Algebra & Number Theory

Volume 18
2024
No. 5

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ALGEBRA AND NUMBER THEORY 18:5 (2024)
https://doi.org/10.2140/ant.2024.18.993

Equidistribution theorems for holomorphic Siegel cusp forms of general degree: the level aspect

Henry H. Kim, Satoshi Wakatsuki and Takuya Yamauchi

This paper is an extension of Kim et al. (2020a), and we prove equidistribution theorems for families of holomorphic Siegel cusp forms of general degree in the level aspect. Our main contribution is to estimate unipotent contributions for general degree in the geometric side of Arthur's invariant trace formula in terms of Shintani zeta functions in a uniform way. Several applications, including the vertical Sato–Tate theorem and low-lying zeros for standard L-functions of holomorphic Siegel cusp forms, are discussed. We also show that the "nongenuine forms", which come from nontrivial endoscopic contributions by Langlands functoriality classified by Arthur, are negligible.

1. Introduction

Let G be a connected reductive group over $\mathbb{Q}$ and $\mathbb{A}$ the ring of adeles of $\mathbb{Q}$. An equidistribution theorem for a family of automorphic representations of $G(\mathbb{A})$ is one of recent topics in number theory and automorphic representations. After Sauvageot's important results [1997], Shin [2012] proved a so-called limit multiplicity formula which shows that the limit of an automorphic counting measure is the Plancherel measure. It implies the equidistribution of Hecke eigenvalues or Satake parameters at a

Kim was partially supported by NSERC grant No. 482564. Wakatsuki was partially supported by JSPS Grant-in-Aid for Scientific Research (C) No. 20K03565 and (B) No. 21H00972. Yamauchi was partially supported by JSPS KAKENHI Grant (B) No. 19H01778.
MSC2020: 11F46, 11F70, 22E55, 11R45.
Keywords: trace formula, holomorphic Siegel modular forms, equidistribution theorems, standard L-functions.

fixed prime in a family of cohomological automorphic forms on $G(\mathbb{A})$. A quantitative version of Shin's result is given by Shin and Templier [2016]. A different approach is discussed in [Finis et al. 2015] for $G = \mathrm{GL}_n$ or SL_n, treating more general automorphic forms which are not necessarily cohomological. Note that in the works of Shin and Shin and Templier, one needs to consider all cuspidal representations in the L-packets. Shin [2012, second paragraph on p. 88] suggested that one can isolate just holomorphic discrete series at infinity. In [Kim et al. 2020a; 2020b], we carried out his suggestion and established equidistribution theorems for holomorphic Siegel cusp forms of degree 2. We should also mention Dalal's work [2022]; see Remark 3.12. See also the related works [Knightly and Li 2019; Kowalski et al. 2012].

In this paper we generalize several equidistribution theorems to holomorphic Siegel cusp forms of general degree. A main tool is Arthur's invariant trace formula, as used in the previous work, but we need a more careful analysis in the computation of unipotent contributions. Let us prepare some notations to explain our results.

Let $G = \mathrm{Sp}(2n)$ be the symplectic group of rank n defined over $\mathbb{Q}$. For an n-tuple of integers $\underline{k} = (k_1, \dots, k_n)$ with $k_1 \geq \cdots \geq k_n > n+1$, let $D_{\underline{l}}^{\mathrm{hol}} = \sigma_{\underline{k}}$ be the holomorphic discrete series representation of $G(\mathbb{R})$ with the Harish-Chandra parameter $\underline{l} = (k_1 - 1, \dots, k_n - n)$ or the Blattner parameter $\underline{k}$.

Let $\mathbb{A}$ (respectively, $\mathbb{A}_f$) be the ring of (respectively, finite) adeles of $\mathbb{Q}$, and $\hat{\mathbb{Z}}$ be the profinite completion of $\mathbb{Z}$. For S_1 a finite set of rational primes, let $S = \{\infty\} \cup S_1$, $\mathbb{Q}_{S_1} = \prod_{p \in S_1} \mathbb{Q}_p$, $\mathbb{A}^S$ be the ring of adeles outside S and $\hat{\mathbb{Z}}^S = \prod_{p \notin S_1} \mathbb{Z}_p$. We denote by $\widehat{G(\mathbb{Q}_{S_1})}$ the unitary dual of $G(\mathbb{Q}_{S_1}) = \prod_{p \in S_1} G(\mathbb{Q}_p)$ equipped with the Fell topology. Fix a Haar measure μ^S on $G(\mathbb{A}^S)$ so that $\mu^S(G(\hat{\mathbb{Z}}^S)) = 1$, and let U be a compact open subgroup of $G(\mathbb{A}^S)$. Consider the algebraic representation $\xi = \xi_{\underline{k}}$ of the highest weight $\underline{k}$ so that it is isomorphic to the minimal K_∞-type of $D_{\underline{l}}^{\mathrm{hol}}$. Let h_U denote the characteristic function of U. Then we define a measure on $\widehat{G(\mathbb{Q}_{S_1})}$ by

$$\hat{\mu}_{U, S_1, \xi, D_{\underline{l}}^{\mathrm{hol}}} := \frac{1}{\mathrm{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A})) \cdot \dim \xi} \sum_{\pi_{S_1}^0 \in \widehat{G(\mathbb{Q}_{S_1})}} \mu^S(U)^{-1} \, m_{\mathrm{cusp}}\big(\pi_{S_1}^0; U, \xi, D_{\underline{l}}^{\mathrm{hol}}\big) \delta_{\pi_{S_1}^0}, \tag{1-1}$$

where $\delta_{\pi_{S_1}^0}$ is the Dirac delta measure supported at $\pi_{S_1}^0$, a unitary representation of $G(\mathbb{Q}_{S_1})$, and

$$m_{\mathrm{cusp}}(\pi_{S_1}^0; U, \xi, D_{\underline{l}}^{\mathrm{hol}}) = \sum_{\substack{\pi \in \Pi(G(\mathbb{A}))^0 \\ \pi_{S_1} \simeq \pi_{S_1}^0, \, \pi_\infty \simeq D_{\underline{l}}^{\mathrm{hol}}}} m_{\mathrm{cusp}}(\pi) \ \mathrm{tr}(\pi^S(h_U)), \tag{1-2}$$

where $\Pi(G(\mathbb{A}))^0$ stands for the isomorphism classes of all irreducible unitary cuspidal representations of $G(\mathbb{A})$ and $\pi^S = \otimes'_{p \notin S} \pi_p$.

To state the equidistribution theorem, we need to introduce the Hecke algebra $C_c^\infty(G(\mathbb{Q}_{S_1}))$ which is dense under the map $h \mapsto \hat{h}$, where $\hat{h}(\pi_{S_1}) = \mathrm{tr}(\pi_{S_1}(h))$ is in $\mathcal{F}(\widehat{G(\mathbb{Q}_{S_1})})$ consisting of suitable $\hat{\mu}_{S_1}^{\mathrm{pl}}$-measurable functions on $\widehat{G(\mathbb{Q}_{S_1})}$. (See [Shin 2012, Section 2.3] for that space.)

Let N be a positive integer. Put $S_N = \{p \text{ prime} : p \mid N\}$. We assume that $S_1 \cap S_N = \varnothing$. We denote by $K_p(N)$ the principal congruence subgroup of level N for $G(\mathbb{Z}_p)$ (see (2-3) for the definition), and set $K^S(N) = \prod_{p \notin S} K_p(N)$. For each rational prime p, let us consider the unramified Hecke algebra $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p)) \subset C_c^\infty(\mathbb{Q}_p)$, and for each $\kappa > 0$, $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))^\kappa$, the linear subspace of $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))$ consisting of all Hecke elements whose heights are less than κ. (See (2-2).) Let $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))^{\kappa}_{\le 1}$ be the subset of $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))^{\kappa}$ consisting of all Hecke elements whose complex values have absolute values less than 1. Our first main result is

Theorem 1.1. *Fix $\underline{k} = (k_1, \dots, k_n)$ satisfying $k_1 \ge \cdots \ge k_n > n+1$. Fix a positive integer κ. Then there exist constants a, b and $c_0 > 0$ depending only on G such that for each $h_1 = \otimes_{p \in S_1} h_{1,p}$, where $h_{1,p} \in \mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))^{\kappa}_{\le 1}$, we have*

$$\hat{\mu}_{K^S(N), S_1, \xi, D^{\mathrm{hol}}_{\underline{l}}}(\widehat{h_1}) = \hat{\mu}^{\mathrm{pl}}_{S_1}(\widehat{h_1}) + O\biggl(\Bigl(\prod_{p \in S_1} p\Bigr)^{a\kappa+b} N^{-n}\biggr),$$

if $N \ge c_0 \prod_{p \in S_1} p^{2n\kappa}$. Note that the implicit constant of the Landau O-notation is independent of S_1, N and h_1.

Let us apply this theorem to the vertical Sato–Tate theorem and higher level density theorem for standard L-functions of holomorphic Siegel cusp forms.

The principal congruence subgroup $\Gamma(N)$ of level N for $G(\mathbb{Z})$ is obtained by

$$\Gamma(N) = G(\mathbb{Q}) \cap G(\mathbb{R}) K(N),$$

where $K(N) = \prod_{p<\infty} K_p(N)$. Let $S_{\underline{k}}(\Gamma(N))$ be the space of holomorphic Siegel cusp forms of weight $\underline{k}$ with respect to $\Gamma(N)$ (see the next section for a precise definition), and let $HE_{\underline{k}}(N)$ be a basis consisting of all Hecke eigenforms outside N. We can identify $HE_{\underline{k}}(N)$ with a basis of $K(N)$-fixed vectors in the set of cuspidal representations of $G(\mathbb{A})$ whose infinity component is (isomorphic to) $D^{\mathrm{hol}}_{\underline{l}}$. (See the next section for the details.) Put $d_{\underline{k}}(N) = |HE_{\underline{k}}(N)|$. Then we have [Wakatsuki 2018], for some constant $C_{\underline{k}} > 0$,

$$d_{\underline{k}}(N) = C_{\underline{k}} C_N N^{2n^2+n} + O_{\underline{k}}\bigl(N^{2n^2}\bigr), \tag{1-3}$$

where $C_N = \prod_{p \mid N} \prod_{i=1}^n (1 - p^{-2i})$. Note that $\prod_{i=1}^n \zeta(2i)^{-1} < C_N < 1$.

For each $F \in HE_{\underline{k}}(N)$, we denote by $\pi_F = \pi_\infty \otimes \otimes'_p \pi_{F,p}$ the corresponding automorphic cuspidal representation of $G(\mathbb{A})$. Henceforth, we assume that

$$k_1 > \cdots > k_n > n+1. \tag{1-4}$$

Then the Ramanujan conjecture is true, namely, $\pi_{F,p}$ is tempered for any p; see Theorem 4.3. Unfortunately, this assumption forces us to exclude the scalar-valued Siegel cusp forms.

Let $\widehat{G(\mathbb{Q}_p)}^{\mathrm{ur,temp}}$ be the subspace of $\widehat{G(\mathbb{Q}_p)}$ consisting of all unramified tempered classes. We denote by $(\theta_1(\pi_{F,p}), \dots, \theta_n(\pi_{F,p}))$ the element of Ω corresponding to $\pi_{F,p}$ under the isomorphism $\widehat{G(\mathbb{Q}_p)}^{\mathrm{ur,temp}} \simeq [0,\pi]^n / \mathfrak{S}_n =: \Omega$. Let μ_p be the measure on Ω defined in Section 7.

Theorem 1.2. *Assume* (1-4). *Fix a prime p. Then the set*

$$\bigl\{(\theta_1(\pi_{F,p}), \dots, \theta_n(\pi_{F,p})) \in \Omega : F \in HE_{\underline{k}}(N)\bigr\}$$

is μ_p-equidistributed in Ω, namely, for each continuous function f on Ω,

$$\lim_{\substack{N \to \infty \\ (p,N)=1}} \frac{1}{d_{\underline{k}}(N)} \sum_{F \in HE_{\underline{k}}(N)} f\bigl(\theta_1(\pi_{F,p}), \dots, \theta_n(\pi_{F,p})\bigr) = \int_\Omega f(\theta_1, \dots, \theta_n) \mu_p.$$

By using Arthur's endoscopic classification, we have a finer version of the above theorem. Under the assumption (1-4), the global A-parameter describing π_F, for $F \in HE_{\underline{k}}(N)$, is always semisimple. (See Definition 4.1.) Let $HE_{\underline{k}}(N)^g$ be the subset of $HE_{\underline{k}}(N)$ consisting of F such that the global A-packet containing π_F is associated to a simple global A-parameter. They are Siegel cusp forms which do not come from smaller groups by Langlands functoriality in Arthur's classification. In this paper, we call them genuine forms. Let $HE_{\underline{k}}(N)^{ng}$ be the subset of $HE_{\underline{k}}(N)$ consisting of F such that the global A-packet containing π_F is associated to a nonsimple global A-parameter, i.e., they are Siegel cusp forms which come from smaller groups by Langlands functoriality in Arthur's classification. We call them nongenuine forms. We show that nongenuine forms are negligible. The following result is interesting in its own right. For this, we need some further assumptions on the level N.

Theorem 1.3. *Assume* (1-4). *We also assume*

(1) *N is an odd prime or*

(2) *N is odd and all prime divisors $p_1, \dots, p_r$ ($r \geq 2$) of N are congruent to* 1 *modulo* 4 *such that $\left(\frac{p_i}{p_j}\right) = 1$ for $i \neq j$, where $\left(\frac{*}{*}\right)$ denotes the Legendre symbol.*

Then

(1) $|HE_{\underline{k}}(N)^g| = C_{\underline{k}} C_N N^{2n^2+n} + O_{n,\underline{k},\epsilon}\big(N^{2n^2+n-1+\epsilon}\big)$ *for any* $\epsilon > 0$;

(2) $|HE_{\underline{k}}(N)^{ng}| = O_{n,\underline{k},\epsilon}\big(N^{2n^2+n-1+\epsilon}\big)$ *for any* $\epsilon > 0$;

(3) *for a fixed prime p, the set $\{(\theta_1(\pi_{F,p}), \dots, \theta_n(\pi_{F,p})) \in \Omega : F \in HE_{\underline{k}}(N)^g\}$ is μ_p-equidistributed in Ω.*

The above assumptions on the level N are necessary in order to estimate nongenuine forms related to nonsplit but quasisplit orthogonal groups in the Arthur's classification by using the transfer theorems for some Hecke elements in the quadratic base change in the ramified case [Yamauchi 2021]. (See Proposition 4.12 for the details.)

Next, we discuss ℓ-level density (where ℓ is a positive integer) for standard L-functions in the level aspect. Let us denote by $\Pi(\mathrm{GL}_n(\mathbb{A}))^0$ the set of all isomorphism classes of irreducible unitary cuspidal representations of $\mathrm{GL}_n(\mathbb{A})$. Keep the assumption on $\underline{k}$ as in (1-4) and the above assumption on the level N. Then F can be described by a global A-parameter $\boxplus_{i=1}^r \pi_i$ with $\pi_i \in \Pi(\mathrm{GL}_{m_i}(\mathbb{A}))^0$ and $\sum_{i=1}^r m_i = 2n+1$. Then we may define the standard L-function of $F \in HE_{\underline{k}}(N)$ by

$$L(s, \pi_F, \mathrm{St}) := \prod_{i=1}^r L(s, \pi_i),$$

which coincides with the classical definition in terms of Satake parameters of F outside N. Then we show unconditionally that the ℓ-level density of the standard L-functions of the family $HE_{\underline{k}}(N)$ has the symmetry type Sp in the level aspect. (See Section 9 for the precise statement. Shin and Templier [2016] showed it under several hypotheses for a family which includes nonholomorphic forms.) Here, in order to obtain lower bounds for conductors, it is necessary to introduce a concept of newforms. This may be of independent interest. Since any local newform theory for $\mathrm{Sp}(2n)$ is unavailable except for $n = 1, 2$, we define the old space $S_{\underline{k}}^{\mathrm{old}}(\Gamma(N))$ to be the intersection of $S_{\underline{k}}(\Gamma(N))$ with the smallest $G(\mathbb{A}_f)$-invariant space of functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ containing $S_{\underline{k}}(\Gamma(M))$ for all proper divisors M of N. The new space $S_{\underline{k}}^{\mathrm{new}}(\Gamma(N))$ is the orthogonal complement of $S_{\underline{k}}^{\mathrm{old}}(\Gamma(N))$ in $S_{\underline{k}}(\Gamma(N))$ with respect to the Petersson inner product. Then if $F \in S_{\underline{k}}^{\mathrm{new}}(\Gamma(N))$, $q(F) \geq N^{1/2}$ (Theorem 8.3), and if N is squarefree, we can show that $\dim S_{\underline{k}}^{\mathrm{new}}(\Gamma(N)) \geq \zeta(n^2)^{-1} d_{\underline{k}}(N)$ if $n \geq 2$ (Theorem 5.4).

As a corollary, we obtain a result on the order of vanishing of $L(s, \pi_F, \mathrm{St})$ at $s = \frac{1}{2}$, the center of symmetry of the L-function, by using the method of Iwaniec et al. [2000] for holomorphic cusp forms on $\mathrm{GL}_2(\mathbb{A})$ (see also [Brumer 1995] for another formulation related to the Birch–Swinnerton–Dyer conjecture): Let r_F be the order of vanishing of $L(s, \pi_F, \mathrm{St})$ at $s = \frac{1}{2}$. Then we show that under the GRH (generalized Riemann hypothesis), $\sum_{F \in HE_{\underline{k}}(N)} r_F \leq C d_{\underline{k}}(N)$ for some constant $C > 0$. This would be the first result of this kind in Siegel modular forms. We can also show a similar result for the degree 4 spinor L-functions of $\mathrm{GSp}(4)$.

Let us explain our strategy in comparison with the previous works. We choose a test function

$$f = \mu^S(K(N))^{-1} f_\xi h_1 h_{K^S(N)} \in C_c^\infty(G(\mathbb{R})) \otimes \left(\otimes_{p \in S_1} \mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))^\kappa_{\leq 1}\right) \otimes C_c^\infty(G(\mathbb{A}^S))$$

such that f_ξ is a pseudocoefficient of $D_{\underline{l}}^{\mathrm{hol}}$ normalized as $\mathrm{tr}(\pi_\infty(f_\xi)) = 1$. A starting main equality is

$$I_{\mathrm{spec}}(f) = I(f) = I_{\mathrm{geom}}(f),$$

where $I_{\mathrm{spec}}(f)$ (respectively, $I_{\mathrm{geom}}(f)$) is the spectral (respectively, the geometric) side of Arthur's invariant trace $I(f)$. Under the assumption $k_n > n+1$, the spectral side becomes simple by the results of Arthur [1989] and Hiraga [1996], and it is directly related to $S_{\underline{k}}(\Gamma(N))$ because of the choice of a pseudocoefficient of $D_{\underline{l}}^{\mathrm{hol}}$. Now the geometric side is given by

$$I_{\mathrm{geom}}(f) = \sum_{M \in \mathcal{L}} (-1)^{\dim(A_M/A_G)} \frac{|W_0^M|}{|W_0^G|} \sum_{\gamma \in (M(\mathbb{Q}))_{M,\tilde{S}}} a^M(\tilde{S}, \gamma) I_M^G(\gamma, f_\xi) J_M^M(\gamma, h_P), \tag{1-5}$$

where $\tilde{S} = \{\infty\} \sqcup S_N \sqcup S_1$ and $(M(\mathbb{Q}))_{M,\tilde{S}}$ denotes the set of $(M, \tilde{S})$-equivalence classes in $M(\mathbb{Q})$ (see [Arthur 2005, p. 113]); for each M in a finite set $\mathcal{L}$, we choose a parabolic subgroup P such that M is a Levi subgroup of P. (See loc. cit. for details.) Roughly speaking:

- If the test function f is fixed, the terms on (1-5) vanish except for a finite number of $(M, \tilde{S})$-equivalence classes.
- The factor $a^M(\tilde{S}, \gamma)$ is called a global coefficient and it is almost the volume of the centralizer of γ in M if γ is semisimple. The general properties are unknown.
- The factor $I_M^G(\gamma, f_\xi)$ is called an invariant weighted orbital integral, and as the notation shows, it strongly depends on the weight $\underline{k}$ of $\xi = \xi_{\underline{k}}$. Therefore, it is negligible when we consider the level aspect.
- The factor $J_M^M(\gamma, h_P)$ is an orbital integral of γ for $h = \mu^S(K(N))^{-1} h_1 h_{K^S(N)}$.

According to the types of conjugacy classes and M, the geometric side is divided into the terms

$$I_{\mathrm{geom}}(f) = I_1(f) + I_2(f) + I_3(f) + I_4(f),$$

where

- $I_1(f)$: $M = G$ and $\gamma = 1$;
- $I_2(f)$: $M \neq G$ and $\gamma = 1$;
- $I_3(f)$: γ is unipotent, but $\gamma \neq 1$;
- $I_4(f)$: the other contributions.

The first term $I_1(f)$ is $f(1)$ up to constant factors, and the Plancherel formula $\hat{\mu}^{\mathrm{pl}}_{S_1}(\hat{f}) = f(1)$ yields the first term of the equality in Theorem 1.1. The condition $N \geq c_0 \prod_{p \in S_1} p^{2n\kappa}$ in Theorem 1.1 implies that the nonunipotent contribution $I_4(f)$ vanishes by [Shin and Templier 2016, Lemma 8.4]. Therefore, everything is reduced to studying the unipotent contributions $I_2(f)$ and $I_3(f)$. An explicit bound for $I_2(f)$ was given by [Shin and Templier 2016, proof of Theorem 9.16]. However, as for $I_3(f)$, since the number of $(M, \tilde{S})$-equivalence classes in the geometric unipotent conjugacy class of each γ is increasing when N goes to infinity, it is difficult to estimate $I_3(f)$ directly. In the case of GSp(4), we computed unipotent contributions by using case-by-case analysis as in [Kim et al. 2020a]. Here we give a new uniform way to estimate all the unipotent contributions. It is given by a sum of special values of zeta integrals with real characters for spaces of symmetric matrices; see Lemma 3.3 and Theorem 3.7. This formula is a generalization of the dimension formula (see [Shintani 1975; Wakatsuki 2018]) to the trace formula of Hecke operators. By using their explicit formulas [Saito 1999] and analyzing Shintani double zeta functions [Kim et al. 2022], we express the geometric side as a finite sum of products of local integrals and special values of the Hecke L functions with real characters, and then obtain the estimates of the geometric side; see Theorem 3.10.

This paper is organized as follows. In Section 2, we set up some notations. In Section 3, we give key results (see Theorem 3.7 and Theorem 3.10) in estimating trace formulas of Hecke elements. In Section 4, we study Siegel modular forms in terms of Arthur's classification and show that nongenuine forms are negligible. In Section 5, we give a notion of newforms which is necessary to estimate conductors. Sections 6–10 are devoted to proving the main theorems. Finally, in the Appendix, we give an explicit computation of the convolution product of some Hecke elements, which is needed in the computation of ℓ-level density of standard L-functions.

