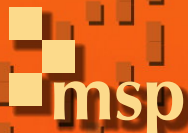


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on exponential sums and monodromy
with semicontinuity of the minimal exponent**

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We combine two of Igusa’s conjectures with recent semicontinuity results by Mustață and Popa to form a new, natural conjecture about bounds for exponential sums. These bounds have a deceptively simple and general formulation in terms of degrees and dimensions only. We provide evidence consisting partly of adaptations of already known results about Igusa’s conjecture on exponential sums, but also some new evidence like for all polynomials in up to 4 variables. We show that, in turn, these bounds imply consequences for Igusa’s (strong) monodromy conjecture. The bounds are related to estimates for major arcs appearing in the circle method for local-global principles.

1. Introduction

Let f be a polynomial in n variables over \mathbb{Z} and of degree $d > 1$, and let s be the (complex affine) dimension of the critical locus of the degree d homogeneous part of f . The main objects of our study are the finite exponential sums from (1) and their estimates in terms of n , d , and s as in Conjecture 1 below. For any positive integer N and any complex primitive N -th root of unity ξ , consider the exponential sum

$$\sum_{x \in (\mathbb{Z}/N\mathbb{Z})^n} \xi^{f(x)}. \quad (1)$$

When N runs over the set of prime numbers, the sums from (1) fall under the scope of works by Grothendieck, Deligne, Katz, Laumon, and others, building in particular on the Weil conjectures. We don’t pursue new results for N running over the set of prime numbers. Instead we put forward new bounds for these sums uniformly in general N with, roughly, a win of a factor

$$N^{-(n-s)/d}$$

on the trivial bound; see Conjecture 1 below. In this context, Birch [1962] proved and used bounds with exponent

$$(n-s)/2^{d-1}(d-1)$$

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instead of our projected and stronger $(n - s)/d$; see (5) below. The bounds in the conjecture look deceptively simple, but a reduction argument to, say, the case $s = 0$, turns out to be surprisingly hard in general, and moreover, the case $s = 0$ for nonhomogeneous f is surprisingly hard as well. As evidence for Conjecture 1 we prove an almost generic case (based on the Newton polyhedron of f), as well as the case with up to 4 variables, and, the cases restricted to those N which are cube free and more generally $(d+2)$ -th power free.

Conjecture 1 combines two of Igusa's conjectures, namely on exponential sums and on monodromy, and represents an update of these conjectures in line with the recently proved semicontinuity result for the minimal exponent by Mustața and Popa [2020] and the conjectured equality of the minimal exponent with the motivic oscillation index; see [Cluckers et al. 2019] and Section 2.7 below.

1.1. Let us make all this more precise, for f a polynomial over \mathbb{Z} in n variables. For an integer $N > 0$ and a complex primitive N -th root of unity ξ , put

$$E_f(N, \xi) := \left| \frac{1}{N^n} \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^n} \xi^{f(x)} \right|, \quad (2)$$

which is simply the complex modulus of the sum in (1) normalized by the number of terms. Write d for the degree of f and f_d for the homogeneous degree d part of f . We assume that $d > 1$. Write $s = s(f)$ for the dimension of the critical locus of f_d , namely, of the solution set in \mathbb{C}^n of the equations

$$0 = \frac{\partial f_d}{\partial x_1}(x) = \dots = \frac{\partial f_d}{\partial x_n}(x). \quad (3)$$

Note that $0 \leq s \leq n - 1$. Our projected bounds are as follows:

Conjecture 1. *Given f , n , s , and d as above and any $\varepsilon > 0$, one has*

$$E_f(N, \xi) \ll N^{-(n-s)/d + \varepsilon}. \quad (4)$$

In this context, note that Birch [1962, Lemma 5.4] obtained the following bound, based on the very same data of f , n , s , and d (and, assuming f to be homogeneous):

$$E_f(N, \xi) \ll N^{-\frac{n-s}{2^{d-1}(d-1)} + \varepsilon}, \quad (5)$$

which he used to estimate major arcs to obtain general local-global principles (see Section 1.9 below).

Remarkably, the weakening of Conjecture 1 with

$$-\frac{n-s}{2(d-1)} + \varepsilon \quad (6)$$

in the exponent of N in (4) instead of $-(n - s)/d + \varepsilon$ has just been shown in [Nguyen 2021], vastly improving Birch's bounds (5). The case of Conjecture 1 with $d = 3$ is in line with the resembling (but averaged) bounds (170) of [Hooley 1988]. In the one variable case, similar bounds to (4) have already been studied; see, e.g., [Chalk 1987; Hua 1959] and some generalizations in [Cochrane and Zheng 1999;

2000]. Knowing only n, s and d , the exponent $-(n-s)/d$ is optimal in (4), as witnessed by $f = \sum_{i=1}^{n-s} x_i^d$ and $N = p^d$ for primes p ; see also the example in (20).

Remark 1.2. The notation in (4) means that, given f and $\varepsilon > 0$, there is a constant $c = c(f, \varepsilon)$ such that, for all integers $N \geq 1$ and all primitive N -th roots of unity ξ , the value $E_f(N, \xi)$ is no larger than $cN^{-(n-s)/d+\varepsilon}$.

Remark 1.3. The critical case of Conjecture 1 is with N having a single prime divisor. Indeed, by the Chinese remainder theorem, if one writes $N = \prod_i p_i^{e_i}$ for distinct prime numbers p_i and integers $e_i > 0$, then one has

$$E_f(N, \xi) = \prod_i E_f(p_i^{e_i}, \xi_i) \tag{7}$$

for some primitive $p_i^{e_i}$ -th roots of unity ξ_i . In detail, if one writes $1/N = \sum_i a_i/p_i^{e_i}$ with $(a_i, p_i) = 1$, then one takes $\xi_i = \xi^{b_i}$ with $b_i = a_i N/p_i^{e_i}$.

1.4. Conjecture 1 simplifies Igusa’s original question on exponential sums (recalled in Section 1.6) to bounds involving only n, d , and s . It opens a way to proceed with Igusa’s conjecture on exponential sums beyond the case of nonrational singularities that is obtained recently in [Cluckers et al. 2019].

In most of the evidence that we provide below, one can furthermore take $\varepsilon = 0$ and one may wonder to which extent this sharpening of Conjecture 1 holds. Such a sharpening with $\varepsilon = 0$ goes beyond Igusa’s conjectures in ways explained in Section 2.5. One may also wonder whether the implied constant c can be taken depending only on f_d and n (and ε , but not on f). If one excludes a finite set S (depending on f or just on f_d) of prime divisors of N , then it seems furthermore possible that the implied constant can be taken depending only on d and n (and ε); see Remark 5.7 for more details.

1.5. In Section 2 we relate the bounds from Conjecture 1 to Igusa’s monodromy conjecture. Conjecture 1 implies the strong monodromy conjecture for poles of local zeta functions with real part in the range strictly between $-(n-s)/d$ and zero. More precisely, we show under Conjecture 1 that there are no poles (of a local zeta function of f) with real part in this range except -1 (see Proposition 2.3); from [Mustařa and Popa 2020] it follows correspondingly that there are no zeros of the Bernstein–Sato polynomial of f in this range other than -1 (see Proposition 2.4). Note that Conjecture 1 is much stronger than the strong monodromy conjecture in the mentioned range, as the latter implies merely a much weaker variant of Conjecture 1, namely the bounds from (15) instead of (4), where the constant c_p is allowed to depend on p .

1.6. Igusa’s original question on exponential sums predicts upper bounds with a noncanonical exponent coming from a choice of log resolution for homogeneous f (with $f = f_d$); see [Igusa 1978]. More precisely, let $h : Y \rightarrow X = \mathbb{A}_{\mathbb{Q}}^n$ be a log resolution of $D = f^{-1}(0) := \text{Spec}(\mathbb{Q}[x_1, \dots, x_n]/(f))$, i.e., Y is an integral smooth scheme, $h : Y \rightarrow X$ is a proper map, the restriction $h : Y \setminus h^{-1}(D) \rightarrow X \setminus D$ is an isomorphism, and $(h^{-1}(D))_{\text{red}}$ has simple normal crossings as subscheme of Y . Such a log resolution exists by the work of Hironaka [1964]. Write $h^{-1}(D) = \sum_{i \in I} N_i E_i$ and $\text{Div}(h^*(dx_1 \wedge \dots \wedge dx_n)) = \sum_{i \in I} (v_i - 1) E_i$

for irreducible components E_i of $(h^{-1}(D))_{\text{red}}$ and positive integers N_i, v_i . By blowing up further, one may suppose that $E_j \cap E_i = \emptyset$ whenever $(v_i, N_i) = (v_j, N_j) = (1, 1)$ and $i \neq j$. Put

$$J = \{i \in I \mid (v_i, N_i) \neq (1, 1)\} \quad \text{and} \quad \sigma_0 = \sigma_0(h) := \min_{i \in J} \frac{v_i}{N_i}.$$

Note that σ_0 depends on the choice of h in general. Igusa originally conjectured, for any $\sigma < \sigma_0$, and under a few extra conditions that are most likely superfluous (namely, that $\sigma_0 > 2$ and that f is homogeneous), that one has a bound

$$E_f(N, \xi) \leq cN^{-\sigma} \tag{8}$$

for all $N > 0$, all primitive N -th roots of unity ξ and a constant c independent of N, ξ . In the case that $\sigma_0 \leq 1$, the bounds (8) are proved even more generally than in Igusa's original conjecture in [Cluckers et al. 2019]; see Section 3.2 below. Furthermore, precisely (and only) in the case that $\sigma_0 \leq 1$ holds, the value σ_0 is independent of the choice of h , and, is called the log canonical threshold of f .

When one takes a fixed prime number p , Igusa [1978] proves that inequality (8) holds for N of the form $N = p^m$ with $m \geq 1$, primitive N -th root of unity ξ and a constant $c = c_p$ depending on p and $\sigma < \sigma_0$ (but not on m, ξ).

When f satisfies the nondegeneracy condition of Section 4, there is a toric log resolution h of D related to the Newton polyhedron of f at zero. In this case, $\sigma_0(h) = \sigma_f$ with σ_f defined again from the Newton polyhedron (see Section 4 for the definition of σ_f). Denef and Sperber [2001] conjectured that when f is nondegenerate, one can replace inequality (8) in Igusa's conjecture by

$$E_f(p^m, \xi) \leq cm^{\kappa-1} p^{-\sigma_f m}, \tag{9}$$

where κ is an invariant coming from the Newton polyhedron of f at zero (see Section 4) and c is independent of p, m, ξ . Thus, the Denef–Sperber conjecture is a bit stronger than Igusa's conjecture in the case of nondegenerate polynomials, by the more explicit form of the exponents σ_f and κ ; it has been proved and generalized in [Denef and Sperber 2001; Cluckers 2008a; 2010; Castryck and Nguyen 2019]. See Proposition 4.1 below.

In [Cluckers and Veys 2016], some of Igusa's original conditions, like homogeneity for f , were dropped with some care, namely by focusing on squareful integers; see Section 2.7 for more details. Igusa's condition that $\sigma_0(h) > 2$ for some h was already dropped before; it was more relevant for his intended application of his conjecture to local-global principles than for the content of the conjecture itself. Additionally, Igusa's noncanonical exponent $\sigma_0(h)$ was replaced by canonical candidates for the exponent: the motivic oscillation index of f , and, (expected to be equal) the minimal exponent of $f - v$ with v a well-chosen critical value of $f : \mathbb{C}^n \rightarrow \mathbb{C}$, see [Cluckers and Veys 2016; Cluckers et al. 2019] and Section 2.7 below. Our suggested bounds encompass several issues related to the minimal exponent (and, the motivic oscillation index), by replacing them by the much simpler and natural value $(n - s)/d$, yielding Conjecture 1 as new variant of (8). As an extra upshot, Conjecture 1 makes sense again for all positive integers N , and not only for squareful integers. Although the bounds from Conjecture 1 seem simple and very natural, they appear surprisingly hard to show in general, and even the much weaker

bounds with constants depending on p and N running over powers of p as in (15), remain elusive in general up to date, even in the case with $s = 0$.

1.7. From Section 3 on we develop evidence for Conjecture 1. We first rephrase some well-known results as evidence, namely, Igusa's treatment of the smooth homogeneous case (with $f = f_d$ and $s = 0$), the case with degree $d = 2$, the case with $(n - s)/d \leq 1$, the case of at most 3 variables, and, the case with cube free N . We then generalize this further to new evidence for all N which are $(d+2)$ -th power free (see Section 3.6). This treatment of the $(d+2)$ -th power free case is mainly provided for expository reasons, as it uses some recent results on bounds of [Cluckers et al. 2016] in the context of motivic integration and uniform p -adic integration as in [Cluckers et al. 2018]; it indicates that the case $N = p^e$ with p prime and e small is generally more easy than with e large.

