Locally analytic vector bundles
on the Fargues–Fontaine curve

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We develop a version of Sen theory for equivariant vector bundles on the Fargues–Fontaine curve. We show that every equivariant vector bundle canonically descends to a locally analytic vector bundle. A comparison with the theory of (φ, Γ)-modules in the cyclotomic case then recovers the Cherbonnier–Colmez decompletion theorem. Next, we focus on the subcategory of de Rham locally analytic vector bundles. Using the p-adic monodromy theorem, we show that each locally analytic vector bundle $\mathcal{E}$ has a canonical differential equation for which the space of solutions has full rank. As a consequence, $\mathcal{E}$ and its sheaf of solutions $\mathrm{Sol}(\mathcal{E})$ are in a natural correspondence, which gives a geometric interpretation of a result of Berger on (φ, Γ)-modules. In particular, if V is a de Rham Galois representation, its associated filtered (φ, N, G_K)-module is realized as the space of global solutions to the differential equation. A key to our approach is a vanishing result for the higher locally analytic vectors of representations satisfying the Tate–Sen formalism, which is also of independent interest.

1. Introduction

The study of p-adic Galois representations has been conditioned to an extent by two dogmas. One is the *analytic* dogma; its main idea is to associate to every such representation a (φ, Γ)-module over the Robba ring and to study these objects using p-adic analysis. The other dogma is *geometric*: to every p-adic Galois representation one associates an equivariant vector bundle over the Fargues–Fontaine curve. The aim of this article is, roughly speaking, to find a framework where both analysis and geometry can be carried out. In recent years, much of the theory of p-adic Galois representations has been understood in terms of the

geometry of the Fargues–Fontaine curve. A notable exception has been the p-adic Langlands program, where the analytic approach plays a crucial role. Thus we are motivated to reduce this discrepancy by introducing corresponding objects on the Fargues–Fontaine curve which are also amenable to analytic methods. These are the locally analytic vector bundles, the main new objects introduced in this article.

We shall now explain this in more detail. Let K be a finite extension of $\mathbb{Q}_p$ with absolute Galois group G_K. Let K_{cyc} be the cyclotomic extension of K and write $\Gamma = \text{Gal}(K_{\text{cyc}}/K)$. For the sake of simplifying the introduction, we shall focus now on the cyclotomic setting, though as we shall explain later, the content of this paper will apply to a wider class of Galois extensions K_∞/K. We have the category $\text{Rep}_{\mathbb{Q}_p}(G_K)$ of finite dimensional $\mathbb{Q}_p$-representations of G_K.

On the one hand, $\text{Rep}_{\mathbb{Q}_p}(G_K)$ can be studied via p-adic analysis. To do this, one introduces the Robba ring $\mathcal{R}$, which is the ring of power series over a certain finite extension of $\mathbb{Q}_p$ in a variable T which converge in some annuli $r \leq |T| < 1$. It has an action of a Frobenius operator φ as well as an action of Γ. By work of Cherbonnier–Colmez, Fontaine and Kedlaya, it is known that there is a fully faithful embedding

$$\text{Rep}_{\mathbb{Q}_p}(G_K) \hookrightarrow \{(\varphi, \Gamma)\text{-modules over } \mathcal{R}\},$$

with the essential image consisting of the semistable slope 0 objects. If D is a (φ, Γ)-module over $\mathcal{R}$, a fundamental fact is that the Γ-action on D can be differentiated, namely, there is a well defined action of $\text{Lie}(\Gamma)$ on D. Since $\text{Lie}(\Gamma)$ is 1-dimensional, this data is the same as that of a connection ∇ which acts on functions of T by a multiple of d/dT. It is precisely this structure which allows the introduction of p-adic analysis into the picture. For example, in the construction of the p-adic Langlands correspondence for $\text{GL}_2(\mathbb{Q}_p)$ given in [Colmez 2010], the use of this analytic structure is ubiquitous.

On the other hand, $\text{Rep}_{\mathbb{Q}_p}(G_K)$ can be studied via geometry. The Fargues–Fontaine curve, studied extensively in [Fargues and Fontaine 2018], is defined as the analytic adic space

$$\mathcal{X} = \mathcal{X}(\widehat{K}_{\text{cyc}}) := (\text{SpaA}_{\text{inf}} - \{p[p^\flat] = 0\})/(\varphi^{\mathbb{Z}}, \text{Gal}(\overline{K}/K_{\text{cyc}}))$$

(see Section 3) and has a natural action of Γ. By the work of Fargues and Fontaine, there is a fully faithful embedding

$$\text{Rep}_{\mathbb{Q}_p}(G_K) \hookrightarrow \{\Gamma\text{-equivariant vector bundles on } \mathcal{X}\},$$

again with the essential image consisting of the semistable slope 0 objects. In fact, Fargues and Fontaine show there is an equivalence

$$\{(\varphi, \Gamma)\text{-modules over } \mathcal{R}\} \cong \{\Gamma\text{-equivariant vector bundles on } \mathcal{X}\},$$

compatible with each of the aforementioned embeddings of $\text{Rep}_{\mathbb{Q}_p}(G_K)$.

Unfortunately, the action of Γ on an equivariant vector bundle on $\mathcal{X}$ *cannot* be differentiated. This is already true for the structure sheaf $\mathcal{O}_{\mathcal{X}}$. Here is a simplified model of the situation which illustrates why there is no action of $\text{Lie}(\Gamma)$ on $\mathcal{O}_{\mathcal{X}}$. The functions on an open subset of $\mathcal{X}$ can roughly be thought of

as power series in T^{1/p^∞} satisfying certain convergence conditions. When we try to apply the operator d/dT to such a power series, the result will often not converge since the derivative

$$d(T^{1/p^n})/dT = (1/p^n)T^{1/p^n-1}$$

grows exponentially larger p-adically as n goes to infinity. Nevertheless, there is a way to single out the sections for which the action of $\mathrm{Lie}(\Gamma)$ does not explode. This is achieved by considering only those sections on which the action of Γ is regular enough. In this toy model picture, this will amount to considering only the power series where the coefficient of the exponent of T^{k/p^n} will decay proportionally to p^n.

More canonically and more generally, these elements for which differentiation is possible are precisely the locally analytic elements. Given an equivariant vector bundle $\tilde{\mathcal{E}}$ on $\mathcal{X}$, there is a subsheaf of locally analytic sections $\tilde{\mathcal{E}}^{\mathrm{la}} \subset \tilde{\mathcal{E}}$. This sheaf is a module over $\mathcal{O}_\mathcal{X}^{\mathrm{la}}$ which is preserved under the Γ-action, and, crucially, $\mathrm{Lie}(\Gamma)$ acts on $\tilde{\mathcal{E}}^{\mathrm{la}}$. We are thus naturally led to the definition of a *locally analytic vector bundle on $\mathcal{X}$*: by this we shall mean a locally free $\mathcal{O}_\mathcal{X}^{\mathrm{la}}$-module together with a Γ-action. The point is that locally analytic vector bundles capture both analytic and geometric information, both of which has proven important for the study of $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$.

Our first main result is saying that there is no loss of information in this process: each equivariant vector bundle canonically descends to a locally analytic vector bundle.

Theorem A. *The functor $\tilde{\mathcal{E}} \mapsto \tilde{\mathcal{E}}^{\mathrm{la}}$ gives rise to an equivalence of categories from the category of Γ-equivariant vector bundles on $\mathcal{X}$ to the category of locally analytic vector bundles on $\mathcal{X}$. Its inverse is given by the functor $\mathcal{E} \mapsto \mathcal{O}_\mathcal{X} \otimes_{\mathcal{O}_\mathcal{X}^{\mathrm{la}}} \mathcal{E}$.*

This theorem fits naturally into the framework of Sen theory, as we shall now explain. Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. Then according to Sen's theory, proven in [Sen 1980], there is a canonical isomorphism

$$(V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{\mathrm{Gal}(\bar{K}/K_{\mathrm{cyc}})} \cong \widehat{K}_{\mathrm{cyc}} \otimes_{K_{\mathrm{cyc}}} \boldsymbol{D}_{\mathrm{Sen}}(V),$$

where $\boldsymbol{D}_{\mathrm{Sen}}(V)$ is the K_{cyc}-subspace of elements with finite Γ-orbit in $V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$. Later, Fontaine [2004, §3.4] proved an analogue of this theorem for $\boldsymbol{B}_{\mathrm{dR}}^+$: he showed there is an isomorphism

$$(V \otimes_{\mathbb{Q}_p} \boldsymbol{B}_{\mathrm{dR}}^+)^{\mathrm{Gal}(\bar{K}/K_{\mathrm{cyc}})} \cong (\boldsymbol{B}_{\mathrm{dR}}^+)^{\mathrm{Gal}(\bar{K}/K_{\mathrm{cyc}})} \otimes_{K_{\mathrm{cyc}}[\![t]\!]} \boldsymbol{D}_{\mathrm{dif}}^+(V),$$

where $\boldsymbol{D}_{\mathrm{dif}}^+(V)$ is a canonical $K_{\mathrm{cyc}}[\![t]\!]$-submodule of $V \otimes_{\mathbb{Q}_p} \boldsymbol{B}_{\mathrm{dR}}^+$.

In fact, both of these results are implied by Theorem A by specializing at the "point at infinity" $x_\infty \in \mathcal{X}$. Indeed, when $\tilde{\mathcal{E}}$ is the equivariant vector bundle associated to $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ and $\mathcal{E} = \tilde{\mathcal{E}}^{\mathrm{la}}$, specializing the isomorphism $\tilde{\mathcal{E}} \cong \mathcal{O}_\mathcal{X} \otimes_{\mathcal{O}_\mathcal{X}^{\mathrm{la}}} \mathcal{E}$ at the fiber of x_∞ gives rise to an isomorphism

$$\tilde{\mathcal{E}}_{k(x_\infty)} \cong \mathcal{O}_{\mathcal{X}, k(x_\infty)} \otimes_{\mathcal{O}_{\mathcal{X}, k(x_\infty)}^{\mathrm{la}}} \mathcal{E}_{k(x_\infty)},$$

which is none other than Sen's theorem. Similarly, there is an isomorphism of the completed stalks at x_∞,

$$\tilde{\mathcal{E}}_{x_\infty}^{\wedge, +} \cong \mathcal{O}_{\mathcal{X}, x_\infty}^{\wedge, +} \otimes_{\mathcal{O}_{\mathcal{X}, x_\infty}^{\mathrm{la}, \wedge, +}} \mathcal{E}_{x_\infty}^{\wedge, +},$$

which recovers Fontaine's theorem. In this way, Theorem A is a sheaf theoretic version of Sen theory on $\mathcal{X}$ which specializes at x_∞ to classical Sen theory.

In the interest of applications, we give a proof of this equivalence not just for the cyclotomic extension, but more generally for any p-adic Lie group $\Gamma = \mathrm{Gal}(K_\infty/K)$, where K_∞ is an infinitely ramified Galois extension of K which contains an unramified twist of the cyclotomic extension. Notably, this condition holds when K_∞ is the extension generated by the torsion points of a formal group.

As we shall explain in the article, these ideas are closely related to the decompletion of (φ, Γ)-modules, especially in the case $K_\infty = K_{\mathrm{cyc}}$. This is not too surprising, because such (φ, Γ)-modules are also obtained by a Sen theory type of idea through the theorem of Cherbonnier and Colmez [1998], and further, these objects relate to $\boldsymbol{D}_{\mathrm{Sen}}$ and $\boldsymbol{D}_{\mathrm{dif}}^+$ in a similar way. In fact, Theorem A is equivalent to the Cherbonnier–Colmez theorem on decompletion of (φ, Γ)-modules (after inverting p). Our proof is not independent from the ideas of Cherbonnier–Colmez, since we still use their trace maps in our arguments. However, it is logically different — more on this below.

First, let us discuss an application of Theorem A, which was a major source of motivation for this work. We give a geometric reinterpretation of Berger's work [2008b] on p-adic differential equations and filtered (φ, N)-modules. In that article, Berger establishes several results regarding de Rham (φ, Γ)-modules (for example, these (φ, Γ)-modules arising from de Rham p-adic Galois representations). To such a (φ, Γ)-module D, Berger associates another (φ, Γ)-module $\mathrm{N}_{\mathrm{dR}}(D)$ (a so called p-adic differential equation), and a $\overline{K}$-vector space of solutions

$$\mathrm{Sol}(D) := \varinjlim_{[L:K]<\infty} (\mathcal{R}_L[\log T] \otimes_{\mathcal{R}} \mathrm{N}_{\mathrm{dR}}(V))^{G_L},$$

where $\mathcal{R}_L$ is the Robba ring with respect to L. The following results can be derived from the main results of [Berger 2008b], for D a de Rham (φ, Γ)-module:

  (i) $\mathrm{Sol}(D)$ is a $\overline{K}$-vector space of rank equal to the rank of D.

 (ii) There is a canonical isomorphism

$$\mathcal{R}_{\overline{K}}[\log T] \otimes_{K^{\mathrm{un}}} \mathrm{Sol}(D) \cong \mathcal{R}_{\overline{K}}[\log T] \otimes_{\mathcal{R}} \mathrm{N}_{\mathrm{dR}}(V).$$

(iii) $\overline{K} \otimes_{K^{\mathrm{un}}} \mathrm{Sol}(D)$ is canonically isomorphic to $\overline{K} \otimes_K \boldsymbol{D}_{\mathrm{dR}}(D)$.

(iv) $\mathrm{Sol}(D)$ is naturally a filtered (φ, N, G_K)-module.

Furthermore, the functor $D \mapsto \mathrm{Sol}(D)$ gives rise to an equivalence of categories from the category of de Rham (φ, Γ)-modules over $\mathcal{R}$ to the category of filtered (φ, N, G_K)-modules.

The functor of solutions is ultimately understood in [Berger 2008b] by solving the differential equation $\mathrm{Lie}(\Gamma) = 0$, and as such, uses p-adic analysis in a crucial way. It is therefore natural to apply Theorem A to give a geometric interpretation of these results, something previously inaccessible in the framework of vector bundles on the Fargues–Fontaine curve. In fact, when interpreted in a geometric way, [Berger 2008b, théorème A] turns out to be reminiscent of the Riemann–Hilbert correspondence.

Our second main result is the desired geometric interpretation of Berger's results. To describe it, we need to introduce some notation. We have

$$\mathcal{X}_{\log,\bar{K}} := \varprojlim_{[L:K]<\infty} \mathcal{X}_{\log,L},$$

where each $\mathcal{X}_{\log,L}$ is a the analytic line bundle over $\mathcal{X}_L := \mathcal{X}(\hat{L}_{\text{cyc}})$ corresponding to $\mathcal{O}_{\mathcal{X}_L}(1)$, endowed with the projection $p_{\log,L} : \mathcal{X}_{\log,L} \to \mathcal{X}_L$ (see Section 8C). Essentially, $\mathcal{X}_{\log,\bar{K}}$ is obtained by adjoining all $\bar{K}$-scalars and a logarithm to the functions on $\mathcal{X}$. Now let $\mathcal{E}$ be a de Rham locally analytic vector bundle, i.e., suppose that $\dim_K \hat{\mathcal{E}}_{x_\infty}^{\Gamma=1} = \text{rank}(\mathcal{E})$ (see Section 8B). For example, if V is a de Rham p-adic Galois representation, then its associated locally analytic vector bundle is de Rham. To such $\mathcal{E}$, we associate a sheaf $\text{Sol}(\mathcal{E})$ on $\mathcal{X}$, given by

$$\text{Sol}(\mathcal{E}) := \varinjlim_{[L:K]<\infty} p_{\log,L,*}(p_{\log,L}^* \mathcal{N}_{\text{dR}}(\mathcal{E}))^{\text{Lie}(\Gamma)=0},$$

where $\mathcal{N}_{\text{dR}}(\mathcal{E})$ is a modification of $\mathcal{E}$ corresponding to the de Rham lattice of $\mathcal{E}$ at x_∞. Roughly speaking, $\text{Sol}(\mathcal{E})$ is the sheaf of solutions to the differential equation $\nabla = 0$ on the modification $\mathcal{N}_{\text{dR}}(\mathcal{E})$. We shall also consider a variant $\text{Sol}^\varphi(\mathcal{E})$, which are the solutions on the pullback of $\mathcal{E}$ along the usual covering $\mathcal{Y}_{(0,\infty)} \to \mathcal{X}$ for $\mathcal{Y}_{(0,\infty)} = \text{SpaA}_{\inf} - \{p[p^\flat] = 0\}/\text{Gal}(\bar{K}/K_{\text{cyc}})$. We then have the following result, by analogy with the results of [Berger 2008b] (see Section 8 for yet more precise statements).

Theorem B. *Let $\mathcal{E}$ be a de Rham locally analytic vector bundle.*

(i) *The sheaf of solutions $\text{Sol}(\mathcal{E})$ is locally free over the subsheaf of potentially log smooth sections $\mathcal{O}_{\mathcal{X}}^{\text{plsm}} \subset \mathcal{O}_{\mathcal{X}}^{\text{la}}$ and its rank is equal to the rank of $\mathcal{E}$.*

(ii) *There is a canonical isomorphism*

$$\mathcal{O}_{\mathcal{X}_{\log,\bar{K}}}^{\text{la}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\text{plsm}}} \text{Sol}(\mathcal{E}) \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}_{\log,\bar{K}}}^{\text{la}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\text{la}}} \mathcal{N}_{\text{dR}}(\mathcal{E}).$$

(iii) *The stalk of $\text{Sol}(\mathcal{E})$ at x_∞ is canonically isomorphic to $\bar{K} \otimes_K \boldsymbol{D}_{\text{dR}}(\mathcal{E})$.*

(iv) *The space of global solutions $\text{H}^0(\mathcal{Y}_{(0,\infty)}, \text{Sol}^\varphi(\mathcal{E}))$ is naturally a filtered (φ, N, G_K)-module.*

Furthermore, the functor $\mathcal{E} \mapsto \text{H}^0(\mathcal{Y}_{(0,\infty)}, \text{Sol}^\varphi(\mathcal{E}))$ gives rise to an equivalence of categories from the category of de Rham locally analytic vector bundles to the category of filtered (φ, N, G_K)-modules.

Remark 1.1. (1) In particular, if V is a de Rham representation of G_K with associated locally analytic vector bundle $\mathcal{E}$, then $\text{H}^0(\mathcal{Y}_{(0,\infty)}, \text{Sol}^\varphi(\mathcal{E})) = \boldsymbol{D}_{\text{pst}}(V)$ and the stalk $\text{Sol}(\mathcal{E})_{x_\infty}$ is identified with $\bar{K} \otimes_K \boldsymbol{D}_{\text{dR}}(V)$. The localization map corresponds to the natural map $\boldsymbol{D}_{\text{pst}}(V) \to \bar{K} \otimes_K \boldsymbol{D}_{\text{dR}}(V)$.

(2) If $\mathcal{E}$ becomes crystalline after extending K to a finite extension $L \subset K_\infty$, the sheaf $\mathcal{N}_{\text{dR}}(\mathcal{E})^{\nabla=0} \subset \text{Sol}(\mathcal{E})$ is locally free over the subsheaf of smooth sections $\mathcal{O}_{\mathcal{X}}^{\text{sm}} \subset \mathcal{O}_{\mathcal{X}}^{\text{la}}$ of rank equal to the rank of $\mathcal{E}$, and there is a simpler canonical isomorphism

$$\mathcal{O}_{\mathcal{X}}^{\text{la}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\text{sm}}} \mathcal{N}_{\text{dR}}(\mathcal{E})^{\nabla=0} \xrightarrow{\sim} \mathcal{N}_{\text{dR}}(\mathcal{E}).$$

(3) The sheaf $\mathcal{O}_{\mathcal{X}}^{\mathrm{plsm}}$ is much smaller than $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$. Though we have not been quite able to show this, $\mathcal{O}_{\mathcal{X}}^{\mathrm{plsm}}$ seems to be "almost" a locally constant sheaf except that the base field becomes slightly larger when localizing; for that reason, we think of $\mathrm{Sol}(\mathcal{E})$ as morally being close to a local system on $\mathcal{X}$. In this sense the (φ, N, G_K)-structure is related to the monodromy of the p-adic differential equation $\nabla = 0$.

Finally, let us discuss the proof of Theorem A. The essential point is to show that if $\tilde{\mathcal{E}}$ is an equivariant vector bundle on $\mathcal{X}$, the natural map $\mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \tilde{\mathcal{E}}^{\mathrm{la}} \to \tilde{\mathcal{E}}$ is an isomorphism. Fargues and Fontaine observe that the only point of $\mathcal{X}$ with finite Γ-orbit is x_∞. The idea is then to use a very simple geometric argument: once one knows that $\mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \tilde{\mathcal{E}}^{\mathrm{la}} \to \tilde{\mathcal{E}}$ is injective, everything can be understood by arguing locally at x_∞. Indeed, if this map is an isomorphism after localizing and completing along $\mathcal{O}_{\mathcal{X}} \to \widehat{\mathcal{O}}_{\mathcal{X},x_\infty}^+$, then the cokernel has to be supported at finitely many points outside x_∞. But these points also form a finite Γ-orbit, so the cokernel cannot be supported anywhere.

It therefore remains to understand the properties of our spaces of locally analytic vectors under certain localizations and completions. To do this, we are naturally led to consider higher locally analytic vectors and their vanishing, and we prove a representation-theoretic result which is of independent interest. To state the result, let G be a p-adic Lie group and let $\widetilde{\Lambda}$ be a Banach ring with a continuous action of G. Assume the topology on $\widetilde{\Lambda}$ is p-adic.

Theorem C. *Suppose G and $\widetilde{\Lambda}$ satisfy the Tate–Sen axioms* (TS1)–(TS3) *of* [Berger and Colmez 2008] *as well as an additional axiom* (TS4). *Then for any finite free $\widetilde{\Lambda}$-semilinear representation M of G, the higher locally analytic vectors $\mathrm{R}_{G\text{-la}}^i(M)$ are zero for $i \geq 1$.*

Here are two special cases of the theorem where we conclude that $\mathrm{R}_{G\text{-la}}^i(M) = 0$ for $i \geq 1$:

(1) If M is a finite dimensional $\widehat{K}_\infty$-module with a semilinear action of Γ, for K_∞ containing an unramified twist of K_{cyc}. In fact, the vanishing of $\mathrm{R}_{G\text{-la}}^i(M)$ can be established for arbitrary K_∞, see Section 5.

(2) If M a finite free $\widetilde{B}_I(\widehat{K}_\infty)$-module with a semilinear action of Γ, under the same assumptions on K_∞.

Note that the vanishing of higher locally analytic vectors is automatic for admissible representations, but the examples above are not admissible. Theorem C illustrates how the Tate–Sen axioms can serve as a substitute for admissibility.

Theorem C is especially useful for making cohomological computations. Here is an example application, which follows directly from the main results of [Rodrigues Jacinto and Rodríguez Camargo 2022] (see Section 5): if M satisfies assumptions of the theorem, then for $i \geq 0$ we have natural isomorphisms

$$\mathrm{H}^i(G, M) \cong \mathrm{H}^i(G, M^{\mathrm{la}}) \cong \mathrm{H}^i(\mathrm{Lie}\, G, M^{\mathrm{la}})^G.$$

Finally, let us mention that in recent work Juan Esteban Rodríguez Camargo [2022] proves similar results to our Theorem C. He then applies them in the setting of rigid adic spaces with fantastic applications to the Calegari–Emerton conjecture, among others.

1A. *Structure of the article.* Section 2 contains reminders on locally analytic vectors and their derived functors. In Section 3 we give reminders on the Fargues–Fontaine curve and equivariant vector bundles. In Section 4 we introduce locally analytic bundles and we discuss their basic properties. Section 5 is the longest and most technical section of the paper, in which we prove Theorem C. Theorem A is proved in Section 6. In Section 7 we compare our results to the theory of (φ, Γ)-modules. Finally, in Section 8 we discuss p-adic differential equations on the Fargues–Fontaine curve and explain Theorem B.

At several points in the article we have taken the liberty to raise speculations and ask questions to which we do not yet know the answer.

1B. *Notation and conventions.* The field K denotes a finite extension of $\mathbb{Q}_p$. We write $K_{\mathrm{cyc}} = K(\mu_{p^\infty})$ for the cyclotomic extension. Its Galois group $\Gamma_{\mathrm{cyc}} = \mathrm{Gal}(K_{\mathrm{cyc}}/K)$ is an open subgroup of $\mathbb{Z}_p^\times$. We denote by K_∞ an infinitely ramified Galois extension of K with $\Gamma = \mathrm{Gal}(K_\infty/K)$ a p-adic Lie group. If $\overline{K}$ denotes the algebraic closure of K, we let $G_K = \mathrm{Gal}(\overline{K}/K)$ and $H = \mathrm{Gal}(\overline{K}/K_\infty)$ so that $G_K/H = \Gamma$.

The p-adic completion $\widehat{K}_\infty$ of K_∞ is a perfectoid field. Write ϖ for a pseudouniformizer of $\widehat{K}_\infty$ with valuation $\mathrm{val}(\varpi) = p$ that admits a sequence of p-th power roots ϖ^{1/p^n} (such a choice is always possible, and the constructions in this paper never depend on this choice). Let $\varpi^\flat = (\varpi, \varpi^{1/p}, \dots)$ be the corresponding pseudouniformizer of the tilt $\widehat{K}_\infty^\flat$.

Denote by $\mathrm{Lie}(\Gamma)$ the Lie algebra of Γ. It is a finite dimensional $\mathbb{Q}_p$-vector space, and if $v \in \mathrm{Lie}(\Gamma)$ is sufficiently small, we have a corresponding element $\exp(v) \in \Gamma$.

All representations and group actions appearing in this article are assumed to be continuous. Galois cohomology groups are always taken in the continuous sense.

If W is a Banach space over $\mathbb{Q}_p$ we write W^+ for its unit ball.

All completed tensor products appearing in this article are projective. In other words, if V^+ and W^+ are unit balls of two Banach spaces V and W over $\mathbb{Q}_p$, then

$$V^+ \widehat{\otimes}_{\mathbb{Z}_p} W^+ = \varprojlim_n (V^+ \otimes_{\mathbb{Z}_p} W^+)/p^n \quad \text{and} \quad V \widehat{\otimes}_{\mathbb{Q}_p} W = (V^+ \widehat{\otimes}_{\mathbb{Z}_p} W^+)[1/p].$$

2. Locally analytic and pro-analytic vectors

In this section we give reminders on locally analytic and pro-analytic vectors and quote results that will be used in Sections 4–6. We shall freely use our conventions in Section 1B regarding Banach spaces.

2A. *Locally analytic and pro-analytic vectors.* We shall say a compact p-adic Lie group G is *small* if there exists a saturated integral valued p-valuation on G which defines its topology and if for some $N \in \mathbb{Z}_{\geq 1}$ there exists an embedding of G into $1 + p^2 M_N(\mathbb{Z}_p)$, the group of N by N matrices congruent to 1 mod p^2. See Sections 23 and 26 of [Schneider 2011] for the first condition. If G is small, there exists an ordered basis $g_1, \dots, g_d$ such that $(x_1, \dots, x_d) \mapsto g_1^{x_1} \cdot \dots \cdot g_1^{x_d}$ gives a homeomorphism of $\mathbb{Z}_p^d$ with G. We then have coordinates on G

$$c = (c_1, \dots, c_d) : G \xrightarrow{\sim} \mathbb{Z}_p^d$$

defined by the inverse map where $c_i(g_1^{x_1} \cdot \ldots \cdot g_1^{x_d}) = x_i$.

