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Differentially large fields

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We introduce the notion of *differential largeness* for fields equipped with several commuting derivations (as an analogue to largeness of fields). We lay out the foundations of this new class of "tame" differential fields. We state several characterizations and exhibit plenty of examples and applications. Our results strongly indicate that differentially large fields will play a key role in differential field arithmetic. For instance, we characterize differential largeness in terms of being existentially closed in their power series field (furnished with natural derivations), we give explicit constructions of differentially large fields in terms of iterated powers series, we prove that the class of differentially large fields is elementary, and we show that differential largeness is preserved under algebraic extensions, therefore showing that their algebraic closure is differentially closed.

1. Introduction

Recall that a field K is called *large* (or *ample*) if every irreducible variety defined over K with a smooth K-rational point has a Zariski-dense set of K-rational points. Equivalently, every variety defined over K that has a $K((t))$-rational point also has a K-rational point. Large fields constitute one of the widest classes of *tame fields*: namely, every class of fields that serves as a *locality*, in the sense that universal local-global principles hold, consists entirely of large fields; see [3; 28]. For example, all local fields are large and so are pseudoclassically closed fields (like PAC or PRC fields), the field of totally real numbers, as well as the quotient field of any local Henselian domain [27]. On the other hand, number fields and algebraic function fields are not large by Faltings' theorem and its function field version.

One of the most remarkable Galois-theoretic applications of large fields, due to Pop [26], states every finite split embedding problem over large fields has proper regular solutions. In particular, the regular

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inverse Galois problem is solvable over all large fields. Pop's work (and the work of many others) demonstrates that "over large fields one can do a lot of interesting mathematics". For instance, large fields have been widely used to tackle long-standing problems in field arithmetic: inverse Galois theory, torsors of finite groups, elementary theory of function fields, extremal-valued fields, to name a few. We refer the reader to Pop's survey [28] for earlier and current developments on the subject, and to [3] for a list of open problems.

In this paper we introduce the notion of *differential largeness* in the class of differential fields of characteristic zero in several commuting derivations. We lay out the foundations of this new and exciting class of "tame" differential fields, prove several characterizations (see Theorem 4.3, Proposition 4.7), and exhibit many examples (see Proposition 4.7, Corollary 4.8(ii), Theorems 5.12 and 5.18, and 5.2) and applications (see Corollaries 4.8(iii) and 5.13, Theorems 5.7 and 5.12, Lemma 5.9, Proposition 5.16, and 5.14 and 5.19). An outline of these is given in the rest of the introduction.

In order to give the definition of a differentially large field we need one piece of terminology. We say that a field K is *existentially closed* (e.c.) in L if every variety defined over K that has an L-rational point also has a K-rational point. Hence, a field is large just if it is e.c. in its Laurent series field. Similarly, a differential field K (of characteristic zero throughout, in $m \geq 1$ commuting derivations) is e.c. in a differential field extension L if every differential variety defined over K that has an L-rational differential point also has a K-rational differential point. (See Proposition 2.2 for other characterizations of this property.)

A differential field is *differentially large* if it is large as a pure field, and for every differential field extension L/K, if K is e.c. in L as a field, then it is e.c. in L as a differential field. For example differentially closed fields (a.k.a. *constrainedly closed* in Kolchin terminology) and closed ordered differential fields in the sense of [32] are differentially large.

In Theorem 4.3, we establish several equivalent formulations of differential largeness that justify why indeed this is the right differential analogue of largeness. For instance, we characterize them in terms of differential varieties having a Kolchin-dense set of rational points as long as they have suitable "smooth" rational points. In addition, we prove (in analogy to the characterization of largeness in terms of being e.c. in its Laurent series field) that a differential field K is differentially large just if it is e.c. in its power series field $K((t_1, \dots, t_m))$ as differential fields. The derivations on the power series field are given by the unique commuting derivations $\delta_1, \dots, \delta_m$ extending the ones on K that are compatible with infinite sums and satisfy $\delta_i(t_j) = dt_j/dt_i$.

A key tool in establishing our formulations of differential largeness (and further results) is the introduction of a twisted version of the classical Taylor morphism associated to a ring homomorphism $\varphi : A \to B$ for a given differential ring A. We explain this briefly in the case of one derivation δ. Recall that the Taylor morphism $T_\varphi(a) = \sum_{k\geq 0}\big(\varphi(\delta^k(a))/k!\big)\, t^k$ defines a differential ring homomorphism $(A, \delta) \to (B[\![t]\!], \mathrm{d}/\mathrm{d}t)$. Typically this is applied when A is a differential K-algebra for a differential field K and φ is a (not necessarily differential) K-algebra homomorphism $A \to K$ (so $B = K$). If the derivation on K is trivial, then T_φ is in fact a (differential) K-algebra homomorphism and in this context it was used by Seidenberg, for example, to establish his embedding theorem for differential fields into meromorphic functions. However, if the derivation on K is not trivial, then T_φ is not a K-algebra homomorphism, i.e, it is not an extension of φ. On the other hand, T_φ can be "twisted" in order to obtain a natural differential K-algebra homomorphism $T_\varphi^* : (A, \delta) \to (K[\![t]\!], \partial)$, where ∂ is the natural derivation extending the given one on K and satisfying $\partial(t) = 1$. This is established in Proposition 3.5, where we use it to derive the following result that is of independent interest (for instance, in the analysis of formal solutions to PDEs; see [30]), and is deployed in most parts of this article (in the more general form Corollary 3.6).

Theorem. *Let (K, δ) be a differential field of characteristic zero that is large as a field and let (S, δ) be a differentially finitely generated K-algebra. If there is a K-algebra homomorphism $S \to L$ for some field extension L/K in which K is e.c. (as a field), then there is a differential K-algebra homomorphism $(S, \delta) \to (K[\![\boldsymbol{t}]\!], \partial)$.*

Differentially large fields will play a very similar role in differential field arithmetic to that played by large fields in field arithmetic (of characteristic zero). The principal indicators for this are established in this paper (in Sections 4 and 5). We show that:

(a) A differential field K is differentially large if and only if it is existentially closed in its power series field $K((t_1, \ldots, t_m))$ furnished with m natural derivations extending those on K satisfying $\partial_i(t_j) = dt_j/dt_i$; see Theorem 4.3.

(b) Every large field equipped with commuting derivations has an extension to a differentially large field L such that K is e.c. in L as a pure field; see Corollary 4.8.

(c) Differentially large fields are first-order axiomatizable (see Proposition 4.7 and also Theorem 6.4 for a concrete algebro-geometric description), and the elimination theory of the underlying field transfers to the differential field; see Corollary 4.8.

(d) Differential largeness is preserved under algebraic extensions. Thus, the algebraic closure of a differentially large field is differentially closed. This provides many new differential fields with minimal differential closures; see Theorem 5.12.

(e) Differentially large fields (and differentially closed fields) can be produced by iterated power series constructions; see 5.2.

(f) The existential theory of the class of differentially large fields is the existential theory of the differential field $\mathbb{Q}((\boldsymbol{t}_1))((\boldsymbol{t}_2))$ equipped with its natural derivations; see Theorem 5.7.

(g) Differentially large fields are Picard–Vessiot closed; see Lemma 5.9.

(h) Connected differential algebraic groups defined over differentially large fields have a Kolchin-dense set of rational points; see 5.15.

(i) A differentially large field is PAC (at the field level) if and only if it is pseudodifferentially closed; see Theorem 5.18.

Large fields have also made an appearance in the (inverse) Picard–Vessiot theory of linear ordinary differential equations. In [2], it is shown that if K is a large field of infinite transcendence degree, then every linear algebraic group over K is a Picard–Vessiot group over $(K(x), \mathrm{d}/\mathrm{d}x)$. We envisage that differentially large fields will make a similar appearance in the parameterized Picard–Vessiot theory and its differential (constrained) coholomogy. The first application in this direction already appears in a paper of the first author with A. Pillay [18] using an earlier draft of the present paper. They show that if an ordinary differential field (K, δ) is differentially large and bounded as a field (that is, has finitely many extensions of degree n, for each $n \in \mathbb{N}$), then for any linear differential algebraic group G over K the differential Galois cohomology $H^1_\delta(K, G)$ is finite. This can be thought of as a differential analogue of the classical result of Serre stating that if a field K is bounded then the Galois cohomology $H^1(K, G)$ is finite for any linear algebraic group over K.

2. Preliminaries

All rings and algebras in this article are assumed to be commutative and unital. We also assume that all our fields are of characteristic zero.

We briefly summarize the key notions and terminology, mostly from differential algebra, that we will freely use throughout the paper (especially in Section 4 where we give several equivalent formulations of differential largeness). We make a few remarks on the notion of existentially closed differential ring extensions, we recall the structure theorem for finitely generated differential algebras, and give a quick review of jets and prolongation spaces.

Recall that a derivation on a ring R is an additive map $\delta : R \to R$ satisfying the Leibniz rule

$$\delta(rs) = \delta(r)s + r\delta(s) \quad \text{for all } r, s \in R.$$

Throughout, a differential ring $R = (R, \Delta)$ is a ring R equipped with a distinguished set of *commuting* derivations $\Delta = \{\delta_1, \dots, \delta_m\}$. Usually the order of the derivations does matter, but it will either be clear from the context or we will make it explicit. We also allow the case when $m = 0$, in which case we are simply talking of rings with no additional structure.

Given a differential ring R, a differential R-algebra A is an R-algebra equipped with derivations $\Delta = \{\delta_1, \dots, \delta_m\}$ such that the structure map $R \to A$ is a differential ring homomorphism. If L is another differential R-algebra which is also a field, then an *L-rational point* of A is a differential R-algebra homomorphism $A \to L$. This terminology is in line with the standard language of algebraic geometry, where A is thought of as $R\{x_1, \dots, x_n\}/I$, with I a differential ideal of the differential polynomial ring $R\{x_1, \dots, x_n\}$, and the differential R-algebra homomorphisms $A \to L$ are coordinate free descriptions of the common differential zeroes $a \in L^n$ of the polynomials from I (via evaluation at a).

For the basics in differential algebra, such as differential field extensions and differentially closed fields (also called constrainedly closed which is the differential analogue of algebraically closed), we refer the reader to the excellent book of Kolchin [12].

2.1. Definition (existentially closed extensions). Fix $m \geq 0$. Let $B = (B, \delta_1, \dots, \delta_m)$ be a differential ring and let A be a differential subring of B. (If $m = 0$, B is just a ring and A is a subring.) Then A is said to be *existentially closed (e.c.) in* B if for every $n \in \mathbb{N}$ and all finite collections $\Sigma, \Gamma \subseteq A\{x_1, \ldots, x_n\}$ of differential polynomials in m derivations and n differential variables, if there is a common solution in B^n of $P = 0$ and $Q \neq 0$ ($P \in \Sigma$, $Q \in \Gamma$), then such a solution may also be found in A^n.

We are mainly interested in the case when $A = K$ is a differential field and in this case we will use the following properties (in the case $m = 0$, differentially finitely generated, differential field, etc. should be understood as finitely generated, field, etc., and differentially closed field should be understood as algebraically closed field and Kolchin topology should be understood as Zariski topology). We make heavy use of the following properties.

2.2. Proposition. *In the notation of Definition 2.1*:

(i) *If K is e.c. in B, then one easily checks that B is a domain and that K is also e.c. in* $\mathrm{qf}(B)$.

(ii) *If B is also a differential field, then K is e.c. in B if and only if every differentially finitely generated K-algebra S that possesses a differential point $S \to B$, also possesses a differential point $S \to K$. The reason is that if B is a field then the inequalities $Q \neq 0$ in the definition of existentially closed above may be replaced by the equality $y \cdot Q(x) - 1 = 0$, where y is a new variable.*

(iii) *If B is a differentially finitely generated K-algebra then the following are equivalent*:

(a) *K is e.c. in B.*

(b) *B is a domain and for each $b \in B$, if $f(b) = 0$ for every differential K-rational point $f : B \to K$, then $b = 0$.*[1] *(In particular B has a differential K-rational point.) We refer to this property as **B has a Kolchin-dense set of differential K-rational points**.*

(c) *For all $n \in \mathbb{N}$, each differential prime ideal $\mathfrak{p}$ of $K\{x\}$, $x = (x_1, \ldots, x_n)$, with $B \cong_K K\{x\}/\mathfrak{p}$ and each differential field L containing K, the set $V_K = \{a \in K^n \mid \mathfrak{p}(a) = 0\}$ is dense in $V_L = \{a \in L^n \mid \mathfrak{p}(a) = 0\}$ for the **Kolchin topology** of L^n (having zero sets of differential polynomials from $L\{x\}$ as a basis of closed sets).*

(d) *There is some $n \in \mathbb{N}$, a differential prime ideal $\mathfrak{p}$ of $K\{x\}$, $x = (x_1, \ldots, x_n)$, with $B \cong_K K\{x\}/\mathfrak{p}$ and a differentially closed field M containing K such that the set V_K is dense in V_M for the K-Kolchin topology of M^n (having the zero sets in M^n of differential polynomials from $K\{x\}$ as a basis of closed sets).*

(iv) *If $m = 0$ and B is a finitely generated K-algebra then K is e.c. in B if and only if B is a domain and the set of smooth K-rational points of B is Zariski dense in the L-rational points for any field L containing K. This is a statement in classical algebraic geometry (using the formulation (c) of e.c. in* (iii)). *If in addition K is a large field, then K is e.c. in B if and only if B is a domain that has a smooth K-rational point.*

Proof of (iii). We may assume that B is a domain throughout and write $B = K\{x\}/\mathfrak{p}$, $x = (x_1, \ldots, x_n)$. The arguments below go through for any choice of these data. By the differential basis theorem there is

[1] In other words, in the subspace of $\mathrm{Spec}(B)$ consisting of differential prime ideals, the set of maximal and differential ideals with residue field K, is dense.

some finite $\Phi \subseteq K\{x\}$ such that $\mathfrak{p}$ is the radical differential ideal $\sqrt[d]{\Phi}$ generated by Φ. For a differential field L containing K we write $V_L = \{a \in L^n \mid \mathfrak{p}(a) = 0\}$ and $I_L = \{Q \in L\{x\} \mid Q|_{V_L} = 0\}$. If L is differentially closed, then the differential Nullstellensatz [12, Chapter IV, section 3, Theorem 2, p. 147] says $I_L = \sqrt[d]{\mathfrak{p}}$ (in $L\{x\}$).

