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# Enumeration of conjugacy classes in affine groups

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*Dedicated to Pham Huu Tiep on the occasion of his 60th birthday*

We study the conjugacy classes of the classical affine groups. We derive generating functions for the number of classes analogous to formulas of Wall and the authors for the classical groups. We use these to get good upper bounds for the number of classes. These naturally come up as difficult cases in the study of the noncoprime  $k(GV)$  problem of Brauer.

## 1. Introduction

Let  $G$  be the group of affine transformations of a vector space  $V$  over a finite field. In this paper we derive generating functions for the number of conjugacy classes in this group and in the analogs for the other classical groups. For finite classical groups (not their affine versions), such generating functions were mostly obtained by Wall [1963] (see also [Fulman and Guralnick 2012] for orthogonal and symplectic groups in even characteristic). Besides the natural motivation for considering this, this is one of the most difficult cases in the noncoprime  $k(GV)$  problem introduced by Brauer to obtain results about characters. This asks for bounds on the number of conjugacy classes  $k(H)$ , where  $H$  is a group with a normal abelian subgroup  $V$ . One of the major results in this area, based on work of many authors over a long period, is that  $k(H) \leq |V|$  if  $V$  is its own centralizer in  $H$  and  $\gcd(|H/V|, |V|) = 1$ . In fact there is an entire book devoted to this topic [Schmid 2007]. It turns out if we weaken this assumption, the result is no longer true but it still is close. One critical case is when  $L = H/V$  acts irreducibly on  $V$  (see [Guralnick and Tiep 2005] for reductions and for connections with representation theory). See [Guralnick and Maróti 2013; Guralnick and Tiep 2005; Keller 2006; Robinson 2004] for background and other results. One would like to prove that  $k(H) < c|V|$  for some absolute constant  $c$  (under suitable hypotheses). Another motivation for studying this is the relationship with the conjugacy classes of the largest maximal parabolic subgroup of the classical groups. See [Nakada and Shinoda 1990] for the case of  $GL$ .

In [Guralnick and Tiep 2005], the focus was on the important case when  $L$  is close to simple and the same bound was proved in almost all cases studied. One of the main cases left open was the case that  $V$  is the natural module for a classical group  $L$ . It turns out that again aside from the case of  $AGL(n, q)$ ,

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the bound generally holds. We show that  $q^n \leq k(\text{AGL}(n, q)) < (q^{n+1} - 1)/(q - 1) < 2q^n$  and obtain explicit and useful bounds in the analogs for other classical groups.

Variations on this theme and some other small families that were not considered in [Guralnick and Tiep 2005] will be studied in a sequel.

The paper is organized as follows.

Section 2 gives some preliminaries which are fundamental to our two approaches for calculating exact generating functions for  $k(\text{AGL})$ ,  $k(\text{AGU})$  and  $k(\text{ASp})$  and  $k(\text{AO})$ . The first approach writes  $k(\text{AG})$  as a weighted sum over conjugacy classes of  $G$ . We work this out for all cases except for the famously difficult cases of characteristic two symplectic and orthogonal groups. Our second approach enumerates irreducible representations instead of conjugacy classes. This allows us to calculate  $k(\text{AG})$  recursively, and has the additional benefit of working in both odd and even characteristic.

Section 3 treats  $k(\text{AGL}(n, q))$ , and also  $k(\text{AH})$ , where  $H$  is a group between  $\text{GL}(n, q)$  and  $\text{SL}(n, q)$ . Section 4 treats  $k(\text{AGU}(n, q))$  and  $k(\text{AH})$ , where  $H$  is a group between  $\text{GU}(n, q)$  and  $\text{SU}(n, q)$ . Section 5 treats the case  $\text{ASp}(2n, q)$ . Section 6 treats  $\text{AO}(n, q)$ .

We dedicate this paper to Pham Huu Tiep, our friend and colleague, on the occasion of his 60th birthday. We note that he has done substantial work on the noncoprime  $k(\text{GV})$  problem; see [Guralnick and Tiep 2005].

## 2. Preliminaries

Let  $G$  be a finite group and let  $k$  be a finite field with  $A$  a finite dimensional  $kG$ -module. Then we consider the group  $H = \text{AG}$ , the semidirect product of the normal subgroup  $A$  and  $G$ . We say that  $H$  is the corresponding affine group. We will usually take  $A$  to be irreducible (and by replacing  $k$  by  $\text{End}_G(A)$ , we can assume that  $A$  is absolutely irreducible).

Our first approach, which we call the *orbit* approach, expresses  $k(\text{AG})$  as a weighted sum over conjugacy classes of  $G$ . To describe this, let  $[g, A]$  denote  $(I - g)A$ , where  $I$  is the identity map. The number of orbits of the centralizer  $C_G(g)$  on  $A/[g, A]$  depends only on the conjugacy class  $C$  of  $g$ , and we denote it by  $o(C)$ . If  $g$  and  $x$  are elements of a group  $G$ , then we let  $x^g = g^{-1}xg$ .

**Lemma 2.1.** *Let  $G$  and  $A$  be as above. Then*

$$k(\text{AG}) = \sum_C o(C),$$

where the sum is over all conjugacy classes  $C$  of  $G$ .

*Proof.* Let  $g \in C$  with  $C$  a conjugacy class of  $G$ . We need to show that the number of conjugacy classes of elements  $h \in \text{AG}$  such that  $h$  is conjugate to some element of  $gA$  is the number of orbits of  $C_G(g)$  on  $A/[g, A]$ .

Suppose that  $h = ga$ . Suppose that  $gc$  is conjugate to  $ga$ . Note that

$$\{(ga)^b \mid b \in A\} = ga[g, A].$$

Thus if  $a, c \in A$ ,  $ga$  and  $gc$  are conjugate in  $H$  if and only if  $a[g, A]$  and  $c[g, A]$  are in the same  $C_G(g)$  orbit on  $A/[g, A]$ , whence the result.  $\square$

In this paper we find (for all cases except even characteristic symplectic and orthogonal groups) exact formulas for  $o(C)$ , which may be of independent interest. We then use these formulas, together with generating functions for  $k(G)$ , to find exact generating functions for  $k(AG)$ .

Our second approach, which we call the *character* approach, counts irreducible representations instead of conjugacy classes. This leads to recursive expressions for  $k(AG)$ . Together with known generating functions for  $k(G)$ , this enables us to obtain exact generating functions for  $k(AG)$ . One nice feature of the character approach is that it works in both odd and even characteristic.

Crucial to the character approach is the next lemma, which is a well known elementary exercise.

**Lemma 2.2.** *Let  $G$  be a finite group and  $V$  a finite  $G$ -module. Let  $J = VG$  be the semidirect product. Let  $\Delta$  be a set of  $G$ -orbit representatives on the set of irreducible characters of  $V$ . Then*

$$k(J) = \sum_{\delta \in \Delta} k(G_\delta),$$

where  $G_\delta$  is the stabilizer of the character  $\delta$  in  $G$ .

*Proof.* Let  $W$  be an irreducible  $\mathbb{C}J$ -module. Let  $\delta$  be a character of  $V$  that occurs in  $W$  and set  $W_\delta$  to be the  $\delta$  eigenspace of  $V$ . Note that  $\delta$  is unique up to  $G$ -conjugacy and that the stabilizer of  $W_\delta$  in  $J$  is precisely  $J_\delta = VG_\delta$ . Thus,  $G_\delta$  acts irreducibly on  $W_\delta$ . Conversely given any irreducible  $G_\delta$ -module  $U$ , we can extend it to a  $J_\delta$  module by having  $V$  act via  $\delta$ . Then inducing  $U$  from  $J_\delta$  to  $J$  gives an irreducible  $J$ -module. Thus, we see that  $k(J) = \sum_{\delta \in \Delta} k(G_\delta)$  as required.  $\square$

The following lemma is Euler’s pentagonal number theorem (see for instance page 11 of [Andrews 1976]).

**Lemma 2.3.** *For  $q > 1$ ,*

$$\begin{aligned} \prod_{i \geq 1} \left(1 - \frac{1}{q^i}\right) &= 1 + \sum_{n=1}^{\infty} (-1)^n (q^{-n(3n-1)/2} + q^{-n(3n+1)/2}) \\ &= 1 - q^{-1} - q^{-2} + q^{-5} + q^{-7} - q^{-12} - q^{-15} + \dots \end{aligned}$$

A few times in this paper quantities which can be easily re-expressed in terms of the infinite product  $\prod_{i=1}^{\infty} (1 - 1/q^i)$  will arise, and Lemma 2.3 gives arbitrarily accurate upper and lower bounds on these products. Hence we will state bounds like

$$\prod_{i=1}^{\infty} \left(1 + \frac{1}{2^i}\right) = \prod_{i=1}^{\infty} \frac{\left(1 - \frac{1}{4^i}\right)}{\left(1 - \frac{1}{2^i}\right)} \leq 2.4$$

without explicitly mentioning Euler’s pentagonal number theorem on each occasion.

We also use the following well-known lemma (see for instance [Odlyzko 1995]).

**Lemma 2.4.** *Suppose that  $f(u)$  is analytic for  $|u| < R$ . Let  $M(r)$  denote the maximum of  $|f|$  restricted to the circle  $|u| = r$ . Then for any  $0 < r < R$ , the coefficient of  $u^n$  in  $f(u)$  has absolute value at most  $M(r)/r^n$ .*

As a final bit of notation, we let  $|\lambda|$  denote the size of a partition  $\lambda$ .

### 3. AGL and related groups

Section 3A uses the orbit approach to calculate the generating function for  $k(\text{AGL}(n, q))$ . Section 3B uses the character approach to calculate the generating function for  $k(\text{AGL}(n, q))$  and related groups. Section 3C uses these generating functions to obtain bounds on  $k(\text{AGL}(n, q))$  and related groups.

**3A. Orbit approach to  $k(\text{AGL})$ .** We use Lemma 2.1 to determine a generating function for the numbers  $k(\text{AGL}(n, q))$ .

The following lemma calculates  $o(C)$  for a conjugacy class  $C$  of  $\text{GL}(n, q)$ . This formula involves the number of distinct part sizes of a partition  $\lambda$ , which we denote by  $d(\lambda)$ . For example if  $\lambda$  has 5 parts of size 4, 3 parts of size 2, and 4 parts of size 1, then  $d(\lambda) = 3$ . If  $\lambda$  is the empty partition, then  $d(\lambda) = 0$ .

**Lemma 3.1.** *Let  $C$  be a conjugacy class of  $\text{GL}(n, q)$ , and let  $\lambda_{z-1}(C)$  be the partition corresponding to the eigenvalue 1 in the rational canonical form of an element of  $C$ . Then*

$$o(C) = d(\lambda_{z-1}(C)) + 1.$$

*Proof.* Let  $V$  be the natural module for  $\text{GL}(n, q)$ . Let  $g \in C$  and let  $C(g)$  denote the centralizer of  $g$  in  $\text{GL}(V)$ . Write  $V = V_1 \oplus V_2$  where  $V_2 = \ker(g - I)^n$ . Note that  $[g, V] = V_1 \oplus V_2/[g, V_2]$  and the centralizer of  $g$  preserves this decomposition. Thus, we may assume that  $V = V_2$ , i.e., we may assume that  $g$  is unipotent.

Now write  $V = V_1 \oplus \cdots \oplus V_m$ , where  $g|_{V_i}$  has all Jordan blocks of size  $i$ . We only consider the nonzero  $V_i$ . So  $d_i = \dim V_i/[g, V_i]$  is the number of Jordan blocks of size  $i$ . It is well known that the centralizer of  $g$  induces the full  $\text{GL}(d_i, q)$  and in particular any two nonzero elements of  $V_i/[g, V_i]$  are in the same  $C(g)$  orbit.