In Section 4, we show Conjecture 1 when f is nondegenerate with respect to its Newton polyhedron at zero, using recent work from [Castricky and Nguyen 2019] and some elementary reasoning on Newton polyhedrons. This shows that Conjecture 1 holds under often generic conditions, including the generic weighted homogeneous case, see Remark 4.5.

In Section 5, we show Conjecture 1 for all polynomial in up to 4 variables. This uses [Cluckers et al. 2019] to reduce to the case with $n = 4$, $d = 3$ and $s = 0$.

In our final Remark 5.7, we explain that throughout the evidence for Conjecture 1 of this paper, up to excluding a finite set S of primes divisors of N (depending on f), the constant c can be taken depending only on d , n and ε .

Let us finally mention the further evidence of [Nguyen 2021] for Conjecture 1, with the weakened exponent $(n - s)/2(d - 1)$ in the upper bound of (4) instead of $(n - s)/d$.

1.8. In his vast program from [Igusa 1978], Igusa studies a certain adèlic Poisson summation formula related to f , inspired by Weil's work [1965] on the Hasse principle and Birch's work [1962] on more general local-global principles. Conjecture 1 would imply that Igusa's adèlic Poisson summation formula for f holds under the simple condition

$$n - s > 2d \tag{10}$$

which simplifies (and generalizes) the list of conditions put forward by Igusa [1978], and would drop in particular the condition of homogeneity on f .

1.9. Also for obtaining (or just for streamlining) local-global principles, Conjecture 1 may play a role. When f is homogeneous, the sums $E_f(N, \xi)$ appear for estimating the contribution of the major arcs in the circle method to get a local-global principle for f when

$$n - s > (d - 1)2^d, \tag{11}$$

in work by Birch [1962] and in the recent sharpening from [Browning and Prendiville 2017], which both rely on Birch's bounds (5) quoted above. Birch shows that any homogeneous form $f = f_d$ with (11) and having smooth local zeros for each completion of \mathbb{Q} automatically has a nontrivial rational zero. One may

hope one day to replace Condition (11) on homogeneous f by (10), which is in line with a conjectured local-global principle from [Browning and Heath-Brown 2017]. Conjecture 1 would put the remaining obstacle completely with the estimation for the minor arcs (where actually already lie the limits of the current strategies). Other possible applications may be for small solutions of congruences as studied in, e.g., [Baker 1983].

Note that Birch’s method [1962] also helps to understand the distribution of rational points in the projective hypersurface X associated with a homogeneous polynomial f . More precisely, the singular series

$$\mathfrak{S}(f) = \sum_{N \geq 1} N^{-n} \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^n} \exp\left(\frac{2\pi i a f(x)}{N}\right)$$

which is equal to the product of p -adic densities of f will contribute to the dominant term in the asymptotic formula of the number points of X of bounded height. Conjecture 1 implies that the singular series $\mathfrak{S}(f)$ is absolutely convergent if $n - s > 2d$. Thereby Conjecture 1 may be useful for the future research on the distribution of rational points in algebraic hypersurfaces.

1.10. Generalization to a ring of integers. Before giving precise statements and proofs, we formulate a natural generalization to rings of integers (a generality we will not use later in this paper). For a ring of integers \mathcal{O} of a number field and a polynomial g over \mathcal{O} , one can formulate an analogous conjecture with summation sets $(\mathcal{O}/I)^n$ with nonzero ideals I of \mathcal{O} and primitive additive characters $\psi : \mathcal{O}/I \rightarrow \mathbb{C}^\times$. More precisely, let g be a polynomial in n variables of degree $d > 1$ and with coefficients in \mathcal{O} . For any nonzero ideal I of \mathcal{O} and any primitive additive character $\psi : \mathcal{O}/I \rightarrow \mathbb{C}^\times$, let $N_I := [\mathcal{O} : I]$ be the absolute norm of I and consider

$$E_g(I, \psi) := \left| \frac{1}{N_I^n} \sum_{x \in (\mathcal{O}/I)^n} \psi(g(x)) \right|. \tag{12}$$

Write s for the dimension of the critical locus of the degree d homogeneous part g_d of g . As a generalization of the above questions, one may wonder whether for each $\varepsilon > 0$ (or more strongly with $\varepsilon = 0$) one has

$$E_g(I, \psi) \ll N_I^{-\frac{n-s}{d} + \varepsilon}. \tag{13}$$

As above with the Chinese remainder theorem, one can rephrase this using the finite completions of the field of fractions of \mathcal{O} . Furthermore, one can study similar sums for the local fields $\mathbb{F}_q((t))$ (with similar methods in the large characteristic case); see, e.g., [Cluckers and Veys 2016, Section 2.6; Cluckers et al. 2019, Section 1.2].

2. Link with the monodromy conjecture

2.1. Fix a prime number p . For each integer $m \geq 0$ let $a_{p,m}$ be the number of solutions in $(\mathbb{Z}/p^m\mathbb{Z})^n$ of the equation $f(x) \equiv 0 \pmod{p^m}$, and consider the Poincaré series

$$P_{f,p}(T) := \sum_{m \geq 0} \frac{a_{p,m}}{p^{mn}} T^m,$$

in $\mathbb{Z}[[T]]$. Igusa [1975; 1978] showed that $P_{f,p}(T)$ is a rational function in T , using a log resolution of $f^{-1}(0)$. Let T_0 be a complex pole of $P_{f,p}(T)$ and let t_0 be the real part of a complex number s_0 with $p^{-s_0} = T_0$. Let $h : Y \rightarrow \mathbb{A}_{\mathbb{Q}}^n$ be a log resolution of $f^{-1}(0)$ and $(N_i, v_i)_{i \in I}$ as in Section 3. Igusa [1978] showed that t_0 belongs to the set $\mathcal{P}_h = \{-v_i/N_i \mid i \in I\}$. However, \mathcal{P}_h depends on the choice of log resolution h . Igusa [1975, Theorem 2] also showed a strong link between exponential sums and local zeta functions (see also [Denef 1991, Corollary 1.4.5; [Denef and Veys 1995, Proposition 2.7]]), yielding the following corollary.

Corollary 2.2 [Igusa 1975]. *For p, f, T_0 and t_0 as above, if T_0 is furthermore a pole of $(T - p)P_{f,p}(T)$, then*

$$p^{mt_0} \leq c'_p E_f(p^m, \xi) \tag{14}$$

for infinitely many m and ξ and a constant c'_p independent of m, ξ .

Proof. Proposition 2.7 of [Denef and Veys 1995] gives finitely many complex numbers T , finitely many characters $\chi : \mathbb{C}^\times \rightarrow \mathbb{C}^\times$ of finite order, finitely many integers $b \geq 0$, and finitely many complex numbers c such that for large m , $E_f(p^m, \xi)$ is (the complex modulus of) a finite \mathbb{C} -linear combination of the terms

$$\chi(\xi) \cdot T^{-m} m^b \xi^c,$$

where furthermore a term with $T = T_0$ appears nontrivially in this linear combination for each pole T_0 of $(T - p)P_{f,p}(T)$. Now the corollary follows by looking at the dominant terms, namely, with largest occurring real part of T and for such T the largest occurring value for b . □

Denef [1991] formulated a strong variant of Igusa’s monodromy conjecture by asking whether t_0 as above is automatically a zero of the Bernstein–Sato polynomial of f . The following result addresses this question in a range of values for t_0 , namely strictly between $-(n - s)/d$ and zero, assuming Conjecture 1 for f .

Proposition 2.3 (strong monodromy conjecture, in a range). *Let f, n, s , and d be as in the introduction and suppose that Conjecture 1 holds for f . Let t_0 be coming as above from a pole T_0 of $P_{f,p}(T)$ for a prime number p . Suppose that moreover $t_0 > -(n - s)/d$. Then $t_0 = -1$, and hence, t_0 is a zero of the Bernstein–Sato polynomial of f .*

Proposition 2.3 is a form of the strong monodromy conjecture in the range strictly between $-(n - s)/d$ and zero. We don’t pursue the highest generality here, and leave the generalization for other variants of zeta functions like twisted p -adic local zeta (or even motivic) functions to the reader. Proposition 2.4 below gives a related statement for the zeros of the Bernstein–Sato polynomial of f .

Proof of Proposition 2.3. Let p be a prime number. Let t_0 be the real part of a complex number s_0 such that $T_0 := p^{-s_0}$ is a pole of $P_{f,p}(T)$. Suppose that for all $\varepsilon > 0$ there exists $c_p = c_p(f, \varepsilon)$ such that

$$E_f(p^m, \xi) \leq c_p \cdot (p^m)^{-\frac{n-s}{d} + \varepsilon} \text{ for all } m > 0 \text{ and all primitive } \xi. \tag{15}$$

By Corollary 2.2 it follows that t_0 either equals -1 or, one has

$$p^{mt_0} \leq c'_p E_f(p^m, \xi) \tag{16}$$

for infinitely many pairs (m, ξ) and a constant c'_p independent of m, ξ . Clearly the bound from (15) holds if Conjecture 1 holds for f . By (15) and (16), if $t_0 > -(n - s)/d$, then $t_0 = -1$. Since f is nonconstant, the value -1 is automatically a zero of the Bernstein–Sato polynomial of f . This completes the proof of the proposition. \square

Showing the bounds (15) from the above proof for general f does not seem easy, although they are much weaker (and much less useful adelicly) than the bounds from (4), because of the dependence of c_p on p .

In view of the strong monodromy conjecture, Proposition 2.3 should be compared with the following absence of zeros of the Bernstein–Sato polynomial in a similar range, apart from -1 . Recall that the zeros of the Bernstein–Sato polynomial are negative rational numbers.

Proposition 2.4. *Let $f, n, s,$ and d be as in the introduction and let r be any zero of the Bernstein–Sato polynomial of f . Then either $r = -1$, or, $r \leq -(n - s)/d$.*

Proof. We write $f = f_0 + \dots + f_d$ with f_i is the homogeneous part of degree i of f . Item (3) of Theorem E of [Mustařă and Popa 2020] states that the minimal exponent $\tilde{\alpha}_{f,0}$ of f at zero is at least $(n - s)/d$ if f is homogeneous. Recall that the minimal exponent $\tilde{\alpha}_f$ of f is equal to $\min_{x \in f^{-1}(0)} \tilde{\alpha}_{f,x}$. Moreover, if $\varphi : \mathbb{A}^n \rightarrow \mathbb{A}^n$ is a linear change of variables then $\tilde{\alpha}_{f,x} = \tilde{\alpha}_{f \circ \varphi, \varphi^{-1}(x)}$, and, for any constant $\beta \neq 0$ one has $\tilde{\alpha}_{f,x} = \tilde{\alpha}_{\beta f,x}$. Let $g_\lambda(x)$ be $f_d + \sum_{0 \leq i \leq d-1} \lambda^{d-i} f_i$. Then for each $\lambda \neq 0$ we have $g_\lambda(x) = \lambda^d f(x/\lambda)$. Write $X = \mathbb{A}^n \times \mathbb{A}^1, T = \mathbb{A}^1, \pi : \mathbb{A}^n \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ for the projection, $h(x, \lambda) = g_\lambda(x)$ and $D = h^{-1}(0)$. For each $x \in f^{-1}(0)$, we consider the section $s_x : T \rightarrow X$ with $\lambda \mapsto (\lambda x, \lambda)$, then $s_x(\lambda) \in D_\lambda$ since $h(\lambda x, \lambda) = g_\lambda(\lambda x) = \lambda^d f(\lambda x/\lambda) = 0$ if $\lambda \neq 0$ and $h(0, 0) = f_d(0) = 0$. Now we can use item (2) of Theorem E of [Mustařă and Popa 2020] for X, T, π, D and s_x to see that for each $x \in f^{-1}(0)$ we have

$$\tilde{\alpha}_{f,x} = \tilde{\alpha}_{g_\lambda, \lambda x} \geq \tilde{\alpha}_{g_0, 0} = \tilde{\alpha}_{f_d, 0} \geq (n - s)/d$$

for all $\lambda \neq 0$ in a small enough neighborhood of 0. Thus,

$$\tilde{\alpha}_f = \min_{x \in f^{-1}(0)} \tilde{\alpha}_{f,x} \geq (n - s)/d.$$

The proposition now follows directly from the definition of the minimal exponent $\tilde{\alpha}_f$ of f as the smallest zero of $b_f(-s)/(s - 1)$, where $b_f(s)$ is the Bernstein–Sato polynomial of f . \square

2.5. The variant of Conjecture 1 with $\varepsilon = 0$ (or even just the bounds (15) with $\varepsilon = 0$) implies for any pole T_0 of $P_{f,p}(T)$ with corresponding value t_0 the following bound on the order of the pole: If t_0 equals $-(n - s)/d$ and $-(n - s)/d \neq -1$, then the pole T_0 has multiplicity at most one, and, if $t_0 = -1 = -(n - s)/d$, then the pole T_0 has multiplicity at most two, by a similar reasoning as for Corollary 2.2.