(a) $\Rightarrow$ (b). If K is e.c. in B and $b \in B \setminus \{0\}$, then take $Q \in K\{x\}$ with $b = Q(x + \mathfrak{p})$. Since in B we have a solution of $\Phi = 0$ and $Q \neq 0$, there is also a solution $a \in K^n$ and evaluation $K\{x\} \to K$ at a factors through a differential K-rational point $B \to K$ that is nonzero at b.

(b) $\Rightarrow$ (c). Let M be a differentially closed field containing L such that the fixed field of the group of differential K-automorphisms of M is K (for example, M could be a sufficiently saturated differentially closed field or, in Kolchin's terminology, a universal differential extension of K). It suffices to show that V_K is dense in V_M for the Kolchin topology of M^n. Let W be the closure of V_K in M^n for the Kolchin topology of M^n and let $J = \{Q \in M\{x\} \mid Q|_W = 0\}$. Then every differential K-automorphism of M fixes W setwise and so also fixes J setwise. Using our assumption on M we see that the differential field of definition of J is contained in K; hence, the differential ideal J is generated as an ideal by $J \cap K\{x\}$. On the other hand, we have $I_M \cap K\{x\} = \mathfrak{p}$. As $W \subseteq V_M$ we get $\mathfrak{p} \subseteq I_M \subseteq J$ and we claim that $\mathfrak{p} = J \cap K\{x\}$. Take $P \in J \cap K\{x\}$. Then P vanishes on V_K, which says that the element $P + \mathfrak{p} \in B$ is mapped to 0 by all differential K-rational points of B. By (b), this implies $P + \mathfrak{p} = 0$ in B, in other words $P \in \mathfrak{p}$. We have shown that $\mathfrak{p} = I_M \cap K\{x\} = J \cap K\{x\}$, which implies $W = V_M$ as required.

(c) $\Rightarrow$ (d). This is trivial.

(d) $\Rightarrow$ (a). Let $\Sigma = \{P_1, \dots, P_s\}$, $\Gamma \subseteq K\{y\}$ be finite, $y = (y_1, \dots, y_r)$, and assume there is some $c \in B^r$ with $\Sigma(c) = 0$ and $\Gamma(c) \neq 0$. We need to find some $a \in K^r$ with $\Sigma(a) = 0$ and $\Gamma(a) \neq 0$. Since B is a domain we may assume that $\Gamma = \{Q(y)\}$ is a singleton. We write $c_i = H_i(x + \mathfrak{p})$ with $H_i \in K\{x\}$ and $H = (H_1, \dots, H_r)$. Then $\Sigma(c) = 0$ and $Q(c) \neq 0$ means $P_1(H), \dots, P_s(H) \in \mathfrak{p}$ and $Q(H) \notin \mathfrak{p}$. Since M is differentially closed and $Q(H) \notin \mathfrak{p} = K\{x\} = I_M \cap K\{x\}$ there is some $d \in V_M$ with $Q(H(d)) \neq 0$. By (d) and because $Q(H) \in K\{x\}$, there is some $b \in V_K$ with $Q(H(b)) \neq 0$. Since $P_i(H) \in \mathfrak{p}$ we also know $P_i(H(b)) = 0$. Hence, the tuple $a = (H_1(b), \dots, H_r(b)) \in K^r$ solves the given system. □

If B is a differential K-algebra we will say that K *is existentially closed in B as a field* if it is e.c. in B when we forget about the derivations; hence if the above condition holds true for systems Σ, Γ of ordinary (nondifferential) polynomials. If we want to emphasize that the derivations are to be taken into account we say K *is existentially closed in B as a differential field.*

2.3. Theorem (structure theorem for finitely generated differential algebras). *Let K be a differential field (of characteristic zero) and let S be a differential K-algebra that is differentially finitely generated and a domain. Then, by* [33], *there are K-subalgebras A, P of S and an element $h \in A \setminus \{0\}$ such that A is a finitely generated K-algebra, P is a polynomial K-algebra ($P \cong_K K[T]$ for some possibly infinite set T of indeterminates) and the natural homomorphism $A_h \otimes_K P \to S_h$ given by multiplication is an isomorphism. Note that in general neither A_h nor P is differential.*

2.4. Definition. Let K be a differential field and let S be a differentially finitely generated K-algebra that is a domain. A *decomposition* of S consists of (not necessarily differential) K-subalgebras A, P such that

(a) A is a finitely generated K-algebra and P is a polynomial K-algebra, and

(b) the natural map $A \otimes_K P \to S$ given by multiplication is an isomorphism.

If S possesses a decomposition we say that S is *composite* and we indicate the data of a decomposition by writing $S = A \otimes P$.

2.5. Corollary. *Let K be a differential field and let S be a differentially finitely generated K-algebra. Let $f : S \to L$ be a differential K-algebra homomorphism to some differential field extension L of K.*

(i) *There is a differential K-subalgebra $S_0 \subseteq L$ that is composite and contains the image of f.*

(ii) *If K is e.c. in L as a field, then there is a K-algebra homomorphism $S \to K$.*

Proof. (i). Let $\mathfrak{p}$ be the kernel of f. Then $S/\mathfrak{p}$ is again a differentially finitely generated K-algebra and so we may assume that $\mathfrak{p} = 0$ and $S \subseteq L$. By Theorem 2.3, $S_h \cong_K A_h \otimes_K P$ where A is a finitely generated K-subalgebra of S, $h \in A$, and P is a polynomial K-algebra, $P \subseteq S$. As $S \subseteq L$, we have $A_h \subseteq L$. Hence, we may take $S_0 = S_h$.

(ii). Take S_0 as in (i) and A, P for S_0 as in Definition 2.4. Since A is a finitely generated K-subalgebra of L and K is e.c. in L as a field, there is a K-algebra homomorphism $A \to K$. Since P is a polynomial K-algebra there is also a K-algebra homomorphism $P \to K$. Hence, by the universal property of the tensor product there is a K-algebra homomorphism $S \to K$. □

We recall the basic objects of differential algebraic geometry in the sense of Kolchin [12], and the constructions of jets and prolongations. Some parts are notationally heavy but we try to only introduce those that we will need (and freely use) in coming sections.

2.6. Definition (differential varieties, jets and prolongations). We work inside a (sufficiently saturated or universal) differentially closed field $(\mathbb{U}, \Delta)$, and K denotes a differential subfield of $\mathbb{U}$. A *Kolchin-closed* subset of $\mathbb{U}^n$ is the common zero set of a set of differential polynomials over $\mathbb{U}$ in n differential variables; such sets are also called *affine differential varieties*. If the defining polynomials can be chosen with coefficients in K we say the set is *defined over K*.

By a *differential variety V* we mean a topological space which has as finite open cover $V_1, \dots, V_s$ with each V_i homeomorphic to an affine differential variety (inside some power of $\mathbb{U}$) such that the transition maps are regular as differential morphisms; see [15, Chapter 1, section 7]. We will say that the differential variety is over K when all objects and morphisms can be defined over K. This definition also applies to our use of algebraic varieties, replacing Kolchin-closed with Zariski-closed in powers of $\mathbb{U}$ (recall that $\mathbb{U}$ is algebraically closed and a universal domain for algebraic geometry in Weil's "foundations" sense).

2.7. Notation. We fix integers $n > 0$ and $r \geq 0$, and set

$$\Gamma_n(r) = \left\{ (\xi, i) \in \mathbb{N}^m \times \{1, \dots, n\} \mid \sum_{i=1}^{m} \xi_i \leq r \right\}.$$

The *r-th nabla map* $\nabla_r : \mathbb{U}^n \to \mathbb{U}^{\alpha(n,r)}$ with $\alpha(n,r) := |\Gamma_n(r)| = n \cdot \binom{r+m}{m}$ is defined by

$$\nabla_r(x) = (\delta^\xi x_i : (\xi, i) \in \Gamma_n(r)),$$

where $x = (x_1, \dots, x_n)$ and $\delta^\xi = \delta_1^{\xi_1} \cdots \delta_m^{\xi_m}$. We order the elements of the tuple $(\delta^\xi x_i : (\xi, i) \in \Gamma_n(r))$ according to the canonical orderly ranking of the indeterminates $\delta^\xi x_i$; that is,

$$\delta^\xi x_i < \delta^\zeta x_j \iff \Big(\sum \xi_k, i, \xi_1, \dots, \xi_m\Big) <_{\text{lex}} \Big(\sum \zeta_k, j, \zeta_1, \dots, \zeta_m\Big). \tag{2-1}$$

Let $\mathbb{U}_r := \mathbb{U}[\epsilon_1, \dots, \epsilon_m]/(\epsilon_1, \dots, \epsilon_m)^{r+1}$ where the ϵ_i's are indeterminates, and let $e : \mathbb{U} \to \mathbb{U}_r$ denote the ring homomorphism

$$x \mapsto \sum_{\xi \in \Gamma_1(r)} \frac{1}{\xi_1! \cdots \xi_m!} \, \delta^\xi(x) \, \epsilon_1^{\xi_1} \cdots \epsilon_m^{\xi_m}.$$

We call e the exponential $\mathbb{U}$-algebra structure of $\mathbb{U}_r$. To distinguish between the standard and the exponential algebra structure on $\mathbb{U}_r$, we denote the latter by $\mathbb{U}_r^e$.

2.8. Definition. Given an algebraic variety X the r-th *prolongation* τX is the algebraic variety given by taking the $\mathbb{U}$-rational points of the classical Weil descent (or Weil restriction) of $X \times_{\mathbb{U}} \mathbb{U}_r^e$ from $\mathbb{U}_r$ to $\mathbb{U}$. Note that the base change $V \times_{\mathbb{U}} \mathbb{U}_r^e$ is with respect to the exponential structure while the Weil descent is with respect to the standard $\mathbb{U}$-algebra structure.

For details and properties of prolongation spaces we refer to [21, §2]; for a more general presentation, see [20]. In particular, it is pointed out there that the prolongation $\tau_r X$ always exist when X is quasiprojective (an assumption that we will adhere to later on). A characterizing feature of the prolongation is that for each point $a \in X = X(\mathbb{U})$ we have $\nabla_r(a) \in \tau_r X$. Thus, the map $\nabla_r : X \to \tau_r X$ is a differential regular section of $\pi_r : \tau_r X \to X$ the canonical projection induced from the residue map $\mathbb{U}_r \to \mathbb{U}$. We note that if X is defined over the differential field K then $\tau_r X$ is defined over K as well.

In fact, τ_r as defined above is a functor from the category of algebraic varieties over K to itself, and the maps $\pi_r : \tau_r X \to X$ and $\nabla_r : X \to \tau_r X$ are natural. The latter means that for any morphism of algebraic varieties $f : X \to Y$ we get

$$f \circ \pi_{r,X} = \pi_{r,Y} \circ \tau_r f \quad \text{and} \quad \tau_r f \circ \nabla_{r,X} = \nabla_{r,Y} \circ f. \tag{2-2}$$

If G is an algebraic group, then $\tau_r G$ also has the structure of an algebraic group. Indeed, since τ_r commutes with products, the group structure is given by

$$\tau_r(*) : \tau_r G \times \tau_r G \to \tau_r G,$$

where $*$ denotes multiplication in G. By the right-most equality in (2-2), the map $\nabla_r : G \to \tau_r G$ is an injective group homomorphism. Hence, $\nabla_r(G)$ is a differential algebraic subgroup of $\tau_r G$. We will use this in 5.15 below.

Assume that V is a differential variety which is given as a differential subvariety of a quasiprojective algebraic variety X. We define the r-th jet of V to be the Zariski-closure of the image of V under the r-th nabla map $\nabla_r : X \to \tau_r X$; that is,

$$\operatorname{Jet}_r V = \overline{\nabla_r(V)}^{\mathrm{Zar}} \subseteq \tau_r X.^2$$

The jet sequence of V is defined as $(\operatorname{Jet}_r V : r \geq 0)$. Note that this sequence determines V. Indeed,

$$V = \{a \in X : \nabla_r(a) \in \operatorname{Jet}_r V \text{ for all } r \geq 0\}.$$

2.9. General Assumption. Throughout we assume, whenever necessary for the existence of jets, that our differential varieties are given as differential subvarieties of quasiprojective algebraic varieties. Of course, in the affine case this is always the case. It is worth noting, as it will be used in 5.15, that for connected differential algebraic groups this is also true. Indeed, by [23, Corollary 4.2(ii)] every such group embeds into a connected algebraic group and the latter is quasiprojective by Chevalley's theorem.

3. The Taylor morphism

In parallel to the characterization of large fields in terms of being e.c. in Laurent series, we will prove in Theorem 4.3 that differential largeness can be characterized similarly. For this, we will make use of a *twisted* Taylor morphism. In this section, we give a description of this morphism and use it to construct solutions in power series to systems of differential equations (see Corollary 3.6).

3.1. Setup. Let (A, Δ) be a differential ring with commuting derivations $\Delta = \{\delta_1, \dots, \delta_m\}$. Recall that given a ring homomorphism $\varphi : A \to B$ (where B is a $\mathbb{Q}$-algebra), the Taylor morphism. $T^{\varphi}_{\Delta} : A \to B[\![\boldsymbol{t}]\!]$, where $\boldsymbol{t} = (t_1, \dots, t_m)$, is defined as

$$a \mapsto \sum_{\alpha} \frac{\varphi(\delta^{\alpha} a)}{\alpha!} \boldsymbol{t}^{\alpha},$$

where we make use of multi-index notation. Namely, $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$, $\alpha! = \alpha_1! \cdots \alpha_m!$, $\delta^{\alpha} = \delta_1^{\alpha_1} \cdots \delta_m^{\alpha_m}$, and $\boldsymbol{t}^{\alpha} = t_1^{\alpha} \cdots t_m^{\alpha_m}$. It is a straightforward computation to check that T^{φ}_{Δ} is a differential ring homomorphism

$$(A, \Delta) \to \left(B[\![\boldsymbol{t}]\!], \frac{\mathrm{d}}{\mathrm{d}t_1}, \dots, \frac{\mathrm{d}}{\mathrm{d}t_m}\right).$$

For every such family of commuting derivations Δ on A, there is a unique extension to $A[\![\boldsymbol{t}]\!]$ such that the derivations commute with meaningful sums[3] and map all t_i's to 0. We continue to denote these derivations on $A[\![\boldsymbol{t}]\!]$ by $\Delta = \{\delta_1, \dots, \delta_m\}$; note that they still commute with each other. We work with the derivations $\delta_i + \mathrm{d}/\mathrm{d}t_i$, for $i = 1, \dots, m$, on $A[\![\boldsymbol{t}]\!]$; again these commute with each other. Assuming that A is

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[2]Notice that $\operatorname{Jet}_r V$ is not the jet space defined in [20, 5.3].