2. Preliminaries

A split symplectic group $G = \mathrm{Sp}(2n)$ over the rational number field $\mathbb{Q}$ is defined by

$$G = \mathrm{Sp}(2n) = \left\{ g \in \mathrm{GL}_{2n} : g \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix} {}^t g = \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix} \right\}.$$

The compact subgroup

$$K_\infty = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in G(\mathbb{R}) \right\}$$

of $G(\mathbb{R})$ is isomorphic to the unitary group $\mathrm{U}(n)$ via the mapping $\left(\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right) \mapsto A+iB$, where $i=\sqrt{-1}$. For each rational prime p, we also set $K_p = G(\mathbb{Z}_p)$ and put $K=\prod_{p\le\infty} K_p$. The compact groups K_v and K are maximal in $G(\mathbb{Q}_v)$ and $G(\mathbb{A})$, respectively,

Holomorphic discrete series of $G(\mathbb{R})$ are parameterized by n-tuples $\underline{k}=(k_1,\dots,k_n)\in\mathbb{Z}^n$ such that $k_1\ge\cdots\ge k_n>n$, which is called the Blattner parameter. We write $\sigma_{\underline{k}}$ for the holomorphic discrete series corresponding to the Blattner parameter $\underline{k}=(k_1,\dots,k_n)$. We also write $D^{\mathrm{hol}}_{\underline{l}}$ for one corresponding to the Harish-Chandra parameter $\underline{l}=(k_1-1,k_2-2,\dots,k_n-n)$ so that $D^{\mathrm{hol}}_{\underline{l}}=\sigma_{\underline{k}}$.

Let $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))$ denote the unramified Hecke algebra over $G(\mathbb{Q}_p)$, that is,

$$\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))=\bigl\{\varphi\in C_c^\infty(G(\mathbb{Q}_p)):\varphi(k_1xk_2)=\varphi(x)\ \forall k_1,k_2\in K_p,\ \forall x\in G(\mathbb{Q}_p)\bigr\}.$$

Let T denote the maximal split $\mathbb{Q}$-torus of G consisting of diagonal matrices. We denote by $X_*(T)$ the group of cocharacters on T over $\mathbb{Q}$. An element e_j in $X_*(T)$ is defined by

$$e_j(x)=\mathrm{diag}(\overbrace{1,\dots,1}^{j-1},x,\overbrace{1\dots,1}^{n-j+1},\overbrace{1,\dots,1}^{j-1},x^{-1},\overbrace{1,\dots,1}^{n-j+1})\in T,\quad x\in\mathbb{G}_m. \tag{2-1}$$

Then, one has $X_*(T)=\langle e_1,\dots,e_n\rangle$. By the Cartan decomposition, any function in $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))$ is expressed by a linear combination of characteristic functions of double cosets $K_p\lambda(p)K_p$ $(\lambda\in X_*(T))$. A height function $\|\cdot\|$ on $X_*(T)$ is defined by

$$\Bigl\|\prod_{j=1}^n e_j^{m_j}\Bigr\|=\max\bigl\{|m_j|:1\le j\le n\bigr\},\quad m_j\in\mathbb{Z}.$$

For each $\kappa\in\mathbb{N}$, we set

$$\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))^\kappa=\Bigl\{\varphi\in\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p)):\mathrm{Supp}(\varphi)\subset\bigcup_{\mu\in X_*(T),\,\|\mu\|\le\kappa}K_p\mu(p)K_p\Bigr\}. \tag{2-2}$$

Choose a natural number N. We set

$$K_p(N)=\{x\in K_p:x\equiv I_{2n}\mod N\},\quad K(N)=\prod_{p<\infty}K_p(N). \tag{2-3}$$

One gets a congruence subgroup $\Gamma(N)=G(\mathbb{Q})\cap G(\mathbb{R})K(N)$.

Let $\mathfrak{H}_n:=\{Z\in M_n(\mathbb{C}):Z={}^tZ,\ \mathrm{Im}(Z)>0\}$. We write $S_{\underline{k}}(\Gamma(N))$ for the space of Siegel cusp forms of weight $\underline{k}$ for $\Gamma(N)$, i.e., $S_{\underline{k}}(\Gamma(N))$ consists of $V_{\underline{k}}$-valued smooth functions F on $G(\mathbb{A})$ satisfying the following conditions:

(i) $F(\gamma gk_\infty k_f)=\rho_{\underline{k}}(k_\infty)^{-1}F(g),\quad g\in G(\mathbb{A}),\ \gamma\in G(\mathbb{Q}),\ k_\infty\in K_\infty,\ k_f\in K(N)$,

(ii) $\rho_{\underline{k}}(g_\infty,iI_n)F|_{G(\mathbb{R})}(g_\infty)$ is holomorphic for $g_\infty\cdot iI_n\in\mathfrak{H}_n$,

(iii) $\max_{g\in G(\mathbb{A})}|F(g)|\ll 1$,

where $\rho_{\underline{k}}$ denotes the finite dimensional irreducible polynomial representation of $\mathrm{U}(n)$ corresponding to $\underline{k}$ together with the representation space $V_{\underline{k}}$ and we set $\rho_{\underline{k}}(g,iI_n)=\rho_{\underline{k}}(iC+D)$ for $g=\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)\in G(\mathbb{R})$.

Let $\underline{m} = (m_1, \dots, m_n)$, $m_1|m_2|\cdots|m_n$, and $D_{\underline{m}} = \operatorname{diag}(m_1, \dots, m_n)$. Let $T(D_{\underline{m}})$ be the Hecke operator defined by the double coset

$$\Gamma(N)\begin{pmatrix} D_{\underline{m}} & 0 \\ 0 & D_{\underline{m}}^{-1}\end{pmatrix}\Gamma(N).$$

Specifically, for each prime p, let $D_{p,\underline{a}} = \operatorname{diag}(p^{a_1}, \dots, p^{a_n})$, with $\underline{a} = (a_1, \dots, a_n)$ and $0 \le a_1 \le \cdots \le a_n$.

Let F be a Hecke eigenform in $S_{\underline{k}}(\Gamma(N))$ with respect to the Hecke operator $T(D_{p,\underline{a}})$ for all $p \nmid N$. (See [Kim et al. 2020a, Section 2.2] for Hecke eigenforms in the case of $n = 2$. One can generalize the contents there to $n \ge 3$.) Then F gives rise to an adelic automorphic form ϕ_F on $\operatorname{Sp}(2n, \mathbb{Q})\backslash \operatorname{Sp}(2n, \mathbb{A})$, and ϕ_F gives rise to a cuspidal representation π_F which is a direct sum $\pi_F = \pi_1 \oplus \cdots \oplus \pi_r$, where the π_i are irreducible cuspidal representations of $\operatorname{Sp}(2n)$. Since F is an eigenform, the π_i are all near-equivalent to each other. Since we do not have the strong multiplicity one theorem for $\operatorname{Sp}(2n)$, we cannot conclude that π_F is irreducible. However, the strong multiplicity one theorem for GL_n implies that there exists a global A-parameter $\psi \in \Psi(G)$ such that $\pi_i \in \Pi_\psi$ for all i [Schmidt 2018, p. 3088]. (See Section 4 for the definition of the global A-packet.)

On the other hand, given a cuspidal representation π of $\operatorname{Sp}(2n)$ with a $K(N)$-fixed vector and whose infinity component is holomorphic discrete series of lowest weight $\underline{k}$, there exists a holomorphic Siegel cusp form F of weight $\underline{k}$ with respect to $\Gamma(N)$ such that $\pi_F = \pi$. (See [Schmidt 2017, p. 2409] for $n = 2$. One can generalize the contents there to $n \ge 3$.)

We define $HE_{\underline{k}}(N)$ to be a basis of $K(N)$-fixed vectors in the set of cuspidal representations of $\operatorname{Sp}(2n, \mathbb{A})$ whose infinity component is holomorphic discrete series of lowest weight $\underline{k}$, and identify it with a basis consisting of all Hecke eigenforms outside N. In particular, each $F \in HE_{\underline{k}}(N)$ gives rise to an irreducible cuspidal representation π_F of $\operatorname{Sp}(2n)$. Let $\mathcal{F}_{\underline{k}}(N)$ be the set of all isomorphism classes of cuspidal representations of $\operatorname{Sp}(2n)$ such that $\pi^{K(N)} \neq 0$ and $\pi_\infty \simeq \sigma_{\underline{k}}$. Consider the map $\Lambda : HE_{\underline{k}}(N) \longrightarrow \mathcal{F}_{\underline{k}}(N)$, given by $F \longmapsto \pi_F$. It is clearly surjective. For each $\pi = \pi_\infty \otimes \otimes'_p \pi_p \in \mathcal{F}_{\underline{k}}(N)$, set $\pi_f = \otimes'_p \pi_p$. Then we get

$$|\Lambda^{-1}(\pi)| = \dim \pi_f^{K(N)},$$

where $\pi_f^{K(N)} = \{\phi \in \pi_f : \pi_f(k)\phi = \phi \text{ for all } k \in K(N)\}$.

3. Asymptotics of Hecke eigenvalues

For each function $h \in C_c^\infty(K(N)\backslash G(\mathbb{A}_f)/K(N))$, an adelic Hecke operator T_h on $S_{\underline{k}}(\Gamma(N))$ is defined by

$$(T_h F)(g) = \int_{G(\mathbb{A}_f)} F(gx)h(x)\,\mathrm{d}x, \quad F \in S_{\underline{k}}(\Gamma(N)).$$

See [Kim et al. 2020a, pp. 15–16] for the relationship between the classical Hecke operators and adelic Hecke operators for $n = 2$. One can generalize the contents there to $n \ge 3$ easily. Let $f_{\underline{k}}$ denote a pseudocoefficient of $\sigma_{\underline{k}}$ with $\operatorname{tr}\sigma_{\underline{k}}(f_{\underline{k}}) = 1$; see [Clozel and Delorme 1990].

Lemma 3.1. *Suppose $k_n > n+1$ and $h \in C_c^\infty(K(N)\backslash G(\mathbb{A}_f)/K(N))$. The spectral side $I_{\mathrm{spec}}(f_{\underline{k}}h)$ of the invariant trace formula is given by*

$$I_{\mathrm{spec}}(f_{\underline{k}}h) = \sum_{\pi=\sigma_{\underline{k}}\otimes\pi_f\text{, auto. rep. of } G(\mathbb{A})} m_\pi \operatorname{Tr}(\pi_f(h)) = \operatorname{Tr}\big(T_h|_{S_{\underline{k}}(\Gamma(N))}\big),$$

where m_π means the multiplicity of π in the discrete spectrum of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$.

Proof. The second equality follows from [Wallach 1984]. One can prove the first equality by using the arguments in [Arthur 1989] and the main result in [Hiraga 1996], since it follows from [Hiraga 1996] and $k_n > n+1$ that we obtain $\operatorname{Tr}(\pi_\infty(f_{\underline{k}})) = 0$ for any unitary representation $\pi_\infty(\not\cong \sigma_{\underline{k}})$ of $G(\mathbb{R})$. □

We choose two natural numbers N_1 and N, which are mutually coprime. Suppose that N_1 is squarefree. Set $S_1 = \{p : p \mid N_1\}$. We write h_N for the characteristic function of $\prod_{p\notin S_1\sqcup\{\infty\}} K_p(N)$. For each automorphic representation $\pi = \pi_\infty \otimes \otimes'_p \pi_p$, we set $\pi_{S_1} = \otimes_{p\in S_1}\pi_p$.

Lemma 3.2. *Take a test function h on $G(\mathbb{A}_f)$ as*

$$h = \operatorname{vol}(K(N))^{-1} \times h_1 \otimes h_N, \quad \text{where } h_1 \in \otimes_{p\in S_1}\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p)). \tag{3-1}$$

Then

$$I_{\mathrm{spec}}(f_{\underline{k}}h) = \sum_{\pi=\sigma_{\underline{k}}\otimes\pi_f\text{, auto. rep. of } G(\mathbb{A})} m_\pi \dim \pi_f^{K(N)} \operatorname{Tr}(\pi_{S_1}(h_1)) = \operatorname{Tr}\big(T_h|_{S_{\underline{k}}(\Gamma(N))}\big).$$

Proof. This lemma immediately follows from Lemma 3.1. □

Let V_r denote the vector space of symmetric matrices of degree r, and define a rational representation ρ of the group $\mathrm{GL}_1 \times \mathrm{GL}_r$ on V_r by $x\cdot\rho(a,m) = a\,{}^t m x m$, where $x \in V_r$ and $(a,m) \in \mathrm{GL}_1\times\mathrm{GL}_r$. The kernel of ρ is given by $\operatorname{Ker}\rho = \{(a^{-2}, aI_r) : a \in \mathrm{GL}_1\}$, and we set

$$H_r = \operatorname{Ker}\rho\backslash(\mathrm{GL}_1\times\mathrm{GL}_r).$$

Then, the pair (H_r, V_r) is a prehomogeneous vector space over $\mathbb{Q}$. For $1 \le r \le n$ and $f \in C_c^\infty(G(\mathbb{A}))$ (respectively, $f \in C_c^\infty(G(\mathbb{A}_f))$), we define a function $\Phi_{f,r} \in C_c^\infty(V_r(\mathbb{A}))$ (respectively, $\Phi_{f,r} \in C_c^\infty(V_r(\mathbb{A}_f))$) as

$$\Phi_{f,r}(x) = \int_K f\left(k^{-1}\begin{pmatrix} I_n & * \\ O_n & I_n\end{pmatrix}k\right)\mathrm{d}k \quad \left(\text{respectively, } \int_{K_f}\right), \quad \text{where } * = \begin{pmatrix} x & 0 \\ 0 & 0\end{pmatrix} \in V_n.$$

Let $\tilde{f}_{\underline{k}}$ denote the spherical trace function of $\sigma_{\underline{k}}$ with respect to $\rho_{\underline{k}}$ on $G(\mathbb{R})$; see [Wakatsuki 2018, §5.3]. Notice that $\tilde{f}_{\underline{k}}$ is a matrix coefficient of $\sigma_{\underline{k}}$, and so it is not compactly supported. Take a test function $h \in C_c^\infty(G(\mathbb{A}_f))$ and set $\tilde{f} = \tilde{f}_{\underline{k}}h$. Let χ be a real character on $\mathbb{R}_{>0}\mathbb{Q}^\times\backslash\mathbb{A}^\times$. Define a zeta integral $Z_r(\Phi_{\tilde{f},r}, s, \chi)$ by

$$Z_r(\Phi_{\tilde{f},r}, s, \chi) = \int_{H_r(\mathbb{Q})\backslash H_r(\mathbb{A})} |a^r \det(m)^2|^s \chi(a) \sum_{x\in V_r^0(\mathbb{Q})} \Phi_{\tilde{f},r}(x\cdot g)\,\mathrm{d}g, \quad g = \rho(a,m),$$

where $V_r^0=\{x\in V_r:\det(x)\neq 0\}$ and $\mathrm{d}g$ is a Haar measure on $H_r(\mathbb{A})$. The zeta integral $Z_r(\Phi_{\tilde{f},r},s,\chi)$ is absolutely convergent for the range

$$k_n>2n,\quad \mathrm{Re}(s)>\frac{r-1}{2},\quad \begin{cases}\mathrm{Re}(s)<\dfrac{k_n}{2} & \text{if } r=2,\\ \mathrm{Re}(s)<k_n-\dfrac{r-1}{2} & \text{otherwise,}\end{cases} \tag{3-2}$$

see [Wakatsuki 2018, Proposition 5.15], and $Z(\Phi_{\tilde{f},r},s,\chi)$ is meromorphically continued to the whole s-plane; see [Shintani 1975; Wakatsuki 2018; Yukie 1993]. The following lemma associates $Z(\Phi_{\tilde{f},r},s,\chi)$ with the unipotent contribution $I_{\mathrm{unip}}(f)=I_1(f)+I_2(f)+I_3(f)$ of the invariant trace formula.

Lemma 3.3. *Let S_0 be a finite set of finite places of $\mathbb{Q}$. Take a test function $h_{S_0}\in C_c^\infty(G(\mathbb{Q}_{S_0}))$, and let h^{S_0} denote the characteristic function of $\prod_{p\notin S_0\sqcup\{\infty\}}K_p$. Define a test function $\tilde{f}$ as $\tilde{f}=\tilde{f}_{\underline{k}}h_{S_0}h^{S_0}$. If k_n is sufficiently large ($k_n\gg 2n$), then we have*

$$I_{\mathrm{unip}}\big(f_{\underline{k}}h_{S_0}h^{S_0}\big)=\mathrm{vol}_G\ h_{S_0}(1)\,d_{\underline{k}}+\frac{1}{2}\sum_{r=1}^{n}\sum_{\chi\in\mathscr{X}(S_0)}Z_r\Big(\Phi_{\tilde{f},r},n-\frac{r-1}{2},\chi\Big),$$

where $\mathrm{vol}_G=\mathrm{vol}(G(\mathbb{Q})\backslash G(\mathbb{A}))$, $d_{\underline{k}}$ denotes the formal degree of $\sigma_{\underline{k}}$, and $\mathscr{X}(S_0)$ denotes the set consisting of real characters $\chi=\otimes_v\chi_v$ on $\mathbb{R}_{>0}\mathbb{Q}^\times\backslash\mathbb{A}^\times$ such that χ_v is unramified for any $v\notin S_0\sqcup\{\infty\}$. Note that S_0 may contain S_1 and all prime factors of N.

Remark 3.4. Note that the point $s=n-(r-1)/2$, where $1\le r\le n$, is contained in the range (3-2), and we have $Z_r(\Phi_{\tilde{f},r},s,\chi)\equiv 0$ for any real character $\chi\notin\mathscr{X}(S_0)$.

Proof. To study $I_{\mathrm{unip}}(f_{\underline{k}}h_{S_0}h^{S_0})$, we need an additional zeta integral $\tilde{Z}_r(\Phi_{\tilde{f},r},s)$ defined by

$$\tilde{Z}_r(\Phi_{\tilde{f},r},s)=\int_{\mathrm{GL}_r(\mathbb{Q})\backslash\mathrm{GL}_r(\mathbb{A})}|\det(m)|^{2s}\sum_{x\in V_r^0(\mathbb{Q})}\Phi_{\tilde{f},r}({}^tmxm)\,\mathrm{d}m.$$

The zeta integral $\tilde{Z}_r(\Phi_{\tilde{f},r},s)$ is absolutely convergent for the range (3-2), and $\tilde{Z}(\Phi_{\tilde{f},r},s)$ is meromorphically continued to the whole s-plane; see [Shintani 1975; Wakatsuki 2018; Yukie 1993]. Applying [Wakatsuki 2018, Propositions 3.8 and 3.11, Lemmas 5.10 and 5.16] to $I_{\mathrm{unip}}(f)$, we obtain

$$I_{\mathrm{unip}}\big(f_{\underline{k}}h_{S_0}h^{S_0}\big)=\mathrm{vol}_G\ h_{S_0}(1)\,d_{\underline{k}}+\sum_{r=1}^{n}\tilde{Z}_r\Big(\Phi_{\tilde{f},r},n-\frac{r-1}{2}\Big) \tag{3-3}$$

for sufficiently large $k_n\gg 2n$. Notice that $f_{\underline{k}}$ is changed to $\tilde{f}_{\underline{k}}$ in the right-hand side of (3-3), and this change is essentially required for the proof of (3-3).

By the same argument as in [Hoffmann and Wakatsuki 2018, (4.9)], we have

$$\tilde{Z}_r(\Phi_{\tilde{f},r},s)=\frac{1}{2}\sum_{\chi}Z_r(\Phi_{\tilde{f},r},s,\chi),$$

where χ runs over all real characters on $\mathbb{R}_{>0}\mathbb{Q}^\times\backslash\mathbb{A}^\times$. Suppose that $\chi=\otimes_v\chi_v\notin\mathscr{X}(S_0)$. Then, we can take a prime $p\notin S_0$ such that χ_p is ramified and

$$\Phi_{\tilde{f},r}(a_px)=\Phi_{\tilde{f},r}(x),\quad \forall a_p\in\mathbb{Z}_p^\times. \tag{3-4}$$

Hence, we get $Z_r(\Phi_{\tilde{f},r},s,\chi)\equiv 0$, and the proof is completed. □

Remark 3.5. The rational representation ρ of H_r on V_r is faithful, but the representation $x \mapsto {}^t m x m$ of GL_r on V_r is not. Hence, $Z_r(\Phi_{\tilde{f},r}, s, \chi)$ is suitable for Saito's explicit formula [1999], which we use in the proof of Theorem 3.10, but $\tilde{Z}_r(\Phi_{\tilde{f},r}, s)$ is not. This fact is also important for the study of global coefficients in the geometric side; see [Hoffmann and Wakatsuki 2018].

Let ψ be a nontrivial additive character on $\mathbb{Q}\backslash\mathbb{A}$, and a bilinear form $\langle\ ,\ \rangle$ on $V_r(\mathbb{A})$ is defined by $\langle x, y\rangle := \mathrm{Tr}(xy)$. Let $\mathrm{d}x$ denote the self-dual measure on $V_r(\mathbb{A})$ for $\psi(\langle\ ,\ \rangle)$. Then, a Fourier transform of $\Phi \in C^\infty(V_r(\mathbb{A}))$ is defined by

$$\hat{\Phi}(y) = \int_{V_r(\mathbb{A})} \Phi(x)\psi(\langle x, y\rangle)\,\mathrm{d}x, \quad y \in V_r(\mathbb{A}).$$

For each $\Phi_0 \in C_0^\infty(V_r(\mathbb{A}_f))$, we define its Fourier transform $\widehat{\Phi_0}$ in the same manner. The zeta function $Z_r(\Phi_{\tilde{f},r}, s, \mathbb{1})$ satisfies the functional equation [Shintani 1975; Yukie 1993]

$$Z_r(\Phi_{\tilde{f},r}, s, \mathbb{1}) = Z_r\Big(\widehat{\Phi_{\tilde{f},r}}, \frac{r+1}{2} - s, \mathbb{1}\Big), \tag{3-5}$$

where $\mathbb{1}$ denotes the trivial representation on $\mathbb{R}_{>0}\mathbb{Q}^\times\backslash\mathbb{A}^\times$.

Take a test function $\Phi_0 \in C_0^\infty(V_r(\mathbb{A}_f))$ such that $\Phi_0({}^t k x k) = \Phi_0(x)$ holds for any $k \in \prod_{p<\infty} H_r(\mathbb{Z}_p)$ and $x \in V_r(\mathbb{A}_f)$, where $H_r(\mathbb{Z}_p)$ is identified with the projection of $\mathrm{GL}_1(\mathbb{Z}_p) \times \mathrm{GL}_r(\mathbb{Z}_p)$ into $H_r(\mathbb{A}_f)$. We write L_0 for the subset of $V_r(\mathbb{Q})$ which consists of the positive definite symmetric matrices contained in the support of Φ_0. It follows from the condition of Φ_0 that L_0 is invariant for $\Gamma = H_r(\mathbb{Z}) = H_r(\mathbb{Q}) \cap H_r(\hat{\mathbb{Z}})$. Put $\zeta_r(\Phi_0, s) = 1$ for $r = 0$. For $r > 0$, define a Shintani zeta function $\zeta_r(\Phi_0, s)$ as

$$\zeta_r(\Phi_0, s) = \sum_{x \in L_0/\Gamma} \frac{\Phi_0(x)}{\#(\Gamma_x)\det(x)^s},$$

where $\Gamma_x = \{\gamma \in \Gamma : x \cdot \gamma = x\}$. The zeta function $\zeta_r(\Phi_0, s)$ absolutely converges for $\mathrm{Re}(s) > (r+1)/2$, and is meromorphically continued to the whole s-plane; see [Shintani 1975]. Furthermore, $\zeta_r(\Phi_0, s)$ is holomorphic except for possible simple poles at $s = 1, 3/2, \dots (r+1)/2$.