Now let G is an be any compact p-adic Lie group. By [Schneider 2011, Theorem 27.1] and Ado's theorem (see [Pan 2022a, Proposition 2.1.3]), the collection of small open subgroups of G forms a fundamental system of open neighborhoods of the identity element. Let W be a Banach $\mathbb{Q}_p$-linear representation of G (or G-Banach space for short). If H is a small open subgroup of G, choose coordinates c on H and write $c(h)^{\boldsymbol{k}} = \prod_{i=1}^d c_i(h)^{k_i}$ if $\boldsymbol{k} = (k_1, \ldots, k_d)$ for $h \in H$. We have the subspace $W^{H\text{-an}}$ of H-analytic vectors in W; it is the subspace of elements $w \in W$ for which there exists a sequence of vectors $\{w_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathbb{N}^d}$ with $w_{\boldsymbol{k}} \to 0$ and

$$h(w) = \sum_{\boldsymbol{k} \in \mathbb{N}^d} c(h)^{\boldsymbol{k}} w_{\boldsymbol{k}}$$

for all $h \in H$. The norm $\|w\|_{H\text{-an}} = \sup_{\boldsymbol{k}} \|w_{\boldsymbol{k}}\|$ makes $W^{H\text{-an}}$ into a Banach space. Note that $W^{H\text{-an}}$ does not depend on the choice of coordinates. We write $W^{\text{la}} = \bigcup_H W^{H\text{-an}}$ for the subspace of locally analytic vectors of W, and endow it with the inductive limit topology, which makes it into an LB space. If W is a Fréchet space whose topology is defined by a countable sequence of seminorms, let W_i be the Hausdorff completion of W for the i-th seminorm, so that $W = \varprojlim W_i$ is a projective limit of Banach spaces. We write $W^{\text{pa}} = \varprojlim W_i^{\text{la}}$ for the subspace of pro-analytic vectors. Finally, we extend the definitions of locally analytic vectors and pro-analytic vectors to LB and LF spaces in the obvious way.

The Lie algebra $\text{Lie}(G)$ acts on each $W^{H\text{-an}}$ (and hence also on W^{la} and W^{pa}) through derivations. This action is given as follows. If $v \in \text{Lie}(G)$ then $\exp(p^k v) \in H$ for $k \gg 0$, and we define

$$\nabla_v(w) = \lim_{k \to \infty} \frac{\exp(p^k v)(w) - w}{p^k}.$$

The operator $\nabla_v : W^{H\text{-an}} \to W^{H\text{-an}}$ is bounded; see [Berger and Colmez 2016, Lemma 2.6].

Locally analytic and pro-analytic vectors behave well when we have a basis of such vectors [Berger and Colmez 2016, Proposition 2.3; Berger 2016, Proposition 2.4]:

Proposition 2.1. *Let B be a Banach or Fréchet G-ring and let W be a free B-module of finite rank, equipped with a B-semilinear action of G. If the B-module W has a basis $w_1, \ldots, w_d$ in which the function $G \to \text{GL}_d(B) \subset \text{M}_d(B)$, $g \mapsto \text{Mat}(g)$ is H-analytic (resp. locally analytic, pro-analytic), then $W^{H\text{-an}} = \bigoplus_{j=1}^d B^{H\text{-an}} \cdot w_i$ (resp. $W^{\text{la}} = \bigoplus_{j=1}^d B^{\text{la}} \cdot w_i$, $W^{\text{pa}} = \bigoplus_{j=1}^d B^{\text{pa}} \cdot w_i$).*

It will often be useful for us to choose a specific fundamental system of open neighborhoods of G as follows. Fix a small compact open $G_0 \subset G$ which with coordinates c. For $n \geq 0$ we set

$$G_n = G^{p^n} = \{g^{p^n} : g \in G_0\}.$$

These are subgroups ([Schneider 2011, Remark 26.9]) which have induced coordinates

$$c|_{G_n} : G_n \xrightarrow{\sim} (p^n \mathbb{Z}_p)^d.$$

The normalization is such that for $w \in W^{G_n\text{-an}}$ we can write

$$g(w) = \sum_{\boldsymbol{k} \in \mathbb{N}^d} c(g)^{\boldsymbol{k}} w_{\boldsymbol{k}}$$

for $g \in G_n$ and $\{w_{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathbb{N}^d}$ with $p^{n|\boldsymbol{k}|} w_{\boldsymbol{k}} \to 0$, and the Banach norm is given by

$$\|w\|_{G_n\text{-an}} = \sup_{\boldsymbol{k}} \|p^{n|\boldsymbol{k}|} w_{\boldsymbol{k}}\|.$$

It is easy to check if $w \in W^{G_n\text{-an}}$ then $\|w\|_{G_m\text{-an}} \leq \|w\|_{G_{m+1}\text{-an}}$ for $m \geq n$ and $\|w\|_{G_m\text{-an}} = \|w\|$ for $m \gg n$ (see [Berger and Colmez 2016, Lemme 2.4]).

2B. *Rings of analytic functions.* Suppose first that G is small. Let $\mathcal{C}^{\text{an}}(G, \mathbb{Q}_p)$ be the space of analytic functions on G. These are those functions that after pullback by the coordinates $c : G \xrightarrow{\sim} \mathbb{Z}_p^d$ are of the form

$$\boldsymbol{x} = (x_1, \ldots, x_d) \mapsto \sum_{\boldsymbol{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d} b_{\boldsymbol{k}} \boldsymbol{x}^{\boldsymbol{k}},$$

where $b_{\boldsymbol{k}} \to 0$ as $|\boldsymbol{k}| \to \infty$. The norm $\|f\|_G = \sup_{\boldsymbol{k} \in \mathbb{N}^d} \|b_{\boldsymbol{k}}\|$ makes $\mathcal{C}^{\text{an}}(G, \mathbb{Q}_p)$ into a Banach space. We shall regard $\mathcal{C}^{\text{an}}(G, \mathbb{Q}_p)$ as a G-representation through the left regular action of G.

If now G is any compact p-adic Lie group with a system of small neighborhoods $\{G_n\}_{n \geq 0}$ as in Section 2A, we have for each $n \geq 0$ the space of analytic functions $\mathcal{C}^{\text{an}}(G_n, \mathbb{Q}_p)$ on G_n. Using the coordinates $c : G_n \xrightarrow{\sim} (p^n \mathbb{Z}_p)^d$ as in Section 2A, we shall regard $\mathcal{C}^{\text{an}}(G_n, \mathbb{Q}_p)$ as the ring of functions that under the bijection are identified with functions of the form

$$\boldsymbol{x} = (x_1, \ldots, x_d) \mapsto \sum_{\boldsymbol{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d} b_{\boldsymbol{k}} \boldsymbol{x}^{\boldsymbol{k}},$$

where $p^{n|\boldsymbol{k}|} b_{\boldsymbol{k}} \to 0$ as $|\boldsymbol{k}| \to \infty$. Under this normalization

$$\|f\|_{G_n} = \sup_{\boldsymbol{k} \in \mathbb{N}^d} \|p^{n|\boldsymbol{k}|} b_{\boldsymbol{k}}\|$$

for $f \in \mathcal{C}^{\text{an}}(G_n, \mathbb{Q}_p)$.

The following lemma will be used in Section 5.

Lemma 2.2. *For $k \geq 1$ the subgroup G_{n+k} acts trivially on $\mathcal{C}^{\text{an}}(G_n, \mathbb{Q}_p)^+ / p^k$.*

Proof. This is an easy exercise using the coordinates. See [Pan 2022a, Lemma 2.1.2] for the case $k = 1$. $\square$

The following is shown in [Pan 2022a, Proposition 2.1.3] and in its proof (originally in the proof of [Berger and Colmez 2016, théoréme 6.1]).

Proposition 2.3. *Suppose that G is small. There is a dense subspace $\varinjlim_{\ell \in \mathbb{N}} V_\ell \subset \mathcal{C}^{\text{an}}(G, \mathbb{Q}_p)$, where each V_l is a finite-dimensional G-subrepresentation of $\mathcal{C}^{\text{an}}(G, \mathbb{Q}_p)$ with coefficients in $\mathbb{Q}_p$ such that for any $k, \ell \in \mathbb{N}$, we have $V_k \cdot V_\ell \subset V_{k+\ell}$.*

Furthermore, if we fix G and consider small open subgroups $G' \subset G$, we may choose $V_\ell(G') \subset \mathcal{C}^{\mathrm{an}}(G', \mathbb{Q}_p)$ at once for all G' in such a way that the natural map $\mathcal{C}^{\mathrm{an}}(G, \mathbb{Q}_p) \to \mathcal{C}^{\mathrm{an}}(G', \mathbb{Q}_p)$ restricts to $V_\ell(G) \to V_\ell(G')$.

2C. *Higher locally analytic vectors.* Suppose first that G is small and let W be a G-Banach space. There is a G-equivariant isometry
$$W \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{C}^{\mathrm{an}}(G, \mathbb{Q}_p) \cong \mathcal{C}^{\mathrm{an}}(G, W),$$
where $\mathcal{C}^{\mathrm{an}}(G, W)$ is the space of W-valued analytic functions on G, with its G-Banach structure given by the sup norm and the action $(gf)(x) = g(f(g^{-1}(x)))$ for $f \in \mathcal{C}^{\mathrm{an}}(G, W)$. We then have $(\mathcal{C}^{\mathrm{an}}(G, W))^G = W^{G\text{-an}}$, the identification given by $f \mapsto f(1)$. This gives an alternative description of G-analytic vectors that we shall use in what follows.

The functor $W \mapsto W^{G\text{-an}}$ is left exact. Following [Pan 2022a, §2.2; Rodrigues Jacinto and Rodríguez Camargo 2022], define right derived functors for $i \geq 0$:
$$\mathrm{R}^i_{G\text{-an}}(W) = \mathrm{H}^i\big(G, W \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{C}^{\mathrm{an}}(G, \mathbb{Q}_p)\big)$$
(taking continuous cohomology on the right hand side).

If G is a compact p-adic Lie group with subgroups $\{G_n\}_{n \geq 1}$ as in Sections 2A–2B, taking the colimit over n, there are right derived functors for $W \mapsto W^{G\text{-la}}$ given by
$$\mathrm{R}^i_{G\text{-la}}(W) = \varinjlim_n \mathrm{R}^i_{G_n\text{-an}}(W) = \varinjlim_n \mathrm{H}^i\big(G_n, W \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p)\big).$$

We shall call these groups the higher locally analytic vectors of W. If G is understood from the context we shall just write $\mathrm{R}^i_{\mathrm{la}}$ instead of $\mathrm{R}^i_{G\text{-la}}$.

If
$$0 \to V \to W \to X \to 0$$
is a short exact sequence of G-Banach spaces, then it is strict by the open mapping theorem, and so we have a long exact sequence
$$0 \to V^{\mathrm{la}} \to W^{\mathrm{la}} \to X^{\mathrm{la}} \to \mathrm{R}^1_{\mathrm{la}}(V) \to \mathrm{R}^1_{\mathrm{la}}(W) \to \mathrm{R}^1_{\mathrm{la}}(X) \to \cdots.$$

Lemma 2.4. *Let H be an open subgroup of G and let $H_n = G_n \cap H$. Then for $n \gg 0$ and each $i \geq 0$ there are natural isomorphisms $\mathrm{R}^i_{H_n\text{-an}} \cong \mathrm{R}^i_{G_n\text{-an}}$. In particular, $\mathrm{R}^i_{H\text{-la}} \cong \mathrm{R}^i_{G\text{-la}}$.*

Proof. We have $H_n = G_n$ for $n \gg 0$. $\qquad\square$

Suppose that G be a small compact p-adic Lie group, and let H be a small closed normal subgroup. Let W be a G-Banach space. Using the method of Hochshild–Serre we obtain the following spectral sequences.

Proposition 2.5. (i) *There is a spectral sequence*
$$E_2^{ij} = \mathrm{H}^i\big(G/H, \mathrm{H}^j\big(H, W \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{C}^{\mathrm{an}}(G, \mathbb{Q}_p)\big)\big) \Rightarrow \mathrm{R}^{i+j}_{G\text{-an}}(W).$$

(ii) *There is a spectral sequence*

$$E_2^{ij} = \mathrm{R}^i_{G/H\text{-an}}(\mathrm{H}^j(H, W)) \Rightarrow \mathrm{H}^{i+j}\big(G, W \,\widehat{\otimes}_{\mathbb{Q}_p}\, \mathcal{C}^{\mathrm{an}}(G/H, \mathbb{Q}_p)\big).$$

Proof. Apply the Hochschild–Serre spectral sequence to $W \,\widehat{\otimes}_{\mathbb{Q}_p}\, \mathcal{C}^{\mathrm{an}}(G, \mathbb{Q}_p)$ and $W \,\widehat{\otimes}_{\mathbb{Q}_p}\, \mathcal{C}^{\mathrm{an}}(G/H, \mathbb{Q}_p)$ (see [Rodrigues Jacinto and Rodríguez Camargo 2022, Proposition 5.16]). $\qquad\square$

3. Equivariant vector bundles

In this section we give reminders on the Fargues–Fontaine curve and equivariant vector bundles. For more details, see [Fargues and Fontaine 2018, Chapter 9; Scholze and Weinstein 2020, Lectures 12–13].

3A. *The spaces $\mathcal{Y}_{(0,\infty)}$ and $\mathcal{X}$.* Let F be a perfectoid field, with tilt $F^\flat$. We have Fontaine's ring $\mathrm{A}_{\mathrm{inf}} = \mathrm{A}_{\mathrm{inf}}(F)$, defined as the Witt vectors of the ring of integers $\mathcal{O}_F^\flat$ of $F^\flat$. Write $\mathrm{Spa}(\mathrm{A}_{\mathrm{inf}})$ for the adic space associated to the Huber pair $(\mathrm{A}_{\mathrm{inf}}, \mathrm{A}_{\mathrm{inf}})$.

Let ϖ be a pseudouniformizer of F, and let f be the residue field of $\mathcal{O}_F$. Then there is a point $x_f \in \mathrm{Spa}(\mathrm{A}_{\mathrm{inf}})$ with residue field f, which is the intersection of the two closed subspaces $\{p = 0\}$ and $\{[\varpi] = 0\}$. We set

$$\mathcal{Y} = \mathcal{Y}(F) = \mathrm{Spa}\,\mathrm{A}_{\mathrm{inf}} - \{x_f\} \quad \text{and} \quad \mathcal{Y}_{(0,\infty)} = \mathcal{Y}_{(0,\infty)}(F) = \mathrm{Spa}\,\mathrm{A}_{\mathrm{inf}} - \{p[\varpi] = 0\}.$$

The spaces $\mathcal{Y}$ and $\mathcal{Y}_{(0,\infty)}$ have a Frobenius automorphism φ induced from the Witt vector structure of $\mathrm{A}_{\mathrm{inf}}$.

The space $\mathcal{Y}_{(0,\infty)}$ is a preperfectoid space. The (adic) Fargues–Fontaine curve associated to F is defined as the quotient

$$\mathcal{X} = \mathcal{X}(F) = \mathcal{Y}_{(0,\infty)}(F)/\varphi^{\mathbb{Z}},$$

which makes sense because the Frobenius action is proper and discontinuous. The natural projection $\pi : \mathcal{Y}_{(0,\infty)} \to \mathcal{X}$ is a local isomorphism, so $\mathcal{X}$ is a preperfectoid space, by virtue of $\mathcal{Y}_{(0,\infty)}$ being so. The space $\mathcal{Y}_{(0,\infty)}$ has a canonical point called x_∞, the "point at infinity". It corresponds to the kernel of Fontaine's map

$$\theta : \mathrm{A}_{\mathrm{inf}} \to \mathcal{O}_F, \quad \sum_{n \geq 0} [a_n] p^n \mapsto \sum_{n \geq 0} a_n^\sharp p^n,$$

where for $a \in \mathcal{O}_F$, $a^\sharp$ is defined to be the first coordinate of $a \in \mathcal{O}_F^\flat = \varprojlim_{x \mapsto x^p} \mathcal{O}_F$. Identify x_∞ with its image $\pi(x_\infty) \in \mathcal{X}$. We shall sometimes use the fact that $\ker\theta$ is a principal ideal, generated by $\xi = \varpi - [\varpi^\flat]$ (for example).

If $F = \widehat{K}_\infty$, there is an induced action of the group $\Gamma = \mathrm{Gal}(K_\infty/K)$ on each of the spaces mentioned above, and the map $\mathcal{Y}_{(0,\infty)} \to \mathcal{X}$ is Γ-equivariant. The point $x_\infty \in \mathcal{X}$ is the unique Γ-fixed point; in fact, it is the unique point with finite Γ-orbit [Fargues and Fontaine 2018, Proposition 10.1.1]. From now on, if F is omitted from the notation of $\mathcal{Y}_{(0,\infty)}$ and $\mathcal{X}$, we always take $F = \widehat{K}_\infty$.

3B. *The spaces $\mathcal{Y}_I$ and $\mathcal{X}_I$.* It will be fruitful to consider certain open subsets of $\mathcal{Y}_{(0,\infty)}$ and $\mathcal{X}$. By [Scholze and Weinstein 2020, Lecture 12] there is a surjective continuous map $\kappa : \mathcal{Y} \to [0,\infty]$ given by[1]

$$\kappa(x) = \frac{\log|p(\tilde{x})|}{\log|[\varpi^{\flat}](\tilde{x})|},$$

where $\tilde{x}$ is the maximal generization of x. For each interval $I \subset (0,\infty)$, let $\mathcal{Y}_I$ be the interior of the preimage of $\mathcal{Y}$ under κ. These spaces are Γ-stable if such a Γ action is present. Furthermore, the map φ induces isomorphisms $\varphi : \mathcal{Y}_{pI} \xrightarrow{\sim} \mathcal{Y}_I$. Write $\log(I) = \{\log x : x \in I\}$. Whenever I is sufficiently small so that the inequality $|\log(I)| < \log(p)$ holds, we have $\bar{I} \cap p\bar{I} = 0$ and π maps $\mathcal{Y}_I$ isomorphically onto its image $\pi(\mathcal{Y}_I) = \mathcal{X}_I \subset \mathcal{X}$. Note that $x_\infty \in \mathcal{X}_I$ if and only if I contains an element of $(p-1)p^{\mathbb{Z}}$, because $\kappa(x_\infty) = (p-1)/p$.

For $I \subset (0,\infty)$, we have the coordinate rings

$$\widetilde{B}_I = \widetilde{B}_I(\widehat{K}_\infty) = H^0(\mathcal{Y}_I, \mathcal{O}_{\mathcal{Y}_{(0,\infty)}}).$$

If I is compact, the geometry of $\mathcal{Y}_I$ is simple.

Proposition 3.1. *Suppose $I \subset (0,\infty)$ is a compact interval.*

(i) $\mathcal{Y}_I = \mathrm{Spa}(\widetilde{B}_I, \tilde{A}_I)$, *where $\tilde{A}_I$ is the ring of power bounded elements of $\widetilde{B}_I$. In particular, $\mathcal{Y}_I$ is affinoid.*

(ii) $\widetilde{B}_I$ *is a principal ideal domain.*

(iii) *The global sections functor induces an equivalence of categories between vector bundles on $\mathcal{Y}_I$ and finite free $\widetilde{B}_I$-modules.*

Proof. Parts (i) and (ii) follow from [Fargues and Fontaine 2018, théorème 3.5.1]. Part (iii) follows from [Scholze and Weinstein 2020, Theorem 5.2.8] (originally [Kedlaya and Liu 2015, Theorem 2.7.7]), since finite projective $\widetilde{B}_I$-modules are finite free. $\qquad\square$

3C. *Equivariant vector bundles.* The action of Γ on $\mathcal{X}$ gives an automorphism $\gamma : \mathcal{X} \xrightarrow{\sim} \mathcal{X}$ for each $\gamma \in \Gamma$.

Definition 3.2. A Γ-equivariant vector bundle (or simply Γ-vector bundle) on $\mathcal{X}$ is a vector bundle $\tilde{\mathcal{E}}$ on $\mathcal{X}$ equipped with an isomorphism $c_\gamma : \gamma^*\tilde{\mathcal{E}} \xrightarrow{\sim} \tilde{\mathcal{E}}$ for each $\gamma \in \Gamma$ such that the cocycle condition $c_{\gamma_2} \circ \gamma_2^* c_{\gamma_1} = c_{\gamma_1\gamma_2}$ holds for every $\gamma_1, \gamma_2, \in \Gamma$.

Similarly, we have a notion of a (φ, Γ)-vector bundle on $\mathcal{Y}_{(0,\infty)}$. This consists of a Γ-vector bundle $\widetilde{\mathcal{M}}$ on $\mathcal{Y}_{(0,\infty)}$ together with an additional isomorphism $c_\varphi : \varphi^*\widetilde{\mathcal{M}} \xrightarrow{\sim} \widetilde{\mathcal{M}}$ such that $c_\varphi \circ \varphi^* c_\gamma = c_\gamma \circ \gamma^* c_\varphi$ for every $\gamma \in \Gamma$.

Descent along φ gives the following.

Proposition 3.3. *There is an equivalence of categories*

$$\{\Gamma\text{-}vector \; bundles \; on \; \mathcal{X}\} \cong \{(\varphi, \Gamma)\text{-}vector \; bundles \; on \; \mathcal{Y}_{(0,\infty)}\}.$$

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[1]Our normalization of κ is the inverse of [loc. cit.].

The equivalence is given by the following functors: If $\tilde{\mathcal{E}}$ *is an equivariant vector bundle, we map it to* $\mathcal{O}_{\mathcal{Y}_{(0,\infty)}} \otimes_{\mathcal{O}_{\mathcal{X}}} \tilde{\mathcal{E}}$. *Conversely, if* $\tilde{\mathcal{M}}$ *is a* (φ, Γ)*-vector bundle on* $\mathcal{Y}_{(0,\infty)}$, *we map it to* $\pi_*(\tilde{\mathcal{M}})^{\varphi=1}$.

If $\tilde{\mathcal{E}}$ is a Γ-vector bundle on $\mathcal{X}$ and $U \subset \mathcal{X}$ is an open subset stable under Γ, there is an induced action of Γ on $\mathrm{H}^0(U, \tilde{\mathcal{E}})$. In particular, there is a natural action of Γ on $\mathrm{H}^0(\mathcal{X}_I, \tilde{\mathcal{E}})$ when $|\log(I)| < \log(p)$. For a general open subset U, one only has a map

$$c_\gamma : \mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}}) \otimes_{\mathrm{H}^0(\gamma(U), \mathcal{O}_{\mathcal{X}})} \mathrm{H}^0(\gamma(U), \tilde{\mathcal{E}}) \to \mathrm{H}^0(U, \tilde{\mathcal{E}}).$$

Similar remarks apply for (φ, Γ)-equivariant vector bundles on $\mathcal{Y}_{(0,\infty)}$.

Example 3.4. Let V be a finite dimensional $\mathbb{Q}_p$-representation of G_K. Recall that $H = \mathrm{Gal}(K_\infty/K)$. Then by [Fargues and Fontaine 2018, théorème 10.1.5],

$$\tilde{\mathcal{E}}(V) := (V \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathcal{X}(\mathbb{C}_p)})^H$$

is a Γ-vector bundle on $\mathcal{X}$. More generally, by [loc. cit.], the category of finite dimensional G_K-representations embeds fully faithfully to the category of (φ, Γ)-modules, with essential image the subcategory of étale (φ, Γ)-modules. We can extend the domain of the functor $V \mapsto \tilde{\mathcal{E}}(V)$ from G_K-representations to (φ, Γ)-modules. Conversely, any Γ-vector bundle on $\mathcal{X}$ gives rise to a (φ, Γ)-module, and this correspondence results in a equivalence of categories (see [Fargues and Fontaine 2018, préface, Remark 5.10]). This will be discussed in detail in Section 7.

4. Locally analytic vector bundles

In this section, we introduce the category of locally analytic vector bundles and discuss their basic properties.

4A. *Locally analytic functions of $\mathcal{Y}_{(0,\infty)}$ and $\mathcal{X}$.* Let $U \subset \mathcal{X}$ be an open affinoid. Then U is quasicompact and hence stable under the action of a finite index subgroup $\Gamma' \leq \Gamma$. The space of functions $\mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}})$ is a Banach Γ'-ring, and so it makes sense to speak of its subring of Γ'-locally analytic functions. This does not depend on the choice of Γ', and so we shall write $\mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}})^{\mathrm{la}}$ for the Γ'-locally analytic functions in $\mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}})$ for any Γ'. Since taking locally analytic vectors is left exact, these can be glued and we obtain a sheaf of rings $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$ on $\mathcal{X}$ that satisfies

$$\mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}) = \mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}})^{\mathrm{la}}$$

for every open affinoid $U \subset \mathcal{X}$.

More generally, suppose U is an open subset of $\mathcal{X}$ which is not necessarily affinoid, but for which there is an increasing cover $U = \bigcup_i U_i$ with each U_i affinoid and a single finite index subgroup $\Gamma' \leq \Gamma$ stabilizing all of the U_i simultaneously. This condition will be satisfied in any situation we shall consider. Then the sections of $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$ on U are the pro-analytic functions

$$\mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}) = \varprojlim_i \mathrm{H}^0(U_i, \mathcal{O}_{\mathcal{X}})^{\mathrm{la}} = \mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}})^{\mathrm{pa}}.$$

Lemma 4.1. *The sheaf $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$ is stable for the action of Γ on $\mathcal{O}_{\mathcal{X}}$, in the sense that the inclusion $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}} \subset \mathcal{O}_{\mathcal{X}}$ induces isomorphisms*

$$c_{\gamma} : \gamma^* \mathcal{O}_{\mathcal{X}}^{\mathrm{la}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}.$$

Proof. The action of Γ on $\mathcal{O}_{\mathcal{X}}$ gives rise to an isomorphism $c_{\gamma} : \gamma^* \mathcal{O}_{\mathcal{X}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}}$. Upon taking $U \subset \mathcal{X}$ affinoid, evaluating the morphism c_{γ} at U and taking locally analytic vectors, we get an induced map $c_{\gamma}(U) : \mathrm{H}^0(U, \gamma^* \mathcal{O}_{\mathcal{X}})^{\mathrm{la}} \xrightarrow{\sim} \mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}})^{\mathrm{la}}$. But this is the same as $\mathrm{H}^0(U, \gamma^* \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}) \xrightarrow{\sim} \mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}}^{\mathrm{la}})$ because of the equality $\mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}) = \mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}})^{\mathrm{la}}$. By writing an arbitrary open set as a union of affinoids, we get the desired induced isomorphism $c_{\gamma} : \gamma^* \mathcal{O}_{\mathcal{X}}^{\mathrm{la}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$. $\square$

The preceding discussion then applies equally well to $\mathcal{Y}_{(0,\infty)}$, so we have a sheaf $\mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{la}}$ of locally analytic functions on $\mathcal{Y}_{(0,\infty)}$ endowed with isomorphisms c_{γ}. Since the φ-action on $\mathcal{Y}_{(0,\infty)}$ commutes with the Γ-action, it preserves the Γ-locally analytic functions, and this gives an isomorphism

$$c_{\varphi} : \varphi^* \mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{la}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{la}}$$

which commutes with the Γ-action as usual.