Consider  $gv$  with  $v = v_1 + \cdots + v_m$  with  $v_i \in V_i$ . Note that if  $h \in C(g)$ , then  $hV_i \subset V_1 \oplus \cdots \oplus V_i + [g, V]$ . Thus, two elements in  $V$  which are in the same  $C(g)$ -orbit module  $[g, V]$  must have the same highest nonzero (modulo  $[g, V]$ ) term. Conversely, we need to show that any two such vectors are in the same orbit and indeed are in the orbit of  $v_j$  with  $v_j \in V_j \setminus [g, V_j]$ . By induction, we may assume that  $j = m$ . Note that there exists  $h \in C(g)$  so that  $h$  is trivial on  $V/\sum_{e < m} V_e$  and  $hv_m - v_m$  is an arbitrary element in  $\bigoplus_{e < m} V_e/[g, V_e]$ . Thus, we see that  $v$  and  $v_m$  are in the same orbit. Since  $C(g)$  induces  $\text{GL}(d_m, q)$  on  $V_m/[g, V_m]$  we see that orbit representatives for  $C(g)$  on  $V/[g, V]$  are 0 and one vector  $w_i \in V_i$  for each nonzero  $V_i$ . The result follows.  $\square$

The following interesting identity will be helpful.

**Lemma 3.2.**

$$\sum_{\lambda} [d(\lambda) + 1] u^{|\lambda|} = \frac{1}{1-u} \prod_{i \geq 1} \frac{1}{1-u^i}.$$

*Proof.* Clearly

$$\sum_{\lambda} q^{d(\lambda)} u^{|\lambda|} = \prod_{i \geq 1} \left( 1 + \frac{qu^i}{1-u^i} \right).$$

Differentiate this equation with respect to  $q$  and then set  $q = 1$ . The left hand side becomes

$$\sum_{\lambda} d(\lambda) u^{|\lambda|}.$$

By the product rule, the right hand side becomes

$$\sum_{i \geq 1} \frac{u^i}{1-u^i} \prod_{j \neq i} \left( 1 + \frac{u^j}{1-u^j} \right) = \sum_{i \geq 1} \frac{u^i}{1-u^i} \prod_{j \neq i} \left( \frac{1}{1-u^j} \right) = \left( \sum_{i \geq 1} u^i \right) \prod_{j \geq 1} \frac{1}{1-u^j}.$$

Thus

$$\sum_{\lambda} d(\lambda) u^{|\lambda|} = \left( \sum_{i \geq 1} u^i \right) \prod_{j \geq 1} \frac{1}{1-u^j}. \tag{1}$$

Since

$$\sum_{\lambda} u^{|\lambda|} = \prod_{j \geq 1} \frac{1}{1-u^j},$$

it follows from (1) that

$$\sum_{\lambda} [d(\lambda) + 1] u^{|\lambda|} = \left( \sum_{i \geq 0} u^i \right) \prod_{j \geq 1} \frac{1}{1-u^j} = \frac{1}{1-u} \prod_{j \geq 1} \frac{1}{1-u^j},$$

as claimed. □

In what follows, for  $d \geq 1$ , we let  $N(q; d)$  denote the number of monic irreducible polynomials  $\phi(z)$  of degree  $d$  over  $F_q$  for which  $\phi(0) \neq 0$ , that is monic irreducible polynomials other than  $z$ .

The following well known identity (see for example Theorem 3.25 of [Lidl and Niederreiter 1994]) will be useful.

**Lemma 3.3.** 
$$\prod_{d \geq 1} (1 - u^d)^{-N(q; d)} = \frac{1-u}{1-qu}.$$

Theorem 3.4 derives a generating function for the number of conjugacy classes in  $\text{AGL}(n, q)$ .

**Theorem 3.4.** 
$$1 + \sum_{n \geq 1} k(\text{AGL}(n, q)) u^n = \frac{1}{1-u} \prod_{i \geq 1} \frac{1-u^i}{1-qu^i}.$$

*Proof.* By Lemma 2.1,

$$1 + \sum_{n \geq 1} k(\text{AGL}(n, q)) u^n = 1 + \sum_{n \geq 1} u^n \sum_C o(C),$$

where the sum is over all conjugacy classes  $C$  of  $\text{GL}(n, q)$ .

Since conjugacy classes of  $GL(n, q)$  correspond to rational canonical forms, it follows from the previous equation and Lemma 3.1 that

$$\begin{aligned} 1 + \sum_{n \geq 1} k(\text{AGL}(n, q))u^n &= \left( \sum_{\lambda} [d(\lambda) + 1]u^{|\lambda|} \right) \left( \sum_{\lambda} u^{|\lambda|} \right)^{N(q;1)-1} \prod_{d \geq 2} \left( \sum_{\lambda} u^{d|\lambda|} \right)^{N(q;d)} \\ &= \left( \sum_{\lambda} [d(\lambda) + 1]u^{|\lambda|} \right) \prod_{i \geq 1} \left( \frac{1}{1-u^i} \right)^{N(q;1)-1} \prod_{d \geq 2} \prod_{i \geq 1} \left( \frac{1}{1-u^{di}} \right)^{N(q;d)}. \end{aligned}$$

By Lemma 3.2 this is equal to

$$\frac{1}{1-u} \prod_{d \geq 1} \prod_{i \geq 1} \left( \frac{1}{1-u^{di}} \right)^{N(q;d)} = \frac{1}{1-u} \prod_{i \geq 1} \prod_{d \geq 1} \left( \frac{1}{1-u^{di}} \right)^{N(q;d)}.$$

Applying Lemma 3.3, this simplifies to

$$\frac{1}{1-u} \prod_{i \geq 1} \frac{1-u^i}{1-qu^i},$$

as claimed. □

**3B. Character approach to  $k(\text{AGL})$  and related groups.** We apply Lemma 2.2. Note that if  $\delta$  is the trivial character, then  $G_{\delta} = G$ . We recall the case of  $G = GL(n, q)$  with  $V$  the natural module. The group  $J$  is usually denoted as  $\text{AGL}(n, q)$  the affine general linear group. Note that in this case  $|\Delta| = 2$ . Note that the stabilizer of a nontrivial linear character is isomorphic to  $\text{AGL}(n - 1, q)$  and so:

**Lemma 3.5.**  $k(\text{AGL}(n, q)) = k(GL(n, q)) + k(\text{AGL}(n - 1, q)) = 1 + \sum_{m=1}^n k(GL(m, q)).$

As a corollary, we get another proof of Theorem 3.4.

*Proof.* Lemma 3.5 implies that

$$1 + \sum_{n \geq 1} k(\text{AGL}(n, q))u^n = \frac{1}{1-u} \left( 1 + \sum_{n \geq 1} k(GL(n, q))u^n \right).$$

The result now follows from Macdonald’s theorem [1981]

$$1 + \sum_{n \geq 1} k(GL(n, q))u^n = \prod_{i \geq 1} \frac{1-u^i}{1-qu^i}. \quad \square$$

**Lemma 3.6.** Fix  $q$  and let  $n \geq 2$ . Let  $SL(n, q) \leq H = H(n, q) \leq GL(n, q)$  with  $e = [H : SL(n, q)]$ .

- (1)  $k(AH) = k(H) + k(AH(n - 1, q)).$
- (2)  $k(AH) = (q - 1)/e + \sum_{i=1}^n k(H(i, q)).$

*Proof.* The first statement follows exactly as in the proof of the case of  $GL(n, q)$ . Note that  $k(AH(1, q)) = e + (q - 1)/e = k(H(1, q)) + (q - 1)/e.$

So iterating, we see that

$$k(AH) = k(AH(1, q)) + \sum_{j=2}^n k(H(j, q)) = (q - 1)/e + \sum_{i=1}^n k(H(i, q)). \quad \square$$

**3C. Bounds on  $k(\text{AGL})$  and related groups.** There is an interesting corollary of Theorem 3.4. If  $f(u) = \sum_{n \geq 0} f(n)u^n$  and  $g(u) = \sum_{n \geq 0} g(n)u^n$ , we use the notation  $f \gg g$  to mean that  $f(n) \geq g(n)$  for all  $n \geq 0$ .

**Corollary 3.7.**  $k(\text{AGL}(1, q)) = q$  and for  $n \geq 2$ ,

$$q^n < k(\text{AGL}(n, q)) < 2q^n.$$

*Proof.* By Theorem 3.4, the fact that  $q^n \leq k(\text{AGL}(n, q))$  is equivalent to the statement that

$$\frac{1}{1-u} \prod_{i \geq 1} \frac{1-u^i}{1-qu^i} \gg \frac{1}{1-ug}.$$

Now notice that

$$\frac{1}{1-u} \prod_{i \geq 1} \frac{1-u^i}{1-qu^i} = \frac{1}{1-ug} \prod_{i \geq 2} \frac{1-u^i}{1-qu^i} \gg \frac{1}{1-ug},$$

where the last step follows since  $(1 - u^i)/(1 - qu^i) \gg 1$ . In fact this argument shows that the strict inequality  $q^n < k(\text{AGL}(n, q))$  holds for  $n \geq 2$ , since the coefficient of  $u^i$  in  $(1 - u^i)/(1 - qu^i)$  is positive.

For a second proof that  $q^n \leq k(\text{AGL}(n, q))$  with strict inequality if  $n \geq 2$ , note that  $k(\text{GL}(n, q))$  is at least  $q^n - q^{n-1}$  and indeed is strictly greater for  $n > 1$ , since there are  $q^n - q^{n-1}$  semisimple classes (i.e., different characteristic polynomials) and for  $n > 1$ , there are unipotent classes as well. Now use the fact (Lemma 3.5) that

$$k(\text{AGL}(n, q)) = 1 + \sum_{m=1}^n k(\text{GL}(m, q)).$$

For the upper bound, we know from [Maslen and Rockmore 1997] that  $k(\text{GL}(m, q)) < q^m$  for all  $m$ . So again by Lemma 3.5,

$$k(\text{AGL}(n, q)) \leq q^n + q^{n-1} + \dots + 1 < 2q^n. \quad \square$$

Finally, we give a result for  $AH$  where  $H$  is between  $\text{GL}$  and  $\text{SL}$ .

**Theorem 3.8.** Fix  $q$  and let  $\text{SL}(n, q) \leq H = H(n, q) \leq \text{GL}(n, q)$  with  $e = [H : \text{SL}(n, q)] < q - 1$ . Then  $k(AH) < q^n$  except for  $k(\text{ASL}(1, q)) = q$  and  $k(\text{ASL}(2, 3)) = 10$ .

*Proof.* Suppose that  $n = 1$ . Then as noted in Lemma 3.6,  $k(AH(1, q)) = e + (q - 1)/e$ . Now if  $e + (q - 1)/e \geq q$ , then  $e^2 - 1 \geq q(e - 1)$ . So either  $e - 1 = 0$  or  $e + 1 \geq q$ . But  $e < q - 1$  so the only remaining possibility is  $n = 1, e = 1$ , as claimed.