Lemma 3.6. *Let $1 \le r \le n$, $k_n > 2n$, $h \in C_c^\infty(G(\mathbb{A}_f))$, and take a test function $\tilde{f}$ as $\tilde{f} = \tilde{f}_{\underline{k}} h$. Then, there exists a rational function $\boldsymbol{C}_{n,r}(x_1, \dots, x_n)$ over $\mathbb{R}$ such that*

$$Z_r\Big(\Phi_{\tilde{f},r}, n - \frac{r-1}{2}, \mathbb{1}\Big) = \boldsymbol{C}_{n,r}(\underline{k}) \times \zeta_r(\widehat{\Phi_{h,r}}, r-n).$$

Proof. This can be proved by the functional equation (3-5) and the same argument as in [Wakatsuki 2018, proof of Lemma 5.16]. □

Note that $\zeta_r(\widehat{\Phi_{h,r}}, s)$ is holomorphic in $\{s \in \mathbb{C} : \mathrm{Re}(s) \le 0\}$, and $\boldsymbol{C}_{n,r}(x_1, \dots, x_n)$ is explicitly expressed by the Gamma function and the partitions; see [Wakatsuki 2018, (5.17) and Lemma 5.16]. We will use this lemma for the regularization of the range of $\underline{k}$. The zeta integral $Z_r(\Phi_{\tilde{f},r}, n - (r-1)/2, \mathbb{1})$ was defined only for $k_n > 2n$, but the right-hand side of the equality in Lemma 3.6 is available for any $\underline{k}$. In addition, this lemma is necessary to estimate the growth of $I_{\mathrm{unip}}(f)$ with respect to $S = S_1 \sqcup \{\infty\}$. We later define a Dirichlet series $D^S_{m,u_S}(s)$ just before Proposition 3.9, and the series $D^S_{m,u_S}(s)$ appears in the explicit formula of $Z_r(\Phi, s, \mathbb{1})$ when r is even. For the case that r is even and $3 < r < n$, it seems difficult to estimate the growth of its contribution to $Z_r(\Phi_{\tilde{f},r}, n-(r-1)/2, \mathbb{1})$, but we can avoid such difficulty by this lemma, since the part related to $D^S_{m,u_S}(s)$ in Saito's formula [1999, Theorem 3.3] disappears in the special value $\zeta_r(\widehat{\Phi_{h,r}}, r-n)$.

Theorem 3.7. *Suppose $k_n > n+1$. Let $h_1 \in \mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_{S_1}))^\kappa = \otimes_{p\in S_1}\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))^\kappa$, and let h be a test function on $G(\mathbb{A}_f)$ given as* (3-1). *Then there exists a positive constant c_0 such that, if $N \geq c_0 N_1^{2n\kappa}$,*

$$\mathrm{Tr}\big(T_h|_{S_{\underline{k}}(\Gamma(N))}\big) = \mathrm{vol}_G\ \mathrm{vol}(K(N))^{-1}h_1(1)d_{\underline{k}} + \frac{1}{2}\sum_{r=1}^{n} \boldsymbol{C}_{n,r}(\underline{k})\zeta_r(\widehat{\Phi_{h,r}}, r-n). \tag{3-6}$$

Proof. Let $f = f_{\underline{k}}h$ and $\tilde{f} = \tilde{f}_{\underline{k}}h$. By Lemma 3.2, it is sufficient to prove that the geometric side $I_{\mathrm{geom}}(f)$ equals the right-hand side of (3-6). If one uses the results in [Arthur 1989] and applies [Shin and Templier 2016, Lemma 8.4] by putting $\Xi : G \subset \mathrm{GL}_m$, $m = 2n$, $B_\Xi = 1$, $c_\Xi = c_0$ in their notations, then one gets $I_{\mathrm{geom}}(f) = I_{\mathrm{unip}}(f)$. Hence, by Lemma 3.3 and putting $h_{S_0}h^{S_0} = h$, we have

$$\mathrm{Tr}\big(T_h|_{S_{\underline{k}}(\Gamma(N))}\big) = \mathrm{vol}_G\ \mathrm{vol}(K(N))^{-1}h_1(1)d_{\underline{k}} + \frac{1}{2}\sum_{r=1}^{n}\sum_{\chi\in\mathscr{X}(S_0)} Z_r\Big(\Phi_{\tilde{f},r}, n-\frac{r-1}{2}, \chi\Big) \tag{3-7}$$

for sufficiently large k_n. Let $\mathscr{M}(a) := \mathrm{diag}(1,\dots,1,a,\dots,a)$, where there are n entries of both 1 and a, for $a \in \mathbb{A}^\times$. For any $a_p \in \mathbb{Z}_p^\times$, $b_p \in \mathbb{Q}_p^\times$, $\mu \in X_*(T)$, we have

$$\mathscr{M}(a_p)^{-1}K_p(N)\mathscr{M}(a_p) = K_p(N) \quad \text{and} \quad \mathscr{M}(a_p)^{-1}\mu(b_p)\mathscr{M}(a_p) = \mu(b_p).$$

Hence, (3-4) holds for any $p < \infty$, and so $Z_r(\Phi_{\tilde{f},r}, n-(r-1)/2, \chi)$ vanishes for any $\chi \neq \mathbb{1}$. Therefore, by Lemma 3.6 we obtain the assertion (3-6) for sufficiently large k_n. By the same argument as in [Wakatsuki 2018, proof of Theorem 5.17], we can prove that this equality (3-6) holds in the range $k_n > n+1$, because the both sides of (3-6) are rational functions of $\underline{k}$ in that range, see Lemma 3.6 and [Wakatsuki 2018, Proposition 5.3]. Thus, the proof is completed. □

Let S denote a finite subset of places of $\mathbb{Q}$, and suppose $\infty \in S$. For each character $\chi = \otimes_v \chi_v$ on $\mathbb{Q}^\times\mathbb{R}_{>0}\backslash\mathbb{A}^\times$, we set

$$L^S(s,\chi) = \prod_{p\notin S} L_p(s,\chi_p), \quad L(s,\chi) = \prod_{p<\infty} L_p(s,\chi_p),$$
$$\zeta^S(s) = L^S(s,\mathbb{1}) = \prod_{p\notin S}(1-p^{-s})^{-1}, \quad \text{and} \quad \zeta(s) = L(s,\mathbb{1}),$$

where $L_p(s,\chi_p) = (1-\chi_p(p)p^{-s})^{-1}$ if χ_p is unramified, and $L_p(s,\chi_p) = 1$ if χ_p is ramified.

Lemma 3.8. *Let $s \in \mathbb{R}$. For $s > 1$,*

$$\zeta^S(s) \leq \zeta(s) \quad \textit{and} \quad \big(\zeta^S\big)'(s) \ll \frac{2s\zeta(s)}{s-1},$$

where $(\zeta^S)'(s) = \frac{\mathrm{d}}{\mathrm{d}s}\zeta^S(s)$. For $s \leq -1$,

$$|\zeta^S(s)| \leq (N_S)^{-s}|\zeta(s)|,$$

where $N_S = \prod_{p\in S\setminus\{\infty\}} p$.

Proof. First of all, $(1-p^{-s})^{-1}\geq 1$ for $p\in S$. Hence $\zeta^S(s)\leq\zeta(s)$. Let $\log\zeta^S(s)=\sum_{p\notin S}\log(1-p^{-s})^{-1}$. Then

$$\frac{(\zeta^S)'(s)}{\zeta^S(s)}=\sum_{p\notin S}\frac{-p^{-s}\log p}{1-p^{-s}}.$$

If $s>1$, then $1-p^{-s}\geq\frac{1}{2}$. Hence,

$$\left|\frac{(\zeta^S)'(s)}{\zeta^S(s)}\right|\leq 2\sum_{p\notin S}p^{-s}\log p\leq 2\sum_{p}p^{-s}\log p.$$

By partial summation,

$$\sum_p p^{-s}\log p\leq\int_1^\infty\Bigl(\sum_{p\leq x}\log p\Bigr)sx^{-s-1}\,\mathrm{d}x\leq\int_1^\infty sx^{-s}\,\mathrm{d}x=\frac{s}{s-1}.$$

Here we use the prime number theorem: $\sum_{p\leq x}\log p\sim x$. Therefore, $(\zeta^S)'(s)\ll 2s\zeta(s)/(s-1)$. □

Set $\mathfrak{D}=\{d(\mathbb{Q}^\times)^2:d\in\mathbb{Q}^\times\}$. For each $d\in\mathfrak{D}$, we denote by $\chi_d=\prod_v\chi_{d,v}$ the quadratic character on $\mathbb{Q}^\times\mathbb{R}_{>0}\backslash\mathbb{A}^\times$ corresponding to the quadratic field $\mathbb{Q}(\sqrt{d})$ via class field theory. If $d=1$, then χ_d means the trivial character $\mathbb{1}$. For each positive even integer m, we set

$$\varphi_{d,m}^S(s)=\zeta^S(2s-m+1)\zeta^S(2s)\frac{L^S(m/2,\chi_d)}{L^S(2s-m/2+1,\chi_d)}N\bigl(\mathfrak{f}_d^S\bigr)^{(m-1)/2-s},$$

where $\mathfrak{f}_d^S$ denotes the conductor of $\chi_d^S=\prod_{p\notin S}\chi_{d,p}$. For each $u_S\in\mathbb{Q}_S=\prod_{v\in S}\mathbb{Q}_v$, one sets

$$\mathfrak{D}(u_S)=\bigl\{d(\mathbb{Q}^\times)^2:d\in\mathbb{Q}^\times,\ d\in u_S(\mathbb{Q}_S^\times)^2\bigr\}.$$

We need the Dirichlet series

$$D_{m,u_S}^S(s)=\sum_{d(\mathbb{Q}^\times)^2\in\mathfrak{D}(u_S)}\varphi_{d,m}^S(s).$$

The following proposition is a generalization of [Ibukiyama and Saito 2012, Proposition 3.6]:

Proposition 3.9. *Let $m\geq 2$ be an even integer. Suppose $(-1)^{m/2}u_\infty>0$ for $u_S=(u_v)_{v\in S}$ (namely, the term of $d(\mathbb{Q}^\times)^2=(\mathbb{Q}^\times)^2$ does not appear in $D_{m,u_S}^S(s)$ if $(-1)^{m/2}=-1$). The Dirichlet series $D_{m,u_S}^S(s)$ is meromorphically continued to $\mathbb{C}$, and is holomorphic at any $s\in\mathbb{Z}_{\leq 0}$.*

Proof. See [Kim et al. 2022, Corollary 4.23] for the case $m>3$. For $m=2$, this statement can be proved by using [Hoffmann and Wakatsuki 2018; Yukie 1992]. □

Theorem 3.10. *Fix a parameter $\underline{k}$ such that $k_n>n+1$. Let $h_1\in\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_{S_1}))^\kappa$, and let $h\in C_c^\infty(G(\mathbb{A}_f))$ be a test function on $G(\mathbb{A}_f)$ given as* (3-1). *Suppose* $\sup_{x\in G(\mathbb{Q}_{S_1})}|h_1(x)|\leq 1$. *Then, there exist positive constants a, b, and c_0 such that, if $N\geq c_0N_1^{2n\kappa}$,*

$$\mathrm{Tr}\bigl(T_h|_{S_{\underline{k}}(\Gamma(N))}\bigr)=\mathrm{vol}_G\,\mathrm{vol}(K(N))^{-1}h_1(1)d_{\underline{k}}+\mathrm{vol}(K(N))^{-1}O(N_1^{a\kappa+b}N^{-n}).$$

Here the constants a and b do not depend on κ, N_1, or N. See Lemma 3.3 for vol_G and $d_{\underline{k}}$.

Proof. Set

$$I(\tilde{f},r)=\mathrm{vol}(K(N))\times\zeta_r(\widehat{\Phi_{h,r}},r-n),\quad 1\le r\le n.$$

By Theorem 3.7, it is sufficient to prove $I(\tilde{f},r)=O(N_1^{a\kappa+b}N^{-n})$.

Let R be a finite set of places of $\mathbb{Q}$. Take a Haar measure $\mathrm{d}x_\infty$ on $V_r(\mathbb{R})$, and for each prime p, we write $\mathrm{d}x_p$ for the Haar measure on $V_r(\mathbb{Q}_p)$ normalized by $\int_{V_r(\mathbb{Z}_p)}\mathrm{d}x_p=1$. For a test function $\Phi_R\in C_c^\infty(V_r(\mathbb{Q}_R))$ and an $H_r(\mathbb{Q}_R)$-orbit $\mathscr{O}_R\in V_r^0(\mathbb{Q}_R)/H_r(\mathbb{Q}_R)$, we set

$$Z_{r,R}(\Phi_R,s,\mathscr{O}_R)=c_R\int_{\mathscr{O}_R}\Phi_R(x)|\det(x)|_R^{s-(r+1)/2}\,\mathrm{d}x,$$

where $c_R=\prod_{p\in R,\,p<\infty}(1-p^{-1})^{-1}$, $|\cdot|_R=\prod_{v\in R}|\cdot|_v$, and $\mathrm{d}x=\prod_{v\in R}\mathrm{d}x_v$. It is known that $Z_{r,R}(\Phi_R,s,\mathscr{O}_R)$ absolutely converges for $\mathrm{Re}(s)\ge\frac{r+1}{2}$, and is meromorphically continued to the whole s-plane.

Suppose that R does not contain ∞, that is, R consists of primes. Write $\eta_p(x)$ for the Clifford invariant of $x\in V_r^0(\mathbb{Q}_p)$, see [Ikeda 2017, Definition 2.1], and set $\eta_R((x_p)_{p\in R})=\prod_{p\in R}\eta_p(x_p)$. For $\chi=\mathbb{1}_R$ (trivial) or η_R, we put $(\Phi_R\chi)(x)=\Phi_R(x)\,\chi(x)$. It follows from the local functional equation [Ikeda 2017, Theorems 2.1 and 2.2] over $\mathbb{Q}_p$ ($R=\{p\}$) that $Z_{r,p}(\Phi_p\chi,s,\mathscr{O}_p)$ is holomorphic in the range $\mathrm{Re}(s)<0$, and $Z_{r,p}(\Phi_p\chi,s,\mathscr{O}_p)$ possibly has a simple pole at $s=0$. Hence, for any R, $Z_{r,R}(\Phi_R\chi,s,\mathscr{O}_R)$ does not have any pole in the area $\mathrm{Re}(s)<0$, but it may have a pole at $s=0$. Let $\widehat{\Phi_R}$ denote the Fourier transform of $\Phi_R\in C_c^\infty(V_r(\mathbb{Q}_R))$ over $\mathbb{Q}_R$ for $\prod_{v\in R}\psi_v(\langle\ ,\ \rangle)$, where $\psi_v=\psi|_{\mathbb{Q}_v}$.

Define

$$\Phi_{h_1,r}(x)=h_1\left(\begin{pmatrix}I_n & *\\ O_n & I_n\end{pmatrix}\right)\in C_c^\infty(V_r(\mathbb{Q}_{S_1})),$$

where $*=\left(\begin{smallmatrix}x&0\\0&0\end{smallmatrix}\right)\in V_n$. Note that this definition is compatible with $\Phi_{\tilde{f},r}$ since h_1 is spherical for $\prod_{p\in S_1}K_p$. Set

$$\mathscr{Z}_r(S_1,h_1)=\sum_{\mathscr{O}_{S_1}\in V_r^0(\mathbb{Q}_{S_1})/H_r(\mathbb{Q}_{S_1})}\left|Z_{r,S_1}\big(\widehat{\Phi_{h_1,r}}\chi_r,r-n,\mathscr{O}_{S_1}\big)\right|,$$

where

$$\chi_r=\begin{cases}\mathbb{1}_{S_1} & \text{if } (r \text{ is odd and } r<n) \text{ or } r=2<n,\\ \eta_{S_1} & \text{if } r \text{ is even and } 2<r<n,\end{cases}$$

and

$$\mathscr{Z}_n(S_1,h_1)=\sum_{\mathscr{O}_{S_1}\in V_r^0(\mathbb{Q}_{S_1})/H_r(\mathbb{Q}_{S_1})}\left|Z_{n,S_1}\Big(\Phi_{h_1,n},\frac{n+1}{2},\mathscr{O}_{S_1}\Big)\right|\quad\text{if } r=n.$$

It follows from Saito's formula [1999, Theorem 2.1 and §3] that the zeta function $\zeta_r(\widehat{\Phi_{h,r}},s)$ is expressed by a (finite or infinite) sum of Euler products of $Z_{r,p}(\Phi_p\chi_p,s,\mathscr{O}_p)$, with $\chi_p=\mathbb{1}_p$, η_p, or its finite sums, and he explicitly calculated the local zeta function $Z_{r,p}(\Phi_p\chi_p,s,\mathscr{O}_p)$ in [Saito 1997, §2] if Φ_p is the characteristic function of $V(\mathbb{Z}_p)$. We shall prove $I(\tilde{f},r)=O(N_1^{a\kappa+b}N^{-n})$ by using his results.

Case I. Assume r is odd and $r < n$. In the following, we set $S = S_1 \sqcup \{\infty\}$. By Saito's formula, we have

$$I(\tilde{f}, r) = (\text{constant}) \times N^{r(r-1)/2-rn} \times \sum_{\mathscr{O}_{S_1} \in V_r^0(\mathbb{Q}_{S_1})/H_r(\mathbb{Q}_{S_1})} Z_{r,S_1}(\widehat{\Phi_{h_1,r}}, r-n, \mathscr{O}_{S_1}) \times \zeta^S\Big(\frac{r+1}{2}-n\Big) \times \prod_{l=2}^{n} \zeta^S(l)^{-1} \times \prod_{u=1}^{[r/2]} \zeta^S(2u)\zeta^S(2r-2n-2u+1).$$

Therefore, one has

$$|I(\tilde{f}, r)| \ll N^{r(r-1)/2-rn} \times N_1^{2n^3} \times \mathscr{Z}_r(S_1, h_1)$$

by using Lemma 3.8.

Case II. Assume r is even and $3 < r < n$. By Saito's formula, Proposition 3.9, and Lemma 3.8, one can prove that $|I(\tilde{f}, r)|$ is bounded by

$$N^{r(r-1)/2-rn} \times \mathscr{Z}_r(S_1, h_1) \times \left| \zeta^S\Big(\frac{r}{2}\Big) \times \prod_{l=2}^{n} \zeta^S(l)^{-1} \times \prod_{u=1}^{r/2-1} \zeta^S(2u) \times \prod_{u=1}^{r/2} \zeta^S(2r-2n-2u+1) \right| \ll N^{r(r-1)/2-rn} \times N_1^{2n^3} \times \mathscr{Z}_r(S_1, h_1)$$

up to a constant. Note that Proposition 3.9 was used for this estimate, since it is necessary to prove the vanishing of the term including $D^S_{r,u_S}(s)$ in the explicit formula [Saito 1999, Theorem 3.3].

Case III. Assume $r = n$. In this case, we should use a method different from Case I and Case II since $Z_{r,S_1}(\widehat{\Phi_{h_1,r}}\chi, s, \mathscr{O}_{S_1})$ may have a simple pole at $s = r - n = 0$. Take an n-tuple $\underline{l} = (l_1, \dots, l_n)$, with $l_1 \geq \cdots \geq l_n > 2n$, and let $n(x) = \left(\begin{smallmatrix} I_n & x \\ O_n & I_n \end{smallmatrix}\right) \in G$ where $x \in V_n$. Recall that $\tilde{f}_{\underline{l}}$ satisfies the following two properties:

(i) $\tilde{f}_{\underline{l}}(k^{-1}gk) = \tilde{f}_{\underline{l}}(g)$, for all $k \in K_\infty$, $g \in G(\mathbb{R})$; see [Wakatsuki 2018, §5.3].

(ii) $\int_{\mathbb{R}} \tilde{f}_{\underline{l}}(g_1^{-1} n_1(t) g_2)\, \mathrm{d}t = 0$ for all $g_1, g_2 \in G(\mathbb{R})$, where $n_1(t) = n((b_{ij})_{1 \leq i,j \leq n})$, $b_{11} = t$, and $b_{ij} = 0$ for all $(i, j) \neq (1, 1)$; see [Wakatsuki 2018, Lemma 5.9].

By property (i), we can define $\Phi_{\tilde{f}_{\underline{l}},n}(x) = \tilde{f}_{\underline{l}}(n(x))$, where $x \in V_n(\mathbb{R})$.

Lemma 3.11. *For each orbit $\mathscr{O}_\infty \in V_n^0(\mathbb{R})/H_n(\mathbb{R})$, we have $Z_{n,\infty}(\Phi_{\tilde{f}_{\underline{l}},n}, (n+1)/2, \mathscr{O}_\infty) = 0$.*

Proof. Let $\mathscr{O}_\infty \neq I_n \cdot H_n(\mathbb{R})$, and take a representative element A of $\mathscr{O}_\infty$ as

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathscr{A} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{where } \mathscr{A} \in V_{n-2}^0(\mathbb{R}).$$

The orbit $\mathscr{O}_\infty$ is decomposed into $A \cdot \mathrm{GL}_n(\mathbb{R}) \sqcup (-A) \cdot \mathrm{GL}_n(\mathbb{R})$. The centralizer $H_{n(A)}$ of $n(A)$ in $H_n(\mathbb{R})$ is given by

$$H_{n(A)} = \{m(h)n(y) : h \in \mathrm{O}_A(n),\ y \in V_n(\mathbb{R})\},$$

where

$$m(h) = \begin{pmatrix} {}^t h^{-1} & O_n \\ O_n & h \end{pmatrix} \quad \text{and} \quad \mathrm{O}_A(n) = \{h \in \mathrm{GL}_n : {}^t h A h = A\}.$$

Hence, by property (ii), we have

$$
\begin{aligned}
Z_{n,\infty}\Big(\Phi_{\tilde{f}_{\underline{l}},n},\frac{n+1}{2},\mathscr{O}_\infty\Big) &= \sum_{A'=\pm A}\int_{\mathrm{O}_{A'}(n)\backslash \mathrm{GL}_n(\mathbb{R})} \tilde{f}_{\underline{l}}(m(h)^{-1}n(A')m(h))|\det(h)|^{n+1}\,\mathrm{d}h \\
&= \sum_{A'=\pm A}\int_{\mathscr{N}\mathrm{O}_{A'}(n)\backslash \mathrm{GL}_n(\mathbb{R})}\int_{\mathbb{R}} \tilde{f}_{\underline{l}}(m(h)^{-1}n(A')n_1(2t)m(h))|\det(h)|^{n+1}\,\mathrm{d}t\,\mathrm{d}h \\
&= 0,
\end{aligned}
$$

where $\mathscr{N}=\{(b_{ij}) : b_{jj}=1, \text{with } 1\le j\le n,\ b_{n1}\in\mathbb{R}, \text{ and } b_{ij}=0 \text{ otherwise}\}$.

