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A categorical Künneth formula for constructible Weil sheaves

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We prove a Künneth-type equivalence of derived categories of lisse and constructible Weil sheaves on schemes in characteristic $p > 0$ for various coefficients, including finite discrete rings, algebraic field extensions $E \supset \mathbb{Q}_\ell$, $\ell \neq p$, and their rings of integers $\mathcal{O}_E$. We also consider a variant for ind-constructible sheaves which applies to the cohomology of moduli stacks of shtukas over global function fields.

1. Introduction

The classical Künneth formula expresses the (co-)homology of a product of two spaces X_1 and X_2 in terms of the tensor product of the (co-)homology of the individual factors. For two topological spaces, for example, one has under suitable finiteness hypothesis an isomorphism

$$\bigoplus_{i+j=n} \mathrm{H}^i(X_1, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathrm{H}^j(X_2, \mathbb{Q}) \cong \mathrm{H}^n(X_1 \times X_2, \mathbb{Q}) \tag{1-1}$$

on singular cohomology with rational coefficients. Such cohomology groups are naturally morphism groups in the derived categories of sheaves on these spaces. So one may ask whether the Künneth formula can be extended to a categorical level, that is, whether it is possible to relate the derived categories of sheaves on X_1 and X_2 to those on their product $X_1 \times X_2$. Statements in this direction are referred to as *categorical Künneth formulas* and are known in different contexts: for example, for the respective derived

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categories of topological sheaves, for D-modules on varieties in characteristic 0 and for quasicoherent sheaves; see [Gaitsgory et al. 2022, Section A.2].

In addition to (1-1) above, categorical Künneth formulas require decomposing a sheaf on $X_1 \times X_2$ into exterior products $M_1 \boxtimes M_2$, with M_1, M_2 being sheaves on X_1, X_2, respectively. For varieties in characteristic $p > 0$, an analogous decomposition for constructible (pro-)étale sheaves fails in general, and so does a categorical Künneth formula in this context; see Example 1.4 below. The main result of the manuscript at hand (see Theorem 1.3) shows how to rectify the failure by adding equivariance data under partial Frobenius morphisms, that is, one arrives at a categorical Künneth formula for constructible Weil sheaves. Our work relies on the analogous result [Drinfeld 1980, Theorem 2.1] for étale fundamental groups known as Drinfeld's lemma; see Section 5C for details and references.

1A. ***Definitions and results.*** Weil sheaves are defined in [Deligne 1980, Definition 1.1.10]. We start by explaining a site-theoretic approach which slightly differs from [Geisser 2004; Lichtenbaum 2005].

Let X be a scheme over a finite field $\mathbb{F}_q$, where q is a p-power. Fix an algebraic closure $\mathbb{F}/\mathbb{F}_q$, and denote by $X_{\mathbb{F}}$ the base change. The partial (q-)Frobenius $\phi_X := \mathrm{Frob}_X \times \mathrm{id}_{\mathbb{F}}$ defines an endomorphism of $X_{\mathbb{F}}$.

Definition 1.1. The *Weil-proétale site* $X^{\mathrm{Weil}}_{\mathrm{proét}}$ is the following site: Objects are pairs (U, φ) consisting of $U \in (X_{\mathbb{F}})_{\mathrm{proét}}$, the proétale site of $X_{\mathbb{F}}$ [Bhatt and Scholze 2015], equipped with an endomorphism $\varphi\colon U \to U$ of $\mathbb{F}$-schemes covering ϕ_X. Morphisms are given by equivariant maps. A family $\{(U_i, \varphi_i) \to (U, \varphi)\}$ of morphisms is a cover if the family $\{U_i \to U\}$ is a cover in $(X_{\mathbb{F}})_{\mathrm{proét}}$.

The Weil-proétale site sits in the sequence of sites

$$(X_{\mathbb{F}})_{\mathrm{proét}} \to X^{\mathrm{Weil}}_{\mathrm{proét}} \to X_{\mathrm{proét}} \tag{1-2}$$

given by the functors $U \leftarrow\!\shortmid (U, \varphi)$ and $(U_{\mathbb{F}}, \phi_U) \leftarrow\!\shortmid U$ in the opposite direction. The maps (1-2) commute over $*_{\mathrm{proét}}$, the proétale site of the point. Thus, for any condensed ring Λ viewed as a sheaf of rings on $*_{\mathrm{proét}}$, we get pullback functors on derived categories of proétale Λ-sheaves

$$\mathrm{D}(X, \Lambda) \to \mathrm{D}(X^{\mathrm{Weil}}, \Lambda) \to \mathrm{D}(X_{\mathbb{F}}, \Lambda).$$

In analogy with the definition of lisse and constructible sheaves (as recalled in Definition 3.1), we introduce the categories of lisse and constructible Weil sheaves $\mathrm{D}_{\mathrm{lis}}(X^{\mathrm{Weil}}, \Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X^{\mathrm{Weil}}, \Lambda)$ as the full subcategories of $\mathrm{D}(X^{\mathrm{Weil}}, \Lambda)$ that are dualizable, resp. that are Zariski locally on X dualizable along a constructible stratification. These categories are equivalent to the corresponding categories of sheaves on the prestack $X_{\mathbb{F}}/\phi_X$, that is, equivalent to the homotopy fixed points of the induced ϕ_X^*-action.

Proposition 1.2 (Propositions 4.4 and 4.11). *The pullback of sheaves along* $(X_{\mathbb{F}})_{\mathrm{proét}} \to X^{\mathrm{Weil}}_{\mathrm{proét}}$ *induces an equivalence of* Λ_**-linear symmetric monoidal stable* ∞*-categories*

$$\mathrm{D}_\bullet(X^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} \mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda)^{\phi_X^* = \mathrm{id}},$$

for $\bullet \in \{\varnothing, \mathrm{lis}, \mathrm{cons}\}$.

Thus, objects in $\mathrm{D}_\bullet(X^{\mathrm{Weil}}, \Lambda)$ are pairs (M, α) with $M \in \mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda)$ and $\alpha\colon M \cong \phi_X^* M$. On the abelian level, we recover the classical approach [Deligne 1980, Definition 1.1.10]. If Λ is a finite discrete ring, then every Weil descent datum on constructible Λ-sheaves is effective so that $\mathrm{D}_{\mathrm{cons}}(X^{\mathrm{Weil}}, \Lambda) \cong \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$; see Proposition 4.16. However, the categories are not equivalent if $\Lambda = \mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Q}_\ell$, say. This relates to the difference between continuous representations of Galois groups such as $\hat{\mathbb{Z}}$ versus Weil groups such as $\mathbb{Z}$.

For several $\mathbb{F}_q$-schemes $X_1, \dots, X_n$, a similar process is carried out for their product $X := X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n$ equipped with the partial Frobenii $\phi_{X_i}\colon X_{\mathbb{F}} \to X_{\mathbb{F}}$, see Section 4B. Generalizing Proposition 1.2, there is an equivalence of Λ_*-linear symmetric monoidal stable ∞-categories

$$\mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} \mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda)^{\phi_{X_1}^*=\mathrm{id}, \dots, \phi_{X_n}^*=\mathrm{id}} \tag{1-3}$$

for $\bullet \in \{\varnothing, \mathrm{lis}, \mathrm{cons}\}$. The category on the left is defined using the Weil-proétale site $(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}})_{\text{proét}}$ consisting of objects $(U, \varphi_1, \dots, \varphi_n)$ with $U \in (X_{\mathbb{F}})_{\text{proét}}$ and pairwise commuting endomorphisms $\varphi_i\colon U \to U$ covering the partial Frobenii $\phi_{X_i}\colon X_{\mathbb{F}} \to X_{\mathbb{F}}$ for all $i = 1, \dots, n$. The category on the right is the category of simultaneous homotopy fixed points; see Section 2B. For constructible Weil sheaves, (1-3) relies on decompositions of partial Frobenius invariant cycles in $X_{\mathbb{F}}$; see Proposition 4.8.

The following result is referred to as the categorical Künneth formula for constructible Weil sheaves (or, derived Drinfeld's lemma).

Theorem 1.3 (Theorem 5.2, Remark 5.3). *Let $\mathbb{F}_q$ be a finite field of characteristic $p > 0$. Let $X_1, \dots, X_n$ be finite type $\mathbb{F}_q$-schemes. Let Λ be either a finite discrete ring of prime-to-p torsion, or an algebraic field extension $E \supset \mathbb{Q}_\ell$, $\ell \neq p$, or its ring of integers $\mathcal{O}_E$.*

Then the external tensor product of sheaves $(M_1, \dots, M_n) \mapsto M_1 \boxtimes \cdots \boxtimes M_n$ induces an equivalence

$$\mathrm{D}_{\mathrm{cons}}(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \cdots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}(X_n^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} \mathrm{D}_{\mathrm{cons}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda), \tag{1-4}$$

and likewise for the categories of lisse Weil sheaves if, in the case $\Lambda = E$, one assumes the schemes $X_1, \dots, X_n$ to be geometrically unibranch (for example, normal).

This statement can also be recast as the symmetric monoidality of the functor sending a Weil prestack X^{Weil}, which is defined on R-points by $X^{\mathrm{Weil}}(R) := \operatorname{colim}(X(R) \underset{\phi_X}{\overset{\mathrm{id}}{\rightrightarrows}} X(R))$, to its ∞-category of constructible sheaves (Theorem 5.6).

The tensor product of ∞-categories (see Section 2) is formed using the natural Λ_*-linear structures on the categories. We have an analogous equivalence for the categories of lisse Weil sheaves with coefficients Λ in finite discrete p-torsion rings like $\mathbb{Z}/p^m$, $m \geq 1$, see Theorem 5.2. As the following example shows, the use of Weil sheaves is necessary for the essential surjectivity to hold. This behavior is mentioned in the first arXiv version of [Gaitsgory et al. 2022, (0.8)] which is one of the main motivations for our work.

Example 1.4 (compare [SGA 1 2003, Exposé X, Section 1, Remarques 1.10]). Let $X_{1,\mathbb{F}} = X_{2,\mathbb{F}} = \mathbb{A}^1_{\mathbb{F}}$ be the affine line so that $X_{\mathbb{F}} = \mathbb{A}^2_{\mathbb{F}}$ with coordinates denoted by x_1 and x_2. Then

$$U := \{t^p - t = x_1 \cdot x_2\} \longrightarrow \mathbb{A}^2_{\mathbb{F}}$$

defines a finite étale cover with Galois group $\mathbb{Z}/p$. Let $M \in \mathrm{D}_{\mathrm{lis}}(\mathbb{A}^2_{\mathbb{F}}, \Lambda)$ be the sheaf in degree 0 associated with some nontrivial character $\mathbb{Z}/p \to \Lambda_*^\times$. For $\lambda, \mu \in \mathbb{F}$ not differing by a scalar in $\mathbb{F}_p^\times$, the fibers $U|_{\{x_1=\lambda\}}$, $U|_{\{x_1=\mu\}}$ are not isomorphic over $\mathbb{A}^1_{\mathbb{F}}$ by Artin–Schreier theory. Hence, $M \not\cong \phi_{X_i}^* M$ and one can show that $M \not\cong M_1 \boxtimes M_2$ for any $M_i \in \mathrm{D}(\mathbb{A}^1_{\mathbb{F}}, \Lambda)$.

If Λ as above is p-torsion free, then the full faithfulness of (1-4) is a direct consequence of the Künneth formula for $X_{i,\mathbb{F}}$, $i = 1, \dots, n$. For $\Lambda = \mathbb{Z}/p^m$, we use Artin–Schreier theory instead. It would be interesting to see whether the lisse p-torsion case can be extended to constructible sheaves. In both cases, the essential surjectivity relies on a variant of Drinfeld's lemma for Weil group representations, see Theorem 5.9, together with a characterization of partial-Frobenius stable algebraic cycles (Proposition 4.8) as well as a decomposition argument for representations of a product of abstract groups (Proposition 5.12).

With a view towards [Lafforgue 2018], we consider Weil sheaves whose underlying sheaf is ind-constructible, but where the action of the partial Frobenii do not necessarily preserve the constructible pieces. For finite type $\mathbb{F}_q$-schemes $X_1, \dots, X_n$ and Λ as in Theorem 1.3, we consider the category of simultaneous homotopy fixed points

$$\mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda) \overset{\mathrm{def}}{=} \mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda)^{\phi_{X_1}^*=\mathrm{id}, \dots, \phi_{X_n}^*=\mathrm{id}}$$

for $\bullet \in \{\mathrm{indlis}, \mathrm{indcons}\}$. Then the external tensor product induces a fully faithful functor

$$\mathrm{D}_\bullet(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Mod}_{\Lambda_*}} \cdots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{D}_\bullet(X_n^{\mathrm{Weil}}, \Lambda) \longrightarrow \mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda). \tag{1-5}$$

Unlike the case of lisse or constructible sheaves, the functor is not essentially surjective as one can add freely actions by the partial Frobenii, see Remark 6.6. However, we can identify a large class of objects in the essential image of (1-5). When combined with the smoothness results of Xue [2020c, Theorem 4.2.3], we obtain, for example, that the compactly supported cohomology of moduli stacks of shtukas over global function fields lies in the essential image of (1-5); see Section 6B for details.

Remark 1.5. Another motivation for this work is our (Richarz and Scholbach's) ongoing project aiming for a motivic refinement of [Lafforgue 2018]. In this project, we will need a motivic variant of Drinfeld's lemma. Since triangulated categories of motives such as $\mathrm{DM}(X, \mathbb{Q})$ carry t-structures only conditionally, we need a Drinfeld lemma to be a statement about triangulated categories. In conjunction with the conjecture relating Weil-étale motivic cohomology to Weil-étale cohomology [Kahn 2003; Geisser 2004; Lichtenbaum 2005], our results suggest to look for a Drinfeld lemma for constructible Weil motives.

2. Recollections on ∞-categories

Throughout this section, Λ denotes a unital, commutative ring. We briefly collect some notation pertaining to ∞-categories from [Lurie 2017; 2009]. As in [Lurie 2009, Section 5.5.3], Pr^{L} denotes the ∞-category of presentable ∞-categories with colimit-preserving functors. It contains the subcategory $\mathrm{Pr}^{\mathrm{St}} \subset \mathrm{Pr}^{\mathrm{L}}$ consisting of stable ∞-categories.

2A. ***Monoidal aspects.*** The category Pr^{L} carries the Lurie tensor product [Lurie 2017, Section 4.8.1]. This tensor product induces one on the full subcategory $\mathrm{Pr}^{\mathrm{St}} \subset \mathrm{Pr}^{\mathrm{L}}$ consisting of stable ∞-categories [loc. cit., Proposition 4.8.2.18]. For our commutative ring Λ, the ∞-category Mod_Λ of chain complexes of Λ-modules, up to quasiisomorphism, is a commutative monoid in $\mathrm{Pr}^{\mathrm{St}}$ with respect to this tensor product. This structure includes, in particular, the existence of a functor

$$\mathrm{Mod}_\Lambda \times \mathrm{Mod}_\Lambda \to \mathrm{Mod}_\Lambda$$

which, after passing to the homotopy categories is the classical *derived* tensor product on the unbounded derived category of Λ-modules.

We define $\mathrm{Pr}^{\mathrm{St}}_\Lambda$ to be the category of modules, in $\mathrm{Pr}^{\mathrm{St}}$, over Mod_Λ. Noting that modules over Mod_Λ are in particular modules over Sp, the ∞-category of spectra, $\mathrm{Pr}^{\mathrm{St}}_\Lambda$ can be described as the ∞-category consisting of *stable* presentable ∞-categories together with a Λ-linear structure, such that functors are continuous and Λ-linear. Therefore $\mathrm{Pr}^{\mathrm{St}}_\Lambda$ carries a symmetric monoidal structure, whose unit is Mod_Λ. We will also denote by $\mathrm{Pr}^{\mathrm{St}}_\omega$ the category of compactly generated presentable with functors that send compact objects to compact objects (equivalently, those whose right adjoint is continuous).

In order to express monoidal properties of ∞-categories consisting, say, of bounded complexes, recall from [Lurie 2017, Corollary 4.8.1.4 joint with Lemma 5.3.2.11] or [Ben-Zvi et al. 2010, Proposition 4.4] the symmetric monoidal structure on the ∞-category $\mathrm{Cat}^{\mathrm{Ex}}_\infty(\mathrm{Idem})$ of idempotent complete stable ∞-categories and exact functors: it is characterized by

$$D_1 \otimes D_2 \overset{\mathrm{def}}{=} (\mathrm{Ind}(D_1) \otimes \mathrm{Ind}(D_2))^\omega, \tag{2-1}$$

that is, the compact objects in the Lurie tensor product of the Ind-completions. With respect to these monoidal structures, the Ind-completion functor (taking values in compactly generated presentable ∞-categories with the Lurie tensor product) and the functor forgetting the compact generatedness

$$\mathrm{Cat}^{\mathrm{Ex}}_\infty(\mathrm{Idem}) \xrightarrow[\mathrm{Ind}]{\cong} \mathrm{Pr}^{\mathrm{St}}_\omega \longrightarrow \mathrm{Pr}^{\mathrm{St}} \tag{2-2}$$

are both symmetric monoidal [Lurie 2017, Lemmas 5.3.2.9, 5.3.2.11].

The subcategory of compact objects in Mod_Λ is given by perfect complexes of Λ-modules [loc. cit., Proposition 7.2.4.2.]. It is denoted Perf_Λ. Under the equivalence in (2-2), the category $\mathrm{Perf}_\Lambda \in \mathrm{Cat}^{\mathrm{Ex}}_\infty(\mathrm{Idem})$ corresponds to Mod_Λ. Moreover, Perf_Λ is a commutative monoid in $\mathrm{Cat}^{\mathrm{Ex}}_\infty(\mathrm{Idem})$, so that we can consider its category of modules, denoted as $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem})$. This category inherits a symmetric monoidal structure denoted by $D_1 \otimes_{\mathrm{Perf}_\Lambda} D_2$.

Any stable ∞-category D is canonically enriched over the category of spectra Sp. We write $\mathrm{Hom}_D(\cdot\,,\cdot)$ for the mapping spectrum. Any category in $\mathrm{Pr}^{\mathrm{St}}_\Lambda$ is canonically enriched over Mod_Λ, so that we refer to $\mathrm{Hom}_D(\cdot\,,\cdot) \in \mathrm{Mod}_\Lambda$ as the mapping complex. For example, for $M, N \in \mathrm{Mod}_\Lambda$, then $\mathrm{Hom}_{\mathrm{Mod}_\Lambda}(M, N)$ is commonly also denoted by $\mathrm{RHom}(M, N)$. Its n-th cohomology is the Hom-group $\mathrm{Hom}(M, N[n])$ in the classical derived category.

2B. ***Fixed points of ∞-categories.*** A basic structure in Drinfeld's lemma is the equivariance datum for the partial Frobenii. In this section, we assemble some abstract results where such ∞-categorical constructions are carried out.

