of $\mathcal{X}^{\text{trop}}(\mathbb{Z})$ by mutation invariance).

For any integral point $m \in \mathcal{X}^{\text{trop}}(\mathbb{Z})$, using the scattering diagram $\mathfrak{D}_s^{\text{ord}}$, one can construct the *theta function* ϑ_m, which in general is only a formal power series in a completion $\widehat{\mathbb{k}[M]}_s$ which depends on s. However, it is shown in [Gross et al. 2018, Theorem 4.9] that for $m \in \Delta^+$, ϑ_m is indeed a Laurent polynomial in $\mathbb{k}[M]$ and corresponds to a cluster monomial. Furthermore, there is a canonically defined (i.e., independent of s) subset Θ of $\mathcal{X}^{\text{trop}}(\mathbb{Z})$ containing Δ^+ such that for any $m \in \Theta$, ϑ_m is a Laurent polynomial on every seed torus. It is also shown in [Gross et al. 2018] that the vector space

$$\text{mid}(\mathcal{A}) := \bigoplus_{m \in \Theta} \vartheta_m$$

has an associative algebra structure whose structure constants are defined through *broken lines*. This algebra $\text{mid}(\mathcal{A})$ can be embedded in $\overline{\mathscr{A}}$ so that for $m \in \Delta^+$, ϑ_m is sent to the corresponding cluster monomial. While we do not know in general when $\text{mid}(\mathcal{A})$ equals $\overline{\mathscr{A}}$ (see [Gross et al. 2018, Theorem 0.3]), we do have a basis of $\text{mid}(\mathcal{A})$ parametrized by the subset Θ. Strictly speaking, this process is done through the principal coefficients case.

Our insight is that it is natural to consider the above cluster duality for generalized cluster varieties. In the principal coefficients case, we take a general fiber $\mathcal{X}_\lambda^{\text{prin}} := \pi_{\mathcal{X}}^{-1}(\lambda)$ of the family

$$\pi_{\mathcal{X}} : \mathcal{X}^{\text{prin}} \to \text{Spec}(R).$$

The generalized cluster scattering diagram $\mathfrak{D}_s$ then lives in the tropical set $(\mathcal{X}_\lambda^{\text{prin}})^{\text{trop}}(\mathbb{R})$ which is identified with $M_{\mathbb{R}}$ given a chosen seed s. Towards a generalized version of the cluster duality, we show:

Theorem 1.3 (Theorem 8.12). *For any $m \in \Delta_s^+$, the theta function ϑ_m constructed from the generalized cluster scattering diagram $\mathfrak{D}_s$ corresponds to the cluster monomial of the generalized cluster algebra $\mathscr{A}^{\text{prin}}(s)$ whose g-vector is m. Moreover, it is a Laurent polynomial in the initial cluster variables x_i and coefficients $p_{i,j}$ with nonnegative integer coefficients.*

It follows from the above theorem that the family

$$\pi_{\mathcal{A}} : \mathcal{A}^{\text{prin}} \to \text{Spec}(R)$$

can be reconstructed from a general fiber $\mathcal{X}_\lambda^{\text{prin}}$ (through any of its toric models); see Proposition 8.3.

In principle, in the generalized case, one could consider the subset Θ as in [Gross et al. 2018] and the corresponding generalized middle cluster algebra mid($\mathcal{A}^{\text{prin}}$). This would lead to a formulation of generalized cluster duality similar to the ordinary case in [Gross et al. 2018, Theorem 0.3]. Then the usual problem on when the full Fock–Goncharov conjecture is true remains and can be discussed as in [Gross et al. 2018, Section 8].

1.5. *Relations to other works*. There are examples of generalized cluster scattering diagrams from representation theory, where they are realized as Bridgeland's stability scattering diagrams [2017] for quivers (with loops) with potentials; see [Labardini-Fragoso and Mou 2024] for such examples arising from surfaces with orbifold points.

In rank two, the scattering diagram $\mathfrak{D}_s$ has already appeared in [Gross et al. 2010; Gross and Pandharipande 2010] from origins other than cluster algebras. There the wall-crossing functions are shown to encode relative Gromov–Witten invariants on certain log Calabi–Yau surfaces. Some conjectural wall-crossing functions in [Gross and Pandharipande 2010] were later verified in [Reineke and Weist 2013] using techniques from quiver representations; see Example 6.22.

The recent paper of Cheung, Kelley and Musiker [Cheung et al. 2023] (outlined in [Cheung et al. 2021]) and some part of Kelley's PhD thesis [2021] have significant overlaps with this paper and the author's PhD thesis [Mou 2020, Chapter 8]. We in the following highlight the differences and relationships concerning scattering diagrams.

In [Mou 2020, Chapter 8], a class of generalized cluster scattering diagrams were constructed and properties including mutation invariance and cluster complex structure were proved. In that work, a palindromic and monic restriction (termed *reciprocal* in [Chekhov and Shapiro 2014]) on the coefficients was imposed. Such a scattering diagram can be obtained from applying to $\mathfrak{D}_s$ of the current paper an evaluation λ such that the initial wall-crossings are specialized to reciprocal polynomials, i.e., of the form

$$f = 1 + a_1 z^w + \cdots + a_{r-1} z^{(r-1)w} + z^{rw},$$

where $r \in \mathbb{Z}_{\geq 0}$, $w \in M$, and $a_k = a_{r-k}$ in some ground field $\Bbbk$; see Section 6.4. Scattering diagrams almost identical to these (with the reciprocal restriction) were later considered by Cheung, Kelley and Musiker [Cheung et al. 2021], with more details provided in [Kelley 2021]. The authors treat the coefficients a_i as formal variables. They also outlined the construction of theta functions, following [Gross et al. 2018].

The current paper aims to fill in gaps and missing details in [Mou 2020], enhance the setup therein to include more general coefficients, and discuss the positivity of generalized cluster algebras using scattering diagram techniques. Shortly after this paper was posted on the arXiv, [Cheung et al. 2023] appeared on the arXiv, completing the program [Cheung et al. 2021]. Despite many similarities between

the current paper and [Cheung et al. 2023], our approaches of treating coefficients differ somewhat. In [Cheung et al. 2023], the y-variables in a generalized seed and the coefficients $a = (a_i)$ in a generalized exchange polynomial are treated separately. The coefficients a are assumed to be reciprocal and remain unchanged under mutations. In contrast, we view the coefficients a as deriving from the y-variables (denoted as p in our notation) by factorizing an exchange polynomial into binomials, with each binomial governed by one coefficient in the style of Fomin and Zelevinsky. This approach allows us to realize more general exchange polynomials (beyond just reciprocal ones), at least for an algebraically closed ground field, by specialization from principal coefficients (see Sections 3.5 and 8.5). This setup also enables us to formulate and prove a form of positivity for generalized cluster algebras, a topic not extensively discussed in [Cheung et al. 2023].

2. Preliminaries

In this section, we review some preliminaries commonly used in the theory of cluster algebras [Fomin and Zelevinsky 2002].

2.1. *Semifields.*

Definition 2.1. A semifield $(\mathbb{P}, \oplus, \cdot)$ is a torsion free (multiplicative) abelian group $\mathbb{P}$ with a binary operation *addition* $\oplus$ which is commutative, associative and distributive.

We denote by $\mathbb{Z}\mathbb{P}$ the group ring of $\mathbb{P}$ and by $\mathbb{N}\mathbb{P} \subset \mathbb{Z}\mathbb{P}$ the subset of linear combinations with coefficients in $\mathbb{N}$. Denote by $\mathbb{Q}\mathbb{P}$ the field of fractions of $\mathbb{Z}\mathbb{P}$.

For an element $p \in \mathbb{P}$, we define in $\mathbb{P}$ two elements:

$$p^+ := \frac{p}{p \oplus 1} \quad \text{and} \quad p^- := \frac{1}{p \oplus 1}.$$

Definition 2.2. Let I be a finite set. We define $\mathrm{Trop}(s_i \mid i \in I)$ to be the (multiplicative) abelian group with free generators s_i indexed by I, with the operation addition $\oplus$:

$$\prod_{i \in I} s_i^{a_i} \oplus \prod_{i \in I} s_i^{b_i} := \prod_{i \in I} s_i^{\min\{a_i, b_i\}}.$$

It is clear that $\mathrm{Trop}(s_i \mid i \in I)$ is a semifield. Such a semifield is called a *tropical semifield*.

For $n \in \mathbb{Z}$, we write $[n]_+ := \max\{n, 0\}$. The elements $s^{\pm}$ for

$$s = \prod_{i \in I} s_i^{a_i} \in \mathrm{Trop}(s_i \mid i \in I)$$

has the following simple expressions:

$$s^+ = \prod_{i \in I} s_i^{[a_i]_+} \quad \text{and} \quad s^- = \prod_{i \in I} s_i^{[-a_i]_+}.$$

Definition 2.3. Denote by $\mathbb{Q}_{\mathrm{sf}}(u_1, \ldots, u_l)$ the set of all rational functions in l independent variables which can be written as subtraction-free rational expressions in $u_1, \ldots, u_l$. Then the set $\mathbb{Q}_{\mathrm{sf}}(u_1, \ldots, u_l)$ is a semifield with respect to the usual addition and multiplication.

We call such $\mathbb{Q}_{sf}(u_1, \ldots, u_l)$ a *universal semifield* since for another arbitrary semifield $\mathbb{P}$ and its elements $p_1, \ldots, p_l$, there exists a unique map of semifields from $\mathbb{Q}_{sf}(u_1, \ldots, u_l)$ to $\mathbb{P}$ sending u_i to p_i; see [Berenstein et al. 1996, Lemma 2.1.6].

2.2. *Mutations of matrices.*

Definition 2.4. A matrix $B \in \mathrm{Mat}_{n \times n}(\mathbb{Z})$ is called (left) *skew-symmetrizable* if there exists a diagonal matrix $D = \mathrm{diag}(d_i \mid 1 \le i \le n)$ with $d_i \in \mathbb{Z}_{>0}$ such that

$$DB + (DB)^T = 0.$$

Such a matrix D is called a (left) *skew-symmetrizer* of B.

Definition 2.5 [Fomin and Zelevinsky 2002, Definition 4.2]. Let $B = (b_{ij}) \in \mathrm{Mat}_{n \times n}(\mathbb{Z})$ be a skew-symmetrizable matrix. For $k = 1, \ldots, n$, we define $\mu_k(B) = (b'_{ij}) \in \mathrm{Mat}_{n \times n}(\mathbb{Z})$ the *mutation of B in direction k* by setting

(1) $b'_{ik} = -b_{ik}$ and $b'_{kj} = -b_{kj}$ for $1 \le i, j \le n$;

(2) for $i \ne k$ and $j \ne k$,

$$b'_{ij} = \begin{cases} b_{ij} + b_{ik}b_{kj} & \text{if } b_{ik} > 0 \text{ and } b_{jk} < 0; \\ b_{ij} - b_{ik}b_{kj} & \text{if } b_{ik} < 0 \text{ and } b_{jk} > 0; \\ b_{ij} & \text{otherwise.} \end{cases}$$

It is clear that the matrix $\mu_k(B)$ is again skew-symmetrizable with the same set of skew-symmetrizers of B. One can easily check that a mutation is involutive in the same direction, i.e., $\mu_k \circ \mu_k(B) = B$.

3. Generalized cluster algebras

3.1. *Generalized cluster algebras.* Cluster algebras were originally invented by Fomin and Zelevinsky [2002], which we later refer to as *ordinary cluster algebras*. A generalization of cluster algebras has been provided by Chekhov and Shapiro [2014]. Our definition of generalized cluster algebras below may be considered as a special case (of a slight generalization) of theirs but with a normalization analogous to the one in [Fomin and Zelevinsky 2002, Definition 5.3] for ordinary cluster algebras. The relation and difference will be explained in Section 3.2.

We follow the pattern of [Fomin and Zelevinsky 2007] to define generalized cluster algebras. Most of the key notions here are the generalized versions of their correspondents in the ordinary case.

Definition 3.1. A (*generalized*) *labeled seed* Σ of rank $n \in \mathbb{N}$ is a triple $(\boldsymbol{x}, \boldsymbol{p}, B)$, where:

- $\boldsymbol{p} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n)$ is an n-tuple of collections of elements, where each $\boldsymbol{p}_i = (p_{i,1}, \ldots, p_{i,r_i})$ is a r_i-tuple of elements in a semifield $(\mathbb{P}, \oplus, \cdot)$ for some positive integer r_i.
- $\boldsymbol{x} = \{x_1, \ldots, x_n\}$ is a collection of algebraically independent rational functions of n variables over $\mathbb{Q}\mathbb{P}$. In other words, the $x_1, \ldots x_n$ are elements in some field of rational functions $\mathcal{F} = \mathbb{Q}\mathbb{P}(u_1, \ldots, u_n)$ such that $\mathcal{F} = \mathbb{Q}\mathbb{P}(x_1, \ldots, x_n)$.

- $B \in \mathrm{Mat}_{n \times n}(\mathbb{Z})$ is skew-symmetrizable such that for any $i = 1, \ldots, n$, its i-th column is divisible by r_i. The diagonal matrix $D = \mathrm{diag}(r_i)$ is not necessarily a skew-symmetrizer of B.

For convenience, let I be the index set $\{1, \ldots, n\}$. For an arbitrary positive integer k, we use the interval $[1, k]$ to represent the set $\{1, \ldots, k\}$. We will often call a labeled seed simply a *seed* if there is no confusion.

Associated to a labeled seed $\Sigma = (\boldsymbol{x}, \boldsymbol{p}, B)$, for each $i \in I$, there is the *exchange polynomial*

$$\theta_i(u, v) = \theta[\boldsymbol{p}_i](u, v) := \prod_{l=1}^{r_i} (p_{i,l}^+ u + p_{i,l}^- v) \in \mathbb{ZP}[u, v].$$

Write $\beta_{ij} = b_{ij}/r_j \in \mathbb{Z}$. We put

$$u_{j;+} := \prod_{i:b_{ij}>0} x_i^{\beta_{ij}}, \quad u_{j;-} := \prod_{i:b_{ij}<0} x_i^{-\beta_{ij}}$$

$$p_{i;+} := \prod_{l=1}^{r_i} p_{i,l}^+, \qquad p_{i;-} := \prod_{l=1}^{r_i} p_{i,l}^- \in \mathbb{P}.$$

Note that all the above notions are with respect to Σ.

Definition 3.2. For any $k \in I$, we define the *mutation of a seed* $\Sigma = (\boldsymbol{x}, \boldsymbol{p}, B)$ *in direction* k as a new labeled seed $\mu_k(\boldsymbol{x}, \boldsymbol{p}, B) := ((x_i'), (\boldsymbol{p}_i'), B')$, where $\boldsymbol{p}_i' = (p_{i,j}' \mid j \in [1, r_i])$ in the following way:

(1) $$B' = \mu_k(B);$$

(2) $$p_{k,j}' = p_{k,j}^{-1} \quad \text{for } j \in [1, r_k];$$

(3) $$\text{for } i \neq k, \ j \in [1, r_i], \quad p_{i,j}' = \begin{cases} p_{i,j} \cdot (p_{k;-})^{\beta_{ki}} & \text{if } \beta_{ik} > 0, \\ p_{i,j} \cdot (p_{k;+})^{\beta_{ki}} & \text{if } \beta_{ik} \leq 0, \end{cases}$$

or equivalently

$$\text{for } i \neq k, \quad p_{i,j}' = p_{i,j} \left(\prod_{l=1}^{r_k} (1 \oplus p_{k,l}^{\mathrm{sgn}(\beta_{ik})}) \right)^{-\beta_{ki}};$$

(4) $$x_i' = \begin{cases} x_i & \text{if } i \neq k, \\ x_k^{-1} \theta[\boldsymbol{p}_k](u_{k;+}, u_{k;-}) & \text{if } i = k. \end{cases}$$

Lemma 3.3. *The mutation* μ_k *is involutive, i.e.,* $\mu_k \circ \mu_k(\Sigma) = \Sigma$.

Proof. We check that μ_k is involutive on each component of a seed. We denote

$$\mu_k \circ \mu_k(\Sigma) = ((x_i''), (p_{i,j}'' \mid j \in [1, r_i])_{i \in I}, B'').$$

For this seed, we simply denote the relevant objects appearing in Definition 3.2 by adding a double prime to the old symbols, while for $\mu_k(\Sigma)$, we add a single prime.

(1) First of all, the matrix mutation μ_k is an involution, as shown in [Fomin and Zelevinsky 2002].

(2) We have for $j \in [1, r_k]$,

$$p_{k,j}'' = (p_{k,j}')^{-1} = p_{k,j}.$$

(3) For $i \neq k$, we have for $j \in [1, r_i]$,

$$
\begin{aligned}
p''_{i,j} &= \begin{cases} p'_{i,j} \cdot (p'_{k;-})^{\beta'_{ki}} & \text{if } \beta'_{ik} > 0, \\ p'_{i,j} \cdot (p'_{k;+})^{\beta'_{ki}} & \text{if } \beta'_{ik} \leq 0 \end{cases} \\
&= \begin{cases} p_{i,j} \cdot (p_{k;+})^{\beta_{ki}} \cdot (p'_{k;-})^{-\beta_{ki}} & \text{if } \beta_{ik} < 0, \\ p_{i,j} \cdot (p_{k;-})^{\beta_{ki}} \cdot (p'_{k;+})^{-\beta_{ki}} & \text{if } \beta_{ik} \geq 0 \end{cases} \\
&= p_{i,j}.
\end{aligned}
$$

The last equality is because $p'_{k;+} = p_{k;-}$ and $p'_{k;-} = p_{k;+}$.

(4) Finally for the x part, we have

$$
\begin{aligned}
x''_i &= \begin{cases} x'_i & \text{if } i \neq k, \\ (x'_k)^{-1}\theta[p'_k](u'_{k;+}, u'_{k;-}) & \text{if } i = k \end{cases} \\
&= \begin{cases} x_i & \text{if } i \neq k, \\ x_k \cdot \theta[p_k](u_{k;+}, u_{k;-})^{-1}\theta[p'_k](u'_{k;+}, u'_{k;-}) & \text{if } i = k \end{cases} \\
&= x_i.
\end{aligned}
$$

The last equality is because that $\theta[p'_k](u, v) = \theta[p_k](v, u)$ and $u'_{k;\pm} = u_{k;\mp}$.

So overall we have proven that $\mu_k \circ \mu_k(\Sigma) = \Sigma$. $\qquad\qquad\square$

Fix a positive integer n. We consider the (nonoriented) *n-regular tree* $\mathbb{T}_n$ whose edges are labeled by the numbers $1, \ldots, n$ as in [Fomin and Zelevinsky 2002]. Lemma 3.3 makes the following definition well-defined.

Definition 3.4. A (*generalized*) *cluster pattern* is an assignment of a labeled seed $\Sigma_t = (x_t, p_t, B^t)$ to every vertex $t \in \mathbb{T}_n$, such that for any k-labeled edge with endpoints t and t', the seed $\Sigma_{t'}$ is the mutation in direction k of Σ_t, i.e., $\Sigma_{t'} = \mu_k(\Sigma_t)$.

The elements in Σ_t are written as follows:

$$
x_t = (x_{i;t} \mid i \in I), \quad p_{i;t} = (p_{i,j;t} \mid j \in [1, r_i]), \quad B^t = (b^t_{ij}).
$$

The part x of a labeled seed is called a (labeled) *cluster*, elements x_i are called *cluster variables*, elements $p_{i,j}$ are called *coefficients* and B is called *exchange matrix*.

Two seeds are *mutation-equivalent* if one is obtained from the other by a sequence of mutations. If a seed Σ appears in a cluster pattern, then by definition any seeds mutation-equivalent to Σ must also appear. On the other hand, assigning a seed of rank n to any vertex of $\mathbb{T}_n$ would uniquely determine a cluster pattern.

By definition, all cluster variables live in some ambient field $\mathcal{F}$ of rational functions of n variables. One may identify $\mathcal{F}$ with $\mathbb{QP}(x_1, \cdots, x_n)$ where $(x_1, \ldots, x_n)$ is a cluster.

Definition 3.5. Given a generalized cluster pattern, the (*generalized*) *cluster algebra* $\mathscr{A}$ is defined to be the $\mathbb{ZP}$-subalgebra of the ambient field $\mathcal{F}$ generated by all cluster variables $x_{i;t}$ in all seeds that appear in the cluster pattern. Since a cluster pattern is determined by any of its seed, we denote $\mathscr{A} = \mathscr{A}(\Sigma)$ where $\Sigma = (x, p, B)$ is any seed in this cluster pattern.

Remark 3.6. In the case where $r_i = 1$ for any $i \in I$, all the above generalized notions recover the original versions of [Fomin and Zelevinsky 2007].

For convenience, one can specify one vertex $t_0 \in \mathbb{T}_n$ to be *initial*, thus the associated seed being called the *initial seed* with the *initial* cluster, cluster variables, coefficients and exchange matrix. All other seeds are obtained by applying mutations iteratively to the initial one. For the following two theorems, we denote by $(x_1, \ldots, x_n)$ the initial cluster.

Theorem 3.7 (generalized Laurent phenomenon, cf. [Fomin and Zelevinsky 2002] and [Chekhov and Shapiro 2014]). *Let (x, p, B) be a generalized labeled seed. Then any cluster variable of $\mathscr{A}(x, p, B)$ is a Laurent polynomial over $\mathbb{ZP}$ in the initial cluster variables x_i, i.e., an element in $\mathbb{ZP}[x_1^{\pm}, \ldots, x_n^{\pm}]$.*

Proof. The generalized Laurent phenomenon was already obtained in [Chekhov and Shapiro 2014, Theorem 2.5]. Since our setting is slightly different, we give a proof for completeness.

The proof completely follows from the discussion in [Fomin and Zelevinsky 2002, Section 3]. The generalized Laurent property is a direct corollary of [loc. cit., Theorem 3.2]. One only needs to check the following hypothesis required by [loc. cit., Theorem 3.2]: for any subgraph

$$t_0 \;\overset{i}{\text{———}}\; t_1 \;\overset{j}{\text{———}}\; t_2 \;\overset{i}{\text{———}}\; t_3$$

in the tree $\mathbb{T}_n$, if we define the following three *exchange polynomial* in n variables $x_1, \ldots, x_n$ by writing

$$P(x_{t_0}) = \theta[p_{i;t_0}](u_{i;+}^{t_0}, u_{i;-}^{t_0}), \quad Q(x_{t_1}) = \theta[p_{j;t_1}](u_{j;+}^{t_1}, u_{j;-}^{t_1}), \quad R(x_{t_2}) = \theta[p_{i;t_2}](u_{i;+}^{t_2}, u_{i;-}^{t_2}),$$

then they satisfy $R = C \cdot (P|_{x_j \leftarrow Q_0/x_j})$, where $Q_0 = Q|_{x_i=0}$ for some $C \in \mathbb{NP}[x_1, \ldots x_n]$.

Notice that since $t_0 \;\overset{i}{\text{———}}\; t_1$, we have

$$P = \prod_{l=1}^{r_i} \left(p_{i,l;t_1}^+ \prod_k x_k^{[\beta_{ki}^{t_1}]_+} + p_{i,l;t_1}^- \prod_k x_k^{[-\beta_{ki}^{t_1}]_+} \right).$$

When $\beta_{ij}^{t_1} = 0$, $\beta_{ji}^{t_0} = -\beta_{ij}^{t_1} = 0$. So x_j does not appear in P, implying $P = P|_{x_j \leftarrow Q_0/x_j}$. In this case, we have for any $l \in [1, r_i]$

$$p_{i,l;t_0} = p_{i,l;t_2}^{-1}, \quad \beta_{li}^{t_0} = -\beta_{li}^{t_2}.$$

Thus we have $R = P$.

When $\beta_{ij}^{t_1} < 0$ (implying $\beta_{ji}^{t_1} > 0$), then

$$Q_0/x_j = p_{j;+;t_1} x_j^{-1} \prod_k x_k^{[b_{kj}^{t_1}]_+}$$

and

$$P|_{x_j \leftarrow Q_0/x_j} = \prod_{l=1}^{r_i} \left(p_{i,l;t_1}^+ p_{j;+;t_1}^{\beta_{ji}^{t_1}} x_j^{-\beta_{ji}^{t_1}} \prod_{k \neq j} x_k^{[\beta_{ki}^{t_1}]_+ + \beta_{ji}^{t_1} \cdot [b_{kj}^{t_1}]_+} + p_{i,l;t_1}^- \prod_k x_k^{[-\beta_{ki}^{t_1}]_+} \right)$$

We take the ratio of the two monomials in each factor of the above product to obtain monomials

$$p_{i,l;t_1} \cdot p_{j;+;t_1}^{\beta_{ji}^{t_1}} \cdot \prod_k x_k^{\beta_{ki}^{t_2}}.$$

We get exactly the same monomials if we do the same for R. So R and $P|_{x_j \leftarrow Q_0/x_j}$ only differ by a monomial factor in $\mathbb{NP}[x_1, \ldots, x_n]$. The case when $\beta_{ij}^{t_1} > 0$ is analogous. Hence the hypothesis is checked and the Laurent phenomenon follows from [Fomin and Zelevinsky 2002, Theorem 3.2]. $\qquad\square$

The following Theorem 3.8 is a generalization of the well-known positivity for ordinary cluster algebras. In the case of ordinary cluster algebras, the positivity was originally conjectured by Fomin and Zelevinsky [2002]. It has been proved by Lee and Schiffler [2015] when the exchange matrix B is skew-symmetric and by Gross, Hacking, Keel, and Kontsevich [Gross et al. 2018] when B is more generally skew-symmetrizable.

Theorem 3.8 (positivity). *In a generalized cluster algebra, each of the coefficients in the Laurent polynomial corresponding to any cluster variable from Theorem 3.7 is a nonnegative integer linear combination of elements in $\mathbb{P}$. In other words, any cluster variable is an element in $\mathbb{NP}[x_1^{\pm}, \ldots, x_n^{\pm}]$.*

Proof. By the separation formula Theorem 3.20 and Remark 3.21, we only need to show the positivity in the principal coefficients case (to be defined in Definition 3.13). In this case, we prove the positivity in Theorem 8.12. $\qquad\square$

Remark 3.9. Chekhov and Shapiro [2014, Conjecture 5.1] conjectured a positivity for generalized cluster algebras under a reciprocal condition; see also the formulation in Conjecture 8.13. In the reciprocal case, this positivity implies Theorem 3.8. We do not know how to show this stronger positivity in general; see the discussion in Section 8.5.

3.2. *Relation to Chekhov and Shapiro's definition.* Chekhov and Shapiro [2014] defined generalized cluster algebras by considering more general exchange polynomials. Precisely, a labeled seed in that setting is a triple $(\boldsymbol{x}, (\overline{\alpha}_i \mid i \in I), B)$, where $\boldsymbol{x}$ and B are the same as in Definition 3.1 and $\overline{\alpha}_i = (\alpha_{i,k} \in \mathbb{P} \mid 0 \le k \le r_i)$ for $i \in I$. Here we only take $\mathbb{P}$ as a multiplicative abelian group. The coefficients $\alpha_{i,k}$ are responsible for expressing the exchange polynomial defined as

$$\theta_i(u, v) := \sum_{k=0}^{r_i} \alpha_{i,k} u^{r_i - k} v^k \in \mathbb{ZP}[u, v].$$

The mutation $(\boldsymbol{x}', (\overline{\alpha}_i'), B') = \mu_k(\boldsymbol{x}, (\overline{\alpha}_i), B)$ is defined in the following way:

(1) $B' = \mu_k(B)$.

(2) $\alpha_{k,j}' = \alpha_{k,r_k-j}$ and for $i \neq k$, the coefficients satisfy

$$\alpha_{i,j}'/\alpha_{i,0}' = \begin{cases} \alpha_{k,0}^{j\beta_{ki}} \cdot \alpha_{i,j}/\alpha_{i,0} & \text{if } \beta_{ik} > 0 \\ \alpha_{k,r_k}^{j\beta_{ki}} \cdot \alpha_{i,j}/\alpha_{i,0} & \text{if } \beta_{ik} \le 0. \end{cases}$$

(3) $x_i' = x_i$ for $i \neq k$ and

$$x_k' x_k = \theta_i(u_{k;+}, u_{k;-}).$$

Remark 3.10. In this setting, it does no harm to allow the coefficients $\alpha_{i,k}$ to be elements of $\mathbb{Z}\mathbb{P}$, as long as the above rule (2) is satisfied. For example, one may check that the Laurent phenomenon still holds for cluster variables.

Now assume the multiplicative abelian group $\mathbb{P}$ has an addition $\oplus$ so that it is a semifield. In our setting the exchange polynomials are given by $\theta[\boldsymbol{p}_i](u, v)$, thus the coefficients $\alpha_{i,j}$ corresponding to the coefficients of $\theta[\boldsymbol{p}_i](u, v)$ when expanded as polynomial of u and v. Under Definition 3.2, the polynomials $\theta[\boldsymbol{p}_i](u, v)$ mutate in a way satisfying the rule (2) above. In fact, when talking about coefficients $\alpha_{i,j}/\alpha_{i,0}$, we can normalize our polynomial

$$\tilde{\theta}[\boldsymbol{p}_i](u, v) = \prod_{j \in [1, r_i]} (p_{i,j} u + v).$$

So when expanded as a sum of monomials in u and v, the coefficients of $\tilde{\theta}[\boldsymbol{p}_i]$ are $\prod_{j \in J} p_{i,j}$ for a subset $J \subset [1, r_i]$. According to the mutation formula in Definition 3.2, under μ_k, we have

$$\prod_{j \in J} p'_{i,j} = p_{k;\pm}^{|J|\beta_{ki}} \prod_{j \in J} p_{i,j},$$

which satisfies the rule (2). So our definition of generalized cluster algebras can be viewed as a special case of [Chekhov and Shapiro 2014] if we ease the condition $\alpha_{i,k} \in \mathbb{P}$ into $\alpha_{i,k} \in \mathbb{Z}\mathbb{P}$.

We note that the above rule (2) in [Chekhov and Shapiro 2014] is not enough to uniquely determine the coefficients $(\bar{\alpha}'_i)$ after mutation, whereas the mutation formula in Definition 3.2 is deterministic because of the normalization condition $p_{i,j}^+ \oplus p_{i,j}^- = 1$.

One advantage of our definition is the availability of principal coefficients analogous to [Fomin and Zelevinsky 2007, Definition 3.1], to be discussed in the next section.

3.3. Principal coefficients. As in [Fomin and Zelevinsky 2007] for ordinary cluster algebras, we have the notion of principal coefficients for generalized cluster algebras.

Definition 3.11. We say a generalized cluster algebra $\mathscr{A}$ is of geometric type if $\mathbb{P}$ is a tropical semifield

$$\mathrm{Trop}(s_a \mid a \in I'),$$

where I' is a finite index set.