Now we suppose that  $n \geq 2$ . From [Fulman and Guralnick 2012],  $k(H) \leq e \cdot k(SL(n, q))$ . So from Lemma 3.6,

$$k(AH) \leq \frac{q-1}{e} + e[k(SL(1, q)) + \dots + k(SL(n, q))].$$

From [Fulman and Guralnick 2012],  $k(SL(j, q)) \leq 2.5q^{j-1}$ . Thus

$$k(AH) \leq \frac{q-1}{e} + 2.5e \frac{q^n - 1}{q-1}.$$

We claim that if  $(q-1)/e \geq 3$ , then

$$\frac{q-1}{e} + 2.5e \frac{q^n - 1}{q-1} \leq q^n.$$

Indeed, if  $(q-1)/e \geq 3$ , then

$$\frac{q-1}{e} + 2.5e \frac{q^n - 1}{q-1} \leq \frac{q-1}{e} + (q^n - 1) \frac{2.5}{3}.$$

Since  $(q-1)/e \geq 3$ , we have that  $q \geq 4$ , and it is easy to check that if  $q \geq 4$ , then

$$\frac{q-1}{e} + (q^n - 1) \frac{2.5}{3} \leq q^n.$$

Since  $e < q-1$ , the remaining case is that  $(q-1)/e = 2$ . Since  $(q-1)/e$  is even, we can assume that  $q$  is odd. Then by Proposition 3.8 of [Fulman and Guralnick 2012],

$$k(H) = \begin{cases} \frac{1}{2}k(\text{GL}(n, q)) & \text{if } n \text{ is odd,} \\ \frac{1}{2}k(\text{GL}(n, q)) + \frac{3}{2}k(\text{GL}(n/2, q)) & \text{if } n \text{ is even.} \end{cases} \tag{2}$$

Using the fact that  $k(\text{GL}(j, q)) < q^j$  and Lemma 3.6, one easily checks that if  $q \geq 5$ , then  $k(AH) \leq q^n$ . Similarly if  $q = 3$  (so  $e = 1$  and  $H = \text{SL}$ ), it is not hard to see that  $k(\text{ASL}(2, 3)) = 10$  and that  $k(\text{ASL}(n, 3)) < 3^n$  otherwise. □

#### 4. AGU and related groups

Section 4A uses the orbit approach to calculate the generating function for  $k(\text{AGU}(n, q))$ . Section 4B uses the character approach to calculate the generating function for  $k(\text{AGU}(n, q))$ . Section 4C uses this generating function to obtain bounds on the number of conjugacy classes of  $\text{AGU}(n, q)$  and related groups.

**4A. Orbit approach to  $k(\text{AGU})$ .** This section uses the orbit approach to calculate the generating function for  $k(\text{AGU}(n, q))$ .

The following theorem calculates  $o(C)$  for a conjugacy class  $C$  of  $\text{GU}(n, q)$ . This only involves  $\lambda_{z-1}(C)$ , the partition corresponding to the eigenvalue 1 in the rational canonical form of the conjugacy class  $C$ . As in the GL case, let  $d(\lambda)$  be the number of distinct parts of the partition  $\lambda$ . In what follows we also let  $b(\lambda)$  denote the number of part sizes of  $\lambda$  which have multiplicity exactly 1.

**Theorem 4.1.** *Let  $C$  be a conjugacy class of  $\text{GU}(n, q)$ . Then*

$$o(C) = 1 + q \cdot d(\lambda_{z-1}(C)) - b(\lambda_{z-1}(C)).$$

*Proof.* It suffices to assume that  $C$  consists of unipotent elements and so corresponds to a partition  $\lambda$ . The proof is similar to the case of  $\text{GL}$ .

Now write  $V = V_1 \oplus \cdots \oplus V_m$  where  $g|_{V_i}$  has all Jordan blocks of size  $i$ . We only consider the nonzero  $V_i$ . So  $d_i = \dim V_i/[g, V_i]$  is the number of Jordan blocks of size  $i$ . It is well known that the centralizer of  $g$  induces the full  $\text{GU}(d_i, q)$  and so there are  $q$  orbits of the form  $gv$  with  $0 \neq v \in V_i$  for  $d_i > 1$  and  $q - 1$  orbits if  $d_i = 1$  (there are no nontrivial vectors of norm 0 if  $d_i = 1$ ).

Note that if  $h \in C(g)$ , then  $hV_i \subset V_1 \oplus \cdots \oplus V_i + [g, V]$ . Thus, two elements in  $V$  which are in the same  $C(g)$ -orbit module  $[g, V]$  must have the same highest nonzero (modulo  $[g, V]$ ) term. Conversely, we need to show that any two such vectors are in the same orbit and indeed are in the orbit of  $v_j$  with  $v_j \in V_j \setminus [g, V_j]$ . By induction, we may assume that  $j = m$ . Note that there exists  $h \in C(g)$  so that  $h$  is trivial on  $V/\sum_{e < m} V_e$  and  $hv_m - v_m$  is an arbitrary element in  $\bigoplus_{e < m} V_e/[g, V_e]$ . Thus, we see that the  $v$  and  $v_m$  are in the same orbit. The number of orbits for the nontrivial  $v_m$  is  $q$  or  $q - 1$  as above. The result follows. □

The following combinatorial lemma will also be helpful.

**Lemma 4.2.** (1) *The generating function for the number of unipotent classes of  $\text{GU}(n, q)$  is*

$$\sum_{\lambda} u^{|\lambda|}.$$

*This is equal to*

$$\prod_i \frac{1}{1 - u^i}.$$

(2) *The generating function*

$$\sum_{\lambda} d(\lambda)u^{|\lambda|}$$

*is equal to*

$$\frac{u}{1 - u} \prod_i \frac{1}{1 - u^i}.$$

(3) *The generating function*

$$\sum_{\lambda} b(\lambda)u^{|\lambda|}$$

*is equal to*

$$\frac{u}{1 - u^2} \prod_i \frac{1}{1 - u^i}.$$

*Proof.* The first part is just the well known generating function for the partition function. The second part is in the proof of Lemma 3.2.

For the third assertion, note that

$$\sum_{\lambda} x^{b(\lambda)} u^{|\lambda|}$$

is equal to

$$\prod_i (1 + xu^i + u^{2i} + u^{3i} + \dots).$$

Differentiating with respect to  $x$  and setting  $x = 1$  gives that

$$\sum_{\lambda} b(\lambda) u^{|\lambda|}$$

is equal to

$$\sum_i u^i \prod_{j \neq i} (1 + u^j + u^{2j} + u^{3j} + \dots) = \sum_i u^i \prod_{j \neq i} \frac{1}{1 - u^j} = \sum_i u^i (1 - u^i) \prod_j \frac{1}{1 - u^j} = \frac{u}{1 - u^2} \prod_j \frac{1}{1 - u^j},$$

as claimed. □

Theorem 4.3 gives an exact generating function for  $k(\text{AGU}(n, q))$ .

**Theorem 4.3.**  $k(\text{AGU}(n, q))$  is equal to the coefficient of  $u^n$  in

$$\prod_i \frac{1 + u^i}{1 - qu^i} \cdot \left( 1 + \frac{qu^2 + (q - 1)u}{1 - u^2} \right).$$

*Proof.* By Lemma 2.1 and Theorem 4.1,  $k(\text{AGU}(n, q))$  is equal to  $T_1 + T_2 - T_3$ , where  $T_1$  is  $k(\text{GU}(n, q))$ , and  $T_2, T_3$  are the following sums over conjugacy classes  $C$  of  $\text{GU}(n, q)$ :

$$T_2 = q \sum_C d(\lambda_{z-1}(C)), \quad T_3 = \sum_C b(\lambda_{z-1}(C)).$$

From Wall [1963],  $T_1$  is the coefficient of  $u^n$  in

$$\prod_i \frac{1 + u^i}{1 - qu^i}.$$

To compute the generating function of  $T_2$ , we take Wall’s generating function for  $T_1$ , divide it by the generating function for unipotent conjugacy classes in part (1) of Lemma 4.2, and multiply it by the weighted sum over unipotent classes in part (2) of Lemma 4.2. We conclude that  $T_2$  is the coefficient of  $u^n$  in

$$\frac{qu}{1 - u} \prod_i \frac{1 + u^i}{1 - qu^i}.$$

To compute the generating function of  $T_3$ , we take Wall’s generating function for  $T_1$ , divide it by the generating function for unipotent conjugacy classes in part (1) of Lemma 4.2, and multiply it by the

weighted sum over unipotent classes in part (3) of Lemma 4.2. We conclude that  $T_3$  is the coefficient of  $u^n$  in

$$\frac{u}{1-u^2} \prod_i \frac{1+u^i}{1-qu^i}.$$

Putting the pieces together, we conclude that  $k(\text{AGU}(n, q))$  is the coefficient of  $u^n$  in

$$\prod_i \frac{1+u^i}{1-qu^i} \cdot \left( 1 + \frac{qu}{1-u} - \frac{u}{1-u^2} \right),$$

which simplifies to the desired result. □

**4B. Character approach to  $k(\text{AGU})$ .** We use Lemma 2.2 to find a recursion for  $k(\text{AGU})$ . Then we use this to compute the generating function for  $k(\text{AGU})$ , giving another proof of Theorem 4.3.

Recall that if  $H$  is a finite group and  $p$  is a prime, then  $O_p(H)$  is the (unique) maximal normal  $p$ -subgroup of  $H$ .

**Lemma 4.4.**  $k(\text{AGU}(n, q)) = k(\text{GU}(n, q)) + (q - 1)k(\text{GU}(n - 1, q)) + k(\text{AGU}(n - 2, q)) + (q - 1)k(\text{GU}(n - 2, q)).$

*Proof.* We use the convention that  $\text{GU}(0, q)$  and  $\text{AGU}(0, q)$  are trivial groups and that  $\text{GU}(-1, q)$  and  $\text{AGU}(-1, q)$  are the empty set. We can identify the natural module and the character group of the module because the module is self dual viewed over the field of  $q$ -elements.

Note that  $\text{AGU}(1, q)$  is a semidirect product of an elementary abelian group of order  $q^2$  and  $\text{GU}(1, q)$  which is cyclic of order  $q + 1$ . Thus, it follows that  $k(\text{AGU}(1, q)) = k(\text{GU}(1, q)) + (q - 1)$  as claimed. If  $n = 2$ , we note that  $\text{GU}(2, q)$  has precisely  $q$  nontrivial orbits on the natural module. The stabilizer of a nondegenerate vector is  $\text{GU}(1, q)$  and the stabilizer of a totally singular vector is elementary abelian of order  $q$  and again we see the result holds.

Now suppose that  $n \geq 3$ . Thus, we see that there are  $q - 1$  orbits with stabilizer isomorphic to  $\text{GU}(n - 1, q)$  (corresponding to vectors with a given nonzero norm) and the stabilizer  $H$  of a singular vector. Note that  $H$  has a center  $Z$  of order  $q$  and  $H/Z \cong \text{AGU}(n - 2, q)$ . Also note that any irreducible character of  $U = O_p(H)$  that is nontrivial on  $Z$  has dimension  $q^{n-2}$  and corresponds to one of the  $q - 1$  nontrivial 1-dimensional characters on  $Z$ . Moreover each of these representations extends to a representation of  $H$  (this can be seen by considering the normalizer of  $U$  in the full linear group). Fix a nontrivial linear character of  $Z$  and an irreducible module  $W$  of  $H$  that affords this linear representation. It follows by Clifford theory [Curtis and Reiner 1962, 51.7] that any irreducible representation of  $H$  nontrivial on  $Z$  is of the form  $W \otimes W'$  where  $W'$  is an irreducible  $H/U$ -module. Since there are  $q - 1$  nontrivial central characters of  $U$  and there are  $k(\text{GU}(n - 2, q))$  choices for  $W'$ , the result follows. □

We now give a second proof of Theorem 4.3.

