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Multiplicity structure of the arc space of a fat point

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The equation $x^m = 0$ defines a fat point on a line. The algebra of regular functions on the arc space of this scheme is the quotient of $k[x, x', x^{(2)}, \dots]$ by all differential consequences of $x^m = 0$. This infinite-dimensional algebra admits a natural filtration by finite-dimensional algebras corresponding to the truncations of arcs. We show that the generating series for their dimensions equals $m/(1 - mt)$. We also determine the lexicographic initial ideal of the defining ideal of the arc space. These results are motivated by the nonreduced version of the geometric motivic Poincaré series, multiplicities in differential algebra, and connections between arc spaces and the Rogers–Ramanujan identities. We also prove a recent conjecture put forth by Afsharijoo in the latter context.

1. Introduction

1.1. ***Statement of the main result.*** Let k be a field of characteristic zero. Consider an ideal $I \subset k[\boldsymbol{x}]$, where $\boldsymbol{x} = (x_1, \dots, x_n)$, defining an affine scheme X. We consider the polynomial ring

$$k[\boldsymbol{x}^{(\infty)}] := k[x_i^{(j)} \mid 1 \leqslant i \leqslant n,\ j \geqslant 0]$$

in infinitely many variables $\{x_i^{(j)} \mid 1 \leqslant i \leqslant n,\ j \geqslant 0\}$. This ring is equipped with a k-linear derivation $a \mapsto a'$ defined on the generators by

$$(x_i^{(j)})' = x_i^{(j+1)} \quad \text{for } 1 \leqslant i \leqslant n,\ j \geqslant 0.$$

Then we define the ideal $I^{(\infty)} \subset k[\boldsymbol{x}^{(\infty)}]$ of the arc space of X by

$$I^{(\infty)} := \langle f^{(j)} \mid f \in I,\ j \geqslant 0 \rangle.$$

In this paper, we will focus on the case of a fat point $\mathcal{I}_m := \langle x^m \rangle \subset k[x]$ of multiplicity $m \geqslant 2$. Although the zero set of $\mathcal{I}_m^{(\infty)}$ over k consists of a single point with all the coordinates being zero, the dimension of the corresponding quotient algebra $k[x^{(\infty)}]/\mathcal{I}_m^{(\infty)}$ (the "multiplicity" of the arc space) is infinite.

One can obtain a finer description of the multiplicity structure of $k[x^{(\infty)}]/\mathcal{I}_m^{(\infty)}$ by considering its filtration by finite-dimensional algebras induced by the truncation of arcs

$$k[x^{(\leqslant \ell)}]/\mathcal{I}_m^{(\infty)} := k[x^{(\leqslant \ell)}]/\big(k[x^{(\leqslant \ell)}] \cap \mathcal{I}_m^{(\infty)}\big),$$

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where $x^{(\leqslant \ell)} := \{x, x', \dots, x^{(\ell)}\}$, and arranging the dimensions of these algebras into a generating series

$$D_{\mathcal{I}_m}(t) := \sum_{\ell=0}^{\infty} \dim_k \bigl(k[x^{(\leqslant \ell)}]/\mathcal{I}_m^{(\infty)}\bigr) \cdot t^{\ell}. \tag{1}$$

The main result of this paper is that

$$D_{\mathcal{I}_m}(t) = \frac{m}{1-mt}. \tag{2}$$

1.2. ***Motivations and related results.*** Our motivation for studying the series (1) comes from three different areas: algebraic geometry, differential algebra, and combinatorics.

(1) From the point of view of *algebraic geometry*, $I^{(\infty)}$ defines the *arc space* $\mathcal{L}(X)$ of the scheme X [Denef and Loeser 2001]. Geometrically, the points of the arc space correspond to the Taylor coefficients of the $k[\![t]\!]$-points of X. The arc space of a variety can be viewed as an infinite-order generalization of the tangent bundle or the space of formal trajectories on the variety. For properties and applications of arc spaces, we refer to [Denef and Loeser 2001; Bourqui et al. 2020].

The *reduced structure* of an arc space is often described by means of the geometric motivic Poincaré series [Denef and Loeser 2001, §2.2]

$$P_X(t) := \sum_{\ell=0}^{\infty} [\pi_\ell(\mathcal{L}(X))] \cdot t^{\ell}, \tag{3}$$

where π_ℓ denotes the projection of $\mathcal{L}(X)$ to the affine subspace with the coordinates $\boldsymbol{x}^{(\leqslant \ell)}$ (i.e., the truncation at order ℓ) and $[Z]$ denotes the class of variety Z in the Grothendieck ring [Denef and Loeser 2001, §2.3]. A fundamental result about these series is the Denef–Loeser theorem [1999, Theorem 1.1] saying that $P_X(t)$ is a rational power series.

The arc spaces may also have a rich *scheme* (*i.e.*, *nilpotent*) *structure*, see [Linshaw and Song 2021; Feigin and Makedonskyi 2020; Dumanski and Feigin 2023], reflecting the geometry of the original scheme [Sebag 2011; Bourqui and Haiech 2021]. In the case of a fat point $\mathcal{I}_m = \langle x^m \rangle \subset k[x]$, we will have $\pi_\ell(\mathcal{L}(X)) \cong \mathbb{A}^0$, so the geometric motivic Poincaré series is equal to

$$P(t) = \frac{[\mathbb{A}^0]}{1-t},$$

where $[\mathbb{A}^0]$ is the class of a point. Note that the series does not depend on the multiplicity m of the point. One way to capture the scheme structure of $\mathcal{L}(X)$ could be to take the components of the projections in (3) with their multiplicities. For example, for the case $\mathcal{I}_m$, one will get

$$\sum_{\ell=0}^{\infty} \dim_k \bigl(k[x^{(\leqslant \ell)}]/\mathcal{I}_m^{(\infty)}\bigr) \cdot [\mathbb{A}^0] \cdot t^{\ell} = D_{\mathcal{I}_m}(t)[\mathbb{A}^0].$$

Our result (2) implies that the series above is rational, as in the Denef–Loeser theorem. Interestingly, the shape of the denominator is different from the one in [Denef and Loeser 2001, Theorem 2.2.1]. The formula above is not the only way to take the multiplicities into account. A related and more popular approach is via Arc Hilbert–Poincaré series [Mourtada 2023, §9]; see also [Mourtada 2014; Bruschek et al. 2013].

(2) *Differential algebra* studies, in particular, differential ideals in $k[\boldsymbol{x}^{(\infty)}]$, that is, ideals closed under derivation. From this point of view, $I^{(\infty)}$ is the differential ideal generated by I. Understanding the structure of the differential ideals $\mathcal{I}_m^{(\infty)}$ is a key component of the low power theorem [Levi 1942; 1945] which provides a constructive way to detect singular solutions of algebraic differential equations in one variable. Besides that, various combinatorial properties of $\mathcal{I}_m^{(\infty)}$ have been studied in differential algebra, see [O'Keefe 1960; Pogudin 2014; Arakawa et al. 2021; Zobnin 2005; 2008; Ait El Manssour and Sattelberger 2023].

While there is a rich dimension theory for solution sets of systems of algebraic differential equations [Kondratieva et al. 1999; Pong 2006; Kolchin 1964], we are not aware of a notion of multiplicity of a solution of such a system. In particular, the existing differential analogue of the Bézout theorem [Binyamini 2017] provides only a bound, unlike the equality in classical Bézout theorem [Hartshorne 1977, Theorem 7.7, Chapter 1]. Our result (2) suggests that one possibility is to define the multiplicity of a solution as the growth rate of multiplicities of its truncations, and this definition will be consistent with the usual algebraic multiplicity for the case of a fat point on a line.

(3) Connections between the multiplicity structure of the arc space of a fat point and Rogers–Ramanujan partition identities from *combinatorics* were pointed out by Bruschek, Mourtada, and Schepers in [2013] (for a recent survey, see [Mourtada 2023, §9]). In particular, they used Hilbert–Poincaré series of similar nature to (1) (motivated by the singularity theory [Mourtada 2014, §4]) to obtain new proofs of the Rogers–Ramanujan identities and their generalizations. In this direction, new results have been obtained recently in [Afsharijoo 2021; Afsharijoo et al. 2023; Bai et al. 2020]. Afsharijoo [2021] used computational experiments to conjecture the initial ideal of $\mathcal{I}_m^{(\infty)}$ with respect to the weighted lexicographic ordering [Afsharijoo 2021, §5] (a special case was already conjectured in [Afsharijoo and Mourtada 2020, §1]). This conjecture would imply a new set of partition identities [Afsharijoo 2021, Conjecture 5.1]. Using combinatorial techniques, some of them have been proved in [Afsharijoo 2021], and the rest were established in [Afsharijoo et al. 2023]; see also [Afsharijoo et al. 2022]. However, the original algebraic conjecture about $\mathcal{I}_m^{(\infty)}$ remained open. As a byproduct of our proof of (2), we prove this conjecture (see Theorem 3.3), thus giving a new proof of one of the main results of [Afsharijoo et al. 2023].

Understanding the structure of the ideal $\mathcal{I}_m^{(\infty)}$ is known to be challenging: for example, its Gröbner basis with respect to the lexicographic ordering is not just infinite but even differentially infinite [Zobnin 2005; Afsharijoo and Mourtada 2020], and the question about the nilpotency index of the $x_i^{(j)}$ modulo $\mathcal{I}_m^{(\infty)}$ posed by Ritt [1950, Appendix, Q.5] remained open for sixty years until the paper of Pogudin [2014]; see also [O'Keefe 1960; Arakawa et al. 2021].