Proposition 3.12. *Let $\mathscr{A}$ be a generalized cluster algebra of geometric type. For each seed Σ in $\mathscr{A}$'s cluster pattern and $i \in I$, we introduce matrices*

$$C^{(i)} = C_{\Sigma}^{(i)} = (c_{aj}^{(i)}) \in \mathrm{Mat}_{|I'| \times r_i}(\mathbb{Z})$$

to encode the coefficients $p_{i,j}$ by columns of $C^{(i)}$:

$$p_{i,j} = \prod_{a \in I'} s_a^{c_{aj}^{(i)}} \in \mathbb{P}.$$

Denote by $(\bar{c}_{aj}^{(i)})$ *the matrices corresponding to the seed* $\mu_k(\Sigma)$ *for some* $k \in I$. *Then we have*

$$
\bar{c}_{aj}^{(i)} = \begin{cases} -c_{aj}^{(i)} & \text{if } i = k; \\ c_{aj}^{(i)} + \Big(\sum\limits_{j=1}^{r_k} [-c_{aj}^{(k)}]_+ \Big) \beta_{ki} & \text{if } i \neq k \text{ and } \beta_{ik} > 0; \\ c_{aj}^{(i)} + \Big(\sum\limits_{j=1}^{r_k} [c_{aj}^{(k)}]_+ \Big) \beta_{ki} & \text{if } i \neq k \text{ and } \beta_{ik} \leq 0. \end{cases}
$$

Proof. In the tropical semifield $\mathrm{Trop}(s_a \mid a \in I')$, we have the expressions

$$
p_{i,j}^+ = \prod_{a \in I'} s_a^{[c_{aj}^{(i)}]_+} \quad \text{and} \quad p_{i,j}^- = \prod_{a \in I'} s_a^{[-c_{aj}^{(i)}]_+}.
$$

Then the result follows by spelling out the mutation formula of coefficients ((3) of Definition 3.2). □

The matrices and their dynamics in Proposition 3.12 have led to a remarkable combinatorial phenomenon in cluster theory known as *the sign coherence of c-vectors*. We shall explain it below.

Definition 3.13. We say a generalized cluster algebra $\mathscr{A}$ has *principal coefficients* at a vertex $t_0 \in \mathbb{T}_n$ if $\mathbb{P}$ is the tropical semifield

$$
\mathrm{Trop}(\boldsymbol{p}) := \mathrm{Trop}(p_{i,j} \mid i \in I, j \in [1, r_i]),
$$

and the seed Σ_{t_0} has coefficients $\boldsymbol{p}_i = (p_{i,1}, \dots p_{i,r_i})$. In this case, the cluster algebra, denoted as $\mathscr{A}^{\mathrm{prin}}(\Sigma_{t_0})$, is by definition a subalgebra of

$$
\mathbb{Z}[x_{i;t_0}^\pm, p_{i,j}^\pm \mid i \in I, j \in [1, r_i]].
$$

In the case of principal coefficients, the columns of the matrices $C_{\Sigma_t}^{(i)}$ are called *generalized c-vectors*. At the initial seed $\Sigma = \Sigma_{t_0}$ with principal coefficients, each $C_\Sigma^{(i)}$ is an identity matrix I_{r_i} extended (vertically) by zeros.

Theorem 3.14 (sign coherence of generalized c-vectors). *In the principal coefficients case, for any* $t \in \mathbb{T}_n$, *for any* $i \in I$ *and any* $j \in [1, r_i]$, *the entries of the j-th column of* $C_{\Sigma_t}^{(i)}$ *are either all nonnegative or all nonpositive.*

When $r_i = 1$ for each $i \in I$, i.e., in the case of ordinary cluster algebras, each $C^{(i)} = C_{\Sigma_t}^{(i)}$ is just a column vector with n entries, altogether forming a matrix $C = (C^{(1)}, \dots, C^{(n)})$. They are the so-called C-*matrices* in [Fomin and Zelevinsky 2007] whose columns are *c-vectors*. In this case, Theorem 3.14 then says that each column of any C is either nonnegative or nonpositive. This is well-known in the theory of cluster algebras as *the sign coherence of c-vectors*, which has already been proved by Derksen, Weyman and Zelevinsky [Derksen et al. 2010] for skew-symmetric exchange matrices and by Gross, Hacking, Keel and Kontsevich [Gross et al. 2018] for skew-symmetrizable ones. We will see in Proposition 3.17 that Theorem 3.14 follows from the already established sign coherence of c-vectors.

We set the index set

$$
I' = \bigsqcup_{i \in I} I_i', \quad I_i' := \{(i, j) \mid j \in [1, r_i]\}.
$$

Lemma 3.15. *Let $\Sigma = \Sigma_{t_0}$ be a seed with principal coefficients. We have the following properties for the matrices $C_{\Sigma_t}^{(i)}$ for any seed Σ_t, $t \in \mathbb{T}_n$.*

(1) *Let $i, k \in I$ such that $k \neq i$. Then for any $a, a' \in I_k'$ and any $1 \leq j, j' \leq r_i$, we have*

$$c_{a,j}^{(i)} = c_{a',j'}^{(i)}.$$

(2) *Let $i \in I$. We have*

$$c_{(i,1),1}^{(i)} = c_{(i,2),2}^{(i)} = \cdots = c_{(i,r_i),r_i}^{(i)} = c \pm 1$$

and

$$c_{(i,k),j}^{(i)} = c \quad for \ k \neq j$$

for some integer c.

Proof. We prove this lemma by induction on the distance from t to t_0 in $\mathbb{T}_n$. The base case is for $C_\Sigma^{(i)}$ where the entries in (1) are all zeroes and the ones in (2) are 1 when $k = j$ and 0 otherwise. Then the properties stated in the lemma are preserved under the mutation formula given in Proposition 3.12. □

Let $\overline{\mathbb{P}}$ be the tropical semifield $\mathrm{Trop}(\bar{p}_i \mid i \in I)$. There is a group homomorphism

$$\pi : \mathbb{P} \to \overline{\mathbb{P}}, \quad p_{i,j} \mapsto \bar{p}_i.$$

For $t \in \mathbb{T}_n$, denote the image of $p_{i,j;t}$ in $\overline{\mathbb{P}}$ by $\bar{p}_{i;t}$ (which is independent of j by Lemma 3.15) and the image of $\prod_{j=1}^{r_i} p_{i,j;t}$ in $\overline{\mathbb{P}}$ by $p_{i;t} = \bar{p}_{i;t}^{r_i}$.

Lemma 3.16. *The elements $p_{i;t}$ behave in the following way under the mutation μ_k. If $t' \overset{k}{—} t$ and we write $p_i' = p_{i;t'}$ and $p_i = p_{i;t}$, then we have*

$$p_i' = \begin{cases} p_i^{-1} & \text{if } i = k; \\ p_i \cdot (p_k^-)^{b_{ki}} & \text{if } i \neq k \text{ and } \beta_{ik} > 0; \\ p_i \cdot (p_k^+)^{b_{ki}} & \text{if } i \neq k \text{ and } \beta_{ik} \leq 0. \end{cases}$$

So they behave under mutations in the same way as $p_{i,1;t}$ in the case where $r_i = 1$, $i \in I$, i.e., the case of ordinary cluster algebras.

Proof. By the generalized mutation formula of coefficients, we have

$$\prod_{j=1}^{r_i} p_{i,j}' = \begin{cases} \displaystyle\prod_{j=1}^{r_i} p_{i,j}^{-1} & \text{if } i = k; \\ \displaystyle\prod_{j=1}^{r_i} p_{i,j} \cdot \left(\prod_{j=1}^{r_k} p_{k,j}^- \right)^{b_{ki}} & \text{if } i \neq k \text{ and } \beta_{ik} > 0; \\ \displaystyle\prod_{j=1}^{r_i} p_{i,j} \cdot \left(\prod_{j=1}^{r_k} p_{k,j}^+ \right)^{b_{ki}} & \text{if } i \neq k \text{ and } \beta_{ik} \leq 0. \end{cases}$$

By the matrix description of the elements $p_{k,j}$ in Lemma 3.15, we have that

$$\prod_{j=1}^{r_k} p_{k,j}^\pm = \left(\prod_{j=1}^{r_k} p_{k,j} \right)^\pm \in \mathbb{P}, \quad \pi\left(\prod_{j=1}^{r_k} p_{k,j}^\pm \right) = p_k^\pm \in \overline{\mathbb{P}}.$$

The result then follows. □

Proposition 3.17. *The sign coherence of c-vectors implies the sign coherence of generalized c-vectors.*

Proof. In the case where all $r_i = 1$, the sign coherence then says each column of the matrix $C = (C^{(1)}, \ldots, C^{(n)})$ is either nonnegative or nonpositive.

On the other hand, by Lemma 3.16, the elements p_i behave under mutations in the exact same way as the coefficients in seeds when all $r_i = 1$ (thus we only have one p_i for each i). Thus the column $C^{(i)}$ of Σ_t serves as the coordinates of $p_{i;t}$ in terms of the initial coefficients p_i. Then the sign coherence tells that one of p_i^+ and p_i^- is 1. It then follows from Lemma 3.15 that the corresponding $p_{i,j}^+$ or $p_{i,j}^-$ for each $j \in [1, r_i]$ is also 1, hence the generalized sign coherence. $\qquad\square$

The following lemma will be useful later.

Lemma 3.18. *In the principal coefficient case, for any $t \in \mathbb{T}_n$, the set of coefficients in seed Σ_t*

$$\{p_{i,j;t} \mid i \in I, j \in [1, r_i]\}$$

form a $\mathbb{Z}$-basis of $\mathbb{P} \cong \mathbb{Z}^d$ where $d = \sum_{i \in I} r_i$.

Proof. This follows directly from the mutation formula Proposition 3.12 and Lemma 3.15. $\qquad\square$

3.4. *Separation formula.* In this section, we describe the separation formula for generalized cluster variables, which can be derived in the exact same way as [Fomin and Zelevinsky 2007, Theorem 3.7], so we omit the proof.

Definition 3.19. Let $\mathscr{A}^{\mathrm{prin}}(\Sigma_{t_0})$ be a generalized cluster algebra with principal coefficients at $\Sigma_{t_0} = (\boldsymbol{x}, \boldsymbol{p}, B)$. We define the rational function

$$X_{l;t} \in \mathbb{Q}_{\mathrm{sf}}(\boldsymbol{x}, \boldsymbol{p})$$

corresponding to the subtraction-free rational expression of the cluster variable $x_{l;t}$ by iterating exchange relations. Here $(\boldsymbol{x}, \boldsymbol{p})$ denote the set of all variables in $\boldsymbol{x}$ and $\boldsymbol{p}$.

Define the rational function

$$F_{l;t}(\boldsymbol{p}) = X_{l;t}((1, \ldots, 1), \boldsymbol{p}) \in \mathbb{Q}_{\mathrm{sf}}(\boldsymbol{p}).$$

In general, for a subtraction free expression F in $\mathbb{Q}_{\mathrm{sf}}(x_1, \ldots, x_n)$ and an arbitrary semifield $\mathbb{P}$, we use the notation

$$F \mid_{\mathbb{P}} (y_1, \ldots y_n) \in \mathbb{P}$$

for the evaluation at $x_i = y_i$. This evaluation is well-defined (i.e., independent of the expression used) because of the universal property of the semifield $\mathbb{Q}_{\mathrm{sf}}(x_1, \ldots, x_n)$; see Section 2.1.

Theorem 3.20 (cf. [Fomin and Zelevinsky 2007, Proposition 3.6, Theorem 3.7]).

(1) *We have*

$$X_{l;t} \in \mathbb{Z}[x_i^{\pm}; p_{i,j} \mid i \in I, j \in [1, r_i]], \quad F_{l;t} \in \mathbb{Z}[p_{i,j} \mid i \in I, j \in [1, r_i]].$$

(2) *Let $\mathscr{A}$ be a generalized cluster algebra over an arbitrary semifield $\mathbb{P}'$, with an initial seed $\Sigma_{t_0} = (x, p, B)$. Then the cluster variables in $\mathscr{A}$ can be expressed in the initial cluster as*

$$x_{l;t} = \frac{X_{l;t} \mid_{\mathcal{F}} (x, p)}{F_{l;t} \mid_{\mathbb{P}'} (p)},$$

where $\mathcal{F}$ is the ambient field for $\mathscr{A}$.

Remark 3.21. Suppose the positivity for $x_{l;t}$ in the principal coefficients case (where we denote the semifield by $\mathbb{P}$) has been established. This means $X_{l;t}$ has a subtraction free expression as a Laurent polynomial (i.e., whose coefficients are in $\mathbb{NP}$). Then the evaluation $X_{l;t} \mid_{\mathcal{F}} (x, p)$ also has positive coefficients in $\mathbb{NP}'$, while $F_{l;t} \mid_{\mathbb{P}'} (p)$ is an element in $\mathbb{P}'$. Thus it follows by Theorem 3.20 that $x_{l;t}$ has positive coefficients in the case of arbitrary $\mathbb{P}'$.

3.5. *Cluster algebras with specialized coefficients.* We fix a field $\Bbbk$ of characteristic 0 and consider the case of geometric coefficients. In this case, the generalized cluster algebra $\mathscr{A}(\Sigma)$ for $\Sigma = (x, p, B)$ can be viewed as a subring of $\Bbbk\mathbb{P}[x_1^{\pm}, \ldots, x_n^{\pm}]$ where $\Bbbk\mathbb{P}$ is the group algebra of $\mathbb{P}$ over $\Bbbk$.

Let $\lambda : \mathbb{P} \to \Bbbk^*$ be a group homomorphism (which we will later refer to as an *evaluation*). It extends to a $\Bbbk$-algebra homomorphism

$$\lambda : \Bbbk\mathbb{P}[x_1^{\pm}, \ldots, x_n^{\pm}] \to \Bbbk[x_1^{\pm}, \ldots, x_n^{\pm}].$$

We denote the image of $\mathscr{A}(\Sigma) \otimes \Bbbk$ by $\mathscr{A}(\Sigma, \lambda)$. So we have a family of $\Bbbk$-algebras parametrized by $(\Bbbk^*)^l$ if the free abelian group $\mathbb{P}$ is of rank l. Each $\mathscr{A}(\Sigma, \lambda)$ is in fact the $\Bbbk$-subalgebra generated by cluster variables (with coefficients specialized by λ) within $\Bbbk[x_1^{\pm}, \ldots, x_n^{\pm}]$. These are what we call (generalized) *cluster algebras with specialized coefficients.*

We point out that an ordinary cluster algebra with trivial coefficients (i.e., when $\mathbb{P}$ is trivial) is actually a generalized cluster algebra with specialized coefficients. Suppose B is a skew-symmetrizable matrix and let r_i be the gcd of the i-th column (if that column is nonzero). Let $\mathscr{A}^{\mathrm{prin}}(\Sigma)$ be the generalized cluster algebra with principal coefficients where Σ has exchange matrix B. Choose a group homomorphism $\lambda : \mathrm{Trop}(p) \to \Bbbk^*$ such that the specialized exchange polynomials equals the usual cluster exchange binomial, i.e.,

$$\prod_{j=1}^{r_i} (\lambda(p_{i,j})u + v) = u^{r_i} + v^{r_i}.$$

Of course such λ always exists assuming $\Bbbk$ is algebraically closed. Then it is easy to check that every generalized mutation becomes an ordinary mutation: if $t \overset{k}{\text{---}} t'$,

$$x_{k;t'} = x_{k;t}^{-1} \left(\prod_{i \in I} x_i^{[b_{ik}^t]_+} + \prod_{i \in I} x_i^{[-b_{ik}^t]_+} \right).$$

Thus the algebra $\mathscr{A}^{\mathrm{prin}}(\Sigma, \lambda)$ has the exact same cluster variables as the ordinary cluster algebra with trivial coefficients, and can thus be viewed as an ordinary cluster algebra.

t	B^t	$p_{1,1;t}$	$p_{2,1;t}$	$p_{2,2;t}$	$x_{1;t}$	$x_{2;t}$
0	$\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$	t_{11}	t_{21}	t_{22}	A_1	A_2
1	$\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$	t_{11}^{-1}	t_{21}	t_{22}	$A_1^{-1}(1+t_{11}A_2)$	A_2
2	$\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$	t_{11}^{-1}	t_{21}^{-1}	t_{22}^{-1}	$A_1^{-1}(1+t_{11}A_2)$	$\begin{aligned} &A_2^{-1}\bigl(1+t_{21}A_1^{-1}(1+t_{11}A_2)\bigr) \\ &\times \bigl(1+t_{22}A_1^{-1}(1+t_{11}A_2)\bigr)\end{aligned}$
3	$\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$	t_{11}	$t_{11}^{-1}t_{21}^{-1}$	$t_{11}^{-1}t_{22}^{-1}$	$\begin{aligned}&A_1 A_2^{-1}\bigl((1+t_{21}A_1^{-1})(1+t_{22}A_1^{-1}) \\ &+ t_{11}t_{21}t_{22}A_1^{-2}A_2\bigr)\end{aligned}$	$\begin{aligned} &A_2^{-1}\bigl(1+t_{21}A_1^{-1}(1+t_{11}A_2)\bigr) \\ &\times \bigl(1+t_{22}A_1^{-1}(1+t_{11}A_2)\bigr)\end{aligned}$
4	$\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$	$t_{11}^{-1}t_{21}^{-1}t_{22}^{-1}$	$t_{11}t_{21}$	$t_{11}t_{22}$	$\begin{aligned}&A_1 A_2^{-1}\bigl((1+t_{21}A_1^{-1})(1+t_{22}A_1^{-1}) \\ &+ t_{11}t_{21}t_{22}A_1^{-2}A_2\bigr)\end{aligned}$	$A_2^{-1}(t_{21}+A_1)(t_{22}+A_1)$
5	$\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$	$t_{11}t_{21}t_{22}$	t_{22}^{-1}	t_{21}^{-1}	A_1	$A_2^{-1}(t_{21}+A_1)(t_{22}+A_1)$
6	$\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$	t_{11}	t_{22}	t_{21}	A_1	A_2

Table 1. Labeled seeds of $\mathscr{A}^{\mathrm{prin}}$.

3.6. *An example in type B_2 with principal coefficients.* We consider $\mathscr{A}^{\mathrm{prin}}(\boldsymbol{x}, \boldsymbol{p}, B)$ with principal coefficients for $B = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$ which is of type B_2 in the finite type classification [Fomin and Zelevinsky 2003; Chekhov and Shapiro 2014, Theorem 2.7]. We write $x_{i;t_0} = A_i$, and $p_{i,j;t_0} = t_{ij}$. For the subgraph

$$t_0 \xrightarrow{\ 1\ } t_1 \xrightarrow{\ 2\ } t_2 \xrightarrow{\ 1\ } t_3 \xrightarrow{\ 2\ } t_4 \xrightarrow{\ 1\ } t_5 \xrightarrow{\ 2\ } t_6$$

of $\mathbb{T}_2$, we have the associated labeled seeds calculated in Table 1

We note that the Σ_{t_6} is not exactly the same as the Σ_{t_0} but up to a switch of $p_{2,1}$ and $p_{2,2}$.

3.7. *Generalized Y-seeds.* We define generalized Y-seeds (with coefficients) and their mutations. The formulation to including coefficients in Y-seeds comes from [Bossinger et al. 2020]. The following definition is a generalization of [Bossinger et al. 2020, Definition 2.15], which is an enhancement of a Y-seed of [Fomin and Zelevinsky 2007].

Definition 3.22. A *generalized labeled Y-seed* (with coefficients) Δ is a triple $(\boldsymbol{y}, \boldsymbol{q}, B)$, where

- $\boldsymbol{q} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)$ is an n-tuple of r_i-tuples $\boldsymbol{q}_i = (q_{i,1}, \ldots q_{i,r_i})$ of elements in a semifield $\mathbb{P}$ for positive integers r_i, $1 \le i \le n$;
- $\boldsymbol{y} = \{y_1, \ldots, y_n\}$ is a collection of elements in some universal semifield $\mathbb{QP}_{\mathrm{sf}}(u_1, \ldots, u_l)$;
- B is a left skew-symmetrizable integer matrix such that the i-th column is divisible by r_i for every i.

Definition 3.23. For $k \in \{1, \ldots, n\}$, we define the *mutation* of a Y-seed $(\boldsymbol{y}, \boldsymbol{q}, B)$ in direction k as a new Y-seed $\mu_k(\boldsymbol{y}, \boldsymbol{q}, B) := ((y_i'), (\boldsymbol{q}_i'), B')$ in the following way:

$$B' = \mu_k(B); \tag{3-1}$$

$$q_{k,j}' = q_{k,j}^{-1} \quad \text{for } j \in [1, r_k];$$

$$\text{for } i \neq k, \ j \in [1, r_i], \quad q_{i,j}' = \begin{cases} q_{i,j} \cdot \left(\prod_{l=1}^{r_k} q_{k,l}^- \right)^{-\beta_{ik}} & \text{if } -\beta_{ki} > 0, \\ q_{i,j} \cdot \left(\prod_{l=1}^{r_k} q_{k,l}^+ \right)^{-\beta_{ik}} & \text{if } -\beta_{ki} \leq 0, \end{cases} \tag{3-2}$$

or equivalently

$$\text{for } i \neq k, \quad q_{i,j}' = q_{i,j} \prod_{l=1}^{r_k} (1 \oplus q_{k,l}^{\mathrm{sgn}(-\beta_{ki})})^{\beta_{ik}};$$

$$y_i' = \begin{cases} y_i \prod_{l=1}^{r_k} (q_{k,l}^{\mathrm{sgn}(\beta_{ik})} y_k^{\mathrm{sgn}(\beta_{ik})} + q_{k,l}^{\mathrm{sgn}(-\beta_{ik})})^{\beta_{ik}} & \text{if } i \neq k, \\ y_k^{-1} & \text{if } i = k. \end{cases} \tag{3-3}$$

As in Lemma 3.3, it is straightforward to check that the mutation μ_k on a generalized Y-seed is involutive in the same direction.

Definition 3.24. A *generalized Y-pattern* is an association

$$t \mapsto \Delta_t = (\boldsymbol{y}_t, \boldsymbol{q}_t, B^t)$$

to every vertex $t \in \mathbb{T}_n$ a generalized labeled Y-seed Δ_t such that if t and t' are connected by an edge labeled by $k \in I$, then we have

$$\Delta_{t'} = \mu_k(\Delta_t).$$

Definition 3.25. We say that a generalized Y-pattern has principal coefficients at a vertex $t_0 \in \mathbb{T}_n$ if $\mathbb{P}$ is the tropical semifield

$$\mathrm{Trop}(q_{i,j;t_0} \mid i \in I, j \in [1, r_i]).$$

Given a Y-pattern, the elements $y_{i;t}$ for $t \in \mathbb{T}_n$ are called *Y-variables*.

Remark 3.26. In the case that for any $i \in I$,

$$q_{i,1} = q_{i,2} = \cdots = q_{i,r_i},$$

a generalized Y-seed with coefficients as in Definition 3.22 becomes a labeled Y-seed with coefficients in [Bossinger et al. 2020]. In this case, the mutation formula of Y-variables is independent of the choice r_i. So we get back to the nongeneralized version by letting the coefficients $q_{i,j}, \ j \in [1, r_i]$, equal. While in the cluster case, one recovers the nongeneralized seed mutation by choosing $r_i = 1$. This asymmetry suggests that our generalization is a natural one.

To the best knowledge of the author, the generalized version of Y-patterns has not been considered in the literature. It is interesting to see if these generalized patterns appear naturally anywhere.

4. Generalized cluster varieties

Cluster varieties were introduced by Fock and Goncharov [2009], giving a geometric view to cluster algebras (of geometric types). We follow [Gross et al. 2015] to define relevant notions such as fixed data and seeds. However, in order to deal with generalized coefficients, some new gadgets are needed.

Definition 4.1. We recall the *fixed data* Γ from [Gross et al. 2015]. The fixed data Γ consists of

- a lattice N of finite rank with a skew-symmetric bilinear form $\omega : N \times N \to \mathbb{Q}$;
- an *unfrozen sublattice* $N_{\mathrm{uf}} \subset N$, a saturated sublattice of N;
- an index set $I = \{1, \ldots, \mathrm{rank}\, N\}$ and a subset $I_{\mathrm{uf}} = \{1, \ldots, \mathrm{rank}\, N_{\mathrm{uf}}\}$;
- positive integers d_i for $i \in I$ with greatest common divisor 1;
- a sublattice $N^\circ \subset N$ of finite index such that $\omega(N_{\mathrm{uf}}, N^\circ) \subset \mathbb{Z}$, $\omega(N, N_{\mathrm{uf}} \cap N^\circ) \subset \mathbb{Z}$;
- $M = \mathrm{Hom}(N, \mathbb{Z})$, $M^\circ = \mathrm{Hom}(N^\circ, \mathbb{Z})$;

4.1. *Generalized $\mathcal{A}$-cluster variety.*

Definition 4.2. Given fixed data Γ, an $\mathcal{A}$-*seed with* (*generalized*) *coefficients* is a pair $\boldsymbol{s} = (\boldsymbol{e}, \boldsymbol{p})$ consisting of a *seed* $\boldsymbol{e} = (e_i)_{i \in I}$ which is a labeled collection of elements in N and a labeled collection of tuples of coefficients $\boldsymbol{p} = (\boldsymbol{p}_i)_{i \in I_{\mathrm{uf}}}$, where $\boldsymbol{p}_i = (p_{i,j})_{j \in [1, r_j]}$ and $p_{i,j}$ belongs to some tropical semifield $\mathbb{P}$ such that

(1) $\{e_i \mid i \in I\}$ is a basis for N;

(2) $\{e_i \mid i \in I_{\mathrm{uf}}\}$ is a basis for N_{uf};

(3) $\{d_i e_i \mid i \in I\}$ is a basis for N°;

(4) for $i \in I_{\mathrm{uf}}$, the elements $w_i := \omega(-, d_i e_i)/r_i$ belong to M.

For such a seed $\boldsymbol{s}$, we define two matrices $B = B(\boldsymbol{s}) = (b_{ij})$ and $\tilde{B} = \tilde{B}(\boldsymbol{s}) = (\beta_{ij})$ by setting

$$b_{ij} := \omega(e_i, d_j e_j) \quad \text{and} \quad \beta_{ij} := \langle e_i, w_j \rangle = b_{ij}/r_j.$$

Definition 4.3. Given $\boldsymbol{s}$ an $\mathcal{A}$-seed with coefficients, for $k \in I_{\mathrm{uf}}$, we define the *mutation in direction* k, $\mu_k(\boldsymbol{s}) = (\boldsymbol{e}', \boldsymbol{p}')$ by

$$e_i' = \begin{cases} -e_k & \text{if } i = k, \\ e_i + [\langle e_i, -r_k w_k \rangle]_+ e_k & \text{if } i \neq k; \end{cases}$$

and

$$p_{k,j}' = p_{k,j}^{-1} \quad \text{for } j \in [1, r_k];$$

$$\text{for } i \neq k, \, j \in [1, r_i], \quad p_{i,j}' = \begin{cases} p_{i,j} \cdot (p_{k;-})^{\beta_{ki}} & \text{if } \beta_{ik} > 0, \\ p_{i,j} \cdot (p_{k;+})^{\beta_{ki}} & \text{if } \beta_{ik} \leq 0. \end{cases}$$

Remark 4.4. If we write $w_i' = \omega\left(-, \frac{d_i}{r_i} e_i'\right)$ as the mutations of w_i, then they are given by

$$w_i' = \begin{cases} -w_k, & \text{if } i = k; \\ w_i + [\langle r_k e_k, w_i \rangle]_+ w_k, & \text{if } i \neq k. \end{cases}$$

Denote the dual basis of (e_i) by (e_i^*) and the dual of $(e_i') = \mu_k(e)$ by $(e_i'^*)$. We have

$$e_i'^* = \begin{cases} -e_k^* + \sum_j [\langle e_j, -r_k w_k \rangle]_+ e_j^* & \text{if } i = k; \\ e_i^* & \text{if } i \neq k. \end{cases}$$

If there is no confusion, we will call an $\mathcal{A}$-seed with coefficients simply a seed.

Let $R = \Bbbk \mathbb{P}$, the group algebra of $\mathbb{P}$ over the ground field $\Bbbk$. To any $\mathcal{A}$-seed s, we associate a copy of the R-torus $T_{N,s}(R) := \text{Spec}(\Bbbk[M] \otimes_\Bbbk R)$.

Definition 4.5. To the mutation μ_k from s to $\mu_k(s)$, there is an associated birational morphism (over R)

$$\mu_k : T_{N,s}(R) \dashrightarrow T_{N,\mu_k(s)}(R), \quad \mu_k^*(z^m) = z^m f_k^{-\langle e_k, m \rangle},$$

where

$$f_k := \prod_{j=1}^{r_k} (p_{k,j}^- + p_{k,j}^+ z^{w_k}) \in R[M].$$

We call this birational transformation the *$\mathcal{A}$-cluster mutation* associated to the mutation μ_k of seeds.

Definition 4.6. We define the *oriented rooted tree* $\mathfrak{T}_n$ (where $n = |I_{\text{uf}}|$) as in [Gross et al. 2015]. It is the infinite tree generated from a root v_0 such that

(1) v_0 has outgoing edges labeled by $I_{\text{uf}} = \{1, \ldots, n\}$;

(2) any other vertex has one unique incoming edge, and outgoing edges labeled by I_{uf}.

Let $v_0 \in \mathfrak{T}_n$ be the root. Then for any other vertex $v \in \mathfrak{T}_n$, there is a unique oriented path from v_0 to v. We associate a seed s to the root v_0, the unique path from v to v_0 determines a seed s_v by applying the mutations in directions of the labelings in the path to the initial seed s. Therefore we have an association $v \mapsto s_v$ for $v \in \mathfrak{T}_n \setminus \{v_0\}$ and $v_0 \mapsto s$ such that for an edge $v \xrightarrow{k} v'$ in $\mathfrak{T}_n$, then

$$s_{v'} = \mu_k(s_v).$$

Suppose the unique path from v_0 to v walks through edges labeled by $k_1, k_2, \ldots, k_l$. There is then the birational map

$$\mu_{v_0, v} := \mu_{k_l} \circ \cdots \circ \mu_{k_2} \circ \mu_{k_1} : T_{N,s}(R) \dashrightarrow T_{N,s_v}(R).$$

For arbitrary two vertices v and v' in $\mathfrak{T}_n$, there is the birational map

$$\mu_{v,v'} := \mu_{v_0,v'} \circ \mu_{v_0,v}^{-1} : T_{N,s_v}(R) \dashrightarrow T_{N,s_{v'}}(R).$$

These birational maps surely satisfy the cocycle condition. We use the following lemma to glue T_{N,s_v} together.