Definition 2.1. Let $\phi\colon D \to D$ be an endofunctor in $\mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem})$. The category of *$\phi$-fixed points* is

$$D^{\phi=\mathrm{id}} \overset{\mathrm{def}}{=} \mathrm{Fix}(D,\phi) \overset{\mathrm{def}}{=} \lim(D \underset{\phi}{\overset{\mathrm{id}_D}{\rightrightarrows}} D).$$

Recall that for a symmetric monoidal ∞-category D, a commutative monoid object $\Lambda \in \mathrm{CAlg}(D)$, the forgetful functors $\mathrm{CAlg}(D) \to D$ and $\mathrm{Mod}_\Lambda(D) \to D$ preserve limits [Lurie 2017, Corollaries 3.2.2.5 and 4.2.3.3]. In particular, if D is in addition Λ-linear, that is, an object in $\mathrm{Cat}_{\infty,\Lambda}^{\mathrm{Ex}}(\mathrm{Idem})$, and ϕ is also Λ-linear, then $\mathrm{Fix}(D,\phi)$ admits a natural Λ-linear structure as well.

Because of these facts, we will usually not specify where the limit above is formed. Note that all functors

$$\mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem}) \xrightarrow[\mathrm{Ind}]{\cong} \mathrm{Pr}_\omega^{\mathrm{St}} \xrightarrow{(*)} \mathrm{Pr}^{\mathrm{St}} \longrightarrow \mathrm{Pr}^{\mathrm{L}} \longrightarrow \widehat{\mathrm{Cat}_\infty} \tag{2-3}$$

except for the forgetful functor marked $(*)$ preserve limits; see [Lurie 2017, Corollary 4.2.3.3; 2009, Proposition 5.5.3.13] for the rightmost two functors. To give a concrete example of that failure in our situation, note that $\mathrm{Fix}(D,\mathrm{id}_D) = \mathrm{Fun}(B\mathbb{Z}, D)$, that is, objects are pairs (M,α) consisting of some $M \in D$ and some automorphism $\alpha\colon M \cong M$. Now consider $D = \mathrm{Vect}_\Lambda^{\mathrm{fd}}$, the (abelian) category of finite-dimensional vector spaces over a field Λ. The natural functor

$$\mathrm{Ind}\big(\lim(\mathrm{Vect}_\Lambda^{\mathrm{fd}} \rightrightarrows \mathrm{Vect}_\Lambda^{\mathrm{fd}})\big) \to \lim\big(\mathrm{Ind}(\mathrm{Vect}_\Lambda^{\mathrm{fd}}) \rightrightarrows \mathrm{Ind}(\mathrm{Vect}_\Lambda^{\mathrm{fd}})\big) = \lim\big(\mathrm{Vect}_\Lambda \rightrightarrows \mathrm{Vect}_\Lambda\big)$$

is fully faithful, but *not* essentially surjective: given an automorphism α of an infinite-dimensional vector space M, there need not be a filtration $M = \bigcup M_i$ by finite-dimensional subspaces M_i that is compatible with α.

Fixed point categories inherit t-structures as follows:

Lemma 2.2. *Let $\phi\colon D \to D$ be a functor in $\mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem})$. Suppose D carries a t-structure such that ϕ is t-exact. Then $\mathrm{Fix}(D,\phi)$ carries a unique t-structure such that the evaluation functor is t-exact. There is a natural equivalence*

$$\mathrm{Fix}(D^\heartsuit,\phi) \xrightarrow{\cong} \mathrm{Fix}(D,\phi)^\heartsuit.$$

Proof. Let us abbreviate $\widetilde{D} := \mathrm{Fix}(D,\phi)$. For $\bullet$ being either "≤ 0" or "≥ 0", we put $\widetilde{D}^\bullet := \mathrm{Fix}(D^\bullet,\phi)$, which is a (nonstable) ∞-category. This is clearly the only choice for a t-structure making ev a t-exact functor. It satisfies the claim about the hearts of the t-structure by definition.

We need to show that it is a t-structure. Being a limit of full subcategories, the categories $\widetilde{D}^\bullet$ are full subcategories of $\widetilde{D}$. Since ϕ, being t-exact, commutes with $\tau_D^{\le 0}$ and $\tau_D^{\ge 0}$, these two functors also yield truncation functors for $\widetilde{D}$. For $M \in \widetilde{D}^{\le 0}$, $N \in \widetilde{D}^{\ge 1}$ (we use cohomological conventions), we have

$$\mathrm{Hom}_{\widetilde{D}}(M,N) = \lim(\mathrm{Hom}_D(M,N) \rightrightarrows \mathrm{Hom}_D(M,N)),$$

where on the right hand side M, N denote the underlying objects in D. Since $M \in D^{\leq 0}$, $N \in D^{\geq 1}$, we have $\mathrm{H}^i \operatorname{Hom}_D(M, N) = 0$ for $i = -1, 0$. Thus, $\mathrm{H}^0 \operatorname{Hom}_{\widetilde{D}}(M, N) = 0$ as well. □

Definition 2.1 can be generalized as follows: Let $\varphi\colon B\mathbb{Z}^n \to \mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem})$ be a diagram. For example, for $n = 1$, this amounts to giving $D = \varphi(*) \in \mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem})$ and an equivalence $\phi = \varphi(1)\colon D \to D$. For $n = 2$, such a datum corresponds to giving D, equivalences $\phi_1, \phi_2\colon D \xrightarrow{\cong} D$ together with an equivalence $\phi_1 \circ \phi_2 \xrightarrow{\cong} \phi_2 \circ \phi_1$. So we define the ∞-category of *simultaneous fixed points* as

$$\mathrm{Fix}(D, \phi_1, \dots, \phi_n) \stackrel{\mathrm{def}}{=} \lim \varphi \in \mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem}).$$

Remark 2.3. The statement of Lemma 2.2 carries over verbatim assuming that D has a t-structure and all ϕ_i are t-exact, noting that $B\mathbb{Z}^n = (S^1)^n$ is a finite simplicial set.

Lemma 2.4. *Let $\varphi\colon B\mathbb{Z}^n \to \mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem})$ be a diagram. Denote $D = \varphi(*)$ and $\phi_i = \varphi(e_i)$ for the i-th standard vector $e_i \in \mathbb{Z}^n$. The functor*

$$\mathrm{Fix}(D, \phi_1, \dots, \phi_n) \to \mathrm{Fix}(\mathrm{Ind}(D), \phi_1, \dots, \phi_n)$$

induced from the inclusion $D \subset \mathrm{Ind}(D)$ is fully faithful and takes values in compact objects. In particular, it yields a fully faithful functor

$$\mathrm{Ind}(\mathrm{Fix}(D, \phi_1, \dots, \phi_n)) \to \mathrm{Fix}(\mathrm{Ind}(D), \phi_1, \dots, \phi_n).$$

Proof. Let $M \in \mathrm{Fix}(D, \phi_1, \dots, \phi_n)$ and denote its underlying object in D by the same symbol. For every $N \in \mathrm{Fix}(\mathrm{Ind}(D), \phi_1, \dots,)$, we have a limit diagram of mapping complexes

$$\operatorname{Hom}_{\mathrm{Fix}(\mathrm{Ind}(D))}(M, N) \cong \mathrm{Fix}(\operatorname{Hom}_{\mathrm{Ind}(D)}(M, N), \phi_1, \dots, \phi_n).$$

Since filtered colimits commute with finite limits in the ∞-category of anima (a.k.a. spaces) [Lurie 2009, Proposition 5.3.3.3.], we see that M is compact in $\mathrm{Fix}(\mathrm{Ind}(D))$ because M is so in $\mathrm{Ind}(D)$. □

Lemma 2.5. *Let $\varphi_i\colon B\mathbb{Z} \to \mathrm{Cat}_{\infty,\Lambda}^{\mathrm{Ex}}(\mathrm{Idem})$, $i = 1, \dots, n$ be given. Denote $D_i = \varphi_i(*)$, $\phi_i = \varphi_i(1)$ and $\widetilde{D}_i = \mathrm{Ind}(D_i)$. Then there is a canonical equivalence*

$$\mathrm{Fix}(\widetilde{D}_1, \phi_1) \otimes_{\mathrm{Mod}_\Lambda} \cdots \otimes_{\mathrm{Mod}_\Lambda} \mathrm{Fix}(\widetilde{D}_n, \phi_n) \xrightarrow{\cong} \mathrm{Fix}(\widetilde{D}_1 \otimes_{\mathrm{Mod}_\Lambda} \cdots \otimes_{\mathrm{Mod}_\Lambda} \widetilde{D}_n, \phi_1, \dots, \phi_n).$$

Proof. The categories $\mathrm{Fix}(\widetilde{D}_i, \phi_i)$ are compactly generated: the forgetful functor $U : \mathrm{Fix}(\widetilde{D}_i, \phi_i) \to \widetilde{D}_i = \mathrm{Ind}(D_i)$ preserves colimits, so its left adjoint L preserves compact objects. Moreover, U is conservative, so that the objects $L(d_i)$, for $d_i \in D_i$, form a family of compact generators. Then, we use that any compactly generated category in $\mathrm{Pr}_\Lambda^{\mathrm{St}}$ is dualizable [Lurie 2018, Remark D.7.7.6(1)] so that tensoring with it preserves limits. □

3. Lisse and constructible sheaves

In order to state and prove the categorical Künneth formula for Weil sheaves, we use the framework for lisse and constructible sheaves provided by [Hemo et al. 2023]. For the convenience of the reader, we collect here some basics of the formalism.

Throughout, Λ denotes a condensed ring, for example any T1-topological ring such as discrete rings, algebraic extensions $E/\mathbb{Q}_\ell$ or their ring of integers $\mathcal{O}_E$. In the synopsis below, we refer to the latter choices of Λ as the *standard coefficient rings*. We write Λ_* for the underlying ring. Let $\mathrm{D}(X, \Lambda)$ be the derived category of sheaves of Λ-modules on the proétale site $X_{\text{proét}}$.

Definition 3.1 [Hemo et al. 2023, Definitions 3.3 and 8.1]. For every scheme X and every condensed ring Λ, there are the full subcategories

$$\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X, \Lambda) \subset \mathrm{D}(X, \Lambda). \tag{3-1}$$

By definition, the left hand category of *lisse sheaves* consists of the dualizable objects in the right-most category. An object (henceforth referred to as a sheaf) M in the right hand category is *constructible*, if on any affine $U \subset X$ there is a finite stratification into constructible locally closed subschemes $U_i \subset U$ such that $M|_{U_i}$ is lisse, that is, dualizable. Finally, an *ind-lisse* (respectively, *ind-constructible*) sheaf is a filtered colimit, in the category $\mathrm{D}(X, \Lambda)$, of lisse (respectively, constructible) sheaves. The corresponding full subcategories of $\mathrm{D}(X, \Lambda)$ are denoted by

$$\mathrm{D}_{\mathrm{indlis}}(X, \Lambda) \subset \mathrm{D}_{\mathrm{indcons}}(X, \Lambda) \subset \mathrm{D}(X, \Lambda).$$

For the standard coefficient rings Λ above and quasicompact quasiseparated (qcqs) schemes X, that definition of lisse and constructible sheaves agrees with the classical ones; see [Hemo et al. 2023] for details.

The categories enjoy the following properties:

Synopsis 3.2. (i) Via the natural functor $\mathrm{Mod}_{\Lambda_*} \to \mathrm{D}(X, \Lambda)$, $M \mapsto \underline{M} \otimes_{\underline{\Lambda_*}} \Lambda_X$; see around [Hemo et al. 2023, (3.1)], the category $\mathrm{D}(X, \Lambda)$ is an object in $\mathrm{Pr}^{\mathrm{St}}_{\Lambda_*}$. The functor restricts to a functor $\mathrm{Perf}_{\Lambda_*} \to \mathrm{D}_{\mathrm{lis}}(X, \Lambda)$, and the categories $\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ are objects in $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem})$. In particular, all categories listed in (3-1) are stable idempotent complete Λ_*-linear ∞-categories.

(ii) The extension-by-zero functor along any constructible locally closed immersion and quasicompact étale morphisms preserves constructibility; see [loc. cit., Lemma 3.4, Corollary 4.6].

(iii) The functors $X \mapsto \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ and $X \mapsto \mathrm{D}_{\mathrm{lis}}(X, \Lambda)$ satisfy proétale hyperdescent [loc. cit., Corollary 4.7]. (According to [Hansen and Scholze 2023, Theorem 2.2], it also satisfies v-descent, but we will not need this in this paper.) The functor $X \mapsto \mathrm{D}_{\mathrm{indcons}}(X, \Lambda)$, resp. $X \mapsto \mathrm{D}_{\mathrm{indlis}}(X, \Lambda)$ satisfies hyperdescent for quasicompact étale, resp. finite étale covers; see [Hemo et al. 2023, Corollary 8.7].

(iv) If $\Lambda = \operatorname{colim} \Lambda_i$ is a filtered colimit of condensed rings and X is qcqs, then the natural functors

$$\operatorname{colim} \mathrm{D}_{\mathrm{lis}}(X, \Lambda_i) \xrightarrow{\cong} \mathrm{D}_{\mathrm{lis}}(X, \Lambda), \quad \operatorname{colim} \mathrm{D}_{\mathrm{cons}}(X, \Lambda_i) \xrightarrow{\cong} \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$$

are equivalences [loc. cit., Proposition 5.2].

(v) If X is qcqs, then any constructible sheaf is bounded with respect to the t-structure on $\mathrm{D}(X, \Lambda)$ [loc. cit., Corollary 4.11].

(vi) For X locally Noetherian (and much more generally), the t-structure on $\mathrm{D}(X, \Lambda)$ restricts to one on $\mathrm{D}_{\mathrm{lis}}(X, \Lambda)$ and $\mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ provided that Λ is t-admissible in the sense of [loc. cit., Definition 6.1]. Here, t-admissibility is a combination of an algebraic and a topological condition: first, Λ_* needs to be regular coherent (for example, any regular Noetherian ring of finite Krull dimension, but $\mathbb{Z}/\ell^2$ is excluded). The topological condition on the condensed structure of Λ is satisfied for all the standard coefficient rings listed above; see [loc. cit., Theorem 6.2].

(vii) For X locally Noetherian (and again more generally), a sheaf is lisse if and only if it is proétale locally the constant sheaf associated to a perfect complex of Λ_*-modules; see [loc. cit., Theorem 4.13].

(viii) Let X be a qcqs scheme. If the Λ-cohomological dimension is uniformly bounded for all proétale affines $U = \lim_i U_i$ over X, then $\mathrm{Ind}(\mathrm{D}_{\mathrm{cons}}(X, \Lambda)) = \mathrm{D}_{\mathrm{indcons}}(X, \Lambda)$ and likewise for ind-lisse sheaves. If X is of finite type over $\mathbb{F}_q$ or a separably closed field, this condition holds for any of the above standard rings. For discrete p-torsion rings, algebraic extensions $E/\mathbb{Q}_p$ and their ring of integers $\mathcal{O}_E$, this holds for arbitrary qcqs schemes in characteristic p; see [loc. cit., Lemma 8.6, Proposition 8.2].

For schemes $X_1, \dots, X_n$ over a fixed base scheme S (for example, the spectrum of a field) and a condensed ring Λ, we denote the external product in the usual way

$$\boxtimes\colon \mathrm{D}(X_1, \Lambda) \times \cdots \times \mathrm{D}(X_n, \Lambda) \to \mathrm{D}(X_1 \times_S \cdots \times_S X_n, \Lambda),$$
$$(M_1, \dots, M_n) \mapsto M_1 \boxtimes \cdots \boxtimes M_n := p_1^*(M_1) \otimes_{\Lambda_X} \cdots \otimes_{\Lambda_X} p_n^*(M_n).$$

Here $p_i\colon X := X_1 \times_S \cdots \times_S X_n \to X_i$ are the projections. This functor induces the functor

$$\boxtimes\colon \mathrm{D}(X_1, \Lambda) \otimes_{\mathrm{Mod}_{\Lambda_*}} \cdots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{D}(X_n, \Lambda) \to \mathrm{D}(X_1 \times_S \cdots \times_S X_n, \Lambda), \tag{3-2}$$

in $\mathrm{Pr}^{\mathrm{St}}_{\Lambda_*}$. Here we regard $\mathrm{D}(X_i, \Lambda)$ as objects in $\mathrm{Pr}^{\mathrm{St}}_{\Lambda_*}$, like in (i) in the synopsis above. The external tensor product of constructible sheaves is again constructible, and hence induces a functor

$$\boxtimes\colon \mathrm{D}_{\mathrm{cons}}(X_1, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \cdots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}(X_n, \Lambda) \to \mathrm{D}_{\mathrm{cons}}(X_1 \times_S \cdots \times_S X_n, \Lambda), \tag{3-3}$$

in $\mathrm{Cat}^{\mathrm{Ex}}_{\infty, \Lambda_*}(\mathrm{Idem})$ and likewise for the categories of ind-constructible, resp. (ind-)lisse sheaves.

4. Weil sheaves

In this section, we introduce the categories

$$\mathrm{D}_{\mathrm{lis}}(X^{\mathrm{Weil}}, \Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X^{\mathrm{Weil}}, \Lambda) \subset \mathrm{D}(X^{\mathrm{Weil}}, \Lambda)$$

consisting of lisse, resp. constructible, resp. all Weil sheaves. These are the categories featuring in the categorical Künneth formula (Theorem 1.3).

Throughout this section, X is a scheme over a finite field $\mathbb{F}_q$ of characteristic $p > 0$. Unless the contrary is mentioned, we impose no conditions on X. Moreover, Λ is a condensed ring. We fix an algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$, and denote by $X_{\mathbb{F}} := X \times_{\mathbb{F}_q} \operatorname{Spec} \mathbb{F}$ the base change. Denote by ϕ_X (resp. $\phi_{\mathbb{F}}$) the endomorphism of $X_{\mathbb{F}}$ that is the q-Frobenius on X (resp. $\operatorname{Spec} \mathbb{F}$) and the identity on the other factor.

Let

$$\mathrm{D}_{\mathrm{lis}}(X_{\mathbb{F}}, \Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X_{\mathbb{F}}, \Lambda) \subset \mathrm{D}(X_{\mathbb{F}}, \Lambda)$$

be the categories of lisse, resp. constructible, resp. all proétale sheaves of Λ-modules on $X_{\mathbb{F}}$ (Definition 3.1). These categories are objects in $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda_*}(\mathrm{Idem})$, that is, Λ_*-linear stable idempotent complete symmetric monoidal ∞-categories where $\Lambda_* = \Gamma(*, \Lambda)$ is the underlying ring.