Statement (2) appeared in the Ph.D. thesis of Pogudin [2016, Theorem 3.4.1], but the proof given there was incorrect. We are grateful to Alexey Zobnin for pointing out the error. The proof presented in this paper uses different ideas than the erroneous proof in [Pogudin 2016]. We would like to thank Ilya Dumanski for pointing out that the main dimension result (2) could also be deduced from a combination of Propositions 2.1 and 2.3 from [Feigin and Feigin 2002].

1.3. ***Overview of the proof.*** The key technical tool used in our proofs is a representation of the quotient algebra $k[x^{(\infty)}]/\mathcal{I}_m^{(\infty)}$ as a subalgebra in a certain differential exterior algebra that is constructed in [Pogudin 2014]; see Section 4.1. The injectivity of this representation builds upon the knowledge of a Gröbner basis for $\mathcal{I}_m^{(\infty)}$ with respect to the degree reverse lexicographic ordering [Bruschek et al. 2013; Zobnin 2008; Levi 1942]. We approach (2) as a collection of inequalities

$$m^{\ell+1} \leqslant \dim_k\bigl(k[x^{(\leqslant \ell)}]/\mathcal{I}_m^{(\infty)}\bigr) \leqslant m^{\ell+1} \quad \text{for every } \ell \geqslant 0,\ m \geqslant 1. \tag{4}$$

The starting point of our proof of *the lower bound* uses the insightful conjecture by Afsharijoo [2021, §5] that suggests how the standard monomials of $\mathcal{I}_m^{(\infty)}$ with respect to the lexicographic ordering look like. Using the exterior algebra representation, we prove that these monomials are indeed linearly independent modulo $\mathcal{I}_m^{(\infty)}$, and deduce the lower bound from this; see Section 4.3 and 4.4.

In order to prove *the upper bound* from (4), we represent the image of $k[x^{(\leqslant \ell)}]/\mathcal{I}_m^{(\infty)}$ in the differential exterior algebra as a deformation of an algebra which splits as a direct product of $\ell+1$ algebras of dimension m, thus yielding the desired upper bound; see Section 4.2.

1.4. ***Structure of the paper.*** The rest of the paper is organized as follows: Section 2 contains definitions and notations used to state the main results. Section 3 contains the main results of the paper. The proofs of the results are given in Section 4. Then Section 5 describes computational experiments in [Macaulay2] that we performed to check whether formulas similar to (2) hold for more general fat points in k^n. We formulate some open questions based on the results of these experiments.

2. Preliminaries

Definitions 2.1–2.4 provide necessary background in differential algebra. For further details, we refer to [Kaplansky 1957, Chapter 1] or [Kolchin 1973, §I.1–I.2].

Definition 2.1 (differential rings and fields). A *differential ring* $(R,\,')$ is a commutative ring with a derivation $'\colon R \to R$, that is, a map such that, for all $a, b \in R$, we have $(a+b)' = a'+b'$ and $(ab)' = a'b + ab'$. A *differential field* is a differential ring that is a field. For $i > 0$, $a^{(i)}$ denotes the i-th order derivative of $a \in R$.

Notation 2.2. Let x be an element of a differential ring and $h \in \mathbb{Z}_{\geqslant 0}$. We introduce

$$x^{(<h)} := (x, x', \dots, x^{(h-1)}) \quad \text{and} \quad x^{(\infty)} := (x, x', x'', \dots).$$

Analogously, we can define $x^{(\leqslant h)}$. If $\boldsymbol{x} = (x_1, \dots, x_n)$ is a tuple of elements of a differential ring, then

$$\boldsymbol{x}^{(<h)} := (x_1^{(<h)}, \dots, x_n^{(<h)}) \quad \text{and} \quad \boldsymbol{x}^{(\infty)} := (x_1^{(\infty)}, \dots, x_n^{(\infty)}).$$

Definition 2.3 (differential polynomials). Let R be a differential ring. Consider a ring of polynomials in infinitely many variables

$$R[x^{(\infty)}] := R[x, x', x'', x^{(3)}, \dots],$$

and extend the derivation from R to this ring by $(x^{(j)})' := x^{(j+1)}$. The resulting differential ring is called *the ring of differential polynomials in x over R*. The ring of differential polynomials in several variables is defined by iterating this construction.

Definition 2.4 (differential ideals). Let $S := R[x_1^{(\infty)}, \dots, x_n^{(\infty)}]$ be a ring of differential polynomials over a differential ring R. An ideal $I \subset S$ is called *a differential ideal* if $a' \in I$ for every $a \in I$.

One can verify that, for every $f_1, \dots, f_s \in S$, the ideal

$$\langle f_1^{(\infty)}, \dots, f_s^{(\infty)} \rangle$$

is a differential ideal. Moreover, this is the minimal differential ideal containing $f_1, \dots, f_s$, and we will denote it by $\langle f_1, \dots, f_s \rangle^{(\infty)}$.

Definition 2.5 (fair monomials). (1) For a monomial $m = x^{(h_0)} x^{(h_1)} \cdots x^{(h_\ell)} \in k[x^{(\infty)}]$, we define *the order* and *lowest order*, respectively, as

$$\operatorname{ord} m := \max_{0 \leqslant i \leqslant \ell} h_i \quad \text{and} \quad \operatorname{lord} m := \min_{0 \leqslant i \leqslant \ell} h_i.$$

(2) A monomial $m \in k[x^{(\infty)}]$ is called *fair* (respectively, *strongly fair*) if

$$\operatorname{lord} m \geqslant \deg m - 1 \quad (\text{respectively, } \operatorname{lord} m \geqslant \deg m).$$

We denote the sets of all fair and strongly fair monomials by $\mathcal{F}$ and $\mathcal{F}_s$, respectively. By convention, $1 \in \mathcal{F}$ and $1 \in \mathcal{F}_s$. Note that $\mathcal{F}_s \subset \mathcal{F}$.

(3) For every integers $a, b \geqslant 0$, we define

$$\mathcal{F}_{a,b} := \mathcal{F}^a \cdot \mathcal{F}_s^b,$$

where the product of sets of monomials is the set of pairwise products. In other words, $\mathcal{F}_{a,b}$ is a set of all monomials representable as a product of a fair monomials and b strongly fair monomials.

Remark 2.6. The notion of fair monomials was inspired from the conjectured construction of the initial ideal of $\langle x^i, (x^m)^{(\infty)} \rangle$ given in [Afsharijoo 2021, Conjecture 5.1]. We use the notion to formulate concisely and prove the conjecture (see Theorem 3.3).

Example 2.7. The monomials of order at most two in $\mathcal{F}$ and $\mathcal{F}_s$ are

$$\begin{aligned}\mathcal{F} \cap k[x^{(\leqslant 2)}] &= \{1, x, x', (x')^2, x'x'', x'', (x'')^2, (x'')^3\},\\ \mathcal{F}_s \cap k[x^{(\leqslant 2)}] &= \{1, x', x'', (x'')^2\}.\end{aligned}$$

Using this, one can produce the monomials of order at most one in $\mathcal{F}_{1,1}$ and $\mathcal{F}_{2,0}$

$$\begin{aligned}\mathcal{F}_{1,1} \cap k[x^{(\leqslant 1)}] &= \{1, x, xx', x', (x')^2, (x')^3\},\\ \mathcal{F}_{2,0} \cap k[x^{(\leqslant 1)}] &= \{1, x, x^2, xx', x(x')^2, x', (x')^2, (x')^3, (x')^4\}\end{aligned}$$

For example, $(x')^3 \in \mathcal{F}_{1,1}$ can be written as $(x')^2 \cdot x'$, where $(x')^2 \in \mathcal{F}$ and $x' \in \mathcal{F}_s$. Likewise, for the monomials of order at most two, we can write

$$\begin{aligned}\mathcal{F}_{1,1} \cap k[x^{(\leqslant 2)}] = \big\{&1, x, x', x'', xx', xx'', (x')^2, x'x'', (x'')^2, x(x'')^2, (x')^3, (x')^2x'', x'(x'')^2, (x'')^3,\\ &(x')^2(x'')^2, x'(x'')^3, (x'')^4, (x'')^5\big\}.\end{aligned}$$

3. Main results

The algebra of regular functions on the arc space of a fat point $x^m = 0$ admits a natural filtration by subalgebras induced by the truncation of arcs. Our first main result, Theorem 3.1, gives a simple formula for the dimension of the subalgebra induced by the truncation at order h. Corollary 3.2 gives the generating series for these dimensions, as in (2).

Theorem 3.1. *Let m and h be positive integers and k be a differential field of zero characteristic. Then*

$$\dim_k\bigl(k[x^{(\leqslant h)}]/(k[x^{(\leqslant h)}]\cap\langle x^m\rangle^{(\infty)})\bigr) = m^{h+1}.$$

Corollary 3.2. *Let m be a positive integer and k be a differential field of zero characteristic. Then*

$$\sum_{\ell=0}^{\infty}\dim_k\bigl(k[x^{(\leqslant \ell)}]/\langle x^m\rangle^{(\infty)}\bigr)\cdot t^{\ell} = \frac{m}{1-mt},$$

where $k[x^{(\leqslant \ell)}]/\langle x^m\rangle^{(\infty)} := k[x^{(\leqslant \ell)}]/(k[x^{(\leqslant \ell)}]\cap\langle x^m\rangle^{(\infty)})$.