Lemma 4.7 [Bossinger et al. 2020, Lemma 3.10; Gross et al. 2015, Proposition 2.4]. *Let $\mathcal{I}$ be a set and $\{S_i \mid i \in I\}$ be a collection of integral separated schemes of finite type over a locally Noetherian ring R, with birational maps (of R-schemes) $f_{ij} : S_i \dashrightarrow S_j$ for all i, j, verifying the cocycle condition $f_{jk} \circ f_{ij} = f_{ik}$ as rational maps and such that f_{ii} is the identity map. Let $U_{ij} \subset S_i$ be the largest open subscheme such that $f_{ij} : U_{ij} \to f_{ij}(U_{ij})$ is an isomorphism. Then there is an R-scheme*

$$S = \bigcup_{i \in \mathcal{I}} S_i$$

obtained by gluing the S_i along the open sets U_{ij} via the maps f_{ij}.

Definition 4.8. Let Γ be fixed data and s be an $\mathcal{A}$-seed with coefficients. We apply Lemma 4.7 to glue together the collection of tori indexed by $\mathfrak{T}_n$ to get the *generalized $\mathcal{A}$-cluster variety* associated to s (as an R-scheme)

$$\mathcal{A}_s = \mathcal{A}_{\Gamma,s} := \bigcup_{v \in \mathfrak{T}} T_{N,s_v}(R).$$

We now explain how to obtain a generalized cluster pattern from $\mathcal{A}_s$, justifying the name generalized $\mathcal{A}$-cluster variety. We assume $N_{\mathrm{uf}} = N$, thus $I_{\mathrm{uf}} = I$.[1]

Recall we have the association $v \mapsto s_v = \mu_{v_0,v}(s)$ for $v \in \mathfrak{T}_n$. We write $s_v = (e_v, p_v)$ where $e_v = (e_{i;v} \mid i \in I)$, $p_v = (p_{i;v} \mid i \in I)$ and $p_{i;v} = (p_{i,j;v} \mid j \in [1, r_i])$.

Sending v_0 to any vertex t_0 in the n-regular tree $\mathbb{T}_n$ gives a unique surjective map

$$\pi : \mathfrak{T}_n \to \mathbb{T}_n, \quad v_0 \mapsto t_0$$

such that the labeling on edges is preserved.

For any seed $v \in \mathfrak{T}_n$, there is the corresponding labeled seed with coefficients (in the sense of Definition 3.1)

$$\Sigma_v = \Sigma(s_v) := (x_v, p_v, B^v),$$

where

$$x_{i,v} := \mu_{v_0,v}^*(z^{e_{i;v}^*}) \in \mathbb{QP}(x_1, \ldots, x_n), \quad b_{ij}^v := \omega(e_{i;v}, d_j e_{j;v}),$$

where $x_i = x_{i,v_0}$.

Lemma 4.9. *If two vertices v and v' vertices of $\mathfrak{T}_n$ descend to the same vertex in $\mathbb{T}_n$, i.e., $\pi(v) = \pi(v')$, then their corresponding labeled seeds with coefficients are identical, i.e., $\Sigma_v = \Sigma_{v'}$.*

Proof. Suppose the unique path in $\mathfrak{T}_n$ from v_0 to v goes through edges labeled by $k_1, \ldots, k_l$ in order. We show in the following by induction that

$$\mu_{k_l} \circ \cdots \circ \mu_{k_1}(\Sigma_{v_0}) = \Sigma_v,$$

where the operation μ_k is the mutation in direction k of labeled seeds with coefficients in the sense of Definition 3.2.

---

[1]This is because we do not define cluster patterns with frozen directions. This can be done by making mutations only available at a subset of a given cluster, leaving the rest variables frozen. However, one can always treat the frozen variables as making up coefficients in a cluster pattern.

Let $v_1 \xrightarrow{k} v_2$ be in $\mathfrak{T}_n$. Then one checks $B^{v_2} = \mu_k(B^{v_1})$ using the fact that $\boldsymbol{e}_{v_2} = \mu_k(\boldsymbol{e}_{v_1})$, which is standard from [Gross et al. 2015]. The coefficients parts $\boldsymbol{p}_{v_1}$ and $\boldsymbol{p}_{v_2}$ are related by the mutation μ_k by definition. So we only need to check that $\boldsymbol{x}_{v_1}$ and $\boldsymbol{x}_{v_2}$ are also related by μ_k.

Note that $\mu^*_{v_0,v_2} = \mu^*_{v_0,v_1} \circ \mu^*_k$. So we have for $i \neq k$

$$x_{i;v_2} = \mu^*_{v_0,v_1}(\mu^*_k(z^{e^*_{i;v_2}})) = \mu^*_{v_0,v_1}(z^{e^*_{i;v_1}}) = x_{i;v_1},$$

$$x_{k;v_2} = \mu^*_{v_0,v_1}(\mu^*_k(z^{e^*_{k;v_2}}))$$

$$= \mu^*_{v_0,v_1}\left(z^{-e^*_{k;v_1} + \sum[-b^{v_1}_{ik}]_+ e^*_{i;v_1}} \prod_{j=1}^{r_k}(p^-_{k,j;v_1} + p^+_{k,j;v_1} z^{w_{k;v_1}})\right)$$

$$= \mu^*_{v_0,v_1}\left(z^{-e^*_{k;v_1}} \prod_{j=1}^{r_k}(p^-_{k,j;v_1} z^{w^-_{k;v_1}} + p^+_{k,j;v_1} z^{w^+_{k;v_1}})\right)$$

$$= \mu^*_{v_0,v_1}(z^{-e^*_{k;v_1}}) \prod_{j=1}^{r_k}\left(p^-_{k,j;v_1} \mu^*_{v_0,v_1}(z^{w^-_{k;v_1}}) + p^+_{k,j;v_1} \mu^*_{v_0,v_1}(z^{w^+_{k;v_1}})\right)$$

$$= x^{-1}_{k;v_1} \prod_{j=1}^{r_k}\left(p^-_{k,j;v_1} \prod_{i \in I} x^{[-\beta_{ik}]_+}_{i;v_1} + p^+_{k,j;v_1} \prod_{i \in I} x^{[\beta_{ik}]_+}_{i;v_1}\right).$$

The only unexplained notation in the above equations is that for any $w = \sum_{i \in I} a_i e^*_i \in M$, we write

$$w^- := \sum_{i \in I}[-a_i]_+ e^*_i \quad \text{and} \quad w^+ := \sum_{i \in I}[a_i]_+ e^*_i.$$

Now we have checked that $\mu_k(\Sigma_{v_1}) = \Sigma_{v_2}$. By induction on the distance from v to the root v_0, we conclude that $\mu_{k_l} \circ \cdots \circ \mu_{k_1}(\Sigma_{v_0}) = \Sigma_v$ for any $v \in \mathfrak{T}_n$. Since μ_k is involutive, we can reduce the sequence $(k_1, \cdots, k_l)$ by deleting pairs of consecutive identical indices until there is none. So Σ_v only depends on the reduced sequence of edge labels from v_0 to v. Now notice that two vertices v and v' in $\mathfrak{T}_n$ have the same projection t in $\mathbb{T}_n$ if and only if they have the same reduced sequence of edge labels from v_0, meaning the same labeled seed with coefficients $\Sigma_t := \Sigma_v = \Sigma_{v'}$. $\square$

Proposition 4.10. *According to the above lemma, we have that the labeled seeds Σ_v and $\Sigma_{v'}$ are equal if $\pi(v) = \pi(v') = t \in \mathbb{T}_n$. So we can denote them all by Σ_t. The association $t \mapsto \Sigma_t$ for every $t \in \mathbb{T}_n$ is a cluster pattern.*

Proof. Suppose the unique path from t_0 to some $t \in \mathbb{T}_n$ walks through edges in order of $k_1, \ldots, k_l$. Then already in the proof of the above lemma, we have

$$\Sigma_t = \mu_{k_l} \circ \cdots \circ \mu_{k_1}(\Sigma_{t_0}).$$

This association by definition gives a cluster pattern. $\square$

Definition 4.11. The (generalized) *upper cluster algebra* $\overline{\mathscr{A}}(\boldsymbol{s})$ (of an $\mathcal{A}$-seed $\boldsymbol{s}$ with coefficients) is defined to be the R-algebra

$$H^0(\mathcal{A}_{\boldsymbol{s}}, \mathcal{O}_{\mathcal{A}_{\boldsymbol{s}}}) = \bigcap_{v \in \mathfrak{T}_n} H^0(T_{N,s_v}(R), \mathcal{O}_{T_{N,s_v}(R)}),$$

the ring of regular functions on the (generalized) $\mathcal{A}$-cluster variety $\mathcal{A}_{\boldsymbol{s}}$.

By definition the upper cluster algebra is the algebra of all Laurent polynomials that remains Laurent polynomials after an arbitrary sequence of mutations. It follows from the Laurent phenomenon that all cluster variables are elements in the upper cluster algebra, thus the inclusion

$$\mathscr{A}(s) \subset \overline{\mathscr{A}}(s),$$

where the former denotes the subalgebra generated by cluster variables, i.e., the cluster algebra (over R).

The notion of principal coefficients can be easily translated into the current setting.

Definition 4.12. An $\mathcal{A}$-seed s is said to have *principal coefficients* if the associated labeled seed $\Sigma(s)$ has principal coefficients.

The associated cluster pattern with $t_0 \mapsto \Sigma(s)$, $t \mapsto \Sigma(s_v)$ (where $t = \pi(v)$) then has principal coefficients at t_0. In this case, we denote the corresponding cluster variety by $\mathcal{A}_s^{\mathrm{prin}}$.

4.2. Generalized $\mathcal{X}$-cluster variety. Given fixed data Γ as in the last section, we define the notion of (generalized) $\mathcal{X}$-seeds with coefficients.

Definition 4.13. An $\mathcal{X}$-seed with (generalized) coefficients $s = (e, q)$ is the same as an $\mathcal{A}$-seed. We use the symbol q instead of p to stress that it is an $\mathcal{X}$-seed.

What distinguish $\mathcal{X}$-seeds with $\mathcal{A}$-seeds is the mutation.

Definition 4.14. Given an $\mathcal{X}$-seed $s = (e, q)$, we define the *mutation in direction k*, $\mu_k(s) = (e', q')$ by

$$e'_i = \begin{cases} -e_k & \text{if } i = k, \\ e_i + [\langle e_i, -r_k w_k \rangle]_+ e_k & \text{if } i \neq k; \end{cases}$$

and

$$q'_{k,j} = q_{k,j}^{-1} \quad \text{for } j \in [1, r_k];$$

$$\text{for } i \neq k, \, j \in [1, r_i], \quad q'_{i,j} = \begin{cases} q_{i,j} \cdot (q_{k;-})^{-\beta_{ik}} & \text{if } -\beta_{ki} > 0, \\ q_{i,j} \cdot (q_{k;+})^{-\beta_{ik}} & \text{if } -\beta_{ki} \leq 0, \end{cases}$$

So the pure seed part e behaves in the same way under mutation as in an $\mathcal{A}$-seed while the coefficients part q mutates differently, but same as the coefficients in a labeled Y-seed. Roughly, if in $\mathcal{A}$-seeds, the matrix B governs the mutation of coefficients, then in $\mathcal{X}$-seeds, $-B^T$ does the job.

Definition 4.15. Let $s = (e, q)$ be an $\mathcal{X}$-seed with coefficients. Then there is the associated *$\mathcal{X}$-cluster mutation*

$$\mu_k : T_M(R) \dashrightarrow T_M(R), \quad \mu_k^*(z^n) = z^n \cdot \left(\prod_{l=1}^{r_k} (q_{k,l}^- + q_{k,l}^+ z^{e_k}) \right)^{-\langle n, -w_k \rangle},$$

where $T_M(R)$ is the R-torus $\mathrm{Spec}(\Bbbk[N] \otimes R)$.

Definition 4.16. Let s be an $\mathcal{X}$-seed for Γ. Then there is a unique association $v \mapsto s_v$ for every $v \in \mathfrak{T}_n$ such that $v_0 \mapsto s$ and adjacent associated seeds are related by mutations of $\mathcal{X}$-seeds in corresponding directions. Define the *generalized $\mathcal{X}$-cluster variety* associated to s to be the R-scheme

$$\mathcal{X}_s = \mathcal{X}_{\Gamma,s} := \bigcup_{v \in \mathfrak{T}_n} T_{M,s_v}(R)$$

obtained by gluing $T_{M,s_v}(R)$ via $\mathcal{X}$-cluster mutations using Lemma 4.7.

Write $s_v = ((e_{i;v}), (q_{i;v}))$. Let us keep track of the monomials $z^{e_{i;v}} \in \Bbbk[N]$ (instead of $z^{e_{i;v}^*}$ in the $\mathcal{A}$-case). We define

$$y_{i;v} := \mu_{v,v_0}^*(z^{e_{i;v}}) \in \mathrm{Frac}(\Bbbk[N] \otimes R).$$

It turns out that these $y_{i;v}$ are the Y-variables of the Y-pattern induced by the $\mathcal{X}$-seed s described as follows. We take s as the initial seed. Analogous to the $\mathcal{A}$-situation, any vertex $v \in \mathfrak{T}_n$ descends to a vertex $t = \pi(v) \in \mathbb{T}_n$.

Proposition 4.17. *For $v \in \mathfrak{T}_n$, define the generalized labeled Y-seed $\Delta_v = ((y_{i;v}), (q_{i;v}), B^v)$. Then we have $\Delta_v = \Delta_{v'}$ if $\pi(v) = \pi(v') = t \in \mathbb{T}_n$. Then the association $t \mapsto \Delta_t$ for $t \in \mathbb{T}_n$ is a generalized Y-pattern with coefficients where $\Delta_t := \Delta_v$ for any v such that $t = \pi(v)$.*

Proof. We first note that the Y-variables $y_{i;v}$ live in the universal semifield $\mathbb{QP}_{\mathrm{sf}}(y_1, \ldots, y_n)$ where $y_i = z^{e_i}$ are the initial Y-variables. The proof is completely analogous to Proposition 4.10. We leave the details to the reader. $\qquad\square$

4.3. *Special coefficients.* By construction, given an $\mathcal{A}$-seed (resp. $\mathcal{X}$-seed) s, there is the flat family

$$\pi_{\mathcal{A}} : \mathcal{A}_s \to \mathrm{Spec}\, R \quad (\text{resp. } \pi_{\mathcal{X}} : \mathcal{X}_s \to \mathrm{Spec}\, R).$$

Let λ be a $\Bbbk$-point of $\mathrm{Spec}\, R$. Then the special fiber $\pi^{-1}(\lambda)$ is a $\Bbbk$-scheme and can be viewed as a *generalized cluster variety with special coefficients*, denoted by $\mathcal{A}_{s,\lambda}$ (resp. $\mathcal{X}_{s,\lambda}$). They are also glued together by tori via birational morphisms (namely the $\mathcal{A}$- or $\mathcal{X}$-mutations specialized at λ)

$$\mathcal{A}_{s,\lambda} = \bigcup_{v \in \mathfrak{T}_n} T_{N,v}, \quad \mathcal{X}_{s,\lambda} = \bigcup_{v \in \mathfrak{T}_n} T_{M,v}.$$

The $\mathcal{A}$-type varieties (resp. $\mathcal{X}$-type varieties) lead to cluster patterns (resp. Y-patterns) with specialized coefficients. We have as before in the $\mathcal{A}$-case the inclusion of algebras

$$\mathscr{A}(s, \lambda) \subset \overline{\mathscr{A}}(s, \lambda) := H^0(\mathcal{A}_{s,\lambda}, \mathcal{O}_{\mathcal{A}_{s,\lambda}}).$$

4.4. *Cluster duality.* The cluster duality of Fock and Goncharov predicts, in the ordinary case, that the varieties $\mathcal{A}_s$ and $\mathcal{X}_s$ are dual in the sense that the upper cluster algebra $\overline{\mathscr{A}}(s)$ has a basis parametrized by the tropical set $\mathcal{X}^{\mathrm{trop}}(\mathbb{Z})$ (and vice versa). Note here s is viewed as a seed without coefficients so we do not need to distinguish between $\mathcal{A}$- and $\mathcal{X}$-seeds. Strictly speaking, this statement is not true as in some cases $\mathcal{X}_s$ may have too few regular functions [Gross et al. 2015]. This duality (named the

Fock–Goncharov full conjecture) is the main subject of study (on a precise modified formulation and when it is true) in [Gross et al. 2018].

Our point of view of is that it is more natural to include generalized cluster varieties in cluster dualities, which we will demonstrate in the principal coefficients case. We denote the $\mathcal{X}$-cluster variety with principal coefficients by $\mathcal{X}_s^{\mathrm{prin}}$, where the coefficient group is the tropical semifield

$$\mathbb{P} = \mathrm{Trop}(q_{ij} \mid i \in I, \, j \in [1, r_i]).$$

The scheme $\mathcal{X}_s^{\mathrm{prin}}$ is over $\mathrm{Spec}(R)$ where $R = \Bbbk\mathbb{P}$. There are evaluations λ sending q_{ij} to $\lambda_{ij} \in \Bbbk^*$. Each λ specifies an $\mathcal{X}$-cluster variety with special coefficients as in the following diagram:

$$\begin{array}{ccc}
\mathcal{X}_{s,\lambda}^{\mathrm{prin}} & \hookrightarrow & \mathcal{X}_s^{\mathrm{prin}} \\
\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{X}}} \\
\mathrm{Spec}(\Bbbk) & \xrightarrow{\ \lambda\ } & \mathrm{Spec}(R)
\end{array}$$

With a general choice coefficients, $\mathcal{X}_{s,\lambda}^{\mathrm{prin}}$ should be considered mirror dual to the family

$$\pi_{\mathcal{A}} : \mathcal{A}_s^{\mathrm{prin}} \to \mathrm{Spec}(R),$$

where s is viewed as an $\mathcal{A}$-seed with coefficients. We shall not fully justify this statement in this paper, but instead will show that the family $\pi_{\mathcal{A}} : \mathcal{A}_s^{\mathrm{prin}} \to \mathrm{Spec}(R)$ (as well as the generalized cluster algebra $\mathscr{A}^{\mathrm{prin}}(s)$) can be reconstructed from $\mathcal{X}_{s,\lambda}^{\mathrm{prin}}$, through a consistent wall-crossing structure (or scattering diagram) $\mathfrak{D}_s$ associated to $\mathcal{X}_{s,\lambda}^{\mathrm{prin}}$; see Section 8.

5. Toric models and mutations

This section is a generalization of [Gross et al. 2015, Section 3] aiming for generalized cluster varieties. A *log Calabi–Yau pair* (X, D) is a smooth projective variety X (over an algebraically closed field $\Bbbk$) with a reduced simple normal crossing divisor D such that $K_X + D = 0$ where K_X is the canonical divisor of X. A *log Calabi–Yau variety* U is the interior of a log Calabi–Yau pair (X, D), i.e., $U = X \setminus D$. Described in [Gross et al. 2015], particularly relevant in cluster theory are log Calabi–Yau pairs (X, D) obtained from a blow-up $\pi : X \to X_{\Sigma}$ where X_{Σ} is the toric variety associated to a fan Σ in $\mathbb{R}^n$. The blow-up is along a hypersurface in the toric boundary of X_{Σ}, and D is given by the strict transform of the toric boundary. We will see that both generalized $\mathcal{X}$- and $\mathcal{A}$-varieties can be realized as log Calabi–Yau varieties obtained this way (up to codimension two subsets).

5.1. *Toric models.* Fix a lattice $N \cong \mathbb{Z}^n$ and let M be its dual. Suppose for $i \in I = [1, l]$ we have pairs of vectors $(e_i, w_i) \in N \times M$ such that $\langle e_i, w_i \rangle = 0$. We assume that all nonzero e_i are primitive, but some of them may equal. For each i, we fix a positive integer r_i. We also take functions (elements in $\Bbbk[M]$)

$$f_i = a_{i,0} + a_{i,1} z^{w_i} + \cdots + a_{i,r_i} z^{r_i w_i}$$

with nonzero $a_{i,0}$ and a_{i,r_i}.

We construct in below a log Calabi–Yau variety U_Λ using the data

$$\Lambda := ((e_i)_{i \in I}, (w_i)_{i \in I}, (f_i)_{i \in I}).$$

The following construction is what we mean by a *toric model* for U_Λ and we call such Λ a *toric model data*.

Construction 5.1 (cf. [Gross et al. 2015], Construction 3.4]). Given the data Λ, consider the fan

$$\Sigma = \Sigma_\Lambda := \{\mathbb{R}_{\geq 0} e_i \mid i \in I\} \cup \{0\}$$

in $N_\mathbb{R}$. Let X_Σ be the toric variety defined by Σ, and D_i be the irreducible toric boundary divisor corresponding to $\mathbb{R}_{\geq 0} e_i$. Note that since $\langle e_i, w_i \rangle = 0$, z^{w_i} does not vanish on D_i. Let Z_i be the zero locus of f_i on D_i, i.e., the closed subscheme $\overline{V}(f_i) \cap D_i$, which is a hypersurface. Blow up X_Σ along $\bigcup_{i=1}^l Z_i$ to obtain

$$\pi : \widetilde{X}_\Sigma \to X_\Sigma.$$

Let $\widetilde{D}_i$ be the strict transform of D_i in $\widetilde{X}_\Sigma$. Then the open subscheme $U_\Lambda := \widetilde{X}_\Sigma \setminus \bigcup_i \widetilde{D}_i$ is a log Calabi–Yau variety.

Definition 5.2. For $k \in I$, we say a toric model data Λ *k-mutable* if the pairs (e_i, w_i) satisfy the condition

$$\langle e_i, w_k \rangle = 0 \implies \langle e_k, w_i \rangle = 0$$

for any $i \in I$.

We define mutations of a k-mutable toric model data.

Definition 5.3. Let Λ be a k-mutable toric model data and $\Lambda' = ((e_i'), (w_i'), (f_i'))$ be another set of data. Write $\beta_{ij} = \langle e_i, w_j \rangle$. We write $\Lambda' = \mu_k(\Lambda)$ (or say they are μ_k-*equivalent*) if they satisfy the following conditions:

- $e_k' = -e_k$ and $w_k' = -w_k'$;
- if $i \neq k$ and $\beta_{ik} \geq 0$, $e_i' = e_i$ and $w_i' = w_i$;
- if $i \neq k$ and $\beta_{ik} \leq 0$, $e_i' = e_i - \langle e_i, r_k w_k \rangle e_k$ and $w_i' = w_i + \langle e_k, w_i \rangle r_k w_k$;

and if writing $f_i' = a_{i,0}' + a_{i,1}' z^{w_i'} + \cdots + a_{i,r_i}' z^{r_i w_i'}$,

- $a_{k,j}' = a_{k,r_k - j}$ for $j \in [1, r_k]$;
- for $i \neq k$, $j \in [1, r_i]$,

$$a_{i,j}'/a_{i,0}' = \begin{cases} (a_{k,0})^{j\beta_{ki}} \cdot a_{i,j}/a_{i,0} & \text{if } \beta_{ik} > 0, \\ (a_{k,r_k})^{j\beta_{ki}} \cdot a_{i,j}/a_{i,0} & \text{if } \beta_{ik} \leq 0. \end{cases} \tag{5-1}$$

We note that the mutation μ_k is not deterministic for the (f_i) part, and is not involutive for the $((e_i), (w_i))$ part.

Applying Construction 5.1 to $\Lambda' = \mu_k(\Lambda)$, we obtain another log Calabi–Yau variety $U_{\Lambda'}$. Note that both U_Λ and $U_{\Lambda'}$ contain the torus T_N. Consider the birational morphism

$$\mu_k : T_N \dashrightarrow T_N, \quad \mu_k^*(z^m) = z^m \cdot f_k^{-\langle m, e_k \rangle}.$$

The following theorem is a generalization of the results in [Gross et al. 2015, Section 3].

Theorem 5.4. *The birational morphism μ_k extends to an isomorphism $\mu_k : U_\Lambda \to U_{\Lambda'}$ outside codimension two subsets if $\dim \overline{V}(f_k) \cap Z_i < \dim Z_i$ whenever $\langle e_i, w_k \rangle = 0$ for $i \in I$.*

Proof. We first make up some auxiliary varieties. Let $\Sigma^+ = \Sigma \cup \{\mathbb{R}_{\geq 0} e_k'\}$ and $\Sigma^- = \Sigma' \cup \{\mathbb{R}_{\geq 0} e_k\}$. We can blow up X_{Σ^+} (resp. X_{Σ^-}) in the same way as we do so for X_σ (resp. $X_{\Sigma'}$) to obtain $\widetilde{X}_+$ (resp. $\widetilde{X}_-$). Removing the strict transforms of the toric boundaries, we can still get U_Λ and $U_{\Lambda'}$. Following Lemma 3.6 in [Gross et al. 2015], we show that μ_k extends to an isomorphism (outside codimension two subsets) between $\widetilde{X}_+$ and $\widetilde{X}_-$, mapping the toric boundary of one to that of the other.

Suppose we only blow up X_{Σ^+} along Z_k and X_{Σ^-} along Z_k'. Then the blow-up $\widetilde{X}_+$ has a covering of open subsets

$$\widetilde{X}_+ = \widetilde{\mathbb{P}}_+ \cup \left(\bigcup_{i \neq k} U_i \right) \tag{5-2}$$

where $\widetilde{\mathbb{P}}_+$ is the blow-up along Z_k of the toric variety of the fan $\{\mathbb{R}_{\geq 0} e_k', \mathbb{R}_{\geq 0} e_k\}$ and U_i is the standard open toric chart corresponding to the ray $\mathbb{R}_{\geq 0} e_i$. Replacing U_i with $U_i \setminus \overline{V}(f_k)$ for $i \neq k$, (5-2) is still a covering but up to codimension two (with $\overline{V}(f_k) \cap D_i$ missing). More precisely, f_k is a regular function on U_i if $\langle w_k, e_i \rangle \geq 0$. In this case, $\overline{V}(f_k) \cap D_i$ is just the zero locus of the restriction of f_k on D_i, i.e., $V(f_k) \cap D_i$. As z^{w_k} vanishes on Z_i when $\langle w_k, e_i \rangle > 0$, $\overline{V}(f_k) \cap D_i = \varnothing$ since f_k has nonzero constant term. When $\langle w_k, e_i \rangle < 0$, then $\overline{V}(f_k) \cap D_i = V(z^{-r_k w_k} f_k) \cap D_i$ where $z^{-r_k w_k} f_k = f_k'$ is a regular function on U_i. So $\overline{V}(f_k) \cap D_i$ is still empty since f_k' has nonzero constant. Therefore we only fail to cover $\overline{V}(f_k) \cap D_i$ when $\langle w_k, e_i \rangle = 0$, which is a codimension two subset.

By Lemma 3.2 of [Gross et al. 2015], μ_k extends to a regular isomorphism from $\widetilde{\mathbb{P}}_+$ to $\widetilde{\mathbb{P}}_-$. Here $\widetilde{\mathbb{P}}_-$ is the blow-up along Z_k' of the toric variety defined by the fan $\{\mathbb{R}_{\geq 0} e_k', \mathbb{R}_{\geq 0} e_k\}$. We check that μ_k also extends to a regular isomorphism from $U_i \setminus \overline{V}(f_k)$ to $U_i' \setminus \overline{V}(f_k')$. Note that these are affine schemes so we check that μ_k^* extends to an isomorphism between their rings of regular functions. There are two cases.

(1) If $\langle e_i, w_k \rangle \geq 0$, then $e_i' = e_i$. Note that f_k is a regular function on U_i as well as on U_i'. Thus we have

$$U_i \setminus \overline{V}(f_k) = U_i \setminus V(f_k) \quad \text{and} \quad U_i' \setminus \overline{V}(f_k') = U_i' \setminus V(f_k).$$

For $\langle m, e_i \rangle \geq 0$, z^m defines a regular function on U_i' and

$$\mu_k^*(z^m) = z^m f_k^{-\langle m, e_k \rangle}$$

is a regular function on $U_i \setminus V(f_k)$.

(2) If $\langle e_i, w_k \rangle < 0$, then $e_i' = e_i - \langle e_i, r_k w_k \rangle e_k$. Instead of f_k, the function $f_k' = z^{-r_k w_k} f_k$ is a regular function on U_i and $\overline{V}(f_k) = V(f_k')$. For $\langle m, e_i' \rangle \geq 0$ and z^m a regular function on U_i', we have

$$\mu_k^*(z^m) = z^m f_k^{-\langle m, e_k \rangle} = z^{m - r_k w_k \langle m, e_k \rangle} (f_k')^{-\langle m, e_k \rangle}.$$

We check that $\langle m - r_k w_k \langle m, e_k \rangle, e_i \rangle = \langle m - r_k w_k \langle m, e_k \rangle, e_i' + \langle e_i, r_k w_k \rangle e_k \rangle = \langle m, e_i' \rangle > 0$. Thus $\mu_k^*(z^m)$ is a regular function on $U_i \setminus \overline{V}(f_k) = U_i \setminus V(f_k')$.

Therefore μ_k^* is a morphism between regular functions. In all the cases above, one checks that sending z^m to $z^m f_k^{\langle m, e_k \rangle}$ is the inverse of μ_k^*. Summarizing, we have so far proven that there is an isomorphism

$$\mu_k : U_+ := \widetilde{\mathbb{P}}_+ \cup \Big(\bigcup_{i \neq k} U_i \setminus \overline{V}(f_k) \Big) \to U_- := \widetilde{\mathbb{P}}_- \cup \Big(\bigcup_{i \neq k} U_i' \setminus \overline{V}(f_k') \Big)$$

extending the birational morphism μ_k between tori.

Now we analyze the impact of blowing up the hypersurfaces Z_i (and Z_i') for $i \neq k$. When $\langle w_k, e_i \rangle \neq 0$, as discussed $D_i \cap \overline{V}(f_k) = \varnothing$, so $Z_i \subset D_i$ is contained in U_+. Since $\langle w_k', e_i' \rangle = -\langle w_k, e_i \rangle \neq 0$, the same is true for Z_i', i.e., $Z_i' \subset U_-$. We would like to show that $\mu_k(Z_i) = Z_i'$ when $\langle w_k, e_i \rangle \neq 0$. There are two cases.

