Algebra & Number Theory

Volume 18
2024
No. 4

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ALGEBRA AND NUMBER THEORY 18:4 (2024)
https://doi.org/10.2140/ant.2024.18.735

Wide moments of L-functions I: Twists by class group characters of imaginary quadratic fields

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We calculate certain "wide moments" of central values of Rankin–Selberg L-functions $L\big(\pi\otimes\Omega, \frac{1}{2}\big)$ where π is a cuspidal automorphic representation of GL_2 over $\mathbb{Q}$ and Ω is a Hecke character (of conductor 1) of an imaginary quadratic field. This moment calculation is applied to obtain "weak simultaneous" nonvanishing results, which are nonvanishing results for different Rankin–Selberg L-functions where the product of the twists is trivial.

The proof relies on relating the wide moments of L-functions to the usual moments of automorphic forms evaluated at Heegner points using Waldspurger's formula. To achieve this, a classical version of Waldspurger's formula for general weight automorphic forms is derived, which might be of independent interest. A key input is equidistribution of Heegner points (with explicit error terms), together with nonvanishing results for certain period integrals. In particular, we develop a soft technique for obtaining the nonvanishing of triple convolution L-functions.

1. Introduction

Determining the moments of central values of families of automorphic L-functions has a long history starting with the work of Hardy and Littlewood on the Riemann zeta function

$$\int_0^T \big|\zeta\big(\tfrac{1}{2}+it\big)\big|^2\,dt \sim T\log T,$$

as $T\to\infty$; see [Titchmarsh 1986, Chapter VII]. By now, there exist precise conjectures for all moments of families of L-functions [Conrey et al. 2005] with fascinating connections to random matrix theory [Keating and Snaith 2000]. These moment conjectures are of deep arithmetic importance through their connections to the important topics of nonvanishing and subconvexity (see, e.g., [Blomer et al. 2018]), which in turn are connected to, respectively, rational points on elliptic curves (via the B–S-D conjectures, see [Kolyvagin 1988]) and equidistribution problems (via the Waldspurger formula, see [Michel and Venkatesh 2006]).

In this paper, we will calculate what we call *wide moments* of central values of Rankin–Selberg L-functions $L\big(\pi\otimes\Omega, \frac{1}{2}\big)$, where π is a cuspidal automorphic representation of GL_2 with trivial central character of even lowest weight k_π and Ω is a Hecke character of an imaginary quadratic field K with infinity type $\alpha\mapsto(\alpha/|\alpha|)^k$ for some even integer $k\geq k_\pi$. More precisely, we will study the "canonical" square roots of the central values via their connections to Heegner periods as in the work of

MSC2020: primary 11F67; secondary 11M41.
Keywords: moments of L-functions, periods of automorphic forms.

Waldspurger [1985]. We will use these moment calculations to obtain a number of new nonvanishing results of a certain kind that we call *weak simultaneous nonvanishing*; see Section 1C for the statements. In view of the Bloch–Kato conjectures, these nonvanishing results imply (in the holomorphic case) vanishing for certain twisted Selmer groups; see Corollary 7.6 below.

1A. ***Wide moments of L-functions.*** This paper is the first in a series of papers concerned with obtaining asymptotic evaluations of *wide moments* of automorphic L-function. In all of the cases we will consider, these wide moments are connected to the usual moments of certain underlying periods of automorphic forms (in the case of this paper, through the Waldspurger formula), which are much better behaved than the L-functions themselves. In particular, we can use a variety of more geometrically flavored methods to study the distributional properties of these periods.

The abstract setup is as follows: Given a finite abelian group G with (unitary) dual $\hat{G}$, we define

$$\mathbf{Wide}(\hat{G},n) := \{(\chi_1,\dots,\chi_n)\in(\hat{G})^n : \chi_1\cdots\chi_n = 1\}. \tag{1-1}$$

Given maps $L_1,\dots,L_n : G\to\mathbb{C}$ with Fourier transforms

$$\widehat{L}_i : \hat{G}\to\mathbb{C},\ \chi\mapsto \frac{1}{|G|}\sum_{g\in G} L_i(g)\bar{\chi}(g), \quad \text{for } i=1,\dots,n,$$

we define the *wide moment* of $\widehat{L}_1,\dots,\widehat{L}_n$, as

$$\sum_{(\chi_i)_{1\le i\le n}\in\mathbf{Wide}(\hat{G},n)} \prod_{i=1}^{n} \widehat{L}_i(\chi_i). \tag{1-2}$$

Note that for $n=2$ and $\widehat{L}_1=\widehat{L}_2$ equivariant with respect to inverses $\big($i.e., $\widehat{L}_1(\chi^{-1})=\overline{\widehat{L}_1(\chi)}\big)$, we recover the usual second moment. The key point is that (1-2) is equal to

$$\frac{1}{|G|}\sum_{g\in G}\prod_{i=1}^{n} L_i(g), \tag{1-3}$$

(for $n=2$ this is exactly Plancherel). A nice way to see that (1-2) is equal to (1-3) is to use that the Fourier transform takes products to convolutions, and (1-2) is exactly the n-fold convolution product of $\widehat{L}_1,\dots,\widehat{L}_n$ evaluated at $\chi=1$. In the setting of automorphic L-functions, we can in many cases calculate the wide moments (1-2) using that the dual moments (1-3) are much better behaved.

The first example in the literature of an asymptotic evaluation of a (higher) wide moment of automorphic L-functions seems to be the work of Bettin [2019] on Dirichlet L-functions (note that here the terminology "iterated moments" is used):

$$\frac{1}{(p-2)^{n-1}} \sideset{}{^*}\sum_{(\chi_i)\in\mathbf{Wide}(p,n)} \left|L\Big(\chi_1,\frac{1}{2}\Big)\right|^2\cdots\left|L\Big(\chi_n,\frac{1}{2}\Big)\right|^2 = c_{n,n}(\log p)^n + c_{n,n-1}(\log p)^{n-1}+\cdots+c_{n,0}+O(p^{-\delta}), \tag{1-4}$$

as $p\to\infty$ with p prime, for some $\delta>0$ and $c_{n,i}\in\mathbb{R}$. Here, the asterisks on the sum means that the summation is restricted to primitive Dirichlet characters, and we set $\mathbf{Wide}(p,n) := \mathbf{Wide}(\widehat{(\mathbb{Z}/p\mathbb{Z})^\times},n)$.

This result is a corollary of the moment calculation of the *Estermann function* (which we think of as the underlying automorphic periods in this case). Another related result is the calculation of Chinta [2005] corresponding to a wide moment with $n = 3$ for quadratic Dirichlet L-functions.

The asymptotic evaluation (1-4) was later generalized (with an extra average over the modulus q) by the author [Nordentoft 2021, Corollary 1.9] to the wide moments of

$$\widehat{L}_1(\chi) = \cdots = \widehat{L}_n(\chi) = L\bigl(f \otimes \chi, \tfrac{1}{2}\bigr) \quad \text{for } \chi : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times,$$

with f a fixed holomorphic newform of even weight. In [Nordentoft 2021], the underlying automorphic periods are the *additive twists* of f (which reduces to *modular symbols* for $k = 2$). Furthermore, in a recent joint work between Drappeau and the author, all moments of additive twists of level 1 Maaß forms are calculated [Drappeau and Nordentoft 2022, Corollary 1.9].

The methods used to calculate the wide moments mentioned above are, respectively, a classical approximate functional equation approach [Bettin 2019], multiple Dirichlet series [Chinta 2005], spectral theory [Nordentoft 2021] (see also [Petridis and Risager 2018a]), and dynamical systems [Drappeau and Nordentoft 2022] (building on [Bettin and Drappeau 2022]).

1B. ***Main idea.*** Let us describe the main moment calculation of this paper in the simplest possible setup. Let $f : \mathbb{H} \to \mathbb{C}$ be a classical Hecke–Maaß eigenform of weight 0 and (for simplicity) level 1 (i.e., a real-analytic joint eigenfunction for the hyperbolic Laplace operator and the Hecke operators which is invariant under $\mathrm{PSL}_2(\mathbb{Z})$). Let K be an imaginary quadratic field of discriminant $D_K < -6$ with class group Cl_K. Given a class group character $\chi \in \widehat{\mathrm{Cl}}_K$, we denote by $L(f \otimes \chi, s)$ (the finite part of) the Rankin–Selberg L-function $L(f \otimes \theta_\chi, s)$, where θ_χ is the theta series associated to χ of weight 1 and level $|D_K|$ (equivalently, we have $L(f \otimes \chi, s) = L(\pi_K \otimes \pi_\chi, s)$, where π_K denotes the base change to $\mathrm{GL}_2(\mathbb{A}_K)$ of the automorphic representation corresponding to f and π_χ is the automorphic representation of $\mathrm{GL}_1(\mathbb{A}_K)$ corresponding to χ). A deep formula of Zhang [2001; 2004] gives the relation

$$\Bigl|\sum_{[\mathfrak{a}] \in \mathrm{Cl}_K} f(z_{[\mathfrak{a}]})\chi([\mathfrak{a}])\Bigr|^2 = |c_f|^2 |D_K|^{1/2} L\bigl(f \otimes \chi, \tfrac{1}{2}\bigr), \tag{1-5}$$

where $\chi \in \widehat{\mathrm{Cl}}_K$ is a class group character of K, $z_{[\mathfrak{a}]} \in \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ denotes the Heegner point associated to $[\mathfrak{a}] \in \mathrm{Cl}_K$, and $c_f > 0$ is a constant depending on f (but independent of χ). Using this relation together with orthogonality of characters and equidistribution of Heegner points, Michel and Venkatesh [2007] calculated the first moment of $L\bigl(f \otimes \chi, \tfrac{1}{2}\bigr)$, which they combined with subconvexity to obtain quantitative nonvanishing for these central values. This idea has since been generalized in many directions to obtain a variety of nonvanishing results [Dittmer et al. 2015; Burungale and Hida 2016; Khayutin 2020; Templier 2011a; 2011b].

We observe that (1-5) is exactly saying that the Fourier transform of

$$\mathrm{Cl}_K \ni [\mathfrak{a}] \mapsto |\mathrm{Cl}_K| f(z_{[\mathfrak{a}]})$$

is given by a map of the form

$$\widehat{\mathrm{Cl}}_K \ni \chi \mapsto \varepsilon_{f,\chi} c_f |D_K|^{1/4} \bigl|L\bigl(f \otimes \chi, \tfrac{1}{2}\bigr)\bigr|^{1/2},$$

for some $\varepsilon_{f,\chi}$ of norm 1. Thus, by the Fourier equality (1-2)=(1-3) and equidistribution of Heegner points due to Duke [1988], we conclude that for level 1 Hecke–Maaß eigenforms $f_1, \dots, f_n$, we have

$$\frac{|D_K|^{n/4}}{|\mathrm{Cl}_K|^n} \sum_{(\chi_i)\in\mathbf{Wide}(K,n)} \prod_{i=1}^n \varepsilon_{f_i,\chi_i} c_{f_i} \Big| L\Big(f_i \otimes \chi_i, \frac{1}{2}\Big)\Big|^{1/2} = \frac{1}{|\mathrm{Cl}_K|} \sum_{[\mathfrak{a}]\in\mathrm{Cl}_K} \prod_{i=1}^n f_i(z_{[\mathfrak{a}]}) = \Big\langle \prod_{i=1}^n f_i, \frac{3}{\pi}\Big\rangle + o(1), \tag{1-6}$$

as $|D_K| \to \infty$, where we used the short-hand $\mathbf{Wide}(K,n) := \mathbf{Wide}(\widehat{\mathrm{Cl}}_K, n)$. This shows immediately that if $\langle \prod_{i=1}^n f_i, 1\rangle \neq 0$, then there exists

$$(\chi_1, \dots, \chi_n) \in \mathbf{Wide}(K,n) \text{ such that } \prod_{i=1}^n L\big(f_i \otimes \chi_i, \tfrac{1}{2}\big) \neq 0.$$

We call the above *weak simultaneous nonvanishing*; see Section 2 for some background on this type of nonvanishing.

1C. ***Nonvanishing results.*** The above proof sketch already gives new results. We will, however, push these ideas further in several aspects. First of all, we deal with general weight forms (holomorphic or Maaß), which requires us to develop explicit Waldspurger type formulas in these cases (see Section 4), which might be of independent interest. In particular, this requires studying Hecke characters which ramify at ∞, which leads to some complications. Secondly, we will obtain an explicit error term in (1-6), which requires bounding certain inner-products involving powers of the Laplace operator; see Section 5. This allows us to obtain nonvanishing results with some uniformity in the spectral aspect. In particular, in the case of width $n = 2$, we obtain the following improved version of [Michel and Venkatesh 2006, Theorem 1] allowing general weights and with a uniform lower bound for D_K in terms of the spectral parameter:

Corollary 1.1. *Let f be either a Hecke–Maaß cusp form of spectral parameter t_f and level 1 or a cuspidal holomorphic Hecke eigenform of weight k_f and level 1. Let k be a positive even integer with the further requirement that $k \geq k_f$ if f is holomorphic. Put $T = |t_f| + k + 1$ in the Maaß case and $T = k + 1$ in the holomorphic case.*

Then for any $\varepsilon > 0$, there exists a constant $c = c(\varepsilon) > 0$ such that for any imaginary quadratic field K with discriminant $|D_K| \geq cT^{22+\varepsilon}$, we have

$$\#\big\{\chi \in \widehat{\mathrm{Cl}}_K : L\big(f \otimes \chi\Omega_K, \tfrac{1}{2}\big) \neq 0\big\} \gg_f \begin{cases} |D_K|^{1/1058} & \text{if } f \text{ is holomorphic,} \\ |D_K|^{1/2648} & \text{if } f \text{ is Maaß,} \end{cases}$$

where Ω_K is a Hecke character of K of conductor 1 and ∞-type $\alpha \mapsto (\alpha/|\alpha|)^k$.

Remark 1.2. We obtain similar results for general squarefree levels N; see Corollary 7.1.

The case of width $n = 3$ is also very appealing, as in this case the triple period $\langle f_1 f_2 f_3, 1\rangle$ is related to *triple convolution L-functions* via the Ichino–Watson formula [Watson 2002; Ichino 2008]. This leads to the following nonvanishing result for level 1 Maaß forms:

Corollary 1.3. *Let f_1 be a fixed Hecke–Maaß cusp form of level 1. Then for any $\varepsilon > 0$, there exists a constant $c = c(f_1, \varepsilon) > 0$ such that for any $T \geq c$, we have for all but $O_\varepsilon(T^{2\varepsilon})$ Hecke–Maaß cusp forms f_2 of level 1 with $|t_{f_2} - T| \leq T^\varepsilon$ that there exists a Hecke–Maaß cusp form f_3 not equal to f_2 with $|t_{f_3} - T| \leq T^\varepsilon$ such that the following holds*: *We have $L\big(f_1 \otimes f_2 \otimes f_3, \frac{1}{2}\big) \neq 0$ and for any imaginary quadratic field K with $|D_K| \geq cT^{35+\varepsilon}$,*

$$\#\big\{\chi_1, \chi_2 \in \widehat{\mathrm{Cl}}_K : L\big(f_1 \otimes \chi_1, \tfrac{1}{2}\big) L\big(f_2 \otimes \chi_2, \tfrac{1}{2}\big) L\big(f_3 \otimes \chi_1\chi_2, \tfrac{1}{2}\big) \neq 0\big\} \gg_T |D_K|^{1/1766}.$$

In the case of holomorphic forms, we can obtain nonvanishing for a general width n (stated here in the simplest case of level 1, we refer to Corollary 7.5 for a more general statement).