4A. ***The Weil-proétale site.*** The Weil-étale topology for schemes over finite field is introduced in [Lichtenbaum 2005]; see also [Geisser 2004]. Our approach for the proétale topology is slightly different:

Definition 4.1. The *Weil-proétale site* of X, denoted by $X^{\mathrm{Weil}}_{\mathrm{pro\acute{e}t}}$, is the following site: Objects in $X^{\mathrm{Weil}}_{\mathrm{pro\acute{e}t}}$ are pairs (U, φ) consisting of $U \in (X_{\mathbb{F}})_{\mathrm{pro\acute{e}t}}$ equipped with an endomorphism $\varphi\colon U \to U$ of $\mathbb{F}$-schemes such that the map $U \to X_{\mathbb{F}}$ intertwines φ and ϕ_X. Morphisms in $X^{\mathrm{Weil}}_{\mathrm{pro\acute{e}t}}$ are given by equivariant maps, and a family $\{(U_i, \varphi_i) \to (U, \varphi)\}$ of morphisms is a cover if the family $\{U_i \to U\}$ is a cover in $(X_{\mathbb{F}})_{\mathrm{pro\acute{e}t}}$.

Note that $X^{\mathrm{Weil}}_{\mathrm{pro\acute{e}t}}$ admits small limits formed componentwise as $\lim(U_i, \varphi_i) = (\lim U_i, \lim \varphi_i)$. In particular, there are limit-preserving maps of sites

$$(X_{\mathbb{F}})_{\mathrm{pro\acute{e}t}} \to X^{\mathrm{Weil}}_{\mathrm{pro\acute{e}t}} \to X_{\mathrm{pro\acute{e}t}} \tag{4-1}$$

given by the functors (in the opposite direction) $U \leftarrow\!\shortmid (U, \varphi)$ and $(U_{\mathbb{F}}, \phi_U) \leftarrow\!\shortmid U$. We denote by $\mathrm{D}(X^{\mathrm{Weil}}, \Lambda)$ the unbounded derived category of sheaves of Λ_X-modules on $X^{\mathrm{Weil}}_{\mathrm{pro\acute{e}t}}$. The maps of sites (4-1) induce functors

$$\mathrm{D}(X, \Lambda) \to \mathrm{D}(X^{\mathrm{Weil}}, \Lambda) \to \mathrm{D}(X_{\mathbb{F}}, \Lambda), \tag{4-2}$$

whose composition is the usual pullback functor along $X_{\mathbb{F}} \to X$.

Remark 4.2. The functor $\mathrm{D}(X, \Lambda) \to \mathrm{D}(X^{\mathrm{Weil}}, \Lambda)$ is not an equivalence in general. This relates to the difference between continuous representations Galois versus Weil groups. See, however, Proposition 4.16 for filtered colimits of finite discrete rings Λ.

We have the following basic functoriality: Let $j\colon U \to X$ be a weakly étale morphism and consider the corresponding object $(U_{\mathbb{F}}, \phi_U)$ of $X^{\mathrm{Weil}}_{\mathrm{pro\acute{e}t}}$. Then the slice site $(X^{\mathrm{Weil}}_{\mathrm{pro\acute{e}t}})_{/(U_{\mathbb{F}}, \phi_U)}$ is equivalent to $U^{\mathrm{Weil}}_{\mathrm{pro\acute{e}t}}$. This gives a functor $(X_{\mathrm{pro\acute{e}t}})^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{St}}_{\Lambda}$, $U \mapsto \mathrm{D}(U^{\mathrm{Weil}}, \Lambda)$ which is a hypercomplete sheaf of Λ_*-linear presentable stable categories.

Also, we obtain an adjunction

$$j_!\colon \mathrm{D}(U^{\mathrm{Weil}}, \Lambda) \rightleftarrows \mathrm{D}(X^{\mathrm{Weil}}, \Lambda)\colon j^*$$

that is compatible with the $((j_{\mathbb{F}})_!, (j_{\mathbb{F}})^*)$-adunction under (4-2). The category $\mathrm{D}(X^{\mathrm{Weil}}, \Lambda)$ is equivalent to the category of ϕ_X-equivariant sheaves on $X_{\mathbb{F}}$, as we will now explain.

For each $i \geq 0$, consider the object $(X_i, \Phi_i) \in X^{\mathrm{Weil}}_{\mathrm{proét}}$ with $X_i = \mathbb{Z}^{i+1} \times X_{\mathbb{F}}$ the countably disjoint union of $X_{\mathbb{F}}$, the map $X_i \to X_{\mathbb{F}}$ given by projection and the endomorphism $\Phi_i \colon X_i \to X_i$ given by $(\underline{n}, x) \mapsto (\underline{n} - (1, \dots, 1), \phi_X(x))$ on sections. The inclusion $\mathbb{Z}^i \to \mathbb{Z}^{i+1}$, $\underline{n} \mapsto (0, \underline{n})$ induces a map of schemes $X_{i-1} \to X_i$ where $X_{-1} := X_{\mathbb{F}}$. By pullback, we get a limit-preserving map of sites

$$(X_{i-1})_{\mathrm{proét}} \to (X^{\mathrm{Weil}}_{\mathrm{proét}})_{/(X_i, \Phi_i)}. \tag{4-3}$$

Lemma 4.3. *For each $i \geq 0$, the map* (4-3) *induces an equivalence on the associated* 1*-topoi.*

Proof. As universal homeomorphisms induce equivalences on proétale 1-topoi [Bhatt and Scholze 2015, Lemma 5.4.2], we may assume that X is perfect. In this case, the sites (4-3) are equivalent because ϕ_X is an isomorphism. Explicitly, an inverse is given by sending an object $U \in (X_{i-1})_{\mathrm{proét}}$ to the object $V = \bigsqcup_{\underline{n} \in \mathbb{Z}^{i+1}} V_{\underline{n}}$, $V_{\underline{n}} \to \{\underline{n}\} \times X_{\mathbb{F}}$ defined by

$$V_{\underline{n}} = U_{(n_2 - n_1, \dots, n_{i+1} - n_1)} \times_{X_{\mathbb{F}}, \phi_X^{n_1}} X_{\mathbb{F}},$$

and with endomorphism $\varphi \colon V \to V$ defined by the maps $V_{\underline{n}} = V_{\underline{n} - (1, \dots, 1)} \times_{X_{\mathbb{F}}, \phi_X} X_{\mathbb{F}} \to V_{\underline{n} - (1, \dots, 1)}$. □

Weil sheaves admit the following presentation as the ϕ_X^*-fixed points of $\mathrm{D}(X_{\mathbb{F}}, \Lambda)$, see Definition 2.1.

Proposition 4.4. *The last functor in* (4-2) *induces an equivalence*

$$\mathrm{D}(X^{\mathrm{Weil}}, \Lambda) \cong \lim(\mathrm{D}(X_{\mathbb{F}}, \Lambda) \underset{\phi_X^*}{\overset{\mathrm{id}}{\rightrightarrows}} \mathrm{D}(X_{\mathbb{F}}, \Lambda)). \tag{4-4}$$

Remark 4.5. Objects in (4-4) are pairs (M, α) where $M \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$ and α is an isomorphism $M \cong \phi_X^* M$. Note that the composition $\phi_X \circ \phi_{\mathbb{F}}$ is the absolute q-Frobenius of $X_{\mathbb{F}}$. In particular, it induces the identity on proétale topoi; see [Bhatt and Scholze 2015, Lemma 5.4.2]. Therefore, replacing ϕ_X^* by $\phi_{\mathbb{F}}^*$ in (4-4) yields an equivalent category.

Proof of Proposition 4.4. The structural morphism $(X_0, \Phi_0) \to (X_{\mathbb{F}}, \phi_X)$ is a cover in $X^{\mathrm{Weil}}_{\mathrm{proét}}$. Its Čech nerve has objects $(X_i, \Phi_i) \in X^{\mathrm{Weil}}_{\mathrm{proét}}$, $i \geq 0$ as above. By descent, there is an equivalence

$$\mathrm{D}(X^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} \mathrm{Tot}(\mathrm{D}((X^{\mathrm{Weil}}_{\mathrm{proét}})_{/(X_\bullet, \Phi_\bullet)}, \Lambda)). \tag{4-5}$$

Under Lemma 4.3, the cosimplicial 1-topos associated with $(X^{\mathrm{Weil}}_{\mathrm{proét}})_{/(X_\bullet, \Phi_\bullet)}$ is equivalent to the cosimplicial 1-topos associated with the action of ϕ_X^* on $(X_{\mathbb{F}})_{\mathrm{proét}}$. The equivalence (4-5) then becomes

$$\mathrm{D}(X^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} \lim_{B\mathbb{Z}} \mathrm{D}(X_{\mathbb{F}}, \Lambda),$$

for the diagram $B\mathbb{Z} \to \mathrm{Pr}^{\mathrm{St}}_{\Lambda}$ corresponding to the endomorphism ϕ_X^* of $\mathrm{D}(X_{\mathbb{F}}, \Lambda)$. That is, $\mathrm{D}(X^{\mathrm{Weil}}, \Lambda)$ is equivalent to the homotopy fixed points of $\mathrm{D}(X_{\mathbb{F}}, \Lambda)$ with respect to the action of ϕ_X^*, which is our claim. □

4B. ***Weil sheaves on products.*** The discussion of the previous section generalizes to products of schemes as follows. Let $X_1, \dots, X_n$ be schemes over $\mathbb{F}_q$, and denote by $X := X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n$ their product. For every $1 \le i \le n$, we have a morphism $\phi_{X_i}\colon X_{i,\mathbb{F}} \to X_{i,\mathbb{F}}$ as in the previous section. We use the notation ϕ_{X_i} to also denote the corresponding map on $X_{\mathbb{F}} = X_{1,\mathbb{F}} \times_{\mathbb{F}} \cdots \times_{\mathbb{F}} X_{n,\mathbb{F}}$ which is ϕ_{X_i} on the i-th factor and the identity on the other factors.

We define the site $(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}})_{\text{proét}}$ whose underlying category consists of tuples $(U, \varphi_1, \dots, \varphi_n)$ with $U \in (X_{\mathbb{F}})_{\text{proét}}$ and pairwise commuting endomorphisms $\varphi_i\colon U \to U$ such that the following diagram commutes

$$\begin{tikzcd} U \arrow[r, "\varphi_i"] \arrow[d] & U \arrow[d] \\ X_{\mathbb{F}} \arrow[r, "\phi_{X_i}"] & X_{\mathbb{F}}, \end{tikzcd}$$

for all $1 \le i \le n$. As before, we denote by $\mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$ the corresponding derived category of Λ-sheaves.

Using a similar reasoning as in the previous section, we can identify this category of sheaves with the homotopy fixed points

$$\mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} \mathrm{Fix}(\mathrm{D}(X_{\mathbb{F}}, \Lambda), \phi^*_{X_1}, \dots, \phi^*_{X_n}) \tag{4-6}$$

of the commuting family of the functors $\phi^*_{X_i}$, see Remark 2.3. Explicitly, for $n = 2$, this is the homotopy limit of the diagram:

$$\begin{tikzcd} \mathrm{D}(X_{\mathbb{F}}, \Lambda) \arrow[r, shift left, "\phi^*_{X_1}"] \arrow[r, shift right, "\mathrm{id}"'] \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "\phi^*_{X_2}"] & \mathrm{D}(X_{\mathbb{F}}, \Lambda) \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "\phi^*_{X_2}"] \\ \mathrm{D}(X_{\mathbb{F}}, \Lambda) \arrow[r, shift left, "\phi^*_{X_1}"] \arrow[r, shift right, "\mathrm{id}"'] & \mathrm{D}(X_{\mathbb{F}}, \Lambda) \end{tikzcd}$$

Roughly speaking, objects in the category $\mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$ are given by tuples $(M, \alpha_1, \dots, \alpha_n)$ with $M \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$ and with pairwise commuting equivalences $\alpha_i\colon M \cong \phi^*_{X_i} M$. That is, equipped with a collection of equivalences $\phi^*_{X_j}(\alpha_i) \circ \alpha_j \simeq \phi^*_{X_i}(\alpha_j) \circ \alpha_i$ for all i, j satisfying higher coherence conditions.

4C. ***Partial-Frobenius stability.*** For schemes $X_1, \dots, X_n$ over $\mathbb{F}_q$, we denote by $X := X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n$ their product together with the partial Frobenii $\mathrm{Frob}_{X_i}\colon X \to X$, $1 \le i \le n$. To give a reasonable definition of lisse and constructible Weil sheaves, we need to understand the relation between partial-Frobenius invariant constructible subsets in X and constructible subsets in the single factors X_i.

Definition 4.6. A subset $Z \subset X$ is called *partial-Frobenius invariant* if $\mathrm{Frob}_{X_i}(Z) = Z$ for all $1 \le i \le n$.

The composition $\mathrm{Frob}_{X_1} \circ \cdots \circ \mathrm{Frob}_{X_n}$ is the absolute q-Frobenius on X and thus induces the identity on the topological space underlying X. Therefore, in order to check that $Z \subset X$ is partial-Frobenius invariant, it suffices that, for any fixed i, the subset Z is Frob_{X_j}-invariant for all $j \neq i$. This remark, which also applies to $X_{\mathbb{F}} = X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n \times_{\mathbb{F}_q} \operatorname{Spec} \mathbb{F}$, will be used below without further comment.

We first investigate the case of two factors with one being a separably closed field. This eventually rests on Drinfeld's descent result [1987, Proposition 1.1] for coherent sheaves.

Lemma 4.7. *Let X be a qcqs $\mathbb{F}_q$-scheme, and let $k/\mathbb{F}_q$ be a separably closed field. Denote by $p\colon X_k \to X$ the projection. Then $Z \mapsto p^{-1}(Z)$ induces a bijection*

$$\{\textit{constructible subsets in } X\} \leftrightarrow \{\textit{partial-Frobenius invariant, constructible subsets in } X_k\}.$$

Proof. The injectivity is clear because p is surjective. It remains to check the surjectivity. Without loss of generality we may assume that k is algebraically closed, and replace Frob_X by Frob_k which is an automorphism. Given that $Z \mapsto p^{-1}(Z)$ is compatible with passing to complements, unions and localizations on X, we are reduced to proving the bijection for constructible closed subsets Z and for X affine over $\mathbb{F}_q$. By Noetherian approximation (Lemma 4.9), we reduce further to the case where X is of finite type over $\mathbb{F}_q$ and still affine. Now we choose a locally closed embedding $X \to \mathbb{P}^n_{\mathbb{F}_q}$ into projective space. A closed subset $Z' \subset X_k$ is ϕ_k-invariant if and only if its closure inside $\mathbb{P}^n_k$ is so. Hence, it is enough to consider the case where $X = \mathbb{P}^n_{\mathbb{F}_q}$ is the projective space. Let Z' be a closed Frob_k-invariant subset of X_k. When viewed as a reduced subscheme, the isomorphism ϕ_k restricts to an isomorphism of Z'. In particular, $\mathcal{O}_{Z'}$ is a coherent $\mathcal{O}_{X_k}$-module equipped with an isomorphism $\mathcal{O}_{Z'} \cong \phi_k^* \mathcal{O}_{Z'}$. Hence, Drinfeld's descent result [1987, Proposition 1.1] (see also [Kedlaya 2019, Section 4.2] for a recent exposition) yields $Z' = Z_k$ for a unique closed subscheme $Z \subset X$. □

The following proposition generalizes the results [Lau 2004, Lemma 9.2.1] and [Lafforgue 2018, Lemme 8.12] in the case of curves.

Proposition 4.8. *Let $X_1, \dots, X_n$ be qcqs $\mathbb{F}_q$-schemes, and denote $X = X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n$. Then any partial-Frobenius invariant constructible closed subset $Z \subset X$ is a finite set-theoretic union of subsets of the form $Z_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} Z_n$, for appropriate constructible closed subschemes $Z_i \subset X_i$.*

In particular, any partial-Frobenius invariant constructible open subscheme $U \subset X$ is a finite union of constructible open subschemes of the form $U_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} U_n$, for appropriate constructible open subschemes $U_i \subset X_i$.

Proof. By induction, we may assume $n = 2$. By Noetherian approximation (Lemma 4.9), we reduce to the case where both X_1, X_2 are of finite type over $\mathbb{F}_q$. In the following, all products are formed over $\mathbb{F}_q$, and locally closed subschemes are equipped with their reduced subscheme structure. Let $Z \subset X_1 \times X_2$ be a partial-Frobenius invariant closed subscheme. The complement $U = X_1 \times X_2 \setminus Z$ is also partial-Frobenius invariant.

In the proof, we can replace X_1 (and likewise X_2) by a stratification in the following sense: Suppose $X_1 = A' \sqcup A''$ is a set-theoretic stratification into a closed subset A' with open complement A''. Once we know $Z \cap A' \times X_2 = \bigcup_j Z'_{1j} \times Z'_{2j}$ and $Z \cap A'' \times X_2 = \bigcup_j Z''_{1j} \times Z''_{2j}$ for appropriate closed subschemes

$Z'_{1j} \subset A'$, $Z''_{1j} \subset A''$ and Z'_{2j}, $Z''_{2j} \subset X_2$, we have the set-theoretic equality

$$Z = \bigcup_j Z'_{1i} \times Z'_{2j} \cup \bigcup_j \overline{Z''_{1j}} \times Z''_{2j},$$

where $\overline{Z''_{1j}} \subset X_1$ denotes the scheme-theoretic closure. Here we note that taking scheme-theoretic closures commutes with products because the projections $X_1 \times X_2 \to X_i$ are flat, and that the topological space underlying the scheme-theoretic closure agrees with the topological closure because all schemes involved are of finite type.

The proof is now by Noetherian induction on X_2, the case $X_2 = \varnothing$ being clear (or, if the reader prefers the case where X_2 is zero dimensional reduces to Lemma 4.7). In the induction step, we may assume, using the above stratification argument, that both X_i are irreducible with generic point η_i. We let $\bar{\eta}_i$ be a geometric generic point over η_i, and denote by $p_i : X_1 \times X_2 \to X_i$ the two projections. Both p_i are faithfully flat of finite type and in particular open, so that $p_i(U)$ is open in X_i. We have a set-theoretic equality

$$Z = ((X_1 \setminus p_1(U)) \times X_2) \cup (X_1 \times (X_2 \setminus p_2(U))) \cup (Z \cap p_1(U) \times p_2(U)).$$

Once we know $Z \cap p_1(U) \times p_2(U) = \bigcup_j Z_{1j} \times Z_{2j}$ for appropriate closed $Z_{ij} \subset p_i(U)$, we are done. We can therefore replace X_i by $p_i(U)$ and assume that both $p_i : U \to X_i$ are surjective.

The base change $U \times_{X_2} \bar{\eta}_2$ is a $\phi_{\bar{\eta}_2}$-invariant subset of $X_1 \times \bar{\eta}_2$. By Lemma 4.7, it is thus of the form $U_1 \times \bar{\eta}_2$ for some open subset $U_1 \subset X_1$. There is an inclusion (of open subschemes of $X_1 \times \eta_2$): $U \times_{X_2} \eta_2 \subset U_1 \times \eta_2$. It becomes a set-theoretic equality, and therefore an isomorphism of schemes, after base change along $\bar{\eta}_2 \to \eta_2$. By faithfully flat descent, this implies that the two mentioned subsets of $X_1 \times \eta_2$ agree. We claim $U_1 = X_1$. Since the projection $U \to X_2$ is surjective, in particular its image contains η_2, so that U_1 is a nonempty subset, and therefore open dense in the irreducible scheme X_1. Let $x_1 \in X_1$ be a point. Since the projection $U \to X_1$ is surjective, $U \cap (\{x_1\} \times X_2)$ is a nonempty open subscheme of $\{x_1\} \times X_2$. So it contains a point lying over (x_1, η_2). We conclude $X_1 \times \eta_2 \subset U$.