(1) Suppose $\langle w_k, e_i \rangle > 0$. In this case, $e_i' = e_i$ and $w_i' = w_i$. By definition $Z_i' = D_i' \cap V(f_i') = V(z^{m_0}) \cap V(f_i') \subset U_i'$ for some m_0 such that $\langle m_0, e_i' \rangle = 1$. Now we have $\mu_k^*(z^{m_0}) = z^{m_0} f_k^{-\langle m_0, e_k \rangle}$ and

$$\mu_k^*(f_i') = a_{i,0}' + a_{i,1}' z^{w_i} f_k^{-\langle w_i, e_k \rangle} + \cdots + a_{i,r_i}' z^{r_i w_i} f_k^{-\langle r_i w_i, e_k \rangle}.$$

Note that f_k is invertible on $U_i \setminus \overline{V}(f_k)$ and restricts to constant p_{k0} on D_i. So $V(\mu_k^*(z^{m_0}))$ is just the divisor D_i and

$$\mu_k^*(f_i')|_{D_i} = a_{i,0}' + a_{i,1}' a_{k,0}^{-\beta_{ki}} z^{w_i} + \cdots a_{i,r_i}' a_{k,0}^{-r_i \beta_{ki}} z^{r_i w_i} = \lambda \cdot f_i|_{D_i}.$$

for some nonzero $\lambda \in \Bbbk$ by the μ_k-equivalence assumption on Λ and Λ'. Therefore $\mu_k(Z_i) = Z_i'$.

(2) Suppose $\langle w_k, e_i \rangle < 0$. In this case we have $e_i' = v_i - \langle r_k w_k, e_i \rangle e_k$ and $w_i' = w_i + \langle w_i, e_k \rangle r_k w_k$. Still $Z_i' = V(z^{m_0}) \cap V(f_i')$. Now instead of f_k, the function $f_k' = z^{-r_k w_k} f_k$ is a regular function on U_i and restricts to constant a_{k,r_k} on D_i. First, $\mu_k^*(z^{m_0}) = z^{m_0 - \langle m_0, e_k \rangle r_k w_k} (f_k')^{\langle m_0, e_k \rangle}$. Since f_k' is invertible on $U_i \setminus V(f_k)$, $V(\mu_k^*(z^{m_0})) = D_i$ as $\langle m_0 + \langle m_0, e_k \rangle r_k w_k, e_i \rangle = 1$. Secondly we have

$$\mu_k^*(f_i') = a_{i,0}' + a_{i,1}' z^{w_i'} f_k^{-\langle w_i, e_k \rangle} + \cdots + a_{i,r_i}' z^{r_i w_i'} f_k^{-\langle r_i w_i, e_k \rangle}$$
$$= a_{i,0}' + a_{i,1}' z^{w_i' - \langle w_i, e_k \rangle r_k w_k} (f_k')^{-\langle w_i, e_k \rangle} + \cdots + a_{i,r_i}' z^{r_i w_i' - \langle r_i w_i, e_k \rangle r_k w_k} (f_k')^{-\langle r_i w_i, e_k \rangle}.$$

Hence

$$\mu_k^*(f_i')|_{D_i} = a_{i,0}' + a_{i,1}' a_{k,r_k}^{-\beta_{ki}} z^{w_i} + \cdots + a_{i,r_i}' a_{k,r_k}^{-r_i \beta_{ki}} z^{r_i w_i} = \lambda \cdot f_i|_{D_i}$$

for some nonzero $\lambda \in \Bbbk$ again by the μ_k-equivalence assumption. Therefore in this case we also have $\mu_k(Z_i) = Z_i'$.

Finally we consider the case $\langle w_k, e_i \rangle = 0$. The argument we need is exactly the same as in the last paragraph of the proof in [Gross et al. 2015]. By the assumption $\langle w_k, e_i \rangle = 0 \implies \langle w_i, e_k \rangle = 0$, so we have

$$\mu_k^*(f_i') = f_i,$$

and thus $\mu_k(Z_i) = Z_i'$. The problem is that Z_i may not be fully contained in $D_i \setminus V(f_k)$, with $V(f_k) \cap Z_i$ missing. If $V(f_k) \cap Z_i$ contains a irreducible component of Z_i, then U_Λ would contain the corresponding exceptional divisor while blowing up in U_+ does not. However the isomorphism $\mu_k : U_+ \to U_-$ need not extend as isomorphism across this exceptional divisor. Now we need the further hypothesis $\dim V(f_k) \cap Z_i < \dim Z_i$ so that the missing part in the blow-up center is of at least codimension three in U_i. After blowing up the corresponding locus in U_+ and U_+, we have the diagram

$$\begin{array}{ccc} \widetilde{U}_+ & \xrightarrow{\mu_k} & \widetilde{U}_- \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ U_+ & \xrightarrow{\mu_k} & U_- \end{array}$$

where vertical arrows are blow-ups and horizontal arrows are genuine isomorphisms. Removing the strict transform of the toric boundary, we have immersions

$$\widetilde{U}_+ \setminus \widetilde{D} \subset U_\Lambda \quad \text{and} \quad \widetilde{U}_- \setminus \widetilde{D} \subset U_{\Lambda'}$$

missing codimension two loci. Summarizing, the birational map μ_k can be extended to an isomorphism $\mu_k : U_\Lambda \dashrightarrow U_{\Lambda'}$ outside sets of codimension two. $\qquad\square$

A sufficient condition for the assumption in Theorem 5.4 to hold is

$$\forall \langle e_i, w_k \rangle = 0, \quad \dim \overline{V}(f_k) \cap Z_i < \dim Z_i.$$

Definition 5.5 (cf. [Berenstein et al. 2005, Definition 1.4]). A toric model data $\Lambda = ((e_i), (w_i), (f_i))$ is said to be *coprime* if the functions f_i are pairwise coprime as elements in the ring $\Bbbk[M]$.

Corollary 5.6. *The result in Theorem 5.4 holds if Λ is coprime.*

Proof. Note that $Z_i = \overline{V}(f_i) \cap D_i$. If needed, multiply some monomial z^m to f_i so that $\tilde{f}_i = z^m f_i$ is a regular function on D_i. Do the same to f_k to get $\tilde{f}_k$. By the coprime condition on Λ, $\tilde{f}_i$ and $\tilde{f}_k$ are still coprime, so we have

$$\dim V(\tilde{f}_k) \cap V(\tilde{f}_i) < \dim V(\tilde{f}_i),$$

where the above subschemes are taken inside D_i. $\qquad\square$

The following is an easy-to-check condition on Λ for the coprimeness to hold.

Lemma 5.7. *If the vectors w_i are linear independent, then Λ is coprime.*

5.2. *The upper bound.* Suppose we are given the data $\Lambda = ((e_i), (w_i), (f_i))$. Assume that Λ is i-mutable for any $i \in I$. For $i \in I$, let $T_N^{(i)}$ be a copy of the torus T_N. Then we have birational maps for each $i \in I$,

$$\mu_i : T_N \dashrightarrow T_N^{(i)}, \quad \mu_i^*(z^m) = z^m f_i^{-\langle e_i, m \rangle}.$$

We glue the $|I| + 1$ tori along the maps μ_i to obtain a scheme X_Λ.

In previous section, we know that not only the torus T_N, U_Λ also contains the torus $T_N^{(i)}$, that is, we have the following diagram for every $i \in I$:

$$
\begin{array}{ccc}
T_N & \dashrightarrow{}^{\mu_i} & T_N^{(i)} \\
\downarrow & \swarrow & \\
U_\Sigma & &
\end{array}
$$

These diagrams determine a unique morphism $\psi : X_\Lambda \to U_\Lambda$.

Lemma 5.8 [Gross et al. 2015, Lemma 3.5]. *The morphism $\psi : X_\Lambda \to U_\Lambda$ satisfies the following properties*:

(1) *If $\dim Z_i \cap Z_j < \dim Z_i$ for all $i \neq j$, then ψ is an isomorphism outside a set of codimension at least two.*

(2) *If $Z_i \cap Z_j = \varnothing$ for all $i \neq j$, then ψ is an open immersion. In particular, in this case, X_Λ is separated.*

In the $\mathcal{A}$-cluster case to be explained later, the variety X_Λ may be named *the upper bound* according to [Fomin and Zelevinsky 2007].

5.3. *Toric models for cluster varieties.* In this section, we realize generalize cluster varieties as log Calabi–Yau varieties utilizing Construction 5.1.

5.3.1. *$\mathcal{A}$-cluster cases*. Suppose we have fixed data Γ and an $\mathcal{A}$-seed with coefficients $s = (e, p)$. We further choose an evaluation $\lambda : \mathbb{P} \to \Bbbk^*$. This amounts to pick a $\Bbbk$-point of $\mathrm{Spec}(\Bbbk\mathbb{P})$. These lead to the generalized $\mathcal{A}$-cluster variety $\mathcal{A}_{s,\lambda}$ with special coefficients.

Meanwhile consider the toric model data

$$\Lambda(s, \lambda) := ((e_i)_{i \in I_{\mathrm{uf}}}, (w_i)_{i \in I_{\mathrm{uf}}}, (f_i)_{i \in I_{\mathrm{uf}}})$$

defined as follows. The vectors $(e_i)_{i \in I_{\mathrm{uf}}}$ are taken from the seed s. Recall that we have the exchange matrix $B = (b_{ij})$ where $b_{ij} := \omega(e_i, d_j e_j)$. Write $\beta_{ij} = b_{ij}/r_j$. Note that $\{e_i \mid i \in I\}$ form a basis of the lattice N and we denote by e_i^* the dual basis of M. Then define

$$w_i := \omega(-, d_i e_i / r_i) = \sum_{j \in I} \beta_{ij} e_i^* \in M, \quad f_i := \lambda(\theta[p_i](z^{w_i}, 1)) \in \Bbbk[M].$$

Then Construction 5.1 applies to the toric model data $\Lambda(s, \lambda)$, and thus there is the associated log Calabi–Yau variety $U_{\Lambda(s,\lambda)}$. Recall that we also have the scheme $X_{\Lambda(s,\lambda)}$ obtained by gluing $n + 1$ copies of the torus T_N as in Section 5.2. We call $X_{\Lambda(s,\lambda)}$ *the upper bound* for (s, λ), which by definition is an open subscheme of $\mathcal{A}_{s,\lambda}$.

The following lemma is easy to verify by direct computations.

Lemma 5.9. *We have $\mu_k(\Lambda(s, \lambda)) = \Lambda(\mu_k(s), \lambda)$ in the sense of Definition 5.3. The latter μ_k is the mutation of an $\mathcal{A}$-seed with coefficients.*

Proposition 5.10. (1) *The morphism $\psi : X_{\Lambda(s,\lambda)} \to U_{\Lambda(s,\lambda)}$ is an open immersion with image an open subset whose complement has codimension at least two.*

(2) *The birational map $\mu_k : U_{\Lambda(s,\lambda)} \dashrightarrow U_{\Lambda(\mu_k(s),\lambda)}$ is an isomorphism outside codimension two in each of the listed situations*:

    A. *The functions f_i have general coefficients.*
    B. *The seed s is mutation equivalent to one with principal coefficients, and $\lambda \in (\Bbbk^*)^{|I'|}$ is general enough.*

Proof. (1) follows from Lemma 5.8, part (2) — as we only need to check the hypothesis $Z_i \cap Z_j = \varnothing$ for all $i \neq j$. In fact, in $\mathcal{A}$-cluster case, since $e_i \neq e_j$, we have $T_{N/\langle e_i \rangle} \cap T_{N/\langle e_j \rangle} = \varnothing$ for all $i \neq j$, where $T_{N/\langle e_i \rangle}$ is viewed as the dense torus contained in the divisor D_i. As Z_i is a closed subset of $T_{N/\langle e_i \rangle}$, the hypothesis holds.

(2) follows from Theorem 5.4. We need to check that whenever $\langle e_i, u_k \rangle = 0$,

$$\dim \overline{V}(f_k) \cap \overline{V}(f_i) \cap D_i < \dim \overline{V}(f_i) \cap D_i.$$

A sufficient condition is the functions f_i being coprime. Note that for $i \in I$,

$$f_i = \prod_{j=1}^{r_i} (\lambda(p_{i,j}^+) z^{w_i} + \lambda(p_{i,j}^-)).$$

When these f_i have general coefficients (case A), they are coprime. In case B, one may replace f_i by

$$\tilde{f}_i = \prod_{j=1}^{r_i} (\lambda(p_{i,j}) z^{w_i} + 1).$$

Since the elements $p_{i,j}$ for $i \in I$ and $j \in [1, r_i]$ form a $\mathbb{Z}$ basis in $\mathbb{P}$ (by Lemma 3.18) when s is mutation equivalent to one with principal coefficients, these $\tilde{f}_i$ are coprime as long as λ is general. $\quad\square$

Remark 5.11. Suppose we are in the situation of case B of Proposition 5.10(2). Then we have isomorphisms of the rings of regular functions

$$\Bbbk[X_{\Lambda(s,\lambda)}] \cong \Bbbk[U_{\Lambda(s,\lambda)}] \cong \Bbbk[U_{\Lambda(\mu_k(s),\lambda)}].$$

The equality then extends to any seed s_v that is mutation equivalent to s. It then follows that they are all isomorphic to the upper cluster algebra

$$\mathscr{A}(s, \lambda) = \Bbbk[\mathcal{A}_{s,\lambda}].$$

The cluster variables in seed s are $x_{i,s} := z^{e_i^*}$. Each $x_{i,s}$ extends to a regular function on the toric variety X_Σ corresponding to the toric model data $\Lambda(s, \lambda)$. Then $x_{i,s}$ pulls back to the blow-up $\widetilde{X}_\Sigma$ and restricts to a regular function on the open subvariety $U_{\Lambda(s,\lambda)}$. It follows from (2) of Proposition 5.10 that $x_{i,s}$ is also a regular function on $X_{\Lambda(s_v,\lambda)}$ and in particular is a Laurent polynomial if restricted to T_{N,s_v}. This explains the generalized Laurent phenomenon Theorem 3.7, which was observed in [Gross et al. 2015] for the ordinary case.

5.3.2. $\mathcal{X}$-cluster cases. Suppose we have fixed data Γ and an $\mathcal{X}$-seed with coefficients $s = (e, q)$. Let us make the assumption that for any $j \in I_{\mathrm{uf}}$,

$$r_j = \gcd(b_{ij}, i \in I).$$

This is equivalent to say that each w_j for $j \in I_{\mathrm{uf}}$ is primitive as an element of $M = \mathrm{Hom}(N, \mathbb{Z})$. Switching the roles of (e_i) and (w_i), we obtain the toric model data

$$\Omega(s, \lambda) = ((-w_i), (e_i), (g_i))$$

for M instead of N, where

$$g_i := \lambda(\theta[q_i](z^{e_i}, 1)) \in \mathbb{k}[N]$$

with some chosen evaluation λ. Since the matrix B is skew-symmetrizable, $\Omega(s, \lambda)$ is k-mutable for any $k \in I_{\mathrm{uf}}$.

Lemma 5.12. *The assumption that $r_j = \gcd(b_{ij}, i \in I)$ is invariant under mutations.*

Proof. This is because if the j-th column of B is divisible by r_j then the same is true for the matrix $\mu_k(B) = (b'_{ij})$. Thus we have

$$\gcd(b_{ij}, i \in I) = \gcd(b'_{ij}, i \in I)$$

as μ_k is involutive on B. $\qquad\square$

The above lemma shows that we have well-defined data $\Omega(\mu_k(s), \lambda)$.

Lemma 5.13. *We have $\mu_k(\Omega(s, \lambda)) = \Omega(\mu_k(s), \lambda)$, where the later μ_k is the mutation for an $\mathcal{X}$-seed with coefficients.*

Proof. This lemma is analogous to Lemma 5.9 and is also easy to check. However, to show that the carefully chosen signs and conventions are the correct ones, we record some details here.

In the notations of Definition 5.3, for the data $\Omega(s, \lambda)$, we take $e_i = -w_i$ and $w_i = e_i$. So after the mutation μ_k in sense of Definition 5.3, for $i \neq k$

$$(-w_i)' = \begin{cases} -w_i - \langle (-w_i), r_k e_k \rangle (-w_k) & \text{if } \langle -w_i, e_k \rangle \leq 0, \\ -w_i & \text{if } \langle -w_i, e_k \rangle > 0. \end{cases}$$

Note that the two conditions are equivalent to $\beta_{ik} \leq 0$ and $\beta_{ik} > 0$ respectively. And in these two cases, we have

$$(-w_i)' = -w_i - \langle e_k, w_i \rangle r_k w_k \quad \text{and} \quad -w_i$$

respectively. This is exactly $-w_i'$ for $w_i' = \omega(-, d_i e_i'/r_i)$ from the seed $\mu_k(s)$. Similarly, one checks that the e part is also compatible with mutations.

As for coefficients, for the data $\Omega(\mu_k(s), \lambda)$, we have

$$g_i'(u, v) = \lambda(\theta[\boldsymbol{q}_i'](u, v).$$

Here $q_{i,j}'$ is obtain from $\mathcal{X}$-type mutations for coefficients (see Definition 4.14) which coincides with Definition 5.3. $\qquad \square$

Recall that $X_{\Omega(s,\lambda)}$ is the upper bound for $\Omega(s, \lambda)$ as defined in Section 5.2.

Proposition 5.14. *For the $\mathcal{X}$-type constructions,*

(1) *the morphism $\psi : X_{\Omega(s,\lambda)} \to U_{\Omega(s,\lambda)}$ is an open immersion with image being an open subset whose complement has codimension at least two;*

(2) *the birational map $\mu_k : U_{\Omega(s,\lambda)} \dashrightarrow U_{\Omega(\mu_k(s),\lambda)}$ is an isomorphism outside codimension two subsets.*

Proof. The proof of (1) is completely analogous to that of (1) of Proposition 5.10. For (2), it follows from that for any $\mathcal{X}$-seed s, the data $\Omega(s, \lambda)$ is always coprime by Lemma 5.7 as the vectors e_i form a basis of N. $\qquad \square$

6. Scattering diagrams

This section deals with scattering diagrams. Our main objects of study *generalized cluster scattering diagrams* will be defined in Section 6.2.

6.1. *The tropical vertex.* We start with a more general setup of scattering diagrams as in [Argüz and Gross 2022, Section 5.1.1]. Let N be a lattice of finite rank, $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. Let P be a monoid with a monoid map $r : P \to M$. Denote by $P^{\times}$ the groups of units of P and let $\mathfrak{m}_P = P \setminus P^{\times}$. An ideal of the monoid P induces a monomial ideal of the ring $\Bbbk[P]$, where $\Bbbk$ is a ground field. So we use the same letter to denote both. For any monomial ideal $I \subset \Bbbk[P]$, define the ring

$$R_I := \Bbbk[P]/I.$$

Denote by $\widehat{\Bbbk[P]}$ the completion of $\Bbbk[P]/\mathfrak{m}_P^n$ for $n \in \mathbb{N}$.

For I such that its radical $\sqrt{I}$ is equal to $\mathfrak{m}_P$ (e.g., $I = \mathfrak{m}_P^n$ for some $n \in \mathbb{N}$), define the *module of log derivations* $\Theta(R_I) := R_I \otimes_{\mathbb{Z}} N$ as follows.

If we write the element $z^p \otimes n$ as $z^p \partial_n$ for $p \in P$ and $n \in N$, then it acts on R_I by

$$z^p \partial_n(z^{p'}) = \langle n, r(p') \rangle z^{p+p'}, \quad p' \in P.$$

Then the submodule $\mathfrak{m}_P\Theta(R_I)$ is a Lie algebra with the commutator bracket

$$[z^{p_1}\partial_{n_1}, z^{p_2}\partial_{n_2}] = z^{p_1+p_2}\partial_{\langle r(p_2),n_1\rangle n_2 - \langle r(p_1),n_2\rangle n_1}.$$

Taking exponential of elements in this Lie algebra, we get group elements in $\mathrm{Aut}(R_I)$. There is a nilpotent Lie subalgebra of $\mathfrak{m}_P\Theta(R_I)$ defined by

$$\mathfrak{v}_I := \bigoplus_{m\in P\setminus I, r(m)\neq 0} z^m(\Bbbk \otimes r(m)^\perp).$$

Since it is nilpotent, this Lie subalgebra (as a set) is in bijection with the corresponding algebraic group $\mathbb{V}_I := \exp(\mathfrak{v}_I) \subset \mathrm{Aut}(R_I)$. Taking the projective limit with respect to the ideals $\mathfrak{m}_P^n$ for $n \in \mathbb{N}$, we get a pro-unipotent group $\widehat{\mathbb{V}}$, which is in bijection with the pro-nilpotent Lie algebra $\hat{\mathfrak{v}} := \varprojlim \mathfrak{v}_{\mathfrak{m}_P^n}$. The group $\widehat{\mathbb{V}}$ is called the *higher-dimensional tropical vertex group*, acts by automorphisms on $\Bbbk[\widehat{P}]$. We also denote (without completion)

$$\mathfrak{v} := \bigoplus_{m\in P, \, r(m)\neq 0} z^m(\Bbbk \otimes r(m)^\perp).$$

Definition 6.1. A *scattering diagram* in $M_\mathbb{R}$ over R_I is a finite set $\mathfrak{D}$ of *walls* where each *wall* $(\mathfrak{d}, f_\mathfrak{d})$ is a rational polyhedral cone $\mathfrak{d} \subset M_\mathbb{R}$ of codimension one along with an attached element called *wall-crossing function*

$$f_\mathfrak{d} = \sum_{\substack{m\in P\setminus I \\ r(m)\in \Lambda_\mathfrak{d}}} c_m z^m \in R_I,$$

where $\Lambda_\mathfrak{d} \subset M$ is the integral tangent space of any point in $\mathfrak{d}$, i.e., $\Lambda_\mathfrak{d} = M \cap \mathbb{R}\langle\mathfrak{d}\rangle$. We require that $f_\mathfrak{d} \equiv 1 \bmod \mathfrak{m}_P$.

Remark 6.2. Upon choosing a generator n_0 of $\Lambda_\mathfrak{d}^\perp \cap N$, the wall-crossing function $f_\mathfrak{d}$ induces an element in $\mathbb{V}_I \subset \mathrm{Aut}(R_I)$ by the action

$$z^p \mapsto z^p f_\mathfrak{d}^{\langle r(p),n_0\rangle}.$$

So this wall-crossing automorphism depends on how one crosses the wall. One may view that this wall-crossing automorphism depends on the direction in which one transversally crosses the wall. With n_0 chosen, such an automorphism can be equivalently represented by the corresponding Lie algebra element $\log(f_\mathfrak{d})\partial_{n_0} \in \mathfrak{v}_I$.

Let $\mathrm{Supp}(\mathfrak{D})$ be the union of all walls in $\mathfrak{D}$. Let $\mathrm{Sing}(\mathfrak{D})$ be the union of at least codimension two intersections of every pair of walls and the boundary of every wall. Let $\gamma : [0, 1] \to M_\mathbb{R}$ be a piecewise smooth proper map such that the end points $\gamma(0)$ and $\gamma(1)$ avoid $\mathrm{Supp}(\mathfrak{D})$ and whose image is disjoint from $\mathrm{Sing}(\mathfrak{D})$. We also assume that γ meets walls transversally.

Suppose that γ crosses walls $\mathfrak{d}_1, \ldots, \mathfrak{d}_s$ in $\mathfrak{D}$ at times

$$0 < t_1 \leq t_2 \leq \cdots \leq t_s < 1.$$

These numbers t_i are obtained by considering the finite set $\gamma^{-1}(\mathrm{Supp}(\mathfrak{D})) \subset [0, 1]$ as γ is proper. It is possible that $t_i = t_j$ as walls may overlap. Suppose γ crosses a wall $(\mathfrak{d}, f_\mathfrak{d})$ at time t. Denote by $\xi_{\gamma,\mathfrak{d}}$ the element in $\mathbb{V}_I$ with the action

$$z^p \mapsto z^p f_\mathfrak{d}^{\langle r(p), n_0 \rangle}, \quad p \in P \setminus I$$

where n_0 is chosen such that $\langle n_0, \gamma'(t_i) \rangle > 0$.

Definition 6.3. We define the *path-ordered product* of γ in $\mathfrak{D}$ to be the element

$$\mathfrak{p}_{\gamma,\mathfrak{D}} := \xi_{\gamma,\mathfrak{d}_s} \xi_{\gamma,\mathfrak{d}_{s-1}} \cdots \xi_{\gamma,\mathfrak{d}_1} \in \mathbb{V}_I.$$

Definition 6.4. A scattering diagram $\mathfrak{D}$ over R_I is *consistent* if the path-ordered product $\mathfrak{p}_{\gamma,\mathfrak{D}}$ only depends on the endpoints $\gamma(0)$ and $\gamma(1)$ for any path $\gamma : [0, 1] \to M_\mathbb{R}$ for which $\mathfrak{p}_{\gamma,\mathfrak{D}}$ is well-defined.

Recall that we have the completed algebra $\widehat{\Bbbk[P]} := \varprojlim R_{\mathfrak{m}_P^k}$. For an element $f \in \widehat{\Bbbk[P]}$, denote by $f^{<k}$ its projection in $R_{\mathfrak{m}_P^k}$.

Definition 6.5. A *scattering diagram* in $M_\mathbb{R}$ over $\widehat{\Bbbk[P]}$ is a (possibly infinite) set $\mathfrak{D}$ of walls $(\mathfrak{d}, f_\mathfrak{d})$ with $\mathfrak{d}$ a rational polyhedral cone of codimension one and the wall-crossing function

$$f_\mathfrak{d} = \sum_{\substack{m \in P \\ r(m) \in \Lambda_\mathfrak{d}}} c_m z^m \in \widehat{\Bbbk[P]},$$

such that modulo the ideal $\mathfrak{m}_P^n$, the collection $\mathfrak{D}^{<n} := \{(\mathfrak{d}, f_\mathfrak{d}^{<n})\}$ is a scattering diagram over $R_{\mathfrak{m}_P^n}$. A scattering diagram $\mathfrak{D}$ is consistent if $\mathfrak{D}^{<n}$ is consistent for any $n \in \mathbb{N}$.

The path-ordered product for $\mathfrak{D}$ over $\widehat{\Bbbk[P]}$ is defined through the projective limit of path-ordered products for $\mathfrak{D}^{<n}$:

$$\mathfrak{p}_{\gamma,\mathfrak{D}} := \varprojlim \mathfrak{p}_{\gamma,\mathfrak{D}^{<n}} \in \widehat{\mathbb{V}} \subset \mathrm{Aut}(\widehat{\Bbbk[P]}).$$

Definition 6.6. We say two scattering diagrams $\mathfrak{D}$ and $\mathfrak{D}'$ (over the same algebra) are *equivalent* if for any γ, we have $\mathfrak{p}_{\gamma,\mathfrak{D}} = \mathfrak{p}_{\gamma,\mathfrak{D}'}$ whenever both path-ordered products are well-defined.

Definition 6.7. We say a wall $\mathfrak{d}$ has *direction m_0* for some $m_0 \in M$ if the attached wall-crossing function $f_\mathfrak{d}$ only contains monomials z^p such that $r(p) = -k m_0$ for some $k \in \mathbb{N}$. A wall $(\mathfrak{d}, f_\mathfrak{d})$ with direction m_0 is called *incoming* if $\mathfrak{d} = \mathfrak{d} - \mathbb{R}_{\geq 0} m_0$.

Next we explain how to assign a scattering diagram to an $\mathcal{X}$-type toric model. We are actually in a particular situation within the more general framework of [Argüz and Gross 2022], which works for any log Calabi–Yau variety obtained from blowing-up a toric variety along hypersurfaces in the toric boundary.

Let $s = (e, q)$ be an $\mathcal{X}$-seed with principal coefficients for some fixed data Γ. We assume that $N_{\mathrm{uf}} = N$ to avoid frozen directions. As usual, write $e = (e_i)$. We assume the condition that $r_j = \gcd(b_{ij} \mid i \in I)$ for any $j \in I$. This assumption implies any $w_i := \frac{d_i}{r_i}\omega(-, e_i) \in M$ is primitive. Recall that we have used the fan

$$\Sigma_0 := \{0\} \cup \{-\mathbb{R}_{\geq 0} w_i\}$$

to describe the toric model of $U = U_{\Omega(s,\lambda)}$. The functions (in the data $\Omega(s, \lambda)$ to define U) are then

$$g_i = \prod_{j=1}^{r_i} (1 + \lambda_{ij} z^{e_i}) \in \Bbbk[N].$$

We pick a complete fan Σ in $M_{\mathbb{R}}$ containing Σ_0. For example, we may take a refinement of (the cone complex induced by) the hyperplane arrangement $\{e_i^{\perp} \mid i \in I\}$. Let X_{Σ} be the corresponding (complete) toric variety, with D_i being the boundary toric divisor corresponding to the ray $-\mathbb{R}_{\geq 0} w_i$. Let $H = \bigcup_i H_i$, where

$$H_i = \bigcup_{j \in [1, r_i]} H_{ij} := \bigcup_{j \in [1, r_i]} \overline{V}(1 + \lambda_{ij} z^{e_i}) \cap D_i$$

which is a union of disjoint hypersurfaces in D_i (as the coefficients $\lambda_{ij} \in \Bbbk^*$ are general). These hypersurfaces are exactly where we blow up X_{Σ} to obtain the log Calabi–Yau variety $U_{\Omega(s,\lambda)}$.

Take the monoid

$$P := M \oplus \prod_{i \in I} \mathbb{N}^{r_i},$$

with the natural projection $r : P \to M$. We write multiplicatively $t_{i,1}, t_{i,2}, \ldots t_{i,r_i}$ for the generators of $\mathbb{N}^{r_i}$. For each ray $\rho_i := -\mathbb{R}_{\geq 0} w_i$ and H_{ij}, there is a finite scattering diagram $\mathfrak{D}_{ij}$ called a *widget* from a certain *tropical hypersurface* [Argüz and Gross 2022, Definition 5.3 and Section 5.1.3]. In our case, they are given by:

Lemma 6.8. *The widget $\mathfrak{D}_{ij}$ consists of all codimension one cones of the fan Σ contained in the hyperplane $e_i^{\perp}$ containing ρ_i, with the same wall-crossing function $(1 + t_{i,j} z^{w_i})$. In other words, we have*

$$\mathfrak{D}_{ij} = \{(\sigma, 1 + t_{i,j} z^{w_i}) \mid \sigma \in \Sigma, \ \dim \sigma = n - 1, \ \sigma \subset e_i^{\perp}, \ \rho_i \subset \sigma\}.$$

Proof. By definition [Argüz and Gross 2022, Definition 5.3 and Section 5.1.3], $\mathfrak{D}_{ij}$ consists of walls $(\sigma, (1 + t_{i,j} z^{w_i})^{\omega_{\sigma}})$ where σ runs through all codimension one cones in Σ containing ρ_i and $\omega_{\sigma} = H_{ij} \cdot D_{\sigma}$ is the intersection number computed in the divisor D_i. Here D_{σ} is the one-dimensional toric stratum in D_i corresponding to σ. Note that if $e_i \notin \sigma^{\perp}$, then z^{e_i} or z^{-e_i} vanishes along D_{σ}. So $H_{ij} = \overline{V}(1 + \lambda_{ij} z^{e_i})$ does not intersect D_{σ} and thus $\omega_{\sigma} = 0$. If $\sigma \subset e_i^{\perp}$, as e_i is primitive, the intersection is at the point $z^{e_i} = -1/\lambda_{ij}$, where z^{e_i} can be viewed as the coordinate on D_{σ}. Thus the multiplicity ω_{σ} is 1. $\square$

Note that by Definition 6.7 every wall in $\mathfrak{D}_{ij}$ is incoming since $-w_i$ is contained in every σ.