*Proof.* Let  $k_n = k(\text{GU}(n, q))$  and let  $a_n = k(\text{AGU}(n, q))$ . Then Lemma 4.4 gives

$$a_n = k_n + (q - 1)k_{n-1} + (q - 1)k_{n-2} + a_{n-2}. \tag{3}$$

Let

$$K(u) = 1 + \sum_{n \geq 1} k_n u^n, \quad A(u) = 1 + \sum_{n \geq 1} a_n u^n.$$

Multiplying (3) by  $u^n$  and summing over  $n \geq 1$  gives that

$$A(u) - 1 = K(u) - 1 + (q - 1)uK(u) + (q - 1)u^2K(u) + u^2A(u).$$

Solving for  $A(u)$ , one obtains that

$$A(u) = K(u) \left( \frac{1 + u(q - 1) + u^2(q - 1)}{1 - u^2} \right) = K(u) \left( 1 + \frac{qu^2 + (q - 1)u}{1 - u^2} \right).$$

From Wall [1963],

$$K(u) = \prod_i \frac{1 + u^i}{1 - qu^i},$$

and the theorem follows. □

**4C. Bounds for AGU and related groups.** As a corollary, we obtain the following result.

**Corollary 4.5.**  $k(\text{AGU}(n, q)) \leq 20q^n.$

*Proof.* From Theorem 4.3,  $k(\text{AGU}(n, q))$  is equal to the coefficient of  $u^n$  in

$$\prod_i \frac{1 - u^i}{1 - qu^i} \prod_i \frac{1 + u^i}{1 - u^i} \left( 1 + \frac{qu^2 + (q - 1)u}{1 - u^2} \right).$$

Now all coefficients of powers of  $u$  in

$$\prod_i \frac{1 + u^i}{1 - u^i} \left( 1 + \frac{qu^2 + (q - 1)u}{1 - u^2} \right)$$

are nonnegative. It follows that  $k(\text{AGU}(n, q))$  is at most

$$\sum_{m=0}^n \text{Coef. } u^{n-m} \text{ in } \prod_i \frac{1 - u^i}{1 - qu^i} \text{Coef. } u^m \text{ in } \prod_i \frac{1 + u^i}{1 - u^i} \left( 1 + \frac{qu^2 + (q - 1)u}{1 - u^2} \right).$$

Now  $\prod_i \frac{1 - u^i}{1 - qu^i}$  is the generating function for the number of conjugacy classes of  $\text{GL}(n, q)$ . By [Maslen and Rockmore 1997],  $k(\text{GL}(n, q))$  is at most  $q^n$ . Hence the coefficient of  $u^{n-m}$  in it is at most  $q^{n-m}$ . It follows that  $k(\text{AGU}(n, q))$  is at most

$$q^n \sum_{m=0}^n \frac{1}{q^m} \left( \text{Coef. } u^m \text{ in } \prod_i \frac{1 + u^i}{1 - u^i} \left( 1 + \frac{qu^2 + (q - 1)u}{1 - u^2} \right) \right).$$

Since the coefficients of  $u^m$  in

$$\prod_i \frac{1 + u^i}{1 - u^i} \left( 1 + \frac{qu^2 + (q - 1)u}{1 - u^2} \right)$$

are nonnegative, it follows that  $k(\text{AGU}(n, q))$  is at most

$$q^n \sum_{m=0}^{\infty} \frac{1}{q^m} \left( \text{Coef. } u^m \text{ in } \prod_i \frac{1+u^i}{1-u^i} \left( 1 + \frac{qu^2 + (q-1)u}{1-u^2} \right) \right),$$

which (set  $u = 1/q$ ) is equal to

$$q^n \prod_i \frac{(1+1/q^i)}{(1-1/q^i)} \cdot \left( 1 + \frac{1}{1-1/q^2} \right).$$

The term

$$\prod_i \frac{(1+1/q^i)}{(1-1/q^i)} \cdot \left( 1 + \frac{1}{1-1/q^2} \right)$$

is visibly maximized among prime powers  $q$  when  $q = 2$ , when it is at most 20 (we used the remark after Lemma 2.3 to bound the infinite product). □

**Corollary 4.6.**  $k(\text{AGU}(n, q)) \leq q^{2n}$ .

*Proof.* By the preceding result, this holds if  $20 \leq q^n$ . So we only need to check the cases  $n = 1$ , or  $n = 2$ ,  $q = 2, 3, 4$  or  $n = 3$ ,  $q = 2$  or  $n = 4$ ,  $q = 2$ . From the generating function (Theorem 4.3),  $k(\text{AGU}(1, q)) = 2q$ , and the other finite number of cases are computed easily from the generating function and seen to be at most  $q^{2n}$ . □

We can also use the previous results to get bounds for the groups between  $\text{ASU}(n, q)$  and  $\text{AGU}(n, q)$ . Since  $\text{SL}(2, q) \cong \text{SU}(2, q)$ , we assume that  $n \geq 3$ . With more effort one can get much better bounds as we did in the case of  $\text{SL}(n, q)$ . We just obtain the bound required for the  $k(\text{GV})$  problem.

**Corollary 4.7.** *Let  $n \geq 3$ . Let  $\text{ASU}(n, q) \leq H \leq \text{AGU}(n, q)$ . Then  $k(H) \leq q^{2n}$ .*

*Proof.* Let  $G = \text{AGU}(n, q)$ . Since  $[G : H] \leq q + 1$ ,  $k(H) \leq k(G)(q + 1) \leq 20q^n(q + 1)$ . This is at most  $q^{2n}$  unless  $q = 2$  with  $n \leq 5$  or  $q = 3$  or  $4$  and  $n = 3$ . These cases all follow using the exact values of  $k(G)$  (obtained from our generating function) in the bound  $k(H) \leq k(G)(q + 1)$ , except for the cases  $q = 2$ ,  $n = 3, 4$ . One computes (either using a recursion similar to Lemma 4.4 and exact values of  $k(\text{SU})$  in [Macdonald 1981], or by Magma) that  $k(\text{ASU}(3, 2)) = 24$  and  $k(\text{ASU}(4, 2)) = 49$ , completing the proof. □

### 5. ASp

Section 5A uses the orbit approach to calculate the generating function for  $k(\text{ASp}(2n, q))$ , assuming that the characteristic is odd. Section 5B uses the character approach to calculate the generating function for  $k(\text{ASp}(2n, q))$  in both odd and even characteristic. Section 5C uses these generating functions to obtain bounds on  $k(\text{ASp}(2n, q))$ .

**5A. Orbit approach to  $k(\text{ASp})$ , odd characteristic.** This section treats the affine symplectic groups. We only work in odd characteristic. In this case the conjugacy class of a unipotent element is determined by its Jordan form (over the algebraic closure) and it is much more complicated to deal with the characteristic 2 case. Since our character approach works in characteristic 2, we will not pursue the direct approach in that case. So for this section, let  $q$  be odd.

The following theorem calculates  $o(C)$  for a conjugacy class  $C$  of  $\text{Sp}(2n, q)$ . This only involves the unipotent part of the class  $C$ . Recall that the conjugacy class of a unipotent element is determined (over the algebraic closure) by a partition of  $2n$  with  $a_i$  parts of  $i$ . Moreover,  $a_i$  is even if  $i$  is odd. Over a finite field, we attach a sign  $\epsilon_i$  for each even  $i$  with  $a_i \neq 0$  and this gives a description of all the unipotent conjugacy classes (see [Liebeck and Seitz 2012] for details). We let  $\lambda_{z-1}^\pm(C)$  denote this signed partition for the unipotent part of the class  $C$ .

**Theorem 5.1.** *Suppose that the characteristic is odd. Let  $C$  be a conjugacy class of  $\text{Sp}(2n, q)$ . Let  $a_i$  be the number of parts of  $\lambda_{z-1}^\pm(C)$  of size  $i$ . Then  $o(C)$  is equal to*

$$1 + \sum_{\substack{i \text{ odd} \\ a_i \neq 0}} 1 + \sum_{\substack{i \text{ even} \\ a_i \neq 0}} f_i,$$

where

$$f_i = \begin{cases} q & \text{if } a_i > 2 \text{ (independently of the sign),} \\ q & \text{if } a_i = 2 \text{ and the sign is } +, \\ (q - 1) & \text{if } a_i = 2 \text{ and the sign is } -, \\ (q - 1)/2 & \text{if } a_i = 1 \text{ (independently of the sign).} \end{cases} \tag{4}$$

*Proof.* The proof is similar to the case of GL and GU and reduces to the case of unipotent elements. So assume that  $C$  is a unipotent class. Let  $g \in C$ . Write  $V$  as an orthogonal direct sum of spaces  $V_i$  where  $g$  has  $a_i$  Jordan blocks of size  $i$  on  $V_i$ . As in the previous cases, one can show that  $gv$  is either conjugate to  $g$  or for some  $i$ ,  $g$  is conjugate to  $gv_i$  where  $v_i \in V_i \setminus [g, V_i]$ .

By [Liebeck and Seitz 2012], we see that there is a subgroup of  $C(g)$  acting as  $\text{Sp}(a_i, q)$  for  $i$  odd or  $\text{O}^{\epsilon_i}(a_i, q)$  if  $i$  is even acting naturally on  $V_i/[g, V_i]$ . Thus, the number of classes of the form  $gv_i$  with  $v_i \in V_i \setminus [g, V_i]$  is 1 if  $i$  is odd and  $f_i$  as given above if  $i$  is even. □

The following combinatorial lemma will also be helpful.

**Lemma 5.2.** *Suppose that the characteristic is odd.*

(1) *The generating function for the number of unipotent classes of the groups  $\text{Sp}(2n, q)$  is*

$$\sum_{\lambda^\pm} u^{|\lambda^\pm|/2}.$$

*This is equal to*

$$\prod_{i \text{ odd}} \frac{1}{1 - u^i} \prod_i \left( \frac{1 + u^i}{1 - u^i} \right).$$

(2) *The generating function*

$$\sum_{\lambda^\pm} u^{|\lambda^\pm|/2} \sum_{\substack{j \text{ odd} \\ a_j \neq 0}} 1$$

is equal to

$$\frac{u}{1-u^2} \prod_{i \text{ odd}} \frac{1}{1-u^i} \prod_i \left( \frac{1+u^i}{1-u^i} \right).$$

(3) *Let  $f_j$  be as in Theorem 5.1. The generating function*

$$\sum_{\lambda^\pm} u^{|\lambda^\pm|/2} \sum_{\substack{j \text{ even} \\ a_j \neq 0}} f_j$$

is equal to

$$\left( \frac{(q-1)u}{1-u} + \frac{u^2}{1-u^2} \right) \prod_{i \text{ odd}} \frac{1}{1-u^i} \prod_i \left( \frac{1+u^i}{1-u^i} \right).$$

*Proof.* For the first part, the unipotent conjugacy classes of  $\text{Sp}(2n, q)$  correspond to signed partitions  $\lambda^\pm$  of size  $2n$ . Clearly the generating function for such partitions is equal to

$$\prod_{i \text{ odd}} (1 + u^i + u^{2i} + \dots) \prod_{i \text{ even}} (1 + 2u^{i/2} + 2u^{2i/2} + \dots)$$

which is equal to

$$\prod_{i \text{ odd}} \frac{1}{1-u^i} \prod_i \left( 1 + \frac{2u^i}{1-u^i} \right) = \prod_{i \text{ odd}} \frac{1}{1-u^i} \prod_i \left( \frac{1+u^i}{1-u^i} \right).$$