Corollary 1.4. *Let $n \geq 1$, $k_1, \dots, k_n \in 2\mathbb{Z}_{>0}$, and put $k = \sum_i k_i$. For $i = 1, \dots, n$, let $g_i \in \mathcal{S}_{k_i}(1)$ be a cuspidal holomorphic Hecke eigenform of level 1. Then for each $\varepsilon > 0$, there exists a constant $c = c(\varepsilon) > 0$ such that the following holds*: *For any imaginary quadratic field K with $|D_K| \geq ck^{45+\varepsilon}$,*

$$\begin{aligned}\#\big\{(\chi_1, \dots, \chi_{n+1}) \in \mathbf{Wide}(K, n+1),\ &\text{level 1 Hecke eigenforms } g \in \mathcal{S}_k(1) : \\ &L\big(g_1 \otimes \chi_1 \Omega_{i,K}, \tfrac{1}{2}\big) \cdots L\big(g_n \otimes \chi_n \Omega_{n,K}, \tfrac{1}{2}\big) L\big(g \otimes \chi_{n+1} \Omega_{n+1,K}, \tfrac{1}{2}\big) \neq 0\big\} \\ &\gg_k |D_K|^{(n+1)/2115},\end{aligned}$$

where $\Omega_{i,K}$ are Hecke characters of K of ∞-type $x \mapsto (x/|x|)^{k_i}$ for $i = 1, \dots, n$ and $\Omega_{n+1,K} = \prod_{i=1}^n \Omega_{i,K}$.

Remark 1.5. Note that it follows, in particular, that the respective nonvanishing sets in Corollaries 1.1, 1.3 and 1.4 are *nonempty* as soon as, respectively, $|D_K| \geq cT^{22+\varepsilon}$, $|D_K| \geq cT^{35+\varepsilon}$ and $|D_K| \geq ck^{45+\varepsilon}$.

Remark 1.6. The fact that we can obtain nonvanishing results for general width n in the holomorphic case relies crucially on the finite dimensionality of the space of holomorphic forms of fixed level and weight. This clearly fails for nonholomorphic Maaß forms, which is the reason we cannot obtain nonvanishing results beyond the cases of two and three characters in the Maaß case. Notice that if we apply Corollary 1.4 with $n = 2$, we obtain an improved version of Corollary 1.3 in the case of holomorphic forms.

1D. ***Main moment calculation.*** The above nonvanishing results are all corollaries of our main L-function calculation. To state this, denote by $\mathcal{B}_k^*(N)$ the set of L^2-normalized Hecke–Maaß newforms of level N and even weight $k \geq 0$ (i.e., raising operators applied to either classical Hecke–Maaß newforms of weight 0 and level N or to $y^{k'/2}g$ with $g \in \mathcal{S}_{k'}(N)$ a holomorphic cuspidal newform of even weight $k' \leq k$). Then we have the following moment calculation:

Theorem 1.7. *Let $N \geq 1$ be a fixed squarefree integer and $n \geq 1$. For $i = 1, \dots, n$, let π_i be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ of conductor N with trivial central character, spectral parameter t_{π_i} and even lowest weight k_{π_i}. Let $k_1, \dots, k_n \in 2\mathbb{Z}$ be integers such that $|k_i| \geq k_{\pi_i}$ and $\sum_i k_i = 0$.*

Let $|D_K| \to \infty$ transverse a sequence of discriminants of imaginary quadratic fields K such that all primes dividing N split in K. For each K, pick Hecke characters $\Omega_{i,K}$ with infinite types $x \mapsto (x/|x|)^{k_i}$ such that $\prod_i \Omega_{i,K}$ is the trivial Hecke character.

Then we have for $f_i \in \mathcal{B}^*_{k_i}(N)$ belonging to π_i and any $\varepsilon > 0$,

$$\sum_{(\chi_i)_{1\le i\le n}\in \mathbf{Wide}(K,n)} \prod_{i=1}^{n} \varepsilon_{\chi_i,f_i} c_{f_i} L\big(\pi_i \otimes \chi_i \Omega_{i,K}, \tfrac{1}{2}\big)^{1/2}$$
$$= \frac{|\mathrm{Cl}_K|^n}{|D_K|^{n/4}} \left(\left\langle \prod_{i=1}^{n} f_i, 1 \right\rangle + O_\varepsilon\left(\left\| \prod_{i=1}^{n} f_i \right\|_2 |D_K|^{-1/16} T^{5/2} n^{15/4} (T|D_K|n)^\varepsilon \right) \right), \quad (1\text{-}7)$$

where $T = \max_{i=1,\dots,n} |k_i| + |t_{\pi_i}| + 1$, the weights ε_{χ,f_i} are all of norm 1 and c_{f_i} are certain constants depending only on f_i.

Remark 1.8. We obtain a slightly more general statement that applies to old-forms as well, meaning that we allow for the automorphic representations π_i to have different conductors. Furthermore, we obtain an improved error term in the case of holomorphic forms and/or in the case of level 1. We refer to Theorem 6.3 for details (including the exact values of the constants c_f). As an application, we can also calculate a related "diagonal wide moment"; see Corollary 6.6.

The plan of the paper is as follows. In Section 2, we will introduce the notion of *weak simultaneous nonvanishing*. Section 3 provides the necessary background on imaginary quadratic fields and automorphic forms. Section 4 proves an explicit and classical Waldspurger type formula for general weight automorphic forms. In Section 5, we will prove two technical lemmas: one on the norm of powers of the hyperbolic Laplacian and one on a lower bound for the L^2-norm of a product of automorphic forms. In Section 6, we will prove our main moment calculation. Finally, Section 7 proves the nonvanishing of certain automorphic periods, which combined with our moment calculation, yields weak simultaneous nonvanishing results.

2. Weak simultaneous nonvanishing

We will call the nonvanishing results proved in the present paper *weak simultaneous nonvanishing*. This terminology is referring to the fact that we show nonvanishing of twists of different L-functions with some "algebraic dependence" on the twists (their product is trivial). Ideally, of course we would like to show nonvanishing for the same character. Some results in this direction have been obtained by Saha and Schmidt [2013, Theorem 1] in the case of two holomorphic forms using techniques from Siegel modular forms. Outside of this case, however, simultaneous nonvanishing seems out of reach with current methods.

Let us start by considering the simplest case, $n = 2$. This means that we are studying the nonvanishing of two maps $L_1, L_2 : G \to \mathbb{C}$, where G is a finite abelian group. If both L_1 and L_2 are nonvanishing for more than 50% of $g \in G$, then by the pigeonhole principle there is some $g \in G$ such that $L_1(g)L_2(g) \neq 0$. But clearly we can construct examples where L_1, L_2 vanish for exactly 50% of $g \in G$ but there is no simultaneous nonvanishing.

More generally, consider $L_1, \dots, L_n : G \to \mathbb{C}$. Then we say that $L_1, \dots, L_n$ are *weakly simultaneously nonvanishing* if

$$\big\{(g_1, \dots, g_n) \in \mathbf{Wide}(G, n) : L_i(g_i) \neq 0 \text{ for } i = 1, \dots, n\big\} \neq \varnothing.$$

Recall that by (1-1) this means that there exist $g_1, \dots, g_n \in G$ such that

$$g_1 \cdots g_n = 1_G \quad \text{and} \quad L_1(g_1) \cdots L_n(g_n) \neq 0.$$

We think of this as expressing that we can find nonvanishing for $L_1, \dots, L_n$ with some "algebraic dependence". This is interesting since most nonvanishing results for automorphic L-functions are obtained by using the method of mollification, which gives no information about the algebraic structure of the nonvanishing set. Of course, if all of the $L_1, \dots, L_n$ vanish on a very large percentage of elements of G, then one gets a weak simultaneous nonvanishing for purely combinatorial reasons. In most cases, this is not the case, which we make precise as follows:

Proposition 2.1. *Let $n \geq 2$ be an integer and $0 \leq c \leq 1$. Then there exists a finite abelian group G and maps $L_1, \dots, L_n : G \to \mathbb{C}$ satisfying*

$$\#\{g \in G : L_i(g) \neq 0\} \geq c|G|, \quad \textit{where } i = 1, \dots, n,$$

with no weak simultaneous nonvanishing if and only if $c \leq \frac{1}{2}$.

Proof. Assume first of all that $c > \frac{1}{2}$. Then if $g_1, \dots, g_{n-2}$ are such that $L_i(g_i) \neq 0$ for $i = 1, \dots, n-2$. Then, again by the pigeonhole principle, there is at least one $g \in G$ such that $L_{n-1}(g) \neq 0$ and $L_n((g_1 \cdots g_{n-1} g)^{-1}) \neq 0$ (since all of the elements $(g_1 \cdots g_{n-1} g)^{-1}$ are different as $g \in G$ varies).

On the other hand if $c \leq \frac{1}{2}$, then we can consider any finite abelian group G with a subgroup H of index 2. Now we let $L_i(g) \neq 0$ if and only if $g \in H$ for $i = 1, \dots, n-1$, and let L_n be nonvanishing on the complement of H. In this case, it is easy to check that there is no weak simultaneous nonvanishing. ☐

This shows that we need to know nonvanishing for at least 50% of the maps L_i in order to get weak simultaneous nonvanishing for purely combinatorial reasons. This is very far from being known in the case of the Rankin–Selberg L-functions studied in this paper, as even a positive proportion of nonvanishing seems out of reach with current methods; see [Michel and Venkatesh 2007] and [Templier 2011a].

3. Background

3A. ***Different incarnations of the class group.*** Let K be an imaginary quadratic field of discriminant $D < -6$. Denote by $\mathcal{I}_K$ the group of integral fractional ideals of K, $\mathcal{P}_K$ the subgroup of principal fractional ideals and $\mathrm{Cl}_K = \mathcal{I}_K / \mathcal{P}_K$ the class group of K, which we know from Gauß is a finite group. Furthermore, we have Siegel's bound

$$|\mathrm{Cl}_K| \gg_\varepsilon |D_K|^{1/2-\varepsilon} \tag{3-1}$$

for any $\varepsilon > 0$ where the implied constant is ineffective.

Given a fractional ideal $\mathfrak{a} \in \mathcal{I}_K$, we denote by $[\mathfrak{a}] \in \mathrm{Cl}_K$ the corresponding ideal class. We denote by $[\alpha_1, \alpha_2]$ the ideal generated by $\alpha_1, \alpha_2 \in K$ over $\mathbb{Z}$ and by $\widehat{\mathrm{Cl}}_K$ the group of class group characters, i.e., group homomorphisms $\chi : \mathrm{Cl}_K \to \mathbb{C}^\times$.

Let $\mathbb{A}_K^\times$, respectively, $\mathbb{A}_{K,\mathrm{fin}}^\times$, denote the idéles, respectively, finite idéles of K, and let $\widehat{\mathbb{O}}_K^\times = \prod_p \mathbb{O}_p^\times$ denote the standard maximal compact subgroup of $\mathbb{A}_{K,\mathrm{fin}}^\times$. Then we have the natural isomorphisms

$$\mathcal{I}_K \cong \mathbb{A}_{K,\mathrm{fin}}^\times / \widehat{\mathbb{O}}_K^\times \quad \text{and} \quad \mathrm{Cl}_K \cong K^\times \backslash \mathbb{A}_{K,\mathrm{fin}}^\times / \widehat{\mathbb{O}}_K^\times. \tag{3-2}$$

Given $\mathfrak{a} \in \mathcal{I}_K$, we denote by $\hat{\mathfrak{a}} \in \mathbb{A}_{K,\mathrm{fin}}^\times$ any lift of the corresponding element of $\mathbb{A}_{K,\mathrm{fin}}^\times / \widehat{\mathbb{O}}_K^\times$ under the above isomorphism.

3A1. *Heegner forms.* We refer to [Darmon 1994] for a concise treatment of the following material. Let N be a squarefree integer such that all primes dividing N split completely in K. Consider a residue class $r \bmod 2N$ such that $r^2 \equiv D \bmod 4N$. For $(a,b,c) \in \mathbb{Z}^3$ having greatest common divisor equal to 1 and satisfying $b^2 - 4ac = D$, $a \equiv 0 \bmod N$, and $b \equiv r \bmod 2N$, we denote by $[a,b,c]$ the integral binary quadratic form

$$Q(x,y) = ax^2 + bxy + cy^2. \tag{3-3}$$

We call such a quadratic form a *Heegner form of level N* and *orientation r* and denote by $\mathcal{Q}_D(N,r)$ the set of all such forms, which carries an action of the Hecke congruence subgroup $\Gamma_0(N)$ via coordinate transformation. It is a well-known fact extending Gauß that the map $\Gamma_0(N)\backslash\mathcal{Q}_D(N,r) \to \mathrm{Cl}_K$ defined by

$$[a,b,c] \mapsto \left[a, \frac{-b+\sqrt{D}}{2}\right],$$

is a bijection.

Given a Heegner form $Q = [a,b,c] \in \mathcal{Q}_D(N,r)$, we define the associated *Heegner point* as

$$z_Q := \frac{-b+\sqrt{D}}{2a} \in \mathbb{H}. \tag{3-4}$$

This defines a map $\mathcal{Q}_D(N,r) \to \mathbb{H}$ which is equivariant with respect to the action $\Gamma_0(N)$ (acting via linear fractional transformation on $\mathbb{H}$). In particular, we get a map $\mathrm{Cl}_K \to \Gamma_0(N)\backslash\mathbb{H}$ using the above.

3A2. *Oriented embeddings.* Again let $(a,b,c) \in \mathbb{Z}^3$ have greatest common divisor equal to 1 and satisfy $b^2 - 4ac = D$, $a \equiv 0 \bmod N$, and $b \equiv r \bmod 2N$. Associated to the triple (a,b,c), we define an (algebra) embedding $\Psi : K \hookrightarrow \mathrm{Mat}_{2\times 2}(\mathbb{Q})$ by

$$\Psi(\sqrt{D}) := \begin{pmatrix} b & 2c \\ -2a & -b \end{pmatrix}. \tag{3-5}$$

This embedding satisfies

$$\Psi(K) \cap \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{Z}) : N \mid c \right\} = \Psi(\mathbb{O}_K),$$

where $\mathbb{O}_K$ denotes the ring of integers of K. This means that Ψ is an *optimal embedding of level N* and *orientation r*. Conversely, every oriented optimal embedding of level N arises from such a triple of integers $(a,b,c) \in \mathbb{Z}^3$. Denote by $\mathcal{E}_D(N,r)$ the set of all such embeddings. The congruence subgroup $\Gamma_0(N)$ acts on $\mathcal{E}_D(N,r)$ by conjugation, namely,

$$(\gamma \cdot \Psi)(x + \sqrt{D}y) := \gamma^{-1}\Psi(x + \sqrt{D}y)\gamma$$

for $\gamma \in \Gamma_0(N)$.

There is a natural bijection between oriented optimal embeddings Ψ of level N and orientation r, as in (3-5), and Heegner forms $Q = [a, b, c]$, as in (3-3) (since these are both completely determined by $(a, b, c) \in \mathbb{Z}^3$), which is equivariant with respect to the action of $\Gamma_0(N)$. By the above, we have a bijection

$$\Gamma_0(N)\backslash\mathscr{E}_D(N, r) \to \mathrm{Cl}_K. \tag{3-6}$$

Given an optimal embedding Ψ of level N, we can extend it to an (algebra) embedding

$$\Psi_{\mathbb{A}} : \mathbb{A}_K \to \mathrm{Mat}_{2\times 2}(\mathbb{A})$$

by tensoring (over $\mathbb{Q}$) by $\mathbb{A}$. The local components of $\Psi_{\mathbb{A}}$ are defined as follows: If p is a prime of $\mathbb{Q}$ which is inert in K with $p\mathcal{O}_K = \mathfrak{p}$, then $K \otimes \mathbb{Q}_p \cong K_{\mathfrak{p}}$; and thus we get an embedding $\Psi_p : K_{\mathfrak{p}} \to \mathrm{Mat}_{2\times 2}(\mathbb{Q}_p)$ given by

$$K \otimes \mathbb{Q}_p \ni x \otimes y \mapsto \Psi(x) \otimes y \in \mathrm{Mat}_{2\times 2}(\mathbb{Q}_p),$$

defined up to the choice of isomorphism $K \otimes \mathbb{Q}_p \cong K_p$ (similarly for the inert infinite place). If p is ramified with $p\mathcal{O}_K = \mathfrak{p}^2$, then $K \otimes \mathbb{Q}_p \cong K_{\mathfrak{p}}$; and we get a map $\Psi_p : K_{\mathfrak{p}} \to \mathrm{Mat}_{2\times 2}(\mathbb{Q}_p)$ by tensoring as in the inert case. Finally, if p is split in K with $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$, then we have an algebra isomorphism $K \otimes \mathbb{Q}_p \cong K_{\mathfrak{p}} \times K_{\bar{\mathfrak{p}}}$ given by

$$K \otimes \mathbb{Q}_p \ni j_1 x + j_2 y \mapsto (x, y) \in K_{\mathfrak{p}} \times K_{\bar{\mathfrak{p}}}, \quad \text{with } x, y \in \mathbb{Q}_p, \tag{3-7}$$

where

$$j_1 = \frac{1 \otimes 1 + \sqrt{D} \otimes (\sqrt{D}\,)^{-1}}{2} \quad \text{and} \quad j_2 = \frac{1 \otimes 1 - \sqrt{D} \otimes (\sqrt{D}\,)^{-1}}{2}.$$

Here we consider $\sqrt{D}$ as an element of $\mathbb{Q}_p$ and use that $\mathbb{Q}_p \cong K_{\mathfrak{p}}$ as p splits in K. By using this, we get an algebra embedding $\Psi_p : K_{\mathfrak{p}} \times K_{\bar{\mathfrak{p}}} \to \mathrm{Mat}_{2\times 2}(\mathbb{Q}_p)$ by tensoring. Again this is well defined up to the choice of isomorphism $\mathbb{Q}_p \cong K_{\mathfrak{p}}$.

