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Fundamental exact sequence for the pro-étale fundamental group

Marcin Lara

The pro-étale fundamental group of a scheme, introduced by Bhatt and Scholze, generalizes formerly known fundamental groups — the usual étale fundamental group $\pi_1^{\text{ét}}$ defined in SGA1 and the more general π_1^{SGA3}. It controls local systems in the pro-étale topology and leads to an interesting class of "geometric coverings" of schemes, generalizing finite étale coverings.

We prove exactness of the fundamental sequence for the pro-étale fundamental group of a geometrically connected scheme X of finite type over a field k, i.e., that the sequence

$$1 \to \pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X) \to \mathrm{Gal}_k \to 1$$

is exact as abstract groups and the map $\pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X)$ is a topological embedding.

On the way, we prove a general van Kampen theorem and the Künneth formula for the pro-étale fundamental group.

1. Introduction

Bhatt and Scholze [2015] introduced the pro-étale topology for schemes. The main motivation was that the definitions of ℓ-adic sheaves and cohomologies in the usual étale topology are rather indirect. In contrast, the naive definition of, e.g., a constant $\mathbb{Q}_\ell$-sheaf in the pro-étale topology as $X_{\text{proét}} \ni U \mapsto \mathrm{Maps}_{\text{cts}}(U, \mathbb{Q}_\ell)$ is a sheaf and if X is a variety over an algebraically closed field, then $H^i(X_{\text{ét}}, \mathbb{Q}_\ell) = H^i(X_{\text{proét}}, \mathbb{Q}_\ell)$, where the right-hand side is defined "naively" by applying the derived functor $\mathrm{R}\Gamma(X_{\text{proét}}, -)$ to the described constant sheaf.

Along with the new topology, Bhatt and Scholze [2015] introduced a new fundamental group — *the pro-étale fundamental group*. It is defined for a connected locally topologically noetherian scheme X with

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a geometric point $\bar{x}$ and denoted $\pi_1^{\text{proét}}(X, \bar{x})$. The name "pro-étale" is justified by the fact that there is an equivalence $\pi_1^{\text{proét}}(X, \bar{x}) - \text{Sets} \simeq \text{Loc}_{X_{\text{proét}}}$ between the categories of (possibly infinite) discrete sets with continuous action by $\pi_1^{\text{proét}}(X, \bar{x})$ and locally constant sheaves of (discrete) sets in $X_{\text{proét}}$. This is analogous to the classical fact that $\pi_1^{\text{ét}}(X, \bar{x}) - \text{FSets}$ is equivalent to the category of lcc sheaves on $X_{\text{ét}}$, where $G - \text{FSets}$ denotes *finite* sets with a continuous G action. This is the first striking difference between these fundamental groups: $\pi_1^{\text{proét}}$ allows working with sheaves of infinite sets. In fact, Bhatt and Scholze [2015] study abstract "infinite Galois categories", which are pairs $(\mathcal{C}, F)$ satisfying certain axioms that (together with an additional tameness condition) turn out to be equivalent to a pair $(G - \text{Sets}, F_G : G - \text{Sets} \to \text{Sets})$ for a Hausdorff topological group G and the forgetful functor F_G. In fact, one takes $G = \text{Aut}(F)$ with a suitable topology. This generalizes the usual Galois categories, introduced by Grothendieck to define $\pi_1^{\text{ét}}(X, \bar{x})$. In Grothendieck's approach, one takes the category FÉt_X of finite étale coverings together with the fiber functor $F_{\bar{x}}$ and obtains that $\pi_1^{\text{ét}}(X, \bar{x}) - \text{FSets} \simeq \text{FÉt}_X$. Discrete sets with a continuous $\pi_1^{\text{proét}}(X, \bar{x})$-action correspond to a larger class of coverings, namely "*geometric coverings*", which are defined to be schemes Y over X such that $Y \to X$:

(1) Is étale (not necessarily quasicompact!).

(2) Satisfies the valuative criterion of properness.

We denote the category of geometric coverings by Cov_X (seen as a full subcategory of Sch_X). It is clear that $\text{FÉt} \subset \text{Cov}_X$. As Y is not assumed to be of finite type over X, the valuative criterion does not imply that $Y \to X$ is proper (otherwise we would get finite étale morphisms again) and so in general we get more. A basic example of a nonfinite covering in Cov_X can be obtained by viewing an infinite chain of (suitably glued) $\mathbb{P}^1_k$'s as a covering of the nodal curve $X = \mathbb{P}^1/\{0, 1\}$ obtained by gluing 0 and 1 on $\mathbb{P}^1_k$ (to formalize the gluing one can use [Schwede 2005]). Then, if $k = \bar{k}$, $\pi_1^{\text{proét}}(X, \bar{x}) = \mathbb{Z}$ and $\pi_1^{\text{ét}}(X, \bar{x}) = \widehat{\mathbb{Z}}$. In this example, the prodiscrete group π_1^{SGA3} defined in Chapter X.6 of [SGA 3_{II} 1970] would give the same answer. This is essentially because our infinite covering is a torsor under a discrete group in $X_{\text{ét}}$. However, for more general schemes (e.g., an elliptic curve with two points glued), the category Cov_X contains more. So far, all the new examples were coming from nonnormal schemes. This is not a coincidence, as for a normal scheme X, any $Y \in \text{Cov}_X$ is a (possibly infinite) disjoint union of finite étale coverings. In this case, $\pi_1^{\text{proét}}(X, \bar{x}) = \pi_1^{\text{SGA3}}(X, \bar{x}) = \pi_1^{\text{ét}}(X, \bar{x})$. In general $\pi_1^{\text{ét}}$ can be recovered as the profinite completion of $\pi_1^{\text{proét}}$ and π_1^{SGA3} is the prodiscrete completion of $\pi_1^{\text{proét}}$.

The groups $\pi_1^{\text{proét}}$ belong in general to a class of *Noohi groups*. These can be characterized as Hausdorff topological groups G that are Raĭkov complete and such that the open subgroups form a basis of neighborhoods at 1_G. However, *open **normal** subgroups do not necessarily form a basis* of open neighborhoods of 1_G in a Noohi group. In the case of $\pi_1^{\text{proét}}$, this means that there might exist a connected $Y \in \text{Cov}_X$ that do not have a Galois closure. Examples of Noohi groups include: profinite groups, (pro)discrete groups, but also $\mathbb{Q}_\ell$ and $\text{GL}_n(\mathbb{Q}_\ell)$. A slightly different example would be $\text{Aut}(S)$, where S is a discrete set and Aut has the compact-open topology.

The fact that groups like $\mathrm{GL}_n(\mathbb{Q}_\ell)$ are Noohi (but not profinite or prodiscrete) makes $\pi_1^{\text{proét}}$ better suited to work with $\mathbb{Q}_\ell$ (or $\overline{\mathbb{Q}}_\ell$) local systems. Indeed, denoting by $\mathrm{Loc}_{X_{\text{proét}}}(\mathbb{Q}_\ell)$ the category of $\mathbb{Q}_\ell$-local systems on $X_{\text{proét}}$, i.e., locally constant sheaves of finite-dimensional $\mathbb{Q}_\ell$-vector spaces (again, the "naive" definition works in $X_{\text{proét}}$), one has an equivalence $\mathrm{Rep}_{\text{cts},\mathbb{Q}_\ell}(\pi_1^{\text{proét}}(X,\bar{x})) \simeq \mathrm{Loc}_{X_{\text{proét}}}(\mathbb{Q}_\ell)$. This fails for $\pi_1^{\text{ét}}$, as any $\mathbb{Q}_\ell$-representation of a profinite group must stabilize a $\mathbb{Z}_\ell$-lattice, while $\mathbb{Q}_\ell$-local systems (in the above sense) stabilize lattices only étale locally. The group π_1^{SGA3} is not enough either; as shown by [Bhatt and Scholze 2015, Example 7.4.9] (due to Deligne), if X is the scheme obtained by gluing two points on a smooth projective curve of suitably large genus, there are $\mathbb{Q}_\ell$-local systems on X that do not come from a representation of $\pi_1^{\text{SGA3}}(X)$.

We will often drop $\bar{x}$ from the notation for brevity. This usually does not matter much, as a different choice of the base point leads to an isomorphic group.

Classical results. In [SGA 1 1971], Grothendieck proved some foundational results regarding the étale fundamental group. Among them:

(1) The fundamental exact sequence, i.e., the comparison between the "arithmetic" and "geometric" fundamental groups:

Theorem 1.1 [SGA 1 1971, Exposé IX, Théorème 6.1]. *Let k be a field with algebraic closure $\bar{k}$. Let X be a quasicompact and quasiseparated scheme over k. If the base change $X_{\bar{k}}$ is connected, then there is a short exact sequence*

$$1 \to \pi_1^{\text{ét}}(X_{\bar{k}}) \to \pi_1^{\text{ét}}(X) \to \mathrm{Gal}_k \to 1$$

of profinite topological groups.

(2) The homotopy exact sequence:

Theorem 1.2 [SGA 1 1971, Exposé X, Corollaire 1.4]. *Let $f : X \to S$ be a flat proper morphism of finite presentation whose geometric fibers are connected and reduced. Assume S is connected and let $\bar{s}$ be a geometric point of S. Then there is an exact sequence*

$$\pi_1^{\text{ét}}(X_{\bar{s}}) \to \pi_1^{\text{ét}}(X) \to \pi_1^{\text{ét}}(S) \to 1$$

of fundamental groups.

(3) "Künneth formula":

Proposition 1.3 [SGA 1 1971, Exposé X, Corollary 1.7]. *Let X, Y be two connected schemes locally of finite type over an algebraically closed field k and assume that Y is proper. Let $\bar{x}$, $\bar{y}$ be geometric points of X and Y respectively with values in the same algebraically closed field extension K of k. Then the map induced by the projections is an isomorphism*

$$\pi_1^{\text{ét}}(X \times_k Y, (\bar{x}, \bar{y})) \xrightarrow{\sim} \pi_1^{\text{ét}}(X, \bar{x}) \times \pi_1^{\text{ét}}(Y, \bar{y}).$$

(4) Invariance of $\pi_1^{\text{ét}}$ under extensions of algebraically closed fields for proper schemes [SGA 1 1971, Exposé X, Corollaire 1.8].

(5) General van Kampen theorem (proved in a special case in [SGA 1 1971, IX Section 5] and generalized in [Stix 2006]).

The aim of this and the subsequent article [Lara 2022] is to generalize statements (1) and (2), correspondingly, to the case of $\pi_1^{\text{proét}}$. In the present article, we also establish the generalizations of all the other points besides (2). The main difficulties in trying to directly generalize the proofs of Grothendieck are as follows:

- Geometric coverings of schemes (i.e., elements of Cov_X defined above) are often not quasicompact, unlike elements of $\mathrm{FÉt}_X$. For example, for X a variety over a field k and connected $Y \in \mathrm{Cov}_{X_{\bar{k}}}$, there may be no finite extension l/k such that Y would be defined over l. Similarly, some useful constructions (like Stein factorization) no longer work (at least without significant modifications).
- For a connected geometric covering $Y \in \mathrm{Cov}_X$, there is in general no Galois geometric covering dominating it. Equivalently, there might exist an open subgroup $U < \pi_1^{\text{proét}}(X)$ that does not contain an open normal subgroup. This prevents some proofs that would work for π_1^{SGA3} to carry over to $\pi_1^{\text{proét}}$.
- The topology of $\pi_1^{\text{proét}}$ is more complicated than the one of $\pi_1^{\text{ét}}$, e.g., it is not necessarily compact, which complicates the discussion of exactness of sequences.

Our results. Our main theorem is the generalization of the fundamental exact sequence. More precisely, we prove the following:

Theorem (Theorem 4.14). *Let X be a geometrically connected scheme of finite type over a field k. Then the sequence*

$$1 \to \pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X) \to \mathrm{Gal}_k \to 1$$

is exact as abstract groups.

Moreover, the map $\pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X)$ is a topological embedding and the map $\pi_1^{\text{proét}}(X) \to \mathrm{Gal}_k$ is a quotient map of topological groups.

The *most difficult part is showing that $\pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X)$ is* injective or, more precisely, *a topological embedding*. This is Theorem 4.13.

As in the case of usual Galois categories, statements about exactness of sequences of Noohi groups translate to statements on the corresponding categories of $G-\mathrm{Sets}$. If the groups involved are the pro-étale fundamental groups, this translates to statements about geometric coverings. We give a detailed dictionary in Proposition 2.37. As Noohi groups are not necessarily compact, the statements on coverings are equivalent to some weaker notions of exactness (e.g., preserving connectedness of coverings is equivalent to the map of groups having dense image). In fact, we first prove a "near-exact" version of Theorem 4.14 and obtain the above one as a corollary using an extra argument.

For $\pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X)$ *to be a topological embedding boils down to the following statement*: Every geometric covering Y of $X_{\bar{k}}$ can be dominated by a covering Y' that embeds into a base-change to $\bar{k}$ of a geometric covering Y'' of X (i.e., defined over k):

$$\begin{tikzcd} Y' \subset Y''_{\bar{k}} \arrow[r] \arrow[d, two heads] & Y'' \\ Y & \end{tikzcd}$$

For finite coverings, the analogous statement is easy to prove; by finiteness, the given covering is defined over a finite field extension l/k and one concludes quickly. This is also the case for infinite coverings detected by π_1^{SGA3}, see Proposition 4.8. But for general geometric coverings, the situation is much less obvious; as we show by counterexamples (Examples 4.5 and 4.6), *it is **not** true in general that a connected geometric covering of $X_{\bar{k}}$ is isomorphic to a base-change of a covering of X_l for some finite extension l/k*. This property is crucially used in the proof of [SGA 1 1971, Exposé IX, Theorem 6.1], and thus *trying to carry the classical proof of SGA over to $\pi_1^{\text{proét}}$ fails*. This last statement is, however, stronger than what we need to prove, and so does not contradict our theorem.

A useful technical tool across the article is the *van Kampen theorem* for $\pi_1^{\text{proét}}$. Its abstract form is proven by adapting the proof in [Stix 2006] to the case of Noohi groups and infinite Galois categories. For a morphism of schemes $X' \twoheadrightarrow X$ of effective descent for Cov (satisfying some extra conditions), it allows one to write the pro-étale fundamental group of X in terms of the pro-étale fundamental groups of the connected components of X' and certain relations. By the results of [Rydh 2010], one can take $X' = X^\nu \to X$ to be the normalization morphism of a Nagata scheme X. As $\pi_1^{\text{proét}}$ and $\pi_1^{\text{ét}}$ coincide for normal schemes, this allows us to present $\pi_1^{\text{proét}}(X)$ in terms of $\pi_1^{\text{ét}}(X_w^\nu)$, where $X^\nu = \bigsqcup_w X_w^\nu$, and the (discrete) topological fundamental group of a suitable graph. In this case, the van Kampen theorem takes on concrete form and generalizes [Lavanda 2018, Theorem 1.17].

Theorem (van Kampen theorem, Corollary 3.19, Remark 3.21, Proposition 3.12; compare [Stix 2006]). *Let X be a Nagata scheme and $X^\nu = \bigsqcup_w X_w^\nu$ its normalization written as a union of connected components. Then, after a choice of geometric points, étale paths between them and a maximal tree T within a suitable "intersection" graph Γ, there is an isomorphism*

$$\pi_1^{\text{proét}}(X, \bar{x}) \simeq \left(\left(*_w^{\text{top}} \pi_1^{\text{ét}}(X_w^\nu, \bar{x}_w) *^{\text{top}} \pi_1^{\text{top}}(\Gamma, T)\right)/\langle R_1, R_2\rangle\right)^{\text{Noohi}}$$

*where R_1, R_2 are two sets of relations described in Corollary 3.19 and $(-)^{\text{Noohi}}$ is the **Noohi completion** defined in Section 2.*

In the proof of the main theorem, the van Kampen theorem allows us to construct $\pi_1^{\text{proét}}(X_{\bar{k}})$- and $\pi_1^{\text{proét}}(X)$-sets in more concrete terms of graphs of groups involving the $\pi_1^{\text{ét}}$. We "explicitly" construct a Galois invariant open subgroup of a given open subgroup $U < \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$ in terms of *"regular loops"* (with respect to U), see Definition 4.20.

In fact, the existence of elements that are too far from being a product of regular loops is tacitly behind the counterexamples Examples 4.5 and 4.6, while the fact that, despite this, there is still an abundance of (products of) regular loops (i.e., their closure is open) is behind our main proof. We also sketch a quicker but less constructive approach in Remark 4.27.

Another interesting result proven with the help of the van Kampen theorem is the Künneth formula.

Proposition (Künneth formula for $\pi_1^{\text{proét}}$, Proposition 3.29). *Let X, Y be two connected schemes locally of finite type over an algebraically closed field k and assume that Y is proper. Let $\bar{x}$, $\bar{y}$ be geometric points of X and Y respectively with values in the same algebraically closed field extension K of k. Then the map induced by the projections is an isomorphism*

$$\pi_1^{\text{proét}}(X \times_k Y, (\bar{x}, \bar{y})) \xrightarrow{\sim} \pi_1^{\text{proét}}(X, \bar{x}) \times \pi_1^{\text{proét}}(Y, \bar{y}).$$

Along the way, we prove the invariance of $\pi_1^{\text{proét}}$ under extensions of algebraically closed fields for proper schemes (see Proposition 3.31) and give a short direct proof of the fact that $\pi_1^{\text{SGA3}}(X_{\bar{k}}, \bar{x}) \hookrightarrow \pi_1^{\text{SGA3}}(X, \bar{x})$, see Corollary 4.10.

In a separate article [Lara 2022], we discuss the homotopy exact sequence for $\pi_1^{\text{proét}}$. It is proven by constructing an infinite (i.e., nonquasicompact) analogue of the Stein factorization. Although the construction does not use the main results of this article, the auxiliary results on Noohi groups and $\pi_1^{\text{proét}}$ have proven to be very handy.

We hope that our techniques, with some extra tweaks and work, will allow to draw similar conclusions about other Noohi fundamental groups arising from the infinite Galois formalism. One such example could be the de Jong fundamental group π_1^{dJ}, defined in the rigid-analytic setting in [de Jong 1995]. In a later joint work [Achinger et al. 2022], we have proven the existence of a specialization morphism between $\pi_1^{\text{proét}}$ and π_1^{dJ}, relating $\pi_1^{\text{proét}}$ to this more established fundamental group.

1A. *Conventions and notations.*

- For us, compact = quasicompact + Hausdorff.
- $H <^\circ G$ will mean that H is an open subgroup of G.
- For subgroups $H < G$, H^{nc} will denote the normal closure of H in G, i.e., the smallest normal subgroup of G containing H. We will use $\langle\langle - \rangle\rangle$ to denote the normal closure of the subgroup generated by some subset of G, i.e., $\langle\langle - \rangle\rangle = \langle - \rangle^{nc}$.
- For a field k, we will use $\bar{k}$ to denote its (fixed) algebraic closure and k^{sep} or k^s to denote its separable closure (in $\bar{k}$).
- The topological groups are assumed to be Hausdorff unless specified otherwise or appearing in a context where it is not automatically satisfied (e.g., as a quotient by a subgroup that is not necessarily closed). We will usually comment whenever a non-Hausdorff group appears.
- We assume (almost) every base scheme to be locally topologically noetherian. This does not cause problems when considering geometric coverings, as a geometric covering of a locally topologically

noetherian scheme is locally topologically noetherian again — this is [Bhatt and Scholze 2015, Lemma 6.6.10].

- A "G-set" for a topological group G will mean a discrete set with a continuous action of G unless specified otherwise. We will denote the category of G-sets by $G-\mathrm{Sets}$. We will denote the category of sets by Sets.
- We will often omit the base points from the statements and the discussion; by Corollary 3.18, this usually does not change much. In some proofs (e.g., involving the van Kampen theorem), we keep track of the base points.

2. Infinite Galois categories, Noohi groups and $\pi_1^{\text{proét}}$

2A. *Overview of the results in* [Bhatt and Scholze 2015]. Throughout the entire article we use the language and results of [Bhatt and Scholze 2015], especially of Chapter 7, as this is where the pro-étale fundamental group was defined. Some familiarity with the results of [loc. cit., Section 7] is a prerequisite to read this article. We are going to give a quick overview of some of these results below, but we recommend keeping a copy of [loc. cit.] at hand.

Definition 2.1 [Bhatt and Scholze 2015, Definition 7.1.1]. Fix a topological group G. Let $G-\mathrm{Sets}$ be the category of discrete sets with a continuous G-action, and let $F_G : G-\mathrm{Sets} \to \mathrm{Sets}$ be the forgetful functor. We say that G is a *Noohi group* if the natural map induces an isomorphism $G \to \mathrm{Aut}(F_G)$ of topological groups. Here, $S \in \mathrm{Sets}$ are considered with the discrete topology, $\mathrm{Aut}(S)$ with the compact-open topology and $\mathrm{Aut}(F_G)$ is topologized using $\mathrm{Aut}(F_G(S))$ for $S \in G-\mathrm{Sets}$. More precisely, the stabilizers $\mathrm{Stab}^{\mathrm{Aut}(F_G)}_{F(S),s}$ for connected $S \in G-\mathrm{Sets}$, $s \in F(S)$, form a basis of neighborhoods of $1 \in \mathrm{Aut}(F_G)$.

In particular, it follows from the definition that open subgroups form a basis of neighborhoods of 1 in a Noohi group. Now, by [Bhatt and Scholze 2015, Proposition 7.1.5], it follows that a topological group is Noohi if and only if it satisfies the following conditions:

- Its open subgroups form a basis of open neighborhoods of $1 \in G$.
- It is Raĭkov complete.

A topological group G is Raĭkov complete if it is complete for its two-sided uniformity (see [Dikranjan 2013] or [Arhangel'skii and Tkachenko 2008, Chapter 3.6] for an introduction to the Raĭkov completion). Using the above proposition it is easy to give examples of Noohi groups.

Example 2.2. The following classes of topological groups are Noohi: discrete groups, profinite groups, $\mathrm{Aut}(S)$ with the compact-open topology for S a discrete set (see [Bhatt and Scholze 2015, Lemma 7.1.4]), groups containing an open subgroup which is Noohi; see [loc. cit., Lemma 7.1.8].

The following groups are Noohi: $\mathbb{Q}_\ell$, $\overline{\mathbb{Q}_\ell}$ for the colimit topology induced by expressing $\overline{\mathbb{Q}_\ell}$ as a union of finite extensions (in contrast with the situation for the ℓ-adic topology), $\mathrm{GL}_n(E)$ for any algebraic extension $E/\mathbb{Q}_\ell$ and the colimit topology on $\mathrm{GL}_n(E)$; see [loc. cit., Example 7.1.7].

The notion of a Noohi group is tightly connected to a notion of an infinite Galois category, which we are about to introduce. Here, an object $X \in \mathcal{C}$ is called connected if it is not empty (i.e., initial), and for every subobject $Y \to X$ (i.e., $Y \xrightarrow{\sim} Y \times_X Y$), either Y is empty or $Y = X$.

Definition 2.3 [Bhatt and Scholze 2015, Definition 7.2.1]. An *infinite Galois category* $\mathcal{C}$ is a pair $(\mathcal{C}, F : \mathcal{C} \to \text{Sets})$ satisfying:

(1) $\mathcal{C}$ is a category admitting colimits and finite limits.

(2) Each $X \in \mathcal{C}$ is a disjoint union of connected (in the sense explained above) objects.

(3) $\mathcal{C}$ is generated under colimits by a set of connected objects.

(4) F is faithful, conservative, and commutes with colimits and finite limits.

The *fundamental group of* $(\mathcal{C}, F)$ is the topological group $\pi_1(\mathcal{C}, F) := \text{Aut}(F)$, topologized by the compact-open topology on $\text{Aut}(S)$ for any $S \in \text{Sets}$.

An infinite Galois category $(\mathcal{C}, F)$ is *tame* if for any connected $X \in \mathcal{C}$, $\pi_1(\mathcal{C}, F)$ acts transitively on $F(X)$.

Example 2.4. If G is a topological group, then $(G-\text{Sets}, F_G)$ is a tame infinite Galois category.

Theorem 2.5 [Bhatt and Scholze 2015, Theorem 7.2.5]. *Fix an infinite Galois category* $(\mathcal{C}, F)$ *and a Noohi group* G. *Then*:

(1) $\pi_1(\mathcal{C}, F)$ *is a Noohi group.*

(2) *There is a natural identification of* $\text{Hom}_{\text{cont}}(G, \pi_1(\mathcal{C}, F))$ *with the groupoid of functors* $\mathcal{C} \to G-\text{Sets}$ *that commute with the fiber functors.*

(3) *If* $(\mathcal{C}, F)$ *is tame*, *then* F *induces an equivalence* $\mathcal{C} \simeq \pi_1(\mathcal{C}, F) - \text{Sets}$.

The "tameness" assumption cannot be dropped as there exist infinite Galois categories that are not of the form $(G-\text{Sets}, F_G)$; see [Bhatt and Scholze 2015, Example 7.2.3]. This was overlooked in [Noohi 2008], where a similar formalism was considered.

Remark 2.6. The above formalism was also studied in [Lepage 2010, Chapter 4] under the names of "quasiprodiscrete" groups and "pointed classifying categories".

In Section 2B below we will study "Noohi completion" and the dictionary between Noohi groups and $G-\text{Sets}$ (see Section 2C). For now, let us return to gathering the results from [Bhatt and Scholze 2015].

Pro-étale topology and the definition of $\pi_1^{\text{proét}}(X)$.

Definition 2.7. Let X be a locally topologically noetherian scheme. Let $Y \to X$ be a morphism of schemes such that:

(1) It is étale (not necessarily quasicompact!).

(2) It satisfies the valuative criterion of properness.

We will call Y a *geometric covering* of X. We will denote the category of geometric coverings by Cov_X.

As Y is not assumed to be of finite type over X, the valuative criterion does not imply that $Y \to X$ is proper (otherwise we would simply get a finite étale morphism).

Example 2.8. For an algebraically closed field $\bar{k}$, the category $\mathrm{Cov}_{\mathrm{Spec}(\bar{k})}$ consists of (possibly infinite) disjoint unions of $\mathrm{Spec}(\bar{k})$ and we have $\mathrm{Cov}_{\mathrm{Spec}(\bar{k})} \simeq \mathrm{Sets}$.

More generally, one has:

Lemma 2.9 [Bhatt and Scholze 2015, Lemma 7.3.8]. *If X is a henselian local scheme, then any $Y \in \mathrm{Cov}_X$ is a disjoint union of finite étale X-schemes.*

Let us choose a geometric point $\bar{x} : \mathrm{Spec}(\bar{k}) \to X$ on X. By Example 2.8, this gives a fiber functor $F_{\bar{x}} : \mathrm{Cov}_X \to \mathrm{Sets}$. By [Bhatt and Scholze 2015, Lemma 7.4.1], the pair $(\mathrm{Cov}_X, F_{\bar{x}})$ is a tame infinite Galois category. Then one defines:

Definition 2.10. The *pro-étale fundamental group* is defined as

$$\pi_1^{\text{proét}}(X, \bar{x}) = \pi_1(\mathrm{Cov}_X, F_{\bar{x}}).$$

In other words, $\pi_1^{\text{proét}}(X, \bar{x}) = \mathrm{Aut}(F_{\bar{x}})$ and this group is topologized using the compact-open topology on $\mathrm{Aut}(S)$ for any $S \in \mathrm{Sets}$.

One can compare the groups $\pi_1^{\text{proét}}(X, \bar{x})$, $\pi_1^{\text{ét}}(X, \bar{x})$ and $\pi_1^{\text{SGA3}}(X, \bar{x})$, where the last group is the group introduced in Chapter X.6 of [SGA 3_{II} 1970].

Lemma 2.11. *For a scheme X, the following relations between the fundamental groups hold*:

(1) *The group $\pi_1^{\text{ét}}(X, \bar{x})$ is the profinite completion of $\pi_1^{\text{proét}}(X)$.*

(2) *The group $\pi_1^{\text{SGA3}}(X, \bar{x})$ is the prodiscrete completion of $\pi_1^{\text{proét}}(X, \bar{x})$.*

Proof. This follows from [Bhatt and Scholze 2015, Lemma 7.4.3 and 7.4.6]. □

As shown in [loc. cit., Example 7.4.9], $\pi_1^{\text{proét}}(X, \bar{x})$ is indeed more general than $\pi_1^{\text{SGA3}}(X, \bar{x})$. This can be also seen by combining Example 4.5 with Proposition 4.8 below.

The following lemma is extremely important to keep in mind and will be used many times throughout the paper. Recall that, for example, a normal scheme is geometrically unibranch.

Lemma 2.12 [Bhatt and Scholze 2015, Lemma 7.4.10]. *If X is geometrically unibranch, then*

$$\pi_1^{\text{proét}}(X, \bar{x}) \simeq \pi_1^{\text{ét}}(X, \bar{x}).$$

There is another way of looking at the pro-étale fundamental group, which justifies the name “pro-étale”.

Definition 2.13. (1) A map $f : Y \to X$ of schemes is called *weakly étale* if f is flat and the diagonal $\Delta_f : Y \to Y \times_X Y$ is flat.

(2) The pro-étale site $X_{\text{proét}}$ is the site of weakly étale X-schemes, with covers given by fpqc covers.

This definition of the pro-étale site is justified by a foundational theorem — part (c) of the following fact.

Fact 2.14. Let $f : A \to B$ be a map of rings:

(a) f is étale if and only if f is weakly étale and finitely presented.

(b) If f is ind-étale, i.e., B is a filtered colimit of étale A-algebras, then f is weakly étale.

(c) [Bhatt and Scholze 2015, Theorem 2.3.4] If f is weakly étale, then there exists a faithfully flat ind-étale $g : B \to C$ such that $g \circ f$ is ind-étale.

Definition 2.15 [Bhatt and Scholze 2015, Definition 7.3.1]. We say that $F \in \mathrm{Shv}(X_{\text{proét}})$ is *locally constant* if there exists a cover $\{Y_i \to X\}$ in $X_{\text{proét}}$ with $F|_{Y_i}$ constant. We write Loc_X for the corresponding full subcategory of $\mathrm{Shv}(X_{\text{proét}})$.

We are ready to state the following important result.

Theorem 2.16 [Bhatt and Scholze 2015, Lemma 7.3.9]. *Let X to be locally topologically noetherian scheme. One has* $\mathrm{Loc}_X = \mathrm{Cov}_X$ *as subcategories of* $\mathrm{Shv}(X_{\text{proét}})$.

Topological invariance of the pro-étale fundamental group. We note that universal homeomorphisms of schemes induce equivalences on the corresponding categories of geometric coverings.

Proposition 2.17 [Bhatt and Scholze 2015, Lemma 5.4.2]. *Let $h : X' \to X$ be a universal homeomorphism of topologically noetherian schemes (i.e., induces a homeomorphism of topological spaces after any base-change). Then the pullback*

$$h^* : \mathrm{Cov}_X \to \mathrm{Cov}_{X'}, \quad Y \mapsto Y' = Y \times_X X'$$

is an equivalence of categories.

Proof. As $\mathrm{Cov}_X \simeq \mathrm{Loc}_X$, the theorem follows by the same proof as in [loc. cit., Lemma 5.4.2].

Alternatively, one can argue more directly (i.e., avoiding the equivalence with Loc_X) as follows. By [Stacks 2020, Theorem 04DZ], $V \mapsto V' = V \times_X X'$ induces an equivalence of categories of schemes étale over X and schemes étale over X'. By [Rydh 2010, Proposition 5.4.], this induces an equivalence between schemes étale and separated over respectively X and X'. The only thing left to be shown is that if for an étale separated scheme $Y \to X$, the map $Y \times_X X' \to X'$ satisfies the existence part of the valuative criterion of properness, then so does $Y \to X$. But this property can be characterized in purely topological terms (see [Stacks 2020, Lemma 01KE]) and so the result follows from the fact that h is a universal homeomorphism. □

2B. *Noohi completion.* Let HausdGps denote the category of Hausdorff topological groups (recall that we assume all topological groups to be Hausdorff, unless stated otherwise) and NoohiGps to be the full subcategory of Noohi groups. Let G be a topological group. Denote $\mathcal{C}_G = G-\mathrm{Sets}$ and let $F_G : \mathcal{C}_G \to \mathrm{Sets}$ be the forgetful functor. Observe that $(\mathcal{C}_G, F_G)$ is a tame infinite Galois category. Thus, the group $\mathrm{Aut}(F_G)$ is a Noohi group. It is easy to see that a morphism $G \to H$ defines an induced morphism of groups $\mathrm{Aut}(F_G) \to \mathrm{Aut}(F_H)$ and check that it is continuous. Let $\psi_N : \mathrm{HausdGps} \to \mathrm{NoohiGps}$ be the functor defined by $G \mapsto \mathrm{Aut}(F_G)$. Denote also the inclusion $i_N : \mathrm{NoohiGps} \to \mathrm{HausdGps}$.

Definition 2.18. We call $\psi_N(G)$ the Noohi completion of G and will denote it G^{Noohi}.

Example 2.19. In [Bhatt and Scholze 2015, Example 7.2.6], it was explained that the category of Noohi groups admits coproducts. Let G_1, G_2 be two Noohi groups and let $G_1 *^N G_2$ denote their coproduct as Noohi groups. Let $G_1 *^{\mathrm{top}} G_2$ be their topological coproduct. It exists and it is a Hausdorff group [Graev 1948]. Then $G_1 *^N G_2 = (G_1 *^{\mathrm{top}} G_2)^{\mathrm{Noohi}}$.

Let $\alpha_G : G \to \mathrm{Aut}(F_G) = G^{\mathrm{Noohi}}$ denote the obvious morphism.

Proposition 2.20. *For a topological group G, the functor F_G induces an equivalence of categories*

$$\widetilde{F}_G : G - \mathrm{Sets} \xrightarrow{\sim} G^{\mathrm{Noohi}} - \mathrm{Sets}\,.$$

Moreover, $\alpha_G^* \circ \widetilde{F}_G \simeq \mathrm{id}$, and thus α^* is an equivalence of categories, too.

Proof. The first part follows directly from [Bhatt and Scholze 2015, Theorem 7.2.5]. The natural isomorphism $\alpha_G^* \circ \widetilde{F}_G \simeq \mathrm{id}$ is clear from the definitions. It follows that α_G^* is an equivalence. □

The following lemma is in contrast with [Noohi 2008, Remark 2.13], but agrees with [Lepage 2010, Proposition 4.1.1].

Lemma 2.21. *For any topological group G, the image of $\alpha_G : G \to G^{\mathrm{Noohi}}$ is dense.*

Proof. Let $U \subset G^{\mathrm{Noohi}}$ be open. As G^{Noohi} is Noohi, there exists $q \in G^{\mathrm{Noohi}}$ and an open subgroup $V <^\circ G^{\mathrm{Noohi}}$ such that $qV \subset U$. The quotient G^{Noohi}/V gives a G^{Noohi}-set. It is connected in the category $G^{\mathrm{Noohi}} - \mathrm{Sets}$ and, by Proposition 2.20, $\alpha_G^*(G^{\mathrm{Noohi}}/V)$ is connected. Thus, the action of G on G^{Noohi}/V is transitive and so there exists $g \in G$ such that $\alpha_G(g) \cdot [V] = [qV]$, i.e., $\alpha_G(g) \in qV$. Thus, the image of α_G is dense. □

Observation. Let $f : H \to G$ be a map of topological groups. Directly from the definitions, one sees that the following diagram commutes:

$$\begin{tikzcd} H \arrow[r, "f"] \arrow[d, "\alpha_H"'] & G \arrow[d, "\alpha_G"'] \\ H^{\mathrm{Noohi}} \arrow[r, "f^{\mathrm{Noohi}}"] & G^{\mathrm{Noohi}} \end{tikzcd}$$

Lemma 2.22 (universal property of Noohi completion). *Let $f : H \to G$ be a continuous morphism from a topological group to a Noohi group. Then there exists a unique map $f' : H^{\mathrm{Noohi}} \to G$ such that $f' \circ \alpha_H = f$.*

Proof. By the definition of a Noohi group, α_G is an isomorphism. Defining $f' := \alpha_G^{-1} \circ f^{\mathrm{Noohi}}$ gives the existence. The uniqueness follows from α_H having dense image. Alternatively, one can combine Proposition 2.20 with [Bhatt and Scholze 2015, Theorem 7.2.5(2)]. □

Corollary 2.23. *The functor ψ_N is a left adjoint of i_N.*

Remark 2.24. There are few places, where we write G^{Noohi} for a non-Hausdorff group G. This is mostly to avoid a large overline sign over a subgroup described by generators. In these cases, we mean

$$G^{\mathrm{Noohi}} := (G^{\mathrm{Hausd}})^{\mathrm{Noohi}}$$

where G^{Hausd} is the maximal Hausdorff quotient. As $(-)^{\mathrm{Hausd}}$ is a left adjoint as well, this usually does not cause problems. This also provides a left adjoint to the forgetful functor NoohiGps $\to$ TopGps to all topological groups.

We now move towards a more explicit description of the Noohi completion.

Lemma 2.25. *Let (G, τ) be a topological group. Denote by $\mathcal{B}$ the collection of sets of the form*

$$x_1\Gamma_1 y_1 \cap x_2\Gamma_2 y_2 \cap \cdots \cap x_m\Gamma_m y_m$$

where $m \in \mathbb{N}$, $x_i, y_i \in G$ and $\Gamma_i < G$ are open subgroups of G. Then $\mathcal{B}$ is a basis of a group topology τ' on G that is weaker than τ and open subgroups of (G, τ) form a basis of open neighborhoods of 1_G in (G, τ').

Moreover, the natural map $i' : (G, \tau) \to (G, \tau')$ induces an equivalence of categories $(G, \tau') - \mathrm{Sets} \to (G, \tau) - \mathrm{Sets}$. If $\{1_G\} \subset (G, \tau)$ is thickly closed, i.e., $\bigcap_{U <^{\circ} G} U = \{1_G\}$ (see Definition 2.30), then (G, τ') is Hausdorff and $(G, \tau)^{\mathrm{Noohi}} \xrightarrow{\sim} (G, \tau')^{\mathrm{Noohi}}$ is an isomorphism.

Proof. The first statement follows from [Bourbaki 1966, Proposition III.1.1] by taking the filter of subsets of G containing an open subgroup. It is also proven in [Lavanda 2018, Lemma 1.13] (the proposition is stated there in a particular case, but the proof works for any topological group). The second statement follows from the fact that for a discrete set S, any continuous morphism $(G, \tau) \to \mathrm{Aut}(S)$ factorizes through $i' : (G, \tau) \to (G, \tau')$. □

Fact 2.26 [Bhatt and Scholze 2015, Proposition 7.1.5]. Let G be a topological group such that its open subgroups form a basis of open neighborhoods of 1_G. Then $G^{\mathrm{Noohi}} \simeq \widehat{G}$, where $\widehat{G}$ denotes the Raĭkov completion of G.

Proposition 2.27. *Let (G, τ) be a topological group. Assume that $\{1_G\} \subset (G, \tau)$ is thickly closed (see Definition 2.30). Then there is a natural isomorphism of groups*

$$G^{\mathrm{Noohi}} \simeq \widehat{(G, \tau')},$$

where τ' denotes the topology described in the previous lemma and $\widehat{\cdots}$ denotes the Raĭkov completion.

Proof. We combine Fact 2.26 with the last lemma and get $(G, \tau)^{\mathrm{Noohi}} \simeq (G, \tau')^{\mathrm{Noohi}} \simeq \widehat{(G, \tau')}$. □

Observation 2.28. Let G be a topological group and H a normal subgroup. Then the full subcategory of $G - \mathrm{Sets}$ of objects on which H acts trivially is equal to the full subcategory of $G - \mathrm{Sets}$ on which its closure $\overline{H}$ acts trivially and it is equivalent to the category of $G/\overline{H} - \mathrm{Sets}$. So, it is an infinite Galois category with the fundamental group equal to $(G/\overline{H})^{\mathrm{Noohi}}$.

Lemma 2.29. *Let X be a connected, locally path-connected, semilocally simply connected topological space and $x \in X$ a point. Let F_x be the functor taking a covering space $Y \to X$ to the fiber Y_x over the point $x \in X$. Then* $(\mathrm{TopCov}(X), F_x)$ *is a tame infinite Galois category and* $\pi_1(\mathrm{TopCov}(X), F_x) = \pi_1^{\mathrm{top}}(X, x)$, *where we consider $\pi_1^{\mathrm{top}}(X, x)$ with the discrete topology. Here,* $\mathrm{TopCov}(X)$ *denotes the category of covering spaces of X.*

Proof. We first claim that there is an isomorphism: $(\mathrm{TopCov}(X), F_x) \simeq (\pi_1^{\mathrm{top}}(X, x) - \mathrm{Sets}, F_{\pi_1^{\mathrm{top}}(X,x)})$. This is in fact a classical result in algebraic topology, which can be recovered from [Fulton 1995, Chapter 13] or [Hatcher 2002, Chapter 1] and is stated explicitly in [Çakar 2014, Corollary 4.1]. This finishes the proof, as discrete groups are Noohi. □

2C. *Dictionary between Noohi groups and G* − Sets.

Definition 2.30. Let $H \subset G$ be a subgroup of a topological group G. Then we define a "thick closure" $\overline{\overline{H}}$ of H in G to be the intersection of all open subgroups of G containing H, i.e., $\overline{\overline{H}} := \bigcap_{H \subset U <^{\circ} G} \mathrm{U}$. If a subgroup satisfies $H = \overline{\overline{H}}$ we will call it thickly closed in G.

In a topological group open subgroups are also closed, so a thickly closed subgroup is also an intersection of closed subgroups, so it is closed in G. Observe also that an arbitrary intersection of thickly closed subgroups is thickly closed. This justifies, for example, the existence of the smallest normal thickly closed subgroup containing a given group. In fact, we can formulate a more precise observation.

Observation 2.31. Let $H < G$ be a subgroup of a topological group G. Then the smallest normal thickly closed subgroup of G containing H is equal to $\overline{\overline{(H^{nc})}}$, where H^{nc} is the normal closure of H in G.

Observation 2.32. Let G be a topological group such that the open subgroups form a local base at 1_G. Let $W \subset G$ be a subset. Then the topological closure of W can be written as $\overline{W} = \bigcap_{V <^{\circ} G} WV$.

The following lemma can be found on page 79 of [Lepage 2010].

Lemma 2.33. *Let G be a topological group such that the open subgroups form a basis of neighborhoods of 1_G. Let $H \lhd G$ be a normal subgroup. Then*

$$\overline{H} = \overline{\overline{H}}$$

i.e., the usual topological closure and the thick closure coincide.

Proof. We compute that $\overline{H} = \bigcap_{V <^{\circ} G} HV \overset{(*)}{\supset} \bigcap_{H < U <^{\circ} G} U = \overline{\overline{H}} \supset \overline{H}$. The inclusion $(*)$ follows from the fact that HV is an (open) subgroup of G as H is normal. □

Let us make an easy observation, that will be useful to keep in mind while reading the proof of the technical proposition below.

Observation 2.34. Let $U < G$ be an open subgroup of a topological group and let $g_0 \in G$. Then the mapping $G/g_0Ug_0^{-1} \to G/U$ given by $[gg_0Ug_0^{-1}] \mapsto [gg_0U]$ is an isomorphism of G-sets.

Given open subgroups $U, V < G$ and some surjective map of G-sets $\phi : G/V \twoheadrightarrow G/U$ we can assume that it is the standard quotient map (i.e., $V \subset U$) up to replacing U by a conjugate open subgroup (more precisely by $g_0 U g_0^{-1}$, where g_0 is such that $\phi([V]) = [g_0 U]$).

Remark 2.35. A map $Y' \to Y$ in an infinite Galois category $(\mathcal{C}, F)$ is an epimorphism/monomorphism if and only if the map $F(Y') \to F(Y)$ is surjective/injective. Similarly Y is an initial object if and only if $F(Y) = \varnothing$ and so on. The proofs of those facts are the same as the proofs in [Stacks 2020, Tag 0BN0]. This justifies using words "injective" or "surjective" when speaking about maps in $(\mathcal{C}, F)$.

Recall the following fact.

Observation. Let $f : G' \to G$ be a surjective map of topological groups. Then the induced morphism $G'/\ker(f) \to G$ is an isomorphism if and only if f is open. In such case, we say that f is a quotient map. In the language of [Bourbaki 1966, III.2.8] we would call f *strict* and surjective.

Definition 2.36. We will say that an object of a tame infinite Galois category is *completely decomposed* if it is a (possibly infinite) disjoint union of final objects.

Proposition 2.37. *Let $G'' \xrightarrow{h'} G' \xrightarrow{h} G$ be maps between Noohi groups and $\mathcal{C}_{G''} \xleftarrow{H'} \mathcal{C}_{G'} \xleftarrow{H} \mathcal{C}_G$ the corresponding maps of the infinite Galois categories. Then the following hold*:

(1) *The map $h' : G'' \to G'$ is a topological embedding if and only if for every connected object X in $\mathcal{C}_{G''}$, there exist connected objects $X' \in \mathcal{C}_{G''}$ and $Y \in \mathcal{C}_{G'}$ and maps $X' \twoheadrightarrow X$ and $X' \hookrightarrow H'(Y)$.*

(2) *The following are equivalent*:

  (a) *The morphism $h : G' \to G$ has dense image.*

  (b) *The functor H maps connected objects to connected objects.*

  (c) *The functor H is fully faithful.*

(3) *The thick closure of* $\mathrm{Im}(h') \subset G'$ *is normal if and only if for every connected object Y of $\mathcal{C}_{G'}$ such that $H'(Y)$ contains a final object of $\mathcal{C}_{G''}$, $H'(Y)$ is completely decomposed.*

(4) $h'(G'') \subset \mathrm{Ker}(h)$ *if and only if the composition $H' \circ H$ maps any object to a completely decomposed object.*

(5) *Assume that $h'(G'') \subset \ker(h)$ and that $h : G' \to G$ has dense image. Then the following conditions are equivalent*:

  (a) *The induced map $(G'/\ker(h))^{\mathrm{Noohi}} \to G$ is an isomorphism and the smallest normal thickly closed subgroup containing* $\mathrm{Im}(h')$ *is equal to* $\ker(h)$.

  (b) *For any connected $Y \in \mathcal{C}_{G'}$ such that $H'(Y)$ is completely decomposed, Y is in the essential image of H.*

  (c) *The induced map $(G'/\ker(h))^{\mathrm{Noohi}} \to G$ is an isomorphism and for any connected $Y \in \mathcal{C}_{G'}$ such that $H'(Y)$ is completely decomposed, there exists $Z \in \mathcal{C}_G$ and an epimorphism $H(Z) \twoheadrightarrow Y$.*

Proof. (1) The proof is virtually the same as for usual Galois categories, but there every injective map is automatically a topological embedding (as profinite groups are compact). Assume that $G'' \to G'$

is a topological embedding. Let $X \in \mathcal{C}_{G''}$ be connected and write $X \simeq G''/U$ for an open subgroup $U < G''$. Then there exists an open subset $\widetilde{V} \subset G'$ such that $\widetilde{V} \cap G'' = U$ (as $G'' \to G'$ is a topological embedding) and an open subgroup $V < G'$ such that $V \subset \widetilde{V}$ (as G' is Noohi). Denote $W = V \cap G''$. Then $X' := G''/W \twoheadrightarrow X$ and $X' \hookrightarrow H'(G'/V)$, so we conclude by setting $Y := G'/V$. For the other implication: we want to prove that $G'' \to G'$ is a topological embedding under the assumption from the statement. It is enough to check that the set of preimages $h'^{-1}(\mathcal{B})$ of some basis $\mathcal{B}$ of opens of $e_{G'}$ forms a basis of opens of $e_{G''}$. Indeed, assume that this is the case. Firstly, observe that it implies that h' is injective, as both G'' and G' are Hausdorff (and in particular T_0). If U is an open subset of G'', then we can write $U = \bigcup g''_\alpha U_\alpha$ for some $g''_\alpha \in G''$ and $U_\alpha \in h'^{-1}(\mathcal{B})$. We can write $U_\alpha = h'^{-1}(V_\alpha)$ for some $V_\alpha \in \mathcal{B}$. Then $V = \bigcup h'(g''_\alpha)V_\alpha$ satisfies $h'^{-1}(V) = U$ because $h'^{-1}(h'(g''_\alpha)V_\alpha) = g''_\alpha U_\alpha$ (by injectivity of h'). So this will prove that the topology on G'' is induced from G' via h'. Let $\mathcal{B} = \{U < G \mid U \text{ is open}\}$. This is a basis of opens of $e_{G'}$ (as G' is Noohi). We want to check that $h'^{-1}(\mathcal{B})$ is a basis of opens of $e_{G''}$. As open subgroups of G'' form a basis of opens of $e_{G''}$ it is enough to show that for any open subgroup $U < G''$ there exists an open subgroup $V < G'$ such that $h'^{-1}(V) \subset U$. From the assumption we know that there exist open subgroups $\widetilde{U} < G''$ and $V < G'$ such that $G''/\widetilde{U} \twoheadrightarrow G''/U$ and $G''/\widetilde{U} \hookrightarrow G'/V$. The surjectivity of the first map means that we can assume (up to replacing $\widetilde{U}$ by a conjugate) $\widetilde{U} \subset U$. The injectivity of the second means that we can assume (up to replacing V by a conjugate) that $h'^{-1}(V) \subset \widetilde{U}$. Indeed, the injectivity implies that if $h'(g'')V = V$, then $g''\widetilde{U} = \widetilde{U}$ which translates immediately to $h'^{-1}(V) \subset \widetilde{U}$. So we have also $h'^{-1}(V) \subset U$, which is what we wanted to prove.

(2) The equivalence between (a) and (b) follows from the observation that a map between Noohi groups $G' \to G$ has a dense image if and only if for any open subgroup U of G, the induced map on sets $G' \to G/U$ is surjective. Here, we only use that open subgroups form a basis of open neighborhoods of $1_G \in G$.

Now, the functor H is automatically faithful and conservative (because $F_{G'} \circ H = F_G$ is faithful and conservative). Assume that (b) holds. Let $S, T \in G-\text{Sets}$ and let $g \in \mathrm{Hom}_{G'-\text{Sets}}(H(S), H(T))$. We have to show that g comes from $g_0 \in \mathrm{Hom}_{G-\text{Sets}}(S, T)$. We can and do assume S, T connected for that. Let $\Gamma_g \subset H(S) \times H(T)$ be the graph of g. It is a connected subobject. As $H(S) \times H(T) = H(S \times T)$, the assumption (b) implies that each connected component of $H(S) \times H(T)$ is the pullback of a connected component Γ_0 of $S \times T$. Thus, Γ_g is the pullback of some $\Gamma_0 \subset S \times T$. By conservativity of H, the projection $p_{\Gamma_0} : \Gamma_0 \to S$ is an isomorphism, as this is true for $p_{\Gamma_g} : \Gamma_g \to H(S)$. The composition $q_{\Gamma_0} \circ p_{\Gamma_0}^{-1} : S \to T$ maps via H to g.

Conversely, assume (c) holds. Let $S \in G-\text{Sets}$ be connected. We want to show that $H(S)$ is connected. Suppose on the contrary that $H(S) = A \sqcup B$ with $A, B \in G'-\text{Sets}$. Let $T = \bullet \sqcup \bullet \in G-\text{Sets}$ be a two-element set with a trivial G-action. Then $\mathrm{Hom}_{G-\text{Sets}}(S, T)$ has precisely two elements, while $\mathrm{Hom}_{G'-\text{Sets}}(H(S), H(T)) = \mathrm{Hom}_{G'-\text{Sets}}(A \sqcup B, \bullet \sqcup \bullet)$ has at least four.

(3) Assume first that the thick closure of $\mathrm{im}(h')$ is normal. Let $Y = G'/U$ be an element of $\mathcal{C}_{G'}$ whose pull-back to $G''-\text{Sets}$ contains the final object. This means that G'' fixes one of the classes, let's say $[g'U]$.

This is equivalent to $g'^{-1}h'(G'')g'$ fixing $[U]$, i.e., $g'^{-1}h'(G'')g' \subset U$. But this implies immediately that $\overline{\overline{(g'^{-1}h'(G'')g')}} \subset U$. Let $\widetilde{g} \in G'$ be any element. We have $\overline{\overline{(g'^{-1}h'(G'')g')}} = g'^{-1}\overline{\overline{h'(G'')}}g' = \overline{\overline{h'(G'')}} = \widetilde{g}^{-1}\overline{\overline{h'(G'')}}\widetilde{g}$ from the assumption that $\overline{\overline{h'(G'')}}$ is normal. So $\widetilde{g}^{-1}h'(G'')\widetilde{g} \subset \widetilde{g}^{-1}\overline{\overline{h'(G'')}}\widetilde{g} \subset U$ and we conclude that $h'(G'')$ fixes an arbitrary class $[\widetilde{g}U]$. This shows that G'/U pulls back to a completely decomposed object.

The other way round: assume that for every connected object Y of $\mathcal{C}_{G'}$ such that $H'(Y)$ contains a final object, $H'(Y)$ is completely decomposed. Let U be an open subgroup of G' containing $h'(G'')$. Then G'' fixes $[U] \in G/U$ and so, by assumption, fixes every $[g'U] \in G/U$. This implies that for any $g' \in G'$ $g'^{-1}h'(G'')g' \subset U$ which easily implies that also $h'(G'')^{nc} \subset U$. As this is true for any U containing $h'(G'')$ we get that $\overline{\overline{h'(G'')}} = \overline{\overline{(h'(G'')^{nc})}}$ and the last group is the smallest normal thickly closed subgroup of G' containing $h'(G'')$ (Observation 2.31).

(4) The same as for usual Galois categories, we use that $\bigcap_{U <^{\circ} G} U = 1_G$.

(5) (b) $\Rightarrow$ (c): Assume (b). We only need to show, that $(G'/\ker(h))^{\mathrm{Noohi}} \to G$ is an isomorphism. This is equivalent to showing that H induces an equivalence $G'/\ker(h)-\mathrm{Sets} \simeq G-\mathrm{Sets}$. As $G'/\ker(h)-\mathrm{Sets} \simeq \{S \in G'-\mathrm{Sets} \mid \ker(h) \text{ acts trivially on } S\} \subset \{S \in G-\mathrm{Sets} \mid G'' \text{ acts trivially on } S\}$, the assumption of (b) implies that the functor $G-\mathrm{Sets} \to G'/\ker(h)-\mathrm{Sets}$ is essentially surjective. By the global assumption that $G' \to G$ has dense image, it is fully faithful (see (2)).

(c) $\Rightarrow$ (b): Assume (c). Let $Y \in \mathcal{C}_{G'}$ be connected and such that $H'(Y)$ is completely decomposed. We have $Z \in \mathcal{C}_G$ and an epimorphism $H(Z) \twoheadrightarrow Y$. As $\ker(h)$ acts trivially on $H(Z)$, we conclude that it also acts trivially on Y. Thus, by abuse of notation, $Y \in G'/\ker(h)-\mathrm{Sets}$. But $G'/\ker(h)-\mathrm{Sets} \simeq (G'/\ker(h))^{\mathrm{Noohi}}-\mathrm{Sets} \simeq G-\mathrm{Sets}$ from the assumption. Thus, we see that Y is in the essential image of H.

(b) $\Rightarrow$ (a): Assume (b). We give two proofs of this fact.

First proof. We have proven above that (b) $\Rightarrow (G'/\ker(h))^{\mathrm{Noohi}} \simeq G$. Let N be the smallest normal thickly closed subgroup of G' containing $h'(G'')$. Observe that $N \subset \ker h$ (as $\ker(h)$ is thickly closed). Let U be an open subgroup containing N. We want to show that U contains $\ker h$. This will finish the proof as both N and $\ker h$ are thickly closed. Write $Y = G'/U$. Observe that G'/U pulls back to a completely decomposed G''-set if and only if for any $g' \in G'$ there is $g'h'(G'')g'^{-1} \subset U$. Indeed, $h'(G'')$ fixes $[g'U] \in G'/U$ if and only if $g'h'(G'')g'^{-1}$ fixes $[U]$. So $N \subset U$ implies that Y pulls back to a completely decomposed G''-set and, by assumption, Y is isomorphic to a pull-back of some G-set and so $\ker(h)$ acts trivially on Y. This implies that $\ker h \subset U$, which finishes the proof. □

Alternative proof. We already know that (b) $\Rightarrow (G'/\ker(h))^{\mathrm{Noohi}} \simeq G$. Let $N \subset \ker(h)$ be as in the first proof above. Consider the map $G/N \twoheadrightarrow G/\ker(h)$. The assumption (b) and full faithfulness of H (by the global assumption and using (2)) imply that $(G'/N)^{\mathrm{Noohi}} \to G$ is an isomorphism. Thus, $(G'/N)^{\mathrm{Noohi}} \simeq (G'/\ker(h))^{\mathrm{Noohi}}$. Using Proposition 2.27, we check that the canonical maps $G'/N \to$

$(G'/N)^{\mathrm{Noohi}}$ and $G'/\ker(h) \to (G'/\ker(h))^{\mathrm{Noohi}}$ are injective. Thus, $G'/N \twoheadrightarrow G'/\ker(h)$ is injective and so $N = \ker(h)$. □

(a) ⇒ (b): Assume (a). Let $Y = G'/U$ be a connected G'-set that pulls back via h' to a completely decomposed object. As we have seen while proving "(b) ⇒ (a)", this implies that for any $g' \in G'$ $g'h'(G'')g'^{-1} \subset U$, so $H^{nc} \subset U$ and so also $\overline{\overline{(H^{nc})}} \subset U$. But, by Observation 2.31, there is $N = \overline{\overline{(H^{nc})}}$. By assumption, we have $N = \ker h$ and so we conclude that $\ker h \subset U$. But then, by assumption $(G'/\ker(h))^{\mathrm{Noohi}} \simeq G$, Y is in the essential image of H. □

To distinguish between exactness in the usual sense (i.e., on the level of abstract groups) and notions of exactness appearing in Proposition 2.37, we introduce a new notion. It will be mainly used in the context of Noohi groups.

Definition 2.38. Let $G'' \xrightarrow{h'} G' \xrightarrow{h} G \to 1$ be a sequence of topological groups such that $\mathrm{im}(h') \subset \ker(h)$. Then we will say that the sequence is:

(1) *Nearly exact on the right* if h has dense image,

(2) *Nearly exact in the middle* if $\overline{\overline{\mathrm{im}(h')}} = \ker(h)$, i.e., the thick closure of the image of h' in G' is equal to the kernel of h.

(3) *Nearly exact* if it is both nearly exact on the right and nearly exact in the middle.

We end this subsection with a lemma on topological groups and their Noohi completions that will be used later in the proof of the main theorem.

Lemma 2.39. *Let G be a topological group and $\tilde{G}$ be a subgroup of G^{Noohi} such that the canonical map $G \to G^{\mathrm{Noohi}}$ factorizes through $\tilde{G}$:*

$$G \to \tilde{G} \subset G^{\mathrm{Noohi}}.$$

*Let $V_0 < \tilde{G}$ be a subgroup. Let $S = (\tilde{G}/V_0, \mathrm{discr})$ be the discrete set that comes naturally with an **abstract** action by $\tilde{G}$.*

If the induced abstract G-action on S is continuous, then V_0 is open in $\tilde{G}$. Moreover, in such case, denoting $V = \mathrm{Stab}_{G^{\mathrm{Noohi}}}([V_0] \in \tilde{G}/V_0)$, there is

$$V = V_0^{\mathrm{Noohi}} = \overline{V_0}^{G^{\mathrm{Noohi}}} \quad \textit{and} \quad V_0 = V \cap \tilde{G}.$$

Proof. By the universal property, the G-action on S extends to G^{Noohi} and this action is transitive. Then V_0 is the preimage of the stabilizer $V = \mathrm{Stab}_{G^{\mathrm{Noohi}}}([V_0] \in \tilde{G}/V_0)$, which is open.

The group V is open in a Noohi group, thus Noohi; see [Bhatt and Scholze 2015, Lemma 7.1.8.]. By the universal property, there is a factorization $V_0^{\mathrm{Noohi}} \to V$. But as V_0 is a subgroup of a Noohi group, its open subgroups form a basis of 1_{V_0}. Thus, the Noohi completion of V_0 is just the Raĭkov completion. But as the canonical map from a group to its Raĭkov completion is a topological embedding, [Arhangel'skii and Tkachenko 2008, Corollary 3.6.18] implies that $V_0^{\mathrm{Noohi}} \to V$ is a topological embedding. By a characterization of Raĭkov completeness (see [Dikranjan 2013, Proposition 6.2.7]), it follows that V_0^{Noohi}

is closed in V. But as $\tilde{G}$ contains the image of G, it is dense in G^{Noohi}, and from the definition of V it follows that V_0 has to be dense in V. Putting this together, we get that $V_0^{\mathrm{Noohi}} = V = \overline{V_0}^{G^{\mathrm{Noohi}}}$. □

2D. *A remark on valuative criteria.* We will sometimes shorten "the valuative criterion of properness" to "VCoP". It is useful to keep in mind the precise statements of different parts of the valuative criterion, see [Stacks 2020, Lemmas 01KE, 01KC and Section 01KY]. Let us prove a lemma (which is implicit in [Bhatt and Scholze 2015]), that VCoP can be checked fpqc-locally.

Lemma 2.40. *Let $g : X \to S$ be a map of schemes. The properties*

(a) *g is étale,*

(b) *g is separated,*

(c) *g satisfies the existence part of VCoP,*

can be checked fpqc-locally on S. Moreover, property (c) can be also checked after a surjective proper base-change.

Proof. The cases of étale and separated morphisms are proven in [Stacks 2020, Section 02YJ]. For the last part, satisfying the existence part of VCoP is equivalent to specializations lifting along any base-change of g [Stacks 2020, Lemma 01KE]. It is easy to see that this property can be checked Zariski locally. Thus, if $S' \to S$ is an fpqc cover such that the base-change $g' : X' \to S'$ satisfies specialization lifting for any base-change, we can assume that S, S' are affine with $S' \to S$ faithfully flat. Let $T \to S$ be any morphism. Consider the diagram:

$$\begin{tikzcd} X' \arrow[d] & S' \times_S X \times_S T \arrow[l] \arrow[r] \arrow[d] & T \times_S X \arrow[d] \\ S' & S' \times_S T \arrow[l] \arrow[r] & T \end{tikzcd}$$

Let $\xi' \in T \times_S X$, let ξ be its image in T and let $t \in T$ be such that $\xi \rightsquigarrow t$. We need to find $t' \in T \times_S X$ over t such that $\xi' \rightsquigarrow t'$. Let $Z = \overline{\{\xi'\}} \subset T \times_S X$ be the closure of $\{\xi'\}$. We need to show that the set-theoretic image $W \subset T$ of Z in T contains t. It is enough to show, that W is stable under specialization or, equivalently, that $T \setminus W$ is stable under generalization. But, from flatness [Stacks 2020, Lemma 03HV], generalizations lift along $S' \times_S T \to T$. Thus, it is enough to show that the preimage of $T \setminus W$ in $S' \times_S T$ is stable under generalizations or, equivalently (using the surjectivity of $S' \times_S T \to T$), that the preimage of W in $S' \times_S T$ is closed under specializations. But an easy diagram chasing (using the fact that the right square of the diagram above is cartesian) shows that the preimage of W in $S' \times_S T$ is the image of a closed subset of $S' \times_S X \times T$. We conclude, because specializations lift along $S' \times_S X \times_S T \to S' \times_S T$ by assumption.

The last part of the statement is proven in an analogous way. □

Lemma 2.41. *Let $f : Y \to X$ be a geometric covering of a locally topologically noetherian scheme. Then f is separated.*

Proof. By [Bhatt and Scholze 2015, Remark 7.3.3], f is quasiseparated. A quasiseparated morphism satisfying VCoP is separated; see [Stacks 2020, Tag 01KY]. □

3. Seifert–van Kampen theorem for $\pi_1^{\text{proét}}$ and its applications

3A. ***Abstract Seifert–van Kampen theorem for infinite Galois categories.*** We aim at recovering a general version of van Kampen theorem, proven in [Stix 2006], in the case of the pro-étale fundamental group. Most of the definitions and proofs are virtually the same as in [loc. cit.], after replacing "Galois category" with "(tame) infinite Galois category" and "profinite" with "Noohi", but still some additional technical difficulties appear here and there. We make the necessary changes in the definitions and deal with those difficulties below.

Denote by $\Delta_{\leq 2}$ a category whose objects are $[0] = \{0\}$, $[1] = \{0, 1\}$, $[2] = \{0, 1, 2\}$ and has strictly increasing maps as morphisms. There are face maps $\partial_i : [n-1] \to [n]$ for $n = 1, 2$ and $0 \leq i \leq n$ which omit the value i and vertices $v_i : [0] \to [2]$ with image i.

The category of 2-*complexes in a category* $\mathscr{C}$ is the category of contravariant functors $T_\bullet : \Delta_{\leq 2} \to \mathscr{C}$. We denote $T_n = T_\bullet([n])$ and call it *the n-simplices of $T_\bullet$*. $T(\partial_i)$ is called the *i-th boundary map*.

By a 2-*complex* E we mean a 2-complex in the category of sets. We often think of E as a category: its objects are the elements of E_n for $n = 0, 1, 2$ and its morphisms are obtained by defining $\partial : s \to t$ where $s \in E_n$ and $t = E(\partial)(s)$. Let $\Delta_n = \left\{\sum_{i=0}^n \lambda_i e_i \in \mathbb{R}_{\geq 0}^{n+1} \mid \sum_i \lambda_i = 1\right\}$ denote the topological n-simplex. Then we define $|E| = \bigsqcup E_n \times \Delta_n / \sim$, where $\sim$ identifies $(s, d(x))$ with $(E(\partial)(s), x)$ for all $\partial : [m] \to [n]$ and its corresponding linear map $d : \Delta_m \to \Delta_n$ sending e_i to $e_{\partial(i)}$, and $s \in E_n$ and $x \in \Delta_m$. We call E *connected* if $|E|$ is a connected topological space.

Definition 3.1. *Noohi group data* $(\mathscr{G}, \alpha)$ on a 2-complex E consists of the following:

(1) A mapping (not necessarily a functor!) $\mathscr{G}$ from the category E to the category of Noohi groups: to a complex $s \in E_n$ is attributed a Noohi group $\mathscr{G}(s)$ and to a map $\partial : s \to t$ is attached a continuous morphism $\mathscr{G}(\partial) : \mathscr{G}(s) \to \mathscr{G}(t)$.

(2) For every triple $v \in E_0$, $e \in E_1$, $f \in E_2$ and boundary maps ∂', ∂ such that $\partial'(f) = e$, $\partial(e) = v$, an element $\alpha_{vef} \in \mathscr{G}(v)$ (its existence is a part of the definition) such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{G}(f) \arrow[r, "\mathscr{G}(\partial')"] \arrow[d, "\mathscr{G}(\partial\partial')"'] & \mathscr{G}(e) \arrow[d, "\mathscr{G}(\partial)"] \\ \mathscr{G}(v) \arrow[r, "\alpha_{vef}(\,\cdot\,)\alpha_{vef}^{-1}"] & \mathscr{G}(v) \end{tikzcd}$$

Definition 3.2. Let $(\mathscr{G}, \alpha)$ be Noohi group data on the 2-complex E. A $(\mathscr{G}, \alpha)$-*system* M on E consists of the following:

(1) For every simplex $s \in E$ a $\mathscr{G}(s)$-set M_s.

(2) For every boundary map $\partial : s \to t$ a map of $\mathscr{G}(s)$-sets $m_\partial : M_s \to \mathscr{G}(\partial)^*(M_t)$, such that.

(3) For every triple $v \in E_0$, $e \in E_1$, $f \in E_2$ and boundary maps ∂', ∂ such that $\partial'(f) = e$, $\partial(e) = v$ the following diagram commutes:

$$\begin{tikzcd} M_f \arrow[r, "m_{\partial'}"] \arrow[d, "m_{\partial\partial'}"'] & M_e \arrow[d, "m_\partial"] \\ M_v \arrow[r, "\alpha_{vef}\cdot"] & M_v \end{tikzcd}$$

Definition 3.3. A $(\mathscr{G}, \alpha)$-system is called *locally constant* if all the maps m_∂ are bijections.

Observe that $\alpha\cdot : m \mapsto \alpha m$ is a $\mathscr{G}(v)$-equivariant map $M_v \to (\alpha()\alpha^{-1})^* M_v$. Observe that there is an obvious notion of a morphism of $(\mathscr{G}, \alpha)$-systems: a collection of $\mathscr{G}(s)$-equivariant maps that commute with the m. Let us denote by $\mathrm{lcs}(E, (\mathscr{G}, \alpha))$ the category of locally constant $(\mathscr{G}, \alpha)$-systems.

Let $M \in \mathrm{lcs}(\mathscr{G}, \alpha)$ for Noohi group data $(\mathscr{G}, \alpha)$ on some 2-complex E. We define oriented graphs $E_{\le 1}$ and $M_{\le 1}$ (which will be an oriented graph *over* $E_{\le 1}$) as in [Stix 2006], but our graphs $M_{\le 1}$ are possibly infinite. For $E_{\le 1}$ the vertices are E_0 and edges E_1 such that ∂_0 (resp. ∂_1) map an edge to its target (resp. origin). For $M_{\le 1}$ the vertices are $\bigsqcup_{v \in E_0} M_v$ and edges are $\bigsqcup_{e \in E_1} M_e$ serves as the set of edges. The target/origin maps are induced by the m_∂ and the map $M_{\le 1} \to E_{\le 1}$ is the obvious one.

There is an obvious topological realization functor for graphs $|\cdot|$. By applying this functor to the above construction we get a *topological covering* (because M is locally constant) $|M_{\le 1}| \to |E_{\le 1}|$. This gives a functor

$$|\cdot_{\le 1}| : \mathrm{lcs}(E, (\mathscr{G}, \alpha)) \to \mathrm{TopCov}(|E_{\le 1}|).$$

Choosing a maximal subtree T of $|E_{\le 1}|$ gives a fiber functor $F_T : \mathrm{TopCov}(|E_{\le 1}|) \to \mathrm{Sets}$ by $(p : Y \to |E_{\le 1}|) \mapsto \pi_0(p^{-1}(|T|))$. The pair $(\mathrm{TopCov}(|E_{\le 1}|), F_T)$ is an infinite Galois category and the resulting fundamental group $\pi_1(\mathrm{Cov}(|E_{\le 1}|), F_T))$ is isomorphic to $\pi_1^{\mathrm{top}}(|E_{\le 1}|)$ (see Lemma 2.29) which is in turn isomorphic to $\mathrm{Fr}(E_1)/\langle\langle\{\vec{e} | e \in T\}^{\mathrm{Fr}(E_1)}\rangle\rangle = \mathrm{Fr}(\vec{e} | e \in E_1 \setminus T)$, where $\mathrm{Fr}(\,\cdot\,)$ denotes a free group on the given set of generators and $\langle\langle\{\vec{e} \mid e \in T\}^{Fr(E_1)}\rangle\rangle$ denotes the normal closure in $\mathrm{Fr}(E_1)$ of the subgroup generated by $\{\vec{e} \in T\}$. Here, $\vec{e}$ acts on $F_T(M)$ via

$$\pi_0(p^{-1}(|T|)) \cong \pi_0(p^{-1}(\partial_0(e))) \cong \pi_0(p^{-1}(|e|)) \cong \pi_0(p^{-1}(\partial_1(e)) \cong \pi_0(p^{-1}(|T|)).$$

As in [Stix 2006], for every $s \in E_0$ and $M \in \mathrm{lcs}(E, (\mathscr{G}, \alpha))$ we have that $F_T(M)$ can be seen canonically as a $\mathscr{G}(s)$-module by $M_s = \pi_0(p^{-1}(s)) \cong \pi_0(p^{-1}(T))$. Denote $\pi_1(E_{\le 1}, T) = \mathrm{Fr}(E_1)/\langle\langle\{\vec{e} \mid e \in T\}^{\mathrm{Fr}(E_1)}\rangle\rangle$. Putting the above together we get a functor

$$Q : \mathrm{lcs}(E, (\mathscr{G}, \alpha)) \to (*^N_{v \in E_0} \mathscr{G}(v) *^N \pi_1(E_{\le 1}, T)) - \mathrm{sets}.$$

Remark 3.4. In the setting of usual ("finite") Galois categories, it is usually enough to say that a particular morphism between two Galois categories is exact, because of the following fact [Stacks 2020, Tag 0BMV]:

Let G be a topological group. Let $F : \text{Finite}-G-\text{Sets} \to \text{Sets}$ be an exact functor with $F(X)$ finite for all X. Then F is isomorphic to the forgetful functor.

As we do not know if an analogous fact is true for infinite Galois categories, given two infinite Galois categories $(\mathcal{C}, F)$, $(\mathcal{C}', F')$ and a morphism $\phi : \mathcal{C} \to \mathcal{C}'$, we are usually more interested in checking whether $F \simeq F' \circ \phi$. If ϕ satisfies this condition, it also commutes with finite limits and arbitrary colimits. Indeed, we have a map $\operatorname{colim} \phi(X_i) \to \phi(\operatorname{colim} X_i)$ that becomes an isomorphism after applying F' (as F' and $F = F' \circ \phi$ commute with colimits) and we conclude by conservativity of F'. Similarly for finite limits.

Proposition 3.5. *Let $(E, (\mathscr{G}, \alpha))$ be a connected 2-complex with Noohi group data. Define a functor $F : \operatorname{lcs}(E, (\mathscr{G}, \alpha)) \to$ Sets in the following way: pick any simplex s and define F by $M \mapsto M_s$. Then $(\operatorname{lcs}(E, (\mathscr{G}, \alpha)), F)$ is a tame infinite Galois category.*

Moreover, the obtained functor

$$Q : \operatorname{lcs}(E, (\mathscr{G}, \alpha)) \to (*^N_{v \in E_0} \mathscr{G}(v) *^N \pi_1(E_{\leq 1}, T)) - \textit{sets}$$

satisfies $F \simeq F_{\text{forget}} \circ Q$ and maps connected objects to connected objects.

Proof. We first check conditions (1), (2) and (4) of [Bhatt and Scholze 2015, Definition 7.2.1]. Then we show that Q maps connected objects to connected objects and we use the proof of this last fact to show the condition (3).

Colimits and finite limits: they exist simplex-wise and taking limits and colimits is functorial so we get a system as candidate for a colimit/finite limit. This will be a locally constant system, as the colimit/finite limit of bijections between some G-sets is a bijection.

Each M is a disjoint union of connected objects: let us call $N \in \operatorname{lcs}(\mathscr{G}, \alpha)$ a *subsystem* of M if there exists a morphism $N \to M$ such that for any simplex s the map $N_s \to M_s$ is injective (we then identify, for any simplex s, N_s with a subset of M_s). We can intersect such subsystems in an obvious way and observe that it gives another subsystem. So for any element $a \in M_v$ there exists the smallest subsystem N of M such that $a \in N_v$. We see readily that for any vertices v, v' and $a \in M_v$, $a' \in M_{v'}$ the smallest subsystems N and N' containing one of them are either equal or disjoint (in the sense that, for each simplex s, N_s and N'_s are disjoint as subsets of M_s). It is easy to see that in this way we have obtained a decomposition of M into a disjoint union of connected objects.

F is faithful, conservative and commutes with colimits and finite limits. Observe that $\phi_s : \operatorname{lcs}(E, (\mathscr{G}, \alpha)) \ni M \mapsto M_s \in \mathscr{G}(s)-\text{Sets}$ is faithful, conservative and commutes with colimits and finite limits and $F = F_s \circ \phi_s$, where F_s is the usual forgetful functor on $\mathscr{G}(s) - \text{Sets}$.

It is obvious that $F \simeq F_{\text{forget}} \circ Q$. We are now going to show that Q preserves connected objects. Take a connected object $M \in \operatorname{lcs}(E, (\mathscr{G}, \alpha))$ and suppose that N is a nonempty subset of $F_T(M)$ stable under the action of $\pi_1(E_{\leq 1}, T)$ and $\mathscr{G}(v)$ for $v \in E_0$. Stability under the action of $\pi_1(E_{\leq 1}, T)$ shows that N can be extended to a subgraph $N_{\leq 1} \subset M_{\leq 1}$: for an edge e of $M_{\leq 1}$ we declare it to be an edge of $N_{\leq 1}$ if one of its ends touches a connected component of $p^{-1}(|T|)$ corresponding to an element of N. This is well defined, as in this case both ends touch such a component — this is because the action of $m_{\partial_1} m_{\partial_0}^{-1}$ equals the action of $\vec{e} \in \pi_1(E_{\leq 1}, T)$.

Now we want to show that it extends to 2-simplexes. This is a local question and we can restrict to simplices in the boundary of a given face $f \in E_2$. Define N_f as a preimage of N_s via any ∂ such that $\partial(f) = s$. We see that if the choice is independent of s, then we have extended N to a locally constant system. To see the independence it is enough to prove that if (vef) is a barycentric subdivision (i.e., we have ∂ and ∂' such that $\partial'(f) = e$ and $\partial(e) = v$), then $m_{\partial\partial'}^{-1}(N_v) = m_{\partial'}^{-1}(N_e)$. But from the $\mathscr{G}(v)$-invariance we have $N_v = \alpha_{vef}^{-1}(N_v)$ and so

$$m_{\partial\partial'}^{-1}(N_v) = m_{\partial\partial'}^{-1}(\alpha_{vef}^{-1}(N_v)) = m_{\partial'}^{-1} m_{\partial}^{-1}(N_v) = m_{\partial'}(N_e)$$

and thus N can be seen as an element of $\mathrm{lcs}(E, (\mathscr{G}, \alpha))$ which is a subobject of M, which contradicts connectedness of M.

To see that $\mathrm{lcs}(E, (\mathscr{G}, \alpha))$ is generated under colimits by a set of connected objects, observe that in the above proof of the fact that Q preserves connected objects, we have in fact shown the following statement.

Fact 3.6. Let $M \in \mathrm{lcs}(\mathscr{G}, \alpha)$ and let Z be a connected component of $Q(M)$. Then there exists a subsystem $W \subset M$ such that $Q(W) = Z$.

We want to show that there exists a *set* of connected objects in $\mathrm{lcs}(\mathscr{G}, \alpha)$ such that any connected object of $\mathrm{lcs}(\mathscr{G}, \alpha)$ is isomorphic to an element in that set. As an analogous fact is true in $(*_{v\in E_0}^N \mathscr{G}(v) *^N \pi_1(E_{\leq 1}, T)) - \text{sets}$, it is easy to see that it is enough to check that, for any X, Y, if $QX \simeq QY$, then $X \simeq Y$. Let $X, Y \in \mathrm{lcs}(\mathscr{G}, \alpha)$ be connected. Assume that $QX \simeq QY$. Looking at the graph of this isomorphism, we find a connected subobject $Z \subset QX \times QY$ that maps isomorphically on QX and QY via the respective projections. By the above fact, we know that there exists $W \subset X \times Y$ such that $QW = Z$. Because $F \simeq F_{\mathrm{forget}} \circ Q$ and F is conservative, we see that the projections $W \to X$ and $W \to Y$ must be isomorphisms. This shows $X \simeq Y$ as desired.

The only claim left is that $\mathrm{lcs}(E(\mathscr{G}, \alpha))$ is tame, but this follows from tameness of $(*_{v\in E_0}^N \mathscr{G}(v) *^N \pi_1(E_{\leq 1}, T)) - \text{Sets}$, the equality $F \simeq F_{\mathrm{forget}} \circ Q$ and the fact that Q maps connected objects to connected objects. $\square$

Let us denote by $\pi_1(E, \mathscr{G}, s)$ the fundamental group of the infinite Galois category $(\mathrm{lcs}(E, \mathscr{G}), F_s)$. The proposition above tells us that there is a continuous map of Noohi groups with dense image $*_{v\in E_0}^N \mathscr{G}(v) *^N \pi_1(E_{\leq 1}, T)) \to \pi_1(E, \mathscr{G}, s)$. We now proceed to describe the kernel.

Recall that $\pi_1(E_{\leq 1}, T) = \mathrm{Fr}(E_1)/\langle\langle \{\vec{e} \mid e \in T\}^{\mathrm{Fr}(E_1)} \rangle\rangle$.

Theorem 3.7 (abstract Seifert–van Kampen theorem for infinite Galois categories). *E be a connected 2-complex with group data $(\mathscr{G}, \alpha)$. With notations as above, the functor Q induces an isomorphism of Noohi groups*

$$(*_{v\in E_0}^N \mathscr{G}(v) *^N \pi_1(E_{\leq 1}, T)/\overline{H})^{\mathrm{Noohi}} \to \pi_1(E, \mathscr{G}, s)$$

where $\overline{H}$ is the closure of the group

$$H = \left\langle\!\!\left\langle \begin{array}{c} \mathscr{G}(\partial_1)(g)\vec{e} = \vec{e}\mathscr{G}(\partial_0)(g) \\ \overrightarrow{(\partial_2 f)}\alpha^{(f)}_{102}(\alpha^{(f)}_{120})^{-1}\overrightarrow{(\partial_0 f)}\alpha^{(f)}_{210}(\alpha^{(f)}_{201})^{-1}(\overrightarrow{(\partial_1 f)})^{-1}\alpha^{(f)}_{021}(\alpha^{(f)}_{012})^{-1} \end{array} \;\middle|\; \begin{array}{c} e\in E_1,\, g\in\mathscr{G}(e) \\ f\in E_2 \end{array} \right\rangle\!\!\right\rangle$$

where $\langle\!\langle - \rangle\!\rangle$ denotes the normal closure of the subgroup generated by the indicated elements and the α come from the definition of a $(\mathscr{G}, \alpha)$-system for each given f.

Proof. The same proof as the proof of [Stix 2006, Theorem 3.2(2)] shows that Q induces an equivalence of categories between the infinite Galois categories $(\mathrm{lcs}(E, \mathscr{G}), F_s)$ and the full subcategory of objects of $*^N_{v\in E_0}\mathscr{G}(v) *^N \pi_1(E_{\le 1}, T) - \mathrm{Sets}$ on which H acts trivially. We conclude by Observation 2.28. $\square$

Remark 3.8. It is important to note that we can replace free Noohi products by free topological products in the statement above, as we take the Noohi completion of the quotient anyway. More precisely, the canonical map

$$(*^{\mathrm{top}}_{v\in E_0}\mathscr{G}(v) *^{\mathrm{top}} \pi_1(E_{\le 1}, T)/\overline{H}_1)^{\mathrm{Noohi}} \to (*^N_{v\in E_0}\mathscr{G}(v) *^N \pi_1(E_{\le 1}, T)/\overline{H})^{\mathrm{Noohi}}$$

is an isomorphism, where H_1 is the normal closure in $*^{\mathrm{top}}_{v\in E_0}\mathscr{G}(v) *^{\mathrm{top}} \pi_1(E_{\le 1}, T)$ of a group having the same generators as H. This is because the categories of $G - \mathrm{Sets}$ are the same for those two Noohi groups.

Fact 3.9. The topological free product $*^{\mathrm{top}}_i G_i$ of topological groups has as an underlying space the free product of abstract groups $*_i G_i$. This follows from the original construction of Graev [1948].

3B. *Application to the pro-étale fundamental group.*

Descent data. Let $T_\bullet$ be a 2-complex in a category $\mathscr{C}$ and let $\mathscr{F} \to \mathscr{C}$ be a category fibered over $\mathscr{C}$, with $\mathscr{F}(S)$ as a category of sections above the object S.

Definition 3.10. The category $\mathrm{DD}(T_\bullet, \mathscr{F})$ of *descent data* for $\mathscr{F}/\mathscr{C}$ relative $T_\bullet$ has as objects pairs (X', ϕ) where $X' \in \mathscr{F}(T_0)$ and ϕ is an isomorphism $\partial_0^* X' \xrightarrow{\sim} \partial_1^* X'$ in $\mathscr{F}(T_1)$ such that the *cocycle condition* holds, i.e., the following commutes in $\mathscr{F}(T_2)$:

$$\begin{tikzcd} v_2^* X' \arrow[rr, "\partial_0^*\phi"] \arrow[dr, "\partial_1^*\phi"'] & & v_1^* X' \arrow[dl, "\partial_2^*\phi"] \\ & v_0^* X' & \end{tikzcd}$$

Morphisms $F : (X', \phi) \to (Y', \psi)$ in $\mathrm{DD}(T_\bullet, \mathscr{F})$ are morphisms $F : X' \to Y'$ in $\mathscr{F}(T_0)$ such that its two pullbacks $\partial_0^* f$ and $\partial_1^* f$ are compatible with ϕ, ψ, i.e., $\partial_1^* f \circ \phi = \psi \circ \partial_0^* f$.

Let $h : S' \to S$ be a map of schemes. There is an associated 2-complex of schemes

$$S_\bullet(h) : S' \leftleftarrows S' \times_S S' \mathrel{\substack{\leftarrow\\[-0.3em]\leftarrow\\[-0.3em]\leftarrow}} S' \times_S S' \times_S S'.$$

The value of ∂_i is the projection under omission of the i-th component. We abbreviate $\mathrm{DD}(S_\bullet(h), \mathscr{F})$ by $\mathrm{DD}(h, \mathscr{F})$. Observe that h^* gives a functor $h^* : \mathscr{F}(S) \to \mathrm{DD}(h, \mathscr{F})$.

Definition 3.11. In the above context $h : S' \to S$ is called an *effective descent* morphism for $\mathscr{F}$ if h^* is an equivalence of categories.

Proposition 3.12 [Lavanda 2018, Proposition 1.16]. *Let $g : S' \to S$ be a proper, surjective morphism of finite presentation, then g is a morphism of effective descent for geometric coverings.*

Proof. This was proven by Lavanda and relies on the results of [Rydh 2010]. More precisely, this follows from Proposition 5.4 and Theorem 5.19 of [loc. cit.], then checking that the obtained algebraic space is a scheme (using étaleness and separatedness, see [Stacks 2020, Tag 0417]) and that it still satisfies the valuative criterion (see Lemma 2.40). □

Discretization of descent data. We would like to apply the procedure described in [Stix 2006, Section 4.3] but to the pro-étale fundamental group. However, in the classical setting of Galois categories, given a category $\mathcal{C}$ and functors $F, F' : \mathcal{C} \to \mathrm{Sets}$ such that $(\mathcal{C}, F)$ and $(\mathcal{C}, F')$ are Galois categories (i.e., F, F' are fiber functors), there exists an isomorphism (not unique) between F and F'. Choosing such an isomorphism is called "choosing a path" between F and F'. However, it is not clear whether an analogous statement is true for tame infinite Galois categories as the proof does not carry over to this case (see the proof of [Stacks 2020, Lemma 0BN5] or in [SGA 1 1971] — these proofs are essentially the same and rely on the pro-representability result of Grothendieck [1960, Proposition A.3.3.1]).

Question 3.13. Let $\mathcal{C}$ be a category and $F, F' : \mathcal{C} \to \mathrm{Sets}$ be two functors such that $(\mathcal{C}, F)$ and $(\mathcal{C}, F')$ are tame infinite Galois categories. Is it true that F and F' are isomorphic?

As we do not know the answer to this question, we have to make an additional assumption when trying to discretize the descent data. Fortunately, it will always be satisfied in the geometric setting, which is our main case of interest.

Definition 3.14. Let $(\mathcal{C}, F)$, $(\mathcal{C}', F')$ be two infinite Galois categories and let $\phi : \mathcal{C} \to \mathcal{C}'$ be a functor. We say that ϕ is *compatible* if there exists an isomorphism of functors $F \simeq F' \circ \phi$.

Let $\mathscr{F} \to \mathscr{C}$ be fibered in tame infinite Galois categories. More precisely, we have a notion of connected objects in $\mathscr{C}$ and any $T \in \mathscr{C}$ is a coproduct of connected components. Over connected objects $\mathscr{F}$ takes values in tame infinite Galois categories (i.e., over a connected $Y \in \mathscr{C}$ there exists a functor $F_Y : \mathscr{F}(Y) \to \mathrm{Sets}$ such that $(\mathscr{F}(Y), F_Y)$ is a tame infinite Galois category but we do not fix the functor).

Definition 3.15. Let $T_\bullet$ be a 2-complex in $\mathscr{C}$. Let $E = \pi_0(T_\bullet)$ be its 2-complex of connected components: the 2-complex in Sets built by degree-wise application of the connected component functor. We will say that $T_\bullet$ is a *compatible* 2-complex if one can fix fiber functors F_s of $\mathscr{F}(s)$ for each simplex $s \in E$ such that $(\mathscr{F}(s), F_s)$ is tame and for any boundary map $\partial : s \to s'$ there exists an isomorphism of fiber functors $F_s \circ T(\partial)^* \xrightarrow{\sim} F_{s'}$.

The 2-complexes that will appear in the (geometric) applications below will always be compatible. From now on, we will assume all 2-complexes to be compatible, even if not stated explicitly. Let $T_\bullet$ be a compatible 2-complex in $\mathscr{C}$. Fix fiber functors F_s and isomorphisms between them as in the definition of a compatible 2-complex. For any ∂, denote the fixed isomorphism by $\vec{\partial}$. For a 2-simplex (vef) of the barycentric subdivision with $\partial' : f \to e$ and $\partial : e \to v$ we define

$$\alpha_{vef} = \vec{\partial}'\vec{\partial}(\overrightarrow{\partial\partial'})^{-1}$$

or, more precisely,

$$\alpha_{vef} = T(\partial)(\vec{\partial}')\vec{\partial}(\overrightarrow{\partial\partial'})^{-1} \in \mathrm{Aut}(F_v) = \pi_1(\mathscr{F}(s), F_s).$$

We define Noohi group data $(\mathscr{G}, \alpha)$ on E in the following way: $\mathscr{G}(s) = \pi_1(\mathscr{F}(s), F_s)$ for any simplex $s \in E$ and to $\partial : s \to s'$ is associated $\mathscr{G}(\partial) : \pi_1(\mathscr{F}(s), F_s) \xrightarrow{T(\partial)^*} \pi_1(\mathscr{F}(s'), F_s \circ T(\partial)^*) \xrightarrow{\vec{\partial}(\ )\vec{\partial}^{-1}} \pi_1(\mathscr{F}(s'), F_{s'})$. We define elements α as described above and we easily check that this gives Noohi group data.

Proposition 3.16. *The choice of functors F_s and the choice of $\vec{\partial}$ as above fix a functor*

$$\mathrm{discr} : \mathrm{DD}(T_\bullet, \mathscr{F}) \to \mathrm{lcs}(E, (\mathscr{G}, \alpha))$$

which is an equivalence of categories.

Proof. Given a descent datum (X', ϕ) relative $T_\bullet$ we have to attach a locally constant $(\mathscr{G}, \alpha)$-system on E in a functorial way. For $v \in E_0$, $e \in E_1$ and $f \in E_2$, the definition of suitable $\mathscr{G}(v)$ (or $\mathscr{G}(e)$ or $\mathscr{G}(f)$) sets and maps m_∂ between them can be given by the same formulas as in [Stix 2006, Proposition 4.4] and also the same computations as in [loc. cit., Proposition 4.4] show that we obtain an element of $\mathrm{lcs}(E, (\mathscr{G}, \alpha))$. Again, the reasoning of [loc. cit., Proposition 4.4] gives a functor in the opposite direction: given $M \in \mathrm{lcs}(E, (\mathscr{G}, \alpha))$ we define $X' \in \mathscr{F}(T_0) = \prod_{v \in E_0} \mathscr{F}(v)$ as $X'_{|v}$ corresponding to M_v for all $v \in E_0$. Maps from edges to vertices define a map $\phi : T(\partial_0)^* X' \to T(\partial_1)^* X'$ and to check the cocycle condition one reverses the argument of the proof that discr gives a locally constant system. □

To apply the last proposition we need to know that the compatibility condition holds in the setting we are interested in.

Lemma 3.17 [Bhatt and Scholze 2015, Lemma 7.4.1]. *Let $f : X' \to X$ be a morphism of two connected locally topologically noetherian schemes and let $\bar{x}', \bar{x}$ be geometric points on X', X, correspondingly. Then the functor $f^* : \mathrm{Cov}_X \to \mathrm{Cov}_{X'}$ is a compatible functor between infinite Galois categories* $(\mathrm{Cov}_X, F_{\bar{x}})$ *and* $(\mathrm{Cov}_{X'}, F_{\bar{x}'})$, *i.e., the functors $F_{\bar{x}}$ and $F_{\bar{x}'} \circ f^*$ are isomorphic.*

Proof. Looking at the image of $\bar{x}'$ (as a geometric point) on X, we reduce to the case when both $\bar{x}'$ and $\bar{x}$ lie on the same scheme X. In that case we proceed as in the second part of the proof of [loc. cit., Lemma 7.4.1]. □

Corollary 3.18 [Bhatt and Scholze 2015, Lemma 7.4.1]. *Let X be a connected topologically noetherian scheme. Let $\bar{x}_1, \bar{x}_2$ be two geometric points on X. Then there is an isomorphism $\pi_1^{\text{proét}}(X, \bar{x}_1) \simeq \pi_1^{\text{proét}}(X, \bar{x}_2)$. It is unique* (*only*) *up to an inner automorphism.*

The above results combine to recover the analogue of [Stix 2006, Corollary 5.3] in the pro-étale setting.

Corollary 3.19. *Let $h : S' \to S$ be an effective descent morphism for geometric coverings. Assume that S is connected and S, S', $S' \times_S S'$, $S' \times_S S' \times_S S'$ are locally topologically noetherian. Let $S' = \bigsqcup_v S'_v$ be the decomposition into connected components. Let $\bar{s}$ be a geometric point of S, let $\bar{s}(t)$ be a geometric point of the simplex $t \in \pi_0(S_\bullet(h))$, and let T be a maximal tree in the graph $\Gamma = \pi_0(S_\bullet(h))_{\leq 1}$. For every boundary map $\partial : t \to t'$ let $\gamma_{t',t} : \bar{s}(t') \to S_\bullet(h)(\partial)\bar{s}(t)$ be a fixed path (i.e., an isomorphism of fiber functors as in Lemma 3.17). Then canonically with respect to all these choices*

$$\pi_1^{\text{proét}}(S, \bar{s}) \cong \left(\left(*^N_{v\in E_0}\pi_1^{\text{proét}}(S'_v, \bar{s}(v)) *^N \pi_1(\Gamma, T)\right)/\overline{H}\right)^{\text{Noohi}}$$

where H is the normal subgroup generated by the cocycle and edge relations

$$\pi_1^{\text{proét}}(\partial_1)(g)\vec{e} = \vec{e}\pi_1^{\text{proét}}(\partial_0)(g), \tag{1}$$

$$\overrightarrow{(\partial_2 f)}\alpha_{102}^{(f)}(\alpha_{120}^{(f)})^{-1}\overrightarrow{(\partial_0 f)}\alpha_{210}^{(f)}(\alpha_{201}^{(f)})^{-1}(\overrightarrow{(\partial_1 f)})^{-1}\alpha_{021}^{(f)}(\alpha_{012}^{(f)})^{-1} = 1, \tag{2}$$

for all parameter values $e \in S_1(h)$, $g \in \pi_1^{\text{proét}}(e, \bar{s}(e))$, and $f \in S_2(h)$. The map $\pi_1^{\text{proét}}(\partial_i)$ uses the fixed path $\gamma_{\partial_i(e),e}$ and $\alpha_{ijk}^{(f)}$ is defined using $v \in S_0(h)$ and $e \in S_1(h)$ determined by $v_i(f) = v$, $\{\partial_0(e), \partial_1(e)\} = \{v_i(f), v_j(f)\}$ as

$$\alpha_{ijk}^{(f)} = \gamma_{v,e}\gamma_{e,f}\gamma_{v,f}^{-1} \in \pi_1^{\text{proét}}(v, \bar{s}(v)).$$

Remark 3.20. Similarly as in Remark 3.8, we could replace $*^N$ by $*^{\text{top}}$ in the above, as we take the Noohi completion of the whole quotient anyway.

Remark 3.21. We will often use Corollary 3.19 for h — the normalization map (or similar situations), where the connected components S'_v are normal. In this case $\pi_1^{\text{proét}}(S'_v, \bar{s}(v)) = \pi_1^{\text{ét}}(S'_v, \bar{s}_v)$. This implies that $\pi_1^{\text{proét}}(\partial_1)$ factorizes through the profinite completion of $\pi_1^{\text{proét}}(e, \bar{s}(e))$, which can be identified with $\pi_1^{\text{ét}}(e, \bar{s}(e))$. Moreover, the map $\pi_1^{\text{proét}}(e, \bar{s}(e)) \to \pi_1^{\text{ét}}(e, \bar{s}(e))$ has dense image and, in the end, we take the closure $\overline{H}$ of H. The upshot of this discussion is that in the definition of generators of H we might consider $g \in \pi_1^{\text{ét}}(e, \bar{s}(e))$ instead of $g \in \pi_1^{\text{proét}}(e, \bar{s}(e))$ and $\pi_1^{\text{ét}}(\partial_i)$ instead of $\pi_1^{\text{proét}}(\partial_i)$, $i \in \{0, 1\}$, i.e.,

$$\pi_1^{\text{proét}}(S, \bar{s}) \cong \left(\left(*^{\text{top}}_{v\in E_0}\pi_1^{\text{ét}}(S'_v, \bar{s}(v)) *^{\text{top}} \pi_1(\Gamma, T)\right)/\overline{H}\right)^{\text{Noohi}}$$

where H is the normal subgroup generated by

$$\pi_1^{\text{ét}}(\partial_1)(g)\vec{e}\pi_1^{\text{ét}}(\partial_0)(g)^{-1}\vec{e}^{-1} \quad \text{for all } e \in S_1(h),\ g \in \pi_1^{\text{ét}}(e, \bar{s}(e)) \tag{R_1}$$

and

$$\overrightarrow{(\partial_2 f)}\alpha_{102}^{(f)}(\alpha_{120}^{(f)})^{-1}\overrightarrow{(\partial_0 f)}\alpha_{210}^{(f)}(\alpha_{201}^{(f)})^{-1}(\overrightarrow{(\partial_1 f)})^{-1}\alpha_{021}^{(f)}(\alpha_{012}^{(f)})^{-1} \quad \text{for all } f \in S_2(h). \tag{R_2}$$

Let us move on to some applications.

Ordered descent data. Let $\mathscr{F}$ be a category fibered over $\mathscr{C}$ (with a fixed cleavage, for convenience). Assume that $\mathscr{C}$ is some subcategory of the category of locally topologically noetherian schemes with the property that finite fiber products in $\mathscr{C}$ are the same as the finite fiber products as schemes. Let $h = \bigsqcup_{i\in I} h_i : S' = \bigsqcup_i S'_{i\in I} \to S$ be a morphism of schemes and let $<$ be a total order on the set of indices I. Let $S^<_\bullet(h) \subset S_\bullet(h)$ be the open and closed sub-2-complex of schemes in $\mathscr{C}$ of ordered partial products

$$S^<_0(h) = S', \quad S^<_1(h) = \bigsqcup_{i<j} S'_i \times_S S'_j, \quad S^<_2(h) = \bigsqcup_{i<j<k} S'_i \times_S S'_j \times_S S'_k.$$

Proposition 3.22. *Let $h = \bigsqcup_{i\in I} h_i : S' = \bigsqcup_i S'_{i\in I} \to S$ be a morphism of schemes such that, for every $i, j \in I$, the maps induced by the diagonal morphisms $\Delta_i^* : \mathscr{F}(S'_i \times_S S'_i) \to \mathscr{F}(S'_i)$ and $(\Delta_i \times \mathrm{id}_{S'_j})^* : \mathscr{F}(S'_i \times_S S'_j \times_S S'_i) \to \mathscr{F}(S'_i \times S'_j)$ are fully faithful. Then the natural open and closed immersion $S^<_\bullet(h) \hookrightarrow S_\bullet(h)$ induces an equivalence of categories*

$$\mathrm{DD}(h, \mathscr{F}) \xrightarrow{\cong} \mathrm{DD}(S^<_\bullet(h), \mathscr{F}).$$

Proof. For the problem at hand, we can and do replace $\mathscr{F}$ by an equivalent category that admits a splitting cleavage (i.e., the associated pseudofunctor is a functor). Let $Y \in \mathscr{F}(S'_i)$ and consider $\partial_0^* Y, \partial_1^* Y \in \mathscr{F}(S'_i \times_S S'_i)$ obtained via maps induced by the projections $\mathscr{F}(S'_i) \to \mathscr{F}(S'_i \times_S S'_i)$. We first claim that there is *exactly one* isomorphism $\partial_0|^*_{S_i \times_S S_i} Y \to \partial_1|^*_{S_i \times_S S_i} Y$ as in the definition of descent data. Observe that $\Delta_i^* \partial_0^* Y = Y$, $\Delta_i^* \partial_1^* Y = Y$ and from the assumption any isomorphism $\phi : \partial_0^* Y|_{S_i} \to \partial_1^* Y|_{S_i}$ corresponds to precisely one isomorphism $\psi \in \mathrm{Hom}_{S_i}(Y|_{S_i}, Y|_{S_i})$. Pulling back the cocycle condition via the diagonal $\Delta_{2,i}^* : \mathscr{F}(S'_i \times_S S'_i \times_S S'_i) \to \mathscr{F}(S'_i)$ we get $\psi = \mathrm{id}_{Y|_{S_i}}$, so there is at most one map ϕ as above. Moreover, our assumptions imply that $\Delta_{2,i}^*$ is fully faithful as well, which shows that $\phi : \partial_0^* Y|_{S_i} \to \partial_1^* Y|_{S_i}$ corresponding to $\mathrm{id}_{Y|_{S_i}}$ will satisfy the condition. A similar reasoning shows that if we have ϕ_{ij} specified for $i < j$, then ϕ_{ji} is uniquely determined and if the ϕ_{ij} satisfy the cocycle condition on S_{ijk} for $i < j < k$, then the ϕ_{ij} together with the ϕ_{ji} obtained will satisfy the cocycle condition on any $S_{\alpha\beta\gamma}$, $\alpha, \beta, \gamma \in \{i, j, k\}$. □

Observation 3.23. If the map of schemes $S'_i \to S$ is injective, i.e., if the diagonal map $S'_i \to S'_i \times_S S'_i$ is an isomorphism, then the assumptions of the proposition are satisfied.

Two examples.

Example 3.24. Let k be a field and C be $\mathbb{P}^1_k$ with two k-rational closed points p_0 and p_1 glued (see [Schwede 2005] for results on gluing schemes). Denote by p the node (i.e., the image of the p_i in C). We want to compute $\pi_1^{\text{proét}}(C)$. By the definition of C, we have a map $h : \widetilde{C} = \mathbb{P}^1 \to C$ (which is also the normalization). It is finite, so it is an effective descent map for geometric coverings. Thus, we can use the van Kampen theorem. This goes as follows:

- Check that $\widetilde{C} \times_C \widetilde{C} \simeq \widetilde{C} \sqcup p_{01} \sqcup p_{10}$ as schemes over C, where $p_{\alpha\beta}$ are equal to $\mathrm{Spec}(k)$ and map to the node of C via the structural map. This can be done by checking that $\mathrm{Hom}_C(Y, \widetilde{C} \sqcup p_{01} \sqcup p_{10}) \simeq \mathrm{Hom}_C(Y, \widetilde{C}) \times \mathrm{Hom}_C(Y, \widetilde{C})$.

- Similarly, check that $\widetilde{C} \times_C \widetilde{C} \times_C \widetilde{C} \simeq \widetilde{C} \sqcup p_{001} \sqcup p_{010} \sqcup p_{011} \sqcup p_{100} \sqcup p_{101} \sqcup p_{110}$, where the projection $\widetilde{C} \times_C \widetilde{C} \times_C \widetilde{C} \to \widetilde{C} \times_C \widetilde{C}$ omitting the first factor maps p_{abc} to p_{bc} and so on.

- We fix a geometric point $\bar{b} = \operatorname{Spec}(\bar{k})$ over the base scheme $\operatorname{Spec}(k)$ and fix geometric points $\bar{p}_0$ and $\bar{p}_1$ over p_0 and p_1 that map to $\bar{b}$. Then we fix geometric points on $\widetilde{C}$, $p_{01}, p_{10} \subset \widetilde{C} \sqcup p_{01} \sqcup p_{10} \simeq \widetilde{C} \times_C \widetilde{C}$ in a compatible way and similarly for connected components of $\widetilde{C} \times_C \widetilde{C} \times_C \widetilde{C}$ (i.e., let us say that $\bar{p}_{\alpha\beta\gamma} \mapsto \bar{p}_\alpha$ via v_0 and $\bar{p}_{\alpha\beta} \mapsto \bar{p}_\alpha$). We fix a path γ from $\bar{p}_0$ to $\bar{p}_1$ that becomes trivial on $\operatorname{Spec}(k)$ via the structural map (this can be done by viewing $\bar{p}_0$ and $\bar{p}_1$ as geometric points on $\widetilde{C}_{\bar{k}}$, choosing the path on $\widetilde{C}_{\bar{k}}$ first and defining γ to be its image). Let $\bar{p}$ be the fixed geometric point on C given by the image of $\bar{p}_0$ (or, equivalently, $\bar{p}_1$).

- We want to use Corollary 3.19 to compute $\pi_1^{\text{proét}}(C, \bar{p})$. We choose $\bar{p}_0$ as the base point $\bar{s}(\widetilde{C})$ for $\widetilde{C} \in \pi_0(S_0(h))$, $\widetilde{C} \in \pi_0(S_1(h))$ and $\widetilde{C} \in \pi_0(S_2(h))$. Then for any $t, t' \in \pi_0(S_\bullet(h))$ and the boundary map $\partial : t \to t'$, we use either the identity or γ to define $\gamma_{t',t} : \bar{s}(t') \to S_\bullet(h)(\partial)\bar{s}(t)$ as all the points $\bar{p}_{abc}$ map ultimately either to $\bar{p}_0$ or $\bar{p}_1$.