In the case $s=(n+1)/2$, we note that $|\det(x)|$ vanishes in the integral of $Z_{n,\infty}(\Phi_{\tilde{f}_{\underline{l}},n},(n+1)/2,\mathscr{O}_\infty)$. Hence, it follows from property (ii) that

$$
\sum_{\mathscr{O}_\infty\in V_n^0(\mathbb{R})/H_n(\mathbb{R})} Z_{n,\infty}\Big(\Phi_{\tilde{f}_{\underline{l}},n},\frac{n+1}{2},\mathscr{O}_\infty\Big) = \int_{V_n(\mathbb{R})}\Phi_{\tilde{f}_{\underline{l}},n}(x)\,\mathrm{d}x = 0,
$$

and so we also find $Z_{n,\infty}(\Phi_{\tilde{f}_{\underline{l}},n},(n+1)/2, I_n\cdot H_n(\mathbb{R}))=0$. □

By Lemmas 3.6 and 3.11, the residue formula [Yukie 1993, Chapter 4] of $Z_n(\Phi,s,\mathbb{1})$ and the same argument as in [Hoffmann and Wakatsuki 2018, proof of Theorem 4.22], we obtain

$$
\begin{aligned}
\zeta_r(\widehat{\Phi_{h,r}},0) &= \boldsymbol{C}_{n,n}(\underline{l})^{-1}\,Z_n\Big(\Phi_{\tilde{f}_{\underline{l}}h,r},\frac{n+1}{2},\mathbb{1}\Big) \\
&= \boldsymbol{C}_{n,n}(\underline{l})^{-1}\ \mathrm{vol}(H_n(\mathbb{Q})\backslash H_n(\mathbb{A})^1)\int_{V(\mathbb{R})}\Phi_{\tilde{f}_{\underline{l}},n}(x_\infty)\log|\det(x_\infty)|_\infty\,\mathrm{d}x_\infty \\
&\qquad\qquad\times\int_{V(\mathbb{Q}_{S_1})}\Phi_{h_1,n}(x_{S_1})\,\mathrm{d}x_{S_1}\,N^{-n(n+1)/2},
\end{aligned}
$$

where $H_n(\mathbb{A})=\{(a,m)\in H_n(\mathbb{A}) : |a^n\det(m)^2|=1\}$. From this, we have

$$
|I(\tilde{f},r)| \ll N^{-n(n+1)/2}\times\mathscr{Z}_r(S_1,h_1).
$$

Case IV. Assume $r=2<n$. By Saito's formula [Hoffmann and Wakatsuki 2018, Theorem 4.15], we have

$$
|I(\tilde{f},r)| \ll N^{1-2n}\times\mathscr{Z}_2(S_1,h_1)\times\big|\zeta^S(2)^{-1}\zeta^S(3-2n)\big|\times\max_{u_S\in\mathbb{Q}_S^\times/(\mathbb{Q}_S^\times)^2,\,u_\infty<0}|D^S_{2,u_S}(2-n)|.
$$

Hence, it is enough to give an upper bound of $|D^S_{2,u_S}(2-n)|$ for $u_\infty<0$. Choose a representative element $u_S=(u_v)_{v\in S}$ satisfying $u_p\in\mathbb{Z}_p$, with $p\in S_1$. Take a test function $\Phi=\otimes_v\Phi_v$ such that the support of Φ_∞ is contained in $\{x\in V_2^0(\mathbb{R}) : \det(x)>0\}$ and Φ_p is the characteristic function of $\mathrm{diag}(1,-u_p)+p^2V_2(\mathbb{Z}_p)$ (respectively, $V_2(\mathbb{Z}_p)$) for each $p\in S_1$ (respectively, $p\notin S$). Let

$$
\Psi(y,yu)=\int_{K_2}\hat{\Phi}\Big({}^tk\begin{pmatrix}0 & y\\ y & yu\end{pmatrix}k\Big)\,\mathrm{d}k,\qquad K_2=\mathrm{O}(2,\mathbb{R})\times\prod_p\mathrm{GL}_2(\mathbb{Z}_p),
$$

and we set

$$
T(\Phi,s)=\frac{\mathrm{d}}{\mathrm{d}s_1}T(\Phi,s,s_1)\big|_{s_1=0}\quad\text{and}\quad T(\Phi,s,s_1)=\int_{\mathbb{A}^\times}\int_{\mathbb{A}}|y^2|^s\|(1,u)\|^{s_1}\Psi(y,yu)\,\mathrm{d}u\,\mathrm{d}^\times y.
$$

By [Shintani 1975, Lemma 1], one obtains $Z_{2,\infty}(\widehat{\Phi_\infty}, n-\frac{1}{2}, \mathscr{O}_\infty) = 0$ for any orbit $\mathscr{O}_\infty$ in $V_2^0(\mathbb{R})$. Therefore, from the functional equation [Yukie 1992, Corollary (4.3)], one deduces

$$\left|N_1^{-6} D^S_{2,u_S}(2-n)\right| \ll \left|Z_{2,S}(\Phi_S, 2-n, \mathscr{O}_S)\, D^S_{2,u_S}(2-n)\right| = \left|2^{-1} T\big(\hat{\Phi}, n-\tfrac{1}{2}\big)\right|.$$

By [Yukie 1992, Proposition (2.12) (2)], one gets

$$\left|T\left(\hat{\Phi}, n-\frac{1}{2}\right)\right| \ll N_1^{4n-2} \times \left\{\zeta^S(2n-2) + \left|(\zeta^S)'(2n-2)\right| + \left|\frac{(\zeta^S)'(2n-1)\zeta^S(2n-2)}{\zeta^S(2n-1)}\right|\right\},$$

where $(\zeta^S)'(s) = \frac{\mathrm{d}}{\mathrm{d}s}\zeta^S(s)$, because $\mathrm{Supp}(\hat{\Phi}_p) \subset p^{-2} V(\mathbb{Z}_p)$ for any $p \in S_1$. Therefore, one gets

$$|D^S_{2,u_S}(2-n)| \ll N_1^{4n+4}$$

by Lemma 3.8.

The final task is to prove $\mathscr{Z}_r(S_1, h_1) \ll N_1^{a\kappa+b}$ for some a and b. Using the local functional equations in [Ikeda 2017, Theorem 2.1] (see also [Sweet 1995]), one gets

$$\mathscr{Z}_r(S_1, h_1) \ll N_1^c \times \sum_{\mathscr{O}_{S_1} \in V_r^0(\mathbb{Q}_{S_1})/H_r(\mathbb{Q}_{S_1})} Z_{r,S_1}\left(|\Phi_{h_1,r}|, n-\frac{r-1}{2}, \mathscr{O}_{S_1}\right)$$

for some $c \in \mathbb{N}$. By [Assem 1993, Lemma 2.1.1] and the assumption $\sup_{x \in G(\mathbb{Q}_{S_1})} |h_1(x)| \le 1$, we have

$$|\Phi_{h_1,r}| \le \Phi_{S_1,r,-\kappa},$$

where $\Phi_{S_1,r,-\kappa}$ denotes the characteristic function of $\otimes_{p \in S_1} p^{-\kappa} V_r(\mathbb{Z}_p)$. Hence, by a change of variables, we get

$$\begin{aligned} Z_{r,S_1}\left(|\Phi_{h_1,r}|, n-\frac{r-1}{2}, \mathscr{O}_{S_1}\right) &\le Z_{r,S_1}\left(\Phi_{S_1,r,-\kappa}, n-\frac{r-1}{2}, \mathscr{O}_{S_1}\right) \\ &= N_1^{\kappa n r - \kappa r(r-1)/2} Z_{r,S_1}\left(\Phi_{S_1,r,0}, n-\frac{r-1}{2}, \mathscr{O}_{S_1}\right) \\ &\le N_1^{\kappa n r - \kappa r(r-1)/2}. \end{aligned}$$

It follows from classification theory of quadratic forms that $\#(V_r^0(\mathbb{Q}_{S_1})/H_r(\mathbb{Q}_{S_1})) \ll N_1$. Therefore, we obtain a desired upper bound for $\mathscr{Z}_r(S_1, h_1)$. Thus, we obtain $I(\tilde{f}, r) = O(N_1^{a\kappa+b} N^{-n})$. □

Remark 3.12. We give some remarks on Shin and Templier's work [2016] and Dalal's work [2022]. In the setting of [Shin and Templier 2016], they considered "all" cohomological representations as a family which exhausts an L-packet at infinity since they chose the Euler–Poincaré pseudocoefficient at the infinite place. Then there is no contribution from nontrivial unipotent conjugacy classes. Therefore, our work is different from Shin–Templier's work in that we can consider only holomorphic forms in an L-packet.

Shin suggested to consider a family of automorphic representations whose infinite type is any fixed discrete series representation. Dalal [2022] carried it out in the weight aspect by using the stable trace formula. The stabilization allows us to remove the contribution $I_3(f)$ (see Section 1), but instead of $I_3(f)$, the contributions of endoscopic groups have to enter. Dalal obtained a good bound for them by using the concept of hyperendoscopy introduced by Ferrari [2007]. In studying the level aspect, it seems difficult to directly get a sufficient bound for the growth of the hyperendoscopic groups in question; since $\mathrm{Sp}(2n)$ has infinitely many elliptic endoscopic groups

$$\mathrm{SO}(N_1, N_1) \times \mathrm{Sp}(2N_2) \quad \text{and} \quad \mathrm{SO}(N_1+1, N_1-1, E/\mathbb{Q}) \times \mathrm{Sp}(2N_2), \quad N_1 + N_2 = n,$$

where E runs over quadratic extensions of $\mathbb{Q}$ and $\mathrm{SO}(N_1+1, N_1-1, E/\mathbb{Q})$ is the quasisplit orthogonal group attached to $E/\mathbb{Q}$ (see [Arthur 2013, p. 13–14] and [Assem 1998, §4]), it is quite complicated to count the hyperendoscopic groups. (The referee pointed out to us that the essential difficulty in applying hyperendoscopy techniques is in computing endoscopic transfers of indicators of any level subgroup. In particular, answering the transfer problem is necessary to even know which set of groups we are counting.) We also observe the same complication coming from elliptic endoscopic groups in the unipotent terms of the (unstable) Arthur trace formula; see [Hoffmann and Wakatsuki 2018, p. 8]. Assem's results [1993; 1998] make us expect that, for $1 \le r \le n$, some parts of zeta integrals $Z_r(\Phi_{\tilde{f},r}, s, \chi)$ probably correspond to the central contributions of the endoscopic groups $\mathrm{SO}(n-r+1, n-r+1) \times \mathrm{Sp}(2r-2)$ and $\mathrm{SO}(n-r+2, n-r, E/\mathbb{Q}) \times \mathrm{Sp}(2r-2)$. To avoid such complication, we have simplified the unipotent terms in several steps as follows:

- Our method showed the vanishing of a large part of the unipotent terms; see Lemma 3.3 and [Wakatsuki 2018].
- The contributions of $Z_r(\Phi_{\tilde{f},r}, s, \chi)$ vanish when χ is nontrivial; see Theorem 3.7.
- Our careful analysis estimates upper bounds of the contributions of $Z_r(\Phi_{\tilde{f},r}, s, \mathbb{1})$ by using the functional equations; see the proof of Theorem 3.10.

Analogous simplifications should be required even if we use the stable trace formula.

4. Arthur classification of Siegel modular forms

In this section, we study Siegel modular forms in terms of Arthur's classification [2013]; see §1.4 and §1.5 of loc. cit.. Recall $G = \mathrm{Sp}(2n)/\mathbb{Q}$. We call a Siegel cusp form which comes from smaller groups by Langlands functoriality "a nongenuine form". In this section, we estimate the dimension of the space of nongenuine forms and show that they are negligible. This result is interesting in its own right.

Let $F \in HE_{\underline{k}}(N)$, see Section 2, and $\pi = \pi_F$ be the corresponding automorphic representation of $G(\mathbb{A})$. According to Arthur's classification, π can be described by using the global A-packets. Let us recall some notations. A (discrete) global A-parameter is a symbol

$$\psi = \pi_1[d_1] \boxplus \cdots \boxplus \pi_r[d_r]$$

satisfying the following conditions:

(1) for each i, with $1 \le i \le r$, π_i is an irreducible unitary cuspidal self-dual automorphic representation of $\mathrm{GL}_{m_i}(\mathbb{A})$. In particular, the central character ω_i of π_i is trivial or quadratic;

(2) for each i, $d_i \in \mathbb{Z}_{>0}$ and $\sum_{i=1}^{r} m_i d_i = 2n + 1$;

(3) if d_i is odd, then π_i is orthogonal, i.e., $L(s,\pi_i,\mathrm{Sym}^2)$ has a pole at $s=1$;

(4) if d_i is even, then π_i is symplectic, i.e., $L(s,\pi_i,\wedge^2)$ has a pole at $s=1$;

(5) $\omega_1^{d_1}\cdots\omega_r^{d_r}=\mathbb{1}$;

(6) if $i\neq j$ and $\pi_i\simeq\pi_j$, then $d_i\neq d_j$.

We say that two global A-parameters $\boxplus_{i=1}^r\pi_i[d_i]$ and $\boxplus_{i=1}^{r'}\pi_i'[d_i']$ are equivalent if $r=r'$ and there exists $\sigma\in\mathfrak{S}_r$ such that $d_i'=d_{\sigma(i)}$ and $\pi_i'=\pi_{\sigma(i)}$. Let $\Psi(G)$ be the set of equivalent classes of global A-parameters. For each $\psi\in\Psi(G)$, one can associate a set Π_ψ of equivalent classes of simple admissible $G(\mathbb{A}_f)\times(\mathfrak{g},K_\infty)$-modules; see [Arthur 2013]. The set Π_ψ is called a global A-packet for ψ.

Definition 4.1. Let $\psi=\boxplus_{i=1}^r\pi_i[d_i]$ be a global A-parameter.

- ψ is said to be semisimple if $d_1=\cdots=d_r=1$; otherwise, ψ is said to be nonsemisimple;
- ψ is said to be simple if $r=1$ and $d_1=1$.

By [Arthur 2013, Theorem 1.5.2] (though our formulation is slightly different from the original one), we have a following decomposition

$$L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}))\simeq\bigoplus_{\psi\in\Psi(G)}\bigoplus_{\pi\in\Pi_\psi}m_{\pi,\psi}\pi, \tag{4-1}$$

where $m_{\pi,\psi}\in\{0,1\}$; see [Atobe 2018, Theorem 2.2] for $m_{\pi,\psi}$. We have the following immediate consequence of (4-1):

Proposition 4.2. *Let $1_{K(N)}$ be the characteristic function of $K(N)\subset G(\mathbb{A}_f)$. Then*

$$S_{\underline{k}}(\Gamma(N))=\bigoplus_{\psi\in\Psi(G)}\bigoplus_{\substack{\pi\in\Pi_\psi\\ \pi_\infty\simeq\sigma_{\underline{k}}}}m_{\pi,\psi}\pi_f^{K(N)}$$

and

$$|HE_{\underline{k}}(N)|=\mathrm{vol}(K(N))^{-1}\sum_{\psi\in\Psi(G)}\sum_{\substack{\pi\in\Pi_\psi\\ \pi_\infty\simeq\sigma_{\underline{k}}}}m_{\pi,\psi}\,\mathrm{tr}(\pi_f(1_{K(N)})). \tag{4-2}$$

Theorem 4.3. *Assume* (1-4). *For a global A-parameter $\psi=\boxplus_{i=1}^r\pi_i[d_i]$, suppose that there exists $\pi\in\Pi_\psi$ with $\pi_\infty\simeq\sigma_{\underline{k}}$. Then ψ is semisimple, i.e., $d_i=1$ for all i, and each π_i is regular algebraic and satisfies the Ramanujan conjecture, i.e., $\pi_{i,p}$ is tempered for any p.*

Proof. By the proof of [Chenevier and Lannes 2019, Corollary 8.5.4], we see that $d_1=\cdots=d_r=1$. Hence, ψ is semisimple. Further, by comparing infinitesimal characters $c(\pi_\infty)$, $c(\psi_\infty)$ of π_∞, ψ_∞ respectively, we see that each π_i is regular algebraic by [Chenevier and Lannes 2019, Corollary 6.3.6 and Proposition 8.2.10]. It follows from [Caraiani 2012;2014] that $\pi_{i,p}$ is tempered for any p. □

Therefore, for each finite prime p, the local Langlands parameter at p of π is described as one of the isobaric sum $\boxplus_{i=1}^r\pi_{i,p}$ which is an admissible representation of $\mathrm{GL}_{2n+1}(\mathbb{Q}_p)$.

Definition 4.4. We denote by $HE_{\underline{k}}(N)^{ng}$ the subset of $HE_{\underline{k}}(N)$ consisting of all forms which belong to

$$\bigoplus_{\substack{\psi\in\Psi(G)\\ \psi:\text{nonsimple}}} \bigoplus_{\substack{\pi\in\Pi_\psi\\ \pi_\infty\simeq\sigma_{\underline{k}}}} m_{\pi,\psi}\pi_f^{K(N)},$$

under the isomorphism (4-1). A form in this space is called a nongenuine form.

Similarly, we denote by $HE_{\underline{k}}(N)^{g}$ the subset of $HE_{\underline{k}}(N)$ consisting of all forms which belong to

$$\bigoplus_{\substack{\psi\in\Psi(G)\\ \psi:\text{simple}}} \bigoplus_{\substack{\pi\in\Pi_\psi\\ \pi_\infty\simeq\sigma_{\underline{k}}}} m_{\pi,\psi}\pi_f^{K(N)},$$

under the isomorphism (4-1). A form in this space is called a genuine form.

Definition 4.5. Denote by $\Pi(\mathrm{GL}_n(\mathbb{R}))^c$ the isomorphism classes of all irreducible cohomological admissible $(\mathfrak{gl}_n, O(n))$-modules. For $\tau_\infty\in\Pi(\mathrm{GL}_n(\mathbb{R}))^c$ and a quasicharacter $\chi:\mathbb{Q}^\times\backslash\mathbb{A}^\times\to\mathbb{C}^\times$, we define

$$L^{\mathrm{cusp,ort}}\big(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}),\tau_\infty,\chi\big) := \bigoplus_{\substack{\pi:\text{orthogonal}\\ \pi_\infty\simeq\tau_\infty,\,\omega_\pi=\chi}} m(\pi)\pi$$

and

$$L^{\mathrm{cusp,ort}}\big(K^{\mathrm{GL}_n}(N),\tau_\infty,\chi\big) := \bigoplus_{\substack{\pi:\text{orthogonal}\\ \pi_\infty\simeq\tau_\infty,\,\omega_\pi=\chi}} m(\pi)\pi^{K^{\mathrm{GL}_n}(N)},$$

where the direct sums are taken over the isomorphism classes of all orthogonal cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ and ω_π stands for the central character of π. The constant $m(\pi)$ is the multiplicity of π in $L^2(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}))$ which satisfies $m(\pi)\in\{0,1\}$ by [Shalika 1974]. Here, $K^{\mathrm{GL}_n}(N)$ is the principal congruence subgroup of $\mathrm{GL}_n(\hat{\mathbb{Z}})$ of level N. Put

$$l^{\mathrm{cusp,ort}}(n,N,\tau_\infty,\chi) := \dim_{\mathbb{C}}\big(L^{\mathrm{cusp,ort}}(K^{\mathrm{GL}_n}(N),\tau_\infty,\chi)\big)$$

for simplicity. Clearly, $l^{\mathrm{cusp,ort}}(1,N,\tau_\infty,\chi)=|\hat{\mathbb{Z}}^\times/(1+N\hat{\mathbb{Z}})^\times|=\varphi(N)$, where φ stands for Euler's totient function.

Let $P(2n+1)$ be the set of all partitions of $2n+1$ and $P_{\underline{m}}$ be the standard parabolic subgroup of GL_{2n+1} associated to a partition $2n+1=m_1+\cdots+m_r$, and $\underline{m}=(m_1,\dots,m_r)$.

In order to apply the formula (4-2), it is necessary to study the transfer of Hecke elements in the local Langlands correspondence established by [Arthur 2013, Theorem 1.5.1]. We regard $G=\mathrm{Sp}(2n)$ as a twisted elliptic endoscopic subgroup of GL_{2n+1}; see [Ganapathy and Varma 2017] or [Oi 2023].

Proposition 4.6. *Let N be an odd positive integer. Put $S_N:=\{p\ \text{prime}: p\mid N\}$. For the pair (GL_{2n+1},G), the characteristic function of $\mathrm{vol}(K(N))^{-1}1_{K(N)}$ as an element of $C_c^\infty(G(\mathbb{Q}_{S_N}))$ is transferred to*

$$\mathrm{vol}\left(K^{\mathrm{GL}_{2n+1}}(N)\right)^{-1}1_{K^{\mathrm{GL}_{2n+1}}(N)}$$

as an element of $C_c^\infty(\mathrm{GL}_{2n+1}(\mathbb{Q}_{S_N}))$.

Proof. It follows from [Ganapathy and Varma 2017, Lemma 8.2.1 (i)]. □

Remark 4.7. Keep the notation in the previous proposition. If Π is the twisted endoscopic transfer of π, then the claim immediately implies

$$\dim_{\mathbb{C}} \pi^{K(N)} \le \dim_{\mathbb{C}} \Pi^{K^{\mathrm{GL}_{2n+1}}(N)}.$$

In fact, we have $\dim_{\mathbb{C}} \pi^{K(N)} = \mathrm{tr}(I_\theta : \Pi^{K^{\mathrm{GL}_{2n+1}}(N)} \to \Pi^{K^{\mathrm{GL}_{2n+1}}(N)})$, where $I_\theta : \Pi \to \Pi$ is the intertwining operator defining the twisted trace. Since I_θ is of finite order, we have the above inequality; see the argument for [Yamauchi 2021, Theorem 1.6].

Applying Proposition 4.6, we have the following:

Proposition 4.8. *Assume* (1-4) *and N is odd. Then $|HE_{\underline{k}}(N)^{ng}|$ is bounded by*

$$\frac{A_n(N)}{\varphi(N)} \sum_{\substack{\underline{m}=(m_1,\dots,m_r)\in P(2n+1)\\ r\ge 2}} \sum_{\substack{\tau_i\in\Pi(\mathrm{GL}_{m_i}(\mathbb{R}))^c\\ c(\boxplus_{i=1}^r\tau_i)=c(\sigma_{\underline{k}})}} \sum_{\substack{\chi_i:\mathbb{Q}^\times\backslash\mathbb{A}^\times\to\mathbb{C}^\times\\ \chi_i^2=1,\ c(\chi)\mid N}} d_{P_{\underline{m}}}(N) \prod_{i=1}^r l^{\mathrm{cusp,ort}}(m_i, N, \tau_i, \chi_i),$$

where the second sum is indexed by all r-tuples $(\tau_1, \dots, \tau_r)$ such that $\tau_i \in \Pi(\mathrm{GL}_{m_i}(\mathbb{R}))^c$ and $c(\boxplus_{i=1}^r \tau_i) = c(\sigma_{\underline{k}})$, the equality of the infinitesimal characters. Further $c(\chi)$ stands for the conductor of χ and $\varphi(N) = |(\mathbb{Z}/N\mathbb{Z})^\times|$. Here,

(1) $A_n(N) := 2^{(2n+1)\omega(N)}$ *where* $\omega(N) := |\{p \text{ prime} : p \mid N\}|$;

(2) $d_{P_{\underline{m}}}(N) = |P_{\underline{m}}(\mathbb{Z}/N\mathbb{Z})\backslash \mathrm{GL}_{2n+1}(\mathbb{Z}/N\mathbb{Z})| = \mathrm{vol}(K^{\mathrm{GL}_{2n+1}}(N))^{-1}/|P_{\underline{m}}(\mathbb{Z}/N\mathbb{Z})|$.