3B. ***Hecke characters of imaginary quadratic fields.*** Let K be an imaginary quadratic field of discriminant $D < -6$. In this paper, we will be working with Hecke characters of K of conductor 1, which (in the classical picture) are unitary characters $\chi : \mathcal{I}_K \to \mathbb{C}^\times$ such that for $(\alpha) \in \mathcal{P}_K$, we have $\chi((\alpha)) = \chi_\infty^{-1}(\alpha)$ for some character $\chi_\infty : \mathbb{C}^\times \to \mathbb{C}^\times$, which we call the ∞-type of χ. By considering the induced representation, we can see that given χ_∞ such that $\chi_\infty(-1) = 1$, we have exactly $|\mathrm{Cl}_K|$ Hecke characters of conductor 1 with ∞-type χ_∞; if χ_0 is any such Hecke character with ∞-type χ_∞, then the set of all such Hecke characters is given by $\{\chi_0\chi : \chi \in \widehat{\mathrm{Cl}}_K\}$. We will only be considering the ∞-types $\alpha \mapsto (\alpha/|\alpha|)^k$ for $k \in 2\mathbb{Z}$.

Given a Hecke character χ as above with ∞-type χ_∞, we get, using the isomorphism (3-2), an (idélic) Hecke character

$$\Omega : K^\times\backslash\mathbb{A}_K^\times/\mathcal{O}_K^\times \to \mathbb{C}^\times.$$

The above conditions translates to the fact that Ω is unramified at all finite places of K and the ∞-component Ω_∞ is equal to χ_∞.

Associated to a Hecke character χ as above with ∞-type $\alpha \mapsto (\alpha/|\alpha|)^k$, there is a theta series

$$\theta_\chi(z) := \sum_{\mathfrak{a} \text{ int. ideal of } \mathcal{O}_K} e^{2\pi i (\mathrm{N}\mathfrak{a}) z} (\mathrm{N}\mathfrak{a})^{k/2} \chi(\mathfrak{a}) \in \mathcal{M}_{k+1}(\Gamma_0(|D|), \chi_K),$$

which is a modular form of weight $k+1$, level $|D|$, and nebentypus equal to the quadratic character χ_K associated to K via class field theory. Furthermore, we know that θ_χ is noncuspidal exactly if $k=0$ and χ is a genus character of the class group of K; see [Iwaniec 1997, Theorem 12.5]. Recall that this is an example of automorphic induction from GL_1/K to $\mathrm{GL}_2/\mathbb{Q}$.

3C. *Automorphic forms.* In this section, we follow [Bump 1997, Chapters 2–3]. Let $L^2(\Gamma_0(N), k)$ denote the L^2-space of automorphic functions of level N and weight $k \in 2\mathbb{Z}$. That is, measurable maps $f : \mathbb{H} \to \mathbb{C}$ satisfying:

- The *automorphic condition* of weight k and level N
$$f(\gamma z) = j_\gamma(z)^k f(z),$$
for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, where
$$j_\gamma(z) := \frac{j(\gamma, z)}{|j(\gamma, z)|}, \quad \text{with } j(\gamma, z) = cz + d,$$
and
$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : N \mid c \right\}.$$
- The L^2*-condition*
$$\|f\|_2^2 := \langle f, f \rangle = \int_{\Gamma_0(N)\backslash \mathbb{H}} |f(z)|^2 \, d\mu(z) < \infty,$$
where $d\mu(z) = y^{-2} dx dy$ and $\langle \cdot, \cdot \rangle$ is the Petersson inner-product. Notice that the above integral is well defined since $|j_\gamma(z)| = 1$.

We have the weight k *raising* and *lowering operators* acting on $C^\infty(\mathbb{H})$, the space of smooth functions on $\mathbb{H}$, given by

$$R_k = (z - \bar{z}) \frac{\partial}{\partial z} + \frac{k}{2} \quad \text{and} \quad L_k = -(z - \bar{z}) \frac{\partial}{\partial \bar{z}} - \frac{k}{2}.$$

They define maps

$$R_k : L^2(\Gamma_0(N), k) \cap C^\infty(\mathbb{H}) \to L^2(\Gamma_0(N), k+2) \cap C^\infty(\mathbb{H}),$$
$$L_k : L^2(\Gamma_0(N), k) \cap C^\infty(\mathbb{H}) \to L^2(\Gamma_0(N), k-2) \cap C^\infty(\mathbb{H}),$$

which are adjoint in the sense that

$$\langle R_k f_1, f_2 \rangle = -\langle f_1, L_{k+2} f_2 \rangle \tag{3-8}$$

for $f_1 \in L^2(\Gamma_0(N), k) \cap C^\infty(\mathbb{H})$ and $f_2 \in L^2(\Gamma_0(N), k+2) \cap C^\infty(\mathbb{H})$. Furthermore, we have the product rule

$$R_{k_1+k_2}(f_1 f_2) = (R_{k_1} f_1) f_2 + f_1 (R_{k_2} f_2),$$

for $f_i \in L^2(\Gamma_0(N), k_i) \cap C^\infty(\mathbb{H})$, and similarly for the lowering operator.

The weight k *Laplacian* acting on $L^2(\Gamma_0(N),k)\cap C^\infty(\mathbb{H})$ is defined as

$$\Delta_k=-R_{k-2}L_k+\lambda\Big(\frac{k}{2}\Big)=-L_{k+2}R_k+\lambda\Big(-\frac{k}{2}\Big),$$

where $\lambda(s)=s(1-s)$. On $L^2(\Gamma_0(N),k)$, this defines a symmetric, unbounded operator with a unique self-adjoint extension which we also denote by Δ_k with some dense domain $D(\Delta_k)\subset L^2(\Gamma_0(N),k)$ (suppressing the level N in the notation).

A *Maaß form* of weight k and level N is a (necessarily real analytic) eigenfunction of Δ_k. Given a Maaß form f of eigenvalue λ we denote by $t_f:=\sqrt{\lambda-\frac{1}{4}}$ the *spectral parameter* of f (if $\lambda>\frac{1}{4}$, we always pick the positive square root).

Denote by $\mathcal{S}_k(N)$ the vector space of weight k and level N (classical) holomorphic cusp forms. If $g\in\mathcal{S}_k(N)$, then it is easy to see that $y^{k/2}g$ is a Maaß form of weight k and level N of eigenvalue $\lambda(k/2)$. In fact, it can be show that any Maaß form of weight $k\geq 0$ and level N is of the form

$$R_{k-2}\cdots R_{k_0}y^{k_0/2}g,\quad \text{with } g\in\mathcal{S}_{k_0}(N) \text{ where } k_0\leq k \text{ and } k_0\equiv k \bmod 2$$

or

$$R_{k-2}\cdots R_0 f,\quad \text{with } f \text{ a Maaß form of weight 0 and level } N.$$

And similarly for $k<0$, now with lowering operators and antiholomorphic cusp forms.

Furthermore, we say that a Maaß form of weight k and level N is a *Hecke–Maaß eigenform* if it is an eigenfunction for the Hecke operators T_n with $(N,n)=1$ (which commute with the action of the raising and lowering operators), as well as the reflection operator

$$X:L^2(\Gamma_0(N),k)\to L^2(\Gamma_0(N),k),\quad (Xf)(z):=f(-\bar{z}).$$

Finally, we say that a Hecke–Maaß eigenform is a *Hecke–Maaß newform* if it is an eigenfunction for *all* Hecke operators T_n, with $n\geq 1$.

Denote by $\mathcal{B}^*_{k,\mathrm{hol}}(N)$ the set consisting of $f/\|f\|_2$, where $f=y^{k/2}g$ with $g\in\mathcal{S}_k(N)$ a (Hecke-normalized) holomorphic Hecke newform, and by $\mathcal{B}^*(N)$ the set consisting of $f/\|f\|_2$, with f a *nonconstant* (Hecke-normalized) Hecke–Maaß newform of weight 0 and level N. We will sometimes refer to these simply as (classical) "Maaß forms". It follows from Atkin–Lehner theory that for $k\geq 0$, we have the following orthonormal basis consisting of Hecke–Maaß eigenforms for the subspace of $L^2(\Gamma_0(N),k)$ spanned by *nonconstant* Maaß forms of weight k and level N:

$$\mathcal{B}_k(N):=\bigcup_{dN'\mid N}\nu^*_{d,N'}R_{k-2}\cdots R_0\mathcal{B}^*(N')\cup\bigcup_{dN'\mid N}\ \bigcup_{\substack{0<k_0\leq k\\ k_0\equiv k \bmod 2}}\nu^*_{d,N'}R_{k-2}\cdots R_{k_0}\mathcal{B}^*_{k_0,\mathrm{hol}}(N'),\qquad (3\text{-}9)$$

where $\nu^*_{d,N'}:L^2(\Gamma_0(N'),k)\to L^2(\Gamma_0(N),k)$ are defined by $(\nu^*_{d,N'}f)(z):=f(dz)$. If $k<0$, we have a similar basis now with lowering operators and antiholomorphic cusp forms.

Using (3-8), we see that for any $f\in\mathcal{B}_k(N)$, we have the following useful relation:

$$\|R_{k+2l}\cdots R_k f\|_2^2=\|f\|_2^2\prod_{j=0}^{l}\Big(\frac{k+2j-1}{2}+it_f\Big)\Big(\frac{k+2j-1}{2}-it_f\Big).\qquad (3\text{-}10)$$

3C1. *Adélization of Maaß forms.* Given an element of $f \in L^2(\Gamma_0(N), k)$ we define a lift $\tilde{f}: \mathrm{GL}_2^+(\mathbb{R}) \to \mathbb{C}$ as

$$\tilde{f}(g) := j_g(i)^{-k} f(gi),$$

which satisfies

$$\tilde{f}(gk_\theta) = e^{ik\theta}\tilde{f}(g),$$

for all $\theta \in [0, 2\pi)$, where $k_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$ and $g \in \mathrm{GL}_2^+(\mathbb{R})$.

Now consider the following decomposition of $\mathrm{GL}_2(\mathbb{A})$ coming from strong approximation:

$$\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{Q}) K_0(N)\, \mathrm{GL}_2^+(\mathbb{R}), \tag{3-11}$$

where $\mathrm{GL}_2(\mathbb{Q})$ is embedded diagonally and

$$K_0(N) := \left\{ k \in \mathrm{GL}_2(\mathbb{A}) : k_\infty = 1,\ k_p = \begin{pmatrix} a_p & b_p \\ c_p & d_p \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p),\ c_p \in p^r\mathbb{Z}_p,\ p^r \| N \right\}.$$

Now we define the *adélization* of f as

$$\phi_f(g) = \phi_f(\gamma k g_\infty) := \tilde{f}(g_\infty),$$

which does not depend on the choice of decomposition

$$g = \gamma k g_\infty \in \mathrm{GL}_2(\mathbb{Q}) K_0(N)\, \mathrm{GL}_2^+(\mathbb{R}).$$

Given a Hecke–Maaß newform f, the adélization ϕ_f generates a unique cuspidal automorphic representation $\pi_f = \pi$ of $\mathrm{GL}_2(\mathbb{A})$. The infinity component of this representation π_∞ is a discrete series representation of lowest weight $k_\pi = k$ if f corresponds to a holomorphic Hecke newform of weight k. On the other hand if f is of weight 0 and nonconstant (i.e., corresponds to a classical Maaß form), then π_∞ is a principal series representation of lowest weight $k_\pi = 0$. We denote by t_π the spectral parameter t_f of f.

3C2. *Automorphic L-functions.* In general, associated to an automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$ we can define the (finite part of the) L-function $L(\pi, s)$ as a product over finite primes in terms of the Satake parameters and a completed version $\Lambda(\pi, s)$ satisfying a functional equation $\Lambda(\pi, s) = \varepsilon_\pi \Lambda(\check{\pi}, 1-s)$, where ε_π is of norm 1 (the root number) and $\check{\pi}$ is the contragredient of π. We refer to [Godement and Jacquet 1972] for details. Furthermore, given automorphic representations π_1, π_2, π_3 of $\mathrm{GL}_n(\mathbb{A})$, we will be interested in the Rankin–Selberg convolution L-function $L(\pi_1 \otimes \pi_2, s)$ (see [Jacquet et al. 1983]), the symmetric square L-function $L(\mathrm{sym}^2 \pi_1, s)$ (see [Bump 1997, Chapter 3.8]), and the triple convolution L-function $L(\pi_1 \otimes \pi_2 \otimes \pi_3, s)$ (see [Watson 2002]).

4. A classical version of Waldspurger's formula

In order to make our moment calculations explicit, we will need an explicit version of Waldspurger's formula as developed my Martin and Whitehouse [2009] and, furthermore, translate this to a classical formula. In doing so, we will follow Popa [2006, Chapter 5].

4A. ***A formula of Martin and Whitehouse (following Waldspurger).*** Let π be an automorphic representation of $\mathrm{GL}_2(\mathbb{Q})$ of squarefree conductor N and even lowest weight k_π corresponding to the classical cuspidal newform f (Maaß or holomorphic also of weight k_π). Let $D<-6$ be a negative fundamental discriminant with $(D,2N)=1$ and such that all primes dividing N split in $K=\mathbb{Q}[\sqrt{D}]$. Let $k\geq k_\pi$ be even, and let $\Omega: K^\times\backslash\mathbb{A}_K^\times\to\mathbb{C}^\times$ be an idélic Hecke character of conductor 1 and ∞-type $\Omega_\infty(\alpha)=(\alpha/|\alpha|)^k$. Recall from Section 3B that any two such characters differ by a class group character, and thus there are $|\mathrm{Cl}_K|$ such characters.

We will be interested in obtaining an explicit formula in terms of Heegner points of the central value of the Rankin–Selberg L-function $L\big(\pi\otimes\Omega,\tfrac12\big)$, by which we mean the Rankin–Selberg convolution of the base change π_K of π to $\mathrm{GL}_2(\mathbb{A}_K)$ and the automorphic representation π_Ω of $\mathrm{GL}_1(\mathbb{A}_K)$ corresponding to Ω. We note that the above (Heegner) conditions on D and N imply that the root number of $L(\pi\otimes\Omega,s)$ is equal to $+1$.