Given a polynomial ideal and monomial ordering, the monomials which do not appear as leading terms of the elements of the ideal are called *standard* monomials. Motivated by applications to combinatorics, Afsharijoo [2021, §5] used computations experiment to conjecture a description of the standard monomials of $\langle x^m\rangle^{(\infty)}$ with respect to the degree lexicographic ordering. Our second main result, Theorem 3.3, gives such a description and, combined with Lemma 4.10, establishes the conjecture.

Theorem 3.3. *Let k be a differential field of zero characteristic. Consider a degree lexicographic monomial ordering on $k[x^{(\infty)}]$ with the variables ordered as $x < x' < x'' < \cdots$. Let m and i be positive integers with $1\leqslant i\leqslant m$. Then the set of standard monomials of the ideal $\langle x^i, (x^m)^{(\infty)}\rangle$ is $\mathcal{F}_{i-1,m-i}$; see Definition 2.5. Note that, for $i = m$, we obtain the differential ideal $\langle x^m\rangle^{(\infty)}$.*

Corollary 3.4. *Theorem 3.3 also holds for the following orderings*:

- ***purely lexicographic*** *with the variables ordered as in Theorem 3.3*;
- ***weighted lexicographic***: *monomials are first compared by the sum of the orders and then lexicographically as in Theorem 3.3.*

Remark 3.5. The multiplicity of the scheme of polynomial arcs of degree less than h of $x = 0$, defined by $\langle x^m, x^{(h)}\rangle^{(\infty)}$, has been studied in [Ait El Manssour and Sattelberger 2023]. It was shown that this multiplicity, equal to $\dim_k k[x^{(\infty)}]/\langle x^m, x^{(h)}\rangle^{(\infty)}$, is a polynomial in m of degree h which is the Erhart polynomial of some lattice polytope [Ait El Manssour and Sattelberger 2023, Theorem 2.5]. Theorem 3.1 together with a natural surjective morphism $k[x^{(<h)}]/\langle x^m\rangle^{(\infty)} \to k[x^{(\infty)}]/\langle x^m, x^{(h)}\rangle^{(\infty)}$ implies that this polynomial is bounded by m^h.

4. Proofs

4.1. *Key technical tool: embedding into the exterior algebra.*

Notation 4.1. Let k be a field. Then, for $\boldsymbol{\xi} = (\xi_0, \xi_1, \dots, \xi_n)$, we introduce a countable collection of symbols $\{\xi_i^{(j)} \mid 0 \leqslant i \leqslant n,\ j \geqslant 0\}$, and by $\Lambda_k(\boldsymbol{\xi}^{(\infty)})$, we denote the exterior algebra of a k-vector space spanned by these symbols. $\Lambda_k(\boldsymbol{\xi}^{(\infty)})$ is equipped with a structure of a (noncommutative) differential algebra by

$$\big(\xi_j^{(i)}\big)' := \xi_j^{(i+1)} \quad \text{for every } i \geqslant 0 \text{ and } 0 \leqslant j \leqslant n.$$

The next proposition is a minor modification of [Pogudin 2014, Lemma 1]. The proof we will give is a simplification of the proof in [Pogudin 2014, Lemma 1], which will be extended to a proof of Lemma 4.4.

Proposition 4.2. *Let m be a positive integer. Consider $\boldsymbol{\eta} = (\eta_0, \dots, \eta_{m-2})$ and $\boldsymbol{\xi} = (\xi_0, \dots, \xi_{m-2})$. Let*

$$\Lambda := \Lambda_k(\boldsymbol{\eta}^{(\infty)}) \otimes \Lambda_k(\boldsymbol{\xi}^{(\infty)}),$$

which is equipped with a structure of differential algebra (as a tensor product of differential algebras, using the Leibnitz rule, that is $(a \otimes b)' := a' \otimes b + a \otimes b'$). Consider a differential homomorphism $\varphi\colon k[x^{(\infty)}] \to \Lambda$ defined by

$$\varphi(x) := \sum_{i=0}^{m-2} \eta_i \otimes \xi_i.$$

Then the kernel of φ is $\langle x^m \rangle^{(\infty)}$.

Example 4.3. Consider the case $m = 3$. Then we will have

$$\varphi(x) = \eta_0 \otimes \xi_0 + \eta_1 \otimes \xi_1.$$

The image of x' will then be

$$\varphi(x') = (\varphi(x))' = \eta_0' \otimes \xi_0 + \eta_0 \otimes \xi_0' + \eta_1' \otimes \xi_1 + \eta_1 \otimes \xi_1'.$$

One can show, for example, that $(x')^4 \notin \langle x^3 \rangle^{(\infty)}$ by showing that $\varphi((x')^4) \neq 0$:

$$\varphi((x')^4) = 24(\eta_0 \wedge \eta_0' \wedge \eta_1 \wedge \eta_1') \otimes (\xi_0 \wedge \xi_0' \wedge \xi_1 \wedge \xi_1') \neq 0.$$

Furthermore, a direct computation shows that $\varphi((x')^5) = 0$. Combined with Proposition 4.2, this implies that $(x')^5 \in \langle x^3 \rangle^{(\infty)}$.

Proof of Proposition 4.2. Consider $(\varphi(x))^m$. This is a sum of tensor products of exterior polynomials of degree m in $m-1$ variables, so it must be zero. Since $(\varphi(x))^m = 0$ and φ is a differential homomorphism, we conclude that $\operatorname{Ker} \varphi \supset \langle x^m \rangle^{(\infty)}$.

Now we will prove the inverse inclusion. We define the weighted degree inverse lexicographic ordering $\prec$ on $k[x^{(\infty)}]$ (see [Zobnin 2008, p. 524]): $M \prec N$ if and only if

- $\operatorname{tord} M < \operatorname{tord} N$, where tord is defined as the sum of the orders, or
- $\operatorname{tord} M = \operatorname{tord} N$ and $\deg M < \deg N$, or
- $\operatorname{tord} M = \operatorname{tord} N$, $\deg M = \deg N$, and N is lexicographically *lower* than M, where the variables are ordered $x < x' < x'' < \cdots$.

For example, we will have $x \prec x' \prec x'' \prec \cdots$ and $xx'' \prec (x')^2$. Then, for every $h \geqslant 0$, the leading monomial of $(x^m)^{(h)}$ with respect to $\prec$ is $(x^{(q)})^{m-r}(x^{(q+1)})^r$, where q and r are the quotient and the reminder of the integer division of h by m, respectively. Let $\mathcal{M}$ be the set of all monomials not divisible by any monomial of the form $(x^{(q)})^{m-r}(x^{(q+1)})^r$. Then we can characterize $\mathcal{M}$ as

$$\mathcal{M} = \bigl\{x^{(h_0)} \cdots x^{(h_\ell)} \mid h_0 \leqslant \cdots \leqslant h_\ell,\ \forall\, 0 \leqslant i \leqslant \ell - m + 1\colon h_{i+m-1} > h_i + 1\bigr\}.$$

We will define a linear map ψ from $\mathcal{M}$ to the set of monomials in Λ with the following properties:

(P1) *For every $P \in \mathcal{M}$, we have that $\psi(P) \neq 0$.*

(P2) *For every $P \in \mathcal{M}$, the monomial $\psi(P)$ appears in the polynomial $\varphi(P)$ and, for any $P_0 \in \mathcal{M}$ such that $P_0 \prec P$, the monomial $\psi(P)$ does not appear in the polynomial $\varphi(P_0)$.*

Informally speaking, $\psi(M)$ is the "leading monomial" in $\varphi(M)$. Once such a map ψ has been defined, we can prove the proposition as follows: Let $Q \in \operatorname{Ker}\varphi \setminus \langle x^m \rangle^{(\infty)}$. By replacing Q with the result of the reduction of Q by x^m, $(x^m)'$, ... with respect to $\prec$, we can further assume that all the monomials in Q belong to $\mathcal{M}$[1]. Let Q_0 be the largest of them. By (P1) and (P2), $\varphi(Q_0)$ will involve $\psi(Q_0)$ and $\varphi(Q - Q_0)$ will not, so $\varphi(Q) \neq 0$. This contradicts the assumption that $Q \in \operatorname{Ker}\varphi$. The proposition is proved.

Therefore, it remains to define ψ satisfying (P1) and (P2). We will start with the case $m = 2$ to show the main idea while keeping the notation simple. We define ψ by

$$\psi\bigl(x^{(h_0)} \cdots x^{(h_\ell)}\bigr) := \bigl(\eta^{(0)} \otimes \xi^{(h_0)}\bigr) \wedge \bigl(\eta^{(1)} \otimes \xi^{(h_1 - 1)}\bigr) \wedge \cdots \wedge \bigl(\eta^{(\ell)} \otimes \xi^{(h_\ell - \ell)}\bigr), \tag{5}$$

where $h_0 \leqslant h_1 \leqslant \cdots \leqslant h_\ell$. For proving (P1), we observe that, if $h_{i+1} > h_i + 1$ for all i, then $h_0 < h_1 - 1 < h_2 - 2 < \cdots < h_\ell - \ell$, so there are no coinciding ξ's in (5). The construction for arbitrary m will consist of splitting the monomial into $m - 1$ interlacing submonomials and applying (5) with (η_i, ξ_i) to i-th submonomial. More formally, if $h_0 \leqslant h_1 \leqslant \cdots \leqslant h_\ell$, we define

$$\psi\bigl(x^{(h_0)} \cdots x^{(h_\ell)}\bigr) := \prod_{i=0}^{\ell} \Bigl(\eta_{i \,\%\, (m-1)}^{([i/(m-1)])} \otimes \xi_{i \,\%\, (m-1)}^{(h_i - [i/(m-1)])}\Bigr), \tag{6}$$

where $a \,\%\, b$ denotes the remainder of the division of a by b, and $[\alpha]$ denotes the integer part of α. Property (P1) is proved by applying (P1) for $m = 2$ to each submonomial.