Proof. Let $\pi = \pi_\infty \otimes \otimes'_p \pi_p$ be an element of Π_ψ for $\psi = \boxplus_{i=1}^r \pi_i$. Let Π_p be the local Langlands correspondence of π_p to $\mathrm{GL}_{2n+1}(\mathbb{Q}_p)$ established by [Arthur 2013, Theorem 1.5.1], and let $\mathcal{L}(\Pi_p) : L_{\mathbb{Q}_p} \to \mathrm{GL}_{2n+1}(\mathbb{C})$ be the local L-parameter of Π_p, where $L_{\mathbb{Q}_p} = W_{\mathbb{Q}_p}$ for each $p < \infty$ and $L_{\mathbb{R}} = W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C})$. Since the localization ψ_p of the global A-parameter ψ at p is tempered by Theorem 4.3, we see that $\mathcal{L}(\Pi_p)$ is equivalent to ψ_p. Since $\mathcal{L}(\Pi_p)$ is independent of $\pi \in \Pi_\psi$ and multiplicity one for $\mathrm{GL}_{2n+1}(\mathbb{A})$ holds, the isobaric sum $\psi = \boxplus_{i=1}^r \pi_i$ as an automorphic representation of $\mathrm{GL}_{2n+1}(\mathbb{A})$ gives rise to a unique global L-parameter on Π_ψ. On the other hand, it follows from [Arthur 2013, Theorem 1.5.1] that $|\Pi_{\psi_p}| \le 2^{2n+1}$ for the local A-packet Π_{ψ_p} at p if $p \mid N$, and Π_{ψ_p} is a singleton if $p \nmid N$. It yields that $|\Pi_\psi| \le 2^{(2n+1)\omega(N)}$. Since the local Langlands correspondence $\pi_p \mapsto \Pi_p$ satisfies the character relation by [Arthur 2013, Theorem 1.5.1], it follows from Proposition 4.6 with Remark 4.7 that for each $\pi \in \Pi_\psi$,

$$\begin{aligned}\dim\big(\pi_f^{K(N)}\big) &= \mathrm{vol}(K(N))^{-1}\,\mathrm{tr}(\pi(1_{K(N)}))\\ &\le \mathrm{vol}(K^{\mathrm{GL}_{2n+1}}(N))^{-1}\,\mathrm{tr}\big((\boxplus_{i=1}^r \pi_i)(1_{K^{\mathrm{GL}_{2n+1}}(N)})\big)\\ &= \dim\big((\boxplus_{i=1}^r \pi_i)_f^{K^{\mathrm{GL}_{2n+1}}(N)}\big),\end{aligned}$$

where we denote by $\pi_f=\otimes'_{p<\infty}\pi_p$ the finite part of the cuspidal representation π. Plugging this into Proposition 4.2, we have

$$\begin{aligned}|HE_{\underline{k}}(N)^{ng}| &= \mathrm{vol}(K(N))^{-1}\sum_{\substack{\psi=\boxplus_{i=1}^r\pi_i\in\Psi(G),\, r\geq 2\\ c(\psi_\infty)=c(\sigma_{\underline{k}})}}\ \sum_{\pi\in\Pi_\psi} m_{\pi,\psi}\,\mathrm{tr}(\pi_f(1_{K(N)}))\\ &\leq \frac{A_n(N)}{\varphi(N)}\sum_{\substack{\psi=\boxplus_{i=1}^r\pi_i\in\Psi(G),\, r\geq 2\\ c(\psi_\infty)=c(\sigma_{\underline{k}})}} \dim\bigl((\boxplus_{i=1}^r\pi_i)_f^{K^{\mathrm{GL}_{2n+1}}(N)}\bigr),\end{aligned}$$

where $1/\varphi(N)$ is inserted because of the condition on the central characters in global A-parameters. Here, $r\geq 2$ is essential to gain the factor $1/\varphi(N)$; see Remark 4.9.

Next we describe $\dim((\boxplus_{i=1}^r\pi_i)_f^{K^{\mathrm{GL}_{2n+1}}(N)})$ in terms of the data (m_i,N,τ_i,χ_i) with $1\leq i\leq r$. Since

$$P_{\underline{m}}(\mathbb{A}_f)\backslash\mathrm{GL}_{2n+1}(\mathbb{A}_f)/K(N)\simeq P_{\underline{m}}(\hat{\mathbb{Z}})\backslash\mathrm{GL}_{2n+1}(\hat{\mathbb{Z}})/K(N)\simeq P_{\underline{m}}(\mathbb{Z}/N\mathbb{Z})\backslash\mathrm{GL}_{2n+1}(\mathbb{Z}/N\mathbb{Z})$$

and a complete system of the representatives can be taken from elements in $\mathrm{GL}_{2n+1}(\hat{\mathbb{Z}})$, and therefore, they normalize $K(N)$. Then a standard method for fixed vectors of an induced representation shows that

$$\dim\bigl((\boxplus_{i=1}^r\pi_i)_f^{K^{\mathrm{GL}_{2n+1}}(N)}\bigr)=d_{P_{\underline{m}}}(N)\prod_{i=1}^r\dim\bigl(\pi_{i,f}^{K^{\mathrm{GL}_{m_i}}(N)}\bigr),$$

Here, if χ_i is the central character of π_i and $\pi_{i,\infty}\simeq\tau_i$, then $\dim(\pi_{i,f}^{K^{\mathrm{GL}_{m_i}}(N)})=l^{\mathrm{cusp,ort}}(m_i,N,\tau_i,\chi_i)$. Notice that the conductor of χ_i is a divisor of N. Summing up, we have the claim. □

Remark 4.9. Let $r\geq 2$. The group homomorphism $((\mathbb{Z}/N\mathbb{Z})^\times)^r\to(\mathbb{Z}/N\mathbb{Z})^\times$, $(x_1,\dots,x_r)\mapsto x_1\cdots x_r$, is obviously surjective, and it yields

$$\bigl|\{(\chi_1,\dots,\chi_r)\in\widehat{(\mathbb{Z}/N\mathbb{Z})^{\times r}}:\chi_1\cdots\chi_r=1\}\bigr|=\frac{|\widehat{(\mathbb{Z}/N\mathbb{Z})^{\times r}}|}{\varphi(N)}.$$

This trivial equality explains the appearance of the factor $1/\varphi(N)$ in Proposition 4.8.

Next we study $l^{\mathrm{cusp,ort}}(n,N,\tau,\chi)$ for $\tau\in\Pi(\mathrm{GL}_n(\mathbb{R}))^c$ and for $n\geq 2$. Now if π is a cuspidal representation of GL_{2m+1} which is orthogonal, i.e., $L(s,\pi,\mathrm{Sym}^2)$ has a pole at $s=1$, then π comes from a cuspidal representation τ on $\mathrm{Sp}(2m)$. In this case, the central character ω_π of π is trivial.

If π is a cuspidal representation of GL_{2m} which is orthogonal, i.e., $L(s,\pi,\mathrm{Sym}^2)$ has a pole at $s=1$, then $\omega_\pi^2=1$; If $\omega_\pi=1$, π comes from a cuspidal representation τ on the split orthogonal group $\mathrm{SO}(m,m)$; If $\omega_\pi\neq 1$, then π comes from a cuspidal representation τ on the quasisplit orthogonal group $\mathrm{SO}(m+1,m-1)$.

First we consider the case when χ is trivial in estimating $l^{\mathrm{cusp,ort}}(2n+\delta,N,\tau,\chi)$, where $\delta=0$ or 1. For a positive integer n, let

$$H=\begin{cases}\mathrm{SO}(n,n) & \text{if } G'=\mathrm{GL}_{2n},\\ \mathrm{Sp}(2n) & \text{if } G'=\mathrm{GL}_{2n+1}.\end{cases}$$

We regard H as a twisted elliptic endoscopic subgroup G'.

Proposition 4.10. *Let N be an odd positive integer. For the pair (G', H), the characteristic function of $\mathrm{vol}(K^H(N))^{-1}1_{K^H(N)}$ as an element of $C_c^\infty(H(\mathbb{Q}_{S_N}))$ is transferred to*

$$\mathrm{vol}(K^{G'}(N))^{-1}1_{K^{G'}(N)}$$

as an element of $C_c^\infty(G'(\mathbb{Q}_{S_N}))$.

Proof. It follows from [Ganapathy and Varma 2017, Lemma 8.2.1 (i)]. □

Each cuspidal representation π of $G'(\mathbb{A})$ contributing to $l^{\mathrm{cusp,ort}}(N, \tau, \mathbb{1})$ can be regarded as a simple A-parameter. Also as a cuspidal representation, it strongly descends to a generic cuspidal representation Π_π of $H(\mathbb{A})$ whose L-parameter $\mathcal{L}(\Pi_\tau)$ at infinity of Π_π is same as one of π_∞. In this setting, by [Arthur 2013, Proposition 8.3.2 (b)], the problem is reduced to estimate

$$L^{\mathrm{cusp,gen}}(H, N, \mathcal{L}(\Pi_\tau), \mathbb{1}) := \bigoplus_{\pi \subset L^{\mathrm{cusp,generic,ort}}(H(\mathbb{Q})\backslash H(\mathbb{A}), \mathcal{L}(\Pi_\tau), \mathbb{1})} m(\pi)\pi^{K^H(N)}, \quad m(\pi) \in \{0, 1, 2\},$$

where π runs over all irreducible unitary, cohomological orthogonal cuspidal automorphic representations of $H(\mathbb{A})$ whose L-parameter at infinity is isomorphic to $\mathcal{L}(\Pi_\tau)$ with the central character $\chi = \mathbb{1}$.

Proposition 4.11. *Keep the notations as above. Then*

- $l^{\mathrm{cusp,ort}}(2n+\delta, N, \tau, \mathbb{1}) \le C_n(N)\dim(L^{\mathrm{cusp,gen}}(H, N, \mathcal{L}(\Pi_\tau), \mathbb{1}))$, *where* $C_n(N) := 2^{(2n+\delta)\omega(N)}$ *and*
$$\delta = \begin{cases} 0 & \text{if } G' = \mathrm{GL}_{2n}, \\ 1 & \text{if } G' = \mathrm{GL}_{2n+1}. \end{cases}$$
- $\dim(L^{\mathrm{cusp,gen}}(H, N, \mathcal{L}(\Pi_\tau), \mathbb{1})) \ll c \cdot \mathrm{vol}(K^H(N))^{-1} \sim c N^{\dim(H)}$ *for some $c > 0$, when the infinitesimal character of $\mathcal{L}(\Pi_\tau)$ is fixed and $N \to \infty$.*

Proof. The first claim follows from [Arthur 2013, Proposition 8.3.2 (b)] with a completely similar argument of Proposition 4.8.

The second claim follows from [Savin 1989]. □

Next we consider the case when χ is a quadratic character. In this case, a cuspidal representation π contributing to $L^{\mathrm{cusp,ort}}(K^{\mathrm{GL}_n}(N), \tau_\infty, \chi)$ comes from a cuspidal representation of the quasisplit orthogonal group $\mathrm{SO}(m+1, m-1)$ defined over the quadratic extension associated to χ. However any transfer theorem for Hecke elements in $(\mathrm{GL}_{2m}, \mathrm{SO}(m+1, m-1))$ remains open. To get around this situation, we make use of the transfer theorems for some Hecke elements in the quadratic base change due to Yamauchi [2021]. For this, we need the following assumptions on the level N:

(1) N is an odd prime or

(2) N is odd and all prime divisors $p_1, \dots, p_r$ $(r \ge 2)$ of N are congruent to 1 modulo 4 and $\left(\frac{p_i}{p_j}\right) = 1$ for $i \ne j$, where $\left(\frac{*}{*}\right)$ denotes the Legendre symbol.

These conditions are needed in order that for any quadratic extension $M/\mathbb{Q}$ with the conductor d_M dividing N, there exists an integral ideal $\mathfrak{N}$ of M such that $\mathfrak{N}\mathfrak{N}^\theta = (d_M)$ where θ is the generator of $\mathrm{Gal}(M/\mathbb{Q})$.

Proposition 4.12. *Keep the assumptions on N as above. Then*

$$l^{\mathrm{cusp,ort}}(2m, N, \tau, \chi) \le 2^{2m\cdot\omega(N)} \operatorname{vol}(K^H(N))^{-1},$$

where $H = \mathrm{SO}(m, m)$.

Proof. Let $M/\mathbb{Q}$ be the quadratic extension associated to χ and $\mathcal{O}_M$ the ring of integers of M. Let θ be the generator of $\mathrm{Gal}(M/\mathbb{Q})$. Let $K_M^{\mathrm{GL}_{2m}}(\mathfrak{N})$ be the principal congruence subgroup of $\mathrm{GL}_{2m}(\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_M)$ of the level $\mathfrak{N}$. Clearly, the θ-fixed part of $K_M^{\mathrm{GL}_{2m}}(\mathfrak{N})$ is $K^{\mathrm{GL}_{2m}}(d_M)$ where d_M is the conductor of $M/\mathbb{Q}$ and it contains $K^{\mathrm{GL}_{2m}}(N)$ since $d_M|N$. Applying [Yamauchi 2021, Theorem 1.6], we have for a cuspidal representation π of $\mathrm{GL}_{2m}(\mathbb{A})$ and its base change $\Pi := \mathrm{BC}_{M/\mathbb{Q}}(\pi)$ to $\mathrm{GL}_{2m}(\mathbb{A}_M)$,

$$\operatorname{vol}\big(K^{\mathrm{GL}_{2m}}(N)\big)^{-1} \operatorname{tr}\big(\pi(1_{K^{\mathrm{GL}_{2m}}(N)})\big) \le \operatorname{vol}\big(K_M^{\mathrm{GL}_{2m}}(\mathfrak{N})\big)^{-1} \operatorname{tr}\big(\Pi(1_{K_M^{\mathrm{GL}_{2m}}(\mathfrak{N})})\big).$$

Recall that our π contributing to $L^{\mathrm{cusp,ort}}(2m, N, \tau, \chi)$ is orthogonal, namely, $L(s, \pi, \mathrm{Sym}^2)$ has a pole at $s = 1$. Note that $L(s, \Pi, \mathrm{Sym}^2) = L(s, \pi, \mathrm{Sym}^2)L(s, \pi, \mathrm{Sym}^2 \otimes \chi)$. Now, $L(s, \pi \times (\pi \otimes \chi)) = L(s, \pi, \wedge^2 \otimes \chi)L(s, \pi, \mathrm{Sym}^2 \otimes \chi)$. Suppose Π is cuspidal. Then $\pi \not\simeq \pi \otimes \chi$. So the left-hand side has no zero at $s = 1$, and $L(s, \pi, \mathrm{Sym}^2 \otimes \chi)$ has no zero at $s = 1$. Therefore, $L(s, \Pi, \mathrm{Sym}^2)$ has a pole at $s = 1$.

If Π is noncuspidal, then by Arthur and Clozel [1989], there exists a cuspidal representation τ of $\mathrm{GL}_m(\mathbb{A}_M)$ such that

$$\Pi = \tau \boxplus \tau^\theta.$$

In such a case, if $m = 2$, then $\pi = \mathrm{AI}_M^{\mathbb{Q}}\, \tau$ for some cuspidal representation τ of $\mathrm{GL}_2(\mathbb{A}_M)$; an automorphic induction from $\mathrm{GL}_2(\mathbb{A}_M)$ to $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$. Since π is cuspidal and orthogonal, τ has to be dihedral. Such π are counted in [Kim et al. 2020b, Section 2.6] and it amounts to $O(N^{11/2+\varepsilon})$ for any $\varepsilon > 0$. This will be negligible because $\operatorname{vol}(K^H(N)) \sim cN^{m(2m-1)} = cN^6$ for some constant $c > 0$. Assume $m \ge 3$. It is easy to see that the dimension of $\bigoplus_{\Pi:\mathrm{noncuspidal}} \Pi_f^{K_M^{\mathrm{GL}_{2m}}(\mathfrak{N})}$ is bounded by

$$O\big(N^{m^2-1+m(m+1)/2}\big) = O\big(N^{3m^2/2+m/2-1}\big),$$

where the -1 of $m^2 - 1$ in the exponent of left-hand side in the above equation is inserted because of the fixed central character. Since $\dim \mathrm{SO}(m, m) = m(2m - 1)$ and $m \ge 3$, spaces $\Pi_f^{K_M^{\mathrm{GL}_{2m}}(\mathfrak{N})}$ for which Π is noncuspidal are negligible in the estimation. Further, Π is orthogonal with trivial central character. (The central character of Π is $\chi \circ N_{M/\mathbb{Q}} = 1$.) Therefore, we can bound $l^{\mathrm{cusp,ort}}(2m, N, \tau, \chi)$ by

$$l^{\mathrm{cusp,ort}}(2m, \mathfrak{N}, \mathrm{BC}_{M_\infty/\mathbb{R}}(\tau), 1),$$

which is similarly defined for cuspidal representations of $\mathrm{GL}_{2m}(\mathbb{A}_M)$. Applying the argument of the proof of Proposition 4.11 to $(\mathrm{GL}_{2m}/M, \mathrm{SO}(m, m)/M)$, the quantity $l^{\mathrm{cusp,ort}}(2m, N, \tau, \chi)$ is bounded by $2^{2m\omega(\mathfrak{N})} \operatorname{vol}(K^{H_M}(\mathfrak{N}))^{-1}$, where $H_M := \mathrm{SO}(m, m)/M$ and $\omega(\mathfrak{N})$ denotes the number of prime ideals dividing $\mathfrak{N}$. The claim follows from $\mathcal{O}_M/\mathfrak{N} \simeq \mathbb{Z}/N\mathbb{Z}$ since $\operatorname{vol}(K^{H_M}(\mathfrak{N})) = \operatorname{vol}(K^H(N))$ and clearly $\omega(\mathfrak{N}) = \omega(N)$. □

Note that for any split reductive group $\mathcal{G}$ over $\mathbb{Q}$ and the principal congruence subgroup $K^{\mathcal{G}}(N)$ of level N, we have that $\mathrm{vol}(K^{\mathcal{G}}(N)) \sim cN^{-\dim \mathcal{G}}$ for some constant $c > 0$ as $N \to \infty$. Furthermore, $\omega(N) \ll \log N/(\log\log N)$. Hence $2^{\omega(N)} \ll N^{\epsilon}$, and $A_n(N) = O(N^{\varepsilon})$ and $C_{m_i}(N) = O(N^{\varepsilon})$ for each $1 \le i \le r$.

Theorem 4.13. *Assume* (1-4). *Keep the assumptions on N as in Proposition 4.12. Then* $|HE_{\underline{k}}(N)^{ng}| = O_n(N^{2n^2+n-1+\varepsilon})$ *for any* $\varepsilon > 0$. *In particular,*

$$\lim_{N\to\infty} \frac{|HE_{\underline{k}}(N)^{ng}|}{|HE_{\underline{k}}(N)|} = 0.$$

Proof. By Proposition 4.8, for each partition $\underline{m} = (m_1, \dots, m_r)$ of $2n+1$, we must only estimate

$$\frac{A_n(N)}{\varphi(N)} d_{P_{\underline{m}}}(N) \prod_{i=1}^{r} l^{\mathrm{cusp,ort}}(m_i, N, \tau_i, \chi_i).$$

By Proposition 4.11 and Proposition 4.12,

$$l^{\mathrm{cusp,ort}}(m_i, N, \tau_i, \chi_i) \ll N^{m_i(m_i-1)/2+\varepsilon}$$

for any $\varepsilon > 0$. Further, $d_{P_{\underline{m}}(N)} = O(N^{\dim P_{\underline{m}} \backslash \mathrm{GL}_{2n+1}}) = O(N^{\sum_{1\le i<j\le r} m_i m_j})$. Note that $\varphi(N)^{-1} = O(N^{-1+\varepsilon})$ for any $\varepsilon > 0$. Since

$$\sum_{1\le i<j\le r} m_i m_j + \sum_{i=1}^{r} \frac{m_i(m_i-1)}{2} = \frac{1}{2}\Big(\sum_{1\le i,j\le r} m_i m_j\Big) - \frac{1}{2}\sum_{i=1}^{r} m_i = \frac{1}{2}(2n+1)^2 - \frac{1}{2}(2n+1) = 2n^2+n,$$

we have the first claim.

The second claim follows from the dimension formula (1-3). □

5. A notion of newforms in $S_{\underline{k}}(\Gamma(N))$

In this section, we introduce a notion of a newform in $S_{\underline{k}}(\Gamma(N))$ with respect to principal congruence subgroups. Since any local newform theory for $\mathrm{Sp}(2n)$ is unavailable except for $n = 1, 2$, we need a notion of newforms so that we can control a lower bound of conductors for such newforms. This is needed in application to low lying zeros. (See Theorem 8.3 and Lemma 9.3.)

Recall the description

$$S_{\underline{k}}(\Gamma(N)) = \bigoplus_{\psi \in \Psi(G)} \bigoplus_{\substack{\pi \in \Pi_\psi \\ \pi_\infty \simeq \sigma_{\underline{k}}}} m_{\pi,\psi} \pi_f^{K(N)}$$

in terms of Arthur's classification.

Definition 5.1. The new part (space) of $S_{\underline{k}}(\Gamma(N))$ is defined by

$$S_{\underline{k}}^{\mathrm{new}}(\Gamma(N)) = \bigoplus_{\psi \in \Psi(G)} \bigoplus_{\substack{\pi = \pi_f \otimes \sigma_{\underline{k}} \in \Pi_\psi \\ \pi^{K(N)} \neq 0 \text{ but } \pi^{K(d)} = 0 \text{ for any } d|N, d\neq N}} m_{\pi,\psi} \pi_f^{K(N)}.$$

The orthogonal complement $S_{\underline{k}}^{\mathrm{old}}(\Gamma(N))$ of $S_{\underline{k}}^{\mathrm{new}}(\Gamma(N))$ in $S_{\underline{k}}(\Gamma(N))$ with respect to Petersson inner product is said to be the old space. Let $HE_{\underline{k}}^{\mathrm{new}}(N)$ be a subset of $HE_{\underline{k}}(N)$ which is a basis of $S_{\underline{k}}^{\mathrm{new}}(\Gamma(N))$.

Remark 5.2. As the referee pointed out, $S_{\underline{k}}^{\mathrm{old}}(\Gamma(N))$ is the intersection of $S_{\underline{k}}(\Gamma(N))$ with the smallest $G(\mathbb{A}_f)$-invariant space of functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ containing $S_{\underline{k}}(\Gamma(M))$ for all proper divisors M of N.

Set $d_p = (1-p^{-1})^n$, $d_M = \prod_{p|M} d_p$ and $C_p = \prod_{j=1}^n (1-p^{-2j})$, $C_M = \prod_{p|M} C_p$. We set $d_1 = 1$ and $C_1 = 1$.

Recall $d_{\underline{k}}(N) = \dim S_{\underline{k}}(\Gamma(N)) = C_{\underline{k}}\, C_N\, N^{2n^2+n} + O_{\underline{k}}(N^{2n^2})$.