Let $\Psi_{\mathbb{A}}:\mathbb{A}_K\hookrightarrow\mathrm{GL}_2(\mathbb{A})$ be an oriented optimal algebra embedding of level N. Then associated to the triple $(\pi,\Omega,\Psi_{\mathbb{A}})$, Martin and Whitehouse [2009, Theorem 4.1] define a specific test vector $\phi_{\mathrm{MW}}\in\pi$ such that we have the formula

$$\frac{\big|\int_{\mathbb{A}^\times K^\times\backslash\mathbb{A}_K^\times}\phi_{\mathrm{MW}}(\Psi_{\mathbb{A}}(x))\Omega^{-1}(x)\,dx\big|^2}{\int_{Z(\mathbb{A})\,\mathrm{GL}_2(\mathbb{Q})\backslash\,\mathrm{GL}_2(\mathbb{A})}|\phi_{\mathrm{MW}}(g)|^2\,dg}=\frac{L(\pi\otimes\Omega,1/2)}{L(\mathrm{sym}^2\pi,1)}\frac{c_\infty(\pi_\infty,k)}{2\sqrt{|D|}}\prod_{p\mid N}\Big(1-\frac{1}{p}\Big)^{-1},\tag{4-1}$$

where the measure dg is normalized so that the volume of $Z(\mathbb{A})\,\mathrm{GL}_2(\mathbb{Q})\backslash\,\mathrm{GL}_2(\mathbb{A})$ is $(\pi/3)\prod_{p\mid N}(1-p^{-2})$ (here we are using that the Tamagawa number of $\mathrm{GL}_2/\mathbb{Q}$ is 2) and dx is normalized so that $\mathbb{A}^\times K^\times\backslash\mathbb{A}_K^\times$ has volume $2\Lambda(\chi_K,1)$, where χ_K is the quadratic character associated to K via class field theory and

$$\Lambda(\chi_K,s)=\pi^{-(s+1)/2}\Gamma\Big(\frac{s+1}{2}\Big)L(\chi_K,s).$$

The local constants are given by:

$$c_\infty(\pi_\infty,k)=\begin{cases}(2\pi)^k\prod_{j=0}^{k/2-1}\big(\frac14+(t_\pi)^2+j(j+1)\big)^{-1} & \text{if } \pi_\infty \text{ is a p.s,}\\ (2\pi)^{k-k_\pi}\dfrac{\Gamma(k_\pi+1)}{\Gamma\big(\frac12(k+2)\big)B\big(\frac12(k+k_\pi),\frac12(k-k_\pi+2)\big)} & \text{if } \pi_\infty \text{ is a d.s,}\end{cases}$$

where "p.s" and "d.s" refer to "principal series" and "discrete series", respectively, and $B(x,y)$ denotes the Beta function.

To make this formula explicit, we need to specify an embedding $\Psi_{\mathbb{A}}$. To do this, let $[a,b,c]$ be a Heegner form of level N and orientation r and consider the associated optimal embedding $\Psi:K\hookrightarrow\mathrm{Mat}_{2\times2}(\mathbb{Q})$ of level N (as in Section 3A2) satisfying

$$\Psi(K)\cap M_0(N)=\Psi(\mathcal{O}_K),$$

where $M_0(N)=\big\{\big(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\big)\in\mathrm{Mat}_{2\times2}(\mathbb{Z}):N\mid c\big\}$. As described in Section 3A2, we get by tensoring with $\mathbb{A}$ an associated embedding $\Psi_{\mathbb{A}}:\mathbb{A}_K^\times\to\mathrm{GL}_2(\mathbb{A})$. We write Ψ_{fin} for the finite component and Ψ_∞ for the infinite component of this embedding.

Now the recipe described in [Martin and Whitehouse 2009, Chapter 4.2] gives the following characterization of the test vector ϕ_{MW}: the finite component $\phi_{\mathrm{MW},p}$ at a finite prime $p<\infty$ is uniquely determined (up to scaling) by the invariance under a certain Eichler order, which in our setting is exactly the order in $\mathrm{GL}_2(\mathbb{Q}_p)$ of reduced discriminant $p^{\nu_p(N)}$ (using that Ψ is optimal of level N). This means that we can pick $\phi_{\mathrm{MW},p}=\phi_{f,p}=\phi_{f_k,p}$, where ϕ_f (respectively, ϕ_{f_k}) are the lifts to $\mathrm{GL}_2(\mathbb{A})$ of the Hecke–Maaß newform $f\in L^2(\Gamma_0(N),k_\pi)$ corresponding to π (respectively, $f_k=R_{k-2}\cdots R_{k_\pi}f$).

At the infinite place the test vector $\phi_{\mathrm{MW},\infty}$ is characterized by being the vector of the minimal K-type (in the sense of [Popa 2008]) such that

$$\pi_\infty(x)\phi_{\mathrm{MW},\infty}=\Omega_\infty(x)\phi_{\mathrm{MW},\infty}$$

for all $x\in\Psi_\infty(S^1)\cap\mathrm{O}_2(\mathbb{R})$, where $S^1=\{z\in\mathbb{C}^\times:|z|=1\}$ is the maximal compact of $\mathbb{C}^\times$ and $\mathrm{O}_2(\mathbb{R})$ is the maximal compact of $\mathrm{GL}_2(\mathbb{R})$. There is a slight complication due to the fact that the embedding Ψ_∞ defined above does not send the maximal compact $S^1\subset\mathbb{C}^\times$ to $\mathrm{SO}_2(\mathbb{R})$. We can, however, easily check that this is the case after conjugating by

$$\gamma_\infty=\begin{pmatrix}\sqrt{D} & -b\\ 0 & a\end{pmatrix}. \tag{4-2}$$

Thus, we conclude that the following vector satisfies the conditions specified by Martin and Whitehouse:

$$\phi_{\mathrm{MW},\infty}=\pi(\gamma_\infty)\phi_{f_k,\infty},$$

where $\phi_{f_k,\infty}=R_{k-2}\cdots R_{k_\pi}\phi_{f,\infty}$ is a weight k vector in the representation space π_∞. We conclude that we can pick the global test vector as

$$\phi_{\mathrm{MW}}=\pi(\gamma_\infty)\phi_{f_k},$$

where again $f_k=R_{k-2}\cdots R_{k_\pi}f$ and $\gamma_\infty\in\mathrm{GL}_2(\mathbb{R})\subset\mathrm{GL}_2(\mathbb{A})$ as in (4-2).

For ϕ_{MW} as above, we have for $x_{\mathrm{fin}}\in\mathbb{A}^\times_{K,\mathrm{fin}}$ and $x_\infty\in\mathbb{C}^\times$ that

$$\phi_{\mathrm{MW}}\big(\Psi_\infty(x_\infty)\Psi_{\mathrm{fin}}(x_{\mathrm{fin}})\big)\Omega^{-1}\big((x_\infty,x_{\mathrm{fin}})\big)$$

is independent of x_∞. In particular, we get a well-defined map

$$\mathrm{Cl}_K\ni[\mathfrak{a}]\mapsto\phi_{\mathrm{MW}}(\Psi_{\mathbb{A}}(\hat{\mathfrak{a}}))\Omega^{-1}(\hat{\mathfrak{a}}),$$

where $\hat{\mathfrak{a}}\in\mathbb{A}^\times_{K,\mathrm{fin}}$ is any lift of $\mathfrak{a}$ under the first isomorphism in (3-2). By the second isomorphism in (3-2), it follows that we have a bijection

$$K^\times\mathbb{A}^\times\backslash\mathbb{A}_K^\times/\widehat{\mathbb{O}}_K^{\,\times}\xrightarrow{\sim}\bigsqcup_{[\mathfrak{a}]\in\mathrm{Cl}_K}\mathbb{C}^\times/\mathbb{R}^\times, \tag{4-3}$$

from which we conclude that

$$\int_{\mathbb{A}^\times K^\times\backslash\mathbb{A}_K^\times}\phi_{\mathrm{MW}}(\Psi_{\mathbb{A}}(x))\Omega^{-1}(x)\,dx=\frac{2}{|D|^{1/2}}\sum_{[\mathfrak{a}]\in\mathrm{Cl}_K}\phi_{f_k}(\Psi_{\mathrm{fin}}(\hat{\mathfrak{a}})\gamma_\infty)\overline{\Omega(\hat{\mathfrak{a}})}. \tag{4-4}$$

Here we can check the normalization by letting ϕ_{MW} and Ω being constants and recalling that the total measure of $\mathbb{A}^\times K^\times\backslash\mathbb{A}_K^\times$ is $2\Lambda(\chi_K,1)=2|\mathrm{Cl}_K||D|^{-1/2}$ by the class number formula.

4B. ***Explicit representatives of the class group.*** Consider integral prime ideals $\mathfrak{p}_1 = (1), \mathfrak{p}_2, \dots, \mathfrak{p}_h$ which are representatives for the class group Cl_K dividing the rational primes p_i which we assume are coprime to $2Na$ (so that $h = |\mathrm{Cl}_K|$ and $p_i\mathcal{O}_K = \mathfrak{p}_i\overline{\mathfrak{p}_i}$ splits in K for $i = 2, \dots, h$). The ideal class $[\mathfrak{p}_i]$ is represented by the idéle $\widehat{\mathfrak{p}_i} := (p_i)_{\mathfrak{p}_i} \in \mathbb{A}_K^\times$ (where the subscript means that the element is concentrated at the place $\mathfrak{p}_i$). Thus we see using the definition (3-7) of $\Psi_{\mathbb{A}}$ that since

$$j_1 \cdot p_i + j_2 \cdot 1 = 1 \otimes \frac{p_i+1}{2} + \sqrt{D} \otimes \frac{p_i-1}{2\sqrt{D}} \in K \otimes \mathbb{Q}_{p_i},$$

we have that

$$\Psi_{\mathbb{A}}((p_i)_{\mathfrak{p}_i}) = \begin{pmatrix} \frac{p_i+1}{2} + b\frac{p_i-1}{2\sqrt{D}} & c\frac{p_i-1}{\sqrt{D}} \\ -a\frac{p_i-1}{\sqrt{D}} & \frac{p_i+1}{2} - b\frac{p_i-1}{2\sqrt{D}} \end{pmatrix}_{p_i}.$$

For $i = 2, \dots, h$, it is a short computation that for an integer b_i with $b_i \equiv b \bmod 2a$ and $b_i^2 \equiv D \bmod p_i$ (and put also $b_1 = 1$ for completeness), we have

$$\mathfrak{p}_i = \left[\frac{-b_i + \sqrt{D}}{2}, p_i\right]. \tag{4-5}$$

Using the congruences for b_i, it follows that there is $k_i \in K_0(N)$ such that

$$\Psi_{\mathbb{A}}((p_i)_{\mathfrak{p}_i}) = \gamma_i k_i (\gamma_i^{-1})_\infty$$

with $\gamma_i \in M_2(\mathbb{Q})$ given by

$$\gamma_i = \begin{pmatrix} p_i & \frac{b_i - b}{2a} \\ 0 & 1 \end{pmatrix}.$$

Thus we conclude by the definition of adélization that

$$\phi_{f_k}(\Psi_{\mathrm{fin}}(\widehat{\mathfrak{p}}_i)\gamma_\infty) = j_{\gamma_i^{-1}\gamma_\infty}(i)^k f_k(\gamma_i^{-1}\gamma_\infty i) = f_k\left(\frac{-b_i + \sqrt{D}}{2ap_i}\right).$$

To proceed, we need to understand how the Heegner points $(-b_i + \sqrt{-D})/(2ap_i)$ behaves as $i = 1, \dots, h$ varies. Let $I : \Gamma_0(N)\backslash\mathcal{E}_D(N, r) \to \mathrm{Cl}_K$ be the bijection in (3-6). Then we have the following adaption of [Popa 2006, Proposition 6.2.2]:

Lemma 4.1. *We have*

$$\gamma_i^{-1}\gamma_\infty i = z_{Q_{\Psi,i}} \in \mathbb{H},$$

where $z_{Q_{\Psi,i}}$ is the Heegner point of a Heegner form $Q_{\Psi,i}$ of level N and orientation r (depending on Ψ and i) belonging to the class $I([\psi]) \cdot [\mathfrak{p}_i] \in \mathrm{Cl}_K$.

Proof. Consider the binary quadratic form

$$Q(x, y) = ap_i x^2 + b_i xy + c_i y^2,$$

where

$$c_i = \frac{b_i^2 - D}{4ap_i}$$

is an integer by the above congruence conditions. This means that Q is a discriminant D Heegner form of level N and orientation r, with corresponding Heegner point given by

$$\frac{-b_i + \sqrt{D}}{2ap_i}.$$

Thus the lemma reduces to showing the following identity of ideals (modulo principal ideals):

$$\left[ap_i, \frac{-b_i + \sqrt{D}}{2}\right] = \left[\frac{-b_i + \sqrt{D}}{2}, p_i\right] \cdot \left[\frac{-b + \sqrt{D}}{2}, a\right]. \tag{4-6}$$

This follows, as in the proof of [Popa 2006, Proposition 6.2.2], since both sides have the same ideal norm and we can check using the congruence condition on b_i that the right-hand side is contained in the left-hand side. □

This implies that the automorphic period (4-4) depends on the choice of optimal embedding Ψ but only up to a phase. In particular, the absolute square does not depend on the choice of Ψ as should be the case by (4-1).

4C. *An explicit formula.* To simplify matters, we from now on pick our optimal embedding Ψ such that $[a, b, c]$ corresponds to the trivial element of Cl_K and to lighten notation, we write

$$Q_i = ap_i x^2 + b_i xy + c_i y^2, \quad \text{with } i = 1, \dots, h, \tag{4-7}$$

where p_i and b_i are as above. Now if $Q \in \mathcal{Q}_D(N, r)$ is any quadratic form such that $[Q] = [\mathfrak{p}_i]$, then it follows from Lemma 4.1 that there is some $\gamma_Q \in \Gamma_0(N)$ such that $z_Q = \gamma_Q z_{Q_i}$, which implies that

$$f_k(z_Q) = j_{\gamma_Q}(z_{Q_i})^k f(z_{Q_i}) = \Omega_\infty(\alpha_Q)\phi_{f_k}(\Psi_{\mathrm{fin}}(\widehat{\mathfrak{p}}_i)\gamma_\infty),$$

where $\alpha_Q = j(\gamma_Q, z_{Q_i}) \in K^\times$. Similarly if $\mathfrak{a} \in \mathcal{I}_K$ is a different representative of the ideal class $[\mathfrak{p}_i] \in \mathrm{Cl}_K$, then we have

$$\Omega^{-1}(\hat{\mathfrak{a}}) = \Omega_\infty(\alpha_{\mathfrak{a}})\Omega^{-1}(\widehat{\mathfrak{p}}_i)$$

for some $\alpha_{\mathfrak{a}} \in K^\times$.

From this we conclude, by combining (4-4) and Lemma 4.1, that

$$\int_{\mathbb{A}^\times K^\times \backslash \mathbb{A}_K^\times} \phi_{\mathrm{MW}}(\Psi_{\mathbb{A}}(x))\Omega^{-1}(x)\, dx = \sum_{[Q] \in \Gamma_0(N) \backslash \mathcal{Q}_D(N,r)} f_k(z_Q)\overline{\Omega(\widehat{\mathfrak{a}}_Q)}\Omega_\infty(\alpha_{Q,\mathfrak{a}_Q}), \tag{4-8}$$

where z_Q is the Heegner point associated to the Heegner form $Q \in \mathcal{Q}_D(N, r)$, $[\mathfrak{a}_Q] = [Q]$ (under the bijection $\Gamma_0(N)\backslash\mathcal{Q}_D(N, r) \xrightarrow{\sim} \mathrm{Cl}_K$), and $\alpha_{Q,\mathfrak{a}_Q} \in K^\times$ is a complex number depending on the choices of Q and $\mathfrak{a}_Q$ (but not on π, Ω, nor f_k).

4C1. *The case of old forms.* We will now explain how to extend the identity (4-8) to the case of old forms. Let d, N' be positive integers such that $dN' \mid N$, and consider a newform (i.e., new at finite places) $f_k \in \mathcal{B}_k^*(N')$ belonging to the automorphic representation π. Then we get an element $\nu_{d,N'}^* f_k \in \mathcal{B}_k(N)$ given by $z \mapsto f_k(dz)$. Recall the representatives $\mathfrak{p}_1, \ldots, \mathfrak{p}_h \in \mathcal{I}_K$ of the class group Cl_K defined in (4-5) and the associated Heegner forms $Q_i = [a, b_i, c_i]$ defined in (4-7). Then we see directly that

$$dz_{Q_i} = \frac{-b_i + \sqrt{D}}{2p_i a/d} = z_{Q_i'},$$

where $Q_i' = [p_i a/d, b_i, c_i d] \in \mathcal{Q}_D(N', r)$ is a Heegner form of level N' and orientation $r \bmod (2N')$. From this, we see that

$$f_k(dz_{Q_i}) = \phi_{f_k}(\Psi'_{\mathrm{fin}}(\widehat{\mathfrak{p}}_i)\gamma'_\infty), \quad \text{with } i = 1, \ldots, h,$$

where Ψ' is the optimal embedding of level N' corresponding to the triple $[a/d, b, cd]$ and

$$\gamma'_\infty = \begin{pmatrix} \sqrt{D} & -b \\ 0 & \frac{a}{d} \end{pmatrix}.$$

Observe that $[a/d, b, cd]$ might not correspond to the trivial element of the class group. Thus, using (4-8),

$$\begin{aligned}\sum_{[Q]\in\Gamma_0(N)\backslash\mathcal{Q}_D(N,r)} \nu_{d,N'}^* f_k(z_Q)\overline{\Omega(\widehat{\mathfrak{a}}_Q)}\Omega_\infty(\alpha_{Q,\mathfrak{a}_Q}) &= \sum_{i=1}^{h} \nu_{d,N'}^* f_k(z_{Q_i})\overline{\Omega(\widehat{\mathfrak{p}}_i)} \\ &= \int_{\mathbb{A}^\times K^\times\backslash\mathbb{A}_K^\times} \phi'_{\mathrm{MW}}(\Psi'_{\mathbb{A}}(x))\Omega^{-1}(x)\,dx, \qquad (4\text{-}9)\end{aligned}$$

where ϕ'_{MW} is the vector defined by Martin and Whitehouse corresponding to the triple $(\pi, \Omega, \Psi'_{\mathbb{A}})$ and the numbers $\alpha_{Q,\mathfrak{a}_Q}$ are as in (4-8).