We claim that there is a nonempty open subset $A_2 \subset X_2$ such that

$$X_1 \times A_2 \subset U \quad \text{or, equivalently,} \quad X_1 \times (X_2 \setminus A_2) \supset X_1 \times X_2 \setminus U.$$

The underlying topological space of $V = X_1 \times X_2 \setminus U$ is Noetherian and thus has finitely many irreducible components V_j. The closure of the projection $\overline{p_2(V_j)} \subset X_2$ does not contain η_2, since $X_1 \times \eta_2 \subset U$. Thus, $A_2 := \bigcap_j X_1 \setminus \overline{p_2(V_j)}$ satisfies our requirements.

Now we continue by Noetherian induction applied to the stratification $X_2 = A_2 \sqcup (X_2 \setminus A_2)$: We have $Z \cap X_1 \times A_2 = \varnothing$, so that we may replace X_2 by the proper closed subscheme $X_2 \setminus A_2$. Hence, the proposition follows by Noetherian induction. $\square$

The following lemma on Noetherian approximation of partial Frobenius invariant subsets is needed for the reduction to finite type schemes:

Lemma 4.9. *Let $X_1, \dots, X_n$ be qcqs $\mathbb{F}_q$-schemes, and denote $X = X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n$. Let $X_i = \lim_j X_{ij}$ be a cofiltered limit of finite type $\mathbb{F}_q$-schemes with affine transition maps, and write $X = \lim_j X_j$, $X_j := X_{1j} \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_{nj}$; see* [Stacks 2017, Tag 01ZA] *for the existence of such presentations. Let $Z \subset X$ be a constructible closed subset. Then the intersection*

$$Z' = \bigcap_{i=1}^{n} \bigcap_{m \in \mathbb{Z}} \mathrm{Frob}_{X_i}^m(Z)$$

is partial Frobenius invariant, constructible closed and there exists an index j and a partial Frobenius invariant closed subset $Z'_j \subset X_j$ such that $Z' = Z'_j \times_{X_j} X$ as sets.

We note that each Frob_{X_i} induces a homeomorphism on the underlying topological space of X so that Z' is well-defined. This lemma applies, in particular, to partial Frobenius invariant constructible closed subsets $Z \subset X$ in which case we have $Z = Z'$.

Proof. As Z is constructible, there exists an index j and a constructible closed subscheme $Z_j \subset X_j$ such that $Z = Z_j \times_{X_j} X$ as sets. We put $Z'_j = \bigcap_{i=1}^n \bigcap_{m\in\mathbb{Z}} \mathrm{Frob}_{X_{ij}}^m(Z_j)$. As X_j is of finite type over $\mathbb{F}_q$, the subset Z'_j is still constructible closed. As partial Frobenii induce bijections on the underlying topological spaces, one checks that $\mathrm{Frob}_{X_{ij}}^m(Z_j) \times_{X_j} X = \mathrm{Frob}_{X_i}^m(Z)$ as sets for all $m \in \mathbb{Z}$. Thus, $Z' = Z'_j \times_{X_j} X$ which, also, is constructible closed because $X \to X_j$ is affine. $\square$

4D. *Lisse and constructible Weil sheaves.* In this subsection, we define the subcategories of lisse and constructible Weil sheaves and establish a presentation similar to (4-4). Let $X_1, \dots, X_n$ be schemes over $\mathbb{F}_q$, and denote $X := X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n$. Let Λ be a condensed ring.

Definition 4.10. Let $M \in \mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$:

(1) The Weil sheaf M is called *lisse* if it is dualizable. (Here dualizability refers to the symmetric monoidal structure on $\mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$, given by the derived tensor product of Λ-sheaves on the Weil-proétale topos.)

(2) The Weil sheaf M is called *constructible* if for any open affine $U_i \subset X_i$ there exists a finite subdivision into constructible locally closed subschemes $U_{ij} \subseteq U_i$ such that each restriction $M|_{U_{1j}^{\mathrm{Weil}} \times \cdots \times U_{nj}^{\mathrm{Weil}}} \in \mathrm{D}(U_{1j}^{\mathrm{Weil}} \times \cdots \times U_{nj}^{\mathrm{Weil}}, \Lambda)$ is lisse.

The full subcategories of $\mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$ consisting of lisse, resp. constructible Weil sheaves are denoted by

$$\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda).$$

Both categories are idempotent complete stable $\Gamma(X, \Lambda)$-linear symmetric monoidal ∞-categories.

From the presentation (4-6), we get that a Weil sheaf M is lisse if and only if the underlying object $M_{\mathbb{F}} \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$ is lisse. So (4-6) restricts to an equivalence

$$\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda) \cong \mathrm{Fix}(\mathrm{D}_{\mathrm{lis}}(X_{\mathbb{F}}, \Lambda), \phi_{X_1}^*, \dots, \phi_{X_n}^*). \tag{4-7}$$

The same is true for constructible Weil sheaves by the following proposition:

Proposition 4.11. *A Weil sheaf $M \in \mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$ is constructible if and only if the underlying sheaf $M_{\mathbb{F}} \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$ is constructible. Consequently*, (4-6) *restricts to an equivalence*

$$\mathrm{D}_{\mathrm{cons}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda) \cong \mathrm{Fix}(\mathrm{D}_{\mathrm{cons}}(X_{\mathbb{F}}, \Lambda), \phi_{X_1}^*, \ldots, \phi_{X_n}^*). \tag{4-8}$$

Proof. Clearly, if M is constructible, so is $M_{\mathbb{F}}$ by Definition 4.10. Let $M \in \mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$ such that $M_{\mathbb{F}}$ is constructible. We may assume that all X_i are affine. We claim that there is a finite subdivision $X_{\mathbb{F}} = \bigsqcup X_\alpha$ into constructible locally closed subsets such that $M_{\mathbb{F}}|_{X_\alpha}$ is lisse and such that each X_α is partial Frobenius invariant.

Assuming the claim we finish the argument as follows. By Proposition 4.8, any open stratum $U = X_{j_0} \subset X_{\mathbb{F}}$ is a finite union of subsets of the form $U_{1,\mathbb{F}} \times_{\mathbb{F}} \cdots \times_{\mathbb{F}} U_{n,\mathbb{F}}$ and the restriction of M to each of them is lisse. In particular, the complement $X_{\mathbb{F}} \backslash U$ is defined over $\mathbb{F}_q$ and arises as a finite union of schemes of the form $X' = X_1' \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n'$ for suitable qcqs schemes X_i' over $\mathbb{F}_q$. Intersecting each $X_{\mathbb{F}}'$ with the remaining strata $\bigsqcup_{j \neq j_0} X_j$, we conclude by induction on the number of strata.

It remains to prove the claim. We start with any finite subdivision $X_{\mathbb{F}} = \bigsqcup X_j'$ into constructible locally closed subsets such that $M_{\mathbb{F}}|_{X_j'}$ is lisse. Pick an open stratum X_{j_0}', and set

$$X_{j_0} = \bigcup_{i=1}^{n} \bigcup_{m \in \mathbb{Z}} \phi_{X_i}^m (X_{j_0}'). \tag{4-9}$$

This is a constructible open subset of $X_{\mathbb{F}}$ by Lemma 4.9 applied to its closed complement. Furthermore, $M_{\mathbb{F}}|_{X_{j_0}}$ is lisse by its partial Frobenius equivariance, noting that $\phi_{X_i}^*$ induces equivalences on proétale topoi to treat the negative powers in (4-9). As before, $X_{\mathbb{F}} \backslash X_{j_0}$ is defined over $\mathbb{F}_q$. So replacing X_j', $j \neq j_0$ by $X_j' \cap (X_{\mathbb{F}} \backslash X_{j_0})$, the claim follows by induction on the number of strata. □

In the case of a single factor $X = X_1$, the preceding discussion implies

$$\mathrm{D}_\bullet(X^{\mathrm{Weil}}, \Lambda) \cong \lim(\mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda) \underset{\phi_X^*}{\overset{\mathrm{id}}{\rightrightarrows}} \mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda)), \tag{4-10}$$

for $\bullet \in \{\varnothing, \mathrm{lis}, \mathrm{cons}\}$.

4E. ***Relation with the Weil groupoid.*** In this subsection, we relate lisse Weil sheaves with representations of the Weil groupoid. Throughout, we work with étale fundamental groups as opposed to their proétale variants in order to have Drinfeld's lemma available; see Section 5C. The two concepts differ in general, but agree for geometrically unibranch (for example, normal) Noetherian schemes; see [Bhatt and Scholze 2015, Lemma 7.4.10].

For a Noetherian scheme X, let $\pi_1(X)$ be the *étale fundamental groupoid* of X as defined in [SGA 1 2003, Exposé V, Sections 7 and 9]. Its objects are geometric points of X, and its morphisms are isomorphisms of fiber functors on the finite étale site of X. This is an essentially small category. The automorphism group in $\pi_1(X)$ at a geometric point $x \to X$ is profinite. It is denoted $\pi_1(X, x)$ and called the *étale fundamental group* of (X, x). If X is connected, then the natural map $B\pi_1(X, x) \to \pi_1(X)$ is an equivalence for any $x \to X$. If X is the disjoint sum of schemes X_i, $i \in I$, then $\pi_1(X)$ is the disjoint sum of the $\pi_1(X_i)$, $i \in I$. In this case, if $x \to X$ factors through X_i, then $\pi_1(X, x) = \pi_1(X_i, x)$.

Definition 4.12. Let $X_1, \dots, X_n$ be Noetherian schemes over $\mathbb{F}_q$, and write $X = X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n$. The *Frobenius-Weil groupoid* is the stacky quotient

$$\mathrm{FWeil}(X) = \pi_1(X_{\mathbb{F}})/\langle \phi_{X_1}^{\mathbb{Z}}, \dots, \phi_{X_n}^{\mathbb{Z}} \rangle, \tag{4-11}$$

where we use that the partial Frobenii ϕ_{X_i} induce automorphisms on the finite étale site of $X_{\mathbb{F}}$.

For $n = 1$, we denote $\mathrm{FWeil}(X) = \mathrm{Weil}(X)$. Even if X is connected, its base change $X_{\mathbb{F}}$ might be disconnected in which case the action of ϕ_X permutes some connected components. Therefore, fixing a geometric point of $X_{\mathbb{F}}$ is inconvenient, and the reason for us to work with fundamental groupoids as opposed to fundamental groups. The automorphism groups in $\mathrm{Weil}(X)$ carry the structure of locally profinite groups: indeed, if X is connected, then $\mathrm{Weil}(X)$ is, for any choice of a geometric point $x \to X_{\mathbb{F}}$, equivalent to the classifying space of the Weil group $\mathrm{Weil}(X, x)$ from [Deligne 1980, Définition 1.1.10]. Recall that this group sits in an exact sequence of topological groups

$$1 \to \pi_1(X_{\mathbb{F}}, x) \to \mathrm{Weil}(X, x) \to \mathrm{Weil}(\mathbb{F}/\mathbb{F}_q) \simeq \mathbb{Z}, \tag{4-12}$$

where $\pi_1(X_{\mathbb{F}}, x)$ carries its profinite topology and $\mathbb{Z}$ the discrete topology. The topology on the morphism groups in $\mathrm{Weil}(X)$ obtained in this way is independent from the choice of $x \to X_{\mathbb{F}}$. The image of $\mathrm{Weil}(X, x) \to \mathbb{Z}$ is the subgroup $m\mathbb{Z}$ where m is the degree of the largest finite subfield in $\Gamma(X, \mathcal{O}_X)$. In particular, we have $m = 1$ if $X_{\mathbb{F}}$ is connected. Let us add that if $x \to X_{\mathbb{F}}$ is fixed under ϕ_X, then the action of ϕ_X on $\pi_1(X_{\mathbb{F}}, x)$ corresponds by virtue of the formula $\phi_X^* = (\phi_{\mathbb{F}}^*)^{-1}$ to the action of the geometric Frobenius, that is, the inverse of the q-Frobenius in $\mathrm{Weil}(\mathbb{F}/\mathbb{F}_q)$.

Likewise, for every $n \geq 1$, the stabilizers of the Frobenius–Weil groupoid are related to the partial Frobenius–Weil groups introduced in [Drinfeld 1987, Proposition 6.1] and [Lafforgue 2018, Remarque 8.18]. In particular, there is an exact sequence

$$1 \to \pi_1(X_{\mathbb{F}}, x) \to \mathrm{FWeil}(X, x) \to \mathbb{Z}^n,$$

for each geometric point $x \to X_{\mathbb{F}}$. This gives $\mathrm{FWeil}(X)$ the structure of a locally profinite groupoid.

Let Λ be either of the following coherent topological rings: a coherent discrete ring, an algebraic field extension $E \supset \mathbb{Q}_\ell$ for some prime ℓ, or its ring of integers $\mathcal{O}_E \supset \mathbb{Z}_\ell$. For a topological groupoid W, we will denote by $\mathrm{Rep}_\Lambda(W)$ the category of continuous representations of W with values in finitely presented Λ-modules and by $\mathrm{Rep}_\Lambda^{\mathrm{fp}}(W) \subset \mathrm{Rep}_\Lambda(W)$ its full subcategory of representations on finite projective Λ-modules. Here finitely presented Λ-modules M carry the quotient topology induced from the choice of any surjection $\Lambda^n \to M$, $n \geq 0$ and the product topology on Λ^n.

Lemma 4.13. *In the situation above, the category* $\mathrm{Rep}_\Lambda(W)$ *is* Λ_**-linear and abelian. In particular, its full subcategory* $\mathrm{Rep}_\Lambda^{\mathrm{fp}}(W)$ *is* Λ_**-linear and additive.*

Proof. Let W_{disc} be the discrete groupoid underlying W, and denote by $\mathrm{Rep}_\Lambda(W_{\mathrm{disc}})$ the category of W_{disc}-representations on finitely presented Λ-modules. Evidently, this category is Λ_*-linear. It is abelian since Λ is coherent (Synopsis 3.2(vi)); see also [Hemo et al. 2023, Lemma 6.5]. We claim that $\mathrm{Rep}_\Lambda(W) \subset \mathrm{Rep}_\Lambda(W_{\mathrm{disc}})$ is a Λ_*-linear full abelian subcategory. If Λ is discrete (and coherent), then every finitely presented Λ-module carries the discrete topology and the claim is immediate; see also [Stacks 2017, Tag 0A2H]. For $\Lambda = E, \mathcal{O}_E$, one checks that every map of finitely presented Λ-modules is continuous, every surjective map is a topological quotient and every injective map is a closed embedding. For the latter, we use that every finitely presented Λ-module can be written as a countable filtered colimit of compact Hausdorff spaces along injections, and that every injection of compact Hausdorff spaces is a closed embedding. This implies the claim. □

We apply this for W being either of the locally profinite groupoids $\pi_1(X)$, $\pi_1(X_{\mathbb{F}})$ or $\mathrm{FWeil}(X)$. Note that restricting representations along $\pi_1(X_{\mathbb{F}}) \to \mathrm{FWeil}(X)$ induces an equivalence of Λ_*-linear abelian categories

$$\mathrm{Rep}_\Lambda(\mathrm{FWeil}(X)) \cong \mathrm{Fix}(\mathrm{Rep}_\Lambda(\pi_1(X_{\mathbb{F}})), \phi_{X_1}, \dots, \phi_{X_n}), \tag{4-13}$$

and similarly for the Λ_*-linear additive category $\mathrm{Rep}^{\mathrm{fp}}_\Lambda(\mathrm{FWeil}(X))$.

Definition 4.14. For an integer $n \geq 0$, we write $\mathrm{D}^{\{-n,n\}}_{\mathrm{lis}}(X, \Lambda)$ for the full subcategory of $\mathrm{D}_{\mathrm{lis}}(X, \Lambda)$ of objects M such that M and its dual $M^\vee$ lie in degrees $[-n, n]$ with respect to the t-structure on $\mathrm{D}(X, \Lambda)$.

Lemma 4.15. *In the situation above, there is a natural functor*

$$\mathrm{Rep}_\Lambda(\mathrm{FWeil}(X)) \to \mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)^\heartsuit, \tag{4-14}$$

that is fully faithful. Moreover, the following properties hold if Λ is either finite discrete or $\Lambda = \mathcal{O}_E$ for $E \supset \mathbb{Q}_\ell$ finite:

(1) *An object M lies in the essential image of* (4-14) *if and only if its underlying sheaf $M_{\mathbb{F}}$ is locally on $(X_{\mathbb{F}})_{\mathrm{pro\acute{e}t}}$ isomorphic to $\underline{N} \otimes_{\underline{\Lambda_*}} \Lambda_{X_{\mathbb{F}}}$ for some finitely presented Λ_*-module N.*

(2) *The functor* (4-14) *restricts to an equivalence of Λ_*-linear additive categories*

$$\mathrm{Rep}^{\mathrm{fp}}_\Lambda(\mathrm{FWeil}(X)) \xrightarrow{\cong} \mathrm{D}^{\{0,0\}}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda).$$

(3) *If Λ_* is regular* (*so that Λ is t-admissible*, *see Synopsis 3.2*(vi)), *then* (4-14) *restricts to an equivalence of Λ_*-linear abelian categories*

$$\mathrm{Rep}_\Lambda(\mathrm{FWeil}(X)) \xrightarrow{\cong} \mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)^\heartsuit.$$

If all X_i, $i = 1, \dots, n$ are geometrically unibranch, then (1), (2) and (3) hold for general coherent topological rings Λ as above.

Proof. There is a canonical equivalence of topological groupoids $\pi_1(X_{\mathbb{F}}) \cong \widehat{\pi_1^{\mathrm{pro\acute{e}t}}(X_{\mathbb{F}})}$ with the profinite completion of the proétale fundamental groupoid; see [Bhatt and Scholze 2015, Lemma 7.4.3]. It follows from [loc. cit., Lemmas 7.4.5, 7.4.7] that restricting representations along $\pi_1^{\text{proét}}(X_{\mathbb{F}}) \to \pi_1(X_{\mathbb{F}})$ induces full embeddings

$$\mathrm{Rep}_\Lambda(\pi_1(X_{\mathbb{F}})) \hookrightarrow \mathrm{Rep}_\Lambda(\pi_1^{\text{proét}}(X_{\mathbb{F}})) \hookrightarrow \mathrm{D}(X_{\mathbb{F}}, \Lambda)^\heartsuit, \tag{4-15}$$

that are compatible with the action of ϕ_{X_i} for all $i = 1, \dots, n$. So we obtain the fully faithful functor (4-14) by passing to fixed points, see (4-13), (4-7) and Lemma 2.2 (see also Remark 2.3).