[3]In the sense of [12, Chapter 0, section 13, p. 30]; specifically, if $(f_i)_{i \in \mathbb{N}}$ is a sequence that converges to 0 in the $(\boldsymbol{t})$-adic topology of $K[\![\boldsymbol{t}]\!]$, then $\sum_i f_i$ is meaningful.

a $\mathbb{Q}$-algebra, we now study the algebraic properties of the Taylor morphism associated to the evaluation map

$$\mathrm{ev}: A[\![\boldsymbol{t}]\!] \to A[\![\boldsymbol{t}]\!], \quad f \mapsto f(0, \dots, 0).$$

For instance, we show that the map from $\mathrm{Der}(A)$ to ring endomorphisms of $A[\![t]\!]$ given by $\delta \mapsto T^{\mathrm{ev}}_{\delta+\mathrm{d}/\mathrm{d}t}$ is a monoid homomorphism when restricted to any submonoid of commuting derivations. Here the monoid structure on $\mathrm{Der}(A)$ is just addition of derivations (and so is indeed a group), while the monoid structure on ring endomorphisms is composition. Note that as a consequence $T^{\mathrm{ev}}_{\delta+\mathrm{d}/\mathrm{d}t}$ is a differential ring isomorphism, because $T^{\mathrm{ev}}_{\delta+\mathrm{d}/\mathrm{d}t}$ has compositional inverse $T^{\mathrm{ev}}_{-\delta+\mathrm{d}/\mathrm{d}t}$ and $T^{\mathrm{ev}}_{\mathrm{d}/\mathrm{d}t}$ is the identity map on $A[\![t]\!]$. We state all this more generally below.

We first introduce some convenient notation and terminology. Let $\Delta = \{\delta_1, \dots, \delta_m\}$ and $\Omega = \{\partial_1, \dots, \partial_m\}$ be families of commuting derivations on A. We say that these families commute if δ_i commutes with ∂_j for all $1 \le i, j \le m$; when this is the case, we denote by $\Delta + \Omega$ the family of commuting derivations on A given by $\{\delta_1 + \partial_1, \dots, \delta_m + \partial_m\}$. Note that the natural extensions of Δ and Ω to $A[\![\boldsymbol{t}]\!]$, as discussed above, commute with the family

$$\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}} := \left\{\frac{\mathrm{d}}{\mathrm{d}t_1}, \dots, \frac{\mathrm{d}}{\mathrm{d}t_m}\right\}.$$

Therefore, the family of derivations $\Delta + \Omega + \mathrm{d}/\mathrm{d}\boldsymbol{t}$ on $A[\![\boldsymbol{t}]\!]$ is a commuting family.

3.2. Theorem. *Let A be a $\mathbb{Q}$-algebra, and let Δ and Ω be families of m-many commuting derivations on A. If Δ and Ω commute, then*

$$T^{\mathrm{ev}}_{\Delta+\Omega+\mathrm{d}/\mathrm{d}\boldsymbol{t}} = T^{\mathrm{ev}}_{\Delta+\mathrm{d}/\mathrm{d}\boldsymbol{t}} \circ T^{\mathrm{ev}}_{\Omega+\mathrm{d}/\mathrm{d}\boldsymbol{t}}. \tag{3-1}$$

Proof. For $\alpha \in \mathbb{N}^m$ we use the multi-index notation

$$(\delta+\partial)^\alpha = (\delta_1+\partial_1)^{\alpha_1} \cdots (\delta_m+\partial_m)^{\alpha_m},$$
$$\left(\delta+\partial+\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}}\right)^\alpha = \left(\delta_1+\partial_1+\frac{\mathrm{d}}{\mathrm{d}t_1}\right)^{\alpha_1} \cdots \left(\delta_m+\partial_m+\frac{\mathrm{d}}{\mathrm{d}t_m}\right)^{\alpha_m}.$$

We use the product order $\le$ on $\mathbb{N}^m$ given by $\beta \le \alpha$ if and only if $\beta_i \le \alpha_i$ for $1 \le i \le m$. As the derivations commute, we have the usual binomial identities

$$(\delta+\partial)^\alpha = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \delta^\beta \partial^{\alpha-\beta} = \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} \delta^\beta \partial^\gamma,$$
$$\left(\delta+\partial+\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}}\right)^\alpha = \sum_{\xi\le\alpha} \sum_{\beta+\gamma=\xi} \binom{\alpha}{\xi}\binom{\xi}{\beta} \delta^\beta \partial^\gamma \frac{\mathrm{d}^{\alpha-\xi}}{\mathrm{d}\boldsymbol{t}}$$
$$= \sum_{\beta+\gamma\le\alpha} \binom{\alpha}{\beta+\gamma}\binom{\beta+\gamma}{\beta} \delta^\beta \partial^\gamma \frac{\mathrm{d}^{\alpha-\beta-\gamma}}{\mathrm{d}\boldsymbol{t}}.$$

Now take $f = \sum_\xi a_\xi \boldsymbol{t}^\xi \in A[\![\boldsymbol{t}]\!]$. We show that both sides of (3-1) applied to f are equal to

$$\sum_\alpha \Big(\sum_{\beta+\gamma\le\alpha} \frac{1}{\beta!\cdot\gamma!}\, \delta^\beta \partial^\gamma (a_{\alpha-\beta-\gamma}) \Big) \boldsymbol{t}^\alpha. \tag{3-2}$$

We begin with the left-hand-side. By definition, the coefficient at $\boldsymbol{t}^\alpha$ of $T^{\mathrm{ev}}_{\Delta+\Omega+\mathrm{d}/\mathrm{d}\boldsymbol{t}}\big(\sum_\xi a_\xi \boldsymbol{t}^\xi\big)$ is given by

$$\begin{aligned}
\frac{1}{\alpha!}\,\mathrm{ev}\Big[\Big(\delta+\partial+\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}}\Big)^\alpha\Big(\sum_\xi a_\xi \boldsymbol{t}^\xi\Big)\Big] &= \frac{1}{\alpha!}\mathrm{ev}\Big[\sum_{\beta+\gamma\le\alpha}\binom{\alpha}{\beta+\gamma}\binom{\beta+\gamma}{\beta}\delta^\beta\partial^\gamma\frac{\mathrm{d}^{\alpha-\beta-\gamma}}{\mathrm{d}\boldsymbol{t}}\Big(\sum_\xi a_\xi \boldsymbol{t}^\xi\Big)\Big]\\
&= \frac{1}{\alpha!}\mathrm{ev}\Big[\sum_{\beta+\gamma\le\alpha}\sum_\xi\binom{\alpha}{\beta+\gamma}\binom{\beta+\gamma}{\beta}\delta^\beta\partial\gamma(a_\xi)\frac{\mathrm{d}^{\alpha-\beta-\gamma}}{\mathrm{d}\boldsymbol{t}}(\boldsymbol{t}^\xi)\Big]\\
&= \frac{1}{\alpha!}\sum_{\beta+\gamma\le\alpha}\binom{\alpha}{\beta+\gamma}\binom{\beta+\gamma}{\beta}\delta^\beta\partial^\gamma(a_{\alpha-\beta-\gamma})\cdot(\alpha-\beta-\gamma)!\\
&= \sum_{\beta+\gamma\le\alpha}\frac{1}{\beta!\cdot\gamma!}\delta^\beta\partial^\gamma(a_{\alpha-\beta-\gamma}),
\end{aligned}$$

which is the term in (3-2). We now compute the right-hand-side of (3-1), when applied to f. The coefficient at $\boldsymbol{t}^\alpha$ is

$$\begin{aligned}
\frac{1}{\alpha!}\,\mathrm{ev}\Big[\Big(\delta+\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}}\Big)^\alpha\Big(T^{\mathrm{ev}}_{\Omega+\mathrm{d}/\mathrm{d}\boldsymbol{t}}\Big(\sum_\xi a_\xi \boldsymbol{t}^\xi\Big)\Big)\Big] &= \frac{1}{\alpha!}\,\mathrm{ev}\Big[\Big(\delta+\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}}\Big)^\alpha\Big(\sum_\zeta\frac{1}{\zeta!}\mathrm{ev}\Big(\Big(\partial+\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}}\Big)^\zeta\Big(\sum_\xi a_\xi \boldsymbol{t}^\xi\Big)\Big)\boldsymbol{t}^\zeta\Big)\Big]\\
&= \frac{1}{\alpha!}\,\mathrm{ev}\Big[\Big(\delta+\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}}\Big)^\alpha\Big(\sum_\zeta\frac{1}{\zeta!}\mathrm{ev}\Big(\sum_{\gamma\le\zeta}\binom{\zeta}{\gamma}\partial^\gamma\frac{\mathrm{d}^{\zeta-\gamma}}{\mathrm{d}\boldsymbol{t}}\Big(\sum_\xi a_\xi \boldsymbol{t}^\xi\Big)\Big)\boldsymbol{t}^\zeta\Big)\Big]\\
&= \frac{1}{\alpha!}\,\mathrm{ev}\Big[\Big(\delta+\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}}\Big)^\alpha\Big(\sum_\zeta\frac{1}{\zeta!}\Big(\sum_{\gamma\le\zeta}\binom{\zeta}{\gamma}\partial^\gamma(a_{\zeta-\gamma})\cdot(\zeta-\gamma)!\Big)\boldsymbol{t}^\zeta\Big)\Big]\\
&= \frac{1}{\alpha!}\,\mathrm{ev}\Big[\Big(\delta+\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}}\Big)^\alpha\Big(\sum_\zeta\sum_{\gamma\le\zeta}\frac{1}{\gamma!}\partial^\gamma(a_{\zeta-\gamma})\boldsymbol{t}^\zeta\Big)\Big]\\
&= \frac{1}{\alpha!}\,\mathrm{ev}\Big[\sum_\zeta\sum_{\beta\le\alpha}\sum_{\gamma\le\zeta}\frac{1}{\gamma!}\binom{\alpha}{\beta}\delta^\beta\partial^\gamma(a_{\zeta-\gamma})\frac{\mathrm{d}^{\alpha-\beta}}{\mathrm{d}\boldsymbol{t}}(\boldsymbol{t}^\zeta)\Big]\\
&= \frac{1}{\alpha!}\sum_{\beta\le\alpha}\sum_{\gamma\le\alpha-\beta}\frac{1}{\gamma!}\binom{\alpha}{\beta}\delta^\beta\partial^\gamma(a_{\alpha-\beta-\gamma})\cdot(\alpha-\beta)!\\
&= \sum_{\beta\le\alpha}\sum_{\gamma\le\alpha-\beta}\frac{1}{\beta!\cdot\gamma!}\delta^\beta\partial^\gamma(a_{\alpha-\beta-\gamma})\\
&= \sum_{\beta+\gamma\le\alpha}\frac{1}{\beta!\cdot\gamma!}\delta^\beta\partial^\gamma(a_{\alpha-\beta-\gamma}),
\end{aligned}$$

which is the term in (3-2), as required. □

What will be important to us is the following consequence.

3.3. Corollary. *For any family of commuting derivations $\Delta=\{\delta_1,\dots,\delta_m\}$ on a $\mathbb{Q}$-algebra A, the Taylor morphism of the evaluation map* $\mathrm{ev}: A[\![\boldsymbol{t}]\!]\to A$ *at* 0 *is an isomorphism of differential rings*

$$T^{\mathrm{ev}}_{\Delta+\mathrm{d}/\mathrm{d}\boldsymbol{t}}:\Big(A[\![\boldsymbol{t}]\!],\,\Delta+\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}}\Big)\to\Big(A[\![\boldsymbol{t}]\!],\,\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}}\Big).$$

Its compositional inverse is $T^{\mathrm{ev}}_{-\Delta+\mathrm{d}/\mathrm{d}\boldsymbol{t}}$, *where* $-\Delta$ *is the family of commuting derivations* $\{-\delta_1, \ldots, -\delta_m\}$. *Furthermore*, $T^{\mathrm{ev}}_{\Delta+\mathrm{d}/\mathrm{d}\boldsymbol{t}}$ *is a differential isomorphism*

$$(A[\![\boldsymbol{t}]\!], \Delta) \to (A[\![\boldsymbol{t}]\!], \Delta).$$

Proof. We recall that $T^{\mathrm{ev}}_{\Delta+\mathrm{d}/\mathrm{d}\boldsymbol{t}}$ is a differential homomorphism $(A[\![\boldsymbol{t}]\!], \Delta + \mathrm{d}/\mathrm{d}\boldsymbol{t}) \to (A[\![\boldsymbol{t}]\!], \mathrm{d}/\mathrm{d}\boldsymbol{t})$. By Theorem 3.2, we have

$$T^{\mathrm{ev}}_{\Delta+\mathrm{d}/\mathrm{d}\boldsymbol{t}} \circ T^{\mathrm{ev}}_{-\Delta+\mathrm{d}/\mathrm{d}\boldsymbol{t}} = T^{\mathrm{ev}}_{\mathrm{d}/\mathrm{d}\boldsymbol{t}} = T^{\mathrm{ev}}_{-\Delta+\mathrm{d}/\mathrm{d}\boldsymbol{t}} \circ T^{\mathrm{ev}}_{\Delta+\mathrm{d}/\mathrm{d}\boldsymbol{t}}.$$

It is easy to check that $T^{\mathrm{ev}}_{\mathrm{d}/\mathrm{d}\boldsymbol{t}}$ is the identity on $A[\![\boldsymbol{t}]\!]$. Hence, $T^{\mathrm{ev}}_{-\Delta+\mathrm{d}/\mathrm{d}\boldsymbol{t}}$ is the compositional inverse of $T^{\mathrm{ev}}_{\Delta+\mathrm{d}/\mathrm{d}\boldsymbol{t}}$.