Combining (4-9) and (4-1), we arrive at the following result (recalling the definition (3-9) of $\mathcal{B}_k(N)$):

Theorem 4.2. *Let N be a squarefree integer and K be an imaginary quadratic field of discriminant D with $(D, 2N) = 1$ and such that all primes dividing N splits in K. Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ of conductor N' dividing N and even lowest weight k_π. Let $k \geq k_\pi$ be an even integer and $\Omega : K^\times\backslash\mathbb{A}_K^\times/\widehat{\mathcal{O}}_K^\times \to \mathbb{C}^\times$ a Hecke character of K of conductor 1 and ∞-type $\alpha \mapsto (\alpha/|\alpha|)^k$.*

Then for any $f_k \in \mathcal{B}_k(N)$ belonging to the representation space of π, we have

$$\left|\sum_{[Q]\in\Gamma_0(N)\backslash\mathcal{Q}_D(N,r)} f_k(z_Q)\overline{\Omega(\widehat{\mathfrak{a}}_Q)}\Omega_\infty(\alpha_{Q,\mathfrak{a}_Q})\right|^2 = \frac{L(\pi\otimes\Omega, 1/2)}{L(\mathrm{sym}^2\pi, 1)}\frac{|D|^{1/2}}{8N'}c_\infty(\pi_\infty, k), \qquad (4\text{-}10)$$

where z_Q is the Heegner point associated to the Heegner form $Q \in \mathcal{Q}_D(N, r)$, $\mathfrak{a}_Q \in \mathcal{I}_K$ is such that $[Q] = [\mathfrak{a}_Q]$ (under the bijection $\Gamma_0(N)\backslash\mathcal{Q}_D(N, r) \xrightarrow{\sim} \mathrm{Cl}_K$), $\alpha_{Q,\mathfrak{a}_Q} \in K^\times$ is a complex number depending on the choices Q and $\mathfrak{a}_Q$ (but not on π, Ω nor f_k), and

$$c_\infty(\pi_\infty, k) = \begin{cases} (2\pi)^k \prod_{j=0}^{k/2-1}\left(\frac{1}{4} + (t_\pi)^2 + j(j+1)\right)^{-1} & \text{if } \pi_\infty \text{ is a p.s,} \\ (2\pi)^{k-k_\pi-1}\dfrac{\Gamma(k_\pi)}{\left(\Gamma\left(\frac{1}{2}(k-2)\right)B\left(\frac{1}{2}(k+k_\pi+1), \frac{1}{2}(k-k_\pi+1)\right)\right)} & \text{if } \pi_\infty \text{ is a d.s,} \end{cases} \qquad (4\text{-}11)$$

where "p.s" ("d.s") refers to "principal series" ("discrete series") and $B(x, y)$ denotes the Beta function.

Using orthogonality of characters (i.e., Fourier inversion) we conclude the following key identity:

Corollary 4.3. *Let π, Ω, f_k be as in Theorem 4.2. Then given an element of the class group $[\mathfrak{a}] \in \mathrm{Cl}_K$ and a Heegner form $Q \in \mathcal{Q}_D(N, r)$ such that $[Q] = [\mathfrak{a}]$, we have*

$$f_k(z_Q)\Omega(x_Q) = \frac{c_{f_k}|D|^{1/4}}{|\mathrm{Cl}_K|} \sum_{\chi \in \widehat{\mathrm{Cl}}_K} \varepsilon_{\chi, f_k, r} \left| L\left(\pi \otimes \chi\Omega, \tfrac{1}{2}\right)\right|^{1/2} \chi([\mathfrak{a}]), \tag{4-12}$$

where $x_Q \in \mathbb{A}_K^\times$ is some element depending on the choice of Q (but not on π, Ω, nor f_k), $\varepsilon_{\chi, f_k, r}$ are complex numbers of norm 1, and

$$c_{f_k} = \frac{c_\infty(\pi_\infty, k)}{8N' L(\mathrm{sym}^2 \pi, 1)}, \tag{4-13}$$

with $c_\infty(\pi_\infty, k)$ as in (4-11).

5. Some technical lemmas

In this section, we will prove two key estimates. The first is a bound for the norm of Δ^m, which will be key in obtaining explicit error terms in our moment calculation. Similar consideration have been made in a different context in [Petridis and Risager 2018b, Theorem 5.1]. Secondly, we will obtain a lower bound for the L^2-norm of the product of Maaß forms. This is an extremely crude lower bound, which suffices for our purposes.

5A. ***A bound for the norm of Δ^m.*** In the course of proving our bound for the norm of Δ^m applied to certain vectors, we will need the following convenient L^∞-bound for $f \in \mathcal{B}_k(N)$ due to Blomer and Holowinsky [2010]:

$$\frac{\|f\|_\infty}{\|f\|_2} \ll N^{-1/32}(|t_f| + |k| + 1)^A \tag{5-1}$$

for some unspecified constant $A > 0$. The focus of [Blomer and Holowinsky 2010] is the level aspect, which we consider fixed in the present paper. Here the key thing is, however, that we get a polynomial bound for raised (and lowered) Hecke–Maaß forms with the constant being independent of the weight k and the spectral parameter t_f. The specific value of A is not important for our application.

Lemma 5.1. *Let $k_1, \dots, k_n$ be even integers such that $\sum_{i=1}^n k_i = 0$. For $i = 1, \dots, n$, let $f_i \in \mathcal{B}_{k_i}(N)$ be a Hecke–Maaß form of weight k_i, level N, and spectral parameter t_{f_i}. Then we have*

$$\left\| \Delta^m \prod_{i=1}^n f_i \right\|_\infty \ll n^{2m}(m + \max_{i=1,\dots,n} |t_{f_i}| + |k_i|)^{nA+2m} \prod_{i=1}^n \|f_i\|_2 \tag{5-2}$$

for all $m \in \mathbb{Z}_{\geq 0}$. Here the implied constant is allowed to depend on N.

Proof. Recalling that $\Delta = L_2 R_0$, we get, using the product rule for the raising and lowering operators,

$$\left|\Delta^m \prod_{i=1}^n f_i(z)\right| = \left|L_2 R_0 \cdots L_2 R_0 \prod_{i=1}^n f_i(z)\right|$$
$$\leq n^{2m} \max_{\substack{m_1, \dots, m_n \in \mathbb{N} : \Sigma m_i = 2m \\ (U_{i,j})_{i,j} \text{ fifty-fifty raising/lowering operators} \\ \text{where } 1 \leq i \leq n, 1 \leq j \leq m_i}} \prod_{i=1}^n |U_{i,1} \cdots U_{i,m_i} \, f_i(z)|. \tag{5-3}$$

Here the maximum is taken over all combinations of $2m$ operators

$$U_{i,j} : 1 \le i \le n,\ 1 \le j \le m_i,$$

which are all either a raising or a lowering operator of appropriate weight and such that the total number of raising and lowering operators are equal. If we have $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m_i - 1\}$ such that $\{U_{i,j}, U_{i,j+1}\}$ is of the type {raising, lowering}, then we get

$$U_{i,j} U_{i,j+1} = -\Delta_{\pm\kappa} + \lambda\Big(\frac{\kappa}{2}\Big)$$

for some weight κ with $|\kappa| \le 2m + |k_i|$ (since we can have at most m raising respectively, lowering operators). Here the sign corresponds to whether $U_{i,j}$ is a raising or lowering operator. This shows that we can replace $U_{i,j} U_{i,j+1}$ with multiplication by

$$\lambda\Big(\frac{\kappa}{2}\Big) - \lambda_{f_i} = -\Big(\frac{\kappa-1}{2} + i t_{f_i}\Big)\Big(\frac{\kappa-1}{2} - i t_{f_i}\Big).$$

Repeating this, we get

$$|U_{i,1} \cdots U_{i,m_i} f_i(z)| = \left| R_{k+2m_i'-2} \cdots R_k f_i(z) \prod_{j=1}^{(m_i-m_i')/2} \Big(\frac{\kappa_j - 1}{2} + i t_f\Big)\Big(\frac{\kappa_j - 1}{2} - i t_f\Big) \right|$$

for some $0 \le m_i' \le m_i$, where $|\kappa_j| \le 2m + |k_i|$ (or a similar expression with lowering instead of raising operators).

By combining the bound (5-1) and the computation of the L^2-norm (3-10), we conclude that for $f \in \mathcal{B}_k(N)$ and $l \ge 0$

$$\begin{aligned}\|R_{k+2l} R_{k+2l-2} \cdots R_k f\|_\infty &\ll \|f\|_2\big(|t_f|+|k+l|+1\big)^A \prod_{j=0}^{l} \left|\Big(\frac{k+2j-1}{2}+it_f\Big)\Big(\frac{k+2j-1}{2}-it_f\Big)\right|^{1/2} \\ &\ll \|f\|_2\big(|t_f|+|k|+l+1\big)^{l+A},\end{aligned}$$

and similarly in the case of lowering operators. Combining all of the above, we arrive at

$$|U_{i,1} \cdots U_{i,m_i} f_i(z)| \ll \|f_i\|_2(|t_f| + |k_i| + m_i + 1)^{A+m_i},$$

for any sequence of raising and lowering operators $U_{i,1}, \dots, U_{i,m_i}$ as in the maximum in (5-3). Plugging this into (5-3) gives the wanted. □

5B. ***A lower bound for weight k automorphic forms.*** In this subsection, we will prove a lower bound for the L^2-norm of a product of Maaß forms. The idea is to go far up in the cusp so that the first term in the Fourier expansion is the dominating term.

Let $W_{k/2,s} : \mathbb{R}_{>0} \to \mathbb{C}$ be the *Whittaker function* of weight $k/2$ and spectral parameter s, i.e., the unique solution to

$$\frac{d^2W}{dy^2} + \Big(-\frac{1}{4} + \frac{k/2}{y} + \frac{1/4 - s^2}{y^2}\Big)W = 0,$$

satisfying

$$W_{k/2,s}(y) \sim y^{k/2} e^{-y/2},$$

as $y \to \infty$ (with k, s fixed). Then we define $\mathcal{W}_{k/2,s} : \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}$ for $k \in \mathbb{Z}$ as

$$\mathcal{W}_{k/2,s}(z) := \begin{cases} (-1)^{k/2} W_{|k|/2,s}(|y|) e^{ix/2} & \operatorname{sign}(k) y > 0, \\ \dfrac{\Gamma((|k|+1)/2+s)\Gamma((|k|+1)/2-s)}{\Gamma(1/2+s)\Gamma(1/2-s)} W_{-|k|/2,s}(|y|) e^{ix/2} & \operatorname{sign}(k) y < 0, \end{cases}$$

for $z = x + iy \in \mathbb{C}\backslash\mathbb{R}$. We can check that

$$\mathcal{W}_{0,s}(z) = \left(\frac{|y|}{\pi}\right)^{1/2} K_s\left(\frac{|y|}{2}\right) e^{ix/2},$$

where $K_s(y)$ is the K-Bessel function and

$$\mathcal{W}_{k/2,(k-1)/2}(z) = (-1)^{k/2} y^{k/2} e^{iz/2}$$

for $k \in 2\mathbb{Z}_{\geq 0}$ and $y > 0$. Furthermore, for $k \in 2\mathbb{Z}_{\geq 0}$, we can check (see, for instance, [Strömberg 2008, Section 4.4]) that the normalizations match up so that we have

$$R_k \mathcal{W}_{k/2,s} = \mathcal{W}_{k/2+1,s}, \tag{5-4}$$

with

$$R_k = (z - \bar{z}) \frac{\partial}{\partial z} + \frac{k}{2} = iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \frac{k}{2},$$

denoting the weight k raising operator (and similarly for $k \leq 0$ now with lowering operators). We have the following asymptotic expansion (see [Gradshteyn and Ryzhik 2000, (9.227)] or [Whittaker and Watson 1962, Chapter 16.3]) valid for $y > 1$:

$$W_{k/2,s}(y) = e^{-y/2} y^{k/2} \left(1 + \sum_{n \geq 1} \frac{(s^2 - (k/2 - 1/2)^2) \cdots (s^2 - (k/2 - n + 1/2)^2)}{n!\, y^n}\right).$$

In particular, we conclude that

$$\begin{aligned} \mathcal{W}_{k/2,s}(z) &= e^{-y/2} y^{k/2} \left(1 + O\left(\sum_{n \geq 1} \frac{(|s| + |k|/2 + n)^{2n}}{n!\, y^n}\right)\right) \\ &= e^{-y/2} y^{k/2} \left(1 + O\left(\frac{(|s| + |k| + 1)^2}{y}\right)\right) \end{aligned} \tag{5-5}$$

for $y > (|s| + |k| + 1)^2$.

Now, we let $k \geq 0$ and consider an L^2-normalized Hecke–Maaß form $f \in \mathcal{B}_k(N)$ of the form $\nu^*_{d,N'} R_{k-2} \cdots R_{k'} f_0$, with f_0 a Hecke–Maaß newform of weight k' and level N' such that $dN' \mid N$. Combining (5-4) and (3-10) with the well-known Fourier expansions of holomorphic and Maaß forms, we get the following Fourier expansion in the general weight case:

$$f(z) = \frac{c_f}{|L(\operatorname{sym}^2 f, 1) \gamma_\infty(f, k)|^{1/2}} \sum_{n \neq 0} \frac{\lambda_{f_0}(n)}{|n|^{1/2}} \mathcal{W}_{k/2, it_f}(4\pi d n z), \tag{5-6}$$

for some constant c_f bounded uniformly from above and away from 0 in terms of the level N. Here $\lambda_{f_0}(n)$ denotes the Hecke eigenvalues of f (with the convention that $\lambda_{f_0}(-n) = 0$ for $-n < 0$ if f_0 is holomorphic and $\lambda_{f_0}(-n) = \pm\lambda_{f_0}(n)$ according to whether f_0 is an even or odd Maaß form) and

$$\gamma_\infty(f,k) = \begin{cases} \prod_\pm \Gamma\Big(\frac{k+1}{2} \pm it_f\Big) & \text{if } f_0 \text{ is Maaß,} \\ \Gamma(k)\Gamma\Big(\frac{k-k'}{2}+1\Big) & \text{if } f_0 \text{ is holomorphic.} \end{cases}$$

Using this we can prove the following crude lower bound:

Proposition 5.2. *For $i = 1,\dots,n$, let $f_i \in \mathcal{B}_{k_i}(N)$ be an L^2-normalized weight k_i Hecke–Maaß eigenform of level N. Then we have*

$$\Big\|\prod_{i=1}^n f_i\Big\|_2 \gg_\varepsilon e^{-cnT^{2+\varepsilon}}$$

for all $\varepsilon > 0$, where $T = \max_{i=1,\dots,n}|t_{f_i}| + |k_i| + 1$ and $c = c(N,\varepsilon) > 0$ is some positive constant.