With this setup, the $\alpha_{ijk}^{(f)}$ (defined as in Corollary 3.19) are trivial for any f and so the relation (2) in this corollary reads $\overrightarrow{(\partial_2 f)}\overrightarrow{(\partial_0 f)}(\overrightarrow{(\partial_1 f)})^{-1} = 1$. Applying this to different faces $f \in \pi_0(\widetilde{C} \times_C \widetilde{C} \times_C \widetilde{C})$ gives that the image of $\pi_1(\Gamma, T) \simeq \mathbb{Z}^{*3}$ in $\pi_1^{\text{proét}}(C, \bar{p})$ is generated by a single edge (in our case only one maximal tree can be chosen – containing a single vertex). The choice of paths made guarantees $\pi_1^{\text{proét}}(\partial_0)(g) = \pi_1^{\text{proét}}(\partial_1)(g)$ in $\pi_1^{\text{proét}}(\widetilde{C}, \bar{p}_0)$ for any $g \in \pi_1^{\text{proét}}(p_{ab}, \bar{p}_{ab}) = \operatorname{Gal}(k)$. So relation (1) in Corollary 3.19 implies that the image of $\pi_1^{\text{proét}}(\widetilde{C}, \bar{p}_0) \simeq \operatorname{Gal}(k)$ in $\pi_1^{\text{proét}}(C, \bar{p}_0)$ commutes with the elements of the image of $\pi_1(\Gamma, T)$. Putting this together we get

$$\begin{aligned}
&\pi_1^{\text{proét}}(C, \bar{p}) \\
&\quad\simeq \left((\pi_1^{\text{proét}}(\widetilde{C}, \bar{p}_0) *^{\text{top}} \pi_1(\Gamma, T)) / \langle\langle \pi_1^{\text{proét}}(\partial_1)(g)\vec{e} = \vec{e}\pi_1^{\text{proét}}(\partial_2)(g), \overrightarrow{(\partial_2 f)}\overrightarrow{(\partial_0 f)}(\overrightarrow{(\partial_1 f)})^{-1} = 1 \rangle\rangle \right)^{\text{Noohi}} \\
&\quad\simeq (\operatorname{Gal}_k \times \mathbb{Z})^{\text{Noohi}} \\
&\quad= \operatorname{Gal}_k \times \mathbb{Z}.
\end{aligned}$$

Example 3.25. Let $X_1, \dots, X_m$ be geometrically connected normal curves over a field k and let $Y_{m+1}, \dots, Y_n$ be nodal curves over k as in Example 3.24. Let $x_i : \operatorname{Spec}(k) \to X_i$ be rational points and let y_j denote the node of Y_j. Let $X := \cup_\bullet X_i \cup_\bullet Y_j$ be a scheme over k obtained via gluing of the X_i and Y_j along the rational points x_i and y_j (in the sense of [Schwede 2005]). The notation $\cup_\bullet$ denotes gluing along the obvious points. The point of gluing gives a rational point $x : \operatorname{Spec}(k) \to X$. We choose a geometric point $\bar{b} = \operatorname{Spec}(\bar{k})$ over the base $\operatorname{Spec}(k)$ and choose a geometric point $\bar{x}$ over x such that it maps to $\bar{b}$. The maps $X_i \to X$ and $Y_j \to X$ are closed immersions (this is basically [Schwede 2005, Lemma 3.8]). We also get geometric points $\bar{x}_i$ and $\bar{y}_j$ over x_i and y_j that map to $\bar{b}$ as well. Denote $\overline{X}_i = (X_i)_{\bar{k}}$. Let $\operatorname{Gal}_{k,i} = \pi_1^{\text{ét}}(x_i, \bar{x}_i)$. It is a copy of Gal_k in the sense that the induced map $\pi_1^{\text{ét}}(x_i, \bar{x}_i) \to \pi_1^{\text{ét}}(\operatorname{Spec}(k), \bar{b})$ is an isomorphism. Let us denote by $\iota_i : \operatorname{Gal}_k \to \operatorname{Gal}_{k,i}$ the inverse of this isomorphism. The group $\pi_1^{\text{ét}}(x_i, \bar{x}_i)$ acts on $\pi_1^{\text{ét}}(\overline{X}_i, \bar{x}_i)$ and allows to write $\pi_1^{\text{ét}}(X_i, \bar{x}_i) \simeq \pi_1^{\text{ét}}(\overline{X}_i, \bar{x}_i) \rtimes \operatorname{Gal}_{k,i}$.

After some computations (as in the previous example), using Corollary 3.19 and Example 3.24, one gets

$$\pi_1^{\text{proét}}(X,\bar{x}) \simeq \left(*^N_{1\le i\le m}(\pi_1^{\text{ét}}(\overline{X}_i,\bar{x}_i)\rtimes\mathrm{Gal}_{k,i})*^N_{m+1\le j\le n}(\mathbb{Z}\times\mathrm{Gal}_{k,j})/\langle\langle\iota_i(\sigma)=\iota_{i'}(\sigma)\,|\,\sigma\in\mathrm{Gal}_k, i,i'=1,\ldots,n\rangle\rangle\right)^{\text{Noohi}}.$$

Let us describe the category of group-sets:

$$\begin{aligned}
&\left(*^N_{1\le i\le m}(\pi_1^{\text{ét}}(\overline{X}_i,\bar{x}_i)\rtimes\mathrm{Gal}_{k,i})*^N_{m+1\le j\le n}(\mathbb{Z}\times\mathrm{Gal}_{k,j})/\langle\langle\iota_i(\sigma)=\iota_{i'}(\sigma)\,|\,\sigma\in\mathrm{Gal}_k, i,i'=1,\ldots,n\rangle\rangle\right)^{\text{Noohi}}-\text{Sets}\\
&\quad\simeq\left\{S\in\left(*^N_{1\le i\le m}(\pi_1^{\text{ét}}(\overline{X}_i,\bar{x}_i)\rtimes\mathrm{Gal}_{k,i})*^N_{m+1\le j\le n}(\mathbb{Z}\times\mathrm{Gal}_{k,j})\right)^{\text{Noohi}}-\text{Sets}\,|\,\forall_{i,i',\sigma}\forall_{s\in S}\iota_i(\sigma)\cdot s=\iota_{i'}(\sigma)\cdot s\right\}\\
&\quad\simeq\left\{S\in\left(*^N_{1\le i\le m}\pi_1^{\text{ét}}(\overline{X}_i,\bar{x}_i)*^N\mathbb{Z}^{*n-m}*^N\mathrm{Gal}_k\right)^{\text{Noohi}}-\text{Sets}\,|\,(\spadesuit)\right\}
\end{aligned}$$

where the condition $(\spadesuit)$ reads

$$\forall_{\sigma\in\mathrm{Gal}_k,s\in S,1\leqslant i\leqslant m}\forall_{\gamma\in\pi_1^{\text{ét}}(\overline{X}_i,\bar{x}_i),w\in\mathbb{Z}^{*n-m}}\big(\sigma\cdot(\gamma\cdot s)={}^{\sigma}\gamma\cdot(\sigma\cdot s)\text{ and }\sigma\cdot(w\cdot s)=w\cdot(\sigma\cdot s)\big).$$

We have used Observation 2.28 and Lemma 3.26 below.

Lemma 3.26. *Let K and Q be topological groups and assume we have a continuous action $K\times Q\to K$ respecting multiplication in K. Then $K\rtimes Q$ with the product topology (on $K\times Q$) is a topological group and there is an isomorphism*

$$K*^{\text{top}}Q/\langle\langle qkq^{-1}={}^{q}k\rangle\rangle\to K\rtimes Q.$$

Proof. That $K\rtimes Q$ becomes a topological group is easy from the continuity assumption of the action. The isomorphism is obtained as follows: from the universal property we have a continuous homomorphism $K*^{\text{top}}Q\to K\rtimes Q$ and the kernel of this map is the smallest normal subgroup containing the elements $qkq^{-1}({}^{q}k)^{-1}$ (this follows from the fact that the underlying abstract group of $K*^{\text{top}}Q$ is the abstract free product of the underlying abstract groups, similarly for $K\rtimes Q$ and that we know the kernel in this case). So we have a continuous map that is an isomorphism of abstract groups. We have to check that the inverse map $K\rtimes Q\ni kq\mapsto kq\in K*^{\text{top}}Q/\langle\langle qkq^{-1}={}^{q}k\rangle\rangle$ is continuous. It is enough to check that the map $K\times Q\ni(k,q)\mapsto kq\in K*^{\text{top}}Q$ (of topological spaces) is continuous, but this follows from the fact that the maps $K\to K*^{\text{top}}Q$ and $Q\to K*^{\text{top}}Q$ are continuous and that the multiplication map $(K*^{\text{top}}Q)\times(K*^{\text{top}}Q)\to K*^{\text{top}}Q$ is continuous. □

Let us also state a technical lemma concerning the "functoriality" of the van Kampen theorem. It is important that the diagram formed by the schemes X_1, X_2, $\widetilde{X}$, $\widetilde{X}_1$ in the statement is cartesian.

Lemma 3.27. *Let $f:X_1\to X_2$ be a morphism of connected schemes and $h:\widetilde{X}\to X_2$ be a morphism of schemes. Denote by $h_1:\widetilde{X}_1\to X_1$ the base-change of h via f. Assume that h and h_1 are effective descent morphisms for geometric coverings and that local topological noetherianity assumptions are satisfied for the schemes involved as in the statement of Corollary 3.19. Assume that for any connected component $W\in\pi_0(S_\bullet(h))$, the base-change W_1 of W via f is connected. Choose the geometric points on $W_1\in\pi_0(S_\bullet(h_1))$ and paths between the obtained fiber functors as in Corollary 3.19 and choose the* *geometric points and paths on $W \in \pi_0(S_\bullet(h))$ as the images of those chosen for $\widetilde{X}_1$. Identify the graphs $\Gamma = \pi_0(S_\bullet(h))_{\leqslant 1}$ and $\Gamma_1 = \pi_0(S_\bullet(h_1))_{\leqslant 1}$ (it is possible thanks to the assumption made) and choose a maximal tree T in Γ. Using the above choices, use Corollary 3.19. to write the fundamental groups*

$$\pi_1^{\text{proét}}(X_1) \simeq \big((*^{\text{top}}_{W\in\pi_0(\widetilde{X})}\pi_1^{\text{proét}}(W_1)) *^{\text{top}} \pi_1(\Gamma_1, T)/\langle R'\rangle\big)^{\text{Noohi}}$$

and

$$\pi_1^{\text{proét}}(X_2) \simeq \big((*^{\text{top}}_{W\in\pi_0(\widetilde{X})}\pi_1^{\text{proét}}(W)) *^{\text{top}} \pi_1(\Gamma, T)/\langle R\rangle\big)^{\text{Noohi}}.$$

Then the map of fundamental groups $\pi_1^{\text{proét}}(f) : \pi_1^{\text{proét}}(X_1) \to \pi_1^{\text{proét}}(X_2)$ is the Noohi completion of the map

$$\big((*^{\text{top}}_{W\in\pi_0(\widetilde{X})}\pi_1^{\text{proét}}(W_1)) *^{\text{top}} \pi_1(\Gamma_1, T)\big)/\langle R'\rangle \to \big(*^{\text{top}}_{W\in\pi_0(\widetilde{X})}\pi_1^{\text{proét}}(W) *^{\text{top}} \pi_1(\Gamma, T)\big)/\langle R\rangle,$$

which is induced by the maps $\pi_1^{\text{proét}}(W_1) \to \pi_1^{\text{proét}}(W)$ and the identity on $\pi_1(\Gamma_1, T)$ (which makes sense after identification of Γ_1 with Γ).

Proof. It is clear that on (the image of) $\pi_1^{\text{proét}}(W_1)$ (in $\pi_1^{\text{proét}}(X_1)$) the map is the one induced from $f_W : W_1 \to W$. The part about $\pi_1(\Gamma_1, T)$ follows from the fact that $\pi_1(\Gamma_1, T) < \pi_1^{\text{proét}}(X_1)$ acts in the same way as $\pi_1(\Gamma, T) < \pi_1^{\text{proét}}(X_2)$ on any geometric covering of X_2. This follows from the choice of points and paths on $W \in \pi_0(S_\bullet(h))$ as the images of the points and paths on the corresponding connected components $W_1 \in \pi_0(S_\bullet(h_1))$. The maps as in the statement give a morphism $\phi : (*^{\text{top}}_{W\in\pi_0(\widetilde{X})}\pi_1^{\text{proét}}(W_1)) *^{\text{top}} \pi_1(\Gamma_1, T) \to (*^{\text{top}}_{W\in\pi_0(\widetilde{X})}\pi_1^{\text{proét}}(W)) *^{\text{top}} \pi_1(\Gamma, T)$ and it is easy to check that $\phi(R') \subset R$, which finishes the proof. □

3C. *Künneth formula.* In this subsection we use the van Kampen formula to prove the Künneth formula for $\pi_1^{\text{proét}}$.

Let X, Y be two connected schemes locally of finite type over an algebraically closed field k and assume that Y is proper. Let $\bar{x}, \bar{y}$ be geometric points of X and Y respectively with values in the same algebraically closed field extension K of k. With these assumptions, the classical statement says that the "Künneth formula" for $\pi_1^{\text{ét}}$ holds, i.e., the following fact:

Fact 3.28 [SGA 1 1971, Exposé X, Corollary 1.7]. With the above assumptions, the map induced by the projections is an isomorphism

$$\pi_1^{\text{ét}}(X \times_k Y, (\bar{x}, \bar{y})) \xrightarrow{\sim} \pi_1^{\text{ét}}(X, \bar{x}) \times \pi_1^{\text{ét}}(Y, \bar{y}).$$

We want to establish analogous statement for $\pi_1^{\text{proét}}$.

Proposition 3.29. *Let X, Y be two connected schemes locally of finite type over an algebraically closed field k and assume that Y is proper. Let $\bar{x}, \bar{y}$ be geometric points of X and Y respectively with values in the same algebraically closed field extension K of k. Then the map induced by the projections is an isomorphism*

$$\pi_1^{\text{proét}}(X \times_k Y, (\bar{x}, \bar{y})) \xrightarrow{\sim} \pi_1^{\text{proét}}(X, \bar{x}) \times \pi_1^{\text{proét}}(Y, \bar{y}).$$

Choosing a path between $(\bar{x}, \bar{y})$ and some fixed k-point of $X \times_k Y$ (seen as a geometric point) and looking at the images of this path via projections onto X and Y reduces us (by Corollary 3.18 and compatibility of the chosen paths), to the situation where we can assume that $\bar{x}$ and $\bar{y}$ are k-points. We are going to assume this in the proof. Before we start, let us state and prove the surjectivity of the above map as a lemma. Properness is not needed for this.

Lemma 3.30. *Let X, Y be two connected schemes over an algebraically closed field k with k-points on them: $\bar{x}$ on X and $\bar{y}$ on Y. Then the map induced by the projections*

$$\pi_1^{\text{proét}}(X \times_k Y, (\bar{x}, \bar{y})) \to \pi_1^{\text{proét}}(X, \bar{x}) \times \pi_1^{\text{proét}}(Y, \bar{y})$$

is surjective.

Proof. Consider the map $(\mathrm{id}_X, \bar{y}) : X = X \times_k \bar{y} \to X \times_k Y$. It is easy to check that the map induced on fundamental groups $\pi_1^{\text{proét}}(X, \bar{x}) \to \pi_1^{\text{proét}}(X \times_k Y, (\bar{x}, \bar{y})) \to \pi_1^{\text{proét}}(X, \bar{x}) \times \pi_1^{\text{proét}}(Y, \bar{y})$ is given by $(\mathrm{id}_{\pi_1^{\text{proét}}(X,\bar{x})}, 1_{\pi_1^{\text{proét}}(Y,\bar{y})}) : \pi_1^{\text{proét}}(X, \bar{x}) \to \pi_1^{\text{proét}}(X, \bar{x}) \times \pi_1^{\text{proét}}(Y, \bar{y})$. Analogous fact holds if we consider $(\bar{x}, \mathrm{id}_Y) : Y \to X \times_k Y$. As a result, the image $\mathrm{im}(\pi_1^{\text{proét}}(X \times_k Y, (\bar{x}, \bar{y})) \to \pi_1^{\text{proét}}(X, \bar{x}) \times \pi_1^{\text{proét}}(Y, \bar{y}))$ contains the set $(\pi_1^{\text{proét}}(X, \bar{x}) \times \{1_{\pi_1^{\text{proét}}(Y,\bar{y})}\}) \cup (\{1_{\pi_1^{\text{proét}}(X,\bar{x})}\} \times \pi_1^{\text{proét}}(Y, \bar{y}))$. This finishes the proof, as this set generates $\pi_1^{\text{proét}}(X, \bar{x}) \times \pi_1^{\text{proét}}(Y, \bar{y})$. □

Proof of Proposition 3.29. As X, Y are locally of finite type over a field, the normalization maps are finite and we can apply Proposition 3.12. Let $\widetilde{X} \to X$ be the normalization of X and let $\widetilde{X} = \bigsqcup_v \widetilde{X}_v$ be its decomposition into connected components and let us fix a closed point $x_v \in \widetilde{X}_v$ for each v. Similarly, let $\bigsqcup_u \widetilde{Y}_u = \widetilde{Y} \to Y$ be the decomposition into connected components of the normalization of Y with closed points $y_u \in \widetilde{Y}_u$.

We first deal with a particular case.

Claim. *The statement of Proposition 3.29 holds under the additional assumption that*

- *either, for any v, the projections induce isomorphisms*

$$\pi_1^{\text{proét}}(\widetilde{X}_v \times_k Y, (x_v, \bar{y})) \xrightarrow{\sim} \pi_1^{\text{proét}}(\widetilde{X}_v, x_v) \times \pi_1^{\text{proét}}(Y, \bar{y}),$$

- *or, for any u, the projections induce isomorphisms*

$$\pi_1^{\text{proét}}(X \times_k \widetilde{Y}_u, (\bar{x}, y_u)) \xrightarrow{\sim} \pi_1^{\text{proét}}(X, \bar{x}) \times \pi_1^{\text{proét}}(\widetilde{Y}_u, y_u).$$

Proof of the claim. Apply Corollary 3.19 to $h : \widetilde{X} \to X$. We choose $\bar{x}$ and x_v's as geometric points $\bar{s}(t)$ of the corresponding simplexes $t \in \pi_0(S_\bullet(h))_0$ and choose $\bar{s}(t)$ to be arbitrary closed points (of suitable double and triple fiber products) for $t \in \pi_0(S_\bullet(h))_2$. We fix a maximal tree T in $\Gamma = \pi_0(S_\bullet(h))_{\leq 1}$ and fix paths $\gamma_{t',t} : \bar{s}(t') \to S_\bullet(h)(\partial)\bar{s}(t)$. Thus, we get $\pi_1^{\text{proét}}(X, \bar{x}) \cong \left(\left({*}_v^N \pi_1^{\text{proét}}(\widetilde{X}_v, x_v) *^N \pi_1(\Gamma, T)\right)/\overline{H}\right)^{\text{Noohi}}$ where H is defined as in Corollary 3.19.

Observe now that $\widetilde{X}_v \times_k Y$ are connected (as k is algebraically closed) and that $h \times \mathrm{id}_Y : \widetilde{X} \times Y \to X \times Y$ is an effective descent morphism for geometric coverings. So we might use Corollary 3.19 in this setting. As

$(\widetilde{X}_v \times Y) \times_{X\times Y} (\widetilde{X}_w \times Y) = (\widetilde{X}_v \times_X X_w) \times_k Y$, and similarly for triple products, we can identify in a natural way $i^{-1} : \pi_0(S_\bullet(h \times \mathrm{id}_Y)) \xrightarrow{\sim} \pi_0(S_\bullet(h))$. In particular we can identify the graph $\Gamma_Y = \pi_0(S_\bullet(h \times \mathrm{id}_Y))_{\leq 1}$ with Γ and we choose the maximal tree T_Y of Γ_Y as the image of T via this identification. For $t \in \pi_0(S_\bullet(h))$ choose $(\bar{s}(t), \bar{y})$ as the closed base points for $i(t) \in \pi_0(S_\bullet(h \times \mathrm{id}_Y))$. Denote by α_{ijk} elements of various $\pi_1^{\text{proét}}(\widetilde{X}_v)$ defined as in Corollary 3.19 and by $\vec{e}$ elements of $\pi_1(\Gamma, T)$. By the choices and identifications above we can identify $\pi_1(\Gamma_Y, T_Y)$ with $\pi_1(\Gamma, T)$. Using van Kampen and the assumption, we write

$$\begin{aligned}\pi_1^{\text{proét}}(X \times Y, (\bar{x}, \bar{y})) &\cong \Big(\big({*}_v^N \pi_1^{\text{proét}}(\widetilde{X}_v \times Y, (x_v, \bar{y})) *^N \pi_1(\Gamma_Y, T_Y)\big)/\overline{H_Y}\Big)^{\text{Noohi}} \\ &\cong \Big(\big({*}_v^N (\pi_1^{\text{proét}}(\widetilde{X}_v, x_v) \times \pi_1^{\text{proét}}(Y, \bar{y})_v) *^N \pi_1(\Gamma, T)\big)/\overline{H_Y}\Big)^{\text{Noohi}}.\end{aligned}$$

Here $\pi_1^{\text{proét}}(Y, \bar{y})_v$ denotes a "copy" of $\pi_1^{\text{proét}}(Y, \bar{y})$ for each v. By Lemma 3.30, for $T \in \pi_0(S_\bullet(h))$ the natural map $\pi_1^{\text{proét}}(T \times Y, (\bar{s}(T), \bar{y})) \to \pi_1^{\text{proét}}(T, \bar{s}(T)) \times \pi_1^{\text{proét}}(Y, \bar{y})$ is surjective. It follows that the relations defining H_Y (as in Corollary 3.19) can be written as

$$\pi_1^{\text{proét}}(\partial_1)(g) h_{y,1} \vec{e} = \vec{e} \pi_1^{\text{proét}}(\partial_0)(g) h_{y,0},$$

for $e \in e(\Gamma)$, $g \in \pi_1^{\text{proét}}(e, \bar{s}(e))$, $e \in S_1(h)$, $h_y \in \pi_1^{\text{proét}}(Y, \bar{y})$, and

$$\overrightarrow{(\partial_2 f)} \alpha_{102} \alpha_{120}^{-1} \overrightarrow{(\partial_0 f)} \alpha_{210} \alpha_{201}^{-1} (\overrightarrow{(\partial_1 f)})^{-1} \alpha_{021} \alpha_{012}^{-1} = 1,$$

for $f \in S_2(h)$, where the α in the second relation are elements of suitable the $\pi_1^{\text{proét}}(\widetilde{X}_v)$ and are the same as in the corresponding generators of H. The $h_{y,i}$ denotes a copy of element $h_y \in \pi_1^{\text{proét}}(Y, \bar{y})$ in a suitable $\pi_1^{\text{proét}}(Y, \bar{y})_v$. Varying e and h_y while choosing $g = 1 \in \pi_1^{\text{proét}}(e, \bar{s}(e))$ for every e, gives that $h_{y,1}\vec{e} = \vec{e} h_{y,0}$. For $e \in T$ we have $\vec{e} = 1$ and so the first relation reads $h_{y,1} = h_{y,0}$, i.e., it identifies $\pi_1^{\text{proét}}(Y, \bar{y})_v$ with $\pi_1^{\text{proét}}(Y, \bar{y})_w$ for v, w — the ends of the edge e. As T is a maximal tree in Γ, it contains all the vertices, so the first relation identifies $\pi_1^{\text{proét}}(Y, \bar{y})_v = \pi_1^{\text{proét}}(Y, \bar{y})_w$ for any two vertices v, w and we will denote this subgroup (of the quotient) by $\pi_1^{\text{proét}}(Y, \bar{y})$. This way $h_{y,1}\vec{e} = \vec{e} h_{y,0}$ reads simply $h_y \vec{e} = \vec{e} h_y$, so elements of $\pi_1^{\text{proét}}(Y, \bar{y})$ commute with elements of $\pi_1(\Gamma, T)$. Moreover, elements of $\pi_1^{\text{proét}}(Y, \bar{y})$ commute with elements of each $\pi_1^{\text{proét}}(\widetilde{X}_v, x_v)$, as this was true for $\pi_1^{\text{proét}}(Y, \bar{y})_v$. On the other hand, choosing $h_y = 1$ in the first relation and looking at the second relation, we see that H_Y contains all the relations of H. Using notations from the above discussion, we can sum it up by writing

$$H_Y = \langle\!\langle \text{relations generating } H, h_{y,0} = h_{y,1}, h_y \vec{e} = \vec{e} h_y, h_y g = g h_y (g \in \pi_1^{\text{proét}}(\widetilde{X}_v, x_v)) \rangle\!\rangle.$$

Putting this together, we get equivalences of categories

$$\begin{aligned}&\Big(\big({*}_v^N (\pi_1^{\text{proét}}(\widetilde{X}_v, x_v) \times \pi_1^{\text{proét}}(Y, \bar{y})_v) *^N \pi_1(\Gamma, T)\big)/\overline{H_Y}\Big) - \text{Sets} \\ &\qquad\cong \Big\{ S \in \big({*}_v^N (\pi_1^{\text{proét}}(\widetilde{X}_v, x_v) \times \pi_1^{\text{proét}}(Y, \bar{y})_v) *^N \pi_1(\Gamma, T)\big) - \text{Sets} \mid H_Y \text{ acts trivially on } S \Big\} \\ &\qquad\overset{\spadesuit}{\cong} \Big\{ S \in \big(({*}_v^N \pi_1^{\text{proét}}(\widetilde{X}_v, x_v) *^N \pi_1(\Gamma, T)) \times \pi_1^{\text{proét}}(Y, \bar{y})\big) - \text{Sets} \mid H \text{ acts trivially on } S \Big\} \\ &\qquad\cong \Big(\big(({*}_v^N \pi_1^{\text{proét}}(\widetilde{X}_v, x_v) *^N \pi_1(\Gamma, T))/\overline{H}\big) \times \pi_1^{\text{proét}}(Y, \bar{y})\Big) - \text{Sets} \\ &\qquad\cong (\pi_1^{\text{proét}}(X, \bar{x}) \times \pi_1^{\text{proét}}(Y, \bar{y})) - \text{Sets},\end{aligned}$$

where equality ♠ follows from the fact that for topological groups G_1, G_2 there is an equivalence $(G_1 \times G_2) - \mathrm{Sets} \cong \{S \in G_1 *^N G_2 - \mathrm{Sets} \mid \forall_{g_1 \in G_1, g_2 \in G_2} \forall_{s \in S} g_1 g_2 s = g_2 g_1 s\}$ (see Lemma 3.26).

This finishes the proof of the Claim in the "either" case. After noting that each $\tilde{Y}_u$ is still proper, the "or" case follows in a completely symmetrical manner. We have proven a particular case of the proposition. Let us now go ahead and prove the full statement. □

General case. The general case follows from the claim proven above in the following way: Let $\bigsqcup_v \widetilde{X}_v = \widetilde{X} \to X$ and $\bigsqcup_u \widetilde{Y}_u = \widetilde{Y} \to Y$ be decompositions into connected components of the normalizations of X and Y. Fix v and note that $\pi_1^{\text{proét}}(\widetilde{X}_v \times_k Y) = \pi_1^{\text{proét}}(\widetilde{X}_v) \times \pi_1^{\text{proét}}(Y)$ by applying the claim to Y and $\widetilde{X}_v$. This is possible, as the $\widetilde{Y}_u$, $\widetilde{X}_v$ and the products $\widetilde{Y}_u \times_k \widetilde{X}_v$ (for all u) are normal varieties and so their pro-étale fundamental groups are equal to the usual étale fundamental groups (by Lemma 2.12) for which the equality $\pi_1^{\text{ét}}(\widetilde{Y}_u \times_k \widetilde{X}_v) = \pi_1^{\text{ét}}(\widetilde{Y}_u) \times \pi_1^{\text{ét}}(\widetilde{X}_v)$ is known (see Fact 3.28). Thus, for any v, we have that $\pi_1^{\text{proét}}(\widetilde{X}_v \times_k Y) = \pi_1^{\text{proét}}(\widetilde{X}_v) \times \pi_1^{\text{proét}}(Y)$. We can now apply the claim to X and Y and finish the proof in the general case. □

3D. *Invariance of $\pi_1^{\text{proét}}$ of a proper scheme under a base-change $K \supset k$ of algebraically closed fields.*

Proposition 3.31. *Let X be a proper scheme over an algebraically closed field k. Let $K \supset k$ be another algebraically closed field. Then the pullback induces an equivalence of categories*

$$F : \mathrm{Cov}_X \to \mathrm{Cov}_{X_K}.$$

In particular, if X is connected, $X_K \to X$ induces an isomorphism

$$\pi_1^{\text{proét}}(X_K) \xrightarrow{\sim} \pi_1^{\text{proét}}(X).$$

Proof. Let $X^\nu \to X$ be the normalization. It is finite, and thus a morphism of effective descent for geometric coverings. Let us show that the functor F is essentially surjective. Let $Y' \in \mathrm{Cov}_{X_K}$. As k is algebraically closed and X^ν is normal, we conclude that X^ν is geometrically normal, and thus the base change $(X^\nu)_K$ is normal as well; see [Stacks 2020, Tag 038O]. Pulling Y' back to $(X^\nu)_K$ we get a disjoint union of schemes finite étale over $(X^\nu)_K$ with a descent datum. It is a classical result [SGA 1 1971, Exposé X, Corollary 1.8] that the pullback induces an equivalence $\mathrm{Fét}_{X^\nu} \to \mathrm{Fét}_{X^\nu_K}$ of finite étale coverings and similarly for the double and triple products $X_2^\nu = X^\nu \times_X X^\nu$, $X_3^\nu = X^\nu \times_X X^\nu \times_X X^\nu$. These equivalences obviously extend to categories whose objects are (possibly infinite) disjoint unions of finite étale schemes (over X^ν, X_2^ν, X_3^ν respectively) with étale morphisms as arrows. These categories can be seen as subcategories of Cov_{X^ν} and so on. These subcategories are moreover stable under pullbacks between $\mathrm{Cov}_{X_i^\nu}$. Putting this together we see, that $Y'' = Y' \times_{X_K} (X^\nu)_K$ with its descent datum is isomorphic to a pullback of a descent datum from X^ν. Thus, we conclude that there exists $Y \in \mathrm{Cov}_X$ such that $Y' \simeq Y_K$. Full faithfulness of F is shown in the same way. If X is connected, it can be also proven more directly, as F being fully faithful is equivalent to preserving connectedness of geometric coverings, but any connected $Y \in \mathrm{Cov}_X$ is geometrically connected, and thus Y_K remains connected by Proposition 2.37(2). Note that in the above argument we do not claim that the double and triple intersections X_2^ν, X_3^ν are normal, as this is in general false. Instead, we are only using that all the considered geometric coverings of those schemes came as pullbacks from X^{ν}, and thus were already split-to-finite. □

4. Fundamental exact sequence

4A. ***Statement of the results and examples.*** The main result of this chapter is the following theorem.

Theorem (see Theorem 4.14 below). *Let k be a field and fix an algebraic closure $\bar{k}$. Let X be a geometrically connected scheme of finite type over k. Then the sequence of abstract groups*

$$1 \to \pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X) \to \mathrm{Gal}_k \to 1$$

is exact.

Moreover, the map $\pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X)$ is a topological embedding and the map $\pi_1^{\text{proét}}(X) \to \mathrm{Gal}_k$ is a quotient map of topological groups.

One shows the near exactness first and obtains the above version as a corollary with an extra argument. The most difficult part of the sequence is exactness on the left. We will prove it as a separate theorem and its proof occupies an entire subsection.

Theorem (see Theorem 4.13 below). *Let k be a field and fix an algebraic closure $\bar{k}$ of k. Let X be a scheme of finite type over k such that the base change $X_{\bar{k}}$ is connected. Then the induced map*

$$\pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X)$$

is a topological embedding.

By Proposition 2.37, it translates to the following statement in terms of coverings: every geometric covering of $X_{\bar{k}}$ can be dominated by a covering that embeds into a base-change to $\bar{k}$ of a geometric covering of X (i.e., defined over k). In practice, we prove that every connected geometric covering of $X_{\bar{k}}$ can be dominated by a (base-change of a) covering of X_l for l/k finite.

For finite coverings, the analogous statement is very easy to prove simply by finiteness condition. But for general geometric coverings this is nontrivial and maybe even slightly surprising as we show by counterexamples (Examples 4.5 and 4.6) that it is not always true that a connected geometric covering of $X_{\bar{k}}$ is isomorphic to a base-change of a covering of X_l for some finite extension l/k. This last statement is, however, stronger than what we need to prove, and thus does not contradict our theorem. Observe, that the stronger statement is true for finite coverings and, even more generally, whenever $\pi_1^{\text{proét}}(X_{\bar{k}})$ is prodiscrete, as proven in Proposition 4.8.

Let us proceed to proving the easier part of the sequence first.

Observation 4.1. By Proposition 2.17, the category of geometric coverings is invariant under universal homeomorphisms. In particular, for a connected X over a field and k'/k purely inseparable, there is $\pi_1^{\text{proét}}(X_{k'}) = \pi_1^{\text{proét}}(X)$. Similarly, we can replace X by X_{red} and so assume X to be reduced when convenient. In this case, base change to separable closure X_{k^s} is reduced as well. We will often use this observation without an explicit reference.

We start with the following lemmas.

Lemma 4.2. *Let k be a field. Let $k \subset k'$ be a (possibly infinite) Galois extension. Let X be a connected scheme over k. Let $\overline{T}_0 \subset \pi_0(X_{k'})$ be a nonempty closed subset preserved by the* $\mathrm{Gal}(k'/k)$*-action. Then $\overline{T}_0 = \pi_0(X_{k'})$.*

Proof. Let $\overline{T}$ be the preimage of $\overline{T}_0$ in $X_{k'}$ (with the reduced induced structure). By [Stacks 2020, Lemma 038B], $\overline{T}$ is the preimage of a closed subset $T \subset X$ via the projection morphism $p : X_{k'} \to X$. On the other hand, by [loc. cit., Lemma 04PZ], the image $p(\overline{T})$ equals the entire X. Thus, $T = X$ and $\overline{T} = X_{k'}$, and so $\overline{T}_0 = \pi_0(X_{k'})$. □

Lemma 4.3. *Let X be a connected scheme over a field k with an l'-rational point with l'/k a finite field extension. Then $\pi_0(X_{k^{\mathrm{sep}}})$ is finite, the* Gal_k *action on $\pi_0(X_{k^{\mathrm{sep}}})$ is continuous and there exists a finite separable extension l/k such that the induced map $\pi_0(X_{k^{\mathrm{sep}}}) \to \pi_0(X_l)$ is a bijection. Moreover, there exists the smallest field (contained in k^{sep}) with this property and it is Galois over k.*

Proof. Let us first show the continuity of the Gal_k-action. The morphism $\mathrm{Spec}(l') \to X$ gives a Gal_k-equivariant morphism $\mathrm{Spec}(l' \otimes_k k^{\mathrm{sep}}) \to X_{k^{\mathrm{sep}}}$ and a Gal_k-equivariant map $\pi_0(\mathrm{Spec}(l' \otimes_k k^{\mathrm{sep}})) \to \pi_0(X_{k^{\mathrm{sep}}})$. Denote by $M \subset \pi_0(X_{k^{\mathrm{sep}}})$ the image of the last map. It is finite and Gal_k-invariant, and by Lemma 4.2, $M = \pi_0(X_{k'})$. We have tacitly used that M is closed, as $\pi_0(X_{k'})$ is Hausdorff (as the connected components are closed). As Gal_k acts continuously on $\pi_0(\mathrm{Spec}(l' \otimes_k k^{\mathrm{sep}}))$ (for example by [Stacks 2020, Lemma 038E]), we conclude that it acts continuously on $\pi_0(X_{k^{\mathrm{sep}}})$ as well. From Lemma 4.2 again and from [loc. cit., Tag 038D], we easily see that the fields $l \subset k^{\mathrm{sep}}$ such that $\pi_0(X_{k^{\mathrm{sep}}}) \to \pi_0(X_l)$ is a bijection are precisely those that Gal_l acts trivially on $\pi_0(X_{k^{\mathrm{sep}}})$. To get the minimal field with this property we choose l such that $\mathrm{Gal}_l = \ker(\mathrm{Gal}_k \to \mathrm{Aut}(\pi_0(X_{k^{\mathrm{sep}}})))$. □

Theorem 4.4. *Let k be a field and fix an algebraic closure $\bar{k}$. Let X be a geometrically connected scheme of finite type over k. Let $\bar{x} : \mathrm{Spec}(\bar{k}) \to X_{\bar{k}}$ be a geometric point on $X_{\bar{k}}$. Then the induced sequence*

$$\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x}) \xrightarrow{\iota} \pi_1^{\text{proét}}(X, \bar{x}) \xrightarrow{p} \mathrm{Gal}_k \to 1$$

of topological groups is nearly exact in the middle (i.e., the thick closure of $\mathrm{im}(\iota)$ *equals* $\ker(p)$*) and $\pi_1^{\text{proét}}(X) \to \mathrm{Gal}_k$ is a topological quotient map.*

Proof. (1) The map p is surjective and open: let $U < \pi_1^{\text{proét}}(X)$ be an open subgroup. There is a geometric covering Y of X with a $\bar{k}$-point $\bar{y}$ such that the morphism $\pi_1^{\text{proét}}(Y, \bar{y}) \to \pi_1^{\text{proét}}(X, \bar{x})$ is equal to $U \subset \pi_1^{\text{proét}}(X, \bar{x})$. As Y is locally of finite type over k, the image of $\bar{y}$ in Y has a finite extension l of k as the residue field. Thus, we get $\mathrm{Gal}_l \to \pi_1^{\text{proét}}(Y) \to \mathrm{Gal}_k$ and we see that the image $\pi_1^{\text{proét}}(Y) \to \mathrm{Gal}_k$ contains an open subgroup, so is open. We have shown that p is an open morphism. In particular the image of $\pi_1^{\text{proét}}(X)$ in Gal_k is open and so also closed. On the other hand, this image is dense as we have the following diagram:

$$\begin{tikzcd} & \pi_1^{\text{proét}}(X) \arrow[r] \arrow[d] & \pi_1^{\text{proét}}(\mathrm{Spec}(k)) \arrow[d, equal] \\ \widehat{\pi_1^{\text{proét}}(X)}^{\text{prof}} = & \pi_1^{\text{ét}}(X) \arrow[r, two heads] & \pi_1^{\text{ét}}(\mathrm{Spec}(k)) \end{tikzcd}$$

where $\widehat{\cdots}^{\text{prof}}$ means the profinite completion. In the diagram, the left vertical map has dense image and the lower horizontal is surjective. This shows that $\pi_1^{\text{proét}}(X) \to \mathrm{Gal}_k$ is surjective.

(2) The composition $\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x}) \to \pi_1^{\text{proét}}(X, \bar{x}) \to \mathrm{Gal}_k$ is trivial — this is clear thanks to Proposition 2.37 and the fact that the map $X_{\bar{k}} \to \mathrm{Spec}(k)$ factorizes through $\mathrm{Spec}(\bar{k})$.

(3) The thick closure of $\mathrm{im}(\iota)$ is normal: as remarked above, $\pi_1^{\text{proét}}(X_{\bar{k}}) = \pi_1^{\text{proét}}(X_{k^s})$, where k^s denotes the separable closure. Thus, we are allowed to replace $\bar{k}$ with k^s in the proof of this point. Moreover, by the same remark, we can and do assume X to be reduced. Let $Y \to X$ be a connected geometric covering such that there exists a section $s : X_{k^s} \to Y \times_X X_{k^s} = Y_{k^s}$ over X_{k^s}. Observe that any such section is a clopen immersion: this follows immediately from the equivalence of categories of $\pi_1^{\text{proét}}(X_{k^s}) - \mathrm{Sets}$ and geometric coverings. Define $\overline{T} := \bigcup_{\sigma \in \mathrm{Gal}(k)} {}^{\sigma}s(X_{k^s}) \subset Y_{k^s}$. Observe that two images of sections in the sum either coincide or are disjoint as X_{k^s} is connected and they are clopen. Now, $\overline{T}$ is obviously open, but we claim that it is also a closed subset. This follows from Lemma 4.3 (which implies that $\pi_0(Y_{k^s})$ is finite), but one can also argue directly by using that Y_{k^s} is locally noetherian and ${}^{\sigma}s(X_{k^s})$ are clopen. Now by [Stacks 2020, Tag 038B], $\overline{T}$ descends to a closed subset $T \subset Y$. It is also open as T is the image of $\overline{T}$ via projection $Y_{k^s} \to Y$ which is surjective and open map. Indeed, surjectivity is clear and openness is easy as well and is a particular case of a general fact, that any map from a scheme to a field is universally open [loc. cit., Tag 0383] By connectedness of Y we see that $T = Y$. So $Y_{k^s} = \overline{T}$. But this last one is a disjoint union of copies of X_{k^s}, which is what we wanted to show by Proposition 2.37.

(4) The smallest normal thickly closed subgroup of $\pi_1^{\text{proét}}(X)$ containing $\mathrm{im}(\iota)$ is equal to $\ker(p)$: as we already know that this image is contained in the kernel and that the map $\pi_1^{\text{proét}}(X) \to \mathrm{Gal}_k$ is a quotient map of topological groups, we can apply Proposition 2.37. Let Y be a connected geometric covering of X such that $Y_{\bar{k}} = Y \times_X X_{\bar{k}}$ splits completely. Denote $Y_{\bar{k}} = \bigsqcup_\alpha X_{\bar{k},\alpha}$, where by $X_{\bar{k},\alpha}$ we label different copies of $X_{\bar{k}}$. By Lemma 4.3, $\pi_0(Y_{\bar{k}})$ is finite, and thus the indexing set $\{\alpha\}$ and the covering $Y \to X$ are finite. But in this case, the statement follows from the classical exact sequence of étale fundamental groups due to Grothendieck. □

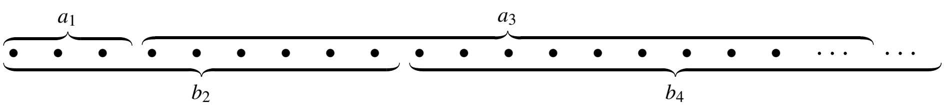
As promised above, we give examples of geometric coverings of $X_{\bar{k}}$ that cannot be defined over any finite field extension l/k.

Example 4.5. Let $X_i = \mathbb{G}_{m,\mathbb{Q}}$, $i = 1, 2$. Define X to be the gluing $X = \cup_\bullet X_i$ of these schemes at the rational points $1_i : \mathrm{Spec}(\mathbb{Q}) \to X_i$ corresponding to 1. Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and so a geometric point $\bar{b}$ over the base $\mathrm{Spec}(\mathbb{Q})$. This gives geometric points $\bar{x}_i$ on $\overline{X}_i = X_{i,\overline{\mathbb{Q}}}$ and X_i lying over 1_i, which we choose as base points for the fundamental groups involved. Similarly, we get a geometric point $\bar{x}$

over the point of gluing x that maps to $\bar{b}$. Then Example 3.25 gives us a description of the fundamental group $\pi_1^{\text{proét}}(X,\bar{x}) \simeq \left(*^{\text{top}}_{i=1,2}(\pi_1^{\text{ét}}(\overline{X}_i,\bar{x}_i) \rtimes \mathrm{Gal}_{\mathbb{Q},i})/\overline{\langle\langle \iota_1(\sigma) = \iota_2(\sigma) \mid \sigma \in \mathrm{Gal}_{\mathbb{Q}}\rangle\rangle}\right)^{\text{Noohi}}$ and of its category of sets:

$$\begin{aligned}&\pi_1^{\text{proét}}(X,\bar{x}) - \text{Sets}\\ &\quad \simeq \left\{S \in \left(*^{\text{top}}\pi_1^{\text{ét}}(\overline{X}_1) *^{\text{top}} \pi_1^{\text{ét}}(\overline{X}_2) *^{\text{top}} \mathrm{Gal}_{\mathbb{Q}}\right) - \text{Sets} \mid \forall_{\sigma\in\mathrm{Gal}_{\mathbb{Q}}}\forall_i\forall_{\gamma\in\pi_1^{\text{ét}}(\overline{X}_i)}\forall_{s\in S}\, \sigma\cdot(\gamma\cdot s) = {}^{\sigma}\gamma\cdot(\sigma\cdot s)\right\}.\end{aligned}$$

For the base change $\overline{X}$ to $\overline{\mathbb{Q}}$, we have $\pi_1^{\text{proét}}(\overline{X},\bar{x}) \simeq \pi_1^{\text{ét}}(\overline{X}_1,\bar{x}_1) *^{N} \pi_1^{\text{ét}}(\overline{X}_2,\bar{x}_2)$. Recall that the groups $\pi_1^{\text{ét}}(\overline{X}_i,\bar{x}_i)$ are isomorphic to $\widehat{\mathbb{Z}}(1) = \varprojlim \mu_n$ as $\mathrm{Gal}_{\mathbb{Q}}$-modules. Fix these isomorphisms. Let $S = \mathbb{N}_{>0}$. Let us define a $\pi_1^{\text{proét}}(\overline{X},\bar{x})$-action on S, which means giving actions by $\pi_1^{\text{ét}}(\overline{X}_1,\bar{x}_1)$ and $\pi_1^{\text{ét}}(\overline{X}_2,\bar{x}_2)$ (no compatibilities of the actions required). Let ℓ be a fixed odd prime number (e.g., $\ell = 3$). We will give two different actions of $\mathbb{Z}_\ell(1)$ on S which will define actions of $\widehat{\mathbb{Z}}(1)$ by projections on $\mathbb{Z}_\ell(1)$. We start by dividing S into consecutive intervals labeled $a_1, a_3, a_5, \dots$ of cardinality $\ell^1, \ell^3, \ell^5, \dots$ respectively. These will be the orbits under the action of $\pi_1^{\text{ét}}(\overline{X}_1,\bar{x}_1)$. Similarly, we divide S into consecutive intervals $b_2, b_4, b_6, \dots$ of cardinality $\ell^2, \ell^4, \dots$.



We still have to define the action on each a_m and b_m. We choose arbitrary identifications $b_m \simeq \mu_{\ell^m}$ as $\mathbb{Z}_\ell(1)$-modules. Now, fix a compatible system of ℓ^n-th primitive roots of unity $\zeta = (\zeta_{\ell^n}) \in \mathbb{Z}(1)$. For a_m's, we choose the identifications with μ_{ℓ^m} arbitrarily with one caveat: we demand that for any even number m, the intersection $b_m \cap a_{m+1}$ contains the elements $1, \zeta_{\ell^{m+1}} \in \mu_{\ell^{m+1}}$ via the chosen identification $a_{m+1} \simeq \mu_{\ell^{m+1}}$. As $|b_m \cap a_{m+1}| > 0$ and $|b_m \cap a_{m+1}| \equiv 0 \mod \ell$, the intersection $b_m \cap a_{m+1}$ contains at least two elements and we see that choosing such a labeling is always possible.

Assume that S corresponds to a covering that can be defined over a finite Galois extension $K/\mathbb{Q}$. Fix $s_0 \in a_1 \cap b_2$. By increasing K, we might and do assume that Gal_K fixes s_0. Let p be a prime number $\neq \ell$ that splits completely in K and $\mathfrak{p}$ be a prime of $\mathcal{O}_K$ lying above p. Let $\phi_{\mathfrak{p}} \in \mathrm{Gal}_K$ be a Frobenius element (which depends on the choice of the decomposition group and the coset of the inertia subgroup). It acts on $\mathbb{Z}_\ell(1)$ via $t \mapsto t^p$ and this action is independent of the choice of $\phi_{\mathfrak{p}}$. Choose $N > 0$ such that $p^N \equiv 1 \mod \ell^2$ and let m be the biggest number such that $p^N \equiv 1 \mod \ell^m$. If m is odd, we look at $p^{\ell N}$ instead. In this case $m+1$ is the biggest number such that $p^{\ell N} \equiv 1 \mod \ell^{m+1}$ and so, by changing N if necessary, we can assume that m is even, > 1. The whole point of the construction is the following: if $s \in a_i \cap b_j$ with $i, j < m$ is fixed by $\phi_{\mathfrak{p}}^N$, then so are $g\cdot s$ and $h\cdot s$ (for $h \in \pi_1^{\text{ét}}(\overline{X}_1,\bar{x}_1)$ and $g \in \pi_1^{\text{ét}}(\overline{X}_2,\bar{x}_2)$). Then moving such s with the g and h to $b_m \cap a_{m+1}$ leads to a contradiction. Indeed, let $s_1 \in b_m \cap a_{m+1} \subset S$ correspond to $1 \in \mu_{\ell^{m+1}} \simeq a_{m+1}$ (it is possible by the choices made in the construction of S). Write $s_1 = g_m h_{m-1}\cdots h_3 g_2 h_1 \cdot s_0$ with $h_i \in \pi_1^{\text{ét}}(\overline{X}_1,\bar{x}_1)$ and $g_j \in \pi_1^{\text{ét}}(\overline{X}_2,\bar{x}_2)$ (this form is not unique, of course). This is possible thanks to the fact that the sets a_i, b_j form consecutive intervals separately such that b_j intersects nontrivially a_{j-1} and a_{j+1}. By the construction of S again, there is an element $s_2 \in b_m \cap a_{m+1}$

corresponding to $\zeta_{\ell^{m+1}} \in \mu_{\ell^{m+1}}$ via $a_{m+1} \simeq \mu_{\ell^{m+1}}$. We can now write s_2 in two ways:

$$s_2 = \zeta \cdot s_1 = g \cdot s_1,$$

where $g \in \pi_1^{\text{ét}}(\bar{X}_2, \bar{x}_2)$ and ζ is the chosen element in $\pi_1^{\text{ét}}(\bar{X}_1, \bar{x}_1) \simeq \hat{\mathbb{Z}}(1)$. By the choices made, the action of $\phi_{\mathfrak{p}}^N$ fixes the elements s_1 and $g \cdot s_1$, while it moves $\zeta \cdot s_1$. Indeed, $\phi_{\mathfrak{p}}^N \cdot (\zeta \cdot s_1) = (\phi_{\mathfrak{p}}^N \zeta \phi_{\mathfrak{p}}^{-N}) \cdot (\phi_{\mathfrak{p}}^N \cdot s_1) = \zeta^{p^N} \cdot (\phi_{\mathfrak{p}}^N \cdot s_1) = \zeta^{p^N} \cdot s_1 = \zeta_{\ell^{m+1}}^{p^N} \neq \zeta_{\ell^{m+1}} = \zeta \cdot s_1 \in \mu_{\ell^{m+1}} \simeq a_{m+1}$ — a contradiction.

Example 4.6. Let $X_i = \mathbb{G}_{m,\mathbb{Q}}$, $i = 1, 2, 3$ and let X_4, X_5 be the nodal curves obtained from gluing 1 and -1 on $\mathbb{P}^1_{\mathbb{Q}}$ (see Example 3.24). Define X to be the gluing $X = \cup_{\bullet} X_i$ of all these schemes at the rational points corresponding to 1 (or the image of 1 in the case of the nodal curves). We fix an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ and so fix a geometric point $\bar{b}$ over the base $\operatorname{Spec}(\mathbb{Q})$. We get geometric points $\bar{x}_i$ on $\bar{X}_i = X_i \times_{\mathbb{Q}} \bar{\mathbb{Q}}$ lying over 1. We have and fix the following isomorphisms of $\mathrm{Gal}_{\mathbb{Q}}$-modules. For $1 \le i \le 3$, $\pi_1^{\text{ét}}(\bar{X}_i, \bar{x}_i) \simeq \widehat{\mathbb{Z}}(1)$ and for $4 \le j \le 5$, we have $\pi_1^{\text{proét}}(\bar{X}_j, \bar{x}_j) \simeq \langle t^{\mathbb{Z}} \rangle$ (i.e., $\mathbb{Z}$ written multiplicatively). Let $t_i \in \pi_1^{\text{proét}}(\bar{X}_i, \bar{x}_i)$ be the elements corresponding via these isomorphisms to a fixed inverse system of primitive roots $\zeta \in \widehat{\mathbb{Z}}(1)$ (for $i = 1, 2, 3$) and to $t \in \langle t^{\mathbb{Z}} \rangle$ (for $i = 4, 5$). Example 3.25 gives a description of the fundamental group

$$\pi_1^{\text{proét}}(X, \bar{x}) \simeq \left(*_{i=1,2,3}^N (\widehat{\mathbb{Z}}(1)_i \rtimes \mathrm{Gal}_{\mathbb{Q},i}) *_{j=4,5}^N (\langle t^{\mathbb{Z}} \rangle \times \mathrm{Gal}_{\mathbb{Q},j}) \Big/ \overline{\left\langle\!\!\left\langle \iota_i(\sigma) = \iota_{i'}(\sigma) \;\middle|\; \begin{matrix} \sigma \in \mathrm{Gal}_{\mathbb{Q}} \\ i, i' = 1, \ldots, 5 \end{matrix} \right\rangle\!\!\right\rangle} \right)^{\text{Noohi}}$$

and of its category of sets:

$$\begin{aligned} \pi_1^{\text{proét}}(X, \bar{x}) - \text{Sets} \simeq \Big\{ S \in \big(*^{\text{top}}_{1 \le i \le 3} \widehat{\mathbb{Z}}(1) *^{\text{top}} \langle t^{\mathbb{Z}} \rangle^{*2} *^{\text{top}} \mathrm{Gal}_{\mathbb{Q}} \big) - \text{Sets} \mid \\ \forall_{\sigma \in \mathrm{Gal}_{\mathbb{Q}}} \forall_{1 \le i \le 3} \forall_{\substack{\gamma \in \mathbb{Z}(1)_i \\ w \in \langle t^{\mathbb{Z}} \rangle^{*2}}} \forall_{s \in S} \sigma \cdot (\gamma \cdot s) =^{\sigma} \gamma \cdot (\sigma \cdot s) \text{ and } \sigma \cdot (w \cdot s) = w \cdot (\sigma \cdot s) \Big\}. \end{aligned}$$

Let $G = \left\{ \left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix} \right) \right\} \subset \mathrm{GL}_2(\mathbb{Q}_\ell)$ be the subgroup of upper triangular matrices. Fix $u_1 \in \mathbb{Z}_\ell^\times$ such that $u_1^p \neq u_1$. Let $H = *_i^{\text{top}} \pi_1^{\text{ét}}(\bar{X}_i, \bar{x}_i)$ and define a continuous homomorphism $\psi : H \to G$ by:

$$\psi(t_1) = \begin{pmatrix} u_1 & \\ & 1 \end{pmatrix}, \quad \psi(t_2) = \begin{pmatrix} 1 & \\ & u_1 \end{pmatrix}, \quad \psi(t_3) = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}, \quad \psi(t_4) = \begin{pmatrix} \ell & \\ & 1 \end{pmatrix}, \quad \psi(t_5) = \begin{pmatrix} 1 & \\ & \ell \end{pmatrix}.$$

It is easy to see that ψ is surjective.

Let $U \subset G$ be the subgroup of matrices with elements in $\mathbb{Z}_\ell$, i.e., $U = \left\{ \left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix} \right) \right\} \subset \mathrm{GL}_2(\mathbb{Z}_\ell)$. It is an open subgroup of G. Thus, using ψ and the fact that $H^{\text{Noohi}} = \pi_1^{\text{proét}}(\bar{X}, \bar{x})$, we get that $S := G/U$ defines a $\pi_1^{\text{proét}}(\bar{X}, \bar{x})$-set. It is connected (i.e., transitive) and so corresponds to a connected geometric covering of $\bar{X}$. Assume that it can be defined over a finite extension L of $\mathbb{Q}$. We can assume $L/\mathbb{Q}$ is Galois. By the description above, it means that there is a compatible action of groups $\mathbb{Z}(1)_i$, $\mathbb{Z}^{*2}$ and Gal_L on S. By increasing L, we can assume moreover that Gal_L fixes $[U]$.

Choose $p \neq \ell$ that splits completely in Gal_L, fix a prime $\mathfrak{p}$ of L dividing p and let $\phi_{\mathfrak{p}} \in \mathrm{Gal}_L$ denote a fixed Frobenius element. Let $t_3^{u_1}$ denote the unique element of $\psi^{-1}_{|\pi_1^{\text{ét}}(\bar{X}_3, \bar{x}_3)}\left(\left(\begin{smallmatrix} 1 & u_1 \\ & 1 \end{smallmatrix} \right) \right)$. Let $n \gg 0$. An easy calculation shows that $\psi(t_4^{-n} t_1 t_3 t_1^{-1} t_3^{-u_1} t_4^n) = 1_{\mathrm{GL}_2(\mathbb{Q}_\ell)} \in U$. Then $\phi_{\mathfrak{p}} \cdot [U] = \phi_{\mathfrak{p}} \cdot (t_4^{-n} t_1 t_3 t_1^{-1} t_3^{-u_1} t_4^n \cdot [U]) =$

${}^{\phi_\mathfrak{p}}(t_4^{-n}t_1t_3t_1^{-1}t_3^{-u_1}t_4^n)\cdot(\phi_\mathfrak{p}\cdot[U]) = t_4^{-n}t_1^pt_3^pt_1^{-p}t_3^{-pu_1}t_4^n\cdot[U]$. But

$$\begin{aligned}\psi(t_4^{-n}t_1^pt_3^pt_1^{-p}t_3^{-pu_1}t_4^n) &= \begin{pmatrix}\ell^{-n} & \\ & 1\end{pmatrix}\begin{pmatrix}u_1^p & \\ & 1\end{pmatrix}\begin{pmatrix}1 & p\\ & 1\end{pmatrix}\begin{pmatrix}u_1^{-p} & \\ & 1\end{pmatrix}\begin{pmatrix}1 & -pu_1\\ & 1\end{pmatrix}\begin{pmatrix}\ell^n & \\ & 1\end{pmatrix}\\ &= \begin{pmatrix}1 & \ell^{-n}p(u_1^p-u_1)\\ & 1\end{pmatrix}\notin U.\end{aligned}$$

As $n\gg 0$ and $u_1^p\neq u_1$, it follows that $\phi_\mathfrak{p}\cdot[U]\neq[U]$ — a contradiction.

It is important to note, that the above (counter)examples are possible only when considering the geometric coverings that are not trivialized by an étale cover (but one really needs to use the pro-étale cover to trivialize them). In [Bhatt and Scholze 2015], the category of geometric coverings trivialized by an étale cover on X is denoted by $\mathrm{Loc}_{X_{\text{ét}}}$ and the authors prove the following

Fact 4.7 [Bhatt and Scholze 2015, Lemma 7.4.5]. Under $\mathrm{Loc}_X\simeq\pi_1^{\text{proét}}(X)-\mathrm{Sets}$, the full subcategory $\mathrm{Loc}_{X_{\text{ét}}}\subset\mathrm{Loc}_X$ corresponds to the full subcategory of those $\pi_1^{\text{proét}}(X)-\mathrm{Sets}$ where an open subgroup acts trivially.

We are now going to prove:

Proposition 4.8. *Let X be a geometrically connected separated scheme of finite type over a field k. Let $Y\in\mathrm{Cov}_{X_{\bar k}}$ be such that $Y\in\mathrm{Loc}_{(X_{\bar k})_{\text{ét}}}$. Then there exists a finite extension l/k such and $Y_0\in\mathrm{Cov}_{X_l}$ such that $Y\simeq Y_0\times_{X_l}X_{\bar k}$.*

Proof. By the topological invariance (Proposition 2.17), we can replace $\bar k$ by k^{sep} if desired. By the assumption $Y\in\mathrm{Loc}_{(X_{\bar k})_{\text{ét}}}$, there exists an étale cover of finite type that trivializes Y. Being of finite type, it is a base-change $X'_{\bar k}=X'\times_{\mathrm{Spec}(l)}\mathrm{Spec}(\bar k)\to X_{\bar k}$ of an étale cover $X'\to X_l$ for some finite extension l/k. Thus, $Y_{|X'_{\bar k}}$ is constant (i.e., $\simeq\bigsqcup_{s\in S}X'=\underline{S}$) and the isomorphism between the pull-backs of $Y_{|X'_{\bar k}}$ via the two projections $X'_{\bar k}\times_{X_{\bar k}}X'_{\bar k}\rightrightarrows X'_{\bar k}$ is expressed by an element of a constant sheaf $\underline{\mathrm{Aut}(S)}(X'_{\bar k}\times_{X_{\bar k}}X'_{\bar k})=\mathrm{Aut}(\underline{S})(X'_{\bar k}\times_{X_{\bar k}}X'_{\bar k})$ (we use the fact that $X'_{\bar k}$ is étale over $X_{\bar k}$, and thus $\pi_0(X'_{\bar k}\times_{X_{\bar k}}X'_{\bar k})$ is discrete, in this case even finite). By enlarging l, we can assume that the connected components of the schemes involved: X', $X'\times_{X_l}X'$ etc. are geometrically connected over l. Define $Y'_0=\bigsqcup_{s\in S}X'$. The discussion above shows that the descent datum on $Y_{|X'_{\bar k}}$ with respect to $X'_{\bar k}\to X_{\bar k}$ is in fact the pull-back of a descent datum on Y'_0 with respect to $X'\to X_l$. As étale covers are morphisms of effective descent for geometric coverings (this follows from the fpqc descent for fpqc sheaves and the equivalence $\mathrm{Cov}_{X_l}\simeq\mathrm{Loc}_{X_l}$ of [Bhatt and Scholze 2015, Lemma 7.3.9]), the proof is finished. □

Remark 4.9. Over a scheme with a nondiscrete set of connected components, $\underline{\mathrm{Aut}(S)}$ might not be equal to $\mathrm{Aut}(\underline{S})$.

Proposition 4.8 shows that our main theorem is significantly easier for π_1^{SGA3}.

Corollary 4.10. *Let X be a geometrically connected separated scheme of finite type over a field k. Fix an algebraic closure $\bar{k}$ of k. Then*

$$\pi_1^{\mathrm{SGA3}}(X_{\bar{k}}) \to \pi_1^{\mathrm{SGA3}}(X)$$

is a topological embedding.

4B. *Preparation for the proof of Theorem 4.13.* We are going to use the following proposition.

Proposition 4.11. *Let X be a scheme of finite type over a field k with a k-rational point x_0 and assume that $X_{\bar{k}}$ is connected. Let $Y_1, \dots, Y_N$ be a set of connected finite étale coverings of $X_{\bar{k}}$. Then there exists a finite Galois covering Y of $X_{\bar{k}}$ that dominates each Y_i and such that the corresponding normal subgroup of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}_0)$ is normalized by $\mathrm{Gal}_k = \pi_1^{\text{ét}}(x_0, \bar{x}_0)$ in $\pi_1^{\text{ét}}(X, \bar{x}_0)$.*

Proof. There is a finite connected Galois covering of $X_{\bar{k}}$ dominating $Y_1, \dots, Y_N$. Thus, we can assume $N = 1$ and Y_1 is Galois. Fix a geometric point $\bar{x}_0$ over x_0. The splitting $s : \mathrm{Gal}_k = \pi_1^{\text{ét}}(x_0, \bar{x}_0) \to \pi_1^{\text{ét}}(X, \bar{x}_0)$ allows us to write $\pi_1^{\text{ét}}(X, \bar{x}_0) \simeq \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}_0) \rtimes \mathrm{Gal}_k$. Fix a geometric point $\bar{y}$ on Y_1 over $\bar{x}_0$. The group $U = \pi_1^{\text{ét}}(Y_1, \bar{y})$ is a normal open subgroup of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}_0)$. As the pair $(Y_1, \bar{y})$ is defined over a finite Galois field extension l/k (contained in $\bar{k}$), it is easy to check that $\mathrm{Gal}_l \subset \mathrm{Gal}_k$ fixes U, i.e., ${}^{\sigma}U = U$ for $\sigma \in \mathrm{Gal}_l$. It follows that the set of conjugates ${}^{\sigma}U$ is finite, of cardinality bounded by $[l : k]$. Define $V = \bigcap_{\sigma \in \mathrm{Gal}_k} {}^{\sigma}U$. It follows that this is an open subgroup of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}_0)$ fixed by the action of Gal_k. Moreover, it is normal in $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}_0)$, as for any $g \in \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}_0)$, there is $g\big(\bigcap_{\sigma \in \mathrm{Gal}_k} {}^{\sigma}U\big)g^{-1} = \bigcap_{\sigma \in \mathrm{Gal}_k} g\,{}^{\sigma}U g^{-1} = \bigcap_{\sigma \in \mathrm{Gal}_k} {}^{\sigma}(({}^{\sigma^{-1}}g)U({}^{\sigma^{-1}}g^{-1})) = \bigcap_{\sigma \in \mathrm{Gal}_k} {}^{\sigma}U$, due to normality of U. This open subgroup $V < \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}_0)$ corresponds to a covering with the desired properties. ☐

Before starting the proof, we need to collect some facts about the Galois action on the geometric $\pi_1^{\text{ét}}$. They are discussed, for example, in [Stix 2013, Chapter 2]. The existence, functoriality and compatibility with compositions of the action can be readily seen to generalize to $\pi_1^{\text{proét}}$ as well, but *note* (*see the last point below*) *that one has to be careful when discussing continuity*. For a connected topologically noetherian scheme W and geometric points $\bar{w}_1, \bar{w}_2$, let $\pi_1^{\text{proét}}(W, \bar{w}_1, \bar{w}_2) = \mathrm{Isom}_{\mathrm{Cov}_{W_{\bar{k}}}}(F_{\bar{w}_1}, F_{\bar{w}_2})$ denote the set of isomorphisms of the two fiber functors, topologized in a way completely analogous to the case when $\bar{w}_1 = \bar{w}_2$. By Corollary 3.18, it is a bitorsor under $\pi_1^{\text{proét}}(W, \bar{w}_1)$ and $\pi_1^{\text{proét}}(W, \bar{w}_2)$. The bitorsors under profinite groups $\pi_1^{\text{ét}}(W, \bar{w}_1, \bar{w}_2)$ are defined similarly and are rather standard. For a geometrically unibranch W, the two notions match.

Lemma 4.12. *For a scheme W of finite type over k and two geometric points $\bar{w}_1, \bar{w}_2$ on W lying over k-points, there is an abstract Gal_k-action on $\pi_1^{\text{ét}}(W_{\bar{k}}, \bar{w}_1, \bar{w}_2)$ and $\pi_1^{\text{proét}}(W_{\bar{k}}, \bar{w}_1, \bar{w}_2)$ such that*:

(a) *It is given by $\psi_\sigma = \pi_1^{\text{ét}}(\mathrm{id}_W \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\sigma^{-1}), \bar{w}_1, \bar{w}_2)$ or an analogously defined automorphism of $\pi_1^{\text{proét}}(W_{\bar{k}}, \bar{w}_1, \bar{w}_2)$. This makes sense as $\bar{w}_1, \bar{w}_2$ are Gal_k-invariant.*

(b) *The morphism $\pi_1^{\text{proét}}(W_{\bar{k}}, \bar{w}_1, \bar{w}_2) \to \pi_1^{\text{ét}}(W_{\bar{k}}, \bar{w}_1, \bar{w}_2)$ is Gal_k-equivariant. Similarly, maps of schemes $(W, \bar{w}_1, \bar{w}_2) \to (W', \bar{w}_1, \bar{w}_2)$ induce Gal_k-equivariant maps on $\pi_1^{\text{ét}}$ and $\pi_1^{\text{proét}}$.*

(c) *For three geometric points $\bar{w}_1, \bar{w}_2, \bar{w}_3$, the Galois action is compatible with the composition maps, i.e., for any $\sigma \in \mathrm{Gal}_k$, the following diagram commutes*:

$$\begin{tikzcd} \pi_1^{\text{proét}}(W_{\bar{k}}, \bar{w}_2, \bar{w}_3) \times \pi_1^{\text{proét}}(W_{\bar{k}}, \bar{w}_1, \bar{w}_2) \arrow[r, "(-)\circ(-)"] \arrow[d, "(\psi_\sigma, \psi_\sigma)"'] & \pi_1^{\text{proét}}(W_{\bar{k}}, \bar{w}_1, \bar{w}_3) \arrow[d, "\psi_\sigma"] \\ \pi_1^{\text{proét}}(W_{\bar{k}}, \bar{w}_2, \bar{w}_3) \times \pi_1^{\text{proét}}(W_{\bar{k}}, \bar{w}_1, \bar{w}_2) \arrow[r, "(-)\circ(-)"] & \pi_1^{\text{proét}}(W_{\bar{k}}, \bar{w}_1, \bar{w}_3) \end{tikzcd}$$

Similarly for $\pi_1^{\text{ét}}$. Inductively, this also holds for arbitrary (finite) composition maps.

(d) *Let s_{w_1}, s_{w_2} be the sections of the maps $\pi_1^{\text{ét}}(W, \bar{w}_i) \to \mathrm{Gal}_k$ coming from rational points w_1, w_2. Then $\psi_\sigma(\gamma) = s_{w_2}(\sigma) \circ \gamma \circ s_{w_1}(\sigma^{-1})$ for $\gamma \in \pi_1^{\text{ét}}(W_{\bar{k}}, \bar{w}_1, \bar{w}_2) \subset \pi_1^{\text{ét}}(W, \bar{w}_1, \bar{w}_2)$. Note that, while ψ_σ is defined for $\pi_1^{\text{proét}}$, the right-hand side of this formula only makes sense thanks to the fundamental exact sequence for $\pi_1^{\text{ét}}$ (and its version for the sets of paths, see* [Stix 2013, Proposition 18]). *Thus, at this stage, we cannot make an analogous statement for $\pi_1^{\text{proét}}$.*

In terms of continuity of ψ_σ, there is a priori a huge difference in how much we can say about $\pi_1^{\text{ét}}$ and $\pi_1^{\text{proét}}$:

(e) *For each σ, the map ψ_σ is continuous as an automorphism of $\pi_1^{\text{ét}}(W_{\bar{k}}, \bar{w}_1, \bar{w}_2)$ and $\pi_1^{\text{proét}}(W_{\bar{k}}, \bar{w}_1, \bar{w}_2)$.*

(f) *The action $\mathrm{Gal}_k \times \pi_1^{\text{ét}}(W_{\bar{k}}, \bar{w}_1, \bar{w}_2) \to \pi_1^{\text{ét}}(W_{\bar{k}}, \bar{w}_1, \bar{w}_2)$ is continuous. Note, however, that* ***at this stage of the proof we do not know whether this is true for $\pi_1^{\text{proét}}$****. In fact, this is closely related to the main result we need to prove.*

4C. ***Proof that $\pi_1^{\text{proét}}(X_{\bar{k}}) \to \pi_1^{\text{proét}}(X)$ is a topological embedding.*** In this subsection we finally prove our main result.

Theorem 4.13. *Let k be a field and fix an algebraic closure $\bar{k}$ of k. Let X be a scheme of finite type over k such that the base-change $X_{\bar{k}}$ is connected. Let $\bar{x}$ be a $\mathrm{Spec}(\bar{k})$-point on $X_{\bar{k}}$. Then the induced map*

$$\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x}) \to \pi_1^{\text{proét}}(X, \bar{x})$$

is a topological embedding.

Then, we will derive the final form of the fundamental exact sequence.

Theorem 4.14. *With the assumptions as in Theorem 4.13, the sequence of abstract groups*

$$1 \to \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x}) \to \pi_1^{\text{proét}}(X, \bar{x}) \to \mathrm{Gal}_k \to 1$$

is exact.

Moreover, the map $\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x}) \to \pi_1^{\text{proét}}(X, \bar{x})$ is a topological embedding and the map $\pi_1^{\text{proét}}(X, \bar{x}) \to \mathrm{Gal}_k$ is a quotient map of topological groups.

In the proof, after some preparatory steps (e.g., extending the field k), we define the set of *regular loops* in $\pi_1^{\text{proét}}(X_{\bar{k}})$ with respect to a fixed open subgroup $U <^\circ \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$ and use it to construct an Galois invariant open subgroup V inside of U (see Steps II and III below). *There is also an alternative approach* to proving the existence of V that avoids the direct construction involving regular loops. *We sketch it in Remark 4.27*. While this latter approach is quicker, it is less instructive: as explained in Remark 4.26 below, the notion of a regular loop provides an insight of what goes wrong in the counterexample Example 4.5. Still, it might be worth having a look at, as our main approach is rather lengthy.

Step I: ***Setting things up and applying van Kampen.*** For any finite extension $\bar{k}/l/k$ of k, the map $\pi_1^{\text{proét}}(X_l, \bar{x}) \to \pi_1^{\text{proét}}(X, \bar{x})$ is an embedding of an open subgroup and we have a factorization $\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x}) \to \pi_1^{\text{proét}}(X_l, \bar{x}) \to \pi_1^{\text{proét}}(X, \bar{x})$. Here, we have tacitly lifted $\bar{x}$ to X_l. Thus, we can start by replacing k by a finite extension. Considering the normalization $X^\nu \to X$, base-changing the whole problem to a finite extension l of k, considering the factorization $l/l'/k$ into separable and purely inseparable extension of fields, and using first that the base-change along a separable field extension of a normal scheme is normal and then the topological invariance of $\pi_1^{\text{proét}}$, we can assume that we have a surjective finite morphism $h : \widetilde{X} \to X$ such that the connected components of $\widetilde{X}$, $\widetilde{X} \times_X \widetilde{X}$, $\widetilde{X} \times_X \widetilde{X} \times_X \widetilde{X}$ are geometrically connected, have rational points and for each $W \in \pi_0(\widetilde{X})$, there is $\pi_1^{\text{proét}}(W) = \pi_1^{\text{ét}}(W)$ and $\pi_1^{\text{proét}}(W_{\bar{k}}) = \pi_1^{\text{ét}}(W_{\bar{k}})$.

Let $\widetilde{X} = \bigsqcup_{v \in \text{Vert}} \widetilde{X}_v$ be the decomposition into connected components. Note that the indexing set Vert is finite. For each $t \in \pi_0(\widetilde{X}) \cup \pi_0(\widetilde{X} \times_X \widetilde{X}) \cup \pi_0(\widetilde{X} \times_X \widetilde{X} \times_X \widetilde{X})$, we fix a k-rational point $x(t)$ on t and a $\bar{k}$-point $\bar{x}(\bar{t})$ on $\bar{t} = t_{\bar{k}}$ lying over $x(t)$. We will often write $\bar{x}_t$ to mean $\bar{x}(t)$. Let us fix $v_{\bar{x}} \in \text{Vert}$ for the rest of the text and say that the image of $\bar{x}(\widetilde{X}_{v_{\bar{x}}, \bar{k}})$ in $X_{\bar{k}}$ will be the fixed geometric point $\bar{x}$ of $X_{\bar{k}}$ and its image in X the fixed geometric point of X. For any $W_{\bar{k}}, W'_{\bar{k}} \in \pi_0(S_\bullet(\bar{h}))$ and every boundary map $\bar{\partial} : W_{\bar{k}} \to W'_{\bar{k}}$, we fix paths $\gamma_{W'_{\bar{k}}, W_{\bar{k}}} \in \pi_1^{\text{proét}}(W'_{\bar{k}}, \bar{x}_{W'_{\bar{k}}}, \bar{\partial}(\bar{x}_{W_{\bar{k}}}))$ between the chosen geometric points, as in Corollary 3.19. This is possible thanks to Lemma 3.17. We define $\gamma_{W',W}$ to be the image of this path.

Let $\bar{h} : \widetilde{X}_{\bar{k}} \to X_{\bar{k}}$ be the base-change of h. We choose a maximal tree T (resp. T') in the graph $\Gamma = \pi_0(S_\bullet(h))_{\leqslant 1}$ (resp. $\Gamma' = \pi_0(S_\bullet(\bar{h}))_{\leqslant 1}$). After making these choices, we can apply Corollary 3.19 with Remark 3.21 to write the fundamental groups of $(X, \bar{x})$ and $(X_{\bar{k}}, \bar{x})$. This way we get a diagram

$$\begin{array}{ccc} \Big(\big((*_v^{\text{top}} \pi_1^{\text{ét}}(\widetilde{X}_{v,\bar{k}}, \bar{x}_v)) *^{\text{top}} \pi_1(\Gamma', T')\big) / \overline{\langle R'_1, R'_2 \rangle^{nc}}\Big)^{\text{Noohi}} & \xrightarrow{\simeq} & \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x}) \\ \big\downarrow & & \big\downarrow \\ \Big(\big((*_v^{\text{top}} \pi_1^{\text{ét}}(\widetilde{X}_v, \bar{x}_v)) *^{\text{top}} \pi_1(\Gamma, T)\big) / \overline{\langle R_1, R_2 \rangle^{nc}}\Big)^{\text{Noohi}} & \xrightarrow{\simeq} & \pi_1^{\text{proét}}(X, \bar{x}) \end{array}$$

where $\overline{(\cdot)}$ denotes the topological closure, $\langle R \rangle^{nc}$ denotes the normal subgroup generated by the set R, and R_1, R'_1, R_2, R'_2 are as in Remark 3.21.

Note that, while the (connected components of the) fiber products $\widetilde{X} \times_X \widetilde{X}$, $\widetilde{X} \times_X \widetilde{X} \times_X \widetilde{X}$ are not necessarily normal nor satisfy $\pi_1^{\text{proét}}(W) = \pi_1^{\text{ét}}(W)$, we can effectively work as if this was the case, see Remark 3.21.

Observation 4.15. The maps and groups above enjoy the following properties:

(a) By Lemma 3.27, the left vertical map is the Noohi completion of the obvious map of the underlying quotients of free topological products.

(b) By geometrical connectedness of the schemes in sight, we can (and do) identify

$$\pi_0(S_\bullet(h)) = \pi_0(S_\bullet(\bar{h})), \quad \Gamma' = \Gamma \quad \text{and} \quad T' = T.$$

(c) As the $\gamma_{W,W}$ are chosen to be the images of the $\gamma_{W'_{\bar{k}},W_{\bar{k}}}$, we see that $\alpha^{(f)}_{abc}$'s appearing in R_2, and so a priori elements of the $\pi_1^{\text{ét}}(\widetilde{X}_v, \bar{x}_v)$, are in fact in $\pi_1^{\text{ét}}(\widetilde{X}_{v,\bar{k}}, \bar{x}_v)$. It follows that

$$R'_2 = R_2.$$

(d) The k-rational points $x(W)$ give identification

$$\pi_1^{\text{ét}}(W, \bar{x}_W) \simeq \pi_1^{\text{ét}}(W_{\bar{k}}, \bar{x}_W) \rtimes \mathrm{Gal}_k\,.$$

When $W = \widetilde{X}_v$ for $v \in \mathrm{Vert}$, we will write $\mathrm{Gal}_{k,v}$ in the identification above to distinguish between different copies of Gal_k in the van Kampen presentation of $\pi_1^{\text{proét}}(X, \bar{x})$.

(e) As $\gamma_{W',W}$ is the image of the path $\gamma_{W'_{\bar{k}},W_{\bar{k}}}$ on $W'_{\bar{k}}$, it maps to the trivial element of $\mathrm{Gal}_k = \pi_1^{\text{ét}}(\mathrm{Spec}(k), \bar{x}(W), \bar{x}(W'))$. It implies, that the following diagram commutes:

$$\begin{tikzcd}
\pi_1^{\text{ét}}(W_{\bar{k}}, \bar{x}(W)) \arrow[r] \arrow[d] & \pi_1^{\text{ét}}(W, \bar{x}(W)) \arrow[d] \arrow[dr] & \\
 & & \mathrm{Gal}_k \\
\pi_1^{\text{ét}}(W'_{\bar{k}}, \bar{x}(W')) \arrow[r] & \pi_1^{\text{ét}}(W', \bar{x}(W')) \arrow[ur] &
\end{tikzcd}$$

Let P be a walk in Γ, i.e., a sequence of consecutive edges (with possible repetitions) $e_1, \dots, e_m$ in Γ with an orientation such that the terminal vertex of e_i is the initial vertex of e_{i+1}. Using the orientation of Γ, it can be written as $\epsilon_1 e_1 \cdots \epsilon_m e_m$ with $\epsilon_i \in \{\pm\}$ indicating whether the orientation agrees or not. This will come handy as follows: define $\partial_0^+ = \partial_0$, $\partial_0^- = \partial_1$, $\partial_1^+ = \partial_1$, $\partial_1^- = \partial_0$.

For each P as above with a vertex sequence $(v_1, v_2, \dots, v_{m+1})$, there is a map

$$\pi_1^{\text{ét}}(\widetilde{X}_{v_{m+1},\bar{k}}, \partial_1^{\epsilon_m}(\bar{x}_{e_m}), \bar{x}_{v_{m+1}}) \times \pi_1^{\text{ét}}(\widetilde{X}_{v_m,\bar{k}}, \bar{x}_{v_m}, \partial_0^{\epsilon_m}(\bar{x}_{e_m})) \times \\ \cdots \times \pi_1^{\text{ét}}(\widetilde{X}_{v_2,\bar{k}}, \partial_1^{\epsilon_1}(\bar{x}_{e_1}), \bar{x}_{v_2}) \times \pi_1^{\text{ét}}(\widetilde{X}_{v_1,\bar{k}}, \bar{x}_{v_1}, \partial_0^{\epsilon_1}(\bar{x}_{e_1})) \to \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x}_{v_1}, \bar{x}_{v_{m+1}})$$

where

$$(\gamma_{2m}, \dots, \gamma_1) \mapsto \gamma_{2m} \circ \cdots \circ \gamma_1.$$

In the following, we will use $\circ_?$ to denote the "composition of étale paths" and $\bullet_?$ to denote the multiplication in some group(oid) ?. When $? = \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$ or $\pi_1^{\text{proét}}(X, \bar{x})$, we will skip the subscript. While we could just use $\circ_?$ everywhere, it is sometimes convenient to keep track of when some paths "have been closed" by using $\bullet_?$.

Step II: ***Defining regular loops in*** $\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$**.**

Definition 4.16. An element $\gamma \in \mathrm{Isom}_{\mathrm{Cov}_{X_{\bar{k}}}}(F_{\bar{x}_w}, F_{\bar{x}_v})$ is called an *étale path of special form supported on P* if it lies in the image of the composition map above for some walk P starting in w and ending in v.

Any element $(\gamma_{2m}, \dots, \gamma_1)$ in the preimage of such γ will be called a *presentation* of γ with respect to P.

For a walk P, denote by $l(P)$ the length of P, i.e., the number of consecutive edges (not necessarily different) it is composed of.

Observation 4.17. A useful example of a path of special form is the following. In the van Kampen presentation, the maps $\pi_1^{\text{ét}}(\widetilde{X}_{v,\bar{k}}, \bar{x}_v) = \pi_1^{\text{proét}}(\widetilde{X}_{v,\bar{k}}, \bar{x}_v) \to \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$ are given by

$$\rho_v(-) = \gamma_v^{-1} \circ (-) \circ \gamma_v$$

where $\gamma_v \in \pi_1^{\text{proét}}(\widetilde{X}_{v,\bar{k}}, \bar{x}, \bar{x}_v)$ is defined as follows: if $P_{v_{\bar{x}},v} \subset T$ denotes the unique shortest path in the tree $T \subset \Gamma$ (forgetting the orientation) from $v_{\bar{x}}$ to v, then the choices of paths $\gamma_{W'_{\bar{k}}, W_{\bar{k}}}$ made when applying the van Kampen theorem give a unique étale path of special form γ_v supported on $P_{v_{\bar{x}},v}$.

Before introducing the main objects of the proof, we note a simple result.

Lemma 4.18. *For a fixed path $\gamma \in \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x}, \bar{y})$ of special form, the map* $\mathrm{Gal}_k \ni \sigma \mapsto \psi_\sigma(\gamma) \in \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x}, \bar{y})$ *is continuous.*

Proof. This follows from the continuity of the composition maps of paths and the fact that the statement is true for $\pi_1^{\text{ét}}$. □

To prove Theorem 4.13, *it is enough to prove the following statement*: any connected geometric covering Y of $X_{\bar{k}}$ can be dominated by a covering defined over a finite separable extension l/k.

Indeed, let $Y' \in \mathrm{Cov}_{X_l}$ be a connected covering that dominates Y after base-change to $\bar{k}$. By looking at the separable closure of k in l and using the topological invariance of $\pi_1^{\text{proét}}$, we can assume l/k is separable. The composition $Y'' = Y' \to X_l \to X$ is an element of Cov_X and there is a diagonal embedding $Y' \times_{\mathrm{Spec}(l)} \mathrm{Spec}(\bar{k}) \to Y'' \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$. By Proposition 2.37(5), the proof will be finished.

Let us fix a connected $Y \in \mathrm{Cov}$ till the end of the proof and denote by $S = Y_{\bar{x}}$ the corresponding $\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$-set. Fix some point $s_0 \in S$ and let $U = \mathrm{Stab}_{\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})}(s_0)$.

Definition 4.19. For each $v \in \mathrm{Vert}$, define

$$O_v^N = \{s \in F_{\bar{x}_v}(Y) \mid \exists_{\substack{\text{walk } P,\\ l(P) \le N}} \exists_{\substack{\gamma \text{ of sp. form,}\\ \text{supp. on } P}} s = \gamma \cdot s_0\}$$

and call it the set of "elements at v reachable in at most N steps".

The following is a crucial observation regarding O_v^N.

Lemma. *For any v and N, the set O_v^N is finite.*

Proof. We proceed by induction on N. For $N = 1$, the walks of length not greater than N starting in v_0 (are either trivial or) consist of a single edge whose initial vertex is necessarily $v_{\bar{x}}$. As Γ is finite, there are only finitely many such edges. Let us fix one, named e, with vertices v_0, w. We need to show that the set

$$\{(\theta \circ \delta) \cdot s_0 \in F_{\bar{x}_w}(Y) \mid \delta \in \pi_1^{\text{ét}}(\widetilde{X}_{v_{\bar{x}},\bar{k}}, \bar{x}, \partial_1^{\epsilon(e)}(\bar{x}_e)), \theta \in \pi_1^{\text{ét}}(\widetilde{X}_{w,\bar{k}}, \partial_0^{\epsilon(e)}(\bar{x}_e), \bar{x}_w)\}$$

is finite. However, as in general the sets $\pi_1^{\text{ét}}(W, \bar{x}_1, \bar{x}_2)$ are (bi)torsors under profinite groups (namely $\pi_1^{\text{ét}}(W, \bar{x}_1)$ and $\pi_1^{\text{ét}}(W, \bar{x}_2)$) and the maps and actions in sight are continuous, we see that the finiteness of this last set follows directly from finiteness of orbits of points in discrete sets under an action by a profinite group.

Now, to see the inductive step, assume that the claim is true for N. To prove it for $N + 1$, note that any element in O_v^{N+1} can be connected by a single edge to an element of O_w^N (for some vertex w). As O_w^N is finite and as we have just explained that, starting from a fixed point, one can only reach finitely many points by applying étale paths of special forms supported on a single edge, the result follows. $\square$

Now, for each $v \in \mathrm{Vert}$ and $N \in \mathbb{N}$, define $C_v^N \in \pi_1^{\text{ét}}(\widetilde{X}_{v,\bar{k}}, \bar{x}_v) - \mathrm{FSets}$ so that:

(1) It is Galois.

(2) It dominates each of the $\pi_1^{\text{ét}}(\widetilde{X}_{v,\bar{k}}, \bar{x}_v)$-orbits of elements of O_v^N.

(3) The corresponding open normal subgroup $\ker(\pi_1^{\text{ét}}(\widetilde{X}_{v,\bar{k}}, \bar{x}_v) \to \mathrm{Aut}_{\mathrm{Sets}}(C_v^N))$ is normalized by $\mathrm{Gal}_{k,v}$, where we use the action $\mathrm{Gal}_{k,v} \ni \sigma \mapsto \psi_\sigma \in \mathrm{Aut}(\pi_1^{\text{ét}}(\widetilde{X}_{v,\bar{k}}, \bar{x}_v))$ or, equivalently by Lemma 4.12(d), conjugation by $s_{\bar{x}_v}(\sigma)$ in $\pi_1^{\text{ét}}(\widetilde{X}_v, \bar{x}_v)$.

(4) There is a surjection $C_v^{N+1} \twoheadrightarrow C_v^N$ of $\pi_1^{\text{ét}}(\widetilde{X}_v, \bar{x}_v)$-sets.

We can find sets satisfying the first three conditions by applying Proposition 4.11, and the last condition can be guaranteed by choosing the C_v^N's inductively (for a given v).

We now proceed to define a subgroup of $\pi_1^{\text{proét}}(X, \bar{x})$ that will lead to the desired $\pi_1^{\text{proét}}(X, \bar{x})$-set. For that we need to find a suitably large subgroup of elements of U that are well behaved under the Galois action.

Definition 4.20. We will call an element $g \in \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$ a *regular loop* (with respect to U) if there exists v, m, a walk P of length m from $v_1 = v_{\bar{x}}$ to $v_{m+1} = v$, étale paths γ, γ' of special form supported on P and P^{-1}, respectively, and $\beta \in \pi_1^{\text{ét}}(\widetilde{X}_{v,\bar{k}}, \bar{x}_v)$ such that:

- $g = \gamma' \circ \beta \circ \gamma$.
- β acts trivially on C_v^m, i.e.,
$$\beta \in \ker(\pi_1^{\text{ét}}(\widetilde{X}_{v,\bar{k}}, \bar{x}_v) \to \mathrm{Aut}(C_v^m)).$$
- There exist presentations $(\gamma_{2m}, \ldots, \gamma_1)$ and $(\gamma'_1, \ldots, \gamma'_{2m})$ of γ and γ' such that the following condition is satisfied. For any $1 \leqslant i \leqslant m$, there is
$$\gamma'_{2i-1} \circ \gamma_{2i-1} \in \ker(\pi_1^{\text{ét}}(\widetilde{X}_{v_i,\bar{k}}, \bar{x}_{v_i}) \to \mathrm{Aut}(C_{v_i}^i))$$
and
$$\gamma_{2i} \circ \gamma'_{2i} \in \ker(\pi_1^{\text{ét}}(\widetilde{X}_{v_{i+1},\bar{k}}, \bar{x}_{v_{i+1}}) \to \mathrm{Aut}(C_{v_{i+1}}^i)).$$

The following picture might be useful to visualize the definition:

$$
\begin{tikzcd}
\bullet \arrow[r, bend left, "\gamma_1"] & \bullet \arrow[l, bend left, "\gamma_1'"] \arrow[r, bend left, "\gamma_2"] & \bullet \arrow[l, bend left, "\gamma_2'"] \cdots \bullet \arrow[r, bend left, "\gamma_{2m-1}"] & \bullet \arrow[l, bend left, "\gamma_{2m-1}'"] \arrow[r, bend left, "\gamma_{2m}"] & \bullet \arrow[l, bend left, "\gamma_{2m}'"] \arrow[loop right, "\beta"]
\end{tikzcd}
$$

Here, the larger bullets correspond to the $\bar{x}_{v_i}$ and the smaller ones to $\partial^{\epsilon}_{0\text{ or }1}(\bar{x}(e_i))$.

Remark 4.21. We find the definition involving the C_v^N quite convenient. One could, however, avoid introducing the C_v^N and make a slightly different definition. Define $O_v^{N,+}$ to be the set of (isomorphism classes of) Gal_k-conjugates of the $\pi_1^{\text{ét}}(\widetilde{X}_{v,\bar{k}}, \bar{x}_v)$-sets in O_v^N. Proposition 4.11 then implies that $O_v^{N,+}$ are finite as well. Moreover, for each v, both O_v^N and $O_v^{N,+}$ are increasing with N. We could then require the β and the $(\gamma'_{2i-1} \circ \gamma_{2i-1})$ as above to act trivially on each element of $O_v^{m,+}$ and $O_{v_i}^{i,+}$, correspondingly.

Step III: ***Defining the desired open subgroup V, checking its properties and finishing the proof.*** We make the following central definition:

> *Let $V_0 < \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$ denote the subgroup generated by the set of regular loops and let V be its topological closure.*

Let $G = \big((*_v^{\text{top}} \pi_1^{\text{ét}}(\widetilde{X}_{v,\bar{k}}, \bar{x}_v)) *^{\text{top}} \pi_1(\Gamma', T')\big)/\overline{\langle R_1', R_2'\rangle^{nc}}$ denote the topological group appearing in the van Kampen presentation above. We have that $G^{\text{Noohi}} = \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$. Let $\tilde{G} \subset \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$ denote the subgroup of all étale paths (or "loops", rather) of special form, supported on walks from $v_{\bar{x}}$ to $v_{\bar{x}}$.

Observation 4.22. By Observation 4.17, the map $G \to \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x}) = G^{\text{Noohi}}$ factorizes through $\tilde{G}$. Directly from the definitions, there is $V_0 < \tilde{G}$. We are thus in the situation of Lemma 2.39. We will use it below.

For brevity, let us denote $\overline{G}_v = \pi_1^{\text{ét}}(\widetilde{X}_{v,\bar{k}}, \bar{x}_v)$ and $G_v = \pi_1^{\text{ét}}(\widetilde{X}_v, \bar{x}_v) \simeq \overline{G}_v \rtimes \mathrm{Gal}_k$ in the proofs below.

Proposition 4.23. *The following statements about the subgroup V hold*:

(1) *There is a containment $V < U$.*

(2) *It is an open subgroup.*

(3) *The groups V_0 and V are invariant under the Galois action, i.e., $\psi_\sigma(V_0) = V_0$ and $\psi_\sigma(V) = V$ for all $\sigma \in \mathrm{Gal}_k$.*

Proof. (1) As any open subgroup is automatically closed, it is enough to show that any regular loop lies in U. Let g be a regular loop and write $g = \gamma' \circ \beta \circ \gamma$ with γ, γ' étale paths of special form supported on some walk (and its inverse) from $v_{\bar{x}}$ to v of length m, with presentations $(\gamma_1, \dots, \gamma_{2m})$ and $(\gamma'_{2m}, \dots, \gamma'_1)$ of γ and γ', and $\beta \in \ker(\pi_1^{\text{ét}}(\widetilde{X}_{v,\bar{k}}, \bar{x}_v) \to C_v^m)$, as in the definition of a regular loop. Let us introduce the following notation (and analogously for γ')

$$\gamma_{i\leftarrow 1} = \gamma_i \circ \cdots \circ \gamma_2 \circ \gamma_1.$$

By definition, there is

$$\gamma_{2i\leftarrow 1} \cdot s_0 \in O_{v_i}^i.$$

For $i = m$, it follows from the condition on β that $(\beta \circ \gamma) \cdot s_0 = \gamma \cdot s_0$. Similarly, the condition on $\gamma_{2m} \circ \gamma'_{2m}$ implies that $(\gamma'^{-1}_{2m} \circ \gamma^{-1}_{2m}) \circ \gamma_{2m\leftarrow 1} \cdot s_0 = \gamma_{2m\leftarrow 1} \cdot s_0$, and thus

$$(\gamma' \circ \beta \circ \gamma) \cdot s_0 = (\gamma' \circ \gamma) \cdot s_0 = ((\gamma'_{1\leftarrow 2m-1} \circ \gamma'_{2m} \circ \gamma_{2m\leftarrow 1})) \cdot s_0 = \gamma'_{1\leftarrow 2m-1} \circ \gamma_{2m-1\leftarrow 1} \cdot s_0.$$

The process continues in a similar fashion to show that g stabilizes s_0, and thus belongs to U.

(2) By Lemma 2.39, it is enough to check that the map $G \to \mathrm{Aut}(\tilde{G}/V_0)$ is continuous when $\tilde{G}/V_0$ is considered with the discrete topology.

Using the universal property of free topological products, continuity can be checked separately for $\overline{G}_v$ and D. For D, this is automatic, as D is discrete. To see the result for $\overline{G}_v$'s, we need to show that the stabilizers of the action of $\overline{G}_v$ on $\tilde{G}/V_0$ induced by $\overline{G}_v \to G$ are open. Fix $[gV_0] \in \tilde{G}/V_0$ and $g \in \tilde{G}$ representing it. The element g is represented by some étale path (or a "loop", in fact) of special form ρ supported on a walk P_ρ of length $l(P_\rho)$. By Observation 4.17, the morphism $\overline{G}_v \to \tilde{G} \subset \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$ is also defined using an étale path of special form γ_v supported on a walk $P_{v_{\bar{x}},v}$ in the tree $T \subset \Gamma$. Let $H_v = \ker(\overline{G}_v \to \mathrm{Aut}(C_v^{l(P_\rho)+l(P_v)})) < \overline{G}_v$. Then H_v is open in $\overline{G}_v$ and its image in $\tilde{G}$ can be written as $\{\gamma_v^{-1} \circ \beta \circ \gamma_v | \beta \in H_v\}$. It follows from the setup that for $\beta \in H_v$,

$$g^{-1} \circ \gamma_v^{-1} \circ \beta \circ \gamma_v \circ g \in V_0$$

and so any element in the image of H_v fixes $[gV_0]$ in G/V. Thus, the stabilizer of $[gV_0]$ in $\overline{G}_v$ is also open, as desired.

(3) For each σ, the map ψ_σ is continuous. As $V = \overline{V_0}^{G^{\text{Noohi}}}$, it is thus enough to prove that V_0 is Gal_k-invariant. By Lemma 4.12, it follows that under the action of Gal_k, an étale path of special form supported on a walk P is mapped again to an étale path of special form supported on P. Consequently, checking that the action of $\sigma \in \mathrm{Gal}_k$ maps a regular loop g to another regular loop boils down to checking the following fact. If g has a presentation $g = \gamma' \circ \beta \circ \gamma$ as in the definition of a regular loop, then

- $\psi_\sigma(\beta)$ still acts trivially on $C_v^{l(P)}$;
- $\psi_\sigma(\gamma'_i \circ \gamma_i)$ or $\psi_\sigma(\gamma_i \circ \gamma'_i)$, depending on parity, still acts trivially on $C^{\lceil i \rceil}_{v_{\lfloor i \rfloor+1}}$ for every i.

However, this follows from property (3) in the definition of the $C^j_{v_j}$. □

Denote by S' the quotient $\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})/V$ considered as a $\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$-set.

Let $\rho_v : \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x}_v) \to \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$ be the isomorphism defined using the fixed (étale) path γ_v between $\bar{x}_v$ and $\bar{x}$, as in Observation 4.17. We have an action given by $\sigma \mapsto \psi_\sigma$ on both $\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$ and $\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x}_v)$. We already know that $V < \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$ is invariant under this action, but this is not necessarily true for $\rho_v^{-1}(V) < \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x}_v)$. This holds after a finite base field extension.

Lemma 4.24. *For each $v \in \mathrm{Vert}$, define an (abstract) $\mathrm{Gal}_{k,v}$-action on $\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$ to be*

$$^{\sigma_v}g = \rho_v(\psi_\sigma(\rho_v^{-1}(g))).$$

Then there exists a finite extension l/k, such that for all $v \in \mathrm{Vert}$, there is:

(a) $\mathrm{Gal}_{l,v}$ *fixes* V.

(b) *The obtained induced* $\mathrm{Gal}_{l,v}$*-action on* S' *can be written as*

$$\sigma_v \cdot [gV] = [(\gamma_v^{-1} \circ \psi_\sigma(\gamma_v)) \bullet \psi_\sigma(g)V].$$

(c) *The induced* $\mathrm{Gal}_{l,v}$ *action on* S' *is continuous and compatible with the* $\overline{G}_v$*-action.*

Proof. As there are finitely many vertices v, it is enough to prove the statements for a single fixed v. Let $g \in V$. By definition of ρ_v, there is

$$^{\sigma_v}g = \gamma_v^{-1} \circ (\psi_\sigma(\gamma_v \circ g \circ \gamma_v^{-1})) \circ \gamma_v = (\gamma_v^{-1} \circ \psi_\sigma(\gamma_v)) \bullet \psi_\sigma(g) \bullet (\psi_\sigma(\gamma_v^{-1}) \circ \gamma_v).$$

By Proposition 4.23, we have $\psi_\sigma(g) \in V$ and we only need to show that $\gamma_v^{-1} \circ \psi_\sigma(\gamma_v) \in V$. By Lemma 4.18 and Observation 4.17, the map $\mathrm{Gal}_k \ni \sigma \mapsto \psi_\sigma(\gamma_v) \in \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x}, \bar{x}_v)$ is continuous, and we conclude that for an open subgroup of $\sigma \in \mathrm{Gal}_k$ we have the desired containment.

It follows from the previous point that we get an induced action of $\mathrm{Gal}_{l,v}$ on S'. Using that $\gamma_v^{-1} \circ \psi_\sigma(\gamma_v) \in V$, the alternative formula in the statement follows from the computation

$$\sigma_v \cdot [gV] = [\rho_v(\psi_\sigma(\rho_v^{-1}(g)))V] = [(\gamma_v^{-1} \circ \psi_\sigma(\gamma_v)) \bullet \psi_\sigma(g) \bullet (\psi_\sigma(\gamma_v)^{-1} \circ \gamma_v)V] = [(\gamma_v^{-1} \circ \psi_\sigma(\gamma_v)) \bullet \psi_\sigma(g)V].$$

Let us move to the last point. Compatibility with the $\overline{G}_v$-action follows from Lemma 4.12(d) and the fact that the map $\overline{G}_v \to \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$ is defined by postcomposing with ρ_v. To check continuity, fix $[gV]$. By Lemma 2.39, this class is represented by a path (loop) of special form, and so we can assume this about g. Checking that the stabilizer of $[gV]$ is open boils down to checking that for an open subgroup of the σ in $\mathrm{Gal}_{l,v}$, one has $g^{-1} \bullet (\gamma_v^{-1} \circ \psi_\sigma(\gamma_v)) \bullet \psi_\sigma(g) \in V$. However, this follows from the openness of V and Lemma 4.18. □

Proposition 4.25. *There is a (continuous)* $\pi_1^{\text{proét}}(X_l, \bar{x})$*-action on* S' *that extends the* $\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$*-action.*

Proof. By the van Kampen theorem for $\pi_1^{\text{proét}}(X_l, \bar{x})$, it is enough to show that there are continuous actions of the $\overline{G}_v \rtimes \mathrm{Gal}_{l,v}$ and D compatible with the $\overline{G}_v$ and D actions that S' is already equipped with, and such that the van Kampen relations are satisfied. We already have a continuous action by D on S', and by Lemma 4.24, we get an action of $\overline{G}_v \rtimes \mathrm{Gal}_{l,v}$.

Let us now check that the van Kampen relations are preserved. In the case of relation R_2, this is automatic by Observation 4.15(c). This is because we have left the $\overline{G}_v$-actions intact. To check that relation R_1 is respected, it suffices to check that $\pi_1^{\text{ét}}(\partial_1)(\sigma)\vec{E} = \vec{E}\pi_1^{\text{ét}}(\partial_0)(\sigma)$ for $\sigma \in \mathrm{Gal}_{l,E}$ and E an edge in Γ with vertices v_-, v_+. Let $\delta_{W',W} = \gamma_{W',W}$ denote the fixed paths from the van Kampen setup in the computation below to make the distinction from the γ_v clearer. Using Lemma 4.12(d), we compute that

$$\pi_1^{\text{ét}}(\partial_0)(\sigma) = \delta_{v+,E}^{-1} \circ_{G_{v+}} \sigma_E \circ_{G_{v+}} \delta_{v+,E} = \delta_{v+,E}^{-1} \circ_{G_{v+}} \psi_\sigma(\delta_{v+,E}) \circ_{G_{v+}} \sigma_{v+} = (\delta_{v+,E}^{-1} \circ_{G_{v+}} \psi_\sigma(\delta_{v+,E})) \bullet_{G_{v+}} \sigma_{v+}.$$

The image of $(\delta_{v+,E}^{-1} \circ_{G_{v+}} \psi_\sigma(\delta_{v+,E}))$ in $\pi_1^{\text{proét}}(X, \bar{x})$ via ρ_{v+} is $\gamma_{v+}^{-1} \circ \delta_{v+,E}^{-1} \circ \psi_\sigma(\delta_{v+,E}) \circ \gamma_{v+}$.

By definition, $\vec{E} \in \pi_1^{\text{proét}}(X, \bar{x})$ can be written as $\vec{E} = \gamma_{v-}^{-1} \circ \delta_{v-,E}^{-1} \circ \delta_{v+,E} \circ \gamma_{v+}$. Putting this together and using the formula of Lemma 4.24, we have that $\vec{E} \bullet \pi_1^{\text{ét}}(\partial_0)(\sigma) \cdot [hV]$ equals

$$(\gamma_{v-}^{-1} \circ \delta_{v-,E}^{-1} \circ \delta_{v+,E} \circ \gamma_{v+}) \circ (\gamma_{v+}^{-1} \circ \delta_{v+,E}^{-1} \circ \psi_\sigma(\delta_{v+,E}) \circ \gamma_{v+}) \cdot [(\gamma_{v+}^{-1} \circ \psi_\sigma(\gamma_{v+})) \bullet \psi_\sigma(h)V]$$
$$= [(\gamma_{v-}^{-1} \circ \delta_{v-,E}^{-1} \circ \psi_\sigma(\delta_{v+,E} \circ \gamma_{v+})) \bullet \psi_\sigma(h)V].$$

A similar computation (left to the reader) shows that

$$\pi_1^{\text{ét}}(\partial_1)(\sigma) \bullet \vec{E} \cdot [hV] = [(\gamma_{v-}^{-1} \circ \delta_{v-,E}^{-1} \circ \psi_\sigma(\delta_{v+,E} \circ \gamma_{v+})) \bullet \psi_\sigma(h)V]$$

as well. This finishes the proof of the proposition. □

End of the proof of Theorem 4.13. We have proven that for a transitive $\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$-set S there exists a finite extension l/k and a transitive $\pi_1^{\text{proét}}(X_l, \bar{x})$-set S' that dominates S as $\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$-sets. As explained above, this finishes the proof. □

We have finished our main proof, and thus the most difficult part of the exact sequence is now proven. We now obtain the final form of the fundamental exact sequence.

End of the proof of Theorem 4.14. We already know the statements of the "moreover" part and the near exactness in the middle of the sequence. All we have to prove is that $\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$ is thickly closed in $\pi_1^{\text{proét}}(X, \bar{x})$. As $\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x}) \to \pi_1^{\text{proét}}(X, \bar{x})$ is a topological embedding of Raĭkov complete groups, $\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$ is a closed subgroup of $\pi_1^{\text{proét}}(X, \bar{x})$; see, e.g., [Dikranjan 2013, Proposition 6.2.7.]. By Lemma 2.33, the proof will be finished if we show that $\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$ is normal in $\pi_1^{\text{proét}}(X, \bar{x})$. Observe that checking whether $\overline{\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})} = \overline{\overline{\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})}}$ can be performed after replacing $\pi_1^{\text{proét}}(X, \bar{x})$ by any open subgroup U such that $\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x}) < U <^\circ \pi_1^{\text{proét}}(X, \bar{x})$. Choosing a suitably large finite field extension l/k and looking at $U = \pi_1^{\text{proét}}(X_l, \bar{x})$, we are reduced to the situation as in the proof of Theorem 4.13, i.e., we have enough rational points on the connected components we are interested in when applying van Kampen. Let $\tilde{G} < \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$ be the dense subgroup defined above Proposition 4.23. Note that by the van Kampen theorem applied to $\pi_1^{\text{proét}}(X, \bar{x})$ together with the observations in Observation 4.15, it follows that the subgroup generated by $\tilde{G}$ and the $\mathrm{Gal}_{k,v}$ is dense in $\pi_1^{\text{proét}}(X, \bar{x})$. Putting this together, it follows that it is enough to check that, for each v, conjugation by elements of $\mathrm{Gal}_{k,v}$ fixes $\tilde{G}$ in $\pi_1^{\text{proét}}(X, \bar{x})$. This, however, follows from Lemma 4.12(c), (d) and the fact that $\mathrm{Gal}_{k,v} \to \pi_1^{\text{proét}}(X, \bar{x})$ is defined as the composition $\mathrm{Gal}_{k,v} \to \pi_1^{\text{proét}}(X, \bar{x}_v) \xrightarrow{\rho_v} \pi_1^{\text{proét}}(X, \bar{x})$, where $\rho_v = \gamma_v^{-1} \circ (-) \circ \gamma_v$ with $\gamma_v \in \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x}, \bar{x}_v)$ of special form. □

Remark 4.26. Let us revisit the counterexample of Example 4.5 from the point of view of the proof above. We will freely use the notation set there. In this example, we have started from the fixed point s_0, and used the group elements to reach the point $s_1 = g_m h_{m-1} \cdots h_3 g_2 h_1 \cdot s_0$. We have then concluded that $s_2 = \zeta_{\ell^{m+1}} \cdot s_1 = g \cdot s_1$ and justified that the setup forces that this equality contradicts the possibility of extending the Galois action to the set S. The problem here is caused by the fact that, denoting

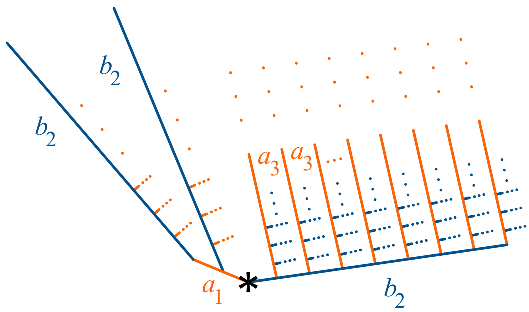


Figure 1. Graphical interpretation of S'.