Theorem 6.9 [Argüz and Gross 2022, Theorem 5.6 and Section 5.1.3]. *Consider the scattering diagram* (*with only incoming walls*)

$$\mathfrak{D}_{(X_{\Sigma}, H), \mathrm{in}} := \bigcup_{i \in I} \bigcup_{j \in [1, r_i]} \mathfrak{D}_{ij}.$$

There exists a unique (*up to equivalence*) *consistent scattering diagram $\mathfrak{D}_{(X_{\Sigma}, H)}$ over $\widehat{\Bbbk[P]}$ containing $\mathfrak{D}_{(X_{\Sigma}, H), \mathrm{in}}$ such that $\mathfrak{D}_{(X_{\Sigma}, H)} \setminus \mathfrak{D}_{(X_{\Sigma}, H), \mathrm{in}}$ consists only of nonincoming walls.*

6.2. Generalized cluster scattering diagrams. Instead of applying Theorem 6.9 to (X_Σ, H), there is another way to obtain the same scattering diagram by generalizing the construction of cluster scattering diagrams in [Gross et al. 2018].

Given fixed data Γ and an $\mathcal{A}$-seed $s = (e, p)$ with principal coefficients, we are going to define the *generalized cluster scattering diagram* $\mathfrak{D}_s$.

Recall that we have the semifield $\mathbb{P} = \mathrm{Trop}(p)$, isomorphic to $\prod_{i \in I} \mathbb{Z}^{r_i}$ as an abelian group. Let $P = P_s$ as before be $M \oplus \prod_{i \in I} \mathbb{N}^{r_i}$, but regarded as a submonoid of $M \oplus \mathbb{P}$ generated by M and p. There is a submonoid $P^\oplus = P_s^\oplus \subset P$ generated by elements

$$\{(w_i, p_{i,j}) \mid i \in I, j \in [1, r_i]\}.$$

One could take the completion of $P^\oplus$ with respect to the ideal $P^+ := P^\oplus \setminus \{0\}$, giving that $\widehat{\Bbbk[P^\oplus]} \subset \widehat{\Bbbk[P]}$. In N, there is a submonoid $N_s^\oplus = N^\oplus$ generated by $\{e_i \mid i \in I\}$. Denote $N^+ = N^\oplus \setminus \{0\}$. We also consider the monoid map

$$\pi : P^\oplus \to N^\oplus, \quad (w_i, p_{i,j}) \mapsto e_i.$$

Let $n = \sum_{i \in I} \alpha_i e_i \in N$. Define

$$\bar{n} := \sum_{i \in I} \alpha_i \cdot \frac{d_i}{r_i} e_i \in N_\mathbb{R}.$$

These $\bar{n}$ form a sublattice $\bar{N}$ of $N_\mathbb{R}$ isomorphic to N. We have the similar notion $\bar{N}^+$, the monoid generated by $\bar{e}_i$.

There is a subspace $\mathfrak{g}$ of the tropical vertex lie algebra $\mathfrak{v}$ defined as

$$\mathfrak{g} = \mathfrak{g}_s := \bigoplus_{n \in N^+} \mathfrak{g}_n, \quad \mathfrak{g}_n := \bigoplus_{\substack{\pi(p)=n \\ p \in P^+}} z^p \cdot (\Bbbk \otimes \bar{n}).$$

Lemma 6.10. *The subspace $\mathfrak{g}$ is an N^+-graded Lie subalgebra of $\mathfrak{v}$.*

Proof. For any $n = \sum_{i \in I} \alpha_i e_i \in N^+$, consider the elements

$$\prod_{i,j} p_{i,j}^{c_{i,j}} \cdot z^{p^*(n)}$$

such that $\sum_{j \in [1, r_i]} c_{i,j} = \alpha_i$ and

$$p^*(n) := \omega(-, \bar{n}) = \sum_{i \in I} \alpha_i \omega(-, d_i e_i / r_i) = \sum_{i \in I} \alpha_i w_i.$$

Those elements form a basis of the vector space $\mathfrak{g}_n$. We check that for two such elements

$$[p_1 z^{p^*(n_1)} \partial_{\bar{n}_1}, p_2 z^{p^*(n_2)} \partial_{\bar{n}_2}] = p_1 p_2 \cdot z^{p^*(n_1+n_2)} \partial_{\omega(\bar{n}_1, \bar{n}_2)\bar{n}_2 - \omega(\bar{n}_2, \bar{n}_1)\bar{n}_1}$$

$$= \omega(\bar{n}_1, \bar{n}_2) p_1 p_2 \cdot z^{p^*(n_1+n_2)} \partial_{\bar{n}_1 + \bar{n}_2} \in \mathfrak{g}_{n_1+n_2}. \qquad \square$$

Remark 6.11. One may also view the above Lie algebra $\mathfrak{g}$ as being $\bar{N}^+$-graded where both $\bar{N}$ and N are sublattices of $N_\mathbb{R}$. When later considering a scattering diagram $\mathfrak{D}$ over an $\bar{N}^+$-graded Lie algebra $\mathfrak{g}$ (instead of N^+-graded), the walls live in $M_\mathbb{R}$ with integral normal vectors in $\bar{N}^+$.

Consider the ideals $(N^+)^k \subset N^+$ for $k \geq 1$. These correspond to the monomial ideals $(P^+)^k$. Then we have quotient Lie algebras (and their corresponding groups $G^{<k}$)

$$\mathfrak{g}^{<k} := \mathfrak{g}/\mathfrak{g}_{(N^+)^k} = \bigoplus_{n \in N^+ \setminus (N^+)^k} \mathfrak{g}_n,$$

and their projective limits

$$\hat{\mathfrak{g}} = \prod_{n \in N^+} \mathfrak{g}_n \quad \text{and} \quad G := \exp(\hat{\mathfrak{g}}).$$

The group $G^{<k}$ acts on $\Bbbk[P^\oplus]/(P^+)^k$ by automorphisms as in Remark 6.2.

For $n_0 \in N^+$ primitive, we define as in [Gross et al. 2018] a Lie algebra (and its corresponding pro-unipotent group)

$$\mathfrak{g}^{\parallel}_{n_0} := \bigoplus_{k>0} \mathfrak{g}_{k \cdot n_0} \subset \mathfrak{g} \quad \text{and} \quad G^{\parallel}_{n_0} := \exp(\hat{\mathfrak{g}}^{\parallel}_{n_0}) \subset G.$$

There is a general framework for scattering diagrams over an N^+-graded Lie algebra (as opposed to the tropical vertex case); see [Kontsevich and Soibelman 2014, Section 2.1; Gross et al. 2018, Section 1.1]. In this case, one could make use of an existence-and-uniqueness theorem of [Kontsevich and Soibelman 2014] (see also [Gross et al. 2018, Theorem 1.21]) to obtain a consistent scattering diagram with certain prescribed *incoming data*. The cluster scattering diagram of [Gross et al. 2018] can be defined this way, which we will extend to the generalized case in Definition 6.17.

Definition 6.12. A *wall* in $M_\mathbb{R}$ (for N^+ and an N^+-graded Lie algebra $\mathfrak{g}$) is a pair $(\mathfrak{d}, g_\mathfrak{d})$ such that

(1) $g_\mathfrak{d}$ belongs to $G^{\parallel}_{n_0}$ for some primitive $n_0 \in N^+$;

(2) $\mathfrak{d} \subset n_0^\perp \subset M_\mathbb{R}$ is a codimension one convex rational polyhedral cone.

Remark 6.13. The above definition works for general N^+-graded Lie algebras. In the case that $\mathfrak{g}$ is a Lie subalgebra of the tropical vertex Lie algebra $\mathfrak{v}$, the group $G^{\parallel}_{n_0}$ is embedded in $\mathrm{Aut}(\widehat{\Bbbk[P^\oplus]})$. Then the wall-crossing element $g_\mathfrak{d}$ can be equivalently represented by a function $f_\mathfrak{d} \in \widehat{\Bbbk[P^\oplus]}$.

Now every wall has a direction $-p^*(n_0) \in M$ in the sense of Definition 6.7. We call a wall $(\mathfrak{d}, g_\mathfrak{d})$ with direction m_0 *incoming* if $\mathfrak{d} = \mathfrak{d} - \mathbb{R}_{\geq 0} m_0$ and *nonincoming* (or *outgoing*) otherwise.

Definition 6.14. A *scattering diagram* over an N^+-graded algebra $\mathfrak{g}$ in $M_\mathbb{R}$ is a collection of walls such that for every degree $k > 0$, there are only a finite number of $(\mathfrak{d}, g_\mathfrak{d}) \in \mathfrak{D}$ with the image of $g_\mathfrak{d}$ in $G^{<k}$ not being identity.

The path-ordered product of a path $\gamma : [0, 1] \to M_\mathbb{R}$ for a scattering diagram $\mathfrak{D}$ over $\mathfrak{g}$ can be defined similarly as in Definition 6.3. We note that when γ crosses a wall $(\mathfrak{d}, g_\mathfrak{d})$ at time t, then the element $\xi_{\gamma, \mathfrak{d}}$ also depends on $\gamma'(t)$:

$$\xi_{\gamma, \mathfrak{d}} = \begin{cases} g_\mathfrak{d} & \text{if } \langle n_0, \gamma'(t) \rangle > 0, \\ g_\mathfrak{d}^{-1} & \text{if } \langle n_0, \gamma'(t) \rangle < 0. \end{cases}$$

The consistency for these scattering diagrams is defined using path-ordered products in the same way as Definition 6.3.

Theorem 6.15 [Kontsevich and Soibelman 2014, Proposition 2.1.12; Gross et al. 2018, Theorem 1.21].
Let $\mathfrak{D}_{\mathrm{in}}$ be a scattering diagram over $\mathfrak{g}$ consisting only of incoming walls. Then there exists a unique (*up to equivalence*) *consistent scattering diagram $\mathfrak{D}$ containing $\mathfrak{D}_{\mathrm{in}}$ such that $\mathfrak{D} \setminus \mathfrak{D}_{\mathrm{in}}$ consists only of outgoing walls.*

Now we get back to the cluster situation. Suppose given fixed data Γ and $\boldsymbol{s}$ an $\mathcal{A}$-seed with principal coefficients. Unlike the previous section, here we do not assume the maximality of the positive integers r_i, i.e., r_i needs not to be $\gcd(b_{ki} \mid k \in I)$.

We calculate in the following how the group $G_{n_0}^{\parallel}$ is embedded in $\mathrm{Aut}(\widehat{\Bbbk[P^{\oplus}]})$. Suppose $n_0 = \sum_{i \in I} \alpha_i e_i$, a primitive element in N^+. Consider any element

$$x = \sum_{k>0} \sum_{\substack{p \in \mathbb{P}^{\oplus} \\ \pi(p)=kn_0}} c_p \cdot p \cdot z^{kp^*(n_0)} \partial_{k\bar{n}_0} \in \hat{\mathfrak{g}}_{n_0}^{\parallel}, \quad c_p \in \Bbbk.$$

For nonzero $n \in N_{\mathbb{Q}}$, denote by $\mathrm{ind}(n)$ the largest number in $\mathbb{Q}_{\geq 0}$ such that $n/\mathrm{ind}(n) \in N$. Thus $n/\mathrm{ind}(n)$ is primitive in N.

Lemma 6.16. *The group element $\exp(x) \in G_{n_0}^{\parallel}$ acts on $\widehat{\Bbbk[P^{\oplus}]}$ as an automorphism by*

$$z^m \mapsto z^m \exp\left(\sum_{k>0} \sum_{\substack{p \in \mathbb{P}^{\oplus} \\ \pi(p)=kn_0}} \mathrm{ind}(\bar{n}_0) k c_p \cdot p \cdot z^{kp^*(n_0)} \right)^{\langle r(m), \bar{n}_0/\mathrm{ind}(\bar{n}_0) \rangle}, \quad m \in P^{\oplus}.$$

Proof. This follows by rewriting x as

$$x = \left(\sum_{k>0} \sum_{\substack{p \in \mathbb{P}^{\oplus} \\ \pi(p)=kn_0}} \mathrm{ind}(\bar{n}_0) k c_p \cdot p \cdot z^{kp^*(n_0)} \right) \partial_{\bar{n}_0/\mathrm{ind}(\bar{n}_0)}. \qquad \square$$

Due to Lemma 6.16, any $\exp(x) \in G_{n_0}^{\parallel}$ can be represented by a function f as in Lemma 6.16 such that the action of $\exp(x)$ sends z^m to $z^m f^{\langle r(m), \bar{n}_0/\mathrm{ind}(\bar{n}_0) \rangle}$.

Given $\boldsymbol{s} = (\boldsymbol{e}, \boldsymbol{p})$, for each $i \in I$, consider the hyperplane $e_i^{\perp}$ with the attached wall-crossing function

$$f_i = \prod_{j=1}^{r_i} (1 + p_{i,j} z^{w_i}) \in \Bbbk[P^{\oplus}].$$

As discussed, the function f_i represents an element in $G_{e_i}^{\parallel}$.

Definition 6.17. Let $\mathfrak{D}_{\boldsymbol{s},\mathrm{in}}$ be the scattering diagram over $\mathfrak{g}$ in $M_{\mathbb{R}}$ consisting only of the incoming walls of the form $\mathfrak{d}_i := (e_i^{\perp}, f_i)$, i.e.,

$$\mathfrak{D}_{\boldsymbol{s},\mathrm{in}} := \{(e_i^{\perp}, f_i) \mid i \in I\}.$$

We define *the generalized cluster scattering* $\mathfrak{D}_{\boldsymbol{s}}$ to be the unique (up to equivalence) consistent scattering diagram associated to $\mathfrak{D}_{\boldsymbol{s},\mathrm{in}}$ guaranteed by Theorem 6.15.

Remark 6.18. One may tend to think of $\mathfrak{D}_s$ as a scattering diagram over $\widehat{\Bbbk[P^\oplus]}$ or over $\widehat{\Bbbk[P]}$ (as $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{v}$) in Definition 6.5. However there is one subtle issue. Suppose that there is a wall $(\mathfrak{d} \subset n_0^\perp, f_\mathfrak{d})$ in $\mathfrak{D}_s$ for some $n_0 \in N^+$ primitive. Then the wall-crossing action is given by

$$\xi_{f_\mathfrak{d}}(z^p) = z^p f_\mathfrak{d}^{\langle \bar{n}_0/\text{ind}(\bar{n}_0), r(p) \rangle}.$$

Since in general $\bar{n}_0$ may not be proportional to n_0, the cone $\mathfrak{d}$ may not be contained in $\bar{n}_0^\perp$. In this case, the wall $(\mathfrak{d}, f_\mathfrak{d})$ does not qualify as a wall in Definition 6.5. This issue can be resolved in the following two ways (so that one can view $\mathfrak{D}_s$ as a scattering diagram of Definition 6.5).

(1) We could regard $\mathfrak{g}$ as graded by $\bar{N}^+ \subset N_\mathbb{Q}$ (rather than N^+-graded) and modify Definition 6.12 (the definition of a wall $(\mathfrak{d}, g_\mathfrak{d})$) so that $\mathfrak{d}$ is a codimension one cone in some hyperplane $n_0^\perp$ for $n_0 \in \bar{N}^+$ and $g_\mathfrak{d}$ belongs to $G_{n_0}^\parallel$.

(2) Another way to resolve the issue is to consider the dual $\eta^* : M_\mathbb{R} \to M_\mathbb{R}$ of the linear map

$$\eta : N_\mathbb{R} \to N_\mathbb{R}, \quad n \mapsto \bar{n}.$$

We then apply $(\eta^*)^{-1}$ to every wall $(\mathfrak{d}, f_\mathfrak{d})$ to get the collection

$$(\eta^*)^{-1}(\mathfrak{D}_s) := \{((\eta^*)^{-1}(\mathfrak{d}), f_\mathfrak{d}) \mid (\mathfrak{d}, f_\mathfrak{d}) \in \mathfrak{D}_s\}$$

Then the cone $(\eta^*)^{-1}(\mathfrak{d})$ is indeed contained in $\bar{n}_0^\perp$. So this collection of walls is a scattering diagram in Definition 6.5.

From now on, to avoid any further confusion, the notation $\mathfrak{D}_s$ is reserved for the consistent scattering diagram $(\eta^*)^{-1}(\mathfrak{D}_s)$ over $\widehat{\Bbbk[P^\oplus]}$.

Lemma 6.19. *Let s be a seed with principal coefficients for some generalized fixed data Γ (viewed of both $\mathcal{A}$- and $\mathcal{X}$-type) with the condition that for each $i \in I$, the element*

$$w_i = \omega(-, d_i e_i/r_i)$$

is primitive in M. In this case, we have defined both scattering diagrams $\mathfrak{D}_{(X_\Sigma, H)}$ (with a chosen general evaluation λ) and $\mathfrak{D}_s$. Identify the parameters $t_{i,j}$ with $p_{i,j}$. Then $\mathfrak{D}_{(X_\Sigma, H)}$ and $\mathfrak{D}_s$ are equivalent as scattering diagrams over $\widehat{\Bbbk[P]}$.

Proof. We require ω_i to be primitive so that $\mathfrak{D}_{(X_\Sigma, H)}$ is defined. According to Remark 6.18, $\mathfrak{D}_s$ is viewed as a scattering diagram over $\widehat{\Bbbk[P]}$ in the same $M_\mathbb{R}$ as $\mathfrak{D}_{(X_\Sigma, H)}$ so it is legitimate to compare them. Let $\widetilde{\mathfrak{D}}$ be the consistent scattering diagram over $\mathfrak{g}$ obtained using the initial data $\mathfrak{D}_{(X_\Sigma, H), \text{in}}$. Notice that the walls in $\mathfrak{D}_{(X_\Sigma, H), \text{in}}$ are parts of the hyperplanes $e_i^\perp$. We then subdivide the walls in $\mathfrak{D}_{s, \text{in}}$ so that $\mathfrak{D}_{(X_\Sigma, H), \text{in}}$ becomes the subset of incoming walls. Thus $\widetilde{\mathfrak{D}}$ is equivalent to $\mathfrak{D}_s$ by Theorem 6.15.

On the other hand, $\widetilde{\mathfrak{D}}$ is also a scattering diagram over $\widehat{\Bbbk[P]}$. By Theorem 6.9, It is also equivalent to $\mathfrak{D}_{(X_\Sigma, H)}$ since they have the same incoming walls. Therefore we have $\mathfrak{D}_{(X_\Sigma, H)} \cong \widetilde{\mathfrak{D}} \cong \mathfrak{D}_s$. $\qquad\square$

6.3. *The cluster scattering diagrams of GHKK.* The ordinary cluster scattering diagram $\mathfrak{D}_s^{\mathrm{ord}}$ corresponds to the case where $r_i = 1$ for each $i \in I$. Thus there is only one parameter $p_i := p_{i,1}$ for each $i \in I$. The lattice $\overline{N}$ is generated by $\bar{e}_i = d_i e_i$. The initial incoming walls are then

$$\{(e_i^\perp, 1 + p_i z^{w_i}) \mid i \in I\},$$

where $w_i = \omega(-, \bar{e}_i) \in M$.

This scattering diagram is closely related to the *cluster scattering diagram* $\mathfrak{D}_s^{\mathrm{GHKK}}$ of Gross, Hacking, Keel and Kontsevich [Gross et al. 2018, Theorem 1.12]. We explain the difference and relation here. The scattering diagram $\mathfrak{D}_s^{\mathrm{GHKK}}$ is actually defined for $\overline{N}$ and $\overline{M} := \mathrm{Hom}(\overline{N}, \mathbb{Z})$ (in the ordinary case equal to N° and M° respectively). Under the injectivity assumption [Gross et al. 2018, Section 1.1], the incoming walls are

$$\{(e_i^\perp, 1 + z^{\omega(e_i, -)}) \mid i \in I\},$$

where $\omega(e_i, -)$ is in M°. The injectivity assumption means that $\omega(e_i, -)$ generate a strict convex cone. If this is not the case, we may extend M° to $M^\circ \oplus \mathbb{P}$ (identified with $M^\circ \oplus N$ in [Gross et al. 2018]) and let incoming walls be

$$\{(e_i^\perp, 1 + p_i z^{\omega(e_i, -)}) \mid i \in I\}.$$

It lives in $(M^\circ \oplus N) \otimes \mathbb{R}$, or in $M^\circ \otimes \mathbb{R}$ if regarding p_i as formal parameters as we do. Then $\mathfrak{D}_s^{\mathrm{GHKK}}$ is defined to be the unique consistent scattering diagram over $\widehat{\mathbb{k}[P]}$ with only these incoming walls, where $P \subset M^\circ \oplus N$ is a submonoid contained in a strictly convex cone and containing the cone generated by $(p_i, \omega(e_i, -))$. The Lie algebra $\mathfrak{g}$, however, is naturally graded by N^+ (generated by e_i's), not $\overline{N}^+$ (generated by $\bar{e}_i$'s). Thus if one uses Theorem 6.15 to define $\mathfrak{D}_s^{\mathrm{GHKK}}$, the same rescaling issue in Remark 6.18 still exists and can be resolved in a similar way. In [Gross et al. 2018], $\mathfrak{D}_s^{\mathrm{GHKK}}$ is regarded as living in $M_{\mathbb{R}}^\circ$ with the integral normal vectors of walls being in N°.

The structures of $\mathfrak{D}_s^{\mathrm{GHKK}}$ and $\mathfrak{D}_s^{\mathrm{ord}}$ are very much alike. For example, they both admit cluster complex structures; see [Gross et al. 2018, Theorem 2.13] and Theorem 7.10. It turns out that in the convention of [Fomin and Zelevinsky 2007] (e.g., the definition of g-vectors), $\mathfrak{D}_s^{\mathrm{GHKK}}$ corresponds to the cluster algebra associated to $-B^T$ while $\mathfrak{D}_s^{\mathrm{ord}}$ corresponds to the one associated to B, where $B = (b_{ij})$ with $b_{ij} = \omega(e_i, \bar{e}_j)$.

6.4. *Scattering diagrams with special coefficients.* Just as specializing a cluster algebra $\mathscr{A}$ at some evaluation $\lambda : \mathbb{P} \to \mathbb{k}^*$, one can do the same to $\mathfrak{D}_s$, obtaining a consistent scattering diagram with special coefficients.

We consider another monoid $Q = M \oplus \prod_{i \in I} \mathbb{N}$ (with t_i being the standard generators of $\prod_{i \in I} \mathbb{N}$). Let $\lambda : \mathbb{P} \to \mathbb{k}^*$, $p_{i,j} \mapsto \lambda_{i,j}$ be an evaluation. Define the map (abusing the same notation λ)

$$\lambda : \mathbb{k}[P] \to \mathbb{k}[Q], \quad z^m \mapsto z^m \text{ for } m \in M, \quad p_{i,j} \mapsto \lambda_{i,j} t_i.$$

Lemma 6.20. *The collection*

$$\lambda(\mathfrak{D}_s) := \{(\mathfrak{d}, \lambda(f_{\mathfrak{d}})) \mid (\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}_s\}$$

obtained by applying the algebra homomorphism λ to every wall-crossing function $f_{\mathfrak{d}}$ is a consistent scattering diagram over $\widehat{\Bbbk[Q]}$.

Proof. The algebra homomorphism λ respects the completions of $\Bbbk[P]$ and $\Bbbk[Q]$. So $\lambda(f_{\mathfrak{d}})$ belongs to $\widehat{\Bbbk[Q]}$. Recall we have the monoid map $r : P \to M$ which forgets the components in $\bigoplus_{i \in I} \mathbb{N}^{r_i}$. We use the same notation $r : Q \to M$ for the analogous map on Q. Then $(\mathfrak{d}, \lambda(f_{\mathfrak{d}}))$ becomes a wall over $\widehat{\Bbbk[Q]}$, and $\lambda(\mathfrak{D}_s)$ is a scattering diagram over $\widehat{\Bbbk[Q]}$.

The consistency of $\lambda(\mathfrak{D}_s)$ follows from the consistency of $\mathfrak{D}_s$ as λ is an algebra homomorphism. $\quad\square$

We call $\lambda(\mathfrak{D}_s)$ the (generalized) cluster scattering diagram of s with special coefficients λ. In fact, the ordinary cluster scattering diagram $\mathfrak{D}_s$ when $r_i = 1$ can be obtained this way. We denote the ordinary one by $\mathfrak{D}_s^{\mathrm{ord}}$. Its incoming walls are

$$(e_i^{\perp}, 1 + p_i z^{\omega(-, d_i e_i)}).$$

If there exist coefficients $\lambda_{ij} \in \Bbbk^*$ such that

$$\prod_{j=1}^{r_i} (1 + \lambda_{ij} t_i z^{w_i}) = 1 + t_i^{r_i} z^{r_i w_i} = 1 + t_i^{r_i} z^{\omega(-, d_i e_i)},$$

then we can apply the corresponding morphism $\lambda : \Bbbk[P] \to \Bbbk[Q]$ to $\mathfrak{D}_s$ so that

$$\lambda(\mathfrak{D}_s) \cong \mathfrak{D}_s^{\mathrm{ord}}$$

as they have the exact same set of incoming walls. Here $t_i^{r_i}$ is identified with p_i. The existence of such an evaluation λ amounts to find the r_i roots of the polynomial $1 + x^{r_i}$ in $\Bbbk$, which is always possible if $\Bbbk$ is algebraically closed.

6.5. *Examples.* We illustrate some examples of generalized cluster scattering diagrams in this section.

Example 6.21. Consider the fixed data Γ consisting of

- the lattice $N = \mathbb{Z}^2$ with the standard basis $e_1 = (1, 0)$ and $e_2 = (0, 1)$, and the skew-symmetric form ω be determined by $\omega(e_1, e_2) = -1$;

- $N_{\mathrm{uf}} = N$;

- the rank $r = 2$ and $I = I_{\mathrm{uf}} = \{1, 2\}$;

- positive integers $d_1 = 1$ and $d_2 = 2$;

- the sublattice N° generated by e_1 and $2e_2$;

- $M = \mathrm{Hom}(N, \mathbb{Z})$, $M^{\circ} = \mathrm{Hom}(N^{\circ}, \mathbb{Z})$.

$$1 + t_{11}A_2$$

$$(1 + t_{21}A_1^{-1})(1 + t_{22}A_1^{-1}) \quad\quad\quad\quad\quad (1 + t_{21}A_1^{-1})(1 + t_{22}A_1^{-1})$$

$$1 + t_{11}A_2 \quad\quad\quad\quad 1 + t_{11}t_{21}t_{22}A_1^{-2}A_2$$

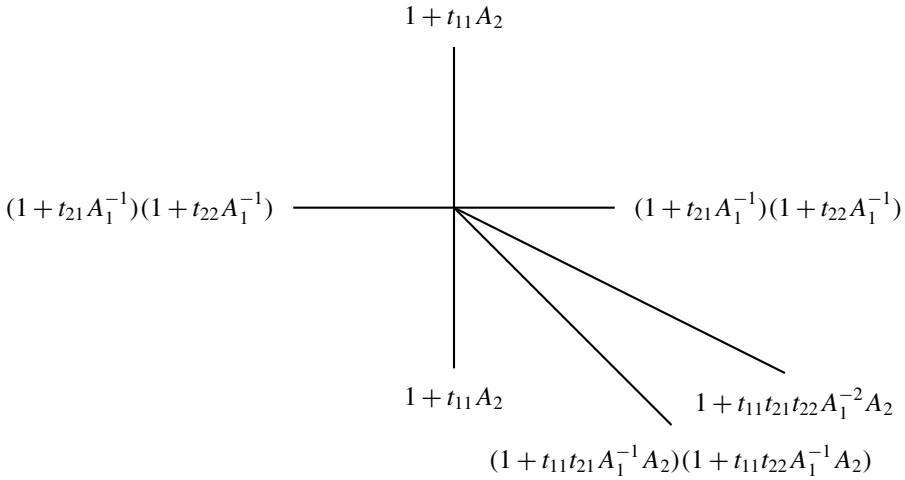$$(1 + t_{11}t_{21}A_1^{-1}A_2)(1 + t_{11}t_{22}A_1^{-1}A_2)$$

Figure 1. The generalized cluster scattering diagram for Example 6.21.

Let s be a seed consisting of $e = (e_1, e_2)$ and $p_1 = (t_{11})$, $p_2 = (t_{21}, t_{22})$. We have matrices

$$B = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In this case we have $\bar{e}_i = d_i e_i / r_i = e_i$. So $\bar{N} = N$ and we shall not worry about the rescaling issue. Then $w_1 = e_2^*$ and $w_2 = -e_1^*$. We write $A_i = z^{e_i^*}$ for $i = 1, 2$. The coefficients group is $\mathbb{P} = \mathbb{Z}^3$ with generators $\{t_{11}, t_{21}, t_{22}\}$. The initial incoming scattering diagram is

$$\mathfrak{D}_{s,\text{in}} = \{(e_1^\perp, 1 + t_{11}A_2), (e_2^\perp, (1 + t_{21}A_1^{-1})(1 + t_{22}A_1^{-1}))\}.$$

The resulting generalized cluster scattering diagram is

$$\mathfrak{D}_s = \mathfrak{D}_{s,\text{in}} \cup \{(\mathbb{R}_{>0}(1, -1), f_{(1,-1)}), (\mathbb{R}_{>0}(2, -1), f_{(2,-1)})\},$$

where

$$f_{(1,-1)} = (1 + t_{11}t_{21}A_1^{-1}A_2)(1 + t_{11}t_{22}A_1^{-1}A_2) \quad \text{and} \quad f_{(2,-1)} = 1 + t_{11}t_{21}t_{22}A_1^{-2}A_2.$$

The scattering diagram $\mathfrak{D}_s$ is depicted in Figure 1.