4B. *A flatness result.* For our application at Section 6 it would be useful to know the inclusion $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}} \subset \mathcal{O}_{\mathcal{X}}$ is flat. We are only able to establish this in the cyclotomic case where $K_{\infty} = K_{\mathrm{cyc}}$, and only for certain open subsets. Nevertheless, this will suffice for our needs.

So in this subsection suppose $K_{\infty} = K_{\mathrm{cyc}}$ and let I be a closed interval of the form $I = [r, s]$ with $r \geq (p-1)/p$. We write $\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}$ for $\widetilde{\boldsymbol{B}}_I(\widehat{K}_{\mathrm{cyc}})$ of Section 3B. Let K_0' be the maximal unramified extension of $\mathbb{Q}_p$ contained in K_{cyc}. Then we write $\boldsymbol{B}_{I,\mathrm{cyc},K}$ for the ring of power series $f(T) = \sum_{k \in \mathbb{Z}} a_k T^k$ with $a_k \in K_0'$, such that $f(T)$ converges on the nonempty annulus where $|T| \in I$. By a classical result, $\boldsymbol{B}_{I,\mathrm{cyc},K}$ is a principal ideal domain [Lazard 1962, corollaire à proposition 4]. There is an embedding $\boldsymbol{B}_{I,\mathrm{cyc},K} \hookrightarrow \widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}$ for which $\boldsymbol{B}_{I,\mathrm{cyc},K}$ is Γ_{cyc}-stable. If K is unramified over $\mathbb{Q}_p$, this embedding can be described as follows: the variable T is mapped to $[\varepsilon] - 1$, where $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \in \widehat{K}_{\mathrm{cyc}}^{\flat}$. Further, one calculates that $\gamma(T) = (1 + T)^{\chi_{\mathrm{cyc}}(\gamma)} - 1$, so $\boldsymbol{B}_{I,\mathrm{cyc},K}$ is indeed stable under the action of Γ_{cyc}.

Proposition 4.2. *Suppose $I = [r, (p-1)p^{k-1}]$ with $k \geq 1$. Then*

(i) $\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}^{\mathrm{la}} = \bigcup_{n \geq 0} \varphi^{-n}(\boldsymbol{B}_{p^n I,\mathrm{cyc},K})$,

(ii) $\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}^{\mathrm{la}}$ *is a Prüfer domain,*

(iii) *the natural ring morphism $\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}^{\mathrm{la}} \to \widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}$ is flat.*

Proof. Part (i) is [Berger 2016, Theorem 4.4 (2)]. Note that in [loc. cit.] this is stated only for I of the form $[(p-1)p^{l-1}, (p-1)p^{k-1}]$, but the argument given there (see also Section 13 of [Berger 2021]) is valid for any interval of the form $[r, (p-1)p^{k-1}]$. Part (ii) follows, because each $\boldsymbol{B}_{p^n I,\mathrm{cyc}}$ is a principal ideal domain, and an increasing union of such rings is a Prüfer domain. Finally, the ring $\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}$ is a domain and hence torsionfree over the subring $\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}^{\mathrm{la}}$. Part (iii) is established by recalling that a torsionfree module over a Prüfer domain is flat [Lam 1999, Proposition 4.20]. $\square$

Question 4.3. *To what extent do* (ii) *and* (iii) *of Proposition 4.2 hold for coordinate rings of general open subsets in* $\mathcal{X}$ *and general* K_∞? *We do not expect* $\widetilde{\boldsymbol{B}}_I^{\mathrm{la}}$ *to be a Prüfer domain when* Γ *has dimension larger than* 1. *Nevertheless, it might still be the case that* $\widetilde{\boldsymbol{B}}_I^{\mathrm{la}} \to \widetilde{\boldsymbol{B}}_I$ *is flat.*

4C. *Locally analytic vector bundles.*

Definition 4.4. A locally analytic vector bundle on $\mathcal{X}$ is a locally finite free $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$-module $\mathcal{E}$ on $\mathcal{X}$ equipped with an isomorphism $c_\gamma : \gamma^* \mathcal{E} \xrightarrow{\sim} \mathcal{E}$ for each $\gamma \in \Gamma$ such that the cocycle condition $c_{\gamma_2} \circ \gamma_2^* c_{\gamma_1} = c_{\gamma_1 \gamma_2}$ holds for every $\gamma_1, \gamma_2, \in \Gamma$. We require the action to be continuous with respect to the locally analytic topology.

Example 4.5. (1) Let $\tilde{\mathcal{E}}$ be a Γ-vector bundle on $\mathcal{X}$. Define a sheaf $\tilde{\mathcal{E}}^{\mathrm{la}}$ by generalizing the definition of $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$. Namely, for every open affinoid $U \subset \mathcal{X}$ choose $\Gamma' \leq \Gamma$ stabilizing U. Then $\mathrm{H}^0(U, \tilde{\mathcal{E}})$ is a Banach Γ'-ring and it makes sense to speak of $\mathrm{H}^0(U, \tilde{\mathcal{E}})^{\mathrm{la}}$, which does not depend on the choice of Γ'. Glue these together to form a sheaf $\tilde{\mathcal{E}}^{\mathrm{la}}$. The sheaf $\tilde{\mathcal{E}}^{\mathrm{la}}$ is an $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$ -module with a Γ-action. We shall show in Section 6 that $\tilde{\mathcal{E}}^{\mathrm{la}}$ is locally free and therefore an example of a locally analytic vector bundle.

(2) Conversely, if $\mathcal{E}$ is a locally analytic vector bundle, we can associate to it a Γ-vector bundle $\tilde{\mathcal{E}} = \mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \mathcal{E}$. If $U \subset \mathcal{X}$ is an open affinoid such that $\mathcal{E}|_U$ is free, it follows from Proposition 2.1 that

$$\mathrm{H}^0(U, \mathcal{E}) = \mathrm{H}^0(U, \tilde{\mathcal{E}})^{\mathrm{la}},$$

and so $\mathcal{E} = \tilde{\mathcal{E}}^{\mathrm{la}}$. This shows that the functor from Γ-vector bundles to locally analytic vector bundles mapping $\tilde{\mathcal{E}}$ to $\tilde{\mathcal{E}}^{\mathrm{la}}$ is essentially surjective.

It follows from Example 4.5(2) that if $\mathcal{E}$ is a locally analytic vector bundle, we have an action by derivations

$$\mathrm{Lie}(\Gamma) \times \mathcal{E} \to \mathcal{E},$$

or, what amounts to the same, a connection

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathbb{Q}_p} (\mathrm{Lie}\,\Gamma)^\vee$$

satisfying the identity

$$\nabla(fx) = \nabla(f)x + f\nabla(x)$$

for local sections f of $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$ and x of $\mathcal{E}$.

Remark 4.6. We emphasize that if $U \subset \mathcal{X}$ is an arbitrary open subset then we have an induced action of $\mathrm{Lie}(\Gamma)$ on $\mathrm{H}^0(U, \mathcal{E})$. This is unlike the Γ-action, which only maps $\mathrm{H}^0(U, \mathcal{E})$ to itself if U is Γ-stable. This is one pleasant aspect of working with locally analytic vector bundles instead of Γ-vector bundles.

Finally, we have the following propositions computing sections of interest. They will not be used elsewhere in the article. We may define a locally analytic φ-vector bundle on $\mathcal{Y}_{(0,\infty)}$ by imitating Definition 4.4. Then given a (φ, Γ)-vector bundle $\widetilde{\mathcal{M}}$ on $\mathcal{Y}_{(0,\infty)}$, one can define a locally analytic φ-vector bundle $\widetilde{\mathcal{M}}^{\mathrm{la}}$ on $\mathcal{Y}_{(0,\infty)}$ as in Example 4.5.

Proposition 4.7. *Let $\tilde{\mathcal{E}}$ (resp. $\widetilde{\mathcal{M}}$) be a Γ-vector bundle on $\mathcal{X}$ (resp. a (φ, Γ)-vector bundle on $\mathcal{Y}_{(0,\infty)}$) and let $\tilde{\mathcal{E}}^{\mathrm{la}}$ (resp. $\widetilde{\mathcal{M}}^{\mathrm{la}}$) be its associated locally analytic vector bundle (resp. locally analytic φ-vector bundle). There are natural isomorphisms*:

(i) $\mathrm{H}^0(\mathcal{Y}_I, \widetilde{\mathcal{M}}^{\mathrm{la}}) \cong \mathrm{H}^0(\mathcal{Y}_I, \widetilde{\mathcal{M}})^{\mathrm{la}}$ *for I a closed interval.*

(ii) $\mathrm{H}^0(\mathcal{Y}_I, \widetilde{\mathcal{M}}^{\mathrm{la}}) \cong \mathrm{H}^0(\mathcal{Y}_I, \widetilde{\mathcal{M}})^{\mathrm{pa}}$ *for I an open interval.*

(iii) $\mathrm{H}^0(\mathcal{X}_I, \tilde{\mathcal{E}}^{\mathrm{la}}) \cong \mathrm{H}^0(\mathcal{X}_I, \tilde{\mathcal{E}})^{\mathrm{la}}$ *for I a closed interval with $|\log(I)| < \log(p)$.*

(iv) $\mathrm{H}^0(\mathcal{X}_I, \tilde{\mathcal{E}}^{\mathrm{la}}) \cong \mathrm{H}^0(\mathcal{X}_I, \tilde{\mathcal{E}})^{\mathrm{pa}}$ *for I an open interval with $|\log(I)| < \log(p)$.*

(v) $\mathrm{H}^0(\mathcal{X}, \tilde{\mathcal{E}}^{\mathrm{la}}) \cong \mathrm{H}^0(\mathcal{X}, \tilde{\mathcal{E}})^{\mathrm{la}}$.

(vi) $\mathrm{H}^0(\mathcal{X} - x_\infty, \tilde{\mathcal{E}}^{\mathrm{la}}) \cong \mathrm{H}^0(\mathcal{X} - x_\infty, \tilde{\mathcal{E}})^{\mathrm{pa}}$.

Proof. Parts (i) and (iii) are immediate from the definition. For (ii) and (iv), use the coverings $\mathcal{Y}_I = \bigcup_{J \subset I} \mathcal{Y}_J$ and $\mathcal{X}_I = \bigcup_{J \subset I} \mathcal{X}_J$ ranging over $J \subset I$ closed. For (v), consider the covering

$$\mathcal{X} = \mathcal{X}_{[1, \sqrt{p}]} \cup \mathcal{X}_{[\sqrt{p}, p]}$$

with intersection $\mathcal{X}_{[\sqrt{p}, \sqrt{p}]} \sqcup \mathcal{X}_{[1,1]}$ (identifying 1 with p via φ). This yields exact sequences

$$0 \to \mathrm{H}^0(\mathcal{X}, \tilde{\mathcal{E}}^{\mathrm{la}}) \to \mathrm{H}^0(\mathcal{X}_{[1, \sqrt{p}]}, \tilde{\mathcal{E}}^{\mathrm{la}}) \oplus \mathrm{H}^0(\mathcal{X}_{[\sqrt{p}, p]}, \tilde{\mathcal{E}}^{\mathrm{la}}) \to \mathrm{H}^0(\mathcal{X}_{[\sqrt{p}, \sqrt{p}]}, \tilde{\mathcal{E}}^{\mathrm{la}}) \oplus \mathrm{H}^0(\mathcal{X}_{[1,1]}, \tilde{\mathcal{E}}^{\mathrm{la}})$$

and

$$0 \to \mathrm{H}^0(\mathcal{X}, \tilde{\mathcal{E}})^{\mathrm{la}} \to \mathrm{H}^0(\mathcal{X}_{[1, \sqrt{p}]}, \tilde{\mathcal{E}})^{\mathrm{la}} \oplus \mathrm{H}^0(\mathcal{X}_{[\sqrt{p}, p]}, \tilde{\mathcal{E}})^{\mathrm{la}} \to \mathrm{H}^0(\mathcal{X}_{[\sqrt{p}, \sqrt{p}]}, \tilde{\mathcal{E}})^{\mathrm{la}} \oplus \mathrm{H}^0(\mathcal{X}_{[1,1]}, \tilde{\mathcal{E}})^{\mathrm{la}}.$$

By virtue of (iii) the kernels of these sequences are identified. This proves part (v).

For (vi), use the covering

$$\mathcal{X} - x_\infty = \mathcal{X}_{[1, \sqrt{p}]} \cup (\mathcal{X}_{[\sqrt{p}, p]} - x_\infty)$$

with intersection $\mathcal{X}_{[\sqrt{p}, \sqrt{p}]} \sqcup \mathcal{X}_{[1,1]}$. We may write $\mathcal{X}_{[\sqrt{p}, p]} - x_\infty$ as a union of Γ-stable rational open subsets

$$\mathcal{X}_{[\sqrt{p}, p]} - \infty = \cup_{n \geq 1} \mathcal{X}_{[\sqrt{p}, p]}\{|\xi| \geq p^{-n}\}.$$

Thus

$$\mathrm{H}^0(\mathcal{X}_{[\sqrt{p}, p]} - x_\infty, \tilde{\mathcal{E}}^{\mathrm{la}}) \cong \mathrm{H}^0(\mathcal{X}_{[\sqrt{p}, p]} - x_\infty, \tilde{\mathcal{E}})^{\mathrm{pa}}.$$

Repeating the argument which proved part (v), we conclude. $\square$

We place ourselves in the cyclotomic setting so that $\Gamma = \Gamma_{\mathrm{cyc}}$ and $H = \mathrm{Gal}(\bar{K}/K_{\mathrm{cyc}})$, and we write $\boldsymbol{B}_{\mathrm{cris}}^+(\widehat{K}_{\mathrm{cyc}}) = (\boldsymbol{B}_{\mathrm{cris}}^+)^H$. Following Section 10.2 of [Fargues and Fontaine 2018], for $n \in \mathbb{Z}$ take $\tilde{\mathcal{E}} = \mathcal{O}_{\mathcal{X}}(n)$ to be the Γ-line bundle corresponding to the graded module

$$\bigoplus_{m \geq 0} \boldsymbol{B}_{\mathrm{cris}}^+(\widehat{K}_{\mathrm{cyc}})^{\varphi = p^{m+n}}.$$

Proposition 4.8. $\mathrm{H}^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(n)^{\mathrm{la}}) = \begin{cases} 0, & n < 0, \\ \mathbb{Q}_p(n), & n \geq 0. \end{cases}$

Proof. To show this, notice first that

$$\mathrm{H}^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(n)) = \boldsymbol{B}_{\mathrm{cris}}^+(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^n} = \begin{cases} 0, & n < 0, \\ \mathbb{Q}_p, & n = 0, \\ \boldsymbol{B}_{\mathrm{cris}}^+(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^n}, & n > 0. \end{cases}$$

If $n > 0$ then by [Fargues and Fontaine 2018, 6.4.2] there is an exact sequence

$$0 \to \mathbb{Q}_p(n) \to \boldsymbol{B}_{\mathrm{cris}}^{+,\varphi=p^n} \to \boldsymbol{B}_{\mathrm{dR}}^+/t^n\boldsymbol{B}_{\mathrm{dR}}^+ \to 0.$$

Take H-invariants and locally analytic vectors. By [Berger and Colmez 2016, théorème 4.11] we know that $(\boldsymbol{B}_{\mathrm{dR}}^+/t^n\boldsymbol{B}_{\mathrm{dR}}^+)^{H,\mathrm{la}} = K_{\mathrm{cyc}}[\![t]\!]/t^n$, so we are left with an exact sequence

$$0 \to \mathbb{Q}_p(n) \to \boldsymbol{B}_{\mathrm{cris}}^+(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^n,\mathrm{la}} \to K_{\mathrm{cyc}}[\![t]\!]/t^n.$$

Claim. $$\boldsymbol{B}_{\mathrm{cris}}^+(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^n,\mathrm{la}} = \mathbb{Q}_p(n).$$

Note that a similar statement appears in Section 3.3 of [Berger and Colmez 2016] in the case $n = 1$. Given the claim the computation is finished because part (v) of Proposition 4.7 implies that

$$\mathrm{H}^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(n)^{\mathrm{la}}) = \boldsymbol{B}_{\mathrm{cris}}^+(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^n,\mathrm{la}} = \begin{cases} 0, & n < 0, \\ \mathbb{Q}_p(n), & n \geq 0. \end{cases}$$

To show the claim, take $x \in \boldsymbol{B}_{\mathrm{cris}}^+(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^n,\mathrm{la}}$. Its image in $K_{\mathrm{cyc}}[\![t]\!]/t^n$ is killed by the polynomial

$$P_n(\gamma) := \prod_{i=0}^{n-1} (\chi_{\mathrm{cyc}}(\gamma)^{-i}\gamma - 1)$$

for γ which generates an open subgroup of Γ. It follows that $P_n(\gamma)(x) \in \mathbb{Q}_p(n)$ for this γ. Since $P_n(\gamma)$ acts on $\mathbb{Q}_p(n)$ by a nonzero element we reduce to showing that $\boldsymbol{B}_{\mathrm{cris}}^+(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^n,P_n(\gamma)=0}$ is 0. In fact, if K' is the subfield of K_{cyc} corresponding to $\gamma^{\mathbb{Z}_p} \subset \Gamma$ with maximal unramified subextension K_0', we shall compute that

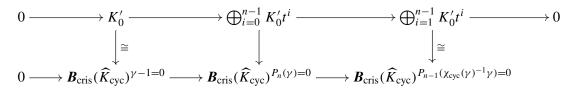$$\boldsymbol{B}_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{P_n(\gamma)=0} = \bigoplus_{i=0}^{n-1} K_0't^i,$$

and in particular there are no nonzero elements with $\varphi = p^n$.

To show this latter description of the elements killed by $P_n(\gamma)$, we argue by induction. If $n = 1$ then $P_n(\gamma) = \gamma - 1$ and the equality follows from the usual description of the Galois invariants of $\boldsymbol{B}_{\mathrm{cris}}$. For $n \geq 2$, we have $P_n(\gamma)/(\gamma - 1) = P_{n-1}(\chi_{\mathrm{cyc}}(\gamma)^{-1}\gamma)$ and

$$\boldsymbol{B}_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{P_{n-1}(\chi_{\mathrm{cyc}}(\gamma)^{-1}\gamma)=0} = t\,\boldsymbol{B}_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{P_{n-1}(\gamma)=0} = \bigoplus_{i=1}^{n-1} K_0't^i.$$

Thus there is a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K_0' & \longrightarrow & \bigoplus_{i=0}^{n-1} K_0' t^i & \longrightarrow & \bigoplus_{i=1}^{n-1} K_0' t^i & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle\cong} & & \downarrow & & \downarrow{\scriptstyle\cong} & & \\
0 & \longrightarrow & \boldsymbol{B}_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{\gamma-1=0} & \longrightarrow & \boldsymbol{B}_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{P_n(\gamma)=0} & \longrightarrow & \boldsymbol{B}_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{P_{n-1}(\chi_{\mathrm{cyc}}(\gamma)^{-1}\gamma)=0} & &
\end{array}
$$

whose rows are exact and whose outer vertical maps are isomorphisms. We conclude by the applying the five lemma. $\qquad\square$

Remark 4.9. Set $\boldsymbol{B}_e(\widehat{K}_{\infty}) = \boldsymbol{B}_e^H$ for the usual ring $\boldsymbol{B}_e = \boldsymbol{B}_{\mathrm{cris}}^{\varphi=1}$, so that $\boldsymbol{B}_e \subset \mathrm{H}^0(\mathcal{X} - x_{\infty}, \mathcal{O}_{\mathcal{X}})$. This inclusion is not an equality: the ring $\boldsymbol{B}_e$ allows only meromorphic functions at x_{∞} while in $\mathrm{H}^0(\mathcal{X} - x_{\infty}, \mathcal{O}_{\mathcal{X}})$ there will be functions with essential singularities. The subring $\boldsymbol{B}_e(\widehat{K}_{\infty})^{\mathrm{pa}} \subset \mathrm{H}^0(\mathcal{X} - x_{\infty}, \mathcal{O}_{\mathcal{X}})^{\mathrm{la}}$ is more tractable and we can understand its structure to an extent. In particular, let us consider the subring $\boldsymbol{B}_e(\widehat{K}_{\infty})^{\mathrm{pa}} = \boldsymbol{B}_e \cap \mathrm{H}^0(\mathcal{X} - x_{\infty}, \mathcal{O}_{\mathcal{X}}^{\mathrm{la}})$ in the case $\Gamma = \Gamma_{\mathrm{cyc}}$. We claim that in fact $\boldsymbol{B}_e(\widehat{K}_{\infty})^{\mathrm{pa}} = \mathbb{Q}_p$. To see this, take $x \in \boldsymbol{B}_e(\widehat{K}_{\infty})^{\mathrm{pa}}$, and restrict it to $\mathcal{X}_{[\sqrt{p}, p]} - x_{\infty}$. Since $\mathcal{Y}_{[\sqrt{p}, p]}$ maps isomorphically onto $\mathcal{X}_{[\sqrt{p}, p]}$, the element t gives an element of $\mathrm{H}^0(\mathcal{X}_{[\sqrt{p}, p]} - x_{\infty}, \mathcal{O}_{\mathcal{X}}^{\mathrm{la}})$. Multiplying by a bounded power of t, the function $t^n x$ extends to an element of

$$
\mathrm{H}^0(\mathcal{X}_{[\sqrt{p}, p]}, \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}) = \mathrm{H}^0(\mathcal{X}_{[\sqrt{p}, p]}, \mathcal{O}_{\mathcal{X}})^{\mathrm{la}},
$$

which shows that x itself is actually an element of $\boldsymbol{B}_e(\widehat{K}_{\infty})^{\mathrm{la}}$, with a pole of order n at x_{∞}. Therefore, $t^n x \in \mathrm{H}^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(n)^{\mathrm{la}})$ which is equal to $\mathbb{Q}_p(n)$ as was shown in Proposition 4.8. This means x is in $\mathbb{Q}_p$ and so $\boldsymbol{B}_e(\widehat{K}_{\infty})^{\mathrm{pa}} = \mathbb{Q}_p$.

Question 4.10. (1) *Is it true that* $\mathrm{H}^0(\mathcal{X} - x_{\infty}, \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}) = \mathbb{Q}_p$ *if* $\Gamma \neq \Gamma_{\mathrm{cyc}}$ *and* $\dim \Gamma = 1$?

(2) *If* $\dim \Gamma > 1$ *then one can sometimes produce elements in* $\boldsymbol{B}_e(\widehat{K}_{\infty})^{\mathrm{la}}$ *which do not belong to* $\mathbb{Q}_p$. *For example, in the Lubin–Tate setting, the element* $(t_{-\sqrt{p}}/t_{\sqrt{p}})^2$ *lies in* $\boldsymbol{B}_e(\widehat{K}_{\infty})^{\mathrm{la}}$, *for* $t_{\pm\sqrt{p}}$ *being the analogue of Fontaine's element attached to the uniformizer* $\pi = \pm\sqrt{p}$ *(see Section 8.3 of* [Colmez 2002] *for the notation appearing here). Is it true that in some generality* $\boldsymbol{B}_e(\widehat{K}_{\infty})^{\mathrm{la}}$ *will be* $d-1$ *dimensional for* $d = \dim \Gamma$? *See* [Berger and Colmez 2016, théoréme 6.1] *for a related statement.*

5. Acyclicity of locally analytic vectors for semilinear representations

In this section, we shall prove vanishing the of $\mathrm{R}_{\mathrm{la}}^i$-groups for certain semilinear representations. These results will be used to prove the descent result in Section 6 but are also of independent interest. We follow the strategy of [Pan 2022a], where the case of a trivial representation and a particular family of algebras $\widetilde{\Lambda}$ is treated.

5A. *Statement of the results.* To state the main result of this section, we recall the Tate–Sen axioms of [Berger and Colmez 2008, 3]. Let G be a profinite group and let $\widetilde{\Lambda}$ be a G-Banach ring endowed with a valuation val for which the G action is continuous and unitary. We suppose there is a character

$\chi : G \to \mathbb{Z}_p^\times$ with open image and let $H = \ker \chi$. Given an open normal subgroup $G_0 \subset G$ we let $H_0 = G_0 \cap H$ and $\Gamma_{H_0} = G/H_0$.

The Tate–Sen axioms are the following.

(TS1) There exists $c_1 > 0$ such that for any open subgroup $H_1 \subset H_2$ of H_0 there exists $\alpha \in \widetilde{\Lambda}^{H_1}$ with $\mathrm{val}(\alpha) > -c_1$ and $\sum_{\tau \in H_2/H_1} \tau(\alpha) = 1$.

(TS2) There exists $c_2 > 0$ and for each H_0 open in H an integer $n(H_0)$ depending on H_0 such that for $n \geq n(H_0)$, we have the extra data of

  - closed subalgebras $\Lambda_{H_0,n} \subset \widetilde{\Lambda}^{H_0}$, and
  - trace maps $\mathrm{R}_{H_0,n} : \widetilde{\Lambda}^{H_0} \to \Lambda_{H_0,n}$

satisfying:

  (1) For $H_1 \subset H_2$ we have $\Lambda_{H_2,n} \subset \Lambda_{H_1,n}$ and $\mathrm{R}_{H_1,n}|_{\Lambda_{H_2,n}} = \mathrm{R}_{H_2,n}$.

  (2) $\mathrm{R}_{H_0,n}$ is $\Lambda_{H_0,n}$-linear and $\mathrm{R}_{H_0,n}(x) = x$ for $x \in \Lambda_{H_0,n}$.

  (3) $g(\Lambda_{H_0,n}) = \Lambda_{gH_0g^{-1},n}$ and $g(\mathrm{R}_{H_0,n}(x)) = \mathrm{R}_{gH_0g^{-1},n}(gx)$ if $g \in G$.

  (4) $\lim_{n\to\infty} \mathrm{R}_{H_0,n}(x) = x$ for $x \in \widetilde{\Lambda}^{H_0}$.

  (5) If $n \geq n(H_0)$ and $x \in \widetilde{\Lambda}^{H_0}$ then $\mathrm{val}(\mathrm{R}_{H_0,n}(x)) \geq \mathrm{val}(x) - c_2$.

(TS3) There exists $c_3 > 0$ and for each open normal subgroup G_0 of G an integer $n(G_0) \geq n(H_0)$ such that if $n \geq n(G_0)$ and $\gamma \in \Gamma_{H_0}$ has $n(\gamma) = \mathrm{val}_p(\chi(\gamma) - 1) \leq n$, then $\gamma - 1$ acts invertibly on $X_{H_0,n} = (1 - \mathrm{R}_{H_0,n})(\widetilde{\Lambda}^{H_0})$ and $\mathrm{val}((\gamma - 1)^{-1}(x)) \geq \mathrm{val}(x) - c_3$.