For proving (P2), consider $P_0 \in \mathcal{M}$ with $P_0 \preceq P$ and $\psi(P)$ appearing in $\varphi(P_0)$. Since ψ preserves the total orders and doubles the degrees, we have $\operatorname{tord} P_0 = \operatorname{tord} P$ and $\deg P_0 = \deg P$. Let $H := \operatorname{ord} P_0$. Since $P_0 \preceq P$, we have $H \geqslant h_\ell$. Since the maximal orders of η and ξ in $\psi(P)$ do not exceed $[\ell/(m-1)]$ and $h_\ell - [\ell/(m-1)]$, respectively, we have $H \leqslant h_\ell$. Thus, $H = h_\ell$. Applying the same argument recursively to $P/x^{(h_\ell)}$ and $P_0/x^{(h_\ell)}$, we conclude that $P = P_0$.

We will prove that $\varphi(P)$ involves $\psi(P)$ by induction on $\deg P$. The case $\deg P = 0$ is clear. Consider P, with $\deg P > 0$. Similarly to the preceding argument, one can obtain $\psi(P)$ (from $\psi(P/x^{(\ell)})$) only by

[1] Interestingly, although it is known that x^m, $(x^m)'$, ... form a Gröbner basis, we do not really need to use this fact here since a reduction with respect to any set of polynomials is well defined.

taking $\eta_{\ell\,\%\,(m-1)}^{([\ell/(m-1)])} \otimes \xi_{\ell\,\%\,(m-1)}^{(h_\ell-[\ell/(m-1)])}$ (i.e., the last term in (6)) from one of the occurrences of $x^{(h_\ell)}$ in P. Therefore, the coefficient in front of $\psi(P)$ in $\varphi(P)$ will be, up to sign, equal to $\deg_{x^{(h_\ell)}}$ times the coefficient in front of $\psi(P/x^{(h_\ell)})$ in $\varphi(P/x^{(h_\ell)})$. The latter is nonzero by the induction hypothesis. $\square$

Lemma 4.4. *In the notation of Proposition 4.2, let $1 \leqslant r < m$. Then the preimage of the ideal in Λ generated by $\eta_{r-1} \otimes \xi_{r-1}, \dots, \eta_{m-2} \otimes \xi_{m-2}$ under φ is equal to $\langle (x^m)^{(\infty)}, x^r \rangle$.*

Proof. We first prove that the image of x^r belongs to $\langle \eta_{r-1} \otimes \xi_{r-1}, \dots, \eta_{m-2} \otimes \xi_{m-2} \rangle$. This is because $\varphi(x^r)$ is the sum of monomials which are products of r different $\eta_i \otimes \xi_i$. Since there are $m-1$ of them, every such monomial will involve at least one of the last $m-r$ of the $\eta_i \otimes \xi_i$.

Let us consider a polynomial $g \in k[x^{(\infty)}] \setminus \langle (x^m)^{(\infty)}, x^r \rangle$ and prove that $\varphi(g)$ does not belong to $\langle \eta_{r-1} \otimes \xi_{r-1}, \dots, \eta_{m-2} \otimes \xi_{m-2} \rangle$. We can assume that each monomial P of g belongs to

$$\mathcal{M}_r = \{M \in \mathcal{M} \mid \deg_x M < r \text{ or } 0 < h_{r-1}\}.$$

We will use the map ψ defined in (6). In fact, $\psi(P)$ does not involve the zero-order derivatives of $\xi_{r-1}, \dots, \xi_{m-2}$, since $h_i - [i/(m-1)]$ can only be zero for a monomial in $\mathcal{M}$ only if $i \leqslant r-2$. Thus,

$$\psi(P) \notin \langle \eta_{r-1} \otimes \xi_{r-1}, \dots, \eta_{m-2} \otimes \xi_{m-2} \rangle.$$

Assume that P_0 is the largest summand that appears in g. Then $\varphi(P_0)$ involves $\psi(P_0)$, but $\varphi(g - P_0)$ does not. Therefore, $\varphi(g)$ does not belong to $\langle \eta_{r-1} \otimes \xi_{r-1}, \dots, \eta_{m-2} \otimes \xi_{m-2} \rangle$. $\square$

4.2. ***Upper bounds for the dimension.*** Throughout the section, we fix a differential field k of zero characteristic.

Proposition 4.5. *Let m, h be positive integers. We denote by $A_{m,h}$ the subalgebra of $k[x^{(\infty)}]/\langle x^m \rangle^{(\infty)}$ generated by the images of $x, x', \dots, x^{(h)}$. Then*

$$\dim A_{m,h} \leqslant m^{h+1}.$$

First we describe a general construction which will be a special case of the so-called associated graded algebra. Let $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ be a $\mathbb{Z}_{\geqslant 0}$-graded algebra over k equipped with a homogeneous derivation of weight one (that is, $A_i' \subseteq A_{i+1}$ for every $i \geqslant 0$). We introduce a map $\operatorname{gr}\colon A \to A$ defined as follows: Consider a nonzero $a \in A$, and let i be the largest index such that $a \in A_i \oplus A_{i+1} \oplus \cdots$. Then we define $\operatorname{gr}(a)$ to be the image of the projection of a onto A_i along $A_{i+1} \oplus A_{i+2} \oplus \cdots$. In other words, we replace each element with its lowest homogeneous component.

Note that gr is not a homomorphism, it is not even a linear map. However, it has two important properties we state as a lemma.

Lemma 4.6. (1) *Let $a_1, \dots, a_n \in A$, and let $p \in k[\mathbf{x}^{(\infty)}]$ be a differential monomial. Then*

$$p(\operatorname{gr}(a_1), \dots, \operatorname{gr}(a_n)) \neq 0 \quad \Longrightarrow \quad \operatorname{gr}(p(a_1, \dots, a_n)) = p(\operatorname{gr}(a_1), \dots, \operatorname{gr}(a_n)).$$

(2) *If $a_1, \dots, a_n \in A$ are k-linearly dependent, then $\operatorname{gr}(a_1), \dots, \operatorname{gr}(a_n)$ also are k-linearly dependent.*

Proof. To prove the first part, one sees that p does not vanish on the lowest homogeneous parts of $a_1, \dots, a_n$, so the homogeneity of the multiplication and derivation imply that taking the lowest homogeneous part commutes with applying p for $a_1, \dots, a_n$.

To prove the second part, let i be the lowest grading appearing among $a_1, \dots, a_n$. Restricting to the component of this weight, one gets a linear relation for $\operatorname{gr}(a_1), \dots, \operatorname{gr}(a_n)$. □

Lemma 4.7. *Let A be a graded differential algebra as above. Consider elements $a_1, \dots, a_n$ in A, and denote the algebras (not differential) generated by $a_1, \dots, a_n$ and $\operatorname{gr}(a_1), \dots, \operatorname{gr}(a_n)$ by B and B_{gr}, respectively. Then* $\dim B_{\mathrm{gr}} \leqslant \dim B$.

Proof. The algebra B_{gr} is spanned by all the monomials in $\operatorname{gr}(a_1), \dots, \operatorname{gr}(a_n)$. We choose a basis in this spanning set, that is, we consider monomials $p_1, \dots, p_N \in k[x_1, \dots, x_n]$ such that

$$p_1\bigl(\operatorname{gr}(a_1), \dots, \operatorname{gr}(a_n)\bigr), \dots, p_N\bigl(\operatorname{gr}(a_1), \dots, \operatorname{gr}(a_n)\bigr)$$

form a basis of B_{gr}. The first part of Lemma 4.6 implies that

$$\operatorname{gr}\bigl(p_i(a_1, \dots, a_n)\bigr) = p_i\bigl(\operatorname{gr}(a_1), \dots, \operatorname{gr}(a_n)\bigr) \quad \text{for every } 1 \leqslant i \leqslant N.$$

Then the second part of Lemma 4.6 implies that $p_1(a_1, \dots, a_n), \dots, p_N(a_1, \dots, a_n)$ are linearly independent. Since they belong to B, we have $\dim B \geqslant N = \dim B_{\mathrm{gr}}$. □

Proof of Proposition 4.5. Let Λ and φ be the exterior algebra and the homomorphism from Proposition 4.2. Proposition 4.2 implies that $A_{m,h}$ is isomorphic to the subalgebra of Λ generated by

$$\sum_{i=0}^{m-2} \eta_i \otimes \xi_i, \ \sum_{i=0}^{m-2} (\eta_i \otimes \xi_i)', \ \sum_{i=0}^{m-2} (\eta_i \otimes \xi_i)'', \dots, \sum_{i=0}^{m-2} (\eta_i \otimes \xi_i)^{(h)}.$$

We define a grading on Λ by setting the weights of $\eta_j^{(i)}$ and $\xi_j^{(i)}$ to be equal to i for every $i \geqslant 0$ and $0 \leqslant j < m-1$. The exterior algebra Λ becomes a graded algebra, and the derivation is homogeneous of weight one.