Example 6.22. Consider the fixed data Γ consisting of

- the lattice $N = \mathbb{Z}^2$ with the standard basis $e_1 = (1, 0)$ and $e_2 = (0, 1)$, and the skew-symmetric form ω be determined by $\omega(e_1, e_2) = -1$;

- $N_{\text{uf}} = N$;

- the rank $r = 2$ and $I = I_{\text{uf}} = \{1, 2\}$;

- positive integers $\lambda_1 = 1$ and $\lambda_2 = 1$;

- the sublattice N° generated by e_1 and e_2;

- $M = \text{Hom}(N, \mathbb{Z})$, $M^\circ = \text{Hom}(N^\circ, \mathbb{Z})$.
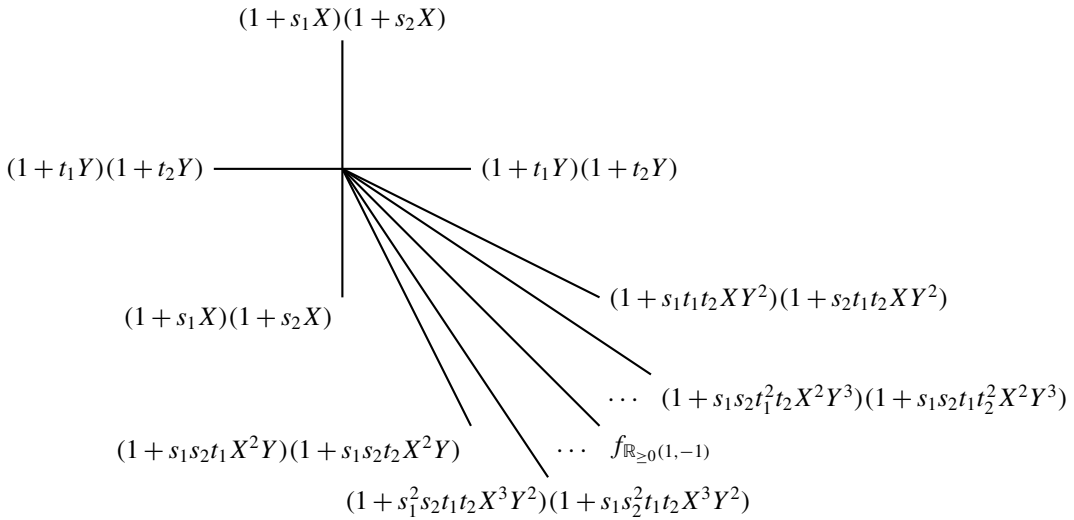
Figure 2. The generalized cluster scattering diagram for Example 6.22.

The seed is given by $\boldsymbol{e} = (e_1, e_2)$ and $\boldsymbol{p}_1 = (s_1, s_2)$, $\boldsymbol{p}_2 = (t_1, t_2)$. The corresponding $\mathfrak{D}_s$ is depicted in Figure 2. We write $X = z^{e_2^*}$ and $Y = z^{-e_1^*}$. The five rays depicted in the fourth quadrant are in the directions $(2, -1)$, $(3, -2)$, $(1, -1)$, $(2, -3)$ and $(1, -2)$ in clockwise order. In fact, in the fourth quadrant there are additional nontrivial walls whose underlying cones are $\mathbb{R}_{\geq 0}(n, -(n+1))$ and $\mathbb{R}_{\geq 0}(n+1, -n)$ for each positive integer $n \geq 3$ (which we omit in the figure below). The wall-crossing function, for example for $\mathbb{R}_{\geq 0}(2k, -(2k+1))$ for $k \in \mathbb{Z}_{>0}$, is

$$f_{\mathbb{R}_{\geq 0}(2k, -(2k+1))} = (1 + s_1^{k+1} s_2^k t_1^k t_2^k X^{2k+1} Y^{2k})(1 + s_1^k s_2^{k+1} t_1^k t_2^k X^{2k+1} Y^{2k}),$$

which can be obtained using Theorem 7.10.

The wall-crossing function attached to the ray $\mathbb{R}_{\geq 0}(1, -1)$

$$f_{\mathbb{R}_{\geq 0}(1, -1)} = \frac{(1 + s_1 t_1 XY)(1 + s_1 t_2 XY)(1 + s_2 t_1 XY)(1 + s_2 t_2 XY)}{(1 - s_1 s_2 t_1 t_2 X^2 Y^2)^4}$$

is much more difficult to calculate. This was explicitly obtained by Reineke and Weist [2013] by relating the wall-crossing functions to quiver representations.

6.6. *Mutation invariance of $\mathfrak{D}_s$.* A first step to investigate the structure of $\mathfrak{D}_s$ is through a comparison with $\mathfrak{D}_{\mu_k(s)}$. For the ordinary case, this is called the *mutation invariance* in [Gross et al. 2018]. In the generalized situation, we show an analogous mutation invariance still holds. One just needs to take care of the generalized coefficients $p_{i,j}$.

Notice that the definition of $\mathfrak{D}_s$ does not involve the semifield structure of $\mathbb{P}$. So one can view that the coefficients part $\boldsymbol{p}$ actually provides a $\mathbb{Z}$-basis of the multiplicative abelian group $\mathbb{P}$ (grouped and labeled in a certain way). Thus even though $\mu_k(s)$ no longer has principal coefficients in $\mathbb{P}$, $\mathfrak{D}_{\mu_k(s)}$ is still defined. To stress that the coefficients are no longer semifield elements, we use $t_{i,j}$ instead of $p_{i,j}$.

Now $s = (e, t)$ consists of e a labeled basis of N and tuples of coefficients $t = (t_i)$.

Definition 6.23. Define the mutation $\mu_k^+(s) = (e', t')$ such that $e' = \mu_k(e)$ as before and for the coefficients,

$$t'_{i,j} = \begin{cases} t_{k,j}^{-1} & \text{if } i = k, \\ t_{i,j} \cdot \prod\limits_{l=1}^{r_k} t_{k,l}^{[\beta_{ki}]_+} & \text{if } i \neq k. \end{cases}$$

Remark 6.24. Note that this mutation does not depend on any semifield structure on $\mathbb{P}$. So it is different from the μ_k from Definition 4.3 for mutations of many steps. For this reason, we call $s = (e, p)$ a seed with coefficients (avoiding the type $\mathcal{A}$- or $\mathcal{X}$-) and use the new symbol μ_k^+ for mutations in this context (as we will see in Section 7.1 the meaning of the sign $+$).

Definition 6.25. We set

$$\mathcal{H}_{k,+} := \{m \in M_{\mathbb{R}} \mid \langle e_k, m \rangle \geq 0\}, \quad \mathcal{H}_{k,-} := \{m \in M_{\mathbb{R}} \mid \langle e_k, m \rangle \leq 0\}.$$

For $k \in I$, define the piecewise linear transformation $T_k : M_{\mathbb{R}} \to M_{\mathbb{R}}$ by

$$T_k(m) := \begin{cases} m + \langle e_k, m \rangle r_k w_k, & m \in \mathcal{H}_{k,+}, \\ m, & m \in H_{k,-}. \end{cases}$$

One sees that in the two half spaces, the map T_k is actually the restriction of two linear maps $T_{k,+}$ and $T_{k,-}$ respectively. The map T_k is with respect to the seed s and thus sometimes will be denoted as T_k^s. The vector $r_k w_k$ can also be expressed as $r_k w_k = \omega(-, d_k e_k) = \sum_{i=1}^n b_{ik} e_i^*$. One checks that

$$T_{k,+}(w_i) = w_i + \beta_{ki} r_k w_k.$$

Recall we have the projection $r : M \oplus \mathbb{P} \to M$. The transformation T_k can be lifted to $M \oplus \mathbb{P}$ by

$$\widetilde{T}_k(m, p) := \begin{cases} (m + \langle e_k, m \rangle r_k w_k, \ p \cdot t_k^{\langle e_k, m \rangle}), & m \in \mathcal{H}_{k,+}, \\ (m, p), & m \in \mathcal{H}_{k,-}, \end{cases}$$

where $t_k = \prod_{l=1}^{r_k} t_{k,l}$. Note that $\widetilde{T}_k$ on its domain of linearity is the restriction of two linear transformations $\widetilde{T}_{k,\varepsilon}$ respectively.

Construction 6.26. We define the scattering diagram $T_k(\mathfrak{D}_s)$ as in [Gross et al. 2018, Definition 1.22] (but taking care of the parameters $t_{i,j}$ here) in the following steps.

(1) Replace each wall in $\mathfrak{D}_s$ not fully contained in $e_k^\perp$ if necessary by splitting it into two new walls

$$(\mathfrak{d} \cap \mathcal{H}_{k,+}, f_{\mathfrak{d}}) \quad \text{and} \quad (\mathfrak{d} \cap \mathcal{H}_{k,-}, f_{\mathfrak{d}}).$$

Regard this new collection of walls as the current representative of $\mathfrak{D}_s$.

(2) For a wall $(\mathfrak{d}, f_{\mathfrak{d}})$ contained in $\mathcal{H}_{k,\varepsilon}$, define the wall $T_{k,\varepsilon}(\mathfrak{d}, f_{\mathfrak{d}}) = (T_{k,\varepsilon}(\mathfrak{d}), \widetilde{T}_{k,\varepsilon}(f_{\mathfrak{d}}))$ where the new wall-crossing function $\widetilde{T}_{k,\varepsilon}(f_{\mathfrak{d}})$ is the one obtained from $f_{\mathfrak{d}}$ by replacing each monomial of the form

$$pz^m \quad \text{by} \quad \widetilde{T}_{k,\varepsilon}(pz^m),$$

where the later is the monomial corresponding to $\widetilde{T}_{k,\varepsilon}(m, p) \in M \oplus \mathbb{P}$. For example, we have

$$\widetilde{T}_{k,+}(t_{i,j}z^{w_i}) = t_{i,j}t_k^{\beta_{ki}}z^{w_i+\beta_{ki}r_k w_k}, \quad \text{while} \quad \widetilde{T}_{k,-}(t_{i,j}z^{w_i}) = t_{i,j}z^{w_i}.$$

We call these walls uniformly by $T_k(\mathfrak{d}, f_{\mathfrak{d}})$ no matter which half they belong to. We stress that the sign ε in $T_{k,\varepsilon}$ is determined by which half space the wall $\mathfrak{d}$ lies in.

(3) Consider the collection of walls

$$T_k(\mathfrak{D}_s) := \left\{ T_k(\mathfrak{d}, f_{\mathfrak{d}}) \,\middle|\, (\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}(s) \setminus \left(e_k^{\perp}, \prod_{j=1}^{r_k}(1+t_{k,j}z^{w_k}) \right) \right\} \cup \left\{ \left(e_k^{\perp}, \prod_{j=1}^{r_k}(1+t_{k,j}^{-1}z^{-w_k}) \right) \right\}.$$

Denote the monoid $(P')^{\oplus} := P_{\mu_k^+(s)}^{\oplus} \subset M \oplus \mathbb{P}$. While $\mathfrak{D}_s$ is over $\widehat{\Bbbk[P_s^{\oplus}]}$, $\mathfrak{D}_{\mu_k^+(s)}$ is over $\widehat{\Bbbk[(P')^{\oplus}]}$.

Theorem 6.27 (cf. [Gross et al. 2018, Theorem 1.24]). *The set of walls $T_k(\mathfrak{D}_s)$ is indeed a consistent scattering diagram over $\widehat{\Bbbk[(P')^{\oplus}]}$, and furthermore is equivalent to $\mathfrak{D}_{\mu_k^+(s)}$.*

We find it most natural to understand the mutation invariance by making connection to the *canonical wall structure* (or canonical scattering diagram) [Gross and Siebert 2022] via [Argüz and Gross 2022, Theorem 6.1], where $\mathfrak{D}_s$ can be viewed as associated to the toric model $U_{\Omega(s,\lambda)}$ for general λ. However, as in Section 5.3.2, this would require the condition

$$r_i = \gcd(b_{ij}, i \in I).$$

Fortunately, we can prove the mutation invariance following the same strategy in [Gross et al. 2018] without this condition. The proof occupies the rest of the section.

First define a monoid $\bar{P}$ containing both $P^{\oplus}$ and $(P')^{\oplus}$. Let σ be the cone in $(M \oplus \mathbb{P})_{\mathbb{R}}$ generated by

$$\{(w_i, t_{i,j}) \mid i \in I, j \in [1, r_i]\} \cup \{(-w_k, -t_{k,j}) \mid 1 \le j \le r_k\}.$$

Take $\bar{P} = \sigma \cap (M \oplus \mathbb{P})$ and we tend to talk about scattering diagrams over $\widehat{\Bbbk[\bar{P}]}$. However the ideal $\mathfrak{m}_{\bar{P}}$ misses the elements $(w_k, t_{k,j})$. This means a wall such as

$$(e_k^{\perp}, (1+t_{k,j}z^{w_k}))$$

in $\mathfrak{D}_s$ does not qualify as a wall over $\widehat{\Bbbk[\bar{P}]}$. For this reason, we extend the definition of scattering diagram as in [Gross et al. 2018, Definition 1.27] (slightly generalizing the *slab* for our needs).

Define

$$\bar{N}_s^{+,k} := \left\{ \sum_{i \in I} a_i \bar{e}_i \,\middle|\, a_i \in \mathbb{Z}_{\ge 0} \text{ for } i \ne k, a_k \in \mathbb{Z}, \text{ and } \sum_{i \in I \setminus \{k\}} a_i > 0 \right\} \subset \bar{N}.$$

Since $\bar{N}_s^{+,k} = \bar{N}_{\mu_k^+(s)}^{+,k}$, we denote them by $\bar{N}^{+,k}$.

Definition 6.28 (cf. [Gross et al. 2018, Definition 1.27]). A *wall* for $\bar{P}$ is a pair $(\mathfrak{d}, f_{\mathfrak{d}})$ with $\mathfrak{d}$ as before but with primitive normal vector n_0 in $\bar{N}^{+,k}$ and

$$f_{\mathfrak{d}} = 1 + \sum_{k \ge 1, \pi(t)=kn_0} c_{k,t} \cdot t z^{k\omega(-,n_0)} \equiv 1 \mod \mathfrak{m}_{\bar{P}}.$$

The *slab* for s and $k \in I$ means the pair

$$\mathfrak{d}_k := \left(e_k^\perp, \prod_{j=1}^{r_k} (1 + t_{k,j} z^{w_k}) \right).$$

A scattering diagram $\mathfrak{D}$ for $\overline{P}$ is a collection of walls and possibly this single slab, with the condition that for each $k > 0$, $f_\mathfrak{d} \equiv 1 \mod \mathfrak{m}_{\overline{P}}^k$ for all but finitely many walls in $\mathfrak{D}$.

We quote the following very hard theorem from [Gross et al. 2018]. The objects here are understood in our definitions so there are minor differences. However, one can still prove the theorem in the exact same way. So we omit its proof here.

Theorem 6.29 [Gross et al. 2018, Theorem 1.28]. *There exists a unique* (*up to equivalence*) *consistent scattering diagram $\overline{\mathfrak{D}}_s$ in the sense of Definition 6.28 such that*

(1) $\overline{\mathfrak{D}}_s \supseteq \mathfrak{D}_{s,\mathrm{in}}$,

(2) $\overline{\mathfrak{D}}_s \setminus \mathfrak{D}_{s,\mathrm{in}}$ *consists only of outgoing walls.*

Furthermore, $\overline{\mathfrak{D}}_s$ is also a scattering diagram for the $\overline{N}_s^+$-graded Lie algebra $\mathfrak{g}_s$. As such, it is equivalent to $\mathfrak{D}_s$.

Proof of Theorem 6.27. First we choose a representative for $\mathfrak{D}_s$ given by Theorem 6.29. Now $T_k(\mathfrak{D}_s)$ becomes a scattering diagram in the sense of Definition 6.28 for the seed $s' = \mu_k^+(s)$. This is because

(1) the operation T_k removes the old slab $\mathfrak{d}_k$ and adds the new slab

$$\mathfrak{d}_k' := \left(e_k^\perp, \prod_{j=1}^{r_k} (1 + t_{k,j}^{-1} z^{-w_k}) \right);$$

(2) for a wall (contained in either $\mathcal{H}_{k,+}$ or $\mathcal{H}_{k,-}$), $\widetilde{T}_k$ sends a monomial of the form $\prod_{i,j} (t_{i,j} z^{w_i})^{a_{ij}}$ in its wall-crossing function to

$$\prod_{i,j} (t_{i,j} t_k^{\beta_{ki}} z^{w_i + \beta_{ki} r_k w_k})^{a_{ij}} \quad \text{or} \quad \prod_{i,j} (t_{i,j} z^{w_i})^{a_{ij}}.$$

So if $t z^m \in \mathfrak{m}_{\overline{P}}^i$ for some i, so is $\widetilde{T}_k(t z^m)$.

We next show that

(1) $T_k(\mathfrak{D}_s)$ and $\mathfrak{D}_{s'}$ have the same set of slabs and incoming walls;

(2) $T_k(\mathfrak{D}_s)$ is consistent as a scattering diagram with a slab.

Then by the uniqueness statement of Theorem 6.29, $T_k(\mathfrak{D}_s)$ and $\mathfrak{D}_{s'}$ are equivalent.

Statement (1) follows from the same argument in *Step I* of [Gross et al. 2018, Proof of Theorem 1.24]. For (2), we check the consistency of $T_k(\mathfrak{D}_s)$, that is, for any loop γ, $\mathfrak{p}_{\gamma, T_k(\mathfrak{D}_s)} = \mathrm{id}$ whenever it is defined.

If γ is confined in one of the half spaces, the path-ordered product is identity because of the consistency of $\mathfrak{D}_s$. So we assume that γ crosses the slab $\mathfrak{d}'_k$. Split γ into four subpaths γ_1, γ_2, γ_3 and γ_4 such that

(1) γ_1 starts at a point in $\mathcal{H}_{k,-}$ and only crosses the slab $\mathfrak{d}'_k$;

(2) γ_2 is contained entirely in $\mathcal{H}_{k,+}$;

(3) γ_3 only crosses $\mathfrak{d}'_k$ back to $\mathcal{H}_{k,-}$;

(4) γ_4 is contained entirely in $\mathcal{H}_{k,-}$.

Let $\widetilde{T}_{k,+} : \Bbbk[M \oplus \mathbb{P}] \to \Bbbk[M \oplus \mathbb{P}]$ be the algebra automorphism induced by $\widetilde{T}_{k,+}$ (see (2) in the Construction 6.26 the action of $\widetilde{T}_{k,+}$ on monomials). Denote by $\mathfrak{p}_{\mathfrak{d}'_k}$ the wall-crossing automorphism

$$z^m \mapsto z^m \prod_{j=1}^{r_k} (1 + t_{k,j}^{-1} z^{-w_k})^{-\langle e_k, m \rangle}.$$

So we have

$$\mathfrak{p}_{\gamma_1, T_k(\mathfrak{D}_s)} = \mathfrak{p}_{\mathfrak{d}'_k}, \tag{6-1}$$

$$\mathfrak{p}_{\gamma_2, T_k(\mathfrak{D}_s)} = \widetilde{T}_{k,+} \circ \mathfrak{p}_{\gamma_2, \mathfrak{D}_s} \circ \widetilde{T}_{k,+}^{-1}, \tag{6-2}$$

$$\mathfrak{p}_{\gamma_3, T_k(\mathfrak{D}_s)} = \mathfrak{p}_{\mathfrak{d}'_k}^{-1}, \tag{6-3}$$

$$\mathfrak{p}_{\gamma_4, T_k(\mathfrak{D}_s)} = \mathfrak{p}_{\gamma_4, \mathfrak{D}_s}. \tag{6-4}$$

All the above equalities except (6-2) are by definitions. To show (6-2), we see that it suffices to show the case where γ_2 only crosses one wall $\mathfrak{d}$ contained in $n_0^\perp$ with the wall-crossing function $f(m_0)$. We write $\widetilde{T} = \widetilde{T}_{k,+}$ and $T = T_{k,+}$. Then we compute the action of the right-hand side of (6-2) on z^m:

$$z^m \mapsto \widetilde{T}^{-1}(z^m) \mapsto \widetilde{T}^{-1}(z^m) f(z^{m_0})^{\langle T^{-1}(m), n_0 \rangle} \mapsto z^m f(\widetilde{T}(z^{m_0}))^{\langle m, (T^{-1})^*(n_0) \rangle}.$$

Note that the wall $\mathfrak{d}$ gets transformed under T_k to be contained in $(T^{-1})^*(n_0)$ with $f(\widetilde{T}(z^{m_0}))$. So the above action is the same as $\mathfrak{p}_{\gamma_2, T_k(\mathfrak{D}_s)}(z^m)$.

To show $\mathfrak{p}_{\gamma, T_k(\mathfrak{D}_s)} = \mathrm{id}$, it suffices to show that

$$\widetilde{T}_{k,+}^{-1} \circ \mathfrak{p}_{\mathfrak{d}'_k} = \mathfrak{p}_{\mathfrak{d}_k}, \tag{6-5}$$

so that $\mathfrak{p}_{\gamma, T_k(\mathfrak{D}_s)} = \mathfrak{p}_{\gamma, T_k(\mathfrak{D}_s)} = \mathrm{id}$.

Letting the left-hand side act on some monomial, we have

$$\widetilde{T}_{k,+}^{-1} \circ \mathfrak{p}_{\mathfrak{d}'_k}(t z^m) = \widetilde{T}_{k,+}^{-1}\left(t z^m \prod_{j=1}^{r_k} (1 + t_{k,j}^{-1} z^{-w_k})^{-\langle e_k, m \rangle}\right)$$

$$= t \cdot t_k^{-\langle e_k, m \rangle} \cdot z^{m - \langle e_k, m \rangle r_k w_k} \prod_{j=1}^{r_k} (1 + t_{k,j}^{-1} z^{-w_k})^{-\langle e_k, m \rangle}$$

$$= t z^m \prod_{j=1}^{r_k} (1 + t_{k,j}^{-1} z^{w_k})^{-\langle e_k, m \rangle}$$

$$= \mathfrak{p}_{\mathfrak{d}_k}(t z^m). \tag{6-6}$$

This finishes the proof. □

$$1 + t_{11}t_{21}t_{22}A_1^{-2}A_2$$

$$(1 + t_{21}^{-1}A_1)(1 + t_{22}^{-1}A_1)$$

$$(1 + t_{21}^{-1}A_1)(1 + t_{22}^{-1}A_1)$$

$$1 + t_{11}A_2$$

$$1 + t_{11}t_{21}t_{22}A_1^{-2}A_2$$

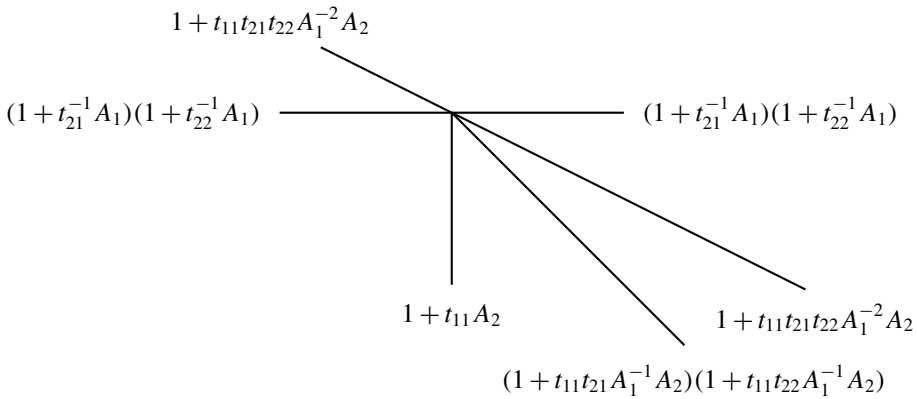$$(1 + t_{11}t_{21}A_1^{-1}A_2)(1 + t_{11}t_{22}A_1^{-1}A_2)$$

Figure 3. The generalized cluster scattering diagram for Example 6.30.

Example 6.30. In this example we compute $T_2(\mathfrak{D}_s)$ for the scattering diagram $\mathfrak{D}_s$ in Example 6.21. Recall that the exchange matrix for s is $B = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$. So we have $T_{2,+}(e_2^*) = e_2^* - 2e_1^*$, which determines the ray $\mathbb{R}_{\geq 0}(e_2^* - 2e_1^*)$ of Figure 3.

6.7. Positivity. The scattering diagram $\mathfrak{D}_s$ has the following positivity.

Theorem 6.31 (cf. [Gross et al. 2018, Theorem 1.28]). *The scattering diagram $\mathfrak{D}_s$ is equivalent to a scattering diagram all of whose walls $(\mathfrak{d}, f_\mathfrak{d})$ satisfy $f_\mathfrak{d} = (1 + t z^m)^c$ for some $m = \omega(-, \bar{n})$, $n \in N^+$, some $t \in \mathbb{P}$ such that $\pi(t) = n$, and c being a positive integer. In other words, if we write $n = \sum_{i \in I} \alpha_i e_i$, then*

(1) *$\mathfrak{d}$ is contained in $\bar{n}^\perp \subset M_\mathbb{R}$ where $\bar{n} = \sum_{i \in I} \alpha_i \frac{d_i}{r_i} e_i$;*

(2) *$m = \sum_{i \in I} \alpha_i w_i = \omega(-, \bar{n})$;*

(3) *if writing $t = \prod_{i,j} t_{i,j}^{\alpha_{i,j}}$, then $\sum_{j=1}^{r_i} \alpha_{ij} = \alpha_i$.*

Proof. This theorem essentially follows from [Gross et al. 2018, Appendix C.3], the proof of the positivity of $\mathfrak{D}_s^{\mathrm{GHKK}}$. We use a representative of $\mathfrak{D}_s$ constructed in the same algorithm used to produce $\mathfrak{D}_s^{\mathrm{GHKK}}$ in the proof of [Gross et al. 2018, Theorem 1.28]. We will construct order by order a sequence of finite scattering diagrams $\mathfrak{D}_1 \subset \mathfrak{D}_2 \subset \cdots$ (over $\widehat{k[P_s^\oplus]}$ or the graded Lie algebra $\mathfrak{g}_s$) such that their union

$$\mathfrak{D} = \bigcup_{k=1}^{\infty} \mathfrak{D}_k$$

is equivalent to $\mathfrak{D}_s$. We then prove inductively that every wall in $\mathfrak{D}_k$ has the positivity property.

Let $\mathfrak{D}_1 = \mathfrak{D}_{s,\mathrm{in}}$. Note that $\mathfrak{D}_1$ is equivalent to $\mathfrak{D}_s$ modulo $(P^+)^2$. Suppose that we have defined up to $\mathfrak{D}_k$ which is equivalent to $\mathfrak{D}$ modulo $(P^+)^{k+1}$, and assume that every wall in $\mathfrak{D}_k$ has wall-crossing function of the form $(1 + t z^m)^c$ for some positive integer c. We construct $\mathfrak{D}_{k+1}$ as follows, and show that it is equivalent to $\mathfrak{D}$ modulo $(P^+)^{k+2}$ and furthermore that it still has the same positivity property for its wall-crossing functions.

There is a finite rational polyhedral cone complex that underlies the support of $\mathfrak{D}_k$ (which is true for any scattering diagram with finitely many walls). We call the codimension two cells *joints*. Let j be a joint of $\mathfrak{D}_k$. Then by [Gross et al. 2018, Definition-Lemma C.2], it falls into two classes:

(1) *Parallel*, if every wall with the normal vector n containing j has $\omega(-, n)$ tangent to j.

(2) *Perpendicular*, if every wall with the normal vector n containing j has $\omega(-, n)$ not tangent to j.

Let γ_j be a simple loop around j small enough so that it only intersects walls containing j. By our assumption, the path-ordered product $\mathfrak{p}_{\gamma_j, \mathfrak{D}_k}$ is identity modulo $(P^+)^{k+1}$, but modulo $(P^+)^{k+2}$, it can be written as

$$\mathfrak{p}_{\gamma_j, \mathfrak{D}_k} = \exp\Big(\sum_{d(t,m)=k+1} c_{t,m} t z^m \partial_{n(t,m)} \Big),$$

where $c_{t,m} \in \Bbbk$. Here we define the degree $d(t, m) := k + 1$ if $(t, m) \in (P^+)^{k+1} \setminus (P^+)^{k+2}$, and $n(t, m)$ is primitive in N^+ uniquely determined by (t, m).

If j is perpendicular, we define a set of walls

$$\mathfrak{D}[j] := \{ (j - \mathbb{R}_{\geq 0} m, (1 + t z^m)^{\pm c_{t,m}}) \mid d(t, m) = k + 1 \},$$

where $j - \mathbb{R}_{\geq 0} m$ is of codimension one since m is not tangent to j. Here the function $(1 + t z^m)^{\pm c_{t,m}}$ makes sense as a power series. The sign $\pm$ in the power is chosen so that when γ_j crosses $j - \mathbb{R}_{\geq 0} m$, the wall-crossing automorphism is

$$\exp(-c_{t,m} t z^m \partial_{n(t,m)}).$$

In this way, if we add the walls in $\mathfrak{D}[j]$ to $\mathfrak{D}_k$, we have the path-ordered product $\mathfrak{p}_{\gamma_j, \mathfrak{D}_k \cup \mathfrak{D}[j]} = \mathrm{id}$ modulo $(P^+)^{k+2}$. We then define

$$\mathfrak{D}_{k+1} = \mathfrak{D}_k \cup \bigcup_j \mathfrak{D}[j],$$

where the union is over all perpendicular joints of $\mathfrak{D}_k$.

There are two things we need to show in the induction:

(1) $\mathfrak{D}_{k+1}$ is equivalent to $\mathfrak{D}_s$ modulo $(P^+)^{k+2}$.

(2) All the walls in $\mathfrak{D}_{k+1}$ have wall-crossing functions of the form $(1 + t z^m)^c$ for some positive integer c.

Part (1) follows from the argument in [Gross et al. 2018, Lemma C.6 and Lemma C.7]. This part guarantees that the constructed union $\mathfrak{D}$ is equivalent to $\mathfrak{D}_s$.

Part (2) is about the positivity of wall-crossings. By the construction of $\mathfrak{D}_{k+1}$, we only need to examine the new walls emerging from perpendicular joints of $\mathfrak{D}_k$. Let j be a perpendicular joint of $\mathfrak{D}_k$. The integral normal space $j^\perp \cap N$ is a rank two saturated sublattice O of N. Locally at j, $\mathfrak{D}_k \cup \mathfrak{D}[j]$ induces a scattering diagram living in $O_\mathbb{R}^\vee = M_\mathbb{R}/(\Lambda_j \otimes \mathbb{R})$. Precisely, consider the set of walls

$$\mathfrak{D}' = \{ ((\mathfrak{d} + \Lambda_j \otimes \mathbb{R})/(\Lambda_j \otimes \mathbb{R}), f_\mathfrak{d}) \mid j \subset \mathfrak{d}, \ (\mathfrak{d}, f_\mathfrak{d}) \in \mathfrak{D}_k \cup \mathfrak{D}[j] \}.$$

The wall-crossing functions $f_{\mathfrak{d}}$ are all of the form

$$(1+tz^m)^c,$$

$c \in \Bbbk$ ($f_{\mathfrak{d}}$ makes sense as a power series). The wall $\mathfrak{d}$ has some primitive normal vector $o \in O \cap N^+$, and m is proportional to $\omega(-, o)$. We also know since j is perpendicular, $\bar{m} \neq 0$ (the image of m under the quotient $M \to O^\vee$) in $O_{\mathbb{R}}^\vee$. And the one-dimensional wall $\bar{\mathfrak{d}} = (\mathfrak{d} + \Lambda_j \otimes \mathbb{R})/(\Lambda_j \otimes \mathbb{R})$ is contained in $\mathbb{R}(\bar{m})$, orthogonal to the normal vector o. Then $\mathfrak{D}'$ is a rank two scattering diagram in $O_{\mathbb{R}}^\vee$ over $\widehat{\Bbbk[P^+]}$, with the monoid map from P^+ to $O^\vee$ being $r : P \to M$ postcomposed by the quotient from M to $O^\vee$. It is consistent up to modulo $(P^+)^{k+2}$. Then by [Gross et al. 2018, Proposition C.13], the wall-crossing functions admit the positivity property, i.e., the power c is always a positive integer. This shows the positivity for $\mathfrak{D}_{k+1}$ assuming that of $\mathfrak{D}_k$. Therefore, the union $\mathfrak{D}$ is also positive by induction; hence so is $\mathfrak{D}_s$. $\square$

7. The cluster complex structure

In this section, we study the *cluster complex structure* of the scattering diagram $\mathfrak{D}_s$, which is a description of parts of the walls of $\mathfrak{D}_s$. The construction of such a structure of $\mathfrak{D}_s$ is analogous to [Gross et al. 2018, Construction 1.30].