Proof. Clearly we may assume that $k \geq 0$. Given $f \in \mathcal{B}_k(N)$, we write

$$f = \nu^*_{d,N'} R_{k-2} \cdots R_{k'} f_0$$

for a Hecke–Maaß newform f_0 of weight k' (with $k' \leq k$ and $k' \equiv k \mod 2$) and level N' with $dN' \mid N$. We have, by a standard bound for the Hecke eigenvalues (see, for instance, [Iwaniec 2002, (8.7)] in the Maaß case) and by bounding the quotient of Γ-factors trivially, that

$$\begin{aligned}\sum_{n\neq 0} \frac{\lambda_f(n)}{|n|^{1/2}} &\mathcal{W}_{k/2,it_f}(4\pi nz) \\ &= e^{2\pi i dx} W_{k/2,it_f}(4\pi dy) + \varepsilon_f e^{-2\pi i dx} \frac{\Gamma\big((k+1)/2+s\big)\Gamma\big((k+1)/2-s\big)}{\Gamma(1/2+s)\Gamma(1/2-s)} W_{-k/2,it_f}(4\pi dy) \\ &\qquad + O\Big(|t_f|^{1/2} \sum_{n\geq 2} \big|W_{k/2,it_f}(4\pi dny)\big| + (k+|t_f|+1)^k \big|W_{-k/2,it_f}(4\pi dny)\big|\Big), \end{aligned} \tag{5-7}$$

where $\varepsilon_f = 0$ if f_0 is holomorphic and if f_0 is a Maaß form we have $\varepsilon_f = \pm 1$ where ± 1 is the sign of f_0 under the reflection operator X defined in Section 3C. By the asymptotics (5-5) we see easily that

$$\sum_{n\geq 2} |W_{k/2,it_f}(4\pi dny)| + (k+|t_f|+1)^k |W_{-k/2,it_f}(4\pi dny)| \ll e^{-3d\pi y}$$

for $y \geq (|t_f| + k + 1)^{2+\varepsilon}$. For $k = 0$ we conclude from the asymptotic (5-5) that (5-7) is equal to

$$(e^{2\pi i dx} + \varepsilon_f e^{-2\pi i dx}) e^{-2\pi dy} + O(y^{-\varepsilon} e^{-2\pi dy}),$$

for $y \geq (|t_f| + k + 1)^{2+\varepsilon}$. Similarly, for $k > 0$, we see that (5-7) is equal to

$$e^{2\pi i dx}(4\pi dy)^{k/2} e^{-2\pi dy} + O((4\pi dy)^{k/2-\varepsilon} e^{-2\pi dy})$$

for $y \geq (|t_f|+k+1)^{2+\varepsilon}$, using the bound

$$\frac{\Gamma\big((k+1)/2+s\big)\Gamma\big((k+1)/2-s\big)}{\Gamma(1/2+s)\Gamma(1/2-s)} W_{-k/2,it_f}(4\pi dy) \ll (k+|t_f|+1)^k (4\pi dy)^{-k/2} e^{-2\pi dy}.$$

By Stirling's approximation, we have the crude bound

$$\gamma_\infty(f,k) \ll e^{O((|t_f|+k)\log(|t_f|+k))},$$

and we also have $|t_f|^{-\varepsilon} \ll_\varepsilon L(\mathrm{sym}^2 f, 1) \ll_\varepsilon |t_f|^\varepsilon$. Thus we conclude from (5-6) that for $k=0$,

$$|f(z)| \gg e^{-3\pi dy} \tag{5-8}$$

for $y \geq (|t_f|+k+1)^{2+\varepsilon}$ and x such that $e^{2\pi idx} + \varepsilon_f e^{-2\pi idx} \gg 1$. Similarly if $k>0$, we have

$$|f(z)| \gg e^{-3\pi dy} \tag{5-9}$$

for $y \geq (|t_f|+k+1)^{2+\varepsilon}$ (and any x). Now we easily conclude the wanted lower bound for the L^2-norm of the product by computing the contribution from the range $x \in [0,1]$ and $y \asymp (|t_f|+k+1)^{2+\varepsilon}$. □

In the holomorphic case, we can do slightly better since the Fourier expansion is better behaved.

Proposition 5.3. *For $i = 1, \ldots, n$, let $f_i \in \mathcal{B}_{k_i,\mathrm{hol}}(N)$ be a weight k_i holomorphic Hecke–Maaß eigenform of level N (L^2-normalized). Then we have*

$$\Big\| \prod_{i=1}^n f_i \Big\|_2 \gg_\varepsilon e^{-cnT^{1+\varepsilon}}$$

for all $\varepsilon > 0$, where $T = \max_{i=1,\ldots,n} |k_i|$ and $c(N,\varepsilon) = c > 0$ is some positive constant.

Proof. Let $f \in \mathcal{B}_{k,\mathrm{hol}}(N)$ be of the form $\nu^*_{d,N'} y^{k/2} g$ with $g \in \mathcal{S}_k(N')$ a holomorphic Hecke newform. By the Fourier expansion (5-6), we have

$$f(z) = \frac{c_f}{|L(\mathrm{sym}^2 f,1)\Gamma(k)|^{1/2}} \sum_{n\geq 1} \frac{\lambda_g(n)}{n^{1/2}} (4\pi dny)^{k/2} e^{2\pi idnz}.$$

By bounding everything trivially, it is easy to see that for $y \gg k^{1+\varepsilon}$,

$$\sum_{n\geq 1} \frac{\lambda_g(n)}{n^{1/2}} (4\pi dny)^{k/2} e^{2\pi idnz} = (4\pi dy)^{k/2} e^{2\pi idz} + O_\varepsilon(e^{-3\pi dy}).$$

Now the lower bound for $\|\prod_{i=1}^n f_i\|_2$ follows as above. □

Remark 5.4. It seems quite hard to obtain strong lower bounds for $\|\prod_i f_i\|_2$ as this is related to the deep problem of nonlocalization of the eigenfunctions f_i (such as L^∞-bounds), see, for instance, [Sarnak 1995]. In particular, it is very hard to rule out that the f_i localize in disjoint regions.

6. Proof of the main theorem

We will now use the results proved in the previous sections to obtain our wide moment calculation. First of all, we will use the above to obtain a version of equidistribution of Heegner points with explicit error terms. For this, we will need the following convenient basis for the space spanned by Maaß forms of squarefree level N (see [Humphries and Khan 2020, Lemma 3.1]):

$$\mathcal{B}'(N) := \{u_d \in C^\infty(\mathbb{H}) \cap L^2(\Gamma_0(N)\backslash\mathbb{H}) : N'd \mid N, u \in \mathcal{B}^*(N')\},$$

(recall that we denote by $\mathcal{B}^*(N')$ all Hecke–Maaß newforms f of weight 0 and level N') where

$$u_d(z) := \left(L_d(\operatorname{sym}^2 u, 1)\frac{\varphi(d)}{d\nu(N/N')}\right)^{1/2} \sum_{vw=d} \frac{\nu(v)}{v}\frac{\mu(w)\lambda_u(w)}{\sqrt{w}} u(vz). \tag{6-1}$$

Here,

$$L_d(\operatorname{sym}^2 u, s) := \prod_{p|d} \frac{1}{1-\lambda_u(p^2)p^{-s}+\lambda_u(p^2)p^{-2s}-p^{-3s}}.$$

There is a similar basis for the Eisenstein part of the spectrum (see [Humphries and Khan 2020, Section 3.2]). Given $u \in \mathcal{B}'(N)$, we put

$$L(\operatorname{sym}^2 u, s) := L(\operatorname{sym}^2 u', s)$$

and

$$L(u, s) := L(u', s),$$

where $u = (u')_d$ with $u' \in \mathcal{B}^*(N')$ and $dN' \mid N$.

Theorem 6.1. *Let $k_1, \dots, k_n \in 2\mathbb{Z}$ be even integers such that $\sum k_i = 0$. For $i = 1, \dots, n$, let $f_i \in \mathcal{B}_{k_i}(N)$ be a Hecke–Maaß eigenform of fixed level N, weight k_i, and spectral parameter t_{f_i}. Let $|D_K| \to \infty$ transverse a sequence of discriminants of imaginary quadratic fields K such that all primes dividing N split in K. Then we have*

$$\frac{1}{|\mathrm{Cl}_K|} \sum_{[Q]\in\Gamma_0(N)\backslash\mathcal{Q}_{D_K}(N,r)} \prod_{i=1}^n f_i(z_Q) = \left\langle \prod_{i=1}^n f_i, \frac{1}{\operatorname{vol}(\Gamma_0(N))}\right\rangle + O_\varepsilon\left(\left\|\prod_{i=1}^n f_i\right\|_2 |D_K|^{-1/16}T^5 n^5 (T|D_K|n)^\varepsilon\right),$$

where $T = \max_{i=1,\dots,n} |t_{f_i}| + |k_i| + 1$.

We have the following improvements for the exponents in the error term:

$$\begin{cases} |D_K|^{-1/16}T^{5/2}n^5 & \text{if all } f_i \text{ are holomorphic},\\ |D_K|^{-1/12}T^2n^2 & \text{if the level is } N = 1,\\ |D_K|^{-1/12}Tn^2 & \text{if all } f_i \text{ are holomorphic of level } 1. \end{cases} \tag{6-2}$$

Proof. We put $D = |D_K|$ to lighten notation. By the spectral expansion for $\Gamma_0(N)\backslash\mathbb{H}$, see [Iwaniec 2002, Theorem 7.3], we have

$$\sum_{[Q]\in\Gamma_0(N)\backslash\mathcal{Q}_{-D}(N,r)} \prod_{i=1}^{n} f_i(z_Q) = |\mathrm{Cl}_K|\left\langle \prod_{i=1}^{n} f_i, \frac{1}{\mathrm{vol}(\Gamma_0(N))}\right\rangle + \sum_{u\in\mathcal{B}'(N)}\left\langle \prod_{i=1}^{n} f_i, u\right\rangle W_{u,K} + (\text{Eisenstein}), \quad (6\text{-}3)$$

where

$$W_{u,K} := \sum_{[Q]\in\Gamma_0(N)\backslash\mathcal{Q}_{-D}(N,r)} u(z_Q)$$

is the Weyl sum of level N corresponding to u, and the Eisenstein contribution is given by

$$(\text{Eisenstein}) := \sum_{\mathfrak{a}} \frac{1}{4\pi}\int_{\mathbb{R}}\left\langle \prod_{i=1}^{n} f_i, E_{\mathfrak{a}}\left(\cdot\,, \tfrac{1}{2}+it\right)\right\rangle W_{\mathfrak{a},t,K}\, dt,$$

where the sum runs over the set of inequivalent cusps of $\Gamma_0(N)$, $E_{\mathfrak{a}}\left(z, \frac{1}{2}+it\right)$ denotes the Eisenstein series at the cusp $\mathfrak{a}$ (see [Iwaniec 2002, (3.11)]), and

$$W_{\mathfrak{a},t,K} := \sum_{[Q]\in\Gamma_0(N)\backslash\mathcal{Q}_{-D}(N,r)} E_{\mathfrak{a}}\left(z_Q, \tfrac{1}{2}+it\right)$$

is the corresponding Weyl sum.

We will now bound the cuspidal contribution in (6-3), and as usual the Eisenstein contribution can be bounded similarly. By Theorem 4.2, we have

$$|W_{u,K}|^2 \ll_N \frac{D^{1/2}L(u,1/2)L(u\otimes\chi_K,1/2)}{L(\mathrm{sym}^2 u, 1)} \quad (6\text{-}4)$$

for $u\in\mathcal{B}'(N)$. Here the case when u is a linear combination of old forms as in (6-1) follows by linearity. Now we observe that for $u\in\mathcal{B}^*(N)$, we have using the self adjointness of Δ,

$$\left\langle \prod_{i=1}^{n} f_i, u\right\rangle \left(t_u^2+\tfrac{1}{4}\right)^m = \left\langle \prod_{i=1}^{n} f_i, \Delta^m u\right\rangle = \left\langle \Delta^m\prod_{i=1}^{n} f_i, u\right\rangle.$$

Applying the Cauchy–Schwarz inequality and Lemma 5.1, this implies

$$\left\langle \prod_{i=1}^{n} f_i, u\right\rangle \ll \prod_{i=1}^{n}\|f_i\|_2 \frac{n^{2m}(m+T)^{nA+2m}}{(|t_u|^2+1)^m} \quad (6\text{-}5)$$

for any $m\geq 0$, where $T=\max_{i=1,\dots,n}|t_{f_i}|+|k_i|+1$. Putting $m=(nT^2)^{1+\varepsilon}$ in the estimate (6-5), we see that we can truncate the spectral expansion (6-3) at $t_u \ll (Tn)^2(TDn)^{\varepsilon}$ at the cost of an error of size

$$\ll_{\varepsilon} (TDn)^{-c(nT^2)^{1+\varepsilon}} \prod_{i=1}^{n}\|f_i\|_2,$$

for some constant $c=c(N,\varepsilon)>0$. By Proposition 5.2, this error is negligible.

To estimate the remaining terms, we use the bound (6-4) together with Cauchy–Schwarz and Bessel's inequality, nonnegativity, and standard bounds for symmetric square L-functions. This gives

$$\sum_{\substack{u\in\mathcal{B}'(N)\\ t_u\ll(Tn)^2(TDn)^\varepsilon}} \Big\langle \prod_{i=1}^n f_i, u\Big\rangle W_{u,K} \ll_\varepsilon \Big\|\prod_{i=1}^n f_i\Big\|_2 D^{1/4}\Big(\sum_{N'\mid N}\sum_{\substack{u\in\mathcal{B}^*(N')\\ t_u\ll(Tn)^2(TDn)^\varepsilon}} L\big(u,\tfrac12\big)L\big(u\otimes\chi_K,\tfrac12\big)\Big)^{1/2}(TDn)^\varepsilon, \tag{6-6}$$

where χ_K is the quadratic character corresponding to K via class field theory (recall that $\mathcal{B}^*(N')$ denotes the set of all Hecke–Maaß newforms of weight 0 and level N').

From here on, we distinguish between the case of level 1 and higher (square free) level N. In the case of general level N, we use the GL_2 subconvexity bound due to Blomer and Harcos [2008]

$$L\big(u\otimes\chi_K,\tfrac12\big)\ll(1+|t_u|)^{3+\varepsilon}D^{3/8+\varepsilon},$$

which gives

$$\begin{aligned}\sum_{\substack{u\in\mathcal{B}'(N)\\ t_u\ll(Tn)^2(TDn)^\varepsilon}} \Big\langle \prod_{i=1}^n f_i, u\Big\rangle W_{u,K} &\ll \Big\|\prod_{i=1}^n f_i\Big\|_2 D^{1/4+3/16}(Tn)^3(TDn)^\varepsilon\Big(\sum_{N'\mid N}\sum_{\substack{u\in\mathcal{B}^*(N')\\ t_u\ll(Tn)^2(TDn)^\varepsilon}} L\big(u,\tfrac12\big)\Big)^{1/2}\\ &\ll \Big\|\prod_{i=1}^n f_i\Big\|_2 D^{1/2-1/16}(Tn)^5(TDn)^\varepsilon,\end{aligned}$$

using a standard first-moment bound for $L\big(u,\tfrac12\big)$ (for instance, using a spectral large sieve).

If the level is 1, we follow Young [2017] and use Hölder's inequality together with his Lindelöf strength third moment bound [Young 2017, Theorem 1.1] to estimate the above by

$$\ll_\varepsilon \Big\|\prod_{i=1}^n f_i\Big\|_2 D^{5/12}(Tn)^2(TDn)^\varepsilon.$$

Finally, if all of the f_i are holomorphic, then by Proposition 5.3 we can use the estimate (6-5) with $m=nT^{1+\varepsilon}$ instead, which leads to the improved exponents. □

Remark 6.2. Alternatively, we can estimate (6-6) by using the bound

$$\Big\langle\prod_i f_i,u\Big\rangle\ll_\varepsilon t_u^{5/12+\varepsilon}\Big\|\prod_i f_i\Big\|_1,$$

where $\|\cdot\|_1$ denotes the L^1-norm, using here the L^∞-bound of Iwaniec and Sarnak [1995]. This leads to the error term

$$O_\varepsilon\Big(\Big\|\prod_{i=1}^n f_i\Big\|_1|D_K|^{-1/16}T^{35/6}n^{35/6}(T|D_K|n)^\varepsilon\Big),$$

which is more convenient in some cases (with similar improvements in the special cases of holomorphic and / or level 1 as in (6-2)).