We fix $h \geqslant 0$ and consider the following elements of Λ:

$$\tilde{\alpha}_{j,i} := (1+\partial)^i \alpha_j \quad \text{for } i \geqslant 0, \ 0 \leqslant j < m-1, \text{ and } \alpha \in \{\eta, \xi\},$$

where ∂ is the operator of differentiation. We introduce

$$v_i := \sum_{j=0}^{m-2} \tilde{\eta}_{j,i} \otimes \tilde{\xi}_{j,i} \quad \text{for } 0 \leqslant i \leqslant h,$$

and let Y_h be the algebra generated by $v_0, \dots, v_h$. For every $0 \leqslant i \leqslant h$, we have $v_i^m = 0$, so Y_h is spanned by the products of the form

$$v_0^{d_0} v_1^{d_1} \dots v_h^{d_h}, \quad \text{where } 0 \leqslant d_0, \dots, d_h < m.$$

Therefore, $\dim Y_h \leqslant m^{h+1}$.

Claim. *There is an invertible $(h+1)\times(h+1)$ matrix M over $\mathbb{Q}$ such that, for $u_0, \ldots, u_h$ defined by*

$$(u_0, \ldots, u_h)^T := M(v_0, \ldots, v_h)^T, \tag{7}$$

we have

$$\operatorname{gr}(u_i) = \sum_{j=0}^{m-2} (\eta_j \otimes \xi_j)^{(i)} \quad \textit{for every } 0 \leqslant i \leqslant h.$$

We will first demonstrate how the proposition follows from the claim, and then we prove the claim. Since M is invertible, $u_0, \ldots, u_h$ generate Y_h as well. Since $\operatorname{gr}(u_0), \ldots, \operatorname{gr}(u_h)$ generate $A_{m,h}$, Lemma 4.7 implies that $m^{h+1} \geqslant \dim Y_h \geqslant \dim A_{m,h}$.

Therefore, it remains to prove the claim. For every $0 \leqslant i \leqslant h$, we can write

$$v_i = (1\otimes 1 + 1 \otimes \partial)^i (1 \otimes 1 + \partial \otimes 1)^i v_0 = (1 \otimes 1 + 1 \otimes \partial + \partial \otimes 1 + \partial \otimes \partial)^i v_0.$$

We set $u_i := (1 \otimes \partial + \partial \otimes 1 + \partial \otimes \partial)^i v_0$ for every $0 \leqslant i \leqslant h$. Note that, since $1 \otimes \partial + \partial \otimes 1$ is just the original derivation on Λ, we have

$$\operatorname{gr}(u_i) = (1 \otimes \partial + \partial \otimes 1)^i v_0 = v_0^{(i)} = \sum_{j=0}^{m-2} (\eta_j \otimes \xi_j)^{(i)}. \tag{8}$$

By expanding the binomial $(1\otimes 1 + (1 \otimes \partial + \partial \otimes 1 + \partial \otimes \partial))^i$, we can write $v_i = \sum_{j=0}^{i} \binom{i}{j} u_j$. Then we have

$$(v_0, \ldots, v_h)^T = \widetilde{M}(u_0, \ldots, u_h)^T, \tag{9}$$

where $\widetilde{M}$ is the $(h+1)\times(h+1)$-matrix with the (i, j)-th entry being $\binom{i}{j}$. Since $\widetilde{M}$ is lower-triangular with ones on the diagonal, it is invertible. We set $M := \widetilde{M}^{-1}$. So we have $(u_0, \ldots, u_h)^T := M(v_0, \ldots, v_h)^T$, which together with (8) finishes the proof of the claim. $\square$

By combining the proof of Proposition 4.5 with Lemma 4.4, we can extend Proposition 4.5 as follows:

Corollary 4.8. *Let m, h, i be positive integers with $1 \leqslant i \leqslant m$. By $A_{(m,i),h}$ we denote the subalgebra of $k[x^{(\infty)}]/\langle x^i, (x^m)^{(\infty)}\rangle$ generated by the images of $x, x', \ldots, x^{(h)}$. Then*

$$\dim A_{(m,i),h} \leqslant i \cdot m^h.$$

Proof. The proof will be a refinement of the proof of Proposition 4.5, and we will use the notation from there. Let π be the canonical homomorphism $\pi\colon \Lambda \to \Lambda_i := \Lambda/\langle \xi_{i-1}\otimes \eta_{i-1}, \ldots, \xi_{m-2}\otimes\eta_{m-2}\rangle$. Since the ideal $\langle \xi_{i-1}\otimes \eta_{i-1}, \ldots, \xi_{m-2}\otimes\eta_{m-2}\rangle$ is homogeneous with respect to the grading on Λ, there is a natural grading on Λ_i.

We have $A_{(m,i),h} \cong \pi(A_{m,h})$. Since π is a homogeneous homomorphism, $\pi(A_{m,h})$ is generated by $\pi(\operatorname{gr}(u_0)), \ldots, \pi(\operatorname{gr}(u_h))$ from (7), so $\dim A_{(m,i),h} = \dim \pi(A_{m,h}) \leqslant \dim \pi(Y_h)$. We observe that $\pi(v_0)^i = 0$, so $\pi(Y_h)$ is spanned by products of the form

$$\pi(v_0)^{d_0}\pi(v_1)^{d_1}\cdots\pi(v_h)^{d_h},$$

where $0 \leqslant d_0 < i$ and $0 \leqslant d_1, \ldots, d_h < m$. Therefore, $\dim \pi(Y_h) \leqslant i \cdot m^h$. $\square$

4.3. ***Combinatorial properties of fair monomials.***

Definition 4.9 (nonoverlapping monomials). We say that two monomials $m_1, m_2 \in k[x^{(\infty)}]$ *do not overlap* if $\operatorname{ord} m_1 \leqslant \operatorname{lord} m_2$ or $\operatorname{ord} m_2 \leqslant \operatorname{lord} m_1$.

Lemma 4.10. *Let m, i be integers with $0 \leqslant i \leqslant m$. Let $P \in \mathcal{F}_{i,m-i}$. Then there exist $P_1, \dots, P_i \in \mathcal{F}$ and $P_{i+1}, \dots, P_m \in \mathcal{F}_s$ such that*

$$P = P_1 \cdots P_m \quad \textit{and, for every } 1 \leqslant i < m, \quad \operatorname{ord} P_i \leqslant \operatorname{lord} P_{i+1}.$$

Remark 4.11. Lemma 4.10 implies that the set $\mathcal{F}_{i-1,m-i}$ from Theorem 3.3 coincides with the set of standard monomials conjectured by Afsharijoo [2021, §5].

Proof. Suppose that P can be written as

$$P = \big(x^{(h_{1,0})} \cdots x^{(h_{1,\ell_1})}\big) \cdots \big(x^{(h_{m,0})} \cdots x^{(h_{m,\ell_m})}\big),$$

where each $(x^{(h_{i,0})} \cdots x^{(h_{i,\ell_i})})$ belongs to $\mathcal{F}$ or $\mathcal{F}_s$ and $h_{1,0} \leqslant h_{2,0} \leqslant \cdots \leqslant h_{m,0}$. We first prove that we can make the product to be a product of nonoverlapping monomials.

Let us sort the orders $h_{1,0}, h_{1,1}, \dots, h_{m,\ell_m}$ in the ascending order

$$\big\{(r_{1,0}, \dots, r_{1,\ell_1}); (r_{2,0}, \dots, r_{2,\ell_2}); \dots ; (r_{m,0}, \dots, r_{m,\ell_m})\big\}.$$

Claim. *For all $0 \leqslant i \leqslant m$, we have $h_{i,0} \leqslant r_{i,0}$.*

In the whole list of the $h_{i,j}$, all the numbers to the right from $h_{i,0}$ are $\geqslant h_{i,0}$. Therefore, after sorting, $h_{i,0}$ will either stay or move to the left. Thus, $h_{i,0} \leqslant r_{i,0}$, so the claim is proved.

Hence if $x^{(h_{i,0})} \cdots x^{(h_{i,\ell_i})}$ was a fair (respectively, strongly fair) monomial then $x^{(r_{i,0})} \cdots x^{(r_{i,\ell_i})}$ is a fair (respectively, strongly fair) monomial.

Now we will move all the strongly fair monomials to the right in the decomposition of P. We first prove that, for every $Q = Q_1 Q_2$ such that $Q_1 \in \mathcal{F}_s$, $Q_2 \in \mathcal{F}$, and $\operatorname{ord} Q_1 \leqslant \operatorname{lord} Q_2$, there exist $\widetilde{Q}_1 \in \mathcal{F}_s$ and $\widetilde{Q}_2 \in \mathcal{F}$ such that $Q = \widetilde{Q}_1 \widetilde{Q}_2$ and $\operatorname{ord} \widetilde{Q}_1 \leqslant \operatorname{lord} \widetilde{Q}_2$. Let

$$Q_1 = x^{(h_{1,0})} \cdots x^{(h_{1,\ell_1})} \quad \text{and} \quad Q_2 = x^{(h_{2,0})} \cdots x^{(h_{2,\ell_2})},$$

where $\ell_1 < h_{1,0}$ and $\ell_2 \leqslant h_{2,0}$. If $\ell_2 < h_{2,0}$, then $Q_2 \in \mathcal{F}_s$; so we are done. Otherwise, $\ell_1 + 1 \leqslant h_{1,0}$ implies that $Q_1 x^{(h_{1,0})}$ is a fair monomial, and $\ell_2 - 1 < h_{2,0}$ implies that $Q_2 / x^{(h_{1,0})} \in \mathcal{F}_s$. Thus, we can take $\widetilde{Q}_1 := Q_1 x^{(h_{1,0})}$ and $\widetilde{Q}_2 := Q_2 / x^{(h_{1,0})}$.