$\gamma = g_m h_{m-1} \cdots h_3 g_2 h_1$, the element

$$\gamma^{-1} \circ g^{-1} \circ \zeta_{\ell m+1} \circ \gamma$$

stabilizes s_0, but it is not a "regular loop" in the language introduced above. Of course, this only means that this particular "obvious" presentation is not as in the definition of a regular loop. But, by now, we know that it provably cannot be a regular loop with any presentation.

Let us now apply the construction of our main proof in the context of this Example. Let X, $\bar{x}$ and the $\pi_1^{\text{proét}}(\overline{X}, \bar{x})$-set S be as in Example 4.5. Recall that S decomposes $S = \bigsqcup_{i \geq 1} a_{2i-1}$ (resp. $S = \bigsqcup_{i \geq 1} b_{2i}$) as $\pi_1^{\text{ét}}(\overline{X}_1, \bar{x}_1)$-set (resp. $\pi_1^{\text{ét}}(\overline{X}_2, \bar{x}_2)$-set), where we have fixed identifications $\pi_1^{\text{ét}}(\overline{X}_i, \bar{x}_i) \simeq \widehat{\mathbb{Z}}(1)$, $a_j = \mu_{\ell^j}$, $b_j = \mu_{\ell^j}$.

The 2-simplex obtained from the normalization $\widetilde{X} = X_1 \sqcup X_2$ of X has two vertices. Let O_1^N, O_2^N denote the corresponding sets of "elements reachable in at most N steps", as in Definition 4.19. Then one checks that

$$O_1^N = a_1 \cup \cdots \cup a_{2N+1}, \qquad O_2^N = b_2 \cup \cdots \cup b_{2N}.$$

Let us denote $a_i' := a_{2i-1} \setminus \{1\}$, $b_i' := b_{2i} \setminus \{1\}$ (using the fixed identifications with the μ_j). Let $\Sigma = \bigcup_{i \geq 1} (a_i' \cup b_i')$ be the alphabet consisting of all the elements of all the a_j' and b_j'. Let $S' \subset \operatorname{words}(\Sigma)$ be the subset of words on Σ of the following form:

$$S' = \{\varnothing\} \cup \{\beta_m \alpha_{m-1} \cdots \beta_3 \alpha_2 \beta_2 \alpha_1\} \cup \{\alpha_m \beta_m \cdots \beta_3 \alpha_2 \beta_2 \alpha_1\} \cup \{\alpha_m \beta_{m-1} \cdots \alpha_3 \beta_2 \alpha_2 \beta_1\} \cup \{\beta_m \alpha_m \cdots \alpha_2 \beta_2 \alpha_1\},$$

where m runs over $\mathbb{N}_{\geqslant 1}$ and $\alpha_j \in a_j'$, $\beta_j \in b_j'$ for each j. Geometrically, S' can be thought of as an infinite tree: at the element $\varnothing$ we glue copies of a_1 and b_2 so that $1 \in a_1$ and $1 \in b_2$ are identified at $\varnothing$. Now, to each element of $a_1 \setminus 1$ we glue a copy of b_2 at $1 \in b_2$ and to each element of the initial $b_2 \setminus \{1\}$, we glue a copy of a_3 at $1 \in a_3$. Now, to each copy of the recently glued b_2's, we glue a copy of a_3, and to each copy of previously glued a_3's, we glue b_4. The procedure continues; see Figure 1.

There is an obvious action on such a tree by Gal_k, compatible with the $\pi_1^{\text{proét}}(\overline{X}, \bar{x})$-action; via the description of S' in terms of words on Σ, it corresponds to applying the Gal_k-action to each letter via the identifications with the μ_j. There is moreover a $\pi_1^{\text{proét}}(\overline{X}, \bar{x})$-equivariant surjective map $S' \to S$; Indeed, an element of the form, for example, $\alpha_m\beta_m\cdots\beta_3\alpha_2\beta_2\alpha_1 \in S'$ is mapped to the element $\alpha_m\beta_m\cdots\beta_3\alpha_2\beta_2\alpha_1\cdot s_0 \in S$, where the multiplication makes sense thanks to how the a_i, b_j grow with i and j. The constructed set S' thus satisfies the desired properties of the set sought in the proof of Theorem 4.13. Up to some minor tweaking, it will correspond to the set obtained by following the proof of the theorem. We will not, however, try to give a precise proof of that last claim here.

While this "tree construction" example is much more enlightening in the simple cases of schemes glued at one point, it proved to be rather difficult to turn this intuition into a formal proof that would work for arbitrary schemes (i.e., where the normalization might no longer have such a pleasant form). For that reason, we have opted for a proof that is less geometric in nature.

Remark 4.27. We sketch a slightly different approach to the central part of the main proof. It is a bit quicker, but less constructive, i.e., does not "explicitly" construct the desired Galois invariant open subgroup in terms of regular loops. We will freely use the fact that a surjective map from a compact space onto a Hausdorff space is a quotient map.

Assume that we have already done the preparatory steps of the main proof, i.e., we have increased the base field to have many rational points and applied the van Kampen theorem. We want to prove that the action

$$\mathrm{Gal}_k \times \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x}) \to \pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$$

given by ψ_σ is continuous. Let $G, \tilde{G}$ be as introduced above Observation 4.22.

Firstly, one checks that any element of $\tilde{G}$, so a path of special form, can be in fact rewritten with a presentation that makes it visibly an image of an element of G, at the expense of the presentation possibly getting longer. Another words, the map $G \to \tilde{G}$ is surjective. By default, $\tilde{G}$ is considered with the subspace topology from $\pi_1^{\text{proét}}(X_{\bar{k}}, \bar{x})$. Let us denote $(\tilde{G}, \text{quot})$ the same group but considered with the quotient topology from G. We thus have a continuous bijection $(\tilde{G}, \text{quot}) \to \tilde{G}$.

The group G is a topological quotient of the free topological product of finitely many compact groups G_v and a finitely generated free group $D \simeq \mathbb{Z}^{*r}$. One checks from the universal properties that this free product can be written as a quotient of the free topological group $F(Z)$ (see [Arhangel'skii and Tkachenko 2008, Chapter 7]) on a compact space of generators $Z = \bigsqcup_v G_v \bigsqcup_{\{1,\dots,r\}} *$, i.e., the disjoint union of the G_v and r singletons.

By [Arhangel'skii and Tkachenko 2008, Theorem 7.4.1], $F(Z)$ is, as a topological space, a colimit of an increasing union $\cdots \subset B_n \subset B_{n+1} \subset \cdots$ of compact subspaces. These spaces are explicitly described as words of bounded length in $F(Z)$ (this makes sense, as the underlying group of $F(Z)$ is the abstract free group on Z). From this, it follows that (as a topological space) $(\tilde{G}, \text{quot}) = \operatorname{colim} K_n$, with $K_n = \operatorname{im}(B_n)$.

Working directly with the K_n is inconvenient for our purposes, as these sets are not necessarily preserved by the Galois action. The reason is that the van Kampen presentation as a quotient of a free product uses fixed paths, while applying Galois action will usually move the paths. One then has to conjugate by a suitable element to "return" to the paths fixed in van Kampen, possibly increasing the length of the word.

Instead, we can consider subsets $K'_n \subset \tilde{G}$ of elements that are paths of special form of length $\leqslant n$, i.e., possessing a presentation as a path of special form of length $\leqslant n$ (see Definition 4.16). By a reasonably simple combinatorics, one can cook up "brute force" bounds $f(n,d), g(n,d) \in \mathbb{N}$ in terms of n and the diameter $d = \mathrm{diam}(\Gamma)$ of Γ such that there is

$$K_n \subset K'_{f(n,d)} \quad \text{and} \quad K'_n \subset K_{g(n,d)}.$$

In conclusion, $(\tilde{G}, \mathrm{quot}) = \mathrm{colim}\, K'_n$ in Top.

By Lemma 4.12, the Gal_k-action preserves the sets K'_n and $\mathrm{Gal}_k \times K'_n \to K'_n$ is continuous. As Gal_k is compact, $\mathrm{Gal}_k \times (-)$ has a right adjoint $\mathrm{Maps}_{\mathrm{cts}}(\mathrm{Gal}_k, -)$ in Top and so $\mathrm{Gal}_k \times (\mathrm{colim}_{n\in\mathbb{N}} K'_n) = \mathrm{colim}_{n\in\mathbb{N}}(\mathrm{Gal}_k \times K'_n)$. From this, we immediately get that $\mathrm{Gal}_k \times (\tilde{G}, \mathrm{quot}) \to (\tilde{G}, \mathrm{quot})$ is continuous. As Gal_k-action respects the group action of $\tilde{G}$, it quickly follows that the action is still continuous when $(\tilde{G}, \mathrm{quot})$ is equipped with the weakened topology τ making open subgroups a base at 1, as in Lemma 2.25. By (the easier part of) Lemma 2.39, this weakened topology on $(\tilde{G}, \mathrm{quot})$ matches that of $\tilde{G}$. It follows that $\mathrm{Gal}_k \times \tilde{G} \to \tilde{G}$ is continuous.

By Lemma 2.25 again, one has to check that the continuity is not lost when passing to the Raĭkov completion of the maximal Hausdorff quotient of (G, τ). This in turn can be justified by similar arguments as in the proof of Lemma 2.39. This finishes the sketch. See also [Bhatt and Scholze 2015, Proposition 4.3.3].

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Infinitesimal dilogarithm on curves over truncated polynomial rings

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We construct infinitesimal invariants of thickened one dimensional cycles in three dimensional space, which are the simplest cycles that are not in the Milnor range. This generalizes Park's work on the regulators of additive cycles. The construction also allows us to prove the infinitesimal version of the strong reciprocity conjecture for thickenings of all orders. Classical analogs of our invariants are based on the dilogarithm function and our invariant could be seen as their infinitesimal version. Despite this analogy, the infinitesimal version cannot be obtained from their classical counterparts through a limiting process.

1. Introduction

1.1. ***Statement of the main technical result.*** For a scheme X, one expects an abelian category $\mathcal{M}_X$ of mixed motivic $\mathbb{Q}$-sheaves on X, such that the extensions groups $\mathrm{H}^i_{\mathcal{M}}(X, \mathbb{Q}(n)) := \mathrm{Ext}^i_{\mathcal{M}_X}(\mathbb{Q}(0), \mathbb{Q}(n))$ of the Tate sheaves are computed in terms of the K-groups as $K_{2n-i}(X)^{(n)}_{\mathbb{Q}}$. We emphasize that we do not assume that X is smooth over a field or even reduced. At present, such a category has not been constructed. When X is a smooth and projective curve over a base scheme S, which in our context will be the spectrum of an artin ring, the conjectural Leray–Serre spectral sequence would give a map

$$K_3(X)^{(3)}_{\mathbb{Q}} = \mathrm{H}^3_{\mathcal{M}}(X, \mathbb{Q}(3)) \to \mathrm{H}^1_{\mathcal{M}}(S, \mathbb{Q}(2)) = K_3(S)^{(2)}_{\mathbb{Q}}.$$

In certain cases, there are regulator maps from $K_3(S)^{(2)}_{\mathbb{Q}}$ to an abelian group A. The composition with the above map would induce a map from $K_3(X)^{(3)}_{\mathbb{Q}}$ to A. In case $S = \operatorname{Spec} k[t]/(t^m)$, such a map $K_3(S)^{(2)}_{\mathbb{Q}} \to \bigoplus_{m<r<2m} k$ was constructed in [Hesselholt 2005]. One of our aims in this paper is to give an analog of the induced map

$$\mathrm{H}^3_{\mathcal{M}}(X, \mathbb{Q}(3)) = K_3(X)^{(3)}_{\mathbb{Q}} \to \bigoplus_{m<r<2m} k,$$

which does not depend on the conjectural category of motives. This map is an infinitesimal analog of a real analytic regulator as we will describe in Section 2.2. This makes this paper a continuation of our project started in [Ünver 2021] and followed up in [Ünver 2020], which aim to give infinitesimal analogs of real analytic regulators.

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First, let us state the main technical result on which all the applications are based. Let k be a field of characteristic 0, $k_m := k[t]/(t^m)$, for $m \geq 2$, and C/k_m be a smooth and projective curve. We denote the underlying reduced scheme of C by $\underline{C}$. We will need a variant of the Bloch complex; see [Goncharov 1995, Sections 1.8 and 1.9] and Section 2.1. If X/k is a smooth and projective curve, then the part of the classical Bloch complex relevant for us is

$$B_2(k(X)) \otimes k(X)^\times \to \bigoplus_{x \in |X|} B_2(k(x)) \oplus \Lambda^3 k(X)^\times \to \bigoplus_{x \in |X|} \Lambda^2 k(x)^\times. \tag{1.1.1}$$

Here the summations are over the set closed points $|X|$ of X and B_2 denotes the Bloch group (Section 2.1.A).

In order to define a variant of the above complex for C, we first need to make a choice of smooth liftings. By a smooth lifting $\mathfrak{c}$ of a closed point $c \in |C|$, we mean a closed subscheme $\mathfrak{c}$ of C, which is supported on c and is smooth over k_m (see Section 7). For each point $c \in |C|$, fix once and for all a smooth lifting $\mathfrak{c}$ and let $\mathscr{P}$ denote the set of all of these liftings. Let η be the generic point of C, for a function $f \in \mathcal{O}_{C,\eta}^\times$ and $\mathfrak{c} \in \mathscr{P}$, we define a notion of f being *good with respect to* $\mathfrak{c}$ or equivalently of being $\mathfrak{c}$-*good* in Section 7. We then define the sheaf $(\mathcal{O}_C, \underline{\mathscr{P}})^\times$ in Section 8.1, by requiring that its sections on an open set U to be those $f \in \mathcal{O}_{C,\eta}^\times$ which are $\mathfrak{c}$-good for all $\mathfrak{c} \in \mathscr{P}$ with $c := |\mathfrak{c}| \in U$. We similarly define a sheaf $B_2(\mathcal{O}_C, \underline{\mathscr{P}})$ in Section 8.1, which is a generalization of the Bloch group but which also encodes the notion of goodness with respect to elements of $\mathscr{P}$. This gives us a complex $\mathscr{C}(C, \mathscr{P})$ of sheaves on C which are concentrated in degrees 2 and 3:

$$B_2(\mathcal{O}_C, \underline{\mathscr{P}}) \otimes (\mathcal{O}_C, \underline{\mathscr{P}})^\times \to \bigoplus_{\mathfrak{c} \in \mathscr{P}} i_{\mathfrak{c}*}(B_2(k(\mathfrak{c}))) \oplus \Lambda^3 (\mathcal{O}_C, \underline{\mathscr{P}})^\times. \tag{1.1.2}$$

Here $k(\mathfrak{c})$ denotes the artin ring which is the ring of regular functions on the affine scheme $\mathfrak{c}$, and $i_\mathfrak{c}$ denotes the imbedding from $\mathfrak{c}$ to C. Since we fixed a single lifting $\mathfrak{c} \in \mathscr{P}$ for each point c in $|C|$, the sum above can also be thought of as a sum over $|C|$. The main technical result is the following construction of infinitesimal Chow dilogarithms:

Theorem 8.1.1. *Let k be a field of characteristic* 0, *C be a smooth and projective curve over $k_m := k[t]/(t^m)$, with $m \geq 2$ and $\mathscr{P}$ be a choice of a smooth lifting for each closed point of C. For each $m < r < 2m$, there is an infinitesimal regulator*

$$\rho_{m,r} : \mathrm{H}^3(\mathscr{C}(C, \mathscr{P})) \to k. \tag{1.1.3}$$

Specializing to the case when C is the projective line $\mathbb{P}^1_{k_m}$, with coordinate function z, we fix an $a \in k_m^\times$ such that $1 - a \in k_m^\times$. If we choose $\mathscr{P}$ such that z, $1 - z$ and $z - a$ are all good with respect to $\mathscr{P}$, then $(1 - z) \wedge z \wedge (z - a) \in \Gamma(\Lambda^3(\mathcal{O}_{\mathbb{P}^1}, \underline{\mathscr{P}})^\times)$ and

$$\rho_{m,r}((1 - z) \wedge z \wedge (z - a)) = \ell i_{m,r}([a]),$$

where $\ell i_{m,r} : B_2(k_m) \to k$ is the additive dilogarithm defined in [Ünver 2009] (*see Section 3*).

The notation of the theorem in the main body of the paper is slightly different but equivalent.

This generalizes the construction in [Ünver 2021] in two different ways: we sheafify the previous construction and we construct the regulator for any $m < r < 2m$, rather than only for $m = 2$. More precisely, if we let $k(C, \underline{\mathscr{P}})^\times$ denote the set of global sections $\Gamma(C, (\mathcal{O}_C, \underline{\mathscr{P}})^\times)$ of $(\mathcal{O}_C, \underline{\mathscr{P}})^\times$, the construction of [loc. cit.] only gives a map from $\Lambda^3 k(C, \underline{\mathscr{P}})^\times$ and only in the case when $m = 2$ and $r = 3$. We will sketch the main idea of the construction in the section below, but let us mention here that the construction of a map $\rho_{m,m+1}$ from $\Lambda^3 k(C, \underline{\mathscr{P}})^\times$ to k can be done by the methods of [loc. cit.]. On the other hand, the construction of $\rho_{m,r}$ for $m + 1 < r < 2m$ requires the new methods that we introduce in this paper.

1.2. ***Applications.*** As we described above, specializing to triples of functions gives us the infinitesimal Chow dilogarithm

$$\rho_{m,r} : \Lambda^3 k(C, \underline{\mathscr{P}})^\times \to k. \tag{1.2.1}$$

which we will denote by the same symbol.

1.2.A. *Infinitesimal strong reciprocity conjecture.* The first application of this construction will be to an infinitesimal analog of the strong reciprocity conjecture of Goncharov [2005]. If X/k is a smooth and projective curve over an algebraically closed field k, the Suslin reciprocity theorem states that the sum of the residue maps

$$\sum_{x \in |X|} \mathrm{res}_x : K_3^M(k(X)) \to K_2^M(k) \tag{1.2.2}$$

at all the closed points of X is equal to 0. Goncharov [2005] conjectured that the map of complexes

$$\begin{tikzcd} B_2(k(X)) \otimes k(X)^\times \arrow[r] \arrow[d] & \Lambda^3 k(X)^\times \arrow[d] \\ B_2(k) \arrow[r] & \Lambda^2 k^\times \end{tikzcd}$$

obtained from (1.1.1) by taking sums of the maps $B_2(k(x)) \to B_2(k)$ and $\Lambda^2 k(x)^\times \to \Lambda^2 k^\times$ is homotopic to 0. More precisely, he conjectures that there is a canonical map $h : \Lambda^3 k(X)^\times \to B_2(k)$ which makes the diagram

$$\begin{tikzcd} B_2(k(X)) \otimes k(X)^\times \arrow[r] \arrow[d] & \Lambda^3 k(X)^\times \arrow[d] \arrow[dl, "h"'] \\ B_2(k) \arrow[r] & \Lambda^2 k^\times \end{tikzcd}$$

commute and has the property that $h(\lambda \wedge f \wedge g) = 0$, if $\lambda \in k^\times$ and $f, g \in k(X)^\times$. Note that this is a stronger version of the Suslin reciprocity theorem since the cokernel of the horizontal maps in the diagram above are $K_3^M(k(X))$ and $K_2^M(k)$. This original version of the conjecture is proved by Rudenko [2021], by using homotopy invariance.

We prove an infinitesimal version of this conjecture using the infinitesimal Chow dilogarithm above and the determination of the structure of the Bloch group over k_m which was done in [Ünver 2009]. Our method is entirely different from Rudenko's, since homotopy invariance is no longer true in the infinitesimal world. Let $B_2(k(C, \underline{\mathscr{P}}))$ denote the set of global sections of $B_2(\mathcal{O}_C, \underline{\mathscr{P}})$, then the infinitesimal version of the strong reciprocity conjecture states:

Theorem 9.1.1. *There is a map $h : \Lambda^3 k(C, \underline{\mathscr{P}})^\times \to B_2(k_m)$, which makes the diagram*

$$\begin{tikzcd} B_2(k(C, \underline{\mathscr{P}})) \otimes k(C, \underline{\mathscr{P}})^\times \arrow[r] \arrow[d] & \Lambda^3 k(C, \underline{\mathscr{P}})^\times \arrow[d] \arrow[dl, dotted, "h"'] \\ B_2(k_m) \arrow[r] & \Lambda^2 k_m^\times \end{tikzcd}$$

commute and has the property that $h(k_m^\times \wedge \Lambda^2 k(C, \underline{\mathscr{P}})^\times) = 0$.

1.2.B. *Application to algebraic cycles.* As another application of the infinitesimal Chow dilogarithm, we construct invariants of higher algebraic cycles up to rational equivalence. In principle the group of algebraic cycles that we are interested in should be denoted by $\mathrm{CH}^2(k_m, 3)$. However, since k_m is far from being smooth over k, such a group of cycles which can be expressed in terms of K-theory is not defined.

One way to overcome this problem is to use the additive Chow groups of Bloch and Esnault [2003]. Additive Chow groups were defined in order to give a cycle theoretic interpretation of the motivic cohomology groups of k_m. One can think of additive Chow cycles as those cycles which are very close to the 0 cycle, the closeness to 0 being defined via the modulus (t^m). A regulator on this group was defined by Park [2009] for $r = m+1$. We think that additive Chow groups tell only part of the story when we try to understand higher cycles on k_m. For this reason we define a somewhat bigger class of higher cycles over k_m. We do this by defining a group $\underline{z}^2_f(k_\infty, 3)$ of codimension 2 cycles on $\mathbb{A}^3_{k_\infty}$, where $k_\infty := k[\![t]\!]$. The main theorem is then a reciprocity theorem.

Theorem 9.4.2. *For $m < r < 2m$, we define a regulator $\rho_{m,r} : \underline{z}^2_f(k_\infty, 3) \to k$. If Z_a, with $a, 1-a \in k_\infty^\times$ is the dilogarithmic cycle given by the parametric equation $(1-z, z, z-a)$ then*

$$\rho_{m,r}(Z_a) = \ell i_{m,r}([a]).$$

If $Z_i \in \underline{z}^2_f(k_\infty, 3)$, for $i = 1, 2$, satisfy the condition (M_m), then they have the same infinitesimal regulator value

$$\rho_{m,r}(Z_1) = \rho_{m,r}(Z_2).$$

This essentially states that if two cycles are the same modulo (t^m) then they have the same value under the regulator. Note the similarity of this to the definition of de Rham cohomology on singular schemes by first imbedding them in a smooth scheme. The precise definition of $\underline{z}^2_f(k_\infty, 3)$ and the condition (M_m) can be found in Section 9.4. After the category of motives over nonreduced rings is constructed, we expect these invariants to induce the regulators in this category.

1.3. ***Main ideas behind the construction.*** In this section, we will try to illustrate the ideas behind the construction in Theorem 8.1.1. For each $2 \le m < r < 2m$, we will construct a regulator whose source is the degree 3 cohomology of the complex of sheaves

$$B_2(\mathcal{O}_C, \underline{\mathscr{P}}) \otimes (\mathcal{O}_C, \underline{\mathscr{P}})^\times \to \bigoplus_{\mathfrak{c} \in \mathscr{P}} i_{\mathfrak{c}*}(B_2(k(\mathfrak{c}))) \oplus \Lambda^3(\mathcal{O}_C, \underline{\mathscr{P}})^\times$$

concentrated in the degrees [2, 3]. Suppose that we are given a Zariski open cover $\{U_i\}_{i\in I}$ of $\underline{C}$ and a corresponding cocyle γ, given by the following data: $\gamma_i \in \Lambda^3(\mathcal{O}_C, \underline{\mathscr{P}})^\times(U_i)$, $\varepsilon_{i,c} \in B_2(k(\mathfrak{c}))$ for every $c \in U_i$ and $\beta_{ij} \in (B_2(\mathcal{O}_C, \underline{\mathscr{P}}) \otimes (\mathcal{O}_C, \underline{\mathscr{P}})^\times)(U_{ij})$. We will define $\rho_{m,r}(\gamma) \in k$, by first making many choices and then showing that the construction is independent of all the choices:

(i) Let $\tilde{\mathcal{A}}_\eta$ be a lifting of $\mathcal{O}_{C,\eta}$ to a smooth k_∞-algebra and for every $c \in |C|$, let $\tilde{\mathcal{A}}_c$ be a lifting of the completion $\hat{\mathcal{O}}_{C,c}$ of the local ring of C at c, to a smooth k_∞-algebra, together with a smooth lifting $\tilde{\mathfrak{c}}$ of $\mathfrak{c}$.

(ii) Let an $i \in I$ be arbitrary and for each c choose a $j_c \in I$ such that $c \in U_{j_c}$.

(iii) Choose an arbitrary lifting $\tilde{\gamma}_{i\eta} \in \Lambda^3 \tilde{\mathcal{A}}_\eta^\times$ of the germ $\gamma_{i\eta} \in \Lambda^3 \mathcal{O}_{C,\eta}^\times$.

(iv) Choose a good lifting $\tilde{\gamma}_{j_c} \in \Lambda^3(\tilde{\mathcal{A}}_c, \tilde{\mathfrak{c}})^\times$ of the image $\hat{\gamma}_{j_c,c}$ of γ_{j_c} in $\Lambda^3(\hat{\mathcal{O}}_{C,c}, \mathfrak{c})^\times$, for every $c \in |C|$.

(v) Choose an arbitrary lifting $\tilde{\beta}_{j_ci,\eta} \in B_2(\tilde{\mathcal{A}}_\eta) \otimes \tilde{\mathcal{A}}_\eta$ of the image $\beta_{j_ci,\eta} \in B_2(\mathcal{O}_{C,\eta}) \otimes \mathcal{O}_{C,\eta}^\times$ of β_{j_ci}, for every $c \in |C|$.

We then define the value of the regulator $\rho_{m,r}$ on the above element by the expression

$$\rho_{m,r}(\gamma) := \sum_{c\in|C|} \mathrm{Tr}_k\big(\ell_{m,r}(\mathrm{res}_{\tilde{\mathfrak{c}}}\, \tilde{\gamma}_{j_c}) - \ell i_{m,r}(\varepsilon_{j_c,c}) + \mathrm{res}_c\, \omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}_{j_ci,\eta}), \tilde{\gamma}_{j_c})\big). \tag{1.3.1}$$

We continue with the description of this expression.

The starting point for the above definition is our construction of the additive dilogarithm in [Ünver 2009]. For a regular local $\mathbb{Q}$-algebra R, letting $R_m := R[t]/(t^m)$, for every $2 \le m < r < 2m$, we have an additive dilogarithm map $\ell i_{m,r} : B_2(R_m) \to R$ that satisfies all the analogous properties of the Bloch–Wigner dilogarithm function. Most importantly, the direct sums of these maps over all the possible r give an isomorphism between the infinitesimal part of the K-group $K_3(R_m)^{(2)}_{\mathbb{Q}}$ and $\bigoplus_{m<r<2m} R$. We explain this in detail in Section 3 and give explicit formulas for these functions $\ell i_{m,r}$. The function $\ell i_{m,r}$ can also be described in terms of the differential δ in the Bloch complex of $B_2(R_\infty)$, with $R_\infty := R[\![t]\!]$, by the following commutative diagram:

$$\begin{CD} B_2(R_\infty) @>{\delta}>> \Lambda^2 R_\infty^\times \\ @VVV @VV{\ell_{m,r}}V \\ B_2(R_m) @>{\ell i_{m,r}}>> R \end{CD}$$

Where $\ell_{m,r}$ is given explicitly in Definition 3.0.2 below.

We can then describe the first two terms in (1.3.1) as follows. For a connected, étale k_m-algebra (resp. k_∞-algebra) A, there is a canonical isomorphism $A \simeq k'_m$ (resp. $A \simeq k'_\infty$). Using this isomorphism for $k(\mathfrak{c})$, we get a canonical identification $B_2(k(\mathfrak{c})) = B_2(k(c)_m)$. Therefore, $\ell i_{m,r}(\varepsilon_{j,c}) \in k(c)$ is unambiguously defined using the map $\ell i_{m,r} : B_2(k(c)_m) \to k(c)$. Since the element $\tilde{\gamma}_{j_c} \in \Lambda^3(\tilde{\mathcal{A}}_c, \tilde{\mathfrak{c}})^\times$ is assumed to be $\tilde{\mathfrak{c}}$-good, the residue $\mathrm{res}_{\tilde{\mathfrak{c}}}\, \tilde{\gamma}_{j_c}$ is defined as an element of $\Lambda^2 k(\tilde{\mathfrak{c}})^\times$ in the beginning of Section 7. Using the identification $\Lambda^2 k(\tilde{\mathfrak{c}})^\times = \Lambda^2 k(c)^\times_\infty$ and the map $\ell_{m,r} : \Lambda^2 k(c)^\times_\infty \to k(c)$, we define the element $\ell_{m,r}(\mathrm{res}_{\tilde{\mathfrak{c}}}\, \tilde{\gamma}_{j_c}) \in k(c)$. Defining the last term $\mathrm{res}_c\, \omega_{m,r}$ and proving its properties will constitute a large proportion of the paper. If R is smooth of relative dimension 1 over k, we construct a map $\omega_{m,r} : \Lambda^3(R_r, (t^m))^\times \to \Omega^1_{R/k}$. Here $(R_r, (t^m))^\times$ denotes $\{(a, b) \mid a, b \in R_r^\times, ab^{-1} \in 1 + (t^m)\}$. Since we do not fix a lifting of our curve in the construction of $\rho_{m,r}$, defining $\omega_{m,r}$ on this group is not enough. More precisely, we need to extend $\omega_{m,r}$ to the following context. Suppose that $\mathcal{R}$ and $\mathcal{R}'$ are smooth of relative dimension 1 over k_r together with a fixed isomorphism

$$\chi : \mathcal{R}/(t^m) \to \mathcal{R}'/(t^m),$$

of k_m-algebras between their reductions modulo (t^m). Let

$$(\mathcal{R}, \mathcal{R}', \chi)^\times := \{(a, b) \mid a \in \mathcal{R}^\times, b \in \mathcal{R}'^\times, \chi(a + (t^m)) = b + (t^m) \text{ in } \mathcal{R}'/(t^m)\}.$$

Ideally, we would like to extend the definition of $\omega_{m,r}$ to a map from $\Lambda^3(\mathcal{R}, \mathcal{R}', \chi)^\times$ to $\Omega^1_{\underline{\mathcal{R}}/k}$. This can be done when $r = m + 1$ but it is not true if $m + 1 < r$.

However, it turns out that for us purposes, we do not need these 1-forms themselves but only their residues and we can construct these residues independently of all the choices. Suppose that $\mathcal{S}$ is a smooth k_m-algebra of relative dimension 1, with x a closed point and η the generic point of its spectrum. Suppose that $\mathcal{R}, \mathcal{R}'$ are liftings of $\mathcal{S}_\eta$ to k_r, with χ the corresponding isomorphism from $\mathcal{R}/(t^m)$ to $\mathcal{R}'/(t^m)$. We construct a map

$$\mathrm{res}_x\, \omega_{m,r} : \Lambda^3(\mathcal{R}, \mathcal{R}', \chi)^\times \to k',$$

where k' is the residue field of x, which is functorial and independent of all the choices. Let $\tilde{\chi} : \mathcal{R} \to \mathcal{R}'$ be an isomorphism of k_r-algebras which is a lifting of χ. Choosing also an isomorphism $\underline{\mathcal{R}}_r \simeq \mathcal{R}$ of k_r-algebras, provides us with an identification

$$(\mathcal{R}, \mathcal{R}', \chi)^\times \xrightarrow{\tilde{\chi}^*} (\underline{\mathcal{R}}_r, (t^m))^\times.$$

Let $\underline{\psi}$ denote the isomorphism $\underline{\mathcal{R}} \to \underline{\mathcal{S}}_\eta$ induced by the one from $\mathcal{R}/(t^m)$ to $\mathcal{S}_\eta$. Then we define $\mathrm{res}_x\, \omega_{m,r}$ by the composition

$$\Lambda^3(\mathcal{R}, \mathcal{R}', \chi)^\times \xrightarrow{\Lambda^3 \tilde{\chi}^*} \Lambda^3(\underline{\mathcal{R}}_r, (t^m))^\times \xrightarrow{\omega_{m,r}} \Omega^1_{\underline{\mathcal{R}}/k} \xrightarrow{d\underline{\psi}} \Omega^1_{\underline{\mathcal{S}}_\eta/k} \xrightarrow{\mathrm{res}_x} k'.$$

We prove that this composition is independent of all the choices. Applying this construction in the above context, we see that $\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}_{j_c i})$ and $\tilde{\gamma}_{j_c}$ are two liftings of the same object γ_{j_c} to two different generic liftings of $\hat{\mathcal{O}}_{C,c}$. Therefore, the expression

$$\mathrm{res}_c\, \omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}_{j_c i,\eta}), \tilde{\gamma}_{j_c}) \in k(c)$$

is defined.

Applying traces and taking the sum over all the closed points, we obtain the expression in (1.3.1). Next we show that the sum is in fact a finite sum. The above construction involves many choices and would be completely useless if it depended on anything other than the initial data. This is the content of Theorem 8.1.1, our main theorem. Because of its basic properties that we prove below, $\rho_{m,r}$ deserves to be called a regulator.

Finally, let us mention that in [Ünver 2021], where the case $m = 2$ was handled, the only possible r is 3 and hence satisfies $r = m+1$. In this case, the map $\omega_{2,3}$ can be defined as a map from $\Lambda^3(\mathcal{R}, \mathcal{R}', \chi)^\times \to \Omega^1_{\underline{\mathcal{R}}/k}$. This is not true in general and this is why we have to pursue a different approach in this paper which is based on defining only the residue of the differential rather than the differential itself.

1.4. ***Outline.*** We give an outline of the paper. In Section 2, we describe the complex analytic version of our construction for motivation. In Section 3, we give a review of the construction in [Ünver 2009] of the additive dilogarithm on the Bloch group of a truncated polynomial ring. In Section 4, we describe the infinitesimal part of the Milnor K-theory of a local $\mathbb{Q}$-algebra endowed with a nilpotent ideal, which is split, in terms of Kähler differentials. Without any doubt the results in this section are known to the experts and we do not claim originality. The reason for our inclusion of this section is first that we could not find an easily quotable statement in the full generality which we will need in our later work, and second that we found a short argument which is in line with the general set-up of this paper. In Section 5, for a regular local $\mathbb{Q}$-algebra R, we define regulators $B_2(R_m) \otimes R_m^\times$ to Ω^1_R for every $m < r < 2m$, which vanishes on boundaries. This construction depends on the splitting of R_m in an essential way. In Section 6, we introduce the main object of this paper: for a smooth algebra R of relative dimension 1 over k, we define regulators $\omega_{m,r} : \Lambda^3(R_r, (t^m))^\times \to \Omega^1_{R/k}$, for each $m < r < 2m$. In Section 7, we compute the residues of the value of $\omega_{m,r}$ on good liftings. In Section 8, we use the results of the previous sections to construct the regulator from $\mathrm{H}^3_{\mathcal{B}}(C, \mathbb{Q}(3))$ and specializing to triples of rational functions we obtain the infinitesimal Chow dilogarithm of higher modulus. In Section 9, we give examples of the infinitesimal Chow dilogarithm in the cases of the projective curve and elliptic curves and also give the applications to the strong reciprocity conjecture and the invariants of cycles.

Conventions and notation. We are interested in everything modulo torsion. Therefore, *we tensor all abelian groups under consideration with $\mathbb{Q}$ without explicitly signifying this in the notation.* For example, $K_n^M(A)$ denotes Milnor K-theory of A tensored with $\mathbb{Q}$ etc.

For a ring R, we let $R_\infty := R[\![t]\!]$ (resp. $R((t))$) be the formal power series (resp. the formal Laurent series) ring over R. For $m \geq 1$, we let $R_m := R[t]/(t^m)$, be the truncated polynomial ring over R of modulus m. If R is a $\mathbb{Q}$-algebra then we write $\exp(\alpha) := \sum_{0 \leq n}(\alpha^n/n!)$ for $\alpha \in (t) \subseteq R_\infty$. The same formula is used for $\alpha \in (t) \subseteq R_m$. For an appropriate functor F, we let $F(R_\infty)^\circ := \ker(F(R_\infty) \to F(R))$ (resp. $F(R_m)^\circ := \ker(F(R_m) \to F(R))$), denote the infinitesimal part of $F(R_\infty)$ (resp. $F(R_m)$).

For any set X, let $\mathbb{Q}[X]$ denote the vector space over $\mathbb{Q}$ with basis $\{[x] \mid x \in X\}$. We denote both the differential $B_2(A) \to \Lambda^2 A^\times$ in the Bloch complex of weight 2 (see Section 2.1.A) and the differential $B_2(A) \otimes A^\times \to \Lambda^3 A^\times$ in the Bloch complex of weight 3 (see Section 2.1.B) by δ. Since the sources of the maps are different, this will not cause any confusion. When we use these maps in the case when $A = R_r$ and we want to emphasize dependence on r in the notation, we denote both of these differentials by δ_r.

2. The analogy with the complex case

In this section we will describe the analogy with the complex case after recalling some of the standard definitions. Our aim is to give a flavor of the concepts before going into the technical details.

2.1. ***Basic definitions.*** Here we collect some of the basic definitions that are standard in the literature. We will use these definitions in this section and generalize them in the later sections.

2.1.A. *The Bloch group B_2 and the Bloch complex of weight 2.* For any ring A, we let $A^\flat := \{a \in A \mid a(1-a) \in A^\times\}$. For a local $\mathbb{Q}$-algebra R, the Bloch group $B_2(R)$ is the quotient of $\mathbb{Q}[R^\flat]$ by the subspace generated by

$$[x] - [y] + [y/x] - [(1-x^{-1})/(1-y^{-1})] + [(1-x)/(1-y)],$$

for all $x, y \in R^\flat$ such that $x - y \in R^\times$. There is a map $\delta : B_2(R) \to \Lambda^2 R^\times$, which is defined on the generators by letting $\delta([x]) := (1-x) \wedge x$. The corresponding complex obtained by putting $B_2(R)$ in degree 1 and $\Lambda^2 R^\times$ is degree 2 is called *the Bloch complex of weight* 2. This complex computes the weight 2 motivic cohomology of R, when R is a field. We refer to [Ünver 2009] for details about the Bloch group and the Bloch complex of weight 2. We denote the cohomology of this complex with $\mathrm{H}^i(R, \mathbb{Q}(2))$.

2.1.B. *The pre-Bloch complex of weight 3.* Continuing with the notation above, we have a complex

$$\mathbb{Q}[R^\flat] \to B_2(R) \otimes R^\times \to \Lambda^3 R^\times, \tag{2.1.1}$$

concentrated in degrees $[1, 3]$, where the first map sends a basis element $[x]$ to $[x] \otimes x$ and the second one sends $[x] \otimes y$ to $\delta(x) \wedge y$. Abusing the notation, we denote all the differentials in this complex by δ. This complex and its variants are defined and studied in detail in [Goncharov 1995]. We will call this complex *the pre-Bloch complex of weight* 3.

The first group $\mathbb{Q}[R^\flat]$ when divided by the appropriate relations is denoted by $B_3(R)$. At this stage of this theory, the exact type of these relations are not clear. There are several different candidates and it is not known that they give the same answer [Goncharov 1995]. The corresponding sequence obtained is a candidate for the weight 3 motivic cohomology complex [loc. cit.]. Since we will only deal with the cohomology groups in degrees 2 and 3, we will only work with the pre-Bloch complex (2.1.1) above and the precise relations in order to define $B_3(R)$ will not be important for us. The reason that we do not call this complex the Bloch complex is that we do not use a version of the group $B_3(R)$ and instead use $\mathbb{Q}[R^\flat]$. For R equal to the dual numbers of a field, this complex and its higher weight analogs were used in [Ünver 2010] to construct the additive polylogarithms.

For the degrees $i = 2$ and 3, we will denote the cohomology of the pre-Bloch complex in (2.1.1) by $\mathrm{H}^i(R, \mathbb{Q}(3))$.

2.1.C. *Residue map between the Bloch complexes.* Suppose that R is a discrete valuation ring with residue field k and with field of fractions K. There is a canonical residue homomorphism $K_n^M(K) \to K_{n-1}^M(k)$ constructed by Milnor [1970] between the Milnor K-groups. Goncharov [1995, Section 1.14] generalized to a map between the Bloch complexes.

For us, the only parts of this construction that will be relevant are

$$\mathrm{res} : \Lambda^n K^\times \to \Lambda^{n-1} k^\times \quad \text{and} \quad \mathrm{res} : B_2(K) \otimes K^\times \to B_2(k).$$

To describe these maps, let us fix a uniformizer π of R. The map will turn out to be independent of the choice of the uniformizer.

The first map is determined by the following formula

$$\mathrm{res}(u_0\pi^m \wedge u_1 \wedge u_2 \wedge \cdots \wedge u_{n-1}) = m \cdot \underline{u}_1 \wedge \underline{u}_2 \wedge \cdots \wedge \underline{u}_{n-1},$$

where $m \in \mathbb{Z}$, u_i, for $1 \le i < n$ are units in R and $\underline{u}_i$ for $1 \le i < n$ are the images of u_i in k. The second map is determined by the formulas that

$$\mathrm{res}([a] \otimes b) = 0,$$

if $a \in K^\flat \setminus R^\flat$ or $b \in R^\times$, and

$$\mathrm{res}([u] \otimes \pi) = [\underline{u}],$$

if $u \in R^\flat$ and $\underline{u}$ is the image of u in $k^\flat$. These maps give a commutative diagram

$$\begin{tikzcd} B_2(K) \otimes K^\times \arrow[r] \arrow[d, "\mathrm{res}"] & \Lambda^3 K^\times \arrow[d, "\mathrm{res}"] \\ B_2(k) \arrow[r] & \Lambda^2 k^\times \end{tikzcd}$$

and hence a sequence

$$B_2(K) \otimes K^\times \to B_2(k) \oplus \Lambda^3 K^\times \to \Lambda^2 k^\times.$$

If we start with a smooth curve X/k, then taking residues at all the closed points and summing them will give a sequence

$$B_2(k(X)) \otimes k(X)^\times \to \bigoplus_{x \in |X|} B_2(k(x)) \oplus \Lambda^3 k(X)^\times \to \bigoplus_{x \in |X|} \Lambda^2 k(x)^\times.$$

This is part of the motivic complex of weight 3 of the curve X [Goncharov 1995], whose middle cohomology receives a map from the motivic cohomology $\mathrm{H}^3_{\mathcal{M}}(X, \mathbb{Q}(3))$ of X.

2.2. ***Complex analog of the main construction.*** Here we briefly explain the complex analog of our construction, which is one of our main motivations for the infinitesimal case. If $X/\mathbb{C}$ is a smooth projective curve, then as above one expects a map

$$K_3(X)^{(3)}_{\mathbb{Q}} = \mathrm{H}^3_{\mathcal{M}}(X, \mathbb{Q}(3)) \to \mathrm{H}^1_{\mathcal{M}}(\mathbb{C}, \mathbb{Q}(2)) = K_3(\mathbb{C})^{(2)}_{\mathbb{Q}}.$$

Composing with the Borel regulator $K_3(\mathbb{C})^{(2)}_{\mathbb{Q}} \to \mathbb{C}/(2\pi i)^2\mathbb{Q}$ and taking the imaginary part would give a map $K_3(X)^{(3)}_{\mathbb{Q}} \to \mathbb{R}$. Up to normalization, this map can be constructed as follows [Goncharov 2005, Section 6]. For f_1, f_2, and $f_3 \in \mathbb{C}(X)^\times$, let

$$r_2(f_1, f_2, f_3) := \mathrm{Alt}_3\big(\tfrac{1}{6}\log|f_1|\cdot d\log|f_2|\wedge d\log|f_3| - \tfrac{1}{2}\log|f_1|\cdot d\arg f_2\wedge d\arg f_3\big),$$

(such that $dr_2(f_1, f_2, f_3) = \mathrm{Re}(d\log(f_1)\wedge d\log(f_2)\wedge d\log(f_3))$). The Chow dilogarithm map $\rho : \Lambda^3\mathbb{C}(X)^\times \to \mathbb{R}$ is given in terms of this by

$$\rho(f_1\wedge f_2\wedge f_3) := \int_{X(\mathbb{C})} r_2(f_1, f_2, f_3). \tag{2.2.1}$$

In the special case when $X = \mathbb{P}^1$, we have

$$\rho((1-z)\wedge z\wedge(z-a)) = D_2(a), \tag{2.2.2}$$

where $D_2(z) := \mathrm{Im}(\ell i_2(z)) + \arg(1-z)\cdot\log(|z|)$ is the Bloch–Wigner dilogarithm, with $\ell i_2(z)$ the (multivalued) analytic continuation of $\sum_{1\le n} z^n/n^2$. The middle cohomology of

$$\to B_2(\mathbb{C}(X))\otimes\mathbb{C}(X)^\times_{\mathbb{Q}} \to \Big(\bigoplus_{x\in X} B_2(\mathbb{C})\Big)\oplus\Lambda^3\mathbb{C}(X)^\times_{\mathbb{Q}} \to \bigoplus_{x\in X}\Lambda^2\mathbb{C}^\times_{\mathbb{Q}} \to \tag{2.2.3}$$

receives a map from $\mathrm{H}^3_{\mathcal{M}}(X, \mathbb{Q}(3)) \simeq K_3(X)^{(3)}_{\mathbb{Q}}$. Combining D_2 and ρ, if we let

$$\rho_X := -\Big(\bigoplus_{x\in X} D_2\Big)\oplus\rho : \Big(\bigoplus_{x\in X} B_2(\mathbb{C})\Big)\oplus\Lambda^3\mathbb{C}(X)^\times_{\mathbb{Q}} \to \mathbb{R},$$

then ρ_X vanishes on the image of $B_2(\mathbb{C}(X))\otimes\mathbb{C}(X)^\times_{\mathbb{Q}}$ and induces the map $K_3(X)^{(3)}_{\mathbb{Q}} \to \mathbb{R}$, we were looking for above. If one assumes a theory of motivic sheaves then this is the composition of $\mathrm{H}^3_{\mathcal{M}}(X, \mathbb{Q}(3)) \to \mathrm{Ext}^1_{\mathcal{M}_{\mathbb{C}}}(\mathbb{Q}(0), \mathrm{H}^2(X/\mathbb{C})(3)) = \mathrm{H}^1_{\mathcal{M}}(\mathbb{C}, \mathbb{Q}(2)) \to B_2(\mathbb{C})$ and the Bloch–Wigner dilogarithm $D_2 : B_2(\mathbb{C}) \to \mathbb{R}$. Here $\mathcal{M}_{\mathbb{C}}$ denotes the category of motives over $\mathbb{C}$ and $\mathrm{H}^2(X/\mathbb{C}) = \mathbb{Q}(-1)$ denotes the relative motivic cohomology of $X/\mathbb{C}$. In the special case of $X = \mathbb{P}^1$ the map ρ can be made even more explicit [Goncharov 2005, Section 6.3]. Suppose that f_1, f_2, and f_3 are arbitrary rational functions on $\mathbb{P}^1$. By the linearity of ρ and the fact that ρ vanishes on elements of the form $\lambda\wedge f\wedge g$, if $\lambda\in\mathbb{C}^\times$, we notice that in order to determine $\rho(f_1\wedge f_2\wedge f_3)$, it is enough to determine its value for $f_i = z-\alpha_i$, for pairwise distinct α_i. Using functoriality with respect to automorphisms of $\mathbb{P}^1$ fixing ∞, and the formula (2.2.2), we determine that

$$\rho((z-\alpha_1)\wedge(z-\alpha_2)\wedge(z-\alpha_3)) = D_2\bigg(\frac{\alpha_3-\alpha_2}{\alpha_1-\alpha_2}\bigg). \tag{2.2.4}$$

3. Additive dilogarithm of higher modulus

In this section, we review and rephrase the theory of the additive dilogarithm over truncated polynomial rings in a manner which we will need in the remainder of the paper. Further results for this function can be found in [Ünver 2009].

For a $\mathbb{Q}$-algebra R, let $R_\infty := R[\![t]\!]$, denote the formal power series over R and $R_m := R_\infty/(t^m)$ the truncated polynomial ring of modulus m over R. Since R is a $\mathbb{Q}$-algebra we have the logarithm $\log : (1+tR_\infty)^\times \to R_\infty$ given by $\log(1+z) := \sum_{1\le n}(-1)^{n+1}z^n/n$, for $z \in tR_\infty$. Let $\log^\circ : R_\infty^\times \to R_\infty$, be the branch of the logarithm associated to the splitting of $R_\infty \twoheadrightarrow R$ corresponding to the inclusion $R \hookrightarrow R_\infty$, defined as $\log^\circ(\alpha) := \log(\alpha/\alpha(0))$. If $q = \sum_{0\le i} q_i t^i \in R_\infty$ and $1 \le a$ then let $q|_a := \sum_{0\le i<a} q_i t^i \in R_\infty$, denote the truncation of q to the sum of the first a-terms, and $t_a(q) := q_a$, the coefficient of t^a in q. If $u \in tR_\infty$ and $s(1-s) \in R^\times$, we let

$$\ell i_{m,r}(s\exp(u)) := t_{r-1}\Big(\log^\circ(1-s\exp(u|_m))\cdot \tfrac{\partial u}{\partial t}|_{r-m}\Big), \tag{3.0.1}$$

for $m < r < 2m$. Here, and everywhere in the paper, $\exp(z)$ denotes the formal power series $\sum_{0\le n} z^n/n!$. Also note that $(\partial u/\partial t)|_{r-m}$ denotes the truncation of the derivative of u with respect to t, to the sum of its first $(r-m)$-terms. Fixing $m \ge 2$, these $\ell i_{m,r}$, for $m < r < 2m$ together constitute a regulator from $\ker(K_3(R_m)^{(2)} \to K_3(R)^{(2)})$ to $R^{\oplus(m-1)}$. This is exactly analogous to the Bloch–Wigner dilogarithm in the complex case [Bloch 2000; Suslin 1991; Ünver 2009].

Since every element of $R_\infty^\flat$ can be written in the form $s\exp(u)$ as the above, we can linearly extend $\ell i_{m,r}$ to obtain a map from the vector space $\mathbb{Q}[R_\infty^\flat]$ with basis $R_\infty^\flat$. We denote this map by the same symbol.

When we would like to specify the δ defined on $B_2(R_\infty)$ (resp. $B_2(R_m)$), as given in Section 2.1.A, we denote it by δ_∞ (resp. δ_m).

Let V be a free R module with basis $\{e_i\}_{i\in I}$ and $\{e_i^\vee\}_{i\in I}$ the dual basis of $V^\vee$. Given v and $\alpha = \sum_{i\in I} a_i e_i$ in V, we let

$$(v|\alpha) := \sum_{i\in I} a_i e_i^\vee(v) \in R.$$

If there is an ordering on I, we let $\{e_i \wedge e_j\}_{i>j}$ be the corresponding basis of $\Lambda^2 V$. Then, with the above notation, the expression $(w|\beta)$, for $w, \beta \in \Lambda^2 V$, is defined. We consider tR_∞, as a free R-module with basis $\{t^i\}_{1\le i}$. Let us denote the composition of $B_2(R_\infty) \xrightarrow{\delta} \Lambda^2 R_\infty^\times$ with the canonical projection $\mathbb{Q}[R_\infty^\flat] \to B_2(R_\infty)$ also by δ. Also denote the map

$$\Lambda^2 R_\infty^\times \to \Lambda^2 tR_\infty \twoheadrightarrow \Lambda^2_R tR_\infty$$

induced by $\Lambda^2 \log^\circ : \Lambda^2 R_\infty^\times \to \Lambda^2 tR_\infty$, by the same symbol.

Proposition 3.0.1. *With the notation above, for $\alpha \in \mathbb{Q}[R_\infty^\flat]$ and $2 \le m < r < 2m$, we have*

$$\ell i_{m,r}(\alpha) = \Bigg(\Lambda^2 \log^\circ(\delta(\alpha)) \,\Bigg|\, \sum_{1\le i\le r-m} i t^{r-i} \wedge t^i\Bigg), \tag{3.0.2}$$

and this function descends through the canonical projections

$$\mathbb{Q}[R_\infty^\flat] \to B_2(R_\infty) \to B_2(R_m),$$

to define a map from $B_2(R_m)$ to R, denoted by the same notation.

Proof. We proved in [Ünver 2009, Proposition 2.2.1] that the function defined by the right-hand side of (3.0.2), temporarily denote it by $\ell i^*_{m,r}$, descends to give a map from $\mathbb{Q}[R_m^\flat]$ and in [loc. cit., Proposition 2.2.2] that it descends to give a map from $B_2(R_m)$. Therefore it only remains to prove the equality (3.0.2).

With the notation $\ell_i(\alpha) := t_i(\log^\circ(\alpha))$, $\ell i^*_{m,r}$ can be rewritten as

$$\ell i^*_{m,r} = \Bigg(\sum_{1\le i\le r-m} i\cdot \ell_{r-i}\wedge \ell_i\Bigg)\circ\delta.$$

Then we have $\ell i^*_{m,r}(s\exp(u)) = \ell i^*_{m,r}(s\exp(u|_m))$, since we know that $\ell i^*_{m,r}$ descends to $\mathbb{Q}[R_m^\flat]$. We have $\ell_i(s\exp(u|_m)) = u_i$, for $1\le i<m$ and $\ell_i(s\exp(u|_m)) = 0$, for $m\le i$. Using this we obtain that $\ell i^*_{m,r}(s\exp(u|_m)) = \sum_{1\le i\le r-m} i\cdot \ell_{r-i}(1-s\exp(u|_m))\cdot u_i = \ell i_{m,r}(s\exp(u))$. □

Let us give a name to the essential map which constitute $\ell i_{m,r}$.

Definition 3.0.2. We denote the map from $\Lambda^2 R_\infty^\times$ to R which sends $\alpha\wedge\beta$ to

$$\Bigg(\Lambda^2\log^\circ(\alpha\wedge\beta)\,\Bigg|\sum_{1\le i\le r-m} i t^{r-i}\wedge t^i\Bigg)$$

by $\ell_{m,r}$. It is clear that $\ell_{m,r}:\Lambda^2 R_\infty^\times\to R$ factors through the projection $\Lambda^2 R_\infty^\times \to \Lambda^2 R_r^\times$. The additive dilogarithm above is given in terms of this function as

$$\ell i_{m,r} = \ell_{m,r}\circ\delta_\infty = \ell_{m,r}\circ\delta_r.$$

We will use the main result from [Ünver 2009], there it was stated in the case when R is a field of characteristic 0, but the same proof works when R is a regular, local $\mathbb{Q}$-algebra. Let $B_2(R_m)^\circ$ denote the kernel of the natural map from $B_2(R_m)$ to $B_2(R)$, consistent with the notation in the introduction.

Theorem 3.0.3. *The complex $B_2(R_m)^\circ \xrightarrow{\delta^\circ} (\Lambda^2 R_m^\times)^\circ$ computes the infinitesimal part of the weight two motivic cohomology of R_m, and the map $\bigoplus_{m<r<2m} \ell i_{m,r}$ induces an isomorphism*

$$\mathrm{HC}_2^\circ(R_m)^{(1)} \simeq K_3^\circ(R_m)^{(2)} \simeq \ker(\delta^\circ) \xrightarrow{\sim} R^{\oplus(m-1)}$$

from the relative cyclic homology group $\mathrm{HC}_2^\circ(R_m)^{(1)}$ to $R^{\oplus(m-1)}$.

4. Infinitesimal Milnor K-theory of local rings

Suppose that R is a local $\mathbb{Q}$-algebra and A is an R-algebra, together with a nilpotent ideal I such that the natural map $R\to A/I$ is an isomorphism. Then the Milnor K-theory $K_n^M(A)$ of A, naturally splits into a direct sum $K_n^M(A) = K_n^M(R)\oplus K_n^M(A)^\circ$. In this section, we will describe this infinitesimal part

$K_n^M(A)^\circ$ in terms of Kähler differentials. It is easy to find such an isomorphism using Goodwillie's theorem [1986], and standard computations in cyclic homology. However, in the next section, we need an explicit description of this isomorphism in order to determine which symbols vanish in the corresponding Milnor K-group. Fortunately, determining what this isomorphism turns out to be quite easy. By the functoriality and the multiplicativity of the isomorphism, we reduce the computation to the case of K_2^M of the dual numbers over R where the computation is easy.

There is no doubt that the results in this section are well-known and we do not claim any originality. We simply have not been able to find a description of the map φ below which is easily quotable in the literature. Since our discussion is quite short we did not refrain from including it in the present paper. We will only need the result below for $A = R_m$. On the other hand, in a future work we will need this result in full generality which justifies our somewhat more general discussion.

Proposition 4.0.1. *There exists a unique map $\varphi : K_n^M(A)^\circ \to \Omega_A^{n-1}/(d\Omega_A^{n-2} + \Omega_R^{n-1})$ such that*

$$\varphi(\{\alpha, \beta_1, \dots, \beta_{n-1}\}) = \log(\alpha)\frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_{n-1}}{\beta_{n-1}}, \tag{4.0.1}$$

for $\alpha \in 1 + I$ and $\beta_1, \dots, \beta_{n-1} \in A^\times$, and this map is an isomorphism.

Proof. The uniqueness follows from the fact that the infinitesimal part of Milnor K-theory is generated by terms $\{\alpha, \beta_1, \dots, \beta_{n-1}\}$ as in the statement. In order to see this, let $\iota : R \to A$ denote the structure map. Since $R \to A \to A/I$ is an isomorphism, every element in $A^\times$ can be uniquely written as $\iota(r)\alpha$ with $r \in R^\times$ and $\alpha \in 1 + I$. This implies that $K_n^M(A)$ is generated by elements of the form $\{\alpha_1, \dots, \alpha_i, \iota(r_1), \dots, \iota(r_{n-i})\}$, with $0 \le i \le n$ and $r_j \in R$, $\alpha_j \in 1 + I$. The terms with $1 \le i$ are in $K_n^M(A)^\circ = \ker(K_n^M(A) \to K_n^M(A/I))$. They are also of the form $\{\alpha, \beta_1, \dots, \beta_{n-1}\}$. In order to prove the statement, we only need to show that if a linear combination of terms of the form $\{\iota(r_1), \dots, \iota(r_n)\}$ are in $\ker(K_n^M(A) \to K_n^M(A/I))$ then it is in fact 0. This again follows from the fact that the natural map from R to A/I is an isomorphism.

We define a functorial map φ by the following composition:

$$K_n^M(A)^\circ \to K_n^{(n)}(A)^\circ \xrightarrow{\sim} \mathrm{HC}_{n-1}^{(n-1)}(A)^\circ = (\Omega_A^{n-1}/d\Omega_A^{n-2})^\circ = \Omega_A^{n-1}/(d\Omega_A^{n-2} + \Omega_R^{n-1}). \tag{4.0.2}$$

The first map is the multiplicative map induced by the isomorphism when $n = 1$, the second one is the Goodwillie isomorphism [1986], and the last one is given by [Loday 1992, Theorem 4.6.8].

By Nesterenko and Suslin's theorem [1989], Milnor K-theory is the first obstruction to the stability of the homology of general linear groups

$$K_n^M(A) \simeq \mathrm{H}_n(\mathrm{GL}_n(A), \mathbb{Q})/\mathrm{H}_n(\mathrm{GL}_{n-1}(A), \mathbb{Q}).$$

Moreover, the composition

$$K_n^M(A)^\circ \to K_n^{(n)}(A)^\circ \to \mathrm{Prim}(\mathrm{H}_n(\mathrm{GL}(A), \mathbb{Q})) \to \mathrm{H}_n(\mathrm{GL}(A), \mathbb{Q}) \simeq \mathrm{H}_n(\mathrm{GL}_n(A), \mathbb{Q}) \twoheadrightarrow K_n^M(A)$$

is multiplication by $(n-1)!$ by [Nesterenko and Suslin 1989]. This implies the injectivity of φ. It only remains to prove the property (4.0.1), since then the surjectivity of φ also follows.

The multiplicativity of φ takes the following form: for $a, b \in K_m^M(A)^\circ$, $\varphi(a\cdot b) = \varphi(a) \wedge d(\varphi(b))$. We do induction on n. The statement is clear for $n = 1$. We show that we may assume that $\beta_i \in R^\times$.

Lemma 4.0.2. *Suppose that we have the formula (4.0.1) for $\alpha \in 1 + I$ and $\beta_i \in R^\times$, for $1 \le i \le n-1$, then we have the same formula for $\alpha \in 1 + I$ and $\beta_i \in A^\times$, for $1 \le i \le n-1$.*

Proof. We do induction on the number of β_i which are not in $R^\times$. If all of them are in $R^\times$, the hypothesis of the lemma gives the expression. If there is at least one β_i which is not in $R^\times$, without loss of generality assume that $\beta_{n-1} \notin R^\times$. Let us write $\beta_{n-1} := \lambda \cdot \beta$, with $\lambda \in R^\times$ and $\beta \in 1 + I$. Then

$$\varphi(\{\alpha, \beta_1, \dots, \beta_{n-1}\}) = \varphi(\{\alpha, \beta_1, \dots, \beta_{n-2}, \lambda\}) + \varphi(\{\alpha, \beta_1, \dots, \beta_{n-2}, \beta\}).$$

By the multiplicativity of φ, the formula for $n = 1$, and the induction hypothesis on n, we have

$$\varphi(\{\alpha, \beta_1, \dots, \beta_{n-2}, \beta\}) = \varphi(\{\alpha, \beta_1, \dots, \beta_{n-2}\}) \wedge d(\log(\beta)) = \log(\alpha)\frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_{n-2}}{\beta_{n-2}} \wedge \frac{d\beta}{\beta}.$$

By the induction hypothesis on the number of β_i not in $R^\times$, we have

$$\varphi(\{\alpha, \beta_1, \dots, \beta_{m-1}, \lambda\}) = \log(\alpha)\frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_{m-1}}{\beta_{m-1}} \wedge \frac{d\lambda}{\lambda}.$$

Adding these two expressions, we obtain the expression we were looking for. □

The above lemma shows that we may without loss of generality assume that the $\beta_i \in R^\times$. The next lemma shows that we may also assume that $A = R_r$ and $\alpha = 1 + t$.

Lemma 4.0.3. *Suppose that we have the formula (4.0.1) for $\alpha = 1 + t$ and $\beta_i \in R^\times$ for $1 \le i \le n-1$, for the ring $R_r := R[t]/(t^r)$. Then we have the same formula for any A as above.*

Proof. Given $\alpha \in 1 + I \subseteq A^\times$ and $\beta_i \in R^\times$. Since $\alpha - 1$ is nilpotent, we have an R-algebra morphism $\psi : R_r \to A$, for some r, such that $\psi(t) = \alpha - 1$. The result then follows by the functoriality of φ since the map induced by ψ maps $\{1+t, \beta_1, \dots, \beta_{n-1}\}$ to $\{\alpha, \beta_1, \dots, \beta_{n-1}\}$. □

Next we will show that we can also assume that $r = 2$. Note that for each $\lambda \in \mathbb{Q}^\times$, we obtain an R-automorphism ψ_λ of R_r which sends t to $\lambda \cdot t$. If F is any functor from the category rings to the category of $\mathbb{Q}$-vector spaces, this gives us an action of $\mathbb{Q}^\times$ on $F(R_r)$, which we call the $\star$-action of $\mathbb{Q}^\times$ and denote $F(\psi_\lambda)(v)$ by $\lambda \star v$. For $m \in \mathbb{Z}$, we let $F(R_r)^{[m]}$ denote the subspace of elements $v \in F(R_r)$ such that $\lambda \star v = \lambda^m \cdot v$, for every $\lambda \in \mathbb{Q}^\times$. An element $v \in F(R_r)^{[m]}$ is said to be an element of $\star$-weight m.

Lemma 4.0.4. *Suppose that we have the formula (4.0.1) for $\alpha = 1 + t$ and $\beta_i \in R^\times$ for $1 \le i \le n-1$, for the ring R_2. Then we have the same formula for any A as above.*

Proof. We need to prove the result for $1 + t \in R_r$, and $\beta_i \in R^\times$. Since $1 + t = \exp(\log(1+t))$, it is a product of elements of the form $\exp(at^m)$, for $1 \le m < r$, and $a \in \mathbb{Q}$. Therefore it is enough to prove the formula for elements as above with $\alpha = \exp(at^m)$. Since $\{\exp(at^m), \beta_1, \dots, \beta_{n-1}\}$ is of $\star$-weight m, its image under φ is in $(\Omega^{n-1}_{R_r}/d\Omega^{n-2}_{R_r})^{[m]}$, the $\star$-weight m part of $\Omega^{n-1}_{R_r}/d\Omega^{n-2}_{R_r}$. On the other hand the natural surjection $R_r \to R_{m+1}$ induces an isomorphism

$$(\Omega^{n-1}_{R_r}/d\Omega^{n-2}_{R_r})^{[m]} \simeq (\Omega^{n-1}_{R_{m+1}}/d\Omega^{n-2}_{R_{m+1}})^{[m]}.$$

Therefore, without loss of generality, we will assume that $r = m+1$. Then we use the map from R_2 to R_{m+1} that sends t to at^m. This map sends $1+t$ to $\exp(at^m)$ and hence maps $\{1+t, \beta_1, \ldots, \beta_{n-1}\}$ to $\{\exp(at^m), \beta_1, \ldots, \beta_{n-1}\}$. Therefore, again by the functoriality of φ, the result follows from the assumption on R_2. $\square$

To finish the proof, we will need a special identity in $K_2^M(R_3)$.

Lemma 4.0.5. *We have the following relation in $K_2^M(R_3)$:*

$$2\bigl\{1+\tfrac{t^2}{2}, \lambda\bigr\} = \bigl\{1+\tfrac{t}{\lambda}, 1+\lambda t\bigr\},$$

for any $\lambda \in R^\times$.

Proof. It is possible to give a direct computational proof of this statement. We choose to give a proof which is based on the ideas in this section.

First suppose that R is a field. We know that both sides are in $K_2^M(R_3)^\circ$. We know from [Graham 1973] that the map $K_2^M(R_3)^\circ \to (\Omega^1_{R_3}/dR_3)^\circ$ which sends $\{\alpha, \beta\}$ to $\log(\alpha)d\beta/\beta$, where $\alpha - 1 \in (t)$, is an isomorphism.

The left-hand side goes to $t^2 d\lambda/\lambda$, whereas the right-hand side goes to

$$\frac{t}{\lambda}d(\lambda t) = t^2\frac{d\lambda}{\lambda} + t\,dt = t^2\frac{d\lambda}{\lambda} + \tfrac{1}{2}dt^2 = t^2\frac{d\lambda}{\lambda}$$

in $(\Omega^1_{R_3}/dR_3)^\circ$. This proves the statement when R is a field.

In general, the statement for $\mathbb{Q}[x, x^{-1}]$ implies the one for a general R by sending x to λ. Finally, if we can show that $K_2^M(\mathbb{Q}[x, x^{-1}]_3)^\circ \to K_2^M(\mathbb{Q}(x)_3)^\circ$ is an injection, the known statement for $\mathbb{Q}(x)$ implies the one for $\mathbb{Q}[x, x^{-1}]$. This injectivity follows from the commutative diagram:

$$\begin{tikzcd}
K_2^M(\mathbb{Q}[x,x^{-1}]_3)^\circ \arrow[r, hook, "\varphi"] \arrow[d] & (\Omega^1_{\mathbb{Q}[x,x^{-1}]_3}/d(\mathbb{Q}[x,x^{-1}]_3))^\circ \arrow[d] & t\Omega^1_{\mathbb{Q}[x,x^{-1}]} \oplus t^2\Omega^1_{\mathbb{Q}[x,x^{-1}]} \arrow[l, "\sim"'] \arrow[d, hook] \\
K_2^M(\mathbb{Q}(x)_3)^\circ \arrow[r, hook, "\varphi"] & (\Omega^1_{\mathbb{Q}(x)_3}/d(\mathbb{Q}(x)_3))^\circ & t\Omega^1_{\mathbb{Q}(x)} \oplus t^2\Omega^1_{\mathbb{Q}(x)} \arrow[l, "\sim"']
\end{tikzcd}$$

Where the injectivity of φ was proven above. This finishes the proof of the lemma. $\square$

Finally, we prove the result for R_2.

Proposition 4.0.6. *Let $\alpha = 1+t \in R_2^\times$ and $\beta_i \in R^\times$, for $1 \le i \le n-1$, then $\varphi(\{\alpha, \beta_1, \ldots, \beta_{n-1}\})$ is given by (4.0.1).*

Proof. Note the map ψ from R_2 to R_3 that sends t to $t^2/2$. This map induces an isomorphism

$$(\Omega^{n-1}_{R_2}/d\Omega^{n-2}_{R_2})^{[1]} \simeq (\Omega^{n-1}_{R_3}/d\Omega^{n-2}_{R_3})^{[2]}.$$

Therefore we only need to compute the image of

$$\left\{1+\tfrac{t^2}{2}, \beta_1, \dots, \beta_{n-1}\right\} \tag{4.0.3}$$

in $(\Omega^{n-1}_{R_3}/d\Omega^{n-2}_{R_3})^{[2]}$. By the previous lemma, we know that

$$\left\{1+\tfrac{t^2}{2}, \beta_1\right\} = \tfrac{1}{2}\left\{1+\tfrac{t}{\beta_1}, 1+\beta_1 t\right\},$$

which implies that (4.0.3) is equal to

$$\tfrac{1}{2}\left\{1+\tfrac{t}{\beta_1}, 1+\beta_1 t, \beta_2, \dots, \beta_{n-1}\right\}. \tag{4.0.4}$$

This last expression is the $\frac{1}{2}$ times the product of $\{1+t/\beta_1\} \in K^M_1(R_3)^\circ$ and

$$\{1+\beta_1 t, \beta_2, \dots, \beta_{n-1}\} \in K^M_{n-1}(R_3)^\circ.$$

By the induction hypothesis on n,

$$\varphi(\{1+\beta_1 t, \beta_2, \dots, \beta_{n-1}\}) = \log(1+\beta_1 t)\frac{d\beta_2}{\beta_2} \wedge \cdots \wedge \frac{d\beta_{n-1}}{\beta_{n-1}}.$$

Since φ is multiplicative, this implies that (4.0.4) is sent by φ to

$$\tfrac{1}{2}\log\left(1+\frac{t}{\beta_1}\right) d\log(1+\beta_1 t)\frac{d\beta_2}{\beta_2} \wedge \cdots \wedge \frac{d\beta_{n-1}}{\beta_{n-1}} = \log\left(1+\frac{t^2}{2}\right)\frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_{n-1}}{\beta_{n-1}}. \qquad \square$$

This finishes the proof of Proposition 4.0.1. $\square$

In the case of truncated polynomial rings, we can also describe this isomorphism as follows:

Corollary 4.0.7. *The map $\lambda_i : \Lambda^n R^\times_\infty \to \Omega^{n-1}_R$ given by*

$$\lambda_i(a_1 \wedge \cdots \wedge a_n) = \mathrm{res}_{t=0}\,\frac{1}{t^i} d\log(a_1) \wedge \cdots \wedge d\log(a_n) \in \Omega^{n-1}_R,$$

for $1 \le i < r$, descends to give a map $K^M_n(R_r)^\circ \to \Omega^{n-1}_R$. Their sums induce an isomorphism

$$K^M_n(R_r)^\circ \to \bigoplus_{1\le i<r} \Omega^{n-1}_R.$$

Proof. For $1 \le i < r$, we let $\mu_i : (\Omega^{n-1}_{R_r}/d(\Omega^{n-2}_{R_r}))^\circ \to \Omega^{n-1}_R$ be given by $\mu_i(w) := \mathrm{res}_{t=0}\,\frac{1}{t^i} d\omega$. The induced map

$$(\Omega^{n-1}_{R_r}/d(\Omega^{n-2}_{R_r}))^\circ \to \bigoplus_{1\le i<r} \Omega^{n-1}_R$$

is an isomorphism.

The surjectivity can be seen as follows. Given $\omega \in \Omega_R^{n-1}$, and $1 \le i, j < r$,

$$\mu_j(t^i \omega) = \mathrm{res}_{t=0} \frac{1}{t^j} d(t^i \omega) = \delta_{ij} \cdot i \cdot \omega.$$

To prove injectivity, using the notation in the proof of Lemma 4.0.3, we note that $(\Omega_{R_r}^{n-1}/d(\Omega_{R_r}^{n-2}))^\circ$ is the direct sum of its subspaces $((\Omega_{R_r}^{n-1}/d(\Omega_{R_r}^{n-2}))^\circ)^{[i]}$ of $\star$-weight i, for $1 \le i < r$. The subspace $((\Omega_{R_r}^{n-1}/d(\Omega_{R_r}^{n-2}))^\circ)^{[i]}$ consists of elements of the form $t^i\alpha + \beta t^{i-1}dt$, with $\alpha \in \Omega_R^{n-1}$ and $\beta \in \Omega_R^{n-2}$. For $1 \le j < r$, we have

$$\mu_j(t^i\alpha + \beta t^{i-1}dt) = \delta_{ij}(i \cdot \alpha + (-1)^{n-1} d\beta).$$

Therefore, if $\mu_i(t^i\alpha + \beta t^{i-1}dt) = 0$, then $\alpha = ((-1)^n/i)d\beta$ and hence

$$t^i\alpha + \beta t^{i-1}dt = d\left(\frac{(-1)^n\beta}{i} t^i\right).$$

This proves the injectivity of the map

$$(\Omega_{R_r}^{n-1}/d(\Omega_{R_r}^{n-2}))^\circ = \bigoplus_{1\le i<r} ((\Omega_{R_r}^{n-1}/d(\Omega_{R_r}^{n-2}))^\circ)^{[i]} \to \bigoplus_{1\le i<r} \Omega_R^{n-1}.$$

The corollary then follows from Proposition 4.0.1. □

5. Construction of maps from $B_2(R_m) \otimes R_m^\times$ to Ω_R^1

In this section, we assume that R is a regular, local $\mathbb{Q}$-algebra.

5.1. *Preliminaries on the construction.* In this section, we fix m and r such that $2 \le m < r < 2m$. We let $f(s, u) := \log^\circ(1 - s\exp(u)) = \log((1 - s\exp(u))/(1 - s))$. As in the proof of Proposition 3.0.1, we define $\ell_i : R_\infty^\times \to R$, by the formula $\ell_i(a) := t_i(\log^\circ(a))$. Note that t_i is defined in the beginning of Section 3. Let us consider the expression

$$\alpha_j := \sum_{1\le i\le j-1} i d\ell_{j-i} \wedge \ell_i = \sum_{\substack{a+b=j\\1\le a,b}} b d\ell_a \wedge \ell_b, \tag{5.1.1}$$

for $m \le j < r$, which defines a map from $\Lambda^2 R_\infty^\times$ to Ω_R^1. We will use this expression to define a map from $B_2(R_m) \otimes R_m^\times$.

Lemma 5.1.1. *For $s \in R^\flat$, and $u := \sum_{0<i} u_i t^i \in tR_\infty$, letting $f_s := \frac{\partial f}{\partial s}$ and $u_t := \frac{\partial u}{\partial t} = \sum_{0<i} i u_i t^{i-1}$, we have*

$$\alpha_j(\delta(s\exp(u))) = t_{j-1}(f_s u_t)ds = t_{j-1}\left(\frac{\partial \log^\circ(1 - s\exp(u))}{\partial s} \cdot \frac{\partial u}{\partial t}\right)ds.$$

Proof. Let us write $f(s, u) =: f = \sum_{0<i} f_i t^i$. The expression $i d\ell_{j-i} \wedge \ell_i$ evaluated on $\delta(s\exp(u))$ is equal to

$$\mathrm{id}(f_{j-i})u_i - i f_i du_{j-i} = \mathrm{id}(f_{j-i})u_i + (j - i) f_i du_{j-i} - j f_i du_{j-i}.$$

Summing these, we find that

$$\alpha_j(\delta(s\exp(u))) = \sum_{1\le i\le j-1} (\mathrm{id}(f_{j-i})u_i + (j-i) f_i du_{j-i}) - j \sum_{1\le i\le j-1} f_i du_{j-i}.$$

Let $Du := \sum_{1\le i} du_i t^i$ and $u_t := \frac{\partial u}{\partial t}$. Then the last expression can be rewritten as

$$t_{j-1}(D(fu_t)) - jt_j(fDu) = t_{j-1}(D(fu_t) - (fDu)_t) = t_{j-1}(Dfu_t - f_tDu). \tag{5.1.2}$$

We would like to see that the coefficient of du_i in (5.1.2) is equal to 0. The coefficient of du_i in $Df = D\log((1-s\exp(u))/(1-s))$ is equal to $(-s\exp(u))/(1-s\exp(u))t^i$. Therefore, the coefficient of du_i in (5.1.2) is

$$t_{j-1-i}\left(\frac{-s\exp(u)}{1-s\exp(u)}u_t - f_t\right).$$

Since $f_t = \frac{\partial}{\partial t}(\log(1-s\exp(u))) = (-s\exp(u))/(1-s\exp(u))u_t$, the last expression is 0.

Therefore $\alpha_j(\delta(s\exp(u)))$ does not depend on the du_i, and we can rewrite (5.1.2) as

$$\alpha_j(\delta(s\exp(u))) = t_{j-1}(Dfu_t - f_tDu) = t_{j-1}(f_su_t)ds,$$

where $f_s = \frac{\partial f}{\partial s}$. □

Lemma 5.1.2. *If $u = u|_m$ and $m \le j < r$, we have*

$$jt_j(f) = st_{j-1}(f_su_t).$$

Proof. The expression $jt_j(f) - st_{j-1}(f_su_t)$ is equal to

$$t_{j-1}(f_t - s(f_su_t)) = t_{j-1}\left(\frac{\partial}{\partial t}\log(1-s\exp(u)) - s\frac{\partial}{\partial s}\log\left(\frac{1-s\exp(u)}{1-s}\right)\cdot u_t\right).$$

Since

$$\frac{\partial}{\partial t}\log(1-s\exp(u)) = \frac{-s\exp(u)}{1-s\exp(u)}\cdot u_t = s\frac{\partial}{\partial s}\log(1-s\exp(u))\cdot u_t,$$

the above expression is equal to $t_{j-1}((s/(s-1))u_t)$, which is 0, under the assumption that $u = u_1t + \cdots + u_{m-1}t^{m-1}$ and $m \le j$. □

Let $d\ell_0 : R_\infty^\times \to \Omega_R^1$ be defined as $d\ell_0(\alpha) := d\log(\alpha(0))$. Note that ℓ_0 itself is not defined, even though ℓ_i are defined for $i > 0$.

Proposition 5.1.3. *The map $M_{m,r}$ defined as*

$$M_{m,r} := \ell i_{m,r} \otimes d\ell_0 - \sum_{m\le j<r} \frac{r-j}{j}(\alpha_j \circ \delta) \otimes \ell_{r-j}$$

gives a map from $B_2(R_\infty) \otimes R_\infty^\times$ to Ω_R^1, of $\star$-weight r, which vanishes on the image under δ of those $[se^u] \in \mathbb{Q}[R_\infty^\flat]$, with $u = u|_m$. The map $M_{m,m+1}$ descends to a map from $B_2(R_m) \otimes R_m^\times$.

Proof. That $M_{m,r}$ is of $\star$-weight r follows immediately from the expression for $\alpha_j(\delta(s\exp(u)))$ in Lemma 5.1.1, which shows that $\alpha_j(\delta(s\exp(u)))$ is of $\star$-weight j. Let us now show that $M_{m,r}$ evaluated on $[s\exp(u)]\otimes s\exp(u)$, with $u=u|_m$, is equal to 0. By Lemma 5.1.1, $\alpha_j(\delta(s\exp(u)))$ is equal to $t_{j-1}(f_s u_t)ds$. This implies that $\sum_{m\le j<r}(r-j/j)(\alpha_j\circ\delta)\otimes\ell_{r-j}$ evaluated on $[s\exp(u)]\otimes s\exp(u)$ is equal to

$$\sum_{m\le j<r}\frac{r-j}{j}t_{j-1}(f_s u_t)u_{r-j}ds=\frac{1}{s}\sum_{m\le j<r}(r-j)t_j(f)u_{r-j}ds,$$

by Lemma 5.1.2, since $u=u|_m$. The final expression can be rewritten as

$$t_{r-1}(f\cdot u_t|_{r-m})\frac{ds}{s}=\ell i_{m,r}(s\exp(u))\frac{ds}{s}=(\ell i_{m,r}\otimes d\ell_0)([s\exp(u)]\otimes s\exp(u))$$

since $u=u|_m$. This proves the first part of the proposition.

When $r=m+1$, $M_{m,r}$ takes the form

$$\ell i_{m,m+1}\otimes d\ell_0-\tfrac{1}{m}((d\ell_{m-1}\wedge\ell_1+2d\ell_{m-2}\wedge\ell_2+\cdots+(m-1)d\ell_1\wedge\ell_{m-1})\circ\delta)\otimes\ell_1.$$

Since all the functions in this expression depend on the classes of the elements in R_m, the statement easily follows. □

5.2. *The regulator maps from* $\mathrm{H}^2(R_m,\mathbb{Q}(3))$ *to* Ω^1_R. We would like to define maps

$$L_{m,r}:\mathrm{H}^2(R_m,\mathbb{Q}(3))\to\Omega^1_R$$

based on the maps $M_{m,r}$ in Proposition 5.1.3. The problem with $M_{m,r}$ is that it does not descend to a map on $B_2(R_m)\otimes R_m^\times$, if $r\neq m+1$. We will modify $M_{m,r}$ slightly to correct this defect but keep the other properties to obtain $L_{m,r}$. In order to simplify the notation from now on we are going to let $\ell i_{m,m}:=0$. Note that $\ell i_{m,r}$ was previously defined only when $m+1\le r\le 2m-1$ so this will not cause any confusion. We define $\beta_m(j)$, for $m\le j<2m-1$, by

$$\begin{aligned}\beta_m(j)&:=d\ell i_{m,j}+\sum_{\substack{a+b=j\\1\le a,b<m}}b(d\ell_a\wedge\ell_b)\circ\delta=\sum_{\substack{a+b=j\\m\le a,1\le b}}bd(\ell_a\wedge\ell_b)\circ\delta+\sum_{\substack{a+b=j\\1\le a,b<m}}b(d\ell_a\wedge\ell_b)\circ\delta\\&=d\ell i_{m,j}+\alpha_j\circ\delta-\sum_{1\le a\le j-m}((j-a)d\ell_a\wedge\ell_{j-a}+ad\ell_{j-a}\wedge\ell_a)\circ\delta\end{aligned}$$

and

$$L_{m,r}:=\ell i_{m,r}\otimes d\ell_0-\sum_{m\le j<r}\left(\tfrac{r-j}{j}\beta_m(j)\otimes\ell_{r-j}-\ell i_{m,j}\otimes d\ell_{r-j}\right).\tag{5.2.1}$$

We would like to emphasize that, because of our conventions, the summand that corresponds to $j=m$ is equal to $((r-m)/m)\alpha_m\otimes\ell_{r-m}$ exactly as in the case of $M_{m,r}$, the terms corresponding to $m<j$ are modified however.

Lemma 5.2.1. *With the above definition, $L_{m,r}$ defines a map from $B_2(R_m)\otimes R_m^\times$ to Ω^1_R of $\star$-weight r.*

Proof. Since all the terms in the definition of $L_{m,r}$ depend on the variables modulo t^m, we obtain a map from $B_2(R_m) \otimes R_m^\times$ to Ω^1_R.

Since we know that $M_{m,r}$ is of $\star$-weight r, in order to prove that $L_{m,r}$ is of $\star$-weight r, it suffices to prove the same for $L_{m,r} - M_{m,r}$. This difference is equal to the sum of

$$-\sum_{m \le j < r} \big(\tfrac{r-j}{j} d\ell i_{m,j} \otimes \ell_{r-j} - \ell i_{m,j} \otimes d\ell_{r-j}\big) \tag{5.2.2}$$

and

$$\sum_{m \le j < r} \tfrac{r-j}{j} \Bigg(\sum_{1 \le a \le j-m} ((j-a) d\ell_a \wedge \ell_{j-a} + a d\ell_{j-a} \wedge \ell_a) \Bigg) \circ \delta \otimes \ell_{r-j}. \tag{5.2.3}$$

Let us first look at the term $(j-a)d\ell_a \wedge \ell_{j-a} + ad\ell_{j-a} \wedge \ell_a$. For any $u \wedge v \in \Lambda^2 tR_\infty$ and $\lambda \in R^\times$,

$$\begin{aligned}
&((j-a)d\ell_a \wedge \ell_{j-a} + ad\ell_{j-a} \wedge \ell_a)(\lambda \star (u \wedge v)) \\
&\quad = ((j-a)d(\lambda^a \ell_a) \wedge \lambda^{j-a} \ell_{j-a} + ad(\lambda^{j-a} \ell_{j-a}) \wedge \lambda^a \ell_a)((u \wedge v)) \\
&\quad = \big(\lambda^j ((j-a)d\ell_a \wedge \ell_{j-a} + ad\ell_{j-a} \wedge \ell_a) + (j-a)a(\ell_a \wedge \ell_{j-a} + \ell_{j-a} \wedge \ell_a)\lambda^{j-1} d\lambda\big)(u \wedge v) \\
&\quad = \lambda^j ((j-a)d\ell_a \wedge \ell_{j-a} + ad\ell_{j-a} \wedge \ell_a)(u \wedge v).
\end{aligned}$$

Therefore the term (5.2.3) is of $\star$-weight r.

Similarly, $(r-j)d\ell i_{m,j} \otimes \ell_{r-j} - j\ell i_{m,j} \otimes d\ell_{r-j}$ evaluated on $\lambda \star (u \otimes v)$ is equal to

$$\begin{aligned}
&(r-j)d(\lambda^j \ell i_{m,j}) \otimes \lambda^{r-j} \ell_{r-j} - j\lambda^j \ell i_{m,j} \otimes d(\lambda^{r-j} \ell_{r-j}) \\
&\quad = \lambda^r ((r-j)d\ell i_{m,j} \otimes \ell_{r-j} - j\ell i_{m,j} \otimes d\ell_{r-j}) + (r-j)j(\ell i_{m,j} \otimes \ell_{r-j} - \ell i_{m,j} \otimes \ell_{r-j})\lambda^{r-1} d\lambda \\
&\quad = \lambda^r ((r-j)d\ell i_{m,j} \otimes \ell_{r-j} - j\ell i_{m,j} \otimes d\ell_{r-j}).
\end{aligned}$$

evaluated on $u \otimes v$. This implies that the term (5.2.2) is of $\star$-weight r and finishes the proof of the lemma. □

Proposition 5.2.2. *The map $L_{m,r} : B_2(R_m) \otimes R_m^\times \to \Omega^1_R$ vanishes on the boundaries of the elements in $\mathbb{Q}[R_m^\flat]$ and hence induces a map*

$$(B_2(R_m) \otimes R_m^\times)/\operatorname{im}(\delta) \to \Omega^1_R,$$

which by restriction gives the regulator map $\mathrm{H}^2(R_m, \mathbb{Q}(3)) \to \Omega^1_R$ of $\star$-weight r we were looking for. We continue to denote these two induced maps by the same notation $L_{m,r}$.

Proof. We know that $M_{m,r}$ vanishes on the boundary $\delta(s \exp(u))$ of elements $s \exp(u) \in \mathbb{Q}[R_\infty]$, with $u = u|_m$, by Proposition 5.1.3. We also know by the previous lemma that $L_{m,r}$ descends to a map on $B_2(R_m) \otimes R_m^\times$. Therefore, in order to prove the statement, we only need to prove that

$$L_{m,r}(\delta(s \exp(u))) = M_{m,r}(\delta(s \exp(u))),$$

for $u = u|_m$. We first rewrite $L_{m,r}$ as the composition of $\delta \otimes \mathrm{id}$ with

$$\sum_{\substack{a+b=r\\ m\le a,1\le b}} b\cdot \ell_a\wedge\ell_b\otimes d\ell_0 - \sum_{\substack{a+b+c=r\\ m\le a+b,\\ 1\le a,b,c<m}} \frac{c}{a+b} b\cdot d\ell_a\wedge\ell_b\otimes\ell_c \\ - \sum_{\substack{a+b+c=r\\ m\le a,1\le b,c}} \frac{c}{a+b} b\cdot d(\ell_a\wedge\ell_b)\otimes\ell_c + \sum_{\substack{a+b+c=r\\ m\le a,1\le b,c}} b\cdot\ell_a\wedge\ell_b\otimes d\ell_c.$$

On the other hand, recall that $M_{m,r}$ is the composition of $\delta\otimes\mathrm{id}$ with

$$\sum_{\substack{a+b=r\\ m\le a,1\le b}} b\cdot \ell_a\wedge\ell_b\otimes d\ell_0 - \sum_{\substack{a+b+c=r\\ m\le a+b,\\ 1\le a,b,c}} \frac{c}{a+b} b\cdot d\ell_a\wedge\ell_b\otimes\ell_c.$$

If we compare the two expressions we see that all of the terms match above except possibly the ones that correspond to the triples (a,b,c) with $1\le a,b,c$, $a+b+c=r$, and $m\le a$ or $m\le b$. By antisymmetry, we may assume without loss of generality that $m\le a$. We need to compare the coefficients of the terms $d\ell_a\wedge\ell_b\otimes\ell_c$, $\ell_a\wedge d\ell_b\otimes\ell_c$, and $\ell_a\wedge\ell_b\otimes d\ell_c$, subject to the above constraints, in $L_{m,r}$ and $M_{m,r}$. The coefficient of $d\ell_a\wedge\ell_b\otimes\ell_c$ in $L_{m,r}$ and $M_{m,r}$ are both equal to $-cb/(a+b)$. The coefficient of $\ell_a\wedge d\ell_b\otimes\ell_c$ in $L_{m,r}$ is $-cb/(a+b)$ and in $M_{m,r}$, it is $ca/(a+b)$. Finally, the coefficient of $\ell_a\wedge\ell_b\otimes d\ell_c$ in $L_{m,r}$ is b, whereas in $M_{m,r}$ it is 0.

We finally note that the values of $\ell_a\wedge d\ell_b\otimes\ell_c$ and $\ell_a\wedge\ell_c\otimes d\ell_b$ on $\delta(s\exp(u))\otimes s\exp(u)$ are the same when $u=u|_m$. Then the equality $-cb/(a+b)+c=ca/(a+b)$ finishes the proof. □

We can restate Lemma 5.2.1 as follows. First, let

$$\gamma_m(j) := d\ell_{m,j} + \sum_{\substack{a+b=j\\ 1\le a,b<m}} b(d\ell_a\wedge\ell_b).$$

Note that since $\ell i_{m,j} = \ell_{m,j}\circ\delta$ by Definition 3.0.2, we have $\beta_m(j)=\gamma_m(j)\circ\delta$. Finally, if we let

$$N_{m,r} := \ell_{m,r}\otimes d\ell_0 - \sum_{m\le j<r} \left(\tfrac{r-j}{j}\gamma_m(j)\otimes\ell_{r-j} - \ell_{m,j}\otimes d\ell_{r-j}\right),$$

then by (5.2.1), we have the following.

Corollary 5.2.3. *For $2\le m<r<2m$, we have a commutative diagram*:

$$\begin{array}{ccc} B_2(R_r)\otimes R_r^\times & \xrightarrow{\delta\otimes\mathrm{id}} & \Lambda^2 R_r^\times\otimes R_r^\times \\ \big\downarrow & & \big\downarrow{\scriptstyle N_{m,r}} \\ B_2(R_m)\otimes R_m^\times & \xrightarrow{L_{m,r}} & \Omega^1_R \end{array}$$

We expect that the above maps combine to give an isomorphism between the infinitesimal part of the cohomology of R_m and the direct sum of the module of Kähler differentials, justifying the name of the regulator; see [Goncharov 1995, Conjecture 1.15]. However, at this point, we can only prove the surjectivity.

Proposition 5.2.4. *Suppose that R is a regular local $\mathbb{Q}$-algebra and $2 \leq m$ as the above. The direct sum of the $L_{m,r}$ induces a surjection*

$$\bigoplus_{m<r<2m} L_{m,r} : \mathrm{H}^2(R_m, \mathbb{Q}(3))^\circ \twoheadrightarrow \bigoplus_{m<r<2m} \Omega^1_R.$$

Proof. Suppose that $\alpha \in B_2(R_m)^\circ$ is in the part of kernel of the δ° which is of $\star$-weight r. By Theorem 3.0.3, this part is isomorphic to R via the restriction of the map $\ell i_{m,r}$. Computing the value of $L_{m,r}$ on $\alpha \otimes b$, for $b \in R^\times$, we see that $L_{m,r}(\alpha \otimes b) = \ell i_{m,r}(\alpha) db/b$. Since $\alpha \otimes b$ is in the kernel of δ, we see that the image of $L_{m,r}$ above is the additive group generated by the set $Rd\log(R^\times)$. Since R is local this is equal to Ω^1_R. This implies the surjectivity. □

Conjecture 5.2.5. *We conjecture that the map $\bigoplus_{m<r<2m} L_{m,r}$ in Proposition 5.2.4 is injective and hence is an isomorphism.*

6. Construction of the maps from $\Lambda^3(R_{2m-1}, (t^m))^\times$ to $\Omega^1_{R/k}$

For a ring A and ideal I, let $(A, I)^\times := \{(a, b) \mid a, b \in A^\times, a - b \in I\}$, and let $\pi_i : (A, I)^\times \to A^\times$, for $i = 1, 2$ denote the two projections. If R is a k-algebra, in this section we will define a map $\omega_{m,r} : \Lambda^3(R_r, (t^m))^\times \to \Omega^1_{R/k}$.

6.1. *Definition of $\Omega_{m,r}$.* Assume that R is $\mathbb{Q}$-algebra and $2 \leq m < r < 2m$. Let us put $I_{m,r} := \mathrm{im}((1 + (t^m)) \otimes \Lambda^2 R_r^\times) \subseteq (\Lambda^3 R_r^\times)^\circ$.

Definition 6.1.1. We define the map $\Omega_{m,r} : I_{m,r} \to \Omega_R$ by the following formulae:

(i) If $x \geq m$, $x + y + z = r$, $y, z \geq 1$ and $a, b, c \in R$ then

$$\Omega_{m,r}(\exp(at^x) \wedge \exp(bt^y) \wedge \exp(ct^z)) := a(yb \cdot dc - zc \cdot db).$$

(ii) If $x \geq m$, $x + y = r$, $y \geq 1$, $a, b \in R$ and $\gamma \in R^\times$ then

$$\Omega_{m,r}(\exp(at^x) \wedge \exp(bt^y) \wedge \gamma) := a\left(yb \cdot \frac{d\gamma}{\gamma}\right).$$

(iii) If $x \geq m$, $x + z = r$, $z \geq 1$, $a, c \in R$ and $\beta \in R^\times$ then

$$\Omega_{m,r}(\exp(at^x) \wedge \beta \wedge \exp(ct^z)) := a\left(-zc \cdot \frac{d\beta}{\beta}\right).$$

(iv) If $x = r$ and $\beta, \gamma \in R^\times$ then

$$\Omega_{m,r}(\exp(at^x) \wedge \beta \wedge \gamma) := 0.$$

Remark 6.1.2. Notice that in case (ii) of the above definition, using the notation $\exp(ct^z)$ with $z = 0$ instead of γ would not make sense. This is because in order for $\exp(ct^z)$ to be well-defined, we need $ct^z \in (t) \subseteq R_r$. However, if we continue to use this notation $\exp(c)$, without specifying what c is and without the notation making actual sense, we note that the formula (ii) becomes a special case of formula (i) in the following sense. If we formally put $\gamma = \exp(c)$ then again formally $\log(\gamma) = c$ and $\frac{d\gamma}{\gamma} = d\log(\gamma) = dc$. This makes formula (ii) exactly the same as formula (i) if we also note that since we put $z = 0$ the term involving $zc.db$ disappears in (i). We will be using these notations and conventions in order to shorten the expressions in the remaining of the paper. However, we would like to emphasize that when proving the statements under consideration we are always using the Definition 6.1.1 since these notations are only formal and do not make actual sense. Similar comments apply to (iii) when $y = 0$ and to (iv) when $y = z = 0$. To sum up we will write that, if $x \geq m$, $x + y + z = r$ and $y, z \geq 0$ then

$$\Omega_{m,r}(\exp(at^x) \wedge \exp(bt^y) \wedge \exp(ct^z)) = a(yb \cdot dc - zc \cdot db). \tag{6.1.1}$$

In the proof of the next proposition, we will use the following notation. Recall that $\mathbb{Q}_r = \mathbb{Q}[t]/(t^r)$. Since we assume that R contains $\mathbb{Q}$, R_r is a $\mathbb{Q}_r$-algebra. Let $\underline{d} : R_r \to \Omega^1_{R_r/\mathbb{Q}_r}$ denote the canonical differential. Note that $\underline{d}$ has the property that $\underline{d}(t) = 0$. There is a natural isomorphism

$$\bigoplus_{0 \leq i < r} t^i \Omega^1_R \to \Omega^1_{R_r/\mathbb{Q}_r}.$$

Proposition 6.1.3. *Suppose that $\hat{f}, \tilde{f} \in R_r^\flat$ and $\hat{g}, \tilde{g} \in R_r^\times$ have the same reductions modulo (t^m), with $2 \leq m < r < 2m$. Then we have*

$$\Omega_{m,r}(\delta([\hat{f}]) \wedge \hat{g} - \delta([\hat{f}]) \wedge \tilde{g}) = 0 \quad \textit{and} \quad \Omega_{m,r}(\delta([\hat{f}]) \wedge \hat{g} - \delta([\tilde{f}]) \wedge \hat{g}) = 0.$$

Proof. By the assumptions $\hat{g}/\tilde{g}$ is a product of terms of the form $\exp(at^x)$ with $a \in R$ and $m \leq x < r$. Hence, in order to prove the first equality we need to prove that $\Omega_{m,r}$ vanishes on $\delta([\hat{f}]) \wedge \exp(at^x)$. By the definition of $\Omega_{m,r}$ above, we have

$$\Omega_{m,r}(\delta([\hat{f}]) \wedge \exp(at^x)) = -a \cdot \operatorname{res}_{t=0} \frac{1}{t^{r-x}} (d\log \wedge d\log)((1-\hat{f}) \wedge \hat{f}) = 0,$$

since $(d\log \wedge d\log)((1-\hat{f}) \wedge \hat{f}) = 0$. In order to prove the second equality, note that $\hat{g}$ is a product of terms of type $\exp(ct^z)$ with $0 \leq z$, where for $z = 0$, we use the notation in Remark 6.1.2. Then using the first equality, we only need to prove that

$$\Omega_{m,r}(\delta([\hat{f}]) \wedge \exp(ct^z) - \delta([\tilde{f}]) \wedge \exp(ct^z)) = 0, \tag{6.1.2}$$

for $0 \leq z < m$. On the other hand, since $\hat{f}$ and $\tilde{f}$ are equal modulo (t^m), we see that (6.1.2) holds when $r - z < m$. Therefore from now on we assume that $m \leq r - z$. If we knew (6.1.2) in the special case when $\hat{f} = \tilde{f} + at^x$ with $m \leq x$, then by successively using this information we obtain (6.1.2) for any $\hat{f}$ and $\tilde{f}$ which have the same reduction modulo (t^m). Therefore, from now on we assume that $\hat{f} = \tilde{f} + at^x$, and $\tilde{f} = s + b_1 t + b_2 t^2 + \cdots$. The left-hand side of (6.1.2)) is a sum of two terms: one containing the term dc and the other one containing the term c. Let us first consider the term containing dc. The term containing dc is 0 when $r - z = m$ since $m \leq x$. Therefore we assume that $m < r - z$. In this case, we compute that this term containing dc is equal to

$$\left(\left(\sum_{1\le i\le r-z-m} i\cdot \ell_{r-z-i}\wedge \ell_i\right)\circ\delta\right)([\hat{f}]-[\tilde{f}])\cdot dc. \tag{6.1.3}$$

We have $\left(\left(\sum_{1\le i\le r-z-m} i\cdot \ell_{r-z-i}\wedge \ell_i\right)\circ\delta\right)([h]) = \ell i_{m,r-z}([h|_{t^m}])$, by the definition of $\ell i_{m,r-z}$, for any $h\in R_r^\flat$. This implies that (6.1.3) is equal to $(\ell i_{m,r-z}([\hat{f}|_{t^m}]) - \ell i_{m,r-z}([\tilde{f}|_{t^m}]))\cdot dc = 0$. Finally, we consider summand of (6.1.2) which contain the term c. Letting $\beta := b_1 t + b_2 t^2+\cdots$, this term is equal to $-z\cdot c\cdot a$ times the coefficient of t^{r-z-x} in

$$\sum_{0<i,j}(-1)^{i-1}\frac{\beta^{i-1}}{(s-1)^i}\cdot \underline{d}\left(\frac{(-1)^{j-1}}{j}\frac{\beta^j}{s^j}\right) - \left((-1)^{i-1}\frac{\beta^{i-1}}{s^i}\cdot \underline{d}\left(\frac{(-1)^{j-1}}{j}\frac{\beta^j}{(s-1)^j}\right)\right) \\ + \sum_{0<i}(-1)^{i-1}\frac{\beta^{i-1}}{(s-1)^i}\cdot\frac{ds}{s} - \left((-1)^{i-1}\frac{\beta^{i-1}}{s^i}\cdot\frac{ds}{s-1}\right).$$

Let us rewrite the last expression as

$$\sum_{0<i,j}(-1)^{i+j}\beta^{i+j-2}\underline{d}(\beta)\left(\frac{1}{(s-1)^i s^j}-\frac{1}{s^i(s-1)^j}\right) \\ +\sum_{0<i,j}(-1)^{i+j-1}\beta^{i+j-1}ds\left(\frac{1}{(s-1)^i s^{j+1}}-\frac{1}{s^i(s-1)^{j+1}}\right) \\ +\sum_{0<i}(-1)^{i-1}\beta^{i-1}ds\left(\frac{1}{(s-1)^i s}-\frac{1}{s^i(s-1)}\right).$$

In this expression, the first sum is equal to 0. In the second sum only the terms with $i=1$ survive, to make the sum equal to

$$\sum_{0<j}(-1)^j\beta^j ds\left(\frac{1}{(s-1)s^{j+1}}-\frac{1}{s(s-1)^{j+1}}\right).$$

This is precisely the negative of the third sum above. This finishes the proof. □

Let $s:\Lambda^3(R_r,(t^m))^\times\to\Lambda^3 R_r^\times$, be given by $s:=\Lambda^3\pi_1-\Lambda^3\pi_2$. The group $\Lambda^3(R_r,(t^m))^\times$ is generated by elements of the form $(a,a')\wedge(b,b')\wedge(c,c')$ with $a-a', b-b', c-c'\in(t^m)$ and $a,a',b,b',c,c'\in R_r^\times$. If we let $a=\alpha a'$, $b=\beta b'$, and $c=\gamma c'$, then $\alpha,\beta,\gamma\in 1+(t^m)$. We have

$$s((a,a')\wedge(b,b')\wedge(c,c')) = \alpha a'\wedge\beta b'\wedge\gamma c' - a'\wedge b'\wedge c'\in I_{m,r},$$

since the last expression is a sum of elements of the form $\delta\wedge d\wedge e$ with $\delta\in 1+(t^m)$ and $d,e\in R_r^\times$. This implies that s factors through $I_{m,r}\subseteq(\Lambda^3 R_r^\times)^\circ$.

Definition 6.1.4. Suppose that R is a k-algebra. We let $\underline{\Omega}_{m,r}$ denote the composition of $\Omega_{m,r}$ with the canonical projection $\Omega^1_R\to\Omega^1_{R/k}$. We define $\omega_{m,r}:\Lambda^3(R_r,(t^m))^\times\to\Omega^1_{R/k}$ as the composition $\underline{\Omega}_{m,r}\circ s$ of $s:\Lambda^3(R_r,(t^m))^\times\to I_{m,r}$, and $\underline{\Omega}_{m,r}: I_{m,r}\to\Omega^1_{R/k}$.

6.2. ***Relation of $\underline{\Omega}_{m,r}$ to $\underline{L}_{m,r}$.*** In this section, we assume that R is a smooth local k-algebra of relative dimension 1. We will relate the construction $\Omega_{m,r}$ to $L_{m,r}$, assuming Conjecture 5.2.5. Even though the results in this section will not be used in the rest of the paper, we include this section since it gives a more conceptual description of $\Omega_{m,r}$ and it will be referred to in future work. Let us denote the composition of $L_{m,r}$ with the canonical projection $\Omega^1_R \to \Omega^1_{R/k}$ by $\underline{L}_{m,r}$. Given $\alpha \in I_{m,r}$, we will show that there exists

$$\varepsilon \in \operatorname{im}((1+(t^m)) \otimes k^\times \otimes R_r^\times) \subseteq (\Lambda^3 R_r^\times)^\circ,$$

such that $\alpha - \varepsilon = \delta_r(\gamma)$ for some $\gamma \in (B_2(R_r) \otimes R_r^\times)^\circ$. Assuming Conjecture 5.2.5, we will then show that

$$\underline{\Omega}_{m,r}(\alpha) = \underline{L}_{m,r}(\gamma|_{t^m}) \in \Omega^1_{R/k}.$$

Lemma 6.2.1. *For any $\alpha \in (\Lambda^3 R_r^\times)^\circ$, there exists $\varepsilon \in \operatorname{im}(\Lambda^2 R_r^\times \otimes k^\times) \subseteq (\Lambda^3 R_r^\times)^\circ$ such that $\alpha - \varepsilon$ lies in the image of $(B_2(R_r) \otimes R_r^\times)^\circ$ in $(\Lambda^3 R_r^\times)^\circ$. Moreover, if*

$$\alpha \in \operatorname{im}((1+t^m R_r)^\times \otimes \Lambda^2 R_r^\times) \subseteq (\Lambda^3 R_r^\times)^\circ$$

then we can choose

$$\varepsilon \in \operatorname{im}((1+t^m R_r)^\times \otimes R_r^\times \otimes k^\times) \subseteq (\Lambda^3 R_r^\times)^\circ$$

such that $\alpha - \varepsilon$ lies in the image of $(B_2(R_r) \otimes R_r^\times)^\circ$ in $(\Lambda^3 R_r^\times)^\circ$.