We introduce an additional possible axiom which does not appear in [Berger and Colmez 2008].

(TS4) For any sufficiently small open normal $G_0 \subset G$ with $H_0 = G_0 \cap H$ and for any $n \geq n(G_0)$, there exists a positive real number $t = t(H_0, n) > 0$ such that if $\gamma \in G_0/H_0$ and $x \in \Lambda_{H_0,n}$ then

$$\mathrm{val}((\gamma - 1)(x)) \geq \mathrm{val}(x) + t.$$

We then have the following result.

Theorem 5.1. *Let M be a finite free $\widetilde{\Lambda}$-semilinear representation of G. Suppose there exists an open subgroup $G_0 \subset G$, a G-stable $\widetilde{\Lambda}^+$-lattice $M^+ \subset M$ and an integer $k > c_1 + 2c_2 + 2c_3$ such that in some basis of M^+, we have $\mathrm{Mat}(g) \in 1 + p^k \mathrm{Mat}_d(\widetilde{\Lambda}^+)$ for every $g \in G_0$. Then:*

(i) *If* (TS1)–(TS3) *are satisfied then for $i \geq 2$*

$$\mathrm{R}_{G\text{-la}}^i(M) = 0.$$

In fact, $\mathrm{R}_{G_0\text{-an}}^i(M) = 0$ *for any sufficiently small open subgroup $G_0 \subset G$.*

(ii) *If in addition* (TS4) *is satisfied then*

$$\mathrm{R}_{G\text{-la}}^1(M) = 0.$$

In fact, for every sufficiently small open subgroup G_0 there is an open subgroup $G_1 \subset G_0$ such that the map $R^1_{G_0\text{-an}}(M) \to R^1_{G_1\text{-an}}(M)$ is 0.

(iii) *In particular, if* (TS1)–(TS4) *are satisfied then M has no higher locally analytic vectors.*

Remark 5.2. The following was pointed out by the anonymous referee: if the action of G_0 on $\widetilde{\Lambda}$ was locally analytic, then the hypothesis of the existence of M^+ such that G_0 acts trivially mod p^k on it would imply that the action of G_0 on M is locally analytic as well, as it can be deduced from Proposition 2.1 and Lemma 2.2. So the nonlocally analyticity comes only from the coefficients $\widetilde{\Lambda}$.

The following special case is often useful in applications.

Proposition 5.3. *If G and $\widetilde{\Lambda}$ satisfy* (TS1)–(TS4) *and if in addition the topology on $\widetilde{\Lambda}$ is p-adic, and if M is a finite free $\widetilde{\Lambda}$-semilinear representation of G, then the higher locally analytic vectors $R^i_{\text{la}}(M)$ vanish for $i \geq 1$.*

Proof. We shall explain how this follows from Theorem 5.1. Indeed, we claim that any finite free $\widetilde{\Lambda}$-semilinear representation of G satisfies the assumptions of the Theorem 5.1 after possibly replacing G by a smaller open subgroup G'. This suffices because, by Lemma 2.4, higher locally analytic vectors do not change when we replace G by G'.

To see why such a G' exists, suppose M is a finite free $\widetilde{\Lambda}$-semilinear representation of G and choose any $\widetilde{\Lambda}$-basis $e_1, \ldots, e_d$ of M. If we take $M^+ = \bigoplus_{i=1}^d \widetilde{\Lambda}^+ e_i$ then M^+ is a lattice of M, and by continuity we may find an open subgroup $G' \subset G$ so that $\text{Mat}(g) \in \text{GL}_d(\widetilde{\Lambda}^+)$ for $g \in G'$. This implies that M^+ is G'-stable. Since the topology on $\widetilde{\Lambda}$ is p-adic, we can find an open subgroup $G'_0 \subset G'$ such that $\text{Mat}(g) \in 1 + p^k \text{Mat}_d(\widetilde{\Lambda}^+)$ for every $g \in G'_0$. Thus, the assumptions of Theorem 5.1 hold for this M^+, G' and G'_0. $\qquad\square$

Before giving the proof of Theorem 5.1, we record a few applications.

Corollary 5.4. *Suppose G and $\widetilde{\Lambda}$ satisfy* (TS1)–(TS4) *and let M be as in the statement of the theorem. Then for all $i \geq 0$,*
$$H^i(G, M) \cong H^i(G, M^{\text{la}}) \cong H^i(\text{Lie } G, M^{\text{la}})^G.$$

Proof. Apply [Rodrigues Jacinto and Rodríguez Camargo 2022, Corollary 1.6 and Theorem 1.7]. $\qquad\square$

Two main cases of interest are the following. To state them, we set up some notation first. Let F be an infinitely ramified algebraic extension of K which contains an unramified twist of the cyclotomic extension, i.e., the field extension of K cut out by $\eta\chi_{\text{cyc}}$ for η an unramified character. Suppose also that $\text{Gal}(F/K)$ is a p-adic Lie group. For why we allow an unramified twist of the cyclotomic extension on what follows, see Section 8 of [Berger 2016].

Example 5.5. (1) Take $G = \text{Gal}(F/K)$ and $\widetilde{\Lambda} = \widehat{F}$. Then G and $\widetilde{\Lambda}$ satisfy the axioms (TS1)–(TS3) for arbitrary $c_1 > 0$, $c_2 > 0$ and $c_3 > 1/(p-1)$. See [Berger and Colmez 2008, Proposition 4.1.1] for the case $F = \overline{K}$, which goes back to Tate. For general F the same proof works.

In addition, we claim that G and $\widetilde{\Lambda}$ satisfy the axiom (TS4). Indeed, if G_0 is an open subgroup of G corresponding to a finite extension L of K, then $\Lambda_{H_0,n} = L(\zeta_{p^n})$ and $G_0/H_0 = \mathrm{Gal}(L_{\mathrm{cyc}}/L)$. We take G_0 sufficiently small so that L contains ζ_p. Let $\pi = \zeta_{p^n} - 1$ be the uniformizer of L. For $\gamma \in \mathrm{Gal}(L_{\mathrm{cyc}}/L)$, we have

$$\mathrm{val}((\gamma-1)(\pi)) = \mathrm{val}(\zeta_{p^n}^{\gamma-1} - 1) = \frac{1}{(p-1)p^{n-2}}.$$

Using the identity $(\gamma-1)(ab) = (\gamma-1)(a)b + \gamma(a)(\gamma-1)(b)$, one then shows by induction that

$$\mathrm{val}((\gamma-1)(\pi^m)) \geq \mathrm{val}(\pi^m) + \frac{1}{p^{n-2}}.$$

If x is any element of $\Lambda_{H_0,n} = L(\zeta_{p^n})$, we may write $x = p^k \pi^m y$ with $k \in \mathbb{Z}$, $m \geq 1$ and $0 \leq \mathrm{val}(y) < \mathrm{val}(\pi)$. Since $\mathcal{O}_L[\zeta_{p^n}] = \mathcal{O}_L[\pi]$, we see by writing y as a polynomial in π that

$$\mathrm{val}(\gamma-1)(y) \geq \mathrm{val}(\pi) + \frac{1}{p^{n-2}}.$$

Using the identity for $\gamma - 1$, we have

$$\mathrm{val}(\gamma-1)(x) \geq k + \min\big(\mathrm{val}((\gamma-1)(\pi^m)y), \mathrm{val}(\pi^m(\gamma-1)(y))\big)$$
$$\geq k + \min\Big(\mathrm{val}(\pi^m) + \mathrm{val}(y) + \frac{1}{p^{n-2}}, \mathrm{val}(\pi^m) + \mathrm{val}(\pi) + \frac{1}{p^{n-2}}\Big)$$
$$\geq \mathrm{val}(x) + \frac{1}{p^{n-2}},$$

so (TS4) holds with $t = 1/p^{n-2}$.

(2) Take $G = \mathrm{Gal}(F/K)$ and for a closed interval $I \subset (p/p - 1, \infty)$ let $\widetilde{\Lambda} = \widetilde{\boldsymbol{B}}_I(\widehat{F})$. Then again G and $\widetilde{\Lambda}$ satisfy the axioms (TS1)–(TS4) for arbitrary $c_1 > 0$, $c_2 > 0$ and $c_3 > 1/(p-1)$. Here if $G_0 \subset G$ is an open subgroup corresponding a finite extension L of K then one takes $\Lambda_{H_0,n} = \varphi^{-n}(\boldsymbol{B}_{p^n I, \mathrm{cyc}, L})$ with notation as in Section 4B. For (TS1)–(TS3), see [Berger 2008a, Proposition 1.1.12]. Axiom (TS4) follows from [Colmez 2008, Corollary 9.5].

Corollary 5.6. (i) *If M is a finite free $\widehat{F}$-semilinear representation of $\mathrm{Gal}(F/K)$ then $\mathrm{R}_{\mathrm{la}}^i(M) = 0$ for $i \geq 1$.*

(ii) *If $I \subset (p/p - 1, \infty)$ is a closed interval and M is a finite free $\widetilde{\boldsymbol{B}}_I(\widehat{F})$-semilinear representation of $\mathrm{Gal}(F/K)$ then $\mathrm{R}_{\mathrm{la}}^i(M) = 0$ for $i \geq 1$.*

Proof. In both of these cases the topology on $\widetilde{\Lambda}$ is p-adic, so the theorem applies by Proposition 5.3. $\square$

Remark 5.7. Suppose F/K is any infinitely ramified p-adic Lie extension of K (not necessarily containing an unramified twist of the cyclotomic extension), and let M be a finite free $\widehat{F}$-semilinear representation of $\mathrm{Gal}(F/K)$. Then $\mathrm{R}_{\mathrm{la}}^i(M) = 0$ for $i \geq 1$. To prove this, one is always allowed to replace K by a finite extension. Then the extension FK_{cyc}/F can be assumed to be either trivial or infinite. In the first case, the group $\mathrm{R}_{\mathrm{la}}^i(M)$ vanishes by the corollary. In the second case, one can argue as in the proof of [Pan 2022a, Theorem 3.6.1]. We omit the details since this result will not be used in the article.

The rest of the chapter is devoted to the proof of Theorem 5.1. The proof is inspired by that of [Pan 2022a, Theorem 3.6.1]. The strategy is the following:

(1) In Sections 5B and 5C, we establish some results using (TS1), (TS2) and (TS3) that allow us to descend certain infinite rank $\widetilde{\Lambda}$-semilinear representations of G to $\Lambda_{H_k,n}^+$-semilinear representations of G_0, which are fixed by H_k.

(2) In Section 5D, we apply these results to $\mathcal{C}^{\mathrm{an}}(G, M)$.

(3) Using this and the Hochshild–Serre theorem, we show in Section 5E that $\mathrm{R}_{G\text{-la}}^i(M)$ vanishes when $i \geq 2$, and we give an explicit description for $\mathrm{R}_{G\text{-la}}^1(M)$. It remains to show this latter cohomology group vanishes.

(4) To do this, we decompose $\mathrm{R}_{G\text{-la}}^1(M)$ as a sum of two groups. For the first one, we use an explicit calculation in Section 5F and (TS4) to show its vanishing. For the second one, we show it is zero in Section 5G by using again (TS4) and a computation inspired by Berger and Colmez [2016]. Both of these computations are of a p-adic functional analysis flavor.

5B. *Vanishing of H-cohomology.* If $t \in \mathbb{R}$ we write

$$p^{-t}\widetilde{\Lambda}^+ := \text{elements in } \widetilde{\Lambda} \text{ with val} \geq -t.$$

The first result we shall need for the proof of Theorem 5.1 is the following.

Proposition 5.8. *Suppose that* $(G, H, \widetilde{\Lambda})$ *satisfies* (TS1) *for some* $c_1 > 0$. *If* $H_0 \subset H$ *is an open subgroup, and* $r \geq 1$, *we have*

(i) *The natural map* $\mathrm{H}^r(H_0, \widetilde{\Lambda}^+) \to \mathrm{H}^r(H_0, p^{-2c_1}\widetilde{\Lambda}^+)$ *is* 0.

(ii) *Let* M^+ *be a finite free* $\widetilde{\Lambda}^+$-*semilinear representation of* H_0 *which has an* H_0-*fixed basis. Then the map* $\mathrm{H}^r(H_0, M^+) \to \mathrm{H}^r(H_0, p^{-2c_1}M^+)$ *is* 0.

(iii) *Let* $M^+ = \widehat{\bigcup_{k\in\mathbb{N}} M_k^+}$ *be the completion of an increasing union of finite free* $\widetilde{\Lambda}^+$-*semilinear representation of* H_0, *each having an* H_0-*fixed basis. Then the map* $\mathrm{H}^r(H_0, M^+) \to \mathrm{H}^r(H_0, p^{-2c_1}M^+)$ *is* 0.

In particular, in each of the cases (i)–(iii) *the rational cohomology* $\mathrm{H}^r(H_0, M)$ *is equal to zero.*

Proof. We have (i) $\Rightarrow$ (ii), since continuous cohomology commutes with direct sums.

Next, we prove (ii) $\Rightarrow$ (iii). To do this, observe that if $t \in \mathbb{Z}_{\geq 1}$ then $p^t M_k^+$ also a finite free $\widetilde{\Lambda}^+$-semilinear representation of H_0 which has an H_0-fixed basis. Taking long exact cohomologies of the sequences

$$0 \to p^t\left(\bigcup_{k\in\mathbb{N}} M_k^+\right) \to \left(\bigcup_{k\in\mathbb{N}} M_k^+\right) \to M^+/p^t M^+ \to 0$$

and

$$0 \to p^{t-2c_1}\left(\bigcup_{k\in\mathbb{N}} M_k^+\right) \to p^{-2c_1}\left(\bigcup_{k\in\mathbb{N}} M_k^+\right) \to p^{-2c_1}M^+/p^{t-2c_1}M^+ \to 0,$$

we get from (ii) that the natural map

$$\mathrm{H}^r(H_0, M^+/p^t M^+) \to \mathrm{H}^r(H_0, p^{-2c_1} M^+/p^{t-2c_1} M^+)$$

is 0. Now given a cocycle $\xi \in Z^r(H_0, M^+)$, write ξ_0 for its image in $Z^r(H_0, p^{-2c_1} M^+)$. We wish to show that ξ_0 is a coboundary. Choose some fixed $t_0 \geq 3c_1$. Then by virtue of the observation above, the right vertical map of the commutative diagram

$$
\begin{array}{ccccc}
\mathrm{H}^r(H_0, p^{t_0} M^+) & \longrightarrow & \mathrm{H}^r(H_0, M^+) & \longrightarrow & \mathrm{H}^r(H_0, M^+/p^{t_0} M^+) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^r(H_0, p^{t_0-2c_1} M^+) & \longrightarrow & \mathrm{H}^r(H_0, p^{-2c_1} M^+) & \longrightarrow & \mathrm{H}^r(H_0, p^{-2c_1} M^+/p^{t_0-2c_1} M^+)
\end{array}
$$

is 0, which implies that $\xi_0 = \xi_1 + \delta(m_1)$, where m_1 is an $r-1$ cocycle valued in $p^{-2c_1} M^+$ and ξ_1 is an r-cocycle valued in $p^{t_0-2c_1} M^+ \subset p^{c_1} M^+$. Repeating this argument by induction with M^+ replaced with $p^{ic_1} M^+$, we get that we can write $\xi_i = \xi_{i+1} + \delta(m_{i+1})$, where ξ_i is valued in $p^{ic_1} M^+$ and m_{i+1} is valued in $p^{(i-3)c_1} M^+$. Hence the series $\sum_{i=1}^{\infty} m_i$ converges to an $r-1$ cocycle m valued in $p^{-2c_1} M^+$, and we get $\xi_0 = \delta(m)$, as required.

Finally, we prove (i). This statement is probably well known, but for lack of a suitable reference, we provide a proof here. It is essentially a fiber product of the arguments appearing in [Tate 1967, 3.2, Corollary 1; Colmez 2008, Proposition 10.2].

Let $\xi \in Z^r(H_0, \widetilde{\Lambda}^+)$ be an r-cocycle of H_0 valued in $\widetilde{\Lambda}^+$. By a valuation of a cochain we shall mean the infimum of its valuation on elements. Writing δ for the differential, we shall construct a sequence of $r-1$ cochains $x_n \in C^{r-1}(H_0, p^{-2c_1} \widetilde{\Lambda}^+)$ for $n \geq -1$ such that

(1) $\mathrm{val}(\xi - \delta x_n) \geq nc_1$ for $\sigma \in H_0$, and

(2) $\mathrm{val}(x_n - x_{n-1}) \geq (n-2)c_1$ for $n \geq 0$.

This will suffice, since $x_n \to x$ for some $x \in C^{r-1}(H_0, p^{-2c_1} \widetilde{\Lambda}^+)$ which shows that $\xi = \delta x$ is 0 in $\mathrm{H}^r(H_0, p^{-2c_1} \widetilde{\Lambda}^+)$.

To do this, choose $x_{-1} = 0$, which clearly satisfies the first condition. Suppose x_n has been constructed; we construct x_{n+1}. Let ξ_n be the r-cocycle

$$\xi_n := \xi - \delta x_n,$$

which is valued in $p^{nc_1} \widetilde{\Lambda}^+$. Choose $H_1 \subset H_0$ an open subgroup such that for every $\sigma_1, \ldots, \sigma_r \in H_0$ and $\sigma \in H_1$ we have

$$\mathrm{val}(\xi_n(\sigma_1, \ldots, \sigma_r) - \xi_n(\sigma_1, \ldots, \sigma_r \sigma)) \geq (n+2)c_1.$$

Such a choice is possible by the continuity of ξ_n as well as the compactness of H_0.

Now by the axiom (TS1) there is an element $\alpha \in \widetilde{\Lambda}^{H_1}$ such that $\mathrm{val}(\alpha) > -c_1$ and $\sum_{\tau \in H_0/H_1} \tau(\alpha) = 1$. Let S be a system of representatives for H_0/H_1, and define an $r-1$ cochain

$$x_S(\sigma_1, \ldots, \sigma_{r-1}) = (-1)^r \sum_{\tau \in S} (\sigma_1 \sigma_2 \cdot \ldots \cdot \sigma_{r-1}\tau)(\alpha)\xi_n(\sigma_1, \ldots, \sigma_{r-1}, \tau).$$

Each term in the sum has $\mathrm{val} \geq (n-1)c_1$, so $\mathrm{val}(x_S) \geq (n-1)c_1$. In particular, $x_S \in C^{r-1}(H_0, p^{-2c_1}\widetilde{\Lambda}^+)$.

We now compute $(\xi_n - \delta x_S)(\sigma_1, \ldots, \sigma_r)$. We have by definition of δ an equation

$$\delta x_S(\sigma_1, \ldots, \sigma_r) = (-1)^r \sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_r \tau)(\alpha)\sigma_1(\xi_n(\sigma_2, \ldots, \sigma_r, \tau))$$
$$+ \sum_{j=1}^{r-1}(-1)^{j+r} \sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_r \tau)(\alpha)\xi_n(\sigma_1, \ldots, \sigma_j\sigma_{j+1}, \ldots, \sigma_r, \tau)$$
$$+ \sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_{r-1}\tau)(\alpha)\xi_n(\sigma_1, \ldots, \sigma_{r-1}, \tau). \quad (5\text{-}1)$$

On the other hand, ξ_n is an r-cocycle, so that $\delta\xi_n(\sigma_1, \ldots, \sigma_r, \tau) = 0$ for every $\sigma_1, \ldots, \sigma_r$ and τ. Multiplying by $(-1)^r(\sigma_1 \cdot \ldots \cdot \sigma_r\tau)(\alpha)$ and summing over $\tau \in S$, we get the equation

$$0 = (-1)^r \sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_r\tau)(\alpha)\sigma_1(\xi_n(\sigma_2, \ldots, \sigma_r, \tau))$$
$$+ \sum_{j=1}^{r-1}(-1)^{j+r} \sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_r\tau)(\alpha)\xi_n(\sigma_1, \ldots, \sigma_j\sigma_{j+1}, \ldots, \sigma_r, \tau)$$
$$+ \sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_r\tau)(\alpha)\xi_n(\sigma_1, \ldots, \sigma_{r-1}, \sigma_r\tau) - \sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_r\tau)(\alpha)\xi_n(\sigma_1, \ldots, \sigma_r). \quad (5\text{-}2)$$

Subtracting (5-2) from (5-1), we get

$$\delta x_S(\sigma_1, \ldots, \sigma_r) = \sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_{r-1}\tau)(\alpha)\xi_n(\sigma_1, \ldots, \sigma_{r-1}, \tau) - \sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_r\tau)(\alpha)\xi_n(\sigma_1, \ldots, \sigma_{r-1}, \sigma_r\tau)$$
$$+ \sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_r\tau)(\alpha)\xi_n(\sigma_1, \ldots, \sigma_r).$$

Now by choice of α, the last term is simply $\xi_n(\sigma_1, \ldots, \sigma_r)$. Thus after rearranging, we have for every $\sigma_1, \ldots, \sigma_r \in H_0$ the equation

$$(\xi_n - \delta x_S)(\sigma_1, \ldots, \sigma_r) = \sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_{r-1}\tau)(\alpha)\xi_n(\sigma_1, \ldots, \sigma_{r-1}, \tau) - \sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_r\tau)(\alpha)\xi_n(\sigma_1, \ldots, \sigma_r\tau).$$

For each τ in S, let $\sigma_{r,\tau} \in H_1$ be such that $\tau\sigma_{r,\tau} \in \sigma_r S$. Then the term on the right hand side of the previous equation becomes

$$\sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_{r-1}\tau)(\alpha)\big[\xi_n(\sigma_1, \ldots, \sigma_{r-1}, \tau) - \xi_n(\sigma_1, \ldots, \tau\sigma_{r,\tau})\big],$$

so by the choice of H_1 we have

$$\mathrm{val}(\xi - \delta(x_n + x_S)) = \mathrm{val}(\xi_n - \delta x_S) \geq (n+1)c_1.$$

Finally, set $x_{n+1} := x_n + x_S$ where S is arbitrary. The calculations we have done show that $\mathrm{val}(x_{n+1} - x_n) \geq (n-1)c_1$ and $\mathrm{val}(\xi - \delta x_{n+1}) \geq (n+1)c_1$, as required. This concludes the induction and with it the proof. $\qquad\square$

5C. *Descent of semilinear representations.* In this subsection we suppose that G and $\widetilde{\Lambda}$ satisfy the axioms (TS1), (TS2) and (TS3).

Given an integer $k > c_1 + 2c_2 + 2c_3$ and an open subgroup $G_0 \subset G$ we write $\mathrm{Mod}^k_{\widetilde{\Lambda}^+}(G, G_0)$ for the category of finite free $\widetilde{\Lambda}^+$-semilinear representations M^+ of G such that in some basis of M^+, we have $\mathrm{Mat}(g) \in 1 + p^k \mathrm{Mat}_d(\widetilde{\Lambda}^+)$ for every $g \in G_0$.

The following will allow us to descend coefficients from $\widetilde{\Lambda}^+$ to the much smaller ring $\Lambda^+_{H_0,n} = \widetilde{\Lambda}^+ \cap \Lambda_{H_0,n}$. It is a simple modification of [Berger and Colmez 2008, Proposition 3.3.1] and is proved in exactly the same way.

Proposition 5.9. *Let $M^+ \in \mathrm{Mod}^k_{\widetilde{\Lambda}^+}(G, G_0)$. Then for $n \geq n(G_0)$ and $H_0 = H \cap G_0$ there exists a unique finite free $\Lambda^+_{H_0,n}$-submodule $\boldsymbol{D}^+_{H_0,n}(M^+)$ of M^+ such that*:

(1) $\boldsymbol{D}^+_{H_0,n}(M^+)$ *is fixed by H_0 and stable by G.*

(2) *The natural map $\widetilde{\Lambda}^+ \otimes_{\Lambda^+_{H_0,n}} \boldsymbol{D}^+_{H_0,n}(M^+) \to M^+$ is an isomorphism. In particular, $\boldsymbol{D}^+_{H_0,n}(M^+)$ is free of rank $=$ rank M^+.*

(3) $\boldsymbol{D}^+_{H_0,n}(M^+)$ *has a basis which is c_3-fixed by G_0/H_0, meaning that for $\gamma \in G_0/H_0$ we have*

$$\mathrm{val}(\mathrm{Mat}(\gamma) - 1) > c_3.$$

Corollary 5.10. *Let $M^+ \in \mathrm{Mod}^k_{\widetilde{\Lambda}^+}(G, G_0)$, $M = M^+ \otimes_{\widetilde{\Lambda}^+} \widetilde{\Lambda}$ and $r \geq 1$. The map*

$$\mathrm{H}^r(H_0, M^+) \to \mathrm{H}^r(H_0, p^{-2c_1} M^+)$$

is 0 and $H^r(H_0, M) = 0$.

Proof. This follows from Proposition 5.8 since M^+ has a basis fixed by H_0. $\qquad\square$

Lemma 5.11. *Let H_0 be an open subgroup of H, $n \geq n(H_0)$ an integer, $\gamma \in \Gamma_H$ an element such that $n(\gamma) \leq n$ and $B \in \mathrm{M}_{l \times d}(\widetilde{\Lambda}^{H_0})$ a matrix. Let $d \in \mathbb{N} \cup \{\infty\}$. Suppose there are $V_1 \in \mathrm{GL}_l(\Lambda_{H_0,n})$ and $V_2 \in \mathrm{GL}_d(\Lambda_{H_0,n})$ such that $\mathrm{val}(V_1 - 1), \mathrm{val}(V_2 - 1) > c_3$ and $\gamma(B) = V_1 B V_2$. Then $B \in \mathrm{M}_{l \times d}(\Lambda_{H_0,n})$.*

Proof. The proof is exactly the same as that of [Berger and Colmez 2008, Lemma 3.2.5]. The only difference between that lemma and the statement appearing here is that there one further assumes $l = d$ and $B \in \mathrm{GL}_d(\widetilde{\Lambda}^{H_0})$, but these assumptions are not used in the proof. In fact, the very same argument shows the result holds for matrices with $d = \infty$, as long as we understand that an infinite matrix has coefficients which tend to zero as the indexes tend to ∞. Namely, if R is a ring with valuation and $l, d \in \mathbb{N} \cup \{\infty\}$, let $\mathrm{M}_{l \times d}(R)$ be the set of matrices $A = (a_{ij})$ of size $l \times d$ and $a_{ij} \in R$ such that $\mathrm{val}(a_{ij}) \to \infty$ as $i + j \to \infty$. The argument then works in the same way. $\qquad\square$

Using Lemma 5.11, we have the following description of $\boldsymbol{D}^+_{H_0,n}(M^+)$. It explains why $\boldsymbol{D}^+_{H_0,n}(M^+)$ is functorial in M^+.