It follows that $T^{\mathrm{ev}}_{-\Delta+\mathrm{d}/\mathrm{d}\boldsymbol{t}}$ is also a differential isomorphism $(A[\![\boldsymbol{t}]\!], \mathrm{d}/\mathrm{d}\boldsymbol{t}) \to (A[\![\boldsymbol{t}]\!], \Delta+\mathrm{d}/\mathrm{d}\boldsymbol{t})$ — in other words, that $T^{\mathrm{ev}}_{-\Delta+\mathrm{d}/\mathrm{d}\boldsymbol{t}} \circ \mathrm{d}/\mathrm{d}t_i = (\delta_i + \mathrm{d}/\mathrm{d}t_i) \circ T^{\mathrm{ev}}_{-\Delta+\mathrm{d}/\mathrm{d}\boldsymbol{t}}$. Now $\mathrm{d}/\mathrm{d}t_i \circ T^{\mathrm{ev}}_{-\Delta+\mathrm{d}/\mathrm{d}\boldsymbol{t}} = T^{\mathrm{ev}}_{-\Delta+\mathrm{d}/\mathrm{d}\boldsymbol{t}} \circ (-\delta_i + \mathrm{d}/\mathrm{d}t_i)$, because $T^{\mathrm{ev}}_{-\Delta+\mathrm{d}/\mathrm{d}\boldsymbol{t}}$ is a differential isomorphism $(A[\![\boldsymbol{t}]\!], -\Delta + \mathrm{d}/\mathrm{d}\boldsymbol{t}) \to (A[\![\boldsymbol{t}]\!], \mathrm{d}/\mathrm{d}\boldsymbol{t})$. It follows that

$$T^{\mathrm{ev}}_{-\Delta+\mathrm{d}/\mathrm{d}\boldsymbol{t}} \circ \frac{\mathrm{d}}{\mathrm{d}t_i} = \delta_i \circ T^{\mathrm{ev}}_{-\Delta+\mathrm{d}/\mathrm{d}\boldsymbol{t}} + T^{\mathrm{ev}}_{-\Delta+\mathrm{d}/\mathrm{d}\boldsymbol{t}} \circ \left(-\delta_i + \frac{\mathrm{d}}{\mathrm{d}t_i}\right),$$

which implies $T^{\mathrm{ev}}_{-\Delta+\mathrm{d}/\mathrm{d}\boldsymbol{t}} \circ \delta_i = \delta_i \circ T^{\mathrm{ev}}_{-\Delta+\mathrm{d}/\mathrm{d}\boldsymbol{t}}$, as claimed in the "furthermore" part. □

We now use Corollary 3.3 to introduce a twisting of the Taylor morphism.

3.4. Definition (the twisted Taylor morphism). We assume all derivations commute. Let A be a differential ring with derivations $\Delta = \{\delta_1, \ldots, \delta_m\}$ and let B be a $\mathbb{Q}$-algebra and a differential ring with derivations $\Omega = \{\partial_1, \ldots, \partial_m\}$. Let $\varphi : A \to B$ be a (not necessarily differential) ring homomorphism. We write ∂_i again for the extension of ∂_i to $B[\![\boldsymbol{t}]\!]$, $\boldsymbol{t} = (t_1, \ldots, t_m)$, obtained from differentiating coefficients as explained in Setup 3.1. Let $\mathrm{ev} : B[\![\boldsymbol{t}]\!] \to B$ be the evaluation map at 0. If we equip $B[\![\boldsymbol{t}]\!]$ with the derivations $\Omega + \mathrm{d}/\mathrm{d}\boldsymbol{t}$ as in Setup 3.1 and apply Corollary 3.3 for (B, Ω), we get a differential ring isomorphism

$$T^{\mathrm{ev}}_{\Omega+\mathrm{d}/\mathrm{d}\boldsymbol{t}} : \left(B[\![\boldsymbol{t}]\!], \Omega + \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}}\right) \to \left(B[\![\boldsymbol{t}]\!], \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}}\right)$$

with compositional inverse $T^{\mathrm{ev}}_{-\Omega+\mathrm{d}/\mathrm{d}\boldsymbol{t}}$. Consequently, the map

$$T^*_\varphi := T^{\mathrm{ev}}_{-\Omega+\mathrm{d}/\mathrm{d}\boldsymbol{t}} \circ T^\varphi_\Delta : (A, \Delta) \xrightarrow{T^\varphi_\Delta} \left(B[\![\boldsymbol{t}]\!], \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}}\right) \xrightarrow{T^{\mathrm{ev}}_{-\Omega+\mathrm{d}/\mathrm{d}\boldsymbol{t}}} \left(B[\![\boldsymbol{t}]\!], \Omega + \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}}\right)$$

is a differential ring homomorphism $(A, \Delta) \to (B[\![\boldsymbol{t}]\!], \Omega + \mathrm{d}/\mathrm{d}\boldsymbol{t})$, called the *twisted Taylor morphism* of φ. Writing $T^*_\varphi(a) = \sum_\alpha b_\alpha \boldsymbol{t}^\alpha$, the b_α's are explicitly computed as

$$b_\alpha = \frac{1}{\alpha!} \sum_{\beta \le \alpha} (-1)^{\alpha-\beta} \binom{\alpha}{\beta} \partial^{\alpha-\beta} \big(\varphi(\delta^\beta(a))\big).$$

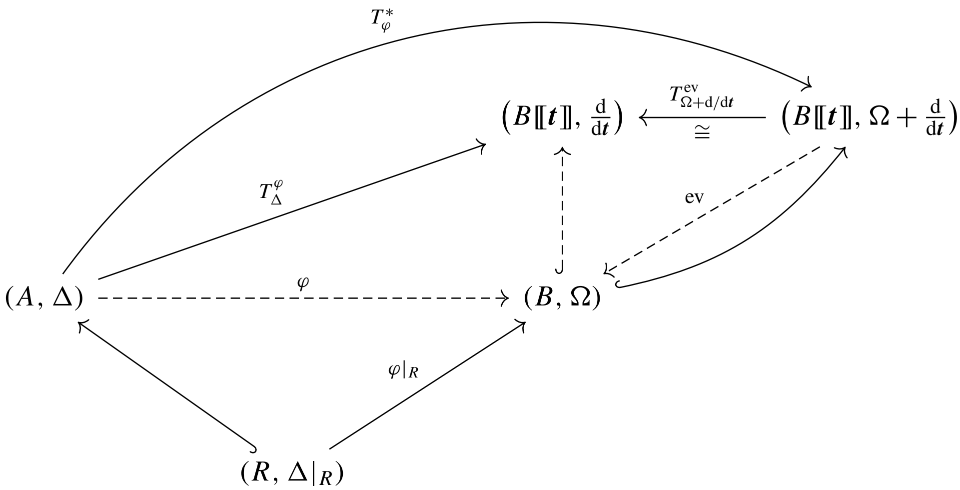
3.5. Proposition. *We use the same assumptions and notation as in Definition 3.4. If* $a \in A$ *and* $\mathbb{Z}\{a\}$ *denotes the differential subring generated by* a *in* A, *one checks readily that*:

(i) $T^\varphi_\Delta(a) = \varphi(a) \iff \delta^\alpha(a) \in \ker(\varphi)$ *for all nonzero* $\alpha \in \mathbb{N}^m$.

(ii) $T^*_\varphi(a) = T^\varphi_\Delta(a) \iff \varphi(\mathbb{Z}\{a\})$ *is contained in the ring of* Ω-*constants of* B.

(iii) $T^*_\varphi(a) = \varphi(a) \iff$ *the restriction of* φ *to* $\mathbb{Z}\{a\}$ *is a differential homomorphism.*

Hence, by the implication $\Leftarrow$ *in* (iii), *if* R *is a differential subring of* A *such that the restriction* $\varphi|_R$ *is a differential ring homomorphism* $(R, \Delta|_R) \to (B, \Omega)$, *then* T^*_φ *extends* φ *and the part showing solid arrows in the following diagram commutes*:



Notice that all solid arrows in this diagram are differential homomorphisms. The main case for us is when $R = K$ *is a field and* B *is a* K-*algebra such that* φ *is a* K-*algebra homomorphism. In this case the twisted Taylor morphism* T^*_φ *is in fact a differential* K-*algebra homomorphism.*

3.6. Corollary. *Let* (K, Δ) *be a differential field that is large as a field and let* S *be a differentially finitely generated* K-*algebra. If there is a* K-*algebra homomorphism* $S \to L$ *for some field extension* L/K *in which* K *is e.c.* (*as a field*, *there are no derivations on* L *given*), *then there is a differential* K-*algebra homomorphism* $S \to K[\![t]\!]$, *where the derivations on* $K[\![t]\!]$ *are* $\Delta + \mathrm{d}/\mathrm{d}t$ *as described above.*

Proof. Since K as a field is e.c. in L, there is a field extension L' of L which is an elementary extension of the field K. We replace L by L' if necessary and assume that L is an elementary extension of the field underlying K. As K is large, also L is large. We equip L with a set of commuting derivations extending those on K (this is chosen arbitrarily and can always be done).

By Proposition 3.5, there is a differential K-algebra homomorphism $S \to L((t))$. As L is large and also an elementary extension of the field K, we know that K is e.c. as a field in $L((t))$. Hence, by Corollary 2.5(ii) there is a K-algebra homomorphism $S \to K$. By Proposition 3.5 there is a differential K-algebra homomorphism $S \to K[\![t]\!]$. □

4. Differentially large fields and algebraic characterizations

We introduce the notion of differential largeness and characterize it in multiple ways; see Theorem 4.3 and Proposition 4.7. First we recall the notion of largeness of fields.

4.1. Definition. A field K is said to be *large* (or *ample* in [6, Remark 16.12.3]) if every irreducible affine algebraic variety V over K with a smooth K-point has a Zariski-dense set of K-points (equivalently, K is e.c. in the function field $K(V)$).

Another equivalent formulation of largeness is that K is e.c. in the formal Laurent series field $K((t))$. Examples of large fields are pseudoalgebraically closed fields, pseudoreal closed fields and pseudo p-adically closed fields. By [27] the fraction field of any Henselian local ring is large; in particular, for every field K and all $n \geq 1$, the power series field $K((t_1, \ldots, t_n))$ is large.

Convention. Recall that for us a differential field always means a differential field in m commuting derivations $\Delta = \{\delta_1, \ldots, \delta_m\}$ and of characteristic zero. For a differential field (K, Δ), we equip the Laurent series field $K((\boldsymbol{t}))$ with the natural derivations extending those on K; namely, $\Delta + \mathrm{d}/\mathrm{d}\boldsymbol{t}$ as described in the previous section.

4.2. Definition. A differential field K is said to be *differentially large* if it is large as a pure field and for every differential field extension L of K the following implication holds:

If K is e.c. in L as a field, then K is e.c. in L as a differential field.

We now provide several algebraic characterizations of differential largeness. These characterizations resemble to some extent the characterizations of largeness of a field and serve as justification for the terminology "differentially large". A further characterization will be given in Proposition 4.7.

4.3. Theorem (characterizations of differential largeness). *Let $K = (K, \Delta)$ be a differential field. The following conditions are equivalent*:

(i) *K is differentially large.*

(ii) *K is e.c. in $K((\boldsymbol{t}))$ as a differential field, where the derivations on $K((\boldsymbol{t}))$ are the natural ones extending those on K.*

(iii) *K is e.c. in $K((\boldsymbol{t}_1)) \ldots ((\boldsymbol{t}_k))$ as a differential field for every $k \geq 1$.*

(iv) *K is large as a field and every differentially finitely generated K-algebra that has a K-rational point also has a differential K-rational point.*

(v) *K is large and every composite K-algebra in which K is e.c. as a field has a differential K-rational point.*

(vi) *Every composite differential K-subalgebra S of $K((\boldsymbol{t}))$ has a differential K-rational point.*

(vii) *K is large as a field and for every composite K-algebra $S = A \otimes_K P$, if A has a K-rational point, then S has a differential K-rational point.*

(viii) *K is large as a field and for every composite K-algebra $S = A \otimes_K P$, if the variety defined by A is smooth and if A has a K-rational point $A \to K$, then S has a differential K-rational point.*

(ix) *K is large as a field and for every composite K-algebra $S = A \otimes_K P$, if A has a smooth K-rational point, then S has a Kolchin-dense set of differential K-rational points* (*see Proposition* 2.2(*iii*)(*b*)).

(x) *K is large as a field and for every irreducible differential variety V over K such that for infinitely many $r \geq 0$ the algebraic variety* $\mathrm{Jet}_r(V)$ *has a smooth K-point, the set of differential K-rational points of V is Kolchin dense in V; in other words, for every proper closed differential subvariety $W \subseteq V$ there is a differential K-point in $V \setminus W$.*

Proof. (i) $\Rightarrow$ (iii). In the tower $K \subseteq K((\boldsymbol{t}_1)) \subseteq K((\boldsymbol{t}_1))((\boldsymbol{t}_2)) \subseteq \cdots \subseteq K((\boldsymbol{t}_1)) \ldots ((\boldsymbol{t}_k))$ all fields are large and therefore K is e.c. in $K((\boldsymbol{t}_1)) \ldots ((\boldsymbol{t}_k))$ as a field. So by definition of differential largeness, K is e.c. in $K((\boldsymbol{t}_1)) \ldots ((\boldsymbol{t}_k))$ as a differential field.

(iii) $\Rightarrow$ (ii). This is trivial.

(ii) $\Rightarrow$ (iv). Since K is e.c. in $K((\boldsymbol{t}))$ as a differential field it is also e.c. in $K((\boldsymbol{t}))$ as a field and so K is large as a field. Let S be a differentially finitely generated K-algebra and assume there is a point $S \to K$. Then by Proposition 3.5, there is a differential K-algebra homomorphism $S \to K[\![\boldsymbol{t}]\!]$. By Proposition 2.2(ii) applied to $K \subseteq K((\boldsymbol{t}))$, (ii) entails a differential K-algebra homomorphism $S \to K$.

(iv) $\Rightarrow$ (v). Take A, P for S as in Definition 2.4. Since K is also e.c. in A as a field and A is a finitely generated K-algebra, there is a K-algebra homomorphism $g : A \to K$. Since $S \cong_K A \otimes_K P$ and P is a polynomial K-algebra, g can be extended to a K-algebra homomorphism $S \to K$. Hence, (iv) applies.

(v) $\Rightarrow$ (i). Let L be a differential field extension of K and suppose K is e.c. in L as a field. Let S be a differentially finitely generated K-algebra, which has a differential point $f : S \to L$. By Proposition 2.2(ii) it suffices to find a differential point $S \to K$. By Corollary 2.5(i) we may replace S by a composite K-algebra contained in L and assume that f is the inclusion map $S \hookrightarrow L$. Now (v) applies.

Hence we know that conditions (i)–(v) are equivalent.

(iv) $\Rightarrow$ (vii). If $S = A \otimes_K P$ is composite and A has a K-rational point, then as P is a polynomial K-algebra we may extend this point to a point $S \to K$. By (iv), S has a differential K-rational point.

(vii) $\Rightarrow$ (vi). If $S = A \otimes_K P$ is a composite K-subalgebra of $K((\boldsymbol{t}))$, then as K is a large field, K is e.c. in A as a field and thus A has a K-rational point. Now (vii) applies.

(vi) $\Rightarrow$ (ii). This follows from Corollary 2.5(i) using the characterization Proposition 2.2(ii) of e.c.

Hence we know that conditions (i)–(vii) are equivalent.