Lemma 5.3. *Assume that* (1-4) *holds and N is squarefree. Then we have*

$$d_{\underline{k}}(N) = \sum_{M|N} \dim S_{\underline{k}}^{\mathrm{new}}(\Gamma(M)) \left(\frac{N}{M}\right)^{n^2} C_{N/M}\, d_{N/M}^{-1}.$$

Proof. Let $M \mid N$. Take an automorphic representation $\pi = \pi_f \otimes \sigma_{\underline{k}}$ such that $\dim \pi_f^{K(M)} > 0$ and $\dim \pi_f^{K(L)} = 0$ for any $L \mid M$, $L < M$. Under this condition, π has an intersection with $S_{\underline{k}}^{\mathrm{new}}(\Gamma(M))$, and also with $S_{\underline{k}}(\Gamma(N))$. Let $\pi_f = \otimes_p \pi_p$. By the assumptions and Theorem 4.3, for any prime $p \nmid M$, π_p is tempered spherical, and so π_p is an irreducible induced representation from a Borel subgroup B of $G(\mathbb{Q}_p)$. So $\dim \pi_p^{K_p} = 1$. Now $K_p/K_p(p) \simeq \mathrm{Sp}_{2n}(\mathbb{F}_p)$, $\#\mathrm{Sp}_{2n}(\mathbb{F}_p) = p^{2n^2+n} C_p$, and $\#B(\mathbb{F}_p) = p^{n^2+n} d_p$. Hence, $\dim \pi_p^{K_p(p)} = p^{n^2} C_p\, d_p^{-1}$ for all $p \nmid M$. Since N is squarefree, this leads to

$$\dim \pi_f^{K(N)} = \dim \pi_f^{K(M)} \times \left(\frac{N}{M}\right)^{n^2} C_{N/M}\, d_{N/M}^{-1}.$$

Thus, we obtain the assertion. □

Theorem 5.4. *Assume that* (1-4) *holds and N is squarefree. Then we have*

$$\dim S_{\underline{k}}^{\mathrm{new}}(\Gamma(N)) = C_{\underline{k}} C_N N^{2n^2+n} \prod_{p|N} \left(1 - d_p^{-1} p^{-n^2-n}\right) + O_{\underline{k}}\left(N^{2n^2}\right).$$

Here, $\zeta(n^2)^{-1} < \prod_{p|N}\left(1-d_p^{-1}\, p^{-n^2-n}\right) < 1$ if $n > 1$. If $n = 1$, we have $\prod_{p|N}\left(1-d_p^{-1}\, p^{-2}\right) > \prod_p \left(1-1/(p(p-1))\right) = 0.374\ldots$.

Proof. Since $C_{N/M} = C_N/C_M$ and $d_{N/M} = d_N/d_M$, from Lemma 5.3, we have

$$d_{\underline{k}}(N) N^{-n^2} C_N^{-1} d_N = \sum_{M|N} \dim S_{\underline{k}}^{\mathrm{new}}(\Gamma(M)) M^{-n^2} C_M^{-1} d_M.$$

The Möbius inversion formula gives

$$\dim S_{\underline{k}}^{\mathrm{new}}(\Gamma(N)) N^{-n^2} C_N^{-1} d_N = \sum_{M|N} \mu(M) d_{\underline{k}}\left(\frac{N}{M}\right)\left(\frac{N}{M}\right)^{-n^2} C_{N/M}^{-1} d_{N/M},$$

where μ denotes the Möbius function. Therefore,

$$\dim S_{\underline{k}}^{\mathrm{new}}(\Gamma(N)) = \sum_{M|N} \mu(M) d_{\underline{k}}\left(\frac{N}{M}\right) M^{n^2} C_M\, d_M^{-1}. \tag{5-1}$$

By [Wakatsuki 2018, Corollary 1.2], there exist constants $C_{\underline{k},r}$ such that $d_{\underline{k}}(N)=\sum_{r=0}^{n} C_{\underline{k},r}C_N\,N^{f(r)}$ if $N>2$, where $f(r)=2n^2+n+\frac{1}{2}r(r-1)-nr$ and $C_{\underline{k},0}=C_{\underline{k}}$. Further, we take two constants D_1 and D_2 so that $d_{\underline{k}}(N)=\sum_{r=0}^{n} C_{\underline{k},r}\,C_N\,N^{f(r)}+D_N$ for $N=1$ or 2. Therefore, by (5-1), we obtain

$$\dim S_{\underline{k}}^{\mathrm{new}}(\Gamma(N))=\sum_{r=0}^{n} C_{\underline{k},r}C_N N^{f(r)}\sum_{M|N}\mu(M)d_M^{-1}M^{n^2-f(r)} \\ +\mu(N)N^{n^2}C_N d_N^{-1}D_1+\begin{cases}\mu\left(\frac{N}{2}\right)\left(\frac{N}{2}\right)^{n^2}C_{N/2}d_{N/2}^{-1}D_2 & \text{if } 2\mid N,\\ 0 & \text{if } 2\nmid N.\end{cases}$$

Since N is squarefree,

$$\sum_{M|N}\mu(M)d_M^{-1}M^{n^2-f(r)}=\prod_{p|N}\bigl(1-d_p^{-1}p^{n^2-f(r)}\bigr).$$

Therefore,

$$\dim S_{\underline{k}}^{\mathrm{new}}(\Gamma(N))=\sum_{r=0}^{n} C_{\underline{k},r}C_N N^{f(r)}\prod_{p|N}\bigl(1-d_p^{-1}p^{-f(r)+n^2}\bigr) \\ +\mu(N)N^{n^2}C_N d_N^{-1}D_1+\begin{cases}\mu\left(\frac{N}{2}\right)\left(\frac{N}{2}\right)^{n^2}C_{N/2}d_{N/2}^{-1}D_2 & \text{if } 2\mid N,\\ 0 & \text{if } 2\nmid N.\end{cases}$$

From this, we obtain the assertion.

Now, $d_p<1$. Hence $\prod_{p|N}\bigl(1-d_p^{-1}p^{n^2-f(r)}\bigr)<1$. Also $d_p^{-1}<p^n$ since $1/(1-p^{-1})<p$. Therefore, $\prod_{p|N}\bigl(1-d_p^{-1}p^{-n^2-n}\bigr)>\prod_{p|N}\bigl(1-p^{-n^2}\bigr)$. Here if $n>1$,

$$\prod_{p|N}\bigl(1-p^{-n^2}\bigr)^{-1}<\prod_{p}\bigl(1-p^{-n^2}\bigr)^{-1}=\zeta(n^2).$$

If $n=1$,

$$\prod_{p|N}\bigl(1-d_p^{-1}p^{-n^2-n}\bigr)=\prod_{p|N}\Bigl(1-\frac{1}{p(p-1)}\Bigr)>\prod_{p}\Bigl(1-\frac{1}{p(p-1)}\Bigr),$$

which is the Artin constant. □

6. Equidistribution theorem of Siegel cusp forms; proof of Theorem 1.1

By the definition in (1-1), we see that

$$\hat{\mu}_{K^S(N),S_1,\xi_{\underline{k}},D_{\underline{l}}^{\mathrm{hol}}}(\widehat{h_1})=\frac{\mathrm{Tr}(T_{h_1}|_{S_{\underline{k}}(\Gamma(N))})}{\mathrm{vol}(G(\mathbb{Q})\backslash G(\mathbb{A}))\cdot\dim\xi_{\underline{k}}}.$$

Notice that $\dim\xi_{\underline{k}}=d_{\underline{k}}$ (under a suitable normalization of the measure). Applying Theorem 3.10 to S_1, we have the claim by the Plancherel formula of Harish-Chandra: $\hat{\mu}_{S_1}^{\mathrm{pl}}(\widehat{h_1})=h_1(1)$.

7. Vertical Sato–Tate theorem for Siegel modular forms: proofs of Theorems 1.2 and 1.3

Suppose that $\underline{k} = (k_1, \dots, k_n)$ satisfies the condition (1-4). Put $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. For $F \in HE_{\underline{k}}(N)$, consider the cuspidal automorphic representation $\pi = \pi_F = \pi_\infty \otimes \otimes'_p \pi_{F,p}$ of $G(\mathbb{A})$ associated to F. As discussed in the previous section, under the condition (1-4), the A-parameter ψ whose A-packet contains π is semisimple and $\pi_{F,p}$ is tempered for all p. Then if $p \nmid N$, $\pi_{F,p}$ is spherical, and we can write $\pi_{F,p}$ as $\pi_{F,p} = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_p$, where $B = TU$ is the upper Borel subgroup and χ_p is a unitary character on $B(\mathbb{Q}_p)$. For each $1 \le j \le n$, put $\alpha_{jp}(\chi_p) := \chi_p(e_j(p^{-1}))$ (see (2-1) for $e_j(p^{-1})$) and by temperedness, we may write $\alpha_{jp}(\chi_p) = e^{\sqrt{-1}\theta_j}$, $\theta_j \in [0, \pi]$. Let $\hat{G} = \mathrm{SO}(2n+1)(\mathbb{C})$ be the complex split orthogonal group over $\mathbb{C}$ associated to the antidiagonal identity matrix. Let $\mathcal{L}(\pi_p) : W_{\mathbb{Q}_p} \to \mathrm{SO}(2n+1)(\mathbb{C})$ be the local Langlands parameter given by

$$\mathcal{L}(\pi_p)(\mathrm{Frob}_p) = (\alpha_{1p}(\chi_p), \dots, \alpha_{np}(\chi_p), 1, \alpha_{1p}(\chi_p)^{-1}, \dots, \alpha_{np}(\chi_p)^{-1}),$$

which is called the p-Satake parameter. Put $a^{(i)}(\chi_p) = a^{(i)}_{F,p}(\chi_p) = \frac{1}{2}(\alpha_{ip}(\chi_p) + \alpha_{ip}(\chi_p)^{-1}) = \cos\theta_i$ for $1 \le i \le n$. Let $\widehat{G(\mathbb{Q}_p)}^{\mathrm{ur,\,temp}}$ be the isomorphism classes of unramified tempered representations of $G(\mathbb{Q}_p)$. By [Shin and Templier 2016, Lemma 3.2], we have a topological isomorphism

$$\widehat{G(\mathbb{Q}_p)}^{\mathrm{ur,\,temp}} \xrightarrow{\sim} [0, \pi]^n / \mathfrak{S}_n =: \Omega$$

given by

$$\pi_p = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{Sp}_{2n}(\mathbb{Q}_p)} \chi_p \mapsto \big(\arg(a^{(1)}(\chi_p)), \dots, \arg(a^{(n)}(\chi_p))\big) =: (\theta_1, \dots, \theta_n).$$

We denote by $(\theta_1(\pi_{F,p}), \dots, \theta_n(\pi_{F,p})) \in \Omega$ the corresponding element to $\pi_{F,p}$ under the above isomorphism. Let $\hat{B} = \hat{T}\hat{U}$ be the upper Borel subgroup of $\hat{G} = \mathrm{SO}(2n+1)(\mathbb{C})$. Let $\Delta^+(\hat{G})$ be the set of all positive roots in $X^*(\hat{T}) = \mathrm{Hom}(\hat{T}, \mathrm{GL}_1)$ with respect to $\hat{B}$. We view $(\theta_1, \dots, \theta_n)$ as parameters of Ω. Let $\mu_p^{\mathrm{pl,\,temp}}$ be the restriction of the Plancherel measure on $\widehat{G(\mathbb{Q}_p)}$ to $\widehat{G(\mathbb{Q}_p)}^{\mathrm{ur,\,temp}}$, and by abusing the notation, we denote by $\mu_p = \mu_p^{\mathrm{pl,\,temp}}$ its pushforward to Ω. Put

$$t := \big(e^{\sqrt{-1}\theta_1}, \dots, e^{\sqrt{-1}\theta_n}, 1, e^{-\sqrt{-1}\theta_1}, \dots, e^{-\sqrt{-1}\theta_n}\big)$$

for simplicity. By [Shin and Templier 2016, Proposition 3.3], we have

$$\mu_p^{\mathrm{pl,\,temp}}(\theta_1, \dots, \theta_n) = W(\theta_1, \dots, \theta_n) d\theta_1 \cdots d\theta_n,$$

where

$$\begin{aligned} W(\theta_1, \dots, \theta_n) &= \frac{1}{(2\pi)^n}\Big(1 + \frac{1}{p}\Big)^{n^2} \frac{\prod_{\alpha \in \Delta^+(\hat{G})} |1 - e^{\sqrt{-1}\alpha(t)}|^2}{\prod_{\alpha \in \Delta^+(\hat{G})} |1 - p^{-1}e^{\sqrt{-1}\alpha(t)}|^2} \\ &= \frac{1}{(2\pi)^n}\Big(1 + \frac{1}{p}\Big)^{n^2} \frac{\prod_{i=1}^n |1 - e^{\sqrt{-1}\theta_i}|^2 \prod_{\substack{1 \le i < j \le n \\ \varepsilon = \pm 1}} |1 - e^{\sqrt{-1}(\theta_i + \varepsilon\theta_j)}|^2}{\prod_{i=1}^n |1 - p^{-1}e^{\sqrt{-1}\theta_i}|^2 \prod_{\substack{1 \le i < j \le n \\ \varepsilon = \pm 1}} |1 - p^{-1}e^{\sqrt{-1}(\theta_i + \varepsilon\theta_j)}|^2}. \end{aligned}$$

By letting $p \to \infty$, we recover the Sato–Tate measure

$$\mu_\infty^{\mathrm{ST}} = \lim_{p\to\infty} \mu_p^{\mathrm{pl,\,temp}} = \frac{1}{(2\pi)^n} \prod_{i=1}^n \big|1 - e^{\sqrt{-1}\theta_i}\big|^2 \prod_{\substack{1\le i<j\le n\\ \varepsilon=\pm 1}} \big|1 - e^{\sqrt{-1}(\theta_i+\varepsilon\theta_j)}\big|^2 d\theta_1 \cdots d\theta_n.$$

Then Theorems 1.2 and 1.3 follow from Theorems 1.1 and 4.13.

8. Standard L-functions of $\mathrm{Sp}(2n)$

Let $\underline{k} = (k_1, \dots, k_n)$ and $F \in HE_{\underline{k}}(N)$, and let π_F be a cuspidal representation of $G(\mathbb{A})$ associated to F.

Assume (1-4) for $\underline{k}$. By (4-1) and the observation there, the global A-packet Π_ψ containing π_F is associated to a semisimple global A parameter $\psi = \boxplus_{i=1}^r \pi_i$ where π_i is an irreducible cuspidal representation of $\mathrm{GL}_{m_i}(\mathbb{A})$. Then the isobaric sum $\Pi := \boxplus_{i=1}^r \pi_i$ is an automorphic representation of $\mathrm{GL}_{2n+1}(\mathbb{A})$. Therefore, we may define

$$L(s, \pi_F, \mathrm{St}) := L(s, \Pi) = \prod_{i=1}^r L(s, \pi_i).$$

Let $L_p(s, \pi_F, \mathrm{St}) := L(s, \Pi_p) = \prod_{i=1}^r L(s, \pi_{ip})$ be the local p-factor of $L(s, \pi_F, \mathrm{St})$ for each rational prime p.

Let $\pi_F = \pi_\infty \otimes \otimes'_p \pi_p$. For $p \nmid N$, we have that π_p is the spherical representation of $G(\mathbb{Q}_p)$ with the Satake parameter $(\alpha_{1p}, \dots, \alpha_{np}, 1, \alpha_{1p}^{-1}, \dots, \alpha_{np}^{-1})$. Then

$$L_p(s, \pi_F, \mathrm{St})^{-1} = (1 - p^{-s}) \prod_{i=1}^n (1 - \alpha_{ip} p^{-s})(1 - \alpha_{ip}^{-1} p^{-s}).$$

We define the conductor $q(F)$ of F to be the product of the conductors $q(\pi_i)$ of π_i, for $1 \le i \le r$.

Theorem 8.1. *Let $F \in HE_{\underline{k}}(N)$. Then the standard L-function $L(s, \pi_F, \mathrm{St})$ has a meromorphic continuation to all of $\mathbb{C}$. Let*

$$\Lambda(s, \pi_F, \mathrm{St}) = q(F)^{s/2} L_\infty(s, \pi_F, \mathrm{St}) L(s, \pi_F, \mathrm{St}),$$

where $L_\infty(s, \pi_F, \mathrm{St}) = \Gamma_{\mathbb{R}}(s+\epsilon)\Gamma_{\mathbb{C}}(s+k_1-1)\cdots\Gamma_{\mathbb{C}}(s+k_n-n)$,

$$\epsilon = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd,} \end{cases}$$

and $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$, $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. Then

$$\Lambda(s, \pi_F, \mathrm{St}) = \epsilon(F)\Lambda(1-s, \pi_F, \mathrm{St}),$$

where $\epsilon(F) \in \{\pm 1\}$.

Proof. It follows from the functional equation of $L(s, \Pi)$ by noting that Π is self-dual, and $L(s, \Pi_\infty) = L_\infty(s, \pi_F, \mathrm{St})$ is the local L-function attached to the holomorphic discrete series of the lowest weight $\underline{k}$; see [Kozima 2002]. □

The epsilon factor $\epsilon(F)$ turns out to be always 1.

Proposition 8.2. *Let π_F be associated to a semisimple A-parameter. Then $\epsilon(F)=1$.*

Proof. Recall the global A-parameter $\psi=\boxplus_{i=1}^r\pi_i$. Let ω_i be the central character of π_i. Since π_i is orthogonal, its epsilon factor is $\omega_i(-1)$ by [Lapid 2004, Theorem 1]. Hence,

$$\epsilon(F)=\prod_{i=1}^r\omega_i(-1)=\left(\prod_{i=1}^r\omega_i\right)(-1)=\mathbb{1}(-1)=1$$

by the condition on the central character. □

Theorem 8.3. *For any $F\in HE_{\underline{k}}(N)$, the conductor $q(F)$ satisfies $q(F)\le N^{2n+1}$. If $F\in HE_{\underline{k}}^{\mathrm{new}}(N)$, then $q(F)\ge\max\{N\prod_{p|N}p^{-1},\prod_{p|N}p\}$. So if $F\in HE_{\underline{k}}^{\mathrm{new}}(N)$, $q(F)\ge N^{1/2}$.*

Proof. Let π_F be associated to a semisimple global A parameter $\psi=\boxplus_{i=1}^r\pi_i$ where π_i is an irreducible cuspidal representation of $\mathrm{GL}_{m_i}(\mathbb{A})$, and let $\Pi:=\boxplus_{i=1}^r\pi_i$. Let $\Pi=\Pi_\infty\otimes\otimes_p'\Pi_p$. By Proposition 4.6, Π has a nonzero fixed vector by $K^{GL_{2n+1}}(p^{e_p})$, where $e_p=\mathrm{ord}_p(N)$. As in the proof of [Kim et al. 2020a, Lemma 8.1], it implies $\mathrm{depth}(\Pi_p)\le e_p-1$. Hence $q(\Pi_p)\le p^{(2n+1)e_p}$ by [Lansky and Raghuram 2003, Proposition 2.2]. Therefore, $q(F)\le N^{2n+1}$.

If $F\in HE_{\underline{k}}^{\mathrm{new}}(N)$, by Definition 5.1, it is not fixed by $K^{GL_{2n+1}}(p^{e_p-1})$ for each $p\mid N$. By [Miyauchi and Yamauchi 2022, Theorem 1.2], we have $q(\Pi_p)\ge p^{m_i(e_p-1)}$ for some i. In particular, $q(\Pi_p)\ge p^{e_p-1}$ for each $p\mid N$. Hence, $q(F)\ge N\prod_{p|N}p^{-1}$. It is clear that $q(\Pi_p)\ge p$ if $p\mid N$. Hence,

$$q(F)\ge\max\left\{N\cdot\prod_{p|N}p^{-1},\ \prod_{p|N}p\right\}.$$

Now, $q(F)^2=q(F)\cdot q(F)\ge N$. Hence our result follows. □

Proposition 8.4. *Keep the assumptions on N as in Proposition 4.12. Let $F\in HE_{\underline{k}}(N)$. Then $L(s,\pi_F,\mathrm{St})$ has a pole at $s=1$ if and only if π_F is associated to a semisimple global A-parameter $\psi=1\boxplus\pi_1\boxplus\cdots\boxplus\pi_r$ where π_i is an orthogonal irreducible cuspidal representation of $\mathrm{GL}_{m_i}(\mathbb{A})$, such that if $m_i=1$, π_i is a nontrivial quadratic character. Let $HE_{\underline{k}}(N)^0$ be the subset of $HE_{\underline{k}}(N)$ such that $L(s,\pi_F,\mathrm{St})$ has a pole at $s=1$. Then $|HE_{\underline{k}}(N)^0|=O(N^{2n^2-n+\epsilon})$. So $|HE_{\underline{k}}(N)^0|/|HE_{\underline{k}}(N)|=O(N^{-2n+\epsilon})$.*

This proves [Shin and Templier 2016, Hypothesis 11.2] in our family.

Proof. This follows from the proof of Theorem 4.13, by noting that partitions $\underline{m}=(m_1,\dots,m_r)$ of $2n$ contribute to $HE_{\underline{k}}(N)^0$. □

Böcherer [1986] gave the relationship between Hecke operators and L-functions for level one and scalar-valued Siegel modular forms and it is extended by Shimura [1994a] to a more general setting.

Let $\underline{a}=(a_1,\dots,a_n)$, $0\le a_1\le\cdots\le a_n$, and $D_{p,\underline{a}}=\mathrm{diag}(p^{a_1},\dots,p^{a_n})$. Let F be an eigenform in $HE_{\underline{k}}(N)$ with respect to the Hecke operator $T(D_{p,\underline{a}})$ for all $p\nmid N$, and let $\lambda(F,D_{p,\underline{a}})$ be the eigenvalue.

Then we have the following identity [Shimura 1994a, Theorem 2.9]:

$$\sum_{\underline{a}} \lambda(F, D_{p,\underline{a}}) X^{\sum_{i=1}^{n} a_i} = \frac{1-X}{1-p^n X} \prod_{i=1}^{n} \frac{(1-p^{2i}X^2)}{(1-\alpha_{ip}p^n X)(1-\alpha_{ip}^{-1}p^n X)}, \tag{8-1}$$

where $\underline{a} = (a_1, \ldots, a_n)$ runs over $0 \le a_1 \le \cdots \le a_n$.

Let $\underline{m} = (m_1, \ldots, m_n)$, $m_1 | m_2 | \cdots | m_n$, and $D_{\underline{m}} = \operatorname{diag}(m_1, \ldots, m_n)$, and let $\lambda(F, D_{\underline{m}})$ be the eigenvalue of the Hecke operator $T(D_{\underline{m}})$. Let

$$L^N(s, F) = \sum_{\underline{m},\,(m_n, N)=1} \lambda(F, D_{\underline{m}}) \det(D_{\underline{m}})^{-s}.$$

Then

$$L^N(s, F) = \prod_{p \nmid N} L(s, F)_p,$$

$$L(s, F)_p = \sum_{\underline{a}} \lambda(F, D_{p,\underline{a}}) \det(D_{p,\underline{a}})^{-s}.$$

It converges for $\operatorname{Re}(s) > 2n + (k_1 + \cdots + k_n)/n + 1$.

Hence, we have

$$\zeta^N(s) \left[\prod_{i=1}^{n} \zeta^N(2s-2i) \right] L^N(s, F) = L^N(s-n, \pi_F, \mathrm{St}),$$

where $L^N(s, \pi_F, \mathrm{St}) = \prod_{p \nmid N} L_p(s, \pi_F, \mathrm{St})$, and $\zeta^N(s) = \prod_{p \nmid N} (1-p^{-s})^{-1}$.

The central value of $L^N(s, F)$ is at $s = n + \frac{1}{2}$, and $L^N(s, F)$ has a zero at $s = n + \frac{1}{2}$ since $L^N(s, \pi_F, \mathrm{St})$ is holomorphic at $s = \frac{1}{2}$. Theorem 3.10 implies

Theorem 8.5. *For $\underline{m} = (m_1, \ldots, m_n)$, $m_1 | m_2 | \cdots | m_n$ with $m_n > 1$ and $(m_n, N) = 1$, $N \gg m_n^{2n}$,*

$$\frac{1}{|HE_{\underline{k}}(N)|} \sum_{F \in HE_{\underline{k}}(N)} \lambda(F, D_{\underline{m}}) = O(m_n^{\alpha} N^{-n}),$$

for some constant α.