Remark 2.6. Conjecture 1 implies the bounds (15) with moreover constants c_p taken independently from p , and, the conjecture in turns would follow from this. By (7), the variant of Conjecture 1 with $\varepsilon = 0$ is equivalent with the bounds (15) with $\varepsilon = 0$ and such that furthermore the products of the constants c_p over any set P of primes is bounded independently of P .

2.7. The minimal exponent of f is defined as the smallest zero of the quotient $b_f(-s)/(s - 1)$ with $b_f(s)$ the Bernstein–Sato polynomial of f if such a zero exists, and it is defined as $+\infty$ otherwise. Write $\hat{\alpha}_f$ for the minimum of the minimal exponents of $f - v$ for v running over the (complex) critical values of f . In a more canonical variant of Igusa’s original question, one may wonder more technically than Conjecture 1 whether for all $\varepsilon > 0$ one has

$$E_f(N, \xi) \ll N^{-\hat{\alpha}_f + \varepsilon} \text{ for all } \xi \text{ and all squareful integers } N, \tag{17}$$

similarly as the question introduced in [Cluckers and Veys 2016] for the motivic oscillation index (and where the necessity of working with squareful integers N is explained). Recall that an integer N is called squareful if for any prime p dividing N also p^2 divides N . In [Castricky and Nguyen 2019; Chambille and Nguyen 2021; Cluckers 2008a; 2010; 2019; Denef and Sperber 2001], evidence is given for this sharper but more technical question. As mentioned above, $\hat{\alpha}_f$ is hard to compute in general, and $(n - s)/d$ is much more transparent. However, $\hat{\alpha}_f$ is supposedly equal to the motivic oscillation index of f , which in turn is optimal as exponent of N^{-1} in the upper bounds for $E_f(N, \xi)$ for squareful N (see the last section of [Cluckers et al. 2019], or, a reasoning as for Corollary 2.2). Note that by Proposition 2.4, one has

$$\hat{\alpha}_f \geq (n - s)/d, \tag{18}$$

which shows that (17) is indeed a sharper (or equally sharp) bound than (4).

3. Some first evidence

In this section we translate some well-known results into evidence for Conjecture 1, and we show the (new) case of $(d+2)$ -th power free N . A key (but hard) case of Conjecture 1 for inhomogeneous f is when f_d is projectively smooth, namely with $s = 0$, since the case of general s can be derived from a sufficiently uniform form of the inhomogeneous case with $s = 0$, see, e.g., how (22) is used below for squarefree N . However, the inhomogeneous case with $s = 0$ seems very hard at the moment. This should not be confused with Igusa’s more basic case recalled in Section 3.1, for homogeneous f with $s = 0$.

3.1. When f itself is smooth homogeneous, namely, $f = f_d$ and $s = 0$, then Conjecture 1 with $\varepsilon = 0$ is known by Igusa’s bounds [1978] by a straightforward computation and reduction to Deligne’s bounds. In detail, if $f = f_d$ and $s = 0$, Igusa [1978] showed (using [Deligne 1974]) that for each prime p there is a constant c_p such that

$$E_f(p^m, \xi) \leq c_p p^{-mn/d} \text{ for all integers } m > 0 \text{ and all choices of } \xi, \tag{19}$$

and, that one can take $c_p = 1$ when p is larger than some value M depending on f . More precisely, one can take $c_p = 1$ when p does not divide d and when the reduction of f modulo p is smooth. Furthermore, Igusa [1978] shows that the exponent $-n/d$ of p^m is optimal in the upper bound of (19) when $m = d$. This easily shows that the exponent $(n - s)/d$ is optimal in Conjecture 1, for example by taking

$$f = (x_1 + \cdots + x_{s+1})^d + x_{s+2}^d + \cdots + x_n^d \quad (20)$$

and $N = p^d$ for all prime numbers p .

3.2. When f is such that

$$(n - s)/d \leq 1, \quad (21)$$

then Conjecture 1 follows from [Cluckers et al. 2019] and its recent solution of Igusa's conjecture for nonrational singularities. Indeed, in [Cluckers et al. 2019] the stronger (and optimal) upper bounds from (17) are shown for all squareful N in the case of nonrational singularities, as well as the case with 1 in the exponent instead of $\hat{\alpha}_f$ in the case of rational singularities. Recall that this is indeed stronger, by (18). The bounds for those integers N that are not squareful are recovered by the treatment of squarefree N below, by writing a general integer as a product of a squareful and a squarefree integer. We mention on the side that $\hat{\alpha}_f \leq 1$ if and only if $f - v = 0$ has nonrational singularities for some critical value $v \in \mathbb{C}$ of f , by [Saito 1993] and that in this case $\hat{\alpha}_f$ equals the minimum of the log canonical thresholds of $f - v$ for v running over the (complex) critical values of f . These results under condition (21) imply that Conjecture 1 holds for all f in three (or less) variables. Indeed, the degree two case is easy by diagonalizing f_2 over \mathbb{Q} , and, (21) holds when $n \leq 3 \leq d$. More surprisingly, Igusa's Conjecture (with the motivic oscillation index in the upper bound) is proved recently in [Nguyen and Veys 2022] for all polynomials in 3 variables. Some related results of the special case with $n \leq 2$ are developed in [Fraser and Wright 2020; Lichtin 2013; Veys 2020]. In Section 5 we will prove that Conjecture 1 holds for all polynomials in up to four variables.

3.3. Although it is classical, let us explain the case of $d = 2$ in more detail, by showing that Conjecture 1 holds with $\varepsilon = 0$ for f of degree $d = 2$. In fact, the argument as in the proof of Lemma 25 of [Heath-Brown 1996] is shorter and simpler for the case $d = 2$, but our treatment will be useful later in this paper. First suppose that the degree two part of f is a diagonal form, namely, $f_2(x) = \sum_{i=1}^n a_i x_i^2$ for some $a_i \in \mathbb{Z}$. In this case it is sufficient to show the case with $n = 1$ and $d = 2$ (indeed, $f = h_1(x_1) + \cdots + h_n(x_n)$ for some polynomials h_j in one variable x_j and of degree ≤ 2). But this case follows readily from Hua's bounds, see [Hua 1959] or [Chalk 1987], and is in fact elementary.

For general f with $d = 2$, by diagonalizing f_2 over \mathbb{Q} and taking a suitable integer multiple, we find a matrix $T \in \mathbb{Z}^{n \times n}$ with nonzero determinant so that $f_2(Tx)$ is a diagonal form over \mathbb{Z} in the variables x , namely, $f_2(Tx) = \sum_{i=1}^n a_i x_i^2$ for some $a_i \in \mathbb{Z}$. The map sending x to Tx transforms \mathbb{Z}_p^n into a set of the form $\prod_{j=1}^n p^{e_{p,j}} \mathbb{Z}_p$ for some integers $e_{p,j} \geq 0$ (called a box). By composing with a map of the form

$(x_j)_j \mapsto (b_j x_j)_j$ for some integers b_j it is clear that we may assume that T is already such that $e_{p,j} = e_p$ for all p and all j and some integers $e_p \geq 0$. Hence, the case $d = 2$ follows from Lemma 3.4.

Lemma 3.4. *Let $f, n, s,$ and d be as in the introduction and let $T \in \mathbb{Z}^{n \times n}$ be a matrix with nonzero determinant and such that, for each prime p , the transformation $x \mapsto Tx$ maps \mathbb{Z}_p^n onto $p^{e_p} \mathbb{Z}_p^n$ for some $e_p \geq 0$. Then, Conjecture 1 for each of the polynomials $g_i(x) := f(i + Tx)$ for $i \in \mathbb{Z}^n$ implies Conjecture 1 for f , and, similarly for Conjecture 1 with $\varepsilon = 0$.*

Proof. For each $i \in \mathbb{Z}^n$, write $g_i(x)$ for the polynomial $f(i + Tx)$. For any prime p , let $\mu_{p,n}$ be the Haar measure on \mathbb{Q}_p^n , normalized so that \mathbb{Z}_p^n has measure 1. For any integer $m > 0$ and any primitive p^m -th root of unity ξ , we have, by the change of variables formula for p -adic integrals, and with $e = e_p$ and with integrals taken against the measure $\mu = \mu_{p,n}$,

$$E_f(p^m, \xi) = \left| \int_{x \in \mathbb{Z}_p^n} \xi^{f(x) \bmod p^m} \mu \right| \leq \sum_{j=1}^n \sum_{i_j=0}^{p^e-1} \left| \int_{x \in i + (p^e \mathbb{Z}_p)^n} \xi^{f(x) \bmod p^m} \mu \right|.$$

For each i we further have

$$\begin{aligned} \left| \int_{x \in i + (p^e \mathbb{Z}_p)^n} \xi^{f(x) \bmod p^m} \mu \right| &= p^{-ne} \left| \int_{x \in \mathbb{Z}_p^n} \xi^{f(i + p^e x) \bmod p^m} \mu \right| \\ &= p^{-ne} \left| \int_{x \in \mathbb{Z}_p^n} \xi^{g_i(x) \bmod p^m} \mu \right| = p^{-ne} E_{g_i}(p^m, \xi). \end{aligned}$$

Since $e_p = 0$ for all but finitely many primes p , we are done. □

3.5. When one restricts to integers N which are squarefree (namely, not divisible by a nontrivial square), then Conjecture 1 with $\varepsilon = 0$ follows from Deligne's bound [1974], as we now explain. The reasoning is classical but also instructive for later use in this paper; a similar induction argument on $s \geq 0$ already appears in [Hooley 1991]. By [Deligne 1974], for each prime number p such that the reduction of f_d modulo p is smooth, one has

$$E_f(p, \xi) \leq (d - 1)^n p^{-n/2} \text{ for each primitive } p\text{-th root of unity } \xi, \tag{22}$$

where smooth means that the reduction modulo p of the equations (3) has 0 as the only solution over an algebraic closure of \mathbb{F}_p . If $s = 0$ then the reduction of f_d modulo p is smooth whenever p is large and thus Conjecture 1 for squarefree N and with $\varepsilon = 0$ follows for f with $s = 0$ (note the different exponent of p in (22) and of N in 1 when $d > 2$). We proceed by induction on s by restricting f to hyperplanes, as follows. The bound (22) for all large p is our base case when $s = 0$. Now suppose that $s > 0$. After a linear coordinate change of $\mathbb{A}_{\mathbb{Z}}^n$, we may suppose that the polynomial $g(\hat{x}) := f(0, \hat{x})$ in the variables $\hat{x} = (x_2, \dots, x_n)$ is still of degree d and that its degree d homogeneous part g_d has critical locus of dimension $s - 1$. Hence, for large prime p , the reduction of g_d modulo p has also critical locus

of dimension $s - 1$, in $\mathbb{A}_{\mathbb{F}_p}^{n-1}$. Hence, for large p , one has by induction on s that

$$\left| \frac{1}{p^{n-1}} \sum_{\hat{x}=(x_2, \dots, x_n) \in \mathbb{F}_p^{n-1}} \xi^{f(a, \hat{x})} \right| \leq (d - 1)^{n-s} p^{-(n-s)/2} \tag{23}$$

for each $a \in \mathbb{F}_p$ and each primitive p -th root of unity ξ . Indeed, the polynomial $f(a, \hat{x})$ has $g_d \pmod p$ as its degree d homogeneous part for each $a \in \mathbb{F}_p$. Now, summing over $a \in \mathbb{F}_p$ and dividing by p gives

$$\left| \frac{1}{p^n} \sum_{x \in \mathbb{F}_p^n} \xi^{f(x)} \right| \leq (d - 1)^{n-s} p^{-(n-s)/2} \tag{24}$$

for large p (coming from the condition that the reduction of g_d modulo p has critical locus of dimension $s - 1$). Conjecture 1 with $\varepsilon = 0$ for squarefree integers N thus follows, by comparing the exponents in the upper bounds of (24) and (4), which allows to swallow the constant $(d - 1)^{n-s}$ when $d > 2$. (Alternatively, one can use the much more general Theorem 5 of [Katz 1999] when $d > 2$ and an argument as in Section 3.3 when $d = 2$.)

3.6. When one restricts to integers N which are cube free (namely, not divisible by a nontrivial cube), then Conjecture 1 with $\varepsilon = 0$ follows exactly in the same way as for squarefree N , but now using both the bounds from [Heath-Brown 1985] and from [Deligne 1974]. Indeed, this similarly gives

$$E_f(p^2, \xi) \leq (d - 1)^{n-s} p^{-(n-s)} \tag{25}$$

for large p and all ξ . Together with the squarefree case, this implies the cube free case of Conjecture 1, with $\varepsilon = 0$ (note again the different exponent of p^2 in (25) and of N in 1, when $d > 2$). In fact, with some more work we can go up to $(d+2)$ -th powers instead of just cubes, as follows.