7.1. *The cluster complex.* Take a representative for the scattering diagram $\mathfrak{D}_s$ with minimal support (which always exists). By Theorem 6.29, one can choose such a representative $\mathfrak{D}_s$ so that there are no other walls contained in the initial incoming ones $\mathfrak{d}_i$.

Define

$$\mathcal{C}^+ = \mathcal{C}_s^+ := \{m \in M_{\mathbb{R}} \mid \langle e_i, m \rangle \geq 0 \ \forall i \in I\},$$
$$\mathcal{C}^- = \mathcal{C}_s^- := \{m \in M_{\mathbb{R}} \mid \langle e_i, m \rangle \leq 0 \ \forall i \in I\}.$$

The closed cones $\mathcal{C}_s^\pm$ are closures of connected components of $M_{\mathbb{R}} \setminus \mathrm{Supp}(\mathfrak{D}_s)$. They are thus called *chambers*. By the mutation invariance Theorem 6.27, we have that the cones

$$T_k^{-1}(\mathcal{C}_{\mu_k^+(s)}^\pm) \subset M_{\mathbb{R}} \setminus \mathrm{Supp}(\mathfrak{D}_s)$$

are also closures of connected components. Applying mutations on seeds provides an iterative way to construct chambers of $M_{\mathbb{R}} \setminus \mathrm{Supp}(\mathfrak{D}_s)$ as follows.

Note again that the coefficients part of $s = (e, t)$ does not mutate as in Definition 4.3, which requires setting the tropical semifield $\mathbb{P}$ from the initial seed and once for all. Instead, we regard the coefficients part t as in the multiplicative group $\mathbb{P}$ and mutates in the way specified by Section 6.6. In this way, we can apply mutations iteratively on s.

Let us consider the rooted tree $\mathfrak{T}_n$ from Definition 4.6. There is an association $v \mapsto s_v$ such that $v_0 \mapsto s$ and adjacent seeds with coefficients are related by the corresponding mutation (in the sense of Section 6.6) of the labeled edges. Once this association is done, we denote the rooted tree by $\mathfrak{T}_s$.

Suppose the unique path from v_0 to a vertex v goes through the arrows labeled by $\{k_1, k_2, \ldots, k_l\}$. Define the piecewise linear map

$$T_{v_0,v} = T_{k_l} \circ \cdots \circ T_{k_2} \circ T_{k_1} : M_{\mathbb{R}} \to M_{\mathbb{R}}.$$

Since $\mathcal{C}_s^{\pm}$ are chambers of the scattering diagram $\mathfrak{D}_s$, then again due to the mutation invariance, we have that

$$\mathcal{C}_v^{\pm} := T_{v_0,v}^{-1}(\mathcal{C}_{s_v}^{\pm})$$

are chambers of $\mathfrak{D}_s$.

Each $\mathcal{C}_v^{\pm}$ is a simplicial (rational polyhedral) cone of maximal dimension, as each T_k is a linear isomorphism on its domains of linearity. The intersection $\mathcal{C}_s^+ \cap \mathcal{C}_{\mu_k^+(s)}^+$ is their common facet generated by $\{e_i^* \mid i \neq k\}$. Each facet of $\mathcal{C}_v$ is canonically labeled by an index $i \in I$. Inductively, for any two vertices v and v' connected by an arrow labeled by $k \in I$, then $\mathcal{C}_v^+$ and $\mathcal{C}_{v'}^+$ share a common facet labeled by k.

We borrow the following notation from [Gross et al. 2018]: we use the short-hand subscription notation $v \in s$ for an object parametrized by a vertex $v \in \mathfrak{T}_s$ with the root v_0 labeled by s. This is done to emphasize the dependence on the initial seed s.

Definition 7.1. We denote by $\mathcal{C}_{v \in s}^{\pm}$ the chambers $\mathcal{C}_v^{\pm}$ of $\subset M_{\mathbb{R}} \setminus \mathrm{Supp}(\mathfrak{D}_s)$. We write $\Delta_s^{\pm}$ for the set of chambers $\mathcal{C}_{v \in s}^{\pm}$ for v running over all vertices of $\mathfrak{T}_s$. We call elements in Δ_s^+ *cluster chambers*.

Remark 7.2. As we have pointed out, $\mathcal{C}_v^+ \cap \mathcal{C}_{v'}^+$ is a common facet if v and v' are adjacent in $\mathfrak{T}_s$. More generally, by adding all the faces of every $\mathcal{C}_v^+$ to the set Δ_s^+, we obtain a collection of cones which form a cone complex, still denoted by Δ_s^+. For this reason, we call Δ_s^+ *the cluster* (*cone*) *complex* and Δ_s^- *the negative cluster* (*cone*) *complex*.

The simplicial cone $\mathcal{C}_{v \in s}^{\pm}$ is determined by (the generators of) its one-dimensional faces. The cone $\mathcal{C}_{s_v}^+$ is generated by the dual vectors $\{e_{i;v}^* \mid i \in I\}$. These are pulled back by $T_{v_0,v}^{-1}$ to be the generators of $\mathcal{C}_{v \in s}^+$.

Definition 7.3. We define the *g-vectors* for $v \in \mathfrak{T}_s$ as a tuple

$$\boldsymbol{g}_v = (g_{i;v} \mid i \in I), \quad \text{where } g_{i;v} := T_{v_0,v}^{-1}(e_{i;v}^*) \in M.$$

We will use the notation $\boldsymbol{g}_{v \in s}$ to emphasize the initial seed s.

Remark 7.4. Denote the dual vectors (in N) of $\boldsymbol{g}_v$ by $\boldsymbol{g}_v^* = (g_{i;v}^* \mid i \in I)$. They are normal vectors of the facets of $\mathcal{C}_v^+$. Since the walls of $\mathfrak{D}_s$ only have normal vectors in N_s^+ or $-N_s^+$, the vector $g_{i;v}^*$ has a well-defined sign

$$\varepsilon_{i;v} = \mathrm{sgn}(g_{i;v}^*) = \begin{cases} + & \text{if } g_{i;v}^* \in N_s^+, \\ - & \text{if } g_{i;v}^* \in N_s^-. \end{cases}$$

We will show later the vectors $\boldsymbol{g}_v$ can be calculated iteratively by a variant of mutations as defined below.

Definition 7.5. Let $e = (e_i \mid i \in I)$ be a seed (without coefficients) for Γ. Define the *signed mutation* $\mu_k^\varepsilon(e) = (e_i' \mid i \in I)$ for $\varepsilon \in \pm$ as follows:

$$e_i' = \begin{cases} -e_k & \text{if } i = k, \\ e_i + [-\varepsilon\omega(e_i, d_k e_k)]_+ e_k & \text{if } i \neq k. \end{cases}$$

So the signed mutation μ_k^+ coincides with our previous Definition 6.23 (ignoring the coefficients part).

On the mutation of the dual of e, we use the same notation $\mu_k^\varepsilon(e^*) = (f_i' \mid i \in I)$ where $e^* = (f_i \mid i \in I)$. Then

$$f_i' = \begin{cases} f_i & \text{if } i \neq k \\ -f_k + \sum_{i \in I}[-\varepsilon\omega(e_i, d_k e_k)]_+ f_k & \text{if } i = k. \end{cases}$$

There is another tuple of vectors in M that changes under signed mutations. For a seed s, let $w = (w_i \mid i \in I)$, where

$$w_i := \omega\left(-, \frac{d_k}{r_k} e_k\right) = \sum_{j \in I} b_{ji} f_i \in M.$$

Let $w' = (w_i')$ associated to $\mu_k^\varepsilon(e)$. Then we have

$$w_i' = \begin{cases} -w_k & \text{if } i = k, \\ w_i + [\varepsilon\omega(e_k, d_k e_i)]_+ w_k & \text{if } i \neq k. \end{cases}$$

We will later denote $\mu_k^\varepsilon(w) = w'$.

There are also signed mutations for coefficients. Recall we have fixed a multiplicative abelian group $\mathbb{P} = \prod_{i \in I} \mathbb{Z}^{r_i}$. The coefficients $t = (t_{i,j} \mid i \in I, j \in [1, r_i])$ are a basis of $\mathbb{P}$.

Definition 7.6. For $s = (e, t)$, a seed e together with coefficients $t = (t_{i,j})$ in $\mathbb{P}$, we define its *signed mutation in direction k*, $\mu_k^\varepsilon(e, (t_{i,j})) = (e', (t_{i,j}'))$ for $\varepsilon \in \pm$ by setting $s' = \mu_k^\varepsilon(s)$ and

$$t_{i,j}' = \begin{cases} t_{k,j}^{-1} & \text{if } i = k, \\ t_{i,j} \cdot \prod_{l=1}^{r_k} t_{k,l}^{[\varepsilon\omega(e_k, e_i)]_+} & \text{if } i \neq k. \end{cases}$$

Proposition 7.7 (cf. [Mou 2020, Proposition 4.4.9]). *For every $v \in \mathfrak{T}_s$, the dual of g-vectors g_v^* is a seed of N. These seeds and their duals, i.e., the g-vectors, can obtained iteratively as follows*:

(1) $g_{v_0} = e^*$ and $g_{v_0}^* = e$.

(2) *For any $v \xrightarrow{k} v'$ in $\mathfrak{T}_s$, we have*

$$g_{v'}^* = \mu_k^{\varepsilon_{k;v}}(g_v^*), \quad g_{v'} = \mu_k^{\varepsilon_{k;v}}(g_v).$$

Proof. We prove this proposition by induction on the distance from v to v_0. The base case is when $v = v_0$, in which we have

$$g_{v'}^* = \mu_k^+(e) = \mu_k^+(g_{v_0}^*), \quad g_{v'} = \mu_k^+(e^*) = \mu_k^+(g_{v_0}).$$

Now assuming that $v \neq v_0$ and suppose that the unique path from v_0 to v starts with $v_0 \xrightarrow{i} v_1$ for some $i \in I$. Write $\boldsymbol{s}_1 = \boldsymbol{s}_{v_1} = \mu_i^+(\boldsymbol{s})$. By induction, we assume that the proposition holds for g-vectors with respect to the seed $\boldsymbol{s}_1$:

$$\boldsymbol{g}_{v' \in \boldsymbol{s}_1} = \mu_k^\varepsilon(\boldsymbol{g}_{v \in \boldsymbol{s}_1}),$$

where $\varepsilon = \varepsilon_{k; v \in \boldsymbol{s}_1} = \operatorname{sgn}(g_{k; v \in \boldsymbol{s}_1}^*)$ with respect to $\boldsymbol{s}_1$. Note that by definition

$$\boldsymbol{g}_{v' \in \boldsymbol{s}} = (T_i^{\boldsymbol{s}})^{-1}(\boldsymbol{g}_{v' \in \boldsymbol{s}_1}), \quad \boldsymbol{g}_{v \in \boldsymbol{s}} = (T_i^{\boldsymbol{s}})^{-1}(\boldsymbol{g}_{v \in \boldsymbol{s}_1}),$$

and we want to prove

$$\boldsymbol{g}_{v' \in \boldsymbol{s}} = \mu_k^\delta(\boldsymbol{g}_{v \in \boldsymbol{s}}),$$

where $\delta = \varepsilon_{k; v \in \boldsymbol{s}}$ with respect to $\boldsymbol{s}$.

Then it amounts to show that

$$(T_i^{\boldsymbol{s}})^{-1} \circ \mu_k^\varepsilon(\boldsymbol{g}_{v \in \boldsymbol{s}_1}) = \mu_k^\delta \circ (T_i^{\boldsymbol{s}})^{-1}(\boldsymbol{g}_{v \in \boldsymbol{s}_1}). \tag{7-1}$$

We split the discussion into the following two cases. The codimension one skeletons of the chambers $\mathcal{C}_{v \in \boldsymbol{s}_1}^+$ and $\mathcal{C}_{v' \in \boldsymbol{s}_1}^+$ are in the essential support of $\mathfrak{D}_{\boldsymbol{s}_1}$. As v and v' are adjacent, these two chambers share a common facet. Therefore they are either separated by the hyperplane $e_i^\perp$ or contained in the same half space (since the hyperplane is also in the essential support).

Case 1. The two groups of g-vectors $\boldsymbol{g}_{v \in \boldsymbol{s}_1}$ and $\boldsymbol{g}_{v' \in \boldsymbol{s}_1}$ are separated by $e_i^\perp$. In this case, the normal vector $g_{k; v \in \boldsymbol{s}_1}^*$ is in the direction of e_i. The signs δ and ε on the two sides of (7-1) are then different. We assume that $\varepsilon = \operatorname{sgn}(g_{k; v \in \boldsymbol{s}_1}^*) = +$; the other case is analogous. By our assumption, $\boldsymbol{g}_{v \in \boldsymbol{s}_1}^*$ qualifies as a seed of fixed data Γ, thus forming a basis of N, which implies $g_{k; v \in \boldsymbol{s}_1}^* = e_i$. Since $\{\lambda_j g_{j; v \in \boldsymbol{s}_1}^* \mid j \in I\}$ form a basis of the sublattice N°, we have $d_i = d_k$. We note that the map $T_i^{\boldsymbol{s}}$ is actually determined by the vectors e_i and $d_i e_i$. On the left-hand side of (7-1), $T_i^{\boldsymbol{s}}$ is the identity, while on the right-hand side, it is $T_{i,+}^{\boldsymbol{s}}$. So we need to show the equality

$$\mu_k^+(\boldsymbol{g}_{v \in \boldsymbol{s}_1}) = \mu_k^- \circ (T_{i,+}^{\boldsymbol{s}})^{-1}(\boldsymbol{g}_{v \in \boldsymbol{s}_1}).$$

To simplify the notation, we denote $\boldsymbol{g} = \boldsymbol{g}_{v \in \boldsymbol{s}_1}$ and $g_i = g_{i; v \in \boldsymbol{s}_1}$. On the left side of the equality, the tuple $\mu_k^+(\boldsymbol{g}) = (g_i')$ differs with $\boldsymbol{g}$ by only one vector

$$g_k' = -g_k + \sum_{i \in I} [-b_{ik}^v]_+ g_i.$$

On the right-hand side, we first have

$$(T_{i,+}^{\boldsymbol{s}})^{-1}(g_k) = -g_k + \sum_{i \in I} -b_{ik}^v g_i,$$

while other g-vectors remain unchanged under $(T_{i,+}^{\boldsymbol{s}})^{-1}$. It is easy to check that the dual of $(T_{i,+}^{\boldsymbol{s}})^{-1}$ is an automorphism of (N, ω), that is, it is a linear automorphism on N preserving the form ω. Thus we have,

if writing $\mu_k^- \circ (T_{i,+}^s)^{-1}(\boldsymbol{g}_{v \in s_1}) = (g_i'')$,

$$g_k'' = -g_k + \sum_{i \in I} -b_{ik}^v g_i + \sum_{i \in I} [b_{ik}^v]_+ g_i = g_k', \quad \text{and} \quad g_i'' = g_i \quad \text{for } i \neq k.$$

This finishes the proof of the desired equality.

Case 2. The g-vectors $\boldsymbol{g}_{v \in s_1}$ and $\boldsymbol{g}_{v' \in s_1}$ are all contained in the same half $\mathcal{H}_{i,+}^s$ or $\mathcal{H}_{i,-}^s$. Again we need to prove (7-1). We observe that the two signs δ and ε are equal. In fact, the sign ε of $g_{k;v \in s_1}^*$ depends on its coordinates in $e_{j;v_1}$ for $j \neq i$ since $g_{k;v \in s_1}^*$ is not purely proportional to e_i. The same is true for the sign δ which only depends on $g_{k;v \in s}^*$'s coordinates in e_j for $j \neq i$. Since $g_{k;v \in s}^*$ only differ in the direction of e_i, and also because $e_{j;v_1}$ and e_j also differ by multiples of e_i, we conclude that $\varepsilon = \delta$. The equality (7-1) then directly follows from a fact we already mentioned in Case 1 that the dual of $(T_{i,\varepsilon}^s)^{-1}$ acts as an automorphism on (N, ω). $\qquad \square$

A direct corollary of Proposition 7.7 is another description of c-vectors mentioned in Section 3.3. Recall that we have $\pi : \mathbb{P} \to \overline{\mathbb{P}}$, $p_{i,j} \mapsto \bar{p}_i$. We write the group operation in $\mathbb{P}$ and $\overline{\mathbb{P}}$ by addition instead of multiplication.

Corollary 7.8. *We identify the lattice $\overline{N}$ with $\overline{\mathbb{P}}$ by $\bar{e}_i = \frac{d_i}{r_i} e_i \mapsto \bar{p}_i$. Then we have for any $i \in I$ and $v \in \mathfrak{T}_s$,*

$$\frac{d_i}{r_i} g_{i;v}^* = \bar{p}_{i;v}, \quad d_i g_{i;v}^* = r_i \bar{p}_{i;v} = p_{i;v}.$$

Proof. For the initial vertex v_0, this is given by the identification $\bar{e}_i \mapsto \bar{p}_i$. The iteration of $g_{i;v}^*$ is provided by signed mutations according to Proposition 7.7. We have if $v \xrightarrow{k} v'$ in $\mathfrak{T}_s$,

$$g_{i;v'}^* = \begin{cases} -g_{k;v}^* & \text{if } i = k, \\ g_{i;v}^* + [-\varepsilon b_{ik}^v]_+ g_{i;v}^* & \text{if } i \neq k, \end{cases}$$

where $\varepsilon = \text{sgn}(g_{k;v}^*)$. What is implicit is that we have already known that $g_{i;k}^*$ is either nonnegative or nonpositive. On the other hand, the mutation of $p_{i;v}$ is given by

$$p_{i;v'} = \begin{cases} -p_{k;v} & \text{if } i = k, \\ p_{i;v} + b_{ki}^v \cdot p_{k;v}^+ & \text{if } i \neq k \text{ and } b_{ik} \leq 0, \\ p_{i;v} + b_{ki}^v \cdot p_{k;v}^- & \text{if } i \neq k \text{ and } b_{ik} > 0. \end{cases}$$

Thus assuming $d_i g_{i;v}^* = p_{i;v}$ for all $i \in I$ would imply $d_i g_{i;v'}^* = p_{i;v'}$ for all $i \in I$ as they have the same mutation formula when $p_{k;v}$ has a well-defined sign. Therefore the result is proved by induction on the distance from v to v_0. $\qquad \square$

Lemma 7.9. *The generalized coefficients $p_{i,j;v}$ have the following signed mutation formula. If $v \xrightarrow{k} v'$ in $\mathfrak{T}_s$, then*

$$p_{i,j;v'} = \begin{cases} -p_{k,j;v} & \text{if } i = k, \\ p_{i,j;v} + [\varepsilon \beta_{ki}^v]_+ \cdot \sum_{j=1}^{r_k} p_{k,j;v} & \text{if } i \neq k \end{cases}$$

where $\varepsilon = \text{sgn}(g_{k;v}^*)$.

Proof. By Corollary 7.8, $p_{i;v}$ is sign coherent because $g^*_{i;v}$ is so. As we have already shown in Proposition 3.17 that the sign coherence of $p_{i;v}$ implies that of $p_{i,j;v}$, the result follows by induction. $\square$

7.2. Wall-crossings. We next study the wall-crossing functions attached to walls of the cluster chambers. Each cluster chamber $\mathcal{C}^+_{v\in s}$ has exactly n facets $\mathfrak{d}_{i;v\in s}$ naturally indexed by I (a facet has the same index as its normal vector $g^*_{i,v\in s}$). The wall $(\mathfrak{d}_{i;v\in s}, f_{i;v\in s})$ is pulled back by $T^{-1}_{v_0,v}$ from the scattering diagram $\mathfrak{D}_{s_v}$ (with coefficients t_v). The wall-crossing function $f_{i;v}$ has the following description. Here we identify the initial coefficients $t_{i,j}$ with $p_{i,j}$, and endow $\mathbb{P}$ the semifield structure $\mathrm{Trop}(p)$.

Theorem 7.10. *The scattering diagram $\mathfrak{D}_s$ has a representative in its equivalent class such that it is the union of the scattering diagram*

$$\mathfrak{D}(\Delta^+_s) := \{(\mathfrak{d}_{i;v}, f_{i;v}) \mid i \in I, v \in \mathfrak{T}_s\}, \quad \text{where } f_{i;v} = \prod_{j=1}^{r_i} \left(1 + p^{\varepsilon_{i;v}}_{i,j;v} \cdot z^{\varepsilon_{i;v} \sum_{j=1}^{n} \beta^v_{ji} g_{j;v}}\right)$$

and another one whose support is disjoint from Δ^+_s.

Proof. We prove this theorem by induction on the distance from v to v_0. We first note that by Lemma 7.9 the coefficients $p_{i,j;v} \in \mathbb{P}$ can be computed iteratively by signed mutations. The vectors

$$w_{i;v} := \sum_{j=1}^{n} \beta^v_{ji} g_{j;v} = \omega\left(-, \frac{d_i}{r_i} g^*_{i;v}\right) \in M$$

can also be computed iteratively by signed mutations since the g-vectors do by Proposition 7.7.

Assume that the result is true for the distance between two vertices no greater v_0 and v. Suppose we have that $v \xrightarrow{k} v' \in \mathfrak{T}_s$ and that the unique path from v_0 to v_1 starts from $v_0 \xrightarrow{i_0} v_1$.

Let's look at the chambers $\tau := \mathcal{C}^+_{v\in s_1}$ and $\tau' := \mathcal{C}^+_{v'\in s_1}$ in $\mathfrak{D}_{s_1}$. They have g-vectors satisfying

$$g_{v'\in s_1} = \mu^\varepsilon_k(g_{v\in s_1}),$$

where $\varepsilon = \varepsilon_{k;v\in s_1} := \mathrm{sgn}(g^*_{k;v\in s_1})$. For the wall-crossing functions, by our assumption, for $i \in I$, we have

$$f_{i;v\in s_1} = \prod_{j=1}^{r_i} (1 + p^{\varepsilon_{i;v\in s_1}}_{i,j;v\in s_1} z^{\varepsilon_{i;v\in s_1} w_{i;v\in s_1}}),$$

$$f_{i;v'\in s_1} = \prod_{j=1}^{r_i} (1 + p^{\varepsilon_{i;v'\in s_1}}_{i,j;v'\in s_1} z^{\varepsilon_{i;v'\in s_1} w_{i;v'\in s_1}}).$$

These two functions are related by the signed mutation μ^ε_k. More precisely, we have

$$\mu^\varepsilon_k(g^*_{v\in s_1}, p_{v\in s_1}) = (g^*_{v'\in s_1}, p_{v'\in s_1}), \quad \mu^\varepsilon_k(w_{v\in s_1}) = w_{v'\in s_1}.$$

We want to pull back the chambers $\mathcal{C}^+_{v\in s_1}$ and $\mathcal{C}^+_{v'\in s_1}$, as well as the wall-crossing functions $f_{i;v\in s_1}$ and $f_{i;v'\in s_1}$ to $\mathfrak{D}_s$ via the operation $(T^s_{i_0})^{-1}$ to get the chambers $\sigma := \mathcal{C}^+_{v\in s}$, $\sigma' := \mathcal{C}^+_{v'\in s}$ and the wall-crossing functions $f_i := f_{i;v\in s}$ and $f'_i := f_{i;v'\in s}$ by the mutation invariance Theorem 6.27. We want to show that f_i and f'_i are also related by signed mutations. In the following, we calculate f_i and f'_i in detail by applying $\widetilde{T}^{-1}_{i_0}$ to $f_{i;v\in s_1}$ and $f_{i;v'\in s_1}$. This depends on the following two cases as in the proof of Proposition 7.7:

(1) The two chambers τ and τ' are separated by the hyperplane $e_{i_0}^\perp$.

(2) They are contained in the same half space $\mathcal{H}_{i_0,+}$ or $\mathcal{H}_{i_0,-}$.

Case 1. In this case, the normal vector $g^*_{k;v\in s_1}$ is either e_{i_0} or $-e_{i_0}$. Assume it is e_{i_0}; the other case is similar. Then the chamber τ is in $\mathcal{H}_{i_0,+}$ while τ' is in $\mathcal{H}_{i_0,-}$. First of all, we have $f_k = f'_k$ obtained simply by reversing the monomials in $f_{k;v\in s_1} = f_{k;v'\in s_1}$. Since T_{i_0} (as well as $\widetilde{T}_{i_0}$) is identity on $\mathcal{H}_{i_0,-}$, we have for $i \neq k$, $f'_i = f_{i;v'\in s_1}$. Note that for the signs, for $i \in I$,

$$\varepsilon_{i;v'\in s_1} = \varepsilon_{i;v'\in s}$$

unless $g^*_{i;v'\in s_1}$ is proportional to e_{i_0}, which only happens for $g^*_{k;v'\in s_1} = -g^*_{k;v\in s_1}$, where we have

$$\varepsilon_{k;v'\in s_1} = -, \qquad \varepsilon_{k;v'\in s} = +.$$

So we conclude for any $i \in I$,

$$f'_i = \prod_{j=1}^{r_i} (1 + (p_{i,j;v'\in s_1} z^{w_{i;v'\in s_1}})^{\varepsilon_{i;v'\in s}}).$$

For $f_{i;v\in s_1}$ and f_i, we first consider the signs $\varepsilon_{i;v\in s_1}$ and $\varepsilon_{i;v\in s}$. Since the dual of $T_{i_0}^{-1}$ on N only shifts in the direction of e_{i_0}, we have for $i \neq k$

$$\varepsilon_{i;v\in s_1} = \varepsilon_{i;v\in s},$$

as the vectors $g^*_{i;v\in s_1}$ and $g^*_{i;v\in s}$ must have the same sign in all the other directions except for e_{i_0}, and the only one proportional to e_{i_0} is $g^*_{k;v\in s_1}$. Thus we have for $i \neq k$,

$$f_i = \prod_{j=1}^{r_i} (1 + \widetilde{T}_{i_0}^{-1}(p_{i,j;v\in s_1} z^{w_{i;v\in s_1}})^{\varepsilon_{i;v\in s}})$$

We want to show that f_i and f'_i are related by the mutation μ_k^δ. Precisely, it amounts to show that

$$\mu_k^\delta(\widetilde{T}_{i_0}^{-1}(p_{i,j;v\in s_1} z^{w_{i;v\in s_1}} \mid i \in I, j \in [1, r_i])) = \mu_k^\varepsilon(p_{i,j;v\in s_1} z^{w_{i;v\in s_1}} \mid i \in I, j \in [1, r_i]), \qquad (7\text{-}2)$$

where δ is the sign $\varepsilon_{k;v\in s}$. Here we abuse the notation $\mu_k^\pm$ which acts on a tuple of functions, but it should be clear what it means. By our assumption, $\varepsilon = +$ and $\delta = -\varepsilon = -$. Then this follows from the general fact that for any seed (e, t) and $k \in I$, we have

$$\mu_k^-(\widetilde{T}_k^{-1}(t_{i,j} z^{w_i} \mid i \in I, j \in [1, r_i]) = \mu_k^+(t_{i,j} z^{w_i} \mid i \in I, j \in [1, r_i]).$$

Case 2. Suppose τ and τ' are both contained in the same half space. According to our above discussion, as in the notation of (7-2), it then amounts to check that

$$\widetilde{T}_{i_0}^{-1}(\mu_k^\varepsilon(p_{i,j;v\in s_1} z^{w_{i;v\in s_1}} \mid i \in I, j \in [1, r_i])) = \mu_k^\delta(\widetilde{T}_{i_0}^{-1}(p_{i,j;v\in s_1} z^{w_{i;v\in s_1}} \mid i \in I, j \in [1, r_i])),$$

where $\delta = \varepsilon_{k;v\in s}$. As we have discussed in the **Case 2** of the proof of Proposition 7.7, the signs are equal: $\delta = \varepsilon$. Then the rest follows immediately from the fact that the dual of $T_{i_0,\varepsilon}$ acts as an automorphism on the data (N, ω). $\qquad\square$

8. Reconstructing $\mathscr{A}^{\mathrm{prin}}$

In this section, we see how to reconstruct the generalized cluster algebra $\mathscr{A}^{\mathrm{prin}}(s)$ as well as the variety $\mathcal{A}^{\mathrm{prin}}(s)$ from $\mathcal{X}_{s,\lambda}$ through $\mathfrak{D}_s$.

8.1. _Reconstructing $\mathscr{A}^{\mathrm{prin}}(s)$ from $\mathfrak{D}_s$._ Given fixed data Γ and an $\mathcal{A}$-seed with principal coefficients $s = (e, p)$, denote by $\mathscr{A}^{\mathrm{prin}}(s)$ the corresponding generalized cluster algebra. Recall that we denote by $x_{i;v}$ the cluster variables associated to the seed s_v.

Consider the generalized cluster scattering diagram $\mathfrak{D}_s$, whose wall-crossings act on $\Bbbk\widehat{[P]}$ by automorphisms. For two vertices $v, v' \in \mathfrak{T}_s$, let γ be a path from the chamber $\mathcal{C}^+_{v \in s}$ to $\mathcal{C}^+_{v' \in s}$ and consider the path-ordered product

$$\mathfrak{p}_{v,v'} = \mathfrak{p}^s_{v,v'} := \mathfrak{p}_{\gamma,\mathfrak{D}_s} : \Bbbk\widehat{[P]} \to \Bbbk\widehat{[P]}.$$

Since $\mathfrak{D}_s$ is consistent and one can always choose some γ contained in the cluster complex, the path-ordered product $\mathfrak{p}_{v,v'}$ can also be viewed as an automorphism of $\mathrm{Frac}(M \oplus \mathbb{P})$.