Applying the described transformation while possible to the nonoverlapping decomposition of P, one can arrange that the last $m - i$ components are strongly fair. $\square$

Proposition 4.12. *For every positive integers m, h, i with $0 \leqslant i \leqslant m$, the cardinality of $\mathcal{F}_{i,m-i} \cap k[x^{(\leqslant h)}]$ is equal to $(i+1) \cdot (m+1)^h$.*

The proof of the proposition will use the following lemma:

Lemma 4.13. *For every integers h and d, we have*

$$\left|\left\{P \mid P \in \mathcal{F} \cap k[x^{\leqslant h}] \text{ and } \deg P = d\right\}\right| = \binom{h+1}{d}.$$

If one replaces $\mathcal{F}$ with $\mathcal{F}_s$, the cardinality will be $\binom{h}{d}$.

Proof. Let $x^{(h_0)} \cdots x^{(h_\ell)} \in \mathcal{F}$ such that $\ell \leqslant h_0 \leqslant \cdots \leqslant h_\ell$. We define a map

$$(h_0, \ldots, h_\ell) \mapsto (h_0 - \ell, h_1 - \ell - 1, \ldots, h_\ell).$$

The map assigns to the orders of a monomial in $\mathcal{F} \cap k[x^{\leqslant h}]$ a list of strictly increasing nonnegative integers not exceeding h. A direct computation shows that this map is a bijection. Since the number of such sequences of length d is equal to the number of subsets of $[0, 1, \ldots, h]$ of cardinality d, the number of monomials is $\binom{h+1}{d}$.

The case of $\mathcal{F}_s$ is analogous with the only difference being that the subset will be in $[1, 2, \ldots, h]$, thus yielding $\binom{h}{d}$. □

Proof of Proposition 4.12. We will prove the proposition by induction on m. For the base case, we have $\mathcal{F}_{0,0} = \{1\}$, so the statement is true.

Consider $m > 0$, and assume that for all smaller m the proposition is proved. We fix $0 \leqslant i \leqslant m$. Consider a monomial $P \in \mathcal{F}_{i,m-i} \cap k[x^{(\leqslant h)}]$, let $P_1 \cdots P_m$ be a decomposition from Lemma 4.10 with $\deg P_m$ being as large as possible. We denote $\operatorname{tail} P := P_m$ and $\operatorname{head} P := P_1 \cdots P_{m-1}$.

We will show that the map $P \to (\operatorname{head} P, \operatorname{tail} P)$ defines a bijection between $\mathcal{F}_{i,m-i}$ and

$$\begin{aligned} &\text{for } i < m\colon \{(Q_0, Q_1) \in \mathcal{F}_{i,m-i-1} \times \mathcal{F}_s \mid \operatorname{ord} Q_0 \leqslant \deg Q_1\},\\ &\text{for } i = m\colon \{(Q_0, Q_1) \in \mathcal{F}_{m-1,0} \times \mathcal{F} \mid \operatorname{ord} Q_0 < \deg Q_1\}. \end{aligned} \tag{10}$$

We will prove the case $i < m$, as the proof in the case $i = m$ is analogous. First we will show that, for every $P \in \mathcal{F}_{i,m-i}$, we have $\operatorname{ord}\operatorname{head} P \leqslant \deg\operatorname{tail} P$. Assume the contrary, and let $\ell := \operatorname{ord}\operatorname{head} P > \deg\operatorname{tail} P$. Then we will have

$$\operatorname{lord}(x^{(\ell)} \operatorname{tail} P) \geqslant \min(\ell, \operatorname{lord}\operatorname{tail} P) = \ell \geqslant \deg(x^{(\ell)} \operatorname{tail} P).$$

This implies that $x^{(\ell)} \operatorname{tail} P \in \mathcal{F}_s$. Thus, in the decomposition of Lemma 4.10, we could have taken P_m to be $x^{(\ell)} \operatorname{tail} P$. This contradicts the maximality of $\deg\operatorname{tail} P$. In the other direction, if $Q_0 \in \mathcal{F}_{i,m-i-1}$ and $Q_1 \in \mathcal{F}_s$ such that $\operatorname{ord} Q_0 \leqslant \deg Q_1$, then $Q_0 Q_1 \in \mathcal{F}_{i,m-i}$. Moreover, since $x^{(\operatorname{ord} Q_0)} Q_1 \notin \mathcal{F}$, we have $\operatorname{tail}(Q_0 Q_1) = Q_1$.

We will now use the bijection (10) to count the elements in $\mathcal{F}_{i,m-i} \cap k[x^{(\leqslant h)}]$. For $i < m$,

$$\begin{aligned} \left|\mathcal{F}_{i,m-i} \cap k[x^{(\leqslant h)}]\right| &= \sum_{\ell=0}^{h} \left|\mathcal{F}_{i,m-i-1} \cap k[x^{(\leqslant \ell)}]\right| \cdot \left|\{Q_1 \in \mathcal{F}_s \cap k[x^{(\leqslant h)}] \mid \deg Q_1 = \ell\}\right| \\ &= \sum_{\ell=0}^{h} (i+1) \cdot m^\ell \binom{h}{\ell} = (i+1) \cdot (m+1)^h \quad \text{(by Lemma 4.13).} \end{aligned}$$

For $i = m$:

$$\begin{aligned}\left|\mathcal{F}_{m,0} \cap k[x^{(\leqslant h)}]\right| &= \sum_{\ell=0}^{h+1} \left|\mathcal{F}_{m-1,0} \cap k[x^{(<\ell)}]\right| \cdot \left|\{Q_1 \in \mathcal{F}_s \cap k[x^{(\leqslant h)}] \mid \deg Q_1 = \ell\}\right| \\ &= \sum_{\ell=0}^{h+1} m^\ell \binom{h+1}{\ell} = (m+1)^{h+1} \quad \text{(by Lemma 4.13)}.\end{aligned}$$

Thus, the proposition is proved. □

4.4. *Lower bounds for the dimension.*

Notation 4.14. For a differential polynomial $P \in k[\boldsymbol{x}^{(\infty)}]$ and $1 \leqslant i \leqslant n$, we define

- $\operatorname{tord}_{x_i} P$ to be the *total order* of P in x_i, that is, the largest sum of the orders of the derivatives of x_i among the monomials of P;
- $\deg_{x_i^{(\infty)}} P$ to be the *total degree* of P with respect to the variables $x_i, x_i', x_i'', \ldots$.
- We fix a monomial ordering $\prec$ on $k[\boldsymbol{x}^{(\infty)}]$ defined as follows: To each differential monomial $M = x_{i_0}^{(h_0)} x_{i_1}^{(h_1)} \cdots x_{i_\ell}^{(h_\ell)}$ with $(h_0, i_0) \preceq_{\text{lex}} (h_1, i_1) \preceq_{\text{lex}} \cdots \preceq_{\text{lex}} (h_\ell, i_\ell)$, we assign a tuple

$$(\ell,\, h_\ell,\, h_{\ell-1},\, \ldots,\, h_0,\, i_\ell,\, i_{\ell-1},\, \ldots,\, i_0),$$

  and compare monomials by comparing the corresponding tuples lexicographically.

Definition 4.15 (isobaric ideal). An ideal $I \subset k[x^{(\infty)}]$ is called *isobaric* if it can be generated by isobaric polynomials, that is, polynomials with all the monomials having the same total order.

Proposition 4.16. *For $i = 1, 2$, the elements of $\mathcal{F}_{i-1,2-i}$ are the standard monomials modulo $\langle (x^2)^{(\infty)}, x^i \rangle$.*

Proof. We use Proposition 4.2 to obtain the differential homomorphism $\varphi \colon k[x^{(\infty)}] \to \Lambda$ defined by $\varphi(x) = \eta \otimes \xi$ (we will use η and ξ instead of η_0 and ξ_0 for brevity). Let $\tilde{\varphi}$ be the composition of φ with the projection onto $\Lambda / \langle \eta \otimes \xi \rangle$. We will prove the proposition for the elements in $\mathcal{F}_{1,0}$, and the other case can be done in the same way by replacing φ with $\tilde{\varphi}$.

Let $X = x^{(h_0)} \cdots x^{(h_\ell)}$, where $h_0 \leqslant h_1 \leqslant \cdots \leqslant h_\ell$, be an element of $\mathcal{F}_{1,0}$. We will show that a summand

$$B(X) := \left(\eta^{(h_0-\ell)} \wedge \eta^{(h_1-(\ell-1))} \wedge \cdots \wedge \eta^{(h_\ell)}\right) \otimes \left(\xi^{(\ell)} \wedge \xi^{(\ell-1)} \wedge \cdots \wedge \xi' \wedge \xi\right) \tag{11}$$

appears in $\varphi(X)$ with nonzero coefficient. We will prove this by induction on ℓ. The base case $\ell = 0$ is trivial, so let $\ell > 0$. Since $\eta^{(h_0-\ell)}$ may come only from one of the occurrences of $x^{(h_0)}$ in X, we must take $\eta^{(h_0-\ell)} \otimes \xi^{(\ell)}$ from one of the $x^{(h_0)}$. Therefore, the coefficient at $B(X)$ in $\varphi(X)$ is $\deg_{x^{(h_0)}} X$ times the coefficient at $B(X/x^{(h_0)})$ in $\varphi(X/x^{(h_0)})$, which is nonzero by the induction hypothesis.