Part (1) describes the essential image of $\mathrm{Rep}_\Lambda(\pi_1^{\text{proét}}(X_{\mathbb{F}})) \hookrightarrow \mathrm{D}(X_{\mathbb{F}}, \Lambda)^\heartsuit$. So if Λ is finite discrete or profinite, then the first functor in (4-15) is an equivalence, and we are done. Part (2) is immediate from (1), noting that an object in the essential image of (4-15) is lisse if and only if its underlying module is finite projective. Likewise, part (3) is immediate from (1), using Synopsis 3.2(vii). Here we need to exclude rings like $\Lambda = \mathbb{Z}/\ell^2$ in order to have a t-structure on lisse sheaves.

Finally, if all X_i are geometrically unibranch, so is $X_{\mathbb{F}}$ which follows from the characterization [Stacks 2017, Tag 0BQ4]. In this case, we get $\pi_1(X_{\mathbb{F}}) \cong \pi_1^{\text{proét}}(X_{\mathbb{F}})$ by [Bhatt and Scholze 2015, Lemma 7.4.10]. This finishes the proof. □

4F. *Weil-étale versus étale sheaves.* We end this section with the following description of Weil sheaves with (ind-)finite coefficients. Note that such a simplification in terms of ordinary sheaves is not possible for $\Lambda = \mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Q}_\ell$, say.

Proposition 4.16. *Let X be a qcqs $\mathbb{F}_q$-scheme. Let Λ be a finite discrete ring or a filtered colimit of such rings. Then the natural functors*

$$\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \to \mathrm{D}_{\mathrm{lis}}(X^{\mathrm{Weil}}, \Lambda), \quad \mathrm{D}_{\mathrm{cons}}(X, \Lambda) \to \mathrm{D}_{\mathrm{cons}}(X^{\mathrm{Weil}}, \Lambda),$$

are equivalences.

Proof. Throughout, we repeatedly use that filtered colimits commute with finite limits in Cat_∞. Using compatibility of $\mathrm{D}_{\mathrm{cons}}$ with filtered colimits in Λ (Synopsis 3.2(iv)), we may assume that Λ is finite discrete. By the comparison result with the classical bounded derived category of constructible sheaves [Hemo et al. 2023, Proposition 7.1], we can identify the categories $\mathrm{D}_\bullet(X, \Lambda)$, resp. $\mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda)$ for $\bullet \in \{\text{lis}, \text{cons}\}$ with full subcategories of the derived category of étale Λ-sheaves $\mathrm{D}(X_{\text{ét}}, \Lambda)$, resp. $\mathrm{D}(X_{\mathbb{F},\text{ét}}, \Lambda)$. Write $X = \lim X_i$ as a cofiltered limit of finite type $\mathbb{F}_q$-schemes X_i with affine transition maps [Stacks 2017, Tag 01ZA]. Using the continuity of étale sites [Stacks 2017, Tag 03Q4], there are natural equivalences

$$\operatorname{colim} \mathrm{D}_\bullet(X_i, \Lambda) \xrightarrow{\cong} \mathrm{D}_\bullet(X, \Lambda), \quad \operatorname{colim} \mathrm{D}_\bullet(X_i^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} \mathrm{D}_\bullet(X^{\mathrm{Weil}}, \Lambda) \tag{4-16}$$

for $\bullet \in \{\text{lis}, \text{cons}\}$. Hence, we can assume that X is of finite type over $\mathbb{F}_q$.

To show full faithfulness, we claim more generally that the natural map

$$\mathrm{D}(X_{\text{ét}}, \Lambda) \to \lim(\mathrm{D}(X_{\mathbb{F},\text{ét}}, \Lambda) \underset{\phi_X^*}{\overset{\mathrm{id}}{\rightrightarrows}} \mathrm{D}(X_{\mathbb{F},\text{ét}}, \Lambda)) =: \mathrm{D}(X_{\text{ét}}^{\mathrm{Weil}}, \Lambda)$$

is fully faithful. As Λ is torsion, this is immediate from [Geisser 2004, Corollary 5.2] applied to the inner homomorphisms between sheaves. Let us add that this induces fully faithful functors

$$\mathrm{D}^+(X_{\text{ét}}, \Lambda) \to \mathrm{D}^+(X_{\text{ét}}^{\mathrm{Weil}}, \Lambda) \to \mathrm{D}(X^{\mathrm{Weil}}, \Lambda) \tag{4-17}$$

on bounded below objects; see [Bhatt and Scholze 2015, Proposition 5.2.6(1)].

It remains to prove essential surjectivity. Using a stratification as in Definition 4.10, it is enough to consider the lisse case. Pick $M \in \mathrm{D}_{\mathrm{lis}}(X^{\mathrm{Weil}}, \Lambda)$. It is enough to show that M lies is in the essential image of (4-17), noting that the functor detects dualizability. As M is bounded, this will follow from showing that for every $j \in \mathbb{Z}$, the cohomology sheaf $\mathrm{H}^j(M) \in \mathrm{D}(X^{\mathrm{Weil}}, \Lambda)^\heartsuit$ is in the essential image of (4-17).

Fix $j \in \mathbb{Z}$. As M is lisse, the underlying sheaf $\mathrm{H}^j(M)_{\mathbb{F}} \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)^\heartsuit$ is proétale-locally constant (Synopsis 3.2(vii)) and valued in finitely presented Λ-modules. By Lemma 4.15(1), it comes from a representation of $\mathrm{Weil}(X)$. Restriction of representations along $\mathrm{Weil}(X) \to \pi_1(X)$ fits into a commutative diagram:

$$\begin{tikzcd}
\mathrm{Rep}_\Lambda(\pi_1(X)) \arrow[r, "\cong"] \arrow[d] & \mathrm{Rep}_\Lambda(\mathrm{Weil}(X)) \arrow[d] \\
\mathrm{D}(X_{\text{ét}}, \Lambda)^\heartsuit \arrow[r] & \mathrm{D}(X^{\mathrm{Weil}}, \Lambda)^\heartsuit
\end{tikzcd}$$

where the upper horizontal arrow is an equivalence since Λ is finite. In particular, the object $\mathrm{H}^j(M)$ is in the essential image of the fully faithful functor (4-17). $\square$

5. The categorical Künneth formula

We continue with the notation of Section 4. In particular, $\mathbb{F}_q$ denotes a finite field of characteristic $p > 0$. Recall from Section 2 the tensor product of Λ_*-linear idempotent complete stable ∞-categories. The external tensor product of sheaves $(M_1, \dots, M_n) \mapsto M_1 \boxtimes \cdots \boxtimes M_n$ as in (3-2) induces a functor

$$\mathrm{D}_\bullet(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \cdots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_\bullet(X_n^{\mathrm{Weil}}, \Lambda) \to \mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda), \tag{5-1}$$

for $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$. Throughout, we consider the following situation. In Remark 5.3 we explain the compatibility of (5-1) with certain (co-)limits in the schemes X_i and coefficients Λ, which allows to relax these assumptions on X and Λ somewhat.

Situation 5.1. The schemes $X_1, \dots, X_n$ are of finite type over $\mathbb{F}_q$, and Λ is the condensed ring associated with one of the following topological rings:

(a) A finite discrete ring of prime-to-p-torsion.

(b) The ring of integers $\mathcal{O}_E$ of an algebraic field extension $E \supset \mathbb{Q}_\ell$ for $\ell \neq p$ (for example $\overline{\mathbb{Z}}_\ell$).

(c) An algebraic field extension $E \supset \mathbb{Q}_\ell$ for $\ell \neq p$ (for example $\overline{\mathbb{Q}}_\ell$).

(d) A finite discrete p-torsion ring that is flat over $\mathbb{Z}/p^m$ for some $m \geq 1$.

Theorem 5.2. *In Situation 5.1, the functor* (5-1) *is an equivalence in each of the following cases*:

(1) $\bullet = \mathrm{cons}$ *and* Λ *is as in (a), (b) or (c).*

(2) $\bullet = \mathrm{lis}$ *and* Λ *is as in (a), (b), (d) or as in (c) if all* X_i, $i = 1, \dots, n$ *are geometrically unibranch* (*for example*, *normal*).

In the p-torsion free cases (a), (b) and (c), the full faithfulness is a direct consequence of the Künneth formula applied to the $X_{i,\mathbb{F}}$. In the p-torsion case (d), we use Artin–Schreier theory instead. It would be interesting to see whether this part can be extended to constructible sheaves using the mod-p-Riemann–Hilbert correspondence as in, say, [Bhatt and Lurie 2019]. In all cases, the essential surjectivity relies on a variant of Drinfeld's lemma for Weil group representations.

Before turning to the proof of Theorem 5.2, we record the following compatibility of the functor (5-1) with (co-)limits. This can be used to reduce the case of an (infinite) algebraic extension $E \supset \mathbb{Q}_\ell$ in cases (b) and (c) above to the case where $E \supset \mathbb{Q}_\ell$ is finite. In the sequel we will therefore assume E is finite in these cases. Remark 5.3 can further be used to extend Theorem 5.2 to qcqs $\mathbb{F}_q$-schemes X_i and finite discrete rings like $\mathbb{Z}/m$ for any integer $m \geq 1$ in cases (a) and (d).

Remark 5.3 (compatibility of (5-1) with certain (co-)limits). Throughout, we repeatedly use that filtered colimits commute with finite limits in $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda_*}(\mathrm{Idem})$: the forgetful functors $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda_*}(\mathrm{Idem}) \to \mathrm{Cat}^{\mathrm{Ex}}_{\infty}(\mathrm{Idem}) \to \mathrm{Cat}_\infty$ create these (co)limits [Lurie 2017, Theorem 1.1.4.4; 2009, Corollary 4.4.5.21], and the statement holds in any compactly generated ∞-category, such as Cat_∞ [Bhatt and Mathew 2021, Example 3.6(3)]. We will also throughout use that in all the stable ∞-categories encountered below the tensor product preserves colimits and in particular finite limits:

(1) *Filtered colimits in* Λ. First off, extension of scalars along any map of condensed rings $\Lambda \to \Lambda'$ induces a commutative diagram in $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda_*}(\mathrm{Idem})$:

$$\begin{array}{ccc} \mathrm{D}_\bullet(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \cdots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_\bullet(X_n^{\mathrm{Weil}}, \Lambda) & \longrightarrow & \mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \cdots \times X_n^{\mathrm{Weil}}, \Lambda) \\ \downarrow & & \downarrow \\ \mathrm{D}_\bullet(X_1^{\mathrm{Weil}}, \Lambda') \otimes_{\mathrm{Perf}_{\Lambda'_*}} \cdots \otimes_{\mathrm{Perf}_{\Lambda'_*}} \mathrm{D}_\bullet(X_n^{\mathrm{Weil}}, \Lambda') & \longrightarrow & \mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \cdots \times X_n^{\mathrm{Weil}}, \Lambda') \end{array}$$

It follows from the compatibility of $\mathrm{D}_{\mathrm{cons}}$ with filtered colimits in Λ (Synopsis 3.2(iv)) that both sides of (5-1) are compatible with filtered colimits in Λ.

(2) *Finite products in* Λ. Let $\Lambda = \prod \Lambda_i$ be a finite product of condensed rings. For any scheme X, the natural map $\mathrm{D}_\bullet(X, \Lambda) \to \prod \mathrm{D}_\bullet(X, \Lambda_i)$ is an equivalence for $\bullet \in \{\varnothing, \mathrm{lis}, \mathrm{cons}\}$, and likewise for Weil sheaves if X is defined over $\mathbb{F}_q$. As $\Lambda_* = \prod \Lambda_{i,*}$, we see that (5-1) is compatible finite products in the coefficients.

(3) *Limits in* X_i *for discrete* Λ. Assume that Λ is finite discrete; see Situation 5.1(a), (d). Let $X_1, \dots, X_n$ be qcqs $\mathbb{F}_q$-schemes. Write each X_i as a cofiltered limit $X_i = \lim X_{ij}$ of finite type $\mathbb{F}_q$-schemes X_{ij} with

affine transition maps [Stacks 2017, Tag 01ZA]. As Λ is finite discrete, we can use the continuity of étale sites as in (4-16) to show that the natural map

$$\mathrm{colim}_j\, \mathrm{D}_\bullet(X_{1j}^{\mathrm{Weil}} \times \cdots \times X_{nj}^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} \mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \cdots \times X_n^{\mathrm{Weil}}, \Lambda),$$

is an equivalence for $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$. Thus, (5-1) is compatible with cofiltered limits of finite type $\mathbb{F}_q$-schemes with affine transition maps.

5A. ***A formulation in terms of prestacks.*** Before turning to the proof, we point out a formulation of the results of the previous subsection in terms of symmetric monoidality of a certain sheaf theory. This formulation makes the connection with constructions in the geometric approaches to the Langlands program [Gaitsgory et al. 2022; Zhu 2021; Lafforgue and Zhu 2019] more manifest. Readers not familiar with prestacks and formulations of sheaf theories on them can safely skip this section. The categories of constructible, resp. lisse Λ-sheaves assemble into a lax symmetric monoidal functor

$$\mathrm{D}_{\bullet,\Lambda} : (\mathrm{Sch}_{\mathbb{F}})^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem}) \quad (\bullet = \mathrm{lis\ or\ cons}). \tag{5-2}$$

Namely, as a functor it sends a scheme X to the category of constructible, resp. lisse Λ-sheaves on X, and a morphism $f : X \to Y$ to the functor $f^* : \mathrm{D}_\bullet(Y, \Lambda) \to \mathrm{D}_\bullet(X, \Lambda)$. These are objects, resp. maps in the ∞-category $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem}) := \mathrm{Mod}_{\mathrm{Perf}_\Lambda}(\mathrm{Cat}^{\mathrm{Ex}}_\infty(\mathrm{Idem}))$, see Section 2A for notation. The lax monoidal structure is given by the external tensor product of sheaves

$$\boxtimes : \mathrm{D}_\bullet(X_{\mathrm{proét}}, \Lambda) \otimes_{\mathrm{Perf}_\Lambda} \mathrm{D}_\bullet(Y_{\mathrm{proét}}, \Lambda) \to \mathrm{D}_\bullet((X \times_{\mathbb{F}} Y)_{\mathrm{proét}}, \Lambda).$$

That is, we consider the category of schemes as symmetric monoidal with respect to the fiber product over $\mathbb{F}$, and the external tensor product is natural on X and Y in the appropriate sense; see [Gaitsgory and Lurie 2019, Section 3.1; Gaitsgory and Rozenblyum 2017, Section III.2] for details and precise statements. This functor $\boxtimes$ often fails to be an equivalence, so $\mathrm{D}_{\bullet,\Lambda}$ is not symmetric monoidal. The assertion of Theorem 5.2 is that this issue is resolved by replacing sheaves with Weil sheaves. In order to formulate Theorem 5.2 as the monoidality of a certain functor, we need to replace the category of schemes by a category of objects that model Weil sheaves. We will represent these by taking the appropriate formal quotient by the partial Frobenius automorphism. Such formal quotients can be taken in the category of prestacks.

We denote by $\mathrm{PreStk}_{\mathbb{F}}$ the category of (accessible) functors from the category $\mathrm{CAlg}_{\mathbb{F}}$ of commutative algebras over $\mathbb{F}$ to the ∞-category Ani of Anima. The functor of taking points embeds the category of schemes fully faithfully into $\mathrm{PreStk}_{\mathbb{F}}$. We denote by

$$\mathrm{D}_{\bullet,\Lambda} : (\mathrm{PreStk}_{\mathbb{F}})^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem}) \tag{5-3}$$

the functor obtained by right Kan extension [Lurie 2009, Section 4.3.2] along the inclusion $(\mathrm{Sch}^{\mathrm{fp}}_{\mathbb{F}})^{\mathrm{op}} \subset (\mathrm{PreStk}_{\mathbb{F}})^{\mathrm{op}}$. Concretely, [Lurie 2018, Propositions 6.2.1.9 and 6.2.3.1], given a prestack Y which can be written as a colimit of schemes Y_α over some indexing category A we have a canonical equivalence

$$\mathrm{D}_\bullet(Y, \Lambda) \cong \lim_\alpha \mathrm{D}_\bullet(Y_\alpha, \Lambda). \tag{5-4}$$

This limit is formed in $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem})$; recall from around (2-3) that the Ind-completion functor to $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem}) \to \mathrm{Pr}^{\mathrm{St}}_\Lambda$ does *not* preserve (even finite) limits.

With this general sheaf theory in place, we can restrict our attention to the class of prestacks that is relevant to the derived Drinfeld lemma.

Definition 5.4. Let X be a scheme over $\mathbb{F}_q$. The *Weil prestack* is defined as

$$X^{\mathrm{Weil}} := \mathrm{colim}(X \times_{\mathbb{F}_q} \mathbb{F} \overset{\mathrm{id}}{\underset{\phi_X}{\rightrightarrows}} X \times_{\mathbb{F}_q} \mathbb{F}) \in \mathrm{PreStk}_{\mathbb{F}},$$

i.e., it is the prestack sending $R \in \mathrm{CAlg}_{\mathbb{F}}$ to the colimit

$$X^{\mathrm{Weil}}(R) = \mathrm{colim}(X(R) \overset{\mathrm{id}}{\underset{\phi_X}{\rightrightarrows}} X(R)). \tag{5-5}$$

We denote by $\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{Weil}}$ the smallest full monoidal subcategory of $\mathrm{PreStk}_{\mathbb{F}}$ containing the Weil prestacks of finite type schemes $X/\mathbb{F}_q$. Equivalently, this is the full subcategory consisting of finite products of the form $X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}$.

Lemma 5.5. *Let $X_1, \dots, X_n$ be schemes over $\mathbb{F}_q$. There is a canonical equivalence*

$$\mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \times_{\mathbb{F}} \cdots \times_{\mathbb{F}} X_n^{\mathrm{Weil}}) \xrightarrow{\cong} \mathrm{Fix}(\mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda), \phi^*_{X_1}, \dots, \phi^*_{X_n}). \tag{5-6}$$

Proof. Let $\Phi\colon \mathrm{B}\mathbb{Z}^n \to \mathrm{PreStk}_{\mathbb{F}}$ be the functor corresponding to the commuting automorphisms ϕ_{X_i}. Then the claim follows immediately from the identification of $X_1^{\mathrm{Weil}} \times_{\mathbb{F}} \cdots \times_{\mathbb{F}} X_n^{\mathrm{Weil}}$ with the colimit of Φ (as an object in $\mathrm{PreStk}_{\mathbb{F}}$). ☐

Theorem 5.6. *Suppose $\bullet$ and Λ are as in Theorem 5.2. Then the restriction of* $\mathrm{D}_{\bullet,\Lambda}$ *to Weil prestacks, i.e., the following composite*

$$\mathrm{D}_{\bullet,\Lambda}\colon (\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{Weil}})^{\mathrm{op}} \subset \mathrm{PreStk}_{\mathbb{F}} \to \mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem}), \tag{5-7}$$

is symmetric monoidal.