Proposition 5.12. *Given $M^+ \in \mathrm{Mod}^k_{\widetilde{\Lambda}^+}(G, G_0)$, the module $\boldsymbol{D}^+_{H_0,n}(M^+)$ is the union of all finitely generated $\Lambda^+_{H_0,n}$-submodules of M^+ which are G-stable, H_0-fixed and admit a c_3-fixed set of generators.*

Proof. Indeed, if we have a submodule generated by c_3-fixed elements $f_1, \ldots, f_l$ and if $e_1, \ldots, e_d$ is a c_3-fixed basis, write

$$f_i = Be_i$$

for some matrix $B \in \mathrm{M}_{l \times d}(\widetilde{\Lambda}^{H_0,+})$. Then we have

$$\mathrm{Mat}_{f_i}(\gamma)B = \gamma(B)\mathrm{Mat}_{e_i}(\gamma).$$

Here by $\mathrm{Mat}_{f_i}(\gamma)$ we mean any matrix which represents the action in terms of the f_i. It is not a priori unique as the submodule may not be free. Nevertheless, we have $\mathrm{val}(\mathrm{Mat}_{f_i}(\gamma) - 1) > c_3$ by the assumption, and this implies that $\mathrm{Mat}_{f_i}(\gamma)$ is invertible by [Berger and Colmez 2008, Lemma 3.1.2]. So by Lemma 5.11

$$B \in \mathrm{M}_{l \times d}(\Lambda_{H_0,n}) \cap \mathrm{M}_{l \times d}(\widetilde{\Lambda}^{H_0,+}) = \mathrm{M}_{l \times d}(\Lambda^+_{H_0,n}),$$

hence the submodule generated by the f_i is contained in $\boldsymbol{D}^+_{H_0,n}(M^+)$. $\qquad\qquad\square$

Corollary 5.13. *Let $M^+, N^+ \in \mathrm{Mod}^k_{\widetilde{\Lambda}^+}(G, G_0)$. Then for $n \geq n(G_0)$,*

(i) *There are natural isomorphisms*

$$\boldsymbol{D}^+_{H_0,n}(M^+) \otimes_{\Lambda^+_{H_0,n}} \boldsymbol{D}^+_{H_0,n}(N^+) \xrightarrow{\sim} \boldsymbol{D}^+_{H_0,n}(M^+ \otimes_{\widetilde{\Lambda}^+} N^+),$$

$$\boldsymbol{D}^+_{H_0,n}(M^+) \oplus \boldsymbol{D}^+_{H_0,n}(N^+) \xrightarrow{\sim} \boldsymbol{D}^+_{H_0,n}(M^+ \oplus N^+).$$

(ii) *If $M^+ \subset N^+$ then $\boldsymbol{D}^+_{H_0,n}(M^+) = \boldsymbol{D}^+_{H_0,n}(N^+) \cap M^+$.*

5D. Descent of $\mathcal{C}^{\mathrm{an}}(G_0, M)$. From here on G is a compact p-adic Lie group and $G_0 \subset G$ is a small subgroup, as in Section 2. We continue to assume G and $\widetilde{\Lambda}$ satisfy the axioms (TS1), (TS2) and (TS3). The reader may also want to recall our notation and conventions of Section 1B regarding Banach spaces, completions and tensor products.

By Proposition 2.3, we have for $V^+_l = V_l(G_0) \cap \mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+$ an equality

$$\widehat{\varinjlim_{l \in \mathbb{N}} V^+_l} = \mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+.$$

For $M \in \mathrm{Mod}^k_{\widetilde{\Lambda}^+}(G, G_0)$ we have

$$\left(\varinjlim_{l \in \mathbb{N}} M^+ \otimes_{\mathbb{Z}_p} V^+_l\right)^{\wedge} \cong M^+ \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+.$$

Each $M^+ \otimes_{\mathbb{Z}_p} V^+_l$ is a finite free $\widetilde{\Lambda}^+$-semilinear representation of G_0. The action of G_k on each of the V^+_l is trivial mod p^k by Lemma 2.2, and hence its action on $M^+ \otimes V^+_l$ is trivial mod p^k. So if $n \geq n(G_k)$,

we may define using Proposition 5.8 a $\Lambda^+_{H_k,n}$-submodule of $M^+ \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+$ given by

$$\boldsymbol{D}^+_{H_k,n,\infty}(M^+) := \left(\varinjlim_{l \in \mathbb{N}} \boldsymbol{D}^+_{H_k,n}(M^+ \otimes V^+_l) \right)^{\wedge}.$$

The module $\boldsymbol{D}^+_{H_k,n,\infty}(M^+)$ is then G_0-stable and fixed by H_k. By Proposition 5.8 we have natural isomorphisms

$$\widetilde{\Lambda}^+ \otimes_{\Lambda^+_{H_k,n}} \boldsymbol{D}^+_{H_k,n}(M^+ \otimes V^+_l) \xrightarrow{\sim} M^+ \otimes V^+_l.$$

This shows that $\boldsymbol{D}^+_{H_k,n,\infty}(M^+)$ is generated by c_3-fixed elements which give it the sup norm, and there is an isometry

$$\widetilde{\Lambda}^+ \widehat{\otimes}_{\Lambda^+_{H_k,n}} \boldsymbol{D}^+_{H_k,n,\infty}(M^+) \xrightarrow{\sim} M^+ \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+.$$

The next proposition follows from Proposition 5.12

Proposition 5.14. *A finitely generated $\Lambda^+_{H_k,n}$-submodule of $M^+ \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+$ which is stable by G_0, fixed by H_k and is generated by a c_3-fixed set of elements is contained in $\boldsymbol{D}^+_{H_k,n,\infty}(M^+)$.*

In particular, we have the function log defined, by abuse of notation as the composition of

$$\chi : G_0 \twoheadrightarrow G_0/H_0 \hookrightarrow \mathbb{Z}^\times_p \quad \text{and} \quad \log : \mathbb{Z}^\times_p \to \mathbb{Q}_p.$$

It lies in $\mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+$. Note that for $g \in G_0$, we have

$$g(\log) = \log + \log(g^{-1}) = \log - \log(g).$$

Lemma 5.15. *The elements 1 and $\log$ of $\widetilde{\Lambda}^+ \widehat{\otimes} \mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+$ lie in $\boldsymbol{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+)$.*

Proof. The $\Lambda^+_{H_k,n}$-submodule generated by 1 and $\log$ in $\widetilde{\Lambda}^+ \widehat{\otimes} \mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+$ is stable under the G_0 action and fixed by H_k. Furthermore, we claim the elements 1 and $\log$ are c_3-fixed by the action of G_k/H_k. This is clear for 1. To show this for $\log$, notice that if $g^{p^k} \in G_k/H_k$ (recalling that $G_k = G^{p^k}_0$) then

$$\mathrm{val}(g^{p^k} - 1)(\log) \geq k > c_1 + 2c_2 + 2c_3 > c_3.$$

We conclude by Proposition 5.14. $\square$

Proposition 5.16. (i) $\boldsymbol{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+)$ *is a subring of* $\widetilde{\Lambda}^+ \widehat{\otimes} \mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+$.

(ii) *The module structure of $M^+ \widehat{\otimes} \mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+$ over $\widetilde{\Lambda}^+ \widehat{\otimes} \mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+$ restricts to a module structure of $\boldsymbol{D}^+_{H_k,n,\infty}(M^+)$ over $\boldsymbol{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+)$.*

Proof. $\boldsymbol{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+)$ contains 1 by Proposition 5.14. Next, one has the ring and module structure maps

$$\widetilde{\Lambda}^+ \otimes \widetilde{\Lambda}^+ \to \widetilde{\Lambda}^+, \quad \widetilde{\Lambda}^+ \otimes M^+ \to M^+.$$

Applying Proposition 5.12, taking the inductive limit and then taking completions, we get natural maps

$$\boldsymbol{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+) \otimes \boldsymbol{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+) \to \boldsymbol{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+)$$

and

$$\boldsymbol{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+) \otimes \boldsymbol{D}^+_{H_k,n,\infty}(M^+) \to \boldsymbol{D}^+_{H_k,n,\infty}(M^+),$$

giving the desired ring and module structures. $\square$

5E. *Computation of higher locally analytic vectors, I.* Let $M^+ \in \mathrm{Mod}^k_{\widetilde{\Lambda}^+}(G, G_0)$ and $M = M^+ \otimes_{\widetilde{\Lambda}^+} \widetilde{\Lambda}$. In this subsection we shall do a first simplification towards the computation of the groups $\mathrm{R}^i_{G\text{-la}}(M)$ for $i \geq 1$.

If G_0 is any open subgroup of G, we have $\mathrm{R}^i_{G\text{-la}}(M) = \mathrm{R}^i_{G_0\text{-la}}(M)$ so that if $G_n = G_0^{p^n}$ we have

$$\mathrm{R}^i_{G\text{-la}}(M) = \varinjlim_n \mathrm{H}^i\big(G_n, M \,\widehat{\otimes}_{\mathbb{Q}_p} \, \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p)\big).$$

Upon possibly making G_0 smaller, we may assume that G_0 is small and that $\chi : G_0/H_0 \to \mathbb{Z}_p^\times$ has image isomorphic to $\mathbb{Z}_p$. Write $\Gamma_n = G_n/H_n$.

Lemma 5.17. *For $i \geq 1$,*

$$\mathrm{H}^i\big(G_n, M \,\widehat{\otimes}_{\mathbb{Q}_p} \, \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p)\big) \cong \mathrm{H}^i\big(\Gamma_{n+k}, (M \,\widehat{\otimes}_{\mathbb{Q}_p} \, \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p))^{H_{n+k}}\big).$$

Proof. By the Hochshild–Serre spectral sequence and the vanishing of H_{n+k} cohomologies in (iii) of Proposition 5.8 (taking the inductive system $M^+_{k+k'} = M^+ \otimes V^+_{k+k'}$ for $k' \geq 0$), we have

$$\mathrm{H}^i\big(G_n, M \,\widehat{\otimes}_{\mathbb{Q}_p} \, \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p)\big) \cong \mathrm{H}^i\big(G_n/H_{n+k}, (M \,\widehat{\otimes}_{\mathbb{Q}_p} \, \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p))^{H_{n+k}}\big).$$

Now the inclusion $\Gamma_{n+k} \hookrightarrow G_n/H_{n+k}$ induces an isomorphism

$$\mathrm{H}^i\big(G_n/H_{n+k}, (M \,\widehat{\otimes}_{\mathbb{Q}_p} \, \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p))^{H_{n+k}}\big) \cong \mathrm{H}^i\big(\Gamma_{n+k}, (M \,\widehat{\otimes}_{\mathbb{Q}_p} \, \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p))^{H_{n+k}}\big).$$

This again follows from Hochshild–Serre, once we notice all the higher cohomologies of G_n/G_{n+k} appearing vanish. This is because G_n/G_{n+k} is finite and the coefficients are rational. $\qquad\square$

Corollary 5.18. $\qquad\qquad\qquad \mathrm{R}^i_{G_n\text{-an}}(M) = 0 \quad \textit{for } i \geq 2 \textit{ and } n \geq 0.$

Proof. Because $\Gamma_{n+k} \cong \mathbb{Z}_p$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

This proves the first part of Theorem 5.1. It remains to study the 1st derived group

$$\mathrm{R}^1_{G\text{-la}}(M) = \varinjlim_n \mathrm{H}^1\big(\Gamma_{n+k}, (M \,\widehat{\otimes}_{\mathbb{Q}_p} \, \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p))^{H_{n+k}}\big).$$

Now for $m \geq n(G_{n+k})$, we have by Proposition 5.9 a natural isomorphism

$$\widetilde{\Lambda}^+ \otimes \varinjlim_\ell \boldsymbol{D}^+_{H_k,n}(M^+ \otimes V^+_\ell) \cong M^+ \otimes \varinjlim_{\ell \in \mathbb{N}} V^+_\ell.$$

Taking the p-adic completion, we obtain a natural isomorphism

$$\widetilde{\Lambda}^+ \,\widehat{\otimes}_{\Lambda^+_{H_{n+k},m}} \, \boldsymbol{D}^+_{H_{n+k},m,\infty}(M^+) \xrightarrow{\sim} M^+ \,\widehat{\otimes}\, \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p)^+$$

and thus

$$\widetilde{\Lambda}^{+,H_{n+k}} \,\widehat{\otimes}_{\Lambda^+_{H_{n+k},m}} \, \boldsymbol{D}^+_{H_{n+k},m,\infty}(M^+) \xrightarrow{\sim} (M^+ \,\widehat{\otimes}\, \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p)^+)^{H_{n+k}}.$$

On the other hand, recall we have the trace maps

$$\mathrm{R}_{H_{n+k},m} : \widetilde{\Lambda}^{H_{n+k}} \to \Lambda_{H_{n+k},m}$$

which induce for $\mathrm{X}_{H_{n+k},m} = \ker \mathrm{R}_{H_{n+k},m}$ a decomposition

$$\widetilde{\Lambda}^{H_{n+k}} = \Lambda_{H_{n+k},m} \oplus \mathrm{X}_{H_{n+k},m}.$$

Therefore, we can decompose

$$\widetilde{\Lambda}^{H_{n+k}} \,\widehat{\otimes}_{\Lambda_{H_{n+k},m}} \, \boldsymbol{D}_{H_{n+k},m,\infty}(M) \cong \boldsymbol{D}_{H_{n+k},m,\infty}(M) \oplus \big(\mathrm{X}_{H_{n+k},m} \,\widehat{\otimes}_{\Lambda_{H_{n+k},m}} \, \boldsymbol{D}_{H_{n+k},m,\infty}(M)\big),$$

and so we get the description

$$R_{G\text{-la}}^1(M) = \varinjlim_n H^1(\Gamma_{n+k}, \boldsymbol{D}_{H_{n+k},m,\infty}(M)) \oplus H^1\big(\Gamma_{n+k}, X_{H_{n+k},m} \widehat{\otimes}_{\Lambda_{H_{n+k},m}} \boldsymbol{D}_{H_{n+k},m,\infty}^+(M)\big),$$

where in each object of the direct limit, we take $m \geq n(G_{n+k})$.

5F. Computation of higher locally analytic vectors, II. If $m \geq 0$ is an integer and γ is an element of a group, write γ_m for γ^{p^m}. The following simple lemma will be used to compare the behavior of $(\gamma - 1)^m$ and $\gamma_m - 1$.

Lemma 5.19. *Let $\ell \geq 0$. The element $X^{p^\ell} - 1$ of the ring $\mathbb{Z}_p[X]$ is in the ideal generated by the elements $p^i(X-1)^{\ell+1-i}$ for $0 \leq i \leq \ell$.*

Proof. For $\ell \geq 1$ we have

$$X^{p^\ell} - 1 = (X^{p^{\ell-1}} - 1)\bigg(\sum_{i=1}^{p-1} X^{ip^{\ell-1}}\bigg) = (X^{p^{\ell-1}} - 1)\bigg(\sum_{i=1}^{p-1} 1 + (X^{ip^{\ell-1}} - 1)\bigg)$$

$$= (X^{p^{\ell-1}} - 1)\bigg(p + \sum_{i=1}^{p-1}(X^{ip^{\ell-1}} - 1)\bigg),$$

so that $X^{p^\ell} - 1$ lies in the ideal

$$(X^{p^{\ell-1}} - 1)(p, (X^{p^{\ell-1}} - 1)) = (p(X^{p^{\ell-1}} - 1), (X^{p^{\ell-1}} - 1)^2).$$

Let I_ℓ be the ideal generated by the elements $p^i(X-1)^{\ell+1-i}$ for $0 \leq i \leq \ell$. It is easy to check that $(pI_{\ell-1}, I_{\ell-1}^2)$ is contained in I_ℓ. Hence, induction on ℓ shows that $X^{p^\ell} - 1$ belong to I_ℓ. $\square$

So far we have only used the axioms (TS1), (TS2) and (TS3). We shall now use the final axiom (TS4), which proves us with a positive number $t > 0$.

Proposition 5.20. *If (TS4) holds, then*

(i) *$\Lambda_{H,n}$ is Γ_t-analytic for an open subgroup of Γ depending on t.*

(ii) *There exists an element $s = s(t, c_3) = s(n, m, G_0, c_3)$ such that for $\gamma \in G_{n+k}/H_{n+k}$ we have*

$$(\gamma - 1)\boldsymbol{D}_{H_{n+k},m,\infty}^+(M^+) \subset p^s \boldsymbol{D}_{H_{n+k},m,\infty}^+(M^+).$$

(iii) *$\boldsymbol{D}_{H_{n+k},m,\infty}(M)$ is Γ-analytic for some open subgroup Γ of Γ_{n+k} which depends on n, m, G and c_3.*

Proof. Once (ii) is established, we claim parts (i) and (iii) follow from [Pan 2022a, Example 2.1.9]. Let us elaborate a little bit. Take ℓ large enough so that

$$(\ell - i) + (i + 1)t = \ell + t + (t - 1)i \geq 2$$

for each $0 \leq i \leq \ell$. Then for such ℓ (which only depends on t) we have by Lemma 5.19

$$(\gamma_\ell - 1)(\Lambda_{H,n}^+) \subset p^2 \Lambda_{H,n}^+,$$

so that if $b \in \Lambda_{H,n}$, the series

$$\gamma_\ell^x(b) = \sum_{n \geq 0} \binom{x}{n}(\gamma_\ell - 1)^n(b)$$

converges. This shows b is analytic for the subgroup generated by γ_ℓ. The argument for (iii) given (ii) is similar.

To show part (ii), recall the identity

$$(\gamma - 1)(ab) = (\gamma - 1)(a)b + \gamma(a)(\gamma - 1)(b).$$

Axiom (TS4) implies that if $a \in \Lambda_{H,m}^+$ and $b \in \boldsymbol{D}_{H_{n+k},m,\infty}^+(M^+)$ is c_3-fixed, then ab is $\min(c_3, t)$-fixed. Since the c_3-fixed elements topologically generate $\boldsymbol{D}_{H_{n+k},m,\infty}^+(M^+)$, it follows that every element of $\boldsymbol{D}_{H_{n+k},m,\infty}^+(M^+)$ is $s = \min(c_3, t)$-fixed. $\square$

Using this we can show

Lemma 5.21. *Given n there is m sufficiently large depending only on n (and not on M) such that*

$$\mathrm{H}^1\big(\Gamma_{n+k}, \mathrm{X}_{H_{n+k},m} \,\widehat{\otimes}_{\Lambda_{H_{n+k},m}} \boldsymbol{D}_{H_{n+k},m,\infty}(M)\big) = 0.$$

Proof. (This argument is adapted from [Pan 2022a, Lemma 3.6.6].) Fix $m_0 \geq n(G_{n+k})$. From the discussion after Corollary 5.18, for $m \geq m_0$ we have a natural isomorphism

$$\widetilde{\Lambda}^{H_{n+k}} \,\widehat{\otimes}_{\Lambda_{H_{n+k},m}} \boldsymbol{D}_{H_{n+k},m,\infty}(M) \cong \boldsymbol{D}_{H_{n+k},m,\infty}(M) \oplus \big(\mathrm{X}_{H_{n+k},m} \,\widehat{\otimes}_{\Lambda_{H_{n+k},m}} \boldsymbol{D}_{H_{n+k},m,\infty}(M)\big).$$

By Proposition 5.12, we have an isomorphism

$$\Lambda_{H_{n+k},m} \,\widehat{\otimes} \,\boldsymbol{D}_{H_{n+k},m_0,\infty}(M) \cong \boldsymbol{D}_{H_{n+k},m,\infty}(M).$$

Let $\mathrm{X}_{H_{n+k},m}^+ = \mathrm{X}_{H_{n+k},m} \cap \widetilde{\Lambda}^+$. We get an induced isomorphism

$$\mathrm{X}_{H_{n+k},m}^+ \,\widehat{\otimes}_{\Lambda_{H_{n+k},m}^+} \boldsymbol{D}_{H_{n+k},m,\infty}^+(M^+) \cong \mathrm{X}_{H_{n+k},m}^+ \,\widehat{\otimes}_{\Lambda_{H_{n+k},m_0}^+} \boldsymbol{D}_{H_{n+k},m_0,\infty}^+(M).$$

Let γ be a generator of Γ_{n+k}. By Proposition 5.20, there is some s such that

$$(\gamma - 1)\boldsymbol{D}_{H_{n+k},m_0,\infty}^+(M^+) \subset p^s \boldsymbol{D}_{H_{n+k},m_0,\infty}^+(M^+).$$

If ℓ is sufficiently large Proposition 5.20 implies that

$$(\gamma_\ell - 1)\boldsymbol{D}_{H_{n+k},m_0,\infty}^+(M^+) \subset p^{2c_3} \boldsymbol{D}_{H_{n+k},m_0,\infty}^+(M^+)$$

(we take $2c_3$ rather than c_3 to take of convergence later in this argument). Choose such an ℓ, and take m large enough so that $n(\gamma_\ell) \leq m$. Then by (TS3) we have $\mathrm{val}((\gamma_\ell - 1)^{-1}(x)) \geq \mathrm{val}(x) - c_3$ for $x \in \mathrm{X}_{H_{n+k},m}^+$.

We will now show that any element of $\mathrm{X}_{H_{n+k},m} \,\widehat{\otimes}_{\Lambda_{H_{n+k},m}} \boldsymbol{D}_{H_{n+k},m,\infty}(M)$ is in the image of $\gamma_\ell - 1$. This will also imply any element is in the image of $\gamma - 1$, since $\gamma_\ell - 1$ is divisible by $\gamma - 1$, and hence it will further imply that the cohomology

$$\mathrm{H}^1\big(\Gamma_{n+k}, \mathrm{X}_{H_{n+k},m} \,\widehat{\otimes}_{\Lambda_{H_{n+k},m}} \boldsymbol{D}_{H_{n+k},m,\infty}(M)\big) \cong \mathrm{X}_{H_{n+k},m} \,\widehat{\otimes}_{\Lambda_{H_{n+k},m}} \boldsymbol{D}_{H_{n+k},m,\infty}(M)/(\gamma - 1)$$

is 0.

To do this last step, it suffices to show that each simple tensor

$$a \otimes b \in X^+_{H_{n+k},m} \mathbin{\widehat{\otimes}}_{\Lambda^+_{H_{n+k},m_0}} \boldsymbol{D}^+_{H_{n+k},m_0,\infty}(M^+) \cong X^+_{H_{n+k},m} \mathbin{\widehat{\otimes}}_{\Lambda^+_{H_{n+k},m}} \boldsymbol{D}^+_{H_{n+k},m,\infty}(M^+)$$

is in the image of $\gamma_\ell - 1$. Choose an integer r so that $p^r a$ is in the image of $(\gamma_l - 1)^{-1}$ restricted to $X^+_{H_{n+k},m}$ (choose any $r \geq c_3$). It suffices to show $p^r a \otimes b$ is in the image of $\gamma_\ell - 1$. So write $p^r a = (\gamma_\ell - 1)^{-1}(c)$ for $c \in X^+_{H_{n+k},m}$, and consider the series

$$y = \sum_{i=0}^{+\infty} (\gamma_l^{-1} - 1)^{-i}(c) \otimes (\gamma_l - 1)^i(b) = \sum_{i=0}^{+\infty} \gamma_l^i (1 - \gamma_l)^{-i}(c) \otimes (\gamma_l - 1)^i(b).$$

This series converges, because by our choices

$$\mathrm{val}((\gamma_\ell - 1)^{-1}(x)) \geq \mathrm{val}(x) - c_3 \quad \text{on } X^+_{H_{n+k},m} \quad \text{and} \quad (\gamma_\ell - 1)(x) \geq \mathrm{val}(x) + 2c_3 \quad \text{on } \boldsymbol{D}^+_{H_{n+k},m_0,\infty}(M^+)!$$

A direct computation then gives

$$(\gamma_\ell - 1)(y) = (\gamma_\ell - 1)(c) \otimes b = p^r a \otimes b,$$

so $p^r a \otimes b$ is in the image of $\gamma_\ell - 1$, as required. $\qquad\square$

Combing Lemma 5.21 with the discussion after Corollary 5.18, we get the following description of $\mathrm{R}^1_{G\text{-la}}(M)$.

Proposition 5.22. $$\mathrm{R}^1_{G\text{-la}}(M) = \varinjlim_{n,m} \mathrm{H}^1\big(\Gamma_{n+k}, \boldsymbol{D}_{H_{n+k},m,\infty}(M)\big),$$

where the direct limit is taken over pairs n, m.

5G. *Computation of higher locally analytic vectors, III.* We are now almost ready to prove our theorem. First we prove a lemma that will be used.