Proof. The infinitesimal part of the cokernel of the map

$$B_2(R_r) \otimes R_r^\times \to \Lambda^3 R_r^\times$$

is $K_3^M(R_r)^\circ$ which is isomorphic to $\bigoplus_{1\le i<r} t^i \Omega^2_R$ via the map from $\Lambda^3 R_r^\times$, whose i-th coordinate is given by

$$\operatorname{res}_{t=0} \frac{1}{t^i} d\log(y_1) \wedge d\log(y_2) \wedge d\log(y_3), \tag{6.2.1}$$

by Corollary 4.0.7. Further, by the assumption on smoothness of dimension 1, we conclude that the natural map

$$\Omega^1_R \otimes_k \Omega^1_k \to \Omega^2_R$$

is surjective. Since we assume that R is local, the map

$$R \otimes_{\mathbb{Z}} R^\times \to \Omega^1_R,$$

which sends $a \otimes b$ to $a \cdot d\log b$, is surjective.

Note that the image of $\exp(t^i u) \wedge v \wedge \lambda$, with $u \in R$, $v \in R^\times$ and $\lambda \in k^\times$ under the i-th map in (6.2.1) is $i \cdot u \cdot d\log(v) \wedge d\log(\lambda)$ and under the coordinate j maps in (6.2.1) with $j \neq i$, the image is 0. Together with the above, this shows that the map

$$(\Lambda^2 R_r^\times \otimes k^\times)^\circ \to (\Lambda^3 R_r^\times)^\circ \to \bigoplus_{1\le i<r} t^i \Omega^2_R$$

is surjective and hence proves the first statement.

For the second statement, note that if $\alpha \in \mathrm{im}((1+t^m R_r)^\times \otimes \Lambda^2 R_r^\times) \subseteq (\Lambda^3 R_r^\times)^\circ$ then its image in $\bigoplus_{1\le i<r} t^i \Omega_R^2$ lands in the summand $\bigoplus_{m\le i<r} t^i \Omega_R^2$. Since by the above discussion, we also see that the composition

$$(1+t^m R_r)^\times \otimes R_r^\times \otimes k^\times \to (\Lambda^3 R_r^\times)^\circ \to \bigoplus_{m\le i<r} t^i \Omega_R^2$$

is surjective, the second statement similarly follows. □

The next lemma is crucial in relating $\underline{\Omega}_{m,r}$ to $\underline{L}_{m,r}$.

Lemma 6.2.2. *Let $A(r)$ denote the kernel of the differential $B_2(R_r)\otimes R_r^\times \to \Lambda^3 R_r^\times$. Then Conjecture 5.2.5 implies that the composition*

$$A(r) \subseteq B_2(R_r)\otimes R_r^\times \xrightarrow{|_{t^m}} B_2(R_m)\otimes R_m^\times \xrightarrow{L_{m,r}} \Omega_R^1$$

is 0.

Proof. By Proposition 5.2.2, we know that $L_{m,r}$ induces a map from $\mathrm{H}^2(R_m, \mathbb{Q}(3))$ to Ω_R^1 of weight r. The composition of the maps in the statement of the lemma can be rewritten as the composition

$$A(r) \to \mathrm{H}^2(R_r, \mathbb{Q}(3)) \xrightarrow{|_{t^m}} \mathrm{H}^2(R_m, \mathbb{Q}(3)) \xrightarrow{L_{m,r}} \Omega_R^1.$$

By Conjecture 5.2.5 the $\star$-weights of $\mathrm{H}^2(R_r, \mathbb{Q}(3))$ are between $r+1$ and $2r-1$. This implies that the map from $\mathrm{H}^2(R_r, \mathbb{Q}(3)) \to \Omega_R^1$, which is of weight r, is in fact the zero map and finishes the proof. □

Remark 6.2.3. We emphasize that, in the above lemma, we prove that $L_{m,r}(\gamma|_{t^m}) = 0$ for $\gamma \in A(r)$. If γ is only assumed to be in the kernel of the map $B_2(R_m)\otimes R_m^\times \to \Lambda^3 R_m^\times$ then $L_{m,r}(\gamma)$ need not be equal to 0.

Lemma 6.2.4. *Assuming Conjecture 5.2.5, suppose that $\gamma \in B_2(R_r)\otimes R_r^\times$ such that*

$$\delta_r(\gamma) \in \mathrm{im}\Bigg(\bigg(\bigoplus_{0\le s<r} (1+t^s R_r)^\times \otimes (1+t^{r-s} R_r)^\times\bigg) \otimes k^\times\Bigg) \subseteq \Lambda^3 R_r^\times$$

then $\underline{L}_{m,r}(\gamma|_{t^m}) = 0$.

Proof. First let $\alpha := \exp(ut^i)\wedge\exp(vt^j)\wedge\lambda \in \Lambda^3 R_r^\times$ with $u, v \in R$ and $\lambda \in k^\times$ and $r \le i+j$. We have $\exp(ut^i)\wedge\exp(vt^j) \in (\Lambda^2 R_r^\times)^\circ$ and

$$\mathrm{res}_t \frac{1}{t^a} d\log(\exp(ut^i)) \wedge d\log(\exp(vt^j)) = \mathrm{res}_t \frac{1}{t^a} d(ut^i)\wedge d(vt^j) = 0 \in \Omega_R^1,$$

for all $1 \le a < r$, since $r \le i+j$. This implies that there is $\alpha_0 \in B_2(R_r)$ such that $\delta_r(\alpha_0) = \exp(ut^i)\wedge\exp(vt^j)$, and hence $\delta_r(\alpha_0\otimes\lambda) = \alpha$. Let us compute $L_{m,r}((\alpha_0\otimes\lambda)|_{t^m})$. Since $\ell_i(\lambda) = 0$, for $0 < i$ by the formula for $L_{m,r}$ we see that

$$L_{m,r}((\alpha_0\otimes\lambda)|_{t^m}) = \ell i_{m,r}(\alpha_0|_{t^m})\cdot d\log(\lambda) \in \Omega_R^1.$$

This expression vanishes in $\Omega^1_{R/k}$ and therefore $\underline{L}_{m,r}((\alpha_0 \otimes \lambda)|_{t^m}) = 0$. Taking the sum of expressions such as above, we deduce that if

$$\alpha \in \mathrm{im}\Bigg(\Bigg(\bigoplus_{0\le s<r} (1+t^s R_r)^\times \otimes (1+t^{r-s}R_r)^\times\Bigg) \otimes k^\times\Bigg) \subseteq \Lambda^3 R_r^\times$$

then there is a $\tilde{\alpha} \in B_2(R_r) \otimes R_r^\times$ such that $\delta_r(\tilde{\alpha}) = \alpha$ and $\underline{L}_{m,r}(\tilde{\alpha}|_{t^m}) = 0$.

Applying this to $\alpha := \delta_r(\gamma)$ we deduce that there exists $\tilde{\alpha} \in B_2(R_r) \otimes R_r^\times$ such that $\delta_r(\tilde{\alpha}) = \delta_r(\gamma)$ and $\underline{L}_{m,r}(\tilde{\alpha}|_{t^m}) = 0$. Then we have

$$\underline{L}_{m,r}(\gamma|_{t^m}) = \underline{L}_{m,r}(\tilde{\alpha}|_{t^m}) + \underline{L}_{m,r}((\gamma-\tilde{\alpha})|_{t^m}) = \underline{L}_{m,r}((\gamma-\tilde{\alpha})|_{t^m}).$$

Since $\delta_r(\gamma - \tilde{\alpha}) = 0$, the last expression is 0 by Lemma 6.2.2. $\square$

Lemma 6.2.5. *Assuming Conjecture 5.2.5, if $\gamma \in B_2(R_r) \otimes R_r^\times$ such that*

$$\delta_r(\gamma) \in \mathrm{im}((\Lambda^2 R_r^\times)^\circ \otimes k^\times) \subseteq \Lambda^3 R_r^\times$$

then $\underline{L}_{m,r}(\gamma|_{t^m}) = 0$.

Proof. Suppose that γ is as in the statement of the lemma. Fix some $a \in \mathbb{Z}_{>1}$. We inductively define $\gamma^{[i]}$ as follows. Let $\gamma^{[-1]} = \gamma$, and

$$\gamma^{[i]} := a \star \gamma^{[i-1]} - a^i \gamma^{[i-1]},$$

for $0 \le i < r$. Since $L_{m,r}$ is of weight r, $L_{m,r}(\gamma^{[i]}|_{t^m}) = (a^r - a^i) L_{m,r}(\gamma^{[i-1]}|_{t^m})$. Therefore proving that $\underline{L}_{m,r}(\gamma|_{t^m}) = 0$ is equivalent to proving that $\underline{L}_{m,r}(\gamma^{[r-1]}|_{t^m}) = 0$. On the other hand,

$$\delta(\gamma^{[r-1]}) \subseteq \mathrm{im}\Bigg(\Bigg(\bigoplus_{0\le s<r} (1+t^s R_r)^\times \otimes (1+t^{r-s}R_r)^\times\Bigg) \otimes k^\times\Bigg),$$

and therefore the previous lemma implies that $\underline{L}_{m,r}(\gamma^{[r-1]}|_{t^m}) = 0$. $\square$

Assuming Conjecture 5.2.5, we construct a map $\underline{\tilde{\Omega}}_{m,r} : I_{m,r} \to \Omega_{R/k}$, using $\underline{L}_{m,r}$ as follows. Starting with $\alpha \in I_{m,r}$, we know, by Lemma 6.2.1, that there exists

$$\varepsilon \in \mathrm{im}((1+(t^m)) \otimes k^\times \otimes R_r^\times) \subseteq (\Lambda^3 R_r^\times)^\circ,$$

such that $\alpha - \varepsilon = \delta_r(\gamma)$ for some $\gamma \in (B_2(R_r) \otimes R_r^\times)^\circ$. We then define

$$\underline{\tilde{\Omega}}_{m,r}(\alpha) := \underline{L}_{m,r}(\gamma|_{t^m}) \in \Omega^1_{R/k}. \tag{6.2.2}$$

In order to see that $\underline{\tilde{\Omega}}_{m,r}(\alpha) := \underline{L}_{m,r}(\gamma|_{t^m}) \in \Omega^1_{R/k}$ is well-defined, suppose that

$$\varepsilon' \in \mathrm{im}((1+(t^m)) \otimes k^\times \otimes R_r^\times) \subseteq (\Lambda^3 R_r^\times)^\circ$$

and $\gamma' \in (B_2(R_r) \otimes R_r^\times)^\circ$ are other such choices. Then

$$\delta_r(\gamma' - \gamma) = \varepsilon - \varepsilon' \in \mathrm{im}((\Lambda^2 R_r^\times)^\circ \otimes k^\times) \subseteq \Lambda^3 R_r^\times$$

and hence $\underline{L}_{m,r}((\gamma'-\gamma)|_{t^m})=0$ by Lemma 6.2.5. The following proposition is the statement which we were looking for, that relates $\underline{\Omega}_{m,r}$ to $\tilde{\underline{\Omega}}_{m,r}$ and hence to $\underline{L}_{m,r}$.

Proposition 6.2.6. *Assuming Conjecture 5.2.5, we have* $\underline{\Omega}_{m,r}=\tilde{\underline{\Omega}}_{m,r}$.

Proof. Suppose that $x\geq m$ and $x+y+z=r$, with $y,z\geq 0$. We need to check that

$$\tilde{\underline{\Omega}}_{m,r}(\exp(at^x)\wedge\exp(bt^y)\wedge\exp(ct^z))=a(yb\cdot dc-zc\cdot db).$$

(i) Case when $y=z=0$. In this case, we need to compute the image of $\exp(at^r)\wedge\beta\wedge\gamma$ under $\tilde{\underline{\Omega}}_{m,r}$, where $\beta,\gamma\in R^\times$. The image of $\exp(at^r)\wedge\beta$ in $\bigoplus_{1\leq i\leq r-1}t^i\Omega^1_R$ is equal to 0. Therefore, there is $\alpha\in B_2^\circ(R_r)$ such that $\delta_r(\alpha)=\exp(at^r)\wedge\beta$. Then, by definition,

$$\tilde{\underline{\Omega}}_{m,r}(\exp(at^r)\wedge\beta\wedge\gamma)=\underline{L}_{m,r}((\alpha\otimes\gamma)|_{t^m}). \tag{6.2.3}$$

On the other hand, by (5.2.1), $L_{m,r}(\alpha|_{t^m}\otimes\gamma)=\ell i_{m,r}(\alpha|_{t^m})\frac{d\gamma}{\gamma}$. By the expression (3.0.2) for $\ell i_{m,r}$, we have

$$\ell i_{m,r}(\alpha|_{t_m})=\left(\Lambda^2\log^\circ(\delta_r(\alpha))\,\middle|\,\sum_{1\leq i\leq r-m}it^{r-i}\wedge t^i\right)=0,$$

since $\Lambda^2\log^\circ(\delta_r(\alpha))=\Lambda^2\log^\circ(\exp(at^r)\wedge\beta)=0$. By the above formula (6.2.3), this implies that $\tilde{\underline{\Omega}}_{m,r}(\exp(at^r)\wedge\beta\wedge\gamma)=0$ as we wanted to show.

(ii) Case when $y\neq 0$ and $z=0$. In this case we try to compute the image of $\exp(at^x)\wedge\exp(bt^y)\wedge\gamma$ under $\tilde{\underline{\Omega}}_{m,r}$. Here we assume that $\gamma\in R^\times$ and $x+y=r$, with $x\geq m$. By exactly the same argument as above, we deduce that there exists $\alpha\in B_2^\circ(R_r)$ such that $\delta_r(\alpha)=\exp(at^x)\wedge\exp(bt^y)$ and we have

$$\tilde{\underline{\Omega}}_{m,r}(\exp(at^x)\wedge\exp(bt^y)\wedge\gamma)=\underline{L}_{m,r}((\alpha\otimes\gamma)|_{t^m})=\ell i_{m,r}(\alpha|_{t^m})\frac{d\gamma}{\gamma}.$$

Since $\delta_r(\alpha)=\exp(at^x)\wedge\exp(bt^y)$

$$\ell i_{m,r}(\alpha|_{t_m})=\left(\Lambda^2\log^\circ(\delta_r(\alpha))\,\middle|\,\sum_{1\leq i\leq r-m}it^{r-i}\wedge t^i\right)=yab.$$

This exactly coincides with the expression in the statement of the proposition.

(iii) Case when $y\neq 0$ and $z\neq 0$. Note that, by localizing, we may assume that R is local. Moreover, since both sides of the expression are linear in a, b and c, we may assume without loss of generality that $a,b,c\in R^\times$. Since any element in a local ring can be written as a sum of units.

If $\theta:=\exp(at^x)\wedge\exp(bt^y)$ then its image in $\bigoplus_{1\leq i\leq r-1}t^i\Omega^1_R$ is equal to

$$\bigoplus_{1\leq i\leq r-1}\operatorname{res}_t\frac{1}{t^i}(\Lambda^2 d\log(\exp(at^x)\wedge\exp(bt^y))),$$

which only has a nonzero component in degree $x+y$ equal to $yb\cdot da-xa\cdot db$. If we compute the image of $\varphi:=(x/(x+y))\exp(abt^{x+y})\wedge b-(y/(x+y))\exp(abt^{x+y})\wedge a$ in the same group, we obtain the same element. Therefore $\theta-\varphi$ lies in the image of $B_2(R_r)$. Suppose that $\gamma_0\in B_2(R_r)$ such that $\delta(\gamma_0)=\theta-\varphi$.

Since $\exp(abt^{x+y})\wedge\exp(ct^z)$ has weight r, there is $\varepsilon_0\in B_2(R_r)$ such that $\delta(\varepsilon_0)=\exp(abt^{x+y})\wedge\exp(ct^z)$. We now write

$$\begin{aligned}\exp(at^x)\wedge\exp(bt^y)\wedge\exp(ct^z)&=(\theta-\varphi)\wedge\exp(ct^z)+\varphi\wedge\exp(ct^z)\\&=\delta(\gamma_0\otimes\exp(ct^z))-\frac{x}{x+y}\delta(\varepsilon_0\otimes b)+\frac{y}{x+y}\delta(\varepsilon_0\otimes a).\end{aligned}$$

By the definition of $\tilde{\underline{\Omega}}_{m,r}$, we have

$$\begin{aligned}&\tilde{\underline{\Omega}}_{m,r}(\exp(at^x)\wedge\exp(bt^y)\wedge\exp(ct^z))\\&\qquad=\underline{L}_{m,r}(\gamma_0|_{t^m}\otimes\exp(ct^z))-\frac{x}{x+y}\underline{L}_{m,r}(\varepsilon_0|_{t^m}\otimes b)+\frac{y}{x+y}\underline{L}_{m,r}(\varepsilon_0|_{t^m}\otimes a).\end{aligned}$$

By definition,

$$\begin{aligned}\underline{L}_{m,r}(\varepsilon_0|_{t^m}\otimes b)&=\ell i_{m,r}(\varepsilon_0|_{t^m})d\log(b)\\&=\Big(\Lambda^2\log^\circ\delta(\varepsilon_0)\,\Big|\sum_{m\le i<r}(r-i)t^i\wedge t^{r-i}\Big)d\log(b)\\&=zabc\cdot d\log(b)\\&=azc\cdot db.\end{aligned}$$

By the same argument, $\underline{L}_{m,r}(\varepsilon_0|_{t^m}\otimes a)=bzc\cdot da$.

In order to compute $\underline{L}_{m,r}(\gamma_0|_{t^m}\otimes\exp(ct^z))$, first note that, by the definition of $L_{m,r}$, we have

$$\underline{L}_{m,r}(\gamma_0|_{t^m}\otimes\exp(ct^z))=-\frac{z}{x+y}d\ell i_{m,x+y}(\gamma_0|_{t^m})\cdot c+\ell i_{m,x+y}(\gamma_0|_{t^m})\cdot dc.$$

Since $\ell i_{m,x+y}(\gamma_0|_{t^m})=\big(\Lambda^2\log^\circ\delta(\gamma_0)\mid\sum_{m\le i<x+y}(x+y-i)t^i\wedge t^{x+y-i}\big)=yab$, we have

$$\underline{L}_{m,r}(\gamma_0|_{t^m}\otimes\exp(ct^z))=-\frac{zy}{x+y}c\cdot d(ab)+yab\cdot dc.$$

Combining all of these gives,

$$\begin{aligned}\tilde{\underline{\Omega}}_{m,r}(\exp(at^x)\wedge\exp(bt^y)\wedge\exp(ct^z))&=-\frac{zy}{x+y}c\cdot d(ab)+yab\cdot dc-\frac{xzac}{x+y}\cdot db+\frac{yzbc}{x+y}da\\&=a(yb\cdot dc-zc\cdot db).\end{aligned}$$

This finishes the proof of the proposition. □

6.3. ***Behavior of $\omega_{m,r}$ with respect to automorphisms of R_{2m-1} which are identity modulo (t^m).*** In this section, we continue to assume that R is smooth of relative dimension 1 over k. We will show the invariance of $\underline{\Omega}_{m,r}$ with respect to reparametrizations of R_r that are identity on the reduction to R_m. In order to do this, we will need to make an explicit computation on $k'((s))_\infty$, where k' is a finite extension on k. In order to make the formulas concise and intuitive, we will use several notational conventions as follows. If $a\in k'((s))$, we let $a'=a^{(1)}\in k'((s))$ denote its derivative with respect to s and $a^{(n+1)}=(a^{(n)})'$. Similarly, we let $\exp(a)$ denote an arbitrary *nonzero* element in $k'((s))$ and $a'=a^{(1)}:=(\exp(a))'/\exp(a)$

and $a^{(n+1)} = (a^{(n)})'$. This notation is intuitive in the sense that, if one thinks of a as $\log(\exp(a))$ then a' is the logarithmic derivative of $\exp(a)$. With these conventions, we will state the following basic lemma.

Lemma 6.3.1. *Let σ be the automorphism of the k_∞ algebra $k'((s))_\infty$, which is the identity automorphism modulo (t) and has the property that $\sigma(s) = s + \alpha t^w$, with $w \geq 1$ and $\alpha \in k((s))$, then $\sigma(\exp(at^x)) = \exp\big(\sum_{0\leq i} \frac{\alpha^i a^{(i)}}{i!} t^{x+iw}\big)$.*

Proof. Assume that σ is such an automorphism of $k'((s))_\infty$. Let $i : k' \to k'((s))_\infty$ be the standard inclusion, which sends an element of k' to a constant series in s and t. In other words, i is the k'-algebra structure map. Let

$$\pi : k'((s))_\infty \to k'((s))_\infty/(t) = k'((s))$$

denote the canonical projection. By the assumptions, the restriction of $\sigma \circ i$ to k, is the standard inclusion $k \to k'((s))_\infty$. Similarly, by the assumptions, $\pi \circ \sigma \circ i$ is the standard inclusion of k' into $k'((s))$. Since k' is étale over k, these two statements above imply that $\sigma \circ i$ is the same as the inclusion i. Together with the assumption that $\sigma(t) = t$, this implies that σ is an automorphism of k'_∞-algebras.

The rest proof is then separated into two cases, when $x = 0$ and when $x \neq 0$. In both cases, the statement follows from the Taylor expansion formula. $\square$

Lemma 6.3.2. *Let σ be the automorphism of the k_∞ algebra $k'((s))_\infty$, which is the identity automorphism modulo (t) and has the property that $\sigma(s) = s + \alpha t^w$, with $m \leq w$. Then we have,*

$$\underline{\Omega}_{m,r}\left(\frac{\sigma(\exp(at^i) \wedge \exp(bt^j) \wedge \exp(ct^k))}{\exp(at^i) \wedge \exp(bt^j) \wedge \exp(ct^k)}\right) = 0,$$

for $0 < i + j + k$.

Proof. Since $m \leq w$, $0 < i + j + k$ and $m < r < 2m$, the weight r terms of

$$\frac{\sigma(\exp(at^i) \wedge \exp(bt^j) \wedge \exp(ct^k))}{\exp(at^i) \wedge \exp(bt^j) \wedge \exp(ct^k)}$$

are possibly nonzero only when $i + j + k + w = r$ and in this case they are given by

$$\exp(\alpha a' t^{i+w}) \wedge \exp(bt^j) \wedge \exp(ct^k) + \exp(at^i) \wedge \exp(\alpha b' t^{j+w}) \wedge \exp(ct^k) \\ + \exp(at^i) \wedge \exp(bt^j) \wedge \exp(\alpha c' t^{k+w}).$$

By the definition of $\underline{\Omega}_{m,r}$, the above sum is sent to

$$\alpha(a'(jbc' - kcb') - b'(iac' - kca') + c'(iab' - jba'))ds = 0. \qquad \square$$

Corollary 6.3.3. *Let σ be any automorphism of R_r as a k_r-algebra, which reduces to identity on R_m, then $\omega_{m,r} \circ \Lambda^3\sigma = \omega_{m,r}$.*

Proof. This follows by the corresponding statement for $\underline{\Omega}_{m,r}$. This in turn reduces to Lemma 6.3.2 after localizing and completing. $\square$

Definition 6.3.4. If $\mathcal{R}/k_r$ is a smooth k_r-algebra of relative dimension 1. We defined the map

$$\omega_{m,r} : \Lambda^3(\underline{\mathcal{R}}_r, (t^m))^\times \to \Omega^1_{\underline{\mathcal{R}}/k},$$

as the composition $\Omega_{m,r} \circ s$, where $\underline{\mathcal{R}}$ is the reduction of $\mathcal{R}$ modulo (t) and $\underline{\mathcal{R}}_r := \underline{\mathcal{R}} \times_k k_r$. Let $\tau : \underline{\mathcal{R}}_r \to \mathcal{R}$ be a splitting, that is an isomorphism of k_r-algebras which is the identity map modulo (t). By transport of structure, this gives a map

$$\omega_{m,r,\tau} : \Lambda^3(\mathcal{R}, (t^m))^\times \to \Omega^1_{\underline{\mathcal{R}}/k}.$$

Suppose that τ' is another such splitting which agrees with τ modulo (t^m). Applying Corollary 6.3.3 to $\tau'^{-1} \circ \tau$, we deduce that $\omega_{m,r,\tau} = \omega_{m,r,\tau'}$. Therefore, if $\sigma : \underline{\mathcal{R}}_m \to \mathcal{R}/(t^m)$ is a splitting of the reduction $\mathcal{R}/(t^m)$ of $\mathcal{R}$ then $\omega_{m,r,\sigma}$ is unambiguously defined as $\omega_{m,r,\tau}$, where τ is any splitting of $\mathcal{R}$ that reduces to σ modulo (t^m).

Recall the relative version of the Bloch group from [Ünver 2021, Section 2.4.8]. If A is a ring with ideal I, let $(A, I)^\flat := \{(\tilde{a}, \hat{a}) \in (A, I)^\times \mid (1-\tilde{a}, 1-\hat{a}) \in (A, I)^\times\}$. Then the relative Bloch group $B_2(A, I)$ is defined as the abelian group generated by the symbols $[(\tilde{a}, \hat{a})]$ for every $(\tilde{a}, \hat{a}) \in (A, I)^\flat$, modulo the relations generated by the analog of the five term relation for the dilogarithm:

$$[(\tilde{x}, \hat{x})] - [(\tilde{y}, \hat{y})] + [(\tilde{y}/\tilde{x}, \hat{y}/\hat{x})] - \left[\left(\frac{1-\tilde{x}^{-1}}{1-\tilde{y}^{-1}}, \frac{1-\hat{x}^{-1}}{1-\hat{y}^{-1}}\right)\right] + \left[\left(\frac{1-\tilde{x}}{1-\tilde{y}}, \frac{1-\hat{x}}{1-\hat{y}}\right)\right]$$

for every $(\tilde{x}, \hat{x}), (\tilde{y}, \hat{y}) \in (A, I)^\flat$ such that $(\tilde{x}-\tilde{y}, \hat{x}-\hat{y}) \in (A, I)^\times$. As in the classical case, we obtain a complex $\delta : B_2(A, I) \to \Lambda^2(A, I)^\times$, which sends $(\tilde{a}, \hat{a})$ to $(1-\tilde{a}, 1-\hat{a}) \wedge (\tilde{a}, \hat{a})$. As usual, abusing the notation, we will denote the induced map $\delta \otimes \mathrm{id} : B_2(A, I) \otimes (A, I)^\times \to \Lambda^3(A, I)^\times$, also by δ. With these definitions, we have the following expected property of the map $\omega_{m,r,\tau}$.

Proposition 6.3.5. *For a splitting σ of $\mathcal{R}/(t^m)$, the above map $\omega_{m,r,\sigma}$ vanishes on the image of $B_2(\mathcal{R}, (t^m)) \otimes (\mathcal{R}, (t^m))^\times$ in $\Lambda^3(\mathcal{R}, (t^m))^\times$ under δ.*

Proof. By the definition of $\omega_{m,r,\sigma}$, we easily reduce to the split case where $\mathcal{R} = R_r$. It suffices to prove that $\Omega_{m,r}$ vanishes on the following two types of elements:

$$\delta_r([\hat{f}] \otimes \hat{g}) - \delta_r([\tilde{f}] \otimes \hat{g}) \quad \text{and} \quad \delta_r([\hat{f}] \otimes \hat{g}) - \delta_r([\hat{f}] \otimes \tilde{g}),$$

where $\hat{f}, \tilde{f} \in \mathcal{R}^\flat$ have the same reduction modulo (t^m) and $\hat{g}, \tilde{g} \in \mathcal{R}^\times$ have the same reduction modulo (t^m). This precisely the statement of Proposition 6.1.3. ☐

6.4. ***Behavior of*** $\mathbf{res}(\omega_{m,r,\sigma})$ ***with respect to automorphisms of*** R_m**.** In order to proceed with our construction, we need an object such as the 1-form in [Ünver 2021] which controls the effect of changing splittings. This object in $\star$-weight r will be constructed below by using $\omega_{m,r}$. On the other hand, this objects *does depend* on the choice of splittings if these splittings are different modulo (t^m), when $r > m+1$. In the modulus $m = 2$ case the only possible r is 3 so this situation does not occur in [loc. cit.]. In the current case of higher modulus, we will see that the *residues* of the 1-form $\omega_{m,r}$ is invariant under the automorphisms of R_m which are identity modulo (t), which will imply that the residue can be defined independent of various choices. We will see that this will be enough for constructing the Chow dilogarithm of higher modulus. We will again start with an explicit computation on $k'((s))_\infty$.

Proposition 6.4.1. *Suppose that σ is the automorphism of $k'((s))_\infty$ as a k_∞-algebra such that $\sigma(s) = s+\alpha t^w$, with $w \geq 1$ and $\alpha \in k'((s))$, and which is identity modulo (t). Consider the element $\exp(at^x) \wedge \exp(bt^y) \wedge \exp(ct^z)$, with $m \leq x$. If $r-(x+y+z) > 0$, and is divisible by w, let $q = (r-(x+y+z))/w$. Then $\underline{\Omega}_{m,r}((\sigma(\exp(at^x) \wedge \exp(bt^y) \wedge \exp(ct^z)))/(\exp(at^x) \wedge \exp(bt^y) \wedge \exp(ct^z)))$ is equal to*

$$d\left(\frac{\alpha^q}{q!}\sum_{0\leq k\leq q-1} a^{(k)}\binom{q-1}{k}\sum_{i+j=q-k}\left(\binom{q-k-1}{i}yb^{(i)}c^{(j)}-\binom{q-k-1}{j}zb^{(i)}c^{(j)}\right)\right). \tag{6.4.1}$$

Otherwise, $\underline{\Omega}_{m,r}((\sigma(\exp(at^x) \wedge \exp(bt^y) \wedge \exp(ct^z)))/(\exp(at^x) \wedge \exp(bt^y) \wedge \exp(ct^z))) = 0$.

Proof. First note that by Lemma 6.3.1

$$\begin{aligned} Q &:= \frac{\sigma(\exp(at^x) \wedge \exp(bt^y) \wedge \exp(ct^z))}{\exp(at^x) \wedge \exp(bt^y) \wedge \exp(ct^z)} \\ &= \frac{\exp\left(\sum_{0\leq i}\frac{\alpha^i a^{(i)}}{i!}t^{x+iw}\right) \wedge \exp\left(\sum_{0\leq i}\frac{\alpha^i b^{(i)}}{i!}t^{y+iw}\right) \wedge \exp\left(\sum_{0\leq i}\frac{\alpha^i c^{(i)}}{i!}t^{z+iw}\right)}{\exp(at^x) \wedge \exp(bt^y) \wedge \exp(ct^z)} \end{aligned}$$

and hence if $r-(x+y+z) \leq 0$ or $w \nmid r-(x+y+z)$ then Q does not have a component of weight r and $\underline{\Omega}_{m,r}(Q) = 0$.

Suppose then that $r-(x+y+z) > 0$, $w \mid (r-(x+y+z))$ and let $q := (r-(x+y+z))/w$ as in the statement of the proposition. In this case the weight r term of Q is given as

$$\sum_{i+j+k=q}\exp\left(\frac{\alpha^k a^{(k)}}{k!}t^{x+kw}\right) \wedge \exp\left(\frac{\alpha^i b^{(i)}}{i!}t^{y+iw}\right) \wedge \exp\left(\frac{\alpha^j c^{(j)}}{j!}t^{z+jw}\right).$$

This implies that $\underline{\Omega}_{m,r}(Q)$ is equal to

$$\sum_{i+j+k=q}\frac{\alpha^k a^{(k)}}{k!}\left((y+iw)\frac{\alpha^i b^{(i)}}{i!}\left(\frac{\alpha^j c^{(j)}}{j!}\right)'-(z+jw)\left(\frac{\alpha^i b^{(i)}}{i!}\right)'\frac{\alpha^j c^{(j)}}{j!}\right)ds.$$

We first claim that the expression above does not depend on w. The coefficient of $w\cdot(a^{(k)}/k!)\alpha^{q-1}d\alpha$ in this expression is $\sum_{i+j=q-k}(i(b^{(i)}/i!)(jc^{(j)}/j!)-j(ib^{(i)}/i!)(c^{(j)}/j!)) = 0$. The coefficient of $w\cdot(a^{(k)}/k!)\alpha^q ds$ in the same expression is

$$\sum_{i+j=q-k}\left(i\frac{b^{(i)}}{i!}\frac{c^{(j+1)}}{j!}-j\frac{b^{(i+1)}}{i!}\frac{c^{(j)}}{j!}\right) = \sum_{\substack{i+j=q-k+1\\1\leq i,j}}\frac{b^{(i)}}{(i-1)!}\frac{c^{(j)}}{(j-1)!}-\sum_{\substack{i+j=q-k+1\\1\leq i,j}}\frac{b^{(i)}}{(i-1)!}\frac{c^{(j)}}{(j-1)!} = 0.$$

Therefore $\underline{\Omega}_{m,r}(Q)$ can be rewritten as

$$\sum_{i+j+k=q}\frac{\alpha^k a^{(k)}}{k!}\left(y\frac{\alpha^i b^{(i)}}{i!}\left(\frac{\alpha^j c^{(j)}}{j!}\right)'-z\left(\frac{\alpha^i b^{(i)}}{i!}\right)'\frac{\alpha^j c^{(j)}}{j!}\right)ds. \tag{6.4.2}$$

The coefficient of $\alpha'\alpha^{q-1}ds$ in the above expression is equal to

$$\sum_{0\le k\le q-1}\frac{a^{(k)}}{k!}\sum_{i+j=q-k}\left(y\frac{b^{(i)}}{i!}\frac{jc^{(j)}}{j!}-z\frac{ib^{(i)}}{i!}\frac{c^{(j)}}{j!}\right)$$

which agrees with the coefficient of $\alpha'\alpha^{q-1}ds$ in (6.4.1).

Fix i_0, j_0, and k_0 such that $i_0+j_0+k_0=q$. Then the coefficient of $y\alpha^q a^{(k_0)}b^{(i_0)}c^{(j_0+1)}$ in (6.4.1) is equal to

$$\frac{1}{q}\left(\frac{1}{(k_0-1)!}\frac{1}{i_0!}\frac{1}{j_0!}+\frac{1}{k_0!}\frac{1}{i_0!}\frac{1}{(j_0-1)!}+\frac{1}{k_0!}\frac{1}{(i_0-1)!}\frac{1}{j_0!}\right)=\frac{1}{k_0!}\frac{1}{i_0!}\frac{1}{j_0!},$$

which is exactly the same as the coefficient of the same term in (6.4.2). By symmetry, we deduce the same statement for the coefficients of $z\alpha^q a^{(k_0)}b^{(i_0+1)}c^{(j_0)}$. This finishes the proof of the proposition. □

Corollary 6.4.2. *Suppose that σ and $\exp(at^x)\wedge\exp(bt^y)\wedge\exp(ct^z)$ are as above. If $r=m+1$, then*

$$\underline{\Omega}_{m,r}((\sigma(\exp(at^x)\wedge\exp(bt^y)\wedge\exp(ct^z)))/(\exp(at^x)\wedge\exp(bt^y)\wedge\exp(ct^z)))=0.$$

Proof. In this case in order to have $m\le x$ and $(m+1)-(x+y+z)=r-(x+y+z)>0$, we have to have $x=m$ and $y=z=0$. In this case, (6.4.1) is equal to 0. □

Corollary 6.4.3. *If $\mathcal{R}$ is a smooth k_{m+1}-algebra of relative dimension 1 as above, then for $r=m+1$, we have a well-defined map*

$$\omega_{m,m+1}:\Lambda^3(\mathcal{R},(t^m))^\times\to\Omega^1_{\underline{\mathcal{R}}/k}$$

as in Definition 6.3.4, which does not depend on the choice of a splitting of $\mathcal{R}/(t^m)$.

Proof. This follows immediately from Corollary 6.4.2, by reducing to the case $\mathcal{R}=k'((s))_{m+1}$, after localizing and completing. □

For a general r between m and $2m$, the following corollary will be essential.

Corollary 6.4.4. *Fix $m<r<2m$, and let $\mathcal{R}$ be a smooth k_r-algebra of relative dimension 1 as above. Let x be a closed point of the spectrum of $\mathcal{R}$, k' its residue field, and let η be the generic point of $\mathcal{R}$. Then for any two splittings σ and σ' of $\mathcal{R}_\eta/(t^m)$, the reduction modulo (t^m) of the local ring of $\mathcal{R}$ at η, and for any $\alpha\in\Lambda^3(\mathcal{R}_\eta,(t^m))^\times$, the residues of $\omega_{m,r,\sigma}(\alpha)$ and $\omega_{m,r,\sigma'}(\alpha)\in\Omega^1_{\underline{\mathcal{R}}_\eta/k}$ at x are the same:*

$$\mathrm{res}_x\,\omega_{m,r,\sigma'}(\alpha)=\mathrm{res}_x\,\omega_{m,r,\sigma}(\alpha)\in k'.$$

Proof. Again by localizing and completing we reduce to the case of $k'((s))_r$. By Proposition 6.4.1, we see that the difference $\omega_{m,r,\sigma'}(\alpha)-\omega_{m,r,\sigma}(\alpha)$ is the differential of an element in $k'((s))$ and hence has zero residue. □

Remark 6.4.5. Let $\mathcal{R}/k_r$ be as above. Suppose that τ and σ are two splittings $\underline{\mathcal{R}}_m\to\mathcal{R}/(t^m)$. In this case, there should be a map

$$h\omega_{m,r}(\tau,\sigma):\Lambda^3(\mathcal{R},(t^m))^\times\to\underline{\mathcal{R}}$$

such that

$$d(h\omega_{m,r}(\tau,\sigma)) = \omega_{m,r,\tau} - \omega_{m,r,\sigma}.$$

Moreover, $h\omega_{m,r}(\tau,\sigma)$ should vanish on the image of $B_2(\mathcal{R},(t^m)) \otimes (\mathcal{R},(t^m))^\times$. In case $r = m+1$, $h\omega_{m,m+1} = 0$ does satisfy the properties above. Let us look at the first nontrivial case when $m = 3$ and $r = 5$. Note that the reduction modulo (t^2) of the automorphism $\tau^{-1}\circ\sigma : \underline{\mathcal{R}}_3 \to \underline{\mathcal{R}}_3$, which lifts the identity map on $\underline{\mathcal{R}}$, is determined by a k-derivation $\theta : \underline{\mathcal{R}} \to \underline{\mathcal{R}}$. Define $h\underline{\Omega}_{3,5}(\theta) : I_{3,5} \subseteq (\Lambda^3 \underline{\mathcal{R}}_5^\times)^\circ \to \underline{\mathcal{R}}$, as

$$h\underline{\Omega}_{3,5}(\theta)(\exp(at^3)\wedge\exp(bt)\wedge c) = ab\theta\left(\frac{dc}{c}\right),$$

where $a, b, \in \underline{\mathcal{R}}$ and $c \in \underline{\mathcal{R}}^\times$. Let $h\underline{\Omega}_{3,5}(\theta)$ be defined as 0 on all the other type of elements in $I_{3,5}$. Then $h\omega_{3,5}(\tau,\sigma) : \Lambda^3(\mathcal{R},(t^3))^\times \to \underline{\mathcal{R}}$ defined by

$$h\omega_{3,5}(\tau,\sigma)(\alpha) := -h\underline{\Omega}_{3,5}(\theta)(s(\sigma^{-1}(\alpha)))$$

satisfies the desired properties above. An analog of this construction is one of the main tools in defining an infinitesimal version of the Bloch regulator in [Ünver 2020].

Definition 6.4.6. Let $\mathcal{R}$ be a smooth k_r-algebra of relative dimension 1 as above. Let η be the generic point and x be a closed point of the spectrum of $\mathcal{R}$. Then we have a canonical map

$$\mathrm{res}_x\, \omega_{m,r} : \Lambda^3(\mathcal{R}_\eta,(t^m))^\times \to k',$$

where k' is the residue field of x. The map is defined by choosing any splitting σ of $\mathcal{R}_\eta/(t^m)$ and letting $\mathrm{res}_x\, \omega_{m,r} := \mathrm{res}_x\, \omega_{m,r,\sigma}$. This is independent of the choice of the splitting σ, by Corollary 6.4.4.

6.5. ***Variant of the residue map for different liftings.*** For the construction of the infinitesimal Chow dilogarithm, we need a variant of Definition 6.4.6. Fortunately, we do not need to do extra work, Corollary 6.3.3 and Proposition 6.4.1 will still be sufficient to give us what we are looking for.

Suppose that A is a ring with an ideal I and B and B' are two A-algebras together with an isomorphism $\chi : B/IB \simeq B'/IB'$ of A-algebras. We let

$$(B, B', \chi)^\times := \{(p, p') \mid p \in B^\times \text{ and } p' \in B'^\times \text{ such that } \chi(p|_I) = p'|_I\},$$

where $p|_I$ denotes the image of p in $(B/IB)^\times$. Similarly, we define $(B, B', \chi)^\flat$ and $B_2(B, B', \chi)$ and obtain maps, $B_2(B, B', \chi) \to \Lambda^2(B, B', \chi)^\times$ and $B_2(B, B', \chi) \otimes (B, B', \chi)^\times \to \Lambda^3(B, B', \chi)^\times$. We will use these definitions below with $A = k_\infty$ and $I = (t^m)$. In fact the following variant will be essential in what follows.

Suppose that $\mathcal{S}/k_m$ is a smooth algebra of relative dimension 1, with x a closed point and η the generic point of its spectrum. Suppose that $\mathcal{R}, \mathcal{R}'/k_r$ are liftings of $\mathcal{S}_\eta$ to k_r. In other words, we have fixed isomorphisms

$$\psi : \mathcal{R}/(t^m) \to \mathcal{S}_\eta \quad \text{and} \quad \psi' : \mathcal{R}'/(t^m) \to \mathcal{S}_\eta.$$

Letting $\chi := \psi'^{-1} \circ \psi$, we would like to construct a map

$$\operatorname{res}_x \omega_{m,r} : \Lambda^3(\mathcal{R}, \mathcal{R}', \chi)^\times \to k',$$

where k' is the residue field of x. Note that $(\mathcal{R}, \mathcal{R}', \chi)^\times$ consists of pairs of (p, p') with $p \in \mathcal{R}^\times$ and $p' \in \mathcal{R}'^\times$ such that $\psi(p|_{t^m}) = \psi'(p'|_{t^m})$. In other words, it consists of different liftings of elements of $\mathcal{S}_\eta^\times$. We sometimes use the notation $(\mathcal{R}, \mathcal{R}', \psi, \psi')^\times$ to denote the same set.

In order to construct this map, let

$$\tilde{\chi} : \mathcal{R} \to \mathcal{R}'$$

be an isomorphism of k_r-algebras which is a lifting of χ. This provides us with a map

$$(\mathcal{R}, \mathcal{R}', \chi)^\times \xrightarrow{\tilde{\chi}^*} (\mathcal{R}, (t^m))^\times.$$

Choosing a splitting $\sigma : \underline{\mathcal{R}}_m \to \mathcal{R}/(t^m)$, by Definition 6.3.4 we obtain the map $\omega_{m,r,\sigma}$, composing this with the map induced by the reduction $\underline{\psi}$ of ψ, we obtain

$$\Lambda^3(\mathcal{R}, \mathcal{R}', \chi)^\times \xrightarrow{\Lambda^3 \tilde{\chi}^*} \Lambda^3(\mathcal{R}, (t^m))^\times \xrightarrow{\omega_{m,r,\sigma}} \Omega^1_{\underline{\mathcal{R}}/k} \xrightarrow{d\underline{\psi}} \Omega^1_{\underline{\mathcal{S}}_\eta/k} \xrightarrow{\operatorname{res}_x} k'. \tag{6.5.1}$$

Proposition 6.5.1. *The map (6.5.1) above is independent of the choices of the lifting $\tilde{\chi}$ of χ and the choice of the splitting σ of $\mathcal{R}/(t^m)$.*

Proof. That the composition is independent of the choice of $\tilde{\chi}$ follows from Corollary 6.3.3 and Definition 6.3.4. That it is independent of the choice of the splitting σ follows from Proposition 6.4.1. $\square$

Definition 6.5.2. We denote the composition (6.5.1) above by

$$\operatorname{res}_x \omega_{m,r}(\psi, \psi') : \Lambda^3(\mathcal{R}, \mathcal{R}', \psi, \psi')^\times \to k'.$$

If ψ and ψ' are clear from the context, we denote this map by $\operatorname{res}_x \omega_{m,r}$, and $(\mathcal{R}, \mathcal{R}', \psi, \psi')^\times$ by $(\mathcal{R}, \mathcal{R}', (t^m))^\times$. Depending on the context, we also use the notation $\operatorname{res}_x \omega_{m,r}(\chi) : \Lambda^3(\mathcal{R}, \mathcal{R}', \chi) \to k'$ for the same map, with $\chi = \psi'^{-1} \circ \psi$.

With these definitions, we have the following corollary.

Corollary 6.5.3. *Suppose that $\mathcal{R}$ and $\mathcal{R}'$ are smooth k_r-algebras of dimension 1 as above which are liftings of the generic local ring $\mathcal{S}_\eta$ of a smooth k_m-algebra $\mathcal{S}$. Let $\chi : \mathcal{R}/(t^m) \to \mathcal{R}'/(t^m)$ be the corresponding isomorphism of k_m-algebras. Let x be a closed point of $\underline{\mathcal{S}}$. Then the map*

$$\operatorname{res}_x \omega_{m,r}(\chi) : \Lambda^3(\mathcal{R}, \mathcal{R}', \chi)^\times \to k'$$

vanishes on the image of $B_2(\mathcal{R}, \mathcal{R}', \chi) \otimes (\mathcal{R}, \mathcal{R}', \chi)^\times$.

Proof. Follows from Proposition 6.3.5. $\square$

7. The residue of $\omega_{m,r}$ on good liftings

Suppose that $\mathcal{R}/k_r$ is as above. Moreover, we assume that the reduction $\underline{\mathcal{R}}$ of $\mathcal{R}$ modulo (t) is a discrete valuation ring with x being the closed point. We let $\tilde{x}/k_r$ be a *lifting* of x to $\mathcal{R}$. By this what we mean is as follows. Let s be a uniformizer at x, and let $\tilde{s}$ be any lifting of s to $\mathcal{R}$, we call $\tilde{s}$ also a uniformizer at x on $\mathcal{R}$. The associated scheme $\tilde{x}$, which is smooth over k_r, is what we call a lifting of x. In other words a lifting of x is a 0-dimensional closed subscheme $\tilde{x}$ of $\mathcal{R}$ such that its ideal is generated by a single element which reduces to a uniformizer on $\underline{\mathcal{R}}$. Note that if we are given $\tilde{x}$, then $\tilde{s}$ is determined up to a unit in $\mathcal{R}$. Sometimes we will abuse the notation and write $(\tilde{s})$ instead of $\tilde{x}$. Let η denote the generic point of $\mathcal{R}$. We let

$$(\mathcal{R}, \tilde{x})^\times := \{\alpha \in \mathcal{R}_\eta^\times \mid \alpha = u\tilde{s}^n, \text{ for some } u \in \mathcal{R}^\times \text{ and } n \in \mathbb{Z}\}.$$

We say that an element $\alpha \in \mathcal{R}_\eta^\times$ is *good with respect to* $\tilde{x}$, if $\alpha \in (\mathcal{R}, \tilde{x})^\times$. Note that this property depends only on $\tilde{x}$, and not on $\tilde{s}$. The importance of this notion for us is that for wedge products of good liftings, we can define their residue along (s) as in [Ünver 2021, Section 2.4.5]. Namely, there is a map

$$\mathrm{res}_{\tilde{x}} : \Lambda^n(\mathcal{R}, \tilde{x})^\times \to \Lambda^{n-1}(\mathcal{R}/(s))^\times,$$

with the properties that it vanishes on $\Lambda^n\mathcal{R}^\times$ and $s \wedge \alpha_1 \wedge \cdots \wedge \alpha_{n-1}$ is mapped to $\underline{\alpha}_1 \wedge \cdots \wedge \underline{\alpha}_{n-1}$, if $\alpha_i \in \mathcal{R}^\times$ and $\underline{\alpha}_i$ denotes the image of α_i in $(R/(s))^\times$, for $1 \le i \le n-1$. Suppose that $\mathcal{R}'/k_r$ is another such ring, and $\tilde{x}'$ a lifting of the closed point of $\underline{\mathcal{R}}'$. Suppose that there is an isomorphism $\chi : \mathcal{R}/(t^m) \to \mathcal{R}'/(t^m)$ which identifies the reduction of $\tilde{x}$ modulo (t^m) with the reduction of $\tilde{x}'$ modulo (t^m). Then we let

$$(\mathcal{R}, \mathcal{R}', \tilde{x}, \tilde{x}', \chi)^\times := \{(p, p') \mid p \in (\mathcal{R}, \tilde{x})^\times \text{ and } p' \in (\mathcal{R}', \tilde{x}')^\times \text{ such that } \chi(p|_{t^m}) = p'|_{t^m}\}.$$

Note that clearly $(\mathcal{R}, \mathcal{R}', \tilde{x}, \tilde{x}', \chi)^\times \subseteq (\mathcal{R}, \mathcal{R}', \chi)^\times$. In case $\mathcal{R}' = \mathcal{R}$ with χ the identity map, we denote the corresponding group by $(\mathcal{R}, \tilde{x}, (t^m))^\times$. Denote the natural maps $(\mathcal{R}, \mathcal{R}', \tilde{x}, \tilde{x}', \chi)^\times \to (\mathcal{R}, \tilde{x})^\times$ and $(\mathcal{R}, \mathcal{R}', \tilde{x}, \tilde{x}', \chi)^\times \to (\mathcal{R}', \tilde{x}')^\times$ by π_1 and π_2. In this section, we would like to compute $\mathrm{res}_x\, \omega_{m,r}(\chi)(\alpha)$ for $\alpha \in \Lambda^3(\mathcal{R}, \mathcal{R}', \tilde{x}, \tilde{x}', \chi)^\times$ in terms of the value of $\ell_{m,r}$ on the residue of α. The main result of this section is Proposition 7.0.3. We will first start with certain explicit computations on the formal power series rings and then finally reduce our general statement to these special cases. Let us immediately remark that in order to compute the residues, we immediately reduce to the case when $\mathcal{R}$ and $\mathcal{R}'$ are complete with respect to the ideal which correspond to their closed points.

We will first consider the case of $R = k'[\![s]\!]$ and that of the same uniformizer on both of the liftings as follows.

Note that

$$\mathrm{res}_{(s)}(s \wedge \alpha \wedge \beta) = \underline{\alpha} \wedge \underline{\beta} \in \Lambda^2 k_r'^{\times},$$

where $\underline{\alpha}$ and $\underline{\beta}$ are the images of α and β under the natural projection $\mathcal{R}^\times \to (\mathcal{R}/(s))^\times = k_r'^{\times}$. Similarly for p'.

Lemma 7.0.1. *Suppose that $R = k'[[s]]$, and $\mathcal{R} := R_r = k'[[s, t]]/(t^r)$. Suppose that α, α', and $\beta \in \mathcal{R}^\times$ such that $\alpha'|_{t^m} = \alpha|_{t^m} \in \mathcal{R}/(t^m)$. Let $p' := s \wedge \alpha' \wedge \beta$, $p := s \wedge \alpha \wedge \beta$ and $(p, p') := (s, s) \wedge (\alpha, \alpha') \wedge (\beta, \beta) \in \Lambda^3(\mathcal{R}, (s), (t^m))^\times \subseteq \Lambda^3(\mathcal{R}_\eta, (t^m))^\times$. Then the residue of $\omega_{m,r}(p, p')$ at the closed point of $\underline{\mathcal{R}} = k'[[s]]$ is given by*

$$\mathrm{res}_{s=0}\, \omega_{m,r}(p, p') = \ell_{m,r}(\mathrm{res}_{(s)}(p)) - \ell_{m,r}(\mathrm{res}_{(s)}(p')).$$

Proof. By Definition 6.1.4, we see that $\omega_{m,r}(p, p') = \underline{\Omega}_{m,r}(p - p') = \underline{\Omega}_{m,r}(s \wedge (\alpha/\alpha') \wedge \beta)$. Let us compute the residue at $s = 0$ of an expression of the type $\underline{\Omega}_{m,r}(s \wedge \exp(at^i) \wedge \exp(bt^j))$, with $a, b \in k[[s]]$ and $i \geq m$, such that if $j = 0$. We use the notation in Remark 6.1.2. Since

$$\underline{\Omega}_{m,r}(s \wedge \exp(at^i) \wedge \exp(bt^j)) = jab\frac{ds}{s},$$

when $i + j = r$ and is 0 otherwise, we conclude that its residue is equal to $ja(0)b(0)$ if $i + j = r$, and is 0 otherwise. Since, for $i \geq m$,

$$\ell_{m,r}(\mathrm{res}_{(s)}(s \wedge \exp(at^i) \wedge \exp(bt^j))) = \ell_{m,r}(\exp(a(0)t^i) \wedge \exp(b(0)t^j))$$

is equal to $ja(0)b(0)$ if $i + j = r$, and is 0 otherwise, we conclude that

$$\mathrm{res}_{s=0}(\underline{\Omega}_{m,r}(s \wedge \exp(at^i) \wedge \exp(bt^j))) = \ell_{m,r}(\mathrm{res}_{(s)}(s \wedge \exp(at^i) \wedge \exp(bt^j))). \tag{7.0.1}$$

On the other hand, since $\alpha'|_{t^m} = \alpha|_{t^m}$, $s \wedge (\alpha/\alpha') \wedge \beta$ is a sum of terms of the above type, and the linearity of both sides of (7.0.1) imply that (7.0.1) is also valid for $s \wedge (\alpha/\alpha') \wedge \beta$. By linearity of $\ell_{m,r}$ and $\mathrm{res}_{(s)}$, we have

$$\ell_{m,r}\Bigl(\mathrm{res}_{(s)}\Bigl(s \wedge \frac{\alpha}{\alpha'} \wedge \beta\Bigr)\Bigr) = \ell_{m,r}(\mathrm{res}_{(s)}(s \wedge \alpha \wedge \beta)) - \ell_{m,r}(\mathrm{res}_{(s)}(s \wedge \alpha' \wedge \beta)),$$

which together with the above proves the lemma. □

Let us now try to prove the same formula when the choice of the uniformizer is not the same. In other words, with notation as above let $s' \in \mathcal{R}$ such that $s'|_{t^m} = s|_{t^m}$. For simplicity, let us temporarily use the notation $(\mathcal{R}, (s), (s'), (t^m))^\times := (\mathcal{R}, \mathcal{R}, (s), (s'), \mathrm{id}_{\mathcal{R}/(t^m)})^\times$. Let $p' := s' \wedge \alpha \wedge \beta$, $p := s \wedge \alpha \wedge \beta$ and $(p, p') := (s, s') \wedge (\alpha, \alpha) \wedge (\beta, \beta) \in \Lambda^3(\mathcal{R}, (s), (s'), (t^m))^\times$.

Lemma 7.0.2. *With notation as above, the residue of $\omega_{m,r}(p, p')$ at the closed point of $k'[[s]]$ is given by the following formula:*

$$\mathrm{res}_{s=0}\, \omega_{m,r}(p, p') = \ell_{m,r}(\mathrm{res}_{(s)}(p)) - \ell_{m,r}(\mathrm{res}_{(s')}(p')).$$

Proof. If s'' is another lift of the uniformizer s, in other words $s'' \in \mathcal{R}$ with $s''|_{t^m} = s|_{t^m}$ then

$$\mathrm{res}_{s=0}\, \omega_{m,r}(p, p'') = \mathrm{res}_{s=0}\, \omega_{m,r}(p, p') + \mathrm{res}_{s=0}\, \omega_{m,r}(p', p'')$$

and

$$\begin{aligned}&\ell_{m,r}(\mathrm{res}_{(s)}(p)) - \ell_{m,r}(\mathrm{res}_{(s'')}(p''))\\ &\qquad = (\ell_{m,r}(\mathrm{res}_{(s)}(p)) - \ell_{m,r}(\mathrm{res}_{(s')}(p'))) + (\ell_{m,r}(\mathrm{res}_{(s')}(p')) - \ell_{m,r}(\mathrm{res}_{(s'')}(p''))).\end{aligned}$$

Therefore in order to prove the lemma we may assume without loss of generality that $s' = s + at^i$, with $a \in k'[\![s]\!]$ and $m \leq i$. Note that in $\mathcal{R}$, we have $s + at^i = s\exp((a/s)t^i)$, since $r < 2m$. Letting $\alpha = \exp(bt^j)$ and $\beta = \exp(ct^k)$, we can rewrite $\omega_{m,r}(p, p')$ as

$$\underline{\Omega}_{m,r}(p - p') = \underline{\Omega}_{m,r}\Big(\exp\Big(-\frac{a}{s}t^i\Big) \wedge \exp(bt^j) \wedge \exp(ct^k)\Big) = -\frac{a}{s}(jb \cdot dc - kc \cdot db),$$

if $i + j + k = r$ and 0 otherwise. Its residue is

$$-a(0)(jb(0)c'(0) - kc(0)b'(0)) \tag{7.0.2}$$

if $i + j + k = r$ and 0 otherwise, with the usual conventions if j or k is 0.

On the other hand, $\mathrm{res}_{(s)}(p) = \exp(b(0)t^j) \wedge \exp(c(0)t^k)$ and

$$\mathrm{res}_{(s')}(p') = \exp(b(0)t^j - a(0)b'(0)t^{i+j}) \wedge \exp(c(0)t^k - a(0)c'(0)t^{i+k}) \in \Lambda^2 k_r'^{\times}.$$

By the linearity of $\ell_{m,r}$, the right-hand side of the expression in the statement of the lemma is then equal to

$$-\ell_{m,r}(\exp(b(0)t^j) \wedge \exp(-a(0)c'(0)t^{i+k})) - \ell_{m,r}(\exp(-a(0)b'(0)t^{i+j}) \wedge \exp(c(0)t^k)) \\ - \ell_{m,r}(\exp(-a(0)b'(0)t^{i+j}) \wedge \exp(-a(0)c'(0)t^{i+k})).$$

The last summand is equal to 0 since $\ell_{m,r}$ is of weight r and $i + j + i + k \geq 2i \geq 2m > r$. For the same reason, the first two summands are 0 if $i + j + k \neq r$ and if $i + j + k = r$, then the total expression is equal to $-a(0)c'(0)jb(0) + a(0)b'(0)kc(0)$, which agrees with the formula (7.0.2) for the residue of $\Omega_{m,r}$. Since α and β are sums of the terms of the above type, this proves the lemma. □

Proposition 7.0.3. *Suppose that $\mathcal{R}$, $\mathcal{R}'$ are local algebras which are smooth of relative dimension* 1 *over k_r, together with liftings $\tilde{x}$, $\tilde{x}'$ of their closed points and a k_m-isomorphism $\chi : \mathcal{R}/(t^m) \to \mathcal{R}'/(t^m)$ which maps the reductions of $\tilde{x}$ and $\tilde{x}'$ to each other. Then for $q \in \Lambda^3(\mathcal{R}, \mathcal{R}', \tilde{x}, \tilde{x}', \chi)^{\times}$, we have the following formula for the closed point x,*

$$\mathrm{res}_x\, \omega_{m,r}(q) = \ell_{m,r}(\mathrm{res}_{\tilde{x}}(\Lambda^3\pi_1)(q)) - \ell_{m,r}(\mathrm{res}_{\tilde{x}'}(\Lambda^3\pi_2)(q)).$$

Proof. In order to prove the statement, we can replace $\mathcal{R}$ and $\mathcal{R}'$ with their completions at their closed points. Let k' be the residue field at the closed point x. Without loss of generality, we will assume that $\mathcal{R} = \mathcal{R}' = k'[\![s]\!]_r$, $\tilde{x}$ is given by $s = 0$ and $\tilde{x}'$ is given by $s' = 0$ for some $s' \in k'[\![s]\!]_r$ with $s'|_{t^m} = s|_{t^m}$, and χ is the map which is identity on $k'[\![s]\!]_m$.

In order to make the computations we need to choose a lifting $\tilde{\chi}$ of χ from $\mathcal{R}$ to $\mathcal{R}'$. We choose this lifting to be the one that sends s to s' and is identity on k'. Note that $\tilde{\chi}$ being a map of k_r algebras has to satisfy $\tilde{\chi}(t) = t$. The statement above then reduces to the following: suppose that $\alpha, \beta, \gamma \in (k'[\![s]\!]_r, (s))^{\times}$ and $\alpha', \beta', \gamma' \in (k'[\![s]\!]_r, (s'))^{\times}$ such that $\alpha|_{t^m} = \alpha'|_{t^m}$, $\beta|_{t^m} = \beta'|_{t^m}$, and $\gamma|_{t^m} = \gamma'|_{t^m}$, and $p = \alpha \wedge \beta \wedge \gamma$, $p' = \alpha' \wedge \beta' \wedge \gamma'$, and $(p, p') = (\alpha, \alpha') \wedge (\beta, \beta') \wedge (\gamma, \gamma')$, then

$$\mathrm{res}_{s=0}\, \Omega_{m,r}(p - p') = \ell_{m,r}(\mathrm{res}_{(s)}(p)) - \ell_{m,r}(\mathrm{res}_{(s')}(p')).$$

By assumption α is of the form us^n for some $u \in k'[\![s]\!]_r^\times$ and $n \in \mathbb{Z}$. Similarly, α' is of the form $u's'^{n'}$, with $u' \in k'[\![s]\!]_r^\times$. The condition that $\alpha|_{t^m} = \alpha'|_{t^m}$ implies that $n = n'$. The same is true for β, β', and γ, γ'. By multilinearity and antisymmetry, we reduce to checking the above identity in the following two cases: in the first case where $\alpha, \beta, \gamma \in k'[\![s]\!]^\times$ and in the second case where $\alpha = s$, $\alpha' = s'$ and $\beta, \gamma \in k'[\![s]\!]^\times$. If $\alpha, \beta, \gamma \in k'[\![s]\!]^\times$, then $\alpha', \beta', \gamma' \in k'[\![s]\!]^\times$. This implies on the one hand that $\mathrm{res}_{(s)}(p) = 0$ and $\mathrm{res}_{(s')}(p') = 0$, and on the other that $p - p' \in I_{m,r} = (1 + (t^m) \otimes \Lambda^2 k'[\![s]\!]_r^\times) \subseteq (\Lambda^3 k'[\![s]\!]_r^\times)^\circ$, which implies that $\underline{\Omega}_{m,r}(p - p') \in \Omega^1_{k'[\![s]\!]/k}$. Therefore $\mathrm{res}_{s=0}\, \underline{\Omega}_{m,r}(p - p') = 0 = \ell_{m,r}(\mathrm{res}_{(s)}(p)) - \ell_{m,r}(\mathrm{res}_{(s')}(p'))$ in this case.

Let us now consider the more interesting case of $\alpha = s$, $\alpha' = s'$ and $\beta, \gamma, \beta', \gamma' \in k'[\![s]\!]^\times$, with $\beta|_{t^m} = \beta'|_{t^m}$ and $\gamma|_{t^m} = \gamma'|_{t^m}$. Applying Lemma 7.0.1 first with $p = (s, \alpha, \beta)$ and $p' = (s, \alpha', \beta)$ then with $p = (s, \alpha', \beta)$ and $p' = (s, \alpha', \beta')$ and then applying Lemma 7.0.2 with $p = (s, \alpha', \beta')$ and $p' = (s', \alpha', \beta')$ and adding all the equalities finishes the proof of the proposition. □

8. Construction of ρ and a regulator on curves

8.1. ***Regulators on curves.*** Let $\mathcal{R}/k_m$ be smooth of relative dimension 1, as in the previous section but without the assumption that $\underline{\mathcal{R}}$ is a discrete valuation ring. Choose and fix a lifting $\mathfrak{c}$ of c to $\mathcal{R}$ for every closed point c of $\underline{\mathcal{R}}$ as in the previous section. We denote the set of these liftings by $\mathscr{P}$. We let $k(c)$ denote the residue field of c and $k(\mathfrak{c})$ denote the artin ring which is the ring of regular functions on the affine scheme $\mathfrak{c}$. Let $|\underline{\mathcal{R}}| = |\mathcal{R}|$ denote the set of closed points of $\underline{\mathcal{R}}$, or equivalently of $\mathcal{R}$. Note that the reductions of the localizations $\mathcal{R}_c$ of $\mathcal{R}$ are discrete valuation rings. We let

$$(\mathcal{R}, \mathscr{P})^\times := \bigcap_{c \in |\mathcal{R}|} (\mathcal{R}_c, \mathfrak{c})^\times$$

and $(\mathcal{R}, \mathscr{P})^\flat := \{f \in (\mathcal{R}, \mathscr{P})^\times \mid 1 - f \in (\mathcal{R}, \mathscr{P})^\times\}$. We define $B_2(\mathcal{R}, \mathscr{P})$ to be the vector space over $\mathbb{Q}$ generated by the symbols $[f]$ with $f \in (\mathcal{R}, \mathscr{P})^\flat$ modulo the five term relations associated to pairs f and g in $(\mathcal{R}, \mathscr{P})^\flat$ which have the property that $f - g \in (\mathcal{R}, \mathscr{P})^\times$. As usual we have maps $B_2(\mathcal{R}, \mathscr{P}) \to \Lambda^2(\mathcal{R}, \mathscr{P})^\times$ and $B_2(\mathcal{R}, \mathscr{P}) \otimes (\mathcal{R}, \mathscr{P})^\times \to \Lambda^3(\mathcal{R}, \mathscr{P})^\times$. We also have a residue map $\mathrm{res}_{\mathfrak{c}} : B_2(\mathcal{R}, \mathscr{P}) \otimes (\mathcal{R}, \mathscr{P})^\times \to B_2(k(\mathfrak{c}))$ that is defined exactly as in [Ünver 2021, Section 3.3.1] and which gives a commutative diagram:

$$\begin{tikzcd}
B_2(\mathcal{R}, \mathscr{P}) \otimes (\mathcal{R}, \mathscr{P})^\times \arrow[r, "\delta"] \arrow[d, "\mathrm{res}_{\mathfrak{c}}"] & \Lambda^3(\mathcal{R}, \mathscr{P})^\times \arrow[d, "\mathrm{res}_{\mathfrak{c}}"] \\
B_2(k(\mathfrak{c})) \arrow[r, "\delta"] & \Lambda^2 k(\mathfrak{c})^\times
\end{tikzcd}$$

Suppose that C/k_m is a smooth and projective curve. For every closed point c of $\underline{C}$, choose and fix a smooth lifting of $\mathfrak{c}$ of c to C. We denote $\mathscr{P}$ to be the set of these liftings. We let $(\mathcal{O}_C, \underline{\mathscr{P}})^\times$ denote the sheaf on $\underline{C}$ which associates to an open set U of $\underline{C}$, the group $(\mathcal{O}_C(U), \mathscr{P}|_U)^\times$. Similarly, $B_2(\mathcal{O}_C, \underline{\mathscr{P}})$ is the sheaf associated to the presheaf, which associates to U the group $B_2(\mathcal{O}_C(U), \mathcal{P}|_U)$. For each $c \in |C|$, let i_c denote the imbedding of c in $\underline{C}$. The commutative diagram above gives us a complex $\underline{\Gamma}'_B(C, \mathscr{P}, 3)$ of sheaves

$$B_2(\mathcal{O}_C, \underline{\mathscr{P}}) \otimes (\mathcal{O}_C, \underline{\mathscr{P}})^\times \to \bigoplus_{c \in |C|} i_{c*}(B_2(k(\mathfrak{c}))) \oplus \Lambda^3(\mathcal{O}_C, \underline{\mathscr{P}})^\times \to \bigoplus_{c \in |C|} i_{c*}(\Lambda^2 k(\mathfrak{c})^\times),$$

concentrated in degrees [2, 4]. We use the following sign conventions in the above complex: the first map is (δ, res) and the second one is $-\delta + \mathrm{res}$. We will be interested in the infinitesimal part of the degree 3 cohomology $\mathrm{H}^3_B(C, \mathbb{Q}(3)) := \mathrm{H}^3(\underline{C}, \underline{\Gamma}'_B(C, \mathscr{P}, 3))$ of the complex $\underline{\Gamma}'_B(C, \mathscr{P}, 3)$. More precisely, we will be interested in defining regulator maps from $\mathrm{H}^3_B(C, \mathbb{Q}(3))$ to k for every $m < r < 2m$. The above cohomology group is a candidate for the motivic cohomology group $\mathrm{H}^3_{\mathcal{M}}(C, \mathbb{Q}(3))$. To be more precise, we would expect a sheaf $B_3(\mathcal{O}_C, \underline{\mathscr{P}})$ of Bloch groups of weight 3 as in [Goncharov 1995], which would fit into a complex $\underline{\Gamma}_B(C, \mathscr{P}, 3)$ of sheaves on $\underline{C}$

$$B_3(\mathcal{O}_C, \underline{\mathscr{P}}) \to B_2(\mathcal{O}_C, \underline{\mathscr{P}}) \otimes (\mathcal{O}_C, \underline{\mathscr{P}})^\times \to \oplus i_{c*}(B_2(k(\mathfrak{c}))) \oplus \Lambda^3(\mathcal{O}_C, \underline{\mathscr{P}})^\times \to \oplus i_{c*}(\Lambda^2 k(\mathfrak{c})^\times),$$

and which would compute motivic cohomology of weight 3. Since we are only interested in $\mathrm{H}^3(\underline{C}, \underline{\Gamma}_B(C, \mathscr{P}, 3))$ and since on a curve, by Grothendieck's vanishing theorem, the cohomology of any sheaf vanishes in degree greater than 1, we have an isomorphism

$$\mathrm{H}^3(\underline{C}, \underline{\Gamma}'_B(C, \mathscr{P}, 3)) \simeq \mathrm{H}^3(\underline{C}, \underline{\Gamma}_B(C, \mathscr{P}, 3)).$$

For a sheaf of complexes $\mathscr{F}_\bullet$, let $\check{\mathrm{H}}^\bullet(\underline{C}, \mathscr{F}_\bullet)$ denote the colimit of all the Čech cohomology groups over all Zariski covers of $\underline{C}$. For a sheaf $\mathscr{F}$, the natural map $\check{\mathrm{H}}^i(\underline{C}, \mathscr{F}) \to \mathrm{H}^i(\underline{C}, \mathscr{F})$ is an isomorphism for $i = 0, 1$. By the same argument, it follows that the same is true for a complex of sheaves $\mathscr{F}_\bullet$ which is concentrated in degrees 0 and 1. This applied to the complex above implies that the natural map

$$\check{\mathrm{H}}^3(\underline{C}, \underline{\Gamma}'_B(C, \mathscr{P}, 3)) \simeq \mathrm{H}^3(\underline{C}, \underline{\Gamma}'_B(C, \mathscr{P}, 3))$$

is an isomorphism. Therefore, it is enough to construct the map $\check{\mathrm{H}}^3(\underline{C}, \underline{\Gamma}'_B(C, \mathscr{P}, 3)) \to k$. We will in fact construct the map as the composition

$$\check{\mathrm{H}}^3(\underline{C}, \underline{\Gamma}'_B(C, \mathscr{P}, 3)) \hookrightarrow \check{\mathrm{H}}^3(\underline{C}, \underline{\Gamma}''_B(C, \mathscr{P}, 3)) \to k,$$

where $\underline{\Gamma}''_B(C, \mathscr{P}, 3)$ is the quotient complex

$$B_2(\mathcal{O}_C, \underline{\mathscr{P}}) \otimes (\mathcal{O}_C, \underline{\mathscr{P}})^\times \to \bigoplus_{c \in |C|} i_{c*}(B_2(k(\mathfrak{c}))) \oplus \Lambda^3(\mathcal{O}_C, \underline{\mathscr{P}})^\times$$

of $\underline{\Gamma}'_B(C, \mathscr{P}, 3)$.

Suppose that we are given a Zariski open cover $U_\bullet$ of $\underline{C}$, we will define a map from the corresponding cocycle group $\check{\mathrm{Z}}^3(U_\bullet, \underline{\Gamma}''_B(C, \mathscr{P}, 3))$ to k, which will vanish on the coboundaries and hence induce the map in the cohomology group that we are looking for. Suppose that we start with a cocyle as above, given by the data:

(i) $\gamma_i \in \Lambda^3(\mathcal{O}_C, \underline{\mathscr{P}})^\times(U_i)$, for all $i \in I$.

(ii) $\varepsilon_{i,c} \in B_2(k(\mathfrak{c}))$ for every $c \in U_i$ all but finitely many of which are 0, for all $i \in I$.

(iii) $\beta_{ij} \in (B_2(\mathcal{O}_C, \underline{\mathscr{P}}) \otimes (\mathcal{O}_C, \underline{\mathscr{P}})^\times)(U_{ij})$, for all $i, j \in I$.

These data are supposed to satisfy the following properties:

(i) $\delta(\beta_{ij}) = \gamma_j|_{U_{ij}} - \gamma_i|_{U_{ij}}$.

(ii) $\mathrm{res}_{\mathfrak{c}}(\beta_{ij}) = \varepsilon_{j,c} - \varepsilon_{i,c}$, for $c \in U_{ij}$.

(iii) $\beta_{jk}|_{U_{ijk}} - \beta_{ik}|_{U_{ijk}} + \beta_{ij}|_{U_{ijk}} = 0$.

We will construct the image of the above element by making several choices and then proving that the construction is independent of all the choices:

(i) Let $\tilde{\mathcal{A}}_\eta/k_\infty$ be a smooth lifting of $\mathcal{O}_{C,\eta}$ and for every $c \in |C|$, let $\tilde{\mathcal{A}}_c/k_\infty$ be a smooth lifting of the completion $\hat{\mathcal{O}}_{C,c}$ of the local ring of C at c, together with a smooth lifting $\tilde{\mathfrak{c}}$ of $\mathfrak{c}$ as in the previous section.

Moreover, choose:

(ii) an arbitrary $i \in I$ and for each c choose a $j_c \in I$ such that $c \in U_{j_c}$.

(iii) An arbitrary lifting $\tilde{\gamma}_{i\eta} \in \Lambda^3\tilde{\mathcal{A}}_\eta^\times$ of the germ $\gamma_{i\eta} \in \Lambda^3\mathcal{O}_{C,\eta}^\times$ of γ_i at the generic point η.

(iv) A good lifting $\tilde{\gamma}_{j_c} \in \Lambda^3(\tilde{\mathcal{A}}_c, \tilde{\mathfrak{c}})^\times$ of the image $\hat{\gamma}_{j_c,c}$ of γ_{j_c} in $\Lambda^3(\hat{\mathcal{O}}_{C,c}, \mathfrak{c})^\times$, for every $c \in |C|$.

(v) An arbitrary lifting $\tilde{\beta}_{j_c i,\eta} \in B_2(\tilde{\mathcal{A}}_\eta) \otimes \tilde{\mathcal{A}}_\eta$ of the image $\beta_{j_c i,\eta} \in B_2(\mathcal{O}_{C,\eta}) \otimes \mathcal{O}_{C,\eta}^\times$ of $\beta_{j_c i}$, for every $c \in |C|$.

Note that it does not make sense to require that $\tilde{\gamma}_{i\eta}$ be a good lifting since in this context there is no a fixed specialization of the generic point. Similarly, we cannot require that $\tilde{\beta}_{j_c i,\eta}$ be a good lifting, since we know that $\delta(\tilde{\beta}_{j_c i,\eta})$ is a lifting of $\delta(\beta_{j_c i,\eta}) = \gamma_i - \gamma_{j_c}$ and even this last expression need not be good at c as γ_i need not be good at c. We define the value of the regulator $\rho_{m,r}$ on the above element by the expression

$$\sum_{c\in|C|} \mathrm{Tr}_k\big(\ell_{m,r}(\mathrm{res}_{\tilde{\mathfrak{c}}}\, \tilde{\gamma}_{j_c}) - \ell i_{m,r}(\varepsilon_{j_c,c}) + \mathrm{res}_c\, \omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}_{j_c i,\eta}), \tilde{\gamma}_{j_c})\big). \tag{8.1.1}$$

Let us first explain what we mean by the above expression. Since $\tilde{\gamma}_{j_c}$ is $\tilde{\mathfrak{c}}$-good, the residue $\mathrm{res}_{\tilde{\mathfrak{c}}}\, \tilde{\gamma}_{j_c}$ along $\tilde{\mathfrak{c}}$ is defined as an element of $\Lambda^2 k(\tilde{\mathfrak{c}})^\times$. The étaleness of $\tilde{\mathfrak{c}}$ over k_∞, implies that we have a canonical isomorphism $k(\tilde{\mathfrak{c}}) \simeq k(c)_\infty$ of k_∞-algebras. Using this isomorphism and the map $\ell_{m,r} : \Lambda^2 k(c)_\infty^\times \to k(c)$ in Definition 3.0.2, we obtain $\ell_{m,r}(\mathrm{res}_{\tilde{\mathfrak{c}}}\, \tilde{\gamma}_{j_c}) \in k(c)$. For the second term, note that, as above, there is a canonical isomorphism $k(\mathfrak{c}) \simeq k(c)_m$ of k_m algebras using which we can view $\varepsilon_{j_c,c} \in B_2(k(c)_m)$. Applying $\ell i_{m,r} : B_2(k(c)_m) \to k(c)$ to this element gives $\ell i_{m,r}(\varepsilon_{j_c,c}) \in k(c)$. For the last term, note that $\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}_{j_c i,\eta})$ is a lifting of $\gamma_{i\eta} - \delta(\beta_{j_c i,\eta}) = \gamma_{j_c}$ to $\Lambda^3\tilde{\mathcal{A}}_\eta^\times$ and so is $\tilde{\gamma}_{j_c}$ a lifting of γ_{j_c} to $\Lambda^3\tilde{\mathcal{A}}_c^\times$. Using the theory of Section 6.5, we see that the last term $\mathrm{res}_c\, \omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}_{j_c i,\eta}), \tilde{\gamma}_{j_c}) \in k(c)$ is unambiguously defined. Letting Tr_k denote the normalized trace to k, the summands above are defined.

In order to show that the sum makes sense, we also need to show that the sum is finite. Below we will show that the sum is independent of all the choices, therefore it will be enough to show that the sum is finite for a particular choice. First by shrinking U_i if necessary, and choosing a refinement of the cover, we will assume that $\gamma_i \in \Lambda^3 \mathcal{O}_C^\times(U_i)$. Similarly, by shrinking U_i even further, we will assume that the lifting $\tilde{\gamma}_i$ is good on U_i. Therefore, for $c \in U_i$, we can choose $j_c = i$ and $\tilde{\gamma}_{j_c} = \tilde{\gamma}_i$. Since for these c, $\beta_{j_c i} = 0$ we can choose $\tilde{\beta}_{j_c i} = 0$. In order to show that the sum in (8.1.1) is finite, we can concentrate on $c \in U_i$, as $|C| \setminus |U_i|$ is finite. For $c \in U_i$, $\mathrm{res}_c\, \tilde{\gamma}_{j_c} = \mathrm{res}_c\, \tilde{\gamma}_i = 0$, since γ_i is invertible on U_i by assumption. Also for the residues we have $\mathrm{res}_c\, \omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}_{j_c i,\eta}), \tilde{\gamma}_{j_c}) = \mathrm{res}_c\, \omega_{m,r}(\tilde{\gamma}_i, \tilde{\gamma}_i) = 0$ since $i = j_c$, $\tilde{\gamma}_{j_c} = \tilde{\gamma}_i$ and $\tilde{\beta}_{j_c i} = 0$. Therefore the summand, for $c \in U_i$, is equal to $\mathrm{Tr}_k(-\ell i_{m,r}(\varepsilon_{j_c,c})) = \mathrm{Tr}_k(-\ell i_{m,r}(\varepsilon_{i,c}))$. Since $\varepsilon_{i,c} = 0$, for all but a finite number of $c \in U_i$, we are done.

We now show that the expression makes sense and is independent of all the choices. Note that there are many of them.

Theorem 8.1.1. *For every $m < r < 2m$, the above formula (8.1.1) gives a well-defined regulator map $\rho_{m,r} : \check{\mathrm{Z}}^3(U_\bullet, \underline{\Gamma}''_B(C, \mathscr{P}, 3)) \to k$, independent of all the choices. This map vanishes on the coboundaries and hence induces the regulator map*

$$\rho_{m,r} : \mathrm{H}^3_B(C, \mathbb{Q}(3)) \to k$$

of $\star$-weight r.

Specializing to the case when C is the projective line $\mathbb{P}^1_{k_m}$, with coordinate function z, we fix an $a \in k_m^\flat$. If we choose $\mathscr{P}$ such that z, $1-z$ and $z-a$ are all good with respect to $\mathscr{P}$, then $(1-z) \wedge z \wedge (z-a) \in \Gamma(\Lambda^3(\mathcal{O}_{\mathbb{P}^1}, \underline{\mathscr{P}})^\times)$ and

$$\rho_{m,r}((1-z) \wedge z \wedge (z-a)) = \ell i_{m,r}([a]).$$

Proof. We first show the independence of the definition from the various choices. For readability, we separate these into parts.

Independence of the choice of j_c and the liftings $\tilde{\beta}_{j_c i}$ and $\tilde{\gamma}_{j_c}$. Suppose that we choose a different j'_c with $c \in U_{j'_c}$, a different lifting $\tilde{\mathcal{A}}'_c$ of $\hat{\mathcal{O}}_{C,c}$, together with $\tilde{\mathfrak{c}}'$ as above, a $\tilde{\mathfrak{c}}'$-good lifting $\tilde{\gamma}'_{j'_c}$ of $\gamma_{j'_c}$ to $\tilde{\mathcal{A}}'_c$ and a lifting $\tilde{\beta}'_{j'_c i}$ of $\beta_{j'_c i}$ to $\tilde{\mathcal{A}}_\eta$. Since $\tilde{\mathcal{A}}_c \simeq k(c)[\![\tilde{\mathfrak{s}}]\!]_\infty$, where $\tilde{\mathfrak{s}}$ is a choice of a uniformizer associated to $\tilde{\mathfrak{c}}$ and similarly for $\tilde{\mathcal{A}}'_c$, we choose and fix a k_∞-algebra isomorphism between $\tilde{\mathcal{A}}_c$ and $\tilde{\mathcal{A}}'_c$ which is identity modulo (t^m) and which sends $\tilde{\mathfrak{s}}$ to $\tilde{\mathfrak{s}}'$. This last condition is possible to impose since both $\tilde{\mathfrak{s}}$ and $\tilde{\mathfrak{s}}'$ lift $\mathfrak{s}$ by assumption. Below we identify these two algebras using this isomorphism.

We need to compare the two expressions

$$\ell_{m,r}(\mathrm{res}_{\tilde{\mathfrak{c}}}\, \tilde{\gamma}_{j_c}) - \ell i_{m,r}(\varepsilon_{j_c,c}) + \mathrm{res}_c\, \omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}_{j_c i}), \tilde{\gamma}_{j_c}) \tag{8.1.2}$$

and

$$\ell_{m,r}(\mathrm{res}_{\tilde{\mathfrak{c}}'}\, \tilde{\gamma}'_{j'_c}) - \ell i_{m,r}(\varepsilon_{j'_c,c}) + \mathrm{res}_c\, \omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}'_{j'_c i}), \tilde{\gamma}'_{j'_c}). \tag{8.1.3}$$

By linearity we have

$$\mathrm{res}_c\, \omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}'_{j'_c i}), \tilde{\gamma}'_{j'_c}) - \mathrm{res}_c\, \omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}_{j_c i}), \tilde{\gamma}_{j_c}) = \mathrm{res}_c\, \omega_{m,r}(\tilde{\gamma}_{j_c} - \tilde{\gamma}'_{j'_c}, \delta(\tilde{\beta}'_{j'_c i} - \tilde{\beta}_{j_c i})).$$

Let $\tilde{\beta}_{j'_c j_c}$ be a $\tilde{\mathfrak{c}}$-good lifting of $\beta_{j'_c j_c}$ to $\tilde{\mathcal{A}}_c$. Since $\beta_{j'_c j_c}$ itself is $\mathfrak{c}$-good such a lifting exists. We have the identity $\beta_{j'_c j_c} = \beta_{j'_c i} - \beta_{j_c i}$ on $U_{i j_c j'_c}$, which might not contain c, but does of course contain the generic point η. We deduce that $\tilde{\beta}_{j'_c j_c,\eta}$ and $\tilde{\beta}'_{j'_c i,\eta} - \tilde{\beta}_{j_c i,\eta}$ have the same reduction $\beta_{j'_c j_c,\eta}$. Now by Corollary 6.5.3, we conclude that

$$\mathrm{res}_c\, \omega_{m,r}(\delta(\tilde{\beta}_{j'_c j_c,\eta}), \delta(\tilde{\beta}'_{j'_c i,\eta} - \tilde{\beta}_{j_c i,\eta})) = 0.$$

This implies that, using transitivity and linearity, we have:

$$\mathrm{res}_c\, \omega_{m,r}(\tilde{\gamma}_{j_c} - \tilde{\gamma}'_{j'_c}, \delta(\tilde{\beta}'_{j'_c i} - \tilde{\beta}_{j_c i})) = \mathrm{res}_c\, \omega_{m,r}(\tilde{\gamma}_{j_c} - \tilde{\gamma}'_{j'_c}, \delta(\tilde{\beta}_{j'_c j_c,\eta})) = \mathrm{res}_c\, \omega_{m,r}(\tilde{\gamma}_{j_c} - \delta(\tilde{\beta}_{j'_c j_c,\eta}), \tilde{\gamma}'_{j'_c}).$$

In this expression, $\tilde{\gamma}_{j_c} - \delta(\tilde{\beta}_{j'_c j_c})$ is a $\tilde{\mathfrak{c}}$-good lifting to $\tilde{\mathcal{A}}_c$ and $\tilde{\gamma}'_{j'_c}$ is a $\tilde{\mathfrak{c}}'$-good lifting to $\tilde{\mathcal{A}}'_c$. Then Proposition 7.0.3 implies that

$$\mathrm{res}_c\, \omega_{m,r}(\tilde{\gamma}_{j_c} - \delta(\tilde{\beta}_{j'_c j_c}), \tilde{\gamma}'_{j'_c}) = \ell_{m,r}(\mathrm{res}_{\tilde{\mathfrak{c}}}(\tilde{\gamma}_{j_c} - \delta(\tilde{\beta}_{j'_c j_c}))) - \ell_{m,r}(\mathrm{res}_{\tilde{\mathfrak{c}}'}(\tilde{\gamma}'_{j'_c})). \tag{8.1.4}$$

On the other hand,

$$\ell_{m,r}(\mathrm{res}_{\tilde{\mathfrak{c}}}(\delta(\tilde{\beta}_{j'_c j_c}))) = \ell_{m,r}(\delta(\mathrm{res}_{\tilde{\mathfrak{c}}}(\tilde{\beta}_{j'_c j_c}))) = \ell i_{m,r}(\mathrm{res}_c(\beta_{j'_c j_c})),$$

by the definition of $\ell i_{m,r}$. Since by assumption $\mathrm{res}_c(\beta_{j'_c j_c}) = \varepsilon_{j_c,c} - \varepsilon_{j'_c,c}$, we can rewrite the right-hand side of (8.1.4) as

$$\ell_{m,r}(\mathrm{res}_{\tilde{\mathfrak{c}}}(\tilde{\gamma}_{j_c})) - \ell_{m,r}(\mathrm{res}_{\tilde{\mathfrak{c}}'}(\tilde{\gamma}'_{j'_c})) - \ell i_{m,r}(\varepsilon_{j_c,c}) + \ell i_{m,r}(\varepsilon_{j'_c,c}).$$

Combining all of the above, we see that the last expression is equal to the difference

$$\mathrm{res}_c\, \omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}'_{j'_c i}), \tilde{\gamma}'_{j'_c}) - \mathrm{res}_c\, \omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}_{j_c i}), \tilde{\gamma}_{j_c}),$$

which implies the equality of the two expressions (8.1.2) and (8.1.3) and thus proves the independence we were looking for.

Independence of the choice of i and the liftings $\tilde{\gamma}_{i\eta}$ and $\tilde{\beta}_{j_c i}$. Let us choose an i', a lifting $\tilde{\mathcal{A}}'_\eta$ of $\mathcal{O}_{C,\eta}$ and liftings $\tilde{\gamma}'_{i'\eta}$ and $\tilde{\beta}'_{j_c i'}$ to $\tilde{\mathcal{A}}'_\eta$, for each $c \in |C|$. We need to compare

$$\sum_{c\in|C|} \mathrm{Tr}_k(\ell_{m,r}(\mathrm{res}_{\tilde{\mathfrak{c}}}\, \tilde{\gamma}_{j_c}) - \ell i_{m,r}(\varepsilon_{j_c,c}) + \mathrm{res}_c\, \omega_{m,r}(\tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}_{j_c i}), \tilde{\gamma}_{j_c})) \tag{8.1.5}$$

and

$$\sum_{c\in|C|} \mathrm{Tr}_k(\ell_{m,r}(\mathrm{res}_{\tilde{\mathfrak{c}}}\, \tilde{\gamma}_{j_c}) - \ell i_{m,r}(\varepsilon_{j_c,c}) + \mathrm{res}_c\, \omega_{m,r}(\tilde{\gamma}'_{i'\eta} - \delta(\tilde{\beta}'_{j_c i'}), \tilde{\gamma}_{j_c})). \tag{8.1.6}$$

The difference between the above expressions is

$$\sum_{c\in|C|} \mathrm{Tr}_k\, \mathrm{res}_c\, \omega_{m,r}(\tilde{\gamma}'_{i'\eta} - \delta(\tilde{\beta}'_{j_c i'}), \tilde{\gamma}_{i\eta} - \delta(\tilde{\beta}_{j_c i})).$$

Choosing an isomorphism $\tilde{\mathcal{A}}_\eta \simeq \tilde{\mathcal{A}}'_\eta$ of k_∞-algebras which lifts the given one modulo (t^m), we identify $\tilde{\mathcal{A}}_\eta$ and $\tilde{\mathcal{A}}'_\eta$. The above sum can then be rewritten as

$$\sum_{c\in|C|} \operatorname{Tr}_k \operatorname{res}_c \omega_{m,r}(\tilde{\gamma}'_{i'\eta}-\tilde{\gamma}_{i\eta}, \delta(\tilde{\beta}'_{j_c i'}-\tilde{\beta}_{j_c i})).$$

As in the above argument since $\tilde{\beta}_{ii'}$ has the same reduction modulo (t^m) as $\tilde{\beta}'_{j_c i'}-\tilde{\beta}_{j_c i}$ for any j_c, we have $\operatorname{res}_c \omega_{m,r}(\delta(\tilde{\beta}'_{j_c i'}-\tilde{\beta}_{j_c i}), \delta(\tilde{\beta}_{ii'}))=0$ by Corollary 6.5.3. So we can rewrite the above sum as

$$\sum_{c\in|C|} \operatorname{Tr}_k \operatorname{res}_c \omega_{m,r}(\tilde{\gamma}'_{i'\eta}-\tilde{\gamma}_{i\eta}, \delta(\tilde{\beta}_{ii'})).$$

Choosing a splitting of $\tilde{\mathcal{A}}_\eta$, we identify this algebra with $(\underline{\mathcal{A}}_\eta)_\infty=(\mathcal{O}_{\underline{C},\eta})_\infty$. Using this identification, the last expression is the sum of residues of the meromorphic 1-form $\Omega_{m,r}(\tilde{\gamma}'_{i'\eta}-\tilde{\gamma}_{i\eta}-\delta(\tilde{\beta}_{ii'}))$ on $\underline{C}$ and therefore is equal to 0.

Vanishing on coboundaries. Suppose that we start with sections

$$\alpha_i \in (B_2(\mathcal{O}_C,\mathscr{P})\otimes(\mathcal{O}_C,\mathscr{P})^\times)(U_i),$$

for all $i\in I$. Then we need to show that the value of the regulator on the data

$$(\{\gamma_i\}_{i\in I}, \{\varepsilon_{i,c} \mid i\in I, c\in U_i\}, \{\beta_{ij}\}_{i,j\in I})$$

is 0. Here $\gamma_i := \delta(\alpha_i)$, $\varepsilon_{i,c} := \operatorname{res}_{\mathfrak{c}}(\alpha_i)$ and $\beta_{ij} := \alpha_j|_{U_{ij}}-\alpha_i|_{U_{ij}}$.

We fix an $i\in I$ and $j_c\in I$, with $c\in U_{j_c}$, for every $c\in|C|$, and local and generic liftings $\tilde{\mathcal{A}}_c$, and $\tilde{\mathcal{A}}_\eta$ of the curve, as above, together with liftings $\tilde{\mathfrak{c}}$ of $\mathfrak{c}$ to $\tilde{\mathcal{A}}_c$. We need to choose liftings of the data in order to compute the value of the regulator on the above element.

We choose a lifting $\tilde{\alpha}_{i\eta}$ of $\alpha_{i\eta}$ to $\tilde{\mathcal{A}}_\eta$ and let $\tilde{\gamma}_{i\eta} := \delta(\tilde{\alpha}_{i\eta})$. For each $c\in|C|$, we choose a $\tilde{\mathfrak{c}}$-good lifting $\tilde{\alpha}_{j_c}$ of α_{j_c} and let $\tilde{\gamma}_{j_c} := \delta(\tilde{\alpha}_{j_c})$. Finally, we choose an arbitrary lifting $\tilde{\alpha}_{j_c\eta}$ of $\alpha_{j_c\eta}$ to $\tilde{\mathcal{A}}_\eta$, for every $c\in|C|$, and let $\tilde{\beta}_{j_c i,\eta} := \tilde{\alpha}_{i\eta}-\tilde{\alpha}_{j_c\eta}$. Then the value of the regulator (8.1.1) is the sum of traces of the terms

$$\begin{aligned}\ell_{m,r}(\operatorname{res}_{\tilde{\mathfrak{c}}}\tilde{\gamma}_{j_c}) - \ell i_{m,r}(\varepsilon_{j_c,c}) + \operatorname{res}_c \omega_{m,r}(\tilde{\gamma}_{i\eta}-\delta(\tilde{\beta}_{j_c i,\eta}), \tilde{\gamma}_{j_c})&\\ = \ell_{m,r}(\operatorname{res}_{\tilde{\mathfrak{c}}}\delta(\tilde{\alpha}_{j_c})) - \ell i_{m,r}(\varepsilon_{j_c,c}) &+ \operatorname{res}_c \omega_{m,r}(\delta(\tilde{\alpha}_{j_c\eta}), \delta(\tilde{\alpha}_{j_c}))\\ = \ell_{m,r}(\operatorname{res}_{\tilde{\mathfrak{c}}}\delta(\tilde{\alpha}_{j_c})) - \ell i_{m,r}(\varepsilon_{j_c,c})&\end{aligned}$$

by Corollary 6.5.3. Since $\operatorname{res}_{\tilde{\mathfrak{c}}}\delta(\tilde{\alpha}_{j_c})=\delta(\operatorname{res}_{\tilde{\mathfrak{c}}}\tilde{\alpha}_{j_c})$, we have $\ell_{m,r}(\operatorname{res}_{\tilde{\mathfrak{c}}}\delta(\tilde{\alpha}_{j_c}))=\ell_{m,r}(\delta(\operatorname{res}_{\tilde{\mathfrak{c}}}\tilde{\alpha}_{j_c}))$. By the definition of $\ell i_{m,r}$, we have $\ell_{m,r}(\delta(\operatorname{res}_{\tilde{\mathfrak{c}}}\tilde{\alpha}_{j_c}))=\ell i_{m,r}(\operatorname{res}_{\mathfrak{c}}\alpha_{j_c})=\ell i_{m,r}(\varepsilon_{j_c,c})$. This implies that all the summands in the formula for the regulator (8.1.1) are 0 finishing the proof of the first part of the theorem.

For the proof of the second part of the theorem, we note that a more general version of the computation for $\mathbb{P}^1_{k_m}$ will be done in Section 9.2. □

8.2. *Infinitesimal Chow dilogarithm.* Specializing the above construction to global sections of $\Lambda^3(\mathcal{O}_C,\mathscr{P})^\times$ gives us the generalization of the infinitesimal Chow dilogarithm in [Ünver 2021] to higher moduli.

Let us denote the global sections $\Gamma(\underline{C}, (\mathcal{O}_C, \underline{\mathscr{P}})^{\times})$ of $(\mathcal{O}_C, \underline{\mathscr{P}})^{\times}$ by $k(C, \underline{\mathscr{P}})^{\times}$. Suppose that we start with $\gamma \in \Lambda^3 k(C, \underline{\mathscr{P}})^{\times}$. Specializing the construction in the previous section, we have $\rho_{m,r}(\gamma) \in k$, which can be computed as follows.

Choose a lifting $\tilde{\mathcal{A}}_\eta / k_\infty$ of $\mathcal{O}_{C,\eta}$ and local liftings $\tilde{\mathcal{A}}_c$ of $\hat{\mathcal{O}}_{C,c}$, for every $c \in |C|$, together with liftings $\tilde{\mathfrak{c}}$ of $\mathfrak{c}$. Choose an arbitrary lifting $\tilde{\gamma}_\eta$ of γ_η to $\tilde{\mathcal{A}}_\eta$ and $\tilde{\mathfrak{c}}$-good liftings $\tilde{\gamma}_c$ of the germ of γ at c to $\tilde{\mathcal{A}}_c$, for every $c \in |C|$. By the definition in the previous section, we have

$$\rho_{m,r}(\gamma) := \sum_{c \in |C|} \mathrm{Tr}_k(\ell_{m,r}(\mathrm{res}_{\tilde{\mathfrak{c}}}(\tilde{\gamma}_c)) + \mathrm{res}_c\, \omega_{m,r}(\tilde{\gamma}_\eta, \tilde{\gamma}_c)), \tag{8.2.1}$$

for every $m < r < 2m$.

Corollary 8.2.1. *The definition in (8.2.1) of the infinitesimal Chow dilogarithm of modulus m and $\star$-weight r gives a map*

$$\rho_{m,r} : \Lambda^3 k(C, \underline{\mathscr{P}})^{\times} \to k,$$

independent of all the choices and generalizing the construction in [Ünver 2021] to arbitrary m and r with $2 \le m < r < 2m$.

9. Applications and examples

9.1. ***Strong reciprocity conjecture.*** The infinitesimal Chow dilogarithm can be used to give a proof of an infinitesimal version of Goncharov's strong reciprocity conjecture for the curve C/k_m, exactly as in [Ünver 2021, Theorem 3.4.4]. In this section, in addition to our previous hypotheses, we assume that k is algebraically closed.

Taking the infinitesimal part of the sum of the residues at all closed points of C, we have a map:

$$\mathrm{res}_C : \Lambda^3 k(C, \underline{\mathscr{P}})^{\times} \to (\Lambda^2 k_m^{\times})^{\circ}.$$

Similarly, letting $B_2(k(C, \underline{\mathscr{P}}))$ denote the set of global sections of $B_2(\mathcal{O}_C, \underline{\mathscr{P}})$, we have a map

$$\mathrm{res}_C : B_2(k(C, \underline{\mathscr{P}})) \otimes k(C, \underline{\mathscr{P}})^{\times} \to B_2(k_m)^{\circ}.$$

The explicit version of the strong reciprocity conjecture expresses both of these maps in terms of a single map h.

Theorem 9.1.1. *There is a map $h : \Lambda^3 k(C, \underline{\mathscr{P}})^{\times} \to B_2(k_m)^{\circ}$, which makes the diagram*

$$\begin{tikzcd} B_2(k(C, \underline{\mathscr{P}})) \otimes k(C, \underline{\mathscr{P}})^{\times} \arrow[r, "\delta"] \arrow[d, "\mathrm{res}_C"] & \Lambda^3 k(C, \underline{\mathscr{P}})^{\times} \arrow[d, "\mathrm{res}_C"] \arrow[dl, dotted, "h"'] \\ B_2(k_m)^{\circ} \arrow[r, "\delta^{\circ}"] & (\Lambda^2 k_m^{\times})^{\circ} \end{tikzcd}$$

commute and has the property that $h(k_m^{\times} \wedge \Lambda^2 k(C, \underline{\mathscr{P}})^{\times}) = 0$.

Proof. The proof, using the maps $\rho_{m,r}$ constructed above, is exactly the same as that of [Ünver 2021, Theorem 3.4.4] and is omitted. □

The theorem above, in essence, states that the residue map from the Bloch complex of weight 3 on C to the Bloch complex of weight 2 on k_m is homotopic to 0; see [Ünver 2021, Section 3.4].

9.2. ***The special case of the projective line.*** As a first example, let us look at the infinitesimal Chow dilogarithm in the case of the projective line $\mathbb{P}^1$ over k_m with k algebraically closed.

For each point $c \in \mathbb{P}^1_k$ let us choose a smooth lifting $\mathfrak{c} \in \mathbb{P}^1(k_m)$. Considering a lifting as a map $\mathrm{Spec}(k_m) \to \mathbb{P}^1_{k_m}$, if the projection $\mathrm{Spec}(k_m) \to \mathbb{P}^1_k$ factors through the structure map $\mathrm{Spec}(k_m) \to \mathrm{Spec}(k)$, we call that lifting a constant lifting. In the following, we will always choose the constant liftings of the points 0, 1 and ∞, for the other points in $\mathbb{P}^1_k$ the choices will be arbitrary. We denote the set of these liftings by $\mathscr{P}$ as usual.

Letting $a \in \mathscr{P}$ be the chosen lifting of an element in $k^\flat$, the element $(1-z) \wedge z \wedge (z-a)$ satisfies our goodness hypothesis. We will compute the value of $\rho_{m,r}$ on this element.

We will use the formula (1.3.1) directly. In order to do this first let us choose a set $\tilde{\mathscr{P}}$ of liftings to $\mathbb{P}^1_{k_\infty}$ of elements in $\mathscr{P}$. Again for the elements 0, 1, and ∞, we choose the constant liftings. Let us denote the lifting of $a \in \mathscr{P}$ by $\tilde{a} \in \tilde{\mathscr{P}}$. Then the functions z, $1-z$, and $z-\tilde{a}$ are liftings of z, $1-z$, and $z-a$ to functions on $\mathbb{P}^1_{k_\infty}$, which are good with respect to $\tilde{\mathscr{P}}$. Using the definition of $\rho_{m,r}$, we obtain

$$\rho_{m,r}((1-z) \wedge z \wedge (z-a)) = \sum_{\tilde{\mathfrak{c}} \in \tilde{\mathscr{P}}} \ell_{m,r}(\mathrm{res}_{\tilde{\mathfrak{c}}}((1-z) \wedge z \wedge (z-\tilde{a}))).$$

The only contribution to the sum above comes from the residue at $\tilde{a}$. Therefore the last expression is equal to $\ell_{m,r}((1-\tilde{a}) \wedge \tilde{a})$. Using the definition of $\ell i_{m,r}$ we can rewrite this as $\ell_{m,r}((1-\tilde{a}) \wedge \tilde{a}) = \ell_{m,r}(\delta[\tilde{a}]) = \ell i_{m,r}([a])$. Let f, g and h be arbitrary functions on $\mathbb{P}^1_{k_m}$ which are good with respect to $\mathscr{P}$. Then we can write

$$f = \lambda \prod_{1 \le i \le l} (z-\alpha_i)^{\delta_i}, \quad g = \mu \prod_{1 \le j \le m} (z-\beta_j)^{\varepsilon_j}, \quad \text{and} \quad h = \nu \prod_{1 \le k \le n} (z-\gamma_k)^{\eta_k},$$

for some α_i, β_j, $\gamma_k \in \mathscr{P}$, λ, μ, $\nu \in k_m^\times$ and δ_i, ε_j, $\eta_k \in \mathbb{Z}$. Using exactly the same argument at the end of Section 2.2, we find that $\rho_{m,r}(f \wedge g \wedge h)$ is equal to

$$\sum_{i,j,k} \delta_i \varepsilon_j \eta_k \cdot \ell i_{m,r}\left(\left[\frac{\gamma_k - \beta_j}{\alpha_i - \beta_j}\right]\right) \tag{9.2.1}$$

where the summation is on $1 \le i \le l$, $1 \le j \le m$ and $1 \le k \le n$. In this notation, we use the convention that $\ell i_{m,r}([(\gamma_k - \beta_j)/(\alpha_i - \beta_j)]) = 0$, if at least two of α_i, β_j and γ_k are the same. Note that because of the goodness with respect to $\mathscr{P}$ hypothesis, if α_i, β_j and γ_k are pairwise distinct then their reductions to k have to be distinct as well. This, then, implies that $(\gamma_k - \beta_j)/(\alpha_i - \beta_j) \in k_m^\flat$ and the expression (9.2.1) is well-defined.

9.3. ***The special case of elliptic curves.*** As a second example, we will consider the infinitesimal Chow dilogarithm in the case of an elliptic curve. Again, for simplicity, we assume that k is algebraically closed. Suppose that E/k_m is an elliptic curve. Suppose that $E \subseteq \mathbb{P}^2_{k_m}$ is given by a Weierstrass equation $y^2 = x^3 + Ax + B$, with $A, B \in k_m$, in the affine coordinates with $x = X/Z$ and $y = Y/Z$, where X, Y, and Z are the homogeneous coordinates on $\mathbb{P}^2$. Suppose further that we have a lifting $\tilde{E} \subseteq \mathbb{P}^2_{k_\infty}$ which is also given by a Weierstrass equation $y^2 = x^3 + \tilde{A}x + \tilde{B}$, with $\tilde{A}, \tilde{B} \in k_\infty$. Note that these hypotheses are satisfied when E is constant curve, in other words $E = \underline{E} \times_k k_m$, for some elliptic curve $\underline{E}/k$. In this case, we can of course choose the lifting as $\underline{E} \times_k k_\infty$. In the following, we will not assume that our curve is a constant curve. Suppose we fix a choice of smooth liftings $\mathscr{P}$ as above for each point in $E(k)$ such that the lifting of the origin in $E(k)$ is the origin in $E(k_m)$.