Proof. Let S_1 be the set of all prime divisors of m_n. Since $m_n > 1$, S_1 is nonempty. The main term of right-hand side in Theorem 3.10 includes $h_1(1)$. Clearly, $h_1(1) = 0$ because the double coset defining the Hecke operator h_1 does not contain any central elements. Since the automorphic counting measure is supported on cuspidal representations, Theorem 3.10 implies the claim. □

Write

$$L^N(s, F) = \sum_{\substack{m=1 \\ (m,N)=1}}^{\infty} a_F(m) m^{-s} \quad \text{and} \quad L(s, F)_p = \sum_{k=0}^{\infty} a_F(p^k) p^{-ks}$$

for each prime $p \nmid N$. Here $a_F(p^k) = \sum_{\underline{a}} \lambda(F, D_{p,\underline{a}})$, where the sum is over all $\underline{a} = (a_1, \ldots, a_n)$ such that $0 \le a_1 \le \cdots \le a_n$, $a_1 + \cdots + a_n = k$. Hence, for $k > 0$ and $p \nmid N$,

$$\frac{1}{d_{\underline{k}}(N)} \sum_{F \in HE_{\underline{k}}(N)} a_F(p^k) = O(p^{ka} N^{-n}).$$

More generally:

Corollary 8.6. *For $m > 1$, with $(m, N) = 1$, $N \gg m^{2n}$,*

$$\frac{1}{d_{\underline{k}}(N)} \sum_{F \in HE_{\underline{k}}(N)} a_F(m) = O(m^\alpha N^{-n}).$$

Proof. We have $a_F(m) = \sum_{\underline{m}} \lambda(F, D_{\underline{m}})$, where the sum is over all $\underline{m} = (m_1, \dots, m_n)$, $m_1 | m_2 | \cdots | m_n$, $m_1 \cdots m_n = m$. Our assertion follows from Theorem 8.5. □

Write

$$L^N(s, \pi_F, \mathrm{St}) = \sum_{\substack{m=1 \\ (m,N)=1}}^{\infty} \mu_F(m) m^{-s}.$$

Then from (8-1), we have, for $p \nmid N$,

$$\mu_F(p) = \big(a_F(p) + 1\big) p^{-n} \quad \text{and} \quad \mu_F(p^2) = 1 + p^{-2} + \cdots + p^{-2n} + \big(a_F(p^2) + a_F(p)\big) p^{-2n}.$$

More generally, for $p \nmid N$,

$$\mu_F(p^k) = \begin{cases} 1 + p^{-2} h_k(p^{-2}) + p^{-n} \sum_{i=1}^k h_{ik}(p^{-1}) a_F(p^i) & \text{if } k \text{ is even,} \\ p^{-n} h'_k(p^{-2}) + p^{-n} \sum_{i=1}^k h'_{ik}(p^{-1}) a_F(p^i) & \text{if } k \text{ is odd,} \end{cases}$$

where $h_k, h'_k, h_{ik}, h'_{ik} \in \mathbb{Z}[x]$. Therefore, for $(m, N) = 1$,

$$\mu_F(m) = \prod_{p|m} \big(\delta_{p,m} + p^{-2} h_m^\delta(p^{-1})\big) + \sum_{\substack{u|m \\ u>1}} A_u a_F(u),$$

where

$$A_u \in \mathbb{Q}, \quad h_m^\delta \in \mathbb{Z}[x], \quad \text{and} \quad \delta = \delta_{p,m} = \begin{cases} 1 & \text{if } v_p(m) \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, by Corollary 8.6, we have

Theorem 8.7. *Fix $\underline{k} = (k_1, \dots, k_n)$, and let $m = \prod_{p|m} p^{v_p(m)}$ which is coprime to N. Then*

$$\frac{1}{d_{\underline{k}}(N)} \sum_{F \in HE_{\underline{k}}(N)} \mu_F(m) = \prod_{p|m} \big(\delta_{p,m} + p^{-2} h_m^\delta(p^{-1})\big) + O(N^{-n} m^c).$$

This proves [Kim et al. 2020b, Conjecture 6.1 in level aspect] for the Sp(4) case.

9. ℓ-level density of standard L-functions

In this section, we assume (1-4) and keep the assumptions on N in Proposition 4.12. Then we show unconditionally that the ℓ-level density (ℓ a positive integer) of the standard L-functions of the family $HE_{\underline{k}}(N)$ has the symmetry type Sp in the level aspect. Shin and Templier [2016] showed it under several hypotheses with a family which includes nonholomorphic forms.

Under assumption (1-4), F satisfies the Ramanujan conjecture, namely, $|\alpha_{ip}| = 1$ for each i. Let

$$-\frac{L'}{L}(s, \pi_F, \mathrm{St}) = \sum_{m=1}^{\infty} \Lambda(m) b_F(m) m^{-s},$$

where $b_F(p^m) = 1 + \alpha_{1p}^m + \cdots + \alpha_{np}^m + \alpha_{1p}^{-m} + \cdots + \alpha_{np}^{-m}$ when π_p is spherical.

For $F \in HE_{\underline{k}}(N)$, let Π be the Langlands transfer of π_F to GL_{2n+1}. If $F \in HE_{\underline{k}}(N)^g$, then $L(s, \Pi, \wedge^2)$ has no pole at $s = 1$, and $L(s, \Pi, \mathrm{Sym}^2)$ has a simple pole at $s = 1$. Let

$$L(s, \Pi \times \Pi) = \sum \lambda_{\Pi\times\Pi}(n) n^{-s},$$
$$L(s, \Pi, \wedge^2) = \sum \lambda_{\wedge^2(\Pi)}(n) n^{-s},$$
$$L(s, \Pi, \mathrm{Sym}^2) = \sum \lambda_{\mathrm{Sym}^2(\Pi)}(n) n^{-s}.$$

Then $\mu_F(p^2) = \lambda_{\mathrm{Sym}^2(\Pi)}(p)$ and $\mu_F(p)^2 = \lambda_{\Pi\times\Pi}(p) = \lambda_{\wedge^2(\Pi)}(p) + \lambda_{\mathrm{Sym}^2(\Pi)}(p)$.

Note that $\mu_F(p) = b_F(p)$, and $b_F(p^2) = 2\mu_F(p^2) - \mu_F(p)^2$. Let

$$T(p, \underline{a}) = \Gamma(N) \begin{pmatrix} D_{p,\underline{a}} & 0 \\ 0 & D_{p,\underline{a}}^{-1} \end{pmatrix} \Gamma(N).$$

By Theorem A.1, $T(p, (0, \dots, 0, 1))^2$, where there are $n-1$ entries of 0, is a linear combination of

$$T\big(p, (\overbrace{0, \dots, 0}^{n-1}, 2)\big),\ T\big(p, (\overbrace{0, \dots, 0}^{n-2}, 1, 1)\big),\ T\big(p, (\overbrace{0, \dots, 0}^{n-1}, 1)\big),\ T\big(p, (\overbrace{0, \dots, 0}^{n})\big) = \Gamma(N) I_{2n} \Gamma(N).$$

Therefore, by Theorem 8.7, if $p \nmid N$,

$$\frac{1}{d_{\underline{k}}(N)} \sum_{F \in HE_{\underline{k}}(N)} \mu_F(p)^2$$

is of the form

$$1 + p^{-1} g(p^{-1}) + O(p^c N^{-n})$$

for some polynomial $g \in \mathbb{Z}[x]$ and $c > 0$. Here the main term $1 + p^{-1} g(p^{-1})$ comes from the coefficient

$$p \sum_{i=0}^{2n-1} p^i \text{ of } T\big(p, (\overbrace{0, \dots, 0}^{n})\big)$$

in the linear combination. Here the explicit determination of the coefficient is necessary in our application. Hence, we have

Proposition 9.1. *For some $\alpha > 0$ and $p \nmid N$,*

$$\frac{1}{d_{\underline{k}}(N)} \sum_{F \in HE_{\underline{k}}(N)} b_F(p) = O(p^{-1}) + O(p^{\alpha} N^{-n}), \quad \textit{for } N \gg p^{2n}$$
$$\frac{1}{d_{\underline{k}}(N)} \sum_{F \in HE_{\underline{k}}(N)} b_F(p^2) = 1 + O(p^{-1}) + O(p^{\alpha} N^{-n}), \quad \textit{for } N \gg p^{4n}.$$

Remark 9.2. By a more careful analysis, we can replace the error term $O(N_1^{a\kappa+b}N^{-n})$ in Theorem 3.10 by

$$O\biggl(N_1^{n(n+1)/2\kappa+\epsilon}N^{-n(n+1)/2}+N_1^{(2n-1)\kappa+8n-4+\epsilon}N^{1-2n}+N_1^{n\kappa+2n^3+2n-3+\epsilon}N^{-n}\\ +\sum_{r=3}^{n-1}N_1^{\kappa(nr-r(r-1)/2)+(2n-r-1)[r/2]+2n-2r-1+2n^3+\epsilon}N^{r(r-1)/2-nr}\biggr),$$

for any $\epsilon>0$. Hence, the first error term $O(p^\alpha N^{-n})$ in Proposition 9.1 can be replaced (by taking $\kappa=1$) by

$$O\biggl(p^{n(n+1)/2+\epsilon}N^{-(n^2+n)/2}+p^{10n-5+\epsilon}N^{1-2n}+p^{2n^3+3n-3+\epsilon}N^{-n}\\ +\sum_{r=3}^{n-1}p^{2n^3+2n-1+2nr-r^2-2r+\epsilon}N^{r(r-1)/2-nr}\biggr).$$

The second error term $O(p^\alpha N^{-n})$ in Proposition 9.1 can be replaced (by taking $\kappa=2$) by

$$O\biggl(p^{n(n+1)+\epsilon}N^{-(n^2+n)/2}+p^{12n-6+\epsilon}N^{1-2n}+p^{2n^3+4n-3+\epsilon}N^{-n}\\ +\sum_{r=3}^{n-1}p^{2n^3+2n-1+3nr-(3/2)(r^2+r)+\epsilon}N^{r(r-1)/2-nr}\biggr).$$

We denote the nontrivial zeros of $L(s,\pi_F,\mathrm{St})$ by $\sigma_{F,j}=\frac{1}{2}+\sqrt{-1}\,\gamma_{F,j}$. Without assuming the GRH for $L(s,\pi_F,\mathrm{St})$, we can order them as

$$\cdots\le Re(\gamma_{F,-2})\le Re(\gamma_{F,-1})\le 0\le Re(\gamma_{F,1})\le Re(\gamma_{F,2})\le\cdots.$$

Let $c(F)=q(F)(k_1\cdots k_n)^2$ be the analytic conductor, and let

$$\log c_{\underline{k},N}=\frac{1}{d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)}\log c(F).$$

From Theorems 5.4 and 8.3, we have

Lemma 9.3. *Let $n>1$. We assume that N is squarefree. Then*

$$(k_1\cdots k_n)^2N^{1/(2\zeta(n^2))}\le c_{k,N}\le (k_1\cdots k_n)^2N^{2n+1}.$$

This proves [Shin and Templier 2016, Hypothesis 11.4] in our family. It is used in the proof of (9-1).

Proof. By Theorem 8.3, we have $q(F)\le N^{2n+1}$. It gives rise to the upper bound. If $F\in HE_{\underline{k}}^{\mathrm{new}}(N)$, $q(F)\ge N^{1/2}$ by Theorem 8.3. By Theorem 5.4, $|HE_{\underline{k}}^{\mathrm{new}}(N)|\ge\zeta(n^2)^{-1}|HE_{\underline{k}}(N)|$. Hence,

$$\log c_{\underline{k},N}\ge\log(k_1\cdots k_n)^2+\frac{1}{d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}^{\mathrm{new}}(N)}\log q(F)\ge\log(k_1\cdots k_n)^2+\frac{1}{2\zeta(n^2)}\log N. \qquad\square$$

Consider, for an even Paley–Wiener function ϕ,

$$D(F,\phi)=\sum_{\gamma_{F,j}}\phi\biggl(\frac{\gamma_{F,j}}{2\pi}\log c_{\underline{k},N}\biggr).$$

Then as in [Kim et al. 2020a, (9.1)],

$$\frac{1}{d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)} D(F,\phi)=\hat{\phi}(0)-\frac{1}{2}\phi(0)-\frac{2}{(\log c_{\underline{k},N})d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)}\sum_{p}\frac{b_F(p)\log p}{\sqrt{p}}\hat{\phi}\Big(\frac{\log p}{\log c_{\underline{k},N}}\Big)$$
$$-\frac{2}{(\log c_{\underline{k},N})d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)}\sum_{p}\frac{(b_F(p^2)-1)\log p}{p}\hat{\phi}\Big(\frac{2\log p}{\log c_{\underline{k},N}}\Big)$$
$$+O\Big(\frac{|HE_{\underline{k}}(N)^0|}{d_{\underline{k}}(N)}\Big)+O\Big(\frac{1}{\log c_{\underline{k},N}}\Big),$$

where $HE_{\underline{k}}(N)^0$ is in Proposition 8.4. (In [Kim et al. 2020a, (9.4)], the term $O\big(|HE_{\underline{k}}(N)^0|/d_{\underline{k}}(N)\big)$ was omitted.)

By Proposition 9.1, we can show as in [Kim et al. 2020a] that for an even Paley–Wiener function ϕ such that the Fourier transform $\hat{\phi}$ of ϕ is supported in $(-\beta,\beta)$, for some $\beta>0$,

$$\frac{1}{d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)} D(F,\phi)=\hat{\phi}(0)-\frac{1}{2}\phi(0)+O\Big(\frac{1}{\log c_{\underline{k},N}}\Big)=\int_{\mathbb{R}}\phi(x)W(\mathrm{Sp})(x)\,\mathrm{d}x+O\Big(\frac{\omega(N)}{\log N}\Big), \tag{9-1}$$

where $\omega(N)$ is the number of prime factors of N and $W(\mathrm{Sp})(x)=1-(\sin 2\pi x)/(2\pi x)$. (When we exchange two sums, if $p\nmid N$, we use Proposition 9.1. If $p\mid N$, by the Ramanujan bound, $|b_F(p)|\le n$ and $|b_F(p^2)|\le n$. Hence by the trivial bound, we would obtain $\sum_{p|N} b_F(p)\log p/\sqrt{p}\ll\omega(N)$ and $\sum_{p|N} b_F(p^2)\log p/p\ll\omega(N)$.)

In fact, by Remark 9.2, we can take β to be the minimum of

$$\frac{n^2+n}{(2n+1)(n^2+n+1)}-\epsilon,\quad \frac{2n-1}{(2n+1)(10n-9/2)}-\epsilon,\quad \frac{n}{(2n+1)(2n^3+3n-5/2)}-\epsilon,\quad \frac{1}{2n(2n+1)},$$
$$\min_{3\le r\le n-1}\Big\{\frac{nr-r(r-1)/2}{(2n+1)(2nr-r^2-2r+2n^3+2n-1/2)}-\epsilon\Big\}.$$

Namely,

$$\beta=\frac{n}{(2n+1)(2n^3+3n-5/2)}-\epsilon. \tag{9-2}$$

For a general ℓ, let

$$W(\mathrm{Sp})(x)=\det(K_{-1}(x_j,x_k))_{1\le j\le\ell,\,1\le k\le\ell},$$

where $K_{-1}(x,y)=\sin\pi(x-y)/\pi(x-y)-\sin\pi(x+y)/\pi(x+y)$. Let $\phi(x_1,\dots,x_\ell)=\phi_1(x_1)\cdots\phi_\ell(x_\ell)$, where each ϕ_i is an even Paley–Wiener function and $\hat{\phi}(u_1,\dots,u_\ell)=\hat{\phi}_1(u_1)\cdots\hat{\phi}_\ell(u_\ell)$. We assume that the Fourier transform $\hat{\phi}_i$ of ϕ_i is supported in $(-\beta,\beta)$ for $i=1,\dots,\ell$. The ℓ-level density function is

$$D^{(\ell)}(F,\phi)=\sum\nolimits^{*}_{j_1,\cdots,j_\ell}\phi\Big(\gamma_{j_1}\frac{\log c_{\underline{k},N}}{2\pi},\gamma_{j_2}\frac{\log c_{\underline{k},N}}{2\pi},\dots,\gamma_{j_\ell}\frac{\log c_{\underline{k},N}}{2\pi}\Big),$$

where $\sum^{*}_{j_1,\dots,j_\ell}$ is over $j_i=\pm1,\pm2,\dots$ with $j_a\ne\pm j_b$ for $a\ne b$. Then as in [Kim et al. 2020b], using Theorem 8.7, we can show

Theorem 9.4. *We assume that N is squarefree. Let $\phi(x_1,\dots,x_\ell)=\phi_1(x_1)\cdots\phi_\ell(x_\ell)$, where each ϕ_i is an even Paley–Wiener function and $\hat{\phi}(u_1,\dots,u_\ell)=\hat{\phi}_1(u_1)\cdots\hat{\phi}_\ell(u_\ell)$. Assume the Fourier transform $\hat{\phi}_i$ of ϕ_i is supported in $(-\beta,\beta)$ for $i=1,\cdots,\ell$. (See* (9-1) *for the value of β.) Then*

$$\frac{1}{d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)} D^{(\ell)}(F,\phi)=\int_{\mathbb{R}^\ell}\phi(x)W(\mathrm{Sp})(x)\,\mathrm{d}x+O\biggl(\frac{\omega(N)}{\log N}\biggr).$$

Remark 9.5. The above theorem is usually stated for Schwartz functions in the literature. But since Schwartz functions approximate any function in L^2-space, the above theorem holds for Payley–Wiener functions, which are in $L^2(\mathbb{R}^n)$, and whose Fourier transforms have compact supports.

10. The order of vanishing of standard L-functions at $s=\frac{1}{2}$

In this section, we show that the average order of vanishing of standard L-functions at $s=\frac{1}{2}$ is bounded under GRH; see [Iwaniec et al. 2000; Brumer 1995]. Under GRH on $L(s,\pi_F,\mathrm{St})$, its zeros are $\frac{1}{2}+\sqrt{-1}\gamma_F$ with $\gamma_F\in\mathbb{R}$.

Theorem 10.1. *Assume the GRH. Assume* (1-4) *and N is squarefree. Let $r_F=\mathrm{ord}_{s=\frac{1}{2}}L(s,\pi_F,\mathrm{St})$. Then*

$$\frac{1}{d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)} r_F\le C,$$

where $C=\frac{1}{n}(2n+1)\bigl(2n^3+3n-\frac{5}{2}\bigr)-\frac{1}{2}+\epsilon$.

Proof. Choose $\phi(x)=\bigl(2\sin(x\beta/2)/x\bigr)^2$ for $x\in\mathbb{R}$, where β is from (9-2). Then

$$\hat{\phi}(x)=\begin{cases}\beta-|x| & \text{if } |x|<\beta,\\ 0 & \text{otherwise.}\end{cases}$$

Since $\phi(x)\ge 0$ for $x\in\mathbb{R}$, from (9-1), we have

$$\frac{1}{d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)} r_F\phi(0)\le\hat{\phi}(0)-\frac{1}{2}\phi(0)+O\Bigl(\frac{1}{\log\log N}\Bigr).$$

Hence, we have

$$\frac{1}{d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)} r_F\le\frac{1}{\beta}-\frac{1}{2}+O\Bigl(\frac{1}{\log\log N}\Bigr).$$ □

We can show a similar result for the spinor L-function of GSp(4). Recall the following from [Kim et al. 2020a]:

Proposition 10.2. *Assume* $(N,11!)=1$.

(1) (*level aspect*) *Fix k_1,k_2. Then for ϕ whose Fourier transform $\hat{\phi}$ has support in $(-u,u)$ for some $0<u<1$, as $N\to\infty$ (See* [Kim et al. 2020a, Proposition 9.1] *for the value of u),*

$$\frac{1}{d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)} D(\pi_F,\phi,\mathrm{Spin})=\hat{\phi}(0)+\frac{1}{2}\phi(0)+O\Bigl(\frac{1}{\log\log N}\Bigr).$$

(2) (*weight aspect*) *Fix* N. *Then for* ϕ *whose Fourier transform* $\hat{\phi}$ *has support in* $(-u,u)$ *for some* $0<u<1$, *as* $k_1+k_2\to\infty$,

$$\frac{1}{d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)} D(\pi_F,\phi,\mathrm{Spin})=\hat{\phi}(0)+\frac{1}{2}\phi(0)+O\Big(\frac{1}{\log((k_1-k_2+2)k_1k_2)}\Big).$$

By a careful analysis, we can show that $v_1=3, w_1=6$ in [Kim et al. 2020a, Proposition 8.2] in the level aspect. Hence $u=\frac{1}{40}$ in the level aspect. As in Theorem 10.1, we have

Theorem 10.3. *Let* $G=\mathrm{GSp}(4)$. *Assume the GRH, and let* $r_F=\mathrm{ord}_{s=\frac{1}{2}}L(s,\pi_F,\mathrm{Spin})$. *Then*

$$\frac{1}{d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)} r_F\le\begin{cases}\dfrac{1}{u}+\dfrac{1}{2}+O\Big(\dfrac{1}{\log\log N}\Big) & \text{level aspect,}\\[2mm] \dfrac{1}{u}+\dfrac{1}{2}+O\Big(\dfrac{1}{\log((k_1-k_2+2)k_1k_2)}\Big) & \text{weight aspect.}\end{cases}$$

Appendix

In this appendix we compute the product $T(p,(0,\dots,0,1))^2$, with $n-1$ entries of 0, from Section 9.

Theorem A.1. *For the Hecke operators, we have*

$$T(p,(\overbrace{0,\dots,0}^{n-1},1))^2=T(p,(\overbrace{0,\dots,0}^{n-1},2))+(p+1)T(p,(\overbrace{0,\dots,0}^{n-2},1,1))+(p^n-1)T(p,(\overbrace{0,\dots,0}^{n-1},1))$$
$$+\Big(p\sum_{i=0}^{2n-1}p^i\Big)T(p,(\overbrace{0,\dots,0}^{n})).$$

This agrees with [Kim et al. 2020a, (2.7)] when $n=2$. [Note that the coefficient of R_{p^2} there should be replaced with $p^4+p^3+p^2+p$.]

Since $p\nmid N$, we work on $K=\mathrm{Sp}(2n,\mathbb{Z}_p)$ instead of $\Gamma(N)$. Put

$$T_{p,n-1}:=pT(p,(0,\dots,0,1))=K\,\mathrm{diag}(1,\overbrace{p,\dots,p}^{n-1},p^2,\overbrace{p,\dots,p}^{n-1})K\in\mathrm{GSp}(2n,\mathbb{Q}_p).$$

It suffices to consider $T_{p,n-1}^2$. Let us first compute the coset decomposition. Put $\Lambda=\mathrm{GL}_n(\mathbb{Z}_p)$ where the identity element is denoted by 1_n. For any ring R, let $S_n(R)$ be the set of all symmetric matrices of size n defined over R and $M_{m\times n}(R)$ be the set of matrices of size $m\times n$ defined over R. Put

$$M_n(R)=M_{n\times n}(R)$$

for simplicity. For each $D\in M_n(\mathbb{Z}_p)$, we define

$$B(D):=\{B\in M_n(\mathbb{Z}_p)\mid {}^tBD={}^tDB\}.$$

For each $B_1,B_2\in B(D)$, we write $B_1\sim B_2$ if there exists $M\in M_n(\mathbb{Z}_p)$ such that $B_1-B_2=MD$. We denote by $B(D)/\sim$ the set of all equivalence classes of $B(D)$ by the relation $\sim$. We regard $\mathbb{F}_p$

(respectively, $\mathbb{Z}/p^2\mathbb{Z}$) as the subset $\{0, 1, \dots, p-1\}$ (respectively, $\{0, 1, \dots, p^2-1\}$) of $\mathbb{Z}$. Let D_I be the set of the following matrices in $M_n(\mathbb{Z}_p)$:

$$D^I_{n-1} = \mathrm{diag}(\overbrace{p, \dots, p}^{n-1}, 1),$$

$$D^I_s = D^I_s(x) := \left(\begin{array}{c|c|c} p\cdot 1_s & & \\ \hline & 1 & x \\ \hline & & p\cdot 1_{n-1-s} \end{array}\right), \quad 0 \le s \le n-2,\ x \in M_{1\times(n-1-s)}(\mathbb{F}_p),$$

where we fill out zeros in the blank blocks. The cardinality of D_I is $1+p+\dots+p^{n-1} = (p^n-1)/(p-1)$ which is equal to that of $\Lambda\backslash\Lambda d_{n-1}\Lambda$, where $d_{n-1} = \mathrm{diag}(1, p, \dots, p)$ containing $n-1$ entries of p. Similarly, let D_{II} be the set of the following matrices:

$$D^{II}_{n-1} = \mathrm{diag}(p, \overbrace{1, \dots, 1}^{n-1}),$$

$$D^{II}_s = D^{II}_s(y) := \left(\begin{array}{c|c|c} 1_s & y & \\ \hline & p & \\ \hline & & 1_{n-1-s} \end{array}\right), \quad 1 \le s \le n-1,\ y \in M_{s\times 1}(\mathbb{F}_p).$$

The cardinality of D_{II} is $1 + p + \dots + p^{n-1} = (p^n - 1)/(p-1)$ which is equal to that of $\Lambda\backslash\Lambda d_1\Lambda$, where $d_1 = \mathrm{diag}(1, \dots, 1, p)$ containing $n-1$ entries of 1. Finally for each $M \in M_n(\mathbb{Z}_p)$ we denote by $r_p(M)$ the rank of M mod $p\mathbb{Z}_p$.