6A. ***A wide moment of L-functions.*** Combining this with our explicit formula, we arrive at our main L-function computation. We will use the following shorthand for K an imaginary quadratic field with class group Cl_K:

$$\mathbf{Wide}(K,n) := \mathbf{Wide}(\widehat{\mathrm{Cl}}_K, n),$$

with $\mathbf{Wide}(G,n)$ as in (1-1). Note that the following statement is a slight generalization of Theorem 1.7 (allowing for the representations not to have the same conductor):

Theorem 6.3. *Let $N \geq 1$ be a fixed squarefree integer. For $i = 1, \dots, n$, let π_i be a cuspidal automorphic representation of* $\mathrm{GL}_2(\mathbb{A})$ *with trivial central character of conductor $N_i \mid N$, spectral parameter t_{π_i}, and even lowest weight k_{π_i}. Let $k_1, \dots, k_n \in 2\mathbb{Z}$ be integers such that $|k_i| \geq k_{\pi_i}$ and $\sum_i k_i = 0$.*

Let $|D_K| \to \infty$ transverse a sequence of discriminants of imaginary quadratic fields K such that all primes dividing N split in K. For each K, pick Hecke characters $\Omega_{i,K}$ with ∞-type $x \mapsto (x/|x|)^{k_i}$ such that $\prod_i \Omega_{i,K}$ is the trivial Hecke character (*notice that this is always possible since*, *we know that $\prod_i \Omega_{i,K}$ is a class group character*).

Then we have for $f_i \in \mathcal{B}_{k_i}(N)$ in the representation space of π_i,

$$\sum_{(\chi_i)\in \mathbf{Wide}(K,n)} \prod_{i=1}^{n} \Bigl(c_{f_i} \varepsilon_{\chi_i, f_i} L\bigl(\pi_i \otimes \chi_i \Omega_{i,K}, \tfrac{1}{2}\bigr)^{1/2} \Bigr) = \frac{|\mathrm{Cl}_K|^n}{|D_K|^{n/4}} \Biggl(\Biggl\langle \prod_{i=1}^{n} f_i, \frac{1}{\mathrm{vol}(\Gamma_0(N))} \Biggr\rangle + O_\varepsilon\Biggl(\Biggl\| \prod_{i=1}^{n} f_i \Biggr\|_2 |D_K|^{-1/16} T^5 n^5 (T|D_K|n)^{\varepsilon} \Biggr) \Biggr), \tag{6-7}$$

where $T = \max_{i=1,\dots,n} |k_i| + |t_{f_i}| + 1$, $c_{f_i} = (8N_i)^{-1} c_\infty(\pi_{i,\infty}, k_i)$ with c_∞ as in (4-11), *and ε_{χ, f_i} are complex numbers of absolute value* 1.

We have the following improvements for the exponents in the error term:

$$\begin{cases} |D_K|^{-1/16} T^{5/2} n^5 & \text{if } \pi_i \text{ are discrete series of weight } k_{\pi_i} = k_i, \\ |D_K|^{-1/12} T^2 n^2 & \text{if the level } N = 1 \text{ is trivial}, \\ |D_K|^{-1/12} T n^2 & \text{if } N = 1 \text{ and } \pi_i \text{ are discrete series of weight } k_{\pi_i} = k_i. \end{cases} \tag{6-8}$$

Proof. By the fact that $\prod_i \Omega_{i,K}$ is trivial, we see that

$$\prod_{i=1}^{n} f_i(z) = \prod_{i=1}^{n} \bigl(\Omega_{i,K}(x) f_i(z)\bigr)$$

for any $x \in \mathbb{A}_K^\times$. In particular, if we fix a quadratic form $Q \in \mathcal{Q}_{D_K}(N,r)$ and choose $x = x_Q \in \mathbb{A}_K^\times$ as in Corollary 4.3, then we get

$$\prod_{i=1}^{n} f_i(z_Q) = \prod_{i=1}^{n} \bigl(\Omega_{i,K}(x_Q) f_i(z_Q)\bigr) = \sum_{\chi_1,\dots,\chi_n \in \widehat{\mathrm{Cl}}_K} \prod_{i=1}^{n} \biggl(\varepsilon_{\chi_i, i} c_{f_i} \frac{|D_K|^{1/4}}{|\mathrm{Cl}_K|} \Bigl| L\Bigl(\pi_i \otimes \chi_i \Omega_{i,K}, \frac{1}{2}\Bigr)\Bigr|^{1/2} \chi_i([\mathfrak{a}]) \biggr).$$

Summing this identity over a set of representatives for $\Gamma_0(N)\backslash \mathcal{Q}_{D_K}(N,r) \cong \mathrm{Cl}_K$, applying Theorem 6.1, and using orthogonality of class group characters (i.e., the Fourier theoretic equality (1-2)=(1-3)), we arrive at the conclusion. ∎

Remark 6.4. The fact that we have $\|\prod_i f_i\|_2$ in the error term and not, say, L^∞-norms, turns out to be crucial for applications to nonvanishing; see Section 7C.

6B. ***The diagonal case.*** In this subsection, we will use Theorem 6.3 to calculate another family of moments. For this consider the following "nontrivial diagonal":

$$\mathbf{Wide}_{\mathrm{ntd}}(\hat{G},2n) := \left\{(\chi_1,\psi_1,\dots,\chi_n,\psi_n)\in(\hat{G})^{2n} : \chi_i\neq\psi_i,\ \prod_{i=1}^n \chi_i = \prod_{i=1}^n \psi_i\right\}.$$

The starting point is the following lemma:

Lemma 6.5. *Let G be a finite abelian group and $L_1,\dots,L_n : G\to\mathbb{C}$ maps. Then we have*

$$\sum_{(\chi_i,\psi_i)\in\mathbf{Wide}_{\mathrm{ntd}}(\hat{G},2n)} \prod_{i=1}^n \widehat{L}_i(\chi_i)\overline{\widehat{L}_i(\psi_i)} = \frac{1}{|G|}\sum_{M\subset\{1,\dots,n\}}(-1)^{|M|}\Bigl(\sum_{g\in G}\prod_{i\notin M}|L_i(g)|^2\Bigr)\prod_{i\in M}\Bigl(\sum_{g\in G}|L_i(g)|^2\Bigr).$$

Here $\widehat{L} : \hat{G}\to\mathbb{C}$ denotes the Fourier transform given by $\chi\mapsto(1/|G|)\sum_{g\in G}L(g)\bar{\chi}(g)$.

Proof. By the principle of inclusion and exclusion, we have

$$\sum_{(\chi_i,\psi_i)\in\mathbf{Wide}_{\mathrm{ntd}}(\hat{G},2n)} \prod_{i=1}^n \widehat{L}_i(\chi_i)\overline{\widehat{L}_i(\psi_i)} = \sum_{M\subset\{1,\dots,n\}}(-1)^{|M|} \sum_{\substack{(\chi_1,\overline{\psi_1},\dots,\chi_n,\overline{\psi_n})\in\mathbf{Wide}(\hat{G},2n)\\ \chi_i=\psi_i,\, i\in M}} \prod_{i=1}^n \widehat{L}_i(\chi_i)\overline{\widehat{L}_i(\psi_i)}, \tag{6-9}$$

where the sum is over all subsets M of $\{1,\dots,n\}$. Furthermore, we have

$$\sum_{\substack{(\chi_1,\overline{\psi_1},\dots,\chi_n,\overline{\psi_n})\in\mathbf{Wide}(\hat{G},2n)\\ \chi_i=\psi_i,\, i\in M}} \prod_{i=1}^n \widehat{L}_i(\chi_i)\overline{\widehat{L}_i(\psi_i)} = \Bigl(\sum_{(\chi_i,\overline{\psi_i})_{i\notin M}\in\mathbf{Wide}(\hat{G},2(n-|M|))} \prod_{i\notin M}\widehat{L}_i(\chi_i)\overline{\widehat{L}_i(\psi_i)}\Bigr)\prod_{i\in M}\Bigl(\sum_{\chi\in\hat{G}}|\widehat{L}_i(\chi)|^2\Bigr), \tag{6-10}$$

from which the result follows using the Fourier theoretic equality (1-2)=(1-3). □

From this we get the following corollary:

Corollary 6.6. *Let π_i, K, k_i be as in Theorem 6.3. For $i=1,\dots,n$, let $\Omega_{i,K}$ be a Hecke character of K of ∞-type $\alpha\mapsto(\alpha/|\alpha|)^{k_i}$ and $f_i\in\mathcal{B}_{k_i}(N)$ in the representation space of π_i. Then we have*

$$\sum_{(\chi_i,\psi_i)\in\mathbf{Wide}_{\mathrm{ntd}}(K,2n)} \prod_{i=1}^n \varepsilon_{\chi_i,\psi_i,f_i}|c_{f_i}|^2 L\bigl(\pi_i\otimes\chi_i\Omega_{i,K},\tfrac12\bigr)^{1/2} L\bigl(\pi_i\otimes\psi_i\Omega_{i,K},\tfrac12\bigr)^{1/2}$$
$$= \frac{|\mathrm{Cl}_K|^{2n}}{|D_K|^{n/2}}\Bigl(\sum_{M\subset\{1,\dots,n\}}(-1)^{|M|}\Bigl\|\prod_{i\notin M} f_i\Bigr\|^2\cdot\prod_{i\in M}\|f_i\|^2 + O_\varepsilon\Bigl(\prod_{i=1}^n\|f_i\|_\infty|D_K|^{-1/16}T^5n^52^n(T|D_K|n)^\varepsilon\Bigr)\Bigr),$$

as $|D_K|\to\infty$, where $c_{f_i}=(8N_i)^{-1}c_\infty(\pi_{i,\infty},k_i)$ with $c_\infty(\pi_{i,\infty},k_i)$ as in (4-11) *and $\varepsilon_{\chi,\psi,f_i}$ complex numbers of norm* 1.

Proof. The result follows from Lemma 6.5 combined with Theorem 6.3 by bounding the norms in the error terms by the L^∞-norms of the f_i. □

7. Applications to nonvanishing

Clearly, Theorem 6.3 gives a way to produce weak simultaneous nonvanishing results (in the sense of Section 2) given that we have

$$\left\langle \prod_{i=1}^{n} f_i, 1 \right\rangle \neq 0. \tag{7-1}$$

In this section, we show nonvanishing as in (7-1) in a number of different cases.

The simplest case is $n = 2$ and $f_1 = \overline{f_2}$ (which is the one considered by Michel and Venkatesh [2006]) where the period is the L^2-norm and thus automatically nonzero. Using our quantitative moment calculation in Theorem 6.3, we obtain a uniform version of [Michel and Venkatesh 2006, Theorem 1] in the general weight case.

The case $n = 3$ is also very appealing since the corresponding triple periods are connected to triple convolution L-functions via the Ichino–Watson formula [Ichino 2008; Watson 2002]. There are some prior work obtaining nonvanishing of triple periods, which immediately give weak simultaneous nonvanishing using Theorem 6.3. Reznikov [2001] showed using representation theory that for any Maaß form f of level N, there are infinitely many Maaß forms f_1 of level dividing N such that $\langle f^2, f_1\rangle \neq 0$ (in the level 1 case, this was reproved by Li [2009] using more analytic methods). Similarly, the quantum variance computation of Luo and Sarnak [2004] implies the following: for any Hecke–Maaß eigenform f with $L\left(f, \frac{1}{2}\right) \neq 0$, there are $\gg K$ many holomorphic newforms $g \in \mathcal{S}_k(1)$ with $K \leq k \leq 2K$ such that $\langle y^k |g|^2, f\rangle \neq 0$; see also [Sugiyama and Tsuzuki 2022]. We get similar nonvanishing with f a Hecke–Maaß newform using the corresponding quantum variance computation by Zhao and Sarnak [2019]. Note that the nonvanishing results for triple periods $\langle f_1 f_2 f_3, 1\rangle$ obtained in the above mentioned papers all have two of the forms equal. In terms of applications to nonvanishing these result are not that interesting. Motivated by this, we introduce below a method for obtaining nonvanishing for $n = 3$ where all of the forms f_1, f_2, f_3 are different.

Finally in the holomorphic case, we can show nonvanishing of periods for general n using a very soft argument.

7A. ***The second moment case.*** In this subsection, we consider the simplest case of $n = 2$ in which the nonvanishing of the main term in (6-7) is automatic. In particular, this gives an improved version of [Michel and Venkatesh 2006, Theorem 1] with uniformity in the spectral aspect and generalizes the results to general weights.

Corollary 7.1. *Let N be a fixed squarefree integer and $\varepsilon > 0$. Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ of level N, spectral parameter t_π, and even lowest weight k_π. Let k be an even integer such that $|k| \geq k_\pi$, and put $T = |t_\pi| + |k| + 1$.*

Then there exists a constant $c = c(N, \varepsilon) > 0$ such that for any imaginary quadratic field K such that all primes dividing N splits in K with discriminant $|D_K| \geq cT^{160/3+\varepsilon}$ (respectively, $|D_K| \geq cT^{22+\varepsilon}$ if $N = 1$), we have

$$\#\{\chi \in \widehat{\mathrm{Cl}}_K : L(\pi \otimes \chi\Omega_K, \tfrac{1}{2}) \neq 0\} \gg_\pi \begin{cases} |D_K|^{1/1058} & \text{if } \pi \text{ is d.s,} \\ |D_K|^{1/2648} & \text{if } \pi \text{ is p.s,} \end{cases}$$

where Ω_K is a Hecke character of K of conductor 1 and ∞-type $\alpha \mapsto (\alpha/|\alpha|)^k$.

Proof. Let π be as in the corollary above. We apply Theorem 6.3 with the error term coming from Remark 6.2 and with $\pi_1 = \pi_2 = \pi$ and $f_1 = \overline{f_2}$ belonging to π of weight $k \geq k_\pi$. In this special case, it is clear that we can truncate the spectral expansion (6-3) at $t_u \ll T^{1+\varepsilon}|D_K|^\varepsilon$ at a negligible error since we have

$$\|f_1 f_2\|_1 = \|f_1\|_2^2 = 1$$

(for any f_1 as above). Thus, both in the (raised) holomorphic and Maaß case, we have the error terms

$$O_\varepsilon\big(|D_K|^{-1/16} T^{20/6} (|D_K|T)^\varepsilon\big) \quad \text{for general level } N$$

and

$$O_\varepsilon\big(|D_K|^{-1/12} T^{11/6} (|D_K|T)^\varepsilon\big) \quad \text{for level } N = 1.$$

From this, we see that for $|D_K| \geq cT^{160/3+\varepsilon}$ (respectively, $|D_K| \geq cT^{22+\varepsilon}$), the RHS of (6-7) is nonzero. Thus, the LHS (6-7) is also nonzero and satisfies $\gg_{\varepsilon,k} |D_K|^{1/4-\varepsilon}$ using Siegel's lower bound (3-1). Now the result follows directly using the subconvexity bounds for Rankin–Selberg L-functions due to Michel [2004] and Harcos and Michel [2006]. ☐

7B. ***Triple products of Maaß forms.*** A very attractive case of Theorem 6.3 is $n = 3$, where the nonvanishing of $\langle f_1 f_2 f_3, 1\rangle$ is equivalent to the nonvanishing of the triple convolution L-function $L(\pi_1 \otimes \pi_2 \otimes \pi_3, \tfrac{1}{2})$ due to the Ichino–Watson formula [Ichino 2008; Watson 2002]. In this section, we introduce a soft method (relying on results of Lindenstrauss and Jutila–Motohashi) to derive nonvanishing results in the case where f_1, f_2, f_3 are all Maaß forms of level 1.