Let l_0 be the line which intersects the curve at the origin O with multiplicity 3. It is the line given by the equation $Z = 0$. For each $1 \le i \le 3$ let l_i be the line given by $a_i X + b_i Y + c_i Z = 0$, with $a_i, b_i, c_i \in k_m$. Denote the intersection points of the line l_i with the elliptic curve by α_{i1}, α_{i2} and α_{i3}. Note that the group law on E gives that $\alpha_{i1} + \alpha_{i2} + \alpha_{i3} = 0$. Suppose that the intersection points $\alpha_{i1}, \alpha_{i2}, \alpha_{i3}$ lie in the chosen set of liftings $\mathscr{P}$. Let f_i denote the function on E given by l_i/l_0. For a generic choice of the lines, let us compute $\rho_{m,r}(f_1 \wedge f_2 \wedge f_3)$. Let $\tilde{l}_0$ be the line in $\mathbb{P}^2_{k_\infty}$ given by $Z = 0$ and $\tilde{l}_i$ be the line given by $\tilde{a}_i X + \tilde{b}_i Y + \tilde{c}_i Z = 0$ for some liftings $\tilde{a}_i, \tilde{b}_i, \tilde{c}_i \in k_\infty$ of $a_i, b_i, c_i \in k_m$. Then the functions $\tilde{f}_i := \tilde{l}_i/\tilde{l}_0$ are liftings of f_i. If $\tilde{\alpha}_{i1}, \tilde{\alpha}_{i2}, \tilde{\alpha}_{i3}$ are the intersections of $\tilde{l}_i$ with $\tilde{E}$, then we can compute the above regulator as follows. Choose a smooth lifting to $\tilde{E}$ of each element in $\mathscr{P}$, such that:

(i) The origin in $\tilde{E}(k_\infty)$ is the lifting of the origin in $E(k_m)$.

(ii) The elements $\tilde{\alpha}_{i1}$, $\tilde{\alpha}_{i2}$, $\tilde{\alpha}_{i3}$ are the liftings of α_{i1}, α_{i2}, α_{i3} for $1 \le i \le 3$.

Denote the set of these liftings by $\tilde{\mathscr{P}}$. By our formula, we have

$$\rho_{m,r}(f_1 \wedge f_2 \wedge f_3) = \sum_{\tilde{\mathfrak{c}} \in \tilde{\mathscr{P}}} \ell_{m,r}(\mathrm{res}_{\tilde{\mathfrak{c}}}(\tilde{f}_1 \wedge \tilde{f}_2 \wedge \tilde{f}_3)).$$

Since we assume that the lines are generic, there are no common zeros of the functions $\tilde{f}_i$. On the other hand,

$$\mathrm{res}_O(\tilde{f}_1 \wedge \tilde{f}_2 \wedge \tilde{f}_3) = -3(\tilde{b}_2 \wedge \tilde{b}_3 - \tilde{b}_1 \wedge \tilde{b}_3 + \tilde{b}_1 \wedge \tilde{b}_2).$$

Combining these, we obtain that the value $\rho_{m,r}(f_1 \wedge f_2 \wedge f_3)$ is equal to

$$\ell_{m,r}\Bigg(\sum_{\sigma \in C_3} (-1)^{|\sigma|}\bigg(-3\tilde{b}_{\sigma(2)} \wedge \tilde{b}_{\sigma(3)} + \sum_{1 \le j \le 3} \tilde{f}_{\sigma(2)}(\tilde{\alpha}_{\sigma(1)j}) \wedge \tilde{f}_{\sigma(3)}(\tilde{\alpha}_{\sigma(1)j})\bigg)\Bigg), \tag{9.3.1}$$

where C_3 is the subgroup of S_3 generated by the 3-cycle (123).

In fact, the above expression gives an explicit computation for $\rho_{m,r}(f_1 \wedge f_2 \wedge f_3)$ for any generic choice of functions f_1, f_2, f_3 which are good with respect to $\mathscr{P}$. The group law on the elliptic curve has the property that a divisor of degree 0 is the divisor of a rational function if and only if the divisor adds to 0 under the group law. This implies that the functions f_i can be written as products of functions of the form l/l_0, where l is a line, and an element in $k_m^\times$. Since $\rho_{m,r}$ vanishes on elements of the form $\lambda \wedge f \wedge g$, with $\lambda \in k_m^\times$, using the additivity of $\rho_{m,r}$ we obtain the expression for $\rho_{m,r}(f_1 \wedge f_2 \wedge f_3)$ using the one for $\rho_{m,r}((l_1/l_0) \wedge (l_2/l_0) \wedge (l_3/l_0))$. Our main theorem implies that the expression (9.3.1) is independent of the choices of the liftings of both E and the f_i's. When such global liftings of either the elliptic curve or the functions do not exist, we need to choose local liftings and add the defect for choosing different liftings on the intersections as in the formula (1.3.1).

9.4. ***Invariants of cycles on k_m.*** As in [Ünver 2021], this construction gives us an invariant of cycles. For a cycle of modulus m we expect the combination of $\rho_{m,r}$ for all $m < r < 2m$ to be a complete set of invariants for the rational equivalence class of a cycle. For the appropriate, yet to be defined, Chow group $\mathrm{CH}^2(k_m, 3)$, we expect that our regulators $\rho_{m,r}$ to give complete invariants for the infinitesimal part $\mathrm{CH}^2(k_m, 3)^\circ$. This section also generalizes Park's [2009] construction of regulators where the case of $r = m + 1$ is dealt with. Since this section is more or less a generalization of [Ünver 2021, Section 4], we do not go into the details and explain certain constructions in a slightly alternate way.

First, let us recall the definition of cubical higher Chow groups over a smooth k-scheme X/k [Bloch 1986]. Let $\square_k := \mathbb{P}^1_k \setminus \{1\}$ and $\square_k^n$ the n-fold product of $\square_k$ with itself over k, with the coordinate functions $y_1, \ldots, y_n$. For a smooth k-scheme X, we let $\square_X^n := X \times_k \square_k^n$. A codimension 1 face of $\square_X^n$ is a divisor F_i^a of the form $y_i = a$, for $1 \le i \le n$, and $a \in \{0, \infty\}$. A face of $\square_X^n$ is either the whole scheme $\square_X^n$ or an arbitrary intersection of codimension 1 faces. Let $\underline{z}^q(X, n)$ be the free abelian group on the set of codimension q, integral, closed subschemes $Z \subseteq \square_X^n$ which are *admissible*, i.e., which intersect each face properly on $\square_X^n$. For each codimension one face F_i^a, and irreducible $Z \in \underline{z}^q(X, n)$, we let $\partial_i^a(Z)$ be the cycle associated to the scheme $Z \cap F_i^a$. We let $\partial := \sum_{i=1}^n (-1)^n (\partial_i^\infty - \partial_i^0)$ on $\underline{z}^q(X, n)$, which gives a complex $(\underline{z}^q(X, \cdot), \partial)$. Dividing this complex by the subcomplex of degenerate cycles, we obtain Bloch's higher Chow group complex whose homology $\mathrm{CH}^q(X, n) := \mathrm{H}_n(z^q(X, \cdot))$ is the higher Chow group of X.

In order to work with a candidate for Chow groups of cycles on k_m, we need to work with cycles over k_∞ which have a certain finite reduction property. The following definitions are essentially from [Ünver 2021, Section 4.2]. Let $\overline{\square}_k := \mathbb{P}^1_k$, $\overline{\square}_k^n$, the n-fold product of $\overline{\square}_k$ with itself over k, and $\overline{\square}_{k_\infty}^n := \overline{\square}_k^n \times_k k_\infty$. We define a subcomplex $\underline{z}_f^q(k_\infty, \cdot) \subseteq \underline{z}^q(k_\infty, \cdot)$, as the subgroup generated by integral, closed subschemes $Z \subseteq \square_{k_\infty}^n$ which are admissible in the above sense and have *finite reduction*, i.e., $\bar{Z}$ intersects each $s \times \bar{F}$ properly on $\overline{\square}_{k_\infty}^n$, for every face F of $\square_{k_\infty}^n$. Here s denotes the closed point of the spectrum of k_∞ and for a subscheme $Y \subseteq \square_{k_\infty}^n$, $\bar{Y}$ denotes its closure in $\overline{\square}_{k_\infty}^n$. Modding out by degenerate cycles, we have a complex $z_f^q(k_\infty, \cdot)$. Fix $2 \le m < r < 2m$. Let η denote the generic point of the spectrum of k_∞. An irreducible cycle p in $\underline{z}_f^2(k_\infty, 2)$ is given by a closed point p_η of $\square_\eta^2$ whose closure $\bar{p}$ in $\overline{\square}_{k_\infty}^2$ does not meet $(\{0, \infty\} \times \overline{\square}_{k_\infty}) \cup (\overline{\square}_{k_\infty} \times \{0, \infty\})$. Let $\tilde{p}$ denote the normalization of $\bar{p}$ and T denote the underlying set of the closed fiber $\tilde{p} \times_{k_\infty} s$ of $\tilde{p}$. For every $s' \in T$, and $1 \le i$, define $\ell_{\tilde{p},s',i} : \hat{\mathcal{O}}_{\tilde{p},s'}^\times \to k(s')$ by the formula

$$\ell_{\tilde{p},s',i}(y) := \frac{1}{i} \operatorname{res}_{\tilde{p},s'} \frac{1}{t^i} d\log(y).$$

Let

$$l_{m,r}(p) := \sum_{s' \in T} \mathrm{Tr}_k \sum_{1 \le i \le r-m} i \cdot (\ell_{\tilde{p},s',r-i} \wedge \ell_{\tilde{p},s',i})(y_1 \wedge y_2). \tag{9.4.1}$$

Note the similarity with Definition 3.0.2.

Definition 9.4.1. We define the regulator $\rho_{m,r} : \underline{z}_f^2(k_\infty, 3) \to k$ as the composition $l_{m,r} \circ \partial$.

Exactly as in [Ünver 2021], one proves that the regulator above vanishes on boundaries and products, is alternating and has the same value on cycles which are congruent modulo (t^m). We state only this last property, which is the most important one, in detail.

Suppose that Z_i for $i = 1, 2$ are two irreducible cycles in $\underline{z}_f^2(k_\infty, 3)$. We say that Z_1 and Z_2 are equivalent modulo t^m if the following conditions (M_m) hold:

(i) $\bar{Z}_i/k_\infty$ are smooth with $(\bar{Z}_i)_s \cup \left(\bigcup_{j,a} |\partial_j^a Z_i|\right)$ a strict normal crossings divisor on $\bar{Z}_i$.

(ii) $\bar{Z}_1|_{t^m} = \bar{Z}_2|_{t^m}$.

Then we have:

Theorem 9.4.2. *For $m < r < 2m$, we define a regulator $\rho_{m,r} : \underline{z}_f^2(k_\infty, 3) \to k$. If Z_a, for $a \in k_\infty^\flat$, is the dilogarithmic cycle given by the parametric equation $(1-z, z, z-a)$ then*

$$\rho_{m,r}(Z_a) = \ell i_{m,r}([a]).$$

If $Z_i \in \underline{z}_f^2(k_\infty, 3)$, for $i = 1, 2$, satisfy the condition (M_m), then they have the same infinitesimal regulator value

$$\rho_{m,r}(Z_1) = \rho_{m,r}(Z_2).$$

Proof. The second part of the proof is exactly as in [Ünver 2021] and is based on Corollary 8.2.1.

In order to compute $\rho_{m,r}(Z_a)$, we note that $\partial(Z_a) = (1-a, a)$ and

$$\rho_{m,r}(Z_a) = (l_{m,r} \circ \partial)(Z_a) = l_{m,r}(1-a, a) = \sum_{1 \le i \le r-m} i(\ell_{r-i} \wedge \ell_i)(1-a, a) = (\ell_{m,r} \circ \delta)(a) = \ell i_{m,r}([a]). \qquad \square$$

As we remarked above, we expect the invariants $\rho_{m,r}$ for $m < r < 2m$ to give a full set of invariants in the infinitesimal part of a yet to be defined Chow group $\mathrm{CH}^2(k_m, 3)$.

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Wide moments of L-functions I: Twists by class group characters of imaginary quadratic fields

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We calculate certain "wide moments" of central values of Rankin–Selberg L-functions $L(\pi \otimes \Omega, \frac{1}{2})$ where π is a cuspidal automorphic representation of GL_2 over $\mathbb{Q}$ and Ω is a Hecke character (of conductor 1) of an imaginary quadratic field. This moment calculation is applied to obtain "weak simultaneous" nonvanishing results, which are nonvanishing results for different Rankin–Selberg L-functions where the product of the twists is trivial.

The proof relies on relating the wide moments of L-functions to the usual moments of automorphic forms evaluated at Heegner points using Waldspurger's formula. To achieve this, a classical version of Waldspurger's formula for general weight automorphic forms is derived, which might be of independent interest. A key input is equidistribution of Heegner points (with explicit error terms), together with nonvanishing results for certain period integrals. In particular, we develop a soft technique for obtaining the nonvanishing of triple convolution L-functions.

1. Introduction

Determining the moments of central values of families of automorphic L-functions has a long history starting with the work of Hardy and Littlewood on the Riemann zeta function

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^2 dt \sim T \log T,$$

as $T \to \infty$; see [Titchmarsh 1986, Chapter VII]. By now, there exist precise conjectures for all moments of families of L-functions [Conrey et al. 2005] with fascinating connections to random matrix theory [Keating and Snaith 2000]. These moment conjectures are of deep arithmetic importance through their connections to the important topics of nonvanishing and subconvexity (see, e.g., [Blomer et al. 2018]), which in turn are connected to, respectively, rational points on elliptic curves (via the B–S-D conjectures, see [Kolyvagin 1988]) and equidistribution problems (via the Waldspurger formula, see [Michel and Venkatesh 2006]).

In this paper, we will calculate what we call *wide moments* of central values of Rankin–Selberg L-functions $L(\pi \otimes \Omega, \frac{1}{2})$, where π is a cuspidal automorphic representation of GL_2 with trivial central character of even lowest weight k_π and Ω is a Hecke character of an imaginary quadratic field K with infinity type $\alpha \mapsto (\alpha/|\alpha|)^k$ for some even integer $k \geq k_\pi$. More precisely, we will study the "canonical" square roots of the central values via their connections to Heegner periods as in the work of

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Waldspurger [1985]. We will use these moment calculations to obtain a number of new nonvanishing results of a certain kind that we call *weak simultaneous nonvanishing*; see Section 1C for the statements. In view of the Bloch–Kato conjectures, these nonvanishing results imply (in the holomorphic case) vanishing for certain twisted Selmer groups; see Corollary 7.6 below.

1A. ***Wide moments of L-functions.*** This paper is the first in a series of papers concerned with obtaining asymptotic evaluations of *wide moments* of automorphic L-function. In all of the cases we will consider, these wide moments are connected to the usual moments of certain underlying periods of automorphic forms (in the case of this paper, through the Waldspurger formula), which are much better behaved than the L-functions themselves. In particular, we can use a variety of more geometrically flavored methods to study the distributional properties of these periods.

The abstract setup is as follows: Given a finite abelian group G with (unitary) dual $\hat{G}$, we define

$$\mathbf{Wide}(\hat{G}, n) := \{(\chi_1, \dots, \chi_n) \in (\hat{G})^n : \chi_1 \cdots \chi_n = 1\}. \tag{1-1}$$

Given maps $L_1, \dots, L_n : G \to \mathbb{C}$ with Fourier transforms

$$\widehat{L}_i : \hat{G} \to \mathbb{C},\ \chi \mapsto \frac{1}{|G|} \sum_{g \in G} L_i(g)\bar{\chi}(g), \quad \text{for } i = 1, \dots, n,$$

we define the *wide moment* of $\widehat{L}_1, \dots, \widehat{L}_n$, as

$$\sum_{(\chi_i)_{1 \le i \le n} \in \mathbf{Wide}(\hat{G}, n)} \prod_{i=1}^{n} \widehat{L}_i(\chi_i). \tag{1-2}$$

Note that for $n = 2$ and $\widehat{L}_1 = \widehat{L}_2$ equivariant with respect to inverses $\big($i.e., $\widehat{L}_1(\chi^{-1}) = \overline{\widehat{L}_1(\chi)}\big)$, we recover the usual second moment. The key point is that (1-2) is equal to

$$\frac{1}{|G|} \sum_{g \in G} \prod_{i=1}^{n} L_i(g), \tag{1-3}$$

(for $n = 2$ this is exactly Plancherel). A nice way to see that (1-2) is equal to (1-3) is to use that the Fourier transform takes products to convolutions, and (1-2) is exactly the n-fold convolution product of $\widehat{L}_1, \dots, \widehat{L}_n$ evaluated at $\chi = 1$. In the setting of automorphic L-functions, we can in many cases calculate the wide moments (1-2) using that the dual moments (1-3) are much better behaved.

The first example in the literature of an asymptotic evaluation of a (higher) wide moment of automorphic L-functions seems to be the work of Bettin [2019] on Dirichlet L-functions (note that here the terminology "iterated moments" is used):

$$\frac{1}{(p-2)^{n-1}} \sideset{}{^*}\sum_{(\chi_i) \in \mathbf{Wide}(p,n)} \left| L\Big(\chi_1, \frac{1}{2}\Big) \right|^2 \cdots \left| L\Big(\chi_n, \frac{1}{2}\Big) \right|^2 \\ = c_{n,n}(\log p)^n + c_{n,n-1}(\log p)^{n-1} + \cdots + c_{n,0} + O(p^{-\delta}), \tag{1-4}$$

as $p \to \infty$ with p prime, for some $\delta > 0$ and $c_{n,i} \in \mathbb{R}$. Here, the asterisks on the sum means that the summation is restricted to primitive Dirichlet characters, and we set $\mathbf{Wide}(p, n) := \mathbf{Wide}(\widehat{(\mathbb{Z}/p\mathbb{Z})^\times}, n)$.

This result is a corollary of the moment calculation of the *Estermann function* (which we think of as the underlying automorphic periods in this case). Another related result is the calculation of Chinta [2005] corresponding to a wide moment with $n = 3$ for quadratic Dirichlet L-functions.

The asymptotic evaluation (1-4) was later generalized (with an extra average over the modulus q) by the author [Nordentoft 2021, Corollary 1.9] to the wide moments of

$$\widehat{L}_1(\chi) = \cdots = \widehat{L}_n(\chi) = L\big(f \otimes \chi, \tfrac{1}{2}\big) \quad \text{for } \chi : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times,$$

with f a fixed holomorphic newform of even weight. In [Nordentoft 2021], the underlying automorphic periods are the *additive twists* of f (which reduces to *modular symbols* for $k = 2$). Furthermore, in a recent joint work between Drappeau and the author, all moments of additive twists of level 1 Maaß forms are calculated [Drappeau and Nordentoft 2022, Corollary 1.9].

The methods used to calculate the wide moments mentioned above are, respectively, a classical approximate functional equation approach [Bettin 2019], multiple Dirichlet series [Chinta 2005], spectral theory [Nordentoft 2021] (see also [Petridis and Risager 2018a]), and dynamical systems [Drappeau and Nordentoft 2022] (building on [Bettin and Drappeau 2022]).

1B. ***Main idea.*** Let us describe the main moment calculation of this paper in the simplest possible setup. Let $f : \mathbb{H} \to \mathbb{C}$ be a classical Hecke–Maaß eigenform of weight 0 and (for simplicity) level 1 (i.e., a real-analytic joint eigenfunction for the hyperbolic Laplace operator and the Hecke operators which is invariant under $\mathrm{PSL}_2(\mathbb{Z})$). Let K be an imaginary quadratic field of discriminant $D_K < -6$ with class group Cl_K. Given a class group character $\chi \in \widehat{\mathrm{Cl}}_K$, we denote by $L(f \otimes \chi, s)$ (the finite part of) the Rankin–Selberg L-function $L(f \otimes \theta_\chi, s)$, where θ_χ is the theta series associated to χ of weight 1 and level $|D_K|$ (equivalently, we have $L(f \otimes \chi, s) = L(\pi_K \otimes \pi_\chi, s)$, where π_K denotes the base change to $\mathrm{GL}_2(\mathbb{A}_K)$ of the automorphic representation corresponding to f and π_χ is the automorphic representation of $\mathrm{GL}_1(\mathbb{A}_K)$ corresponding to χ). A deep formula of Zhang [2001; 2004] gives the relation

$$\Bigg| \sum_{[\mathfrak{a}] \in \mathrm{Cl}_K} f(z_{[\mathfrak{a}]})\chi([\mathfrak{a}]) \Bigg|^2 = |c_f|^2 |D_K|^{1/2} L\big(f \otimes \chi, \tfrac{1}{2}\big), \tag{1-5}$$

where $\chi \in \widehat{\mathrm{Cl}}_K$ is a class group character of K, $z_{[\mathfrak{a}]} \in \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ denotes the Heegner point associated to $[\mathfrak{a}] \in \mathrm{Cl}_K$, and $c_f > 0$ is a constant depending on f (but independent of χ). Using this relation together with orthogonality of characters and equidistribution of Heegner points, Michel and Venkatesh [2007] calculated the first moment of $L\big(f \otimes \chi, \frac{1}{2}\big)$, which they combined with subconvexity to obtain quantitative nonvanishing for these central values. This idea has since been generalized in many directions to obtain a variety of nonvanishing results [Dittmer et al. 2015; Burungale and Hida 2016; Khayutin 2020; Templier 2011a; 2011b].

We observe that (1-5) is exactly saying that the Fourier transform of

$$\mathrm{Cl}_K \ni [\mathfrak{a}] \mapsto |\mathrm{Cl}_K| f(z_{[\mathfrak{a}]})$$

is given by a map of the form

$$\widehat{\mathrm{Cl}}_K \ni \chi \mapsto \varepsilon_{f,\chi} c_f |D_K|^{1/4} \big| L\big(f \otimes \chi, \tfrac{1}{2}\big)\big|^{1/2},$$

for some $\varepsilon_{f,\chi}$ of norm 1. Thus, by the Fourier equality (1-2)=(1-3) and equidistribution of Heegner points due to Duke [1988], we conclude that for level 1 Hecke–Maaß eigenforms $f_1, \dots, f_n$, we have

$$\begin{aligned}\frac{|D_K|^{n/4}}{|\mathrm{Cl}_K|^n} \sum_{(\chi_i)\in\mathbf{Wide}(K,n)} \prod_{i=1}^{n} \varepsilon_{f_i,\chi_i} c_{f_i} \Big|L\Big(f_i\otimes\chi_i, \frac{1}{2}\Big)\Big|^{1/2} &= \frac{1}{|\mathrm{Cl}_K|} \sum_{[\mathfrak{a}]\in\mathrm{Cl}_K} \prod_{i=1}^{n} f_i(z_{[\mathfrak{a}]}) \\ &= \Big\langle \prod_{i=1}^{n} f_i, \frac{3}{\pi} \Big\rangle + o(1), \end{aligned} \tag{1-6}$$

as $|D_K| \to \infty$, where we used the short-hand $\mathbf{Wide}(K,n) := \mathbf{Wide}(\widehat{\mathrm{Cl}}_K, n)$. This shows immediately that if $\langle \prod_{i=1}^{n} f_i, 1\rangle \neq 0$, then there exists

$$(\chi_1, \dots, \chi_n) \in \mathbf{Wide}(K,n) \text{ such that } \prod_{i=1}^{n} L\big(f_i\otimes\chi_i, \tfrac{1}{2}\big) \neq 0.$$

We call the above *weak simultaneous nonvanishing*; see Section 2 for some background on this type of nonvanishing.

1C. *Nonvanishing results.* The above proof sketch already gives new results. We will, however, push these ideas further in several aspects. First of all, we deal with general weight forms (holomorphic or Maaß), which requires us to develop explicit Waldspurger type formulas in these cases (see Section 4), which might be of independent interest. In particular, this requires studying Hecke characters which ramify at ∞, which leads to some complications. Secondly, we will obtain an explicit error term in (1-6), which requires bounding certain inner-products involving powers of the Laplace operator; see Section 5. This allows us to obtain nonvanishing results with some uniformity in the spectral aspect. In particular, in the case of width $n = 2$, we obtain the following improved version of [Michel and Venkatesh 2006, Theorem 1] allowing general weights and with a uniform lower bound for D_K in terms of the spectral parameter:

Corollary 1.1. *Let f be either a Hecke–Maaß cusp form of spectral parameter t_f and level 1 or a cuspidal holomorphic Hecke eigenform of weight k_f and level 1. Let k be a positive even integer with the further requirement that $k \geq k_f$ if f is holomorphic. Put $T = |t_f| + k + 1$ in the Maaß case and $T = k + 1$ in the holomorphic case.*

Then for any $\varepsilon > 0$, there exists a constant $c = c(\varepsilon) > 0$ such that for any imaginary quadratic field K with discriminant $|D_K| \geq cT^{22+\varepsilon}$, we have

$$\#\{\chi \in \widehat{\mathrm{Cl}}_K : L\big(f\otimes\chi\Omega_K, \tfrac{1}{2}\big) \neq 0\} \gg_f \begin{cases} |D_K|^{1/1058} & \text{if } f \text{ is holomorphic,} \\ |D_K|^{1/2648} & \text{if } f \text{ is Maaß,} \end{cases}$$

where Ω_K is a Hecke character of K of conductor 1 and ∞-type $\alpha \mapsto (\alpha/|\alpha|)^k$.

Remark 1.2. We obtain similar results for general squarefree levels N; see Corollary 7.1.

The case of width $n = 3$ is also very appealing, as in this case the triple period $\langle f_1 f_2 f_3, 1\rangle$ is related to *triple convolution L-functions* via the Ichino–Watson formula [Watson 2002; Ichino 2008]. This leads to the following nonvanishing result for level 1 Maaß forms:

Corollary 1.3. *Let f_1 be a fixed Hecke–Maaß cusp form of level 1. Then for any $\varepsilon > 0$, there exists a constant $c = c(f_1, \varepsilon) > 0$ such that for any $T \geq c$, we have for all but $O_\varepsilon(T^{2\varepsilon})$ Hecke–Maaß cusp forms f_2 of level 1 with $|t_{f_2} - T| \leq T^\varepsilon$ that there exists a Hecke–Maaß cusp form f_3 not equal to f_2 with $|t_{f_3} - T| \leq T^\varepsilon$ such that the following holds: We have $L\big(f_1 \otimes f_2 \otimes f_3, \tfrac{1}{2}\big) \neq 0$ and for any imaginary quadratic field K with $|D_K| \geq cT^{35+\varepsilon}$,*

$$\#\big\{\chi_1, \chi_2 \in \widehat{\mathrm{Cl}}_K : L\big(f_1 \otimes \chi_1, \tfrac{1}{2}\big) L\big(f_2 \otimes \chi_2, \tfrac{1}{2}\big) L\big(f_3 \otimes \chi_1\chi_2, \tfrac{1}{2}\big) \neq 0\big\} \gg_T |D_K|^{1/1766}.$$

In the case of holomorphic forms, we can obtain nonvanishing for a general width n (stated here in the simplest case of level 1, we refer to Corollary 7.5 for a more general statement).

Corollary 1.4. *Let $n \geq 1$, $k_1, \dots, k_n \in 2\mathbb{Z}_{>0}$, and put $k = \sum_i k_i$. For $i = 1, \dots, n$, let $g_i \in \mathcal{S}_{k_i}(1)$ be a cuspidal holomorphic Hecke eigenform of level 1. Then for each $\varepsilon > 0$, there exists a constant $c = c(\varepsilon) > 0$ such that the following holds: For any imaginary quadratic field K with $|D_K| \geq ck^{45+\varepsilon}$,*

$$\begin{aligned} \#\big\{(\chi_1, \dots, \chi_{n+1}) \in \mathbf{Wide}(K, n+1),\ &\text{level 1 Hecke eigenforms } g \in \mathcal{S}_k(1) : \\ &L\big(g_1 \otimes \chi_1\Omega_{i,K}, \tfrac{1}{2}\big) \cdots L\big(g_n \otimes \chi_n\Omega_{n,K}, \tfrac{1}{2}\big) L\big(g \otimes \chi_{n+1}\Omega_{n+1,K}, \tfrac{1}{2}\big) \neq 0\big\} \\ \gg_k |D_K|^{(n+1)/2115}, \end{aligned}$$

where $\Omega_{i,K}$ are Hecke characters of K of ∞-type $x \mapsto (x/|x|)^{k_i}$ for $i = 1, \dots, n$ and $\Omega_{n+1,K} = \prod_{i=1}^n \Omega_{i,K}$.

Remark 1.5. Note that it follows, in particular, that the respective nonvanishing sets in Corollaries 1.1, 1.3 and 1.4 are *nonempty* as soon as, respectively, $|D_K| \geq cT^{22+\varepsilon}$, $|D_K| \geq cT^{35+\varepsilon}$ and $|D_K| \geq ck^{45+\varepsilon}$.

Remark 1.6. The fact that we can obtain nonvanishing results for general width n in the holomorphic case relies crucially on the finite dimensionality of the space of holomorphic forms of fixed level and weight. This clearly fails for nonholomorphic Maaß forms, which is the reason we cannot obtain nonvanishing results beyond the cases of two and three characters in the Maaß case. Notice that if we apply Corollary 1.4 with $n = 2$, we obtain an improved version of Corollary 1.3 in the case of holomorphic forms.

1D. *Main moment calculation.* The above nonvanishing results are all corollaries of our main L-function calculation. To state this, denote by $\mathcal{B}_k^*(N)$ the set of L^2-normalized Hecke–Maaß newforms of level N and even weight $k \geq 0$ (i.e., raising operators applied to either classical Hecke–Maaß newforms of weight 0 and level N or to $y^{k'/2}g$ with $g \in \mathcal{S}_{k'}(N)$ a holomorphic cuspidal newform of even weight $k' \leq k$). Then we have the following moment calculation:

Theorem 1.7. *Let $N \geq 1$ be a fixed squarefree integer and $n \geq 1$. For $i = 1, \dots, n$, let π_i be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ of conductor N with trivial central character, spectral parameter t_{π_i} and even lowest weight k_{π_i}. Let $k_1, \dots, k_n \in 2\mathbb{Z}$ be integers such that $|k_i| \geq k_{\pi_i}$ and $\sum_i k_i = 0$.*

Let $|D_K| \to \infty$ transverse a sequence of discriminants of imaginary quadratic fields K such that all primes dividing N split in K. For each K, pick Hecke characters $\Omega_{i,K}$ with infinite types $x \mapsto (x/|x|)^{k_i}$ such that $\prod_i \Omega_{i,K}$ is the trivial Hecke character.

Then we have for $f_i \in \mathcal{B}^*_{k_i}(N)$ *belonging to* π_i *and any* $\varepsilon > 0$,

$$\sum_{(\chi_i)_{1\le i\le n}\in \mathbf{Wide}(K,n)} \prod_{i=1}^{n} \varepsilon_{\chi_i,f_i} c_{f_i} L\big(\pi_i \otimes \chi_i \Omega_{i,K}, \tfrac{1}{2}\big)^{1/2} = \frac{|\mathrm{Cl}_K|^n}{|D_K|^{n/4}}\bigg(\bigg\langle \prod_{i=1}^{n} f_i, 1\bigg\rangle + O_\varepsilon\bigg(\bigg\| \prod_{i=1}^{n} f_i \bigg\|_2 |D_K|^{-1/16} T^{5/2} n^{15/4} (T|D_K|n)^{\varepsilon}\bigg)\bigg), \tag{1-7}$$

where $T = \max_{i=1,\dots,n} |k_i| + |t_{\pi_i}| + 1$, *the weights* ε_{χ,f_i} *are all of norm* 1 *and* c_{f_i} *are certain constants depending only on* f_i.

Remark 1.8. We obtain a slightly more general statement that applies to old-forms as well, meaning that we allow for the automorphic representations π_i to have different conductors. Furthermore, we obtain an improved error term in the case of holomorphic forms and/or in the case of level 1. We refer to Theorem 6.3 for details (including the exact values of the constants c_f). As an application, we can also calculate a related "diagonal wide moment"; see Corollary 6.6.

The plan of the paper is as follows. In Section 2, we will introduce the notion of *weak simultaneous nonvanishing*. Section 3 provides the necessary background on imaginary quadratic fields and automorphic forms. Section 4 proves an explicit and classical Waldspurger type formula for general weight automorphic forms. In Section 5, we will prove two technical lemmas: one on the norm of powers of the hyperbolic Laplacian and one on a lower bound for the L^2-norm of a product of automorphic forms. In Section 6, we will prove our main moment calculation. Finally, Section 7 proves the nonvanishing of certain automorphic periods, which combined with our moment calculation, yields weak simultaneous nonvanishing results.

2. Weak simultaneous nonvanishing

We will call the nonvanishing results proved in the present paper *weak simultaneous nonvanishing*. This terminology is referring to the fact that we show nonvanishing of twists of different L-functions with some "algebraic dependence" on the twists (their product is trivial). Ideally, of course we would like to show nonvanishing for the same character. Some results in this direction have been obtained by Saha and Schmidt [2013, Theorem 1] in the case of two holomorphic forms using techniques from Siegel modular forms. Outside of this case, however, simultaneous nonvanishing seems out of reach with current methods.

Let us start by considering the simplest case, $n = 2$. This means that we are studying the nonvanishing of two maps $L_1, L_2 : G \to \mathbb{C}$, where G is a finite abelian group. If both L_1 and L_2 are nonvanishing for more than 50% of $g \in G$, then by the pigeonhole principle there is some $g \in G$ such that $L_1(g)L_2(g) \neq 0$. But clearly we can construct examples where L_1, L_2 vanish for exactly 50% of $g \in G$ but there is no simultaneous nonvanishing.

More generally, consider $L_1, \dots, L_n : G \to \mathbb{C}$. Then we say that $L_1, \dots, L_n$ are *weakly simultaneously nonvanishing* if

$$\big\{(g_1, \dots, g_n) \in \mathbf{Wide}(G, n) : L_i(g_i) \neq 0 \text{ for } i = 1, \dots, n\big\} \neq \varnothing.$$

Recall that by (1-1) this means that there exist $g_1, \dots, g_n \in G$ such that

$$g_1 \cdots g_n = 1_G \quad \text{and} \quad L_1(g_1) \cdots L_n(g_n) \neq 0.$$

We think of this as expressing that we can find nonvanishing for $L_1, \dots, L_n$ with some “algebraic dependence”. This is interesting since most nonvanishing results for automorphic L-functions are obtained by using the method of mollification, which gives no information about the algebraic structure of the nonvanishing set. Of course, if all of the $L_1, \dots, L_n$ vanish on a very large percentage of elements of G, then one gets a weak simultaneous nonvanishing for purely combinatorial reasons. In most cases, this is not the case, which we make precise as follows:

Proposition 2.1. *Let $n \geq 2$ be an integer and $0 \leq c \leq 1$. Then there exists a finite abelian group G and maps $L_1, \dots, L_n : G \to \mathbb{C}$ satisfying*

$$\#\{g \in G : L_i(g) \neq 0\} \geq c|G|, \quad \text{where } i = 1, \dots, n,$$

with no weak simultaneous nonvanishing if and only if $c \leq \frac{1}{2}$.

Proof. Assume first of all that $c > \frac{1}{2}$. Then if $g_1, \dots, g_{n-2}$ are such that $L_i(g_i) \neq 0$ for $i = 1, \dots, n-2$. Then, again by the pigeonhole principle, there is at least one $g \in G$ such that $L_{n-1}(g) \neq 0$ and $L_n((g_1 \cdots g_{n-1}g)^{-1}) \neq 0$ (since all of the elements $(g_1 \cdots g_{n-1}g)^{-1}$ are different as $g \in G$ varies).

On the other hand if $c \leq \frac{1}{2}$, then we can consider any finite abelian group G with a subgroup H of index 2. Now we let $L_i(g) \neq 0$ if and only if $g \in H$ for $i = 1, \dots, n-1$, and let L_n be nonvanishing on the complement of H. In this case, it is easy to check that there is no weak simultaneous nonvanishing. $\square$

This shows that we need to know nonvanishing for at least 50% of the maps L_i in order to get weak simultaneous nonvanishing for purely combinatorial reasons. This is very far from being known in the case of the Rankin–Selberg L-functions studied in this paper, as even a positive proportion of nonvanishing seems out of reach with current methods; see [Michel and Venkatesh 2007] and [Templier 2011a].

3. Background

3A. ***Different incarnations of the class group.*** Let K be an imaginary quadratic field of discriminant $D < -6$. Denote by $\mathcal{I}_K$ the group of integral fractional ideals of K, $\mathcal{P}_K$ the subgroup of principal fractional ideals and $\mathrm{Cl}_K = \mathcal{I}_K/\mathcal{P}_K$ the class group of K, which we know from Gauß is a finite group. Furthermore, we have Siegel's bound

$$|\mathrm{Cl}_K| \gg_\varepsilon |D_K|^{1/2-\varepsilon} \tag{3-1}$$

for any $\varepsilon > 0$ where the implied constant is ineffective.

Given a fractional ideal $\mathfrak{a} \in \mathcal{I}_K$, we denote by $[\mathfrak{a}] \in \mathrm{Cl}_K$ the corresponding ideal class. We denote by $[\alpha_1, \alpha_2]$ the ideal generated by $\alpha_1, \alpha_2 \in K$ over $\mathbb{Z}$ and by $\widehat{\mathrm{Cl}}_K$ the group of class group characters, i.e., group homomorphisms $\chi : \mathrm{Cl}_K \to \mathbb{C}^\times$.

Let $\mathbb{A}_K^\times$, respectively, $\mathbb{A}_{K,\mathrm{fin}}^\times$, denote the idéles, respectively, finite idéles of K, and let $\widehat{\mathbb{O}}_K^\times = \prod_p \mathbb{O}_p^\times$ denote the standard maximal compact subgroup of $\mathbb{A}_{K,\mathrm{fin}}^\times$. Then we have the natural isomorphisms

$$\mathcal{I}_K \cong \mathbb{A}_{K,\mathrm{fin}}^\times / \widehat{\mathbb{O}}_K^\times \quad \text{and} \quad \mathrm{Cl}_K \cong K^\times \backslash \mathbb{A}_{K,\mathrm{fin}}^\times / \widehat{\mathbb{O}}_K^\times. \tag{3-2}$$

Given $\mathfrak{a} \in \mathcal{I}_K$, we denote by $\hat{\mathfrak{a}} \in \mathbb{A}_{K,\mathrm{fin}}^\times$ any lift of the corresponding element of $\mathbb{A}_{K,\mathrm{fin}}^\times / \widehat{\mathbb{O}}_K^\times$ under the above isomorphism.

3A1. *Heegner forms.* We refer to [Darmon 1994] for a concise treatment of the following material. Let N be a squarefree integer such that all primes dividing N split completely in K. Consider a residue class $r \bmod 2N$ such that $r^2 \equiv D \bmod 4N$. For $(a, b, c) \in \mathbb{Z}^3$ having greatest common divisor equal to 1 and satisfying $b^2 - 4ac = D$, $a \equiv 0 \bmod N$, and $b \equiv r \bmod 2N$, we denote by $[a, b, c]$ the integral binary quadratic form

$$Q(x, y) = ax^2 + bxy + cy^2. \tag{3-3}$$

We call such a quadratic form a *Heegner form of level N* and *orientation r* and denote by $\mathcal{Q}_D(N, r)$ the set of all such forms, which carries an action of the Hecke congruence subgroup $\Gamma_0(N)$ via coordinate transformation. It is a well-known fact extending Gauß that the map $\Gamma_0(N)\backslash\mathcal{Q}_D(N, r) \to \mathrm{Cl}_K$ defined by

$$[a, b, c] \mapsto \left[a, \frac{-b + \sqrt{D}}{2}\right],$$

is a bijection.

Given a Heegner form $Q = [a, b, c] \in \mathcal{Q}_D(N, r)$, we define the associated *Heegner point* as

$$z_Q := \frac{-b + \sqrt{D}}{2a} \in \mathbb{H}. \tag{3-4}$$

This defines a map $\mathcal{Q}_D(N, r) \to \mathbb{H}$ which is equivariant with respect to the action $\Gamma_0(N)$ (acting via linear fractional transformation on $\mathbb{H}$). In particular, we get a map $\mathrm{Cl}_K \to \Gamma_0(N)\backslash\mathbb{H}$ using the above.

3A2. *Oriented embeddings.* Again let $(a, b, c) \in \mathbb{Z}^3$ have greatest common divisor equal to 1 and satisfy $b^2 - 4ac = D$, $a \equiv 0 \bmod N$, and $b \equiv r \bmod 2N$. Associated to the triple (a, b, c), we define an (algebra) embedding $\Psi : K \hookrightarrow \mathrm{Mat}_{2\times 2}(\mathbb{Q})$ by

$$\Psi(\sqrt{D}) := \begin{pmatrix} b & 2c \\ -2a & -b \end{pmatrix}. \tag{3-5}$$

This embedding satisfies

$$\Psi(K) \cap \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{Z}) : N \mid c \right\} = \Psi(\mathbb{O}_K),$$

where $\mathbb{O}_K$ denotes the ring of integers of K. This means that Ψ is an *optimal embedding of level N* and *orientation r*. Conversely, every oriented optimal embedding of level N arises from such a triple of integers $(a, b, c) \in \mathbb{Z}^3$. Denote by $\mathcal{E}_D(N, r)$ the set of all such embeddings. The congruence subgroup $\Gamma_0(N)$ acts on $\mathcal{E}_D(N, r)$ by conjugation, namely,

$$(\gamma \cdot \Psi)(x + \sqrt{D}y) := \gamma^{-1} \Psi(x + \sqrt{D}y)\gamma$$

for $\gamma \in \Gamma_0(N)$.

There is a natural bijection between oriented optimal embeddings Ψ of level N and orientation r, as in (3-5), and Heegner forms $Q = [a, b, c]$, as in (3-3) (since these are both completely determined by $(a, b, c) \in \mathbb{Z}^3$), which is equivariant with respect to the action of $\Gamma_0(N)$. By the above, we have a bijection

$$\Gamma_0(N)\backslash\mathscr{E}_D(N, r) \to \mathrm{Cl}_K. \tag{3-6}$$

Given an optimal embedding Ψ of level N, we can extend it to an (algebra) embedding

$$\Psi_{\mathbb{A}} : \mathbb{A}_K \to \mathrm{Mat}_{2\times 2}(\mathbb{A})$$

by tensoring (over $\mathbb{Q}$) by $\mathbb{A}$. The local components of $\Psi_{\mathbb{A}}$ are defined as follows: If p is a prime of $\mathbb{Q}$ which is inert in K with $p\mathbb{O}_K = \mathfrak{p}$, then $K \otimes \mathbb{Q}_p \cong K_{\mathfrak{p}}$; and thus we get an embedding $\Psi_p : K_{\mathfrak{p}} \to \mathrm{Mat}_{2\times 2}(\mathbb{Q}_p)$ given by

$$K \otimes \mathbb{Q}_p \ni x \otimes y \mapsto \Psi(x) \otimes y \in \mathrm{Mat}_{2\times 2}(\mathbb{Q}_p),$$

defined up to the choice of isomorphism $K \otimes \mathbb{Q}_p \cong K_p$ (similarly for the inert infinite place). If p is ramified with $p\mathbb{O}_K = \mathfrak{p}^2$, then $K \otimes \mathbb{Q}_p \cong K_{\mathfrak{p}}$; and we get a map $\Psi_p : K_{\mathfrak{p}} \to \mathrm{Mat}_{2\times 2}(\mathbb{Q}_p)$ by tensoring as in the inert case. Finally, if p is split in K with $p\mathbb{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$, then we have an algebra isomorphism $K \otimes \mathbb{Q}_p \cong K_{\mathfrak{p}} \times K_{\bar{\mathfrak{p}}}$ given by

$$K \otimes \mathbb{Q}_p \ni j_1 x + j_2 y \mapsto (x, y) \in K_{\mathfrak{p}} \times K_{\bar{\mathfrak{p}}}, \quad \text{with } x, y \in \mathbb{Q}_p, \tag{3-7}$$

where

$$j_1 = \frac{1 \otimes 1 + \sqrt{D} \otimes (\sqrt{D}\,)^{-1}}{2} \quad \text{and} \quad j_2 = \frac{1 \otimes 1 - \sqrt{D} \otimes (\sqrt{D}\,)^{-1}}{2}.$$

Here we consider $\sqrt{D}$ as an element of $\mathbb{Q}_p$ and use that $\mathbb{Q}_p \cong K_{\mathfrak{p}}$ as p splits in K. By using this, we get an algebra embedding $\Psi_p : K_{\mathfrak{p}} \times K_{\bar{\mathfrak{p}}} \to \mathrm{Mat}_{2\times 2}(\mathbb{Q}_p)$ by tensoring. Again this is well defined up to the choice of isomorphism $\mathbb{Q}_p \cong K_{\mathfrak{p}}$.

3B. ***Hecke characters of imaginary quadratic fields.*** Let K be an imaginary quadratic field of discriminant $D < -6$. In this paper, we will be working with Hecke characters of K of conductor 1, which (in the classical picture) are unitary characters $\chi : \mathscr{I}_K \to \mathbb{C}^\times$ such that for $(\alpha) \in \mathscr{P}_K$, we have $\chi((\alpha)) = \chi_\infty^{-1}(\alpha)$ for some character $\chi_\infty : \mathbb{C}^\times \to \mathbb{C}^\times$, which we call the ∞-type of χ. By considering the induced representation, we can see that given χ_∞ such that $\chi_\infty(-1) = 1$, we have exactly $|\mathrm{Cl}_K|$ Hecke characters of conductor 1 with ∞-type χ_∞; if χ_0 is any such Hecke character with ∞-type χ_∞, then the set of all such Hecke characters is given by $\{\chi_0\chi : \chi \in \widehat{\mathrm{Cl}}_K\}$. We will only be considering the ∞-types $\alpha \mapsto (\alpha/|\alpha|)^k$ for $k \in 2\mathbb{Z}$.

Given a Hecke character χ as above with ∞-type χ_∞, we get, using the isomorphism (3-2), an (idélic) Hecke character

$$\Omega : K^\times\backslash\mathbb{A}_K^\times/\mathbb{O}_K^\times \to \mathbb{C}^\times.$$

The above conditions translates to the fact that Ω is unramified at all finite places of K and the ∞-component Ω_∞ is equal to χ_∞.

Associated to a Hecke character χ as above with ∞-type $\alpha \mapsto (\alpha/|\alpha|)^k$, there is a theta series

$$\theta_\chi(z) := \sum_{\mathfrak{a} \text{ int. ideal of } \mathcal{O}_K} e^{2\pi i (\mathrm{N}\mathfrak{a}) z} (\mathrm{N}\mathfrak{a})^{k/2} \chi(\mathfrak{a}) \in \mathcal{M}_{k+1}(\Gamma_0(|D|), \chi_K),$$

which is a modular form of weight $k+1$, level $|D|$, and nebentypus equal to the quadratic character χ_K associated to K via class field theory. Furthermore, we know that θ_χ is noncuspidal exactly if $k = 0$ and χ is a genus character of the class group of K; see [Iwaniec 1997, Theorem 12.5]. Recall that this is an example of automorphic induction from GL_1/K to $\mathrm{GL}_2/\mathbb{Q}$.

3C. ***Automorphic forms.*** In this section, we follow [Bump 1997, Chapters 2–3]. Let $L^2(\Gamma_0(N), k)$ denote the L^2-space of automorphic functions of level N and weight $k \in 2\mathbb{Z}$. That is, measurable maps $f : \mathbb{H} \to \mathbb{C}$ satisfying:

- The *automorphic condition* of weight k and level N
$$f(\gamma z) = j_\gamma(z)^k f(z),$$
for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, where
$$j_\gamma(z) := \frac{j(\gamma, z)}{|j(\gamma, z)|}, \quad \text{with } j(\gamma, z) = cz + d,$$
and
$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : N \mid c \right\}.$$
- The L^2*-condition*
$$\|f\|_2^2 := \langle f, f \rangle = \int_{\Gamma_0(N)\backslash\mathbb{H}} |f(z)|^2 \, d\mu(z) < \infty,$$
where $d\mu(z) = y^{-2} dx dy$ and $\langle \cdot, \cdot \rangle$ is the Petersson inner-product. Notice that the above integral is well defined since $|j_\gamma(z)| = 1$.

We have the weight k *raising* and *lowering operators* acting on $C^\infty(\mathbb{H})$, the space of smooth functions on $\mathbb{H}$, given by

$$R_k = (z - \bar{z}) \frac{\partial}{\partial z} + \frac{k}{2} \quad \text{and} \quad L_k = -(z - \bar{z}) \frac{\partial}{\partial \bar{z}} - \frac{k}{2}.$$

They define maps

$$R_k : L^2(\Gamma_0(N), k) \cap C^\infty(\mathbb{H}) \to L^2(\Gamma_0(N), k+2) \cap C^\infty(\mathbb{H}),$$
$$L_k : L^2(\Gamma_0(N), k) \cap C^\infty(\mathbb{H}) \to L^2(\Gamma_0(N), k-2) \cap C^\infty(\mathbb{H}),$$

which are adjoint in the sense that

$$\langle R_k f_1, f_2 \rangle = -\langle f_1, L_{k+2} f_2 \rangle \tag{3-8}$$

for $f_1 \in L^2(\Gamma_0(N), k) \cap C^\infty(\mathbb{H})$ and $f_2 \in L^2(\Gamma_0(N), k+2) \cap C^\infty(\mathbb{H})$. Furthermore, we have the product rule

$$R_{k_1+k_2}(f_1 f_2) = (R_{k_1} f_1) f_2 + f_1 (R_{k_2} f_2),$$

for $f_i \in L^2(\Gamma_0(N), k_i) \cap C^\infty(\mathbb{H})$, and similarly for the lowering operator.

The weight k *Laplacian* acting on $L^2(\Gamma_0(N),k)\cap C^\infty(\mathbb{H})$ is defined as

$$\Delta_k = -R_{k-2}L_k + \lambda\Big(\frac{k}{2}\Big) = -L_{k+2}R_k + \lambda\Big(-\frac{k}{2}\Big),$$

where $\lambda(s)=s(1-s)$. On $L^2(\Gamma_0(N),k)$, this defines a symmetric, unbounded operator with a unique self-adjoint extension which we also denote by Δ_k with some dense domain $D(\Delta_k)\subset L^2(\Gamma_0(N),k)$ (suppressing the level N in the notation).

A *Maaß form* of weight k and level N is a (necessarily real analytic) eigenfunction of Δ_k. Given a Maaß form f of eigenvalue λ we denote by $t_f := \sqrt{\lambda-\frac{1}{4}}$ the *spectral parameter* of f (if $\lambda > \frac{1}{4}$, we always pick the positive square root).

Denote by $\mathcal{S}_k(N)$ the vector space of weight k and level N (classical) holomorphic cusp forms. If $g\in\mathcal{S}_k(N)$, then it is easy to see that $y^{k/2}g$ is a Maaß form of weight k and level N of eigenvalue $\lambda(k/2)$. In fact, it can be show that any Maaß form of weight $k\geq 0$ and level N is of the form

$$R_{k-2}\cdots R_{k_0}y^{k_0/2}g, \quad \text{with } g\in\mathcal{S}_{k_0}(N) \text{ where } k_0\leq k \text{ and } k_0\equiv k \bmod 2$$

or

$$R_{k-2}\cdots R_0 f, \quad \text{with } f \text{ a Maaß form of weight 0 and level } N.$$

And similarly for $k<0$, now with lowering operators and antiholomorphic cusp forms.

Furthermore, we say that a Maaß form of weight k and level N is a *Hecke–Maaß eigenform* if it is an eigenfunction for the Hecke operators T_n with $(N,n)=1$ (which commute with the action of the raising and lowering operators), as well as the reflection operator

$$X: L^2(\Gamma_0(N),k)\to L^2(\Gamma_0(N),k), \quad (Xf)(z):=f(-\bar{z}).$$

Finally, we say that a Hecke–Maaß eigenform is a *Hecke–Maaß newform* if it is an eigenfunction for *all* Hecke operators T_n, with $n\geq 1$.

Denote by $\mathcal{B}^*_{k,\mathrm{hol}}(N)$ the set consisting of $f/\|f\|_2$, where $f=y^{k/2}g$ with $g\in\mathcal{S}_k(N)$ a (Hecke-normalized) holomorphic Hecke newform, and by $\mathcal{B}^*(N)$ the set consisting of $f/\|f\|_2$, with f a *nonconstant* (Hecke-normalized) Hecke–Maaß newform of weight 0 and level N. We will sometimes refer to these simply as (classical) “Maaß forms”. It follows from Atkin–Lehner theory that for $k\geq 0$, we have the following orthonormal basis consisting of Hecke–Maaß eigenforms for the subspace of $L^2(\Gamma_0(N),k)$ spanned by *nonconstant* Maaß forms of weight k and level N:

$$\mathcal{B}_k(N) := \bigcup_{dN'\mid N} \nu^*_{d,N'}R_{k-2}\cdots R_0\mathcal{B}^*(N') \cup \bigcup_{dN'\mid N}\ \bigcup_{\substack{0<k_0\leq k\\ k_0\equiv k \bmod 2}} \nu^*_{d,N'}R_{k-2}\cdots R_{k_0}\mathcal{B}^*_{k_0,\mathrm{hol}}(N'), \tag{3-9}$$

where $\nu^*_{d,N'}: L^2(\Gamma_0(N'),k)\to L^2(\Gamma_0(N),k)$ are defined by $(\nu^*_{d,N'}f)(z):=f(dz)$. If $k<0$, we have a similar basis now with lowering operators and antiholomorphic cusp forms.

Using (3-8), we see that for any $f\in\mathcal{B}_k(N)$, we have the following useful relation:

$$\|R_{k+2l}\cdots R_k f\|_2^2 = \|f\|_2^2 \prod_{j=0}^{l}\Big(\frac{k+2j-1}{2}+it_f\Big)\Big(\frac{k+2j-1}{2}-it_f\Big). \tag{3-10}$$

3C1. *Adélization of Maaß forms.* Given an element of $f \in L^2(\Gamma_0(N), k)$ we define a lift $\tilde{f} : \mathrm{GL}_2^+(\mathbb{R}) \to \mathbb{C}$ as

$$\tilde{f}(g) := j_g(i)^{-k} f(gi),$$

which satisfies

$$\tilde{f}(gk_\theta) = e^{ik\theta} \tilde{f}(g),$$

for all $\theta \in [0, 2\pi)$, where $k_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$ and $g \in \mathrm{GL}_2^+(\mathbb{R})$.

Now consider the following decomposition of $\mathrm{GL}_2(\mathbb{A})$ coming from strong approximation:

$$\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{Q}) K_0(N) \,\mathrm{GL}_2^+(\mathbb{R}), \tag{3-11}$$

where $\mathrm{GL}_2(\mathbb{Q})$ is embedded diagonally and

$$K_0(N) := \left\{ k \in \mathrm{GL}_2(\mathbb{A}) : k_\infty = 1,\ k_p = \begin{pmatrix} a_p & b_p \\ c_p & d_p \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p),\ c_p \in p^r \mathbb{Z}_p,\ p^r \| N \right\}.$$

Now we define the *adélization* of f as

$$\phi_f(g) = \phi_f(\gamma k g_\infty) := \tilde{f}(g_\infty),$$

which does not depend on the choice of decomposition

$$g = \gamma k g_\infty \in \mathrm{GL}_2(\mathbb{Q}) K_0(N) \,\mathrm{GL}_2^+(\mathbb{R}).$$

Given a Hecke–Maaß newform f, the adélization ϕ_f generates a unique cuspidal automorphic representation $\pi_f = \pi$ of $\mathrm{GL}_2(\mathbb{A})$. The infinity component of this representation π_∞ is a discrete series representation of lowest weight $k_\pi = k$ if f corresponds to a holomorphic Hecke newform of weight k. On the other hand if f is of weight 0 and nonconstant (i.e., corresponds to a classical Maaß form), then π_∞ is a principal series representation of lowest weight $k_\pi = 0$. We denote by t_π the spectral parameter t_f of f.

3C2. *Automorphic L-functions.* In general, associated to an automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$ we can define the (finite part of the) L-function $L(\pi, s)$ as a product over finite primes in terms of the Satake parameters and a completed version $\Lambda(\pi, s)$ satisfying a functional equation $\Lambda(\pi, s) = \varepsilon_\pi \Lambda(\check{\pi}, 1-s)$, where ε_π is of norm 1 (the root number) and $\check{\pi}$ is the contragredient of π. We refer to [Godement and Jacquet 1972] for details. Furthermore, given automorphic representations π_1, π_2, π_3 of $\mathrm{GL}_n(\mathbb{A})$, we will be interested in the Rankin–Selberg convolution L-function $L(\pi_1 \otimes \pi_2, s)$ (see [Jacquet et al. 1983]), the symmetric square L-function $L(\mathrm{sym}^2 \pi_1, s)$ (see [Bump 1997, Chapter 3.8]), and the triple convolution L-function $L(\pi_1 \otimes \pi_2 \otimes \pi_3, s)$ (see [Watson 2002]).

4. A classical version of Waldspurger's formula

In order to make our moment calculations explicit, we will need an explicit version of Waldspurger's formula as developed my Martin and Whitehouse [2009] and, furthermore, translate this to a classical formula. In doing so, we will follow Popa [2006, Chapter 5].

4A. ***A formula of Martin and Whitehouse (following Waldspurger).*** Let π be an automorphic representation of $\mathrm{GL}_2(\mathbb{Q})$ of squarefree conductor N and even lowest weight k_π corresponding to the classical cuspidal newform f (Maaß or holomorphic also of weight k_π). Let $D < -6$ be a negative fundamental discriminant with $(D, 2N) = 1$ and such that all primes dividing N split in $K = \mathbb{Q}[\sqrt{D}]$. Let $k \geq k_\pi$ be even, and let $\Omega : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$ be an idélic Hecke character of conductor 1 and ∞-type $\Omega_\infty(\alpha) = (\alpha/|\alpha|)^k$. Recall from Section 3B that any two such characters differ by a class group character, and thus there are $|\mathrm{Cl}_K|$ such characters.

We will be interested in obtaining an explicit formula in terms of Heegner points of the central value of the Rankin–Selberg L-function $L\big(\pi \otimes \Omega, \frac{1}{2}\big)$, by which we mean the Rankin–Selberg convolution of the base change π_K of π to $\mathrm{GL}_2(\mathbb{A}_K)$ and the automorphic representation π_Ω of $\mathrm{GL}_1(\mathbb{A}_K)$ corresponding to Ω. We note that the above (Heegner) conditions on D and N imply that the root number of $L(\pi \otimes \Omega, s)$ is equal to $+1$.

Let $\Psi_{\mathbb{A}} : \mathbb{A}_K \hookrightarrow \mathrm{GL}_2(\mathbb{A})$ be an oriented optimal algebra embedding of level N. Then associated to the triple $(\pi, \Omega, \Psi_{\mathbb{A}})$, Martin and Whitehouse [2009, Theorem 4.1] define a specific test vector $\phi_{\mathrm{MW}} \in \pi$ such that we have the formula

$$\frac{\big|\int_{\mathbb{A}^\times K^\times \backslash \mathbb{A}_K^\times} \phi_{\mathrm{MW}}(\Psi_{\mathbb{A}}(x))\Omega^{-1}(x)\,dx\big|^2}{\int_{Z(\mathbb{A})\,\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})} |\phi_{\mathrm{MW}}(g)|^2\,dg} = \frac{L(\pi \otimes \Omega, 1/2)}{L(\mathrm{sym}^2 \pi, 1)} \frac{c_\infty(\pi_\infty, k)}{2\sqrt{|D|}} \prod_{p|N} \Big(1 - \frac{1}{p}\Big)^{-1}, \tag{4-1}$$

where the measure dg is normalized so that the volume of $Z(\mathbb{A})\,\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})$ is $(\pi/3)\prod_{p|N}(1-p^{-2})$ (here we are using that the Tamagawa number of $\mathrm{GL}_2/\mathbb{Q}$ is 2) and dx is normalized so that $\mathbb{A}^\times K^\times \backslash \mathbb{A}_K^\times$ has volume $2\Lambda(\chi_K, 1)$, where χ_K is the quadratic character associated to K via class field theory and

$$\Lambda(\chi_K, s) = \pi^{-(s+1)/2}\Gamma\Big(\frac{s+1}{2}\Big)L(\chi_K, s).$$

The local constants are given by:

$$c_\infty(\pi_\infty, k) = \begin{cases} (2\pi)^k \prod_{j=0}^{k/2-1}\big(\frac{1}{4}+(t_\pi)^2+j(j+1)\big)^{-1} & \text{if } \pi_\infty \text{ is a p.s,} \\ (2\pi)^{k-k_\pi} \dfrac{\Gamma(k_\pi+1)}{\Gamma\big(\frac{1}{2}(k+2)\big)B\big(\frac{1}{2}(k+k_\pi), \frac{1}{2}(k-k_\pi+2)\big)} & \text{if } \pi_\infty \text{ is a d.s,} \end{cases}$$

where "p.s" and "d.s" refer to "principal series" and "discrete series", respectively, and $B(x, y)$ denotes the Beta function.

To make this formula explicit, we need to specify an embedding $\Psi_{\mathbb{A}}$. To do this, let $[a, b, c]$ be a Heegner form of level N and orientation r and consider the associated optimal embedding $\Psi : K \hookrightarrow \mathrm{Mat}_{2\times 2}(\mathbb{Q})$ of level N (as in Section 3A2) satisfying

$$\Psi(K) \cap M_0(N) = \Psi(\mathcal{O}_K),$$

where $M_0(N) = \big\{\big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big) \in \mathrm{Mat}_{2\times 2}(\mathbb{Z}) : N \mid c\big\}$. As described in Section 3A2, we get by tensoring with $\mathbb{A}$ an associated embedding $\Psi_{\mathbb{A}} : \mathbb{A}_K^\times \to \mathrm{GL}_2(\mathbb{A})$. We write Ψ_{fin} for the finite component and Ψ_∞ for the infinite component of this embedding.

Now the recipe described in [Martin and Whitehouse 2009, Chapter 4.2] gives the following characterization of the test vector ϕ_{MW}: the finite component $\phi_{\mathrm{MW},p}$ at a finite prime $p<\infty$ is uniquely determined (up to scaling) by the invariance under a certain Eichler order, which in our setting is exactly the order in $\mathrm{GL}_2(\mathbb{Q}_p)$ of reduced discriminant $p^{v_p(N)}$ (using that Ψ is optimal of level N). This means that we can pick $\phi_{\mathrm{MW},p}=\phi_{f,p}=\phi_{f_k,p}$, where ϕ_f (respectively, ϕ_{f_k}) are the lifts to $\mathrm{GL}_2(\mathbb{A})$ of the Hecke–Maaß newform $f\in L^2(\Gamma_0(N),k_\pi)$ corresponding to π (respectively, $f_k=R_{k-2}\cdots R_{k_\pi}f$).

At the infinite place the test vector $\phi_{\mathrm{MW},\infty}$ is characterized by being the vector of the minimal K-type (in the sense of [Popa 2008]) such that

$$\pi_\infty(x)\phi_{\mathrm{MW},\infty}=\Omega_\infty(x)\phi_{\mathrm{MW},\infty}$$

for all $x\in\Psi_\infty(S^1)\cap \mathrm{O}_2(\mathbb{R})$, where $S^1=\{z\in\mathbb{C}^\times:|z|=1\}$ is the maximal compact of $\mathbb{C}^\times$ and $\mathrm{O}_2(\mathbb{R})$ is the maximal compact of $\mathrm{GL}_2(\mathbb{R})$. There is a slight complication due to the fact that the embedding Ψ_∞ defined above does not send the maximal compact $S^1\subset\mathbb{C}^\times$ to $\mathrm{SO}_2(\mathbb{R})$. We can, however, easily check that this is the case after conjugating by

$$\gamma_\infty=\begin{pmatrix}\sqrt{D} & -b\\ 0 & a\end{pmatrix}. \tag{4-2}$$

Thus, we conclude that the following vector satisfies the conditions specified by Martin and Whitehouse:

$$\phi_{\mathrm{MW},\infty}=\pi(\gamma_\infty)\phi_{f_k,\infty},$$

where $\phi_{f_k,\infty}=R_{k-2}\cdots R_{k_\pi}\phi_{f,\infty}$ is a weight k vector in the representation space π_∞. We conclude that we can pick the global test vector as

$$\phi_{\mathrm{MW}}=\pi(\gamma_\infty)\phi_{f_k},$$

where again $f_k=R_{k-2}\cdots R_{k_\pi}f$ and $\gamma_\infty\in\mathrm{GL}_2(\mathbb{R})\subset\mathrm{GL}_2(\mathbb{A})$ as in (4-2).

For ϕ_{MW} as above, we have for $x_{\mathrm{fin}}\in\mathbb{A}_{K,\mathrm{fin}}^\times$ and $x_\infty\in\mathbb{C}^\times$ that

$$\phi_{\mathrm{MW}}\big(\Psi_\infty(x_\infty)\Psi_{\mathrm{fin}}(x_{\mathrm{fin}})\big)\Omega^{-1}\big((x_\infty,x_{\mathrm{fin}})\big)$$

is independent of x_∞. In particular, we get a well-defined map

$$\mathrm{Cl}_K\ni[\mathfrak{a}]\mapsto\phi_{\mathrm{MW}}(\Psi_{\mathbb{A}}(\hat{\mathfrak{a}}))\Omega^{-1}(\hat{\mathfrak{a}}),$$

where $\hat{\mathfrak{a}}\in\mathbb{A}_{K,\mathrm{fin}}^\times$ is any lift of $\mathfrak{a}$ under the first isomorphism in (3-2). By the second isomorphism in (3-2), it follows that we have a bijection

$$K^\times\mathbb{A}^\times\backslash\mathbb{A}_K^\times/\widehat{\mathcal{O}}_K^{\times}\xrightarrow{\sim}\bigsqcup_{[\mathfrak{a}]\in\mathrm{Cl}_K}\mathbb{C}^\times/\mathbb{R}^\times, \tag{4-3}$$

from which we conclude that

$$\int_{\mathbb{A}^\times K^\times\backslash\mathbb{A}_K^\times}\phi_{\mathrm{MW}}(\Psi_{\mathbb{A}}(x))\Omega^{-1}(x)\,dx=\frac{2}{|D|^{1/2}}\sum_{[\mathfrak{a}]\in\mathrm{Cl}_K}\phi_{f_k}(\Psi_{\mathrm{fin}}(\hat{\mathfrak{a}})\gamma_\infty)\overline{\Omega(\hat{\mathfrak{a}})}. \tag{4-4}$$

Here we can check the normalization by letting ϕ_{MW} and Ω being constants and recalling that the total measure of $\mathbb{A}^\times K^\times\backslash\mathbb{A}_K^\times$ is $2\Lambda(\chi_K,1)=2|\mathrm{Cl}_K||D|^{-1/2}$ by the class number formula.

4B. ***Explicit representatives of the class group.*** Consider integral prime ideals $\mathfrak{p}_1 = (1), \mathfrak{p}_2, \dots, \mathfrak{p}_h$ which are representatives for the class group Cl_K dividing the rational primes p_i which we assume are coprime to $2Na$ (so that $h = |\mathrm{Cl}_K|$ and $p_i \mathcal{O}_K = \mathfrak{p}_i \overline{\mathfrak{p}_i}$ splits in K for $i = 2, \dots, h$). The ideal class $[\mathfrak{p}_i]$ is represented by the idéle $\widehat{\mathfrak{p}}_i := (p_i)_{\mathfrak{p}_i} \in \mathbb{A}_K^\times$ (where the subscript means that the element is concentrated at the place $\mathfrak{p}_i$). Thus we see using the definition (3-7) of $\Psi_{\mathbb{A}}$ that since

$$j_1 \cdot p_i + j_2 \cdot 1 = 1 \otimes \frac{p_i+1}{2} + \sqrt{D} \otimes \frac{p_i-1}{2\sqrt{D}} \in K \otimes \mathbb{Q}_{p_i},$$

we have that

$$\Psi_{\mathbb{A}}((p_i)_{\mathfrak{p}_i}) = \begin{pmatrix} \frac{p_i+1}{2} + b\frac{p_i-1}{2\sqrt{D}} & c\frac{p_i-1}{\sqrt{D}} \\ -a\frac{p_i-1}{\sqrt{D}} & \frac{p_i+1}{2} - b\frac{p_i-1}{2\sqrt{D}} \end{pmatrix}_{p_i}.$$

For $i = 2, \dots, h$, it is a short computation that for an integer b_i with $b_i \equiv b \mod 2a$ and $b_i^2 \equiv D \mod p_i$ (and put also $b_1 = 1$ for completeness), we have

$$\mathfrak{p}_{\mathrm{i}} = \left[\frac{-b_i + \sqrt{D}}{2}, p_i\right]. \tag{4-5}$$

Using the congruences for b_i, it follows that there is $k_i \in K_0(N)$ such that

$$\Psi_{\mathbb{A}}((p_i)_{\mathfrak{p}_i}) = \gamma_i k_i (\gamma_i^{-1})_\infty$$

with $\gamma_i \in M_2(\mathbb{Q})$ given by

$$\gamma_i = \begin{pmatrix} p_i & \frac{b_i - b}{2a} \\ 0 & 1 \end{pmatrix}.$$

Thus we conclude by the definition of adélization that

$$\phi_{f_k}(\Psi_{\mathrm{fin}}(\widehat{\mathfrak{p}}_i)\gamma_\infty) = j_{\gamma_i^{-1}\gamma_\infty}(i)^k f_k(\gamma_i^{-1}\gamma_\infty i) = f_k\left(\frac{-b_i + \sqrt{D}}{2ap_i}\right).$$

To proceed, we need to understand how the Heegner points $(-b_i + \sqrt{-D})/(2ap_i)$ behaves as $i = 1, \dots, h$ varies. Let $I : \Gamma_0(N)\backslash \mathscr{E}_D(N, r) \to \mathrm{Cl}_K$ be the bijection in (3-6). Then we have the following adaption of [Popa 2006, Proposition 6.2.2]:

Lemma 4.1. *We have*

$$\gamma_i^{-1}\gamma_\infty i = z_{Q_{\Psi,i}} \in \mathbb{H},$$

where $z_{Q_{\Psi,i}}$ is the Heegner point of a Heegner form $Q_{\Psi,i}$ of level N and orientation r (depending on Ψ and i) belonging to the class $I([\psi]) \cdot [\mathfrak{p}_i] \in \mathrm{Cl}_K$.

Proof. Consider the binary quadratic form

$$Q(x, y) = ap_i x^2 + b_i xy + c_i y^2,$$

where

$$c_i = \frac{b_i^2 - D}{4ap_i}$$

is an integer by the above congruence conditions. This means that Q is a discriminant D Heegner form of level N and orientation r, with corresponding Heegner point given by

$$\frac{-b_i + \sqrt{D}}{2ap_i}.$$

Thus the lemma reduces to showing the following identity of ideals (modulo principal ideals):

$$\left[ap_i, \frac{-b_i + \sqrt{D}}{2}\right] = \left[\frac{-b_i + \sqrt{D}}{2}, p_i\right] \cdot \left[\frac{-b + \sqrt{D}}{2}, a\right]. \tag{4-6}$$

This follows, as in the proof of [Popa 2006, Proposition 6.2.2], since both sides have the same ideal norm and we can check using the congruence condition on b_i that the right-hand side is contained in the left-hand side. □

This implies that the automorphic period (4-4) depends on the choice of optimal embedding Ψ but only up to a phase. In particular, the absolute square does not depend on the choice of Ψ as should be the case by (4-1).

4C. *An explicit formula.* To simplify matters, we from now on pick our optimal embedding Ψ such that $[a, b, c]$ corresponds to the trivial element of Cl_K and to lighten notation, we write

$$Q_i = ap_i x^2 + b_i xy + c_i y^2, \quad \text{with } i = 1, \dots, h, \tag{4-7}$$

where p_i and b_i are as above. Now if $Q \in \mathcal{Q}_D(N, r)$ is any quadratic form such that $[Q] = [\mathfrak{p}_i]$, then it follows from Lemma 4.1 that there is some $\gamma_Q \in \Gamma_0(N)$ such that $z_Q = \gamma_Q z_{Q_i}$, which implies that

$$f_k(z_Q) = j_{\gamma_Q}(z_{Q_i})^k f(z_{Q_i}) = \Omega_\infty(\alpha_Q)\phi_{f_k}(\Psi_{\mathrm{fin}}(\widehat{\mathfrak{p}}_i)\gamma_\infty),$$

where $\alpha_Q = j(\gamma_Q, z_{Q_i}) \in K^\times$. Similarly if $\mathfrak{a} \in \mathcal{I}_K$ is a different representative of the ideal class $[\mathfrak{p}_i] \in \mathrm{Cl}_K$, then we have

$$\Omega^{-1}(\hat{\mathfrak{a}}) = \Omega_\infty(\alpha_{\mathfrak{a}})\Omega^{-1}(\widehat{\mathfrak{p}}_i)$$

for some $\alpha_{\mathfrak{a}} \in K^\times$.

From this we conclude, by combining (4-4) and Lemma 4.1, that

$$\int_{\mathbb{A}^\times K^\times \backslash \mathbb{A}_K^\times} \phi_{\mathrm{MW}}(\Psi_{\mathbb{A}}(x))\Omega^{-1}(x)\,dx = \sum_{[Q] \in \Gamma_0(N) \backslash \mathcal{Q}_D(N,r)} f_k(z_Q)\overline{\Omega(\widehat{\mathfrak{a}}_Q)}\Omega_\infty(\alpha_{Q,\mathfrak{a}_Q}), \tag{4-8}$$

where z_Q is the Heegner point associated to the Heegner form $Q \in \mathcal{Q}_D(N, r)$, $[\mathfrak{a}_Q] = [Q]$ (under the bijection $\Gamma_0(N)\backslash\mathcal{Q}_D(N, r) \xrightarrow{\sim} \mathrm{Cl}_K$), and $\alpha_{Q,\mathfrak{a}_Q} \in K^\times$ is a complex number depending on the choices of Q and $\mathfrak{a}_Q$ (but not on π, Ω, nor f_k).

4C1. *The case of old forms.* We will now explain how to extend the identity (4-8) to the case of old forms. Let d, N' be positive integers such that $dN' \mid N$, and consider a newform (i.e., new at finite places) $f_k \in \mathcal{B}_k^*(N')$ belonging to the automorphic representation π. Then we get an element $\nu_{d,N'}^* f_k \in \mathcal{B}_k(N)$ given by $z \mapsto f_k(dz)$. Recall the representatives $\mathfrak{p}_1, \dots, \mathfrak{p}_h \in \mathcal{I}_K$ of the class group Cl_K defined in (4-5) and the associated Heegner forms $Q_i = [a, b_i, c_i]$ defined in (4-7). Then we see directly that

$$dz_{Q_i} = \frac{-b_i + \sqrt{D}}{2p_i a/d} = z_{Q_i'},$$

where $Q_i' = [p_i a/d, b_i, c_i d] \in \mathcal{Q}_D(N', r)$ is a Heegner form of level N' and orientation $r \bmod (2N')$. From this, we see that

$$f_k(dz_{Q_i}) = \phi_{f_k}(\Psi'_{\mathrm{fin}}(\widehat{\mathfrak{p}}_i)\gamma'_\infty), \quad \text{with } i = 1, \dots, h,$$

where Ψ' is the optimal embedding of level N' corresponding to the triple $[a/d, b, cd]$ and

$$\gamma'_\infty = \begin{pmatrix} \sqrt{D} & -b \\ 0 & \frac{a}{d} \end{pmatrix}.$$

Observe that $[a/d, b, cd]$ might not correspond to the trivial element of the class group. Thus, using (4-8),

$$\begin{aligned}\sum_{[Q] \in \Gamma_0(N) \backslash \mathcal{Q}_D(N,r)} \nu_{d,N'}^* f_k(z_Q)\overline{\Omega(\widehat{\mathfrak{a}}_Q)}\Omega_\infty(\alpha_{Q,\mathfrak{a}_Q}) &= \sum_{i=1}^{h} \nu_{d,N'}^* f_k(z_{Q_i})\overline{\Omega(\widehat{\mathfrak{p}}_i)} \\ &= \int_{\mathbb{A}^\times K^\times \backslash \mathbb{A}_K^\times} \phi'_{\mathrm{MW}}(\Psi'_{\mathbb{A}}(x))\Omega^{-1}(x)\, dx, \qquad (4\text{-}9)\end{aligned}$$

where ϕ'_{MW} is the vector defined by Martin and Whitehouse corresponding to the triple $(\pi, \Omega, \Psi'_{\mathbb{A}})$ and the numbers $\alpha_{Q,\mathfrak{a}_Q}$ are as in (4-8).

Combining (4-9) and (4-1), we arrive at the following result (recalling the definition (3-9) of $\mathcal{B}_k(N)$):

Theorem 4.2. *Let N be a squarefree integer and K be an imaginary quadratic field of discriminant D with $(D, 2N) = 1$ and such that all primes dividing N splits in K. Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ of conductor N' dividing N and even lowest weight k_π. Let $k \geq k_\pi$ be an even integer and $\Omega : K^\times \backslash \mathbb{A}_K^\times / \widehat{\mathcal{O}}_K^\times \to \mathbb{C}^\times$ a Hecke character of K of conductor 1 and ∞-type $\alpha \mapsto (\alpha/|\alpha|)^k$.*

Then for any $f_k \in \mathcal{B}_k(N)$ belonging to the representation space of π, we have

$$\left| \sum_{[Q] \in \Gamma_0(N) \backslash \mathcal{Q}_D(N,r)} f_k(z_Q)\overline{\Omega(\widehat{\mathfrak{a}}_Q)}\Omega_\infty(\alpha_{Q,\mathfrak{a}_Q}) \right|^2 = \frac{L(\pi \otimes \Omega, 1/2)}{L(\mathrm{sym}^2 \pi, 1)} \frac{|D|^{1/2}}{8N'} c_\infty(\pi_\infty, k), \qquad (4\text{-}10)$$

where z_Q is the Heegner point associated to the Heegner form $Q \in \mathcal{Q}_D(N, r)$, $\mathfrak{a}_Q \in \mathcal{I}_K$ is such that $[Q] = [\mathfrak{a}_Q]$ (under the bijection $\Gamma_0(N) \backslash \mathcal{Q}_D(N, r) \xrightarrow{\sim} \mathrm{Cl}_K$), $\alpha_{Q,\mathfrak{a}_Q} \in K^\times$ is a complex number depending on the choices Q and $\mathfrak{a}_Q$ (but not on π, Ω nor f_k), and

$$c_\infty(\pi_\infty, k) = \begin{cases} (2\pi)^k \prod_{j=0}^{k/2-1} \left(\frac{1}{4} + (t_\pi)^2 + j(j+1)\right)^{-1} & \text{if } \pi_\infty \text{ is a p.s,} \\ (2\pi)^{k-k_\pi-1} \dfrac{\Gamma(k_\pi)}{\left(\Gamma\left(\frac{1}{2}(k-2)\right) B\left(\frac{1}{2}(k+k_\pi+1), \frac{1}{2}(k-k_\pi+1)\right)\right)} & \text{if } \pi_\infty \text{ is a d.s,} \end{cases} \qquad (4\text{-}11)$$

where "p.s" ("d.s") refers to "principal series" ("discrete series") and $B(x, y)$ denotes the Beta function.

Using orthogonality of characters (i.e., Fourier inversion) we conclude the following key identity:

Corollary 4.3. *Let π, Ω, f_k be as in Theorem 4.2. Then given an element of the class group $[\mathfrak{a}] \in \mathrm{Cl}_K$ and a Heegner form $Q \in \mathcal{Q}_D(N, r)$ such that $[Q] = [\mathfrak{a}]$, we have*

$$f_k(z_Q)\Omega(x_Q) = \frac{c_{f_k}|D|^{1/4}}{|\mathrm{Cl}_K|} \sum_{\chi \in \widehat{\mathrm{Cl}}_K} \varepsilon_{\chi, f_k, r} \left|L\left(\pi \otimes \chi\Omega, \tfrac{1}{2}\right)\right|^{1/2} \chi([\mathfrak{a}]), \tag{4-12}$$

where $x_Q \in \mathbb{A}_K^\times$ is some element depending on the choice of Q (but not on π, Ω, nor f_k), $\varepsilon_{\chi, f_k, r}$ are complex numbers of norm 1, and

$$c_{f_k} = \frac{c_\infty(\pi_\infty, k)}{8N' L(\mathrm{sym}^2 \pi, 1)}, \tag{4-13}$$

with $c_\infty(\pi_\infty, k)$ as in (4-11).

5. Some technical lemmas

In this section, we will prove two key estimates. The first is a bound for the norm of Δ^m, which will be key in obtaining explicit error terms in our moment calculation. Similar consideration have been made in a different context in [Petridis and Risager 2018b, Theorem 5.1]. Secondly, we will obtain a lower bound for the L^2-norm of the product of Maaß forms. This is an extremely crude lower bound, which suffices for our purposes.

5A. ***A bound for the norm of Δ^m.*** In the course of proving our bound for the norm of Δ^m applied to certain vectors, we will need the following convenient L^∞-bound for $f \in \mathcal{B}_k(N)$ due to Blomer and Holowinsky [2010]:

$$\frac{\|f\|_\infty}{\|f\|_2} \ll N^{-1/32}(|t_f| + |k| + 1)^A \tag{5-1}$$

for some unspecified constant $A > 0$. The focus of [Blomer and Holowinsky 2010] is the level aspect, which we consider fixed in the present paper. Here the key thing is, however, that we get a polynomial bound for raised (and lowered) Hecke–Maaß forms with the constant being independent of the weight k and the spectral parameter t_f. The specific value of A is not important for our application.

Lemma 5.1. *Let $k_1, \dots, k_n$ be even integers such that $\sum_{i=1}^n k_i = 0$. For $i = 1, \dots, n$, let $f_i \in \mathcal{B}_{k_i}(N)$ be a Hecke–Maaß form of weight k_i, level N, and spectral parameter t_{f_i}. Then we have*

$$\left\| \Delta^m \prod_{i=1}^n f_i \right\|_\infty \ll n^{2m} (m + \max_{i=1,\dots,n} |t_{f_i}| + |k_i|)^{nA+2m} \prod_{i=1}^n \|f_i\|_2 \tag{5-2}$$

for all $m \in \mathbb{Z}_{\geq 0}$. Here the implied constant is allowed to depend on N.

Proof. Recalling that $\Delta = L_2 R_0$, we get, using the product rule for the raising and lowering operators,

$$\begin{aligned}\left|\Delta^m \prod_{i=1}^n f_i(z)\right| &= \left|L_2 R_0 \cdots L_2 R_0 \prod_{i=1}^n f_i(z)\right| \\ &\leq n^{2m} \max_{\substack{m_1,\dots,m_n \in \mathbb{N} : \sum m_i = 2m \\ (U_{i,j})_{i,j} \text{ fifty-fifty raising/lowering operators} \\ \text{where } 1 \leq i \leq n, 1 \leq j \leq m_i}} \prod_{i=1}^n |U_{i,1} \cdots U_{i,m_i} f_i(z)|. \end{aligned} \tag{5-3}$$

Here the maximum is taken over all combinations of $2m$ operators

$$U_{i,j} : 1 \leq i \leq n,\ 1 \leq j \leq m_i,$$

which are all either a raising or a lowering operator of appropriate weight and such that the total number of raising and lowering operators are equal. If we have $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m_i - 1\}$ such that $\{U_{i,j}, U_{i,j+1}\}$ is of the type {raising, lowering}, then we get

$$U_{i,j} U_{i,j+1} = -\Delta_{\pm\kappa} + \lambda\Big(\frac{\kappa}{2}\Big)$$

for some weight κ with $|\kappa| \leq 2m + |k_i|$ (since we can have at most m raising respectively, lowering operators). Here the sign corresponds to whether $U_{i,j}$ is a raising or lowering operator. This shows that we can replace $U_{i,j} U_{i,j+1}$ with multiplication by

$$\lambda\Big(\frac{\kappa}{2}\Big) - \lambda_{f_i} = -\Big(\frac{\kappa-1}{2} + i t_{f_i}\Big)\Big(\frac{\kappa-1}{2} - i t_{f_i}\Big).$$

Repeating this, we get

$$|U_{i,1} \cdots U_{i,m_i} f_i(z)| = \left| R_{k+2m_i'-2} \cdots R_k f_i(z) \prod_{j=1}^{(m_i - m_i')/2} \Big(\frac{\kappa_j - 1}{2} + i t_f\Big)\Big(\frac{\kappa_j - 1}{2} - i t_f\Big)\right|$$

for some $0 \leq m_i' \leq m_i$, where $|\kappa_j| \leq 2m + |k_i|$ (or a similar expression with lowering instead of raising operators).

By combining the bound (5-1) and the computation of the L^2-norm (3-10), we conclude that for $f \in \mathcal{B}_k(N)$ and $l \geq 0$

$$\begin{aligned}\|R_{k+2l} R_{k+2l-2} \cdots R_k f\|_\infty &\ll \|f\|_2\big(|t_f| + |k+l| + 1\big)^A \prod_{j=0}^{l} \left|\Big(\frac{k+2j-1}{2} + i t_f\Big)\Big(\frac{k+2j-1}{2} - i t_f\Big)\right|^{1/2} \\ &\ll \|f\|_2\big(|t_f| + |k| + l + 1\big)^{l+A},\end{aligned}$$

and similarly in the case of lowering operators. Combining all of the above, we arrive at

$$|U_{i,1} \cdots U_{i,m_i} f_i(z)| \ll \|f_i\|_2 (|t_f| + |k_i| + m_i + 1)^{A+m_i},$$

for any sequence of raising and lowering operators $U_{i,1}, \dots, U_{i,m_i}$ as in the maximum in (5-3). Plugging this into (5-3) gives the wanted. □

5B. ***A lower bound for weight k automorphic forms.*** In this subsection, we will prove a lower bound for the L^2-norm of a product of Maaß forms. The idea is to go far up in the cusp so that the first term in the Fourier expansion is the dominating term.

Let $W_{k/2,s} : \mathbb{R}_{>0} \to \mathbb{C}$ be the *Whittaker function* of weight $k/2$ and spectral parameter s, i.e., the unique solution to

$$\frac{d^2 W}{dy^2} + \Big(-\frac{1}{4} + \frac{k/2}{y} + \frac{1/4 - s^2}{y^2}\Big) W = 0,$$

satisfying

$$W_{k/2,s}(y) \sim y^{k/2} e^{-y/2},$$

as $y \to \infty$ (with k, s fixed). Then we define $\mathcal{W}_{k/2,s} : \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}$ for $k \in \mathbb{Z}$ as

$$\mathcal{W}_{k/2,s}(z) := \begin{cases} (-1)^{k/2} W_{|k|/2,s}(|y|) e^{ix/2} & \operatorname{sign}(k) y > 0, \\ \dfrac{\Gamma((|k|+1)/2+s)\Gamma((|k|+1)/2-s)}{\Gamma(1/2+s)\Gamma(1/2-s)} W_{-|k|/2,s}(|y|) e^{ix/2} & \operatorname{sign}(k) y < 0, \end{cases}$$

for $z = x + iy \in \mathbb{C}\backslash\mathbb{R}$. We can check that

$$\mathcal{W}_{0,s}(z) = \left(\frac{|y|}{\pi}\right)^{1/2} K_s\left(\frac{|y|}{2}\right) e^{ix/2},$$

where $K_s(y)$ is the K-Bessel function and

$$\mathcal{W}_{k/2,(k-1)/2}(z) = (-1)^{k/2} y^{k/2} e^{iz/2}$$

for $k \in 2\mathbb{Z}_{\geq 0}$ and $y > 0$. Furthermore, for $k \in 2\mathbb{Z}_{\geq 0}$, we can check (see, for instance, [Strömberg 2008, Section 4.4]) that the normalizations match up so that we have

$$R_k \mathcal{W}_{k/2,s} = \mathcal{W}_{k/2+1,s}, \tag{5-4}$$

with

$$R_k = (z - \bar{z}) \frac{\partial}{\partial z} + \frac{k}{2} = iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \frac{k}{2},$$

denoting the weight k raising operator (and similarly for $k \leq 0$ now with lowering operators). We have the following asymptotic expansion (see [Gradshteyn and Ryzhik 2000, (9.227)] or [Whittaker and Watson 1962, Chapter 16.3]) valid for $y > 1$:

$$W_{k/2,s}(y) = e^{-y/2} y^{k/2} \left(1 + \sum_{n \geq 1} \frac{\left(s^2 - (k/2 - 1/2)^2\right) \cdots \left(s^2 - (k/2 - n + 1/2)^2\right)}{n!\, y^n}\right).$$

In particular, we conclude that

$$\begin{aligned} \mathcal{W}_{k/2,s}(z) &= e^{-y/2} y^{k/2} \left(1 + O\left(\sum_{n \geq 1} \frac{\left(|s| + |k|/2 + n\right)^{2n}}{n!\, y^n}\right)\right) \\ &= e^{-y/2} y^{k/2} \left(1 + O\left(\frac{(|s| + |k| + 1)^2}{y}\right)\right) \end{aligned} \tag{5-5}$$

for $y > (|s| + |k| + 1)^2$.

Now, we let $k \geq 0$ and consider an L^2-normalized Hecke–Maaß form $f \in \mathcal{B}_k(N)$ of the form $\nu^*_{d,N'} R_{k-2} \cdots R_{k'} f_0$, with f_0 a Hecke–Maaß newform of weight k' and level N' such that $dN' \mid N$. Combining (5-4) and (3-10) with the well-known Fourier expansions of holomorphic and Maaß forms, we get the following Fourier expansion in the general weight case:

$$f(z) = \frac{c_f}{|L(\operatorname{sym}^2 f, 1)\gamma_\infty(f,k)|^{1/2}} \sum_{n \neq 0} \frac{\lambda_{f_0}(n)}{|n|^{1/2}} \mathcal{W}_{k/2,it_f}(4\pi d n z), \tag{5-6}$$

for some constant c_f bounded uniformly from above and away from 0 in terms of the level N. Here $\lambda_{f_0}(n)$ denotes the Hecke eigenvalues of f (with the convention that $\lambda_{f_0}(-n) = 0$ for $-n < 0$ if f_0 is holomorphic and $\lambda_{f_0}(-n) = \pm\lambda_{f_0}(n)$ according to whether f_0 is an even or odd Maaß form) and

$$\gamma_\infty(f,k) = \begin{cases} \prod_\pm \Gamma\Big(\dfrac{k+1}{2} \pm it_f\Big) & \text{if } f_0 \text{ is Maaß,} \\ \Gamma(k)\Gamma\Big(\dfrac{k-k'}{2}+1\Big) & \text{if } f_0 \text{ is holomorphic.} \end{cases}$$

Using this we can prove the following crude lower bound:

Proposition 5.2. *For $i = 1, \dots, n$, let $f_i \in \mathcal{B}_{k_i}(N)$ be an L^2-normalized weight k_i Hecke–Maaß eigenform of level N. Then we have*

$$\Big\| \prod_{i=1}^n f_i \Big\|_2 \gg_\varepsilon e^{-cnT^{2+\varepsilon}}$$

for all $\varepsilon > 0$, where $T = \max_{i=1,\dots,n} |t_{f_i}| + |k_i| + 1$ and $c = c(N, \varepsilon) > 0$ is some positive constant.

Proof. Clearly we may assume that $k \geq 0$. Given $f \in \mathcal{B}_k(N)$, we write

$$f = \nu^*_{d,N'} R_{k-2} \cdots R_{k'} f_0$$

for a Hecke–Maaß newform f_0 of weight k' (with $k' \leq k$ and $k' \equiv k \mod 2$) and level N' with $dN' \mid N$. We have, by a standard bound for the Hecke eigenvalues (see, for instance, [Iwaniec 2002, (8.7)] in the Maaß case) and by bounding the quotient of Γ-factors trivially, that

$$\begin{aligned} &\sum_{n \neq 0} \frac{\lambda_f(n)}{|n|^{1/2}} \mathcal{W}_{k/2,it_f}(4\pi n z) \\ &\qquad = e^{2\pi i dx} W_{k/2,it_f}(4\pi dy) + \varepsilon_f e^{-2\pi i dx} \frac{\Gamma\big((k+1)/2+s\big)\Gamma\big((k+1)/2-s\big)}{\Gamma(1/2+s)\Gamma(1/2-s)} W_{-k/2,it_f}(4\pi dy) \\ &\qquad\qquad + O\bigg(|t_f|^{1/2} \sum_{n \geq 2} \big| W_{k/2,it_f}(4\pi dny) \big| + (k+|t_f|+1)^k \big| W_{-k/2,it_f}(4\pi dny) \big| \bigg), \end{aligned} \tag{5-7}$$

where $\varepsilon_f = 0$ if f_0 is holomorphic and if f_0 is a Maaß form we have $\varepsilon_f = \pm 1$ where ± 1 is the sign of f_0 under the reflection operator X defined in Section 3C. By the asymptotics (5-5) we see easily that

$$\sum_{n \geq 2} |W_{k/2,it_f}(4\pi dny)| + (k + |t_f| + 1)^k |W_{-k/2,it_f}(4\pi dny)| \ll e^{-3d\pi y}$$

for $y \geq (|t_f| + k + 1)^{2+\varepsilon}$. For $k = 0$ we conclude from the asymptotic (5-5) that (5-7) is equal to

$$(e^{2\pi i dx} + \varepsilon_f e^{-2\pi i dx}) e^{-2\pi dy} + O(y^{-\varepsilon} e^{-2\pi dy}),$$

for $y \geq (|t_f| + k + 1)^{2+\varepsilon}$. Similarly, for $k > 0$, we see that (5-7) is equal to

$$e^{2\pi i dx} (4\pi dy)^{k/2} e^{-2\pi dy} + O((4\pi dy)^{k/2-\varepsilon} e^{-2\pi dy})$$

for $y \geq (|t_f|+k+1)^{2+\varepsilon}$, using the bound

$$\frac{\Gamma\big((k+1)/2+s\big)\Gamma\big((k+1)/2-s\big)}{\Gamma(1/2+s)\Gamma(1/2-s)} W_{-k/2,it_f}(4\pi dy) \ll (k+|t_f|+1)^k (4\pi dy)^{-k/2} e^{-2\pi dy}.$$

By Stirling's approximation, we have the crude bound

$$\gamma_\infty(f,k) \ll e^{O((|t_f|+k)\log(|t_f|+k))},$$

and we also have $|t_f|^{-\varepsilon} \ll_\varepsilon L(\mathrm{sym}^2 f, 1) \ll_\varepsilon |t_f|^\varepsilon$. Thus we conclude from (5-6) that for $k=0$,

$$|f(z)| \gg e^{-3\pi dy} \tag{5-8}$$

for $y \geq (|t_f|+k+1)^{2+\varepsilon}$ and x such that $e^{2\pi idx}+\varepsilon_f e^{-2\pi idx} \gg 1$. Similarly if $k>0$, we have

$$|f(z)| \gg e^{-3\pi dy} \tag{5-9}$$

for $y \geq (|t_f|+k+1)^{2+\varepsilon}$ (and any x). Now we easily conclude the wanted lower bound for the L^2-norm of the product by computing the contribution from the range $x \in [0,1]$ and $y \asymp (|t_f|+k+1)^{2+\varepsilon}$. □

In the holomorphic case, we can do slightly better since the Fourier expansion is better behaved.

Proposition 5.3. *For $i=1,\dots,n$, let $f_i \in \mathcal{B}_{k_i,\mathrm{hol}}(N)$ be a weight k_i holomorphic Hecke–Maaß eigenform of level N (L^2-normalized). Then we have*

$$\bigg\| \prod_{i=1}^n f_i \bigg\|_2 \gg_\varepsilon e^{-cnT^{1+\varepsilon}}$$

for all $\varepsilon>0$, where $T=\max_{i=1,\dots,n}|k_i|$ and $c(N,\varepsilon)=c>0$ is some positive constant.

Proof. Let $f \in \mathcal{B}_{k,\mathrm{hol}}(N)$ be of the form $v^*_{d,N'} y^{k/2} g$ with $g \in \mathcal{S}_k(N')$ a holomorphic Hecke newform. By the Fourier expansion (5-6), we have

$$f(z) = \frac{c_f}{|L(\mathrm{sym}^2 f,1)\Gamma(k)|^{1/2}} \sum_{n\geq 1} \frac{\lambda_g(n)}{n^{1/2}} (4\pi dny)^{k/2} e^{2\pi idnz}.$$

By bounding everything trivially, it is easy to see that for $y \gg k^{1+\varepsilon}$,

$$\sum_{n\geq 1} \frac{\lambda_g(n)}{n^{1/2}} (4\pi dny)^{k/2} e^{2\pi idnz} = (4\pi dy)^{k/2} e^{2\pi idz} + O_\varepsilon(e^{-3\pi dy}).$$

Now the lower bound for $\|\prod_{i=1}^n f_i\|_2$ follows as above. □

Remark 5.4. It seems quite hard to obtain strong lower bounds for $\|\prod_i f_i\|_2$ as this is related to the deep problem of nonlocalization of the eigenfunctions f_i (such as L^∞-bounds), see, for instance, [Sarnak 1995]. In particular, it is very hard to rule out that the f_i localize in disjoint regions.

6. Proof of the main theorem

We will now use the results proved in the previous sections to obtain our wide moment calculation. First of all, we will use the above to obtain a version of equidistribution of Heegner points with explicit error terms. For this, we will need the following convenient basis for the space spanned by Maaß forms of squarefree level N (see [Humphries and Khan 2020, Lemma 3.1]):

$$\mathcal{B}'(N) := \{u_d \in C^\infty(\mathbb{H}) \cap L^2(\Gamma_0(N)\backslash\mathbb{H}) : N'd \mid N, u \in \mathcal{B}^*(N')\},$$

(recall that we denote by $\mathcal{B}^*(N')$ all Hecke–Maaß newforms f of weight 0 and level N') where

$$u_d(z) := \left(L_d(\operatorname{sym}^2 u, 1)\frac{\varphi(d)}{d\nu(N/N')}\right)^{1/2} \sum_{vw=d} \frac{\nu(v)}{v} \frac{\mu(w)\lambda_u(w)}{\sqrt{w}} u(vz). \tag{6-1}$$

Here,

$$L_d(\operatorname{sym}^2 u, s) := \prod_{p|d} \frac{1}{1-\lambda_u(p^2)p^{-s}+\lambda_u(p^2)p^{-2s}-p^{-3s}}.$$

There is a similar basis for the Eisenstein part of the spectrum (see [Humphries and Khan 2020, Section 3.2]). Given $u \in \mathcal{B}'(N)$, we put

$$L(\operatorname{sym}^2 u, s) := L(\operatorname{sym}^2 u', s)$$

and

$$L(u, s) := L(u', s),$$

where $u = (u')_d$ with $u' \in \mathcal{B}^*(N')$ and $dN' \mid N$.

Theorem 6.1. *Let $k_1, \dots, k_n \in 2\mathbb{Z}$ be even integers such that $\sum k_i = 0$. For $i = 1, \dots, n$, let $f_i \in \mathcal{B}_{k_i}(N)$ be a Hecke–Maaß eigenform of fixed level N, weight k_i, and spectral parameter t_{f_i}. Let $|D_K| \to \infty$ transverse a sequence of discriminants of imaginary quadratic fields K such that all primes dividing N split in K. Then we have*

$$\frac{1}{|\mathrm{Cl}_K|} \sum_{[Q]\in\Gamma_0(N)\backslash\mathcal{Q}_{D_K}(N,r)} \prod_{i=1}^n f_i(z_Q) = \left\langle \prod_{i=1}^n f_i, \frac{1}{\operatorname{vol}(\Gamma_0(N))}\right\rangle + O_\varepsilon\left(\left\|\prod_{i=1}^n f_i\right\|_2 |D_K|^{-1/16} T^5 n^5 (T|D_K|n)^\varepsilon\right),$$

where $T = \max_{i=1,\dots,n} |t_{f_i}| + |k_i| + 1$.

We have the following improvements for the exponents in the error term:

$$\begin{cases} |D_K|^{-1/16}T^{5/2}n^5 & \text{if all } f_i \text{ are holomorphic,} \\ |D_K|^{-1/12}T^2n^2 & \text{if the level is } N=1, \\ |D_K|^{-1/12}Tn^2 & \text{if all } f_i \text{ are holomorphic of level } 1. \end{cases} \tag{6-2}$$

Proof. We put $D = |D_K|$ to lighten notation. By the spectral expansion for $\Gamma_0(N)\backslash\mathbb{H}$, see [Iwaniec 2002, Theorem 7.3], we have

$$\sum_{[Q]\in\Gamma_0(N)\backslash\mathcal{Q}_{-D}(N,r)} \prod_{i=1}^{n} f_i(z_Q) \\ = |\mathrm{Cl}_K| \left\langle \prod_{i=1}^{n} f_i, \frac{1}{\mathrm{vol}(\Gamma_0(N))} \right\rangle + \sum_{u\in\mathcal{B}'(N)} \left\langle \prod_{i=1}^{n} f_i, u \right\rangle W_{u,K} + (\text{Eisenstein}), \tag{6-3}$$

where

$$W_{u,K} := \sum_{[Q]\in\Gamma_0(N)\backslash\mathcal{Q}_{-D}(N,r)} u(z_Q)$$

is the Weyl sum of level N corresponding to u, and the Eisenstein contribution is given by

$$(\text{Eisenstein}) := \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{\mathbb{R}} \left\langle \prod_{i=1}^{n} f_i, E_{\mathfrak{a}}\left(\cdot, \frac{1}{2} + it\right) \right\rangle W_{\mathfrak{a},t,K}\, dt,$$

where the sum runs over the set of inequivalent cusps of $\Gamma_0(N)$, $E_{\mathfrak{a}}\left(z, \tfrac{1}{2} + it\right)$ denotes the Eisenstein series at the cusp $\mathfrak{a}$ (see [Iwaniec 2002, (3.11)]), and

$$W_{\mathfrak{a},t,K} := \sum_{[Q]\in\Gamma_0(N)\backslash\mathcal{Q}_{-D}(N,r)} E_{\mathfrak{a}}\left(z_Q, \tfrac{1}{2} + it\right)$$

is the corresponding Weyl sum.

We will now bound the cuspidal contribution in (6-3), and as usual the Eisenstein contribution can be bounded similarly. By Theorem 4.2, we have

$$|W_{u,K}|^2 \ll_N \frac{D^{1/2} L(u, 1/2) L(u \otimes \chi_K, 1/2)}{L(\mathrm{sym}^2 u, 1)} \tag{6-4}$$

for $u \in \mathcal{B}'(N)$. Here the case when u is a linear combination of old forms as in (6-1) follows by linearity. Now we observe that for $u \in \mathcal{B}^*(N)$, we have using the self adjointness of Δ,

$$\left\langle \prod_{i=1}^{n} f_i, u \right\rangle \left(t_u^2 + \tfrac{1}{4}\right)^m = \left\langle \prod_{i=1}^{n} f_i, \Delta^m u \right\rangle = \left\langle \Delta^m \prod_{i=1}^{n} f_i, u \right\rangle.$$

Applying the Cauchy–Schwarz inequality and Lemma 5.1, this implies

$$\left\langle \prod_{i=1}^{n} f_i, u \right\rangle \ll \prod_{i=1}^{n} \|f_i\|_2 \frac{n^{2m}(m+T)^{nA+2m}}{(|t_u|^2+1)^m} \tag{6-5}$$

for any $m \geq 0$, where $T = \max_{i=1,\dots,n} |t_{f_i}| + |k_i| + 1$. Putting $m = (nT^2)^{1+\varepsilon}$ in the estimate (6-5), we see that we can truncate the spectral expansion (6-3) at $t_u \ll (Tn)^2 (TDn)^{\varepsilon}$ at the cost of an error of size

$$\ll_{\varepsilon} (TDn)^{-c(nT^2)^{1+\varepsilon}} \prod_{i=1}^{n} \|f_i\|_2,$$

for some constant $c = c(N, \varepsilon) > 0$. By Proposition 5.2, this error is negligible.

To estimate the remaining terms, we use the bound (6-4) together with Cauchy–Schwarz and Bessel's inequality, nonnegativity, and standard bounds for symmetric square L-functions. This gives

$$\sum_{\substack{u\in\mathcal{B}'(N)\\ t_u\ll(Tn)^2(TDn)^\varepsilon}} \Big\langle \prod_{i=1}^n f_i, u\Big\rangle W_{u,K}$$

$$\ll_\varepsilon \Big\|\prod_{i=1}^n f_i\Big\|_2 D^{1/4}\Bigg(\sum_{N'|N}\sum_{\substack{u\in\mathcal{B}^*(N')\\ t_u\ll(Tn)^2(TDn)^\varepsilon}} L\big(u,\tfrac12\big)L\big(u\otimes\chi_K,\tfrac12\big)\Bigg)^{1/2}(TDn)^\varepsilon, \tag{6-6}$$

where χ_K is the quadratic character corresponding to K via class field theory (recall that $\mathcal{B}^*(N')$ denotes the set of all Hecke–Maaß newforms of weight 0 and level N').

From here on, we distinguish between the case of level 1 and higher (square free) level N. In the case of general level N, we use the GL_2 subconvexity bound due to Blomer and Harcos [2008]

$$L\big(u\otimes\chi_K,\tfrac12\big) \ll (1+|t_u|)^{3+\varepsilon}D^{3/8+\varepsilon},$$

which gives

$$\sum_{\substack{u\in\mathcal{B}'(N)\\ t_u\ll(Tn)^2(TDn)^\varepsilon}} \Big\langle \prod_{i=1}^n f_i, u\Big\rangle W_{u,K} \ll \Big\|\prod_{i=1}^n f_i\Big\|_2 D^{1/4+3/16}(Tn)^3(TDn)^\varepsilon\Bigg(\sum_{N'|N}\sum_{\substack{u\in\mathcal{B}^*(N')\\ t_u\ll(Tn)^2(TDn)^\varepsilon}} L\big(u,\tfrac12\big)\Bigg)^{1/2}$$

$$\ll \Big\|\prod_{i=1}^n f_i\Big\|_2 D^{1/2-1/16}(Tn)^5(TDn)^\varepsilon,$$

using a standard first-moment bound for $L\big(u,\tfrac12\big)$ (for instance, using a spectral large sieve).

If the level is 1, we follow Young [2017] and use Hölder's inequality together with his Lindelöf strength third moment bound [Young 2017, Theorem 1.1] to estimate the above by

$$\ll_\varepsilon \Big\|\prod_{i=1}^n f_i\Big\|_2 D^{5/12}(Tn)^2(TDn)^\varepsilon.$$

Finally, if all of the f_i are holomorphic, then by Proposition 5.3 we can use the estimate (6-5) with $m=nT^{1+\varepsilon}$ instead, which leads to the improved exponents. □

Remark 6.2. Alternatively, we can estimate (6-6) by using the bound

$$\Big\langle \prod_i f_i, u\Big\rangle \ll_\varepsilon t_u^{5/12+\varepsilon}\Big\|\prod_i f_i\Big\|_1,$$

where $\|\cdot\|_1$ denotes the L^1-norm, using here the L^∞-bound of Iwaniec and Sarnak [1995]. This leads to the error term

$$O_\varepsilon\Bigg(\Big\|\prod_{i=1}^n f_i\Big\|_1 |D_K|^{-1/16}T^{35/6}n^{35/6}(T|D_K|n)^\varepsilon\Bigg),$$

which is more convenient in some cases (with similar improvements in the special cases of holomorphic and / or level 1 as in (6-2)).