For the second part, first note that arguing as in the first part, one has that

$$\sum_{\lambda^\pm} u^{|\lambda^\pm|/2} \sum_{\substack{j \text{ odd} \\ a_j \neq 0}} 1$$

is equal to

$$\begin{aligned} \sum_{\substack{j \text{ odd} \\ a_j \neq 0}} \sum_{\lambda^\pm} u^{|\lambda^\pm|/2} &= \sum_{j \text{ odd}} (u^j + u^{2j} + \dots) \prod_{\substack{i \text{ odd} \\ i \neq j}} (1 + u^i + u^{2i} + \dots) \prod_{i \text{ even}} (1 + 2u^{i/2} + 2u^{2i/2} + \dots) \\ &= \sum_{j \text{ odd}} u^j \prod_{i \text{ odd}} (1 + u^i + u^{2i} + \dots) \prod_{i \text{ even}} (1 + 2u^{i/2} + 2u^{2i/2} + \dots) \\ &= \frac{u}{1-u^2} \prod_{i \text{ odd}} \frac{1}{1-u^i} \prod_i \left( \frac{1+u^i}{1-u^i} \right). \end{aligned}$$

For the third part,

$$\sum_{\lambda^\pm} u^{|\lambda^\pm|/2} \sum_{\substack{j \text{ even} \\ a_j \neq 0}} f_j$$

is equal to

$$\sum_{j \text{ even}} (2u^{j/2} \frac{1}{2}(q-1) + u^{2j/2}(q+q-1) + 2q(u^{3j/2} + u^{4j/2} + \dots)) \prod_{i \text{ odd}} (1+u^i + u^{2i} + \dots) \prod_{\substack{i \text{ even} \\ i \neq j}} \frac{1+u^{i/2}}{1-u^{i/2}}.$$

This is equal to

$$\sum_{j \text{ even}} \frac{1-u^{j/2}}{1+u^{j/2}} (u^{j/2}(q-1) + u^{2j/2}(2q-1) + 2q(u^{3j/2} + u^{4j/2} + \dots)) \prod_{i \text{ odd}} \frac{1}{1-u^i} \prod_i \frac{1+u^i}{1-u^i}.$$

Now clearly

$$\sum_{j \text{ even}} \frac{1-u^{j/2}}{1+u^{j/2}} (u^{j/2}(q-1) + u^{2j/2}(2q-1) + 2q(u^{3j/2} + u^{4j/2} + \dots))$$

is equal to

$$\begin{aligned} \sum_j \frac{1-u^j}{1+u^j} (u^j(q-1) + u^{2j}(2q-1) + 2q(u^{3j} + u^{4j} + \dots)) &= \sum_j \frac{1}{1+u^j} (qu^j - u^j + qu^{2j} + u^{3j}) \\ &= \sum_j (qu^j + u^{2j} - u^j) \\ &= \frac{(q-1)u}{1-u} + \frac{u^2}{1-u^2}, \end{aligned}$$

and the third part of the lemma follows. □

**Theorem 5.3.** *In odd characteristic,  $k(\text{ASp}(2n, q))$  is equal to the coefficient of  $u^n$  in*

$$\prod_i \frac{(1+u^i)^4}{1-qu^i} \left(1 + \frac{qu}{1-u}\right).$$

*Proof.* By Lemma 2.1 and Theorem 5.1,  $k(\text{ASp}(2n, q))$  is equal to  $T_1 + T_2 + T_3$ , where  $T_1$  is  $k(\text{Sp}(2n, q))$ , and  $T_2, T_3$  are the following sums over conjugacy classes  $C$  of  $\text{Sp}(2n, q)$ :

$$T_2 = \sum_C \sum_{\substack{i \text{ odd} \\ a_i \neq 0}} 1, \quad T_3 = \sum_C \sum_{\substack{i \text{ even} \\ a_i \neq 0}} f_i.$$

From Wall [1963],  $T_1$  is the coefficient of  $u^n$  in

$$\prod_i \frac{(1+u^i)^4}{1-qu^i}.$$

To compute the generating function of  $T_2$ , we take Wall’s generating function for  $T_1$ , divide it by the generating function for unipotent conjugacy classes in part (1) of Lemma 5.2, and multiply it by the generating function for the weighted sum over unipotent classes in part (2) of Lemma 5.2. We conclude

that  $T_2$  is the coefficient of  $u^n$  in

$$\frac{u}{1-u^2} \prod_i \frac{(1+u^i)^4}{1-qu^i}.$$

To compute the generating function of  $T_3$ , we take Wall’s generating function for  $T_1$ , divide it by the generating function for unipotent conjugacy classes in part (1) of Lemma 5.2, and multiply it by the generating function for the weighted sum over unipotent classes in part (3) of Lemma 5.2. We conclude that  $T_3$  is the coefficient of  $u^n$  in

$$\left( \frac{(q-1)u}{1-u} + \frac{u^2}{1-u^2} \right) \prod_i \frac{(1+u^i)^4}{1-qu^i}.$$

Since

$$1 + \frac{u}{1-u^2} + \frac{(q-1)u}{1-u} + \frac{u^2}{1-u^2} = 1 + \frac{qu}{1-u},$$

the proof of the theorem is complete. □

**5B. Character approach to  $k(\text{ASp}(2n, q))$ , any characteristic.** We apply Lemma 2.2, as in the other cases.

To begin we treat the case of odd characteristic.

**Lemma 5.4.** *Let  $q$  be odd and  $G = \text{Sp}(2n, q)$ . Then*

$$k(AG) = k(\text{Sp}(2n, q)) + k(\text{ASp}(2n - 2, q)) + (q - 1)k(\text{Sp}(2n - 2, q)).$$

*Proof.* We take  $\text{ASp}(0, q)$  and  $\text{Sp}(0, q)$  to be the trivial group. If  $n = 1$ , then  $G = \text{SL}(2, q)$ . It is straightforward to see that  $k(\text{SL}(2, q)) = q + 4$  and that  $k(\text{ASL}(2, q)) = 2q + 4$  and so the formula holds.

So suppose that  $n \geq 2$ . Let  $V$  be the natural module for  $G$ . Note that in this case  $G$  acts transitively on the nontrivial characters of  $V$  and the stabilizer of such a character is the stabilizer  $H$  of a vector in  $\text{Sp}(2n, q)$ . Let  $U = O_p(H)$  and let  $Z = Z(H)$ . Then  $H/Z \cong \text{ASp}(2n - 2, q)$ . If an irreducible character of  $H$  does not vanish on  $Z$ , then there are  $q - 1$  possibilities (depending on the restriction to  $Z$ ) and arguing as in the unitary case, we see that the number of such characters of  $H$  is  $(q - 1)k(\text{Sp}(2n - 2, q))$ . This gives  $k(\text{ASp}(2n, q)) = k(\text{Sp}(2n, q)) + k(\text{ASp}(2n - 2, q)) + (q - 1)k(\text{Sp}(2n - 2, q))$  as desired. □

We use this recursion to give another proof of the generating function for  $k(\text{Sp}(2n, q))$  in odd characteristic.

*Second proof of Theorem 5.3.* Let  $k_n = k(\text{Sp}(2n, q))$  and let  $a_n = k(\text{ASp}(2n, q))$ . Lemma 5.4 gives that

$$a_n = k_n + (q - 1)k_{n-1} + a_{n-1}. \tag{5}$$

Let

$$K(u) = 1 + \sum_{n \geq 1} k_n u^n, \quad A(u) = 1 + \sum_{n \geq 1} a_n u^n.$$

Multiplying (5) by  $u^n$  and summing over  $n \geq 1$  gives that

$$A(u) - 1 = K(u) - 1 + (q - 1)uK(u) + uA(u).$$

Solving for  $A(u)$  gives

$$A(u) = K(u) \left( \frac{1 + u(q - 1)}{1 - u} \right) = K(u) \left( 1 + \frac{qu}{1 - u} \right).$$

From Wall [1963],

$$K(u) = \prod_i \frac{(1 + u^i)^4}{1 - qu^i},$$

and the result follows. □

In even characteristic, the unipotent radical is abelian but not irreducible. So let  $G = \text{Sp}(2n, q)$  with  $q$  even. Let  $BG$  denote the semidirect product  $WG$ , where  $W$  is the  $2n + 1$  dimensional indecomposable module with  $G$  having a one dimensional fixed space  $W_0$  and  $W/W_0 \cong V$ .

Note that the  $G$ -orbits of characters of  $B$  consist of the trivial character, one orbit of nontrivial characters with  $W_0$  contained in the kernel and  $2(q - 1)$  orbits of characters which are nontrivial on  $W_0$ . The stabilizer of a character in the second orbit is isomorphic to  $B \text{Sp}(2n - 2, q)$  while in the final case the stabilizers are  $O^\pm(2n, q)$  (with  $q - 1$  of each type). This gives the following:

**Lemma 5.5.** *Let  $q$  be even.*

- (1)  $k(B \text{Sp}(2n, q)) = k(\text{ASp}(2n, q)) + (q - 1)(k(O^+(2n, q)) + k(O^-(2n, q)))$
- (2)  $k(\text{ASp}(2n, q)) = k(\text{Sp}(2n, q)) + k(B \text{Sp}(2n - 2, q))$ .

The next lemma follows immediately from the previous lemma. We use the convention that  $\text{ASp}(0, q)$  and  $O^+(0, q)$  are the trivial groups and that  $O^-(0, q)$  is the empty set. So

$$k(\text{ASp}(0, q)) = 1, \quad k(O^+(0, q)) = 1, \quad \text{and} \quad k(O^-(0, q)) = 0.$$

**Lemma 5.6.** *For all  $n \geq 1$ ,*

$$k(\text{ASp}(2n, q)) = k(\text{Sp}(2n, q)) + k(\text{ASp}(2n - 2, q)) + (q - 1)[k(O^+(2n - 2, q)) + k(O^-(2n - 2, q))].$$

Now we obtain the generating function for  $k(\text{ASp}(2n, q))$  in even characteristic.

**Theorem 5.7.** *In even characteristic,  $k(\text{ASp}(2n, q))$  is equal to the coefficient of  $u^n$  in*

$$\frac{1}{1 - u} \prod_i \frac{1 + u^i}{1 - qu^i} \left[ \prod_i \frac{1}{(1 - u^{4i-2})^2} + (q - 1)u \prod_i (1 + u^{2i-1})^2 \right].$$

*Proof.* We define three generating functions:

$$\begin{aligned}
 K_{\text{Sp}}(u) &= 1 + \sum_{n \geq 1} k(\text{Sp}(2n, q))u^n, \\
 K_{\text{O}}(u) &= 1 + \sum_{n \geq 1} [k(\text{O}^+(2n, q)) + k(\text{O}^-(2n, q))]u^n, \\
 A(u) &= 1 + \sum_{n \geq 1} k(\text{ASp}(2n, q))u^n.
 \end{aligned}$$

Multiplying the recursion from Lemma 5.6 by  $u^n$  and summing over  $n \geq 1$  gives that

$$A(u) - 1 = K_{\text{Sp}}(u) - 1 + uA(u) + (q - 1)uK_{\text{O}}(u).$$

Thus

$$A(u) = \frac{K_{\text{Sp}}(u) + (q - 1)uK_{\text{O}}(u)}{1 - u}.$$

From Theorems 3.13 and Theorem 3.21 of [Fulman and Guralnick 2012], elementary manipulations, give that

$$K_{\text{Sp}}(u) = \prod_i \frac{1 + u^i}{1 - qu^i} \prod_i \frac{1}{(1 - u^{4i-2})^2}, \quad K_{\text{O}}(u) = \prod_i \frac{1 + u^i}{1 - qu^i} \prod_i (1 + u^{2i-1})^2,$$

and the result follows. □

**5C. Bounds on  $k(\text{ASp}(2n, q))$ .** As a corollary, we obtain the following results.

**Corollary 5.8.** *In odd characteristic,  $k(\text{ASp}(2n, q)) \leq 27q^n$ .*

*Proof.* From Theorem 5.3,  $k(\text{ASp}(2n, q))$  is the coefficient of  $u^n$  in

$$\prod_i \frac{1 - u^i}{1 - qu^i} \prod_i \frac{(1 + u^i)^4}{1 - u^i} \left(1 + \frac{qu}{1 - u}\right).$$