By the spectral expansion for $L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$ [Iwaniec 2002, Theorem 7.3], we have

$$\|f_1 f_2\|_2^2 = \langle f_1 f_2, f_1 f_2\rangle = \sum_{f \in \mathcal{B}_0(1)} |\langle f_1 f_2, f\rangle|^2 + \frac{1}{4\pi}\int_{\mathbb{R}} |\langle f_1 f_2, E_t\rangle|^2\, dt, \tag{7-2}$$

where $E_t(z) = E(z, \tfrac{1}{2} + it)$ is the nonholomorphic Eisenstein series of level 1. Using the Ichino–Watson formula [Ichino 2008; Watson 2002] (which in the Eisenstein case reduces to Rankin–Selberg), we have

$$|\langle f_1 f_2, f\rangle|^2 = \frac{L(f_1 \otimes f_2 \otimes f, 1/2)}{8L(\mathrm{sym}^2 f_1, 1)L(\mathrm{sym}^2 f_2, 1)L(\mathrm{sym}^2 f, 1)} h(t_{f_1}, t_{f_2}, t_f)$$

and

$$|\langle f_1 f_2, E_t\rangle|^2 = \frac{|L(f_1 \otimes f_2, 1/2 + it)|^2}{4L(\mathrm{sym}^2 f_1, 1)L(\mathrm{sym}^2 f_2, 1)|\zeta(1+2it)|^2} h(t_{f_1}, t_{f_2}, t),$$

where

$$h(t_1,t_2,t_3)=\frac{\prod_{\pm}\Gamma(1/4\pm it_1/2\pm it_2/2\pm it_3/2)}{|\Gamma(1/2+it_1)|^2|\Gamma(1/2+it_2)|^2|\Gamma(1/2+it_3)|^2}.$$

Here the product is over all 8 combinations of signs. If we fix t_1, then it is standard using Stirling's approximation to prove that for $t_2, t_3 \gg 1$, we have

$$h(t_1,t_2,t_3)\ll_{t_1} e^{-\pi|t_2-t_3|}(1+|t_2-t_3|)^{-1}(1+t_2+t_3)^{-1}.$$

This shows that the contribution from respectively, $|t-t_{f_2}|\geq (t_{f_2})^{\varepsilon}$ and $|t_f-t_{f_2}|\geq (t_{f_2})^{\varepsilon}$ in (7-2) is negligible.

We would like to show that actually all of the contribution from the Eisenstein part in (7-2) is negligible. This is connected to the subconvexity problem for Rankin–Selberg L-functions in a conductor dropping region, and is thus very difficult. We can however get unconditional results if we keep f_1 fixed and average over f_2 using the following result due to Jutila and Motohashi [2005, (3.50)]:

Theorem 7.2 (Jutila–Motohashi). *Let $f_1\in\mathcal{B}_0(1)$ be fixed. Then we have*

$$\sum_{|t_{f_2}-T|\leq T^{\varepsilon}}\bigl|L\bigl(f_1\otimes f_2,\tfrac{1}{2}+it\bigr)\bigr|^2\ll_{\varepsilon} T^{1+\varepsilon} \tag{7-3}$$

uniformly for $|t-T|\ll T^{\varepsilon}$.

Strictly speaking [Jutila and Motohashi 2005] only deals with the case where f_1 is an Eisenstein series, but (as remarked in [Blomer and Holowinsky 2010, p. 3]) the same estimate follows in the case of Maaß forms using the exact same argument relying on the spectral large sieve.

From Theorem 7.2, it follows that for any $\delta>0$, we have that

$$\int_{|t-t_{f_2}|\leq (t_{f_2})^{\varepsilon}}\bigl|L\bigl(f_1\otimes f_2,\tfrac{1}{2}+it\bigr)\bigr|^2\,dt\leq T^{1-\delta} \tag{7-4}$$

for all but at most $O_{\varepsilon}(T^{\delta+\varepsilon})$ Maaß forms f_2 with $|t_{f_2}-T|\leq T^{\varepsilon}$.

Recalling the estimates $t_f^{-\varepsilon}\ll_{\varepsilon} L(\mathrm{sym}^2 f,1)\ll_{\varepsilon} t_f^{\varepsilon}$, we conclude combining all of the above that for any f_2 satisfying (7-4), we have

$$\|f_1f_2\|_2^2=\sum_{|t_f-T|\leq T^{\varepsilon}}|\langle f_1f_2,f\rangle|^2+O_{\varepsilon}(T^{-\delta+\varepsilon}). \tag{7-5}$$

By QUE for Maaß forms due to Lindenstrauss [2006] (with key input by Soundararajan [2010]), we know that

$$\|f_1f_2\|_2\to\|f_1\|_2\neq 0,\quad\text{and}\quad \langle f_1f_2,f_2\rangle\to\Bigl\langle f_1,\frac{3}{\pi}\Bigr\rangle=0,$$

as $t_{f_2}\to\infty$. Thus we conclude from (7-5) that for T large enough there is some $f_3\neq f_2$ with $|t_{f_3}-T|\leq T^{\varepsilon}$ such that $\langle f_1f_2,f_3\rangle\neq 0$. Furthermore, we obtain a lower bound for free using Weyl's law,

$$\#\{f\in\mathcal{B}_0(1):|t_f-T|\leq T^{\varepsilon}\}\asymp T^{1+\varepsilon}.$$

From this we obtain the following result:

Proposition 7.3. *Let $f_1 \in \mathcal{B}_0(1)$ be fixed and $\varepsilon > 0$. Then for $T > 0$ large enough (depending on f_1 and ε), we have that for all but $O_\varepsilon(T^{2\varepsilon})$ of $f_2 \in \mathcal{B}_0(1)$ satisfying $|t_{f_2} - T| \leq T^\varepsilon$, there exists some $f_3 \in \mathcal{B}_0(1)$ not equal to f_2 with $|t_{f_3} - T| \leq T^\varepsilon$ such that*

$$|\langle f_1 f_2, f_3\rangle| \gg_\varepsilon \|f_1 f_2\|_2 / T^{1/2+\varepsilon}.$$

From this, we deduce the nonvanishing result in Corollary 1.3.

Proof of Corollary 1.3. Let f_2, f_3 be as in Proposition 7.3. Then we apply Theorem 6.3 (in the level 1 case) with $n = 3$, $k_1 = k_2 = k_3 = 0$, and test vectors f_1, f_2, f_3. We observe that

$$\|f_1 f_2 f_3\|_2 \, |D_K|^{-1/12} T^2 (|D_K| T)^\varepsilon \ll \|f_1 f_2\|_2 \, |t_{f_3}|^{5/12+\varepsilon} |D_K|^{-1/12} T^2 (|D_K| T)^\varepsilon,$$

by the sup-norm bound due to Iwaniec and Sarnak [1995]. Thus we see that if $|D_K| \gg_{f_1,\varepsilon} T^{35+\varepsilon}$, the error term in the asymptotic (6-7) (with exponents as in (6-8)) is strictly less than $\langle f_1 f_2 f_3, 3/\pi\rangle$. Thus we conclude that the LHS of (6-7) is nonvanishing and satisfies $\gg_{\varepsilon,T} |D_K|^{3/4-\varepsilon}$ (using Siegel's lower bound (3-1) again). Now by the subconvexity estimate for $L(f_i \otimes \theta_{\chi_i}, 1/2)$ due to Harcos and Michel [2006, Theorem 1] (where θ_{χ_i} is the holomorphic theta series associated to the Hecke character χ_i), we get the wanted quantitative nonvanishing result as $|D_K| \to \infty$. □

7C. ***The holomorphic case.*** Consider Theorem 6.3 in the case where $\pi_1, \dots, \pi_n$ are all holomorphic discrete series representations of GL_2 and $k_i = k_{\pi_i} > 0$. Furthermore, pick $f_i = y^{k_i/2} g_i$, with $g_i \in \mathcal{S}_{k_i}(N)$ a holomorphic Hecke newform. Then we know that

$$\prod_{i=1}^n g_i \in \mathcal{S}_k(N),$$

where $k = \sum_i k_i$ (which might not be a Hecke–Maaß eigenform(!)). A basis $\mathcal{B}_{k,\mathrm{hol}}(N)$ for $\mathcal{S}_k(N)$ is given by $\nu^*_{d,N'} y^{k/2} g$, where $g \in \mathcal{S}_k(N')$ is a Hecke newform and $dN' \mid N$. This implies that

$$\Bigl\| y^k \prod_{i=1}^n g_i \Bigr\|_2^2 = \sum_{u_1, u_2 \in \mathcal{B}_{k,\mathrm{hol}}(N)} \langle u_1, u_2\rangle \Bigl\langle y^{k/2} \prod_{i=1}^n g_i, u_1 \Bigr\rangle \overline{\Bigl\langle y^{k/2} \prod_{i=1}^n g_i, u_2 \Bigr\rangle}.$$

Since any two $u_1, u_2 \in \mathcal{B}_{k,\mathrm{hol}}(N)$ are orthogonal (with respect to the Petersson inner product) if the underlying Hecke newforms are different and since the dimension of $\mathcal{S}_k(N')$ is $\ll_N k$, we conclude the following:

Proposition 7.4. *Let N be a fixed positive integer, and let $k_1, \dots, k_n \in 2\mathbb{Z}_{>0}$ be even integers. For $i = 1, \dots, n$, let $g_i \in \mathcal{S}_{k_i}(N)$ be a holomorphic Hecke newform of level N and weight k_i. Then there exists some $\nu^*_{d,N'} y^{k/2} g \in \mathcal{B}_{k,\mathrm{hol}}(N)$ with $k = k_1 + \cdots + k_n$ such that*

$$\Bigl\langle \prod_{i=1}^n y^{k_i/2} g_i, \nu^*_{d,N'} y^{k/2} g \Bigr\rangle \gg \frac{\|\prod_{i=1}^n y^{k_i/2} g_i\|_2}{k^{1/2}}. \tag{7-6}$$

Combining this with Theorem 6.3, we obtain the following nonvanishing result:

Corollary 7.5. *Let N be a fixed squarefree integer, and let $k_1, \dots, k_n \in 2\mathbb{Z}_{>0}$ be even integers. For $i = 1, \dots, n$, let π_i be automorphic representations corresponding to holomorphic newforms $g_i \in \mathcal{S}_{k_i}(N)$ and put $k = \sum k_i$. Then there exists a constant $c = c(N, \varepsilon) > 0$ such that for any imaginary quadratic field K such that all primes dividing N split in K and the discriminant satisfies $|D_K| \geq c(\max_i k_i)^{40} n^{80} k^{12+\varepsilon}$, we have*

$$\#\{(\chi_1, \dots, \chi_{n+1}) \in \mathbf{Wide}(K, n+1), g \in \mathcal{B}_{k,\mathrm{hol}}(\Gamma_0(N)) : \\ L(\pi_1 \otimes \chi_1 \Omega_{1,K}, \tfrac{1}{2}) \cdots L(\pi_n \otimes \chi_n \Omega_{n,K}, \tfrac{1}{2}) L(\pi_g \otimes \chi_{n+1} \Omega_{n+1,K}, \tfrac{1}{2}) \neq 0\} \\ \gg_k |D_K|^{(n+1)/2115},$$

where $k = \sum_i k_i$ and $\Omega_{i,K}$ are Hecke characters of K with ∞-types $x \mapsto (x/|x|)^{k_i}$ and $\Omega_{n+1,K} = \prod_{i=1}^n \Omega_{i,K}$.

Proof. For $i = 1, \dots, n$, let $f_i = y^{k_i/2} g_i$, and let $f = \nu^*_{d,N'} y^{k/2} g \in \mathcal{B}_{k,\mathrm{hol}}(\Gamma_0(N))$ be as in Proposition 7.4.

We have the following sup-norm bound due to Xia [2007] (or more precisely the natural extension to general level):

$$\|f\|_\infty \ll_\varepsilon k^{1/4+\varepsilon}.$$

Thus, we conclude that

$$\Big\| f \prod_{i=1}^n f_i \Big\|_2 \ll_\varepsilon k^{1/4+\varepsilon} \Big\| \prod_{i=1}^n f_i \Big\|_2.$$

Combining the above with Theorem 6.3 (using the improved error term (6-8)) and the lower bound (7-6), we conclude that there is some constant depending only on N and $\varepsilon > 0$ such that as soon as

$$|D_K|^{1/16} \gg_{N,\varepsilon} \big(\max_{i=1,\dots,n} k_i\big)^{5/2} n^5 k^{1/4+1/2+\varepsilon},$$

then the RHS of (6-7) is nonzero. Thus the LHS (6-7) is also nonzero and is $\gg_{\varepsilon,k} |D_K|^{n/4-\varepsilon}$ using Siegel's lower bound (3-1).

Finally, since all of the f_i are holomorphic we can employ the subconvexity bound for Rankin–Selberg L-functions $L(f_i \otimes \theta_{\chi_i \Omega_{i,K}}, \tfrac{1}{2})$ due to Michel [2004], where $\theta_{\chi_i \Omega_{i,K}}$ is the holomorphic theta series associated to the Hecke character $\chi_i \Omega_{i,K}$ defined in Section 3B. Finally, we use that

$$L(f \otimes \theta_{\chi \Omega_{n+1,K}}, \tfrac{1}{2}) = \overline{L(f \otimes \theta_{\bar{\chi} \bar{\Omega}_{n+1,K}}, \tfrac{1}{2})}$$

to get rid of the conjugate in the last Rankin–Selberg L-functions. This gives the wanted qualitative lower bound for the nonvanishing. □

In the special case of level 1, we can do slightly better.

Proof of Corollary 1.4. Using the improved error term in Theorem 6.3 in the case of level 1 holomorphic forms, we see that the RHS of (6-7) is nonzero as soon as

$$|D_K|^{1/12} \gg_{N,\varepsilon} (\max_{i=1,\dots,n} k_i) n^2 k^{3/4+\varepsilon}.$$

Using the trivial estimates $n \leq k$ and $\max_i k_i \leq k$, we conclude Corollary 1.4. □

7D. ***Applications to Selmer groups.*** In this last section, we will give applications of our results in the holomorphic case to triviality of the ranks of Bloch–Kato Selmer groups. We will restrict to level 1 for simplicity of exposition.

The setting is as follows: given a holomorphic Hecke eigenform f of weight k and level 1, a Hecke character Ω of an imaginary quadratic field $K/\mathbb{Q}$ of conductor 1 and infinity type $\alpha \mapsto (\alpha/|\alpha|)^k$, and a prime number $p > 2$, we have an associated *Bloch–Kato Selmer group*

$$\mathrm{Sel}(K, V_{f,\Omega}/\Lambda_{f,\Omega}),$$

where $V_{f,\Omega} := V_{f_p}|_{G_K} \otimes \Omega$ denotes the p-adic Galois representation associated to $f \otimes \Omega$ and $\Lambda_{f,\Omega} \subset V_{f,\Omega}$ is a certain lattice. For details and exact definitions, we refer to [Castella 2020, Definition 5.1]. The Bloch–Kato conjecture predicts that the rank of $\mathrm{Sel}(K, V_{f,\Omega}/\Lambda_{f,\Omega})$ is zero exactly if $L\big(\pi_f \otimes \Omega, \frac{1}{2}\big) \neq 0$. This conjecture has been proved under mild assumptions by Castella [2020, Theorem A]. In order to state these assumptions, we will need some notation. Given f as above, we denote by L_f the p-adic Hecke field of f and $\rho_f : G_{\mathbb{Q}} \to \mathrm{Aut}_{L_f}(V_f)$ the p-adic Galois representation associated to f and $\overline{\rho_f}$ the mod p reduction of ρ_f. We denote by Θ the set of all imaginary quadratic fields $K/\mathbb{Q}$ of odd discriminant D_K satisfying the following hypotheses:

(1) The prime p splits in K,

(2) $p \nmid h_K$,

(3) $\overline{\rho_f}|_{G_K}$ is absolutely irreducible.

Then we can rephrase our results in the following way:

Corollary 7.6. *Let f be a holomorphic Hecke eigenform of even weight k and level* 1. *Let $p > 5$ be a prime such that $p - 1 \mid k - 2$ and f is p-ordinary.*

Then there exists a constant $c = c(\varepsilon) > 0$ such that for any imaginary quadratic field $K \in \Theta$ with discriminant $|D_K| \geq ck^{22+\varepsilon}$, we have

$$\#\big\{\chi \in \widehat{\mathrm{Cl}}_K : \mathrm{rank}_{\mathbb{Z}}\big(\mathrm{Sel}(K, V_{f,\chi\Omega_K}/\Lambda_{f,\chi\Omega_K})\big) = 0\big\} \gg_f |D_K|^{1/1058},$$

where Ω_K is a Hecke character of K of conductor 1 *and ∞-type $\alpha \mapsto (\alpha/|\alpha|)^k$.*

Proof. This follows directly from Corollary 7.1 combined with the explicit reciprocity law [Castella 2020, Theorem A] and the arguments in [Castella 2020, Section 6.3]. □

Acknowledgements

We would like to thank Valentin Blomer for useful feedback on an earlier version of the paper. This research was supported by the German Research Foundation under Germany's Excellence Strategy EXC-2047/1-390685813.

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Communicated by Philippe Michel
Received 2022-01-17 Revised 2023-01-30 Accepted 2023-05-13

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

ANT peer review and production are managed by EditFLOW® from MSP.

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Algebra & Number Theory

Volume 18 No. 4 2024