Proposition 8.1. _Let $\mathcal{C}^+_{v \in s}$ be a cluster chamber and $\boldsymbol{g}_v$ the set of g-vectors. Then for any $i \in I$,_

$$x_{i;v} = \mathfrak{p}_{v,v_0}(z^{g_{i;v}}) \in \mathrm{Frac}(M \oplus \mathbb{P}).$$

Proof. We prove this by induction on the distance from v to v_0 in $\mathfrak{T}_s$. Suppose the statement is true for a vertex $v \in \mathfrak{T}_s$ and we have $v \xrightarrow{i} v'$ in $\mathfrak{T}_s$. Then the chambers $\mathcal{C}^+_v$ and $\mathcal{C}^+_{v'}$ are separated by the wall $\mathfrak{d}_{i;v}$ with the wall-crossing $f_{i;v}$ given in Theorem 7.10. Denote $\varepsilon = \mathrm{sgn}(g^*_{i;v}) \in \{+, -\}$. Then we have

$$\mathfrak{p}_{v',v}(z^{g_{i;v'}}) = z^{g_{i;v'}} \prod_{j=1}^{r_i} \left(1 + p^{\varepsilon}_{i,j;s_v} \cdot z^{\sum_{j=1}^{n} \varepsilon \beta^v_{ji} g_{j;v}}\right)^{-\langle g_{i;v'}, g^*_{i;v}\rangle}.$$

By Proposition 7.7, we have

$$g_{i;v'} = -g_{i;v} + \sum_{j=1}^{n} [-\varepsilon r_i \beta^v_{ji}]_+ g_{j;v}.$$

This leads to

$$\mathfrak{p}_{v',v}(z^{g_{i;v'}}) = z^{-g_{i;v}} \prod_{j=1}^{r_i} \left(z^{\sum_{j \in I}[-\varepsilon \beta^v_{ji}]_+ g_{j;v}} + p^{\varepsilon}_{i,j;s_v} \cdot z^{\sum_{j \in I}[\varepsilon \beta^v_{ji}]_+ g_{j;v}}\right).$$

Note that by sign coherence, $p_{i,j;s_v}$ has the same sign as ε. So the above equation is exactly the exchange relation of cluster variables. Applying the path-ordered product $\mathfrak{p}_{v,v_0}$ on both sides of the above equality finishes the induction. $\qquad\square$

By the generalized Laurent phenomenon Theorem 3.7, we know that $x_{i;v}$ actually lives in $\Bbbk[M \oplus \mathbb{P}]$.

Corollary 8.2. _The set of cluster variables of $\mathscr{A}^{\mathrm{prin}}(s)$ is in bijection with the set of g-vectors._

Proof. We send a cluster variable $x_{i;v}$ to the g-vector $g_{i;v}$. To show that $x_{i;v}$ is uniquely determined by $g_{i;v}$, we observe that the formula $\mathfrak{p}_{v,v_0}(z^{g_{i;v}})$ is independent of the choice of v. Suppose there is another chamber $\mathcal{C}^+_{v' \in s}$ such that $g_{i;v}$ is one of the generators. Choose a path γ from $\mathcal{C}^+_{v \in s}$ to $\mathcal{C}^+_{v' \in s}$ close enough

to the ray $\mathbb{R}_+ g_{i;v}$ so that it only crosses walls containing $\mathbb{R}_+ g_{i;v}$. The two path-ordered products $\mathfrak{p}_{v,v_0}$ and $\mathfrak{p}_{v',v_0}$ differ by $\mathfrak{p}_\gamma$, which acts on $z^{g_{i;v}}$ by identity. Thus $\mathfrak{p}_{v,v_0}(z^{g_{i;v}}) = \mathfrak{p}_{v',v_0}(z^{g_{i;v}})$. □

8.2. Reconstructing $\mathcal{A}_s^{\mathrm{prin}}$ from $\mathfrak{D}_s$.

Recall that there is a surjective map from $\mathfrak{T}_s$ to Δ_s^+ (the set of cluster chambers) sending v to $\mathcal{C}_{v\in s}^+$. For each vertex $v \in \mathfrak{T}_s$, we associate a torus $T_{N,v}(R) = T_N(R)$. To a pair of vertices v and v', we associate the birational morphism

$$\mathfrak{q}_{v,v'} = \mathfrak{q}_{v,v'}^s : T_{N,v}(R) \dashrightarrow T_{N,v'}(R), \quad \mathfrak{q}_{v,v'}^* := \mathfrak{p}_{v',v}.$$

Then there is an R-scheme obtained by gluing $T_{N,v}(R)$, $v \in \mathfrak{T}_s$ via these birational morphisms

$$\mathcal{A}_{\mathrm{scat},s}^{\mathrm{prin}} := \bigcup_{v\in\mathfrak{T}_s} T_{N,v}(R).$$

One can actually relate $\mathcal{A}_{\mathrm{scat},s}^{\mathrm{prin}}$ to the previously defined cluster variety

$$\mathcal{A}_s^{\mathrm{prin}} := \bigcup_{v\in\mathfrak{T}_s} T_{N,s_v}(R),$$

which is obtained by gluing together the same set of tori via $\mathcal{A}$-cluster mutations.

Recall the piecewise linear map $T_{v_0,v} : M_\mathbb{R} \to M_\mathbb{R}$ that sends the cluster chamber $\mathcal{C}_{v\in s}^+$ to $\mathcal{C}_{s_v}^+$. When restricted to a domain of linearity, $T_{v_0,v}$ becomes a linear automorphism on M. Denote the restriction of $T_{v_0,v}$ on $\mathcal{C}_{v\in s}^+$ by $T_{v_0,v}|_{\mathcal{C}_{v\in s}^+}$. In particular, $T_{v_0,v_0}|_{\mathcal{C}_s^+}$ is the identity map. These linear isomorphisms induce isomorphisms (or R-schemes) between tori

$$\psi_{v_0,v} : T_{N,s_v}(R) \to T_{N,v\in s}(R), \quad \psi_{v_0,v}^*(z^m) = z^{T_{v_0,v}|_{\mathcal{C}_{v\in s}^+}(m)}.$$

Proposition 8.3. *The isomorphisms $\psi_{v_0,v}$ glue to be an isomorphism*

$$\psi_{v_0} : \mathcal{A}_s^{\mathrm{prin}} \to \mathcal{A}_{\mathrm{scat},s}^{\mathrm{prin}}.$$

Proof. The morphisms $\mu_{v,v'}$ (resp. $\mathfrak{q}_{v,v'}$) are generated $\mu_{v_0,v}$ (resp. $\mathfrak{q}_{v_0,v}$) for all v in $\mathfrak{T}_s$. So the statement is equivalent to the commutativity of the following diagram (for any v).

$$
\begin{array}{ccc}
T_{N,s} & \xrightarrow{\psi_{v_0,v_0}=\mathrm{id}} & T_{N,v_0\in s} \\
\Big\downarrow{\scriptstyle\mu_{v_0,v}} & & \Big\downarrow{\scriptstyle\mathfrak{q}_{v_0,v}} \\
T_{N,s_v} & \xrightarrow{\psi_{v_0,v}} & T_{N,v\in s}
\end{array}
$$

To show $\mathfrak{q}_{v_0,v} = \psi_{v_0,v} \circ \mu_{v_0,v}$, we pull back the functions $z^{g_{i;v}}$ (for all $i \in I$) via these birational morphisms. On the left-hand side, we get the cluster variables

$$x_{i;v} = \mathfrak{q}_{v_0,v}^*(z^{g_{i;v}})$$

by Proposition 8.1. On the right-hand side, these $z^{g_{i;v}}$ get pulled back to $z^{e^*_{i;v}}$ by $\psi^*_{v_0,v}$ as $T_{v_0,v}|_{\mathcal{C}^+_{v\in s}}$ sends the chamber $\mathcal{C}^+_{v\in s}$ to the chamber $\mathcal{C}^+_{s_v}$. Then via $\mu^*_{v_0,v}$, we still get cluster variables

$$x_{i;v} = \mu^*_{v_0,v}(z^{e^*_{i;v}}).$$

As $\{g_{i;v} \mid i \in I\}$ form a basis of M, we conclude that $\mathfrak{q}_{v_0,v} = \psi_{v_0,v} \circ \mu_{v_0,v}$, which finishes the proof. $\square$

We next see in a certain sense the variety $\mathcal{A}^{\mathrm{prin}}_s$ is independent of s. This is a subtle issue as for the cluster algebra $\mathscr{A}^{\mathrm{prin}}(s)$, the initial seed $\Sigma(s)$ is distinguished from others since it has principal coefficients.

To resolve this, we again treat $\mathbb{P}$ as only a multiplicative abelian group. Consider $s' = \mu^+_k(s)$ in the sense of Theorem 6.27. The tree $\mathfrak{T}_{s'}$ is naturally embedded in $\mathfrak{T}_s$, along with the association of seeds with coefficients. First of all, it is clear that the inclusion

$$\bigcup_{v \in \mathfrak{T}_{s'}} T_{N,v\in s} \subset \mathcal{A}^{\mathrm{prin}}_{\mathrm{scat},s}$$

is an equality. The gluing maps are given by path-ordered products of $\mathfrak{D}_s$.

Consider for $v \in \mathfrak{T}_{s'}$, the isomorphism (of R-schemes)

$$\varphi_v : T_{N,v\in s'} \to T_{N,v\in s}$$

such that $\varphi^*_v : \Bbbk[M \oplus \mathbb{P}] \to \Bbbk[M \oplus \mathbb{P}]$ is given by the linear transformation

$$T_k \mid_{\mathcal{C}^+_{v\in s}} : M \oplus \mathbb{P} \to M \oplus \mathbb{P}.$$

Proposition 8.4. *The maps φ_v for $v \in \mathfrak{T}_{s'}$ glue together to have an isomorphism of $\Bbbk[\mathbb{P}]$-schemes*

$$\varphi : \mathcal{A}^{\mathrm{prin}}_{\mathrm{scat},s'} \to \mathcal{A}^{\mathrm{prin}}_{\mathrm{scat},s}.$$

Proof. Let v and v' be two vertices in $\mathfrak{T}_{s'}$. Since each φ_v is an isomorphism, the statement is equivalent to the commutativity of the following diagram (for any v and v').

$$
\begin{array}{ccc}
T_{N,v\in s'} & \xrightarrow{\quad\varphi_v\quad} & T_{N,v\in s} \\
\Big\downarrow{\scriptstyle \mathfrak{q}^{s'}_{v,v'}} & & \Big\downarrow{\scriptstyle \mathfrak{q}^{s}_{v,v'}} \\
T_{N,v'\in s'} & \xrightarrow{\quad\varphi_{v'}\quad} & T_{N,v'\in s}
\end{array}
$$

In terms of algebras, this amounts to showing that

$$T_k \mid_{\mathcal{C}^+_{v\in s}} \circ \mathfrak{p}^s_{v,v'} = \mathfrak{p}^{s'}_{v,v'} \circ T_k \mid_{\mathcal{C}^+_{v'\in s}} : \Bbbk[M \oplus \mathbb{P}] \dashrightarrow \Bbbk[M \oplus \mathbb{P}].$$

If the two chambers $\mathcal{C}^+_{v\in s}$ and $\mathcal{C}^+_{v'\in s}$ are on the same side of the hyperplane $e^\perp_k$, the above equality is just (6-2). If they are separated by $e^\perp_k$, it is the same as (6-5) and has been checked in (6-6). $\square$

Combined with Proposition 8.3, we see that the construction $\mathcal{A}_s^{\mathrm{prin}}$ is independent of s. In terms of the corresponding cluster algebra $\mathscr{A}^{\mathrm{prin}}(s)$, once it has principal coefficients on some seed s, it can be made to do so at any seed mutation equivalent to s.

8.3. *Broken lines and theta functions.* This section is a recast of [Gross et al. 2018, Section 3] in the generalized situation. Recall the setting of scattering diagrams in Definition 6.5.

Definition 8.5 (broken line, cf. [Gross et al. 2018, Definition 3.1]). Let $\mathfrak{D}$ be a scattering diagram over $\widehat{\Bbbk[P]}$ with a monoid map $r : P \to M$. Let $p_0 \in P \setminus \ker(r)$ and $Q \in M_\mathbb{R} \setminus \mathrm{Supp}(\mathfrak{D})$. A *broken line* for p_0 with endpoint Q is a piecewise linear continuous proper map

$$\gamma : (-\infty, 0] \to M_\mathbb{R} \setminus \mathrm{Sing}(\mathfrak{D})$$

with a finite number of domains of linearity $L_1, L_2, \dots, L_k$ (open intervals in $(-\infty, 0]$), where each $L = L_i \subset (-\infty, 0]$ is labeled by a monomial $c_L z^{p_L} \in \Bbbk[P]$ with $p_L \in P$. This data should satisfy:

(1) $\gamma(0) = Q$.

(2) If $L = L_1$ is the first domain of linearity of γ, i.e., $L = (-\infty, t)$ for some $t \leq 0$, then $c_L z^{p_L} = z^{p_0}$.

(3) For $t \in L$ any domain of linearity, $m_L := r(p_L) = -\gamma'(t)$.

(4) For two consecutive domains of linearity $L = (a, t)$ (a can be $-\infty$) and $L' = (t, b)$, the monomial $c_{L'} z^{p_{L'}}$ is a term in the formal power series

$$\mathfrak{p}_{\gamma(t),\mathfrak{D}}(c_L z^{p_L}) = c_L z^{p_L} \prod_{\substack{(\mathfrak{d}, f_\mathfrak{d}) \\ \gamma(t) \in \mathfrak{d}}} f_\mathfrak{d}^{-\langle n_0, m_L \rangle}.$$

Here $n_0 \in N$ is primitive, serving as a normal vector of every $\mathfrak{d}$ appearing in the product such that $\langle n_0, \gamma'(t) \rangle > 0$. So the power $-\langle n_0, m_L \rangle$ is always a positive integer.

Definition 8.6 (theta function, [Gross et al. 2018, Definition 3.3]). Let $\mathfrak{D}$ be a scattering diagram over $\widehat{\Bbbk[P]}$. Let $p_0 \in P \setminus \ker(r)$ and $Q \in M_\mathbb{R} \setminus \mathrm{Supp}(\mathfrak{D})$. For a broken line γ for p_0 with end point Q, define

$$\mathrm{Mono}(\gamma) := c_Q z^{p_Q},$$

where (by abuse of notation) Q stands for the last linear segment of γ. We define the *theta function* for p_0 with endpoint Q as the formal sum

$$\vartheta_{Q,p_0} := \sum_\gamma \mathrm{Mono}(\gamma),$$

where the sum is over the set of all broken lines for p_0 with endpoint Q.

For $p_0 = \ker(r)$, we define for any endpoint Q

$$\vartheta_{Q,p_0} = z^{p_0}.$$

We collect some important properties for theta functions from [Gross et al. 2018].

Theorem 8.7. (1) *The theta function ϑ_{Q,p_0} is in $\widehat{\Bbbk[P]}$.*

(2) *Suppose that $\mathfrak{D}$ is consistent. Then for $Q, Q' \in M_{\mathbb{R}} \setminus \operatorname{Supp}(\mathfrak{D})$ whose coordinates are linearly independent over $\mathbb{Q}$, and $p_0 \in P$,*

$$\vartheta_{Q',p_0} = \mathfrak{p}_{\gamma,\mathfrak{D}}(\vartheta_{Q,p_0}),$$

where γ is a path in $\mathfrak{D}$ from Q to Q' such that its path-ordered product is well-defined.

Proof. Part (1) essentially follows from the proof of [Gross et al. 2018, Proposition 3.4]. We are using a different monoid P here, but the same proof still works with $J := \mathfrak{m}_P = P \setminus M$.

Part (2), as pointed out in the proof of [Gross et al. 2018, Theorem 3.5], is again a special case of [Carl et al. 2024, Section 4]. Here the generic condition on the coordinates of Q and Q' is just to make sure that any broken line does not cross any joint of $\mathfrak{D}$. Modulo $\mathfrak{m}_P^k$, the independence of ϑ_{Q,m_0} on Q within one chamber follows from [Carl et al. 2024, Lemma 4.7]. The compatibility between Q and Q' in different chambers follows from [Carl et al. 2024, Lemma 4.9]. See also a more general discussion on the global property of theta functions in [Gross et al. 2022, Section 3.3]. □

In the case of generalized cluster scattering diagrams $\mathfrak{D}_s$ (see Definition 6.17), the monoid P is $M \oplus \bigoplus_{i \in I} \mathbb{N}^{r_i}$ (contained in $M \oplus \mathbb{P}$) with the natural projection r to the direct summand M. We have the following properties of theta functions.

Proposition 8.8 (mutation invariance of broken line, cf. [Gross et al. 2018, Proposition 3.6]). *The piecewise linear transformation $T_k : M_{\mathbb{R}} \to M_{\mathbb{R}}$ (with a lift on $M \oplus \mathbb{P}$) defines a one-to-one correspondence $\gamma \mapsto T_k(\gamma)$ between broken lines for p_0 with endpoint Q for $\mathfrak{D}_s$ and broken lines for $T_k(p_0)$ with endpoint $T_k(Q)$ for $\mathfrak{D}_{\mu_k(s)}$. This correspondence satisfies, depending on whether $Q \in \mathcal{H}_{k,+}$ or $\mathcal{H}_{k,-}$,*

$$\operatorname{Mono}(T_k(\gamma)) = T_{k,\pm}(\operatorname{Mono}(\gamma)),$$

where $T_{k,\pm}$ acts on a monomial as in Theorem 6.27. In particular, we have

$$\vartheta^{\mu_k(s)}_{T_k(Q),T_k(p_0)} = T_{k,\pm}(\vartheta^s_{Q,p_0}).$$

Proof. We use $T_k(\gamma)$ to denote the piecewise linear map $T_k \circ \gamma : (-\infty, 0] \to M_{\mathbb{R}}$. Suppose L is a domain of linearity of γ labeled with monomial $c_L z^{p_L}$. If $\gamma(L)$ is contained in one of the half spaces $\mathcal{H}_{k,\pm}$, L is also a domain of linearity for $T_k(\gamma)$. We apply the action of $T_{k,\pm}$ on the monomial $c_L z^{p_L}$ (where the sign is chosen depending on which half space L is in). If $\gamma(L)$ crosses $e_k^{\perp}$, split L into L^+ and L^-, and apply $T_{k,\pm}$ respectively to the monomial $c_L z^{p_L}$. One then needs to check the piecewise linear path $T_k \circ \gamma$ together with the new monomial data we just obtained is a broken line for $T_k(p_0)$ with endpoint $T_k(Q)$ in $\mathfrak{D}_{\mu_k(s)}$ as in [Gross et al. 2018, Proposition 3.6]. The inverse of the operation $\gamma \mapsto T_k \circ \gamma$ is also clear. The rest of the statement follows easily. □

Proposition 8.9 (cf. [Gross et al. 2018, Proposition 3.8]). *Consider the scattering diagram $\mathfrak{D}_s$.*

(1) *Let $Q \in \text{Int}(\mathcal{C}_s^+)$ be an end point, and let $p \in P$ be such that $r(p) \in \mathcal{C}_s^+ \cap M$. Then $\vartheta_{Q,p} = z^p$.*

(2) *Let $\mathcal{C}_v^+ \in \Delta_s^+$ be a cluster chamber for some $v \in \mathfrak{T}_s$, and $Q \in \text{int}(\mathcal{C}_v^+)$ and $m \in \mathcal{C}_v^+ \cap M$. Then $\vartheta_{Q,p} = z^p$ if $r(p) = m$.*

Proof. Part (1) is essentially [Gross et al. 2018, Proposition 3.8], although there the scattering diagram is actually different from $\mathfrak{D}_s$ in terms of wall-crossing functions. However, the bending behavior of a broken line on a wall is totally analogous, so the exact same argument still applies.

Part (2) is the generalized version of [Gross et al. 2018, Corollary 3.9]. By Proposition 8.8, the transformation $T_{v_0,v} : M_\mathbb{R} \to M_\mathbb{R}$ defines a one-to-one correspondence between the broken lines for p with $r(p) \in \mathcal{C}_v^+ \cap M$ and $Q \in \text{int}(\mathcal{C}_v^+)$ in $\mathfrak{D}_s$, and the ones for $T_{v_0,v}(p)$ with $r(T_{v_0,v}(p)) \in \mathcal{C}_{s_v}^+ \cap M$ and $T_{v_0,v}(Q) \in \text{int}(\mathcal{C}_{s_v}^+)$. However the only broken lines of the later is labeled by the final monomial $z^{p'}$ for $p' = T_{v_0,v}(p)$ by part (1). The result follows. $\square$

8.4. *Cluster monomials as theta functions.*

Definition 8.10. Let s be a generalized $\mathcal{A}$-seed with principal coefficients. Then for $v \in \mathfrak{T}_s$, a cluster monomial in this seed is a monomial on the torus $T_{N,v}(R) \subset \mathcal{A}_s^{\text{prin}}$ of the form z^m where m is a nonnegative $\mathbb{N}$-linear combination of $\{e_{i;v}^* \mid i \in I\}$. By the Laurent phenomenon, such a monomial extends to a regular function on the whole cluster variety $\mathcal{A}_s^{\text{prin}}$.

Remark 8.11. One may regard a cluster monomial as a function on the initial torus $T_{N,v_0}(R)$. While being a monomial on the cluster variables $x_{i;v}$, it is also a Laurent polynomial in the initial cluster variables x_i by the Laurent phenomenon.

The following description of cluster monomials is a generalized version of [Gross et al. 2018, Theorem 4.9]. It proves the positivity (see Theorem 3.8) of generalized cluster monomials.

Theorem 8.12. *Let $\mathfrak{D}_s$ be the generalized cluster scattering diagram of a seed s. Let $Q \in \text{int}(\mathcal{C}_s^+)$ a general end point and $m \in \mathcal{C}_v^+ \cap M$ for some $v \in \mathfrak{T}_s$. Then the theta function $\vartheta_{Q,m}$ is an element in $z^m \cdot \mathbb{N}[P]$ which expresses the cluster monomial associated to m of the algebra $\mathscr{A}^{\text{prin}}(s)$ in the initial seed s.*

Proof. We first note that m is regarded as a point in P through the inclusion of M in P. Let Q' be a base point in $\text{int}(\mathcal{C}_v^+)$ and γ be a path going from Q' to Q. By part (2) of Theorem 8.7, we have

$$\vartheta_{m,Q} = \mathfrak{p}_\gamma(\vartheta_{m,Q'}).$$

As a theta function, $\vartheta_{m,Q}$ is a (formal) sum of monomials belonging to $z^m \widehat{\mathbb{k}[P]}$. By the positivity Theorem 6.31 of $\mathfrak{D}_s$, $\vartheta_{m,Q}$ has positive integer coefficients, thus an element in $z^m \widehat{\mathbb{N}[P]}$. By part (2) of Proposition 8.9, $\vartheta_{m,Q'} = z^m$. We know that the cone $\mathcal{C}_v^+$ has integral generators $\{g_{i;v} \mid i \in I\}$ in M. Thus m is a nonnegative linear combination of these g-vectors.

On the other hand, by Proposition 8.1, we have the following expression of a cluster variable

$$x_{i;v} = \mathfrak{p}_\gamma(z^{g_{i;v}}).$$

It follows immediately that $\vartheta_{m,Q}$ is a monomial of these $x_{i;v}$, thus expressing a cluster monomial. Finally by the generalized Laurent phenomenon Theorem 3.7, we have $\vartheta_{m,Q} \in z^m \cdot \mathbb{N}[P]$. $\qquad\square$

Since $\vartheta_{m,Q}$ does not depend on Q as long as it is chosen generally in the positive chamber, we simply write it as ϑ_m. Consider the set of functions

$$\{\vartheta_m \mid m \in \Delta_s^+(\mathbb{Z})\},$$

where $\Delta_s^+(\mathbb{Z}) = \bigcup_{v \in \mathfrak{T}_s} \mathcal{C}_v^+ \cap M$. These are all cluster monomials. In general, they do not form an R-basis of the cluster algebra $\mathscr{A}^{\mathrm{prin}}(s)$ or the upper cluster algebra $\overline{\mathscr{A}}^{\mathrm{prin}}(s)$. But one can follow [Gross et al. 2018, Section 7.1] to define the set $\Theta \subset M$ such that for any $m \in \Theta$, ϑ_m is only a sum of monomials from finitely many broken lines. Consider the free R-module

$$\mathrm{mid}(\mathcal{A}_s^{\mathrm{prin}}) := \bigoplus_{m \in \Theta} R \cdot \vartheta_m.$$

It is shown in [Gross et al. 2018, Theorem 7.5] that in the ordinary case there are natural inclusions of R-modules

$$\mathscr{A}^{\mathrm{prin}}(s) \subset \mathrm{mid}(\mathcal{A}_s^{\mathrm{prin}}) \subset \overline{\mathscr{A}}^{\mathrm{prin}}(s)$$

such that for the first inclusion, cluster monomials are sent to the corresponding theta functions, and for the second inclusion, any theta function is sent to the corresponding universal Laurent polynomials on $\mathcal{A}_s^{\mathrm{prin}}$ (see [Gross et al. 2018, Proposition 7.1]). We expect that this is also true in the generalized case.

8.5. *More on positivity.* Chekhov and Shapiro [2014] proposed a positivity conjecture which is stronger than Theorem 3.8. We formulate a version here.

A generalized cluster algebra in the sense of [Chekhov and Shapiro 2014] (see Section 3.2) is called *reciprocal* if any of its exchange polynomials $\theta_i(u, v)$ is monic and palindromic, i.e., $\theta_i(u, v) = \theta_i(v, u)$ and has leading coefficient 1. In this way, the exchange polynomials do not change under mutations. Note that $\theta_i(u, v)$ can have coefficients in $\mathbb{ZP}$ (rather than just in $\mathbb{P}$) in general.

Conjecture 8.13 (cf. [Chekhov and Shapiro 2014, Conjecture 5.1]). *Any cluster variable of a reciprocal generalized cluster algebra whose exchange polynomials have coefficients in $\mathbb{P}$ (or more generally in $\mathbb{NP}$) is expressed as a positive Laurent polynomial in the initial cluster, i.e., an element in $\mathbb{NP}[x_1^{\pm}, \ldots, x_n^{\pm}]$ where the x_i's are the initial cluster variables.*

Chekhov and Shapiro [2014, Section 5] pointed out that this conjecture is true for any generalized cluster algebra associated to a surface with arbitrary orbifold points (see also [Banaian and Kelley 2020] for a proof using snake graphs). The rank two case of this conjecture has been resolved in [Rupel 2013].

We consider here a related situation where the reciprocal assumption is not required. Let $\mathbb{P}$ be an abelian group of finite rank. Consider an algebraic closure $\Bbbk = \overline{\mathbb{QP}}$ of the field of rational functions $\mathbb{QP}$. Let $\mathscr{A}^{\mathrm{prin}}(\Sigma)$ be a generalized cluster algebra with principal coefficients as of Definition 3.13. The coefficients group is the tropical semifield $\mathrm{Trop}(\boldsymbol{p})$. Recall that the initial exchange polynomials have the form

$$\theta_i(u, v) = \prod_{j=1}^{r_i} (p_{i,j} u + v).$$

Let $\lambda : \mathrm{Trop}(\boldsymbol{p}) \to \Bbbk^*$ be an evaluation (as in Section 3.5) such that each $\lambda(\theta_i(u, v))$ satisfies:

(**A**) All its coefficients are in $\mathbb{ZP}$ (in $\mathbb{NP}$ if assuming positivity).

(**B**) $\lambda\left(\prod_{j=1}^{r_i} p_{i,j}\right)$ is an element in $\mathbb{P}$.

By the mutation formula of coefficients, the exchange polynomials after any steps of mutations still satisfy these two conditions. Therefore the cluster algebra with special coefficients $\mathscr{A}^{\mathrm{prin}}(\Sigma, \lambda)$ can be viewed as a generalized cluster algebra of [Chekhov and Shapiro 2014] (with the coefficients group $\mathbb{P}$). Note that any reciprocal generalized cluster algebra can be obtained this way.

The scattering diagram $\lambda(\mathfrak{D}_s)$ (see Section 6.4) is responsible for $\mathscr{A}^{\mathrm{prin}}(\Sigma, \lambda)$. It is over $\Bbbk\widehat{\left[M \oplus \prod_{i \in I} \mathbb{N}\right]}$ with formal parameters t_i. Note that by the generalized Laurent phenomenon, the cluster variables of $\mathscr{A}^{\mathrm{prin}}(\Sigma, \lambda)$ are all in $\mathbb{ZP}[x_1^{\pm}, \ldots, x_n^{\pm}]$.

Theorem 8.14. *Let $\mathscr{A}^{\mathrm{prin}}(\Sigma, \lambda)$ be a generalized cluster algebra as above assuming* (**A**), (**B**), *and that the initial exchange polynomials have coefficients in $\mathbb{NP}$. Let $\boldsymbol{s}$ be an $\mathcal{A}$-seed such that $\Sigma(\boldsymbol{s}) = \Sigma$. If there exists a representative of $\lambda(\mathfrak{D}_s)$ such that every wall-crossing function is in $\mathbb{NP}\widehat{\left[M \oplus \prod_{i \in I} \mathbb{N}\right]}$, then any cluster variable is expressed as a positive Laurent polynomial in the initial cluster, i.e., an element in $\mathbb{NP}[x_1^{\pm}, \ldots, x_n^{\pm}]$.*

Proof. As in Theorem 8.12, the positivity of cluster variables follows from the positivity of the scattering diagram $\lambda(\mathfrak{D}_s)$ since every broken line ends with a monomial with coefficients in $\mathbb{NP} \subset \Bbbk$. Expressing a cluster variable as a theta function for $\lambda(\mathfrak{D}_s)$ (and evaluated at $t_i = 1$ where the t_i's are the standard generators of $\prod_{i \in I} \mathbb{N}$), the result follows. $\qquad\square$

If $\mathscr{A}^{\mathrm{prin}}(\Sigma, \lambda)$ is of finite type (i.e. there are only finitely many distinguished cluster variables), then the cluster complex Δ_s^+ is finite and complete in $M_{\mathbb{R}}$ by Corollary 8.2. By Theorem 7.10, we have that $\mathfrak{D}_s = \mathfrak{D}(\Delta_s^+)$ and the wall-crossing function on any facet of any cluster chamber has coefficients in $\mathbb{NP}$ under the evaluation λ if assuming so for the initial ones. Then the positivity follows in this case from Theorem 8.14. It is not hard to check that in Example 6.22 the expansion of the wall-crossing function $f_{\mathbb{R}_{\geq 0}(1,-1)}$ has every coefficient in $\mathbb{N}[s_1 s_2, s_1 + s_2, t_1 t_2, t_1 + t_2]$. By the description in Example 6.22 of all other walls, all wall-crossings functions in this scattering diagram are positive in this sense. This then implies all cluster variables are positive, i.e., have coefficients in $\mathbb{N}[s_1 s_2, s_1 + s_2, t_1 t_2, t_1 + t_2]$.

Acknowledgements

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