Proof. As was noted above, the functor in (5-2) is lax symmetric monoidal. By [Torii 2023, Proposition 2.7], the Kan extension in (5-3) is still lax symmetric monoidal. To check its restriction to the (symmetric monoidal) subcategory $\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{Weil}}$ is symmetric monoidal it suffices to show that the lax monoidal maps are in fact isomorphisms. This is precisely the content of Theorem 5.2. ☐

5B. *Full faithfulness.* In this section, we prove that the functor (5-1) is fully faithful under the conditions of Theorem 5.2. We first consider the p-torsion free cases:

Proposition 5.7. *Let $X_1, \dots, X_n$ and Λ be as in Situation 5.1(a), (b) or (c). Then the functor* (5-1) *is fully faithful for $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$.*

Proof. For constructible sheaves on $X_{i,\mathbb{F}}$ (as opposed to X_i^{Weil}), this interpretation of the Künneth formula appears already in [Gaitsgory et al. 2022, Section A.2]. Throughout, we drop Λ from the notation. It is enough to verify that for all $M_i, N_i \in \mathrm{D}_{\mathrm{cons}}(X_i^{\mathrm{Weil}})$ the natural map

$$\bigotimes_{i=1}^{n} \mathrm{Hom}_{\mathrm{D}(X_i^{\mathrm{Weil}})}(M_i, N_i) \to \mathrm{Hom}_{\mathrm{D}(X_1^{\mathrm{Weil}}\times\cdots\times X_n^{\mathrm{Weil}})}(M_1 \boxtimes \cdots \boxtimes M_n, N_1 \boxtimes \cdots \boxtimes N_n) \tag{5-8}$$

is an equivalence. As (5-8) is functorial in the objects and compatible with shifts, it suffices, by Definition 4.10, to consider the case where M_i, $i = 1, \dots, n$ is the extension by zero of a lisse Weil Λ-sheaf on some locally closed subscheme $Z_i \subset X_i$. Using the adjunction

$$(\iota_i)_! : \mathrm{D}_{\mathrm{cons}}(Z_i^{\mathrm{Weil}}) \rightleftarrows \mathrm{D}_{\mathrm{cons}}(X_i^{\mathrm{Weil}}) : (\iota_i)^!,$$

and the dualizability of lisse sheaves, we reduce to the case $M_i = \Lambda_{X_i}$, $i = 1, \dots, n$. That is, (5-8) becomes a map of cohomology complexes. By Proposition 4.4, we have

$$\mathrm{R}\Gamma(X_i^{\mathrm{Weil}}, N_i) = \mathrm{Fib}(\mathrm{R}\Gamma(X_{i,\mathbb{F}}, N_i) \xrightarrow{\phi_{X_i}^* - \mathrm{id}} \mathrm{R}\Gamma(X_{i,\mathbb{F}}, N_i)). \tag{5-9}$$

A similar computation holds for the mapping complexes in $\mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}})$; see (4-6). Such finite limits commute with the tensor product in Mod_Λ. Thus, (5-9) reduces to the Künneth formula

$$\mathrm{R}\Gamma(X_{1,\mathbb{F}}, N_1) \otimes \cdots \otimes \mathrm{R}\Gamma(X_{n,\mathbb{F}}, N_n) \xrightarrow{\cong} \mathrm{R}\Gamma(X_{1,\mathbb{F}} \times_{\mathbb{F}} \cdots \times_{\mathbb{F}} X_{n,\mathbb{F}}, N_1 \boxtimes \cdots \boxtimes N_n),$$

where we use that the X_i are of finite type and the coprimality assumptions on Λ; see [Stacks 2017, Tag 0F1P]. □

Next, we consider the p-torsion case.

Proposition 5.8. *Let $X_1, \dots, X_n$ and Λ be as in Situation 5.1(d). Then the functor* (5-1) *is fully faithful for* $\bullet = \mathrm{lis}$.

Proof. As in the proof of Proposition 5.7, we need to show that the map

$$\bigotimes_{i=1}^{n} \mathrm{R}\Gamma(X_i^{\mathrm{Weil}}, N_i) \to \mathrm{R}\Gamma(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, N_1 \boxtimes \cdots \boxtimes N_n) \tag{5-10}$$

is an equivalence for any $N_i \in \mathrm{D}_{\mathrm{lis}}(X_i^{\mathrm{Weil}})$. Using Zariski descent for both sides, we may assume that each X_i is affine. As Λ is finite discrete (see also the discussion around (4-16)), the invariance of the étale site under perfection reduces us to the case where each X_i is perfect. The proof now proceeds by several reduction steps: (1) Reduce to $N_i = \Lambda_{X_i}$. (2) Reduce to $\Lambda = \mathbb{Z}/p$. (3) Reduce to $q = p$ being a prime. (4) The last step is then an easy computation.

Step 1 (*we may assume $N_i = \Lambda_{X_i}$*). In order to show (5-10) is a quasiisomorphism, it suffices to show this after applying $\tau^{\leq r}$ for arbitrary r. The complexes N_i are bounded (Synopsis 3.2(v)). By shifting them appropriately, we may assume $r = 0$. Note that $\mathrm{R}\Gamma(X_i^{\mathrm{Weil}}, N_i) \cong \mathrm{R}\Gamma(X_i, N_i)$; see Proposition 4.16. By right exactness of the tensor product, we have $\tau^{\leq 0}\big(\bigotimes_i \mathrm{R}\Gamma(X_i, N_i)\big) = \bigotimes_i \tau^{\leq 0}\mathrm{R}\Gamma(X_i, N_i)$. By the comparison with the classical notion of constructible sheaves (for discrete coefficients, see [Hemo et al. 2023, Proposition 7.1] and the discussion preceding it), there is an étale covering $U_i \to X_i$ such that $N_i|_{U_i}$ is perfect-constant. Let $U_{i,\bullet}$ be the Čech nerve of this covering. By étale descent, we have

$$\mathrm{R}\Gamma(X_i, N_i) = \lim_{[j]\in\Delta} \mathrm{R}\Gamma(U_{i,j}, N_i).$$

For each $r \in \mathbb{Z}$, there is some j_r such that

$$\tau^{\leq r} \lim_{[j]\in\Delta} \mathrm{R}\Gamma(U_{i,j}, N_i) = \lim_{[j]\in\Delta,\, j\leq j_r} \tau^{\leq r}\mathrm{R}\Gamma(U_{i,j}, N_i).$$

This can be seen from the spectral sequence (note that it is concentrated in degrees $j \geq 0$ and degrees $j' \geq r$ for some r, since the complexes N_i are bounded from below)

$$\mathrm{H}^{j'}(U_{i,j}, N_i) \Rightarrow \mathrm{H}^{j'+j} \lim_{j\in\Delta} \mathrm{R}\Gamma(U_{i,j}, N_i) = \mathrm{H}^{j'+j}(X_i, N_i).$$

As the tensor product in (5-10) commutes with *finite* limits, we may thus assume that each N_i is perfect-constant. Another dévissage reduces us to the case $N_i = \Lambda_{X_i}$, the constant sheaf itself.

Step 2 (*we may assume* $\Lambda = \mathbb{Z}/p$). By assumption, Λ is flat over $\mathbb{Z}/p^m$ for some $m \geq 1$. We immediately reduce to $\Lambda = \mathbb{Z}/p^m$. For any perfect affine scheme $X = \operatorname{Spec} R$ in characteristic $p > 0$, we claim that $\mathrm{R}\Gamma(X, \mathbb{Z}/p^m) \otimes_{\mathbb{Z}/p^m} \mathbb{Z}/p^r \cong \mathrm{R}\Gamma(X, \mathbb{Z}/p^r)$. Assuming the claim, we finish the reduction step by tensoring (5-10) with the short exact sequence of $\mathbb{Z}/p^m$-modules $0 \to \mathbb{Z}/p^{m-1} \to \mathbb{Z}/p^m \to \mathbb{Z}/p \to 0$, using that finite limits commutes with tensor products. It remains to prove the claim. The Artin–Schreier–Witt exact sequence of sheaves on $X_{\text{ét}}$ yields

$$\mathrm{R}\Gamma(X, \mathbb{Z}/p^m) = [W_m(R) \xrightarrow{F-\mathrm{id}} W_m(R)].$$

Now we use that $W_m(R) \otimes_{\mathbb{Z}/p^m} \mathbb{Z}/p^r \xrightarrow{\cong} W_r(R)$ compatibly with F, which holds since R is perfect. This shows the claim, and we have accomplished Step 2.

Step 3 (*we may assume* q *is prime*). Recall that $q = p^r$ is a prime power. In order to reduce to the case $r = 1$, let $X_i' := X_i$, but now regarded as a scheme over $\mathbb{F}_p$. We have $X'_{i,\mathbb{F}} = \bigsqcup_{i=1}^r X_{i,\mathbb{F}}$. The Galois group $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ is generated by the p-Frobenius, which acts by permuting the components in this disjoint union. Thus, we have $\mathrm{D}((X_i')^{\mathrm{Weil}}) = \mathrm{D}(X_i^{\mathrm{Weil}})$. The same reasoning also applies to several factors X_i^{Weil}, so we may assume our ground field to be $\mathbb{F}_p$.

Step 4. Set $R := \bigotimes_{i,\mathbb{F}_p} R_i$, $R_{\mathbb{F}} := R \otimes_{\mathbb{F}_p} \mathbb{F}$. We write ϕ_i for the p-Frobenius on R_i and also for any map on a tensor product involving R_i, by taking the identity on the remaining tensor factors. By Artin–Schreier theory, we have

$$\mathrm{R}\Gamma(X_i^{\mathrm{Weil}}, \mathbb{Z}/p) \overset{*}{=} \mathrm{R}\Gamma(X_i, \mathbb{Z}/p) = [R_i \xrightarrow{\phi_i - \mathrm{id}} R_i],$$

$$\mathrm{R}\Gamma(X_{1,\mathbb{F}} \times_{\mathbb{F}} \cdots \times_{\mathbb{F}} X_{n,\mathbb{F}}, \mathbb{Z}/p) = [R_{\mathbb{F}} \xrightarrow{\phi - \mathrm{id}} R_{\mathbb{F}}],$$

where the equality $*$ follows from Proposition 4.16 and ϕ is the absolute p-Frobenius of $R_{\mathbb{F}}$. Thus, the right hand side in (5-10) is the homotopy orbits of the action of $\mathbb{Z}^{n+1}$ on $R_{\mathbb{F}}$, whose basis vectors act as $\phi_1, \dots, \phi_n$ and ϕ. Note that ϕ is the composite $\phi_{\mathbb{F}} \circ \phi_1 \circ \dots \circ \phi_n$, where $\phi_{\mathbb{F}}$ is the Frobenius on $\mathbb{F}$. Thus, the previously mentioned $\mathbb{Z}^{n+1}$-action on $R_{\mathbb{F}}$ is equivalent to the one where the basis vectors act as $\phi_1, \dots, \phi_n$ and $\phi_{\mathbb{F}}$. We conclude our claim by using that $[R_{\mathbb{F}} \xrightarrow{\mathrm{id} - \phi_{\mathbb{F}}} R_{\mathbb{F}}]$ is quasiisomorphic to $R[0]$. □

5C. *Drinfeld's lemma.* The essential surjectivity in Theorem 5.2 is based on the following variant of Drinfeld's lemma [1980, Theorem 2.1]; see also [Lafforgue 1997, IV.2, Theorem 4; 2018, Lemme 8.11; Lau 2004, Theorem 8.1.4; Kedlaya 2019, Theorem 4.2.12; Heinloth 2018, Lemma 6.3; Scholze and Weinstein 2020, Theorem 16.2.4] for expositions. Its formulation is close to [Lau 2004, Theorem 8.1.4], and in this form is a slight extension of [Lafforgue 2018, Lemme 8.2] for $\mathbb{Z}_\ell$-coefficients and [Xue 2020b, Lemma 3.3.2] for $\mathbb{Q}_\ell$-coefficients. We will drop the coefficient ring Λ from the notation whenever convenient.

Let $X_1, \dots, X_n$ be Noetherian schemes over $\mathbb{F}_q$, and denote $X = X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n$. Recall the Frobenius–Weil groupoid $\mathrm{FWeil}(X)$, see Definition 4.12. The projections $X_{\mathbb{F}} \to X_{i,\mathbb{F}}$ onto the single factors induce a continuous map of locally profinite groupoids

$$\mu\colon \mathrm{FWeil}(X) \to \mathrm{Weil}(X_1) \times \cdots \times \mathrm{Weil}(X_n). \tag{5-11}$$

Theorem 5.9 (version of Drinfelds's lemma). *Let Λ be as in Situation 5.1. Restriction along the map* (5-11) *induces an equivalence*

$$\mathrm{Rep}_\Lambda(\mathrm{Weil}(X_1) \times \cdots \times \mathrm{Weil}(X_n)) \xrightarrow{\cong} \mathrm{Rep}_\Lambda(\mathrm{FWeil}(X)), \tag{5-12}$$

between the abelian categories of continuous representations on finitely presented Λ-modules.

Proof. For all objects $x \in \mathrm{FWeil}(X)$, that is, all geometric points $x \to X_{\mathbb{F}}$, passing to the automorphism groups induces a commutative diagram of locally profinite groups:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X_{\mathbb{F}}, x) \arrow[r] \arrow[d] & \mathrm{FWeil}(X, x) \arrow[r] \arrow[d, "\mu_x"] & \mathbb{Z}^n \arrow[d, equal] \\
1 \arrow[r] & \prod_{i=1}^n \pi_1(X_{i,\mathbb{F}}, x) \arrow[r] & \prod_{i=1}^n \mathrm{Weil}(X_i, x) \arrow[r] & \mathbb{Z}^n
\end{tikzcd}$$

The left vertical arrow is surjective [Stacks 2017, Tags 0BN6, 0385]. Thus μ_x is surjective as well and hence (5-12) is fully faithful. For essential surjectivity, it remains to show that any continuous representation $\mathrm{FWeil}(X, x) \to \mathrm{GL}(M)$ on a finitely presented Λ-module M factors through μ_x. The key input is Drinfeld's lemma: it implies that μ_x induces an isomorphism on profinite completions. Therefore, it is enough to apply Lemma 5.10 below with $H := \mathrm{FWeil}(X, x) \to \mathrm{Weil}(X_1) \times \cdots \times \mathrm{Weil}(X_n) =: G$ and $K := \pi_1(X_{\mathbb{F}}, x)$. This completes the proof of (5-12). □

The following lemma formalizes a few arguments from [Xue 2020b, Section 3.2.3], and we reproduce the proof for the convenience of the reader.

Lemma 5.10 (Drinfeld, Xue). *Let Λ be as in Situation 5.1. Let $\mu\colon H \to G$ be a continuous surjection of locally profinite groups that induces an isomorphism on profinite completions. Assume that there exists a compact open normal subgroup $K \subset H$ containing* $\ker\mu$ *such that H/K is finitely generated and injects into its profinite completion. Then μ induces an equivalence*

$$\mathrm{Rep}_\Lambda(G) \cong \mathrm{Rep}_\Lambda(H)$$

between their categories of continuous representations on finitely presented Λ-modules.

Proof. The case where Λ is finite discrete is obvious, and hence so is the case $\Lambda = \mathcal{O}_E$ for some finite field extension $E \supset \mathbb{Q}_\ell$. The case $\Lambda = E$ is reduced to $\Lambda = \mathbb{Q}_\ell$. As μ is surjective, it remains to show that every continuous representation $\rho\colon H \to \mathrm{GL}(M)$ on a finite-dimensional $\mathbb{Q}_\ell$-vector space factors through G, that is, $\ker\mu \subset \ker\rho$. One shows the following properties:

(1) The group $\ker\mu$ is the intersection over all open subgroups in K which are normal in H.

(2) The group $\ker\rho \cap K$ is a closed normal subgroup in H such that $K/\ker\rho\cap K \cong \rho(K)$ is topologically finitely generated.

These properties imply $\ker\mu \subset \ker\rho\cap K$ as follows: For a finite group L, let $U_L := \cap\ker(K \to L)$ where the intersection is over all continuous morphisms $K \to L$ that are trivial on $\ker\rho\cap K$. Because of the topologically finitely generatedness in (2), this is a finite intersection so that U_L is open in K. Also, it is normal in H, and hence $\ker\mu \subset U_L$ by (1). On the other hand, it is evident that $\ker\rho\cap K = \bigcap_L U_L$ because K is profinite.

For the proof of (1) observe that $\ker\mu$ agrees with the kernel of $H \to H^\wedge \cong G^\wedge$ by our assumption on the profinite completions. Using $\ker\mu \subset K$ and the injection $H/K \to (H/K)^\wedge$ implies (1).

For (2) it is evident that $\ker\rho\cap K$ is a closed normal subgroup in H. Since K is compact, its image $\rho(K)$ is a closed subgroup of the ℓ-adic Lie group $\mathrm{GL}(M)$, hence an ℓ-adic Lie group itself. The final assertion follows from [Serre 1966, Théorème 2]. □

For the overall goal of proving essential surjectivity in Theorem 5.2, we need to investigate how representations of product groups factorize into external tensor products of representations. In view of Lemma 4.13 and its proof, it is enough to consider representations of abstract groups, disregarding the topology. This is done in the next section.

5D. ***Factorizing representations.*** In this subsection, let Λ be a Dedekind domain [Stacks 2017, Tag 034X]. Thus, any submodule N of a finite projective Λ-module M is again finite projective.

Given any group W, we write $\mathrm{Rep}^{\mathrm{fp}}_\Lambda(W)$ for the category of W-representations on finite projective Λ-modules. As in [Curtis and Reiner 1962, Sections 73.8 and 75], we say that such a W-representation M is *fp-simple* if any subrepresentation $0 \neq N \subset M$ has maximal rank. By induction on the rank, every nonzero representation in $\mathrm{Rep}^{\mathrm{fp}}_\Lambda(W)$ admits a nonzero fp-simple subrepresentation. The proof of the following lemma is left to the reader. It parallels [loc. cit., Theorem 75.6].

Lemma 5.11. *A representation $M \in \mathrm{Rep}^{\mathrm{fp}}_{\Lambda}(W)$ is fp-simple if and only if $M \otimes_{\Lambda} \mathrm{Frac}(\Lambda)$ is fp-simple (hence, simple).*

The following proposition will serve in the proof of Theorem 5.2 using Theorem 5.9, where we will need to decompose representations of a product of Weil groups into decompositions of the individual Weil groups.

Proposition 5.12. *Let $W = W_1 \times W_2$ be a product of two groups. Let $M \in \mathrm{Rep}^{\mathrm{fp}}_{\Lambda}(W)$ be fp-simple. Fix a W_1-subrepresentation $M_1 \subset M$ that is fp-simple. Consider the W_2-representation $M_2 := \mathrm{Hom}_{W_1}(M_1, M)$ and the associated evaluation map*

$$\mathrm{ev}\colon M_1 \boxtimes M_2 \to M.$$

(1) *If Λ is an algebraically closed field, then* ev *is an isomorphism and M_2 is simple.*

(2) *If Λ is a perfect field, then* ev *is a split surjection and M_2 is semisimple.*

(3) *If Λ is a Dedekind domain of Krull dimension* 1 *with perfect fraction field, then there is a short exact sequence*

$$0 \to M \oplus \ker \mathrm{ev} \to M_1 \boxtimes M_2 \to T \to 0, \tag{5-13}$$

where T is Λ-torsion.