6A. ***A wide moment of L-functions.*** Combining this with our explicit formula, we arrive at our main L-function computation. We will use the following shorthand for K an imaginary quadratic field with class group Cl_K:

$$\mathbf{Wide}(K,n) := \mathbf{Wide}(\widehat{\mathrm{Cl}}_K, n),$$

with $\mathbf{Wide}(G,n)$ as in (1-1). Note that the following statement is a slight generalization of Theorem 1.7 (allowing for the representations not to have the same conductor):

Theorem 6.3. *Let $N \geq 1$ be a fixed squarefree integer. For $i = 1, \dots, n$, let π_i be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ with trivial central character of conductor $N_i \mid N$, spectral parameter t_{π_i}, and even lowest weight k_{π_i}. Let $k_1, \dots, k_n \in 2\mathbb{Z}$ be integers such that $|k_i| \geq k_{\pi_i}$ and $\sum_i k_i = 0$.*

Let $|D_K| \to \infty$ transverse a sequence of discriminants of imaginary quadratic fields K such that all primes dividing N split in K. For each K, pick Hecke characters $\Omega_{i,K}$ with ∞-type $x \mapsto (x/|x|)^{k_i}$ such that $\prod_i \Omega_{i,K}$ is the trivial Hecke character (notice that this is always possible since, we know that $\prod_i \Omega_{i,K}$ is a class group character).

Then we have for $f_i \in \mathcal{B}_{k_i}(N)$ in the representation space of π_i,

$$\sum_{(\chi_i) \in \mathbf{Wide}(K,n)} \prod_{i=1}^{n} \Big(c_{f_i} \varepsilon_{\chi_i, f_i} L\big(\pi_i \otimes \chi_i \Omega_{i,K}, \tfrac{1}{2}\big)^{1/2} \Big)$$
$$= \frac{|\mathrm{Cl}_K|^n}{|D_K|^{n/4}} \left(\left\langle \prod_{i=1}^{n} f_i, \frac{1}{\mathrm{vol}(\Gamma_0(N))} \right\rangle + O_\varepsilon\left(\left\| \prod_{i=1}^{n} f_i \right\|_2 |D_K|^{-1/16} T^5 n^5 (T|D_K|n)^{\varepsilon} \right) \right), \tag{6-7}$$

where $T = \max_{i=1,\dots,n} |k_i| + |t_{f_i}| + 1$, $c_{f_i} = (8N_i)^{-1} c_\infty(\pi_{i,\infty}, k_i)$ with c_∞ as in (4-11), and ε_{χ, f_i} are complex numbers of absolute value 1.

We have the following improvements for the exponents in the error term:

$$\begin{cases} |D_K|^{-1/16} T^{5/2} n^5 & \text{if } \pi_i \text{ are discrete series of weight } k_{\pi_i} = k_i, \\ |D_K|^{-1/12} T^2 n^2 & \text{if the level } N = 1 \text{ is trivial}, \\ |D_K|^{-1/12} T n^2 & \text{if } N = 1 \text{ and } \pi_i \text{ are discrete series of weight } k_{\pi_i} = k_i. \end{cases} \tag{6-8}$$

Proof. By the fact that $\prod_i \Omega_{i,K}$ is trivial, we see that

$$\prod_{i=1}^{n} f_i(z) = \prod_{i=1}^{n} \big(\Omega_{i,K}(x) f_i(z)\big)$$

for any $x \in \mathbb{A}_K^\times$. In particular, if we fix a quadratic form $Q \in \mathcal{Q}_{D_K}(N,r)$ and choose $x = x_Q \in \mathbb{A}_K^\times$ as in Corollary 4.3, then we get

$$\prod_{i=1}^{n} f_i(z_Q) = \prod_{i=1}^{n} \big(\Omega_{i,K}(x_Q) f_i(z_Q)\big) = \sum_{\chi_1, \dots, \chi_n \in \widehat{\mathrm{Cl}}_K} \prod_{i=1}^{n} \left(\varepsilon_{\chi_i, i} c_{f_i} \frac{|D_K|^{1/4}}{|\mathrm{Cl}_K|} \left| L\Big(\pi_i \otimes \chi_i \Omega_{i,K}, \frac{1}{2}\Big) \right|^{1/2} \chi_i([\mathfrak{a}]) \right).$$

Summing this identity over a set of representatives for $\Gamma_0(N) \backslash \mathcal{Q}_{D_K}(N,r) \cong \mathrm{Cl}_K$, applying Theorem 6.1, and using orthogonality of class group characters (i.e., the Fourier theoretic equality (1-2)=(1-3)), we arrive at the conclusion. $\square$

Remark 6.4. The fact that we have $\|\prod_i f_i\|_2$ in the error term and not, say, L^∞-norms, turns out to be crucial for applications to nonvanishing; see Section 7C.

6B. ***The diagonal case.*** In this subsection, we will use Theorem 6.3 to calculate another family of moments. For this consider the following "nontrivial diagonal":

$$\mathbf{Wide}_{\mathrm{ntd}}(\hat{G},2n) := \Big\{(\chi_1,\psi_1,\dots,\chi_n,\psi_n)\in(\hat{G})^{2n} : \chi_i\neq\psi_i,\ \prod_{i=1}^n\chi_i=\prod_{i=1}^n\psi_i\Big\}.$$

The starting point is the following lemma:

Lemma 6.5. *Let G be a finite abelian group and $L_1,\dots,L_n : G\to\mathbb{C}$ maps. Then we have*

$$\sum_{(\chi_i,\psi_i)\in\mathbf{Wide}_{\mathrm{ntd}}(\hat{G},2n)}\prod_{i=1}^n\widehat{L}_i(\chi_i)\overline{\widehat{L}_i(\psi_i)}=\frac{1}{|G|}\sum_{M\subset\{1,\dots,n\}}(-1)^{|M|}\Big(\sum_{g\in G}\prod_{i\notin M}|L_i(g)|^2\Big)\prod_{i\in M}\Big(\sum_{g\in G}|L_i(g)|^2\Big).$$

Here $\widehat{L}:\hat{G}\to\mathbb{C}$ denotes the Fourier transform given by $\chi\mapsto(1/|G|)\sum_{g\in G}L(g)\bar{\chi}(g)$.

Proof. By the principle of inclusion and exclusion, we have

$$\sum_{(\chi_i,\psi_i)\in\mathbf{Wide}_{\mathrm{ntd}}(\hat{G},2n)}\prod_{i=1}^n\widehat{L}_i(\chi_i)\overline{\widehat{L}_i(\psi_i)}$$
$$=\sum_{M\subset\{1,\dots,n\}}(-1)^{|M|}\sum_{\substack{(\chi_1,\overline{\psi_1},\dots,\chi_n,\overline{\psi_n})\in\mathbf{Wide}(\hat{G},2n)\\ \chi_i=\psi_i,i\in M}}\prod_{i=1}^n\widehat{L}_i(\chi_i)\overline{\widehat{L}_i(\psi_i)},\tag{6-9}$$

where the sum is over all subsets M of $\{1,\dots,n\}$. Furthermore, we have

$$\sum_{\substack{(\chi_1,\overline{\psi_1},\dots,\chi_n,\overline{\psi_n})\in\mathbf{Wide}(\hat{G},2n)\\ \chi_i=\psi_i,i\in M}}\prod_{i=1}^n\widehat{L}_i(\chi_i)\overline{\widehat{L}_i(\psi_i)}$$
$$=\Big(\sum_{(\chi_i,\overline{\psi_i})_{i\notin M}\in\mathbf{Wide}(\hat{G},2(n-|M|))}\prod_{i\notin M}\widehat{L}_i(\chi_i)\overline{\widehat{L}_i(\psi_i)}\Big)\prod_{i\in M}\Big(\sum_{\chi\in\hat{G}}|\widehat{L}_i(\chi)|^2\Big),\tag{6-10}$$

from which the result follows using the Fourier theoretic equality (1-2)=(1-3). ☐

From this we get the following corollary:

Corollary 6.6. *Let π_i,K,k_i be as in Theorem 6.3. For $i=1,\dots,n$, let $\Omega_{i,K}$ be a Hecke character of K of ∞-type $\alpha\mapsto(\alpha/|\alpha|)^{k_i}$ and $f_i\in\mathcal{B}_{k_i}(N)$ in the representation space of π_i. Then we have*

$$\sum_{(\chi_i,\psi_i)\in\mathbf{Wide}_{\mathrm{ntd}}(K,2n)}\prod_{i=1}^n\varepsilon_{\chi_i,\psi_i,f_i}|c_{f_i}|^2L\big(\pi_i\otimes\chi_i\Omega_{i,K},\tfrac12\big)^{1/2}L\big(\pi_i\otimes\psi_i\Omega_{i,K},\tfrac12\big)^{1/2}$$
$$=\frac{|\mathrm{Cl}_K|^{2n}}{|D_K|^{n/2}}\Big(\sum_{M\subset\{1,\dots,n\}}(-1)^{|M|}\Big\|\prod_{i\notin M}f_i\Big\|^2\cdot\prod_{i\in M}\|f_i\|^2+O_\varepsilon\Big(\prod_{i=1}^n\|f_i\|_\infty|D_K|^{-1/16}T^5n^52^n(T|D_K|n)^\varepsilon\Big)\Big),$$

as $|D_K|\to\infty$, where $c_{f_i}=(8N_i)^{-1}c_\infty(\pi_{i,\infty},k_i)$ with $c_\infty(\pi_{i,\infty},k_i)$ as in (4-11) and $\varepsilon_{\chi,\psi,f_i}$ complex numbers of norm 1.

Proof. The result follows from Lemma 6.5 combined with Theorem 6.3 by bounding the norms in the error terms by the L^∞-norms of the f_i. □

7. Applications to nonvanishing

Clearly, Theorem 6.3 gives a way to produce weak simultaneous nonvanishing results (in the sense of Section 2) given that we have

$$\left\langle \prod_{i=1}^{n} f_i, 1 \right\rangle \neq 0. \tag{7-1}$$

In this section, we show nonvanishing as in (7-1) in a number of different cases.

The simplest case is $n = 2$ and $f_1 = \overline{f_2}$ (which is the one considered by Michel and Venkatesh [2006]) where the period is the L^2-norm and thus automatically nonzero. Using our quantitative moment calculation in Theorem 6.3, we obtain a uniform version of [Michel and Venkatesh 2006, Theorem 1] in the general weight case.

The case $n = 3$ is also very appealing since the corresponding triple periods are connected to triple convolution L-functions via the Ichino–Watson formula [Ichino 2008; Watson 2002]. There are some prior work obtaining nonvanishing of triple periods, which immediately give weak simultaneous nonvanishing using Theorem 6.3. Reznikov [2001] showed using representation theory that for any Maaß form f of level N, there are infinitely many Maaß forms f_1 of level dividing N such that $\langle f^2, f_1\rangle \neq 0$ (in the level 1 case, this was reproved by Li [2009] using more analytic methods). Similarly, the quantum variance computation of Luo and Sarnak [2004] implies the following: for any Hecke–Maaß eigenform f with $L\big(f, \frac{1}{2}\big) \neq 0$, there are $\gg K$ many holomorphic newforms $g \in \mathcal{S}_k(1)$ with $K \leq k \leq 2K$ such that $\langle y^k |g|^2, f\rangle \neq 0$; see also [Sugiyama and Tsuzuki 2022]. We get similar nonvanishing with f a Hecke–Maaß newform using the corresponding quantum variance computation by Zhao and Sarnak [2019]. Note that the nonvanishing results for triple periods $\langle f_1 f_2 f_3, 1\rangle$ obtained in the above mentioned papers all have two of the forms equal. In terms of applications to nonvanishing these result are not that interesting. Motivated by this, we introduce below a method for obtaining nonvanishing for $n = 3$ where all of the forms f_1, f_2, f_3 are different.

Finally in the holomorphic case, we can show nonvanishing of periods for general n using a very soft argument.

7A. ***The second moment case.*** In this subsection, we consider the simplest case of $n = 2$ in which the nonvanishing of the main term in (6-7) is automatic. In particular, this gives an improved version of [Michel and Venkatesh 2006, Theorem 1] with uniformity in the spectral aspect and generalizes the results to general weights.

Corollary 7.1. *Let N be a fixed squarefree integer and $\varepsilon > 0$. Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ of level N, spectral parameter t_π, and even lowest weight k_π. Let k be an even integer such that $|k| \geq k_\pi$, and put $T = |t_\pi| + |k| + 1$.*

Then there exists a constant $c = c(N,\varepsilon) > 0$ such that for any imaginary quadratic field K such that all primes dividing N splits in K with discriminant $|D_K| \geq cT^{160/3+\varepsilon}$ (respectively, $|D_K| \geq cT^{22+\varepsilon}$ if $N = 1$), we have

$$\#\{\chi \in \widehat{\mathrm{Cl}}_K : L\big(\pi \otimes \chi\Omega_K, \tfrac{1}{2}\big) \neq 0\} \gg_\pi \begin{cases} |D_K|^{1/1058} & \text{if } \pi \text{ is d.s,} \\ |D_K|^{1/2648} & \text{if } \pi \text{ is p.s,} \end{cases}$$

where Ω_K is a Hecke character of K of conductor 1 and ∞-type $\alpha \mapsto (\alpha/|\alpha|)^k$.

Proof. Let π be as in the corollary above. We apply Theorem 6.3 with the error term coming from Remark 6.2 and with $\pi_1 = \pi_2 = \pi$ and $f_1 = \overline{f_2}$ belonging to π of weight $k \geq k_\pi$. In this special case, it is clear that we can truncate the spectral expansion (6-3) at $t_u \ll T^{1+\varepsilon}|D_K|^\varepsilon$ at a negligible error since we have

$$\|f_1 f_2\|_1 = \|f_1\|_2^2 = 1$$

(for any f_1 as above). Thus, both in the (raised) holomorphic and Maaß case, we have the error terms

$$O_\varepsilon\big(|D_K|^{-1/16}T^{20/6}(|D_K|T)^\varepsilon\big) \quad \text{for general level } N$$

and

$$O_\varepsilon\big(|D_K|^{-1/12}T^{11/6}(|D_K|T)^\varepsilon\big) \quad \text{for level } N = 1.$$

From this, we see that for $|D_K| \geq cT^{160/3+\varepsilon}$ (respectively, $|D_K| \geq cT^{22+\varepsilon}$), the RHS of (6-7) is nonzero. Thus, the LHS (6-7) is also nonzero and satisfies $\gg_{\varepsilon,k} |D_K|^{1/4-\varepsilon}$ using Siegel's lower bound (3-1). Now the result follows directly using the subconvexity bounds for Rankin–Selberg L-functions due to Michel [2004] and Harcos and Michel [2006]. □

7B. *Triple products of Maaß forms.* A very attractive case of Theorem 6.3 is $n = 3$, where the nonvanishing of $\langle f_1 f_2 f_3, 1\rangle$ is equivalent to the nonvanishing of the triple convolution L-function $L\big(\pi_1 \otimes \pi_2 \otimes \pi_3, \tfrac{1}{2}\big)$ due to the Ichino–Watson formula [Ichino 2008; Watson 2002]. In this section, we introduce a soft method (relying on results of Lindenstrauss and Jutila–Motohashi) to derive nonvanishing results in the case where f_1, f_2, f_3 are all Maaß forms of level 1.

By the spectral expansion for $L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$ [Iwaniec 2002, Theorem 7.3], we have

$$\|f_1 f_2\|_2^2 = \langle f_1 f_2, f_1 f_2\rangle = \sum_{f \in \mathcal{B}_0(1)} |\langle f_1 f_2, f\rangle|^2 + \frac{1}{4\pi}\int_{\mathbb{R}} |\langle f_1 f_2, E_t\rangle|^2\, dt, \tag{7-2}$$

where $E_t(z) = E\big(z, \tfrac{1}{2} + it\big)$ is the nonholomorphic Eisenstein series of level 1. Using the Ichino–Watson formula [Ichino 2008; Watson 2002] (which in the Eisenstein case reduces to Rankin–Selberg), we have

$$|\langle f_1 f_2, f\rangle|^2 = \frac{L(f_1 \otimes f_2 \otimes f, 1/2)}{8L(\mathrm{sym}^2 f_1, 1)L(\mathrm{sym}^2 f_2, 1)L(\mathrm{sym}^2 f, 1)} h(t_{f_1}, t_{f_2}, t_f)$$

and

$$|\langle f_1 f_2, E_t\rangle|^2 = \frac{|L(f_1 \otimes f_2, 1/2 + it)|^2}{4L(\mathrm{sym}^2 f_1, 1)L(\mathrm{sym}^2 f_2, 1)|\zeta(1+2it)|^2} h(t_{f_1}, t_{f_2}, t),$$

where

$$h(t_1,t_2,t_3)=\frac{\prod_{\pm}\Gamma(1/4\pm it_1/2\pm it_2/2\pm it_3/2)}{|\Gamma(1/2+it_1)|^2|\Gamma(1/2+it_2)|^2|\Gamma(1/2+it_3)|^2}.$$

Here the product is over all 8 combinations of signs. If we fix t_1, then it is standard using Stirling's approximation to prove that for $t_2,t_3\gg 1$, we have

$$h(t_1,t_2,t_3)\ll_{t_1} e^{-\pi|t_2-t_3|}(1+|t_2-t_3|)^{-1}(1+t_2+t_3)^{-1}.$$

This shows that the contribution from respectively, $|t-t_{f_2}|\geq (t_{f_2})^{\varepsilon}$ and $|t_f-t_{f_2}|\geq (t_{f_2})^{\varepsilon}$ in (7-2) is negligible.

We would like to show that actually all of the contribution from the Eisenstein part in (7-2) is negligible. This is connected to the subconvexity problem for Rankin–Selberg L-functions in a conductor dropping region, and is thus very difficult. We can however get unconditional results if we keep f_1 fixed and average over f_2 using the following result due to Jutila and Motohashi [2005, (3.50)]:

Theorem 7.2 (Jutila–Motohashi). *Let $f_1\in\mathcal{B}_0(1)$ be fixed. Then we have*

$$\sum_{|t_{f_2}-T|\leq T^{\varepsilon}}\left|L\left(f_1\otimes f_2,\tfrac{1}{2}+it\right)\right|^2\ll_{\varepsilon} T^{1+\varepsilon} \tag{7-3}$$

uniformly for $|t-T|\ll T^{\varepsilon}$.

Strictly speaking [Jutila and Motohashi 2005] only deals with the case where f_1 is an Eisenstein series, but (as remarked in [Blomer and Holowinsky 2010, p. 3]) the same estimate follows in the case of Maaß forms using the exact same argument relying on the spectral large sieve.

From Theorem 7.2, it follows that for any $\delta>0$, we have that

$$\int_{|t-t_{f_2}|\leq (t_{f_2})^{\varepsilon}}\left|L\left(f_1\otimes f_2,\tfrac{1}{2}+it\right)\right|^2dt\leq T^{1-\delta} \tag{7-4}$$

for all but at most $O_{\varepsilon}(T^{\delta+\varepsilon})$ Maaß forms f_2 with $|t_{f_2}-T|\leq T^{\varepsilon}$.

Recalling the estimates $t_f^{-\varepsilon}\ll_{\varepsilon} L(\mathrm{sym}^2 f,1)\ll_{\varepsilon} t_f^{\varepsilon}$, we conclude combining all of the above that for any f_2 satisfying (7-4), we have

$$\|f_1f_2\|_2^2=\sum_{|t_f-T|\leq T^{\varepsilon}}|\langle f_1f_2,f\rangle|^2+O_{\varepsilon}(T^{-\delta+\varepsilon}). \tag{7-5}$$

By QUE for Maaß forms due to Lindenstrauss [2006] (with key input by Soundararajan [2010]), we know that

$$\|f_1f_2\|_2\to\|f_1\|_2\neq 0,\quad\text{and}\quad \langle f_1f_2,f_2\rangle\to\left\langle f_1,\frac{3}{\pi}\right\rangle=0,$$

as $t_{f_2}\to\infty$. Thus we conclude from (7-5) that for T large enough there is some $f_3\neq f_2$ with $|t_{f_3}-T|\leq T^{\varepsilon}$ such that $\langle f_1f_2,f_3\rangle\neq 0$. Furthermore, we obtain a lower bound for free using Weyl's law,

$$\#\{f\in\mathcal{B}_0(1):|t_f-T|\leq T^{\varepsilon}\}\asymp T^{1+\varepsilon}.$$

From this we obtain the following result:

Proposition 7.3. *Let $f_1 \in \mathcal{B}_0(1)$ be fixed and $\varepsilon > 0$. Then for $T > 0$ large enough (depending on f_1 and ε), we have that for all but $O_\varepsilon(T^{2\varepsilon})$ of $f_2 \in \mathcal{B}_0(1)$ satisfying $|t_{f_2} - T| \le T^\varepsilon$, there exists some $f_3 \in \mathcal{B}_0(1)$ not equal to f_2 with $|t_{f_3} - T| \le T^\varepsilon$ such that*

$$|\langle f_1 f_2, f_3\rangle| \gg_\varepsilon \|f_1 f_2\|_2 / T^{1/2+\varepsilon}.$$

From this, we deduce the nonvanishing result in Corollary 1.3.

Proof of Corollary 1.3. Let f_2, f_3 be as in Proposition 7.3. Then we apply Theorem 6.3 (in the level 1 case) with $n = 3$, $k_1 = k_2 = k_3 = 0$, and test vectors f_1, f_2, f_3. We observe that

$$\|f_1 f_2 f_3\|_2 \, |D_K|^{-1/12} T^2 (|D_K| T)^\varepsilon \ll \|f_1 f_2\|_2 \, |t_{f_3}|^{5/12+\varepsilon} |D_K|^{-1/12} T^2 (|D_K| T)^\varepsilon,$$

by the sup-norm bound due to Iwaniec and Sarnak [1995]. Thus we see that if $|D_K| \gg_{f_1,\varepsilon} T^{35+\varepsilon}$, the error term in the asymptotic (6-7) (with exponents as in (6-8)) is strictly less than $\langle f_1 f_2 f_3, 3/\pi\rangle$. Thus we conclude that the LHS of (6-7) is nonvanishing and satisfies $\gg_{\varepsilon,T} |D_K|^{3/4-\varepsilon}$ (using Siegel's lower bound (3-1) again). Now by the subconvexity estimate for $L(f_i \otimes \theta_{\chi_i}, 1/2)$ due to Harcos and Michel [2006, Theorem 1] (where θ_{χ_i} is the holomorphic theta series associated to the Hecke character χ_i), we get the wanted quantitative nonvanishing result as $|D_K| \to \infty$. $\square$

7C. *The holomorphic case.* Consider Theorem 6.3 in the case where $\pi_1, \dots, \pi_n$ are all holomorphic discrete series representations of GL_2 and $k_i = k_{\pi_i} > 0$. Furthermore, pick $f_i = y^{k_i/2} g_i$, with $g_i \in \mathcal{S}_{k_i}(N)$ a holomorphic Hecke newform. Then we know that

$$\prod_{i=1}^n g_i \in \mathcal{S}_k(N),$$

where $k = \sum_i k_i$ (which might not be a Hecke–Maaß eigenform(!)). A basis $\mathcal{B}_{k,\mathrm{hol}}(N)$ for $\mathcal{S}_k(N)$ is given by $\nu^*_{d,N'} y^{k/2} g$, where $g \in \mathcal{S}_k(N')$ is a Hecke newform and $dN' \mid N$. This implies that

$$\left\| y^k \prod_{i=1}^n g_i \right\|_2^2 = \sum_{u_1, u_2 \in \mathcal{B}_{k,\mathrm{hol}}(N)} \langle u_1, u_2\rangle \left\langle y^{k/2} \prod_{i=1}^n g_i, u_1 \right\rangle \overline{\left\langle y^{k/2} \prod_{i=1}^n g_i, u_2 \right\rangle}.$$

Since any two $u_1, u_2 \in \mathcal{B}_{k,\mathrm{hol}}(N)$ are orthogonal (with respect to the Petersson inner product) if the underlying Hecke newforms are different and since the dimension of $\mathcal{S}_k(N')$ is $\ll_N k$, we conclude the following:

Proposition 7.4. *Let N be a fixed positive integer, and let $k_1, \dots, k_n \in 2\mathbb{Z}_{>0}$ be even integers. For $i = 1, \dots, n$, let $g_i \in \mathcal{S}_{k_i}(N)$ be a holomorphic Hecke newform of level N and weight k_i. Then there exists some $\nu^*_{d,N'} y^{k/2} g \in \mathcal{B}_{k,\mathrm{hol}}(N)$ with $k = k_1 + \dots + k_n$ such that*

$$\left\langle \prod_{i=1}^n y^{k_i/2} g_i, \nu^*_{d,N'} y^{k/2} g \right\rangle \gg \frac{\|\prod_{i=1}^n y^{k_i/2} g_i\|_2}{k^{1/2}}. \tag{7-6}$$

Combining this with Theorem 6.3, we obtain the following nonvanishing result:

Corollary 7.5. *Let N be a fixed squarefree integer, and let $k_1,\dots,k_n \in 2\mathbb{Z}_{>0}$ be even integers. For $i=1,\dots,n$, let π_i be automorphic representations corresponding to holomorphic newforms $g_i \in \mathcal{S}_{k_i}(N)$ and put $k=\sum k_i$. Then there exists a constant $c=c(N,\varepsilon)>0$ such that for any imaginary quadratic field K such that all primes dividing N split in K and the discriminant satisfies $|D_K|\geq c(\max_i k_i)^{40}n^{80}k^{12+\varepsilon}$, we have*

$$\#\{(\chi_1,\dots,\chi_{n+1})\in \mathbf{Wide}(K,n+1),\, g\in \mathcal{B}_{k,\mathrm{hol}}(\Gamma_0(N)) : \\ L\big(\pi_1\otimes\chi_1\Omega_{1,K},\tfrac12\big)\cdots L\big(\pi_n\otimes\chi_n\Omega_{n,K},\tfrac12\big)L\big(\pi_g\otimes\chi_{n+1}\Omega_{n+1,K},\tfrac12\big)\neq 0\} \\ \gg_k |D_K|^{(n+1)/2115},$$

where $k=\sum_i k_i$ and $\Omega_{i,K}$ are Hecke characters of K with ∞-types $x\mapsto (x/|x|)^{k_i}$ and $\Omega_{n+1,K}=\prod_{i=1}^n \Omega_{i,K}$.

Proof. For $i=1,\dots,n$, let $f_i=y^{k_i/2}g_i$, and let $f=\nu^*_{d,N'}y^{k/2}g\in\mathcal{B}_{k,\mathrm{hol}}(\Gamma_0(N))$ be as in Proposition 7.4.

We have the following sup-norm bound due to Xia [2007] (or more precisely the natural extension to general level):

$$\|f\|_\infty \ll_\varepsilon k^{1/4+\varepsilon}.$$

Thus, we conclude that

$$\Big\| f\prod_{i=1}^n f_i\Big\|_2 \ll_\varepsilon k^{1/4+\varepsilon}\Big\|\prod_{i=1}^n f_i\Big\|_2.$$

Combining the above with Theorem 6.3 (using the improved error term (6-8)) and the lower bound (7-6), we conclude that there is some constant depending only on N and $\varepsilon>0$ such that as soon as

$$|D_K|^{1/16}\gg_{N,\varepsilon}\big(\max_{i=1,\dots,n} k_i\big)^{5/2}n^5k^{1/4+1/2+\varepsilon},$$

then the RHS of (6-7) is nonzero. Thus the LHS (6-7) is also nonzero and is $\gg_{\varepsilon,k}|D_K|^{n/4-\varepsilon}$ using Siegel's lower bound (3-1).

Finally, since all of the f_i are holomorphic we can employ the subconvexity bound for Rankin–Selberg L-functions $L\big(f_i\otimes\theta_{\chi_i\Omega_{i,K}},\tfrac12\big)$ due to Michel [2004], where $\theta_{\chi_i\Omega_{i,K}}$ is the holomorphic theta series associated to the Hecke character $\chi_i\Omega_{i,K}$ defined in Section 3B. Finally, we use that

$$L\big(f\otimes\theta_{\chi\Omega_{n+1,K}},\tfrac12\big)=\overline{L\big(f\otimes\theta_{\bar\chi\bar\Omega_{n+1,K}},\tfrac12\big)}$$

to get rid of the conjugate in the last Rankin–Selberg L-functions. This gives the wanted qualitative lower bound for the nonvanishing. ☐

In the special case of level 1, we can do slightly better.

Proof of Corollary 1.4. Using the improved error term in Theorem 6.3 in the case of level 1 holomorphic forms, we see that the RHS of (6-7) is nonzero as soon as

$$|D_K|^{1/12}\gg_{N,\varepsilon}\big(\max_{i=1,\dots,n} k_i\big)n^2k^{3/4+\varepsilon}.$$

Using the trivial estimates $n\leq k$ and $\max_i k_i\leq k$, we conclude Corollary 1.4. ☐

7D. *Applications to Selmer groups.* In this last section, we will give applications of our results in the holomorphic case to triviality of the ranks of Bloch–Kato Selmer groups. We will restrict to level 1 for simplicity of exposition.

The setting is as follows: given a holomorphic Hecke eigenform f of weight k and level 1, a Hecke character Ω of an imaginary quadratic field $K/\mathbb{Q}$ of conductor 1 and infinity type $\alpha \mapsto (\alpha/|\alpha|)^k$, and a prime number $p > 2$, we have an associated *Bloch–Kato Selmer group*

$$\mathrm{Sel}(K, V_{f,\Omega}/\Lambda_{f,\Omega}),$$

where $V_{f,\Omega} := V_{f_p}|_{G_K} \otimes \Omega$ denotes the p-adic Galois representation associated to $f \otimes \Omega$ and $\Lambda_{f,\Omega} \subset V_{f,\Omega}$ is a certain lattice. For details and exact definitions, we refer to [Castella 2020, Definition 5.1]. The Bloch–Kato conjecture predicts that the rank of $\mathrm{Sel}(K, V_{f,\Omega}/\Lambda_{f,\Omega})$ is zero exactly if $L\big(\pi_f \otimes \Omega, \frac{1}{2}\big) \neq 0$. This conjecture has been proved under mild assumptions by Castella [2020, Theorem A]. In order to state these assumptions, we will need some notation. Given f as above, we denote by L_f the p-adic Hecke field of f and $\rho_f : G_{\mathbb{Q}} \to \mathrm{Aut}_{L_f}(V_f)$ the p-adic Galois representation associated to f and $\overline{\rho_f}$ the mod p reduction of ρ_f. We denote by Θ the set of all imaginary quadratic fields $K/\mathbb{Q}$ of odd discriminant D_K satisfying the following hypotheses:

(1) The prime p splits in K,

(2) $p \nmid h_K$,

(3) $\overline{\rho_f}|_{G_K}$ is absolutely irreducible.

Then we can rephrase our results in the following way:

Corollary 7.6. *Let f be a holomorphic Hecke eigenform of even weight k and level* 1. *Let $p > 5$ be a prime such that $p - 1 \mid k - 2$ and f is p-ordinary.*

Then there exists a constant $c = c(\varepsilon) > 0$ such that for any imaginary quadratic field $K \in \Theta$ with discriminant $|D_K| \geq ck^{22+\varepsilon}$, we have

$$\#\big\{\chi \in \widehat{\mathrm{Cl}}_K : \mathrm{rank}_{\mathbb{Z}}\big(\mathrm{Sel}(K, V_{f,\chi\Omega_K}/\Lambda_{f,\chi\Omega_K})\big) = 0\big\} \gg_f |D_K|^{1/1058},$$

where Ω_K is a Hecke character of K of conductor 1 *and ∞-type $\alpha \mapsto (\alpha/|\alpha|)^k$.*

Proof. This follows directly from Corollary 7.1 combined with the explicit reciprocity law [Castella 2020, Theorem A] and the arguments in [Castella 2020, Section 6.3]. ☐

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On Ozaki's theorem realizing prescribed p-groups as p-class tower groups

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We give a streamlined and effective proof of Ozaki's theorem that any finite p-group Γ is the Galois group of the p-Hilbert class field tower of some number field F. Our work is inspired by Ozaki's and applies in broader circumstances. While his theorem is in the totally complex setting, we obtain the result in any mixed signature setting for which there exists a number field k_0 with class number prime to p. We construct F/k_0 by a sequence of $\mathbb{Z}/p$-extensions ramified only at finite tame primes and also give explicit bounds on $[\mathrm{F} : \mathrm{k}_0]$ and the number of ramified primes of F/k_0 in terms of $\#\Gamma$.

1. Introduction

For a number field k, define $\mathrm{L}_p(\mathrm{k})$ to be the compositum of all finite unramified Galois p-extensions of k. The extension $\mathrm{L}_p(\mathrm{k})/\mathrm{k}$ is called the p-Hilbert class field tower of k, and its Galois group $\mathrm{Gal}(\mathrm{L}_p(\mathrm{k})/\mathrm{k})$ is its p-class tower group. Ozaki [2011] proved that every finite p-group Γ occurs as $\mathrm{Gal}(\mathrm{L}_p(\mathrm{F})/\mathrm{F})$ for some totally complex number field F. His strategy is as follows.

As finite p-groups are solvable, it is natural to proceed by induction. After establishing the base case (realizing $\mathbb{Z}/p$ as a p-class tower group), it remains to show that given any short exact sequence of finite p-groups

$$1 \to \mathbb{Z}/p \to \mathrm{G}' \to \mathrm{G} \to 1 \tag{1}$$

where $\mathrm{G} := \mathrm{Gal}(\mathrm{L}_p(\mathrm{k})/\mathrm{k})$, one can realize G' as $\mathrm{Gal}(\mathrm{L}_p(\mathrm{k}')/\mathrm{k}')$ for some number field k'. Ozaki constructs such a k'/k via a sequence of carefully chosen $\mathbb{Z}/p$-extensions.

In this paper, we provide a streamlined and effective proof of Ozaki's theorem. Some differences between our work and Ozaki's are:

- He must start with a totally complex k_0 and then construct a field F/k_0 whose p-Hilbert class field tower has the given Γ as its Galois group, while we start with a number field k_0 of arbitrary signature whose class number is prime to p.

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- Our result is effective and we are able to obtain explicit upper bounds on $[\mathrm{F}:\mathrm{k}_0]$ and the number of ramified primes in F/k_0, all of which are tame and finite.
- Moreover, we bypass some of the most delicate and involved arguments of [Ozaki 2011].

We prove:

Theorem. *Let Γ be a finite p-group and k_0 a number field with $(\#\mathrm{Cl}_{\mathrm{k}_0}, p) = 1$. There exist infinitely many number fields F/k_0 such that $\mathrm{Gal}(\mathrm{L}_p(\mathrm{F})/\mathrm{F}) \simeq \Gamma$ and*

- *if $\mu_p \not\subset \mathrm{k}_0$ then F/k_0 is of degree at most $p^2 \cdot \#\Gamma$ and is ramified at at most $2+2\log_p(\#\Gamma)$ finite tame primes,*
- *if $\mu_p \subset \mathrm{k}_0$ then F/k_0 is of degree at most $p\cdot(\#\Gamma)^2$ and is ramified at at most $1+3\log_p(\#\Gamma)$ finite tame primes.*

Remark. If our starting field k_0 has infinite p-Hilbert class field tower, there is no hope of solving the problem with a finite extension of k_0. If on the other hand the tower is finite, one can simply pass to the number field $\mathrm{L}_p(\mathrm{k}_0)$, which has the same signature ratio as k_0, and use that as the starting point to realize Γ.

As any (topologically) countably generated pro-p group Γ is the inverse limit of finite p-groups, Ozaki shows any such Γ is the Galois group of the maximal unramified p-extension of some infinite extension of $\mathbb{Q}$. The corresponding corollary of our theorem is:

Corollary. *Any (topologically) countably generated pro-p group Γ, including p-adic analytic Γ, can be realized as $\mathrm{Gal}(\mathrm{L}_p(\mathrm{F})/\mathrm{F})$ for a totally real tamely ramified infinite extension $\mathrm{F}/\mathbb{Q}$.*

We now give details about the structure of our proof and the difference between our methods and Ozaki's, though we were very much inspired by Ozaki's beautiful theorem and techniques.

We start the base case of the inductive process with any number field k_0, of any signature, whose class number is prime to p. Referring to the group extension (1) with G being trivial, one has to find an extension k'/k_0 such that k' has p-class group tower exactly $\mathbb{Z}/p$, which is equivalent to the p-class group being $\mathbb{Z}/p$. This is a standard argument and is part of Proposition 2.15.

The base case being done, we proceed to the inductive step (with our base field relabeled k). There are two cases, depending on whether (1) splits or not. For the sake of brevity, we only outline the nonsplit case in this introduction; the split case is handled similarly. For a set of places of k, we say that an extension k'/k is exactly ramified at S if it is ramified at all the places in S and nowhere else.

We need to find a suitable tame prime v_1 of k such that:

- v_1 splits completely in $\mathrm{L}_p(\mathrm{k})/\mathrm{k}$.
- There is no $\mathbb{Z}/p$-extension of k exactly ramified at v_1.
- The maximal p-extension $\mathrm{L}_p(\mathrm{k})_{\{v_1\}}/\mathrm{L}_p(\mathrm{k})$ exactly ramified at the primes of $\mathrm{L}_p(\mathrm{k})$ above v_1 is of degree p and solves the embedding problem (1).

Arranging this and its split analog are the main technical difficulties. One then chooses a second prime v_2 that also solves the embedding problem as above and remains prime in $\mathrm{L}_p(\mathrm{k})_{\{v_1\}}/\mathrm{L}_p(\mathrm{k})$. The existence of v_1 and v_2 will follow from Chebotarev's theorem. The compositum of these two solutions, after a $\mathbb{Z}/p$-base change k'/k ramified at both v_1 and v_2 (which exists!), gives the unramified solution to the embedding problem (1) which we show is $\mathrm{L}_p(\mathrm{k}')$. This is done in the proof of Theorem 3.3.

Our ability to choose primes v_i as above depends upon the existence of *Minkowski units* in the tower $\mathrm{L}_p(\mathrm{k})/\mathrm{k}$, namely on the condition that $\mathcal{O}^\times_{\mathrm{L}_p(\mathrm{k})} \otimes \mathbb{F}_p \simeq \mathbb{F}_p[\mathrm{G}]^\lambda \oplus N$ where N is an $\mathbb{F}_p[\mathrm{G}]$-torsion module and λ is a large enough integer. In some situations, Minkowski units are rare; see Section 5.3 of [Hajir et al. 2021]. By contrast, both for Ozaki's proof (implicitly) and ours (explicitly), much of the work involves seeking fields for which they exist in abundance.

If $\mu_p \subset \mathrm{k}$, we may not be able to make our choices of v_i as above to both split completely in $\mathrm{L}_p(\mathrm{k})/\mathrm{k}$ and solve the nonsplit embedding problem (1). In this case we need to perform an extra base change $\tilde{\mathrm{k}}/\mathrm{k}$ to shift the obstruction to the embedding problem so that we can proceed as above. The base change $\tilde{\mathrm{k}}/\mathrm{k}$ must preserve the tower, that is $\mathrm{L}_p(\tilde{\mathrm{k}}) = \mathrm{L}_p(\mathrm{k})\tilde{\mathrm{k}}$. Theorem 3.2 provides such a $\tilde{\mathrm{k}}$.

Finally we check that the condition "λ is large enough" persists, that is there are enough Minkowski units to keep the induction going. Proposition 2.14 guarantees this. To sum up, the key ingredients of the proof of the above theorem and corollary are Theorems 3.2 and 3.3 and Proposition 2.14.

We now explain in some detail Ozaki's approach and our simplifications:

- Using a result of Horie [1987], Ozaki starts with a quadratic imaginary field with class number prime to p in which p is inert. He then chooses a suitable layer k in the cyclotomic $\mathbb{Z}_p$-extension as the starting point of his induction. Assuming the problem solved for G in (1) and relabelling k as his base field, he proceeds inductively with the goal to find a $\mathrm{k}' \supset \mathrm{k}$ whose p-Hilbert class field tower has Galois group G'. For the induction to go forward, Ozaki needs $r_2(\mathrm{k}) \geq B_p(\mathrm{k})$ (implicit in this inequality is the existence of enough Minkowski units) where $B_p(\mathrm{k})$ is a certain explicit quantity depending on k, G and the p-part of the class group of $\mathrm{K} := \mathrm{L}_p(\mathrm{k})(\mu_p)$. This involves delicate estimates in Section 4 of [Ozaki 2011]. We replace $r_2(\mathrm{k}) \geq B_p(\mathrm{k})$ with $f(\mathrm{k}) \geq 2h^1(\mathrm{G}) + 3$ where $h^i(\mathrm{G}) := \dim H^i(\mathrm{G}, \mathbb{Z}/p)$ and $f(\mathrm{k})$, which is a lower bound for the number of Minkowski units in $\mathrm{L}_p(\mathrm{k})/\mathrm{k}$, depends only on $h^1(\mathrm{G})$, $h^2(\mathrm{G})$ and the signature of k. We neither consider K nor invoke the estimates of Section 4 of [loc. cit.].

- Ozaki [2011, Section 6] proves his base change Proposition 1, namely he shows there exists a ramified $\mathbb{Z}/p$-extension $\tilde{\mathrm{k}}/\mathrm{k}$ such that $\mathrm{Gal}(\mathrm{L}_p(\tilde{\mathrm{k}})/\tilde{\mathrm{k}}) \simeq \mathrm{Gal}(\mathrm{L}_p(\mathrm{k})/\mathrm{k})$. He uses this repeatedly when solving each embedding problem (1). Several tame primes are ramified in $\tilde{\mathrm{k}}/\mathrm{k}$ and he also needs that K and $\mathrm{K}\tilde{\mathrm{k}}$ have the same p-class group. This makes the proof significantly more involved. Theorem 3.2 of this paper, our version of his Proposition 1, has only one tame prime of ramification and K plays no role. We only invoke Theorem 3.2 when $\mu_p \subset \mathrm{k}$. In particular, for p odd, our Corollary above makes no use of Theorem 3.2.

- To solve the embedding problem (1), Ozaki base changes several times (to a field relabeled k) and then uses a wildly ramified $\mathbb{Z}/p$-extension $\mathrm{L}/\mathrm{L}_p(\mathrm{k})$ to solve (1). After more base changes this is switched to a solution ramified at one tame prime. He then proceeds as in the description of this work using two such solutions and a base change that absorbs the ramification at both tame primes to find a k' such that $\mathrm{Gal}(\mathrm{L}_p(\mathrm{k}')/\mathrm{k}') = \mathrm{G}'$. *We go directly to this last step and require at most two $\mathbb{Z}/p$-base changes to solve the embedding problem. This allows us to quantify explicitly both the degree and number of ramified primes of* F/k_0.

Notations. Let p be a prime number:

- L is a number field, $\mathcal{O}_{\mathrm{L}}$ its ring of integers, $\mathcal{O}_{\mathrm{L}}^\times$ its units and Cl_{L} and $\mathrm{Cl}_{\mathrm{L}}[p^\infty]$ are, respectively, the class group of L and its p-Sylow subgroup.
- For a finite set S of primes of L, set

$$V_{\mathrm{L},S} = \{x \in \mathrm{L}^\times, (x) = \mathcal{I}^p, x \in (\mathrm{L}_v^\times)^p \forall v \in S\}.$$

In particular, one has the exact sequence

$$1 \to \mathcal{O}_{\mathrm{L}}^\times \otimes \mathbb{F}_p \to V_{\mathrm{L},\varnothing}/(\mathrm{L}^\times)^p \to \mathrm{Cl}_{\mathrm{L}}[p] \to 1.$$

- The superscript $^\wedge$ indicates the Kummer dual of an object Z defined over a number field L, though we never work with the $\mathrm{Gal}(\mathrm{L}(\mu_p)/\mathrm{L})$ action on $Z^\wedge$.
- L_S is the maximal pro-p-extension of L unramified outside S, $G_S := \mathrm{Gal}(\mathrm{L}_S/\mathrm{L})$ and $\mathrm{L}_p(\mathrm{L}) := \mathrm{L}_\varnothing$, the maximal unramified pro-p-extension of L, as it will ease notation at various points.
- $h^i(H) := \dim H^i(H, \mathbb{Z}/p)$.
- $\mathrm{Gov}(\mathrm{L}) := \mathrm{L}(\mu_p)(\sqrt[p]{V_{\mathrm{L},\varnothing}})$, the governing field of L. The span of $\{\mathrm{Fr}_v\}_{v\in S}$ in $M(\mathrm{L}) := \mathrm{Gal}(\mathrm{Gov}(\mathrm{L})/\mathrm{L}(\mu_p))$ controls $\dim H^1(G_S)$.

The following may be helpful in orienting the reader:

- We frequently use finite tame primes with desired splitting properties in number field extensions. We *always* use Chebotarev's theorem for the existence of such primes.
- Our $\mathbb{Z}/p$-extensions L'/L of number fields are only ramified at (one or two) finite tame primes so $r_i(\mathrm{L}') = p \cdot r_i(\mathrm{L})$ and $\mu_p \subset \mathrm{L}' \Leftrightarrow \mu_p \subset \mathrm{L}$.
- Note that k_0 is our given base field, whereas k is a field used in the inductive process with p-class tower group G from (1). Our task is to construct k' with p-class tower group G'. Finally, $\tilde{\mathrm{k}}/\mathrm{k}$ is an extension having p-class tower group G, the same as for k.

2. Tools for the proof

2A. $\mathbb{F}_p$[G]-*modules and Minkowski units.* Let G be a finite group, a p-group in our situation. We record a few basic facts about finitely generated $\mathbb{F}_p[\mathrm{G}]$-modules M; see [Curtis and Reiner 1962, Section 62].

Fact 2.1. *Any finitely generated $\mathbb{F}_p[\mathrm{G}]$-module M is isomorphic to $\mathbb{F}_p[\mathrm{G}]^\lambda \oplus N$ where N is a torsion $\mathbb{F}_p[\mathrm{G}]$-module (every $n \in N$ is a torsion element) and where λ depends only on M.*

Proof. As free modules are clearly projective, Theorem 62.3 of [Curtis and Reiner 1962] implies they are injective. It follows immediately that if $\mathbb{F}_p[\mathrm{G}]$ is a submodule of an $\mathbb{F}_p[\mathrm{G}]$-module M, we have the $\mathbb{F}_p[\mathrm{G}]$-module decomposition $M = \mathbb{F}_p[\mathrm{G}] \oplus M^{(1)}$. Apply the same argument to $M^{(1)}$ and iterate until, at the λ-th stage there are no copies of $\mathbb{F}_p[\mathrm{G}]$ in $M^{(\lambda)}$. Thus for every $m_0 \in M^{(\lambda)}$ we have $\mathbb{F}_p[\mathrm{G}] \cdot m_0 \neq \mathbb{F}_p[\mathrm{G}]$ and thus m_0 has nontrivial annihilator. The result is established. $\square$

Set $T_{\mathrm{G}} := \sum_{g \in \mathrm{G}} g$. Denote by I_{G} the augmentation ideal of $\mathbb{F}_p[\mathrm{G}]$. For $x \in M$ set $\mathrm{Ann}_{\mathrm{G}}(x) := \{\alpha \in \mathbb{F}_p[\mathrm{G}] \mid \alpha \cdot x = 0\}$. Let $\{s_1, \dots, s_{h^1(\mathrm{G})}\}$ be a system of minimal generators of G. By Nakayama's lemma and the fact that $I_{\mathrm{G}}/I_{\mathrm{G}}^2 \simeq \mathrm{G}/\mathrm{G}^p[\mathrm{G}, \mathrm{G}]$, I_{G} can be generated, as G-(right or left)-module, by the elements $x_i := s_i - 1$.

Proposition 2.2. *With the x_i as above, let $M = \mathbb{F}_p[\mathrm{G}]^{h^1(\mathrm{G})}$ and $x = (x_1, x_2, \dots, x_{h^1(\mathrm{G})}) \in M$. Then* $\mathrm{Ann}_{\mathrm{G}}(x) = \mathbb{F}_p T_{\mathrm{G}}$.

Proof. $\mathrm{Ann}_{\mathrm{G}}(x) = \bigcap_i \mathrm{Ann}_{\mathrm{G}}(x_i) = \mathrm{Ann}_{\mathrm{G}}(\langle x_i \rangle_{i=1}^{h^1(\mathrm{G})}) = \mathrm{Ann}_{\mathrm{G}}(I_{\mathrm{G}}) = \mathbb{F}_p T_{\mathrm{G}}$. $\square$

Proposition 2.3. *Let $M = \mathbb{F}_p[\mathrm{G}]^\lambda \oplus N$ be a finitely generated $\mathbb{F}_p[\mathrm{G}]$-module where N is torsion. Then* $T_{\mathrm{G}}(M) \simeq \mathbb{F}_p^\lambda$.

Proof. It is clear that $T_{\mathrm{G}}(\mathbb{F}_p[\mathrm{G}]^\lambda) \simeq \mathbb{F}_p^\lambda$. We now show $T_{\mathrm{G}}(N) = 0$.

Let $n \in N$ so $\mathrm{Ann}_{\mathrm{G}}(n) \neq 0$. Note that $\mathrm{Ann}_{\mathrm{G}}(n) \subset \mathbb{F}_p[\mathrm{G}]$ is a p-group stable under the action of the p-group G and thus has a fixed point. But it is easy to see the only fixed points of $\mathbb{F}_p[\mathrm{G}]$ are multiples of T_{G} so $T_{\mathrm{G}} \in \mathrm{Ann}_{\mathrm{G}}(n)$ as desired. $\square$

Definition 2.4. We say the tower $\mathrm{L}_p(\mathrm{k})/\mathrm{k}$ with Galois group G has λ Minkowski units if, as $\mathbb{F}_p[\mathrm{G}]$-modules, $V_{\mathrm{L}_p(\mathrm{k}),\varnothing}/\mathrm{L}_p(\mathrm{k})^{\times p} = \mathcal{O}_{\mathrm{L}_p(\mathrm{k})}^\times \otimes \mathbb{F}_p \simeq \mathbb{F}_p[\mathrm{G}]^\lambda \oplus N$ where N is an $\mathbb{F}_p[\mathrm{G}]$-torsion module.

2B. ***Extensions ramified at a tame set of primes.*** We recall a standard formula on the number of $\mathbb{Z}/p$-extensions of a number field with given tame ramification; see Section 11.3 of [Koch 2002] for a proof. Recall that for a field L,

$$\delta(\mathrm{L}) = \begin{cases} 0, & \mu_p \not\subset \mathrm{L}, \\ 1, & \mu_p \subset \mathrm{L}. \end{cases}$$

Proposition 2.5. *Let* L *be a number field, p a prime number and X a set of tame primes of* L *prime to p. Then*

$$\dim H^1(\mathrm{G}_{\mathrm{L},X}, \mathbb{Z}/p) = \dim(V_{\mathrm{L},X}/\mathrm{L}^{\times p}) - r_1(\mathrm{L}) - r_2(\mathrm{L}) - \delta(\mathrm{L}) + 1 + \sum_{v \in X} \delta(\mathrm{L}_v). \tag{2}$$

Our $v \in X$ are always finite and have norm congruent to 1 mod p so $\delta(\mathrm{L}_v) = 1$.

Fact 2.6. *Let S be a set of tame primes of* L *as above. For each $v \in S$ let* $\mathrm{Fr}_v \in M(\mathrm{L}) := \mathrm{Gal}(\mathrm{Gov}(\mathrm{L})/\mathrm{L}(\mu_p))$. *If the set* $\{\mathrm{Fr}_v, v \in S\}$ *spans an $(\#S - d)$-dimensional subspace of* $M(\mathrm{L})$, *then*

$$\dim H^1(\mathrm{G}_{\mathrm{L},S}, \mathbb{Z}/p) = d + \dim H^1(\mathrm{G}_{\mathrm{L},\varnothing}, \mathbb{Z}/p).$$

When $\mu_p \not\subset \mathrm{L}$, Fr_v *is only well-defined up to nonzero scalar multiplication.*

Proof. In (2), as we vary X from $\varnothing$ to S, we are adding $\sum_{v\in S}\delta(\mathrm{L}_v)=\#S$ to the right side, but also subtracting $\dim(V_{\mathrm{L},\varnothing}/\mathrm{L}^{\times p})-\dim(V_{\mathrm{L},X}/\mathrm{L}^{\times p})$ from the right side. This last quantity is $\#S-d$. $\square$

Fact 2.7. *Let* L *be a number field such that* $(\#\mathrm{Cl}_{\mathrm{L}},p)=1$. *Let* L'/L *be a* $\mathbb{Z}/p$*-extension exactly ramified at* $S=\{v_1,\ldots,v_r\}$ *where the* v_i *are finite and tame. Then* $(\#\mathrm{Cl}_{\mathrm{L}'},p)=1$ *if and only if* L'/L *is the* ***unique*** $\mathbb{Z}/p$*-extension of* L *unramified outside* S. *In particular, that is the case when* $|S|=1$.

Proof. Indeed, $(\#\mathrm{Cl}_{\mathrm{L}'},p)\neq 1$ if and only if there exists an unramified $\mathbb{Z}/p$-extension H/L' such that H/L is Galois (use the fact the action of a p-group on a p-group always has fixed points). Observe that H/L cannot be cyclic of degree p^2 as all inertial elements of $\mathrm{Gal}(H/\mathrm{L})$ have order p and they would thus fix an unramified extension of L, a contradiction. So $\mathrm{Gal}(H/\mathrm{L})\simeq\mathbb{Z}/p\times\mathbb{Z}/p$, and L has at least two disjoints $\mathbb{Z}/p$-extension unramified outside S, also a contradiction. $\square$

Set $Б_{\mathrm{L},S}=(V_{\mathrm{L},S}/\mathrm{L}^{\times p})^\wedge$. Recall $Ш^2_{\mathrm{L},S}:=\mathrm{Ker}(H^2(G_S,\mathbb{Z}/p)\to\oplus_{v\in S}H^2(G_v,\mathbb{Z}/p))$. Fact 2.8 below is well-known; see Theorem 11.3 of [Koch 2002].

Fact 2.8. $Ш^2_{\mathrm{L},S}\hookrightarrow Б_{\mathrm{L},S}$.

Let λ_{L} be the number of Minkowski units in $\mathrm{L}_p(\mathrm{L})/\mathrm{L}$.

Fact 2.9. *If* $\mu_p\not\subset\mathrm{L}$ *then* $\lambda_{\mathrm{L}}=r_1(\mathrm{L})+r_2(\mathrm{L})-1+h^1(\mathrm{G})-h^2(\mathrm{G})$. *If* $\mu_p\subset\mathrm{L}$ *then* $\lambda_{\mathrm{L}}\geq r_1(\mathrm{L})+r_2(\mathrm{L})-h^2(\mathrm{G})$.

This result is Theorem 2.9 of [Hajir et al. 2021], but we sketch the proof for the sake of keeping this paper self-contained.

Proof. Set $\mathrm{G}=\mathrm{Gal}(\mathrm{L}_p(\mathrm{L})/\mathrm{L})$. We consider two "norm maps" induced by the norm map on units $\mathcal{O}^\times_{\mathrm{L}_p(\mathrm{L})}\to\mathcal{O}^\times_{\mathrm{L}}$:

- N_{G} sending $\mathcal{O}^\times_{\mathrm{L}_p(\mathrm{L})}\otimes\mathbb{F}_p$ to $\mathcal{O}^\times_{\mathrm{L}}/(\mathcal{O}^\times_{\mathrm{L}}\cap(\mathcal{O}^\times_{\mathrm{L}_p(\mathrm{L})})^p)\subset\mathcal{O}^\times_{\mathrm{L}_p(\mathrm{L})}\otimes\mathbb{F}_p$.
- $N'_{\mathrm{G}}:\mathcal{O}^\times_{\mathrm{L}_p(\mathrm{L})}\otimes\mathbb{F}_p\to\mathcal{O}^\times_{\mathrm{L}}\otimes\mathbb{F}_p$.

One easily sees $N'_{\mathrm{G}}(\mathcal{O}^\times_{\mathrm{L}_p(\mathrm{L})}\otimes\mathbb{F}_p)\twoheadrightarrow N_{\mathrm{G}}(\mathcal{O}^\times_{\mathrm{L}_p(\mathrm{L})}\otimes\mathbb{F}_p)$ and this is an isomorphism provided $\mathcal{O}^\times_{\mathrm{L}}\cap(\mathcal{O}^\times_{\mathrm{L}_p(\mathrm{L})})^p=(\mathcal{O}^\times_{\mathrm{L}})^p$: in particular this is the case when $\mu_p\not\subset\mathrm{L}$; see Proposition 2.8 of [Hajir et al. 2021].

Write $\mathcal{O}^\times_{\mathrm{L}_p(\mathrm{L})}\otimes\mathbb{F}_p\simeq\mathbb{F}_p[\mathrm{G}]^{\lambda_{\mathrm{L}}}\oplus N$, where N is an $\mathbb{F}_p[\mathrm{G}]$-torsion module. By Proposition 2.3 one has $N_{\mathrm{G}}(\mathcal{O}^\times_{\mathrm{L}_p(\mathrm{L})}\otimes\mathbb{F}_p)\simeq\mathbb{F}_p^{\lambda_{\mathrm{L}}}$. Hence, when $\mu_p\not\subset\mathrm{L}$

$$\dim\left(\frac{\mathcal{O}^\times_{\mathrm{L}}\otimes\mathbb{F}_p}{N'_{\mathrm{G}}(\mathcal{O}^\times_{\mathrm{L}_p(\mathrm{L})}\otimes\mathbb{F}_p)}\right)=\dim(\mathcal{O}^\times_{\mathrm{L}}\otimes\mathbb{F}_p)-\lambda_{\mathrm{L}}.$$

When $\mu_p\subset\mathrm{L}$, note that the "difference" between the images of N_{G} and N'_{G} has p-rank at most $\dim(\mathcal{O}^\times_{\mathrm{L}}\cap\mathcal{O}^{\times p}_{\mathrm{L}_p(\mathrm{L})}/(\mathcal{O}^\times_{\mathrm{L}})^p)\leq h^1(\mathrm{G})$, so

$$\dim\left(\frac{\mathcal{O}^\times_{\mathrm{L}}\otimes\mathbb{F}_p}{N'_{\mathrm{G}}(\mathcal{O}^\times_{\mathrm{L}_p(\mathrm{L})}\otimes\mathbb{F}_p)}\right)\geq\dim(\mathcal{O}^\times_{\mathrm{L}}\otimes\mathbb{F}_p)-\lambda_{\mathrm{L}}-h^1(\mathrm{G}).$$

To conclude, we use the well-known equality (see [Roquette 1967, Lemma 9])

$$h^2(\mathrm{G}) - h^1(\mathrm{G}) = \dim\left(\frac{\mathcal{O}_{\mathrm{L}}^{\times} \otimes \mathbb{F}_p}{N'_{\mathrm{G}}(\mathcal{O}_{\mathrm{L}_p(\mathrm{L})}^{\times} \otimes \mathbb{F}_p)}\right). \qquad \square$$

2C. *Solving the ramified embedding problem with one tame prime.* We start with our nonsplit exact sequence

$$1 \to \mathbb{Z}/p \to \mathrm{G}' \to \mathrm{G} \to 1. \tag{3}$$

given by the element $0 \neq \varepsilon \in H^2(\mathrm{G}, \mathbb{Z}/p)$.

We assume that $\mathrm{G} = \mathrm{Gal}(\mathrm{L}_p(\mathrm{k})/\mathrm{k})$.

Set $S = \{v\}$ where v is a finite tame prime of k. We first show the existence of a lift of G to G′ in some k_S/k for certain v of k. We call this solving the embedding problem (3) in k_S.

Recall that $Ш^2_{\mathrm{k},S} \hookrightarrow Б_{\mathrm{k},S}$ by Fact 2.8. Here $Ш^2_{\mathrm{k},\varnothing} \simeq H^2(\mathrm{G}_{\mathrm{k},\varnothing}, \mathbb{Z}/p) \simeq H^2(\mathrm{G}, \mathbb{Z}/p)$. Let $\mathrm{Inf}_S : H^2(\mathrm{G}_{\mathrm{k},\varnothing}, \mathbb{Z}/p) \to H^2(\mathrm{G}_{\mathrm{k},S}, \mathbb{Z}/p)$ be the inflation map. We have the commutative diagram:

$$\begin{tikzcd}
 & Ш^2_{\mathrm{k},\varnothing} \arrow[r, "\mathrm{Inf}_S"] \arrow[d, hook, "h"] & Ш^2_{\mathrm{k},S} \arrow[d, hook, "g"] \\
(\mathrm{k}_v^{\times} \otimes \mathbb{F}_p)^{\wedge} \arrow[r] & Б_{\mathrm{k},\varnothing} \arrow[r, two heads, "f_S"] & Б_{\mathrm{k},S}
\end{tikzcd}$$

By Hoeschmann's criteria (see [Neukirch et al. 2008, Chapter 3, Section 5]), the embedding problem has a solution in k_S if and only if $\mathrm{Inf}_S(\varepsilon) = 0$. As $\mathrm{L}_p(\mathrm{k})/\mathrm{k}$ is unramified, $\mathrm{Inf}_S(\varepsilon) \in Ш^2_{\mathrm{k},S}$ and as $g(\mathrm{Inf}_S(\varepsilon)) = f_S(h(\varepsilon)) \in Б_{\mathrm{k},S}$, the embedding problem has a solution if and only if $h(\varepsilon) \in \mathrm{Ker}(f_S)$.

Set $\mathrm{Gov}_S(\mathrm{k}) := \mathrm{k}(\mu_p)(\sqrt[p]{V_{\mathrm{k},S}})$. In the governing extensions $\mathrm{k}(\mu_p) \subset \mathrm{Gov}_S(\mathrm{k}) \subset \mathrm{Gov}(\mathrm{k})$, one sees that the kernel of the map $f_S : Б_{\mathrm{k},\varnothing} \twoheadrightarrow Б_{\mathrm{k},S}$ is exactly the (unramified) decomposition group D_v of the prime v. As noted in Fact 2.6, if $w_1, w_2 \mid v$ are two primes of $\mathrm{k}(\mu_p)$, their Frobenius elements in $\mathrm{Gal}(\mathrm{Gov}(\mathrm{k})/\mathrm{k}(\mu_p))$ differ by a nonzero scalar multiple.

We have proved:

Lemma 2.10. *The embedding problem* (3) *has a solution in* k_S/k *if and only if* $h(\varepsilon) \in D_v$. *Thus it has a solution in* k_S/k *if we choose the prime v such that* $\langle \mathrm{Fr}_v \rangle = \langle h(\varepsilon) \rangle$ *in* $M(\mathrm{k})$, *that is the lines spanned by these elements in* $M(\mathrm{k})$ *are equal. This is always possible by Chebotarev's theorem.*

2D. *Cohomological facts implying the persistence of Minkowski units.* Our main aim in this paper is to show that given a short exact sequence

$$1 \to \mathbb{Z}/p \to \mathrm{G}' \to \mathrm{G} \to 1$$

of finite p-groups where $\mathrm{G} = \mathrm{Gal}(\mathrm{L}_p(\mathrm{k})/\mathrm{k})$, there exists a finite tamely ramified extension k'/k with $\mathrm{G}' = \mathrm{Gal}(\mathrm{L}_p(\mathrm{k}')/\mathrm{k}')$. To solve this embedding problem using Theorem 3.3, the tower $\mathrm{L}_p(\mathrm{k})/\mathrm{k}$ must have $2h^1(\mathrm{G})$ Minkowski units. Proposition 2.14 below shows that if we start with enough Minkowski units, after a base change that realizes G′, we will be able to continue the induction. Proposition 2.13, which is only needed in the case when $\mu_p \subset \mathrm{k}$, shows that given at least $h^1(\mathrm{G})$ Minkowski units, we can perform a base change that preserves the tower and the number of Minkowski units increases. Proposition 2.11 is a basic group theory result bounding $h^1(\mathrm{G}')$ and $h^2(\mathrm{G}')$ in terms of $h^1(\mathrm{G})$ and $h^2(\mathrm{G})$. Furuta proves a similar result in Lemma 2 of [Furuta 1972].

Set $H^2(\mathrm{G}', \mathbb{Z}/p)_1 := \mathrm{Ker}(H^2(\mathrm{G}', \mathbb{Z}/p) \xrightarrow{\mathrm{Res}} H^2(\mathbb{Z}/p, \mathbb{Z}/p))$ and $h^2(\mathrm{G}')_1 := \dim H^2(\mathrm{G}', \mathbb{Z}/p)_1$. Note $h^2(\mathbb{Z}/p) = 1$ so $h^2(\mathrm{G}')_1$ is either $h^2(\mathrm{G}')$ or $h^2(\mathrm{G}') - 1$ and in either case $h^2(\mathrm{G}')_1 \geq h^2(\mathrm{G}') - 1$.

Proposition 2.11. *Let*

$$1 \to \mathbb{Z}/p \to \mathrm{G}' \to \mathrm{G} \to 1$$

be a short exact sequence of finite p-groups. Then $h^1(\mathrm{G}') \leq h^1(\mathrm{G}) + 1$ and $h^2(\mathrm{G}') \leq h^1(\mathrm{G}) + h^2(\mathrm{G}) + 1$.

Proof. The h^1 result is clear. For the h^2 statement we have the long exact sequence (see for instance [Dekimpe et al. 2012])

$$\begin{aligned} 0 \to H^1(\mathrm{G}, \mathbb{Z}/p) \to H^1(\mathrm{G}', \mathbb{Z}/p) \to H^1(\mathbb{Z}/p, \mathbb{Z}/p)^G& \\ \to H^2(\mathrm{G}, \mathbb{Z}/p) \to H^2(\mathrm{G}', \mathbb{Z}/p)_1 \to H^1(\mathrm{G}, H^1(\mathbb{Z}/p, \mathbb{Z}/p))&. \end{aligned}$$

If $\mathrm{G}' \to \mathrm{G}$ splits, we have

$$0 \to H^2(\mathrm{G}, \mathbb{Z}/p) \to H^2(\mathrm{G}', \mathbb{Z}/p)_1 \to H^1(\mathrm{G}, H^1(\mathbb{Z}/p, \mathbb{Z}/p))$$

so $h^2(\mathrm{G}')_1 \leq h^2(\mathrm{G}) + h^1(\mathrm{G})$ and since $h^2(\mathrm{G}')_1 \geq h^2(\mathrm{G}') - 1$ the result follows.

In the nonsplit case we have

$$0 \to H^1(\mathbb{Z}/p, \mathbb{Z}/p)^{\mathrm{G}} \to H^2(\mathrm{G}, \mathbb{Z}/p) \to H^2(\mathrm{G}', \mathbb{Z}/p)_1 \to H^1(\mathrm{G}, H^1(\mathbb{Z}/p, \mathbb{Z}/p))$$

so $h^2(\mathrm{G}')_1 \leq h^2(\mathrm{G}) - 1 + h^1(\mathrm{G})$ so $h^2(\mathrm{G}') \leq h^1(\mathrm{G}) + h^2(\mathrm{G})$. □

Definition 2.12. For a number field L set $\mathrm{G} = \mathrm{Gal}(\mathrm{L}_p(\mathrm{L})/\mathrm{L})$. Define f as follows:

$$f(\mathrm{L}) = \begin{cases} r_1(\mathrm{L}) + r_2(\mathrm{L}) - h^2(\mathrm{G}) + h^1(\mathrm{G}) - 1, & \mu_p \not\subset \mathrm{L}, \\ r_1(\mathrm{L}) + r_2(\mathrm{L}) - h^2(\mathrm{G}), & \mu_p \subset \mathrm{L}. \end{cases}$$

Fact 2.9 implies $f(\mathrm{L})$ is a lower bound on the number of Minkowski units of $\mathrm{L}_p(\mathrm{L})/\mathrm{L}$.

Proposition 2.13. *Let $\tilde{\mathrm{k}}/\mathrm{k}$ be a $\mathbb{Z}/p$-extension ramified at finite tame primes such that* $\mathrm{G} = \mathrm{Gal}(\mathrm{L}_p(\mathrm{k})/\mathrm{k}) = \mathrm{Gal}(\mathrm{L}_p(\tilde{\mathrm{k}})/\tilde{\mathrm{k}})$. *Then* $f(\tilde{\mathrm{k}}) = f(\mathrm{k}) + (p-1)(r_1(\mathrm{k}) + r_2(\mathrm{k}))$.

Proof. This follows immediately as we have the same group G for k and $\tilde{\mathrm{k}}$, $\mu_p \subset \tilde{\mathrm{k}} \iff \mu_p \subset \mathrm{k}$ and $r_i(\tilde{\mathrm{k}}) = p \cdot r_i(\mathrm{k})$. □

Proposition 2.14. *Let k'/k be a tamely ramified $\mathbb{Z}/p$-extension such that* $\mathrm{G} = \mathrm{Gal}(\mathrm{L}_p(\mathrm{k})/\mathrm{k})$ *and* $\mathrm{G}' = \mathrm{Gal}(\mathrm{L}_p(\mathrm{k}')/\mathrm{k}')$ *where*

$$1 \to \mathbb{Z}/p \to \mathrm{G}' \to \mathrm{G} \to 1.$$

Let $f(\mathrm{k})$ be as in Definition 2.12. Then

$$f(\mathrm{k}) \geq 2h^1(\mathrm{G}) + 3 \implies f(\mathrm{k}') \geq 2h^1(\mathrm{G}') + 3.$$

Proof. We do the case $\mu_p \not\subset \mathrm{k}$ first. We need to prove

$$r_1(\mathrm{k})+r_2(\mathrm{k})-h^2(\mathrm{G})+h^1(\mathrm{G})-1 \geq 2h^1(\mathrm{G})+3 \Longrightarrow r_1(\mathrm{k}')+r_2(\mathrm{k}')-h^2(\mathrm{G}')+h^1(\mathrm{G}')-1 \geq 2h^1(\mathrm{G}')+3,$$

that is

$$r_1(\mathrm{k}')+r_2(\mathrm{k}') \overset{?}{\geq} h^1(\mathrm{G}')+h^2(\mathrm{G}')+4.$$

Clearly

$$r_1(\mathrm{k}')+r_2(\mathrm{k}') = p(r_1(\mathrm{k})+r_2(\mathrm{k})) \geq p(h^1(\mathrm{G})+h^2(\mathrm{G})+4)$$

and by Proposition 2.11 we have

$$h^2(\mathrm{G}')+h^1(\mathrm{G}')+4 \leq (h^1(\mathrm{G})+h^2(\mathrm{G})+1)+(h^1(\mathrm{G})+1)+4 = 2h^1(\mathrm{G})+h^2(\mathrm{G})+6$$

so it suffices to show

$$(p-1)h^2(\mathrm{G})+(p-2)h^1(\mathrm{G})+4p \overset{\checkmark}{\geq} 6.$$

This holds for all p.

When $\mu_p \subset \mathrm{k}$. We need to prove

$$r_1(\mathrm{k})+r_2(\mathrm{k})-h^2(\mathrm{G}) \geq 2h^1(\mathrm{G})+3 \Longrightarrow r_1(\mathrm{k}')+r_2(\mathrm{k}')-h^2(\mathrm{G}') \geq 2h^1(\mathrm{G}')+3,$$

that is

$$r_1(\mathrm{k}')+r_2(\mathrm{k}') \overset{?}{\geq} 2h^1(\mathrm{G}')+h^2(\mathrm{G}')+3.$$

Again using Proposition 2.11 and that $r_i(\mathrm{k}') = p \cdot r_i(\mathrm{k})$ it suffices to show

$$(p-1)h^2(\mathrm{G})+(2p-3)h^1(\mathrm{G})+3p \overset{\checkmark}{\geq} 6$$

which holds for all p. $\square$

Proposition 2.15 below provides the base case of the induction.

Proposition 2.15. *Recall* $(\#\mathrm{Cl}_{\mathrm{k}_0}, p) = 1$. *There exists a tamely ramified extension* k'/k_0 *such that*

- *the p-part of the class group of* k' *is* $\mathbb{Z}/p$,
- $[\mathrm{k}' : \mathrm{k}_0] = p^3$,
- *and* $f(\mathrm{k}') > 2h^1(\mathbb{Z}/p)+3 = 5$.

Proof. Since $\mathrm{L}_p(\mathrm{k}_0) = \mathrm{k}_0$, we see $\mathrm{G} = \{e\}$. Choose a tame prime v of k whose Frobenius is trivial in the governing Galois group $M(\mathrm{k})$. By Fact 2.6 there is a unique $\mathbb{Z}/p$-extension $\mathrm{k}_1/\mathrm{k}_0$ unramified outside v. That $(\#\mathrm{Cl}_{\mathrm{k}_1}, p) = 1$ follows from Fact 2.7. Repeat this process with k_1 to get a field k_2 with $(\#\mathrm{Cl}_{\mathrm{k}_2}, p) = 1$.

We do one more base change to find a field k' with class group $\mathbb{Z}/p$. This is proved more generally as part of Theorem 3.3, but we include a short proof here.

Choose v_1 a finite tame prime of k_2 with trivial Frobenius in $M(\mathrm{k}_2)$ so that by Fact 2.6 there exists a unique D_1/k_2 ramified at v_1. As $D_1 \cap \mathrm{Gov}(\mathrm{k}_2) = \mathrm{k}_2$, we may choose v_2 a finite tame prime of k_2 with trivial Frobenius in $\mathrm{Gov}(\mathrm{k}_2)$ such that v_2 remains prime in D_1/k_2. Again by Fact 2.6 there exists a unique D_2/k_2 ramified at v_2.

Let D/k_2 be any of the $p-1$ "diagonal" $\mathbb{Z}/p$-extensions of k_2 between D_1 and D_2 so D_1D_2/D is everywhere unramified. We claim $D_1D_2 = \mathrm{L}_p(D)$. Indeed, by Fact 2.7 applied to D_1/k_2 we see $(\#\mathrm{Cl}_{D_1}, p) = 1$. As v_2 is inert in D_1/k_2, the extension D_2D_1/D_1 is ramified only at v_2 and Fact 2.7 applied to D_2D_1/D_1 implies $(\#\mathrm{Cl}_{D_1D_2}, p) = 1$. Whether or not $\mu_p \subset \mathrm{k}_0$, we have $\mathrm{k}' := D$, $\mathrm{Cl}_{\mathrm{k}'}[p^\infty] = \mathbb{Z}/p$ and

$$f(\mathrm{k}') \geq r_1(\mathrm{k}') + r_2(\mathrm{k}') - h^2(\mathbb{Z}/p) = p^3 r_1(\mathrm{k}_0) + p^3 r_2(\mathrm{k}_0) - 1 > 5 = 2h^1(\mathbb{Z}/p) + 3. \qquad \square$$

Depending on p and the signature of k_0 one can decrease the number of base changes, but this analysis complicates the statement of the main theorem without significant gain.

3. Solving the embedding problem

Having established the base case of our induction, we now prove Theorem 3.3.

Inductive Step. *Let*

$$1 \to \mathbb{Z}/p \to \mathrm{G}' \to \mathrm{G} \to 1$$

be exact and let k *be a number field with* $\mathrm{Gal}(\mathrm{L}_p(\mathrm{k})/\mathrm{k}) = \mathrm{G}$ *and* $f(\mathrm{k}) \geq 2h^1(\mathrm{G}) + 3$. *Then there exists a number field* k'/k *with* $\mathrm{Gal}(\mathrm{L}_p(\mathrm{k}')/\mathrm{k}') = \mathrm{G}'$ *and* $f(\mathrm{k}') \geq 2h^1(\mathrm{G}') + 3$.

Theorem 3.2 below is only necessary for the key inductive step, Theorem 3.3, when $\mu_p \subset \mathrm{k}$.

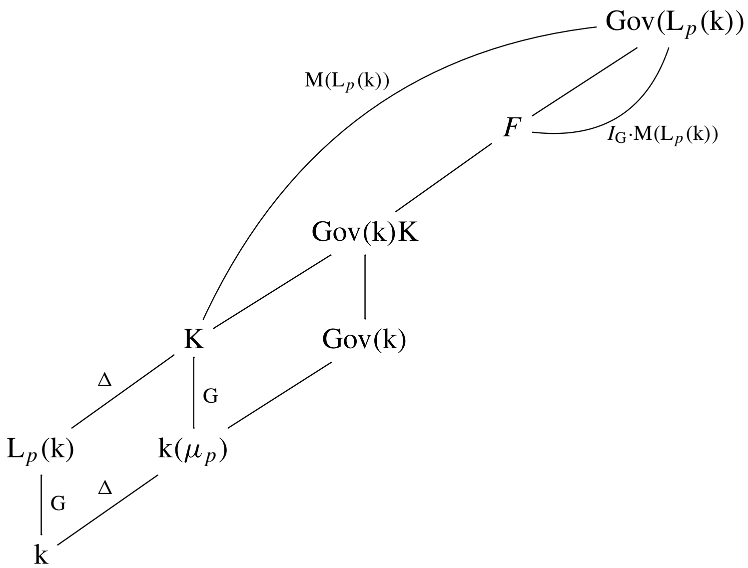
Set $\mathrm{K} := \mathrm{L}_p(\mathrm{k})(\mu_p)$. We only consider finite tame primes v of k that split completely in K/k. When $\mu_p \not\subset \mathrm{k}$, our Frobenius elements in governing fields (or their subfields) are only defined up to scalar multiples. We write $\langle \mathrm{Fr}_v \rangle_{\mathrm{Gov}(\mathrm{k})/\mathrm{k}(\mu_p)}$ for the well-defined line spanned by Frobenius at v in $\mathrm{Gal}(\mathrm{Gov}(\mathrm{k})/\mathrm{k}(\mu_p))$. When the Frobenius is trivial there is no ambiguity so we write $\langle \mathrm{Fr}_v \rangle_{\mathrm{Gov}(\mathrm{k})/\mathrm{k}(\mu_p)} = 0$.

We need primes v of k that let us control $h^1(\mathrm{Gal}(\mathrm{k}_{\{v\}}/\mathrm{k}))$ and $h^1(\mathrm{Gal}(\mathrm{L}_p(\mathrm{k})_{\{v\}}/\mathrm{L}_p(\mathrm{k})))$ simultaneously via Fact 2.6. Recall $M(\mathrm{L}_p(\mathrm{k})) := \mathrm{Gal}(\mathrm{Gov}(\mathrm{L}_p(\mathrm{k}))/\mathrm{L}_p(\mathrm{k})(\mu_p)) \simeq \mathbb{F}_p[\mathrm{G}]^{\lambda_\mathrm{k}} \oplus N$ where N is a torsion module over $\mathbb{F}_p[\mathrm{G}]$. We have no knowledge of N and must work with the free part to control things over $\mathrm{L}_p(\mathrm{k})$. We then use Proposition 3.1 to control things over k.

3A. *The stability theorem.*

Proposition 3.1. *Let* $F \subset \mathrm{Gov}(\mathrm{L}_p(\mathrm{k}))$ *be the field fixed by* $I_\mathrm{G} \cdot M(\mathrm{L}_p(\mathrm{k}))$. *For* v *of* k *splitting completely in* K *and* $w \mid v$ *in* K, *the lines* $\langle \mathrm{Fr}_w \rangle_{F/\mathrm{K}}$ *do not depend on* w *so we may write* $\langle \mathrm{Fr}_v \rangle_{F/\mathrm{K}}$. *Then* $\langle \mathrm{Fr}_{v_1} \rangle_{F/\mathrm{K}} = \langle \mathrm{Fr}_{v_2} \rangle_{F/\mathrm{K}}$ *implies* $\langle \mathrm{Fr}_{v_1} \rangle_{\mathrm{Gov}(\mathrm{k})/\mathrm{k}(\mu_p)} = \langle \mathrm{Fr}_{v_2} \rangle_{\mathrm{Gov}(\mathrm{k})/\mathrm{k}(\mu_p)}$. *If* $\langle \mathrm{Fr}_{v_1} \rangle_{F/\mathrm{K}} = 0$ *then* $\langle \mathrm{Fr}_{v_1} \rangle_{\mathrm{Gov}(\mathrm{k})/\mathrm{k}(\mu_p)} = 0$.

Proof. This diagram is useful in Theorems 3.2 and 3.3 as well:



Let $\Delta = \mathrm{Gal}(\mathrm{k}(\mu_p)/\mathrm{k}) = \mathrm{Gal}(\mathrm{K}/\mathrm{L}_p(\mathrm{k}))$. As $\mathrm{Gal}(F/\mathrm{K}) := M(\mathrm{L}_p(\mathrm{k}))/I_{\mathrm{G}} \cdot M(\mathrm{L}_p(\mathrm{k}))$ is the maximal quotient of $M(\mathrm{L}_p(\mathrm{k}))$ on which G acts trivially, and Δ acts on $\mathrm{Gal}(F/\mathrm{K})$ by scalars, the line $\langle \mathrm{Fr}_w \rangle_{F/\mathrm{K}}$ is invariant under the action of $\mathrm{Gal}(\mathrm{K}/\mathrm{k}) = \mathrm{G} \times \Delta$. Since the $w \mid v$ form an orbit under this action of $\mathrm{Gal}(\mathrm{K}/\mathrm{k})$, this line is independent of the choice of $w \mid v$ as desired.

As $\mathrm{Gov}(\mathrm{k})\mathrm{K}/\mathrm{K}$ ascends from $\mathrm{Gov}(\mathrm{k})/\mathrm{k}(\mu_p)$, we see G acts trivially on $\mathrm{Gal}(\mathrm{Gov}(\mathrm{k})\mathrm{K}/\mathrm{K})$ so

$$\mathrm{Gov}(\mathrm{k})\mathrm{K} \subset F.$$

Below, we implicitly use that our primes of k split completely in K. If $\langle \mathrm{Fr}_{v_1} \rangle_{F/\mathrm{K}} = \langle \mathrm{Fr}_{v_2} \rangle_{F/\mathrm{K}}$, these lines are equal when projected to $\mathrm{Gal}(Gov(\mathrm{k})\mathrm{K}/\mathrm{K}) \subset \mathrm{Gal}(\mathrm{Gov}(\mathrm{k})\mathrm{K}/\mathrm{k}(\mu_p))$ and they are again equal in $\mathrm{Gal}(\mathrm{Gov}(\mathrm{k})/\mathrm{k}(\mu_p))$ so $\langle \mathrm{Fr}_{v_1} \rangle_{\mathrm{Gov}(\mathrm{k})/\mathrm{k}(\mu_p)} = \langle \mathrm{Fr}_{v_2} \rangle_{\mathrm{Gov}(\mathrm{k})/\mathrm{k}(\mu_p)}$. The last statement is clear. □

Theorem 3.2. *Recall $\{x_i\}_{i=1}^{h^1(\mathrm{G})}$ is a minimal set of generators of I_{G}. Assume that $f(\mathrm{k}) \geq h^1(\mathrm{G})$. Let w be a degree one prime of* K *such that*

$$\mathrm{Fr}_w = ((x_1, x_2, \ldots, x_{h^1(\mathrm{G})}, 0, \ldots, 0), 0) \in M(\mathrm{L}_p(\mathrm{k})) \simeq \mathbb{F}_p[\mathrm{G}]^{\lambda_{\mathrm{k}}} \oplus N.$$

Then for v of k *below w,*

$$\langle \mathrm{Fr}_v \rangle_{\mathrm{Gov}(\mathrm{k})/\mathrm{k}(\mu_p)} = 0$$

so there exists a $\mathbb{Z}/p$-extension $\tilde{\mathrm{k}}/\mathrm{k}$ ramified at only v. Furthermore,

$$\mathrm{L}_p(\tilde{\mathrm{k}}) = \mathrm{L}_p(\mathrm{k})\tilde{\mathrm{k}} \quad \textit{and} \quad f(\tilde{\mathrm{k}}) > f(\mathrm{k}).$$

Proof. As Fr_w projects to 0 in the $\mathbb{F}_p$-vector space $\mathrm{Gal}(F/\mathrm{K})$, Proposition 3.1 implies $\langle \mathrm{Fr}_v \rangle_{\mathrm{Gov}(\mathrm{k})/\mathrm{k}(\mu_p)} = 0$ so $\tilde{k}$ exists by Fact 2.6. We show the $\mathbb{F}_p[\mathrm{G}]$-span of $(x_1, \ldots, x_{h^1(\mathrm{G})}) \in \mathbb{F}_p[\mathrm{G}]^{h^1(\mathrm{G})}$ has dimension $\#\mathrm{G} - 1$

by computing the dimension of $\bigcap_{i=1}^{h^1(\mathrm{G})} \mathrm{Ann}(x_i)$. This intersection is the annihilator of I_{G} which by Proposition 2.2 is just $\mathbb{F}_p T_{\mathrm{G}}$, establishing our dimension result. By Fact 2.6 there is a unique extension over $\mathrm{L}_p(\mathrm{k})$ ramified at v and thus it must be $\mathrm{L}_p(\mathrm{k})\tilde{\mathrm{k}}$. Fact 2.7 applied to $\mathrm{L}_p(\mathrm{k})\tilde{\mathrm{k}}/\mathrm{L}_p(\mathrm{k})$ implies $(\#\mathrm{Cl}_{\mathrm{L}_p(\mathrm{k})\tilde{\mathrm{k}}}, p) = 1$ so

$$\mathrm{L}_p(\tilde{\mathrm{k}}) = \mathrm{L}_p(\mathrm{k})\tilde{\mathrm{k}}.$$

Proposition 2.13 gives

$$f(\tilde{\mathrm{k}}) > f(\mathrm{k}). \qquad \square$$

3B. ***The inductive step.***

Theorem 3.3. *Assume that $\mathrm{L}_p(\mathrm{k})/\mathrm{k}$ has $\lambda_{\mathrm{k}} \geq 2h^1(\mathrm{G}) + 3$ Minkowski units. Let $1 \to \mathbb{Z}/p \to \mathrm{G}' \to \mathrm{G} \to 1$. If the extension splits or $\mu_p \not\subset \mathrm{k}$, there exists a $\mathbb{Z}/p$-extension k'/k such that $\mathrm{Gal}(\mathrm{L}_p(\mathrm{k}')/\mathrm{k}') \simeq \mathrm{G}'$ and $\mathrm{L}_p(\mathrm{k}')/\mathrm{k}'$ has at least $2h^1(\mathrm{G}') + 3$ Minkowski units. If $\mu_p \subset \mathrm{k}$ and the extension is nonsplit, k' can be realized as a compositum of two successive $\mathbb{Z}/p$-extensions and $\mathrm{L}_p(\mathrm{k}')/\mathrm{k}'$ has at least $2h^1(\mathrm{G}') + 3$ Minkowski units.*

Proof. Recall that our finite tame primes split completely in K/k. We first treat the split case. This is independent of whether or not $\mu_p \subset \mathrm{k}$.

Split case. Choose tame degree one primes w_1 and w_2 of $\mathrm{Gov}(\mathrm{k})\mathrm{K}$ such that

- $\mathrm{Fr}_{w_1} = ((x_1, x_2, \dots, x_{h^1(\mathrm{G})}, 0, \dots, 0), 0) \in \mathrm{Gal}(\mathrm{Gov}(\mathrm{L}_p(\mathrm{k}))/\mathrm{Gov}(\mathrm{k})\mathrm{K}) \subset M(\mathrm{L}_p(\mathrm{k}))$. This is possible as the tuple lies in $I_{\mathrm{G}} \cdot M(\mathrm{L}_p(\mathrm{k}))$ and $\mathrm{Gov}(\mathrm{k})\mathrm{K} \subset F$. As Fr_{w_1} projects to 0 in $\mathrm{Gal}(F/\mathrm{K})$, we see for v_1 of k below w_1 that $\langle \mathrm{Fr}_{v_1} \rangle_{F/\mathrm{K}} = 0$ so by Proposition 3.1 $\langle \mathrm{Fr}_{v_1} \rangle_{\mathrm{Gov}(\mathrm{k})/\mathrm{k}(\mu_p)} = 0$. By Fact 2.6 applied to k there is one $\mathbb{Z}/p$-extension D_1/k ramified at v_1. Fact 2.6 also gives (see the proof of Theorem 3.2 as well) a unique $\mathbb{Z}/p$-extension of $\mathrm{L}_p(\mathrm{k})$ ramified at v_1, namely $D_1\mathrm{L}_p(\mathrm{k})/\mathrm{L}_p(\mathrm{k})$.
- $\mathrm{Fr}_{w_2} = ((0, 0, \dots, 0_{h^1(\mathrm{G})}, x_1, x_2, \dots, x_{h^1(\mathrm{G})}, 0, 0, 0, \dots, 0), 0)$ so for v_2 of k below w_2, $\langle \mathrm{Fr}_{v_2} \rangle_{F/\mathrm{K}} = 0$. We also insist that v_2 remains prime in D_1/k. This last condition is linearly disjoint from the rest of the defining splitting conditions on v_2 and imposes no contradiction. Again, there are unique $\mathbb{Z}/p$-extensions of both k and $\mathrm{L}_p(\mathrm{k})$ ramified at v_2, namely D_2/k and $D_2\mathrm{L}_p(\mathrm{k})/\mathrm{L}_p(\mathrm{k})$. Let D/k be a "diagonal" extension between D_1 and D_2 ramified at both v_1 and v_2. There are $p - 1$ of these.

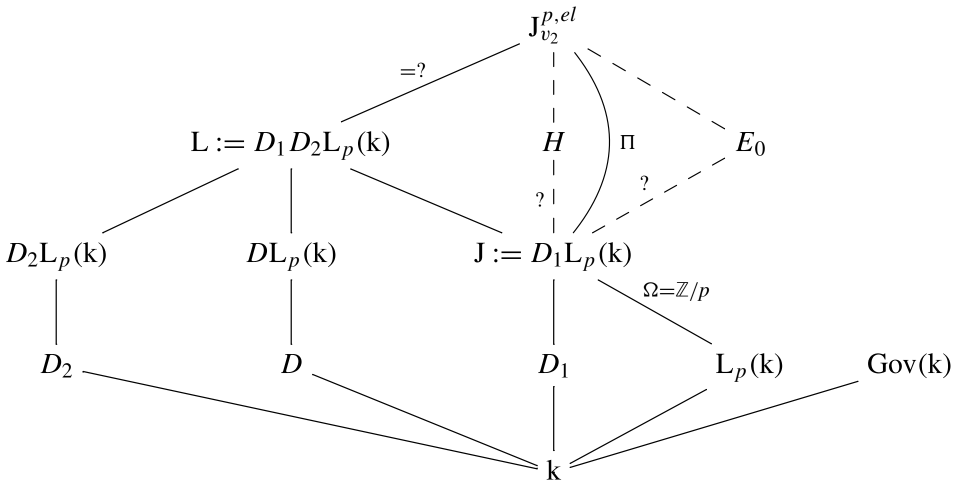
Fact 2.6 and our choices of the Frobenius elements of v_1 and v_2 imply

$$h^1(\mathrm{Gal}(\mathrm{L}_p(\mathrm{k})_{\{v_1,v_2\}}/\mathrm{L}_p(\mathrm{k}))) = 2$$

using that the span of the Frobenius elements above them in $\mathrm{Gal}(\mathrm{Gov}(\mathrm{L}_p(\mathrm{k}))/\mathrm{Gov}(\mathrm{k})\mathrm{K}) \subset M(\mathrm{L}_p(\mathrm{k}))$ has dimension $2\#\mathrm{G} - 2$ and Fact 2.6. (With only $h^1(\mathrm{G})$ Minkowski units, we would again have had

$$h^1(\mathrm{Gal}(\mathrm{L}_p(\mathrm{k})_{\{v_1\}}/\mathrm{L}_p(\mathrm{k}))) = h^1(\mathrm{Gal}(\mathrm{L}_p(\mathrm{k})_{\{v_2\}}/\mathrm{L}_p(\mathrm{k}))) = 1.$$

In this case the span of the Frobenius elements above $\{v_1, v_2\}$ in $\mathrm{Gal}(\mathrm{Gov}(\mathrm{L}_p(\mathrm{k}))/\,\mathrm{Gov}(\mathrm{k})\mathrm{K}) \subset M(\mathrm{L}_p(\mathrm{k}))$ would have been $\#\mathrm{G}-1$ so by Fact 2.6, $h^1(\mathrm{Gal}(\mathrm{L}_p(\mathrm{k})_{\{v_1,v_2\}}/\mathrm{L}_p(\mathrm{k})))$ would have been $2\#\mathrm{G}-(\#\mathrm{G}-1) = \#\mathrm{G}+1$.)



Set $\mathrm{L} := D_1 D_2 \mathrm{L}_p(\mathrm{k})$, $\mathrm{J} := D_1\mathrm{L}_p(\mathrm{k})$ and note L/D is unramified as D/k has absorbed all ramification at $\{v_1, v_2\}$. We will solve the problem by showing $(\#\,\mathrm{Cl}_{D_1 D_2 \mathrm{L}_p(\mathrm{k})}, p) = 1$.

Since $(\#\,\mathrm{Cl}_{\mathrm{L}_p(\mathrm{k})}, p) = 1$ and our choice of v_1 is such that

$$h^1(\mathrm{Gal}(\mathrm{L}_p(\mathrm{k})_{\{v_1\}}/\mathrm{L}_p(\mathrm{k})) = 1,$$

Fact 2.7 applied to $\mathrm{J}/\mathrm{L}_p(k)$ implies $(\#\,\mathrm{Cl}_{\mathrm{J}}, p) = 1$.

We now prove that there exists a unique $\mathbb{Z}/p$-extension over J unramified outside v_2, namely L. Set $\Omega = \mathrm{Gal}(\mathrm{J}/\mathrm{L}_p(\mathrm{k}))$, $\mathrm{J}_{\{v_2\}}^{p,el}$ to be the maximal elementary p-abelian extension of J inside $\mathrm{J}_{\{v_2\}}$, and $\Pi = \mathrm{Gal}(\mathrm{J}_{\{v_2\}}^{p,el}/\mathrm{J})$. Then Ω acts on Π and trivially on $\mathrm{Gal}(\mathrm{L}/\mathrm{J})$. We claim this is the only $\mathbb{Z}/p$-extension of J in $\mathrm{J}_{\{v_2\}}^{p,el}/\mathrm{J}$ on which Ω acts trivially: If not, there exists another $\mathbb{Z}/p$-extension H/J unramified outside v_2 and Galois over $\mathrm{L}_p(\mathrm{k})$. Hence $\mathrm{Gal}(H/\mathrm{L}_p(\mathrm{k}))$ has order p^2 and is abelian. The extension $H/\mathrm{L}_p(\mathrm{k})$ cannot be cyclic because all inertia elements have order p and would then fix an everywhere unramified extension of $\mathrm{L}_p(\mathrm{k})$, a contradiction. Suppose now that $\mathrm{Gal}(H/\mathrm{L}_p(\mathrm{k})) \simeq \mathbb{Z}/p \times \mathbb{Z}/p$, with $H \neq \mathrm{J}D_2 = \mathrm{L}$. Then $\mathrm{Gal}(HD_2/\mathrm{L}_p(\mathrm{k})) \simeq (\mathbb{Z}/p)^3$: this contradicts the already established fact that $h^1(\mathrm{Gal}(\mathrm{L}_p(\mathrm{k})_{\{v_1,v_2\}}/\mathrm{L}_p(\mathrm{k})) = 2$.

The final possibility is that there exists a $\mathbb{Z}/p$-extension E_0/J unramified outside v_2, different from L/J and not fixed by Ω; let S_0 be the set of ramification of E_0/J. As primes above v_2 in $\mathrm{L}_p(\mathrm{k})$ are inert in $\mathrm{J}/\mathrm{L}_p(\mathrm{k})$, $\Omega(S_0) = S_0$: then Ω takes E_0 to another $\mathbb{Z}/p$-extension E_1/J exactly ramified at S_0 and such that $E_1 \neq E_0$. The compositum E_1E_0/J contains a $\mathbb{Z}/p$-extension E_0'/J exactly ramified at a set $S_0' \subsetneq S_0$. Observe that $E_0' \neq \mathrm{L}$ since L/J is totally ramified at every prime above v_2. Continuing the process, we obtain an unramified $\mathbb{Z}/p$-extension H/J, which is impossible since $(\#\,\mathrm{Cl}_{\mathrm{J}}, p) = 1$. Thus L/J is the unique $\mathbb{Z}/p$-extension unramified outside v_2. Fact 2.7 applied to L/J implies $(\#\,\mathrm{Cl}_L, p) = 1$.

We have solved the split embedding problem with $\mathrm{k}' = D$ and $\mathrm{Gal}(\mathrm{L}_p(\mathrm{k}')/\mathrm{k}') = \mathrm{G} \times \mathbb{Z}/p$. It required one base change ramified at two tame finite primes. Proposition 2.14 implies $f(\mathrm{k}') \geq 2h^1(\mathrm{G}') + 3$ so the induction can proceed.

For the nonsplit case we treat $\mu_p \not\subset \mathrm{k}$ and $\mu_p \subset \mathrm{k}$ separately. Theorem 3.2 is only used in the nonsplit case when $\mu_p \subset \mathrm{k}$.

The nonsplit case, $\mu_p \not\subset \mathrm{k}$. By Lemma 2.10 we may use one tame prime v of k to find a *ramified* solution to the embedding problem. As $\mu_p \not\subset \mathrm{k}$ implies $\mathrm{Gov}(\mathrm{k}) \cap \mathrm{L}_p(\mathrm{k}) = \mathrm{k}$, we can assume v splits completely in K/k. Choosing any $w \mid v$ of K we set $\mathrm{Fr}_w = ((z_1, z_2, \ldots, z_{\lambda_{\mathrm{k}}}), n_0) \in M(\mathrm{L}_p(\mathrm{k}))$ where we claim $n_0 \notin I_{\mathrm{G}} \cdot N$ and $z_i \in I_{\mathrm{G}} \subset \mathbb{F}_p[\mathrm{G}]$. Indeed, if any $z_i \notin I_{\mathrm{G}}$, its $\mathbb{F}_p[\mathrm{G}]$-span is all of $\mathbb{F}_p[\mathrm{G}]$ and by Fact 2.6 there is no $\mathbb{Z}/p$-extension of $\mathrm{L}_p(\mathrm{k})$ ramified at the $w \mid v$, contradicting that we are solving an embedding problem with v. If $n_0 \in I_{\mathrm{G}} \cdot N$, then the projection of Fr_w to $\mathrm{Gal}(F/\mathrm{K})$ is trivial so Proposition 3.1 implies $\langle \mathrm{Fr}_v \rangle_{\mathrm{Gov}(\mathrm{k})/\mathrm{k}(\mu_p)} = 0$ and the embedding problem we are solving is split, also a contradiction.

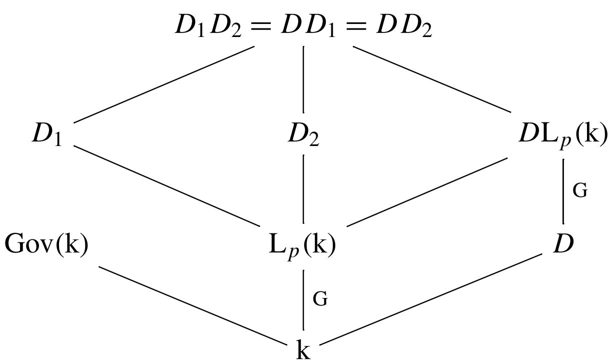
Choose a degree one w_1 of K with $\mathrm{Fr}_{w_1} = ((x_1, x_2, \ldots, x_{h^1(\mathrm{G})}, 0, 0, 0, \ldots, 0), n_0) \in M(\mathrm{L}_p(\mathrm{k}))$ where n_0 is as in the previous paragraph. Let v_1 be the prime of k below w_1. By Fact 2.6 (also see the proof of Theorem 3.2) there is one $\mathbb{Z}/p$-extension $D_1/\mathrm{L}_p(\mathrm{k})$ ramified at v_1.

Choose a degree one w_2 of K with $\mathrm{Fr}_{w_2} = ((0, 0, \ldots, 0, x_1, x_2, \ldots, x_{h^1(\mathrm{G})}, 0, 0, 0, \ldots, 0), n_0) \in M(\mathrm{L}_p(\mathrm{k}))$ and the primes of $\mathrm{L}_p(\mathrm{k})$ above v_2 remain prime in $D_1/\mathrm{L}_p(\mathrm{k})$. This last condition is linearly disjoint from the splitting conditions defining v_2 and imposes no contradiction. Again by Fact 2.6 there is one $\mathbb{Z}/p$-extension $D_2/\mathrm{L}_p(\mathrm{k})$ ramified at v_2.

As the free components of Fr_w, Fr_{w_1} and Fr_{w_2} are all in $I_{\mathrm{G}}^{\lambda_{\mathrm{k}}}$, their projections to $\mathrm{Gal}(F/\mathrm{K})$ depend only on n_0 and Proposition 3.1 implies

$$0 \neq \langle \mathrm{Fr}_v \rangle_{\mathrm{Gov}(\mathrm{k})/\mathrm{k}(\mu_p)} = \langle \mathrm{Fr}_{v_1} \rangle_{\mathrm{Gov}(\mathrm{k})/\mathrm{k}(\mu_p)} = \langle \mathrm{Fr}_{v_2} \rangle_{\mathrm{Gov}(\mathrm{k})/\mathrm{k}(\mu_p)}.$$

Thus there is no extension of k ramified at either v_1 or v_2, but, by Fact 2.6, there is a $\mathbb{Z}/p$-extension of k ramified at $\{v_1, v_2\}$. Call it D. Note $\mathrm{G}' \simeq \mathrm{Gal}(D_1/\mathrm{k}) \simeq \mathrm{Gal}(D_2/\mathrm{k}) \simeq \mathrm{Gal}(D_1 D_2/D)$:



That $D_1 D_2$ has trivial p-class group follows exactly as it did in the split case and we may set $\mathrm{k}' = D$ so $\mathrm{L}_p(\mathrm{k}') = D_1 D_2$ and $\mathrm{Gal}(\mathrm{L}_p(\mathrm{k}')/\mathrm{k}') \simeq \mathrm{G}'$.

We have solved the embedding problem in the nonsplit case when $\mu_p \not\subset \mathrm{k}$. We performed one base change ramified at two tame finite primes and Proposition 2.14 implies $f(\mathrm{k}') \geq 2h^1(\mathrm{G}') + 3$ so the induction can proceed.

The nonsplit case, $\mu_p \subset \mathrm{k}$. We can no longer assume $\mathrm{L}_p(\mathrm{k}) \cap \mathrm{Gov}(\mathrm{k}) = \mathrm{k}$.

Let $0 \neq \varepsilon \in Ш^2_{\mathrm{k},\varnothing}$ be the obstruction to our embedding problem $\mathrm{G}' \twoheadrightarrow \mathrm{G}$. Using Lemma 2.10, let v of k be a tame prime annihilating ε. The difficulty is that in the diagram below we may have $\mathrm{L}_p(\mathrm{k}) \cap \mathrm{Gov}(\mathrm{k}) \supsetneq \mathrm{k}$ *and* that Fr_v, which is necessarily nonzero in $M(\mathrm{k})$, may also be nonzero in $\mathrm{Gal}((\mathrm{L}_p(\mathrm{k}) \cap \mathrm{Gov}(\mathrm{k}))/\mathrm{k})$. This prevents us from also choosing v to split completely in $\mathrm{L}_p(\mathrm{k})/\mathrm{k}$ and as we need in $\mathrm{Gov}(\mathrm{L}_p(\mathrm{k}))/\mathrm{L}_p(\mathrm{k})$ to ensure there is only one extension of $\mathrm{L}_p(\mathrm{k})$ ramified at the primes of $\mathrm{L}_p(\mathrm{k})$ above v. If we could choose v to annihilate ε such that $\mathrm{Fr}_v = 0 \in \mathrm{Gal}(\mathrm{L}_p(\mathrm{k})/\mathrm{k})$, we would be able to proceed as in the $\mu_p \not\subset k$ case. We get around this by a base change.

By Kummer theory and the definition of governing fields, $\mathrm{Gal}(\mathrm{Gov}(\mathrm{L})/\mathrm{L}(\mu_p))$ is an elementary p-abelian group. Let $\tilde{\mathrm{k}}/\mathrm{k}$ be a tamely ramified $\mathbb{Z}/p$-extension as given by Theorem 3.2 so $\mathrm{Gal}(\mathrm{L}_p(\tilde{\mathrm{k}})/\tilde{\mathrm{k}}) = \mathrm{G}$. By Proposition 2.13 we have $\lambda_{\tilde{\mathrm{k}}} \geq 2h^1(\mathrm{G}) + 3$:

$$\begin{tikzcd}
 & L_p(\tilde{\mathrm{k}}) & \mathrm{Gov}(\tilde{\mathrm{k}}) \\
L_p(\mathrm{k}) & L_p(\tilde{\mathrm{k}}) \cap \mathrm{Gov}(\tilde{\mathrm{k}}) \arrow[u, dash] \arrow[ur, dash] & \\
L_p(\mathrm{k}) \cap \mathrm{Gov}(\mathrm{k}) \arrow[u, dash] & \tilde{\mathrm{k}} \arrow[u, dash] & \\
\mathrm{k} \arrow[u, dash] \arrow[u, dash, bend left, "(\mathbb{Z}/p)^r"] \arrow[ur, dash] \arrow[ur, dash, bend right, "\mathbb{Z}/p"'] & &
\end{tikzcd}$$

As $\mathrm{Gov}(\mathrm{k}) \cap \tilde{\mathrm{k}} = \mathrm{k}$, we may choose a prime v to solve the embedding problem for k whose Frobenius is nontrivial in $\mathrm{Gal}(\tilde{\mathrm{k}}/\mathrm{k})$, that is v remains prime in $\tilde{\mathrm{k}}/\mathrm{k}$. As observed above, $\mathrm{L}_p(\tilde{\mathrm{k}}) \cap \mathrm{Gov}(\tilde{\mathrm{k}})/\tilde{\mathrm{k}}$ is a $(\mathbb{Z}/p)^r$-extension for some r and, as $\mathrm{Gal}(\mathrm{L}_p(\mathrm{k})/\mathrm{k}) = \mathrm{Gal}(\mathrm{L}_p(\tilde{\mathrm{k}})/\tilde{\mathrm{k}}) = \mathrm{G}$, it is the base change of such a subextension of $\mathrm{L}_p(\mathrm{k})/\mathrm{k}$ from k so $\mathrm{L}_p(\tilde{\mathrm{k}}) \cap \mathrm{Gov}(\tilde{\mathrm{k}})/\mathrm{k}$ is a $(\mathbb{Z}/p)^{r+1}$-extension. Since v remains prime in $\tilde{\mathrm{k}}/\mathrm{k}$ and residue field extensions are cyclic, it splits completely in $L_p(\tilde{\mathrm{k}}) \cap \mathrm{Gov}(\tilde{\mathrm{k}})/\tilde{\mathrm{k}}$. As the embedding problem is solvable over k by allowing ramification at v, it is also solvable over $\tilde{\mathrm{k}}$ by allowing ramification at the unique prime of $\tilde{\mathrm{k}}$ above v. Thus $\varepsilon \in Ш^2_{\tilde{\mathrm{k}},\varnothing} \hookrightarrow Б_{\tilde{\mathrm{k}},\varnothing} = M(\tilde{\mathrm{k}})$ actually lies in $\mathrm{Gal}(\mathrm{Gov}(\tilde{\mathrm{k}})/(\mathrm{L}_p(\tilde{\mathrm{k}}) \cap \mathrm{Gov}(\tilde{\mathrm{k}})))$. The base change shifted the obstruction to outside of our p-Hilbert class field tower! The rest of the proof is identical to the $\mu_p \not\subset \mathrm{k}$ case. □

We now prove the main theorem of the introduction.