Lemma A.2. *Assume p is odd. The right coset decomposition $T_{p,n-1} = \coprod_{\alpha\in J} K\alpha$ consists of the following elements*:

(1) (*type I*) *We have*

$$\alpha = \alpha_I(D, B) = \begin{pmatrix} p^2\cdot {}^tD^{-1} & B \\ 0_n & D \end{pmatrix},$$

where D runs over the set D_I and B runs over complete representatives of $B(D)/\sim$ such that $r_p(\alpha) = 1$. Further, for each D^I_s, B can be taken over

- *if $s \ne 0$, then $x \ne 0$ and $B = 0$;*
- *if $s = 0$, then $x = 0$ and $B = 0$.*

(2) (*type II*) *We have*

$$\alpha = \alpha_{II}(D, B) = \begin{pmatrix} p\cdot {}^tD^{-1} & B \\ 0_n & pD \end{pmatrix},$$

where D runs over the set D_{II} and B runs over complete representatives of $B(D)/\sim$ such that $r_p(\alpha) = 1$. Further, for each D^{II}_s, B can be taken over.

- *If $s = 0$, then*

$$\begin{pmatrix} B_{22} & B_{23} \\ p\cdot {}^tB_{23} & 0_{n-1} \end{pmatrix},$$

where B_{22} runs over $\mathbb{Z}/p^2\mathbb{Z}$ and B_{23} runs over $M_{1\times(n-1)}(\mathbb{F}_p)$;

- *If $s \neq 0$, for $D_s^{II}(y)$, $y \in M_{s\times 1}(\mathbb{F}_p)$,*

$$\left(\begin{array}{c|c|c} 0_s & p\cdot {}^tB_{21} & 0_{s\times(n-1-s)} \\ \hline B_{21} & B_{22} & B_{23} \\ \hline 0_{(n-1-s)\times s} & p\cdot {}^tB_{23} & 0_{n-1-s} \end{array}\right),$$

where B_{21}, B_{22} and B_{23} run over $M_{1\times s}(\mathbb{F}_p)$, $\mathbb{Z}/p^2\mathbb{Z}$, and $M_{1\times t}(\mathbb{F}_p)$, respectively.

(3) (*type III*) *We have*

$$\alpha = \alpha_{III}(B) = \begin{pmatrix} p1_n & B \\ 0_n & p1_n \end{pmatrix},$$

where B runs over $S_n(\mathbb{F}_p)$ with $r_p(B) = 1$. The number of such B's is $p^n - 1$.

Proof. We just apply the formula [Andrianov 2009, (3.94)]. First we need to compute a complete system of representatives of $\Lambda\backslash\Lambda t\Lambda \simeq (t^{-1}\Lambda t)\cap\Lambda\backslash\Lambda$ for each $t \in \{d_{n-1}, d_1, p1_n\}$ where $d_{n-1} = \mathrm{diag}(1, p, \dots, p)$ and $d_1 = \mathrm{diag}(1, \dots, 1, p)$ containing $n-1$ entries of p and 1, respectively. By direct computation, for $t = d_{n-1}$ (respectively, $t = d_1$), it is given by D^I (respectively, D^{II}). For $t = p\cdot 1_n$, it is obviously a singleton.

As for the computation of $B(D)/\sim$, we give details only for $D \in D^I$, and the case of D^{II} is similarly handled. For each $D = D_s^I(x)$, $0 \le s \le n-2$, put

$$A_s = \left(\begin{array}{c|c|c} 1_s & & \\ \hline & 1 & -px \\ \hline & & 1_{n-1-s} \end{array}\right),$$

so that

$$DA_s = \left(\begin{array}{c|c|c} p\cdot 1_s & & \\ \hline & 1 & \\ \hline & & p\cdot 1_{n-1-s} \end{array}\right).$$

Put $A_{n-1} = 1_{2n}$ for $D = D_{n-1}^I$. Then for each $D = D_s^I$, we have a bijection

$$B(D)/\sim \xrightarrow{\simeq} B(DA_s)/\sim, \quad B \mapsto BA_s.$$

Therefore, we may compute $B(DA_s)/\sim$ and convert them by multiplying A_s^{-1} on the right.

We write $B \in B(DA_s)$ as a block matrix

$$B = \left(\begin{array}{c|c|c} \overbrace{B_{11}}^{s} & \overbrace{B_{12}}^{1} & \overbrace{B_{13}}^{n-1-s} \\ \hline B_{21} & B_{22} & B_{23} \\ \hline B_{31} & B_{32} & B_{33} \end{array}\right)$$

with respect to the partition $s + 1 + (n-1-s)$ of n where the column is also decomposed as in the row. The relation yields

$$B = \left(\begin{array}{c|c|c} B_{12} & B_{12} & B_{13} \\ \hline p\cdot {}^tB_{12} & B_{22} & p\cdot {}^tB_{32} \\ \hline {}^tB_{13} & B_{32} & B_{33} \end{array}\right),$$

where $B_{11} \in S_s(\mathbb{Z}_p)$, $B_{22} \in \mathbb{Z}_p$, and $B_{33} \in S_{n-1-s}(\mathbb{Z}_p)$. We write $X \in M_n(\mathbb{Z}_p)$ as

$$\begin{pmatrix} \overbrace{X_{11}}^{s} & \overbrace{X_{12}}^{1} & \overbrace{X_{13}}^{n-1-s} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix}$$

with respect to the partition $s+1+(n-1-s)$ of n as we have done for B. Then

$$XDA_s = \begin{pmatrix} pX_{11} & X_{12} & pX_{13} \\ pX_{21} & X_{22} & pX_{23} \\ pX_{31} & X_{32} & pX_{33} \end{pmatrix}.$$

Our matrix B in $B(DA_s)/\sim$ is considered by taking modulo XDA_s for any $X \in M_n(\mathbb{Z}_p)$. Hence B can be, up to equivalence, of the form

$$B = \begin{pmatrix} B_{11} & 0_{s\times 1} & B_{13} \\ 0_{1\times s} & 0 & 0_{1\times(n-1-s)} \\ {}^tB_{13} & 0_{(n-1-s)\times 1} & B_{33} \end{pmatrix}, \tag{A-1}$$

where B_{11}, B_{33}, and B_{13} belong to $S_s(\mathbb{F}_p)$, $S_{n-1-s}(\mathbb{F}_p)$, and $M_{s\times(n-1-s)}(\mathbb{F}_p)$, respectively. Further, to multiply A_s^{-1} on the right never change anything. Therefore, (A-1) gives a complete system of representatives of $B(D)/\sim$ for $D = D_s^I$. The condition $r_p(\alpha_I(D, B)) = 1$ and the modulo K on the left yield the desired result. For each $D \in D_s^{II}$, a similar computation shows any element of $S(p\cdot D)/\sim$ is given by

$$\begin{pmatrix} \overbrace{B_{11}}^{s} & \overbrace{p\cdot {}^tB_{21}}^{1} & \overbrace{B_{13}}^{n-1-s} \\ B_{21} & B_{22} & B_{23} \\ {}^tB_{13} & p\cdot {}^tB_{23} & B_{33} \end{pmatrix}$$

modulo the matrices of forms

$$\begin{pmatrix} pX_{11} & p^2X_{12} & pX_{13} \\ pX_{21} & p^2X_{22} & pX_{23} \\ pX_{31} & p^2X_{32} & pX_{33} \end{pmatrix}.$$

Therefore, $B_{11}, B_{13}, B_{21}, B_{22}, B_{23}$, and B_{33} run over

$$M_s(\mathbb{F}_p), \quad M_{s\times(n-1-s)(\mathbb{F}_p)}, \quad M_{1\times s(\mathbb{F}_p)}, \quad \mathbb{Z}/p^2\mathbb{Z}, \quad M_{1\times(n-1-s)(\mathbb{F}_p)}, \quad \text{and} \quad M_{n-1-s}(\mathbb{F}_p),$$

respectively. The claim now follows from the rank condition $r_p(\alpha_{II}(D, B)) = 1$ and the modulo K on the left again.

As for $D = p1_n$ in the case of type III, it is easy to see that $S(D)/\sim$ is naturally identified with $S_n(\mathbb{F}_p)$. Recall p is an odd prime by assumption. The number of matrices in $S_n(\mathbb{F}_p)$ of rank 1 is given in [MacWilliams 1969, Theorem 2]. □

Recall the right coset decomposition $T_{p,n-1} := K\,\mathrm{diag}(1, p, \ldots, p, p^2, p, \ldots, p)K = \coprod_{\alpha\in J} K\alpha$, containing two instances of $n-1$ entries of p. For each $\alpha, \beta \in J$, we observe that any element of $K\alpha\beta K$

is of mod p rank at most two and has the similitude p^4. Hence the double coset $K\alpha\beta K$ satisfies $K\alpha\beta K = K\gamma K$, where γ is one of the following four elements:

$$\gamma_1 := \mathrm{diag}(1, \overbrace{p^2, \dots, p^2}^{n-1}, p^4, \overbrace{p^2, \dots, p^2}^{n-1}), \qquad \gamma_3 := \mathrm{diag}(p, \overbrace{p^2, \dots, p^2}^{n-1}, p^3, \overbrace{p^2, \dots, p^2}^{n-1}),$$

$$\gamma_2 := \mathrm{diag}(p, p, \overbrace{p^2, \dots, p^2}^{n-2}, p^3, p^3, \overbrace{p^2, \dots, p^2}^{n-2}), \qquad \gamma_4 := p^2 \cdot I_{2n}$$

Here we use the Weyl elements in K to renormalize the order of entries. Then

$$T_{p,n-1} \cdot T_{p,n-1} = \sum_{i=1}^{4} m(\gamma_i) K \gamma_i K, \tag{A-2}$$

where $m(\gamma_i)$ is defined by

$$m(\gamma_i) := \left|\{(\alpha, \beta) \in J \times J : K\alpha\beta = K\gamma_i\}\right| \tag{A-3}$$

for each $1 \le i \le 4$; see [Shimura 1994b, p. 52]. Let us compute $m(\gamma_i)$ for each γ_i.

Let J_I be the subset of J consisting of the elements

$$\alpha_I^s(x) = \left(\begin{array}{ccc|ccc} p \cdot 1_s & & & & & \\ & p^2 & & & & \\ & -p \cdot {}^t x & p \cdot 1_{n-1-s} & & & \\ \hline & & & p \cdot 1_s & & \\ & & & & 1 & x \\ & & & & & p \cdot 1_{n-1-s} \end{array}\right), \quad 0 \le s \le n-2,\ x \in M_{1\times(n-1-s)}(\mathbb{F}_p)$$

and $\alpha_I^{n-1} = \mathrm{diag}(p^2, p, \dots, p, 1, p, \dots, p)$ containing $n-1$ entries of p both times.

Similarly, let J_{II} be the subset of J consisting of the elements

$$\alpha_{II}^s(y, B_{21}, B_{22}, B_{33}) = \left(\begin{array}{ccc|ccc} p \cdot 1_s & & & 0_s & p \cdot {}^t B_{21} & 0_{s\times(n-1-s)} \\ -{}^t y & 1 & & B_{21} & B_{22} & B_{23} \\ & & p \cdot 1_{n-1-s} & 0_{(n-1-s)\times s} & p \cdot {}^t B_{23} & 0_{n-1-s} \\ \hline & & & p \cdot 1_s & py & \\ & & & & p^2 & \\ & & & & & p \cdot 1_{n-1-s} \end{array}\right),$$

where $1 \le s \le n-1$, $y \in M_{s\times 1}(\mathbb{F}_p)$, and B_{21}, B_{23}, and B_{22} run over $M_{1\times s}(\mathbb{F}_p)$, $M_{1\times(n-1-s)}(\mathbb{F}_p)$, and $\mathbb{Z}/p^2\mathbb{Z}$, respectively. In addition,

$$\alpha_{II}^0(C_{22}, C_{23}) = \left(\begin{array}{cc|cc} 1 & & C_{22} & C_{23} \\ & p \cdot 1_{n-1} & p \cdot {}^t C_{23} & 0_{n-1} \\ \hline & & p^2 & \\ & & & p \cdot 1_{n-1} \end{array}\right), \quad C_{22} \in \mathbb{Z}/p^2\mathbb{Z},\ C_{23} \in M_{1\times(n-1)}(\mathbb{F}_p).$$

Finally, let J_{III} be the subset of J consisting of the elements

$$\alpha_{III}(B)=\left(\begin{array}{c|c} p\cdot 1_n & B \\ \hline & p\cdot 1_n\end{array}\right),\quad B\in S_n(\mathbb{F}_p)\text{ with } r_p(B)=1.$$

Lemma A.3. *For each* $\alpha\in J$,

$$K\alpha K=K\operatorname{diag}(1,\overbrace{p,\dots,p}^{n-1},p^2,\overbrace{p,\dots,p}^{n-1})K,$$

and

$$\operatorname{vol}\bigl(K\operatorname{diag}(1,\overbrace{p,\dots,p}^{n-1},p^2,\overbrace{p,\dots,p}^{n-1})K\bigr)=p\sum_{i=0}^{2n-1}p^i,$$

where the measure is normalized as $\operatorname{vol}(K)=1$.

Proof. Except for the case of type III, it follows from elementary divisor theory. For type III, it follows from [MacWilliams 1969] that the action of $\mathrm{GL}_n(\mathbb{F}_p)$ on the set of all matrices of rank 1 in $S_n(\mathbb{F}_p)$ given by $B\mapsto {}^tXBX,\ X\in\mathrm{GL}_n(\mathbb{F}_p)$ and such a symmetric matrix B has two orbits $O(\operatorname{diag}(1,0,\dots,0))$ and $O(\operatorname{diag}(g,0,\dots,0))$, both containing $n-1$ entries of 0, where g is a generator of $\mathbb{F}_p^\times$. The claim follows from this and elementary divisor theorem again.

For the latter claim, it is nothing but $|J|$, and we may compute the number of each type. □

Remark A.4. Since $K=\mathrm{Sp}_{2n}(\mathbb{Z}_p)$ contains Weyl elements,

$$\begin{aligned}K\operatorname{diag}(1,\overbrace{p,\dots,p}^{n-1},p^2,\overbrace{p,\dots,p}^{n-1})K&=K\operatorname{diag}(\overbrace{p,\dots,p}^{i},1,\overbrace{p,\dots,p}^{n-i-1},\overbrace{p,\dots,p}^{i},p^2,\overbrace{p,\dots,p}^{n-i-1})K\\&=K\operatorname{diag}(\overbrace{p,\dots,p}^{i},p^2,\overbrace{p,\dots,p}^{n-i-1},\overbrace{p,\dots,p}^{i},1,\overbrace{p,\dots,p}^{n-i-1})K\end{aligned}$$

for $0\le i\le n-1$.

Notice that

$$Kd_{n-1}(p)K=K(p^2\cdot d_{n-1}(p)^{-1})K,$$

where $d_{n-1}(p):=\operatorname{diag}(1,p,\dots,p,p^2,p,\dots,p)$ with $n-1$ entries of p both times.. By definition and Lemma A.3 with Remark A.4, it is easy to see that

$$\begin{aligned}m(\gamma_i)&=\bigl|\{\beta\in J:\gamma_i\beta^{-1}\in Kd_{n-1}(p)K\}\bigr|\\&=\bigl|\{\beta\in J:\beta\cdot(p^2\cdot\gamma_i^{-1})\in Kd_{n-1}(p)K\}\bigr|\\&=\bigl|\{\beta\in J:\beta\cdot(p^2\cdot\gamma_i^{-1})\text{ is }p\text{-integral and }r_p(\beta\cdot(p^2\cdot\gamma_i^{-1}))=1\}\bigr|\end{aligned};$$

see [Shimura 1994b, p. 52] for the first equality.

We are now ready to compute the coefficients. For $m(\gamma_1)$, we observe the p-integrality. We see that only $\alpha_{II}^0(C_{22},C_{23})$ with $C_{22}=0$ and $C_{23}=0_{1\times(n-1)}$ can contribute there. Hence, $m(\gamma_1)=1$.

For $m(\gamma_2)$, we observe the p-integrality and the rank condition. Then only $\alpha_{II}^0(0,0_{1\times(n-1)})$ and $\alpha_{II}^1(y,0,0,0_{1\times(n-2)})$, with $y\in\mathbb{F}_p$, can do there. Hence $m(\gamma_2)=1+p$. For $m(\gamma_3)$, only $\alpha_{III}(B)$, where $B\in S_n(\mathbb{F}_p)$ with $r_p(B)=1$ contribute. By Lemma A.2-(3), we have $m(\gamma_3)=p^n-1$.

Finally, we compute $m(\gamma_4)$. Since $p^{-2}\gamma_4 = I_4$, the condition is checked easily. All members of $J = J_I \cup J_{II} \cup J_{III}$ can contribute there. Therefore, we have only to count the number of each type. Hence, we have

$$m(\gamma_4) = \overbrace{1 + p + \cdots + p^{n-1}}^{\text{type I}} + \overbrace{p^{n+1} + p^{n+2} + \cdots + p^{2n}}^{\text{type II}} + \overbrace{p^n - 1}^{\text{type III}} = p \sum_{i=0}^{2n-1} p^i,$$

as desired. Note that $m(\gamma_4)$ is nothing but the volume of $K d_{n-1}(p) K$; see Lemma A.3.

Recalling $T_{p,n-1} := pT(p,(0,\dots,0,1))$, we have

$$T(p,(0,\dots,0,1))^2 = \sum_{i=1}^{4} m(\gamma_i) K(p^{-2}\gamma_i)K.$$

Note that

$$K(p^{-2}\gamma_1)K = T(p,(\overbrace{0,\dots,0}^{n-1},2)), \quad K(p^{-2}\gamma_2)K = T(p,(\overbrace{0,\dots,0}^{n-2},1,1)),$$
$$K(p^{-2}\gamma_3)K = T(p,(\overbrace{0,\dots,0}^{n-1},1)), \quad K(p^{-2}\gamma_4)K = T(p,(\overbrace{0,\dots,0}^{n})) = K I_{2n} K.$$

We can take K back to $\Gamma(N)$ without changing anything since $p \nmid N$. This proves Theorem A.1.

Remark A.5. We would like to make corrections to [Kim et al. 2020a].

(1) On page 356, line 1, $\mathrm{d}x\,\mathrm{d}y$ is missing in μ_∞^{ST}. In [25, page 929, line 3], the same typo is repeated.

(2) On page 362, line 12-13, $T_{2,p}^2$ should be a linear combination of four double cosets KMK, where M runs over $\mathrm{diag}(1,p^2,p^4,p^2)$, $\mathrm{diag}(p,p,p^3,p^3)$, $\mathrm{diag}(p,p^2,p^3,p^2)$, and $\mathrm{diag}(p^2,p^2,p^2,p^2)$.

(3) On page 362, the coefficient of R_{p^2} should be $p^4+p^3+p^2+p = p\sum_{i=0}^{3} p^i$ which is the volume of $\mathrm{Sp}(4,\mathbb{Z}_p)\,\mathrm{diag}(1,p^2,p^4,p^2)\,\mathrm{Sp}(4,\mathbb{Z}_p)$ explained in [Roberts and Schmidt 2007, p. 190].

(4) On page 403, Lemma 8.1, the inequality $q(F) \ge N$ is not valid. Similarly, on page 405, Lemma 8.3, the inequality $q(F) \ge N$ is not valid. We need to consider newforms as in Section 5 of this paper. Then for a newform, we obtain the inequality $q(F) \ge N^{1/2}$ and $\log c_{\underline{k},N} \asymp \log N$ is valid as in Lemma 9.3 of this paper.

(5) On page 404, line -5, $N \gg p^{10}$ should be $N \gg p^{20}$.

(6) On page 407, line 3, $N \gg p^{30}$ should be $N \gg p^{10}$.

(7) On page 407, line 8: $N \gg p^{10}$ should be $N \gg p^{20}$.

(8) On page 409, line 10, we need to add $-2\big(G\big(\frac{3}{2}\big) + G\big(-\frac{1}{2}\big)\big)$, in order to account for the poles of $\Lambda(s,\pi_F,\mathrm{Spin})$, and the contour integral is over $\mathrm{Re}(s) = 2$. So, in (9.3), we need to add $O\big(|HE_{\underline{k}}(N)^0|/|HE_{\underline{k}}(N)|\big)$. However, only CAP forms give rise to a pole, and the number of CAP forms in $HE_{\underline{k}}(N)$ is $O(N^{8+\epsilon})$. So it is negligible.

In the case of standard L-functions, the non-CAP and nongenuine forms which give rise to poles are: $1 \boxplus \pi$, where π is an orthogonal cuspidal representation of $GL(4)$ with trivial central character, or $1 \boxplus \pi_1 \boxplus \pi_2$, where the π_i are dihedral cuspidal representations of $GL(2)$. In those cases, by

Proposition 4.11 and [Kim et al. 2020b, Theorem 2.9], we can count such forms without extra conditions on N in Proposition 4.12. So our result is valid as it is written.

Remark A.6. The referee brought to our attention a possible gap in [Sauvageot 1997, p. 181]; see [Dalal 2022, p. 129] and [Nelson and Venkatesh 2021, p. 159]. S.W. Shin communicated to us that the issue has not been fixed at this writing. However, we do not use the result in [Sauvageot 1997], nor any other later results [Dalal 2022; Shin 2012; Shin and Templier 2016] which depend on [Sauvageot 1997].

Acknowledgments

We would like to thank M. Miyauchi, M. Oi, S. Sugiyama, S. W. Shin, and M. Tsuzuki for helpful discussions. We thank KIAS in Seoul and RIMS in Kyoto for their incredible hospitality during this research. We thank the referees for their helpful remarks and corrections.

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Communicated by Philippe Michel
Received 2022-10-26 Revised 2023-04-17 Accepted 2023-05-29

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

ANT peer review and production are managed by EditFLOW® from MSP.

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Algebra & Number Theory

Volume 18 No. 5 2024