Now all coefficients of powers of  $u$  in

$$\prod_i \frac{(1 + u^i)^4}{1 - u^i} \left(1 + \frac{qu}{1 - u}\right)$$

are nonnegative. It follows that  $k(\text{ASp}(2n, q))$  is at most

$$\sum_{m=0}^n \left( \text{Coef. } u^{n-m} \text{ in } \prod_i \frac{1 - u^i}{1 - qu^i} \right) \left( \text{Coef. } u^m \text{ in } \prod_i \frac{(1 + u^i)^4}{1 - u^i} \left(1 + \frac{qu}{1 - u}\right) \right).$$

Now  $\prod_i (1 - u^i)/(1 - qu^i)$  is the generating function for the number of conjugacy classes in  $\text{GL}(n, q)$ . By [Maslen and Rockmore 1997],  $k(\text{GL}(n, q))$  is at most  $q^n$ . Hence the coefficient of  $u^{n-m}$  in it is at most  $q^{n-m}$ . It follows that  $k(\text{ASp}(2n, q))$  is at most

$$q^n \sum_{m=0}^n \frac{1}{q^m} \left( \text{Coef. } u^m \text{ in } \prod_i \frac{(1 + u^i)^4}{1 - u^i} \left(1 + \frac{qu}{1 - u}\right) \right).$$

Since the coefficients of  $u^m$  in

$$\prod_i \frac{(1+u^i)^4}{1-u^i} \left(1 + \frac{qu}{1-u}\right)$$

are nonnegative, it follows that  $k(\text{ASp}(2n, q))$  is at most

$$q^n \sum_{m=0}^{\infty} \frac{1}{q^m} \left( \text{Coef. } u^m \text{ in } \prod_i \frac{(1+u^i)^4}{1-u^i} \left(1 + \frac{qu}{1-u}\right) \right),$$

which is equal to

$$q^n \prod_i \frac{(1+1/q^i)^4}{1-1/q^i} \left(1 + \frac{1}{1-1/q}\right).$$

The term

$$\prod_i \frac{(1+1/q^i)^4}{1-1/q^i} \left(1 + \frac{1}{1-1/q}\right)$$

is visibly maximized among odd prime powers  $q$  when  $q = 3$ , when it is at most 27 (we bounded the infinite product  $\prod_i (1+1/q^i)^4 / (1-1/q^i)$  using the remark after Lemma 2.3). □

**Corollary 5.9.** *In odd characteristic,*

$$k(\text{ASp}(2n, q)) \leq q^{2n},$$

except for  $k(\text{ASp}(2, 3)) = 10$ .

*Proof.* From the previous result,  $k(\text{ASp}(2n, q)) \leq 27q^n$ . This immediately implies that  $k(\text{ASp}(2n, q)) \leq q^{2n}$  except possibly for  $\text{ASp}(2, q)$ ,  $\text{ASp}(4, 3)$ , or  $\text{ASp}(4, 5)$ .

From our generating function for  $k(\text{ASp}(2n, q))$  (Theorem 5.3), we see that  $k(\text{ASp}(4, 3)) = 58$ ,  $k(\text{ASp}(4, 5)) = 110$ , and  $k(\text{ASp}(2, q)) = 2q + 4$ , and the result follows. □

Next we move to even characteristic.

**Corollary 5.10.** *In even characteristic,  $k(\text{ASp}(2n, q)) \leq 56q^n$ .*

*Proof.* We rewrite the generating function for  $k(\text{ASp}(2n, q))$  in Theorem 5.7 as

$$\prod_i \frac{1-u^i}{1-qu^i} \frac{1}{1-u} \prod_i \frac{1+u^i}{1-u^i} \left[ \prod_i \frac{1}{(1-u^{4i-2})^2} + (q-1)u \prod_i (1+u^{2i-1})^2 \right].$$

Now arguing exactly as in the odd characteristic case (Corollary 5.8), one sees that  $k(\text{ASp}(2n, q))$  is at most

$$q^n \cdot \frac{1}{1-1/q} \prod_i \frac{1+1/q^i}{1-1/q^i} \left[ \prod_i \frac{1}{(1-1/q^{4i-2})^2} + (1-1/q) \prod_i (1+1/q^{2i-1})^2 \right],$$

and the result follows. □

Next we classify when  $k(\text{ASp}(2n, q)) \leq q^{2n}$ .

**Corollary 5.11.** *In even characteristic,*

$$k(\mathrm{ASp}(2n, q)) \leq q^{2n},$$

except for  $k(\mathrm{ASp}(2, 2)) = 5$ ,  $k(\mathrm{ASp}(4, 2)) = 21$ ,  $k(\mathrm{ASp}(6, 2)) = 67$ .

*Proof.* From the previous result,  $k(\mathrm{ASp}(2n, q)) \leq 56q^n$ . This immediately implies that  $k(\mathrm{ASp}(2n, q)) \leq q^{2n}$  except possibly for  $q = 2$ ,  $1 \leq n \leq 5$ , or  $q = 4$ ,  $n = 1, 2$  or  $q = 8$ ,  $n = 1$ . For these  $q, n$  values one calculates  $k(\mathrm{ASp}(2n, q))$  from the generating function in Theorem 5.7, and the result follows.  $\square$

### 6. Orthogonal Groups

Section 6A uses the orbit approach to calculate the generating function for  $k(\mathrm{AO})$  when the characteristic is odd. Section 6B uses the character approach to calculate the generating function of  $k(\mathrm{AO})$  in any characteristic. To be more precise, we actually derive two generating functions, one for  $k(\mathrm{AO}^+) + k(\mathrm{AO}^-)$  and one for  $k(\mathrm{AO}^+) - k(\mathrm{AO}^-)$ . Clearly this is equivalent to deriving generating functions for  $k(\mathrm{AO}^+)$  and  $k(\mathrm{AO}^-)$ .

Section 6C derives some bounds on  $k(\mathrm{AO})$ .

**6A. Orbit approach for  $k(\mathrm{AO})$ , odd characteristic.** For the orbit approach we assume the characteristic is odd. It is somewhat more convenient to work in orthogonal groups than the special orthogonal group (there is essentially no difference in the result below for  $\mathrm{SO}$ ). The conjugacy class of a unipotent element in  $\mathrm{O}^\epsilon(m, q)$  gives rise to a partition of  $m$  with  $a_i$  pieces of size  $i$ . Moreover,  $a_i$  is even for  $i$  even. This determines the conjugacy class over the algebraic closure. Over the finite field, we attach a sign  $\epsilon_i$  for each odd  $i$  with  $a_i$  nonzero and this determines the class (see [Liebeck and Seitz 2012]). We let  $\lambda_{z-1}^\pm(C)$  denote this signed partition corresponding to the unipotent part of a conjugacy class  $C$ .

The proof of the next result is essentially identical to the case of symplectic groups and so we omit the details (and we can also use the character theory approach below).

**Theorem 6.1.** *Suppose that the characteristic is odd. Let  $C$  be a conjugacy class of  $\mathrm{O}^\epsilon(n, q)$ . Let  $a_i$  be the number of parts of  $\lambda_{z-1}^\pm(C)$  of size  $i$ . Then  $o(C)$  is equal to*

$$1 + \sum_{\substack{i \text{ even} \\ a_i \neq 0}} 1 + \sum_{\substack{i \text{ odd} \\ a_i \neq 0}} f_i,$$

where

$$f_i = \begin{cases} q & \text{if } a_i > 2 \text{ (independently of the sign),} \\ q & \text{if } a_i = 2 \text{ and the sign is } +, \\ (q - 1) & \text{if } a_i = 2 \text{ and the sign is } -, \\ (q - 1)/2 & \text{if } a_i = 1 \text{ (independently of the sign).} \end{cases} \tag{6}$$

The following combinatorial lemma will also be helpful.

**Lemma 6.2.** *Suppose that the characteristic is odd.*

(1) The generating function for the number of unipotent classes of the groups  $O(n, q)$  is

$$\sum_{\lambda^\pm} u^{|\lambda^\pm|}.$$

This is equal to

$$\prod_i \frac{1}{1-u^{4i}} \prod_{i \text{ odd}} \left( \frac{1+u^i}{1-u^i} \right).$$

(2) The generating function

$$\sum_{\lambda^\pm} u^{|\lambda^\pm|} \sum_{\substack{j \text{ even} \\ a_j \neq 0}} 1$$

is equal to

$$\frac{u^4}{1-u^4} \prod_i \frac{1}{1-u^{4i}} \prod_{i \text{ odd}} \left( \frac{1+u^i}{1-u^i} \right).$$

(3) Let  $f_i$  be as in Theorem 6.1. Then

$$\sum_{\lambda^\pm} u^{|\lambda^\pm|} \sum_{\substack{j \text{ odd} \\ a_j \neq 0}} f_j$$

is equal to

$$\left( \frac{(q-1)u}{1-u^2} + \frac{u^2}{1-u^4} \right) \prod_i \frac{1}{1-u^{4i}} \prod_{i \text{ odd}} \left( \frac{1+u^i}{1-u^i} \right).$$

*Proof.* For the first part, the unipotent conjugacy classes of the groups  $O(n, q)$  correspond to signed partitions  $\lambda^\pm$  of size  $n$ . The generating function for such partitions is clearly equal to

$$\prod_{i \text{ odd}} (1 + 2u^i + 2u^{2i} + \dots) \prod_{i \text{ even}} (1 + u^{2i} + u^{4i} + \dots),$$

which is equal to

$$\prod_i \frac{1}{1-u^{4i}} \prod_{i \text{ odd}} \frac{1+u^i}{1-u^i}.$$

For the second part, first note that arguing as in the first part, one has that

$$\sum_{\lambda^\pm} u^{|\lambda^\pm|} \sum_{\substack{j \text{ even} \\ a_j \neq 0}} 1$$

is equal to

$$\begin{aligned} \sum_{j \text{ even}} \sum_{\substack{\lambda^\pm \\ a_j \neq 0}} u^{|\lambda^\pm|} &= \sum_{j \text{ even}} (u^{2j} + u^{4j} + \dots) \prod_{\substack{i \text{ even} \\ i \neq j}} (1 + u^{2i} + u^{4i} + \dots) \prod_{i \text{ odd}} (1 + 2u^i + 2u^{2i} + \dots) \\ &= \sum_{j \text{ even}} u^{2j} \prod_{i \text{ even}} (1 + u^{2i} + u^{4i} + \dots) \prod_{i \text{ odd}} (1 + 2u^i + 2u^{2i} + \dots) \\ &= \frac{u^4}{1-u^4} \prod_i \frac{1}{1-u^{4i}} \prod_{i \text{ odd}} \left( \frac{1+u^i}{1-u^i} \right). \end{aligned}$$

For the third part,

$$\sum_{\lambda^\pm} u^{|\lambda^\pm|} \sum_{\substack{j \text{ odd} \\ a_j \neq 0}} f_j$$

is equal to

$$\sum_{j \text{ odd}} (2u^{j\frac{1}{2}}(q-1) + u^{2j}(q+q-1) + 2q(u^{3j} + u^{4j} + \dots)) \prod_{\substack{i \neq j \\ i \text{ odd}}} \left( \frac{1+u^i}{1-u^i} \right) \prod_{i \text{ even}} (1 + u^{2i} + u^{4i} + \dots).$$

This is equal to

$$\sum_{j \text{ odd}} \frac{1-u^j}{1+u^j} (u^j(q-1) + u^{2j}(2q-1) + 2q(u^{3j} + u^{4j} + \dots)) \prod_{i \text{ odd}} \left( \frac{1+u^i}{1-u^i} \right) \prod_i \frac{1}{1-u^{4i}}.$$

Now, as in the proof of part (3) of Lemma 5.2,

$$\sum_{j \text{ odd}} \frac{1-u^j}{1+u^j} (u^j(q-1) + u^{2j}(2q-1) + 2q(u^{3j} + u^{4j} + \dots))$$

simplifies to

$$\frac{(q-1)u}{1-u^2} + \frac{u^2}{1-u^4},$$

and the result follows. □

As a corollary, we derive a generating function for  $k(\text{AO}^+) + k(\text{AO}^-)$ .