Proof. We have verified the base case of the induction in Proposition 2.15 and the inductive step with Theorem 3.3. It remains to count degrees and ramified primes. Proposition 2.15 involved three $\mathbb{Z}/p$-base changes, the first two ramified at one tame prime and the last at two tame primes. The inductive steps breaks into cases as follows:

- $\mu_p \not\subset \mathrm{k}_0$: At each of the $\log_p(\#\Gamma) - 1$ inductive stages we need one base change ramified at two primes for a total of $3 + (\log_p(\#\Gamma) - 1)$ base changes ramified at $4 + 2(\log_p(\#\Gamma) - 1)$ primes.
- $\mu_p \subset \mathrm{k}_0$: At each of the $\log_p(\#\Gamma) - 1$ inductive stages we need at most two base changes and at most three ramified tame primes so in total there are at most $3 + 2(\log_p(\#\Gamma) - 1)$ base changes ramified at most $4 + 3(\log_p(\#\Gamma) - 1)$ primes. □

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Supersolvable descent for rational points

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We construct an analogue of the classical descent theory of Colliot-Thélène and Sansuc in which algebraic tori are replaced with finite supersolvable groups. As an application, we show that rational points are dense in the Brauer–Manin set for smooth compactifications of certain quotients of homogeneous spaces by finite supersolvable groups. For suitably chosen homogeneous spaces, this implies the existence of supersolvable Galois extensions of number fields with prescribed norms, generalising work of Frei, Loughran and Newton.

1. Introduction

Let X be a smooth and irreducible variety over a number field k. The study of rational points on X often begins by embedding the set $X(k)$ diagonally into the product $X(k_\Omega) = \prod_{v\in\Omega} X(k_v)$, where Ω denotes the set of places of k and k_v denotes the completion of k at v. We endow $X(k_\Omega)$ with the product of the v-adic topologies. The weak approximation property, that is, the density of $X(k)$ in $X(k_\Omega)$, frequently fails. Following Manin [1971], one can attempt to explain such failures by considering the Brauer–Manin set $X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)}$, defined as the set of elements of $X(k_\Omega)$ that are orthogonal, with respect to the Brauer–Manin pairing, to the unramified Brauer group $\mathrm{Br}_{\mathrm{nr}}(X)$ of X. We recall that $\mathrm{Br}_{\mathrm{nr}}(X)$ is the subgroup of $\mathrm{Br}(X)$ formed by those classes that extend to any (equivalently, to some) smooth compactification of X, and that the Brauer–Manin set $X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)}$ is a closed subset of $X(k_\Omega)$ that satisfies the inclusions $X(k) \subseteq X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)} \subseteq X(k_\Omega)$; see [Skorobogatov 2001, §5.2]. A conjecture of Colliot-Thélène predicts that the Brauer–Manin set is enough to fully account for the gap between the topological closure of $X(k)$ and $X(k_\Omega)$ when X is *rationally connected* — that is, when for any algebraically closed field extension K of k, two general K-points of X can be joined by a rational curve over K:

Conjecture 1.1 [Colliot-Thélène 2003]. *Let X be a smooth and rationally connected variety over a number field k. The set $X(k)$ is a dense subset of $X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)}$.*

Though this conjecture is wide open in general, it has been established in several special cases. One approach is via the theory of descent developed by Colliot-Thélène and Sansuc. To explain it, let us assume for the moment that X is proper. Given an algebraic torus T over k, this theory considers torsors $Y \to X$ under T. The *type* of such a torsor is the isomorphism class of the torsor obtained from it by extending the

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scalars from k to an algebraic closure $\bar{k}$ of k; a *type*, in this context, is defined to be an isomorphism class of torsors over $X_{\bar{k}}$, under $T_{\bar{k}}$, that is invariant under $\mathrm{Gal}(\bar{k}/k)$. Torsors $Y \to X$ under T are classified, up to isomorphism, by the étale cohomology group $H^1_{\text{ét}}(X, T)$, and types are the elements of the abelian group $H^1_{\text{ét}}(X_{\bar{k}}, T_{\bar{k}})^{\mathrm{Gal}(\bar{k}/k)}$. As X is proper, these groups fit into the exact sequence

$$H^1(k, T) \to H^1_{\text{ét}}(X, T) \to H^1_{\text{ét}}(X_{\bar{k}}, T_{\bar{k}})^{\mathrm{Gal}(\bar{k}/k)} \to H^2(k, T) \to H^2_{\text{ét}}(X, T) \tag{1-1}$$

induced by the Hochschild–Serre spectral sequence. As can be seen from this sequence, if a type comes from a torsor $Y \to X$ defined over k, then the isomorphism class of this torsor is unique up to a twist by an element of $H^1(k, T)$. In general, not every type comes from a torsor over k: the map $H^1_{\text{ét}}(X, T) \to H^1_{\text{ét}}(X_{\bar{k}}, T_{\bar{k}})^{\mathrm{Gal}(\bar{k}/k)}$ need not be surjective. It is surjective if $X(k) \neq \varnothing$, as the map $H^2(k, T) \to H^2_{\text{ét}}(X, T)$ then possesses a retraction.

The following foundational theorem fully describes the *algebraic* Brauer–Manin set $X(k_\Omega)^{\mathrm{Br}_{1,\mathrm{nr}}(X)}$ in terms of the arithmetic of torsors under a torus, of a given type, over X. By definition $X(k_\Omega)^{\mathrm{Br}_{1,\mathrm{nr}}(X)}$ is the subset of $X(k_\Omega)$ consisting of those collections of local points that are orthogonal, with respect to the Brauer–Manin pairing, to the *algebraic* unramified Brauer group $\mathrm{Br}_{1,\mathrm{nr}}(X) = \mathrm{Ker}(\mathrm{Br}_{\mathrm{nr}}(X) \to \mathrm{Br}(X_{\bar{k}}))$.

Theorem 1.2 [Colliot-Thélène and Sansuc 1987]. *Let X be a smooth, proper and geometrically irreducible variety over a number field k such that $\mathrm{Pic}(X_{\bar{k}})$ is torsion-free. Let T be an algebraic torus over k. Let $\lambda \in H^1_{\acute{e}t}(X_{\bar{k}}, T_{\bar{k}})^{\mathrm{Gal}(\bar{k}/k)}$. Then*

$$X(k_\Omega)^{\mathrm{Br}_{1,\mathrm{nr}}(X)} = \bigcup_{f:Y\to X} f\big(Y(k_\Omega)^{\mathrm{Br}_{1,\mathrm{nr}}(Y)}\big), \tag{1-2}$$

where $f : Y \to X$ ranges over the isomorphism classes of torsors $Y \to X$ of type λ. In particular, if $Y(k)$ is a dense subset of $Y(k_\Omega)^{\mathrm{Br}_{1,\mathrm{nr}}(Y)}$ for every torsor $Y \to X$ of type λ, then $X(k)$ is a dense subset of $X(k_\Omega)^{\mathrm{Br}_{1,\mathrm{nr}}(X)}$.

As X is assumed to be proper in Theorem 1.2, we have $\mathrm{Br}_{1,\mathrm{nr}}(X) = \mathrm{Br}_1(X)$. The group $\mathrm{Br}_{1,\mathrm{nr}}(Y)$, on the other hand, may be smaller than $\mathrm{Br}_1(Y)$, since Y is not proper.

It is by now understood that Theorem 1.2 still holds if $\mathrm{Pic}(X_{\bar{k}})$ is allowed to contain torsion; see, e.g., [Wei 2016]. When $\mathrm{Pic}(X_{\bar{k}})$ is torsion-free, however, there exists a privileged type of torsors over X: denoting by T' the algebraic torus over k with character group $\mathrm{Pic}(X_{\bar{k}})$, there is a canonical isomorphism $H^1_{\text{ét}}(X_{\bar{k}}, T'_{\bar{k}}) = \mathrm{End}(\mathrm{Pic}(X_{\bar{k}}))$; the torsors $Y' \to X$ under T' whose type is classified by the identity endomorphism of $\mathrm{Pic}(X_{\bar{k}})$ are called *universal torsors*. They enjoy the special property that $\mathrm{Pic}(Y'_{\bar{k}}) = 0$. By the Hochschild–Serre spectral sequence, it follows that the natural map $\mathrm{Br}(k) \to \mathrm{Br}_1(Y')$ is surjective, so that $Y'(k_\Omega)^{\mathrm{Br}_{1,\mathrm{nr}}(Y')} = Y'(k_\Omega)$. As a consequence, Theorem 1.2 effectively reduces the statement that $X(k)$ is dense in $X(k_\Omega)^{\mathrm{Br}_{1,\mathrm{nr}}(X)}$ to the (in principle, simpler) weak approximation property for the universal torsors of X. This approach was fruitfully carried out in many special cases, notably for Châtelet surfaces in the influential two-part work [Colliot-Thélène et al. 1987a; 1987b], and later for various other types of varieties [Heath-Brown and Skorobogatov 2002; Colliot-Thélène et al. 2003; Browning and Matthiesen 2017; Browning et al. 2014; Derenthal et al. 2015; Skorobogatov 2015].

We note that even though Theorem 1.2 is stated and proved in [Colliot-Thélène and Sansuc 1987] only in the case of universal torsors, the general case follows. Indeed, for any T and λ as in Theorem 1.2, the type λ determines a morphism $T' \to T$, so that any universal torsor $Y' \to X$ factors through a torsor $Y \to X$ of type λ, and the image of $Y'(k_\Omega) = Y'(k_\Omega)^{\mathrm{Br}_{1,\mathrm{nr}}(Y')}$ in $Y(k_\Omega)$ is then contained in $Y(k_\Omega)^{\mathrm{Br}_{1,\mathrm{nr}}(Y)}$ by the projection formula.

The classical theory of descent of Colliot-Thélène and Sansuc allows one to neatly capture the algebraic part of the Brauer–Manin obstruction in terms of universal torsors and their local points. Formulated as above, it admits, however, a limitation: when the strategy consisting in applying Theorem 1.2 to verify Conjecture 1.1 for a given X works, we find that a stronger claim than Conjecture 1.1 is in fact being proved, namely, the density of $X(k)$ not only in $X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)}$ but also in the larger set $X(k_\Omega)^{\mathrm{Br}_{1,\mathrm{nr}}(X)}$. The two sets $X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)}$ and $X(k_\Omega)^{\mathrm{Br}_{1,\mathrm{nr}}(X)}$ coincide for geometrically rational varieties, but among rationally connected varieties it does happen that $X(k)$ fails to be dense in the larger one (see [Harari 1996, §2] or [Demarche et al. 2017, Example 5.4]), thus limiting the scope of applicability of the method. A way to overcome this issue was suggested in [Harpaz and Wittenberg 2020], where the following variant of Theorem 1.2 is proved — to be precise, Theorem 1.3 results from combining [Harpaz and Wittenberg 2020, Théorème 2.1] in the proper case with [Skorobogatov 2001, Theorem 6.1.2 (a)]:

Theorem 1.3 [Harpaz and Wittenberg 2020]. *Let X be a smooth, proper and geometrically irreducible variety over a number field k. Let T be an algebraic torus over k. Let $\lambda \in H^1_{\acute{e}t}(X_{\bar{k}}, T_{\bar{k}})^{\mathrm{Gal}(\bar{k}/k)}$. Then*

$$X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)} = \bigcup_{f:Y\to X} f\big(Y(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y)}\big), \tag{1-3}$$

where $f: Y \to X$ ranges over the isomorphism classes of torsors $Y \to X$ of type λ. In particular, if $Y(k)$ is a dense subset of $Y(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y)}$ for every torsor $Y \to X$ of type λ, then $X(k)$ is a dense subset of $X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)}$.

The theorems discussed so far admit generalisations to open varieties in which $X(k_\Omega)$ is replaced with the space of adelic points $X(\boldsymbol{A}_k)$, and $\mathrm{Br}_{\mathrm{nr}}(X)$ with the larger group $\mathrm{Br}(X)$; for instance, Theorem 1.2 was generalised in the work of Wei [2016], which builds on [Skorobogatov 2001, Theorem 6.1.2] and on [Harari and Skorobogatov 2013, Proposition 8.12], and later in the work of Cao, Demarche and Xu [2019, Theorem 1.2]. We shall not consider such generalisations in this article. Theorems 1.2 and 1.3 can also be extended to open varieties while keeping focus on the set $X(k_\Omega)$ and on the group $\mathrm{Br}_{\mathrm{nr}}(X)$. In particular, it follows from [Harari and Skorobogatov 2013, Proposition 8.12 and Corollary 8.17], from Harari's "formal lemma" [Colliot-Thélène and Skorobogatov 2021, Theorem 13.4.3] and from [Harpaz and Wittenberg 2020, Théorème 2.1], as in the proof of [Harpaz and Wittenberg 2020, Corollaire 2.2], that the statements of Theorems 1.2 and 1.3 remain valid without the properness assumption, provided we assume, in the case of Theorem 1.3, that the quotient of $\mathrm{Br}_{\mathrm{nr}}(X)$ by the subgroup of constant classes is finite (as is the case when X is rationally connected), and provided we replace, on the one hand, the notion of type due to Colliot-Thélène and Sansuc by that of extended type introduced by Harari and Skorobogatov [2013], and on the other hand, the right-hand sides of the asserted equalities by their topological closure in $X(k_\Omega)$.

Thus, for instance, equation (1-3) becomes

$$X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)} = \overline{\bigcup_{f:Y\to X} f\big(Y(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y)}\big)}, \tag{1-4}$$

where $\overline{M}$ denotes the topological closure of a subset M of $X(k_\Omega)$.

Pursuing the line of thought explored in [Harpaz and Wittenberg 2020], the goal of the present article is to prove an analogue of Theorem 1.3 in the case where the torus T is replaced with a *supersolvable finite group*. We shall adapt the notion of a type to the nonabelian setting as follows: for a smooth and geometrically irreducible variety X, we define a finite descent type on X to be a variety $\overline{Y}$ over $\bar{k}$ equipped with a finite étale map $\overline{Y} \to X_{\bar{k}}$ such that the composed map $\overline{Y} \to X_{\bar{k}} \to X$ is Galois, in the sense that the function field extension $\bar{k}(\overline{Y})/k(X)$ is Galois; and we define a torsor of type $\overline{Y}$ to be an étale map $Y \to X$ such that the $X_{\bar{k}}$-schemes $Y_{\bar{k}}$ and $\overline{Y}$ are isomorphic. A torsor of type $\overline{Y}$ is then naturally a torsor under a finite group scheme over k whose group of $\bar{k}$-points is the finite group $\mathrm{Aut}(\overline{Y}/X_{\bar{k}})$ canonically associated with the type $\overline{Y}$ (although the group scheme itself is not canonically associated with $\overline{Y}$). Any finite descent type determines not only a finite group $\overline{G} = \mathrm{Aut}(\overline{Y}/X_{\bar{k}})$, but also an outer Galois action $\mathrm{Gal}(\bar{k}/k) \to \mathrm{Out}(\overline{G})$ on $\overline{G}$. We say that a finite descent type $\overline{Y}$ is supersolvable if $\overline{G}$ admits a filtration

$$\{1\} = \overline{G}_0 \subseteq \overline{G}_1 \subseteq \cdots \subseteq \overline{G}_n = \overline{G}$$

such that each $\overline{G}_i$ is a normal subgroup of $\overline{G}$ stable under the outer Galois action, and each successive quotient $\overline{G}_{i+1}/\overline{G}_i$ is cyclic. We say that it is rationally connected if $\overline{Y}$, as a variety over $\bar{k}$, is rationally connected (in the sense made explicit just before Conjecture 1.1).

Our main result in this article is the following:

Theorem 1.4 (supersolvable descent, see Theorem 3.1). *Let X be a smooth and geometrically irreducible variety over a number field k. Let $\overline{Y}$ be a rationally connected supersolvable finite descent type on X. Then*

$$X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)} = \overline{\bigcup_{f:Y\to X} f\big(Y(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y)}\big)}, \tag{1-5}$$

where $f : Y \to X$ ranges over the isomorphism classes of torsors $Y \to X$ of type $\overline{Y}$. In particular, Conjecture 1.1 holds for X if it holds for Y for every torsor $Y \to X$ of type $\overline{Y}$.

It is now crucial that X is not assumed proper in the statement of Theorem 1.4: indeed, the hypothesis that $\overline{Y}$ is rationally connected implies that $X_{\bar{k}}$ is rationally connected, and proper smooth rationally connected varieties are simply connected and hence do not possess nontrivial finite descent types.

Theorem 1.4 can be used to prove Conjecture 1.1 for quotient varieties Y/G, where G is a finite supersolvable group acting freely on a quasiprojective rationally connected variety Y, when Conjecture 1.1 is already known for Y and for all of its twists. In particular, the following case of Conjecture 1.1 can be proved via supersolvable descent (see Example 3.4):

Corollary 1.5. *Let Y be a homogeneous space of a connected linear algebraic group L with connected geometric stabilisers. Suppose that a finite supersolvable group G acts on L (as an algebraic group) and on Y (as a homogeneous space of L, compatibly with its action on L), and that the action on Y is free. Then Conjecture 1.1 holds for the variety $X = Y/G$.*

As an application, we prove the following generalisation of a theorem of Frei, Loughran and Newton [2022]:

Corollary 1.6. *Let k be a number field and $\mathcal{A} \subset k^*$ be a finitely generated subgroup. Let G be a supersolvable finite group. Then there exists a Galois extension K/k with Galois group isomorphic to G such that every element of $\mathcal{A}$ is a norm from K. Moreover, given a finite set of places S of k, we can require that the places of S split in K.*

More applications are described in Section 4.

1A. ***Notation and terminology.*** We fix once and for all a field k of characteristic 0 and an algebraic closure $\bar{k}$ of k. A *variety* is a separated scheme of finite type over a field. Somewhat unconventionally, we shall say that a variety X over k is *rationally connected* if the smooth proper varieties over $\bar{k}$ that are birationally equivalent to $X_{\bar{k}}$ are rationally connected in the sense of Campana, Kollár, Miyaoka and Mori; see [Kollár 1996, Chapter IV]. If X is a smooth irreducible variety over k, we denote by $\mathrm{Br}_{\mathrm{nr}}(X) \subseteq \mathrm{Br}(X)$ the unramified Brauer group of $k(X)/k$; see [Colliot-Thélène and Skorobogatov 2021, §6.2].

The words "torsor", "action" and "homogeneous space" will refer to left torsors, left actions and left homogeneous spaces, unless indicated otherwise. An *outer action* of a profinite group Γ on a discrete group H is a continuous group homomorphism $\Gamma \to \mathrm{Out}(H)$, where we endow the group $\mathrm{Out}(H)$ of outer automorphisms of H with the topology induced by the compact-open topology on $\mathrm{Aut}(H)$.

When G is an algebraic group over k, we denote by $H^1(k, G)$ the first nonabelian Galois cohomology pointed set; see [Serre 1994]. When we write $[\sigma]$ for an element of this set, we mean that σ is a cocycle representing the cohomology class $[\sigma]$.

When k is a number field, we denote by Ω the set of places of k and by k_v the completion of k at $v \in \Omega$. For any variety X over k, we let $X(k_\Omega) = \prod_{v\in\Omega} X(k_v)$, endow this set with the product of the v-adic topologies, and when X is smooth and irreducible, we denote by $X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)} \subseteq X(k_\Omega)$ the Brauer–Manin set; see [Skorobogatov 2001, §5.2].

2. Finite descent types

We recall that k denotes a field of characteristic 0. We fix, until the end of Section 2, a smooth and geometrically irreducible variety X over k.

2A. ***Finite descent types and torsors.*** If G is a finite étale group scheme over k and $Y \to X$ is a torsor under G such that Y is geometrically irreducible over k, the function field extension $\bar{k}(Y)/k(X)$ is Galois (with Galois group $G(\bar{k}) \rtimes \mathrm{Gal}(\bar{k}/k)$). This remark motivates the following definition:

Definition 2.1. A *finite descent type on X* is an irreducible finite étale $X_{\bar{k}}$-scheme $\bar{Y}$ that is Galois over X (i.e., such that the function field extension $\bar{k}(\bar{Y})/k(X)$ is Galois).

Equivalently, a finite descent type on X is an irreducible finite étale Galois $X_{\bar{k}}$-scheme $\overline{Y}$ such that the natural morphism $\mathrm{Aut}(\overline{Y}/X) \to \mathrm{Gal}(\bar{k}/k)$ is surjective.

Definition 2.2. Given a finite descent type $\overline{Y}$ on X, a *torsor of type* $\overline{Y}$ is an X-scheme Y such that the $X_{\bar{k}}$-schemes $Y_{\bar{k}}$ and $\overline{Y}$ are isomorphic.

Although Definition 2.2 does not make reference to a group scheme, the name "torsor" is justified by the remark that any torsor $Y \to X$ of type $\overline{Y}$ in the above sense is canonically a torsor under the finite étale group scheme G over k defined by $G(\bar{k}) = \mathrm{Aut}(Y_{\bar{k}}/X_{\bar{k}})$. We warn the reader, though, that two torsors of type $\overline{Y}$ are not, in general, torsors under isomorphic group schemes: the underlying finite group is always isomorphic to $\mathrm{Aut}(\overline{Y}/X_{\bar{k}})$, but the action of $\mathrm{Gal}(\bar{k}/k)$ on $\mathrm{Aut}(\overline{Y}/X_{\bar{k}})$ depends, in general, on the choice of $Y \to X$. As we shall see in Proposition 2.4 (ii), it is nevertheless true that two torsors of type $\overline{Y}$ are torsors under group schemes that are inner forms of each other.

Let $\overline{Y}$ be a finite descent type on X. The short exact sequence of profinite groups

$$1 \to \mathrm{Aut}(\overline{Y}/X_{\bar{k}}) \to \mathrm{Aut}(\overline{Y}/X) \to \mathrm{Gal}(\bar{k}/k) \to 1 \tag{2-1}$$

induces a continuous outer action of $\mathrm{Gal}(\bar{k}/k)$ on the finite group $\overline{G} = \mathrm{Aut}(\overline{Y}/X_{\bar{k}})$. This outer action, together with the natural action of $\mathrm{Gal}(\bar{k}/k)$ on the scheme $X_{\bar{k}}$, induces, in turn, a continuous action of $\mathrm{Gal}(\bar{k}/k)$ on the pointed set $H^1_{\text{ét}}(X_{\bar{k}}, \overline{G})$ of isomorphism classes of torsors under $\overline{G}$ over $X_{\bar{k}}$ (notation justified by [Milne 1980, Chapter III, Corollary 4.7]). With respect to this action, the class of the $\overline{G}$-torsor $\overline{Y} \to X_{\bar{k}}$ is invariant.

The next two propositions provide a group-theoretic point of view on Definition 2.2 in terms of the short exact sequence (2-1). When we speak of a *splitting* of (2-1), we shall always mean a continuous homomorphism $\mathrm{Gal}(\bar{k}/k) \to \mathrm{Aut}(\overline{Y}/X)$ which is a section of the projection $\mathrm{Aut}(\overline{Y}/X) \to \mathrm{Gal}(\bar{k}/k)$.

Proposition 2.3. *Let $\overline{Y}$ be a finite descent type on X.*

(i) *Splittings of (2-1) are in one-to-one correspondence with isomorphism classes of torsors $Y \to X$ of type $\overline{Y}$ endowed with an $X_{\bar{k}}$-isomorphism $\iota : Y_{\bar{k}} \xrightarrow{\sim} \overline{Y}$.*

(ii) *Splittings of (2-1) up to conjugation by $\mathrm{Aut}(\overline{Y}/X_{\bar{k}})$ are in one-to-one correspondence with isomorphism classes of torsors $Y \to X$ of type $\overline{Y}$.*

(iii) *Splittings of (2-1) exist if $X(k) \neq \varnothing$.*

Proof. Assertion (i) results from the fact that such pairs (Y, ι) correspond to subextensions $k(X) \subset L \subset \bar{k}(\overline{Y})$ such that $L \cap \bar{k} = k$ and $\bar{k}(\overline{Y}) = L\bar{k}$, and from Galois theory. Assertion (ii) follows from (i). For (iii), see [Wittenberg 2018, Proposition 2.5]. □

Proposition 2.4. *Let $\overline{Y}$ be a finite descent type on X, and set $\overline{G} = \mathrm{Aut}(\overline{Y}/X_{\bar{k}})$.*

(i) *Let us fix a pair (Y, ι) as in Proposition 2.3 (i) corresponding to a splitting s of (2-1). Let G be the finite étale group scheme over k defined by $G(\bar{k}) = \overline{G}$, with the continuous action of $\mathrm{Gal}(\bar{k}/k)$ on $\overline{G}$ by conjugation through s. Then the natural action of $\overline{G}$ on $\overline{Y}$ descends to an action of G on the X-scheme Y, making it a torsor under G.*

(ii) *Let us fix another pair (Y', ι'), corresponding to another splitting s' of* (2-1). *Let G' denote the corresponding group scheme, as in* (*i*). *Let σ be the cocycle*

$$s's^{-1} : \mathrm{Gal}(\bar{k}/k) \to G(\bar{k}).$$

There are compatible canonical isomorphisms of k-group schemes $G' \simeq G^\sigma$ and of X-schemes $Y' \simeq Y^\sigma$, where G^σ denotes the inner twist of the group scheme G by σ and Y^σ denotes the twist of the torsor Y by σ; see [Skorobogatov 2001, Lemma 2.2.3].

Proof. Both assertions follow from unwinding the correspondence of Proposition 2.3 (i) and noting that the natural action of $\mathrm{Gal}(\bar{k}/k)$ on the scheme $Y_{\bar{k}}$ coincides with the action obtained by transport of structure, via ι, from the action of $\mathrm{Gal}(\bar{k}/k)$ on $\overline{Y}$ given by s. $\square$

Remark 2.5. Let us assume that $\overline{G}$ is abelian. In this case, the outer action of $\mathrm{Gal}(\bar{k}/k)$ on $\overline{G}$ induced by (2-1) is an actual action, thanks to which $\overline{G}$ canonically descends to a finite étale group scheme G over k. In addition, any torsor $Y \to X$ of type $\overline{Y}$ is canonically a torsor under G since any two $X_{\bar{k}}$-isomorphisms $Y_{\bar{k}} \xrightarrow{\sim} \overline{Y}$ give rise to the same isomorphism $\mathrm{Aut}(\overline{Y}/X_{\bar{k}}) \simeq \mathrm{Aut}(Y_{\bar{k}}/X_{\bar{k}})$. Thus, the abelian situation is summarised by the natural map

$$H^1_{\text{ét}}(X, G) \to H^0(k, H^1_{\text{ét}}(X_{\bar{k}}, \overline{G})), \tag{2-2}$$

which appears in the Hochschild–Serre spectral sequence and which sends the isomorphism class of a G-torsor over X to (the isomorphism class of) its *type*. We see, in particular, that the terminology of Definition 2.2 is consistent with the one introduced by Colliot-Thélène and Sansuc [1987, §2] in the theory of descent under groups of multiplicative type, as well as with the notion of extended type of Harari and Skorobogatov [2013] in the finite case.

2B. ***Supersolvability.*** The notion of supersolvability for finite groups endowed with an outer action of $\mathrm{Gal}(\bar{k}/k)$, first introduced in [Harpaz and Wittenberg 2020, Définition 6.4], plays a central rôle in the present article. We recall it below.

Definition 2.6. A finite group $\overline{G}$ endowed with an outer action of $\mathrm{Gal}(\bar{k}/k)$ is said to be *supersolvable* if there exist an integer n and a sequence

$$\{1\} = \overline{G}_0 \subseteq \overline{G}_1 \subseteq \cdots \subseteq \overline{G}_n = \overline{G}$$

of normal subgroups of $\overline{G}$ such that for all $i \in \{1, \ldots, n\}$, the quotient $\overline{G}_i/\overline{G}_{i-1}$ is cyclic and the subgroup $\overline{G}_i$ is stable under the outer action of $\mathrm{Gal}(\bar{k}/k)$. (A normal subgroup is said to be stable under an outer automorphism if it is stable under any automorphism that lifts the given outer automorphism. This is independent of the choice of the lift.)

A finite descent type $\overline{Y}$ on X is said to be *supersolvable* if the finite group $\mathrm{Aut}(\overline{Y}/X_{\bar{k}})$, endowed with the outer action of $\mathrm{Gal}(\bar{k}/k)$ induced by (2-1), is supersolvable in this sense.

When the integer n is fixed, we say that $\overline{G}$, or $\overline{Y}$, is *supersolvable of class n*.

3. Supersolvable descent

We are now in a position to state and prove our main theorem. We say that a finite descent type $\overline{Y}$ on X is *rationally connected* if $\overline{Y}$ is a rationally connected variety over $\bar{k}$ in the sense of Section 1A.

Theorem 3.1. *Let X be a smooth and geometrically irreducible variety over a number field k. Let $\overline{Y}$ be a rationally connected supersolvable finite descent type on X. Then*

$$X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)} = \overline{\bigcup_{f:Y\to X} f\big(Y(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y)}\big)}, \tag{3-1}$$

where $f : Y \to X$ ranges over the isomorphism classes of torsors $Y \to X$ of type $\overline{Y}$ and $\overline{M}$ denotes the topological closure of a subset M of $X(k_\Omega)$.

Remarks 3.2. (i) The existence of a rationally connected finite descent type on X, assumed in Theorem 3.1, implies that X itself is rationally connected.

(ii) When X is rationally connected, the statement of Theorem 3.1 can be expected to hold for an arbitrary finite descent type $\overline{Y}$ on X. Indeed, the equality (3-1) would result from Conjecture 1.1; see [Wittenberg 2018, Proposition 2.5].

(iii) By Proposition 2.3 (ii) and Proposition 2.4, equation (3-1) can be reformulated as the following two-part statement:

- if $X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)} \neq \varnothing$, then the natural outer action of $\mathrm{Gal}(\bar{k}/k)$ on $\overline{G} = \mathrm{Aut}(\overline{Y}/X_{\bar{k}})$ can be lifted to an actual continuous action, in such a way that if G denotes the resulting finite étale group scheme over k, the $\overline{G}$-torsor $\overline{Y} \to X_{\bar{k}}$ descends to a G-torsor $f : Y \to X$;
- fixing such G and Y and denoting by $f^\sigma : Y^\sigma \to X$ the twist of the torsor $f : Y \to X$ by a cocycle σ, we have an equality

$$X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)} = \overline{\bigcup_{[\sigma]\in H^1(k,G)} f^\sigma\big(Y^\sigma(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y^\sigma)}\big)}.$$

The proof of Theorem 3.1 will be given in Sections 3B–3C, first in the case where G is cyclic in Section 3B and then in the general case in Section 3C. Before going into the proof, let us illustrate Theorem 3.1 with the following special case. We say that a finite étale group scheme G over k is *supersolvable* if $G(\bar{k})$ is supersolvable in the sense of Definition 2.6 with respect to the natural outer action of $\mathrm{Gal}(\bar{k}/k)$ (which happens in this case to be an actual action).

Corollary 3.3. *Let Y be a smooth, quasiprojective rationally connected variety over a number field k. Let G be a supersolvable finite étale group scheme over k, acting freely on Y. Let $X = Y/G$ denote the quotient. If Conjecture 1.1 holds for Y^σ for all $[\sigma] \in H^1(k, G)$, then it holds for X.*

Proof. The projection $Y \to X$ is a G-torsor, hence $Y_{\bar{k}}$ is a supersolvable finite descent type on X, as remarked at the beginning of Section 2A. The conclusion now follows from Theorem 3.1, in view of Remark 3.2 (iii). □

Example 3.4. Suppose that Y is a homogeneous space of a connected linear algebraic group L with connected geometric stabilisers, and that the finite supersolvable group G acts compatibly on L as an algebraic group and on Y as a homogeneous space of L, with the action on Y being free. For each $[\sigma] \in H^1(k, G)$, the twisted variety Y^σ is then a homogeneous space of the twisted algebraic group G^σ, with connected geometric stabilisers. As such, the variety Y^σ satisfies Conjecture 1.1, according to Borovoi [1996]. It now follows from Corollary 3.3 that Conjecture 1.1 holds for the quotient variety $X = Y/G$. It should be noted that X is not itself, in general, a homogeneous space of a linear group. This example will play a rôle in Section 4B below.

3A. ***Fibrations over tori.*** The proof of Theorem 3.1 rests on the fibration method via the following theorem, which is a slightly more precise version of [Harpaz and Wittenberg 2020, Théorème 4.2 (ii)]. We recall that a variety is *split* if it possesses an irreducible component of multiplicity 1 that is geometrically irreducible.

Theorem 3.5 (see [Harpaz and Wittenberg 2020, Théorème 4.2 (ii)]). *Let Q be a quasitrivial torus over a number field k. Let Z be a smooth irreducible variety over k. Let $\pi : Z \to Q$ be a dominant morphism satisfying the following assumptions*:

(1) *the geometric generic fibre of π is rationally connected*;

(2) *the fibres of π above the codimension 1 points of Q are split*;

(3) *the morphism $\pi_{\bar{k}} : Z_{\bar{k}} \to Q_{\bar{k}}$ admits a rational section.*

For any dense open subset U of Q such that π is smooth over U, and for any Hilbert subset H of U, we have the equality

$$Z(k_\Omega)^{\mathrm{Br_{nr}}(Z)} = \overline{\bigcup_{q \in U(k) \cap H} Z_q(k_\Omega)^{\mathrm{Br_{nr}}(Z_q)}} \tag{3-2}$$

of subsets of $Z(k_\Omega)$, where $Z_q = \pi^{-1}(q)$.

Hypothesis (3) of Theorem 3.5 is stronger than the hypothesis that appears in [Harpaz and Wittenberg 2020, Théorème 4.2 (ii)]. The proof, however, only depends on the hypothesis formulated in *loc. cit.*; we have opted for stating the theorem in this way for the sake of simplicity. With this minor difference put aside, Theorem 3.5 implies and refines [Harpaz and Wittenberg 2020, Théorème 4.2 (ii)].

The exact argument used in *loc. cit.* to deduce [Harpaz and Wittenberg 2020, Théorème 4.2 (ii)] from [Harpaz and Wittenberg 2020, Théorème 4.1 (ii)] also reduces Theorem 3.5 to the following theorem:

Theorem 3.6. *Let Z be a smooth irreducible variety over a number field k, endowed with a dominant morphism $\pi : Z \to \mathbf{A}^n_k$, for some $n \geq 1$, such that*

(1) *the geometric generic fibre of π is rationally connected*;

(2) *the fibres of π above the codimension 1 points of $\mathbf{A}^n_k$ are split.*

For any dense open subset U of $\boldsymbol{A}_k^n$ such that π is smooth over U, and for any Hilbert subset H of U, we have the equality

$$Z(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Z)} = \overline{\bigcup_{q \in U(k) \cap H} Z_q(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Z_q)}} \tag{3-3}$$

of subsets of $Z(k_\Omega)$, where $Z_q = \pi^{-1}(q)$.

In turn, Theorem 3.6 is essentially contained in the work of Harari [1994; 1997], though its statement appears not to have been written down. We provide a short proof based on the available literature. We shall use Theorem 3.6 only when $H = U$. Assuming that $H = U$, however, would not lead to any significant simplification in the proof.

Proof of Theorem 3.6. Assume, first, that $n = 1$. By the theorems of Nagata and Hironaka, the morphism π extends to a proper morphism $\pi' : Z' \to \boldsymbol{P}_k^1$ for some smooth variety Z' that contains Z as a dense open subset. Applying [Harpaz et al. 2022, Corollary 4.7, Remarks 4.8 (i)–(ii) and Corollary 6.2 (i)] to π' now yields (3-3). For $n \geq 2$, we argue by induction. Fix U and H as in the statement of the theorem, fix a collection of local points $z_\Omega \in Z(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Z)}$ and fix a neighbourhood $\mathcal{U}$ of z_Ω in $Z(k_\Omega)$. We need to show the existence of $q \in U(k) \cap H$ such that $\mathcal{U} \cap Z_q(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Z_q)} \neq \varnothing$.

Let $p : \boldsymbol{A}_k^n \to \boldsymbol{A}_k^1$ be the first projection. For $h \in \boldsymbol{A}_k^1$, set $Z_h = (p \circ \pi)^{-1}(h)$ and let $\pi_h : Z_h \to p^{-1}(h)$ denote the restriction of π. Let $U_0 \subset \boldsymbol{A}_k^1$ be a dense open subset over which $p \circ \pi$ is smooth, small enough that for any $h \in U_0$, the generic fibre of π_h is rationally connected; see [Kollár 1996, Chapter IV, Theorem 3.5.3]. By assumption, there exists a closed subset of codimension ≥ 2 in $\boldsymbol{A}_k^n$ outside of which the fibres of π are split. After shrinking U_0, we may assume that the images, by p, of the irreducible components of this closed subset are either dense in $\boldsymbol{A}_k^1$ or disjoint from U_0. For $h \in U_0$, the fibres of π_h above the codimension 1 points of $p^{-1}(h)$ are then split. After further shrinking U_0, we may also assume that $p^{-1}(h) \cap U \neq \varnothing$ for every $h \in U_0$.

By [Harpaz and Wittenberg 2016, Lemma 8.12], there exists a Hilbert subset $H_0 \subset \boldsymbol{A}_k^1$ such that for every $h \in U_0(k) \cap H_0$, the set $H \cap p^{-1}(h)$ contains a Hilbert subset, say H_h, of $p^{-1}(h) = \boldsymbol{A}_k^{n-1}$. Let η denote the generic point of $\boldsymbol{A}_k^1$. The generic fibre of $p \circ \pi$ is endowed with a map to $\boldsymbol{A}^{n-1}$ with rationally connected generic fibre, hence it is itself rationally connected; see [Graber et al. 2003, Corollary 1.3]. The closed fibres of $p \circ \pi$ are each endowed with a map to $\boldsymbol{A}^{n-1}$ whose generic fibre is, by assumption, split; hence they are themselves split. By the case $n = 1$ of Theorem 3.6 applied to $p \circ \pi$, we deduce the existence of $h \in U_0(k) \cap H_0$ such that $\mathcal{U} \cap Z_h(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Z_h)} \neq \varnothing$. By the induction hypothesis, we can then apply Theorem 3.6 to π_h and finally deduce the existence of $q \in U(k) \cap H_h$ such that $\mathcal{U} \cap Z_q(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Z_q)} \neq \varnothing$. As $H_h \subset H$, this completes the proof. □

3B. ***Cyclic descent.*** We now establish Theorem 3.1 in the case where $\overline{G} = \mathrm{Aut}(\overline{Y}/X_{\bar{k}})$ is a cyclic group. Since most of the proof works in a slightly greater generality, we only assume, for now, that $\overline{G}$ is an abelian group (and drop the supersolvability assumption on $\overline{Y}$). We shall restrict to the cyclic case only at the end of Section 3B.

As $\overline{G}$ is abelian, the exact sequence (2-1) induces a continuous action of $\mathrm{Gal}(\bar{k}/k)$ on $\overline{G}$. Let G be the finite étale group scheme over k defined by $G(\bar{k}) = \overline{G}$. In the next lemma, the symbol $\text{Б}(X)$ denotes the subgroup of $\mathrm{Br}_{\mathrm{nr}}(X)$ consisting of the locally constant classes (i.e., the classes whose image in $\mathrm{Br}(X_{k_v})$ comes from $\mathrm{Br}(k_v)$ for all $v \in \Omega$).

Lemma 3.7. *If $X(k_\Omega)^{\text{Б}(X)} \neq \varnothing$, then $\overline{Y} \to X_{\bar{k}}$ descends to a torsor $f : Y \to X$ under G.*

Proof. Under the given assumption, Colliot-Thélène and Sansuc have shown that the natural map $H^2(k, G) \to H^2_{\text{ét}}(X, G)$ is injective (combine [Colliot-Thélène and Sansuc 1987, Proposition 2.2.5] for X with [Wittenberg 2008, Theorem 3.3.1] for a smooth compactification of X; we note that in the case of a smooth, proper, rationally connected variety, the quoted theorem from [Wittenberg 2008] goes back to [Colliot-Thélène and Sansuc 1987], see [Borovoi et al. 2008, §2.3, Remark]). By the Hochschild–Serre spectral sequence, it follows that the natural map $H^1_{\text{ét}}(X, G) \to H^0(k, H^1_{\text{ét}}(X_{\bar{k}}, G_{\bar{k}}))$ is surjective. $\square$

Lemma 3.8. *Over any field, any algebraic group of multiplicative type R fits into a short exact sequence $1 \to R \to T \to Q \to 1$, where T is a torus and Q is a quasitrivial torus.*

Proof. The character group M of R fits into a short exact sequence of Galois modules $0 \to K \to L \to M \to 0$, with K and L torsion-free and finitely generated as abelian groups; we can even choose L to be a permutation Galois module. Doing the same with $\mathrm{Hom}(K, \mathbf{Z})$ and then dualising, we find that K also fits into an exact sequence of Galois modules $0 \to K \to P \to C \to 0$, with C torsion-free and P permutation. Let $S = L \oplus_K P$ be the amalgamated sum relative to K. The exact sequence $0 \to L \to S \to C \to 0$ shows that S is torsion-free. Dualising the short exact sequence $0 \to P \to S \to M \to 0$, therefore, provides the desired resolution. $\square$

Let us fix a torsor $f : Y \to X$ as in Lemma 3.7 and a resolution

$$1 \to G \to T \to Q \to 1 \tag{3-4}$$

given by Lemma 3.8 applied to $R = G$. Let $Z = Y \times_k^G T$ be the contracted product of Y and T under G (i.e., the quotient of $Y \times_k T$ by the action of G given by $g \cdot (y, t) = (gy, g^{-1}t)$). Let $g : Z \to X$ and $\pi : Z \to Q$ be the morphisms induced by the two projections.

Let us also fix a collection of local points $x_\Omega \in X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)}$ and a neighbourhood $\mathcal{U}$ of x_Ω in $X(k_\Omega)$. We shall show that

$$\mathcal{U} \cap f^\sigma\left(Y^\sigma(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y^\sigma)}\right) \neq \varnothing \tag{3-5}$$

for some $[\sigma] \in H^1(k, G)$. By Remark 3.2 (iii), this will prove the desired equality (3-1).

To this end, we apply [Harpaz and Wittenberg 2020, Corollaire 2.2] to g. This yields a $[\tau] \in H^1(k, T)$ such that $\mathcal{U} \cap g^\tau\left(Z^\tau(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Z^\tau)}\right) \neq \varnothing$. As the torus Q is quasitrivial, Hilbert's theorem 90 implies that $H^1(k, Q) = 0$; the cohomology class $[\tau]$ can therefore be lifted to some $[\sigma_0] \in H^1(k, G)$, and we may assume that the cocycle σ_0 lifts τ. After replacing Y with Y^{σ_0}, which has the effect of replacing Z with Z^τ, we may then assume that $[\tau]$ is the trivial class, i.e., that $\mathcal{U} \cap g\left(Z(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Z)}\right) \neq \varnothing$.

To conclude the proof that (3-5) holds for some $[\sigma] \in H^1(k, G)$, we shall now exploit the structure of a fibration over a quasitrivial torus given by the morphism $\pi : Z \to Q$. For any field extension k'/k and any $q \in Q(k')$, we let $Z_q = \pi^{-1}(q)$, which we view as a torsor under G, over $X_{k'}$, via g. The inverse image T_q of q by the projection $T \to Q$ is a torsor under G, over k', whose cohomology class $[\sigma]$ in $H^1(k', G)$ is the image of q by the boundary map of the exact sequence (3-4). As $Z = Y \times_k^G T$, we have $Z_q = Y \times_k^G T_q$, and hence Z_q and Y^σ are isomorphic as torsors under G, over $X_{k'}$. Taking for k' an algebraically closed field extension of $k(Q)$ and for q a geometric generic point of Q, it follows, first, that the geometric generic fibre of π is isomorphic to a variety obtained from $\overline{Y}$ by an extension of scalars. By our assumption on $\overline{Y}$, we deduce that the generic fibre of π is rationally connected. Taking $k' = k$, it also follows that

$$g\big(Z_q(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Z_q)}\big) = f^\sigma\big(Y^\sigma(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y^\sigma)}\big) \tag{3-6}$$

for every $q \in Q(k)$, where $[\sigma] \in H^1(k, G)$ is the image of q by the boundary map of (3-4). In view of (3-6), we will be done if we show the equality

$$Z(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Z)} = \overline{\bigcup_{q \in Q(k)} Z_q(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Z_q)}} \tag{3-7}$$

of subsets of $Z(k_\Omega)$. Thus, the problem that we need to solve has been reduced to the question of making the fibration method work for $\pi : Z \to Q$, a fibration over a quasitrivial torus whose generic fibre is rationally connected and all of whose fibres are split (even geometrically integral). When $\overline{G}$ is cyclic, a positive answer is given by Theorem 3.5, thanks to the next lemma.

Lemma 3.9. *If $\overline{G}$ is a cyclic group, the morphism $\pi_{\bar{k}} : Z_{\bar{k}} \to Q_{\bar{k}}$ admits a rational section.*

Proof. If $\overline{G}$ is cyclic, we can fit the exact sequence (3-4) over $\bar{k}$ and the Kummer exact sequence into a commutative diagram as pictured below:

$$\begin{tikzcd} 1 \arrow[r] & G_{\bar{k}} \arrow[r] \arrow[d, "\wr"] & T_{\bar{k}} \arrow[r] \arrow[d] & Q_{\bar{k}} \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \boldsymbol{\mu}_{n,\bar{k}} \arrow[r] & \boldsymbol{G}_{\mathrm{m},\bar{k}} \arrow[r, "\times n"] & \boldsymbol{G}_{\mathrm{m},\bar{k}} \arrow[r] & 1 \end{tikzcd} \tag{3-8}$$

As the right-hand side square of (3-8) is cartesian, so is the square obtained by applying the functor $Y \times_k^G -$ to it. Hence $\pi_{\bar{k}} : Z_{\bar{k}} \to Q_{\bar{k}}$ comes, by a dominant base change, from the morphism $\pi_0 : Y \times_k^G \boldsymbol{G}_{\mathrm{m},\bar{k}} \to \boldsymbol{G}_{\mathrm{m},\bar{k}}$ induced by the multiplication by n map $\boldsymbol{G}_{\mathrm{m},\bar{k}} \to \boldsymbol{G}_{\mathrm{m},\bar{k}}$. In particular, it suffices to check that π_0 admits a rational section. Now, the proper models of the geometric generic fibre of π_0 are rationally connected, since $\pi_{\bar{k}}$ and π_0 have the same geometric generic fibre. As the target of π_0 is a curve over an algebraically closed field of characteristic 0, the Graber–Harris–Starr theorem [Graber et al. 2003, Theorem 1.1], combined with [Kollár 1996, Chapter IV, Theorem 6.10], does imply the existence of a rational section of π_0. □

3C. *Proof of Theorem 3.1 in the general case.* We assume that $\overline{Y}$ is a supersolvable finite descent type on X of class n and argue by induction on n. If $n = 0$, there is nothing to prove. Assume that $n > 0$ and that the statement of Theorem 3.1 holds for supersolvable finite descent types of class $n-1$.

Let $\overline{G} = \mathrm{Aut}(\overline{Y}/X_{\bar{k}})$, and let $\{1\} = \overline{G}_0 \subseteq \overline{G}_1 \subseteq \cdots \subseteq \overline{G}_n = \overline{G}$ be a filtration satisfying the requirements of Definition 2.6. The subgroup $\overline{G}_{n-1}$ of $\mathrm{Aut}(\overline{Y}/X)$ is normal since it is stabilised by the outer action of $\mathrm{Gal}(\bar{k}/k)$ on $\overline{G}$ induced by (2-1). Thus $\overline{Y}' = \overline{Y}/\overline{G}_{n-1}$ is a finite descent type on X. As $\mathrm{Aut}(\overline{Y}'/X_{\bar{k}}) = \overline{G}_n/\overline{G}_{n-1}$ is cyclic, we can apply the case of Theorem 3.1 already established in Section 3B, and deduce that

$$X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)} = \overline{\bigcup_{f':Y'\to X} f\big(Y'(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y')}\big)}, \tag{3-9}$$

where $f' : Y' \to X$ ranges over the isomorphism classes of torsors $Y' \to X$ of type $\overline{Y}'$.

Lemma 3.10. *Let $f' : Y' \to X$ be a torsor of type $\overline{Y}'$ and $\iota : Y'_{\bar{k}} \xrightarrow{\sim} \overline{Y}'$ be an isomorphism of $X_{\bar{k}}$-schemes. Viewing the scheme $\overline{Y}$ as a $Y'_{\bar{k}}$-scheme via ι, it is a supersolvable finite descent type on Y' of class $n-1$.*

Proof. As $\overline{Y}$ is Galois over X, it is Galois over Y'. Moreover, the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \overline{G} & \longrightarrow & \mathrm{Aut}(\overline{Y}/X) & \longrightarrow & \mathrm{Gal}(\bar{k}/k) & \longrightarrow & 1 \\
 & & \cup & & \cup & & \| & & \\
1 & \longrightarrow & \overline{G}_{n-1} & \longrightarrow & \mathrm{Aut}(\overline{Y}/Y') & \longrightarrow & \mathrm{Gal}(\bar{k}/k) & \longrightarrow & 1
\end{array}$$

shows that the outer action of $\mathrm{Gal}(\bar{k}/k)$ on $\overline{G}_{n-1}$ coming from the bottom row stabilises the subgroups $\overline{G}_1, \ldots, \overline{G}_{n-2}$ of $\overline{G}_{n-1}$, since these subgroups are stable under the outer action of $\mathrm{Gal}(\bar{k}/k)$ on $\overline{G}$ coming from the top row. □

For any torsor $f' : Y' \to X$ of type $\overline{Y}'$ and for any $X_{\bar{k}}$-isomorphism $\iota : Y'_{\bar{k}} \xrightarrow{\sim} \overline{Y}'$, Lemma 3.10 and the induction hypothesis imply the equality

$$Y'(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y')} = \overline{\bigcup_{f'':Y\to Y'} f''\big(Y(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y)}\big)} \tag{3-10}$$

of subsets of $Y'(k_\Omega)$, where $f'' : Y \to Y'$ ranges over the isomorphism classes of torsors $Y \to Y'$ of type $\overline{Y}$ (viewing $\overline{Y}$ as a $Y'_{\bar{k}}$-scheme via ι). Now for any such f', ι and f'', the composition $f' \circ f'' : Y \to X$ is a torsor of type $\overline{Y}$. Hence combining (3-9) with (3-10) yields (3-1). This completes the proof of Theorem 3.1.

4. Applications

We now discuss applications of supersolvable descent to rational points on homogeneous spaces and to Galois theory, pursuing and expanding the investigations in [Harpaz and Wittenberg 2020]. Unless otherwise noted, the field k will be assumed in Section 4 to be a number field.

4A. ***Homogeneous spaces of linear algebraic groups.*** In Theorem 4.5 below, we apply supersolvable descent to the validity of Conjecture 1.1 for homogeneous spaces of linear algebraic groups. Additional notation and terminology that is useful for dealing with stabilisers of geometric points on such homogeneous spaces will first be introduced in Section 4A1. Theorem 4.5 is stated in Section 4A2 and proved in Section 4A3.

4A1. *Outer Galois actions and σ-algebraic maps.* We first introduce *σ-algebraic* maps, following Borovoi [1993, §1.1].

Definition 4.1. Given a field automorphism σ of $\bar{k}$, a *σ-algebraic map* between two varieties V, W over $\bar{k}$ is a morphism of schemes $f : V \to W$ that makes the square

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\varepsilon_V"] & W \arrow[d, "\varepsilon_W"] \\ \mathrm{Spec}(\bar{k}) \arrow[r, "\mathrm{Spec}(\sigma^{-1})"] & \mathrm{Spec}(\bar{k}) \end{tikzcd}$$

commute, where ε_V and ε_W are the structure morphisms of V and W. Equivalently, if σ^*W denotes the variety over $\bar{k}$ with underlying scheme W and structure morphism $\mathrm{Spec}(\sigma) \circ \varepsilon_W$, a σ-algebraic map $f : V \to W$ is a morphism of varieties $V \to \sigma^*W$.

A σ-algebraic map is generally not a morphism of varieties. Nonetheless, any σ-algebraic map $f : V \to W$ induces a map $f_* : V(\bar{k}) \to W(\bar{k})$, since the sets $V(\bar{k})$ and $W(\bar{k})$ can be identified with the sets of closed points of the schemes V and W. We shall say that a map $V(\bar{k}) \to W(\bar{k})$ is *σ-algebraic* if it coincides with f_* for a σ-algebraic map $f : V \to W$, and that it is *algebraic* if it is σ-algebraic with $\sigma = \mathrm{Id}_{\bar{k}}$.

Remarks 4.2. (i) If V and W are nonempty varieties over $\bar{k}$, a morphism of schemes $f : V \to W$ can be a σ-algebraic map for at most one automorphism σ of $\bar{k}$.

(ii) As $\mathrm{Spec}(\tau^{-1}) \circ \mathrm{Spec}(\sigma^{-1}) = \mathrm{Spec}(\sigma^{-1}\tau^{-1}) = \mathrm{Spec}((\tau\sigma)^{-1})$, precomposing a τ-algebraic map with a σ-algebraic map yields a $\tau\sigma$-algebraic map. In particular, the class of σ-algebraic maps is closed under composition with algebraic maps.

(iii) If $V = V_0 \times_k \mathrm{Spec}(\bar{k})$ for a variety V_0 over k, then for any $\sigma \in \mathrm{Gal}(\bar{k}/k)$, the morphism of schemes $f : V \to V$ given by $\mathrm{Id}_{V_0} \times_k \mathrm{Spec}(\sigma^{-1})$ is a σ-algebraic map. The map $f_* : V(\bar{k}) \to V(\bar{k})$ that it induces is $v \mapsto \sigma(v)$.

(iv) Let $V = V_0 \times_k \mathrm{Spec}(\bar{k})$ for a variety V_0 over k and U be an irreducible finite étale V-scheme. Let $\rho : \mathrm{Aut}(U/V_0) \to \mathrm{Gal}(\bar{k}/k)$ denote the natural map, which factors through $\mathrm{Aut}(V/V_0) = \mathrm{Gal}(\bar{k}/k)$. Then $a : U \to U$ is a $\rho(a)$-algebraic map for any $a \in \mathrm{Aut}(U/V_0)$.

We recall that if X is a (left) homogeneous space of a connected linear algebraic group L over k and if $H_{\bar{x}} \subset L_{\bar{k}}$ denotes the stabiliser of a point $\bar{x} \in X(\bar{k})$, viewed as an algebraic group over $\bar{k}$, the exact sequence

$$1 \to H_{\bar{x}}(\bar{k}) \to G_{\bar{x}} \to \mathrm{Gal}(\bar{k}/k) \to 1, \tag{4-1}$$

where $G_{\bar{x}} = \{(\ell, \sigma) \in L(\bar{k}) \rtimes \mathrm{Gal}(\bar{k}/k)\,;\ \ell\sigma(\bar{x}) = \bar{x}\}$, induces a continuous outer action of the profinite group $\mathrm{Gal}(\bar{k}/k)$ on the discrete group $H_{\bar{x}}(\bar{k})$; see [Demarche and Lucchini Arteche 2019, §2.3]. Thus, the discrete group $H_{\bar{x}}(\bar{k})$ receives a continuous outer action of $\mathrm{Gal}(\bar{k}/k)$ while the algebraic group $H_{\bar{x}}$ is only defined over $\bar{k}$. The notion of σ-algebraic map allows us to reconcile this outer action with the algebraic structure of $H_{\bar{x}}$, as shown by the following proposition:

Proposition 4.3. *Let $\sigma \in \mathrm{Gal}(\bar{k}/k)$. Any group automorphism of $H_{\bar{x}}(\bar{k})$ that represents the outer action of σ is induced by a σ-algebraic map $H_{\bar{x}} \to H_{\bar{x}}$.*

Proof. Let $(\ell, \sigma) \in G_{\bar{x}}$. The automorphism $m \mapsto \ell\sigma(m)\ell^{-1}$ of $L(\bar{k})$, being the composition of the σ-algebraic map $m \mapsto \sigma(m)$ with the algebraic map $m \mapsto \ell m \ell^{-1}$, is itself σ-algebraic (see Remarks 4.2 (ii)–(iii)), i.e., it equals f_* for a σ-algebraic map $f : L_{\bar{k}} \to L_{\bar{k}}$. As f_* stabilises $H_{\bar{x}}(\bar{k})$ and as $H_{\bar{x}}$ is a reduced closed subscheme of $L_{\bar{k}}$, the scheme morphism f stabilises $H_{\bar{x}}$. As the resulting scheme morphism $g : H_{\bar{x}} \to H_{\bar{x}}$ is a σ-algebraic map and as the automorphism g_* coincides with conjugation by (ℓ, σ), the proposition is proved. □

Corollary 4.4. *Let $H^0_{\bar{x}}$ denote the connected component of the identity in $H_{\bar{x}}$. The outer action of $\mathrm{Gal}(\bar{k}/k)$ on $H_{\bar{x}}(\bar{k})$ induced by (4-1) stabilises $H^0_{\bar{x}}(\bar{k})$, and hence induces an outer action of $\mathrm{Gal}(\bar{k}/k)$ on the finite group $\pi_0(H_{\bar{x}})$.*

Proof. This follows from Proposition 4.3, as any scheme morphism $H_{\bar{x}} \to H_{\bar{x}}$ that preserves the identity point must stabilise the open subscheme $H^0_{\bar{x}}$. □

4A2. *Statement.* We now formulate Theorem 4.5, our main application of supersolvable descent to homogeneous spaces of linear algebraic groups, and discuss its first consequences.

Theorem 4.5. *Let X be a homogeneous space of a connected linear algebraic group L over a number field k. Let $\bar{x} \in X(\bar{k})$. Let $H_{\bar{x}}$ denote the stabiliser of $\bar{x}$ and $N \subset H_{\bar{x}}$ be a normal algebraic subgroup of finite index satisfying the following two assumptions*:

(1) *the outer action of $\mathrm{Gal}(\bar{k}/k)$ on $H_{\bar{x}}(\bar{k})$ induced by (4-1) stabilises $N(\bar{k})$*;

(2) *the quotient $H_{\bar{x}}(\bar{k})/N(\bar{k})$ is supersolvable in the sense of Definition 2.6, with respect to the outer action of $\mathrm{Gal}(\bar{k}/k)$ on $H_{\bar{x}}(\bar{k})/N(\bar{k})$ induced by (4-1).*

Let Y range over the homogeneous spaces of L over k that satisfy the following condition:

($\star$) *there exist an L-equivariant map $Y \to X$ and a lifting $\bar{y} \in Y(\bar{k})$ of $\bar{x}$ whose stabiliser, as an algebraic subgroup of $L_{\bar{k}}$, is equal to N.*

If Conjecture 1.1 (respectively, the implication $Y(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y)} \neq \varnothing \Longrightarrow Y(k) \neq \varnothing$) holds for all such Y, then Conjecture 1.1 holds for X (respectively, then $X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)} \neq \varnothing \Longrightarrow X(k) \neq \varnothing$).

Remark 4.6. The weaker statement obtained by allowing Y to range over all homogeneous spaces of L over k whose geometric stabilisers are isomorphic to N as algebraic groups over $\bar{k}$ is sufficient for the applications of Theorem 4.5 considered in this article.

When $N = H^0_{\bar{x}}$, the first hypothesis of Theorem 4.5 is satisfied, by Corollary 4.4. On the other hand, Conjecture 1.1 holds for homogeneous spaces of L with connected geometric stabilisers, by a theorem of Borovoi [1996, Corollary 2.5]. Thus, we deduce:

Corollary 4.7. *Let X be a homogeneous space of a connected linear algebraic group L over a number field k. Let $\bar{x} \in X(\bar{k})$. Assume that the group of connected components of the stabiliser of $\bar{x}$ is supersolvable in the sense of Definition 2.6, with respect to the outer action of* $\mathrm{Gal}(\bar{k}/k)$ *given by Corollary 4.4. Then Conjecture 1.1 holds for X.*

Corollary 4.7 simultaneously generalises Borovoi's theorem mentioned above (where the geometric stabilisers are connected) and [Harpaz and Wittenberg 2020, Théorème B] (where the geometric stabilisers are finite and supersolvable). In fact, even in the particular case of finite and supersolvable geometric stabilisers, Corollary 4.7 strictly generalises [Harpaz and Wittenberg 2020, Théorème B], as it relaxes all hypotheses on the ambient linear group L, assumed in *loc. cit.* to be semisimple and simply connected. What is more, when L is semisimple and simply connected, Corollary 4.7 can be used to ensure the validity of Conjecture 1.1 even in cases where the geometric stabilisers are not supersolvable, as the following example shows:

Example 4.8. Assume that L is semisimple and simply connected. Then, by a theorem of Borovoi [1996, Corollary 2.5], Conjecture 1.1 holds for all Y as in Theorem 4.5 if N is abelian. Thus, Theorem 4.5 implies the validity of Conjecture 1.1 for any homogeneous space of L whose geometric stabilisers are extensions of a supersolvable finite group by an abelian algebraic subgroup (compatibly with the outer Galois action, as stated in Theorem 4.5 (1)–(2)).

Combining Theorem 4.5 with the work of Neukirch [1979] also yields Conjecture 1.1 for homogeneous spaces of SL_n whose geometric stabilisers can be written, compatibly with the outer action of $\mathrm{Gal}(\bar{k}/k)$, as extensions of a supersolvable finite group by a solvable finite group whose order is coprime to the number of roots of unity in k.

Nonsolvable examples where Theorem 4.5 can be applied will be discussed in Section 4A4.

4A3. *Proof of Theorem 4.5.* Set $\bar{Y} = L_{\bar{k}}/N$. We view $\bar{Y}$ as an $X_{\bar{k}}$-scheme through the projection

$$\bar{Y} = L_{\bar{k}}/N \to L_{\bar{k}}/H_{\bar{x}} = X_{\bar{k}}. \tag{4-2}$$

This projection is a torsor under $H_{\bar{x}}/N$, so that there is a natural short exact sequence

$$1 \to N(\bar{k}) \to H_{\bar{x}}(\bar{k}) \xrightarrow{\varphi} \mathrm{Aut}(\bar{Y}/X_{\bar{k}}) \to 1. \tag{4-3}$$

Explicitly, the map φ sends any $\ell \in H_{\bar{x}}(\bar{k})$ to the automorphism of the variety $\bar{Y}$ over $\bar{k}$ which on $\bar{k}$-points, i.e., on the quotient set $L(\bar{k})/N(\bar{k})$, is given by $mN(\bar{k}) \mapsto m\ell^{-1}N(\bar{k})$.

For the statement of the next lemma, we recall that $N(\bar{k})$ is a normal subgroup of the middle term $G_{\bar{x}}$ of (4-1), as a consequence of assumption (1) of Theorem 4.5.

Lemma 4.9. *The $X_{\bar{k}}$-scheme $\overline{Y}$ is a finite descent type on X. In addition, the short exact sequence (2-1) can be identified with the sequence obtained from (4-1) by replacing the first two terms of (4-1) with their quotients by the normal subgroup $N(\bar{k})$.*

Proof. Let $\sigma \in \mathrm{Gal}(\bar{k}/k)$ and $\ell \in L(\bar{k})$ be such that $\ell\sigma(\bar{x}) = \bar{x}$. By assumption (1) of Theorem 4.5, the automorphism $m \mapsto \ell\sigma(m)\ell^{-1}$ of $L(\bar{k})$ stabilises the subgroup $N(\bar{k})$. We deduce that the σ-algebraic map $L(\bar{k}) \to L(\bar{k})$, $m \mapsto \sigma(m)\ell^{-1}$ induces a σ-algebraic map $\overline{Y}(\bar{k}) \to \overline{Y}(\bar{k})$. The latter is the top horizontal arrow of a commutative square

$$\begin{tikzcd} \overline{Y}(\bar{k}) \arrow[r] \arrow[d] & \overline{Y}(\bar{k}) \arrow[d] \\ X(\bar{k}) \arrow[r] & X(\bar{k}) \end{tikzcd} \tag{4-4}$$

whose lower horizontal arrow is the σ-algebraic map $m \mapsto \sigma(m)$ and whose vertical arrows are given by $m \mapsto m\bar{x}$ (i.e., are induced by (4-2)). As the horizontal arrows are σ-algebraic and the vertical ones are algebraic, the square (4-4) is induced on closed points by a commutative square of schemes

$$\begin{tikzcd} \overline{Y} \arrow[r] \arrow[d] & \overline{Y} \arrow[d] \\ X_{\bar{k}} \arrow[r, "\mathrm{Id}_X \times_k \mathrm{Spec}(\sigma^{-1})"] & X_{\bar{k}} \end{tikzcd} \tag{4-5}$$

whose horizontal arrows are σ-algebraic maps and both of whose vertical arrows are the projection (4-2). Hence the top horizontal arrow is an automorphism of the X-scheme $\overline{Y}$.

With $G_{\bar{x}}$ as in (4-1), let $\psi : G_{\bar{x}} \to \mathrm{Aut}(\overline{Y}/X)$ denote the map that sends (ℓ, σ) to this X-scheme automorphism of $\overline{Y}$, and let $\rho : \mathrm{Aut}(\overline{Y}/X) \to \mathrm{Aut}(X_{\bar{k}}/X) = \mathrm{Gal}(\bar{k}/k)$ denote the natural morphism. One readily checks that ψ is a homomorphism and that the exact sequence (4-1) fits into a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & H_{\bar{x}}(\bar{k}) \arrow[r] \arrow[d, "\varphi"] & G_{\bar{x}} \arrow[r] \arrow[d, "\psi"] & \mathrm{Gal}(\bar{k}/k) \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mathrm{Aut}(\overline{Y}/X_{\bar{k}}) \arrow[r] & \mathrm{Aut}(\overline{Y}/X) \arrow[r, "\rho"] & \mathrm{Gal}(\bar{k}/k) & \end{tikzcd}$$

where φ comes from (4-3). (The commutativity of the left square follows from the explicit descriptions of ψ and φ; that of the right square follows from Remarks 4.2 (iv) and (i).) This diagram shows that ρ is surjective, so that $\overline{Y}$ is indeed a finite descent type on X. In addition, it follows from this diagram and from the exact sequence (4-3) that the exact sequence (2-1) can be identified as indicated in the statement of the lemma. □

Lemma 4.10. *Let $Y \to X$ be a torsor of type $\overline{Y}$. There exists a unique action of L on Y such that the morphism $Y \to X$ is L-equivariant. With respect to this action, the variety Y is a homogeneous space of L, and there exists a lifting $\bar{y} \in Y(\bar{k})$ of $\bar{x}$ whose stabiliser, as an algebraic subgroup of $L_{\bar{k}}$, is equal to N.*

Proof. This lemma is valid over any field k of characteristic 0. In order to prove it, we may and will assume that $k = \bar{k}$. Indeed, by Galois descent, the existence of the action of L on Y follows from its existence and uniqueness over $\bar{k}$; and all other conclusions of the lemma are of a geometric nature. Let us write $X = L/H_{\bar{x}}$ and fix an X-isomorphism $Y \simeq \overline{Y} = L/N$. The existence of an action of L on Y satisfying all of the conclusions of the lemma is now obvious, and we need only check its uniqueness. For the latter, the first paragraph of the proof of [Harpaz and Wittenberg 2020, Proposition 5.1] applies verbatim (and it does not depend on the hypotheses of semisimplicity and simple connectedness made in *loc. cit.*). □

By Lemma 4.9 and by assumption (2) of Theorem 4.5, the $X_{\bar{k}}$-scheme $\overline{Y}$ is a supersolvable finite descent type on X. Moreover, Lemma 4.10 and the final assumption of Theorem 4.5 ensure that for any torsor $Y \to X$ of type $\overline{Y}$, the set $Y(k)$ is a dense subset of $Y(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y)}$ (respectively, the implication $Y(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y)} \neq \varnothing \Longrightarrow Y(k) \neq \varnothing$ holds). Theorem 3.1 now implies that the set $X(k)$ is a dense subset of $X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)}$ (respectively, that $X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)} \neq \varnothing \Longrightarrow X(k) \neq \varnothing$). Thus, Theorem 4.5 is proved.

4A4. *A special case: homogeneous spaces of* SL_n *with finite stabilisers.* We now spell out a useful corollary of Theorem 4.5 in the special case where $L = \mathrm{SL}_n$ (Corollary 4.11 below). We shall apply it in Section 4A5 to the inverse Galois problem and to the Grunwald problem.

To prepare for the statement of Corollary 4.11, let us recall that a finite group is said to be *complete* if its centre is trivial and all its automorphisms are inner. We shall say that a finite group N is *almost complete* if its centre is trivial and the homomorphism $\mathrm{Aut}(N) \to \mathrm{Out}(N)$ admits a section.

Corollary 4.11. *Let G be a finite group equipped with an outer action of* $\mathrm{Gal}(\bar{k}/k)$. *Let X be a homogeneous space of* SL_n *over a number field k, with geometric stabilisers isomorphic to G as groups endowed with an outer action of* $\mathrm{Gal}(\bar{k}/k)$. *Let $N \subseteq G$ be a normal subgroup stable under the outer action of* $\mathrm{Gal}(\bar{k}/k)$. *Assume that the group G/N is supersolvable, in the sense of Definition 2.6, with respect to the induced outer action of* $\mathrm{Gal}(\bar{k}/k)$. *Then*:

(i) *If the finite group N is almost complete, then* $X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)} \neq \varnothing \Longrightarrow X(k) \neq \varnothing$.

(ii) *If the finite group N is almost complete and if, for any finite étale subgroup scheme $\tilde{N}$ of* SL_n *over k such that the groups $\tilde{N}(\bar{k})$ and N are isomorphic, the weak approximation property holds for the quotient variety* $\mathrm{SL}_n/\tilde{N}$, *then Conjecture 1.1 holds for X.*

(iii) *If the finite group N is complete and if, for any embedding $N \hookrightarrow \mathrm{SL}_n(k)$, letting $\tilde{N}$ denote the constant subgroup scheme of* SL_n *with $\tilde{N}(k) = N$, the weak approximation property holds for the quotient variety* $\mathrm{SL}_n/\tilde{N}$, *then Conjecture 1.1 holds for X.*

In Corollary 4.11 (ii), we do not require any compatibility between the Galois action on $\tilde{N}(\bar{k})$ and the given outer Galois action on G. One could obtain a slightly more precise statement by doing so; see Remark 4.6.

Proof of Corollary 4.11. By Theorem 4.5, it is enough to prove that for any homogeneous space Y of SL_n over k whose geometric stabilisers are isomorphic, as abstract groups, to N, Conjecture 1.1 holds for Y (respectively, the implication $Y(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y)} \neq \varnothing \Longrightarrow Y(k) \neq \varnothing$) holds) if the assumptions of (ii) and of (iii) (respectively, of (i)) are satisfied. We shall see that Y even satisfies the weak approximation property (respectively, that $Y(k) \neq \varnothing$ unconditionally).

Let us fix a point $\bar{y} \in Y(\bar{k})$ and a group isomorphism $H_{\bar{y}}(\bar{k}) \simeq N$, and consider the resulting exact sequence

$$1 \to N \to G_{\bar{y}} \to \mathrm{Gal}(\bar{k}/k) \to 1, \tag{4-6}$$

where $G_{\bar{y}} = \{(\ell, \sigma) \in \mathrm{SL}_n(\bar{k}) \rtimes \mathrm{Gal}(\bar{k}/k)\,;\ \ell\sigma(\bar{y}) = \bar{y}\}$. We endow $\mathrm{SL}_n(\bar{k}) \rtimes \mathrm{Gal}(\bar{k}/k)$ with the product of the discrete topology on $\mathrm{SL}_n(\bar{k})$ and the Krull topology on $\mathrm{Gal}(\bar{k}/k)$, and $G_{\bar{y}}$ with the induced topology, so that (4-6) becomes an exact sequence of profinite groups. By Lemma 4.12 below and by the hypothesis that N is almost complete, this sequence admits a continuous homomorphic splitting $s : \mathrm{Gal}(\bar{k}/k) \to G_{\bar{y}}$ and we can assume that the image of s commutes with $N \subset G_{\bar{y}}$ if in addition N is complete. Composing s with the projection $G_{\bar{y}} \to \mathrm{SL}_n(\bar{k})$ yields a continuous cocycle $\mathrm{Gal}(\bar{k}/k) \to \mathrm{SL}_n(\bar{k})$. As the Galois cohomology set $H^1(k, \mathrm{SL}_n)$ is a singleton (Hilbert's theorem 90), there exists $b \in \mathrm{SL}_n(\bar{k})$ such that $s(\sigma) = (b^{-1}\sigma(b), \sigma)$ for all $\sigma \in \mathrm{Gal}(\bar{k}/k)$. The very definition of $G_{\bar{y}}$ now shows that $\sigma(b\bar{y}) = b\bar{y}$ for all $\sigma \in \mathrm{Gal}(\bar{k}/k)$, in other words $b\bar{y} \in Y(k)$. This already proves that $Y(k) \neq \varnothing$, and hence takes care of Corollary 4.11 (i). Let us denote by $\tilde{N} \subset \mathrm{SL}_n$ the stabiliser of the rational point $b\bar{y} \in Y(k)$, so that $Y = \mathrm{SL}_n / \tilde{N}$. As $\tilde{N}(\bar{k}) = bH_{\bar{y}}(\bar{k})b^{-1}$, the groups $\tilde{N}(\bar{k})$ and N are isomorphic, and Corollary 4.11 (ii) follows. Finally, the condition that the image of s commutes with N is equivalent to $\sigma(bhb^{-1}) = bhb^{-1}$ for all $\sigma \in \mathrm{Gal}(\bar{k}/k)$ and all $h \in H_{\bar{y}}(\bar{k})$; therefore this condition implies that $\tilde{N}$ is a constant group scheme over k, and Corollary 4.11 (iii) is proved. □

Lemma 4.12. *Let N be a finite group with trivial centre. Then*

(1) *N is almost complete if and only if every short exact sequence of profinite groups*

$$1 \to N \to G \to H \to 1$$

splits as a semidirect product of profinite groups $G \cong N \rtimes H$;

(2) *N is complete if and only if every short exact sequence of profinite groups*

$$1 \to N \to G \to H \to 1$$

splits as a direct product of profinite groups $G \cong N \times H$.

Proof. As the centre of N is trivial, the group of inner automorphisms of N can be identified with N and we have a short exact sequence of finite groups

$$1 \to N \to \mathrm{Aut}(N) \to \mathrm{Out}(N) \to 1. \tag{4-7}$$

If (4-7) splits as a semidirect product, then N is almost complete, by definition. If (4-7) splits as a direct product, then the outer action of $\mathrm{Out}(N)$ on N induced by (4-7) is trivial. As this outer action coincides with the canonical outer action of $\mathrm{Out}(N)$ on N, it follows that $\mathrm{Out}(N)$ is trivial, i.e., N is complete.

Conversely, let us assume that N is almost complete (respectively, complete). Any short exact sequence as in the statement of the lemma canonically fits into a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & N \arrow[r] & \mathrm{Aut}(N) \arrow[r] & \mathrm{Out}(N) \arrow[r] & 1 \end{tikzcd}$$

where the middle vertical arrow sends $g \in G$ to the automorphism $z \mapsto gzg^{-1}$ of N. Thus, the upper row is obtained by pull-back from the lower row, and the upper row splits as a semidirect (respectively, direct) product if so does the lower row. □

A full characterisation of almost complete simple groups is established in [Lucchini et al. 2003]. These include for example all the simple alternating groups A_n for $n \neq 6$, all the sporadic simple groups and all Chevalley groups $L(\boldsymbol{F}_p)/Z(L(\boldsymbol{F}_p))$ where $p \geq 5$ is a prime and L is a split simple simply connected algebraic group over $\boldsymbol{F}_p$; see [Lucchini et al. 2003] and [Borel 1970, p. A-14]. In addition, if G is a finite group all of whose composition factors are almost complete simple groups, then G itself is almost complete, as can be seen by mimicking the proof of [Lucchini Arteche 2022, Theorem 2] and exploiting Lemma 4.12 above.

These remarks already provide many examples to which Corollary 4.11 (i) can be applied. We now illustrate, in Corollary 4.13, cases (ii) and (iii) of Corollary 4.11.

Corollary 4.13. *Let G be a finite group equipped with an outer action of* $\mathrm{Gal}(\bar{k}/k)$. *Let $N \subseteq G$ be a normal subgroup stable under this outer action. Assume that the group G/N is supersolvable, in the sense of Definition 2.6, with respect to the induced outer action of* $\mathrm{Gal}(\bar{k}/k)$. *Assume that one of the following two conditions holds*:

(1) *N is isomorphic to the symmetric group S_m with $m \neq 6$;*

(2) *N is isomorphic to the alternating group A_5.*

Then Conjecture 1.1 holds for any homogeneous space of SL_n *whose geometric stabilisers are isomorphic, as groups endowed with an outer action of* $\mathrm{Gal}(\bar{k}/k)$, *to G.*

It should be noted that Corollary 4.13 in case (2) with $G = N$ was first established by Boughattas and Neftin [2023], who gave in this way the first example of a nonabelian simple group N such that Conjecture 1.1 holds for any homogeneous space of SL_n with geometric stabilisers isomorphic, as abstract groups, to N. We provide an alternative proof, based on the special properties of del Pezzo surfaces of degree 5.

Proof of Corollary 4.13. If $N = S_2$, the supersolvability of G/N implies that of G itself, and the conclusion results from Corollary 4.7. If $N = S_m$ with $m \notin \{2, 6\}$, then N is a complete group for which the Noether problem has a positive answer. In this case, Corollary 4.11 (iii) can be applied: the variety $\mathrm{SL}_n/\tilde{N}$ appearing in its statement is stably rational and, therefore, satisfies the weak approximation property.

It only remains to treat the case $N = A_5$, which is an almost complete finite group. We shall prove that Corollary 4.11 (ii) can be applied, i.e., that for any finite étale subgroup scheme $\tilde{N}$ of SL_n over k such that $\tilde{N}(\bar{k})$ is isomorphic to A_5, the variety $\mathrm{SL}_n/\tilde{N}$ satisfies the weak approximation property; in fact, we shall even prove that $\mathrm{SL}_n/\tilde{N}$ is stably rational.

Let Y denote the split del Pezzo surface of degree 5 over k, i.e., the blow-up of $\boldsymbol{P}^2_k$ along four rational points in general position, and fix group isomorphisms $\tilde{N}(\bar{k}) \simeq A_5$ and $\mathrm{Aut}(Y) \simeq S_5$; see [Dolgachev 2012, Theorem 8.5.8]. As $\mathrm{Aut}(A_5) = S_5$, the natural action of $\mathrm{Gal}(\bar{k}/k)$ on $\tilde{N}(\bar{k})$ determines a homomorphism $\chi : \mathrm{Gal}(\bar{k}/k) \to S_5$. Letting S_5 act on A_5 by conjugation, the twist by χ of the constant group scheme over k associated with A_5 is $\tilde{N}$. Let Y' denote the twist of Y by χ. As the action of A_5 on Y is S_5-equivariant, it gives rise, upon twisting, to an action of $\tilde{N}$ on Y'. Let us now consider the diagonal right action of the group scheme $\tilde{N}$ on $\mathrm{SL}_n \times_k Y$ and the two projections $\mathrm{pr}_1 : (\mathrm{SL}_n \times_k Y')/\tilde{N} \to \mathrm{SL}_n/\tilde{N}$ and $\mathrm{pr}_2 : (\mathrm{SL}_n \times_k Y')/\tilde{N} \to Y'/\tilde{N}$. As the generic fibre of pr_2 is a torsor under the rational algebraic group SL_n, it is itself rational, by Hilbert's theorem 90. As the generic fibre of pr_1 is a del Pezzo surface of degree 5, it is also rational, by the work of Enriques [1897], Manin [1966] and Swinnerton-Dyer [1972]. The varieties $\mathrm{SL}_n/\tilde{N}$ and $Y'/\tilde{N}$ are therefore stably birationally equivalent. To conclude the proof, let us check that the surface $Y'/\tilde{N}$ is rational. Let $Z \to Y'/\tilde{N}$ denote its minimal resolution of singularities. As Y' is a del Pezzo surface of degree 5, the above-cited work of Enriques, Manin and Swinnerton-Dyer implies that Y' is rational and hence that $Z(k) \neq \varnothing$. On the other hand, according to Trepalin [2018, end of proof of Lemma 4.5], the smooth projective surface $Z_{\bar{k}}$ is isomorphic to the Hirzebruch surface $\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(3))$ over $\bar{k}$. In particular, $K_Z^2 = 8$ and Z is geometrically minimal, and hence minimal. All in all Z is a minimal smooth projective geometrically rational surface with $K_Z^2 \geq 5$ and $Z(k) \neq \varnothing$; by the work of Iskovskikh and Manin, it follows that it is rational (see [Benoist and Wittenberg 2023, Proposition 4.16]), as desired. □

4A5. *Inverse Galois problem.* In the situation of Theorem 4.5, let us assume that X is the quotient of L by a finite subgroup $\Gamma \subseteq L(k)$ viewed as a constant group scheme over k, and let $\bar{x}$ be the image of $1 \in L(k)$, so that $H_{\bar{x}} = \Gamma$. Then any normal subgroup $N \subseteq \Gamma$ is stable under the (trivial) outer action of $\mathrm{Gal}(\bar{k}/k)$, and the supersolvability condition on Γ/N that appears in the statement of Theorem 4.5 reduces to the usual notion of supersolvability for abstract groups. When in addition $L = \mathrm{SL}_n$, the conclusion of Theorem 4.5 implies a positive answer to the inverse Galois problem for Γ (see [Harari 2007, §4, Proposition 1]) and, by a theorem of Lucchini Arteche, to the Grunwald problem outside of the finite places of k dividing the order of Γ; see [Lucchini Arteche 2019, §6]. We shall refer to this version of the Grunwald problem as the *tame* Grunwald problem, following [Demarche et al. 2017, §1.2]. Thus, we obtain:

Corollary 4.14. *Let Γ be a finite group and $N \subset \Gamma$ be a normal subgroup such that Γ/N is supersolvable. Let k be a number field. Assume that the set $Y(k)$ is dense in $Y(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(Y)}$ for any $n \geq 1$ and any homogeneous space Y of SL_n over k whose geometric stabilisers are isomorphic, as groups, to N. Then Γ is a Galois group over k and the tame Grunwald problem has a positive solution for Γ over k.*

The second assertion of the corollary means that if S is a finite set of places of k none of which divides the order of Γ and if, for each $v \in S$, a Galois extension K_v/k_v whose Galois group can be embedded into Γ is given, then there exists a Galois extension K/k with Galois group Γ such that for each $v \in S$, the completion of K at a place dividing v is isomorphic, as a field extension of k_v, to K_v.

When the subgroup N is assumed to be trivial, Corollary 4.14 recovers the positive answer to the tame Grunwald problem for supersolvable finite groups obtained in [Harpaz and Wittenberg 2020, Corollaire au théorème B]. Indeed, in this case, Hilbert's theorem 90 guarantees that $Y \simeq \mathrm{SL}_n$, so that Y is rational over k and satisfies the weak approximation property.

Combining Corollary 4.13 for $G = N = A_5$ (due to Boughattas and Neftin [2023]) with Corollary 4.14 yields the following case of the tame Grunwald problem, which to our knowledge is new:

Corollary 4.15. *Let $\Gamma = A_5 \rtimes G$ be a semidirect product of A_5 with a finite supersolvable group G. Then Γ is a Galois group over k and the tame Grunwald problem has a positive solution for Γ over k.*

A more precise understanding of the unramified Brauer group of SL_n/Γ leads, for some groups Γ, to a positive answer to the Grunwald problem with a smaller exceptional set of bad primes than in the conclusion of Corollary 4.14; see [Harpaz and Wittenberg 2020, p. 778]. If we do not exclude any prime, however, the Grunwald problem can have a negative answer under the assumptions of Corollary 4.14, as is well known; see [Wang 1948].

4B. ***Norms from supersolvable extensions.*** The following refinement of the inverse Galois problem was formulated by Frei, Loughran and Newton [2022]: given a number field k, a finite group G and a finitely generated subgroup $\mathcal{A} \subset k^*$, does there exist a Galois extension K/k such that $\mathrm{Gal}(K/k) \simeq G$ and $\mathcal{A} \subset N_{K/k}(K^*)$? We provide a positive answer when G is supersolvable.

Theorem 4.16. *Let k be a number field and $\mathcal{A} \subset k^*$ be a finitely generated subgroup. Let G be a supersolvable finite group. There exists a Galois extension K/k with Galois group isomorphic to G such that every element of $\mathcal{A}$ is a norm from K. Moreover, given a finite set of places S of k, we can require that the places of S split in K.*

Proof. Let us fix an embedding $G \hookrightarrow \mathrm{SL}_n(k)$ for some $n \geq 1$ and a finite system of generators $\alpha_1, \dots, \alpha_m$ of $\mathcal{A}$. As in [Frei et al. 2022, §A.1], we let $T^\alpha \subset \prod_{g \in G} \boldsymbol{G}_\mathrm{m}$, for $\alpha \in k^*$, denote the subvariety, over k, whose $\bar{k}$-points are the maps $t : G \to \bar{k}^*$ such that $\prod_{g \in G} t(g) = \alpha$. Let us set

$$Y = \mathrm{SL}_n \times T^{\alpha_1} \times \cdots \times T^{\alpha_m} \quad \text{and} \quad L = \mathrm{SL}_n \times T^1 \times \cdots \times T^1,$$

with m copies of T^1. Let G act on the right on T^α (for any α) by $(t \cdot g)(g') = t(gg')$. Let G act on the right on Y and on L by the diagonal actions coming from the action just defined on the T^α, from the right multiplication action on the copy of SL_n appearing in Y, and from the trivial action on the copy of SL_n appearing in L.

Set $X = Y/G$. The projection $\pi : Y \to X$ is a right torsor under G since the action of G on SL_n by right multiplication is free. We view it as a left torsor by setting $g \cdot y = y \cdot g^{-1}$. Applying Corollary 3.3 as in Example 3.4 shows that $X(k)$ is a dense subset of $X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)}$. It follows that Ekedahl's version of Hilbert's irreducibility theorem, in the form spelled out in [Harari 2007, Lemme 1], can be applied to π.

Let us fix $y_0 \in Y(k)$. Recall that there exists a finite subset $S_0 \subset \Omega$ such that any $(x_v)_{v\in\Omega} \in X(k_\Omega)$ with $x_v = \pi(y_0)$ for all $v \in S_0$ belongs to $X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)}$; see [Wittenberg 2018, Remarks 2.4 (i)–(ii)]. Let us fix such an S_0, and an S as in the statement of Theorem 4.16. By [Harari 2007, Lemme 1], there exists $x \in X(k)$ such that the scheme $\pi^{-1}(x)$ is irreducible, with x arbitrarily close to $\pi(y_0) \in X(k_v)$ for all $v \in S$.

The function field K of $\pi^{-1}(x)$ is then a Galois extension of k with group G, and the restriction to $\pi^{-1}(x)$ of the invertible function $(s, t_1, \ldots, t_m) \mapsto t_i(1)$ on Y, where 1 denotes the identity element of G, is an element of K^* with norm α_i. Moreover, as π is étale, the implicit function theorem guarantees that if x is sufficiently close to $\pi(y_0)$ in k_v for $v \in S_0$, then $\pi^{-1}(x)$ possesses a k_v-point (close to y_0), so that v splits in K. □

Remarks 4.17. (i) In the particular case where the group G is abelian, the existence of Galois extensions K/k such that $\mathrm{Gal}(K/k) \simeq G$ and $\mathcal{A} \subset N_{K/k}(K^*)$ was first established by Frei, Loughran and Newton [2022, Theorem 1.1], who gave a quantitative estimate for the number of such field extensions with bounded conductor. In *op. cit.*, Appendix, we offered an alternative algebro-geometric proof of their result. A third proof was later given by Frei and Richard [2021].

(ii) In the particular case where the group G is abelian, the proof of [Frei et al. 2022, Appendix] is simpler than the one obtained by specialising the above proof of Theorem 4.16. Indeed, the latter chooses a filtration of G with cyclic quotients and proceeds by induction along this filtration, while the former consists in one step only.

(iii) When formulating Theorem 4.16, we might consider local conditions at a finite set of places S more general than the condition that these places split in K; in the abelian case, this was done in [Frei et al. 2022, Corollary 4.12] and [Frei and Richard 2021, §2.4]. Even when a Galois extension of k with group G that satisfies the given local conditions is assumed to exist, arbitrary local conditions cannot always be satisfied when the group G is a nonabelian finite supersolvable group; see [Frei et al. 2022, Proposition A.9] for an example. According to the proof of Theorem 4.16, this phenomenon is fully controlled by the Brauer–Manin obstruction to weak approximation on the variety X that appears in this proof. In particular, there always exists a finite subset $T \subset \Omega$ such that arbitrary local conditions can be imposed at the places of S as soon as $S \cap T = \varnothing$. To identify T explicitly, however, it would be necessary to analyse further the group $\mathrm{Br}_{\mathrm{nr}}(X)$.

The ideas underlying the proof of Theorem 4.16 can be applied to other similar problems. To conclude the article, we explain a slightly more general framework and give one example.

We fix a number field k, a finite group G and a subgroup $H \subseteq G$ such that the only normal subgroup of G contained in H is the trivial subgroup, and we let G act on the polynomial ring $k[(x_\gamma)_{\gamma\in G/H}]$ by permuting the variables via $g(x_\gamma) = x_{g\gamma}$. We also fix a nonconstant invariant polynomial $\theta \in k[(x_\gamma)_{\gamma\in G/H}]^G$.

Let us consider a k-algebra $\tilde{K}$ endowed with an action of G that turns the morphism $\mathrm{Spec}(\tilde{K}) \to \mathrm{Spec}(k)$ into a G-torsor. Let $K = \tilde{K}^H$. When $\tilde{K}$ is a field, this amounts to specifying a field extension K/k, with Galois closure $\tilde{K}/k$, together with a group isomorphism $p : G \xrightarrow{\sim} \mathrm{Gal}(\tilde{K}/k)$ such that $p(H) = \mathrm{Gal}(\tilde{K}/K)$.

For $z \in K$ and $\gamma \in G/H$, if $\tilde{\gamma} \in G$ stands for a lift of γ, the element $\tilde{\gamma}(z) \in \tilde{K}$ does not depend on the choice of $\tilde{\gamma}$. We denote it by $\gamma(z)$. For $z \in K$, substituting $\gamma(z)$ for x_γ in θ yields an element of $\tilde{K}$ that is invariant under G and hence belongs to k. We denote it by $N_\theta(z)$. This defines a map $N_\theta : K \to k$. For example, if $\theta = \prod_\gamma x_\gamma$ (respectively, $\theta = \sum_\gamma x_\gamma$), we recover the norm (respectively, trace) map from K to k.

Given a finite subset $\mathcal{A} \subset k$, we can now ask: do there exist a field extension K/k, a Galois closure $\tilde{K}/k$ and an isomorphism p as above, such that $\mathcal{A} \subset N_\theta(K)$? When H is trivial and $\theta = \prod_\gamma x_\gamma$, this is exactly the question considered in [Frei et al. 2022; 2021] and in Theorem 4.16. The proof of Theorem 4.16 extends to a positive answer to this more general question under some assumptions on G, H and θ, which we present in Theorem 4.18 below. To prepare for its statement, let us introduce, for $\alpha \in k$, the affine variety $V^\alpha = \operatorname{Spec}\big(k[(x_\gamma)_{\gamma \in G/H}]/(\theta - \alpha)\big)$. As θ is invariant under G, the group G naturally acts on V^α. We set $V = \prod_{\alpha \in \mathcal{A}} V^\alpha$ and equip this product with the diagonal action of G.

Theorem 4.18. *Let k be a number field. Let G be a supersolvable finite group and $H \subseteq G$ be a subgroup such that the only normal subgroup of G contained in H is the trivial subgroup. Let $\theta \in k[(x_\gamma)_{\gamma \in G/H}]^G$ be a nonconstant invariant polynomial. Let $\mathcal{A} \subset k$ be a finite subset. Let V be the variety associated with $H, G, \theta, \mathcal{A}$ as above. Let $\operatorname{Spec}(\tilde{K}_0) \to \operatorname{Spec}(k)$ be a G-torsor. Assume that the following conditions are satisfied*:

(1) *letting $K_0 = \tilde{K}_0^H$, the inclusion $\mathcal{A} \subset N_\theta(K_0)$ holds*;

(2) *the variety V is smooth and rationally connected*;

(3) *for every $[\sigma] \in H^1(k, G)$, Conjecture 1.1 holds for the twisted variety V^σ.*

Then there exist a field extension K/k, a Galois closure $\tilde{K}/k$ of K/k, and a group isomorphism $p : G \xrightarrow{\sim} \operatorname{Gal}(\tilde{K}/k)$, such that $p(H) = \operatorname{Gal}(\tilde{K}/K)$ and $\mathcal{A} \subset N_\theta(K)$. If, moreover, a finite set of places S of k is given, we can require that for all $v \in S$, the k_v-algebras $\tilde{K} \otimes_k k_v$ and $\tilde{K}_0 \otimes_k k_v$ are G-equivariantly isomorphic (*so that the k_v-algebras $K \otimes_k k_v$ and $K_0 \otimes_k k_v$ are isomorphic*).

Remarks 4.19. (i) If $[\sigma_0] \in H^1(k, G)$ denotes the class of the torsor $\operatorname{Spec}(\tilde{K}_0) \to \operatorname{Spec}(k)$, the twisted variety V^{σ_0} admits a rational point if and only if $\mathcal{A} \subset N_\theta(K_0)$. To explain why, we first note that the twist of the affine space $\operatorname{Spec}\big(k[(x_\gamma)_{\gamma \in G/H}]\big)$ by σ_0 can be identified with the Weil restriction $R_{K_0/k}\mathbf{A}^1_{K_0}$. As the regular function θ on this affine space is invariant under G, it induces a regular function on its twist. We view it as a morphism $N_\theta : R_{K_0/k}\mathbf{A}^1_{K_0} \to \mathbf{A}^1_k$. For $\alpha \in \mathcal{A}$, the twist of V^α by σ_0 is then the fibre $N_\theta^{-1}(\alpha)$. The latter possesses a rational point if and only if $\alpha \in N_\theta(K_0)$; hence the claim.

(ii) The group G acts faithfully, and therefore generically freely, on V. Indeed G acts faithfully on G/H by our assumption on H, hence it also acts faithfully on $k[(x_\gamma)_{\gamma \in G/H}]$, while $\theta - \alpha$ is not a scalar multiple of $x_{\gamma_1} - x_{\gamma_2}$ for any $\gamma_1, \gamma_2 \in G/H$.

(iii) Let V' be the largest open subset of V on which G acts freely. If $(V'/G)(k) \neq \varnothing$, then a G-torsor $\operatorname{Spec}(\tilde{K}_0) \to \operatorname{Spec}(k)$ satisfying assumption (1) of Theorem 4.18 exists. Indeed, for $c \in (V'/G)(k)$, twisting the G-torsor $V' \to V'/G$ by its fibre $\operatorname{Spec}(\tilde{K}_0) \to \operatorname{Spec}(k)$ above c yields a G-torsor $(V')^{\sigma_0} \to V'/G$ whose total space admits rational points (namely, rational points above c). Assumption (1) then holds by Remark 4.19 (i).

Proof of Theorem 4.18. We adapt the proof of Theorem 4.16 as follows. Fix an embedding $G \hookrightarrow \mathrm{SL}_n(k)$ for some $n \geq 1$. Set $Y = \mathrm{SL}_n \times V$. Let G act by right multiplication on SL_n, by the given action on V, and diagonally on Y. Set $X = Y/G$. Let $\tilde{\pi} : Y \to X$ and $\pi : Y/H \to X$ denote the quotient maps.

Lemma 4.20. *There exists $x_0 \in X(k)$ such that $\tilde{\pi}^{-1}(x_0)$ and $\mathrm{Spec}(\tilde{K}_0)$ are G-equivariantly isomorphic over k.*

Proof. Let us consider the cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{\tilde{\pi}} & X \\ \Big\downarrow{\scriptstyle \mathrm{pr}_1} & & \Big\downarrow{\scriptstyle \mathrm{pr}_1/G} \\ \mathrm{SL}_n & \xrightarrow{\tilde{\rho}} & \mathrm{SL}_n/G \end{array}$$

As the set $H^1(k, \mathrm{SL}_n)$ is a singleton (Hilbert's theorem 90), there exists $b \in (\mathrm{SL}_n/G)(k)$ such that $\tilde{\rho}^{-1}(b)$ is G-equivariantly isomorphic to $\mathrm{Spec}(\tilde{K}_0)$ (apply [Serre 1994, Chapitre I, §5.4, Corollaire 1] to the inclusion $G \hookrightarrow \mathrm{SL}_n(\bar{k})$). The fibre $(\mathrm{pr}_1/G)^{-1}(b)$ is then isomorphic to the twist of V by the torsor $\mathrm{Spec}(\tilde{K}_0) \to \mathrm{Spec}(k)$. By Remark 4.19 (i), we deduce from this and from our assumption (1) that $(\mathrm{pr}_1/G)^{-1}(b)$ possesses a rational point, say x_0. As $\tilde{\pi}^{-1}(x_0) = \tilde{\rho}^{-1}(b)$, the lemma is proved. □

Assumptions (2) and (3) allow us to deduce from Corollary 3.3 that $X(k)$ is a dense subset of $X(k_\Omega)^{\mathrm{Br}_{\mathrm{nr}}(X)}$. Therefore, there exists $x \in X(k)$ such that the scheme $\tilde{\pi}^{-1}(x)$ is irreducible, with x arbitrarily close to $x_0 \in X(k_v)$ for all $v \in S$; see [Harari 2007, Lemme 1]. Let $\tilde{K}$ and K denote the function fields of $\tilde{\pi}^{-1}(x)$ and $\pi^{-1}(x)$, respectively. The field extension $\tilde{K}/k$ is Galois with group G, and we have $\mathrm{Gal}(\tilde{K}/K) = H$ by construction. By choosing x sufficiently close to x_0 for $v \in S$, we can ensure that for all $v \in S$, the k_v-algebras $\tilde{K} \otimes_k k_v$ and $\tilde{K}_0 \otimes_k k_v$ are G-equivariantly isomorphic; see [Harari and Skorobogatov 2002, Lemma 4.6]. It remains to check that $\mathcal{A} \subset N_\theta(K_0)$. For $\alpha \in \mathcal{A}$, composing the projection map $Y \to V^\alpha$ with the regular function on V^α given by x_H (where H denotes the canonical point of G/H) yields a regular function on Y that is invariant under H, hence descends to a regular function on Y/H. Its restriction $z \in K$ to $\pi^{-1}(x)$ satisfies $N_\theta(z) = \alpha$, as desired. □

Example 4.21. When $\theta = \prod_\gamma x_\gamma$ and $\mathcal{A} \subset k^*$, the varieties V^α are trivial torsors under trivial tori, so that V is rational over k and assumptions (1) and (2) of Theorem 4.18 both hold, in view of Remark 4.19 (i), if we take for $\mathrm{Spec}(\tilde{K}_0) \to \mathrm{Spec}(k)$ the trivial torsor. Assumption (3) holds as well, as the twisted varieties $(V^\alpha)^\sigma$ are torsors under tori; see Example 3.4. When in addition H is the trivial subgroup, this recovers Theorem 4.16.

Example 4.22. Let k be a number field and $\alpha \in k^*$. Then there exists a cubic extension K/k such that the equation $\alpha = \mathrm{Tr}_{K/k}(\beta^2)$ has a solution $\beta \in K$.

To see this, we apply Theorem 4.18 to the symmetric group $G = S_3$, to the subgroup H generated by a transposition, to $\theta = \sum_\gamma x_\gamma^2$ and to $\mathcal{A} = \{\alpha\}$. Let us check that its hypotheses hold. To this end, we identify

G/H with $\{1, 2, 3\}$, so that V^α is the smooth affine quadric surface defined by the equation $x_1^2+x_2^2+x_3^2=\alpha$. The twisted varieties $(V^\alpha)^\sigma \subset (\boldsymbol{A}^3_k)^\sigma$ are also smooth affine quadric surfaces, since $(\boldsymbol{A}^3_k)^\sigma \simeq \boldsymbol{A}^3_k$ (Hilbert's theorem 90). Smooth quadric surfaces are rationally connected and satisfy the weak approximation property, hence assumptions (2) and (3) of Theorem 4.18 are satisfied. To verify the existence of a torsor $\mathrm{Spec}(\tilde{K}_0) \to \mathrm{Spec}(k)$ satisfying (1), we consider the point $(x_1, x_2, x_3) = (0, \sqrt{\alpha/2}, -\sqrt{\alpha/2})$. In the notation of Remark 4.19 (iii), this point belongs to $V'(\bar{k}) \subset V^\alpha(\bar{k})$ since its coordinates are pairwise distinct. As its orbit under $\mathrm{Gal}(\bar{k}/k)$ is contained in its orbit under G, we have $(V'/G)(k) \neq \varnothing$; Remark 4.19 (iii) can be applied.

Remark 4.23. We note that the cubic extensions constructed in Example 4.22 are noncyclic. A cyclic cubic extension K/k such that the equation $\alpha = \mathrm{Tr}_{K/k}(\beta^2)$ has a solution $\beta \in K$ need not exist: for instance, it cannot exist if α is not totally positive. This is a situation where Theorem 4.18 does not apply because there is no torsor $\mathrm{Spec}(\tilde{K}_0) \to \mathrm{Spec}(k)$ satisfying its assumption (1) (taking $G = \boldsymbol{Z}/3\boldsymbol{Z}$ and letting H be trivial). Similarly, even in the noncyclic case, it is not always possible to ensure that the places of a finite set $S \subset \Omega$ split completely in K; for instance, this cannot be achieved if S contains a real place at which α is negative. Here $(V'/G)(k) \neq \varnothing$, as shown in Example 4.22, but $V(k) = \varnothing$. These two observations exhibit a marked contrast with the situation considered in Theorem 4.16 and in Example 4.21, where $V(k) \neq \varnothing$ automatically.

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On Kato and Kuzumaki's properties for the Milnor K_2 of function fields of p-adic curves

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Let K be the function field of a curve C over a p-adic field k. We prove that, for each $n, d \geq 1$ and for each hypersurface Z in $\mathbb{P}^n_K$ of degree d with $d^2 \leq n$, the second Milnor K-theory group of K is spanned by the images of the norms coming from finite extensions L of K over which Z has a rational point. When the curve C has a point in the maximal unramified extension of k, we generalize this result to hypersurfaces Z in $\mathbb{P}^n_K$ of degree d with $d \leq n$.

1. Introduction

Kato and Kuzumaki [1986] stated a set of conjectures which aimed at giving a diophantine characterization of cohomological dimension of fields. For this purpose, they introduced some properties of fields which are variants of the classical C_i-property and which involve Milnor K-theory and projective hypersurfaces of small degree. They hoped that those properties would characterize fields of small cohomological dimension.

More precisely, fix a field K and two nonnegative integers q and i. Let $\mathrm{K}_q(K)$ be the q-th Milnor K-group of K. For each finite extension L of K, one can define a norm morphism $N_{L/K} : \mathrm{K}_q(L) \to \mathrm{K}_q(K)$; see Section 1.7 of [Kato 1980]. Thus, if Z is a scheme of finite type over K, one can introduce the subgroup $N_q(Z/K)$ of $\mathrm{K}_q(K)$ generated by the images of the norm morphisms $N_{L/K}$ when L runs through the finite extensions of K such that $Z(L) \neq \varnothing$. One then says that the field K is C_i^q if, for each $n \geq 1$, for each finite extension L of K and for each hypersurface Z in $\mathbb{P}^n_L$ of degree d with $d^i \leq n$, one has $N_q(Z/L) = \mathrm{K}_q(L)$. For example, the field K is C_i^0 if, for each finite extension L of K, every hypersurface Z in $\mathbb{P}^n_L$ of degree d with $d^i \leq n$ has a 0-cycle of degree 1. The field K is C_0^q if, for each tower of finite extensions $M/L/K$, the norm morphism $N_{M/L} : \mathrm{K}_q(M) \to \mathrm{K}_q(L)$ is surjective.

Kato and Kuzumaki conjectured that, for $i \geq 0$ and $q \geq 0$, a perfect field is C_i^q if, and only if, it is of cohomological dimension at most $i+q$. This conjecture generalizes a question raised by Serre [1965] asking whether the cohomological dimension of a C_i-field is at most i. As it was already pointed out at the end of Kato and Kuzumaki's original paper [1986], Kato and Kuzumaki's conjecture for $i = 0$ follows from the Bloch–Kato conjecture (which has been established by Rost and Voevodsky [2014]); in other words, a perfect field is C_0^q if, and only if, it is of cohomological dimension at most q. However,

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it turns out that the conjectures of Kato and Kuzumaki are wrong in general. For example, Merkur'ev [1991] constructed a field of characteristic 0 and of cohomological dimension 2 which does not satisfy property C_2^0. Similarly, Colliot-Thélène and Madore [2004] produced a field of characteristic 0 and of cohomological dimension 1 which did not satisfy property C_1^0. These counterexamples were all constructed by a method using transfinite induction due to Merkurjev and Suslin. The conjecture of Kato and Kuzumaki is therefore still completely open for fields that usually appear in number theory or in algebraic geometry.

Wittenberg [2015] proved that totally imaginary number fields and p-adic fields have the C_1^1 property. Izquierdo [2018] also proved that, given a positive integer n, finite extensions of $\mathbb{C}(x_1, \dots, x_n)$ and of $\mathbb{C}(x_1, \dots, x_{n-1})((t))$ are C_i^q for any $i, q \geq 0$ such that $i + q = n$. These are essentially the only known cases of Kato and Kuzumaki's conjectures. Note however that a variant of the C_1^q-property involving homogeneous spaces under connected linear groups is proved to characterize fields with cohomological dimension at most $q + 1$ in [Izquierdo and Lucchini Arteche 2022].

In the present article, we are interested in Kato and Kuzumaki's conjectures for the function field K of a smooth projective curve C defined over a p-adic field k. The field K has cohomological dimension 3, and hence it is expected to satisfy the C_i^q-property for $i + q \geq 3$. As already mentioned, the Bloch–Kato conjecture implies this result when $q \geq 3$. The cases $q = 0$ and $q = 1$ seem out of reach with the current knowledge, since they likely imply the C_2^0-property for p-adic fields, which is a widely open question. In this article, we make progress in the case $q = 2$.