Proposition 3.7. *Conjecture 1 with $\varepsilon = 0$ holds when restricted to integers N which are not divisible by a nontrivial $(d+2)$ -th power. In detail, let f, n, s , and d be as in the introduction. Then there is a constant $c = c(f_d)$ (depending only on f_d) such that for all integers $N > 0$ which are not divisible by a nontrivial $(d+2)$ -th power and all primitive N -th roots ξ of 1, one has*

$$E_f(N, \xi) \leq cN^{-(n-s)/d}. \tag{26}$$

We will prove Proposition 3.7 by making a link between $E_f(p^m, \xi)$ and finite field exponential sums, as follows. For any prime p , any $m > 0$, any point P in \mathbb{F}_p^n and any ξ , write

$$E_f^P(p^m, \xi) := \left| \frac{1}{p^{mn}} \sum_{x \in P + (p\mathbb{Z}/p^m\mathbb{Z})^n} \xi^{f(x)} \right|. \tag{27}$$

Compared to $E_f(p^m, \xi)$, the summation set for $E_f^P(p^m, \xi)$ has p -adically zoomed in around the point P .

Let us consider the following positive characteristic analogues,

$$E_f(t^m, \psi) := \left| \frac{1}{p^{mn}} \sum_{x \in (\mathbb{F}_p[t]/(t^m))^n} \psi(f(x)) \right|, \tag{28}$$

and

$$E_f^P(t^m, \psi) := \left| \frac{1}{p^{mn}} \sum_{x \in P+(t\mathbb{F}_p[t]/(t^m))^n} \psi(f(x)) \right|, \tag{29}$$

for any primitive additive character

$$\psi : \mathbb{F}_p[t]/(t^m) \rightarrow \mathbb{C}^\times,$$

where primitive means that ψ does not factor through the projection

$$\mathbb{F}_p[t]/(t^m) \rightarrow \mathbb{F}_p[t]/(t^{m-1}).$$

The sums of (28), resp. (29), can be rewritten as finite field exponential sums, to which classical bounds like (24) apply. This is done by identifying the summation set with \mathbb{F}_p^{mn} , resp. with $\mathbb{F}_p^{(m-1)n}$, namely by sending a polynomial in t to its coefficients, while forgetting the constant terms in the second case.

We first prove the following variant of Proposition 3.7.

Proposition 3.8. *Let f , n , s , and d be as in the introduction. Then there is a constant M (depending only on f_d) such that for all primes p with $p > M$, all integers $m > 0$ with $m \leq d + 1$, and all primitive additive characters $\psi : \mathbb{F}_p[t]/(t^m) \rightarrow \mathbb{C}^\times$ one has*

$$E_f(t^m, \psi) \leq p^{-m \cdot \frac{n-s}{d}}. \tag{30}$$

Proof. By a reasoning as for the squarefree case, it is sufficient to treat the case with $s = 0$ for large p , while letting the f_i for $i < d$ vary over homogeneous polynomials in $\mathbb{F}_p[t, x]$ of degree i in x , and while keeping f_d fixed in $\mathbb{Z}[x]$. So, we may assume that $s = 0$, and, by the squarefree case treated above, that $m > 1$. We also may assume that $d \geq 3$ by the above treatment of the case $d = 2$. For each p , let C_p be the set of critical points of the reduction of f modulo p . Since $s = 0$, one has $\#C_p \leq c_1$ for some constant c_1 depending only on n and d , see for example the final inequality of [Heath-Brown 1985], or Lemma 5.3 below. Clearly we have

$$E_f(t^m, \psi) = \sum_{P \in C_p} E_f^P(t^m, \psi) \tag{31}$$

for all primes $p > d$, all $m > 1$ and all primitive $\psi : \mathbb{F}_p[t]/(t^m) \rightarrow \mathbb{C}^\times$. For $m < d$, note that

$$\frac{1}{p^{mn}} \cdot \#(t\mathbb{F}[t]/(t^m))^n = p^{n(m-1)-mn} = p^{-n} < p^{-mn/d}. \tag{32}$$

For $m < d$, we thus find by (31) that

$$E_f(t^m, \psi) \leq c_1 p^{-n}, \tag{33}$$

and (30) follows when $m < d$ and p is large enough so that the constant c_1 is eaten by the extra saved power of p coming from (32). We now treat the case that $m = d$. If $p > d$ is such that the reduction of f modulo t is smooth homogeneous of degree d , then $C_p = \{0\}$ and there is nothing left to prove since then $E_f(t^m, \psi) = E_f^{P_0}(t^m, \psi) \leq p^{-n} = p^{-mn/d}$, with $P_0 = \{0\}$. If the reduction of f modulo t is not

homogeneous of degree d , and, p is larger than d , then there is a constant c_2 (depending only on n and d) such that

$$E_f^P(t^m, \psi) \leq c_2 p^{-n-1/2} \tag{34}$$

for all P in C_p and all primitive ψ . Indeed, this follows from the worst case of (23) applied to $E_f^P(t^m, \psi)$, after rewriting it as a finite field exponential sum as explained just above the proposition. The case $m = d$ for (30) follows, where the constant c_2 is eaten by the extra saved power of p coming from $d \geq 3$ and (34). For $d = m + 1$, when we rewrite $E_f^P(t^m, \psi)$ for $P \in C_p$ as a finite field exponential sum over $(m - 1)n$ variables running over \mathbb{F}_p , we can again apply (23), now in $n(m - 1)$ variables and with highest degree part f_d which has singular locus of dimension $n(m - 2)$ inside $\mathbb{A}^{n(m-1)}$. Since $d \geq 3$, we again can use the extra saved power of p to obtain (30) and the proposition is proved. \square

The transfer principle for bounds from Theorem 3.1 of [Cluckers et al. 2016] can be applied to compare the exponential sums $E_f(p^m, \xi)$ and $E_f(t^m, \psi)$; we will use it in the following basic form. Recall that a Presburger subset A of \mathbb{N} is a Boolean combination of congruence classes and subintervals of \mathbb{N} .

Corollary 3.9 [Cluckers et al. 2016]. *Let g be a homogeneous polynomial over \mathbb{Z} , of degree $d > 1$ and in n variables. Consider a real number $\sigma > 0$ and a Presburger subset $A \subset \mathbb{N}$. Then the following two statements are equivalent.*

(1) *There exist constants c and M such that for all primes $p > M$ and all polynomials f in $\mathbb{Z}_p[x_1, \dots, x_n]$ of degree d and with $f_d = g$, one has*

$$E_f(p^m, \xi) \leq c p^{-\sigma m}$$

for all $m \in A$ and all primitive p^m -th roots of unity ξ .

(2) *There exist constants c' and M' such that for all primes $p > M'$ and all polynomials f in $\mathbb{F}_p[[t]][x_1, \dots, x_n]$ of degree d and such that $f_d = g \pmod{p}$ holds in $\mathbb{F}_p[x]$, one has*

$$E_f(t^m, \psi) \leq c' p^{-\sigma m}$$

for all $m \in A$ and all primitive characters $\psi : \mathbb{F}_p[t]/(t^m) \rightarrow \mathbb{C}^\times$.

In the corollary, we have extended the notation E_f to more general f , namely with more general coefficients than just in \mathbb{Z} , in the obvious way. More generally than Corollary 3.9, Theorem 3.1 of [Cluckers et al. 2016] allows to transfer bounds that hold for motivic families of functions, and, the families in the corollary are a special case of such family.

Proof of Corollary 3.9. Clearly the left hand sides and of the right hand sides of the inequalities come from motivic functions H and G as required in Theorem 3.1 of [Cluckers et al. 2016]. Now the corollary readily follows from the conclusion of Theorem 3.1 of [Cluckers et al. 2016] for such H and G . \square

Proof of Proposition 3.7. We show for all large primes p , all integers $m > 0$ with $m \leq d + 1$, and all primitive p^m -th roots of 1, that

$$E_f(p^m, \xi) \leq p^{-m \cdot \frac{n-s}{d}}, \tag{35}$$

where moreover the lower bound on p depends only on f_d . The case $d = 2$ is already shown. For $m \neq d \geq 3$ this follows at once from Corollary 3.9 and the corresponding extra power savings when $m \neq d$ in the proof of Proposition 3.8. Indeed, the transfer principle holds uniformly in f as long as f_d and n are fixed, since these bounds (with the extra power savings) from the proof of Proposition 3.8 depend only on f_d and n . Let us now treat the remaining case of $m = d$. It is again enough to treat the case $s = 0$. For a prime $p > d$ such that the reduction of f modulo p is not smooth homogeneous of degree d , we are done similarly by the transfer principle for bounds from [Cluckers et al. 2016] and the corresponding power savings in the proof Proposition 3.8. If $m = d$ and $p > d$ is such that the reduction of f modulo p is smooth homogeneous of degree d , then we have that $P_0 = \{0\}$ is the unique critical point of the reduction of f modulo p , and thus

$$E_f(p^m, \xi) = E_f^{P_0}(p^m, \xi) \leq p^{-n} = p^{-mn/d}.$$

The proof of Proposition 3.7 is thus finished. □

4. The nondegenerate case

In this section we show that Conjecture 1 with $\varepsilon = 0$ holds for nondegenerate polynomials, where the nondegeneracy condition is with respect to the Newton polyhedron of $f - f(0)$ at zero as in [Castrick and Nguyen 2019] (which is slightly different than the nondegeneracy notion of [Kouchnirenko 1976; Varčenko 1976]). The nondegeneracy condition generalizes the situation where f is a sum of monomials in separate variables, like $x_1x_2 + x_3x_4$. In detail, writing $f(x) - f(0) = \sum_{i \in \mathbb{N}^n} \beta_i x^i$ in multi-index notation, let

$$\text{Supp}_f := \{i \in \mathbb{N}^n \mid \beta_i \neq 0\}$$

be the support of $f - f(0)$. Further, let

$$\Delta_0(f) := \text{Conv}(\text{Supp}_f + (\mathbb{R}_{\geq 0})^n)$$

be the convex hull of the sum-set of Supp_f with $(\mathbb{R}_{\geq 0})^n$ where $\mathbb{R}_{\geq 0}$ is $\{x \in \mathbb{R} \mid x \geq 0\}$. The set $\Delta_0(f)$ is called the Newton polyhedron of $f - f(0)$ at zero. Let σ_f be the unique real value such that $(1/\sigma_f, \dots, 1/\sigma_f)$ is contained in a proper face of $\Delta_0(f)$. Further, let κ denote the maximal codimension in \mathbb{R}^n of τ when τ varies over the faces of $\Delta_0(f)$ containing $(1/\sigma_f, \dots, 1/\sigma_f)$. For each face τ of the polyhedron $\Delta_0(f)$, consider the polynomial

$$f_\tau := \sum_{i \in \tau} \beta_i x^i.$$

Call f nondegenerate with respect to $\Delta_0(f)$ when for each face τ of $\Delta_0(f)$ and each critical point P of $f_\tau : \mathbb{C}^n \rightarrow \mathbb{C}$, at least one coordinate of P equals zero. Recall that a complex point $P \in \mathbb{C}^n$ is called a critical point of f_τ if $\partial f_\tau / \partial x_i(P) = 0$ for all $i = 1, \dots, n$.

The following proposition slightly extends the main result of [Castricky and Nguyen 2019] as it covers small primes as well. Note that [Castricky and Nguyen 2019, Theorem 1.4.1] gives evidence for Igusa’s conjecture on exponential sums in the variant of [Cluckers and Veys 2016, Conjecture 1.2].

Proposition 4.1 [Castricky and Nguyen 2019, Theorem 1.4.1]. *Suppose that f is nondegenerate with respect to $\Delta_0(f)$. Then, there is a constant c such that for all primes p , all integers $m \geq 2$ and all primitive p^m -th roots of unity ξ one has*

$$E_f(p^m, \xi) \leq cp^{-m\sigma_f} m^{\kappa-1}. \tag{36}$$

From Proposition 4.1 we will derive the following evidence for Conjecture 1.