**Theorem 6.3.** *In odd characteristic,*

$$1 + \sum_{n \geq 1} u^n [k(\text{AO}^+(n, q)) + k(\text{AO}^-(n, q))]$$

is equal to

$$\prod_i \frac{(1+u^{2i-1})^4}{1-qu^{2i}} \cdot \left( 1 + \frac{u^2 + (q-1)u}{1-u^2} \right).$$

*Proof.* By Lemma 2.1 and Theorem 6.1,

$$k(\text{AO}^+(n, q)) + k(\text{AO}^-(n, q))$$

is equal to  $T_1 + T_2 + T_3$ , where  $T_1$  is  $k(\text{O}^+(n, q)) + k(\text{O}^-(n, q))$ , and  $T_2, T_3$  are the following sums over conjugacy classes  $C$  of  $\text{O}^+(n, q)$  and  $\text{O}^-(n, q)$ :

$$T_2 = \sum_C \sum_{\substack{i \text{ even} \\ a_i \neq 0}} 1, \quad T_3 = \sum_C \sum_{\substack{i \text{ odd} \\ a_i \neq 0}} f_i.$$

From [Wall 1963],  $T_1$  is the coefficient of  $u^n$  in

$$\prod_i \frac{(1+u^{2i-1})^4}{1-qu^{2i}}.$$

To compute the generating function for  $T_2$ , we take Wall’s generating function for  $T_1$ , divide it by the generating function for unipotent conjugacy classes in part (1) of Lemma 6.2, and multiply it by the generating function for the weighted sum over unipotent classes in part (2) of Lemma 6.2. We conclude that  $T_2$  is the coefficient of  $u^n$  in

$$\frac{u^4}{1-u^4} \prod_i \frac{(1+u^{2i-1})^4}{1-qu^{2i}}.$$

To compute the generating function for  $T_3$ , we take Wall’s generating function for  $T_1$ , divide it by the generating function for unipotent conjugacy classes in part (1) of Lemma 6.2 and multiply it by the generating function for the weighted sum over unipotent classes in part (3) of Lemma 6.2. We conclude that  $T_3$  is the coefficient of  $u^n$  in

$$\left( \frac{(q-1)u}{1-u^2} + \frac{u^2}{1-u^4} \right) \prod_i \frac{(1+u^{2i-1})^4}{1-qu^{2i}}.$$

Since

$$1 + \frac{u^4}{1-u^4} + \frac{(q-1)u}{1-u^2} + \frac{u^2}{1-u^4} = 1 + \frac{u^2 + (q-1)u}{1-u^2},$$

the result follows. □

Next, we derive a generating function for  $k(\text{AO}^+) - k(\text{AO}^-)$ .

**Theorem 6.4.** *In odd characteristic,*

$$1 + \sum_{n \geq 1} u^n [k(\text{AO}^+(n, q)) - k(\text{AO}^-(n, q))]$$

is equal to

$$\frac{1}{1-u^2} \prod_i \frac{(1-u^{4i-2})}{1-qu^{4i}}.$$

*Proof.* By Lemma 2.1 and Theorem 6.1,

$$k(\text{AO}^+(n, q)) - k(\text{AO}^-(n, q))$$

is equal to  $T_1 + T_2 + T_3$ , where  $T_1$  is  $k(\text{O}^+(n, q)) - k(\text{O}^-(n, q))$ ,

$$T_2 = \sum_{\substack{C^+ \\ a_i \neq 0}} \sum_{\substack{i \text{ even} \\ a_i \neq 0}} 1 - \sum_{\substack{C^- \\ a_i \neq 0}} \sum_{\substack{i \text{ even} \\ a_i \neq 0}} 1, \quad T_3 = \sum_{\substack{C^+ \\ a_i \neq 0}} \sum_{\substack{i \text{ odd} \\ a_i \neq 0}} f_i - \sum_{\substack{C^- \\ a_i \neq 0}} \sum_{\substack{i \text{ odd} \\ a_i \neq 0}} f_i.$$

Here  $C^+$  ranges over conjugacy classes of  $\text{O}^+(n, q)$ , and  $C^-$  ranges over conjugacy classes of  $\text{O}^-(n, q)$ .

From [Wall 1963],  $T_1$  is the coefficient of  $u^n$  in

$$\prod_i \frac{1-u^{4i-2}}{1-qu^{4i}}.$$

To compute the generating function of  $T_2$ , we take the generating function for  $T_1$ , multiply it by  $\prod_i (1 - u^{4i})$  (which corresponds to removing the unipotent part). Then to add in the weighted unipotent part, one multiplies by

$$\sum_{j \text{ even}} (u^{2j} + u^{4j} + \dots) \prod_{\substack{i \neq j \\ i \text{ even}}} (1 + u^{2i} + u^{4i} + \dots),$$

which is equal to

$$\frac{u^4}{1 - u^4} \prod_i \frac{1}{1 - u^{4i}}.$$

We conclude that  $T_2$  is the coefficient of  $u^n$  in

$$\frac{u^4}{1 - u^4} \prod_i \frac{(1 - u^{4i-2})}{1 - qu^{4i}}.$$

To compute the generating function of  $T_3$ , we take the generating function for  $T_1$ , multiply it by  $\prod_i (1 - u^{4i})$  (which corresponds to removing the unipotent part). Then to add in the weighted unipotent part, one multiplies by

$$\sum_{j \text{ odd}} u^{2j} \prod_{i \text{ even}} (1 + u^{2i} + u^{4i} + \dots).$$

Note that the terms involving  $f_i$  canceled out (except for the  $a_i = 2$  case). The upshot is that the generating function for  $T_3$  is

$$\frac{u^2}{1 - u^4} \prod_i \frac{(1 - u^{4i-2})}{1 - qu^{4i}}.$$

Since

$$1 + \frac{u^4}{1 - u^4} + \frac{u^2}{1 - u^4} = \frac{1}{1 - u^2},$$

the proof is complete. □

**6B. Character approach for  $k(\text{AO})$ , any characteristic.** Next we consider orthogonal groups. In this case, the natural module  $V$  can be identified with its character group and the nontrivial  $G$ -orbits correspond to nonzero vectors of  $V$  of a given norm.

First consider the case  $G = \text{O}^\epsilon(n, q)$  with  $q$  odd. The stabilizers are thus  $\text{AO}^\epsilon(m - 2, q)$  (for an isotropic vector) and  $(q - 1)/2$  copies each of  $\text{O}^+(n - 2, q)$  and  $\text{O}^-(n - 2, q)$ . Note that we use the convention that  $\text{O}^\epsilon(0, q)$  and  $\text{AO}^\epsilon(0, q)$  are empty if  $\epsilon = -$  and are the trivial group if  $\epsilon = +$ . Similarly,  $\text{AO}^\epsilon(-1, q)$  is the empty set. And as in earlier cases, the trivial group has one conjugacy class and the empty set has zero conjugacy classes. This yields the following result.

**Lemma 6.5.** *Let  $q$  be odd and  $n \geq 1$ . Then*

$$k(\text{AO}^\epsilon(n, q)) = k(\text{O}^\epsilon(n, q)) + k(\text{AO}^\epsilon(n - 2, q)) + (q - 1)(k(\text{O}^+(n - 1, q)) + k(\text{O}^-(n - 1, q)))/2.$$

As a corollary, we obtain a second proof of Theorems 6.3 and 6.4.

Second proof of Theorem 6.3. Define

$$K_O(u) = 1 + \sum_{n \geq 1} u^n [k(O^+(n, q)) + k(O^-(n, q))],$$

$$A_O(u) = 1 + \sum_{n \geq 1} u^n [k(AO^+(n, q)) + k(AO^-(n, q))].$$

By the above recursion, we have that for all  $n$ ,

$$k(AO^+(n, q)) = k(O^+(n, q)) + k(AO^+(n-2, q)) + \frac{1}{2}(q-1)[k(O^+(n-1, q)) + k(O^-(n-1, q))],$$

$$k(AO^-(n, q)) = k(O^-(n, q)) + k(AO^-(n-2, q)) + \frac{1}{2}(q-1)[k(O^+(n-1, q)) + k(O^-(n-1, q))].$$

Adding these two equations gives

$$k(AO^+(n, q)) + k(AO^-(n, q))$$

$$= k(O^+(n, q)) + k(O^-(n, q)) + k(AO^+(n-2, q)) + k(AO^-(n-2, q))$$

$$+ (q-1)[k(O^+(n-1, q)) + k(O^-(n-1, q))].$$

Multiplying this by  $u^n$  and summing over  $n \geq 0$  gives that

$$A_O(u) = K_O(u) + u^2 A_O(u) + u(q-1)K_O(u).$$

Thus

$$A_O(u) = \frac{K_O(u)}{1-u^2}(1+u(q-1)).$$

The result now follows from Wall's formula

$$K_O(u) = \prod_i \frac{(1+u^{2i-1})^4}{1-qu^{2i}}. \quad \square$$

Second proof of Theorem 6.4. Let

$$D(u) = 1 + \sum_{n \geq 1} u^n [k(O^+(n, q)) - k(O^-(n, q))],$$

$$B(u) = 1 + \sum_{n \geq 1} u^n [k(AO^+(n, q)) - k(AO^-(n, q))].$$

From Lemmas 6.5, we have that

$$k(AO^+(n, q)) - k(AO^-(n, q)) = k(O^+(n, q)) - k(O^-(n, q)) + k(AO^+(n-2, q)) - k(AO^-(n-2, q)).$$

Multiplying this equation by  $u^n$  and summing over all  $n \geq 0$  gives that

$$B(u) = D(u) + u^2 B(u).$$

Thus  $B(u) = D(u)/(1-u^2)$ , and the result follows from Wall's formula

$$D(u) = \prod_i \frac{(1-u^{4i-2})}{1-qu^{4i}}. \quad \square$$

Finally we turn to characteristic 2. In this case odd dimensional orthogonal groups are isomorphic to symplectic groups, so we need only consider the even dimensional case. So consider  $G = O^\epsilon(n, q)$  with  $q$  and  $n$  both even. The argument is similar. The only difference is that the stabilizer of a vector of nonzero norm in  $AO^\epsilon(n, q)$  is  $Sp(n - 2, q) \times \mathbb{Z}/2$  and so:

**Lemma 6.6.** *Let  $q$  be even and  $n \geq 2$  be even. Then  $k(AO^\epsilon(n, q))$  is equal to*

$$k(O^\epsilon(n, q)) + k(AO^\epsilon(n - 2, q)) + 2(q - 1)(k(Sp(n - 2, q))).$$

For  $n = 2$  we used the convention that  $k(AO^+(0, q)) = 1$  and  $k(Sp(0, q)) = 1$ , and that  $k(AO^-(0, q)) = 0$ .