Proof. Note that ev is a map in $\mathrm{Rep}^{\mathrm{fp}}_{\Lambda}(W)$. Its image has maximal rank by the fp-simplicity of M. Thus, if Λ is a field, then it is surjective.

In case (1), we claim that ev is an isomorphism. The following argument was explained to us by Jean-François Dat: For injectivity, observe that $M_1 \boxtimes M_2 = M_1^{\oplus \dim M_2}$ as W_1-representations. Hence, if the kernel of ev is nontrivial, then it contains M_1 as an irreducible constituent. Therefore, it suffices to prove that $\mathrm{Hom}_{W_1}(M_1, \mathrm{ev})$ is injective. Since Λ is algebraically closed, we have $\mathrm{End}_{W_1}(M_1) = \Lambda$ by Schur's lemma. Hence, the composition

$$M_2 = \mathrm{Hom}_{W_1}(\mathrm{End}_{W_1}(M_1), M_2) \cong \mathrm{Hom}_{W_1}(M_1, M_1 \boxtimes M_2) \to \mathrm{Hom}_{W_1}(M_1, M) = M_2$$

is the identity. This shows that $\mathrm{Hom}_{W_1}(M_1, \mathrm{ev})$ is an isomorphism.

In case (2), we claim that $M_1 \boxtimes M_2$ is semisimple, and hence that M appears as a direct summand. Using [Bourbaki 2012, Section 13.4 Corollaire] applied to the group algebras it is enough to show that M_1 and M_2 are absolutely semisimple. Since Λ is perfect, any finite-dimensional representation is semisimple if and only if it is absolutely semisimple; see [loc. cit., Section 13.1]. Hence, it remains to check that $M_{2,\overline{\Lambda}} = M_2 \otimes_{\Lambda} \overline{\Lambda}$ is semisimple where $\overline{\Lambda}/\Lambda$ is an algebraic closure. The module $M_{2,\overline{\Lambda}} = \mathrm{Hom}_{W_1}(M_{1,\overline{\Lambda}}, M_{\overline{\Lambda}})$ splits as a direct sum according to the simple constituents $\overline{M}_1 \subset M_{1,\overline{\Lambda}}$ and $\overline{M} \subset M_{\overline{\Lambda}}$. Finally, each $\overline{M}_2 = \mathrm{Hom}_{W_1}(\overline{M}_1, \overline{M})$ is either simple or vanishes: If there exists a nonzero W_1-equivariant map $\overline{M}_1 \to \overline{M}$, then it must be injective by the simplicity of $\overline{M}_1$. As $\overline{\Lambda}$ is algebraically closed, the proof of (1) shows that $\overline{M} \cong \overline{M}_1 \boxtimes \overline{M}_2$ so that $\overline{M}_2$ must be simple because $\overline{M}$ is so. This shows that M_2 is absolutely semisimple as well.

In case (3), abbreviate $\Lambda' := \operatorname{Frac}\Lambda$, $M' := M \otimes_\Lambda \Lambda'$ and so on. We will repeatedly use that $(-)\otimes_\Lambda \Lambda'$ preserves and detects fp-simplicity of representations, see Lemma 5.11. By (2), the evaluation map $\mathrm{ev}' := \mathrm{ev}\otimes\Lambda'$ admits a Λ'-linear section $\tilde{i}\colon M' \to (M_1 \boxtimes M_2)'$. As M' is finitely presented, there is some $0 \neq \lambda \in \Lambda$ such that $\lambda\tilde{i}$ arises by scalar extension of a map $i\colon M \to M_1 \boxtimes M_2$. By construction, the map $i \oplus \mathrm{incl}\colon M \oplus \ker(\mathrm{ev}) \to M_1 \boxtimes M_2$ is an isomorphism after tensoring with Λ'. So its cokernel is Λ-torsion, and it is injective as both modules at the left are projective (hence Λ-torsion free). This finishes the proof of the proposition. □

5E. *Essential surjectivity.* In this section, we prove the essential surjectivity asserted in Theorem 5.2. Throughout, we freely use the full faithfulness proven in Propositions 5.7 and 5.8.

Recall that $X_1, \dots, X_n$ are finite type $\mathbb{F}_q$-schemes, and write $X := X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n$. Let Λ be either a finite discrete ring, a finite field extension $E \supset \mathbb{Q}_\ell$ for $\ell \neq p$ or its ring of integers $\mathcal{O}_E$. Note that this covers all cases from Situation 5.1.

First, we show that it suffices to prove containment in the essential image étale locally.

Lemma 5.13. *Let $U_i \to X_i$ be quasicompact étale surjections for $i = 1, \dots, n$. Then the following properties hold*:

(1) *An object $M \in \mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$ belongs to the full subcategory*

$$\mathrm{D}_{\mathrm{cons}}(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \cdots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}(X_n^{\mathrm{Weil}}, \Lambda)$$

if and only if its restriction $M|_{U_1^{\mathrm{Weil}} \times \cdots \times U_n^{\mathrm{Weil}}}$ belongs to the full subcategory

$$\mathrm{D}_{\mathrm{cons}}(U_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \cdots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}(U_n^{\mathrm{Weil}}, \Lambda) \subset \mathrm{D}(U_1^{\mathrm{Weil}} \times \cdots \times U_n^{\mathrm{Weil}}, \Lambda).$$

(2) *Assume that all $U_i \to X_i$ are finite étale. Then (1) holds for the categories of lisse sheaves.*

Proof. The only if direction in part (1) is clear. Conversely, assume that $M|_{U_1^{\mathrm{Weil}} \times \cdots \times U_n^{\mathrm{Weil}}}$ lies in the essential image of the external tensor product. By étale descent, we have an equivalence

$$\mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} \mathrm{Tot}(\mathrm{D}(U_{1,\bullet}^{\mathrm{Weil}} \times \cdots \times U_{n,\bullet}^{\mathrm{Weil}}, \Lambda)).$$

In particular, we get an equivalence $|(j_\bullet)_! \circ j_\bullet^* M| \xrightarrow{\sim} M$ where $j_\bullet := j_{1,\bullet} \times \cdots \times j_{n,\bullet}$ with $j_{i,\bullet}\colon U_{i,\bullet} \to X_i$ for $i = 1, \dots, n$. For each $m \geq 0$, the object $j_m^* M$ lies in

$$\mathrm{D}_{\mathrm{cons}}(U_{1,m}^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \cdots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}(U_{n,m}^{\mathrm{Weil}}, \Lambda).$$

It follows from Synopsis 3.2(ii) that these subcategories are preserved under $(j_m)_!$. So we see

$$(j_m)_! j_m^*(M) \in \mathrm{D}_{\mathrm{cons}}(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \cdots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}(X_n^{\mathrm{Weil}}, \Lambda)$$

for all $m \geq 0$. For every $m \geq 0$, let M_m denote the realization of the m-th skeleton of the simplicial object $(j_\bullet)_! \circ j_\bullet^* M$ so that we have a natural equivalence $\operatorname{colim} M_m \xrightarrow{\cong} M$ in $\mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$. We claim that M is a retract of some M_m, and hence lies in $\mathrm{D}_{\mathrm{cons}}(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \cdots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}(X_n^{\mathrm{Weil}}, \Lambda)$ by idempotent completeness. To prove the claim, note that the sheaf $M_{\mathbb{F}} \in \mathrm{D}_{\mathrm{cons}}(X_{\mathbb{F}}, \Lambda)$ underlying M is compact in the category of ind-constructible sheaves $\mathrm{D}_{\mathrm{indcons}}(X_{\bar{\mathbb{F}}}, \Lambda)$, see Synopsis 3.2(viii). As taking partial Frobenius fixed points is a finite limit, so commutes with filtered colimits, we see that the natural map of mapping complexes

$$\operatorname{colim} \mathrm{Hom}_{\mathrm{D}(X_1^{\mathrm{Weil}}\times\cdots\times X_n^{\mathrm{Weil}},\Lambda)}(M, M_m) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{D}(X_1^{\mathrm{Weil}}\times\cdots\times X_n^{\mathrm{Weil}},\Lambda)}(M, \operatorname{colim} M_m)$$

is an equivalence. In particular, the inverse equivalence $M \xrightarrow{\cong} \operatorname{colim} M_m$ factors through some M_m, presenting M as a retract of M_m. This proves the claim, and hence (1).

For (2), note that if $U_i \to X_i$ are finite étale, then the functors $(j_m)_!$ preserve the lisse categories; see Synopsis 3.2(ii). In particular, for every $m \geq 0$ the object $(j_m)_! j_m^*(M)$ is lisse and so is M_m. We conclude using compactness as before. □

Using Lemma 2.4 and Synopsis 3.2(viii), the fully faithful functor (5-1) uniquely extends to a fully faithful functor

$$\mathrm{Ind}(\mathrm{D}_\bullet(X_1^{\mathrm{Weil}}, \Lambda)) \otimes_{\mathrm{Mod}_{\Lambda_*}} \cdots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{Ind}(\mathrm{D}_\bullet(X_n^{\mathrm{Weil}}, \Lambda)) \to \mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda) \qquad (5\text{-}14)$$

for $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$. We use this in the following variant of Lemma 5.13.

Lemma 5.14. *The statements (1) and (2) of Lemma 5.13 hold for the functor* (5-14) *with* $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$. *Namely, to check that an object lies in the essential image of* (5-14), *one can pass to a quasicompact étale cover if* $\bullet = \mathrm{cons}$, *and to a finite étale cover if* $\bullet = \mathrm{lis}$.

Proof. This is immediate from the proof of Lemma 5.13: Arguing as above and using étale descent for ind-constructible, resp. ind-lisse sheaves (Synopsis 3.2(iii)), we see that $M \cong \operatorname{colim} M_m$ with

$$M_m \in \mathrm{Ind}(\mathrm{D}_\bullet(X_1^{\mathrm{Weil}}, \Lambda)) \otimes_{\mathrm{Mod}_{\Lambda_*}} \cdots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{Ind}(\mathrm{D}_\bullet(X_n^{\mathrm{Weil}}, \Lambda))$$

for all $m \geq 0$ and $\bullet = \mathrm{cons}$, resp. $\bullet = \mathrm{lis}$. As the essential image of (5-14) is closed under colimits, M lies in the corresponding subcategory as well. □

Now we have enough tools to prove the categorical Künneth formula alias derived Drinfeld's lemma:

Proof of Theorem 5.2. In view of Propositions 5.7 and 5.8, it remains to show the essential surjectivity of the external tensor product functor on Weil sheaves (5-1) under the assumptions in Theorem 5.2. Part (1), the case of constructible sheaves, is reduced to part (2), the case of lisse sheaves, by taking a stratification as in Definition 4.10(2) and using the full faithfulness already proven. Here we note that by refining the stratification witnessing the constructibility if necessary, we can even assume all strata to be smooth, so in particular geometrically unibranch. Hence, it remains to prove part (2), that is, the essential surjectivity of the fully faithful functor

$$\boxtimes\colon \mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \cdots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{lis}}(X_n^{\mathrm{Weil}}, \Lambda) \to \mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda), \qquad (5\text{-}15)$$

when either Λ is finite discrete as in cases (a), (d) in Theorem 5.2(2), or $\Lambda = \mathcal{O}_E$ for a finite field extension $E \supset \mathbb{Q}_\ell$, $\ell \neq p$ as in case (b), or $\Lambda = E$ and the X_i are geometrically unibranch as in the remaining case (c). In fact, the latter two cases are easier to handle due to the presence of natural t-structures on the categories of lisse sheaves (Synopsis 3.2(vi)). So we will distinguish two cases below: (1) $\Lambda = \mathcal{O}_E$, or $\Lambda = E$ and all X_i geometrically unibranch. (2) Λ is finite discrete.

Now pick $M \in \mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$. By Synopsis 3.2(v), M is bounded in the standard t-structure on $\mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$. So M is a successive extension of its cohomology sheaves $\mathrm{H}^j(M)$, $j \in \mathbb{Z}$. As M is lisse, Lemma 4.15(1) shows in both cases (1) and (2) that each $\mathrm{H}^j(M)$ comes from a continuous representation on a finitely presented Λ-module in

$$\mathrm{Rep}_\Lambda(\mathrm{FWeil}(X)) \cong \mathrm{Rep}_\Lambda(W), \tag{5-16}$$

where we denote $W := W_1 \times \cdots \times W_n$ with $W_i := \mathrm{Weil}(X_i)$ and the equivalence follows from Theorem 5.9.

Throughout, we repeatedly use that the functor (5-15) is fully faithful, commutes with finite (co-)limits and shifts, and that its essential image is closed under retracts (as the source category is idempotent complete, by definition) and contains all perfect-constant sheaves.

Case 1 (*assume $\Lambda = \mathcal{O}_E$, or $\Lambda = E$ and all X_i geometrically unibranch*). In this case, we have a t-structure on lisse Weil sheaves so that each $\mathrm{H}^j(M)$ belongs to $\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)^\heartsuit$. By induction on the length of M, using the full faithfulness of (5-15), we reduce to the case where M is abelian, that is, a continuous W-representation on a finitely presented Λ-module. The external tensor product induces a commutative diagram:

$$\begin{tikzcd}
\mathrm{Rep}_\Lambda(W_1) \times \cdots \times \mathrm{Rep}_\Lambda(W_n) \arrow[r, "\boxtimes"] \arrow[d, "\cong"] & \mathrm{Rep}_\Lambda(W) \arrow[d, "\cong"] \\
\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}}, \Lambda)^\heartsuit \times \cdots \times \mathrm{D}_{\mathrm{lis}}(X_n^{\mathrm{Weil}}, \Lambda)^\heartsuit \arrow[r, "\boxtimes"] & \mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)^\heartsuit
\end{tikzcd}$$

where the vertical equivalences are given by Lemma 4.15. Note that M splits into a direct sum $M_{\mathrm{tor}} \oplus M_{\mathrm{fp}}$ where the finitely presented Λ-module underlying M_{tor} is Λ-torsion and M_{fp} is projective. So we can treat either case separately. Using that the essential image of (5-15) is closed under extensions (by full faithfulness) and retracts, the finite projective case is reduced to the fp-simple case and, by Proposition 5.12, to the finite torsion case. Note that the W_i-representations constructed in, say (5-13), are obtained from M_{fp} by taking subquotients and tensor products, so are automatically continuous. Next, as the Λ-module underlying M_{tor} is finite torsion, the Λ-sheaf M_{tor} is perfect-constant along some finite étale cover. So we conclude by Lemma 5.13(2).

Case 2 (*assume Λ is finite discrete as above*). In a nutshell, the argument is similar to the last step in Case 1, but a little more involved due to the absence of natural t-structures on the categories of lisse sheaves in general, see Synopsis 3.2(vi) and [Hemo et al. 2023, Remark 6.9]. More precisely, in the special case, where Λ is a finite field, the argument of case 1) applies, but not so if $\Lambda = \mathbb{Z}/\ell^2$, say. So, instead, we extend (5-15) by passing to Ind-completions to a commutative diagram:

$$\begin{tikzcd}
\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}},\Lambda)\otimes_{\mathrm{Perf}_{\Lambda_*}}\cdots\otimes_{\mathrm{Perf}_{\Lambda_*}}\mathrm{D}_{\mathrm{lis}}(X_n^{\mathrm{Weil}},\Lambda) \arrow[r, "\boxtimes"] \arrow[d] & \mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}}\times\cdots\times X_n^{\mathrm{Weil}},\Lambda) \arrow[d] \\
\mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}},\Lambda))\otimes_{\mathrm{Mod}_{\Lambda_*}}\cdots\otimes_{\mathrm{Mod}_{\Lambda_*}}\mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(X_n^{\mathrm{Weil}},\Lambda)) \arrow[r, "\mathrm{Ind}(\boxtimes)"] & \mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}}\times\cdots\times X_n^{\mathrm{Weil}},\Lambda))
\end{tikzcd}$$

of full subcategories of $\mathrm{D}(X_1^{\mathrm{Weil}}\times\cdots\times X_n^{\mathrm{Weil}},\Lambda)$, see the discussion around (5-14). Note that the fully faithful embedding (5-14) factors through $\mathrm{Ind}(\boxtimes)$. Both vertical arrows are the inclusion of the subcategories of compact objects by idempotent completeness of the involved categories and (2-1). Thus, if M lies in the essential image of $\mathrm{Ind}(\boxtimes)$, then it is a retract of a finite colimit of objects in the essential image of $\boxtimes$, so lies itself in this essential image. As M is a successive extension of its cohomology sheaves $\mathrm{H}^j(M)$, it suffices to show

$$\mathrm{H}^j(M)\in\mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}},\Lambda))\otimes_{\mathrm{Mod}_{\Lambda_*}}\cdots\otimes_{\mathrm{Mod}_{\Lambda_*}}\mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(X_n^{\mathrm{Weil}},\Lambda)),$$

for all $j\in\mathbb{Z}$. So fix j and denote $N:=\mathrm{H}^j(M)$ viewed as a continuous W-representation on a finitely presented Λ-module. As Λ is finite, N comes from a continuous representation of $\pi_1(X_1)\times\cdots\times\pi_1(X_n)$ on which some open subgroup acts trivially. Hence, there exist finite étale surjections $U_i\to X_i$ such that the subgroup $\pi_1(U_1)\times\cdots\times\pi_1(U_n)$ acts trivially on N. In particular, $N|_{U_1^{\mathrm{Weil}}\times\cdots\times U_n^{\mathrm{Weil}}}$ is constant, and hence lies in the essential image of the functor

$$\mathrm{Mod}_R\cong\mathrm{Ind}(\mathrm{Perf}_R)\to\mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(U_1^{\mathrm{Weil}}\times\cdots\times U_n^{\mathrm{Weil}},\Lambda)),$$

where $R:=\Gamma(\pi_0(U_1)\times\cdots\times\pi_0(U_n),\Lambda)$. As the sets $\pi_0(U_i)$ are finite discrete, each $R_i:=\Gamma(\pi_0(U_i),\Lambda)$ is a finite free Λ_*-algebra, and we have $R\cong R_1\otimes_{\Lambda_*}\cdots\otimes_{\Lambda_*}R_n$. Thus, the external tensor product induces a commutative diagram:

$$\begin{tikzcd}
\mathrm{Mod}_{R_1}\otimes_{\mathrm{Mod}_{\Lambda_*}}\cdots\otimes_{\mathrm{Mod}_{\Lambda_*}}\mathrm{Mod}_{R_n} \arrow[r, "\cong"] \arrow[d] & \mathrm{Mod}_R \arrow[d] \\
\mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(U_1^{\mathrm{Weil}},\Lambda))\otimes_{\mathrm{Mod}_{\Lambda_*}}\cdots\otimes_{\mathrm{Mod}_{\Lambda_*}}\mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(U_n^{\mathrm{Weil}},\Lambda)) \arrow[r, "\mathrm{Ind}(\boxtimes)"] & \mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(U_1^{\mathrm{Weil}}\times\cdots\times U_n^{\mathrm{Weil}},\Lambda))
\end{tikzcd}$$

where the upper horizontal arrow is an equivalence. So $N|_{U_1^{\mathrm{Weil}}\times\cdots\times U_n^{\mathrm{Weil}}}$ lies in the essential image of $\mathrm{Ind}(\boxtimes)$, and we conclude by Lemma 5.14 applied to the finite étale covers $U_i\to X_i$ and $\bullet=\mathrm{lis}$. □

6. Ind-constructible Weil sheaves

In this section, we introduce the full subcategories

$$\mathrm{D}_{\mathrm{indlis}}(X^{\mathrm{Weil}},\Lambda)\subset\mathrm{D}_{\mathrm{indcons}}(X^{\mathrm{Weil}},\Lambda)$$

of $\mathrm{D}(X^{\mathrm{Weil}},\Lambda)$ consisting of ind-objects of lisse, resp. constructible sheaves equipped with partial Frobenius action. That is, the partial Frobenius only preserves the ind-system of objects, but not necessarily each member. We will define analogous categories for a product of schemes. Similarly to the lisse, resp. constructible case, there is a fully faithful functor

$$\mathrm{D}_{\mathrm{indcons}}(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Mod}_{\Lambda_*}} \cdots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{D}_{\mathrm{indcons}}(X_n^{\mathrm{Weil}}, \Lambda) \to \mathrm{D}_{\mathrm{indcons}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda),$$

which, however, will not be an equivalence in general; see Remark 6.6. Nevertheless, we can identify a class of objects that lie in the essential image and that include many cases of interest such as the shtuka cohomology studied in [Lafforgue 2018; Lafforgue and Zhu 2019; Xue 2020b; 2020c].