Theorem 4.2. *Let f, n, s , and d be as in the introduction. Suppose that f is nondegenerate with respect to $\Delta_0(f)$. Then Conjecture 1 with $\varepsilon = 0$ holds for f . Namely, there is c such that for all integers $N > 0$ and all primitive N -th roots of unity ξ , one has*

$$E_f(N, \xi) \leq cN^{-\frac{n-s}{d}}.$$

Furthermore, for all large primes p (with “large” depending on f), all integers $m > 0$ and all primitive p^m -th roots of unity ξ one has

$$E_f(p^m, \xi) \leq p^{-m\frac{n-s}{d}}.$$

Proof of Proposition 4.1. In [Castricky and Nguyen 2019] it is shown that one can take a constant c so that (36) holds for all large primes p and all $m \geq 2$. So, there is only left to prove that for each prime p there is a constant c_p (depending on p) such that for each integer $m \geq 2$ one has

$$E_f(p^m, \xi) \leq c_p p^{-m\sigma_f} m^{\kappa-1}. \tag{37}$$

Indeed, (37) is only used for the finitely many remaining primes. First, if f is nondegenerate with respect to $\Delta_0(f)$ we show that $f(0)$ is the only possible critical value of f , by induction on n . If $n = 1$, by the nondegeneracy of f , we get that f has no critical point in \mathbb{C}^\times and we are done. Now suppose that $n > 1$. Let f be a polynomial in n variables which is nondegenerate with respect to $\Delta_0(f)$. Suppose that $u = (u_1, \dots, u_n)$ is a critical point of f . By the nondegeneracy of f there exists $1 \leq j \leq n$ such that $u_j = 0$. Without loss of generality we can suppose that $j = n$. We can write $f = f(0) + \sum_{i=0}^d g_i(x_1, \dots, x_{n-1})x_n^i$ for some polynomials g_i with furthermore $g_0(0) = 0$. It is sufficient to show that $f(u_1, \dots, u_n) = f(0)$. Since $u_n = 0$, it suffices to show that $g_0(u_1, \dots, u_{n-1}) = 0$. By the fact that f is nondegenerate with respect to $\Delta_0(f)$ we get that g_0 is nondegenerate with respect to $\Delta_0(g_0)$. It is clear that (u_1, \dots, u_{n-1}) is a critical point of g_0 . So, we can use the inductive hypothesis to deduce that $g_0(u_1, \dots, u_{n-1}) = g_0(0) = 0$. Now, since f has no other possible critical value than $f(0)$ and since there exists a toric log resolution of $f - f(0)$ whose numerical properties (in particular its discrepancy numbers) are controlled by $\Delta_0(f)$ (see for example [Varčenko 1976]), inequality (37) follows from Igusa’s work [1978]. Here, we use the following information on the discrepancy numbers coming from the toric log resolution π of $f - f(0)$, in relation to $\Delta_0(f)$. If E is an irreducible component of the exceptional locus of π and if one writes N_E for

the multiplicity of E in the divisor $\pi^*(f - f(0))$ and $\nu_E - 1$ for the multiplicity of E in $\pi^*(dx_1 \wedge \cdots \wedge dx_n)$, then one has $\nu_E/N_E \geq \sigma$. Furthermore, any intersection of $\kappa + 1$ many such E for which the equality $\nu_E/N_E = \sigma$ holds is empty. Since $f(0)$ is the only critical value of f , we are now done by Igusa's work [1978]. □

The proof of Theorem 4.2 relies on Proposition 4.1 and Lemma 4.3. Note that the following Lemmas 4.3 and 4.4 do not require f to be nondegenerate.

Lemma 4.3. *Let $f, n, s,$ and d be as in the introduction. (In particular, f is allowed to be inhomogeneous and there is no condition on nondegeneracy.) Suppose that $d \geq 3$. Then one has $\sigma_f \geq (n - s)/d$, and, equality holds if and only if there is a smooth form g of degree d in $n - s$ variables such that*

$$f(x) - f(0) = g(x_{i_1}, \dots, x_{i_{n-s}})$$

for some i_j with $1 \leq i_1 < i_2 < \dots < i_{n-s} \leq n$.

We will first prove Lemma 4.3 in the case that $s = 0$, using the following lemma. We write $\text{Conv}(\text{Supp}_f)$ for the convex hull of Supp_f in \mathbb{R}^n .

Lemma 4.4. *Let $f, n, s,$ and d be as in the introduction. Suppose furthermore that $d \geq 3, s = 0$ and that $f = f_d$, namely, f is smooth homogeneous of degree at least 3. Then*

$$\dim(\text{Conv}(\text{Supp}_f)) = n - 1,$$

and, the point $(d/n, \dots, d/n)$ belongs to the interior of $\text{Conv}(\text{Supp}_f)$. In particular, $\sigma_f = n/d$ and $\kappa = 1$.

Proof. Since $f = f_d$, it is clear that

$$\dim(\text{Conv}(\text{Supp}_f)) \leq n - 1.$$

Suppose now that either $\dim(\text{Conv}(\text{Supp}_f)) < n - 1$, or, that $(d/n, \dots, d/n)$ does not belong to the interior of $\text{Conv}(\text{Supp}_f)$. We try to find a contradiction. By our assumptions, there exists a hyperplane $H = \{a \in \mathbb{R}^n \mid k \cdot a = 0\}$ for some $k \in \mathbb{R}^n \setminus \{0\}$ such that the point $(d/n, \dots, d/n)$ belongs to H and such that Supp_f belongs to the half space $H_+ := \{a \in \mathbb{R}^n \mid k \cdot a \geq 0\}$. Let I be the subset of $\{1, \dots, n\}$ consisting of i with $k_i > 0$ and let J be the subset of $\{1, \dots, n\}$ consisting of j with $k_j < 0$. Clearly I and J are disjoint. Since $(d/n, \dots, d/n)$ belongs to H , it follows that I and J are both nonempty and that

$$\sum_{i \in I} k_i = \sum_{j \in J} |k_j|. \tag{38}$$

Furthermore, the inclusion $\text{Supp}_f \subset H_+$ implies that

$$\sum_{i \in I} k_i a_i \geq \sum_{j \in J} |k_j| a_j \text{ for all } a \in \text{Supp}_f. \tag{39}$$

Consider the set Supp_f^0 consisting of those $a \in \text{Supp}_f$ with moreover $a_i = 1$ for some $i \in I$ and $a_{i'} = 0$ for all $i' \notin J \cup \{i\}$. For $a \in \text{Supp}_f^0$ write $t(a)$ for the unique $i \in I$ with $a_i = 1$ and write

$$I_0 := \{i \in I \mid \exists a \in \text{Supp}_f^0 \text{ with } t(a) = i\}.$$

Clearly we can write

$$f(x) = \sum_{i \in I_0} x_i g_i(x_j)_{j \in J} + \sum_{a \in \text{Supp}_f \setminus \text{Supp}_f^0} \beta_a x^a$$

for some polynomials g_i in the variables $(x_j)_{j \in J}$. Also, the algebraic set

$$\bigcap_{i \notin J} \{x_i = 0\} \bigcap_{i \in I_0} \{g_i = 0\}$$

in $\mathbb{A}_{\mathbb{C}}^n$ has dimension at least $|J| - |I_0|$ and is contained in Crit_f , the critical locus of f . By our smoothness condition $s = 0$, this implies

$$|I_0| \geq |J|. \tag{40}$$

Hence, we can write $I_0 = \{i_1, \dots, i_\ell\}$ and $J = \{j_1, \dots, j_m\}$ with $m \leq \ell$ and with

$$k_{i_1} \geq k_{i_2} \geq \dots \geq k_{i_\ell} \quad \text{and} \quad |k_{j_1}| \geq |k_{j_2}| \geq \dots \geq |k_{j_m}|. \tag{41}$$

To prove the lemma it is now sufficient to show that

$$k_{i_r} > |k_{j_r}| \quad \text{for all } r \text{ with } 1 \leq r \leq m. \tag{42}$$

Indeed, (42) gives a contradiction with (38). To prove (42), we suppose that there is r_0 with $1 \leq r_0 \leq m$ and with

$$k_{i_{r_0}} \leq |k_{j_{r_0}}| \tag{43}$$

and we need to find a contradiction. If there exists $a \in \text{Supp}_f^0$ such that $a_{j_{r_1}} \geq 1$ for some $r_1 \leq r_0$, then let a be such an element and let t be $t(a)$; otherwise, let a be arbitrary and put $t = 0$. We will now show that $t < r_0$. If $t = 0$ this is clear, so, suppose that $t > 0$. Since $d \geq 3$ and $a \in \text{Supp}_f^0$, we find by (39) and (41) that

$$k_{i_t} = \sum_{i \in I} k_i a_i \geq \sum_{j \in J} a_j |k_j| > |k_{j_{r_1}}| \geq |k_{j_{r_0}}|. \tag{44}$$

Together with (41) and (43), this implies that $t < r_0$ as desired. We can thus write

$$f = \sum_{1 \leq \ell \leq r_0 - 1} x_{i_\ell} h_\ell(x_j)_{j \in J} + \sum_{a \in A} \beta_a x^a \tag{45}$$

with

$$A = \left\{ a \in \text{Supp}_f \mid \sum_{i \notin \{j_1, \dots, j_{r_0}\}} a_i \geq 2 \right\}$$

and with some polynomials h_ℓ in the variables $(x_j)_{j \in J}$. It follows that the algebraic set

$$\bigcap_{i \notin \{j_1, \dots, j_{r_0}\}} \{x_i = 0\} \bigcap_{1 \leq \ell \leq r_0 - 1} \{h_\ell = 0\}$$

has dimension at least 1 and is contained in Crit_f , again a contradiction with our smoothness assumption $s = 0$. So, relation (42) follows and the lemma is proved. □

The case of Lemma 4.3 with $s = 0$ is derived from Lemma 4.4, as follows.

Proof of Lemma 4.3 with $s = 0$. Let f be of degree $d \geq 3$ and with $s = 0$. We need to show that $\sigma_f \geq n/d$, and, that $\sigma_f = n/d$ if and only if $f = f_d$. Since f_d is smooth, Lemma 4.4 implies that $(d/n, \dots, d/n)$ belongs to $\Delta_0(f)$, and hence, $\sigma_f \geq n/d$. Suppose now that $f \neq f_d$. Then there exists $a \in \text{Supp}_f$ with $\sum_{i=1}^n a_i < d$. Hence, by Lemma 4.4 and the definition of $\Delta_0(f)$, there exists $\varepsilon > 0$ such that

$$\{x \in \mathbb{R}^n \mid \|x - (d/n, \dots, d/n)\| \leq \varepsilon\} \subset \Delta_0(f).$$

Therefore it is clear that $\sigma_f > n/d$. This finishes the proof of Lemma 4.3 with $s = 0$. □

Proof of Lemma 4.3 with $s > 0$. To prove the lemma with $s > 0$ we may suppose that

$$\sigma_f \leq (n - s)/d. \tag{46}$$

By the definition of σ_f we have

$$\min_{a \in \text{Conv}(\text{Supp}_f)} \max(a) = 1/\sigma_f, \tag{47}$$

where $\max(a) = \max_{1 \leq i \leq n} \{a_i\}$ and where $\text{Conv}(\text{Supp}_f)$ is the convex hull of Supp_f . We set

$$k := \min_{\max(a)=1/\sigma_f} \#\{i \mid a_i = 1/\sigma_f\},$$

where the minimum is taken over $a \in \text{Conv}(\text{Supp}_f)$. Let $a \in \text{Conv}(\text{Supp}_f)$ realize this minimum, namely, with $\#\{i \mid a_i = 1/\sigma_f\} = k$ and with $\max(a) = 1/\sigma_f$. We may suppose that

$$a_1 = \dots = a_k = 1/\sigma_f \quad \text{and} \quad a_i < 1/\sigma_f \quad \text{if } i > k.$$

Let $b \in \text{Conv}(\text{Supp}_f)$ be such that $\max(b) = 1/\sigma_f$. Then, for each $\lambda \in [0, 1]$, the point $c_\lambda := \lambda a + (1 - \lambda)b$ lies in $\text{Conv}(\text{Supp}_f)$. When λ is sufficiently close to 1, then we have $c_{\lambda,i} < 1/\sigma_f$ for all $i > k$, and, the definition of k implies that $b_i = 1/\sigma_f$ for all $1 \leq i \leq k$. By the same reasoning, for each $b \in \text{Conv}(\text{Supp}_f)$ one has $b_i \geq 1/\sigma_f$ for some i with $1 \leq i \leq k$. The definition of k and (47) also tell us that $k/\sigma_f \leq d$, and thus we find

$$k \leq n - s \tag{48}$$

from (46). For any tuple of complex numbers $C = (c_{i,j})_{1 \leq i,j \leq s}$ we consider the polynomial

$$g_C = f\left(x_1, \dots, x_{n-s}, x_{n-s+1} + \sum_{1 \leq j \leq n-s} c_{1,j}x_j, \dots, x_n + \sum_{1 \leq j \leq n-s} c_{s,j}x_j\right).$$

For a generic choice of C one has $\text{Supp}_f \subset \text{Supp}_{g_C}$. Furthermore, we show that for a generic choice of C the polynomial

$$h_C = f_d\left(x_1, \dots, x_{n-s}, \sum_{1 \leq j \leq n-s} c_{1,j}x_j, \dots, \sum_{1 \leq j \leq n-s} c_{s,j}x_j\right)$$

is smooth homogeneous in $n - s$ variables, where f_d is the degree d homogeneous part of f . For a generic choice of $e_n = (e_{n,i})_{i < n}$ one has

$$\dim(\text{Sing}(f_{d,e_n})) = n - s,$$

where

$$f_{d,e_n}(x_1, \dots, x_{n-1}) := f_d\left(x_1, \dots, x_{n-1}, \sum_{i=1}^{n-1} e_{n,i}x_i\right),$$

considered as a polynomial in $n - 1$ variables x_i with $i < n$. We repeat this argument to see that for a generic choice $E = (e_{n-s+1}, \dots, e_n)$ with $e_j = (e_{j,i})_{i < j}$ one has that

$$\dim(\text{Sing}(f_d|_{V_E})) = n - s,$$

where

$$V_E = \left\{x \mid x_n = \sum_{i < n} e_{n,i}x_i, \dots, x_{n-s+1} = \sum_{i \leq n-s} e_{n-s+1,i}x_i\right\}.$$