Next using Lemma 6.6 (and generating functions for  $k(Sp)$  and  $k(O)$ ) we derive generating functions for  $k(AO^\pm(2n, q))$  in even characteristic.

**Theorem 6.7.** *Let  $q$  be even. Then  $k(AO^+(2n, q)) + k(AO^-(2n, q))$  is the coefficient of  $u^n$  in*

$$\frac{1}{1-u} (K_O(u) + 4(q-1)uK_{Sp}(u)),$$

where

$$K_O(u) = \prod_{i \geq 1} \frac{(1+u^i)(1+u^{2i-1})^2}{1-qu^i}, \quad K_{Sp}(u) = \prod_{i \geq 1} \frac{(1-u^{4i})}{(1-u^{4i-2})(1-u^i)(1-qu^i)}.$$

*Proof.* Define generating functions,

$$\begin{aligned} K_{Sp}(u) &= 1 + \sum_{n \geq 1} k(Sp(2n, q))u^n, \\ K_O(u) &= 1 + \sum_{n \geq 1} [k(O^+(2n, q)) + k(O^-(2n, q))]u^n, \\ A_O(u) &= 1 + \sum_{n \geq 1} [k(AO^+(2n, q)) + k(AO^-(2n, q))]u^n. \end{aligned}$$

Now take the recursions for  $k(AO^+(2n, q))$  and  $k(AO^-(2n, q))$  in Lemma 6.6, multiply them by  $u^n$  and sum over all  $n \geq 0$ . We conclude that

$$A_O(u) = K_O(u) + uA_O(u) + 4(q-1)uK_{Sp}(u).$$

Thus

$$A_O(u) = \frac{1}{1-u} (K_O(u) + 4(q-1)uK_{Sp}(u)).$$

From [Fulman and Guralnick 2012],

$$K_O(u) = \prod_i \frac{(1+u^i)(1+u^{2i-1})^2}{(1-qu^i)}, \quad K_{Sp}(u) = \prod_i \frac{(1-u^{4i})}{(1-u^{4i-2})(1-u^i)(1-qu^i)}. \quad \square$$

**Theorem 6.8.** *Let  $q$  be even. Then  $k(AO^+(2n, q)) - k(AO^-(2n, q))$  is the coefficient of  $u^n$  in*

$$\frac{1}{1-u} \prod_{i \geq 1} \frac{1-u^{2i-1}}{1-qu^{2i}}.$$

*Proof.* Define generating functions

$$D(u) = 1 + \sum_{n \geq 1} u^n [k(\mathcal{O}^+(2n, q)) - k(\mathcal{O}^-(2n, q))],$$

$$B(u) = 1 + \sum_{n \geq 1} u^n [k(\mathcal{AO}^+(2n, q)) - k(\mathcal{AO}^-(2n, q))].$$

Multiply the recursions for  $k(\mathcal{AO}^+(2n, q))$  and  $k(\mathcal{AO}^-(2n, q))$  in Lemma 6.6 by  $u^n$ , sum over all  $n \geq 0$ , and subtract to obtain

$$B(u) = D(u) + uB(u).$$

Using Wall’s formula [1963] for  $D(u)$ , we conclude that

$$B(u) = \frac{1}{1-u} D(u) = \frac{1}{1-u} \prod_i \frac{1 - u^{2i-1}}{1 - qu^{2i}}. \quad \square$$

**6C. Bounds on  $k(\mathcal{AO})$ .** This section derives bounds on  $k(\mathcal{AO})$ .

We begin with the case of odd characteristic and even dimension.

**Corollary 6.9.** *Let  $q$  be odd. Then  $k(\mathcal{AO}^\pm(2n, q)) \leq 29q^n$ .*

*Proof.* From Theorem 6.3,

$$k(\mathcal{AO}^+(2n, q)) + k(\mathcal{AO}^-(2n, q))$$

is the coefficient of  $u^{2n}$  in

$$\prod_i \frac{(1 + u^{2i-1})^4}{1 - qu^{2i}} \cdot \left( 1 + \frac{u^2 + (q-1)u}{1 - u^2} \right).$$

Rewrite this as

$$\prod_i \frac{1 - u^{2i}}{1 - qu^{2i}} \prod_i \frac{(1 + u^{2i-1})^4}{1 - u^{2i}} \cdot \left( 1 + \frac{u^2 + (q-1)u}{1 - u^2} \right).$$

As in the symplectic case, the coefficient of  $u^{2n-2m}$  in  $\prod_i (1 - u^{2i}) / (1 - qu^{2i})$  is at most  $q^{n-m}$ . Thus

$$k(\mathcal{AO}^+(2n, q)) + k(\mathcal{AO}^-(2n, q))$$

is at most

$$q^n \sum_{m \geq 0} \frac{1}{q^m} \text{Coef. } u^{2m} \text{ in } \prod_i \frac{(1 + u^{2i-1})^4}{1 - u^{2i}} \cdot \left( 1 + \frac{u^2 + (q-1)u}{1 - u^2} \right),$$

which is equal to  $q^n/2$  multiplied by

$$\prod_i \frac{(1 + u^{2i-1})^4}{1 - u^{2i}} \cdot \left( 1 + \frac{u^2 + (q-1)u}{1 - u^2} \right) + \prod_i \frac{(1 - u^{2i-1})^4}{1 - u^{2i}} \cdot \left( 1 + \frac{u^2 - (q-1)u}{1 - u^2} \right)$$

evaluated at  $u = 1/\sqrt{q}$ . Since  $q \geq 3$ , we conclude that

$$k(\mathcal{AO}^+(2n, q)) + k(\mathcal{AO}^-(2n, q)) \leq 53q^n.$$

From Theorem 6.4,

$$k(\text{AO}^+(2n, q)) - k(\text{AO}^-(2n, q))$$

is the coefficient of  $u^n$  in

$$\frac{1}{1-u} \prod_i \frac{1-u^{2i-1}}{1-qu^{2i}}.$$

This is analytic for  $|u| < \frac{1}{q} + \epsilon$ , so Lemmas 2.4 and 2.3 imply an upper bound of

$$q^n \frac{1}{1-1/q} \prod_i \frac{1+1/q^{2i-1}}{1-1/q^{2i-1}} \leq 3.3q^n.$$

Combining the results of the previous two paragraphs proves the corollary, as  $(53 + 3.3)/2 \leq 29$ .  $\square$

**Corollary 6.10.** *Let  $q$  be odd. Then  $k(\text{AO}^\pm(2n, q)) \leq q^{2n}$ .*

*Proof.* The result follows from the previous corollary whenever  $29q^n \leq q^{2n}$ . So we only need to check the cases  $n = 1$ , or  $n = 2$ ,  $q = 3, 5$ , or  $n = 3$ ,  $q = 3$ . These cases are easily checked from our generating function for  $k(\text{AO}^\pm(2n, q))$ .  $\square$

Next we treat the case of odd dimensional groups in odd characteristic. In this case, the upper bound is not of the form constant times  $q^{\text{rank}}$ . This is because every element in the classical group has eigenvalue 1.

**Corollary 6.11.** *Let  $q$  be odd. Then  $k(\text{AO}(2n + 1, q)) \leq 20q^{n+1}$ .*

*Proof.* We prove this by induction on  $n$ . By our earlier recursion,

$$k(\text{AO}(2n + 1, q)) = k(\text{O}(2n + 1, q)) + k(\text{AO}(2n - 1, q)) + \frac{1}{2}(q - 1)[k(\text{O}^+(2n, q)) + k(\text{O}^-(2n, q))].$$

By [Fulman and Guralnick 2012],

$$k(\text{O}(2n + 1, q)) \leq 14.2q^n$$

and

$$k(\text{O}^+(2n, q)) + k(\text{O}^-(2n, q)) \leq 16.3q^n.$$

Thus

$$k(\text{AO}(2n + 1, q)) \leq k(\text{AO}(2n - 1, q)) + 14.2q^n + 8.2q^{n+1}.$$

By induction,  $k(\text{AO}(2n - 1, q)) \leq 20q^n$ , so the result follows since

$$20q^n + 14.2q^n + 8.2q^{n+1} \leq 20q^{n+1}$$

for  $q \geq 3$ .  $\square$

**Corollary 6.12.** *Let  $q$  be odd. Then  $k(\text{AO}(2n + 1, q)) \leq q^{2n+1}$ .*

*Proof.* By the previous corollary, the result holds if  $20 \leq q^n$ . So we need only check the cases  $n = 0$ ,  $n = 1$ , or  $n = 2$ ,  $q = 3$ . The generating function (Theorem 6.3) implies that  $k(\text{AO}(1, q)) = (q + 3)/2$  and  $k(\text{AO}(3, q)) = (q^2 + 10q + 5)/2$ , and shows that the exact value of  $k(\text{AO}(5, 3))$  is less than 243.  $\square$

Next we turn to the case of even characteristic.

**Corollary 6.13.** *Let  $q$  be even. Then  $k(\text{AO}^\pm(2n, q)) \leq 60q^n$ .*

*Proof.* From Theorem 6.7,  $k(\text{AO}^+(2n, q)) + k(\text{AO}^-(2n, q))$  is equal to the coefficient of  $u^n$  in

$$\prod_i \frac{1-u^i}{1-qu^i} \frac{1}{1-u} \left[ \prod_i \frac{(1+u^i)(1+u^{2i-1})^2}{(1-u^i)} + 4(q-1)u \prod_i \frac{(1-u^{4i})}{(1-u^{4i-2})(1-u^i)^2} \right].$$

Arguing as for the symplectic groups, this is at most  $q^n$  multiplied by

$$\frac{1}{1-1/q} \left[ \prod_i \frac{(1+1/q^i)(1+1/q^{2i-1})^2}{(1-1/q^i)} + \frac{4(q-1)}{q} \prod_i \frac{(1-1/q^{4i})}{(1-1/q^{4i-2})(1-1/q^i)^2} \right],$$

which is at most  $111.6q^n$  since  $q \geq 2$ .

From Theorem 6.8,  $k(\text{AO}^+(2n, q)) - k(\text{AO}^-(2n, q))$  is equal to the coefficient of  $u^n$  in

$$\frac{1}{1-u} \prod_i \frac{1-u^{2i-1}}{1-qu^{2i}}.$$

Since this is analytic for  $|u| < q^{-1} + \epsilon$ , Lemma 2.4 gives that  $k(\text{AO}^+(2n, q)) - k(\text{AO}^-(2n, q))$  is at most

$$q^n \frac{1}{1-1/q} \prod_i \frac{1+1/q^{2i-1}}{1-1/q^{2i-1}} \leq 8.4q^n.$$

The corollary now follows since  $(111.6 + 8.4)/2 = 60$ . □

**Corollary 6.14.** *Let  $q$  be even. Then  $k(\text{AO}^\pm(2n, q)) \leq q^{2n}$  except for*

$$\begin{aligned} k(\text{AO}^+(2, 2)) = 5, \quad k(\text{AO}^-(2, 2)) = 5, \quad k(\text{AO}^+(4, 2)) = 20, \\ k(\text{AO}^-(4, 2)) = 18, \quad \text{and} \quad k(\text{AO}^-(6, 2)) = 65. \end{aligned}$$

*Proof.* By the previous corollary,  $k(\text{AO}^\pm(2n, q)) \leq q^{2n}$  if  $60 \leq q^n$ . So we need only check the cases  $n = 1$  or  $q = 2$ ,  $2 \leq n \leq 5$ , or  $q = 4$ ,  $n = 2$ . So the only infinite family of cases to check is when  $n = 1$ , in which case the generating function gives  $k(\text{AO}^\pm(2, q)) = 5q/2$ . The remaining finite number of cases can be checked immediately from the generating function. □

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