Lemma 5.23. *Let* $\Gamma = \gamma^{\mathbb{Z}_p}$ *and let B be a Banach representation of Γ. Suppose $B = B^{\Gamma\text{-an}}$, and that*

$$\|\gamma - 1\| < p^{-1/(p-1)}.$$

Then $\|b\| = \|b\|_{\Gamma\text{-an}}$ *for any $b \in B$.*

Proof. We have for $x \in \mathbb{Z}_p$ that

$$\gamma^x(b) = \sum \frac{\nabla_\gamma^k(b)}{k!} x^k$$

where $\nabla_\gamma = \log(\gamma)$. By definition

$$\|b\|_{\Gamma\text{-an}} = \sup_{k \geq 0} \{\|\nabla_\gamma^k(b)/k!\|\}.$$

Now recall we have

$$\nabla_\gamma = (\gamma - 1) \sum_{m \geq 0} (-1)^m \frac{(\gamma - 1)^m}{m+1},$$

so $\|\nabla_\gamma(b)\| \leq \|\gamma - 1\|\|b\|$, and more generally

$$\|\nabla_\gamma^k(b)\| \leq \|\gamma - 1\|^k \|b\|.$$

It follows that for $k \geq 1$ we have

$$\|\nabla_\gamma^k(b)/k!\| \leq p^{-k/(p-1)}\|\gamma-1\|^k\|b\| < \|b\|,$$

so that $\|b\|_{\Gamma\text{-an}} = \|b\|$. $\square$

Proof of Theorem 5.1. By Proposition 5.22, $\mathrm{R}_{G\text{-la}}^1(M) = \varinjlim_{n,m} \mathrm{H}^1(\Gamma_{n+k}, \boldsymbol{D}_{H_{n+k},m,\infty}(M))$. Fix n and m. Given $b \in \boldsymbol{D}_{H_{n+k},m,\infty}(M)$ we shall show it becomes zero in some $\mathrm{H}^1(\Gamma_{l+k}, \boldsymbol{D}_{H_{l+k},m',\infty}(M))$ for some $\ell \geq n$, $m' \geq m$ — this will show the direct limit is zero. By Proposition 5.20 we know there is an open subgroup $\Gamma \subset \Gamma_{n+k}$ such that $\boldsymbol{D}_{H_{n+k},m,\infty}(M)$ is Γ-analytic. Writing γ for a generator of Γ, we may take Γ small enough so that $\|\gamma-1\| < p^{-1/(p-1)}$, and hence Lemma 5.23 applies. Thus, writing $\|\cdot\|_n$ for the norm on $\boldsymbol{D}_{H_{n+k},m,\infty}(M)$ induced from its inclusion into $M \widehat{\otimes} \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p)$, we have $\|b\|_n = \|b\|_{\Gamma\text{-an}}$ for $b \in \boldsymbol{D}_{H_{n+k},m,\infty}(M)$. We know there is a real number $D > 0$ such that if $b \in \boldsymbol{D}_{H_{n+k},m,\infty}(M)$ then

$$\|\nabla_\gamma(b)\|_n = \|\nabla_\gamma(b)\|_{\Gamma\text{-an}} \leq D\|b\|_{\Gamma\text{-an}} = D\|b\|_n.$$

Now choose $\ell \geq n$ such that Γ_l has index p^t in Γ, where t is taken large enough so that

$$2p^{1/(p-1)}D \leq p^t.$$

Let $\gamma_t = \gamma^{p^t}$ be the generator of Γ_ℓ, and let $\log_\ell \in \mathcal{C}^{\mathrm{an}}(G_\ell, \mathbb{Q}_p) : G_\ell \twoheadrightarrow G_\ell/H_\ell \to \mathbb{Z}_p$ be the logarithm so that $\log_\ell(\gamma_t) = 1$. Now let $m' \geq m$ be large enough so that $\boldsymbol{D}_{H_{\ell+k},m',\infty}(M)$ is defined. Recall that by Lemma 5.15, $\log_\ell \in \boldsymbol{D}_{H_{\ell+k},m',\infty}(\widetilde{\Lambda}^+)$. Let $\Gamma' \subset \Gamma_{\ell+k}$ be an open subgroup so that $\boldsymbol{D}_{H_{\ell+k},m',\infty}(M)$ is Γ'-analytic and write p^q for the index of Γ' in $\Gamma_{\ell+k}$. Finally, write γ' for the generator of Γ'. Again by making Γ' smaller we may assume $\|\gamma'-1\| < p^{-1/(p-1)}$ on $\boldsymbol{D}_{H_{\ell+k},m',\infty}(M)$. We have

$$\gamma' = (\gamma_t^{p^k})^{p^q} = \gamma^{p^{t+k+q}}.$$

Let $z_\ell = \log_\ell/p^{k+q} \in \boldsymbol{D}_{H_{\ell+k},m',\infty}(\widetilde{\Lambda})$, the one computes that $\gamma'(z_\ell) = z_\ell + 1$. Therefore, $\nabla_{\gamma'}(z_\ell) = 1$. Now consider the series

$$bz_\ell - \nabla_{\gamma'}(b)\frac{z_\ell^2}{2!} + \nabla_{\gamma'}^2(b)\frac{z_\ell^3}{3!} - \cdots$$

in $\boldsymbol{D}_{H_{\ell+k},m',\infty}(M)$. We claim first it converges with respect to the norm $\|\cdot\|_\ell$ of $\boldsymbol{D}_{H_{\ell+k},m',\infty}(M)$. Indeed, we have

$$\|z_\ell\|_\ell = p^{k+q}$$

and (noting that $\nabla_{\gamma'}^i = p^{i(t+k+q)}\nabla_\gamma^i$)

$$\|\nabla_{\gamma'}^i(b)\|_\ell = p^{-i(t+k+q)}\|\nabla_\gamma^i(b)\|_\ell \leq p^{-i(t+k+q)}\|\nabla_\gamma^i(b)\|_n \leq p^{-i(t+k+q)}D^i\|b\|_n,$$

so the general term of series has size

$$\|\nabla_{\gamma'}^i(b)/(i+1)! \cdot z_\ell^{i+1}\|_\ell \ll p^{-i(t+k+q)}D^i p^{i(k+q)}p^{i/(p-1)} = (p^{-t}Dp^{1/(p-1)})^i \leq 2^{-i},$$

so the series converges in the in the $\|\cdot\|_\ell$ norm. But then the series must also converge with respect to $\|\cdot\|_{\Gamma'\text{-an}}$ because of Lemma 5.23. So if we write y for the sum of the series, it makes sense to speak of

the derivative $\nabla_{\gamma'}(y)$, and one computes that $\nabla_{\gamma'}(y) = b$. So b is in the image of

$$\nabla_{\gamma'} : \boldsymbol{D}_{H_{\ell+k},m',\infty}(M) \to \boldsymbol{D}_{H_{\ell+k},m',\infty}(M),$$

hence also in the image of $\gamma' - 1$, which divides $\nabla_{\gamma'}$. But $\gamma' = \gamma_{t+k}^{p^q}$ so $\gamma_{t+k} - 1$ divides $\gamma' - 1$. It follows that b is also in the image of $\gamma_{t+k} - 1$. This means that b is 0 in

$$\boldsymbol{D}_{H_{\ell+k},m',\infty}(M)/(\gamma_{t+k} - 1) \cong \mathrm{H}^1(\Gamma_{\ell+k}, \boldsymbol{D}_{H_{\ell+k},m',\infty}(M))$$

and we are done! □

Remark 5.24. (1) Since the choices of ℓ and m' did not depend on b, each $\boldsymbol{D}_{H_{n+k},m,\infty}(M)$ maps in its entirety to 0 in some $\boldsymbol{D}_{H_{l+k},m',\infty}(M)$. This shows that M is strongly $\mathfrak{LA}$-acyclic in the sense of [Pan 2022a, §2.2]. After this work was completed, Pan proved that strong $\mathfrak{LA}$-acyclicity is in fact automatic in this setting, see [Pan 2022b, Proposition 2.3.6].

(2) The proof of Theorem 5.1 shows the vanishing of $\varinjlim_{n,m} \mathrm{H}^1(\mathrm{Lie}(\Gamma_{n+k}), \boldsymbol{D}_{H_{n+k},m,\infty}(M))$, which is a priori stronger than the vanishing of $\varinjlim_{n,m} \mathrm{H}^1(\Gamma_{n+k}, \boldsymbol{D}_{H_{n+k},m,\infty}(M))$.

6. Descent to locally analytic vectors

Work again in the setting of Sections 3–4. We shall assume in this section that K_∞ contains an unramified twist of the cyclotomic extension. The purpose of this section is to prove the following theorem.

Theorem 6.1. *The functor* $\mathcal{E} \mapsto \mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \mathcal{E}$ *gives rise to an equivalence of categories*

$$\{\textit{locally analytic vector bundles on } \mathcal{X}\} \cong \{\Gamma\textit{-vector bundles on } \mathcal{X}\}.$$

The inverse functor is given by $\tilde{\mathcal{E}} \mapsto \tilde{\mathcal{E}}^{\mathrm{la}}$.

In the rest of this section, we shall prove that given a Γ-vector bundle $\tilde{\mathcal{E}}$ on $\mathcal{X}$, the natural map

$$\mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \tilde{\mathcal{E}}^{\mathrm{la}} \to \tilde{\mathcal{E}}$$

is an isomorphism. This is enough for proving Theorem 6.1. Indeed, if this isomorphism is granted, then in particular it follows from Proposition 2.1 that $\tilde{\mathcal{E}}^{\mathrm{la}}$ is locally free over $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$, so that the functor $\tilde{\mathcal{E}} \mapsto \tilde{\mathcal{E}}^{\mathrm{la}}$ is valued in the correct category and is fully faithful. On the other hand, it follows from Example 4.5(2) that it is also essentially surjective.

6A. *Computations at the stalk.* In this section, w let $\tilde{\mathcal{E}}$ be a Γ-vector bundle. We have the fiber $\tilde{\mathcal{E}}_{k(x_\infty)}$ at x_∞, a finite dimensional $\widehat{K}_\infty$-semilinear representation of Γ, and the completed stalk $\tilde{\mathcal{E}}_{x_\infty}^{\wedge,+}$, a finite free $\boldsymbol{B}_{\mathrm{dR}}^+(\widehat{K}_\infty) = \boldsymbol{B}_{\mathrm{dR}}^{+,H}$-module. We define

$$\boldsymbol{D}_{\mathrm{Sen}}(\tilde{\mathcal{E}}) = (\tilde{\mathcal{E}}_{k(x_\infty)})^{\mathrm{la}} \quad \text{and} \quad \boldsymbol{D}_{\mathrm{dif}}^+(\tilde{\mathcal{E}}) = (\tilde{\mathcal{E}}_{x_\infty}^{\wedge,+})^{\mathrm{pa}}.$$

If V is a p-adic representation and $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}(V)$ as in Example 3.4, and if $\Gamma = \Gamma_{\mathrm{cyc}}$, then we recover the classical invariant $\boldsymbol{D}_{\mathrm{Sen}}(V)$ according to [Berger and Colmez 2016, théorème 3.2]. The invariant $\boldsymbol{D}_{\mathrm{dif}}^+(V)$

is also recovered, see [Porat 2022, Proposition 3.3.]. It is therefore natural to extend these definitions to arbitrary $\tilde{\mathcal{E}}$ and Γ as we have done here.

There is the following decompletion result.

Theorem 6.2. (i) *The natural map* $\widehat{K}_\infty \otimes_{\widehat{K}_\infty^{\mathrm{la}}} \boldsymbol{D}_{\mathrm{Sen}}(\tilde{\mathcal{E}}) \to \tilde{\mathcal{E}}_{k(x_\infty)}$ *is an isomorphism.*

(ii) *The natural map* $\boldsymbol{B}_{\mathrm{dR}}^+(\widehat{K}_\infty) \otimes_{\boldsymbol{B}_{\mathrm{dR}}^+(\widehat{K}_\infty)^{\mathrm{pa}}} \boldsymbol{D}_{\mathrm{dif}}^+(\tilde{\mathcal{E}}) \to \tilde{\mathcal{E}}_{x_\infty}^{\wedge,+}$ *is an isomorphism.*

Proof. The fiber $\tilde{\mathcal{E}}_{k(x_\infty)}$ is a finite dimensional $\widehat{K}_\infty$-semilinear representation of Γ. So (i) follows from [Berger and Colmez 2016, théorème 3.4]. For (ii), write I_θ for the maximal ideal of $\boldsymbol{B}_{\mathrm{dR}}^+(\widehat{K}_\infty)$. It suffices to prove that for $n \geq 1$ the natural map

$$\boldsymbol{B}_{\mathrm{dR}}^+(\widehat{K}_\infty)/I_\theta^n \otimes_{(\boldsymbol{B}_{\mathrm{dR}}^+/I_\theta^n)^{\mathrm{la}}} (\tilde{\mathcal{E}}_{x_\infty}/I_\theta^n)^{\mathrm{la}} \to \tilde{\mathcal{E}}_{x_\infty}/I_\theta^n \qquad (*)$$

is an isomorphism.

By Theorem 5.1 (more precisely, Corollary 5.6(i)), we have $\mathrm{R}_{\mathrm{la}}^1(I_\theta^{n-1}\tilde{\mathcal{E}}_{x_\infty}/I_\theta^n) = 0$, so by devissage the map

$$(\tilde{\mathcal{E}}_{x_\infty}/I_\theta^n)^{\mathrm{la}} \to (\tilde{\mathcal{E}}_{x_\infty}/I_\theta)^{\mathrm{la}} = \boldsymbol{D}_{\mathrm{Sen}}(\tilde{\mathcal{E}})$$

is surjective. It follows from the case $n = 1$ and Nakayama's lemma that $(*)$ is surjective too.

For injectivity, we argue as follows. Let $\bar{e}_1, \ldots, \bar{e}_d$ be a basis of $\boldsymbol{D}_{\mathrm{Sen}}(\tilde{\mathcal{E}})$ over the field $\widehat{K}_\infty^{\mathrm{la}}$. By what was just proved, we may choose a lifting $e_1, \ldots, e_d$ of this basis to $(\tilde{\mathcal{E}}_{x_\infty}/I_\theta^n)^{\mathrm{la}}$. Then $1 \otimes e_1, \ldots, 1 \otimes e_d$ generate

$$\boldsymbol{B}_{\mathrm{dR}}^+(\widehat{K}_\infty)/I_\theta^n \otimes_{(\boldsymbol{B}_{\mathrm{dR}}^+/I_\theta^n)^{\mathrm{la}}} (\tilde{\mathcal{E}}_{x_\infty}/I_\theta^n)^{\mathrm{la}}$$

according to Nakayama's lemma.

Now suppose that

$$\sum x_i \otimes e_i \in \boldsymbol{B}_{\mathrm{dR}}^+(\widehat{K}_\infty)/I_\theta^n \otimes_{(\boldsymbol{B}_{\mathrm{dR}}^+/I_\theta^n)^{\mathrm{la}}} (\tilde{\mathcal{E}}_{x_\infty}/I_\theta^n)^{\mathrm{la}}$$

is in the kernel of $(*)$, so its image is 0 mod I_θ^n. Choose a generator ξ of I_θ. Reducing mod I_θ and using the injectivity of $(*)$ for $n = 1$, we get the relation $\sum \bar{x}_i \otimes \bar{e}_i = 0$. As the $\bar{e}_i$ form a basis, each x_i must be divisible by ξ. Writing $x_i = \xi x_i'$, we have

$$\sum x_i \otimes e_i = \sum \xi x_i' \otimes e_i = \xi \sum x_i' \otimes e_i,$$

so the image of

$$\sum x_i' \otimes y_i \in \boldsymbol{B}_{\mathrm{dR}}^+(\widehat{K}_\infty)/I_\theta^{n-1} \otimes_{(\boldsymbol{B}_{\mathrm{dR}}^+/I_\theta^{n-1})^{\mathrm{la}}} (\tilde{\mathcal{E}}_{x_\infty}/I_\theta^{n-1})^{\mathrm{la}}$$

in $\tilde{\mathcal{E}}_{x_\infty}/I_\theta^{n-1}$ is 0. The injectivity now follows from induction. $\qquad \square$

Let I be a closed interval with $|\log(I)| < \log(p)$ and let

$$\widetilde{M}_I = \mathrm{H}^0(\mathcal{X}_I, \tilde{\mathcal{E}}).$$

Theorem 5.1 allows us to prove the following Proposition 6.3; we shall subsequently prove a stronger statement in Theorem 6.5.

Proposition 6.3. *There are natural isomorphisms*

$$\boldsymbol{D}_{\mathrm{Sen}}(\tilde{\mathcal{E}}) \cong \widetilde{M}_I^{\mathrm{la}}/(I_\theta \widetilde{M}_I)^{\mathrm{la}} \quad \text{and} \quad \boldsymbol{D}_{\mathrm{dif}}^+(\tilde{\mathcal{E}}) \cong \varprojlim_n \widetilde{M}_I^{\mathrm{la}}/(I_\theta^n \widetilde{M}_I)^{\mathrm{la}}.$$

Proof. As I_θ is principal, $I_\theta \widetilde{M}_I$ is finite free over $\widetilde{\boldsymbol{B}}_I$. By Corollary 5.6(ii), the cohomology $\mathrm{R}_{\mathrm{la}}^1(I_\theta \widetilde{M}_I)$ vanishes. Applying la to the short exact sequence

$$0 \to I_\theta \widetilde{M}_I \to \widetilde{M}_I \to \widetilde{M}_I I_\theta / \widetilde{M}_I \to 0$$

we get $\widetilde{M}_I^{\mathrm{la}}/(I_\theta \widetilde{M}_I)^{\mathrm{la}} \xrightarrow{\sim} (\widetilde{M}_I/I_\theta \widetilde{M}_I)^{\mathrm{la}} = \boldsymbol{D}_{\mathrm{Sen}}(\tilde{\mathcal{E}})$, which gives the first isomorphism. By the same argument $\widetilde{M}_I^{\mathrm{la}}/(I_\theta^n \widetilde{M}_I)^{\mathrm{la}} \xrightarrow{\sim} (\widetilde{M}_I/I_\theta^n \widetilde{M}_I)^{\mathrm{la}}$ for $n \geq 1$. To get the second isomorphism, take the limit over n. $\qquad\square$

6B. *Descent to locally analytic vectors.* In this subsection we will give a proof of Theorem 6.1. We continue with the notation of Section 6A.

We start with the following key proposition, which builds upon all of the work done in Section 4, Section 5 and the previous subsections of Section 6.

Proposition 6.4. *Let $I = [r, (p-1)p^n]$ be an interval with $n \geq 1$ and $|\log(I)| < \log(p)$. Then the natural map*

$$\widetilde{\boldsymbol{B}}_I \otimes_{\widetilde{\boldsymbol{B}}_I^{\mathrm{la}}} \widetilde{M}_I^{\mathrm{la}} \to \widetilde{M}_I \tag{6-1}$$

is an isomorphism.

Proof. First let us explain how to reduce to the cyclotomic case. After an unramified twist, which causes no obstructions to descent, we may assume $K_{\mathrm{cyc}} \subset K_\infty$. Set

$$\widetilde{M}_{I,\mathrm{cyc}} := \widetilde{M}_I^{\mathrm{Gal}(K_\infty/K_{\mathrm{cyc}})}.$$

We then have

$$\widetilde{M}_I \cong \widetilde{\boldsymbol{B}}_I \otimes_{\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}} \widetilde{M}_{I,\mathrm{cyc}}$$

(see for example [Berger and Colmez 2008, corollarie 3.2.2]), and if the conclusion of the proposition holds for the cyclotomic case, we have

$$\widetilde{M}_{I,\mathrm{cyc}} \cong \widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}} \otimes_{\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}^{\mathrm{la}}} \widetilde{M}_{I,\mathrm{cyc}}^{\mathrm{la}}$$

and hence

$$\widetilde{M}_I \cong \widetilde{\boldsymbol{B}}_I \otimes_{\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}^{\mathrm{la}}} \widetilde{M}_{I,\mathrm{cyc}}^{\mathrm{la}}.$$

This shows that $\widetilde{M}_I$ has a basis of locally analytic vectors and by Proposition 2.1 the map (6-1) is an isomorphism.

It remains to establish the proposition in the cyclotomic case where $\widetilde{\boldsymbol{B}}_I = \widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}$. By Proposition 4.2, $\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}$ is flat as a $\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}^{\mathrm{la}}$-module. Since $\widetilde{M}_{I,\mathrm{cyc}}^{\mathrm{la}}$ is torsionfree as a $\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}^{\mathrm{la}}$-module, it follows from [Stacks

2005–, 0AXM] that $\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}} \otimes_{\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}^{\mathrm{la}}} \widetilde{M}_{I,\mathrm{cyc}}^{\mathrm{la}}$ is also torsionfree. By Proposition 6.3, the completion at $I_\theta \subset \widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}$ of (6-1) is nothing but the map

$$\boldsymbol{B}_{\mathrm{dR}}^+ \otimes_{\boldsymbol{B}_{\mathrm{dR}}^{+,\mathrm{pa}}} \boldsymbol{D}_{\mathrm{dif}}^+(\tilde{\mathcal{E}}) \to \tilde{\mathcal{E}}_{x_\infty}^{\wedge,+},$$

so by Theorem 6.2, the map (6-1) is an isomorphism at least after taking this completion. As $\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}$ is a PID (see Proposition 3.1), it follows that (6-1) is injective with cokernel supported at finitely many maximal ideals. These maximal ideals correspond to a finite set of points on $\mathcal{X}$, and this set must form a finite orbit under the action of Γ. But by [Fargues and Fontaine 2018, Proposition 10.1.1], the only point with finite orbit under the Γ-action is x_∞! Thus the cokernel of (6-1) is supported at I_θ. But then it must be 0, as we have just shown the completion at I_θ is an isomorphism. $\square$

Proof of Theorem 6.1. Let U be an open subaffinoid of $\mathcal{X}_I$ for $I = [r, (p-1)p^n]$. Then we claim that the natural map

$$\mathcal{O}_{\mathcal{X}}(U) \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}(U)} \mathrm{H}^0(U, \tilde{\mathcal{E}}^{\mathrm{la}}) \to \mathrm{H}^0(U, \tilde{\mathcal{E}})$$

is an isomorphism. Indeed, we have

$$\mathrm{H}^0(U, \tilde{\mathcal{E}}) \cong \mathcal{O}_{\mathcal{X}}(U) \otimes_{\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}} \widetilde{M}_{I,\mathrm{cyc}} \cong \mathcal{O}_{\mathcal{X}}(U) \otimes_{\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}^{\mathrm{la}}} \widetilde{M}_{I,\mathrm{cyc}}^{\mathrm{la}}.$$

Thus $\mathrm{H}^0(U, \tilde{\mathcal{E}})$ has a basis of locally analytic elements. By Proposition 2.1, we have an isomorphism

$$\mathcal{O}_{\mathcal{X}}(U) \otimes_{\mathcal{O}_{\mathcal{X}}(U)^{\mathrm{la}}} \mathrm{H}^0(U, \tilde{\mathcal{E}})^{\mathrm{la}} \to \mathrm{H}^0(U, \tilde{\mathcal{E}}),$$

from which the claim follows.

Now let $(\mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \tilde{\mathcal{E}}^{\mathrm{la}})^\circ$ be the presheaf on $\mathcal{X}$ sending

$$U \mapsto \mathcal{O}_{\mathcal{X}}(U) \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}(U)} \tilde{\mathcal{E}}^{\mathrm{la}}(U).$$

The $\mathcal{X}_I$ for various I of the form $I = [r, (p-1)p^n]$ with $|\log(I)| < \log(p)$ give a covering of $\mathcal{X}$, so the claim shows that the natural map

$$(\mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \tilde{\mathcal{E}}^{\mathrm{la}})^\circ \to \tilde{\mathcal{E}}$$

is an isomorphism on stalks. Theorem 6.1 follows. $\square$

The proof of Theorem 6.1 essentially shows that $\mathcal{E}$ is quasicoherent. This leads to a simple interpretation of $\boldsymbol{D}_{\mathrm{Sen}}$ and $\boldsymbol{D}_{\mathrm{dif}}^+$ in terms of $\mathcal{E}$ as follows. Given a locally analytic vector bundle define

$$\boldsymbol{D}_{\mathrm{Sen}}(\mathcal{E}) = \mathcal{E}_{k(x_\infty)},$$

the fiber of $\mathcal{E}$ at x_∞, and

$$\boldsymbol{D}_{\mathrm{dif}}^+(\mathcal{E}) = \hat{\mathcal{E}}_{x_\infty}^+,$$

the completed stalk of $\mathcal{E}$ at x_∞. These would not *a priori* be the same as $\boldsymbol{D}_{\mathrm{Sen}}(\tilde{\mathcal{E}})$ and $\boldsymbol{D}_{\mathrm{dif}}^+(\tilde{\mathcal{E}})$, because quotients in general do not commute with locally analytic vectors, but they do in this case.

Theorem 6.5. *Let $\tilde{\mathcal{E}} = \mathcal{O}_\mathcal{X} \otimes_{\mathcal{O}_\mathcal{X}^{\mathrm{la}}} \mathcal{E}$. There are natural isomorphisms*

$$\boldsymbol{D}_{\mathrm{Sen}}(\tilde{\mathcal{E}}) \cong \boldsymbol{D}_{\mathrm{Sen}}(\mathcal{E}) \quad and \quad \boldsymbol{D}_{\mathrm{dif}}^+(\tilde{\mathcal{E}}) \cong \boldsymbol{D}_{\mathrm{dif}}^+(\mathcal{E}).$$

Proof. For $I = [r, (p-1)p^n]$ with $|\log(I)| < \log(p)$ write $\widetilde{M}_I = \mathrm{H}^0(\mathcal{X}_I, \tilde{\mathcal{E}})$. For any sufficiently small U containing x_∞, the proof of Theorem 6.1 shows that

$$\mathrm{H}^0(U, \mathcal{E}) \cong \mathcal{O}_\mathcal{X}(U)^{\mathrm{la}} \otimes_{\widetilde{\boldsymbol{B}}_I^{\mathrm{la}}} \widetilde{M}_I^{\mathrm{la}}.$$

It follows that the quotient $\mathcal{E}_{x_\infty}/m_{x_\infty}^n \mathcal{E}_{x_\infty}$ of the stalk $\mathcal{E}_{x_\infty}$ by the n-th power of the maximal ideal $m_{x_\infty} \subset \mathcal{O}_{\mathcal{X},x_\infty}^{\mathrm{la}}$ is identified with the quotient $\widetilde{M}_I^{\mathrm{la}}/(I_\theta^n \widetilde{M}_I)^{\mathrm{la}}$. Now use Proposition 6.3. $\qquad\square$

7. The comparison with (φ, Γ)-modules

In this section, we give reminders on (φ, Γ)-modules and compare them to locally analytic vector bundles. We keep the notation from Section 6 and the assumption that $K_{\mathrm{cyc}}^\eta \subset K_\infty$ for some η.

7A. *Galois representations and (φ, Γ)-modules.* Recall the notation from Section 3 and let

$$\widetilde{\boldsymbol{B}}_{\mathrm{rig}}^\dagger = \widetilde{\boldsymbol{B}}_{\mathrm{rig}}^\dagger(\widehat{K}_\infty) = \varinjlim_r \mathrm{H}^0(\mathcal{Y}_{[r,\infty)}, \mathcal{O}_\mathcal{Y}) = \varinjlim_r \varprojlim_{s \geq r} \mathrm{H}^0(\mathcal{Y}_{[r,s]}, \mathcal{O}_\mathcal{Y})$$

be the extended Robba ring. The (φ, Γ)-actions on $\mathcal{Y}$ induce actions on $\widetilde{\boldsymbol{B}}_{\mathrm{rig}}^\dagger$.

Definition 7.1. A (φ, Γ)-module over $\widetilde{\boldsymbol{B}}_{\mathrm{rig}}^\dagger$ is a finite free $\widetilde{\boldsymbol{B}}_{\mathrm{rig}}^\dagger$-module with commuting semilinear (φ, Γ)-actions such that in some basis $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\widetilde{\boldsymbol{B}}_{\mathrm{rig}}^\dagger)$.

We can compare these objects to (φ, Γ)-vector bundles using two functors. On the one hand, if $\widetilde{\mathcal{M}}$ is a (φ, Γ)-vector bundle, then $\widetilde{\mathrm{M}}_{\mathrm{rig}}^\dagger = \varinjlim_r \mathrm{H}^0(\mathcal{Y}_{[r,\infty)}, \widetilde{\mathcal{M}})$ is a (φ, Γ)-module. Here, the nontrivial thing one needs to check is that $\widetilde{\mathrm{M}}_{\mathrm{rig}}^\dagger$ is free, and this follows from $\widetilde{\boldsymbol{B}}_{\mathrm{rig}}^\dagger$ being Bézout [Kedlaya 2004, Theorem 3.20].