(i) $\Rightarrow$ (ix). If $S = A \otimes_K P$ is composite and A has a smooth K-rational point, then as a large field, K is e.c. in A as a field. Since P is a polynomial K-algebra we know that S is a polynomial A-algebra and so A is e.c. in S as a ring. It follows that K is e.c. in S as a field and by (i) (invoke Proposition 2.2(i)) it is then also e.c. in S as a differential field. By Proposition 2.2(iii) we see that S has a Kolchin-dense set of differential K-rational points.

(ix) $\Rightarrow$ (viii). This is trivial.

(viii) $\Rightarrow$ (v). Let $S = A \otimes_K P$ be a composite K-algebra in which K is e.c. as a field. Then K is also e.c. in A as a field and therefore it possesses a smooth K-rational point $f : A \to K$ (see Proposition 2.2(iv)).

Pick $h \in A$ with $f(h) \neq 0$ such that the variety defined by the localization A_h is smooth. We may now apply (viii) to the composite algebra $S_h = A_h \otimes_K P$.

Hence we know that (i)–(ix) are equivalent. Property (x) is just a reformulation of the definition of differential largeness in geometric form as follows. Let $S = K\{x\}/\mathfrak{p}$, $x = (x_1, \dots, x_n)$, be a differentially finitely generated K-algebra and a domain with quotient map $\pi : K\{x\} \to S$. Let V be the differential variety defined by S. Hence, $V = \{a \in M^n \mid \mathfrak{p}(a) = 0\}$, where M is the differential closure of K. Then V is a K-irreducible differential variety defined over K. Now for $r \in \mathbb{N}$, the variety $\mathrm{Jet}_r(V)$ has coordinate ring $A_r := \pi(K\{x\}_{\leq r})$ and S is the union of the chain $(A_r)_r$ of K-subalgebras of S.[4] Clearly K is e.c. in S as a field if and only if K is e.c. in A_r as a field for all (or infinitely many) r. Since K is large, this is equivalent to saying that $\mathrm{Jet}_r(V)$ has a smooth K-point for all (or infinitely many) r. Hence, the assumption about V in (x) precisely says that K is e.c. in S as a field.

On the other hand, the conclusion about V in (x) precisely says that K is e.c. in S as a differential field (use Proposition 2.2(iii)).

This shows that differential largeness is equivalent to (x) formulated for affine differential varieties. But obviously the affine case implies (x) in full. □

4.4. Corollary. *If $K = (K, \delta_1, \dots, \delta_m, \partial_1, \dots, \partial_k)$ is a differentially large field, $m, k \geq 0$, then also $K = (K, \delta_1, \dots, \delta_m)$ is differentially large.*

Proof. This is immediate from the power series characterization in Theorem 4.3(ii). □

4.5. Corollary. *Let $K = (K, \delta_1, \dots, \delta_m, \partial_1, \dots, \partial_k)$ be a differentially large field, $m \geq 0$, $k \geq 1$ and let $C = \{a \in K \mid \partial_1(a) = \dots = \partial_k(a) = 0\}$ be the constant field of $(\partial_1, \dots, \partial_k)$.*

(i) *C is closed under the derivations $\delta_1, \dots, \delta_m$ and $(C, \delta_1, \dots, \delta_m)$ is e.c. in $(K, \delta_1, \dots, \delta_m)$.*

(ii) *$(C, \delta_1, \dots, \delta_m)$ is differentially large; when $m = 0$, this just says that C is a large field.*

Proof. We write $\delta = (\delta_1, \dots, \delta_m)$ and by a trivial induction we may assume that $k = 1$. Set $\partial = \partial_1$.

(i). Since all derivations commute, C is closed under all derivations. Let $(S, \hat{\delta})$ be a (C, δ)-algebra that is finitely generated as such. Suppose we are given a differential K-rational point $\lambda : (S, \hat{\delta}) \to (K, \delta)$ (in fact we will only need that S has a K-rational point). It suffices to find a differential C-algebra homomorphism $(S, \hat{\delta}) \to (C, \delta)$. We expand $(S, \hat{\delta})$ by the trivial derivation and obtain a differentially finitely generated (C, δ, ∂)-algebra $(S, \hat{\delta}, 0)$ (note that ∂ is trivial on C).

A straightforward calculation shows that $(S, \hat{\delta}, 0) \otimes_C (K, \delta, \partial)$ is a differential (K, δ, ∂)-algebra (the derivations are given by $\hat{\delta}_i \otimes \delta_i$ and $0 \otimes \partial$) that is finitely generated as such, and $\lambda \otimes \mathrm{id} : S \otimes_C K \to K$ is a (not necessarily differential) K-algebra homomorphism; also see [19, §3.1] for generalities on derivations and tensor products.

Since (K, δ, ∂) is differentially large, there is a differential point $\mu : (S, \hat{\delta}, 0) \otimes_C (K, \delta, \partial) \to (K, \delta, \partial)$ by Theorem 4.3(iv), and we get a C-algebra homomorphism $\mu_0 : S \to S \otimes_C K \xrightarrow{\mu} K$. Since the natural map $S \to S \otimes_C K$ is differential for δ, also μ_0 is a differential homomorphism $(S, \hat{\delta}) \to (K, \delta)$. But

[4] Here $K\{x\}_{\leq r}$ denotes the subring of $K\{x\}$ of all polynomials in θx_i, where $\mathrm{ord}(\theta) \leq r$.

μ_0 has values in C because for $s \in S$ we have $\partial(\mu_0(s)) = \partial(\mu(s \otimes 1)) = \mu((0 \otimes \partial)(s \otimes 1)) = \mu(0) = 0$. Hence, indeed, $\mu_0(s) \in C$ as required.

(ii) Since (K, δ, ∂) is differentially large, it is e.c. in $K((t_1, \ldots, t_{m+1}))$ when the latter is furnished with the natural derivations; see Theorem 4.3(ii). By (i), (C, δ) is e.c. in (K, δ). If $m = 0$ it follows that C is e.c. as a field in $K((t_1))$, hence C is a large field. If $m \geq 1$, we see that (C, δ) is e.c. in $C((t_1, \ldots, t_m))$, which shows that it is differentially large by Theorem 4.3(ii). □

At the end of this section we show that differentially large fields are first-order axiomatizable; in other words, the class of differentially large fields is an elementary class in the language of differential rings. We show this implicitly in Proposition 4.7, by proving that differentially large fields are precisely those large and differential fields satisfying the axiom scheme UC in [34, 4.5]; thus, we refer to this paper for explicit axioms. The proof of Proposition 4.7 only uses properties of models of UC and results from this paper.

4.6. Remark. It is worth mentioning (for the nonlogician) the benefits of knowing that a class of structures is elementary (first-order axiomatizable). In our context this means that two properties hold: (1) ultraproducts of differentially large fields are again differentially large, and (2) differential fields that are existentially closed in some differentially large field are themselves differentially large. Property (2) is obvious from the characterization Theorem 4.3(ii). So it is only property (1) that needs to be established. Being an elementary class opens up the model theoretic toolbox to the analysis of differentially large fields, and it implies, for example, the following transfer principle (phrased in technical terms in Corollary 4.8 below):

> If K is a differentially large field and K as a pure field has "good" elimination theory, then the differential field K also has good elimination theory.

To illustrate what "good" elimination theory means, we look at classical examples of "good" elementary classes of fields. Algebraically closed fields have good elimination theory; this is due to Chevalley's theorem which says that the projection of a variety is constructible. If K is a real closed field or a p-adically closed field, then projections of K-varieties (by which we mean here Zariski closed subset of some K^n) are generally not constructible; however, the following weaker statement holds: the complement of a projection of a K-variety is again the projection of a K-variety (this property of a field is called "model-completeness"; see [10, section 8.3]). So then the transfer principle above says that for a differentially large field K the following holds: if K is algebraically closed as a field, then the projection of a differential variety is differentially constructible (a finite Boolean combination of Kolchin closed sets); if K is real closed or p-adically closed, then the complement of a projection of a differential variety is again the projection of a differential variety.

4.7. Proposition. *Let K be a differential field that is large as a field. Then K is differentially large if and only if it satisfies the axiom scheme* UC *from* [34, 4.5].

Proof. First assume that K is differentially large. By [34, Theorem 6.2(II)], there is a differential field extension L of K such that $L \models \mathrm{UC}$ and such that K is elementary in L as a field. In particular K is e.c. in

L as a field. Since K is differentially large, K is e.c. in L as a differential field. By [34, Proposition 6.3], UC has an inductive axiom system in the language of differential rings. But then K also satisfies these axioms. Hence $K \models \mathrm{UC}$.

For the converse assume that K is a model of UC. We verify the definition of differentially large. Let L be a differential field extension of K such that K is e.c. in L as a field. Then there is a field M extending L such that K is elementary in M as a field. In particular M is a large field. We may now extend the derivations of L arbitrarily to commuting derivations of M. Hence, we may replace L by M furnished with these derivations and assume that L is large as a field. By [34, Theorem 6.2(II)] again, there is a differential field extension F of L such that $F \models \mathrm{UC}$ and such that L is elementary in F as a field. Then K is e.c. in F as a field and $K, F \models \mathrm{UC}$. By [34, Theorem 6.2(I)], this shows that K is e.c. in F (as a differential field), showing the assertion. □

By Proposition 4.7 we may now record important properties of differentially large fields (that follow from being models of UC; see [34]).

4.8. Corollary. (i) *If L and M are differentially large fields and K is a common differential subfield, then L and M have the same existential theory over K (meaning they solve the same systems of differential equations with coefficients in K) if and only if they have the same existential theory over K as fields.*

(ii) *If K is a differential field that is large as a field, then there is a differential field extension L of K such that L is differentially large and an elementary extension of K as a field.*

(iii) *Let K be a differentially large field and let $A \subseteq K$. Suppose K is model complete as a field in the language $\mathscr{L}_{\mathrm{ri}}(A)$ of rings extended by constant symbols naming the elements of A.*

Then also K is model complete in the language $\mathscr{L}_{\mathrm{diff}}(A)$ of differential rings extended by all constant symbols naming the elements of A. If $\hat{\mathscr{L}}$ is a language extending $\mathscr{L}_{\mathrm{ri}}$ and $\hat{K}$ is an expansion of K to $\hat{\mathscr{L}}_{\mathrm{diff}}$ such that the new symbols are A-definable in the field K and such that the restriction of $\hat{K}$ to $\mathscr{L}^*(A)$ has quantifier elimination[5]*, then $\hat{K}$ has quantifier elimination in the language $\hat{\mathscr{L}}_{\mathrm{diff}}(A)$.*

5. Fundamental properties, constructions and applications

We show that algebraic extensions of differentially large fields are again differentially large by invoking the differential Weil descent in 5.12. Specifically differentially closed fields are identified as precisely the algebraic closures of differentially large fields; in a similar way, M. Singer's closed ordered differential fields are characterized; see 5.13. We show that a differentially large field is pseudoalgebraically closed just if it is pseudodifferentially closed; see 5.18. We characterize the existential theory of differentially large fields in 5.7. We show that differentially large fields are Picard–Vessiot closed in 5.9. In 5.15 we establish Kolchin-denseness of rational points in differential algebraic groups.

[5] An example of $\hat{\mathscr{L}}$ is the language $\mathscr{L}_{\mathrm{ri}}(\leq)$ of ordered rings, $K = (K, \delta)$ is a real closed field furnished with commuting derivations, $A = \varnothing$ and $\hat{K} = (K, \leq, \delta)$. The restriction of $\hat{K}$ to $\mathscr{L}_{\mathrm{ri}}(A)$ then is the ordered field $(K, \leq)$.

We start with a concrete method to construct differentially large fields. This is deployed in 5.14 to obtain concrete constructions of differentially closed fields.

5.1. Proposition. *Let $(K_i, f_{ij})_{i,j\in I}$ be a directed system of differential fields and differential embeddings with the following properties*:

(a) *All K_i are large as fields.*

(b) *All embeddings $f_{ij} : K_i \to K_j$ are isomorphisms onto a subfield of K_j that is e.c. in K_j as a field.*

(c) *For all $i \in I$ there exist $j \geq i$ and a differential homomorphism $K_i[\![\boldsymbol{t}]\!] \to K_j$ extending f_{ij}.*

Then the direct limit L of the directed system is a differentially large field.

Proof. We write $f_i : K_i \to L$ for the natural map into the limit, which obviously is a differential homomorphism between differential fields. We use the characterization Theorem 4.3(vii) to show that L is differentially large. Firstly, L is large as a field, because if C is a curve defined over L that has a smooth L-rational point then take $i \in I$ such that C is defined over K_i (via f_i) and such that C has a smooth K_i-rational point. By (a), the curve C has infinitely many K_i-rational points and so it also has infinitely many L-rational points.

Now let S be a differentially finitely generated L-algebra and a domain that has a point $S \to L$. Pick $r \in \mathbb{N}$ and a differential prime ideal $\mathfrak{p}$ of $L\{x\}$, $x = (x_1, \ldots, x_r)$, such that $S = L\{x\}/\mathfrak{p}$. By the Ritt–Raudenbusch basis theorem there is a finite $\Sigma \subseteq \mathfrak{p}$ whose differential radical is $\mathfrak{p}$. By Theorem 4.3(vii) it suffices to find a differential zero of Σ in L. Take $i \in I$ with $\Sigma \subseteq f_i(K_i)\{x\}$ and let $S_0 := K_i\{x\}/f_i^{-1}(\mathfrak{p})$. Then S_0 is a differentially finitely generated K_i-algebra and the composition of the natural embedding $S_0 \to S$ with a point $S \to L$ is a homomorphism $S_0 \to L$ extending f_i. We now want to invoke Corollary 3.6 and here we need (b). Namely, with this condition one readily verifies Proposition 2.2 and checks that K_i is existentially closed in L as a field (via f_i).

Hence, we may apply Corollary 3.6 to obtain a differential K-algebra homomorphism $S_0 \to K_i[\![\boldsymbol{t}]\!]$. Finally (c) gives us a differential K_i-algebra homomorphism $S_0 \to K_j$ for some $j \geq i$. This yields a differential solution of Σ in L. □

Concretely, Proposition 5.1 may be used to produce differentially large fields via iterated power series constructions using standard power series, Puiseux series or generalized power series. Here are a few instances; see 5.14 for applications.

5.2. Differentially large power series fields. Let K be a differential field. We write $K_0 = K$.

(i) We define by induction on $n \geq 0$, the differential field extension K_{n+1} of K_n as $K_{n+1} = K_n((\boldsymbol{t}_n))$, where $\boldsymbol{t}_n = (t_{n1}, \ldots, t_{nm})$; the derivations on K_{n+1} are the natural ones, extending those on K_n and satisfying $\delta_j(t_{nk}) = (\mathrm{d}/\mathrm{d}t_{nj})(t_{nk})$. Then $K_\infty = \bigcup_{n\in\mathbb{N}} K_n$ is differentially large. If K is large as a field, then K is e.c. in K_∞ as a field.