Let $Y := x^{(s_0)} \cdots x^{(s_{\ell'})}$ be a monomial such that $Y \prec X$. We will prove by contradiction that $B(X)$ does not appear in $\varphi(Y)$. If it does, then $\deg(X) = \deg(Y) = \ell + 1 = \ell' + 1$. Moreover, there exists a permutation σ of $\{0, 1, \ldots, \ell\}$ such that

$$s_i - \sigma(i) = h_i - (\ell - i) \quad \text{for every } 0 \leqslant i \leqslant \ell.$$

The inequality $s_\ell \leqslant h_\ell$ implies $\sigma(\ell) = 0$, and thus, $s_\ell = h_\ell$. Therefore, $s_{\ell-1} \leqslant h_{\ell-1}$, which implies $\sigma(\ell - 1) = 1$, and thus, $s_{\ell-1} = h_{\ell-1}$. Continuing in this way, we show that

$$s_i = h_i \quad \text{for all } 0 \leq i \leq \ell,$$

which contradicts $Y \prec X$. Thus $B(X)$ cannot appear in the $\varphi(Y)$.

Assume that $X \in \operatorname{In}_{\prec}\langle x^2\rangle^{(\infty)}$. Then there exist monomials $P_1, \dots P_N$ such that $P_j \prec X$ for all $1 \leq j \leq N$ and

$$X - \sum_{j=1}^{N} \lambda P_j \in \langle x^2\rangle^{(\infty)}.$$

Hence, $\varphi(X) - \sum_{j=1}^{N} \lambda_j \varphi(P_j) = 0$. Since $P_j \prec X$ for all $1 \leq j \leq N$, $B(X)$ cannot be canceled in $\varphi(X) - \sum_{j=1}^{N} \lambda_j \varphi(P_j)$, which is a contradiction. Therefore, X is a standard monomial. □

Lemma 4.17. *Let $I_1 \subset k[y_1^{(\infty)}], \dots, I_s \subset k[y_s^{(\infty)}]$ be ideals, and we denote by M_i the set of the standard monomials modulo I_i with respect to degree lexicographic ordering for $1 \leqslant i \leqslant s$. Then the standard monomials with respect to the ordering $\prec$ (see Notation 4.14) modulo $\langle I_1, \dots, I_s\rangle \subset k[y_1^{(\infty)}, \dots, y_s^{(\infty)}]$ are*

$$M_1 \cdot M_2 \cdots M_s := \{m_1 m_2 \cdots m_s \mid m_1 \in M_1, \dots, m_s \in M_s\}.$$

Proof. For each I_i, consider the reduced Gröbner basis G_i of I_i with respect to the degree lexicographic ordering. For each pair $f, g \in G := G_1 \cup G_2 \cup \ldots \cup G_s$, their S-polynomial is reduced to zero by G

- if f, g belong to the same G_i, due to the fact that G_i is a Gröbner basis;
- otherwise, by the first Buchberger criterion (since f and g have coprime leading monomials). □

Proposition 4.18. *Let $I_1 \subset k[y_1^{(\infty)}], \dots, I_s \subset k[y_s^{(\infty)}]$ be homogeneous and isobaric ideals (not necessarily differential). By M_i we denote the set of standard monomials modulo I_i with respect to the degree lexicographic ordering for $1 \leqslant i \leqslant s$. We define a homomorphism (not necessarily differential)*

$$\varphi\colon k[x^{(\infty)}] \to k[y_1^{(\infty)}, \dots, y_s^{(\infty)}]/\langle I_1, \dots, I_s\rangle$$

by $\varphi(x^{(k)}) := y_1^{(k)} + \cdots + y_s^{(k)}$ and denote $I := \operatorname{Ker}(\varphi)$. Then the elements of

$$M := \left\{m_1 \ldots m_s \mid \forall 1 \leqslant i \leqslant s\colon m_i \in M_i \text{ and } \forall\, 1 \leqslant j < s\colon \operatorname{ord} m_j \leqslant \operatorname{lord} m_{j+1}\right\} \tag{12}$$

are standard monomials modulo I with respect to the ordering $\prec$ (but maybe not all the standard monomials).

Proof. Consider a monomial $P = x^{(h_0)} \cdots x^{(h_\ell)} \in M$, and fix a representation $P = m_1(x), \dots, m_s(x)$ as in (12). Assume that P is a leading monomial of I. Then there exist monomials $P_1, \dots, P_N$ such that

$$P - \sum_{j=1}^{N} \lambda_j P_j \in \operatorname{Ker}\varphi \quad \text{and} \quad \forall\, 1 \leqslant j \leqslant N\colon P_j \prec P.$$

Then $\varphi(P) - \sum \lambda_j \varphi(P_j) \in \langle I_1, \dots I_s\rangle$. We define $m := m_1(y_1) m_2(y_2) \cdots m_s(y_s)$.

Claim. *For every monomial $\widetilde{m} \neq m$ in $\varphi(P)$, there exists $1 \leqslant j \leqslant s$ such that either $\deg_{y_j^{(\infty)}} m \neq \deg_{y_j^{(\infty)}} \widetilde{m}$ or $\operatorname{tord}_{y_j} m \neq \operatorname{tord}_{y_j} \widetilde{m}$.*

Assume the contrary, that there exists $\widetilde{m}$ such that, for every $1 \leqslant j \leqslant s$, we have $d_i := \deg_{y_j^{(\infty)}} m = \deg_{y_j^{(\infty)}} \widetilde{m}$ and $\operatorname{tord}_{y_j} m = \operatorname{tord}_{y_j} \widetilde{m}$. We write $\widetilde{m} = \widetilde{m}_1(y_1) \cdots \widetilde{m}_s(y_s)$. Let $1 \leqslant j \leqslant s$ be the largest index such that $m_j \neq \widetilde{m_j}$. Since m_j contains d_j largest derivatives in $m_1(x) \cdots m_j(x) = \widetilde{m}_1(x) \cdots \widetilde{m}_j(x)$ and has the same total order as $\widetilde{m}_j$, we conclude that $m_j = \widetilde{m}_j$. Thus, the claim is proved.

We write the homogeneous and isobaric component of $\sum_{j=1}^N \lambda_j \varphi(P_j)$ of the same degree and total order in y_i as m for every $1 \leqslant i \leqslant s$ as $\sum_{i=1}^M \mu_i R_i$, where R_i is a differential monomial and $\mu_i \in k$ for every $1 \leqslant i \leqslant M$. Then such a homogeneous and isobaric component of $\varphi(P) - \sum_{j=1}^N \lambda_j \varphi(P_j)$ is $Q := m - \sum_{i=1}^M \mu_i R_i$ due to the claim. Since, for every $1 \leqslant i \leqslant s$, I_s is homogeneous and isobaric, $Q \in \langle I_1, \ldots, I_s \rangle$.

Note that for every $1 \leqslant i \leqslant M$, the differential monomial R_i is a summand of $\varphi(P_j)$ for some $1 \leqslant j \leqslant N$. Thus, if $P_j = x^{(s_0)} \cdots x^{(s_\ell)}$, then the derivatives that appear in the monomial R_i are of orders $s_0, \ldots, s_\ell$. Hence, $P_j \prec P$ implies $R_j \prec m$. Therefore, m is the leading monomial of Q contradicting Lemma 4.17. □

Corollary 4.19. *The elements of $\mathcal{F}_{i-1,m-i}$ are standard monomials modulo $\langle x^i, (x^m)^{(\infty)} \rangle$.*

Proof. We will use Proposition 4.18. Consider the ideals

$$I_1 = \langle y_1^2 \rangle^{(\infty)}, \ldots, I_{i-1} = \langle y_{i-1}^2 \rangle^{(\infty)},\ I_i = \big\langle y_i, (y_i^2)^{(\infty)} \big\rangle, \ldots, I_{m-1} = \big\langle y_{m-1}, (y_{m-1}^2)^{(\infty)} \big\rangle,$$

and define φ as in Proposition 4.18. Lemma 4.4 implies that $\varphi((x^m)^{(k)}) = ((y_1 + \ldots + y_{m-1})^m)^{(k)} = 0$ for every $k \geq 1$ and $\varphi(x^i) = (y_1 + \ldots + y_{i-1})^i = 0$. Therefore, $\langle (x^m)^{(\infty)}, x^i \rangle \subset \operatorname{Ker}(\varphi)$. Proposition 4.16 implies that the standard monomials modulo I_j are the fair monomials for $j < i$ and strongly fair monomials for $i \leqslant j$. Therefore, Proposition 4.18 implies that $\mathcal{F}_{i-1,m-i}$ are standard monomials modulo $\langle x^i, (x^m)^{(\infty)} \rangle$. □

4.5. *Putting everything together: proofs of the main results.*

Proof of Theorem 3.1. Consider the images of $\mathcal{F}_{m-1,0} \cap k[x^{(\leqslant h)}]$ in $k[x^{(\infty)}]/\langle x^m \rangle^{(\infty)}$. By Corollary 4.19, they are linearly independent modulo $\langle x^m \rangle^{(\infty)}$. Then Proposition 4.12 implies that the dimension of $k[x^{(\leqslant h)}]/\langle x^m \rangle^{(\infty)}$ is at least m^{h+1}. Together with Proposition 4.5, this implies

$$\dim\big(k[x^{(\leqslant h)}]/\langle x^m \rangle^{(\infty)}\big) = m^{h+1}.$$ □

Proof of Theorem 3.3. Fix $h \geqslant 0$. Consider $\mathcal{F}_{i-1,m-i} \cap k[x^{(\leqslant h)}]$. Combining Corollary 4.19, Corollary 4.8, and Proposition 4.12, we show that the image of this set in $k[x^{(\leqslant h)}]/\langle (x^m)^{(\infty)}, x^i \rangle$ forms a basis. Thus, the image of the whole $\mathcal{F}_{i-1,m-i}$ is a basis of $k[x^{(\infty)}]/\langle (x^m)^{(\infty)}, x^i \rangle$. Therefore, by Corollary 4.19, $\mathcal{F}_{i-1,m-i}$ coincides with the set of standard monomials modulo $\langle (x^m)^{(\infty)}, x^i \rangle$. □

Proof of Corollary 3.4. Since the ideal $\langle x^i, (x^m)^{(\infty)} \rangle$ is generated by homogeneous and isobaric (that is, weight-homogeneous) polynomials, its Gröbner bases with respect to the purely lexicographic, degree lexicographic, and weighted lexicographic orderings coincide. □

5. Computational experiments for more general fat points

In this section, we consider a more general case of a fat point in a n-dimensional space, not just on a line. We used [Macaulay2], in particular, the package Jets [Galetto and Iammarino 2021; 2022] to explore possible analogues of our Theorem 3.1 for this more general case. A related Sage implementation for computing the arc space of an affine scheme with respect to a fat point can be found in [Stout 2017, §9] and [Stout 2014, §5.4].