One the other hand, given a (φ, Γ)-module $\widetilde{\mathrm{M}}_{\mathrm{rig}}^\dagger$ we define a (φ, Γ)-vector bundle $\mathrm{FT}(\widetilde{\mathrm{M}}_{\mathrm{rig}}^\dagger)$ as follows. If $\widetilde{\mathrm{M}}_{\mathrm{rig}}^\dagger$ is a (φ, Γ)-module then for every $r \gg 0$ we have a finite free $\widetilde{\boldsymbol{B}}_{[r,\infty)}$-semilinear Γ-representation $\widetilde{\mathrm{M}}_{[r,\infty)}$ together with isomorphisms

$$\varphi^* \widetilde{\boldsymbol{B}}_{[r,\infty)} \otimes_{\widetilde{\boldsymbol{B}}_{[r/p,\infty)}} \widetilde{\mathrm{M}}_{[r/p,\infty)} \xrightarrow{\sim} \widetilde{\mathrm{M}}_{[r,\infty)}$$

as well as identifications

$$\widetilde{\boldsymbol{B}}_{\mathrm{rig}}^\dagger \otimes_{\widetilde{\boldsymbol{B}}_{[r,\infty)}} \widetilde{\mathrm{M}}_{[r,\infty)} \xrightarrow{\sim} \widetilde{\mathrm{M}}_{\mathrm{rig}}^\dagger.$$

Using the isomorphisms $\varphi : \widetilde{\boldsymbol{B}}_{[r,\infty)} \xrightarrow{\sim} \widetilde{\boldsymbol{B}}_{[r/p,\infty)}$ we can then uniquely extend this to all $r > 0$ by inductively defining $\widetilde{\mathrm{M}}_{[r/p^n,\infty)}$ through the isomorphisms

$$\varphi^* \widetilde{\boldsymbol{B}}_{[r/p^{n-1},\infty)} \otimes_{\widetilde{\boldsymbol{B}}_{[r/p^n,\infty)}} \widetilde{\mathrm{M}}_{[r/p^n,\infty)} \xrightarrow{\sim} \widetilde{\mathrm{M}}_{[r/p^{n-1},\infty)}.$$

Setting for every $r > 0$

$$\mathrm{H}^0(\mathcal{Y}_{[r,\infty)}, \mathrm{FT}(\widetilde{\mathrm{M}}_{\mathrm{rig}}^\dagger)) := \widetilde{\mathrm{M}}_{[r,\infty)}$$

and for every $s \geq r$

$$\mathrm{H}^0(\mathcal{Y}_{[r,s]}, \mathrm{FT}(\widetilde{\mathrm{M}}^\dagger_{\mathrm{rig}})) := \widetilde{\mathrm{M}}_{[r,\infty)} \otimes_{\widetilde{\boldsymbol{B}}_{[r,\infty)}} \widetilde{\boldsymbol{B}}_{[r,s]}$$

we obtain a (φ, Γ)-vector bundle $\mathrm{FT}(\widetilde{\mathrm{M}}^\dagger_{\mathrm{rig}})$.

Proposition 7.2. *The functors* $\widetilde{\mathcal{M}} \mapsto \varinjlim_r \mathrm{H}^0(\mathcal{Y}_{[r,\infty)}, \widetilde{\mathcal{M}})$ *and* FT *induce an equivalence of categories*

$$\{(\varphi, \Gamma)\text{-vector bundles on } \mathcal{Y}_{(0,\infty)}\} \cong \{(\varphi, \Gamma)\text{-modules over } \widetilde{\boldsymbol{B}}^\dagger_{\mathrm{rig}}\}.$$

Proof. This is well known. See for example the discussion appearing directly after [Scholze and Weinstein 2020, Definition 13.4.3]. The treatment there is given in the situation where there is no Γ-action present, but the same proof works in our setting. $\square$

The following theorem due to Fontaine and Kedlaya gives the relation of these objects with Galois representations. To formulate it, we need to introduce some terminology. Let y be the point of $\mathcal{Y}$ corresponding to $p = 0$. A (φ, Γ)-module over $\widetilde{\boldsymbol{B}}^\dagger_{\mathrm{rig}}$ is called étale if it has a basis for which $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\mathcal{O}_{\mathcal{Y},y})$. We also have the notion of a semistable slope 0 vector bundle on $\mathcal{X}$ — we refer the reader to [Fargues and Fontaine 2018, définition 5.5.1, exemple 5.5.2.1].

Theorem 7.3. *The following categories are equivalent.*

(1) *Finite dimensional* $\mathbb{Q}_p$*-representations of* G_K.

(2) *Étale* (φ, Γ)*-modules over* $\widetilde{\boldsymbol{B}}^\dagger_{\mathrm{rig}}$.

(3) Γ*-vector bundles on* $\mathcal{X}$ *which are semistable of slope* 0.

Proof. The equivalence of (2) and (3) follows from Proposition 7.2 and Proposition 3.3. The category in (1) is equivalent to (φ, Γ)-modules over $\widetilde{\boldsymbol{B}} = \widehat{\mathcal{O}}_{\mathcal{Y},y}[1/p]$, where $\widehat{\mathcal{O}}_{\mathcal{Y},y}$ is the p-adic completion of $\mathcal{O}_{\mathcal{Y},y}$, by the theorem of Fontaine [1990, théorème 3.4.3 and remarque 3.44(c)]. Next, by a relatively elementary argument, this category is equivalent to the category of (φ, Γ)-modules over $\widetilde{\boldsymbol{B}}^\dagger$, see for example [Kedlaya 2015, Theorem 2.4.5] or [de Shalit and Porat 2019, Theorem 4.3]. Finally, one can replace $\widetilde{\boldsymbol{B}}^\dagger$ by $\widetilde{\boldsymbol{B}}^\dagger_{\mathrm{rig}}$ by [Kedlaya 2004, Proposition 5.11, Corollary 5.12]. See also [Fargues and Fontaine 2018, proposition 11.2.24]. $\square$

7B. *The comparison with locally analytic vector bundles.* Let $\widetilde{\boldsymbol{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}}$ be the subring of pro-analytic vectors in $\widetilde{\boldsymbol{B}}^\dagger_{\mathrm{rig}}$ for the action of Γ. We have a corresponding version of (φ, Γ)-modules.

Definition 7.4. A (φ, Γ)-module $\mathrm{M}^\dagger_{\mathrm{rig}}$ over $\widetilde{\boldsymbol{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}}$ is a finite free $\widetilde{\boldsymbol{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}}$-module with commuting semilinear (φ, Γ)-actions such that in some basis $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\widetilde{\boldsymbol{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}})$, and such that the action of Γ is pro-analytic. It is étale if $\widetilde{\boldsymbol{B}}^\dagger_{\mathrm{rig}} \otimes_{\widetilde{\boldsymbol{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}}} \mathrm{M}^\dagger_{\mathrm{rig}}$ is so.

The following theorem explains the relationship between (φ, Γ)-modules and locally analytic vector bundles.

Theorem 7.5. *The following categories are all equivalent.*

(1) (φ, Γ)*-modules over* $\widetilde{\boldsymbol{B}}^\dagger_{\mathrm{rig}}$.

(2) (φ, Γ)-*modules over* $\widetilde{\boldsymbol{B}}_{\mathrm{rig}}^{\dagger,\mathrm{pa}}$.

(3) (φ, Γ)-*vector bundles over* $\mathcal{Y}_{(0,\infty)}$.

(4) *Locally analytic φ-vector bundles on* $\mathcal{Y}_{(0,\infty)}$.

(5) Γ-*vector bundles on* $\mathcal{X}$.

(6) *Locally analytic vector bundles on* $\mathcal{X}$.

Proof. The equivalences (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) are Propositions 7.2 and 3.3. (4) $\Leftrightarrow$ (6) is similar to Proposition 3.3. The proof of (5) $\Leftrightarrow$ (6) was given in Theorem 6.1, and (3) $\Leftrightarrow$ (4) can be proved in a similar way. It remains to give an equivalence between (2) and (4). The Frobenius trick functor of Section 7A induces a functor

$$\mathrm{FT} : \{(\varphi, \Gamma)\text{-modules over } \widetilde{\boldsymbol{B}}_{\mathrm{rig}}^{\dagger,\mathrm{pa}}\} \to \{\text{Locally analytic } \varphi\text{-vector bundles on } \mathcal{Y}_{(0,\infty)}\}.$$

In the other direction we map a locally analytic φ-vector bundle $\mathcal{M}$ to $\mathcal{M}_{\mathrm{rig}}^{\dagger} = \varinjlim_r \mathrm{H}^0(\mathcal{Y}_{[r,\infty)}, \mathcal{M})$. It is easy to check from the definitions these two are inverses to each other once we know that $\mathcal{M} \mapsto \mathcal{M}_{\mathrm{rig}}^{\dagger}$ is valued in the correct category. So it remains to prove the following:

Claim. $\mathcal{M}_{\mathrm{rig}}^{\dagger}$ *is a (φ, Γ)-module over* $\widetilde{\boldsymbol{B}}_{\mathrm{rig}}^{\dagger,\mathrm{pa}}$.

Proof of Claim. We only need to explain why $\mathcal{M}_{\mathrm{rig}}^{\dagger}$ is a free $\widetilde{\boldsymbol{B}}_{\mathrm{rig}}^{\dagger,\mathrm{pa}}$-module. Since we can always descend along unramified extensions, we may assume $K_{\mathrm{cyc}} \subset K_{\infty}$. Then $\mathcal{M}$ and $\mathcal{M}_{\mathrm{rig}}^{\dagger}$ are both base changed from their cyclotomic counterparts $\mathcal{M}^{\mathrm{Gal}(K_{\infty}/K_{\mathrm{cyc}})}$ and $\mathcal{M}_{\mathrm{rig}}^{\dagger,\mathrm{Gal}(K_{\infty}/K_{\mathrm{cyc}})}$, so we reduce to the cyclotomic case.

To deal with this case, recall the rings $\boldsymbol{B}_{I,\mathrm{cyc}}$ from Section 4. The (cyclotomic) Robba ring is defined as

$$\boldsymbol{B}_{\mathrm{rig},\mathrm{cyc}}^{\dagger} = \varinjlim_r \varprojlim_{s \geq r} \boldsymbol{B}_{[r,s],\mathrm{cyc}}.$$

The maps $\boldsymbol{B}_{[r,s],\mathrm{cyc}} \hookrightarrow \widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}$ of Section 4 induce an embedding $\boldsymbol{B}_{\mathrm{rig},\mathrm{cyc}}^{\dagger} \hookrightarrow \widetilde{\boldsymbol{B}}_{\mathrm{rig},\mathrm{cyc}}^{\dagger} = \widetilde{\boldsymbol{B}}_{\mathrm{rig}}^{\dagger}(\widehat{K}_{\mathrm{cyc}})$. By [Berger 2016, Theorem B] we have

$$\widetilde{\boldsymbol{B}}_{\mathrm{rig}}^{\dagger,\mathrm{pa}} = \bigcup_{n \geq 0} \varphi^{-n}(\boldsymbol{B}_{\mathrm{rig},\mathrm{cyc}}^{\dagger}),$$

and since each $\varphi^{-n}(\boldsymbol{B}_{\mathrm{rig},\mathrm{cyc}}^{\dagger})$ is a Bézout domain [Lazard 1962], the conclusion follows. $\qquad\square$

In particular, we recover a decompletion result entirely phrased in terms of (φ, Γ)-modules:

$$\{(\varphi, \Gamma)\text{-modules over } \widetilde{\boldsymbol{B}}_{\mathrm{rig}}^{\dagger}\} \cong \{(\varphi, \Gamma)\text{-modules over } \widetilde{\boldsymbol{B}}_{\mathrm{rig}}^{\dagger,\mathrm{pa}}\}.$$

This result recovers the decompletion theorem of Cherbonnier and Colmez [1998] and Kedlaya [2004].

Theorem 7.6. *If $K_{\infty} = K_{\mathrm{cyc}}$, base extension induces an equivalence of categories*

$$\{(\varphi, \Gamma)\text{-modules over } \boldsymbol{B}_{\mathrm{rig},\mathrm{cyc}}^{\dagger}\} \cong \{(\varphi, \Gamma)\text{-modules over } \widetilde{\boldsymbol{B}}_{\mathrm{rig},\mathrm{cyc}}^{\dagger}\}.$$

Proof. If M is a (φ, Γ)-module over $\widetilde{\boldsymbol{B}}_{\mathrm{rig,cyc}}^{\dagger,\mathrm{pa}} = \bigcup_n \varphi^{-n}(\boldsymbol{B}_{\mathrm{rig,cyc}}^{\dagger})$ then there exists $n \gg 0$ such that M is defined over $\varphi^{-n}(\boldsymbol{B}_{\mathrm{rig,cyc}}^{\dagger})$. If $e_1, \ldots, e_d$ is a basis of M then $\varphi^n(e_1), \ldots, \varphi^n(e_d)$ is a basis defined over $\boldsymbol{B}_{\mathrm{rig,cyc}}^{\dagger}$. Therefore the category of (φ, Γ)-modules over $\boldsymbol{B}_{\mathrm{rig,cyc}}^{\dagger}$ is equivalent to the category of (φ, Γ)-modules over $\widetilde{\boldsymbol{B}}_{\mathrm{rig,cyc}}^{\dagger,\mathrm{pa}}$. But this latter category is equivalent to (φ, Γ)-modules over $\widetilde{\boldsymbol{B}}_{\mathrm{rig,cyc}}^{\dagger}$ by Theorem 7.5. $\qquad\square$

8. Locally analytic vector bundles and *p*-adic differential equations

8A. *Modifications of locally analytic vector bundles.* We first introduce the following category. It is the locally analytic version of Berger's category of $\boldsymbol{B}$-pairs; see [Berger 2008a].

Definition 8.1. A locally analytic $\boldsymbol{B}$-pair is a pair $\mathcal{W} = (\mathcal{W}_e, W_{\mathrm{dR}}^+)$, where $\mathcal{W}_e$ is a locally free $\mathcal{O}_{\mathcal{X}\text{-}\{\infty\}}^{\mathrm{la}} = \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}|_{\mathcal{X}\text{-}\{\infty\}}$-module with a semilinear Γ-action and $W_{\mathrm{dR}}^+ \subset \boldsymbol{B}_{\mathrm{dR}}^{\mathrm{pa}} \otimes_{\mathcal{O}_{\mathcal{X}\text{-}\{\infty\}}^{\mathrm{la}}} \mathcal{W}_e$ is a Γ-stable $\boldsymbol{B}_{\mathrm{dR}}^{+,\mathrm{pa}}$-lattice.

Proposition 8.2. *The functor from locally analytic vector bundles to locally analytic $\boldsymbol{B}$-pairs mapping $\mathcal{E}$ to $(\mathcal{E}|_{\mathcal{X}\text{-}\{\infty\}}, \boldsymbol{D}_{\mathrm{dif}}^+(\mathcal{E}))$ is an equivalence of categories.*

Proof. There is an obvious functor from the category of locally analytic $\boldsymbol{B}$-pairs to the category of $\boldsymbol{B}$-pairs. This leads to a commutative diagram

$$
\begin{array}{ccc}
\{\text{locally analytic vector bundles}\} & \longrightarrow & \{\text{locally analytic } \boldsymbol{B}\text{-pairs}\} \\
\Big\downarrow{\scriptstyle\cong} & & \Big\downarrow \\
\{\Gamma\text{-vector bundles}\} & \xrightarrow{\ \cong\ } & \{\boldsymbol{B}\text{-pairs}\}
\end{array}
$$

The left vertical arrow is an equivalence by Theorem 6.1. The lower horizontal arrow is also an equivalence, as explained in [Fargues and Fontaine 2018, §10.1.2]. It follows that the functor from locally analytic $\boldsymbol{B}$-pairs to $\boldsymbol{B}$-pairs is essentially surjective, so every $\boldsymbol{B}$-pair comes from a locally analytic $\boldsymbol{B}$-pair by extending scalars. It now follows from Proposition 2.1 that such a locally analytic $\boldsymbol{B}$-pair is unique. This allows us to define a functor from $\boldsymbol{B}$-pairs to locally analytic $\boldsymbol{B}$-pairs, which gives a quasi-inverse to right vertical morphism. It therefore has to be an equivalence. By commutativity of the diagram, the upper horizontal arrow is also an equivalence, as required. $\qquad\square$

Definition 8.3. Given two locally analytic vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ we say that $\mathcal{E}_2$ is a modification of $\mathcal{E}_1$ if $\mathcal{E}_1|_{\mathcal{X}\text{-}\{\infty\}} \cong \mathcal{E}_2|_{\mathcal{X}\text{-}\{\infty\}}$.

Note that in particular any Γ-stable $\boldsymbol{B}_{\mathrm{dR}}^{+,\mathrm{pa}}$-lattice $N \subset \boldsymbol{D}_{\mathrm{dif}}(\mathcal{E})$ defines a modification of $\mathcal{E}$ by taking the pair $(\mathcal{E}|_{\mathcal{X}\text{-}\{\infty\}}, N)$.

Remark 8.4. We could have also defined this notion of modification in terms of usual $\boldsymbol{B}$-pairs. Our choice of presentation is meant to illustrate that one can speak of modifications without leaving the locally analytic realm.

8B. *de Rham and $\mathbb{C}_p$-admissible locally analytic vector bundles.* Let $\mathcal{E}$ be a locally analytic vector bundle. We say that:

- $\mathcal{E}$ is $\mathbb{C}_p$-admissible if $\dim_K \mathcal{E}_{x_\infty}^{\Gamma=1} = \operatorname{rank}(\mathcal{E})$.
- $\mathcal{E}$ is de Rham if $\boldsymbol{D}_{\mathrm{dR}}(\mathcal{E}) := \dim_K \hat{\mathcal{E}}_{x_\infty}^{\Gamma=1} = \operatorname{rank}(\mathcal{E})$.

If V is a p-adic representation and $\mathcal{E} = \tilde{\mathcal{E}}(V)^{\mathrm{la}}$ then $\mathcal{E}_{x_\infty}^{\Gamma=1} = (\mathbb{C}_p \otimes V)^{G_K}$ and $\boldsymbol{D}_{\mathrm{dR}}(\mathcal{E}) = \boldsymbol{D}_{\mathrm{dR}}(V)$, so this extends the usual definitions.

In what follows, note that $\boldsymbol{D}_{\mathrm{dR}}(\mathcal{E})$ has a natural filtration induced from the I_θ filtration on $\hat{\mathcal{E}}_{x_\infty}$.

Definition 8.5. Suppose $\mathcal{E}$ is de Rham.

(1) $\mathcal{N}_{\mathrm{dR}}(\mathcal{E})$ is the modification of $\mathcal{E}$ given by the lattice $\boldsymbol{D}_{\mathrm{dR}}(\mathcal{E}) \otimes_K \boldsymbol{B}_{\mathrm{dR}}^{+,\mathrm{pa}} \subset \boldsymbol{D}_{\mathrm{dif}}(\mathcal{E})$. It is $\mathbb{C}_p$-admissible.

(2) $\mathcal{M}_{\mathrm{dR}}(\mathcal{E})$ is the locally analytic φ-vector bundle corresponding to $\mathcal{N}_{\mathrm{dR}}(\mathcal{E})$.

8C. *The surfaces $\mathcal{Y}_{\log,L}$ and $\mathcal{X}_{\log,L}$.* Fargues and Fontaine [2018, §10.3.3] define a scheme $X_{\log}$. It is a line bundle over the schematic Fargues–Fontaine curve $X_{\mathrm{FF}} = X_{\mathrm{FF}}(\mathbb{C}_p)$ with a natural projection $\pi : X_{\log} \to X$; further, it has a G_K-action and π is G_K-equivariant.

We let $\mathcal{X}_{\log}$ be the analytification of $X_{\log}$. If L is a finite extension of K, we set

$$\mathcal{X}_{\log,L} := \mathcal{X}_{\log}/\operatorname{Gal}(\overline{K}/L_\infty).$$

(Alternatively, this can be defined as the analytification of the quotient of $X_{\log}$ by $\operatorname{Gal}(\overline{K}/L_\infty)$). Similarly, write $\mathcal{Y}_{\log} = \mathcal{Y}_{(0,\infty)} \times_{\mathcal{X}} \mathcal{X}_{\log}$ and $\mathcal{Y}_{\log,L} = \mathcal{Y}_{\log}/\operatorname{Gal}(\overline{K}/L_\infty)$; then $\mathcal{Y}_{\log,L}/\varphi = \mathcal{X}_{\log,L}$. These spaces have an action of $\operatorname{Gal}(L_\infty/L)$, an open subgroup of Γ.

Write p_L (resp. $p_{\log,L}$) for the projection maps $\mathcal{Y}_L \to \mathcal{Y}$ or $\mathcal{X}_L \to \mathcal{X}$ (resp. $\mathcal{Y}_{\log,L} \to \mathcal{Y}$ or $\mathcal{X}_{\log,L} \to \mathcal{X}$). If $I \subset (0,\infty)$ is closed interval, let $\mathcal{Y}_{\log,L,I} = p_{\log,L}^{-1}(\mathcal{Y}_I)$ and similarly $\mathcal{X}_{\log,L,I} = p_{\log,L}^{-1}(\mathcal{X}_I)$ for $\mathcal{X}$ if I is sufficiently small.

Define

$$\widetilde{\boldsymbol{B}}_{\log,L,I} = \mathrm{H}^0(\mathcal{Y}_{\log,L,I}, \mathcal{O}_{\mathcal{Y}_{\log,L}}).$$

As explained in [loc. cit.], there is a natural G_K-equivariant morphism of sheaves

$$d : \mathcal{O}_{X_{\log}} \to \Omega^1_{X_{\log}/X} \cong p_{\log}^* \mathcal{O}_X(-1)$$

which for every vector bundle $\mathcal{E}$ over $\mathcal{X}$ induces an $\mathcal{O}_{\mathcal{X}}$-linear morphism

$$N : p_{\log}^* \mathcal{E} \to p_{\log}^* \mathcal{E} \otimes \Omega^1_{\mathcal{X}_{\log}/\mathcal{X}}.$$

See [Fargues and Fontaine 2018, Lemma 10.3.9] and the subsequent discussion. Similarly, N can be pulled back to $\mathcal{Y}_{\log}$. This then further induces a $\widetilde{\boldsymbol{B}}_{L,I}$-linear differential operator $N : \widetilde{\boldsymbol{B}}_{\log,L,I} \to \widetilde{\boldsymbol{B}}_{\log,L,I}$. If $T \in \widetilde{\boldsymbol{B}}_{\log,L,I}$ is such that $N(T) = 1$ then $\widetilde{\boldsymbol{B}}_{\log,L,I} = \widetilde{\boldsymbol{B}}_{L,I}[T]$ and $N = d/dT$. Such a T exists: if ϖ is any nonunit $\varpi \in \hat{L}_\infty^\times$ and $\varpi^\flat = (\varpi, \varpi^{1/p}, \ldots)$, take $T = \log[\varpi^\flat]$.

Lemma 8.6. *There exists $T \in \widetilde{\boldsymbol{B}}_{\log,L,I}^{\mathrm{la}}$ with $N(T) = 1$. Consequently, $\widetilde{\boldsymbol{B}}_{\log,L,I}^{\mathrm{la}} = \widetilde{\boldsymbol{B}}_{L,I}^{\mathrm{la}}[T]$.*

Proof. The second claim follows the first claim, Proposition 2.1 and the fact that taking locally analytic vectors commutes with filtered colimits. To find such an element T, consider the exact sequence

$$0 \to \widetilde{\boldsymbol{B}}_{L,I} \to \widetilde{\boldsymbol{B}}_{\log,L,I}^{N^2=0} \xrightarrow{N} \widetilde{\boldsymbol{B}}_{L,I} \to 0.$$

After taking locally analytic vectors the sequence stays exact by Theorem 5.1. Thus the sequence

$$0 \to \widetilde{\boldsymbol{B}}_{L,I}^{\mathrm{la}} \to \widetilde{\boldsymbol{B}}_{\log,L,I}^{\mathrm{la},N^2=0} \xrightarrow{N} \widetilde{\boldsymbol{B}}_{L,I}^{\mathrm{la}} \to 0$$

is exact. This means we can lift 1 to an element T with $N(T) = 1$, as required.   □

Proposition 8.7. *Suppose $\varphi^{\mathbb{Z}}(x_\infty) \cap \mathcal{Y}_I \neq \varnothing$. Then*

(i) *If M is a finite extension of L contained in L_∞, then $\widetilde{\boldsymbol{B}}_{\log,L,I}^{\mathrm{Gal}(L_\infty/M)} = M_0$, where M_0 is the maximal unramified extension of $\mathbb{Q}_p$ contained in M.*

(ii) *$\widetilde{\boldsymbol{B}}_{\log,L,I}^{\mathrm{la},\mathrm{Lie}\,\Gamma=0} = L_0'$, the maximal unramified extension of $\mathbb{Q}_p$ contained in L_∞.*

Proof. Point (i) follows from [Fargues and Fontaine 2018, proposition 10.3.15] and (ii) follows from (i).   □

One way to construct de Rham locally analytic vector bundles is as follows. Write $\mathrm{Mod}_{\mathbb{Q}_p^{\mathrm{un}}}^{\mathrm{Fil},\varphi,N}(G_K)$ for the category of finite dimensional vector spaces D over $\mathbb{Q}_p^{\mathrm{un}}$ together with a semilinear action of φ, a monodromy operator N with $\varphi N = pN\varphi$, a filtration on $D \otimes_{\mathbb{Q}_p^{\mathrm{un}}} K^{\mathrm{un}}$ and a discrete action of G_K on D which respects the filtration. For example, if V is a potentially semistable representation then $\boldsymbol{D}_{\mathrm{pst}}(V)$ is an object of $\mathrm{Mod}_{\mathbb{Q}_p^{\mathrm{un}}}^{\mathrm{Fil},\varphi}(G_K)$.

There is a functor

$$\mathcal{E} : \mathrm{Mod}_{\mathbb{Q}_p^{\mathrm{un}}}^{\mathrm{Fil},\varphi}(G_K) \to \{\text{de Rham locally analytic vector bundles}\}$$

defined as follows: Given $D \in \mathrm{Mod}_{\mathbb{Q}_p^{\mathrm{un}}}^{\mathrm{Fil},\varphi}(G_K)$, choose L such that D is defined over L, i.e., $D = \mathbb{Q}_p^{\mathrm{un}} \otimes_{L_0} D_0$. Such an L exists because the action of G_K is discrete. Then $\mathcal{E}(D)$ is defined to be the locally analytic vector bundle corresponding to the pair

$$\left((\mathcal{O}_{\mathcal{Y}_{\log,L}-p_{\log,L}^{-1}(\infty)}^{\mathrm{la}} \otimes_{L_0} D)^{\varphi=1,N=0,\mathrm{Gal}(L_\infty/K_\infty)}, \mathrm{Fil}^0(\boldsymbol{B}_{\mathrm{dR}}^{H_L,\mathrm{pa}} \otimes_{L_0} D_0)^{\mathrm{Gal}(L_\infty/K_\infty)} \right).$$

It is de Rham because

$$D \subset \boldsymbol{B}_{\mathrm{dR}}^{H_K,\mathrm{pa}} \otimes \mathrm{Fil}^0(\boldsymbol{B}_{\mathrm{dR}}^{H_L,\mathrm{pa}} \otimes_{L_0} D_0)^{\mathrm{Gal}(L_\infty/K_\infty)}$$

is fixed by an open subgroup of Γ. If we choose any larger L we get the same pair, so the construction $D \mapsto \mathcal{E}(D)$ is independent of the choice of L.

8D. *Sheaves of smooth functions.* In this subsection we introduce certain sheaves of functions on $\mathcal{X}$. All of these can be defined equally well for $\mathcal{Y}_{(0,\infty)}$.

Definition 8.8. We define the following sheaves of functions on $\mathcal{X}$.

(i) Smooth functions: $\mathcal{O}_{\mathcal{X}}^{\mathrm{sm}} = \mathcal{O}_{\mathcal{X}}^{\mathrm{la},\mathrm{Lie}\,\Gamma=0}$.