6A. ***Ind-constructible Weil sheaves.*** Let $\mathbb{F}_q$ be a finite field of characteristic $p > 0$, and fix an algebraic closure $\mathbb{F}$. Let $X_1, \dots, X_n$ be schemes of finite type over $\mathbb{F}_q$. Let Λ be a condensed ring associated with the one of the following topological rings: a discrete coherent torsion ring (for example, a discrete finite ring), an algebraic field extension $E \supset \mathbb{Q}_\ell$, or its ring of integers $\mathcal{O}_E$. We write $X := X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n$, and denote by $X_{i,\mathbb{F}} := X_i \times_{\mathbb{F}_q} \operatorname{Spec} \mathbb{F}$ and $X_{\mathbb{F}} := X \times_{\mathbb{F}_q} \operatorname{Spec} \mathbb{F}$ the base change. Recall that under these assumptions, by Synopsis 3.2(viii), we have a fully faithful embedding

$$\mathrm{Ind}(\mathrm{D}_{\mathrm{cons}}(X_{\mathbb{F}}, \Lambda)) \xrightarrow{\cong} \mathrm{D}_{\mathrm{indcons}}(X_{\mathbb{F}}, \Lambda) \subset \mathrm{D}(X_{\mathbb{F}}, \Lambda), \tag{6-1}$$

and likewise for (ind-)lisse sheaves.

Definition 6.1. An object $M \in \mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$ is called *ind-lisse*, resp. *ind-constructible* if the underlying sheaf $M_{\mathbb{F}} \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$ is ind-lisse, resp. ind-constructible in the sense of Definition 3.1.

We denote by

$$\mathrm{D}_{\mathrm{indlis}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda) \subset \mathrm{D}_{\mathrm{indcons}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$$

the resulting full subcategories of $\mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$ consisting of ind-lisse, resp. ind-constructible objects. Both categories are naturally commutative algebra objects in $\mathrm{Pr}^{\mathrm{St}}_{\Lambda_*}$ (see the notation from Section 2), that is, presentable stable Λ_*-linear symmetric monoidal ∞-categories where $\Lambda_* := \Gamma(*, \Lambda)$ is the ring underlying Λ. It is immediate from Definition 6.1 that the equivalence (4-6) restricts to an equivalence

$$\mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda) \cong \mathrm{Fix}(\mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda), \phi_{X_1}^*, \dots, \phi_{X_n}^*)$$

for $\bullet \in \{\text{indlis}, \text{indcons}\}$.

Remark 6.2. Note that we have a fully faithful embedding of $D_{\mathrm{cons}}(X^{\mathrm{Weil}})$ into $\mathrm{D}_{\mathrm{indcons}}(X^{\mathrm{Weil}})$ whose image consists of compact objects. However, the latter category is not generated by this image. Indeed, even in the case of a point, the ind-cons category consists of Λ-modules with an action of an automorphism. This automorphism does not have to fix any finitely generated submodule, which would be the case for any objects generated by the image of the constructible Weil complexes.

Our goal in this section is to obtain a categorical Künneth formula for the categories of ind-lisse, resp. ind-constructible Weil sheaves. In order to state the result, we need the following terminology. Under our assumptions on Λ, each cohomology sheaf $\mathrm{H}^j(M)$, $j \in \mathbb{Z}$ for $M \in \mathrm{D}_{\mathrm{lis}}(X_{\mathbb{F}}, \Lambda)$ is naturally a continuous representation of the proétale fundamental groupoid $\pi_1^{\text{proét}}(X_{\mathbb{F}})$ on a finitely presented Λ-module; see Lemma 4.15. Further, the projections $X_{\mathbb{F}} \to X_{i,\mathbb{F}}$ induce a full surjective map of topological groupoids

$$\pi_1^{\text{proét}}(X_{\mathbb{F}}) \to \pi_1^{\text{proét}}(X_{1,\mathbb{F}}) \times \cdots \times \pi_1^{\text{proét}}(X_{n,\mathbb{F}}). \tag{6-2}$$

Definition 6.3. Let $M \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$:

(1) The sheaf M is called *split lisse* if it is lisse and the action of $\pi_1^{\text{proét}}(X_{\mathbb{F}})$ on $\mathrm{H}^j(M)$ factors through (6-2) for all $j \in \mathbb{Z}$.

(2) The sheaf M is called *split constructible* if it is constructible and there exists a finite subdivision into locally closed subschemes $X_{i,\alpha} \subseteq X_i$ such that for each $X_\alpha = \prod_i X_{i,\alpha} \subseteq X$, each restriction $M|_{X_\alpha}$ is split lisse.

Definition 6.4. An object $M \in \mathrm{D}(X_1^{\text{Weil}} \times \cdots \times X_n^{\text{Weil}}, \Lambda)$ is called *ind-(split lisse)*, resp. *ind-(split constructible)* if the underlying object $M_{\mathbb{F}} \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$ is a colimit of split lisse, resp. split constructible objects.

As the category $\mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda)$, $\bullet \in \{\text{indlis}, \text{indcons}\}$ is cocomplete, every ind-(split lisse) object is ind-lisse, and likewise, every ind-(split constructible) object is ind-constructible.

Theorem 6.5. *Assume that Λ is either a finite discrete ring of prime-to-p torsion, an algebraic field extension $E \supset \mathbb{Q}_\ell$ for $\ell \neq p$, or its ring of integers $\mathcal{O}_E$. Then the functor induced by the external tensor product*

$$\mathrm{D}_\bullet(X_1^{\text{Weil}}, \Lambda) \otimes_{\mathrm{Mod}_{\Lambda_*}} \cdots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{D}_\bullet(X_n^{\text{Weil}}, \Lambda) \to \mathrm{D}_\bullet(X_1^{\text{Weil}} \times \cdots \times X_n^{\text{Weil}}, \Lambda) \tag{6-3}$$

is fully faithful for $\bullet \in \{\text{indlis}, \text{ indcons}\}$. For $\bullet = \text{indlis}$, resp. $\bullet = \text{indcons}$ the essential image contains the ind-(split lisse), resp. ind-(split constructible) objects.

Proof. For full faithfulness, it is enough to consider the case $\bullet = \text{indcons}$. Using Lemma 2.5, it remains to show that the functor

$$\bigotimes_i \mathrm{D}_{\text{indcons}}(X_{i,\mathbb{F}},) \cong \mathrm{Ind}\Bigl(\bigotimes_i \mathrm{D}_{\text{cons}}(X_{i,\mathbb{F}}, \Lambda)\Bigr) \to \mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda). \tag{6-4}$$

is fully faithful. In view of (6-1), this is immediate from the Künneth formula for constructible Λ-sheaves as explained in Section 5B.

To identify objects in the essential image, we note that the fully faithful functors (6-3) and (6-4) induce a Cartesian diagram (see Lemma 2.5):

$$\begin{tikzcd}
\bigotimes_i \mathrm{D}_\bullet(X_i^{\text{Weil}},) \arrow[r] \arrow[d] & \mathrm{D}_\bullet(X_1^{\text{Weil}} \times \cdots \times X_n^{\text{Weil}}, \Lambda) \arrow[d] \\
\bigotimes_i \mathrm{D}_\bullet(X_{i,\mathbb{F}}, \Lambda) \arrow[r] & \mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda)
\end{tikzcd} \tag{6-5}$$

for $\bullet \in \{\text{indlis}, \text{ indcons}\}$. Thus, it is enough to show that the object $M_{\mathbb{F}}$ underlying an ind-split object M lies in the image of the lower horizontal arrow. Since this essential image is closed under colimits, it remains to show it contains the split lisse objects for $\bullet = \mathrm{indlis}$, resp. the split constructible objects for $\bullet = \mathrm{indcons}$.

By the full faithfulness of (6-4), the split constructible case reduces to the split lisse case, see also the proof of Theorem 5.2 in Section 5E. So assume $\bullet = \mathrm{indlis}$ and let $M_{\mathbb{F}} \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$ be split lisse. As each cohomology sheaf $\mathrm{H}^j(M_{\mathbb{F}})$, $j \in \mathbb{Z}$ is at least ind-lisse (see also [Hemo et al. 2023, Remark 8.4]), an induction on the cohomological length of $M_{\mathbb{F}}$ reduces us to show that $\mathrm{H}^j(M_{\mathbb{F}})$ lies in the essential image. By definition, being split lisse implies that the action of $\pi_1^{\text{proét}}(X_{\mathbb{F}})$ on $\mathrm{H}^j(M_{\mathbb{F}})$ factors through $\pi_1^{\text{proét}}(X_{1,\mathbb{F}}) \times \cdots \times \pi_1^{\text{proét}}(X_{n,\mathbb{F}})$. Then the arguments of Section 5E show that $\mathrm{H}^j(M_{\mathbb{F}})$ lies is in the essential image of the lower horizontal arrow in (6-5). We leave the details to the reader. □

Remark 6.6. The functor (6-3) is not essentially surjective in general. To see this, note that the functor $\mathrm{D}_{\mathrm{indcons}}(X^{\mathrm{Weil}}, \Lambda) \to \mathrm{D}_{\mathrm{indcons}}(X_{\mathbb{F}}, \Lambda)$ admits a left adjoint F that adds a free partial Frobenius action. Explicitly, for an object $M \in \mathrm{D}_{\mathrm{indcons}}(X_{\mathbb{F}}, \Lambda)$ the object $F(M)$ has underlying sheaf $F(M)_{\mathbb{F}}$ given by a countable direct sum of copies of M. If M was not originally in the image of the external tensor product (for example, M as in Example 1.4), then $F(M)$ will not be either. This is, however, the only obstacle for essential surjectivity: as noted in the proof of Theorem 6.5, the diagram (6-5) is Cartesian.

6B. *Cohomology of shtuka spaces.* Finally, let us mention a key application of Theorem 6.5. Let X be a smooth projective geometrically connected curve over $\mathbb{F}_q$. Let $N \subset X$ be a finite subscheme, and denote its complement by $Y = X \backslash N$. Let $E \supset \mathbb{Q}_\ell$, $\ell \neq p$ be an algebraic field extension containing a fixed square root of q. Let $\mathcal{O}_E$ be its ring of integers and denote by k_E the residue field. Let Λ be any of the topological rings $E, \mathcal{O}_E, k_E$. Let G be a split (for simplicity) reductive group over $\mathbb{F}_q$. We denote by $\widehat{G}$ the Langlands dual group of G considered as a split reductive group over Λ.

In the seminal works [Drinfeld 1980; Lafforgue 2002] ($G = \mathrm{GL}_n$) and [Lafforgue 2018; Lafforgue and Zhu 2019] (general reductive G) on the Langlands correspondence over global function fields, the construction of the $\mathrm{Weil}(Y)$-action on automorphic forms of level N is realized using the cohomology sheaves of moduli stacks of shtukas, defined in [Varshavsky 2004] and [Lafforgue 2018, Section 2]. As explained in [Lafforgue and Zhu 2019; Gaitsgory et al. 2022; Zhu 2021], the output of the geometric construction of Lafforgue can be encoded as a natural transformation

$$\mathrm{H}_{N,I}\colon\ \mathrm{Rep}_\Lambda^{\mathrm{fp}}(\widehat{G}^I) \to \mathrm{Rep}_\Lambda^{\mathrm{cts}}(\mathrm{Weil}(Y)^I), \qquad I \in \mathrm{FinSet} \tag{6-6}$$

of functors $\mathrm{FinSet} \to \mathrm{Cat}$ from the category of finite sets to the category of 1-categories. Here the functor $\mathrm{Rep}_\Lambda^{\mathrm{fp}}(\widehat{G}^\bullet)$ assigns to a finite set I the category of algebraic representations of $\widehat{G}^I$ on finite free Λ-modules, and $\mathrm{Rep}_\Lambda^{\mathrm{cts}}(\mathrm{Weil}(Y)^\bullet)$ the category of continuous representations of $\mathrm{Weil}(Y)^I$ in Λ-modules. In both cases, the transition maps are given by restriction of representations.

Let us recall some elements of its construction. For a finite set I, [Varshavsky 2004] and [Lafforgue 2018, Section 2] define the ind-algebraic stack $\mathrm{Cht}_{N,I}$ classifying I-legged G-shtukas on X with full level-N structure. The morphism sending a G-shtuka to its legs

$$\mathfrak{p}_{N,I}\colon\ \mathrm{Cht}_{N,I} \to Y^I, \tag{6-7}$$

is locally of finite presentation. For every $W \in \mathrm{Rep}^{\mathrm{fp}}_{\Lambda}(\widehat{G}^I)$, there is the normalized Satake sheaf $\mathcal{F}_{N,I,W}$ on $\mathrm{Cht}_{N,I}$; see [Lafforgue 2018, Définition 2.14]. Base changing to $\mathbb{F}$ and taking compactly supported cohomology, we obtain the object

$$\mathcal{H}_{N,I}(W) \overset{\mathrm{def}}{=} (\mathfrak{p}_{N,I,\mathbb{F}})_!(\mathcal{F}_{N,I,W,\mathbb{F}}) \in \mathrm{D}_{\mathrm{indcons}}(Y^I_{\mathbb{F}}, \Lambda);$$

see [Lafforgue 2018, Définition 4.7] and [Xue 2020a, Definition 2.5.1]. Under the normalization of the Satake sheaves, the degree 0 cohomology sheaf

$$\mathrm{H}_{N,I}(W) \overset{\mathrm{def}}{=} \mathrm{H}^0(\mathcal{H}_I(W)) \in \mathrm{D}_{\mathrm{indcons}}(Y^I_{\mathbb{F}}, \Lambda)^{\heartsuit}$$

corresponds to the middle degree compactly supported intersection cohomology of $\mathrm{Cht}_{N,I}$. Using the symmetries of the moduli stacks of shtukas, the sheaf $\mathrm{H}_{N,I}(W)$ is endowed with a partial Frobenius equivariant structure [Lafforgue 2002, Section 6]. So we obtain objects

$$\mathrm{H}_{N,I}(W) \in \mathrm{D}_{\mathrm{indcons}}((Y^{\mathrm{Weil}})^I, \Lambda)^{\heartsuit}. \tag{6-8}$$

Next, using the finiteness [Xue 2020b] and smoothness [Xue 2020c, Theorem 4.2.3] results, the classical Drinfeld's lemma (Theorem 5.9) applies to give objects $\mathrm{H}_{N,I}(W) \in \mathrm{Rep}^{\mathrm{cts}}_{\Lambda}(\mathrm{Weil}(Y)^I)$. The construction of the natural transformation (6-6) encodes the functoriality and fusion satisfied by the objects $\{\mathrm{H}_{N,I}(W)\}$ for varying I and W.

However, in order to analyze construction (6-6) further, it is desirable to upgrade the natural transformation of functors (6-6) to the derived level. Namely, to have construction for the complexes $\{\mathcal{H}_I(W)\}_{I,W}$ and not just for their cohomology sheaves; compare with [Zhu 2021]. Such an upgrade is possible using the derived version of Drinfeld's lemma, as given in the following proposition.

Proposition 6.7. *For $\Lambda \in \{E, \mathcal{O}_E, k_E\}$ and any $W \in \mathrm{Rep}_{\Lambda}(\widehat{G}^I)$, the shtuka cohomology* (6-8) *lies in the essential image of the fully faithful functor*

$$\mathrm{D}_{\mathrm{indlis}}(Y^{\mathrm{Weil}}, \Lambda)^{\otimes I} \to \mathrm{D}_{\mathrm{indcons}}((Y^{\mathrm{Weil}})^I, \Lambda). \tag{6-9}$$

Proof. By [Xue 2020c, Theorem 4.2.3], the ind-constructible sheaf $\mathrm{H}_{N,I}(W)$ is ind-lisse. By [Xue 2020b, Proposition 3.2.15], the action of $\mathrm{FWeil}(Y^I)$ on $\mathrm{H}_{N,I}(W)$ factors through the product $\mathrm{Weil}(Y)^I$. In particular, the action of $\pi_1(X^I_{\mathbb{F}})$ on $\mathrm{H}_{N,I}(W)$ factors through the product $\pi_1(X_{\mathbb{F}})^I$. So it is ind-(split lisse) in the sense of Definition 6.4, and we are done by Theorem 6.5. $\square$

Remark 6.8. One can upgrade the above construction in a homotopy coherent way to show that the whole complex $\mathcal{H}_{N,I}(W)$ lies in $\mathrm{D}_{\mathrm{indcons}}((Y^{\mathrm{Weil}})^I, \Lambda)$. If $N \neq \varnothing$ so that $\mathcal{H}_{N,I}(W)$ is known to be bounded, then Proposition 6.7 implies that $\mathcal{H}_{N,I}(W)$ lies in the essential image of (6-9).

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themo@caltech.edu *Department of Mathematics, California Institute of Technology, Pasadena, CA, United States*

richarz@mathematik.tu-darmstadt.de *Department of Mathematics, Technical University of Darmstadt, Darmstadt, Germany*

jakob.scholbach@unipd.it *Dipartimento di Matematica, University of Padova, Padova, Italy*

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