Let $\boldsymbol{x} = (x_1, \dots, x_n)$, and consider a zero-dimensional ideal $I \subset k[\boldsymbol{x}]$. We will be interested in describing (in particular, in computing the dimension of the quotient ring) $I^{(\infty)} \cap k[\boldsymbol{x}^{(\leqslant h)}]$ for a positive integer h. Since this ideal is the union of the following chain

$$I^{(\leqslant 1)} \cap k[\boldsymbol{x}^{(\leqslant h)}] \subseteq I^{(\leqslant 2)} \cap k[\boldsymbol{x}^{(\leqslant h)}] \subseteq I^{(\leqslant 3)} \cap k[\boldsymbol{x}^{(\leqslant h)}] \subseteq \cdots$$

and $k[\boldsymbol{x}^{(\leqslant h)}]$ is Noetherian, one can compute $I^{(\infty)} \cap k[\boldsymbol{x}^{(\leqslant h)}]$ by computing $I^{(\leqslant H)} \cap k[\boldsymbol{x}^{(\leqslant h)}]$ for large enough H. But how do we determine what H is "large enough"?

- For $I = \langle x^m \rangle \subset k[x]$, the answer is given by Theorem 3.1: if the dimension $k[x^{(\leqslant h)}]/(I^{(\leqslant H)} \cap k[x^{(\leqslant h)}])$ is equal to m^{h+1}, then $I^{(\leqslant H)} \cap k[x^{(\leqslant h)}] = I^{(\infty)} \cap k[x^{(\leqslant h)}]$.
- For general I, we take H to be $1, 2, \dots,$ and we stop when we encounter

$$I^{(\leqslant H)} \cap k[\boldsymbol{x}^{(\leqslant h)}] = I^{(\leqslant H+1)} \cap k[\boldsymbol{x}^{(\leqslant h)}].$$

  We conjecture that in this case $I^{(\leqslant H)} \cap k[\boldsymbol{x}^{(\leqslant h)}] = I^{(\infty)} \cap k[\boldsymbol{x}^{(\leqslant h)}]$ (see Question 5.1) but, strictly speaking, we only know that $I^{(\leqslant H)} \cap k[\boldsymbol{x}^{(\leqslant h)}] \subseteq I^{(\infty)} \cap k[\boldsymbol{x}^{(\leqslant h)}]$.

5.1. ***Ideals $I = \langle x^m \rangle$.*** For ideals of the form $\langle x^m \rangle$, the approach outlined above yields a complete algorithm to compute $I^{(\infty)} \cap k[x^{(\leqslant h)}]$ for any given h and m. We use it for computing examples of Gröbner bases for these ideals with respect to the lexicographic ordering, as shown in Table 1.

5.2. ***General fat points.*** In this subsection, we consider a general zero-dimensional $I \subset k[\boldsymbol{x}]$ with the zero set of I being the origin. We use the following algorithm following the approach described in the beginning of the section to obtain an upper bound of the dimensions of $k[\boldsymbol{x}^{(\leqslant h)}]/(I^{(\infty)} \cap k[\boldsymbol{x}^{(\leqslant h)}])$.

Step 1: Set $H = 1$.

Step 2: While the dimension of $I^{(\leqslant H)} \cap k[\boldsymbol{x}^{(\leqslant h)}]$ is not zero or $I^{(\leqslant H)} \cap k[\boldsymbol{x}^{(\leqslant h)}] \neq I^{(\leqslant H+1)} \cap k[\boldsymbol{x}^{(\leqslant h)}]$, set $H = H + 1$.

Ideal	Gröbner basis
$\langle x^2 \rangle^{(\infty)} \cap k[x^{(\leqslant 2)}]$	$(x'')^4;\ x'(x'')^2;\ (x')^2x'';\ (x')^3;\ 2xx'' + (x')^2;\ xx';\ x^2$
$\langle x^3 \rangle^{(\infty)} \cap k[x^{(\leqslant 2)}]$	$(x'')^7;\ x'(x'')^5;\ (x')^2(x'')^4;\ (x')^3(x'')^2;\ (x')^4x'';\ (x')^5;\ x(x'')^4 + 2(x')^2(x'')^3;$ $3xx'(x'')^2 + (x')^3x'';\ 6x(x')^2x'' + (x')^4;\ x(x')^3;\ x^2x'' + x(x')^2;\ x^2x';\ x^3$

Table 1. Gröbner bases for $\langle x^m \rangle^{(\infty)} \cap k[x^{(\leqslant h)}]$, where $m = 2, 3$.

Ideal	$h=0$	$h=1$	$h=2$	$h=3$
$\langle x^2, y^2, xy\rangle$	3	9	27	81
$\langle x^2, y^2, xz, yz, z^2-xy\rangle$	5	25	125	—
$\langle x^3, y^2, x^2y\rangle$	5	25	125	—
$\langle x^3, y^2, xy\rangle$	4	16	64	256
$\langle x^3, y^3, x^2y\rangle$	7	49	—	—
$\langle x^4, y^4, x^2y^3\rangle$	14	196	—	—

Table 2. (Bounds for) the dimensions of the truncations of the arc space.

Step 3: Return $\dim\big(k[\boldsymbol{x}^{(\leqslant h)}]/(I^{(\leqslant H)}\cap k[\boldsymbol{x}^{(\leqslant h)}])\big)$.

We expect the resulting bound to be exact (see also Question 5.1), for example, it is exact for $I=\langle x^m\rangle$.

Our implementation of this algorithm in [Macaulay2] is available for download at the following webpage: https://mathrepo.mis.mpg.de/MultiplicityStructureOfArcSpaces. Table 2 shows some of the results we obtained. One can see that the computed dimensions form geometric series with the exponent being the multiplicity of the original ideal exactly as in Theorem 3.1.

However, we have also found ideals for which the generating series of the dimensions is definitely not equal to $m/(1-mt)$, where m is the multiplicity of the ideal. We show some examples of this type in Table 3.

Note that while Table 2 gives only indication that the generating series of the multiplicities for these ideals may be $m/(1-mt)$, Table 3 gives a proof that this is not the case for all the fat points.

5.3. ***Open questions.*** Based on the results of the computational experiments, we formulate several open questions.

Question 5.1. Let $I\subset k[\boldsymbol{x}]$ be a zero-dimensional ideal with $V(I)$ being a single point. Is it true that, for every integer h

$$\big(I^{(\leqslant H)}\cap k[\boldsymbol{x}^{(\leqslant h)}]=I^{(\leqslant H+1)}\cap k[\boldsymbol{x}^{(\leqslant h)}]\big)\quad\Longrightarrow\quad\big(I^{(\leqslant H)}\cap k[\boldsymbol{x}^{(\leqslant h)}]=I^{(\leqslant\infty)}\cap k[\boldsymbol{x}^{(\leqslant h)}]\big)?$$

Does this statement remain true if we drop the assumption $|V(I)|=1$?

Ideal	$h=0$	$h=1$	$h=2$
$\langle x^3, y^3, xy\rangle$	5	24	115
$\langle x^4, y^3, xy\rangle$	6	33	—
$\langle x^4, y^3, x^2y\rangle$	8	62	—
$\langle x^4, y^4, xy\rangle$	7	42	—
$\langle x^4, y^4, x^2y\rangle$	10	94	—
$\langle x^4, y^4, x^2y^2\rangle$	12	140	—
$\langle x^4, y^6, x^2y^3\rangle$	18	320	—

Table 3. (Bounds for) the dimensions of the truncations of the arc space.

Question 5.2. Let $I \subset k[\boldsymbol{x}]$ be a zero-dimensional ideal with $V(I)$ being a single point of multiplicity m. Is it true that

$$\lim_{h\to\infty} \frac{\dim k[\boldsymbol{x}^{(\leqslant h)}]/I^{(\infty)}}{m^{h+1}} = 1?$$

Question 5.3. Let $I \subset k[\boldsymbol{x}]$ be a zero-dimensional ideal with $V(I)$ being a single point of multiplicity m. Under which conditions it is true that

$$\sum_{h=0}^{\infty} \left(\dim k[\boldsymbol{x}^{(\leqslant h)}]/I^{(\infty)}\right) \cdot t^h = \frac{m}{1-mt}?$$

More generally, what information about the corresponding scheme can be read off the above generating series?

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