(ii) For $[L : K] < \infty$, L-smooth functions: $\mathcal{O}_{\mathcal{X}}^{L\text{-sm}} = p_{L,*}(p_L^* \mathcal{O}_{\mathcal{X}}^{\text{la}})^{\text{Lie}\,\Gamma=0}$.

(iii) For $[L : K] < \infty$, L log-smooth functions: $\mathcal{O}_{\mathcal{X}}^{L\text{-lsm}} = p_{\log,L,*}(p_{\log,L}^* \mathcal{O}_{\mathcal{X}}^{\text{la}})^{\text{Lie}\,\Gamma=0}$.

(iv) Potentially smooth functions: $\mathcal{O}_{\mathcal{X}}^{\text{psm}} = \varinjlim_{[L:K]<\infty} \mathcal{O}_{\mathcal{X}}^{L\text{-sm}}$.

(v) Potentially log-smooth functions: $\mathcal{O}_{\mathcal{X}}^{\text{plsm}} = \varinjlim_{[L:K]<\infty} \mathcal{O}_{\mathcal{X}}^{L\text{-lsm}}$.

The following proposition has been essentially explained to us by Kedlaya.

Proposition 8.9. *Let U be a connected open affinoid subset of $\mathcal{X}$.*

(i) *The sections of each of $\mathcal{O}_{\mathcal{X}}^{\text{sm}}$, $\mathcal{O}_{\mathcal{X}}^{L\text{-sm}}$ and $\mathcal{O}_{\mathcal{X}}^{\text{psm}}$ at U is a field which injects (noncanonically) into $\mathbb{C}_p$.*

(ii) *If $x_\infty \in U$ then there are canonical injections*

$$\mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}}^{\text{sm}}) \hookrightarrow K_\infty, \quad \mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}}^{L\text{-sm}}) \hookrightarrow L_\infty \quad and \quad \mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}}^{\text{psm}}) \hookrightarrow \overline{K}.$$

(iii) *If $x_\infty \in U$ and $U = \mathcal{X}_I$, we have*

$$\mathrm{H}^0(\mathcal{X}_I, \mathcal{O}_{\mathcal{X}}^{\text{sm}}) = K_0', \quad \mathrm{H}^0(\mathcal{X}_I, \mathcal{O}_{\mathcal{X}}^{L\text{-sm}}) = L_0' \quad and \quad \mathrm{H}^0(\mathcal{X}_I, \mathcal{O}_{\mathcal{X}}^{\text{psm}}) = K_0^{\text{un}}.$$

(iv) *We have* $\mathcal{O}_{\mathcal{X}, x_\infty}^{\text{sm}} = K_\infty, \quad \mathcal{O}_{\mathcal{X}, x_\infty}^{L\text{-sm}} = L_\infty \quad and \quad \mathcal{O}_{\mathcal{X}, x_\infty}^{\text{psm}} = \overline{K}.$

Proof. Each of the assertions (i)–(iv) for $\mathcal{O}_{\mathcal{X}}^{\text{psm}}$ follows from the corresponding assertion for $\mathcal{O}_{\mathcal{X}}^{L\text{-sm}}$. We shall give below arguments proving (i)–(iv) for $\mathcal{O}_{\mathcal{X}}^{\text{sm}}$; the proofs for $\mathcal{O}_{\mathcal{X}}^{L\text{-sm}}$ are the same once K is replaced by L.

After passing to an open subgroup of Γ, we may assume Γ stabilizes U. By [Kedlaya 2016, Theorem 8.8], the ring $\mathcal{O}_{\mathcal{X}}(U)$ is a Dedekind domain. Each rank 1 point x of U defines a maximal ideal of $\mathcal{O}_{\mathcal{X}}(U)$, so $f \in \mathcal{O}_{\mathcal{X}}(U)$ can belong to only finitely many of these points. If $f \in \mathcal{O}_{\mathcal{X}}(U)$ is killed by Lie Γ then f is fixed by a finite subgroup of Γ, so these finitely many maximal ideals must form a finite orbit under the Γ-action. But the only rank 1 point with finite orbit is the point x_∞, again by [Fargues and Fontaine 2018, proposition 10.1.1]. So every $f \in \mathcal{O}_{\mathcal{X}}^{\text{sm}}(U)$ either vanishes only at x_∞ or is invertible.

If $x_\infty \notin U$, this proves that $\mathcal{O}_{\mathcal{X}}^{\text{sm}}(U)$ is a field. In particular, it injects into the residue field of each rank 1 point, and there is a dense subset of $\mathcal{X}$ with residue field a subfield of $\mathbb{C}_p$. This proves (i) in this case. On the other hand, if $x_\infty \in U$ then there is a Γ-equivariant embedding of $\mathcal{O}_{\mathcal{X}}^{\text{la}}(U)$ into $\mathbf{B}_{\text{dR}}^+(\widehat{K}_\infty)^{\text{la}}$ which gives an embedding of $\mathcal{O}_{\mathcal{X}}^{\text{sm}}(U)$ into $\mathbf{B}_{\text{dR}}^+(\widehat{K}_\infty)^{\text{la},\text{Lie}\,\Gamma=0} = K_\infty$. This simultaneously proves (i) and (ii) for $\mathcal{O}_{\mathcal{X}}^{\text{sm}}$.

Next, (iii) follows immediately from Proposition 8.7. For (iv), we have already shown that $\mathcal{O}_{\mathcal{X}}^{\text{sm}}(U) \subset K_\infty$ for each U which contains x_∞, so $\mathcal{O}_{\mathcal{X}, x_\infty}^{\text{sm}} \subset K_\infty$. To show the converse inclusion, use the henselian property of local rings of adic spaces [Morel 2019, III.6.3.7] to show first that $K_\infty \subset \mathcal{O}_{\mathcal{X}, x_\infty}$. It then follows that $K_\infty \subset \mathcal{O}_{\mathcal{X}, x_\infty}^{\text{sm}}$, which concludes the proof. $\square$

We raise a few questions to which we expect a positive answer but have not answered in this article.

Question 8.10. (1) *We can show that $\overline{K} \subset \mathcal{O}_{\mathcal{X}}^{\text{psm}}$ if x is any rank 1 point. Indeed, any untilt of $\mathbb{C}_p^\flat$ is algebraically closed, and one can use this to show that the completed local rings $\mathbf{B}_{\text{dR},x}^+$ contain $\overline{K}$. This implies by the same argument that $\overline{K} \subset \mathcal{O}_{\mathcal{X},x}$. But every element of $\overline{K}$ has finite degree over K_0, which is fixed by G_K. This implies that every $x \in \overline{K}$ is fixed by an open subgroup G_K so $\overline{K} \subset \mathcal{O}_{\mathcal{X},x}^{\text{psm}}$.*

Is it true that $\overline{K} = \mathcal{O}_{\mathcal{X},x}^{\text{psm}}$ for any rank 1 point x?

(2) *Is it true that for every connected open affinoid $U \subset \mathcal{X}$, the field $\mathcal{O}_{\mathcal{X}}^{\mathrm{psm}}(U)$ is a finite extension of K_0^{un}? In particular, this would imply a positive answer to question* (1).

(3) *Is it true that $\mathcal{O}_{\mathcal{X}}^{L\text{-}\mathrm{sm}} = \mathcal{O}_{\mathcal{X}}^{L\text{-}\mathrm{lsm}}$ (and hence $\mathcal{O}_{\mathcal{X}}^{\mathrm{psm}} = \mathcal{O}_{\mathcal{X}}^{\mathrm{plsm}}$)? If $x_\infty \in U$ then $\mathcal{O}_{\mathcal{X}}^{L\text{-}\mathrm{sm}}(U) = \mathcal{O}_{\mathcal{X}}^{L\text{-}\mathrm{lsm}}(U)$. This can be seen by using the embedding into $\boldsymbol{B}_{\mathrm{dR}}^+$ as in the proof of Proposition 8.7.*

8E. *The solution functor.* In this subsection, we assume $\mathcal{E}$ is a de Rham locally analytic vector bundle. Given L finite over K, we define the sheaves of solutions on $\mathcal{X}$,

(1) $\mathrm{Sol}_L(\mathcal{E}) := p_{L,*}(p_L^* \mathcal{N}_{\mathrm{dR}}(\mathcal{E}))^{\mathrm{Lie}\,\Gamma = 0}$, a module over $\mathcal{O}_{\mathcal{X}}^{L\text{-}\mathrm{sm}}$,

(2) $\mathrm{Sol}_{\log,L}(\mathcal{E}) := p_{\log,L,*}(p_{\log,L}^* \mathcal{N}_{\mathrm{dR}}(\mathcal{E}))^{\mathrm{Lie}\,\Gamma = 0}$, a module over $\mathcal{O}_{\mathcal{X}}^{L\text{-}\mathrm{lsm}}$,

(3) $\mathrm{Sol}(\mathcal{E}) := \varinjlim_{[L:K]<\infty} \mathrm{Sol}_{\log,L}(\mathcal{E})$, a module over $\mathcal{O}_{\mathcal{X}}^{\mathrm{plsm}}$.

We have similar versions of these sheaves on $\mathcal{Y}_{(0,\infty)}$, denoted by $\mathrm{Sol}_*^\varphi(\mathcal{E})$ for $* \in \{L, \{\log, L\}, \varnothing\}$. Since the φ action on $\mathcal{Y}_{\log,L}$ is Γ-equivariant, there are natural identifications $\mathrm{Sol}_*(\mathcal{E}) = (\mathrm{Sol}_*^\varphi(\mathcal{E}))^{\varphi=1}$ and $\mathrm{Sol}_*^\varphi(\mathcal{E}) \cong \mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^\bullet \otimes_{\mathcal{O}_{\mathcal{X}}^\bullet} \mathrm{Sol}_*(\mathcal{E})$, where $(*, \bullet) = \{(L, L\text{-}\mathrm{sm}), (\{\log, L\}, L\text{-}\mathrm{lsm}), (\varnothing, \mathrm{plsm})\}$.

To make the link with $\mathcal{E}$ clear, we shall need the following form of the p-adic monodromy theorem due to André [2002], Kedlaya [2004] and Mebkhout [2002].

Proposition 8.11. *There exists a finite extension L over K such that if U is an open subset of $\mathcal{Y}_{[r,\infty)}$ for some $r \gg 0$ then the natural map*

$$\mathcal{O}_{\mathcal{Y}_{\log,L}}^{\mathrm{la}}(p_{\log,L}^{-1} U) \otimes_{\mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{L\text{-}\mathrm{lsm}}(U)} \mathrm{Sol}_{\log,L}^\varphi(\mathcal{E})(U) \to \mathcal{O}_{\mathcal{Y}_{\log,L}}^{\mathrm{la}}(p_{\log,L}^{-1} U) \otimes_{\mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{la}}(U)} \mathcal{M}_{\mathrm{dR}}(\mathcal{E})(U)$$

is an isomorphism. Consequently, if $U \subset \mathcal{X}_I$ for some I then

$$\mathcal{O}_{\mathcal{X}_{\log,L}}^{\mathrm{la}}(p_{\log,L}^{-1} U) \otimes_{\mathcal{O}_{\mathcal{X}}^{L\text{-}\mathrm{lsm}}(U)} \mathrm{Sol}_{\log,L}(\mathcal{E})(U) \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}_{\log,L}}^{\mathrm{la}}(p_{\log,L}^{-1} U) \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}(U)} \mathcal{N}_{\mathrm{dR}}(\mathcal{E})(U).$$

Proof. Let $\widetilde{\boldsymbol{D}}_{\mathrm{rig}}^\dagger$ be the (φ, Γ)-module corresponding to $\mathcal{M}_{\mathrm{dR}}(\mathcal{E})$. By the p-adic monodromy theorem, we know there is an isomorphism

$$\widetilde{\boldsymbol{B}}_{\log,L}^{\dagger,\mathrm{pa}} \otimes_{L_0'} (\widetilde{\boldsymbol{B}}_{\log,L}^{\dagger,\mathrm{pa}} \otimes_{\widetilde{\boldsymbol{B}}_{\mathrm{rig},K}^{\dagger,\mathrm{pa}}} \widetilde{\boldsymbol{D}}_{\mathrm{rig}}^{\dagger,\mathrm{pa}})^{\mathrm{Lie}\,\Gamma=0} \xrightarrow{\sim} \widetilde{\boldsymbol{B}}_{\log,L}^{\dagger,\mathrm{pa}} \otimes_{\widetilde{\boldsymbol{B}}_{\mathrm{rig},K}^{\dagger,\mathrm{pa}}} \widetilde{\boldsymbol{D}}_{\mathrm{rig}}^{\dagger,\mathrm{pa}}$$

in the cyclotomic setting (see [Berger 2008b, III.2.1]). More generally, we may descend along unramified extensions to give it in the twisted cyclotomic case, and by base changing we get it in our setting as well by the usual argument.

It follows that for $r \gg 0$ we also have an isomorphism

$$\widetilde{\boldsymbol{B}}_{\log,[r,\infty),L}^{\mathrm{pa}} \otimes_{L_0'} (\widetilde{\boldsymbol{B}}_{\log,[r,\infty),L}^{\mathrm{pa}} \otimes_{\widetilde{\boldsymbol{B}}_{[r,\infty),K}^{\mathrm{pa}}} \widetilde{\boldsymbol{D}}_{[r,\infty)}^{\mathrm{pa}})^{\mathrm{Lie}\,\Gamma=0} \xrightarrow{\sim} \widetilde{\boldsymbol{B}}_{\log,[r,\infty),L}^{\mathrm{pa}} \otimes_{\widetilde{\boldsymbol{B}}_{[r,\infty),K}^{\mathrm{pa}}} \widetilde{\boldsymbol{D}}_{[r,\infty)}^{\mathrm{pa}}.$$

Pulling back along Frobenius, we obtain this isomorphism for any r. Then by finding $r \gg 0$ so that $U \subset \mathcal{Y}_{[r,\infty)}$, we can base change the isomorphism along the map $\widetilde{\boldsymbol{B}}_{\log,[r,\infty),L}^{\mathrm{pa}} \to \mathcal{O}_{\mathcal{Y}_{\log,L}}^{\mathrm{la}}(p_{\log,L}^{-1} U)$ to conclude. $\square$

Note that whether we need to adjoin log and/or perform a finite extension L of K depends exactly on whether $\mathcal{E}$ becomes crystalline or semistable after restricting G_K to G_L. Applying this observation and taking Lie $\Gamma = 0$ of both sides of the proposition, we obtain the following.

Theorem 8.12. *The sheaf $\mathrm{Sol}(\mathcal{E})$ is a locally free $\mathcal{O}_{\mathcal{X}}^{\mathrm{plsm}}$-module of rank equal to $\mathrm{rank}(\mathcal{E})$. More precisely*:

(i) *If $\mathcal{E}$ becomes crystalline after restricting G_K to $G_{L'}$ for some $L \subset L' \subset L_\infty$ then $\mathrm{Sol}_L(\mathcal{E})$ is a locally free $\mathcal{O}_{\mathcal{X}}^{L\text{-sm}}$-module of rank equal to $\mathrm{rank}(\mathcal{E})$, and there is a natural isomorphism*

$$\mathcal{O}_{\mathcal{X}_L}^{\mathrm{la}} \otimes_{\mathcal{O}_{\mathcal{X}}^{L\text{-sm}}} \mathrm{Sol}_L(\mathcal{E}) \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}_L}^{\mathrm{la}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \mathcal{N}_{\mathrm{dR}}(\mathcal{E}).$$

(ii) *If $\mathcal{E}$ becomes semistable after restricting G_K to $G_{L'}$ for some $L \subset L' \subset L_\infty$ then $\mathrm{Sol}_{\log, L}(\mathcal{E})$ is a locally free $\mathcal{O}_{\mathcal{X}}^{L\text{-lsm}}$-module of rank equal to $\mathrm{rank}(\mathcal{E})$, and there is a natural isomorphism*

$$\mathcal{O}_{\mathcal{X}_{\log, L}}^{\mathrm{la}} \otimes_{\mathcal{O}_{\mathcal{X}}^{L\text{-lsm}}} \mathrm{Sol}_{\log, L}(\mathcal{E}) \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}_{\log, L}}^{\mathrm{la}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \mathcal{N}_{\mathrm{dR}}(\mathcal{E}).$$

Lemma 8.13. *For each sufficiently small open connected affinoid U of $\mathcal{Y}_{(0,\infty)}$ which contains an element of $\varphi^{\mathbb{Z}}(x_\infty)$, and for L large enough so that G_L stabilizes U, there is a natural G_L-embedding $\mathrm{H}^0(U, \mathrm{Sol}_{\log, L}^{\varphi}(\mathcal{E})) \hookrightarrow L_\infty \otimes_K \boldsymbol{D}_{\mathrm{dR}}(\mathcal{E})$.*

Proof. Taking the completed stalk at a φ-translate of x_∞, we obtain an injection

$$\mathcal{O}_{\mathcal{Y}_{\log, L}}^{\mathrm{la}}(p_{\log, L}^{-1} U) \otimes_{\mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{la}}(U)} \mathcal{M}_{\mathrm{dR}}(\mathcal{E})(U) \hookrightarrow \hat{L}_\infty^{\mathrm{la}} \otimes_{\widehat{K}_\infty^{\mathrm{la}}} \boldsymbol{D}_{\mathrm{dif}}(\mathcal{E}).$$

On the other hand, Proposition 8.7 gives an isomorphism

$$\mathcal{O}_{\mathcal{Y}_{\log, L}}^{\mathrm{la}}(p_{\log, L}^{-1} U) \otimes_{\mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{L\text{-lsm}}(U)} \mathrm{Sol}_{\log, L}^{\varphi}(\mathcal{E})(U) \xrightarrow{\sim} \mathcal{O}_{\mathcal{Y}_{\log, L}}^{\mathrm{la}}(p_{\log, L}^{-1} U) \otimes_{\mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{la}}(U)} \mathcal{M}_{\mathrm{dR}}(\mathcal{E})(U).$$

Applying Lie $\Gamma = 0$ to the composition of these maps gives the desired embedding.   □

We can now give an interpretation of the stalk at x_∞:

Proposition 8.14. *There following are each naturally isomorphic to each other.*

(1) *The stalk $\mathrm{Sol}(\mathcal{E})_{x_\infty}$.*

(2) *The stalk $\mathrm{Sol}(\mathcal{E})_y^{\varphi}$ for any $y \in \varphi^{\mathbb{Z}}(x_\infty)$.*

(3) *$\overline{K} \otimes_K \boldsymbol{D}_{\mathrm{dR}}(\mathcal{E})$.*

In particular, $\mathrm{Sol}(\mathcal{E})_{x_\infty}$ is naturally a filtered $\overline{K}$-representation of G_K of dimension $\mathrm{rank}(\mathcal{E})$ and G_K-fixed points $\boldsymbol{D}_{\mathrm{dR}}(\mathcal{E})$.

Proof. It is clear (1) and (2) are isomorphic. By Lemma 8.13, we have a natural embedding of $\mathrm{Sol}(\mathcal{E})_y$, and hence of $\mathrm{Sol}(\mathcal{E})_{x_\infty}$ into $\overline{K} \otimes_K \boldsymbol{D}_{\mathrm{dR}}(\mathcal{E})$. By Theorem 8.12, $\mathrm{Sol}(\mathcal{E})_{x_\infty}$ is a finite free module of rank equal to $\dim_K \boldsymbol{D}_{\mathrm{dR}}(\mathcal{E})$ over $\mathcal{O}_{\mathcal{X}, x_\infty}^{\mathrm{plsm}}$. But by Proposition 8.7 $\mathcal{O}_{\mathcal{X}, x_\infty}^{\mathrm{plsm}} = \overline{K}$, so this embedding must be an isomorphism.   □

Finally, we consider the global solutions to the differential equation, namely

$$D(\mathcal{E}) = \mathrm{H}^0(\mathcal{Y}_{(0,\infty)}, \mathrm{Sol}^{\varphi}(\mathcal{E})) = \mathrm{H}^0\big(\mathcal{Y}_{(0,\infty)}, \mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{plsm}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{plsm}}} \mathrm{Sol}(\mathcal{E})\big).$$

Proposition 8.15. $D(\mathcal{E})$ *is naturally an object of* $\mathrm{Mod}^{\mathrm{Fil},\varphi,N}_{\mathbb{Q}^{\mathrm{un}}_p}(G_K)$ *and* $\dim_{\mathbb{Q}^{\mathrm{un}}_p} D(\mathcal{E}) = \mathrm{rank}(\mathcal{E})$.

Proof. We know each $\mathrm{H}^0(\mathcal{Y}_{(0,\infty)}, \mathrm{Sol}^{\varphi}_{\log,L}(\mathcal{E}))$ is an L'_0 vector space for U sufficiently small (independent of L), so $D(\mathcal{E})$ is a $\mathbb{Q}^{\mathrm{un}}_p$-vector space. The filtration is induced from the embedding

$$\mathrm{H}^0(\mathcal{Y}_{(0,\infty)}, \mathrm{Sol}^{\varphi}(\mathcal{E})) \hookrightarrow \mathrm{Sol}(\mathcal{E})_{x_\infty} \cong \overline{K} \otimes_K \boldsymbol{D}_{\mathrm{dR}}(\mathcal{E}).$$

The φ-action is induced from the map $\varphi : \mathcal{Y}_{(0,\infty)} \to \mathcal{Y}_{(0,\infty)}$. The monodromy operator N is induced from the equivariant connection $p^*_{\log,L}\mathcal{M}_{\mathrm{dR}}(\mathcal{E}) \to p^*_{\log,L}\mathcal{N}_{\mathrm{dR}}(\mathcal{E}) \otimes \Omega^1_{\mathcal{Y}_{\log}/\mathcal{Y}_{(0,\infty)}}$. Finally, G_K acts on the smooth elements in $p^*_{\log,L}\mathcal{M}_{\mathrm{dR}}(\mathcal{E})$, and this action is discrete because every element is killed by Lie Γ, hence by an open subgroup of $\mathrm{Gal}(L_\infty/L)$. To compute the dimension use Theorem 8.12. $\qquad\square$

Using this language, Berger's theorem [2008b, théoréme III.2.4] admits the following interpretation.

Theorem 8.16. *The functors* $D \mapsto \mathcal{E}(D)$ *and* $\mathcal{E} \mapsto D(\mathcal{E})$ *are mutual inverses and induce an equivalence of categories*

$$\mathrm{Mod}^{\mathrm{Fil},\varphi,N}_{\mathbb{Q}^{\mathrm{un}}_p}(G_K) \cong \{de\ Rham\ locally\ analytic\ vector\ bundles\}.$$

Remark 8.17. If $\mathcal{E}$ is the locally analytic vector bundle associated to a p-adic representation V, we see that the global-to-local map

$$\mathrm{H}^0(\mathcal{Y}_{(0,\infty)}, \mathrm{Sol}^{\varphi}(\mathcal{E})) \hookrightarrow \mathrm{Sol}(\mathcal{E})_{x_\infty}$$

is nothing but the more familiar map

$$\boldsymbol{D}_{\mathrm{pst}}(V) \hookrightarrow \overline{K} \otimes_K \boldsymbol{D}_{\mathrm{dR}}(V).$$

Question 8.18. *Theorem 8.16 allows us to consider objects of* $\mathrm{Mod}^{\mathrm{Fil},\varphi,N}_{\mathbb{Q}^{\mathrm{un}}_p}(G_K)$ *as global solutions to p-adic differential equations. The filtration is coming from the behavior of orders of vanishing at $x_\infty = 0$, while the (φ, N, G_K)-structure comes from some sort of monodromy of the map $\varprojlim_L \mathcal{Y}_{\log,L} \to \mathcal{X}$. In our description the space $\varprojlim_L \mathcal{Y}_{\log,L}$ behaves as a substitute for a universal cover of $\mathcal{X}$. It would be interesting if it can be replaced by a more literal cover of $\mathcal{X}$ for which the (φ, N, G_K)-actions can be interpreted as monodromy actions. One could even speculate that in an appropriate sense, the analytic fundamental group of $\mathcal{X}(\mathbb{C}_p)_{\overline{K}}$ should be a tame Weil group with its two dimensions reflecting the φ and N operators.*

We conclude with an example.

Example 8.19. Take $\alpha \in \mathbb{Z}^\times_p$, and given $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ let $\xi_\alpha(g) \in \mathbb{Z}_p$ be the element such that $\zeta^{\xi_\alpha(g)}_{p^n} = g(\alpha^{1/p^n})/\alpha^{1/p^n}$ for each $n \geq 1$. The Kummer extension

$$0 \to \mathbb{Q}_p(\chi_{\mathrm{cyc}}) \to V = V_\alpha \to \mathbb{Q}_p \to 0$$

is given by mapping in a basis e, f the element g to the matrix

$$\begin{pmatrix} \chi_{\mathrm{cyc}}(g) & \xi_\alpha(g) \\ 0 & 1 \end{pmatrix}.$$

The associated locally analytic vector bundle $\mathcal{E}$ sits in an exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}(\chi_{\mathrm{cyc}}) \to \mathcal{E} \to \mathcal{O}_{\mathcal{X}}^{\mathrm{la}} \to 0.$$

We have

$$\mathcal{N}_{\mathrm{dR}}(\mathcal{E}) = \mathcal{O}_{\mathcal{X}}^{\mathrm{la}} x \oplus \mathcal{O}_{\mathcal{X}}^{\mathrm{la}} y \cong \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}(1) \oplus \mathcal{O}_{\mathcal{X}}^{\mathrm{la}},$$

where at a neighborhood of x_∞ we have $x = t^{-1}e$ and $y = -\log[\alpha^\flat]t^{-1}e + f$. Thus

$$\mathrm{H}^0(\mathcal{Y}_{(0,\infty)}, \mathrm{Sol}_{\mathbb{Q}_p}^\varphi(\mathcal{E})) = \mathrm{H}^0(\mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{sm}} x \oplus \mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{sm}} y) = \mathbb{Q}_p x \oplus \mathbb{Q}_p y.$$

The action of φ is given by $\varphi(x) = p^{-1}x$ and $\varphi(y) = y$. This gives the underlying φ-module of $\boldsymbol{D}_{\mathrm{cris}}(V)$.

To get the filtration, we consider the stalk of $\mathrm{Sol}_{\mathbb{Q}_p}(\mathcal{E})$ at x_∞. Observe that Fil^0 consists exactly of these smooth sections which do not have a pole at x_∞. As $\log[\alpha^\flat] \equiv \log_p \alpha \bmod t$, we have $\mathrm{Fil}^0 \mathrm{Sol}_{\mathbb{Q}_p}(\mathcal{E})_{x_\infty} = \mathbb{Q}_{p,\mathrm{cyc}}(x \log_p \alpha + y)$ and so the filtration on $\boldsymbol{D}_{\mathrm{cris}}(V)$ is given by

$$\mathrm{Fil}^{-1} = \boldsymbol{D}_{\mathrm{cris}}(V) \supset \mathrm{Fil}^0 = \mathbb{Q}_p(x \log_p \alpha + y) \supset \mathrm{Fil}^1 = 0.$$

Acknowledgments

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Volume 18   No. 5   2024