To see this we apply Proposition 5.1 to the family of all K_n, $n > 0$ together with the inclusion maps $K_i \hookrightarrow K_j$ for $i \leq j$. Hence, K_∞ is differentially large. Since all K_n are large fields we know that they are e.c. in K_∞ as a field. Hence, if K happens to be large as a field, then K is also e.c. in K_∞ as a field.

This construction is discussed further in 5.14.

(ii) Assume here that the number m of derivations is 1. Then the generalized power series field $K((t^{\mathbb{Q}}))$ carries a derivation defined by $(\mathrm{d}/\mathrm{d}t)\big(\sum a_\gamma t^\gamma\big) = \sum a_\gamma \cdot \gamma \cdot t^{\gamma-1}$ and the given derivation δ on K can be extended to a derivation ∂ by $\partial\big(\sum a_\gamma t^\gamma\big) = \sum \delta(a_\gamma) t^\gamma$. We consider $K((t^{\mathbb{Q}}))$ as a differential field extension of $K((t))$, equipped with the derivation $\mathrm{d}/\mathrm{d}t + \partial$.

Now define $K_{n+1} = K_n((t_n^{\mathbb{Q}}))$. Since K_n carries a Henselian valuation for $n > 0$ we know that K_n is a large field. Hence, Proposition 5.1(a) and (c) hold for the family of all K_n, $n > 0$, and the inclusion maps $K_i \hookrightarrow K_j$ when $i \leq j$.

If K is algebraically closed, real closed or p-adically closed, then so are all K_n and by standard theorems from model theory, Proposition 5.1(b) holds in each case. Thus $K_\infty = \bigcup_n K_n$ is a differentially large field. Also, $K_\infty = \bigcup_n K_n$ is again algebraically closed, real closed or p-adically closed, respectively. To be precise: if K is algebraically closed, then K_∞ is a differentially closed field; if K is real closed, then K_∞ is a closed ordered differential field in the sense of [32]; and if K is p-adically closed, then K_∞ is an existentially closed differential field in the class of p-adically valued and differential fields as considered in [8].

(iii) The differentially large field K_∞ in (ii) has various interesting differentially large subfields: for example, in each step of the construction we can work with Puiseux series only. More precisely, if P_{n+1} is defined to be the Puiseux series field over P_n, namely

$$P_{n+1} = P_n((t^{1/\infty})) = \bigcup_{k \in \mathbb{N}} P_n((t^{1/k})),$$

then $P_\infty = \bigcup_n P_n$ is a differentially large subfield of K_∞. Another example is given by working with completions of Puiseux series. More precisely, if C_{n+1} is defined to be the completion of the Puiseux series field over C_n, namely

$$C_{n+1} = \{f \in C_n((t^{\mathbb{Q}})) \mid \operatorname{supp}(f) \text{ is finite, or } \operatorname{supp}(f) \text{ is unbounded in } \mathbb{Q} \text{ and of order type } \omega\},$$

then $C_\infty = \bigcup_n C_n$ is a differentially large subfield of K_∞.

Again the fields P_∞ and C_∞ as pure fields, are algebraically closed, real closed, or p-adically closed if K has this property. By applying Corollary 4.8(iii) and model completeness of algebraically closed, real closed, and p-adically closed fields, we see that the differential fields P_∞ and C_∞ are elementary substructures of K_∞ in these cases.

5.3. Counterexample. Let L be the differential subfield $K((\boldsymbol{t}_1, \boldsymbol{t}_2, \ldots))$ of the differential field K_∞ from 5.2(i) and let L^{alg} be its algebraic closure. Then none of the t_{nj}^{-1} has an integral in L^{alg}, and hence L^{alg} is not differentially large and so obviously neither is $K((\boldsymbol{t}_1, \boldsymbol{t}_2, \ldots))$. Notice that the latter is large as a pure field by [27].

For the proof we may restrict to the case of one derivation. For $k \in \mathbb{N}$, the derivation δ of K_∞ restricts to a derivation of $L_0 = K((t_1, \ldots, t_k))$. The definition of the derivation, restricted to $K[\![t_1, \ldots, t_k]\!]$, shows that the K-automorphism of this ring permuting the variables is differential; obviously such an automorphism extends uniquely to a differential automorphism of L_0^{alg}. Hence, in order to show that none of the t_n^{-1}, $n \le k$, has an integral in L_0^{alg} we may assume that $n = k$. We write $t = t_k$ and let F be the algebraic closure of the differential subfield $K((t_1, \dots, t_{k-1}))$. Then L_0^{alg} is a differential subfield of the Puiseux series field $P = F((t^{1/\infty}))$, the latter being equipped with the natural derivation extending the one on F and mapping t to 1. It remains to show that t^{-1} has no integral in P. Suppose for a contradiction that $\delta(f) = t^{-1}$ for some $f \in P$. Then the order of f is $-q$ for some $q \in \mathbb{Q}$, $q > 0$. Hence, $t^q \cdot f$ has order 0 and so by definition of the derivation of P we see that the order of $\delta(t^q \cdot f)$ is > -1. On the other hand $\delta(t^q \cdot f) = q \cdot t^{q-1} \cdot f + t^q \cdot t^{-1} = t^{q-1} \cdot (q \cdot f + 1)$ has order -1, a contradiction.

5.4. Remark. In view of Counterexample 5.3 it is of interest to see integrals of t_1^{-1} in the differential field K_∞ from 5.2(i) in the ordinary case: take $k \ge 2$ and let $f_k = \sum_{n \ge 1} (1/n t_1^n) \cdot t_k^n$ (resembling $-\log(1 - (t_k/t_1))$). One readily checks that $\delta(f_k) = t_1^{-1}$.

5.5. Iterating algebraic power series. A further natural question related to the field K_∞ of 5.2(i) asks what type of differential equations can be solved when we iterate only algebraic Laurent series instead of all Laurent series. Let K be an ordinary differential field and let $L = \bigcup_n K((t_1))_{\mathrm{alg}} \dots ((t_n))_{\mathrm{alg}}$, where the derivation is chosen as in 5.2(i). Thus L is a differential subfield of $K(t_1, t_2, \dots)^{\mathrm{alg}}$, where $\delta(t_i) = 1$ for all i. Notice that L is large as a pure field, because algebraic power series are a local henselian domain and so [27] applies again. Since L is a differential subfield of the algebraic closure of $K((t_1, t_2, \dots))$ we already know from Counterexample 5.3 that L is not differentially large. Here we show that L is not even Picard–Vessiot closed in general.

If L were Picard–Vessiot closed, then L has nontrivial solutions of the differential equation $\delta x = x$. However, we show that this is in general not the case even for $M = K(t_1, t_2, \dots)^{\mathrm{alg}}$. To see this, consider the following property of a differential field F:

$$(\dagger) \qquad \forall x \in F, n \in \mathbb{N} : \delta(x) = n \cdot x \Rightarrow x = 0.$$

Then, if F has property (†) so does its algebraic closure F^{alg} and its function field $F(t)$, where $\delta(t) = 1$. Hence, if we start with K being a differential field with trivial derivation, then by induction, property (†) passes to $K(t_1, \dots, t_n)^{\mathrm{alg}}$ and so also passes to M.

For the proof that (†) passes to $F(t)$, assume that $\delta(f/g) = n \cdot f/g$ with g monic and f with leading coefficient a. Then $nfg = \delta(f)g - f\delta(g) = (f^\delta + f')g - f(g^\delta + g')$ and comparing leading coefficients shows that $n \cdot a = \delta(a)$. Hence, by (†) for F we get $a = 0$ as required.

For the proof that (†) passes to F^{alg}, one first checks that it passes to $F(C)$, where C is the constant field of F^{alg}. Hence, we may replace F by $F(C)$ and assume that F and F^{alg} have the same constant field. Let α be algebraic over F with minimal polynomial f and assume $\delta(\alpha) = n \cdot \alpha$. Then any other root β of f also satisfies this equation, which implies that $\delta(\alpha/\beta)$ is a constant; thus it is in F. Hence, $F(\alpha)$ is the splitting field of f and so $F(\alpha)/F$ is Galois. Let d be the order of the Galois group and let $\sigma \in \mathrm{Gal}(F(\alpha)/F)$. As we have seen, $\sigma(\alpha) = c \cdot \alpha$ for some constant c. Hence, $\alpha = \sigma^d(\alpha) = c^d \cdot \alpha$ and so

$c^d = 1$. But then $\sigma(\alpha^d) = (c\alpha)^d = \alpha^d$, which shows that α^d is in the fixed field F. Since $\delta(\alpha^d) = d{\cdot}n{\cdot}\alpha^d$, we get $\alpha = 0$ from (†) for F.

5.6. The existential theory of differentially large fields. The existential theory of the class of all large fields of characteristic zero is the existential theory of the field $\mathbb{Q}((t))$ (see [29, Proposition 2.25]). This follows essentially from the fact that $\mathbb{Q}((t))$ is itself a large field. Since the existential theory of a differentially large field is uniquely determined by its existential theory of its field structure — in the sense of Corollary 4.8(i) — one is led to the question on whether the existential theory of the class of differentially large fields is the existential theory of $\mathbb{Q}((\boldsymbol{t}))$, equipped with its natural derivations.

However, $\mathbb{Q}((\boldsymbol{t}))$ does not satisfy the existential theory of the class of differentially large fields (and so it is not differentially large either). To see an example, let C be the curve defined by $x^3 + y^3 = 1$. Then $(1, 0)$ and $(0, 1)$ are the only rational points on C (and they are regular points). Hence, the sentence φ saying that there is a point (x, y) on C with $x \neq 0$, $y \neq 0$ and $x' = y' = 0$ fails in the differential field $\mathbb{Q}((t))$ (we work with $m = 1$ here). On the other hand φ is true in every differentially large field K, because the constants of K are large as a field by Corollary 4.5.

On the positive side we now show:

5.7. Theorem. *The existential theory of the class of differentially large fields is the existential theory of* $\mathbb{Q}((\boldsymbol{t}_1))((\boldsymbol{t}_2))$.

Proof. Let $\Sigma \subseteq \mathbb{Z}\{x_1, \dots, x_n\}$ be a system of differential polynomials in n variables and m commuting derivations. If Σ has a solutions in $\mathbb{Q}((\boldsymbol{t}_1))((\boldsymbol{t}_2))$ and K is a differentially field, then Σ also has a solution in $K((\boldsymbol{t}_1))((\boldsymbol{t}_2))$. Hence, if K is differentially large, then by Theorem 4.3(iii), Σ also has a solution in K.

Conversely, suppose Σ has a solution in every differentially large field. By Corollary 4.8(ii) there is a differentially large field K containing $\mathbb{Q}((\boldsymbol{t}_1))$ as a differential subfield such that the extension $K/\mathbb{Q}((\boldsymbol{t}_1))$ of fields is elementary. Let $S_0 = \mathbb{Q}\{x_1, \dots, x_n\}/\sqrt[d]{\Sigma}$ and let $f : S_0 \to K$ be a differential point of S_0. Let $\mathfrak{p} = \operatorname{Ker}(f)$ and let $S = S_0/\mathfrak{p}$. It suffices to find a differential point $S \to \mathbb{Q}((\boldsymbol{t}_1))((\boldsymbol{t}_2))$. Write $S = A_h \otimes_{\mathbb{Q}} P$ as in Theorem 2.3. The restriction $f|_{A_h}$ is a K-rational point of A_h. Since A_h is a finitely generated $\mathbb{Q}$-algebra and $\mathbb{Q}((\boldsymbol{t}_1))$ is e.c. in K as a field, there is also a point $g_0 : A_h \to \mathbb{Q}((\boldsymbol{t}_1))$. Since P is a polynomial $\mathbb{Q}$-algebra, g_0 can be extended to a point $g : S \to \mathbb{Q}((\boldsymbol{t}_1))$. By Corollary 3.6, there is a differential point $S \to \mathbb{Q}((\boldsymbol{t}_1))[[\boldsymbol{t}_2]]$. □

5.8. Differentially large fields are PV-closed. We prove that differentially large fields solve plenty of algebraic differential equations. Namely, we prove that they solve all consistent systems of linear differential equations. We first show that they are Picard–Vessiot closed (or PV-closed).

Let $(K, \delta_1, \dots, \delta_m)$ be a differential field, and let $A_i \in \operatorname{Mat}_n(K)$, for $i = 1, \dots, m$, satisfying what is called the *integrability condition*; namely

$$\delta_i A_j - \delta_j A_i = [A_i, A_j],$$

where $\delta_j A_j$ denotes the $n\times n$ matrix obtained by applying δ_i to A_j entrywise. The differential field K is said to be PV-closed if for each such tuple $(A_1, \dots, A_m)$ of matrices there is a $Z \in GL_n(K)$ such that

$$\delta_i Z = A_i Z_i \quad \text{for } i = 1, \dots, m.$$

5.9. Lemma. *Every differentially large field is PV-closed.*

Proof. Suppose K is a differentially large field. Suppose $A_1, \dots, A_m$ are elements in $\mathrm{Mat}_n(K)$ satisfying the integrability condition. Let X be an $n\times n$ matrix of variables and define derivations on $K(X)$ that extend the ones in K and satisfy

$$\delta_i X = A_i X.$$

Then, by the integrability condition, these derivations commute in all of $K(X)$. Since K is e.c. in $K(X)$ as fields, by differential largeness, it is also e.c. as differential fields. This yields the desired (fundamental) solution in K. □

In differentially large fields, Lemma 5.9 is a special case of a stronger property:

5.10. Proposition. *Let Σ and Γ be finite collections of differential polynomials in $K\{x_1, \dots, x_n\}$. Assume that the system*

$$P = 0 \quad \text{and} \quad Q \neq 0 \qquad \text{for } P \in \Sigma \text{ and } Q \in \Gamma$$

is consistent (i.e., it has a solution in some differential field extension of K). If Σ consists of linear differential polynomials and K is differentially large, then the system has a solution in K.