It is clear that the smoothness of $f_d|_{V_E}$ for generic E corresponds to the smoothness of h_C for generic C . Let us fix such a choice of C with all these properties, namely, that h_C is smooth and that $\text{Supp}_f \subset \text{Supp}_{g_C}$. If $a \in \text{Supp}_{g_C}$, it is easy to see that $a_i \geq b_i$ for all i with $1 \leq i \leq n - s$ and for some $b \in \text{Supp}_f$. Hence, $\sigma_{g_C} \leq \sigma_f$, by the definition of k and our chosen ordering of the coordinates. On the other hand, from $\text{Supp}_f \subset \text{Supp}_{g_C}$ it follows that $\sigma_{g_C} \geq \sigma_f$, and hence, we have

$$\sigma_{g_C} = \sigma_f.$$

Let π be the coordinate projection from \mathbb{R}^n to \mathbb{R}^{n-s} . Then, for any $e = (e_j)_{j=1, \dots, s}$, consider the polynomial

$$g_{C,e}(x_1, \dots, x_{n-s}) := g_C(x_1, \dots, x_{n-s}, e_1, \dots, e_s).$$

Then, for generic choice of e , we have

$$\text{Supp}_{g_{C,e}} = \pi(\text{Supp}_{g_C}).$$

Let us fix such a choice of e . It is clear that

$$\sigma_{g_{C,e}} = \sigma_{g_C},$$

by the definition of k and our ordering of the coordinates. Note that the highest degree homogeneous part of $g_{C,e}$ equals h_C , which is smooth. Thus, we can use Lemma 4.3 with $s = 0$ (which is already proved) for $g_{C,e}$. So, we find

$$\sigma_{g_{C,e}} = (n - s)/d \quad \text{and} \quad g_{C,e} - g_{C,e}(0) = h_C.$$

Hence,

$$\pi(\text{Supp}_f) \subset \pi(\text{Supp}_{g_C}) \subset \{a \in \mathbb{R}^{n-s} \mid a_1 + \dots + a_{n-s} = d\}.$$

This holds if and only if $f - f(0) = f_d = h(x_1, \dots, x_{n-s})$ for some polynomial h , which is smooth homogeneous since $\dim(\text{Crit}_{f_d}) = s$. This finishes the proof of the Lemma 4.3. □

We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2. The case that $d = 2$ is treated in Section 3.3. Hence, we may suppose that $d \geq 3$. By Proposition 4.1, there exists a constant c_2 such that for all integers $m > 1$, all primes p and all primitive p^m -th roots of unity ξ we have

$$E_f(p^m, \xi) \leq c_2 p^{-m\sigma_f} m^{\kappa-1}. \tag{49}$$

By Lemma 4.3 we have $\sigma_f \geq (n - s)/d$. If $\sigma_f > (n - s)/d$, then we have $(n - s)/d < \frac{1}{2}(n - s)$, from using $d \geq 3$ and $s < n$. Conjecture 1 for this case follows by combining (7) and (49) with the squarefree case from Section 3.5. If $\sigma_f = (n - s)/d$, we use Lemma 4.3 again to see that $f = g_d + f(0)$ for a smooth form g_d of degree d in $n - s$ variables. Conjecture 1 for this case follows by Igusa’s case from Section 3.1. \square

Remark 4.5. If f is weighted homogeneous, then the notion of nondegeneracy with respect to $\Delta_0(f)$ is generic, but otherwise the genericity is more subtle, by the difference between “critical points” and “singular points”. In fact, whether or not the notion of nondegeneracy with respect to $\Delta_0(f)$ is generic depends on Supp_f . When Supp_f is contained in a hyperplane which does not contain the origin 0 and has a normal vector with nonnegative coordinates (see [Castricky and Nguyen 2019, Section 2.2]), then the condition of nondegeneracy on the coefficients β_i is generic within this support, that is, for any γ outside a Zariski closed subset of $\mathbb{C}^{\text{Supp}_f}$, the polynomial $\sum_{i \in \text{Supp}_f} \gamma_i x^i$ is nondegenerate with respect to its Newton polyhedron at zero. This hyperplane condition generalizes the case of weighted homogeneous polynomials. However, in the general case, this genericity may be lost since we imposed conditions on critical points of f_τ instead of on singular points as is done more traditionally in [Kouchnirenko 1976], [Varčenko 1976]. Especially for $\tau = \Delta_0(f)$ this makes a difference when the mentioned hyperplane condition is not met. For instance, polynomials of the form $f(x) = ax^3 + by^3 + cxy$ for nonzero a, b , and c are never nondegenerate in our sense, the problem being with $\tau = \Delta_0(f)$.

5. The four variable case

In this final section we prove Conjecture 1 when $n \leq 4$ (Theorem 5.1), and a slightly stronger result when furthermore $d \leq 3$ and $s = 0$ (Proposition 5.2).

Theorem 5.1. *Let f, n, s , and d be as in the introduction and suppose that $n \leq 4$. Then Conjecture 1 holds for f .*

The proof of Theorem 5.1 relies on a concrete lemma inspired by Weierstrass preparation (see Lemma 5.4), properties of $\hat{\alpha}_f$ based on results on minimal exponents from [Mustařa and Popa 2020], bounds from [Cluckers et al. 2019; Nguyen and Veys 2022], and Igusa’s results as summarized in [Denef 1991].

Proof of Theorem 5.1. Suppose that $n \leq 4$. If $(n - s)/d \leq 1$ or $d = 2$, then Conjecture 1 follows by the arguments in Sections 3.2 and 3.3. Hence, we may concentrate on the case that $d = 3$ and $s = 0$, but this follows from Proposition 5.2 below, the squarefree case from Section 3.5, the inequality (18) and the relation (7). \square

The rest of this section will focus on the proof of Proposition 5.2, which says slightly more than Theorem 5.1 in the case that $d \leq 3$ and $s = 0$.

Proposition 5.2. *Let $f, n, s,$ and d be as in the introduction and suppose that $n \leq 4, d \leq 3,$ and $s = 0$. Then for all $\varepsilon > 0$ there is a constant c such that*

$$E_f(p^m, \xi) \leq c(p^m)^{-\hat{\alpha}_f + \varepsilon} \text{ for all primes } p, \text{ all } m > 1 \text{ and all } \xi, \tag{50}$$

with $\hat{\alpha}_f$ as in Section 2.7. Furthermore, the value $\hat{\alpha}_f$ is equal to the motivic oscillation index of f as given in [Cluckers et al. 2019]. Hence, $\hat{\alpha}_f$ is the optimal exponent in (50).

The optimality of the exponent $\hat{\alpha}_f$ in (50) means that there is a constant $c_0 > 0$ such that for infinitely many primes p and (some) arbitrarily large m one has

$$c_0(p^m)^{-\hat{\alpha}_f} \leq E_f(p^m, \xi) \text{ for some } \xi. \tag{51}$$

The motivic oscillation index of f as given in [Cluckers et al. 2019] (which corresponds to the one from [Cluckers 2008b] but without the negative sign) is the optimal exponent of p^{-m} in (50), see Section 3.4 of [Cluckers et al. 2019]; therefore, the equality of $\hat{\alpha}_f$ with the motivic oscillation index is a useful property and implies (51).

The following auxiliary lemma is well known, see for example the final inequality of [Heath-Brown 1985], where furthermore an explicit upper bound on the number of critical points is obtained.

Lemma 5.3. *Suppose that $g = g_0 + \dots + g_d$ is a polynomial in $\mathbb{C}[x_1, \dots, x_n]$ of degree d and with $\dim(\text{Crit}_{g_d}) = 0$, where g_i is the degree i homogeneous part of g , and where Crit_{g_d} is the critical locus of $g_d : \mathbb{C}^n \rightarrow \mathbb{C}$, i.e., the scheme associated with the ideal generated by the $\partial g_d / \partial x_i$ for $1 \leq i \leq n$. Then $\text{Crit}_g(\mathbb{C})$ is a finite set.*

Proof. This is shown by homogenizing g as in the reasoning towards the final inequality of [Heath-Brown 1985], where it is even shown that $\#\text{Crit}_g(\mathbb{C}) \leq (d - 1)^n$, by an application of Bézout’s theorem. □

Lemma 5.4. *Let f in $\mathbb{Z}[x_1, x_2, x_3, x_4]$ be of degree $d = 3$ and with $s = 0$. Suppose that one can write $f = f_2 + f_3$ with f_i homogeneous of degree i and with $f_2 \neq 0$. Then there exists a finite field extension K of \mathbb{Q} and a linear transformation $x_i = \sum_{j=1}^4 a_{ij}y_j$ with $(a_{ij})_{1 \leq i, j \leq 4} \in \text{GL}_4(K)$ such that*

$$f(x_1, x_2, x_3, x_4) = g(y_1, y_2, y_3, y_4) \quad \text{with} \quad \frac{\partial^2 g}{\partial y_1^2}(0, 0, 0, 0) \neq 0 \quad \text{and} \quad \frac{\partial^3 g}{\partial y_1^3}(0, 0, 0, 0) = 0.$$

Proof. By a simple calculation we have

$$\frac{\partial^2 g}{\partial y_1^2}(0, 0, 0, 0) = 2f_2(a_{11}, a_{21}, a_{31}, a_{41}) \quad \text{and} \quad \frac{\partial^3 g}{\partial y_1^3}(0, 0, 0, 0) = 6f_3(a_{11}, a_{21}, a_{31}, a_{41}).$$

So, it suffices to show the following relation on zero loci

$$Z(f_3(a_{11}, a_{21}, a_{31}, a_{41})) \not\subseteq Z(f_2(a_{11}, a_{21}, a_{31}, a_{41})) \cup Z(\det(A)) \tag{52}$$

viewed as algebraic subsets of $\mathbb{A}_{\mathbb{Q}}^{16}$ and with $A = (a_{ij})_{1 \leq i, j \leq 4}$ and $\overline{\mathbb{Q}}$ the algebraic closure of \mathbb{Q} . Since $s = \dim(\text{Sing}(f_3)) = 0$ one has that f_3 is absolutely irreducible. Therefore if (52) does not hold then

$$f_3(a_{11}, a_{21}, a_{31}, a_{41}) \mid f_2(a_{11}, a_{21}, a_{31}, a_{41}) \quad \text{or} \quad f_3(a_{11}, a_{21}, a_{31}, a_{41}) \mid \det(A).$$

It is clear that $f_3 \nmid f_2$ since $f_2 \neq 0$. Also, the polynomial $\det(A)$ is absolutely irreducible of degree 4. So, (52) must hold. The lemma is proved. \square

We first prove the part of Proposition 5.2 for large primes, as follows.

Proposition 5.5. *With notation and assumptions from Proposition 5.2, there exist an integer M and a constant c such that for all $p > M$ all $m > 1$, for all primitive p^m -th roots of unity ξ we have*

$$E_f(p^m, \xi) \leq cm^3 p^{-m\hat{\alpha}_f}. \tag{53}$$

Proof. We may focus on the case $d = 3$. By Lemma 5.3, the set $\text{Crit}_f(\overline{\mathbb{Q}})$ is finite. Because of [Denef 1991, Remark 4.5.3], we are done when f has no critical point. Similarly, when f has a critical point a such that $f - f(a)$ has multiplicity 3 at a the proposition follows by Section 3.1 and the fact that $\hat{\alpha}_f = \frac{4}{3}$ in this case. Now we suppose that if a is a critical point of f then $f - f(a)$ has multiplicity 2 at a . By [Denef 1991, Remark 4.5.3], it suffices to show that if $a \in \text{Crit}_f(\overline{\mathbb{Q}})$ then there exist an integer M and a constant c such that for all $p > M$ with $a \in \mathbb{Z}_p^4$, all $m > 1$, all primitive p^m -th roots of unity ξ we have

$$E_f^{(a)}(p^m, \xi) := \int_{a+p\mathbb{Z}_p^4} \xi^{f(x) \bmod p^m} \mu \leq cm^3 p^{-m\hat{\alpha}_f}. \tag{54}$$

Without loss of generality, we can suppose that $a = 0$ and $f(0) = 0$. Lemma 5.4 gives us a finite field extension K of \mathbb{Q} and a linear transformation $x_i = \sum_{1 \leq j \leq 4} a_{ij} y_j$ with $(a_{ij})_{1 \leq i, j \leq 4} \in \text{GL}_4(K)$ such that if $g(y_1, y_2, y_3, y_4) = f(x_1, x_2, x_3, x_4)$ then

$$g(y_1, y_2, y_3, y_4) = h_2(y_2, y_3, y_4)y_1^2 + h_1(y_2, y_3, y_4)y_1 + h_0(y_2, y_3, y_4)$$

with polynomials $h_0, h_1, h_2 \in K(y_2, y_3, y_4)$ and $h_2(0, 0, 0) \neq 0, h_1(0, 0, 0) = h_0(0, 0, 0) = 0$. We set

$$z_1 = h_2(y_2, y_3, y_4)y_1 + \frac{1}{2}h_1(y_1, y_3, y_4)$$

and $z_2 = y_2, z_3 = y_3, z_4 = y_4$. At the new coordinates (z_1, z_2, z_3, z_4) we have $g(y_1, y_2, y_3, y_4) = h(z_1, z_2, z_3, z_4)$ with

$$h(z_1, z_2, z_3, z_4)h_2(z_2, z_3, z_4) = z_1^2 + r(z_2, z_3, z_4)$$

and

$$r(z_2, z_3, z_4) = h_2(z_2, z_3, z_4)h_0(z_2, z_3, z_4) - \frac{1}{4}h_1^2(z_2, z_3, z_4).$$

Using Lemma 5.3 and the argument from [Nguyen and Veys 2022, Proposition 5.9 and Section 6] (to compare between the weights of suitable ℓ -adic cohomology groups related to f and those related to

$z_1^2 + r$), it now follows that there exist an integer M and a constant c such that for all $p > M$ all $m > 1$, all primitive p^m -th roots of unity ξ we have

$$E_f^{(0)}(p^m, \xi) \leq cm^3 p^{-m\hat{\alpha}_f}. \tag{55}$$

So we are done. □

By Proposition 5.5, in order to prove Proposition 5.2, it remains to prove the following proposition.