Our first main result is the following.

Main Theorem 1. *Function fields of p-adic curves satisfy the C_2^2-property.*

Of course, this implies that function fields of p-adic curves also satisfy the C_i^2-property for each $i \geq 2$. It therefore only remains to prove the C_1^2-property. In that direction, we prove the following main result.

Main Theorem 2. *Let K be the function field of a smooth projective curve C defined over a p-adic field k. Assume that C has a point in the maximal unramified extension of k. Then, for each $n, d \geq 1$ and for each hypersurface Z in $\mathbb{P}_K^n$ of degree d with $d \leq n$, we have* $\mathrm{K}_2(K) = N_2(Z/K)$.

This theorem applies for instance when K is the rational function field $k(t)$ or more generally the function field of a curve that has a rational point.

Since the proofs of these theorems are quite involved, we provide here below an outline with some details. Section 2 introduces all the notations and basic definitions we will need in the sequel. In Section 3, we prove Theorem 3.1, which widely generalizes Main Theorem 1. Finally, in Section 4, we prove Theorem 4.8 and its corollaries, Corollaries 4.9 and 4.10, which widely generalize Main Theorem 2.

Ideas of proof for Main Theorem 1. Let K be the function field of a smooth projective curve C defined over a p-adic field k. Take two integers $n, d \geq 1$ such that $d^2 \leq n$, a hypersurface Z in $\mathbb{P}_K^n$ of degree d and an element $x \in \mathrm{K}_2(K)$. We want to prove that $x \in N_2(Z/K)$. To do so, we roughly proceed in four steps, that are inspired from the proof of the C_1^1 property for number fields in [Izquierdo 2018] but that require to deal with several new difficulties:

(1) *Solve the problem locally*: For each closed point v of C, prove that $x \in N_2(Z_{K_v}/K_v)$. This provides r_v finite extensions $M_i^{(v)}/K_v$ such that $Z(M_i^{(v)}) \neq \varnothing$ and
$$x \in \langle N_{M_i^{(v)}/K_v}(\mathrm{K}_2(M_i^{(v)})) \mid 1 \leq i \leq r_v \rangle.$$

(2) *Globalize the extensions $M_i^{(v)}/K_v$*: For each closed point v of C and each $1 \leq i \leq r_v$, find a finite extension $K_i^{(v)}$ of K contained in $M_i^{(v)}$ such that $Z(K_i^{(v)}) \neq \varnothing$. Then prove that there exists a finite subset of these global extensions, say $K_1, \dots, K_r$, such that for every closed point v of C, x lies in the subgroup of $\mathrm{K}_2(K_v)$ generated by the norms coming from the $(K_i \otimes_K K_v)$.

(3) *Establish a local-to-global principle for norm groups*: Prove the vanishing of the Tate–Shafarevich group
$$\text{Ш}_2 := \ker\left(\frac{\mathrm{K}_2(K)}{\langle N_{K_i/K}(\mathrm{K}_2(K_i)) \mid 1 \leq i \leq r \rangle} \to \prod_{v \in C^{(1)}} \frac{\mathrm{K}_2(K_v)}{\langle N_{K_i \otimes_K K_v/K_v}(\mathrm{K}_2(K_i \otimes_K K_v)) \mid 1 \leq i \leq r \rangle} \right).$$

(4) *Conclude*: By step (2), we have $x \in \text{Ш}_2$. Hence, step (3) implies that
$$x \in \langle N_{K_i/K}(\mathrm{K}_2(K_i)) \mid 1 \leq i \leq r \rangle \subset N_2(Z/K),$$
as wished.

Let us now briefly discuss the proofs of Steps (1), (2) and (3). Step (1) can be proved by combining some results for p-adic fields due to Wittenberg [2015] and the computation of the groups $\mathrm{K}_2(K_v)$ thanks to the residue maps in Milnor K-theory; see Section 3A3.

In the way it is written above, Step (2) can be easily deduced from Greenberg's approximation theorem. However, as we will see below, we will need a stronger version of that step, that will require a completely different proof.

Step (3) is the hardest part of the proof. The first key tool that we use is a Poitou–Tate duality for motivic cohomology over the field K proved by Izquierdo [2016]. This provides a finitely generated free Galois module $\hat{T}$ over K such that the Pontryagin dual of Ш_2 is the quotient of
$$\text{Ш}^2(K, \hat{T}) := \ker\left(H^2(K, \hat{T}) \to \prod_{v \in C^{(1)}} H^2(K_v, \hat{T}) \right)$$
by its maximal divisible subgroup. Now, a result of Demarche and Wei [2014] states that, under some technical linear disjointness assumption for the extensions $K_i^{(v)}/K$, one can find two finite extensions K' and K'' of K such that the restriction
$$\text{Ш}^2(K, \hat{T}) \to \text{Ш}^2(K', \hat{T}) \oplus \text{Ш}^2(K'', \hat{T})$$
is injective and $\hat{T}$ is a permutation Galois module over both K' and K''. If the groups $\text{Ш}^2(K', \hat{T})$ and $\text{Ш}^2(K'', \hat{T})$ were trivial, then we would be done. But that is not the case in our context because the p-adic

function field K has finite extensions K' such that $\text{Ш}^2(K', \mathbb{Z})$ is not trivial; see for instance the appendix of [Colliot-Thélène et al. 2012]. This "failure of Chebotarev's density theorem" makes the computation of $\text{Ш}^2(K, \hat{T})$ very complicated and technical. By carrying out quite subtle Galois cohomology computations and by using some results of Kato [1980], we prove that, under some technical assumptions on the $K_i^{(v)}$ (see below) and another technical assumption on C (which is trivially satisfied when $C(k) \neq \varnothing$), the group $\text{Ш}^2(K, \hat{T})$ is always divisible, even though it might not be trivial; see Section 3A5. This is enough to apply the Poitou–Tate duality and deduce the vanishing of Ш_2.

Now, in order to ensure that the $K_i^{(v)}$ and C fulfill the conditions required to carry out the previous argument, we have to

- add a step (0) in which we reduce to the case where C satisfies a technical assumption close to having a rational point; and
- modify the constructions of the $K_i^{(v)}$ in Step (2), which cannot be done anymore by using Greenberg's approximation theorem.

The reduction to the case where C satisfies the required conditions uses the Beilinson–Lichtenbaum conjecture for motivic cohomology and a local-to-global principle due to Kato [1980] with respect to the places of K that come from a suitable regular model of the curve C; see Section 3A2. As for Step (2), we want to construct the $K_i^{(v)}$ so that they fulfill two extra conditions:

(a) One of the $K_i^{(v)}$ has to be of the form $k_i^{(v)}K$ for some finite unramified extension $k_i^{(v)}/k$. This is achieved by observing that $Z(k^{\mathrm{nr}}(C)) \neq \varnothing$ since the field $k^{\mathrm{nr}}(C)$ is C_2 and Z is a hypersurface in $\mathbb{P}^n_K$ of degree d with $d^2 \leq n$.

(b) The $K_i^{(v)}$ have to satisfy some suitable linear disjointness conditions also involving abelian extensions of K that are locally trivial everywhere. This is achieved by an approximation argument that uses the implicit function theorem for Z over the K_v, weak approximation and an analogue of Hilbert's irreducibility theorem for the field K, see Section 3A4.

Note that, since we use the implicit function theorem, the previous argument only works when the hypersurface Z is smooth. We thus need to add an extra step to the proof in which we reduce to that case. This uses a dévissage technique that is due to Wittenberg [2015] and that requires to work with *all* proper varieties over K (instead of only hypersurfaces); see Section 3A7. For that reason, we need to prove a wide generalization of Main Theorem 1 to all proper varieties. This is the object of Theorem 3.1 in the core of the text. Of course, this requires to modify and generalize the proofs of Steps (1), (2) and (3) so that they can be applied in that more general setting.

Ideas for the proof of Main Theorem 2. The proof of Main Theorem 2 follows by combining Main Theorem 1 with a result roughly stating that every element of $\mathrm{K}_2(K)$ can be written as a product of norms coming from extensions of the form $k'K$ with k' a finite extension of k whose ramification degree is fixed, see Theorem 4.1. The general ideas to prove this last result are similar to (and a bit simpler than) those used in Main Theorem 1.

2. Notations and preliminaries

In this section we fix the notations that will be used throughout this article.

Milnor K-theory. Let K be any field and let q be a nonnegative integer. The q-th Milnor K-group of K is by definition the group $\mathrm{K}_0(K) = \mathbb{Z}$ if $q = 0$ and

$$\mathrm{K}_q(K) := \underbrace{K^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} K^\times}_{q \text{ times}} / \langle x_1 \otimes \cdots \otimes x_q \mid \exists i, j, i \neq j, x_i + x_j = 1 \rangle$$

if $q > 0$. For $x_1, \dots, x_q \in K^\times$, the symbol $\{x_1, \dots, x_q\}$ denotes the class of $x_1 \otimes \cdots \otimes x_q$ in $\mathrm{K}_q(K)$. More generally, for r and s nonnegative integers such that $r + s = q$, there is a natural pairing

$$\mathrm{K}_r(K) \times \mathrm{K}_s(K) \to \mathrm{K}_q(K)$$

which we will denote $\{\cdot, \cdot\}$.

When L is a finite extension of K, one can construct a norm homomorphism

$$N_{L/K} : \mathrm{K}_q(L) \to \mathrm{K}_q(K),$$

satisfying the following properties; see Section 1.7 of [Kato 1980] or Section 7.3 of [Gille and Szamuely 2017]:

- For $q = 0$, the map $N_{L/K} : \mathrm{K}_0(L) \to \mathrm{K}_0(K)$ is given by multiplication by $[L : K]$.
- For $q = 1$, the map $N_{L/K} : \mathrm{K}_1(L) \to \mathrm{K}_1(K)$ coincides with the usual norm $L^\times \to K^\times$.
- If r and s are nonnegative integers such that $r + s = q$, we have $N_{L/K}(\{x, y\}) = \{x, N_{L/K}(y)\}$ for $x \in \mathrm{K}_r(K)$ and $y \in \mathrm{K}_s(L)$.
- If M is a finite extension of L, we have $N_{M/K} = N_{L/K} \circ N_{M/L}$.

Recall also that Milnor K-theory is endowed with residue maps; see Section 7.1 of [Gille and Szamuely 2017]. Indeed, when K is a henselian discrete valuation field with ring of integers R, maximal ideal $\mathfrak{m}$ and residue field κ, there exists a unique residue morphism

$$\partial : \mathrm{K}_q(K) \to \mathrm{K}_{q-1}(\kappa)$$

such that, for each uniformizer π and for all units $u_2, \dots, u_q \in R^\times$ whose images in κ are denoted $\overline{u_2}, \dots, \overline{u_q}$, one has

$$\partial(\{\pi, u_2, \dots, u_q\}) = \{\overline{u_2}, \dots, \overline{u_q}\}.$$

The kernel of ∂ is the subgroup $U_q(K)$ of $\mathrm{K}_q(K)$ generated by symbols of the form $\{x_1, \dots, x_q\}$ with $x_1, \dots, x_q \in R^\times$. If $U_q^1(K)$ stands for the subgroup of $\mathrm{K}_q(K)$ generated by those symbols that lie in $U_q(K)$ and that are of the form $\{x_1, \dots, x_q\}$ with $x_1 \in 1 + \mathfrak{m}$ and $x_2, \dots, x_q \in K^\times$, then $U_q^1(K)$ is ℓ-divisible for each prime ℓ different from the characteristic of κ and $U_q(K)/U_q^1(K)$ is canonically isomorphic to $\mathrm{K}_q(\kappa)$. Moreover, if L/K is a finite extension with ramification degree e and residue field λ, then the norm map $N_{L/K} : \mathrm{K}_q(L) \to \mathrm{K}_q(K)$ sends $U_q(L)$ to $U_q(K)$ and $U_q^1(L)$ to $U_q^1(K)$, and the following diagrams commute:

$$\begin{tikzcd} \mathrm{K}_q(L)/U_q(L) \arrow[r, "\partial_L", "\cong"'] \arrow[d, "N_{L/K}"] & \mathrm{K}_{q-1}(\lambda) \arrow[d, "N_{\lambda/\kappa}"] \\ \mathrm{K}_q(K)/U_q(K) \arrow[r, "\partial_K", "\cong"'] & \mathrm{K}_{q-1}(\kappa) \end{tikzcd} \qquad \begin{tikzcd} U_q(L)/U_q^1(L) \arrow[r, "\cong"'] \arrow[d, "N_{L/K}"] & \mathrm{K}_q(\lambda) \arrow[d, "eN_{\lambda/\kappa}"] \\ U_q(K)/U_q^1(K) \arrow[r, "\cong"'] & \mathrm{K}_q(\kappa) \end{tikzcd} \tag{2-1}$$

The C_i^q properties. Let K be a field and let i and q be two nonnegative integers. For each K-scheme Z of finite type, we denote by $N_q(Z/K)$ the subgroup of $\mathrm{K}_q(K)$ generated by the images of the maps $N_{L/K} : \mathrm{K}_q(L) \to \mathrm{K}_q(K)$ when L runs through the finite extensions of K such that $Z(L) \neq \varnothing$. The field K is said to have the C_i^q property if, for each $n \geq 1$, for each finite extension L of K and for each hypersurface Z in $\mathbb{P}_L^n$ of degree d with $d^i \leq n$, one has $N_q(Z/L) = \mathrm{K}_q(L)$.

Motivic complexes. Let K be a field. For $i \geq 0$, we denote by $z^i(K, \cdot)$ Bloch's cycle complex defined in [Bloch 1986]. The étale motivic complex $\mathbb{Z}(i)$ over K is then defined as the complex of Galois modules $z^i(-, \cdot)[-2i]$. By the Nesterenko–Suslin–Totaro theorem and the Beilinson–Lichtenbaum conjecture, it is known that

$$H^i(K, \mathbb{Z}(i)) \cong \mathrm{K}_i(K), \tag{2-2}$$

and

$$H^{i+1}(K, \mathbb{Z}(i)) = 0, \tag{2-3}$$

for all $i \geq 0$. Statement (2-2) was originally proved in [Nesterenko and Suslin 1989; Totaro 1992], and statement (2-3) was deduced from the Bloch–Kato conjecture in [Suslin and Voevodsky 2000; Geisser and Levine 2000; 2001]. The Bloch–Kato conjecture itself was proved in [Suslin and Joukhovitski 2006; Voevodsky 2011]. For the convenience of the reader, we also provide more tractable references: statement (2-2) follows from Theorem 5.1 of [Haesemeyer and Weibel 2019] and Theorem 1.2(2) of [Geisser 2004], and statement (2-3) can be deduced from the Bloch–Kato conjecture as explained in Lemma 1.6 and Theorem 1.7 of [Haesemeyer and Weibel 2019].

Fields of interest. From now on and until the end of the article, p stands for a prime number and k for a p-adic field with ring of integers $\mathcal{O}_k$. We let C be a smooth projective geometrically integral curve over k, and we let K be its function field. We denote by $C^{(1)}$ the set of closed points in C. The residual index $i_{\mathrm{res}}(C)$ of C is defined to be the g.c.d. of the residual degrees of the $k(v)/k$ with $v \in C^{(1)}$. The ramification index $i_{\mathrm{ram}}(C)$ of C is defined to be the g.c.d. of the ramification degrees of the $k(v)/k$ with $v \in C^{(1)}$.

Tate–Shafarevich groups. When M is a complex of Galois modules over K and $i \geq 0$ is an integer, we define the i-th Tate–Shafarevich group of M as

$$Ш^i(K, M) := \ker\Big(H^i(K, M) \to \prod_{v \in C^{(1)}} H^i(K_v, M)\Big).$$

When a suitable regular model $\mathcal{C}/\mathcal{O}_k$ of C/k is given, we also introduce the following smaller Tate–Shafarevich groups:

$$Ш^i_{\mathcal{C}}(K, M) := \ker\Big(H^i(K, M) \to \prod_{v \in \mathcal{C}^{(1)}} H^i(K_v, M)\Big),$$

where $\mathcal{C}^{(1)}$ is the set of codimension 1 points of $\mathcal{C}$.

Poitou–Tate duality for motivic cohomology. We recall the Poitou–Tate duality for motivic complexes over the field K; Theorem 0.1 of [Izquierdo 2016] in the case $d = 1$. Let $\hat{T}$ be a finitely generated free Galois module over K. Set $\check{T} := \mathrm{Hom}(\hat{T}, \mathbb{Z})$ and $T = \check{T} \otimes \mathbb{Z}(2)$. Then there is a perfect pairing of finite groups

$$\overline{Ш^2(K, \hat{T})} \times Ш^3(K, T) \to \mathbb{Q}/\mathbb{Z}, \tag{2-4}$$

where $\bar{A}$ denotes the quotient of A by its maximal divisible subgroup.

Note that, in the case $\hat{T} = \mathbb{Z}$, the Beilinson–Lichtenbaum conjecture (2-3) implies the vanishing of $Ш^3(K, \mathbb{Z}(2))$ and hence the group $Ш^2(K, \mathbb{Z})$ is divisible. By Shapiro's lemma, the same holds for the group $Ш^2(K, \mathbb{Z}[E/K])$ for every étale K-algebra E.

3. On the C_2^2-property for p-adic function fields

The goal of this section is to prove the following theorem:

Theorem 3.1. *Let l/k be a finite unramified extension and set $L := lK$. Let Z be a proper K-variety. Then the quotient*

$$\mathrm{K}_2(K)/\langle N_{L/K}(\mathrm{K}_2(L)), N_2(Z/K)\rangle$$

is $\chi_K(Z, E)^2$-torsion for each coherent sheaf E on Z.

Here, $\chi_K(Z, E)$ denotes the Euler characteristic of E over Z. Main Theorem 1 can be deduced as a very particular case of Theorem 3.1, in which this characteristic is trivial. We explain this at the end of the section.

3A. *Proof of Theorem 3.1.*

3A1. *Step 0: Interpreting norms in Milnor K-theory in terms of motivic cohomology.* The following lemma, which will be extensively used in the sequel, allows to interpret quotients of $\mathrm{K}_2(K)$ by norm subgroups as twisted motivic cohomology groups.

Lemma 3.2. *Let L be a field and let $L_1, \dots, L_r$ be finite separable extensions of L. Consider the étale L-algebra $E := \prod_{i=1}^r L_i$ and let $\check{T}$ be the Galois module defined by the following exact sequence*

$$0 \to \check{T} \to \mathbb{Z}[E/L] \to \mathbb{Z} \to 0. \tag{3-1}$$

Then

$$H^3(L, \check{T} \otimes \mathbb{Z}(2)) \cong \mathrm{K}_2(L)/\langle N_{L_i/L}(\mathrm{K}_2(L_i)) \mid 1 \le i \le r\rangle.$$

Proof. Exact sequence (3-1) induces a distinguished triangle

$$\check{T} \otimes \mathbb{Z}(2) \to \mathbb{Z}[E/L] \otimes \mathbb{Z}(2) \to \mathbb{Z}(2) \to \check{T} \otimes \mathbb{Z}(2)[1].$$

By taking cohomology, we get an exact sequence

$$H^2(L, \mathbb{Z}[E/L] \otimes \mathbb{Z}(2))) \to H^2(L, \mathbb{Z}(2)) \to H^3(L, \check{T} \otimes \mathbb{Z}(2)) \to H^3(L, \mathbb{Z}[E/L] \otimes \mathbb{Z}(2)).$$

By Shapiro's lemma, we have

$$H^2(L, \mathbb{Z}[E/L] \otimes \mathbb{Z}(2))) \cong H^2(E, \mathbb{Z}(2)), \quad H^3(L, \mathbb{Z}[E/L] \otimes \mathbb{Z}(2))) \cong H^3(E, \mathbb{Z}(2)).$$

Moreover, as recalled in Section 2, the Nesterenko–Suslin–Totaro theorem and the Beilinson–Lichtenbaum conjecture give the following isomorphisms:

$$H^2(L, \mathbb{Z}(2)) \cong \mathrm{K}_2(L), \quad H^2(E, \mathbb{Z}(2)) \cong \prod_{i=1}^r \mathrm{K}_2(L_i), \quad H^3(E, \mathbb{Z}(2)) = 0.$$

We therefore get an exact sequence

$$\prod_{i=1}^r \mathrm{K}_2(L_i) \to \mathrm{K}_2(L) \to H^3(L, \check{T} \otimes \mathbb{Z}(2)) \to 0,$$

in which the first map is the product of the norms. □

3A2. *Step 1: Reducing to curves with residual index* 1. In this step, we prove the following proposition, that allows to reduce to the case when the curve C has residual index 1.

Proposition 3.3. *Let k'/k be the unramified extension of k of degree $i_{\mathrm{res}}(C)$ and set $K' := k'K$. Then the norm morphism $N_{K'/K} : \mathrm{K}_2(K') \to \mathrm{K}_2(K)$ is surjective.*

Proof. Consider the Galois module $\check{T}$ defined by the following exact sequence

$$0 \to \check{T} \to \mathbb{Z}[K'/K] \to \mathbb{Z} \to 0,$$

Since K'/K is cyclic, a $\mathbb{Z}$-basis of $\check{T}$ is given by $s^\alpha - s^{\alpha-1}$ with s a generator of $\mathrm{Gal}(K'/K)$ and $1 \le \alpha \le i_{\mathrm{res}}(C) - 1$. Then the arrow $\mathbb{Z}[K'/K] \to \check{T}$ that sends s to $s - 1$ gives rise to an exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[K'/K] \to \check{T} \to 0,$$

and hence to a distinguished triangle

$$\mathbb{Z}(2) \to \mathbb{Z}[K'/K] \otimes \mathbb{Z}(2) \to \check{T} \otimes \mathbb{Z}(2) \to \mathbb{Z}(2)[1].$$

By the Beilinson–Lichtenbaum conjecture, the group $H^3(K', \mathbb{Z}(2))$ is trivial. Hence we get an inclusion

$$Ш^3_{\mathcal{C}}(K, \check{T} \otimes \mathbb{Z}(2)) \subseteq Ш^4_{\mathcal{C}}(K, \mathbb{Z}(2)),$$

where $\mathcal{C}$ is a fixed regular, proper and flat model of C whose reduced special fiber C_0 is a strict normal crossing divisor. Now, the distinguished triangle

$$\mathbb{Z}(2) \to \mathbb{Q}(2) \to \mathbb{Q}/\mathbb{Z}(2) \to \mathbb{Z}(2)[1],$$

and the vanishing of the groups $H^3(K, \mathbb{Q}(2))$ and $H^4(K, \mathbb{Q}(2)) = 0$ (which follow from Lemma 2.5 and Theorem 2.6.c of [Kahn 2012]) give rise to an isomorphism

$$Ш^3_{\mathcal{C}}(K, \mathbb{Q}/\mathbb{Z}(2)) \cong Ш^4_{\mathcal{C}}(K, \mathbb{Z}(2)),$$

and by Proposition 5.2 of [Kato 1986], the group on the left is trivial, and hence so is the former group.

Now observe that, by Lemma 3.2, we have

$$Ш^3_{\mathcal{C}}(K, \check{T} \otimes \mathbb{Z}(2)) \cong \ker\Big(\mathrm{K}_2(K)/\operatorname{im}(N_{K'/K}) \to \prod_{v \in \mathcal{C}^{(1)}} \mathrm{K}_2(K_v)/\operatorname{im}(N_{K'_v/K_v})\Big).$$

We claim that the extension K'/K totally splits at each place $v \in \mathcal{C}^{(1)}$. From this, we deduce that

$$0 = Ш^3_{\mathcal{C}}(K, \check{T} \otimes \mathbb{Z}(2)) \cong \mathrm{K}_2(K)/\operatorname{im}(N_{K'/K}),$$

and hence the norm morphism $N_{K'/K} : \mathrm{K}_2(K') \to \mathrm{K}_2(K)$ is surjective.

It remains to check the claim. It is obviously satisfied for $v \in C^{(1)}$, so we may and do assume $v \in \mathcal{C}^{(1)} \setminus C^{(1)}$. If κ and κ' denote the residue fields of k and k', we then have to prove that all the irreducible components of C_0 are κ'-curves. To do so, consider an infinite sequence of finite unramified field extensions $k = k_0 \subset k_1 \subset k_2 \subset \cdots$ all with degrees prime to $[k' : k]$ and denote by $\kappa = \kappa_0 \subset \kappa_1 \subset \kappa_2 \subset \cdots$ the corresponding residue fields. Let k_∞ (resp. κ_∞) be the union of all the k_i (resp. κ_i). Since κ_∞ is infinite, Lemma 4.6 of [Wittenberg 2015] and the definition of $i_{\mathrm{res}}(C)$ imply that each irreducible component of $C_0 \times_{\kappa_0} \kappa_\infty$ has index divisible by $[k' : k]$. Hence the same is true for all the irreducible components of C_0. But recall that, by the Lang–Weil estimates, any smooth geometrically integral variety defined over a finite field has a zero-cycle of degree 1. We deduce that the irreducible components of C_0 are κ'-curves. □

3A3. *Step 2: Solving the problem locally.* In this step, we prove that the analogous statement to Theorem 3.1 over the completions of K holds. For that purpose, we first need to settle a simple lemma.

Lemma 3.4. *Let l/k be a finite extension and set $K_0 := k((t))$ and $L_0 := l((t))$. The residue map $\partial : \mathrm{K}_2(K_0) \to k^\times$ induces an isomorphism*

$$\mathrm{K}_2(K_0)/N_{L_0/K_0}(\mathrm{K}_2(L_0)) \cong k^\times/N_{l/k}(l^\times).$$

Proof. We have the following commutative diagram from (2-1):

$$\begin{tikzcd} \mathrm{K}_2(L_0) \arrow[r, "\partial_{L_0}"] \arrow[d, "N_{L_0/K_0}"'] & l^\times \arrow[d, "N_{l/k}"] \\ \mathrm{K}_2(K_0) \arrow[r, "\partial_{K_0}"] & k^\times \end{tikzcd}$$

Recalling that $U_2(K_0)$ is by definition the kernel of ∂_{K_0} (see Section 2), this diagram induces an exact sequence

$$0 \to \frac{U_2(K_0)}{U_2(K_0) \cap N_{L_0/K_0}(\mathrm{K}_2(L_0))} \to \frac{\mathrm{K}_2(K_0)}{N_{L_0/K_0}(\mathrm{K}_2(L_0))} \xrightarrow{\bar{\partial}_{K_0}} \frac{k^\times}{N_{l/k}(l^\times)} \to 0.$$

It therefore suffices to prove that $U_2(K_0) = U_2(K_0) \cap N_{L_0/K_0}(\mathrm{K}_2(L_0))$. For that purpose, recall that we have a commutative diagram with exact lines:

$$\begin{tikzcd} 0 \arrow[r] & U_2^1(L_0) \arrow[r] \arrow[d, "N_{L_0/K_0}"] & U_2(L_0) \arrow[r] \arrow[d, "N_{L_0/K_0}"] & \mathrm{K}_2(l) \arrow[r] \arrow[d, "N_{l/k}"] & 0 \\ 0 \arrow[r] & U_2^1(K_0) \arrow[r] & U_2(K_0) \arrow[r] & \mathrm{K}_2(k) \arrow[r] & 0 \end{tikzcd}$$

But the map $N_{l/k} : \mathrm{K}_2(l) \to \mathrm{K}_2(k)$ is surjective since p-adic fields have the C_0^2-property, and the map $N_{L_0/K_0} : U_2^1(L_0) \to U_2^1(K_0)$ is surjective since the group $U_2^1(K_0)$ is divisible (as explained in Section 2). We deduce that $N_{L_0/K_0} : U_2(L_0) \to U_2(K_0)$ is also surjective, as wished. □

Proposition 3.5. *Let l/k be a finite unramified extension and set $K_0 := k((t))$ and $L_0 := l((t))$. Let Z be a proper K_0-variety. Then the quotient*

$$\mathrm{K}_2(K_0)/\langle N_{L_0/K_0}(\mathrm{K}_2(L_0)), N_2(Z/K_0)\rangle$$

is $\chi_{K_0}(Z, E)$-torsion for each coherent sheaf E on Z.

Proof. For each proper K_0-scheme Z, we denote by n_Z the exponent of the quotient group

$$\mathrm{K}_2(K_0)/\langle N_{L_0/K_0}(\mathrm{K}_2(L_0)), N_2(Z/K_0)\rangle.$$

We say that Z satisfies property (P) if it has a model over $\mathcal{O}_{K_0}$ that is irreducible, regular, proper and flat. To prove the proposition, it suffices to check assumptions (1), (2) and (3) of Proposition 2.1 of [Wittenberg 2015].

Assumption (1) is obvious. Assumption (3) is a direct consequence of Gabber and de Jong's theorem (Theorem 3 of the introduction of [Illusie et al. 2014]). It remains to check assumption (2). For that purpose, we proceed in the same way as in the proof of Theorem 4.2 of [Wittenberg 2015]. Indeed, consider a proper K_0-scheme X together with a model $\mathcal{X}$ that is irreducible, regular, proper and flat and denote by Y its special fiber. Let m be the multiplicity of Y and let D be the effective divisor on $\mathcal{X}$ such that $Y = mD$.

The residue map induces an exact sequence

$$0 \to \frac{U_2(K_0)}{U_2(K_0) \cap N_2(X/K_0)} \to \frac{\mathrm{K}_2(K_0)}{N_2(X/K_0)} \to \frac{\mathrm{K}_1(k)}{\partial(N_2(X/K_0))} \to 0. \tag{3-2}$$

Moreover:

(a) Since k satisfies the C_0^2 property, the proof of Lemma 4.4 of [Wittenberg 2015] still holds in our context, and hence the group $U_2(K_0)/(U_2(K_0) \cap N_2(X/K_0))$ is killed by the multiplicity m of the special fiber Y of $\mathcal{X}$.

(b) The proof of Lemma 4.5 of [Wittenberg 2015] also holds in our context, and hence $\partial(N_2(X/K_0)) = N_1(Y/k) = N_1(D/k)$.

(c) By Corollary 5.4 of [Wittenberg 2015] applied to the proper k-scheme $D \sqcup \mathrm{Spec}(l)$, the group $k^\times/\langle N_{l/k}(l^\times), N_1(D/k)\rangle$ is killed by $\chi_k(D, \mathcal{O}_D)$.

By using exact sequence (3-2), facts (b) and (c) and Lemma 3.4, we deduce that

$$\chi_k(D, \mathcal{O}_D) \cdot \mathrm{K}_2(K_0) \subset \langle N_{L_0/K_0}(\mathrm{K}_2(L_0)), N_2(X/K_0), U_2(K_0)\rangle.$$

Hence, by fact (a), we get

$$m\chi_k(D, \mathcal{O}_D) \cdot \mathrm{K}_2(K_0) \subset \langle N_{L_0/K_0}(\mathrm{K}_2(L_0)), N_2(X/K_0)\rangle.$$

But $m\chi_k(D, \mathcal{O}_D) = \chi_{K_0}(X, \mathcal{O}_X)$ by Proposition 2.4 of [Esnault et al. 2015], and hence the quotient $\mathrm{K}_2(K_0)/\langle N_{L_0/K_0}(\mathrm{K}_2(L_0)), N_2(X/K_0)\rangle$ is killed by $\chi_{K_0}(X, \mathcal{O}_X)$. □

3A4. *Step 3: Globalizing local field extensions.* In the rest of the proof, we will show how one can deduce the global Theorem 3.1 from the local Proposition 3.5. For that purpose, we first need to find a suitable way to globalize local extensions: more precisely, given a place $w \in C^{(1)}$ and a finite extension $M^{(w)}$ of K_w such that $Z(M^{(w)}) \neq \varnothing$, we want to find a suitable finite extension M of K that can be seen as a subfield of $M^{(w)}$ and such that $Z(M) \neq \varnothing$. For technical reasons related to the failure of Chebotarev's theorem over the field K, we also need M to be linearly disjoint from a given finite extension of K. The following proposition is the key statement allowing to do that.

Proposition 3.6. *Let Z be a smooth geometrically integral K-variety. Let T be a finite subset of $C^{(1)}$. Fix a finite extension L of K and, for each $w \in T$, a finite extension $M^{(w)}$ of K_w such that $Z(M^{(w)}) \neq \varnothing$. Then there exists a finite extension M of K satisfying the following properties*:

(i) *$Z(M) \neq \varnothing$.*

(ii) *For each $w \in T$, there exists a K-embedding $M \hookrightarrow M^{(w)}$.*

(iii) *The extensions L/K and M/K are linearly disjoint.*

Proof. Before starting the proof, we introduce the following notations for each $w \in T$:

$$n^{(w)} := [M^{(w)} : K_w], \quad m^{(w)} := \prod_{w' \in T \setminus \{w\}} n^{(w')},$$

so that the integer $n := n^{(w)} m^{(w)}$ is independent of w. We now proceed in three substeps.

Substep 1. By Proposition 4.9 in Chapter I of [Hartshorne 1977], there exists a projective hypersurface Z' in $\mathbb{P}^m_K$ given by a nonzero equation

$$f(x_0, \dots, x_m) = 0$$

that is birationally equivalent to Z. Let U and U' be nonempty open sub-schemes of Z and Z' that are isomorphic. Up to reordering the variables and shrinking U', we may and do assume that the polynomial $\partial f / \partial x_0$ is nonzero and that

$$U' \cap \{\partial f / \partial x_0(x_0, \dots, x_m) = 0\} = \varnothing.$$

Given an element $w \in T$, the variety Z is smooth, $Z(M^{(w)}) \neq \varnothing$ and $M^{(w)}$ is large; for the definition of this notion, please refer to [Pop 2014]. Hence the sets $U(M^{(w)})$ and $U'(M^{(w)})$ are nonempty. We can therefore find a nontrivial solution $(y_0^{(w)}, \dots, y_m^{(w)})$ of the equation $f(x_0, \dots, x_m) = 0$ in $M^{(w)}$ such that

$$\begin{cases} (y_0^{(w)}, \dots, y_m^{(w)}) \in U', \\ \partial f / \partial x_0(y_0^{(w)}, \dots, y_m^{(w)}) \neq 0. \end{cases}$$

Substep 2. Given $w \in T$, there exist $m^{(w)}$ elements $\alpha_1, \dots, \alpha_{m^{(w)}} \in M^{(w)}$ whose respective minimal polynomials $\mu_{\alpha_1}, \dots, \mu_{\alpha_{m^{(w)}}}$ are pairwise distinct and such that $M^{(w)} = K_w(\alpha_i)$ for each $1 \le i \le m^{(w)}$. Recalling that $n = n^{(w)} m^{(w)}$, introduce the degree n monic polynomial $\mu^{(w)} := \prod_{i=1}^{m^{(w)}} \mu_{\alpha_i}$ and consider the set H of n-tuples $(a_0, \dots, a_{n-1}) \in K^n$ such that the polynomial $T^n + \sum_{i=0}^{n-1} a_i T^i$ is irreducible over L. By Corollary 12.2.3 of [Fried and Jarden 2008], the set H contains a Hilbertian subset of K^n, and hence, according to Proposition 19.7 of [Jarden 1991], if we fix some $\epsilon > 0$, we can find an n-tuple $(b_0, \dots, b_{n-1})$ in H such that the polynomial $\mu := T^n + \sum_{i=0}^{n-1} b_i T^i$ is coefficient-wise ϵ-close to $\mu^{(w)}$ for each $w \in T$. Consider the field $K' := K[T]/(\mu)$. If ϵ is chosen small enough, then there exists a K-embedding $K' \hookrightarrow M^{(w)}$ for each $w \in T$ by Krasner's lemma; see Lemma 8.1.6 in [Neukirch et al. 2008]. Moreover, since $(b_0, \dots, b_{n-1}) \in H$, the polynomial μ is irreducible over L, and hence the extensions K'/K and L/K are linearly disjoint.

Substep 3. According to Substep 1, for each $w \in T$, $y_0^{(w)}$ is a simple root of the polynomial

$$g^{(w)}(T) := f(T, y_1^{(w)}, \dots, y_m^{(w)}).$$

Let H' be the set of m-tuples $(z_1, \dots, z_m)$ in K' such that $f(T, z_1, \dots, z_m)$ is irreducible over LK'. By Corollary 12.2.3 of [Fried and Jarden 2008], the set H' contains a Hilbertian subset of K'^m. Hence, by Proposition 19.7 of [Jarden 1991], we can find $(y_1, \dots, y_m)$ in H' such that the polynomial

$$g(T) := f(T, y_1, \dots, y_m)$$

is coefficient-wise ϵ-close to $g^{(w)}$ for each $w \in T$. Introduce the field $M := K'[T]/(g(T))$. We check that M satisfies the conditions of the proposition, provided that ϵ is chosen small enough:

(i) Fix $w \in T$. By Substep 1, the m-tuple $(y_0^{(w)}, \dots, y_m^{(w)})$ lies in U'. Hence, for ϵ small enough, if $y_{0,w}$ stands for the root of g that is closest to $y_0^{(w)}$, then the m-tuple $(y_{0,w}, y_1, \dots, y_m)$ lies in U'. We deduce that $U'(M) \neq \varnothing$, and hence $Z(M) \neq \varnothing$.

(ii) For each $w \in T$, the polynomial $g^{(w)}$ has a simple root in $M^{(w)}$, and hence so does $g(T)$ if ϵ is chosen small enough, again by Krasner's Lemma. The field M can therefore be seen as a subfield of $M^{(w)}$.

(iii) Since $(y_1, \dots, y_m) \in H'$, the polynomial $g(T)$ is irreducible over LK'. Hence the extensions M/K' and LK'/K' are linearly disjoint. Moreover, by Substep 2, the extensions K'/K and L/K are linearly disjoint. We deduce that L/K and M/K are linearly disjoint. □

3A5. *Step 4: Computation of a Tate–Shafarevich group.* This step, which is quite technical, consists in computing the Tate–Shafarevich groups of some finitely generated free Galois modules over K. Recall that for each abelian group A, we denote by $\bar{A}$ the quotient of A by its maximal divisible subgroup.

Proposition 3.7. *Let $r \geq 2$ be an integer and let $L, K_1, \dots, K_r$ be finite extensions of K contained in $\overline{K}$. Consider the composite fields $K_{\mathcal{I}} := K_1 \dots K_r$ and $K_{\hat{i}} := K_1 \dots K_{i-1}K_{i+1} \dots K_r$ for each i, and denote by n the degree of L/K. Consider the Galois module $\hat{T}$ defined by the following exact sequence*

$$0 \to \mathbb{Z} \to \mathbb{Z}[E/K] \to \hat{T} \to 0, \tag{3-3}$$

where $E := L \times K_1 \times \cdots \times K_r$. Given two positive integers m and m', make the following assumptions:

(LD1) *The Galois closure of L/K and the extension $K_{\mathcal{I}}/K$ are linearly disjoint.*

(LD2) *For each $i \in \{1, \dots, r\}$, the fields K_i and $K_{\hat{i}}$ are linearly disjoint over K.*

(H1) *The restriction map*

$$Ш^2(K, \hat{T}) \to Ш^2(L, \hat{T}) \oplus Ш^2(K_{\mathcal{I}}, \hat{T})$$

is injective.

(H2) *The restriction map*

$$\mathrm{Res}_{LK_{\mathcal{I}}/K_{\mathcal{I}}} : Ш^2(K_{\mathcal{I}}, \mathbb{Z}) \to Ш^2(LK_{\mathcal{I}}, \mathbb{Z})$$

is surjective and its kernel is m-torsion.

(H3) *For each i, the restriction maps*

$$\mathrm{Res}_{LK_i/K_i} : Ш^2(K_i, \mathbb{Z}) \to Ш^2(LK_i, \mathbb{Z}) \quad \text{and} \quad \mathrm{Res}_{LK_{\hat{i}}/K_{\hat{i}}} : Ш^2(K_{\hat{i}}, \mathbb{Z}) \to Ш^2(LK_{\hat{i}}, \mathbb{Z})$$

are surjective.

(H4) *For each finite extension L' of L contained in the Galois closure of L/K, the kernel of the restriction map*
$$\mathrm{Res}_{L'K_{\mathcal{I}}/L} : Ш^2(L', \mathbb{Z}) \to Ш^2(L'K_{\mathcal{I}}, \mathbb{Z})$$
is m'-torsion.

Then $\overline{Ш^2(K, \hat{T})}$ is $((m \vee m') \wedge n)$-torsion.

Recall that $\bar{A}$ denotes the quotient of A by its maximal divisible subgroup.

Remark 3.8. In the sequel of the article, we will only use the proposition in the case when L/K is Galois. However, this assumption does not simplify the proof.

Proof. Consider the following sequence:

$$\begin{array}{ccccc} Ш^2(K, \hat{T}) & \xrightarrow{f_0} & Ш^2(L, \hat{T}) \oplus Ш^2(K_{\mathcal{I}}, \hat{T}) & \xrightarrow{g_0} & Ш^2(LK_{\mathcal{I}}, \hat{T}) \\ x & \longmapsto & (\mathrm{Res}_{L/K}(x), \mathrm{Res}_{K_{\mathcal{I}}/K}(x)) & & \\ & & (x, y) & \longmapsto & \mathrm{Res}_{LK_{\mathcal{I}}/L}(x) - \mathrm{Res}_{LK_{\mathcal{I}}/K_{\mathcal{I}}}(y). \end{array} \tag{3-4}$$

It is obviously a complex, and the first arrow is injective by (H1). In order to give further information about the complex (3-4), let us consider the following commutative diagram, in which the first and second rows are obtained in the same way as the third:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] \\ Ш^2(K, \mathbb{Z}) \arrow[r, "f_1"] \arrow[d, "\phi_0"] & Ш^2(L, \mathbb{Z}) \oplus Ш^2(K_{\mathcal{I}}, \mathbb{Z}) \arrow[r, two heads, "g_1"] \arrow[d, "\phi_1"] & Ш^2(LK_{\mathcal{I}}, \mathbb{Z}) \arrow[d, "\phi_2"] \\ Ш^2(K, \mathbb{Z}[E/K]) \arrow[r, "f"] \arrow[d, "\psi_0"] & Ш^2(L, \mathbb{Z}[E/K]) \oplus Ш^2(K_{\mathcal{I}}, \mathbb{Z}[E/K]) \arrow[r, "g"] \arrow[d, "\psi_1"] & Ш^2(LK_{\mathcal{I}}, \mathbb{Z}[E/K]) \arrow[d, "\psi_2"] \\ Ш^2(K, \hat{T}) \arrow[r, hook, "f_0"] & Ш^2(L, \hat{T}) \oplus Ш^2(K_{\mathcal{I}}, \hat{T}) \arrow[r, "g_0"] \arrow[d] & Ш^2(LK_{\mathcal{I}}, \hat{T}) \arrow[d] \\ & 0 & 0 \end{tikzcd} \tag{3-5}$$

The second and third columns are exact since the exact sequence (3-3) splits over L, $K_{\mathcal{I}}$ and $LK_{\mathcal{I}}$. Moreover, all the lines are complexes, and in the first one, the arrow g_1 is surjective since the restriction map
$$Ш^2(K_{\mathcal{I}}, \mathbb{Z}) \to Ш^2(LK_{\mathcal{I}}, \mathbb{Z})$$
is surjective by (H2).

The next two lemmas constitute the core of the proof of Proposition 3.7.

Lemma 3.9. *Let $a \in Ш^2(K, \hat{T})$ and $b = (b_L, b_{K_{\mathcal{I}}}) \in Ш^2(L, \mathbb{Z}[E/K]) \oplus Ш^2(K_{\mathcal{I}}, \mathbb{Z}[E/K]))$ such that $f_0(a) = \psi_1(b)$ and $g(b) = 0$. Then $mb_{K_{\mathcal{I}}}$ comes by restriction from $Ш^2(K_{\hat{i}}, \mathbb{Z}[E/K])$ for each i.*

Proof. Consider the following commutative diagram, constructed exactly in the same way as diagram (3-5):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
Ш^2(K,\mathbb{Z}) \arrow[r, "f_1^i"] \arrow[d, "\phi_0"] & Ш^2(L,\mathbb{Z}) \oplus Ш^2(K_{\hat{i}},\mathbb{Z}) \arrow[r, two heads, "g_1^i"] \arrow[d, "\phi_1^i"] & Ш^2(LK_{\hat{i}},\mathbb{Z}) \arrow[d, "\phi_2^i"] \\
Ш^2(K,\mathbb{Z}[E/K]) \arrow[r, "f^i"] \arrow[d, "\psi_0"] & Ш^2(L,\mathbb{Z}[E/K]) \oplus Ш^2(K_{\hat{i}},\mathbb{Z}[E/K]) \arrow[r, "g^i"] \arrow[d, "\psi_1^i"] & Ш^2(LK_{\hat{i}},\mathbb{Z}[E/K]) \arrow[d, "\psi_2^i"] \\
Ш^2(K,\hat{T}) \arrow[r, hook, "f_0^i"] & Ш^2(L,\hat{T}) \oplus Ш^2(K_{\hat{i}},\hat{T}) \arrow[r, "g_0^i"] \arrow[d] & Ш^2(LK_{\hat{i}},\hat{T}) \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

The last two columns are exact since the exact sequence (3-3) splits over L, $K_{\hat{i}}$ and $LK_{\hat{i}}$, and the restriction morphism $Ш^2(K_{\hat{i}}, \mathbb{Z}) \to Ш^2(LK_{\hat{i}}, \mathbb{Z})$ is surjective by (H3). Hence there exists $b_{K_{\hat{i}}} \in Ш^2(K_{\hat{i}}, \mathbb{Z}[E/K])$ such that $\psi_1^i(b_L, b_{K_{\hat{i}}}) = f_0^i(a)$ and $g^i(b_L, b_{K_{\hat{i}}}) = 0$. The pair

$$(0, b_{K_{\mathcal{I}}} - \mathrm{Res}_{K_{\mathcal{I}}/K_{\hat{i}}}(b_{K_{\hat{i}}})) \in Ш^2(L, \mathbb{Z}[E/K]) \oplus Ш^2(K_{\mathcal{I}}, \mathbb{Z}[E/K])$$

then lies in $\ker(g) \cap \ker(\psi_1)$ and a diagram chase in (3-5) shows that there exists $c \in Ш^2(K_{\mathcal{I}}, \mathbb{Z})$ such that

$$\begin{cases} \phi_1(0, c) = (0, b_{K_{\mathcal{I}}} - \mathrm{Res}_{K_{\mathcal{I}}/K_{\hat{i}}}(b_{K_{\hat{i}}})), \\ \mathrm{Res}_{LK_{\mathcal{I}}/K_{\mathcal{I}}}(c) = 0. \end{cases}$$

By (H2), we have $mc = 0$, and hence $m \cdot (b_{K_{\mathcal{I}}} - \mathrm{Res}_{K_{\mathcal{I}}/K_{\hat{i}}}(b_{K_{\hat{i}}})) = 0$. □

Lemma 3.10. *Set $\mu := m \vee m'$ and take $a \in Ш^2(K, \hat{T})$. Then $\mu a \in \mathrm{Im}(\psi_0)$.*

Before proving the lemma, let us introduce some notation.

Notation 3.11. (i) For each i, we can find a family $(K_{ij})_j$ of finite extensions of $K_{\mathcal{I}}$ together with embeddings $\sigma_{ij} : K_i \hookrightarrow K_{ij}$ so that $K_{i,1} = K_{\mathcal{I}}$, the embedding $\sigma_{i,1}$ is the natural embedding $K_i \hookrightarrow K_{\mathcal{I}}$, and the K-algebra homomorphism

$$K_i \otimes_K K_{\mathcal{I}} \to \prod_j K_{ij}$$
$$x \otimes y \mapsto (\sigma_{ij}(x)y)_j$$

is an isomorphism. We denote by $\tilde{\sigma}_{ij} : K_{\mathcal{I}} \to K_{ij}$ the embedding obtained by tensoring σ_{ij} with the identity of $K_{\hat{i}}$. This is well-defined by (LD2).

(ii) For each i, j, we can find a family $(L_{ijj'})_{j'}$ of finite extensions of K_{ij} together with embeddings $\sigma_{ijj'} : L \hookrightarrow L_{ijj'}$ so that the K-algebra homomorphism

$$\begin{aligned} L \otimes_K K_{ij} &\to \prod_{j'} L_{ijj'} \\ x \otimes y &\mapsto (\sigma_{ijj'}(x)y)_{j'} \end{aligned} \tag{3-6}$$

is an isomorphism. We denote by $\tilde{\sigma}_{ijj'} : LK_i \to L_{ijj'}$ the embedding obtained by tensoring $\sigma_{ijj'}$ with σ_{ij}. Observe that, when $j=1$, the K-algebra homomorphism (3-6) is simply the isomorphism $L \otimes_K K_{\mathcal{I}} \cong LK_{\mathcal{I}}$, so that $\sigma_{i,1,1}$ is none other than the inclusion of L in $LK_{\mathcal{I}}$.

(iii) We can find a family of finite extensions $(L_\alpha)_\alpha$ of L together with embeddings $\tau_\alpha : L \hookrightarrow L_\alpha$ so that $L_1 = L$, the embedding τ_1 is the identity of L, and the K-algebra homomorphism

$$\begin{aligned} L \otimes_K L &\to \prod_{\alpha} L_\alpha \\ x \otimes y &\mapsto (\tau_\alpha(x)y)_\alpha \end{aligned}$$

is an isomorphism. For each α, we denote by $\tilde{\tau}_\alpha : LK_{\mathcal{I}} \to L_\alpha K_{\mathcal{I}}$ the embedding obtained by tensoring τ_α with the identity of $K_{\mathcal{I}}$. This is well-defined by (LD1).

Proof. By Shapiro's lemma, one can identify the second line of diagram (3-5) with the following complex:

$$\begin{tikzcd} Ш^2(L,\mathbb{Z}) \oplus \bigoplus_i Ш^2(K_i,\mathbb{Z}) \arrow[d, "f"] \\ \bigoplus_\alpha Ш^2(L_\alpha,\mathbb{Z}) \oplus \bigoplus_i Ш^2(LK_i,\mathbb{Z}) \oplus Ш^2(LK_{\mathcal{I}},\mathbb{Z}) \oplus \bigoplus_{i,j} Ш^2(K_{ij},\mathbb{Z}) \arrow[d, "g"] \\ \bigoplus_\alpha Ш^2(L_\alpha K_{\mathcal{I}},\mathbb{Z}) \oplus \bigoplus_{i,j,j'} Ш^2(L_{ijj'},\mathbb{Z}) \end{tikzcd}$$

where f is given by

$$(x,(y_i)_i) \mapsto \big((\mathrm{Res}_{\tau_\alpha : L \hookrightarrow L_\alpha}(x))_\alpha, (\mathrm{Res}_{LK_i/K_i}(y_i))_i, \mathrm{Res}_{LK_{\mathcal{I}}/L}(x), (\mathrm{Res}_{\sigma_{ij}:K_i \hookrightarrow K_{ij}}(y_i))_{i,j}\big),$$

and g

$$\begin{aligned} &((x_\alpha)_\alpha,(y_i)_i,z,(t_{ij})_{i,j}) \mapsto \\ &\qquad \big((\mathrm{Res}_{L_\alpha K_{\mathcal{I}}/L_\alpha}(x_\alpha) - \mathrm{Res}_{\tilde{\tau}_\alpha : LK_{\mathcal{I}} \hookrightarrow L_\alpha K_{\mathcal{I}}}(z))_\alpha, (\mathrm{Res}_{\tilde{\sigma}_{ijj'}:LK_i \hookrightarrow L_{ijj'}}(y_i) - \mathrm{Res}_{L_{ijj'}/K_{ij}}(t_{ij}))_{i,j}\big). \end{aligned}$$

Now take

$$((x_\alpha)_\alpha,(y_i)_i,z,(t_{ij})_{i,j}) \in \bigoplus_\alpha Ш^2(L_\alpha,\mathbb{Z}) \oplus \bigoplus_i Ш^2(LK_i,\mathbb{Z}) \oplus Ш^2(LK_{\mathcal{I}},\mathbb{Z}) \oplus \bigoplus_{i,j} Ш^2(K_{ij},\mathbb{Z}) \tag{3-7}$$

such that

$$\psi_1((x_\alpha)_\alpha,(y_i)_i,z,(t_{ij})_{i,j}) = f_0(a). \tag{3-8}$$

Since $g_0(f_0(a)) = 0$ and g_1 is surjective, a diagram chase in (3-5) allows to assume that

$$((x_\alpha)_\alpha, (y_i)_i, z, (t_{ij})_{i,j}) \in \ker(g). \tag{3-9}$$

This implies that

$$\begin{cases} \mathrm{Res}_{L_\alpha K_{\mathcal{I}}/L_\alpha}(x_\alpha) = \mathrm{Res}_{\tilde{\tau}_\alpha : LK_{\mathcal{I}} \hookrightarrow L_\alpha K_{\mathcal{I}}}(z) & \forall \alpha, \\ \mathrm{Res}_{\tilde{\sigma}_{ijj'} : LK_i \hookrightarrow L_{ijj'}}(y_i) = \mathrm{Res}_{L_{ijj'}/K_{ij}}(t_{ij}) & \forall i,\, j,\, j'. \end{cases} \tag{3-10}$$

In particular,

$$\mathrm{Res}_{L_1 K_{\mathcal{I}}/L_1}(x_1) = \mathrm{Res}_{LK_{\mathcal{I}}/L}(x_1) = z, \tag{3-11}$$

and hence the commutativity of the following diagram of field extensions:

$$\begin{tikzcd} & L_\alpha K_{\mathcal{I}} & \\ L_\alpha \arrow[ur, dash] & & LK_{\mathcal{I}} \arrow[ul, dash, "\tilde{\tau}_\alpha"'] \\ & L \arrow[ul, dash, "\tau_\alpha"] \arrow[ur, dash] & \end{tikzcd}$$

shows that

$$\begin{aligned} \mathrm{Res}_{L_\alpha K_{\mathcal{I}}/L_\alpha}(\mathrm{Res}_{\tau_\alpha : L \hookrightarrow L_\alpha}(x_1)) &= \mathrm{Res}_{\tilde{\tau}_\alpha : LK_{\mathcal{I}} \hookrightarrow L_\alpha K_{\mathcal{I}}}(\mathrm{Res}_{LK_{\mathcal{I}}/L}(x_1)) \\ &= \mathrm{Res}_{\tilde{\tau}_\alpha : LK_{\mathcal{I}} \hookrightarrow L_\alpha K_{\mathcal{I}}}(z) \\ &= \mathrm{Res}_{L_\alpha K_{\mathcal{I}}/L_\alpha}(x_\alpha). \end{aligned}$$

Since the kernel of $\mathrm{Res}_{L_\alpha K_{\mathcal{I}}/L_\alpha}$ is m'-torsion by (H4), we have

$$m' \,\mathrm{Res}_{\tau_\alpha : L \hookrightarrow L_\alpha}(x_1) = m' x_\alpha \tag{3-12}$$

for all α. Moreover, by (H3), one can find for each i an element $\tilde{y}_i \in Ш^2(K_i, \mathbb{Z})$ such that

$$y_i = \mathrm{Res}_{LK_i/K_i}(\tilde{y}_i). \tag{3-13}$$

Let us check that

$$\mu((x_\alpha)_\alpha, (y_i)_i, z, (t_{ij})_{i,j}) = \mu f(x_1, (\tilde{y}_i)_i). \tag{3-14}$$

By construction (see Equations (3-12), (3-13) and (3-11)), we have

$$\mu(\mathrm{Res}_{\tau_\alpha : L \hookrightarrow L_\alpha}(x_1))_\alpha = \mu(x_\alpha)_\alpha, \quad (y_i)_i = (\mathrm{Res}_{LK_i/K_i}(\tilde{y}_i))_i, \quad \mu \,\mathrm{Res}_{LK_{\mathcal{I}}/L}(x_1) = \mu z.$$

To finish the proof of (3-14), it is therefore enough to check that

$$m t_{ij} = m \,\mathrm{Res}_{\sigma_{ij} : K_i \hookrightarrow K_{ij}}(\tilde{y}_i) \tag{3-15}$$

for each i and j. For that purpose, fix $i = i_0$, and consider first the case $j = 1$. We then have $K_{i_0,1} = K_{\mathcal{I}}$, and hence, by using (3-10)

$$\mathrm{Res}_{LK_{\mathcal{I}}/K_{\mathcal{I}}}(t_{i_0,1}) = \mathrm{Res}_{L_{i_0,1,1}/LK_{i_0}}(y_{i_0}) = \mathrm{Res}_{L_{i_0,1,1}/K_{i_0}}(\tilde{y}_{i_0}) = \mathrm{Res}_{LK_{\mathcal{I}}/K_{\mathcal{I}}}(\mathrm{Res}_{K_{i_0,1}/K_{i_0}}(\tilde{y}_{i_0})).$$

By (H2), we deduce that

$$mt_{i_0,1} = m\,\mathrm{Res}_{K_{i_0,1}/K_{i_0}}(\tilde{y}_{i_0}) = m\,\mathrm{Res}_{K_{\mathcal{I}}/K_{i_0}}(\tilde{y}_{i_0}).$$

Now move on to case of arbitrary j. By Lemma 3.9 together with Equations (3-8) and (3-9), the element

$$m(z, (t_{i,j})_{i,j}) \in Ш^2(LK_{\mathcal{I}}, \mathbb{Z}) \oplus \bigoplus_{i,j} Ш^2(K_{ij}, \mathbb{Z}) = Ш^2(K_{\mathcal{I}}, \mathbb{Z}[E/K])$$

comes by restriction from $Ш^2(K_{\hat{i}}, \mathbb{Z}[E/K])$ for each i. In particular, the element

$$(mt_{i_0,j})_j \in \bigoplus_j Ш^2(K_{i_0,j}, \mathbb{Z}) = Ш^2(K_{\mathcal{I}}, \mathbb{Z}[K_{i_0}/K])$$

comes by restriction from an element $t_{i_0} \in Ш^2(K_{\mathcal{I}}, \mathbb{Z}) = Ш^2(K_{\hat{i}_0}, \mathbb{Z}[K_{i_0}/K])$. In other words

$$(mt_{i_0,j})_j = (\mathrm{Res}_{\tilde{\sigma}_{i_0,j}:K_{\mathcal{I}}\hookrightarrow K_{i_0,j}}(t_{i_0}))_j.$$

In particular, $mt_{i_0,1} = t_{i_0}$, and hence for each j

$$\begin{aligned} mt_{i_0,j} &= \mathrm{Res}_{\tilde{\sigma}_{i_0,j}:K_{\mathcal{I}}\hookrightarrow K_{i_0,j}}(t_{i_0}) \\ &= \mathrm{Res}_{\tilde{\sigma}_{i_0,j}:K_{\mathcal{I}}\hookrightarrow K_{i_0,j}}(mt_{i_0,1}) \\ &= \mathrm{Res}_{\tilde{\sigma}_{i_0,j}:K_{\mathcal{I}}\hookrightarrow K_{i_0,j}}(m\,\mathrm{Res}_{K_{\mathcal{I}}/K_{i_0}}(\tilde{y}_{i_0})) \\ &= m\,\mathrm{Res}_{\sigma_{i_0,j}:K_{i_0}\hookrightarrow K_{i_0,j}}(\tilde{y}_{i_0}). \end{aligned}$$

This finishes the proofs of equalities (3-15) and (3-14). Applying ψ_1 to (3-14) we deduce that

$$\mu f_0(\alpha) = \mu f_0(\psi_0((x_\alpha)_\alpha, (y_i)_i, z, (t_{ij})_{i,j})).$$

Since f_0 is injective, we get

$$\mu\alpha = \mu\psi_0((x_\alpha)_\alpha, (y_i)_i, z, (t_{ij})_{i,j}),$$

which finishes the proof of the lemma. □

We can now finish the proof of Proposition 3.7. As recalled at the end of Section 2, the group $Ш^2(K, \mathbb{Z}[E/K])$ is divisible and hence, by Lemma 3.10,

$$(m \vee m') \cdot Ш^2(K, \hat{T}) \subseteq Ш^2(K, \hat{T})_{\mathrm{div}}.$$

In other words, the group $\overline{Ш^2(K, \hat{T})}$ is $(m \vee m')$-torsion.

On the other hand, using once again the end of Section 2, the group $\overline{Ш^2(L, \hat{T})}$ vanishes. Hence, by restriction-corestriction, $\overline{Ш^2(K, \hat{T})}$ is n-torsion. We deduce that $\overline{Ш^2(K, \hat{T})}$ is $((m \vee m') \wedge n)$-torsion. □

The following lemma will often allow us to check assumptions (H2) and (H3) of Proposition 3.7:

Lemma 3.12. *Let l be a finite unramified extension of k of degree n and set $L = lK$. The restriction map $\mathrm{Res}_{L/K} : Ш^2(K, \mathbb{Z}) \to Ш^2(L, \mathbb{Z})$ is surjective and its kernel is $(i_{\mathrm{res}}(C) \wedge n)$-torsion.*

Proof. By restriction-corestriction, $\ker(\mathrm{Res}_{L/K})$ is killed by n. Moreover, since $Ш^2(K,\mathbb{Z}) = Ш^1(K,\mathbb{Q}/\mathbb{Z})$, an element in $\ker(\mathrm{Res}_{L/K})$ corresponds to a subextension $K \subset L' \subset L$ that is locally trivial at every closed point of the curve C. Since $L = lK$, we can find an extension $k \subset l' \subset l$ such that $L' = l'K$. By the local triviality of L'/K, the field l' has to be contained in the residue field of $k(v)$ for every $v \in C^{(1)}$. In particular, $[l' : k]$ and $[L' : K]$ divide $i_{\mathrm{res}}(C)$. This shows that $\ker(\mathrm{Res}_{L/K})$ is killed by $i_{\mathrm{res}}(C)$, and hence by $i_{\mathrm{res}}(C) \wedge n$.

In order to prove the surjectivity statement, consider an integral, regular, projective model $\mathcal{C}$ of C such that its reduced special fiber C_0 is an SNC divisor. Let c be the genus of the reduction graph of $\mathcal{C}$. According to Corollary 2.9 of [Kato 1986], for each $m \geq 1$, we have an isomorphism

$$Ш^3(K,\mathbb{Z}/m\mathbb{Z}(2)) \cong (\mathbb{Z}/m\mathbb{Z})^c.$$

Hence, by Poitou–Tate duality, we also have

$$Ш^1(K,\mathbb{Z}/m\mathbb{Z}) \cong (\mathbb{Z}/m\mathbb{Z})^c,$$

so that

$$Ш^2(K,\mathbb{Z}) \cong (\mathbb{Q}/\mathbb{Z})^c.$$

Since l/k is unramified, the scheme $\mathcal{C} \times_{\mathcal{O}_k} \mathcal{O}_l$ is a suitable regular model of $C \times_k l$ and hence $Ш^2(L,\mathbb{Z})$ is also isomorphic to $(\mathbb{Q}/\mathbb{Z})^c$. The surjectivity of $\mathrm{Res}_{L/K}$ then follows from the isomorphism $Ш^2(K,\mathbb{Z}) \cong Ш^2(L,\mathbb{Z}) \cong (\mathbb{Q}/\mathbb{Z})^c$ and the finiteness of the exponent of $\ker(\mathrm{Res}_{L/K})$. □

3A6. *Step 5: Proof of Theorem 3.1 for smooth proper varieties.* In this step, we use Poitou–Tate duality to deduce Theorem 3.1 for smooth proper varieties from the previous steps.

Theorem 3.13. *Let l/k be a finite unramified extension and set $L := lK$. Let Z be a smooth proper integral K-variety. Then the quotient*

$$\mathrm{K}_2(K)/\langle N_{L/K}(\mathrm{K}_2(L)), N_2(Z/K)\rangle$$

is $\chi_K(Z,E)^2$-torsion for every coherent sheaf E on Z.

Proof. Take $x \in \mathrm{K}_2(K)$. We want to prove that

$$\chi_K(Z,E)^2 \cdot x \in \langle N_{L/K}(\mathrm{K}_2(L)), N_2(Z/K)\rangle.$$

First observe that, if K' stands for the algebraic closure of K in the function field of Z, then Z has a structure of a smooth proper K'-variety and that $\chi_{K'}(Z,E) = [K' : K]^{-1}\chi_K(Z,E)$. Therefore, by restriction-corestriction, we can assume that $K = K'$, and hence that Z is geometrically integral. Moreover, by Proposition 3.3, we may and do assume that C has residual index 1.

Let now S be the (finite) set of places $v \in C^{(1)}$ such that $\partial_v x \neq 0$. Given a prime number ℓ, since the curve C has residual index 1 and the field k is large, we can find some point $w_\ell \in C^{(1)} \setminus S$ such that the residual degree $[k(w_\ell) : k]_{\mathrm{res}}$ of $k(w_\ell)/k$ is prime to ℓ. Moreover, by Proposition 3.5, we have

$$\chi_K(Z,E) \cdot \mathrm{K}_2(K_{w_\ell}) \subseteq \langle N_{L_{w_\ell}/K_{w_\ell}}(\mathrm{K}_2(L_{w_\ell})), N_2(Z_{w_\ell}/K_{w_\ell})\rangle. \tag{3-16}$$

Before moving further, we need to prove the following lemma:

Lemma 3.14. *Let $n=[l:k]$ with l/k as in Theorem 3.13. If $v_\ell(n)>v_\ell(\chi_K(Z,E))$, then there exists a finite extension $M^{(w_\ell)}$ of K_{w_ℓ} with residue field $m^{(w_\ell)}$ such that $Z(M^{(w_\ell)})\neq\varnothing$ and $v_\ell([m^{(w_\ell)}:k(w_\ell)]_{\mathrm{res}})\leq v_\ell(\chi_K(Z,E))$.*

Proof. By contradiction, assume that, for each finite extension M of K_{w_ℓ} with residue field m such that $Z(M)\neq\varnothing$, we have $v_\ell([m:k(w_\ell)]_{\mathrm{res}})>v_\ell(\chi_K(Z,E))$. By applying the residue map to (3-16) and by denoting $l(w_\ell)$ the residue field of L_{w_ℓ}, we get

$$(K_{w_\ell}^\times)^{\chi_K(Z,E)}\subseteq\langle N_{l(w_\ell)/k(w_\ell)}(l(w_\ell)^\times);\,N_{m/k(w_\ell)}(m^\times)\mid v_\ell([m:k(w_\ell)]_{\mathrm{res}})>v_\ell(\chi_K(Z,E))\rangle.$$

By applying the valuation w_ℓ, we deduce that

$$\chi_K(Z,E)\in\langle[l(w_\ell):k(w_\ell)]_{\mathrm{res}};\,[m:k(w_\ell)]_{\mathrm{res}}\mid v_\ell([m:k(w_\ell)]_{\mathrm{res}})>v_\ell(\chi_K(Z,E))\rangle\subseteq\mathbb{Z}. \tag{3-17}$$

Now, since l/k is unramified, we have $[l:k]_{\mathrm{res}}=n$. Moreover, since $[k(w_\ell):k]_{\mathrm{res}}\wedge\ell=1$, our hypothesis on $v_\ell(n)$ implies that

$$v_\ell([l(w_\ell):k(w_\ell)]_{\mathrm{res}})\geq v_\ell([l:k]_{\mathrm{res}})>v_\ell(\chi_K(Z,E)).$$

Thus, every integer in

$$\langle[l(w_\ell):k(w_\ell)]_{\mathrm{res}};\,[m:k(w_\ell)]_{\mathrm{res}}\mid v_\ell([m:k(w_\ell)]_{\mathrm{res}})>v_\ell(\chi_K(Z,E))\rangle,$$

is divisible by $\ell^{v_\ell(\chi_K(Z,E))+1}$, which contradicts (3-17). □

We keep the notation $n:=[l:k]$ and resume the proof of Theorem 3.13. For $v\in C^{(1)}\setminus S$, we have

$$x\in N_{L_v/K_v}(\mathrm{K}_2(L_v)) \tag{3-18}$$

by Lemma 3.4. For $v\in S$, Proposition 3.5 shows that we can find $M_1^{(v)},\dots,M_{r_v}^{(v)}$ finite extensions of K_v such that $Z(M_i^{(v)})\neq\varnothing$ for all i and

$$\chi_K(Z,E)\cdot x\in\langle N_{L_v/K_v}(\mathrm{K}_2(L_v));\,N_{M_i^{(v)}/K_v}(\mathrm{K}_2(M_i^{(v)})),\,1\leq i\leq r_v\rangle. \tag{3-19}$$

By applying Proposition 3.6 inductively, we can find, for each $v\in S$ and $1\leq i\leq r_v$, a finite extension $K_i^{(v)}$ of K satisfying the following properties:

(i) $Z(K_i^{(v)})\neq\varnothing$.

(ii) There exists a K-embedding $K_i^{(v)}\hookrightarrow M_i^{(v)}$.

(iii) There also exists a K-embedding $K_i^{(v)}\hookrightarrow M^{(w_\ell)}$, where $M^{(w_\ell)}$ is given by Lemma 3.14, for each prime ℓ such that $v_\ell(n)>v_\ell(\chi_K(Z,E))$.

(iv) For each pair (v_0, i_0), the field $K_{i_0}^{(v_0)}$ is linearly disjoint to the composite field

$$L_n \cdot \prod_{(v,i)\neq(v_0,i_0)} K_i^{(v)},$$

over K, where L_n stands for the composite of all cyclic extensions of L that are locally trivial everywhere and whose degrees divide n. Note that L_n is a finite extension of L since $Ш^1(L, \mathbb{Z}/n\mathbb{Z})$ is finite.

Consider the Galois module $\hat{T}$ defined by the following exact sequence:

$$0 \to \mathbb{Z} \to \mathbb{Z}[E/K] \to \hat{T} \to 0,$$

where $E := L \times \prod_{v,i} K_i^{(v)}$. To conclude, we introduce the composite field $K_{\mathcal{I}} = \prod_{v,i} K_i^{(v)}$ and we check the assumptions (LD1), (LD2), (H1), (H2), (H3) and (H4) of Proposition 3.7 with $m = \chi_K(Z, E)$ and

$$m' = |\ker(\mathrm{Res}_{LK_{\mathcal{I}}/L} : Ш^2(L, \mathbb{Z}) \to Ш^2(LK_{\mathcal{I}}, \mathbb{Z}))|.$$

(LD1) The extension L/K is obviously Galois. The fields L and $K_{\mathcal{I}}$ are linearly disjoint over K by (iv).

(LD2) This immediately follows from (iv).

(H1) By proceeding exactly in the same way as in Lemma 4 of [Demarche and Wei 2014], since we already have (LD1), one gets the injectivity of the restriction map

$$H^2(K, \hat{T}) \to H^2(L, \hat{T}) \oplus H^2(K_{\mathcal{I}}, \hat{T}),$$

and hence of

$$Ш^2(K, \hat{T}) \to Ш^2(L, \hat{T}) \oplus Ш^2(K_{\mathcal{I}}, \hat{T}).$$

(H2) Let $C_{\mathcal{I}}$ be the smooth projective k-curve with fraction field $K_{\mathcal{I}}$. On the one hand, by (iii), given a prime ℓ such that $v_\ell(n) > v_\ell(\chi_K(Z, E))$, the field $K_{\mathcal{I}}$ can be seen as a subfield of $M^{(w_\ell)}$ and the inequality $v_\ell([m^{(w_\ell)} : k]_{\mathrm{res}}) \leq v_\ell(\chi_K(Z, E))$ holds by Lemma 3.14. We deduce that $v_\ell(i_{\mathrm{res}}(C_{\mathcal{I}})) \leq v_\ell(\chi_K(Z, E))$ for such ℓ. On the other hand, for any other prime number ℓ, we have $v_\ell(n) \leq v_\ell(\chi_K(Z, E))$. We deduce that $i_{\mathrm{res}}(C_{\mathcal{I}}) \wedge n$ divides $m = \chi_K(Z, E)$, and hence (H2) follows from Lemma 3.12.

(H3) This immediately follows from Lemma 3.12.

(H4) Since L/K is Galois, (H4) immediately follows from the choice of m'.

By Proposition 3.7, we deduce that the group $\overline{Ш^2(K, \hat{T})}$ is $((m \vee m') \wedge n)$-torsion. But by (iv), the fields $K_{\mathcal{I}}$ and L_n are linearly disjoint over K, and hence, by the definition of m', we have $m' \wedge n = 1$, so that $(m \vee m') \wedge n = m \wedge n$. The group $\overline{Ш^2(K, \hat{T})}$ is therefore m-torsion. If we set $\check{T} := \mathrm{Hom}(\hat{T}, \mathbb{Z})$ and $T := \check{T} \otimes \mathbb{Z}(2)$, that is also the case of $Ш^3(K, T)$ according to Poitou–Tate duality.

Now, by Lemma 3.2, we may interpret x as an element of $H^3(K, T)$. Equations (3-18) and (3-19) together with assertion (ii) imply that $mx \in Ш^3(K, T)$, which is m-torsion. Thus $m^2 x = 0 \in Ш^3(K, T)$.

This amounts to

$$m^2x \in \langle N_{L/K}(\mathrm{K}_2(L)); N_{K_i^{(v)}/K}(\mathrm{K}_2(K_i^{(v)})), v \in S, 1 \le i \le r_v\rangle \subseteq \langle N_{L/K}(\mathrm{K}_2(L)), N_2(Z/K)\rangle,$$

the last inclusion being a consequence of (i). □

3A7. *Step 6: Proof of Theorem 3.1.* In this final step, we remove the smoothness assumption from the previous step and prove Theorem 3.1 for all proper varieties. For that purpose, we use the following variation of the dévissage technique given by Proposition 2.1 of [Wittenberg 2015].

Proposition 3.15 [Wittenberg 2015]. *Let K be a field and r a positive integer. Let (P) be a property of proper K-varieties. Suppose we are given, for each proper K-variety X, an integer m_X. Make the following assumptions*:

(1) *For every morphism of proper K-schemes $Y \to X$, the integer m_X divides m_Y.*

(2) *For every proper K-scheme X satisfying (P), the integer m_X divides $\chi_K(X, \mathcal{O}_X)^r$.*

(3) *For every proper and integral K-scheme X, there exists a proper K-scheme Y satisfying (P) and a K-morphism $f : Y \to X$ with generic fiber Y_η such that m_X and $\chi_{K(X)}(Y_\eta, \mathcal{O}_{Y_\eta})$ are coprime.*

Then for every proper K-scheme X and every coherent sheaf E on X, the integer m_X divides $\chi_K(X, E)^r$.

Proof. One can prove this result by following almost word by word the proof of Proposition 2.1 of [Wittenberg 2015]. Alternatively, for each proper K-scheme X, write the prime decomposition of m_X as

$$m_X = \prod_p p^{\alpha_p},$$

and consider the integer

$$n_X := \prod_p p^{\lceil \alpha_p/r \rceil}.$$

One can then easily check that the correspondence $X \mapsto n_X$ satisfies assumptions (1), (2) and (3) of Proposition 2.1 of [loc. cit.]. We deduce that $n_X | \chi_K(X, E)$, and hence that $m_X | \chi_K(X, E)^r$, for every proper K-scheme X and every coherent sheaf E on X. □

Proof of Theorem 3.1. Given a proper K-variety Z, we denote by m_Z the exponent of the quotient

$$\mathrm{K}_2(K)/\langle N_{L/K}(\mathrm{K}_2(L)), N_2(Z/K)\rangle.$$

We say that Z has property (P) if it is smooth and integral. We have to check the three conditions (1), (2) and (3) of Proposition 3.15. Condition (1) is straightforward. Condition (2) follows from Theorem 3.13. Condition (3) follows from Hironaka's theorem on resolution of singularities; Section 3.3 of [Kollár 2007]. □

3B. *Proof of Main Theorem 1.* We can now deduce Main Theorem 1 from Theorem 3.1.

Proof of Main Theorem 1. Fix two integers $n, d \geq 1$ such that $d^2 \leq n$ and a hypersurface Z in $\mathbb{P}^n_K$ of degree d. By Lang's and Tsen's theorems (Theorem 2a of [Nagata 1957] and Theorem 12 of [Lang 1952]), the field $k^{\mathrm{nr}}(C)$ is C_2. Since $d^2 \leq n$, we deduce that there exists a finite unramified extension l of k such that $Z(lK) \neq \varnothing$. By Theorem 3.1, the quotient

$$\mathrm{K}_2(K)/\langle N_{lK/K}(\mathrm{K}_2(lK)), N_2(Z/K)\rangle = \mathrm{K}_2(K)/N_2(Z/K)$$

is $\chi_K(Z, \mathcal{O}_Z)^2$-torsion. But since $d \leq n$, Theorem III.5.1 of [Hartshorne 1977] implies that

$$\chi_K(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}(-d)) = 0,$$

and hence the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n_K}(-d) \to \mathcal{O}_{\mathbb{P}^n_K} \to i_*\mathcal{O}_Z \to 0,$$

where $i : Z \to \mathbb{P}^n_K$ stands for the closed immersion, gives

$$\chi_K(Z, \mathcal{O}_Z) = \chi_K(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}) - \chi_K(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}(-d)) = 1.$$

Hence $\mathrm{K}_2(K) = N_2(Z/K)$. □

4. On the C_1^2 property for p-adic function fields

The goal of this section is to prove Main Theorem 2. Contrary to Main Theorem 1, for which we needed to deal with unramified extensions of k, here we will have to deal with ramified extensions of k. For that purpose, the key statement is given by the following theorem:

Theorem 4.1. *Assume that C has a rational point, let ℓ be a prime number, and fix a finite Galois totally ramified extension l/k of degree ℓ. Let $\mathcal{E}^0_{l/k}$ be the set of all finite ramified subextensions of l^{nr}/k and let $\mathcal{E}_{l/k}$ be the set of finite extensions K' of K of the form $K' = k'K$ for some $k' \in \mathcal{E}^0_{l/k}$. Then*

$$\mathrm{K}_2(K) = \langle N_{K'/K}(\mathrm{K}_2(K')) \mid K' \in \mathcal{E}_{l/k}\rangle.$$

Note that, given any two extensions k' and k'' in $\mathcal{E}^0_{l/k}$ with $k' \subset k''$, the extension k''/k' is unramified. This observation will be often used in the sequel.

Remark 4.2. We think that the assumption that C has a rational point in Theorem 4.1 cannot be removed. To check that, we invite the reader to assume that $i_{\mathrm{ram}}(C) = \ell$. Then, given an integer $n \geq 1$, consider the set $\mathcal{E}^0_n$ whose elements are extensions of k in $\mathcal{E}^0_{l/k}$ that are contained in the composite $l_n := lk_n$, where k_n is the degree ℓ^n unramified extension of k. Define the set $\mathcal{E}_n$ of finite extensions K' of K contained in $L_n := l_nK$ that are of the form $K' = k'K$ for some $k' \in \mathcal{E}^0_n$ and consider the Galois module $\hat{T}_n$ defined by the exact sequence

$$0 \to \mathbb{Z} \to \bigoplus_{K' \in \mathcal{E}_n} \mathbb{Z}[K'/K] \to \hat{T}_n \to 0.$$

By following the proof of Proposition 4.5, one can check that, if K_1 and K_2 are two distinct degree ℓ extensions of K in $\mathcal{E}_n$, then the Tate–Shafarevich group $Ш^2(K,\hat{T}_n)$ is the direct sum of the kernel of the map

$$(\mathrm{Res}_{K_1/K},\mathrm{Res}_{K_2/K}) : Ш^2(K,\hat{T}_n)\to Ш^2(K_1,\hat{T}_n)\oplus Ш^2(K_2,\hat{T}_n)$$

and of a divisible group, given by the kernel of the map

$$\mathrm{Res}_{K_1K_2/K_1}-\mathrm{Res}_{K_1K_2/K_2} : Ш^2(K_1,\hat{T}_n)\oplus Ш^2(K_2,\hat{T}_n)\to Ш^2(K_1K_2,\hat{T}_n).$$

In particular

$$\overline{Ш^2(K,\hat{T}_n)}\cong \ker\big(Ш^2(K,\hat{T}_n)\to Ш^2(K_1,\hat{T}_n)\oplus Ш^2(K_2,\hat{T}_n)\big).$$

The computation of this kernel is a relatively simple (but a bit technical) exercise in the cohomology of finite groups, since it is contained in the group

$$\ker\big(H^2(K,\hat{T}_n)\to H^2(L_n,\hat{T}_n)\big)\cong H^2(\mathrm{Gal}(L_n/K),\hat{T})\cong H^2(\mathbb{Z}/\ell\mathbb{Z}\times\mathbb{Z}/\ell^n\mathbb{Z},\hat{T}).$$

In that way, one checks that $\overline{Ш^2(K,\hat{T}_n)}$ is an $\mathbb{F}_\ell$-vector space of dimension at least $n\ell-n-1$. Moreover, the computation being very explicit, one can even check that the morphism $\overline{Ш^2(K,\hat{T}_{n+1})}\to\overline{Ш^2(K,\hat{T}_n)}$ induced by the natural projection $\hat{T}_{n+1}\to\hat{T}_n$ is always surjective. But then, by dualizing thanks to Poitou–Tate duality, this shows that the groups

$$Q_n:=\ker(\mathrm{K}_2(K)/\langle N_{K'/K}(\mathrm{K}_2(K'))\mid K'\in\mathcal{E}_n\rangle\to\prod_{v\in C^{(1)}}\mathrm{K}_2(K_v)/\langle N_{K'\otimes K_v/K_v}(\mathrm{K}_2(K'\otimes K_v))\mid K'\in\mathcal{E}_n\rangle)$$

are all nontrivial and that the natural maps $Q_n\to Q_{n+1}$ are all injective. We deduce that the nontrivial elements of Q_1 provide nontrivial elements in the quotient

$$\mathrm{K}_2(K)\Big/\left\langle N_{K'/K}(\mathrm{K}_2(K'))\;\middle|\; K'\in\bigcup_n\mathcal{E}_n\right\rangle=\mathrm{K}_2(K)/\langle N_{K'/K}(\mathrm{K}_2(K'))\mid K'\in\mathcal{E}_{l/k}\rangle.$$

4A. *Proof of Theorem 4.1.*

4A1. *Step 1: Solving the local problem.* The first step to prove Theorem 4.1 consists in settling an analogous statement over the completions of K. We start with the following lemma.

Lemma 4.3. *Let ℓ be a prime number and let l/k be a finite Galois totally ramified extension of degree ℓ. Let m/k be a totally ramified extension such that ml/m is unramified. Then there exists $k'\in\mathcal{E}^0_{l/k}$ such that $k'\subset m$.*

Proof. If ml/m is trivial, then m contains l and we are done. Therefore we may and do assume that ml/m has degree ℓ. Denote by k_ℓ the unramified extension of k with degree ℓ and set $l_\ell:=l\cdot k_\ell$. The extension l_ℓ/k is Galois with Galois group $(\mathbb{Z}/\ell\mathbb{Z})^2$, and since ml is unramified of degree ℓ over m, it contains both k_ℓ and l_ℓ, so that l_ℓ is contained in $m'l$ for some finite subextension m' of m/k. But

$$[m':k]\cdot[l_\ell:k]=\ell^2[m':k]>\ell[m':k]=[m'l:k]=[m'l_\ell:k].$$

Hence the intersection $k':=m'\cap l_\ell$ is a degree ℓ totally ramified extension of k, and $k'\in\mathcal{E}^0_{l/k}$. □

Proposition 4.4. *Let ℓ be a prime number and let l/k be a finite Galois totally ramified extension of degree ℓ. Fix $v \in C^{(1)}$. Then*

$$\mathrm{K}_2(K_v) = \langle N_{K'\otimes_K K_v/K_v}(\mathrm{K}_2(K'\otimes_K K_v)) \mid K' \in \mathcal{E}_{l/k}\rangle.$$

Proof. Three different cases arise:

(1) The field $k(v)$ contains l.

(2) The extension $lk(v)/k(v)$ is unramified of degree ℓ.

(3) The extension $lk(v)/k(v)$ is totally ramified of degree ℓ.

Case 1 is trivial, since

$$\mathrm{K}_2(K_v) = N_{lK\otimes_K K_v/K_v}(\mathrm{K}_2(lK\otimes_K K_v)).$$

Let us now consider case 2, and denote by $k(v)_{\mathrm{nr}}$ the maximal unramified subextension of $k(v)/k$. By Lemma 4.3, since $lk(v)_{\mathrm{nr}}/k(v)_{\mathrm{nr}}$ is a Galois totally ramified extension of degree ℓ and $k(v)/k(v)_{\mathrm{nr}}$ is a totally ramified extension such that $k(v)l/k(v)$ is unramified, there exists a finite extension m of $k(v)_{\mathrm{nr}}$ such that $m \in \mathcal{E}^0_{lk(v)_{\mathrm{nr}}/k(v)_{\mathrm{nr}}} \subset \mathcal{E}^0_{l/k}$ and $m \subset k(v)$. By setting $M := mK$, we get that $M \in \mathcal{E}_{l/k}$ and that

$$\mathrm{K}_2(K_v) = N_{M\otimes_K K_v/K_v}(\mathrm{K}_2(M\otimes_K K_v)) \subset \langle N_{K'\otimes_K K_v/K_v}(\mathrm{K}_2(K'\otimes_K K_v)) \mid K' \in \mathcal{E}_{l/k}\rangle,$$

as wished.

Let us finally consider case 3. To do so, fix a uniformizer π of $k(v)$, and as before, let $k(v)_{\mathrm{nr}}$ be the maximal unramified subextension of $k(v)/k$. Denote by $k(v)^{\mathrm{ram}}_{\pi}$ the maximal abelian totally ramified extension of $k(v)$ associated to π by Lubin–Tate theory. Since l/k is abelian, the extension $lk(v)^{\mathrm{ram}}_{\pi}/k(v)^{\mathrm{ram}}_{\pi}$ must be unramified. Hence, by Lemma 4.3, there exists a finite extension m of $k(v)_{\mathrm{nr}}$ such that $m \in \mathcal{E}^0_{lk(v)_{\mathrm{nr}}/k(v)_{\mathrm{nr}}} \subset \mathcal{E}^0_{l/k}$ and $m \subset k(v)^{\mathrm{ram}}_{\pi}$. We deduce from Corollary 5.12 of [Yoshida 2008] that

$$\pi \in N_{m\otimes_{k(v)_{\mathrm{nr}}}k(v)}((m\otimes_{k(v)_{\mathrm{nr}}} k(v))^{\times}) \subset \langle N_{k'\otimes_k k(v)/k(v)}((k'\otimes_k k(v))^{\times}) \mid k' \in \mathcal{E}^0_{l/k}\rangle.$$

This being true for every uniformizer π of $k(v)$, we deduce that

$$k(v)^{\times} \subset \langle N_{k'\otimes_k k(v)/k(v)}((k'\otimes_k k(v))^{\times}) \mid k' \in \mathcal{E}^0_{l/k}\rangle,$$

and hence, by Lemma 3.4,

$$\mathrm{K}_2(K_v) = \langle N_{K'\otimes_K K_v/K_v}(\mathrm{K}_2(K'\otimes_K K_v)) \mid K' \in \mathcal{E}_{l/k}\rangle. \qquad \square$$

4A2. *Step 2: Computation of a Tate–Shafarevich group.* The second step, which is slightly technical, consists in computing the Tate–Shafarevich groups of some finitely generated free Galois modules over K associated to the fields in $\mathcal{E}_l$. Poitou–Tate duality will then allow us to obtain a local-global principle that will let us deduce Theorem 4.1 from Proposition 4.4.

Proposition 4.5. *Assume that C has a rational point, and let ℓ be a prime number. Fix a finite Galois totally ramified extension l/k of degree ℓ. Given $K_1, \dots, K_r$ in $\mathcal{E}_{l/k}$ so that the fields K_1 and K_2 are linearly disjoint over K, consider the Galois module $\hat{T}$ defined by the following exact sequence*

$$0 \to \mathbb{Z} \to \mathbb{Z}[E/K] \to \hat{T} \to 0, \tag{4-1}$$

where $E := K_1 \times \cdots \times K_r$. Then $Ш^2(K, \hat{T})$ is divisible.

Proof. Consider the following complex:

$$\begin{array}{ccccc} Ш^2(K,\hat{T}) & \xrightarrow{f_0} & Ш^2(K_1,\hat{T}) \oplus Ш^2(K_2,\hat{T}) & \xrightarrow{\quad g_0 \quad} & Ш^2(K_1K_2,\hat{T}) \\ x & \longmapsto & (\mathrm{Res}_{K_1/K}(x), \mathrm{Res}_{K_2/K}(x)) & & \\ & & (x, y) & \longmapsto & \mathrm{Res}_{K_1K_2/K_1}(x) - \mathrm{Res}_{K_1K_2/K_2}(y). \end{array} \tag{4-2}$$

We start by proving the following lemma:

Lemma 4.6. *The morphism f_0 is injective.*

Proof. Let $K_{\mathcal{I}}$ be the Galois closure of the composite of all the K_i's. By inflation-restriction, there is an exact sequence

$$0 \to H^2(K_{\mathcal{I}}/K, \hat{T}) \to H^2(K, \hat{T}) \to H^2(K_{\mathcal{I}}, \hat{T}).$$

Take $v \in C(k)$ a rational point. Since the extension $K_{\mathcal{I}}/K$ is obtained by base change from an extension $k_{\mathcal{I}}$ of k, we have the equalities $\mathrm{Gal}(K_{\mathcal{I}}/K) = \mathrm{Gal}(k_{\mathcal{I}}/k) = \mathrm{Gal}(K_{\mathcal{I},v}/K_v)$. The previous inflation-restriction exact sequence therefore induces a commutative diagram with exact lines:

$$\begin{tikzcd} 0 \arrow[r] & H^2(K_{\mathcal{I}}/K, \hat{T}) \arrow[r] \arrow[d, "\cong"] & H^2(K, \hat{T}) \arrow[r] \arrow[d] & H^2(K_{\mathcal{I}}, \hat{T}) \arrow[d] \\ 0 \arrow[r] & H^2(K_{\mathcal{I},v}/K_v, \hat{T}) \arrow[r] & H^2(K_v, \hat{T}) \arrow[r] & H^2(K_{\mathcal{I},v}, \hat{T}) \end{tikzcd}$$

in which the first vertical map is an isomorphism. We deduce that the restriction map

$$\ker\big(H^2(K, \hat{T}) \to H^2(K_v, \hat{T})\big) \to \ker\big(H^2(K_{\mathcal{I}}, \hat{T}) \to H^2(K_{\mathcal{I},v}, \hat{T})\big)$$

is injective. Hence so is the restriction map

$$\mathrm{Res}_{K_{\mathcal{I}}/K} : Ш^2(K, \hat{T}) \to Ш^2(K_{\mathcal{I}}, \hat{T})$$

as well as the restriction maps

$$\mathrm{Res}_{K_1/K} : Ш^2(K, \hat{T}) \to Ш^2(K_1, \hat{T}), \quad \mathrm{Res}_{K_2/K}\, Ш^2(K, \hat{T}) \to Ш^2(K_2, \hat{T}),$$

since the former factors through these. □

Now observe that the complex (4-2) fits in the following commutative diagram, in which the first and second rows are obtained in the same way as the third:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] \\
Ш^2(K,\mathbb{Z}) \arrow[r] \arrow[d] & Ш^2(K_1,\mathbb{Z}) \oplus Ш^2(K_2,\mathbb{Z}) \arrow[r, two heads] \arrow[d] & Ш^2(K_1K_2,\mathbb{Z}) \arrow[d] \\
Ш^2(K,\mathbb{Z}[E/K]) \arrow[r, "f"] \arrow[d] & Ш^2(K_1,\mathbb{Z}[E/K]) \oplus Ш^2(K_2,\mathbb{Z}[E/K]) \arrow[r, "g"] \arrow[d] & Ш^2(K_1K_2,\mathbb{Z}[E/K]) \arrow[d] \\
Ш^2(K,\hat{T}) \arrow[r, hook, "f_0"] & Ш^2(K_1,\hat{T}) \oplus Ш^2(K_2,\hat{T}) \arrow[r, "g_0"] \arrow[d] & Ш^2(K_1K_2,\hat{T}) \arrow[d] \\
& 0 & 0
\end{tikzcd} \tag{4-3}$$

The second and third columns are exact since the exact sequence (4-1) splits over K_1, K_2 and K_1K_2. The lines are all complexes. In the first one, the second arrow is surjective since the restriction map

$$Ш^2(K_1,\mathbb{Z}) \to Ш^2(K_1K_2,\mathbb{Z})$$

is an isomorphism by Lemma 3.12 and C has a rational point. As for the second line, we have the following lemma.

Lemma 4.7. *The second line of diagram (4-3) is exact.*

Proof. For $1 \leq \alpha \leq r$, write

$$K_1 \otimes_K K_\alpha = \prod_\beta L_{\alpha\beta}, \quad K_2 \otimes_K K_\alpha = \prod_\gamma M_{\alpha\gamma}, \quad L_{\alpha\beta} \otimes_{K_\alpha} M_{\alpha\gamma} = \prod_\delta N_{\alpha\beta\gamma\delta}$$

for some fields $L_{\alpha\beta}$, $M_{\alpha\gamma}$ and $N_{\alpha\beta\gamma\delta}$. By Shapiro's lemma, the second line of (4-3) can be identified with the following complex

$$\begin{tikzcd}
\bigoplus_\alpha Ш^2(K_\alpha,\mathbb{Z}) \arrow[d, "f"] \\
\bigoplus_{\alpha,\beta} Ш^2(L_{\alpha\beta},\mathbb{Z}) \oplus \bigoplus_{\alpha,\gamma} Ш^2(M_{\alpha\gamma},\mathbb{Z}) \arrow[d, "g"] \\
\bigoplus_{\alpha,\beta,\gamma,\delta} Ш^2(N_{\alpha\beta\gamma\delta},\mathbb{Z})
\end{tikzcd}$$

where f is given by

$$(x_\alpha) \mapsto ((\mathrm{Res}_{L_{\alpha\beta}/K_\alpha}(x_\alpha))_{\alpha\beta}, (\mathrm{Res}_{M_{\alpha\gamma}/K_\alpha}(x_\alpha))_{\alpha\gamma}),$$

and g

$$((y_{\alpha\beta})_{\alpha,\beta},(z_{\alpha\gamma})_{\alpha,\gamma})\mapsto(\mathrm{Res}_{N_{\alpha\beta\gamma\delta}/L_{\alpha\beta}}(y_{\alpha\beta})-\mathrm{Res}_{N_{\alpha\beta\gamma\delta}/M_{\alpha\gamma}}(z_{\alpha\gamma}))_{\alpha\beta\gamma\delta},$$

Fix $((y_{\alpha\beta})_{\alpha,\beta},(z_{\alpha\gamma})_{\alpha,\gamma})\in\ker(g)$. Then

$$\mathrm{Res}_{N_{\alpha\beta\gamma\delta}/L_{\alpha\beta}}(y_{\alpha\beta})=\mathrm{Res}_{N_{\alpha\beta\gamma\delta}/M_{\alpha\gamma}}(z_{\alpha\gamma})$$

for all α, β, γ, δ. But the restrictions $\mathrm{Res}_{L_{\alpha\beta}/K_\alpha}$, $\mathrm{Res}_{M_{\alpha\gamma}/K_\alpha}$, $\mathrm{Res}_{N_{\alpha\beta\gamma\delta}/L_{\alpha\beta}}$ and $\mathrm{Res}_{N_{\alpha\beta\gamma\delta}/M_{\alpha\gamma}}$ are all isomorphisms by Lemma 3.12 and they fit into a commutative diagram:

$$\begin{tikzcd}
Ш^2(K_\alpha,\mathbb{Z}) \arrow[r, "\mathrm{Res}_{L_{\alpha\beta}/K_\alpha}"] \arrow[d, "\mathrm{Res}_{M_{\alpha\gamma}/K_\alpha}"'] & Ш^2(L_{\alpha\beta},\mathbb{Z}) \arrow[d, "\mathrm{Res}_{N_{\alpha\beta\gamma\delta}/L_{\alpha\beta}}"] \\
Ш^2(M_{\alpha\gamma},\mathbb{Z}) \arrow[r, "\mathrm{Res}_{N_{\alpha\beta\gamma\delta}/M_{\alpha\gamma}}"'] & Ш^2(N_{\alpha\beta\gamma\delta},\mathbb{Z})
\end{tikzcd}$$

We deduce that, for each α, there exists $x_\alpha\in Ш^2(K_\alpha,\mathbb{Z})$ such that

$$\forall\beta,\ \mathrm{Res}_{L_{\alpha\beta}/K_\alpha}(x_\alpha)=y_{\alpha\beta}\quad\text{and}\quad\forall\gamma,\ \mathrm{Res}_{M_{\alpha\gamma}/K_\alpha}(x_\alpha)=z_{\alpha\gamma}.$$

In other words, $((y_{\alpha\beta})_{\alpha,\beta},(z_{\alpha\gamma})_{\alpha,\gamma})\in\mathrm{im}(f)$. □

With all the gathered information, a simple diagram chase in (4-3) shows that the morphism

$$Ш^2(K,\mathbb{Z}[E/K])\to Ш^2(K,\hat{T})$$

is surjective. But as recalled at the end of Section 2, the group $Ш^2(K,\mathbb{Z}[E/K])$ is divisible. Hence so is $Ш^2(K,\hat{T})$. □

4A3. *Step 3: Proof of Theorem 4.1.* We can finally prove Theorem 4.1 by using Poitou–Tate duality.

Proof of Theorem 4.1. Take $x\in \mathrm{K}_2(K)$. By Proposition 4.4, we have

$$\mathrm{K}_2(K_v)=\langle N_{K'\otimes_K K_v/K_v}(\mathrm{K}_2(K'\otimes_K K_v))\mid K'\in\mathcal{E}_{l/k}\rangle$$

for all $v\in C^{(1)}$. Hence we can find $K_1,\dots,K_r\in\mathcal{E}_{l/k}$ such that

$$\begin{aligned}x\in\ker(\mathrm{K}_2(K)/\langle N_{K_i/K}(\mathrm{K}_2(K_i))\mid 1\le i\le r\rangle\\ \to\prod_{v\in C^{(1)}}\mathrm{K}_2(K_v)/\langle N_{K_i\otimes_K K_v/K_v}(\mathrm{K}_2(K_i\otimes_K K_v))\mid 1\le i\le r\rangle).\end{aligned}\tag{4-4}$$

Moreover, up to enlarging the family $(K_i)_i$, we may and do assume that K_1 and K_2 are linearly disjoint. Consider the étale K-algebra $E:=K_1\times\cdots\times K_r$ and the Galois module $\hat{T}$ defined by the following exact sequence

$$0\to\mathbb{Z}\to\mathbb{Z}[E/K]\to\hat{T}\to 0.$$

Set $\check{T}:=\mathrm{Hom}(\hat{T},\mathbb{Z})$ and $T:=\check{T}\otimes\mathbb{Z}(2)$. By Lemma 3.2, (4-4) can be rewritten as

$$x\in Ш^3(K,T).$$

But, by Poitou–Tate duality, $Ш^3(K, T)$ is dual to $\overline{Ш^2(K, \hat{T})}$, and by Proposition 4.5, the group $Ш^2(K, \hat{T})$ is divisible. We deduce that $Ш^3(K, T) = 0$, and hence that

$$x \in \langle N_{K_i/K}(\mathrm{K}_2(K_i)) \mid 1 \le i \le r\rangle \subset \langle N_{K'/K}(\mathrm{K}_2(K')) \mid K' \in \mathcal{E}_{l/k}\rangle. \qquad \square$$

4B. *Proof of Main Theorem 2.* By combining Theorems 3.1 and 4.1, we can now settle the following theorem, from which we will deduce Main Theorem 2.

Theorem 4.8. *Let K be the function field of a smooth projective curve C defined over a p-adic field k. Let l/k be a finite Galois extension and set $L := lK$. Let Z be a proper K-variety. If $s_{l/k}$ stands for the number of (not necessarily distinct) prime factors of the ramification degree of l/k, then the quotient*

$$\mathrm{K}_2(K)/\langle N_{L/K}(\mathrm{K}_2(L)), N_2(Z/K)\rangle$$

is $i_{\mathrm{ram}}(C) \cdot \chi_K(Z, E)^{2s_{l/k}+4}$-torsion for every coherent sheaf E on Z.

Proof. We first assume that C has a rational point, and we prove that

$$\mathrm{K}_2(K)/\langle N_{L/K}(\mathrm{K}_2(L)), N_2(Z/K)\rangle$$

is $\chi_K(Z, E)^{2s_{l/k}+2}$-torsion for every coherent sheaf E on Z by induction on $s_{l/k}$. The case $s_{l/k} = 0$ immediately follows from Theorem 3.1. We henceforth assume now that $s_{l/k} > 0$. Let l_{nr} be the maximal unramified subextension of l/k and set $L_{\mathrm{nr}} := l_{\mathrm{nr}}K$. Theorem 3.1 ensures then that the quotient

$$\mathrm{K}_2(K)/\langle N_{L_{\mathrm{nr}}/K}(\mathrm{K}_2(L_{\mathrm{nr}})), N_2(Z/K)\rangle$$

is $\chi_K(Z, E)^2$-torsion. Now, the extension l/l_{nr} is Galois and totally ramified. Since finite extensions of local fields are solvable, we can find a Galois totally ramified extension m/l_{nr} contained in l and of prime degree ℓ. Set $M := mK$. By Theorem 4.1, we have

$$\mathrm{K}_2(L_{\mathrm{nr}}) = \langle N_{K'/L_{\mathrm{nr}}}(\mathrm{K}_2(K')) \mid K' \in \mathcal{E}_{m/l_{\mathrm{nr}}}\rangle.$$

But for each $k' \in \mathcal{E}^0_{m/l_{\mathrm{nr}}}$, the ramification degree of lk'/k' strictly divides that of l/k. Hence, by induction, the group

$$\mathrm{K}_2(K')/\langle N_{LK'/K'}(\mathrm{K}_2(LK')), N_2(Z/K')\rangle$$

is $\chi_K(Z, E)^{2s_{l/k}}$-torsion for each $K' \in \mathcal{E}_{m/l_{\mathrm{nr}}}$. We deduce that

$$\mathrm{K}_2(K)/\langle N_{L/K}(\mathrm{K}_2(L)), N_2(Z/K)\rangle$$

is $\chi_K(Z, E)^{2s_{l/k}+2}$-torsion, which finishes the induction.

We do not assume anymore that C has a rational point. Let $k_1, \ldots, k_r$ be finite extensions of k over which C acquires rational points and such that the g.c.d.'s of their ramification degrees is $i_{\mathrm{ram}}(C)$. For each i, let $k_{i,\mathrm{nr}}$ be the maximal unramified extension of k contained in k_i, and set $K_i := k_iK$ and $K_{i,\mathrm{nr}} := k_{i,\mathrm{nr}}K$. Theorem 3.1 ensures that the quotient

$$\mathrm{K}_2(K)/\langle N_{K_{i,\mathrm{nr}}/K}(\mathrm{K}_2(K_{i,\mathrm{nr}})), N_2(Z/K)\rangle$$

is $\chi_K(Z,E)^2$-torsion. Moreover, a restriction-corestriction argument shows that the quotient

$$\mathrm{K}_2(K_{i,\mathrm{nr}})/N_{K_i/K_{i,\mathrm{nr}}}(\mathrm{K}_2(K_i))$$

is $[k_i : k_{i,\mathrm{nr}}]$-torsion. Since $[k_i : k_{i,\mathrm{nr}}]$ is the ramification degree of k_i/k, we deduce that

$$\mathrm{K}_2(K)/\langle N_{K_1/K}(\mathrm{K}_2(K_1)),\dots,N_{K_r/K}(\mathrm{K}_2(K_r)),N_2(Z/K)\rangle$$

is $i_{\mathrm{ram}}(C)\cdot\chi_K(Z,E)^2$-torsion. But C has rational points over all the k_i. Hence, by the first case, the quotients

$$\mathrm{K}_2(K_i)/\langle N_{LK_i/K_i}(\mathrm{K}_2(LK_i)),N_2(Z/K_i)\rangle$$

are all $\chi_K(Z,E)^{2s_{l/k}+2}$-torsion. We deduce that

$$\mathrm{K}_2(K)/\langle N_{L/K}(\mathrm{K}_2(L)),N_2(Z/K)\rangle$$

is $i_{\mathrm{ram}}(C)\cdot\chi_K(Z,E)^{2s_{l/k}+4}$-torsion. □

Applying this to the context of the C_1^2-property, we get the following result.

Corollary 4.9. *Let K be the function field of a smooth projective curve C defined over a p-adic field k. Then, for each $n,d\geq 1$ and for each hypersurface Z in $\mathbb{P}^n_K$ of degree d with $d\leq n$, the quotient $\mathrm{K}_2(K)/N_2(Z/K)$ is killed by $i_{\mathrm{ram}}(C)$.*

Proof. Let Z be a hypersurface in $\mathbb{P}^n_K$ of degree d with $d\leq n$. By Tsen's theorem, the field $\bar{k}(C)$ is C_1. Since $d\leq n$, we deduce that there exists a finite extension l of k such that $Z(lK)\neq\varnothing$. By Theorem 4.8, the quotient

$$\mathrm{K}_2(K)/\langle N_{lK/K}(\mathrm{K}_2(lK)),N_2(Z/K)\rangle=\mathrm{K}_2(K)/N_2(Z/K)$$

is $i_{\mathrm{ram}}(C)\cdot\chi_K(Z,\mathcal{O}_Z)^{2s_{l/k}+4}$-torsion. But since $d\leq n$, Theorem III.5.1 of [Hartshorne 1977] and the exact sequence

$$0\to\mathcal{O}_{\mathbb{P}^n_K}(-d)\to\mathcal{O}_{\mathbb{P}^n_K}\to i_*\mathcal{O}_Z\to 0,$$

where $i:Z\to\mathbb{P}^n_K$ stands for the closed immersion, imply that

$$\chi_K(Z,\mathcal{O}_Z)=\chi_K(\mathbb{P}^n_K,\mathcal{O}_{\mathbb{P}^n_K})-\chi_K(\mathbb{P}^n_K,\mathcal{O}_{\mathbb{P}^n_K}(-d))=1.$$

Hence the quotient $\mathrm{K}_2(K)/N_2(Z/K)$ is $i_{\mathrm{ram}}(C)$-torsion. □

Main Theorem 2 can now be immediately deduced from the following corollary.

Corollary 4.10. *Let K be the function field of a smooth projective curve C defined over a p-adic field k. Assume that $i_{\mathrm{ram}}(C)=1$. Then, for each $n,d\geq 1$ and for each hypersurface Z in $\mathbb{P}^n_K$ of degree d with $d\leq n$, we have $N_2(Z/K)=\mathrm{K}_2(K)$.*

Remark 4.11. By Section 9.1 of [Bosch et al. 1990], the assumption that $i_{\mathrm{ram}}(C)=1$ automatically holds when C has an irreducible proper flat regular model whose special fiber has multiplicity 1.

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