Proof. Since the system is assumed to be consistent, the differential ideal generated by Σ in $K\{x_1, \dots, x_n\}$, denoted by $[\Sigma]$, is prime. Thus, the differential field extension $L = \mathrm{qf}(K\{x_1, \dots, x_n\}/[\Sigma])$ has a solution to the system. Since $[\Sigma]$ is generated, as an ideal of $K\{x_1, \dots, x_n\}$, by linear terms, we get that K is e.c. in L as fields, and, by differential largeness, also as differential fields. The result follows. □

5.11. A glimpse on the differential Weil descent. If K is a large field, then every algebraic field extension of K is again large. This follows from an argument involving Weil descent in the case when L/K is finite; see [29, Theorem 2.14; 26, Proposition 1.2]. For differentially large fields, this can also be carried out. We will explain a special case of the differential Weil descent suitable for our purpose and refer to [19, Theorem 3.4] for the general assertion and for proofs.

We will be working with a finite extension L/K of differential fields and a differential L-algebra S. Then the classical Weil descent $W(S)$ of the underlying L-algebra of S is a K-algebra and there is a "natural" bijection

$$\mathrm{Hom}_{K\text{-}\mathtt{Alg}}(W(S), K) \to \mathrm{Hom}_{L\text{-}\mathtt{Alg}}(S, L).$$

Here homomorphisms are algebra homomorphisms over K and L, respectively. Now in [19, Theorem 3.4] it is shown that the ring $W(S)$ can be naturally expanded to a differential K-algebra $W^{\mathrm{diff}}(S)$ such that the bijection above restricts to a bijection

$$\mathrm{Hom}_{\mathrm{diff.}\ K\text{-}\mathtt{Alg}}(W^{\mathrm{diff}}(S), K) \to \mathrm{Hom}_{\mathrm{diff.}\ L\text{-}\mathtt{Alg}}(S, L).$$

This time, homomorphisms are differential algebra homomorphisms over K and L, respectively. The terminology "natural" in both bijections refers to the fact that W and W^{diff} are indeed functors defined on categories of algebras and differential algebras, respectively. However for our application below only the existence of the bijections above are needed. We refer to [19, Section 3] for a self contained exposition of the matter, where all data are constructed explicitly. In particular the construction there shows that $W^{\mathrm{diff}}(S)$ is a differentially finitely generated K-algebra if S is a differentially finitely generated L-algebra.

5.12. Theorem. *If K is differentially large, then so is every algebraic extension* (*equipped with the induced derivations*).

Proof. Let L/K be an algebraic extension. We first deal with the case when L/K is finite. We verify Theorem 4.3(iv) for L. So let S be a differentially finitely generated L-algebra that has an L-rational point. Let $W^{\mathrm{diff}}(S)$ be the differential Weil descent as explained in 5.11. Thus, $W^{\mathrm{diff}}(S)$ is a differentially finitely generated K-algebra and we have a bijection

$$\mathrm{Hom}_{K\text{-}\mathtt{Alg}}(W(S), K) \to \mathrm{Hom}_{L\text{-}\mathtt{Alg}}(S, L),$$

which restricts to a bijection

$$\mathrm{Hom}_{\mathrm{diff.}\ K\text{-}\mathtt{Alg}}(W^{\mathrm{diff}}(S), K) \to \mathrm{Hom}_{\mathrm{diff.}\ L\text{-}\mathtt{Alg}}(S, L).$$

Since S has an L-rational point we may use the first bijection and see that $W(S)$ has a K-rational point. Since K is differentially large there is a differential K-rational point $W^{\mathrm{diff}}(S) \to K$. Using the second bijection we see that S has a differential L-rational point.

Hence, we know the assertion when L/K is finite. In general, let $S = A \otimes P$ be a composite L-algebra such that the affine variety defined by A is smooth. Suppose there is an L-rational point $A \to L$. By Theorem 4.3(viii) it suffices to show that there is a differential point $S \to L$. Write $S = L\{x\}/\mathfrak{p}$, $x = (x_1, \dots, x_r)$, for a prime differential ideal $\mathfrak{p}$ of $L\{x\}$ and let $\Sigma \subseteq \mathfrak{p}$ be finite with $\mathfrak{p} = \sqrt[d]{\Sigma}$. It suffices to find a differential solution of $\Sigma = 0$ in L. Choose a finite extension K_0/K in L with $\Sigma \subseteq K_0\{x\}$. Let $S_0 = K_0\{x\}/\mathfrak{p} \cap K_0\{x\}$, which we consider as a subring of S. By Theorem 2.3 there are a finitely generated K_0-subalgebra A_0 of S_0, a polynomial K_0-subalgebra P_0 of S_0 and an element $h \in A_0$ such that $(S_0)_h \cong (A_0)_h \otimes_{K_0} P_0$.

Since $A_0 \subseteq S$ is finitely generated we may write $P = P_1 \otimes_L P_2$ for some polynomial L-algebras P_i, P_1 finitely generated such that $A_0 \subseteq A \otimes_L P_1$. Then $A \otimes_L P_1$ is again finitely generated, the affine variety defined by $A \otimes_L P_1$ is again smooth and still has an L-rational point. Since L is large, there is also an L-rational point $(A \otimes_L P_1)_h \to L$. Via restriction we get an L-rational point $f : (A_0)_h \to L$. Since $(A_0)_h$ is finitely generated as a K_0-algebra, there is a finite extension K_1/K_0 contained in L such that f has values in K_1. Since P_0 is a polynomial K_0-algebra, f can be extended to a K_1-rational point $(S_0)_h \to K_1$. Tensoring with K_1 gives a K_1-rational point of $(S_0)_h \otimes_{K_0} K_1$. The latter is a differentially finitely generated K_1-algebra. By what we have shown, K_1 is differentially large. By Theorem 4.3(iv) there is a differential point $(S_0)_h \otimes_{K_0} K_1 \to K_1$. Since $\Sigma \subseteq K_1\{x\}$ this gives rise to a differential solution of $\Sigma = 0$ in $K_1 \subseteq L$. □

As an application, we see from 5.12 and Corollary 4.8(iii) that the algebraic closure of a differentially large field is differentially closed. Hence, differentially large fields have minimal differential closures:

5.13. Corollary. *The algebraic closure of a differentially large field is differentially closed. In particular, if* $K \models \mathrm{CODF}_m$, *the theory of closed ordered differential fields in* m *commuting derivations, then* $K(i) \models \mathrm{DCF}_{0,m}$.

The result above has already been deployed in [1] making reference to an earlier draft of this paper. Previously known examples of differential fields with minimal differential closures are models of CODF (which we denote as CODF_1), see [31], and fixed fields of models of $\mathrm{DCF}_{0,m}$ A, the theory differentially closed fields with a generic differential automorphism; see [16]. The corollary delivers a vast variety of new differential fields with this property, namely all differentially large fields; see also Corollary 4.8(ii).

We also get new and explicit models of $\mathrm{DCF}_{0,m}$ and CODF_m:

5.14. Construction of differentially closed fields. We continue with the constructions in 5.2(i). If K is a differential field, then the algebraic closure of the differentially large field $K_\infty = \bigcup_{n\in\mathbb{N}} K((\boldsymbol{t}_1))\dots((\boldsymbol{t}_n))$ from 5.2(i) is differentially closed. If K is an ordered field and the order is extended to L in some way, then the real closure of L is a model of CODF_m.

Observe that these models are different from those obtained using iterated Puiseux series or generalized power series constructions in 5.2(ii) and (iii).

5.15. Kolchin-denseness of rational points in differential algebraic groups. In the classical case of a connected linear algebraic group G over any field F of characteristic zero, the unirationality theorem implies that the F-rational points of G are Zariski-dense. In the differential situation the corresponding statement does not hold. For example, the linear differential algebraic group defined by $\delta x = x$ does in general not have a Kolchin-dense set of rational points. However, as a further application, we prove that in differentially large fields this is true again:

5.16. Proposition. *Assume* K *is differentially large. If* G *is a connected differential algebraic group over* K, *then the set of* K*-rational points of* G, *denoted* $G(K)$, *is Kolchin dense in* $G = G(\mathbb{U})$.

Proof. We verify Theorem 4.3(x), and hence it suffices to show that for infinitely many values of r the jet $\mathrm{Jet}_r\, G$ has a smooth K-rational point. By [23, Corollary 4.2(ii)], G embeds over K into a connected algebraic group H defined over K. As we saw in Definition 2.6, for each r, $\nabla_r G$ is a differential algebraic subgroup of $\tau_r H$. As a result, $\mathrm{Jet}_r\, G$ is an algebraic subgroup of $\tau_r H$, and so $\mathrm{Jet}_r\, G$ is smooth. If e denotes the identity of G, which is a K-point, then, for each r, the K-point $\nabla_r(e)$ is a smooth point of $\mathrm{Jet}_r\, G$. □

The result above has already been deployed in [18] making reference to an earlier draft of this paper.

5.17. Pseudodifferentially closed fields. Recall that a field K is pseudoalgebraically closed (PAC) if every absolutely irreducible algebraic variety over K has a K-point. It is easy to see and well known that PAC fields are large and that the PAC property is equivalent to saying that K is e.c. in every regular field extension L (meaning that K is algebraically closed in L). From model theoretic literature one can formulate several notions of pseudodifferentially closed fields; see [11; 24]. We show that they are all equivalent to the property "PAC + differentially large".

5.18. Theorem (pseudodifferentially closed fields). *Let K be a differential field. The following are equivalent*:

(i) *K is PAC* (*as a field*) *and K is differentially large.*

(ii) *Every absolutely irreducible differential variety over K has a differential K-point. Recall that a differential variety V over K is absolutely irreducible if it is irreducible in the Kolchin topology of a differential closure K^{diff} and this is equivalent to saying that V is irreducible over K^{alg}.*

(iii) *K is e.c. in every differential field extension L in which K is R-regular* (*that is*, $\text{tp}(a/K)$ *is stationary for every tuple a from L, where the type* $\text{tp}(a/K)$ *is with respect to the stable theory* $\text{DCF}_{0,m}$).

(iv) *K is e.c. in every differential field extension L in which K is H-regular* (*that is*, $K^{\text{alg}} \cap L = K$).

If these equivalent conditions hold we call K pseudodifferentially closed.

Proof. We use [11, Lemma 3.35], which says in our situation that for a tuple a from $\mathcal{U}$ (the monster model of $\text{DCF}_{0,m}$), the type $\text{tp}(a/K)$ is stationary if and only if the differential field extension $K\langle a\rangle$ over K is H-regular. Clearly this characterization implies that H-regularity and R-regularity are equivalent. In particular, (iii) is equivalent to (iv).

(i) $\Rightarrow$ (iv). Let K be PAC and differentially large. Let L/K be an H-regular extension of K. Then K is algebraically closed in L as a field and because K is PAC, it is e.c. in L as a field. Since K is differentially large, it follows that it is e.c. in L as a differential field.

(iv) $\Rightarrow$ (ii). This follows from the fact that a type $\text{tp}(a/K)$ is stationary if and only if the Kolchin-locus of a over K is absolutely irreducible. Let V be an absolutely irreducible differential variety over K. Then the generic type $p = \text{tp}(a/K)$ of V over K is stationary, and hence, by the quoted characterization of stationarity, the differential field $L = K\langle a\rangle$ is an H-regular extension of K. By (iv), K is e.c. in L as a differential field and so there is a differential K-point in V, as required.

(ii) $\Rightarrow$ (i). Suppose V is a K-irreducible differential variety such that all jets of V have a smooth K-point. Then all these jets are absolutely Zariski irreducible (as they are Zariski K-irreducible and contain a smooth K-point). It follows that V is absolutely irreducible. Hence, V has a K-point. In fact, V has Kolchin-dense many K-points; indeed, we can take any open differential subvariety O of V and argue similarly (using the fact that K is large, as it is PAC, to produce smooth K-points in the jets of O). This shows (i) using the equivalence (i) $\Longleftrightarrow$ (x) of Theorem 4.3. □

5.19. Pseudodifferentially closed fields are axiomatizable. An application of Theorem 5.18 is that the class of pseudodifferentially closed fields is first-order axiomatizable (so far this had only been established in the case of one derivation in [24, Proposition 5.6]). Indeed, being a PAC field is a first-order condition (see [6, 11.3.2]) and we have seen in Proposition 4.7 that differential largeness is too. The fact that being pseudodifferentially closed is a first-order property has very interesting model-theoretic consequences: (i) by [25, §3] it implies that the theory of a bounded pseudodifferentially closed field is supersimple, and (ii) by [5, Theorem 5.11] it implies that the elementary equivalence theorem holds for pseudodifferentially closed fields.

6. Algebraic-geometric axioms

We present algebraic-geometric axioms for differentially large fields in the spirit of the classical Pierce–Pillay axioms for differentially closed fields in one derivation [22] (see Remark 6.6(i) below). While this section might seem mostly of interest to model theorists, the general reader should keep in mind that Theorem 6.4 is a general statement on systems of algebraic PDEs that have solutions in differentially large fields.

Our presentation here follows the recent algebraic-geometric axiomatization of differentially closed fields in several commuting derivations established in [17]. In particular, we will use the recently developed theory of differential kernels for fields with several commuting derivations from [7]. One significant difference with the arguments in [17] is that theirs only requires the existence of regular realizations of differential kernels, while here we need the existence of principal realizations; see Remark 6.1 and Fact 6.2. We carry on the notation and conventions from previous sections.

We use two different orders $\leq$ and $\trianglelefteq$ on $\mathbb{N}^m \times \{1, \dots, n\}$. Given two elements (ξ, i) and (τ, j) of $\mathbb{N}^m \times \{1, \dots, n\}$, we set $(\xi, i) \leq (\tau, j)$ if and only if $i = j$ and $\xi \leq \tau$ in the product order of $\mathbb{N}^m$. We set $(\xi, i) \trianglelefteq (\tau, j)$ if and only if

$$\left(\sum \xi_k, i, \xi_1, \dots, \xi_m\right) \leq_{\text{lex}} \left(\sum \tau_k, j, \tau_1, \dots, \tau_m\right)$$

Note that if $x = (x_1, \dots, x_n)$ are differential indeterminates and we identify (ξ, i) with $\delta^\xi x_i := \delta_1^{\xi_1} \cdots \delta_m^{\xi_m} x_i$, then $\leq$ induces an order on the set of algebraic indeterminates given by $\delta^\xi x_i \leq \delta^\tau x_j$ if and only if $\delta^\tau x_j$ is a derivative of $\delta^\xi x_i$ (in particular this implies that $i = j$). On the other hand, the ordering $\trianglelefteq$ induces the canonical orderly ranking on the set of algebraic indeterminates.