Proposition 5.6. *Let f be in $\mathbb{Z}[x_1, x_2, x_3, x_4]$ of degree $d = 3$ and with $s = 0$. Then, for each p and $\varepsilon > 0$ there exist a constant $c_{p,\varepsilon}$ and an integer m_p such that for all integers $m > m_p$ and all primitive p^m -th roots of unity ξ we have*

$$E_f(p^m, \xi) \leq c_{p,\varepsilon} p^{-m(\hat{\alpha}_f - \varepsilon)} \tag{56}$$

Proof. By Lemma 5.3, the set $\text{Crit}_f(\mathbb{Z}_p)$ is finite for each p . If there exists a point $a \in \text{Crit}_f(\mathbb{Z}_p)$ such that the multiplicity of f at a is 3 then we are done as in Proposition 5.5. If $\text{Crit}_f(\mathbb{Z}_p) = \emptyset$ then by a basic argument there exists an integer m_p such that we have $E_f(p^m, \xi) = 0$ for all $m \geq m_p$ and all primitive p^m -th roots of unity ξ .

Let us now suppose that $\text{Crit}_f(\mathbb{Z}_p) \neq \emptyset$ and that the multiplicity of $f - f(a)$ at a is 2 for all $a \in \text{Crit}_f(\mathbb{Z}_p)$. By [Igusa 1974; 1978; Denef and Veys 1995], it is sufficient to show that the real part of every nontrivial pole of the Igusa local zeta functions of f at U_a is at most $-\hat{\alpha}_f$, where U_a is a small enough neighborhood of a in \mathbb{Z}_p^4 . Here, we recall that if L is a finite extension of \mathbb{Q}_p , \mathcal{O}_L is the ring of integers of L , V is an open subset of \mathcal{O}_L^n , F is an analytic function defined on a neighborhood of V and χ is a multiplicative character of \mathcal{O}_L^\times then the twisted Igusa local zeta function of F at V associated with χ given by

$$Z_\chi(L, V, F, s) = \int_V \chi(\text{ac}(F(x))) |F(x)|^s |dx|,$$

where $|dx|$ is the normalized Haar measure on L^n so that the measure of \mathcal{O}_L^n is 1, $\text{ac}(z) = z\varpi_L^{-\text{ord}_L(z)}$ for nonzero z and for a fixed uniformizing element ϖ_L of \mathcal{O}_L and the usual valuation map $\text{ord}_L : L \rightarrow \mathbb{Z} \cup \{+\infty\}$ (see, e.g., [Denef 1991; Veys and Zúñiga Galindo 2008]). We say that s is a nontrivial pole of the Igusa local functions of F at V if s is a pole of $Z_\chi(L, V, F, s)$ when $\chi \neq \chi_{\text{trivial}}$ or s is a pole of $(p^{s+1} - 1)Z_\chi(L, V, F, s)$ when $\chi = \chi_{\text{trivial}}$. Let L be a finite extension of \mathbb{Q}_p , V be an open subset of \mathcal{O}_L^n , F be an analytic function on V and denote by $\text{Pol}_V(F)$ the set of the real parts of the nontrivial poles of the Igusa local zeta functions of F at V . By [Igusa 1974; Igusa 1978; Denef and Veys 1995; Veys and Zúñiga Galindo 2008], $\text{Pol}_V(F)$ is a finite set. To prove the proposition, it thus remains to show for all $a \in \text{Crit}_f(\mathbb{Z}_p)$, all small enough neighborhoods U_a of a in \mathbb{Z}_p^4 , that

$$\alpha \leq -\hat{\alpha}_f \text{ for all } \alpha \in \text{Pol}_{U_a}(f - f(a)). \tag{57}$$

The rest of the proof will show this (57).

Fix $a \in \text{Crit}_f(\mathbb{Z}_p)$; to simplify the notation we suppose that $a = (0, 0, 0, 0)$ and $f(0) = 0$. Up to using a transformation as in Lemma 3.4, we may suppose that f_2 is diagonal, where we write $f = f_2 + f_3$ with f_i homogeneous of degree i for each i . Suppose that the coefficient of x_1^2 is nonzero in f_2 . Using

Weierstrass preparation, we can suppose that for a small enough open (p -adic) neighborhood $U = p^{m_0}\mathbb{Z}_p^4$ of 0, there exist analytic functions u, g_1, g_0 on U such that we have

$$f|_U = u(x)(x_1^2 + 2g_1(x_2, x_3, x_4)x_1 + g_0(x_2, x_3, x_4))$$

and $g_1(0, 0, 0) = g_0(0, 0, 0) = 0 \neq u(0)$, with $x = (x_1, x_2, x_3, x_4)$. By shrinking U we can furthermore suppose that $\chi(\text{ac}(u))$ and $\text{ord}_p(u)$ are constant on U for each multiplicative character $\chi \in \mathfrak{C}$. Moreover, we can suppose that 0 is the only critical point of f in U . With these conditions on U , it is easy to show that $\text{Pol}_U(f) = \text{Pol}_U(g)$, where

$$g(x) = x_1^2 + 2g_1(x_2, x_3, x_4)x_1 + g_0(x_2, x_3, x_4) \tag{58}$$

with g_i as above. By setting $y_1 = x_1 + g_1(x_2, x_3, x_4)$, $y_2 = x_2$, $y_3 = x_3$, $y_4 = x_4$ then at the new coordinates (y_1, y_2, y_3, y_4) we have $g(x_1, x_2, x_3, x_4) = y_1^2 + h_1(y_2, y_3, y_4)$, and hence, by the change of variables formula, we may suppose that $g_1 = 0$ in (58), so that $g(x) = x_1^2 + g_0(x_2, x_3, x_4)$. By using the argument in [Denef 1991, Section 5.1] and by noting that $\text{Pol}_{p^{m_0}\mathbb{Z}_p}(x_1^2) = \{-1/2\}$ and enlarging m_0 if needed, we have $\text{Pol}_U(f) = \text{Pol}_U(g) = -1/2 + \text{Pol}_{U'}(g_0) = \{-1/2 + \alpha \mid \alpha \in \text{Pol}_{U'}(g_0)\}$, where $U' = p^{m_0}\mathbb{Z}_p^3$.

As in the proof of Proposition 5.5, we can find a finite extension L of \mathbb{Q}_p , a Zariski open subset W of \mathbb{A}_L^4 and a change of coordinates such that

$$f|_W = \tilde{u}(x)(x_1^2 + \tilde{g}_0(x_2, x_3, x_4)),$$

where \tilde{u}, \tilde{g}_0 are regular functions on W and \tilde{u} is nonzero on W . Let ϖ_L be a uniformizing element of \mathcal{O}_L . Let m_1 be a large enough integer and $\tilde{U} = \varpi_L^{m_1}\mathcal{O}_L^4$. By the above argument we also have $\text{Pol}_{\tilde{U}'}(\tilde{g}_0) - 1/2 = \text{Pol}_{\tilde{U}'}(f) = \text{Pol}_{\tilde{U}'}(g) = -1/2 + \text{Pol}_{\tilde{U}'}(g_0)$, where $\tilde{U}' = \varpi_L^{m_1}\mathcal{O}_L^3$. Thus $\text{Pol}_{\tilde{U}'}(\tilde{g}_0) = \text{Pol}_{\tilde{U}'}(g_0)$.

Suppose first that \tilde{g}_0 has a nonrational singularity at $0 \in \mathbb{A}^3$, namely, $\tilde{\alpha}_{\tilde{g}_0,0}$ is equal to the log-canonical threshold at 0 of \tilde{g}_0 . Let π be a log resolution of \tilde{g}_0 at 0, let $(\nu_i, N_i)_{i \in I}$ be the numerical data of π so that $1 \geq \tilde{\alpha}_{\tilde{g}_0,0} = \min_{(i, N_i) \neq (1,1)} \nu_i/N_i$ (see [Denef 1991; Mustařa 2012]). Thus, by using [Igusa 1974; Igusa 1978] we have $\alpha \leq -\tilde{\alpha}_{\tilde{g}_0,0}$ for all $\alpha \in \text{Pol}_{\tilde{U}'}(\tilde{g}_0)$. Moreover, it follows from [Yano 1978, Corollary 3.17] that $\tilde{\alpha}_{f,0} = 1/2 + \tilde{\alpha}_{\tilde{g}_0,0}$. If there is $\beta \in \text{Pol}_{U'}(g_0)$ such that $\beta > -\tilde{\alpha}_{f,0} + 1/2 = -\tilde{\alpha}_{\tilde{g}_0,0} \geq -1$ then it follows by [Igusa 1974; Igusa 1978; Veys and Zúñiga Galindo 2008] that there is a log-resolution π' of g_0 with a numerical data (ν, N) satisfying $\nu < \tilde{\alpha}_{\tilde{g}_0,0}N \leq N$. Thus we can use [Veys and Zúñiga Galindo 2008, Theorem 2.7] to see that there is $\beta' \in \text{Pol}_{\tilde{U}'}(g_0)$ such that $\beta' > -\tilde{\alpha}_{\tilde{g}_0,0}$. This contradicts the fact that $\text{Pol}_{\tilde{U}'}(\tilde{g}_0) = \text{Pol}_{\tilde{U}'}(g_0)$ and $\alpha \leq -\tilde{\alpha}_{\tilde{g}_0,0}$ for all $\alpha \in \text{Pol}_{\tilde{U}'}(\tilde{g}_0)$. Thus we can conclude that $\beta \leq -\tilde{\alpha}_{f,0} + 1/2$ for all $\beta \in \text{Pol}_{U'}(g_0)$. Therefore $\alpha \leq -\tilde{\alpha}_{f,0}$ for all $\alpha \in \text{Pol}_U(f)$ as desired.

Suppose now that \tilde{g}_0 has a rational singularity at $0 \in \mathbb{A}^3$, namely $\tilde{\alpha}_{\tilde{g}_0,0}$ is strictly larger than the log-canonical threshold at 0 of \tilde{g}_0 , or equivalently, $\tilde{\alpha}_{\tilde{g}_0,0} > 1$. We can use a classical result from [Durfée 1979] to see that either \tilde{g}_0 is smooth at 0, or, we can use an analytic isomorphism to transform \tilde{g}_0 to one of the following forms:

$$\begin{aligned} &x_2^{d+1} + x_3^2 + x_4^2 \quad \text{with } d \geq 1, && x_2^{d-1} + x_2x_3^2 + x_4^2 \quad \text{with } d \geq 4, \\ &x_2^4 + x_3^3 + x_4^2, && x_2^3 + x_2x_3^3 + x_4^2, && x_2^3 + x_3^5 + x_4^2. \end{aligned}$$

By using [Yano 1978, Corollary 3.17], we can compute $\tilde{\alpha}_{\tilde{g}_0,0}$ in each case. In fact, $\tilde{\alpha}_{\tilde{g}_0,0} \leq \frac{3}{2}$ if \tilde{g}_0 is not smooth at 0. Moreover, it is easy to find a log-resolution π of \tilde{g}_0 such that for all numerical data $(\nu, N) \neq (1, 1)$ we have $\nu \geq \tilde{\alpha}_{\tilde{g}_0,0}N$. For example, in the first form, we have $\tilde{\alpha}_{\tilde{g}_0,0} = 1 + 1/(d + 1)$ and we can choose π such that its numerical data is $\{(1, 1)\} \cup \{(3 + 2m, 2 + 2m) \mid 0 \leq m \leq k - 1\} \cup \{(2 + 2k, 2k + 1)\}$ or $\{(1, 1)\} \cup \{(3 + 2m, 2 + 2m) \mid 0 \leq m \leq k - 1\}$ depending on $d = 2k$ or $d = 2k - 1$, respectively. Thus, it follows from [Igusa 1974; 1978] that $\alpha \leq -\tilde{\alpha}_{\tilde{g}_0,0}$ for all $\alpha \in \text{Pol}_{\tilde{U}'}(\tilde{g}_0)$.