We will look at field extensions of K of the form

$$L := K(a_i^\xi : (\xi, i) \in \Gamma_n(r)) \tag{6-1}$$

for some fixed $r \geq 0$. Here we use a_i^ξ as a way to index the generators of L over K. The element $(\tau, j) \in \mathbb{N}^m \times \{1, \dots, n\}$ is said to be a leader of L if there is $\eta \in \mathbb{N}^m$ with $\eta \leq \tau$ and $\sum \eta_k \leq r$ such that a_j^η is algebraic over $K(a_i^\xi : (\xi, i) \triangleleft (\eta, j))$. A leader (τ, j) is a minimal leader of L if there is no leader (ξ, i) with $(\xi, i) < (\tau, j)$. Observe that the notions of leader and minimal leader make sense even when $r = \infty$.

A (differential) kernel of length r over K is a field extension of the form

$$L = K(a_i^\xi : (\xi, i) \in \Gamma_n(r))$$

such that there exist derivations

$$D_k : K(a_i^\xi : (\xi, i) \in \Gamma_n(r-1)) \to L$$

for $k = 1, \dots, m$ extending δ_k and $D_k a_i^\xi = a_i^{\xi+\boldsymbol{k}}$ for all $(\xi, i) \in \Gamma_n(r-1)$, where $\boldsymbol{k}$ denotes the m-tuple whose k-th entry is one and zeroes elsewhere.

Given a kernel $(L, D_1, \dots, D_k)$ of length r, we say that it has a *prolongation of length* $s \geq r$ if there is a kernel $(L', D_1', \dots, D_k')$ of length s over K such that L' is a field extension of L and each D_k' extends D_k. We say that $(L, D_1, \dots, D_k)$ has a *regular realization* if there is a differential field extension $(M, \Delta' = \{\delta_1', \dots, \delta_m'\})$ of $(K, \Delta = \{\delta_1, \dots, \delta_m\})$ such that M is a field extension of L and $\delta_k' a_i^\xi = a_i^{\xi+\boldsymbol{k}}$ for all $(\xi, i) \in \Gamma_n(r-1)$ and $k = 1, \dots, m$. In this case we say that $g := (a_1^{\boldsymbol{0}}, \dots, a_n^{\boldsymbol{0}})$ is a regular realization of L. If in addition the minimal leaders of L and those of the differential field $K\langle g\rangle$ coincide we say that g is a *principal realization* of L.

6.1. Remark. If g is a principal realization of the differential kernel L, then L is existentially closed in $K\langle g\rangle$ as fields. Indeed, since the minimal leaders of L and $K\langle g\rangle$ coincide, for every $(\xi, i) \in \mathbb{N}^m \times \{1, \dots, n\}$ we have that either $\delta^\xi g_i$ is in L or it is algebraically independent from $K(\delta^\eta g_j : (\eta, j) \lhd (\xi, i))$. In other words, the differential ring generated by g over L, namely $L\{g\}$, is a polynomial ring over L. The claim follows.

In general, it is not the case that every kernel has a principal realization (not even regular). In [7], an upper bound $C_{r,m}^n$ was obtained for the length of a prolongation of a kernel that guarantees the existence of a principal realization. This bound depends only on the data (r, m, n) and is constructed recursively as

$$C_{0,m}^1 = 0, \quad C_{r,m}^1 = A(m-1, C_{r-1,m}^1), \qquad \text{and} \qquad C_{r,m}^n = C^1_{C_{r,m}^{n-1}, m},$$

where $A(x, y)$ is the Ackermann function. For example,

$$C_{r,1}^n = r, \quad C_{r,2}^n = 2^n r \qquad \text{and} \qquad C_{r,3}^1 = 3(2^r - 1).$$

6.2. Fact [7, Theorem 18]. If a differential kernel $L = K(a_i^\xi : (\xi, i) \in \Gamma_n(r))$ of length r has a prolongation of length $C_{r,m}^n$, then there is $r \leq h \leq C_{r,m}^n$ such that the differential kernel $K(a_i^\xi : (\xi, i) \in \Gamma_n(h))$ has a principal realization.

6.3. Remark. Note that in the ordinary case $\Delta = \{\delta\}$ (i.e., $m = 1$), we have $C_{r,1}^n = r$ by definition, and so the fact above shows that in this case every differential kernel has a principal realization (this is a classical result of Lando [13]).

The fact above is the key to our algebraic-geometric axiomatization of differential largeness. We need some additional notation. For a given positive integer n, we set

$$\alpha(n) = n \cdot \binom{C_{1,m}^n + m}{m} \quad \text{and} \quad \beta(n) = n \cdot \binom{C_{1,m}^n - 1 + m}{m}.$$

We write $\pi : \mathbb{U}^{\alpha(n)} \to \mathbb{U}^{\beta(n)}$ for the projection onto the first $\beta(n)$ coordinates; i.e., setting $(x_i^\xi)_{(\xi,i)\in\Gamma_n(C^n_{1,m})}$ to be coordinates for $\mathbb{U}^{\alpha(n)}$ then π is the map

$$(x_i^\xi)_{(\xi,i)\in\Gamma_n(C^n_{1,m})} \mapsto (x_i^\xi)_{(\xi,i)\in\Gamma(C^n_{1,m}-1)}.$$

It is worth noting here that $\alpha(n) = |\Gamma_n(C^n_{1,m})|$ and $\beta(n) = |\Gamma_n(C^n_{1,m} - 1)|$. We also use the projection $\psi : \mathbb{U}^{\alpha(n)} \to \mathbb{U}^{n\cdot(m+1)}$ onto the first $n \cdot (m + 1)$ coordinates, that is,

$$(x_i^\xi)_{(\xi,i)\in\Gamma_n(C^n_{1,m})} \mapsto (x_i^\xi)_{(\xi,i)\in\Gamma_n(1)}.$$

Finally, we use the embedding $\varphi : \mathbb{U}^{\alpha(n)} \to \mathbb{U}^{\beta(n)\cdot(m+1)}$ given by

$$(x_i^\xi)_{(\xi,i)\in\Gamma_n(C^n_{1,m})} \mapsto \big((x_i^\xi)_{(\xi,i)\in\Gamma_n(C^n_{1,m}-1)}, (x_i^{\xi+\mathbf{1}})_{(\xi,i)\in\Gamma_n(C^n_{1,m}-1)}, \ldots\ldots, (x_i^{\xi+\boldsymbol{m}})_{(\xi,i)\in\Gamma_n(C^n_{1,m}-1)}\big).$$

Recall from Definition 2.6 that for a Zariski-constructible set X of $\mathbb{U}^n$, the first prolongation of X is denoted by $\tau X = \tau_1 X \subseteq \mathbb{U}^{n(m+1)}$. For the first prolongation it is easy to give the defining equations: $\tau(X)$ is the Zariski-constructible set given by the conditions

$$x \in X \qquad \text{and} \qquad \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(x) \cdot y_{i,k} + f_j^{\delta_k}(x) = 0 \quad \text{for } 1 \le j \le s,\ 1 \le k \le m,$$

where $f_1, \ldots, f_s$ are generators of the ideal of polynomials over $\mathbb{U}$ vanishing at X, and each $f_j^{\delta_k}$ is obtained by applying δ_k to the coefficients of f_j. Note that $(a, \delta_1 a, \ldots, \delta_m a) \in \tau X$ for all $a \in X$. Further, if X is defined over the differential field K then so is τX.

6.4. Theorem. *Assume K is a differential field that is large as a field. Then, K is differentially large if and only if*

($\diamondsuit$) *for every K-irreducible Zariski-closed set W of $\mathbb{U}^{\alpha(n)}$ with a smooth K-point such that $\varphi(W) \subseteq \tau(\pi(W))$, the set of K-points of $\psi(W)$ of the form $(a, \delta_1 a, \ldots, \delta_m a)$ is Zariski-dense in $\psi(W)$.*

Proof. The proof follows the strategy of [17], but here regular realizations are replaced by principal realizations with the appropriate adaptations. As the set up is technically somewhat intricate we give details.

Assume K is differentially large. Let W be as in condition ($\diamondsuit$), we must find Zariski-dense many K-points in $\psi(W)$ of the form $(a, \delta_1 a, \ldots, \delta_m a)$. Let $b = (b_i^\xi)_{(\xi,i)\in\Gamma_n(C^n_{1,m})}$ be a Zariski-generic point of W over K. Then $(b_i^\xi)_{(\xi,i)\in\Gamma_n(C^n_{1,m}-1)}$ is a Zariski-generic point of $\pi(W)$ over K, and

$$\begin{aligned}\varphi(b) = \big((b_i^\xi)_{(\xi,i)\in\Gamma_n(C^n_{1,m}-1)}, (b_i^{\xi+\mathbf{1}})_{(\xi,i)\in\Gamma_n(C^n_{1,m}-1)}, \ldots, (b_i^{\xi+\boldsymbol{m}})_{(\xi,i)\in\Gamma_n(C^n_{1,m}-1)}\big)\\ \in \tau(\pi(W)).\end{aligned}$$

By the standard argument for extending derivations (see [14, Chapter 7, Theorem 5.1], for instance), there are derivations

$$D'_k : K(b_i^\xi : (\xi, i) \in \Gamma_n(C^n_{1,m} - 1)) \to K(b_i^\xi : (\xi, i) \in \Gamma_n(C^n_{1,m}))$$

for $k = 1, \dots, m$ extending δ_k and such that $D'_k b_i^{\xi} = b_i^{\xi+\boldsymbol{k}}$ for all $(\xi, i) \in \Gamma_n(C^n_{1,m} - 1)$. Thus, $L' = K(b_i^{\xi} : (\xi, i) \in \Gamma_n(C^n_{1,m}))$ is a differential kernel over K and, also, it is a prolongation of length $C^n_{1,m}$ of the differential kernel $L = K(b_i^{\xi} : (\xi, i) \in \Gamma_n(1))$ of length 1 with $D_k = D'_k|_L$. By Fact 6.2, there is $r \le h \le C^n_{1,m}$ such that $L'' = K(b_i^{\xi} : (\xi, i) \in \Gamma_n(h))$ has a principal realization; in particular, there is a differential field extension (M, Δ') of (K, Δ) containing L'' such that $\delta'_k b^{\mathbf{0}} = b^{\boldsymbol{k}}$, where $b^{\mathbf{0}} = (b_1^{\mathbf{0}}, \dots, b_n^{\mathbf{0}})$ and similarly for $b^{\boldsymbol{k}}$. Then

$$(*) \qquad (b^{\mathbf{0}}, \delta'_1 b^{\mathbf{0}}, \dots, \delta'_m b^{\mathbf{0}}) \text{ is a generic point of } \psi(W) \text{ over } K.$$

Now, since W has a smooth K-point and K is large, K is e.c. in L' as fields; in particular, K is e.c. in L'' as fields. By Remark 6.1, L'' is e.c. in the differential field $K\langle b^{\mathbf{0}}\rangle$ as fields, and so K is e.c. in $K\langle b^{\mathbf{0}}\rangle$ as fields. Since K is differentially large, the latter implies that K is e.c. in $K\langle b^{\mathbf{0}}\rangle$ as differential fields as well. The conclusion now follows using $(*)$.

For the converse, assume K is e.c. as a field in a differential field extension F. We must show that K is also e.c. in F as differential field. Let $\rho(x)$ be a quantifier-free formula over K (in the language of differential rings with m derivations) in variables $x = (x_1, \dots, x_t)$ with a realization c in F. We may write

$$\rho(x) = \gamma(\delta^{\xi} x_i : (\xi, i) \in \Gamma_t(r)),$$

where $\gamma((x^{\xi})_{(\xi,i)\in\Gamma_t(r)})$ is a quantifier-free formula in the language of rings over K for some r. If $r = 0$, then ρ is a formula in the language of rings, and so $\rho(x)$ has a realization in K since K is e.c. in F as a field. Now assume $r > 0$. Let $n := t \cdot \binom{r-1+m}{m}$, $d := (\delta^{\xi} c_i)_{(\xi,i)\in\Gamma_t(r-1)}$, and

$$W := \text{Zar-loc}_K(\delta^{\xi} d_i : (\xi, i) \in \Gamma_n(C^n_{1,m})) \subseteq \mathbb{U}^{\alpha(n)}.$$

We have that $\varphi(W) \subseteq \tau(\pi(W))$. Since W has a smooth F-point (namely $(\delta^{\xi} d_i)_{(\xi,i)\in\Gamma_n(C^n_{1,m})}$) and K is e.c. in F as fields, W has a smooth K-point. By $(\diamondsuit)$, there is $a = (a_i^{\xi})_{(\xi,i)\in\Gamma_t(r-1)} \in K^n$ such that $(a, \delta_1 a, \dots, \delta_m a) \in \psi(W)$. This implies that $a_i^{\xi} = \delta^{\xi} a_i^{\mathbf{0}}$ for all $(\xi, i) \in \Gamma_t(r-1)$. Thus,

$$(\delta^{\xi} a_i^{\mathbf{0}})_{(\xi,i)\in\Gamma_t(r)} \in \text{Zar-loc}_K((\delta^{\xi} c_i)_{(\xi,i)\in\Gamma_t(r)}) \subseteq \mathbb{U}^{t\cdot\binom{r+m}{m}},$$

and so, since $(\delta^{\xi} c_i)_{(\xi,i)\in\Gamma_t(r)}$ realizes γ, the point $(\delta^{\xi} a_i^{\mathbf{0}})_{(\xi,i)\in\Gamma_t(r)}$ also realizes γ. Consequently, $K \models \rho(a^{\mathbf{0}})$, as desired. □

In the ordinary case ($m = 1$) we get the values $\alpha(n) = 2n$ and $\beta(n) = n$. Also, in this case, $\pi : \mathbb{U}^{2n} \to \mathbb{U}^n$ is just the projection onto the first n coordinates, and $\psi, \varphi : \mathbb{U}^{2n} \to \mathbb{U}^{2n}$ are both the identity map. We thus get the following:

6.5. Corollary. *Assume that (K, δ) is an ordinary differential field of characteristic zero which is large as a field. Then, (K, δ) is differentially large if and only if*

($\diamondsuit'$) *for every K-irreducible Zariski-closed set W of $\mathbb{U}^{2n}$ with a smooth K-point such that $W \subseteq \tau_\delta(\pi(W))$, the set of K-points of W of the form $(a, \delta a)$ is Zariski dense in W.*

6.6. Remark. (i) If K is algebraically closed of characteristic zero, then Corollary 6.5 yields the classical algebraic-geometric axiomatization of DCF_0 given by Pierce and Pillay in [22].

(ii) If K has a model complete theory T in the language of fields and if K is large, then Corollary 6.5 yields a slight variation of the geometric axiomatization of T_D given by Brouette, Cousins, Pillay and Point in [4, Lemma 1.6].

(iii) For large and topological fields with a single derivation, an alternative description of differentially large fields with reference to the topology may be found in [9].

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