We now show that the multiplicity e of g_0 at 0 is at most 2. If this is not true, we can find a log-resolution π' of g_0 such that there is a numerical data $(\nu, N) \neq (1, 1)$ with $\nu \leq N$ (indeed, we can find a log-resolution of g_0 after using a blowing up at 0, thus the numerical data $(3, e)$ appears in this log-resolution). Thus we can use [Igusa 1974; 1978; Veys and Zúñiga Galindo 2008] as above to contradict the facts that $\text{Pol}_{\tilde{U}'}(\tilde{g}_0) = \text{Pol}_{\tilde{U}'}(g_0)$ and $\alpha \leq -\tilde{\alpha}_{\tilde{g}_0,0}$ for all $\alpha \in \text{Pol}_{\tilde{U}'}(\tilde{g}_0)$. This shows that e is at most 2. If $e = 1$ or \tilde{g}_0 is smooth at 0 then it is easy to have that $\text{Pol}_{\tilde{U}'}(\tilde{g}_0) = \text{Pol}_{\tilde{U}'}(g_0) = \text{Pol}_{U''}(g_0) = \emptyset$, so our claim follows. If $e = 2$ and \tilde{g}_0 is not smooth at 0, we can use Weierstrass preparation again and the above argument to suppose that $g_0(x_2, x_3, x_4) = x_2^2 + h(x_3, x_4)$ for some analytic function h in at most two variables. We also have, as above, that $\text{Pol}_U(f) = -1 + \text{Pol}_{U''}(h)$ and $\text{Pol}_{\tilde{U}'}(\tilde{g}_0) = \text{Pol}_{\tilde{U}'}(g_0) = -\frac{1}{2} + \text{Pol}_{\tilde{U}''}(h)$, where $U'' = p^{m_0}\mathbb{Z}_p^2$ and $\tilde{U}'' = \varpi_L^{m_1}\mathcal{O}_L^2$. If there is $\beta \in \text{Pol}_{U''}(h)$ with $\beta > -\tilde{\alpha}_{f,0} + 1 = -\tilde{\alpha}_{\tilde{g}_0,0} + \frac{1}{2} \geq -1$ then we can repeat the above argument to get a contradiction. Our desired result (57) now follows from $\text{Pol}_U(f) = -1 + \text{Pol}_{U''}(h)$ and $\beta \leq -\tilde{\alpha}_{f,0} + 1$ for all $\beta \in \text{Pol}_{U''}(h)$. □

Remark 5.7. For each of the above cases in which Conjecture 1 is shown in this paper, one moreover sees that, after excluding a finite set S (which depends on f) of prime divisors of N , the implied constant can be taken depending only on d and n (and on ε). The only case where this is not directly clear is for the case with $(n - s)/d \leq 1$, since its treatment in [Cluckers et al. 2019] uses a chosen log resolution which depends on f . However, the complexity of such log resolutions (and of the corresponding proof in [Cluckers et al. 2019]) remains bounded when n and d are fixed. Indeed, one first takes a log resolution of a generic polynomial of degree d in n variables; this then yields a log resolution for polynomials whose coefficients lie in a dense Zariski open subset U of the parameter space. One proceeds similarly for a generic polynomial with parameters in the complement of the dense open U .

Note that the exclusion of a finite list of prime divisors of N is necessary, as can be seen when one replaces a polynomial f by pf for some prime p . It is not clear at the moment whether the finite set S has to depend fully on f in general, or, just on f_d .

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References

- [Baker 1983] R. C. Baker, “Small solutions of congruences”, *Mathematika* **30**:2 (1983), 164–188. MR Zbl
- [Birch 1962] B. J. Birch, “Forms in many variables”, *Proc. Roy. Soc. London Ser. A* **265** (1962), 245–263. MR Zbl
- [Browning and Heath-Brown 2017] T. Browning and R. Heath-Brown, “Forms in many variables and differing degrees”, *J. Eur. Math. Soc. (JEMS)* **19**:2 (2017), 357–394. MR Zbl
- [Browning and Prendiville 2017] T. D. Browning and S. M. Prendiville, “Improvements in Birch’s theorem on forms in many variables”, *J. Reine Angew. Math.* **731** (2017), 203–234. MR Zbl
- [Castrycck and Nguyen 2019] W. Castryck and K. H. Nguyen, “New bounds for exponential sums with a non-degenerate phase polynomial”, *J. Math. Pures Appl.* (9) **130** (2019), 93–111. MR Zbl
- [Chalk 1987] J. H. H. Chalk, “On Hua’s estimates for exponential sums”, *Mathematika* **34**:2 (1987), 115–123. MR Zbl
- [Chambille and Nguyen 2021] S. Chambille and K. H. Nguyen, “Proof of the Cluckers–Veys conjecture on exponential sums for polynomials with log-canonical threshold at most a half”, *Int. Math. Res. Not.* **2021**:9 (2021), 6498–6542. MR Zbl
- [Cluckers 2008a] R. Cluckers, “Igusa and Denef–Sperber conjectures on nondegenerate p -adic exponential sums”, *Duke Math. J.* **141**:1 (2008), 205–216. MR Zbl
- [Cluckers 2008b] R. Cluckers, “Igusa’s conjecture on exponential sums modulo p and p^2 and the motivic oscillation index”, *Int. Math. Res. Not.* **2008**:4 (2008), art. id. rnm118. MR Zbl
- [Cluckers 2010] R. Cluckers, “Exponential sums: questions by Denef, Sperber, and Igusa”, *Trans. Amer. Math. Soc.* **362**:7 (2010), 3745–3756. MR Zbl
- [Cluckers and Veys 2016] R. Cluckers and W. Veys, “Bounds for p -adic exponential sums and log-canonical thresholds”, *Amer. J. Math.* **138**:1 (2016), 61–80. MR Zbl
- [Cluckers et al. 2016] R. Cluckers, J. Gordon, and I. Halupczok, “Transfer principles for bounds of motivic exponential functions”, pp. 111–127 in *Families of automorphic forms and the trace formula*, edited by W. Müller et al., Springer, 2016. MR Zbl
- [Cluckers et al. 2018] R. Cluckers, J. Gordon, and I. Halupczok, “Uniform analysis on local fields and applications to orbital integrals”, *Trans. Amer. Math. Soc. Ser. B* **5** (2018), 125–166. MR Zbl
- [Cluckers et al. 2019] R. Cluckers, M. Mustață, and K. H. Nguyen, “Igusa’s conjecture for exponential sums: optimal estimates for nonrational singularities”, *Forum Math. Pi* **7** (2019), art. id. e3. MR Zbl
- [Cochrane and Zheng 1999] T. Cochrane and Z. Zheng, “Pure and mixed exponential sums”, *Acta Arith.* **91**:3 (1999), 249–278. MR Zbl
- [Cochrane and Zheng 2000] T. Cochrane and Z. Zheng, “Exponential sums with rational function entries”, *Acta Arith.* **95**:1 (2000), 67–95. MR Zbl
- [Deligne 1974] P. Deligne, “La conjecture de Weil, I”, *Inst. Hautes Études Sci. Publ. Math.* **43** (1974), 273–307. MR Zbl
- [Denef 1991] J. Denef, “Report on Igusa’s local zeta function”, exposé 741, pp. 359–386 in *Séminaire Bourbaki*, 1990/1991, Astérisque **201–203**, 1991. MR Zbl
- [Denef and Sperber 2001] J. Denef and S. Sperber, “Exponential sums mod p^n and Newton polyhedra”, *Bull. Belg. Math. Soc. Simon Stevin suppl.* (2001), 55–63. MR Zbl
- [Denef and Veys 1995] J. Denef and W. Veys, “On the holomorphy conjecture for Igusa’s local zeta function”, *Proc. Amer. Math. Soc.* **123**:10 (1995), 2981–2988. MR Zbl
- [Durfee 1979] A. H. Durfee, “Fifteen characterizations of rational double points and simple critical points”, *Enseign. Math.* (2) **25**:1-2 (1979), 131–163. MR Zbl

- [Fraser and Wright 2020] R. Fraser and J. Wright, “The local sum conjecture in two dimensions”, *Int. J. Number Theory* **16**:8 (2020), 1667–1699. MR Zbl
- [Heath-Brown 1985] D. R. Heath-Brown, “A multiple exponential sum to modulus p^2 ”, *Canad. Math. Bull.* **28**:4 (1985), 394–396. MR Zbl
- [Heath-Brown 1996] D. R. Heath-Brown, “A new form of the circle method, and its application to quadratic forms”, *J. Reine Angew. Math.* **481** (1996), 149–206. MR Zbl
- [Hironaka 1964] H. Hironaka, “Resolution of singularities of an algebraic variety over a field of characteristic zero, I”, *Ann. of Math. (2)* **79**:1 (1964), 109–203. MR Zbl
- [Hooley 1988] C. Hooley, “On nonary cubic forms”, *J. Reine Angew. Math.* **386** (1988), 32–98. MR Zbl
- [Hooley 1991] C. Hooley, “On the number of points on a complete intersection over a finite field”, *J. Number Theory* **38**:3 (1991), 338–358. MR Zbl
- [Hua 1959] L.-K. Hua, *Additive Primzahltheorie*, B. G. Teubner, Leipzig, 1959. MR Zbl
- [Igusa 1974] J.-I. Igusa, “Complex powers and asymptotic expansions, I: Functions of certain types”, *J. Reine Angew. Math.* **268/269** (1974), 110–130. MR Zbl
- [Igusa 1975] J.-I. Igusa, “Complex powers and asymptotic expansions, II: Asymptotic expansions”, *J. Reine Angew. Math.* **278/279** (1975), 307–321. MR Zbl
- [Igusa 1978] J.-I. Igusa, *Forms of higher degree*, Lectures on Mathematics and Physics **59**, Tata Institute of Fundamental Research, Bombay, 1978. MR Zbl
- [Katz 1999] N. M. Katz, “Estimates for “singular” exponential sums”, *Internat. Math. Res. Notices* **1999**:16 (1999), 875–899. MR Zbl
- [Kouchnirenko 1976] A. G. Kouchnirenko, “Polyèdres de Newton et nombres de Milnor”, *Invent. Math.* **32**:1 (1976), 1–31. MR Zbl
- [Lichtin 2013] B. Lichtin, “On a conjecture of Igusa”, *Mathematika* **59**:2 (2013), 399–425. MR Zbl
- [Mustață 2012] M. Mustață, “IMPANGA lecture notes on log canonical thresholds”, pp. 407–442 in *Contributions to algebraic geometry*, edited by P. Pragacz, Eur. Math. Soc., Zürich, 2012. MR Zbl
- [Mustață and Popa 2020] M. Mustață and M. Popa, “Hodge ideals for \mathbb{Q} -divisors, V -filtration, and minimal exponent”, *Forum Math. Sigma* **8** (2020), art. id. e19. MR Zbl
- [Nguyen 2021] K. H. Nguyen, “On a uniform bound for exponential sums modulo p^m for Deligne polynomials”, preprint, 2021. arXiv 2111.11898
- [Nguyen and Veys 2022] K. H. Nguyen and W. Veys, “On the motivic oscillation index and bound of exponential sums modulo p^m via analytic isomorphisms”, *J. Math. Pures Appl. (9)* **157** (2022), 211–242. MR Zbl
- [Saito 1993] M. Saito, “On b -function, spectrum and rational singularity”, *Math. Ann.* **295**:1 (1993), 51–74. MR Zbl
- [Varčenko 1976] A. N. Varčenko, “Newton polyhedra and estimates of oscillatory integrals”, *Funkcional. Anal. i Priložen.* **10**:3 (1976), 13–38. In Russian; translated in *Functional Anal. Appl.* **18**:3 (1976), 175–196. MR Zbl
- [Veys 2020] W. Veys, “On the log canonical threshold and numerical data of a resolution in dimension 2”, *Manuscripta Math.* **163**:1-2 (2020), 1–11. MR Zbl
- [Veys and Zúñiga Galindo 2008] W. Veys and W. A. Zúñiga Galindo, “Zeta functions for analytic mappings, log-principalization of ideals, and Newton polyhedra”, *Trans. Amer. Math. Soc.* **360**:4 (2008), 2205–2227. MR Zbl
- [Weil 1965] A. Weil, “Sur la formule de Siegel dans la théorie des groupes classiques”, *Acta Math.* **113** (1965), 1–87. MR Zbl
- [Yano 1978] T. Yano, “On the theory of b -functions”, *Publ. Res. Inst. Math. Sci.* **14**:1 (1978), 111–202. MR Zbl

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