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**Polyhedral and tropical geometry of flag positroids**

Jonathan Boretsky, Christopher Eur and Lauren Williams





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A *flag positroid* of ranks  $\mathbf{r} := (r_1 < \dots < r_k)$  on  $[n]$  is a flag matroid that can be realized by a real  $r_k \times n$  matrix  $A$  such that the  $r_i \times r_i$  minors of  $A$  involving rows  $1, 2, \dots, r_i$  are nonnegative for all  $1 \leq i \leq k$ . In this paper we explore the polyhedral and tropical geometry of flag positroids, particularly when  $\mathbf{r} := (a, a + 1, \dots, b)$  is a sequence of consecutive numbers. In this case we show that the nonnegative tropical flag variety  $\text{TrFl}_{\mathbf{r},n}^{\geq 0}$  equals the nonnegative flag Dressian  $\text{FIDr}_{\mathbf{r},n}^{\geq 0}$ , and that the points  $\boldsymbol{\mu} = (\mu_a, \dots, \mu_b)$  of  $\text{TrFl}_{\mathbf{r},n}^{\geq 0} = \text{FIDr}_{\mathbf{r},n}^{\geq 0}$  give rise to coherent subdivisions of the flag positroid polytope  $P(\boldsymbol{\mu})$  into flag positroid polytopes. Our results have applications to Bruhat interval polytopes: for example, we show that a complete flag matroid polytope is a Bruhat interval polytope if and only if its  $(\leq 2)$ -dimensional faces are Bruhat interval polytopes. Our results also have applications to realizability questions. We define a *positively oriented flag matroid* to be a sequence of positively oriented matroids  $(\chi_1, \dots, \chi_k)$  which is also an oriented flag matroid. We then prove that every positively oriented flag matroid of ranks  $\mathbf{r} = (a, a + 1, \dots, b)$  is realizable.

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## 1. Introduction

In recent years there has been a great deal of interest in the tropical Grassmannian [Speyer and Sturmfels 2004; Herrmann et al. 2009; 2014; Cachazo et al. 2019; Bossinger 2021], and matroid polytopes and their subdivisions [Speyer 2008; Ardila et al. 2010; Early 2022], as well as “positive” [Postnikov 2007; Speyer and Williams 2005; 2021; Oh 2008; Ardila et al. 2016; Le and Fraser 2019; Lukowski et al. 2023; Arkani-Hamed et al. 2021b] and “flag” [Tsukerman and Williams 2015; Brandt et al. 2021; Bossinger et al. 2017; Jarra and Lorscheid 2024; Joswig et al. 2023; Boretsky 2022] versions of the above objects. The aim

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of this paper is to illustrate the beautiful relationships between the nonnegative tropical flag variety, the nonnegative flag Dressian, and flag positroid polytopes and their subdivisions, unifying and generalizing some of the existing results. We will particularly focus on the case of flag varieties (respectively, flag positroids) consisting of subspaces (respectively, matroids) of *consecutive* ranks. This case includes both Grassmannians and complete flag varieties.

For positive integers  $n$  and  $d$  with  $d < n$ , we let  $[n]$  denote the set  $\{1, \dots, n\}$  and we let  $\binom{[n]}{d}$  denote the collection of all  $d$ -element subsets of  $[n]$ . Given a subset  $S \subseteq [n]$  we let  $\mathbf{e}_S$  denote the sum of standard basis vectors  $\sum_{i \in S} \mathbf{e}_i$ . For a collection  $\mathcal{B} \subset \binom{[n]}{d}$ , we let

$$P(\mathcal{B}) = \text{the convex hull of } \{\mathbf{e}_B : B \in \mathcal{B}\} \text{ in } \mathbb{R}^n.$$

The collection  $\mathcal{B}$  is said to define a *matroid*  $M$  of rank  $d$  on  $[n]$  if every edge of the polytope  $P(\mathcal{B})$  is parallel to  $\mathbf{e}_i - \mathbf{e}_j$  for some  $i \neq j \in [n]$ . In this case, we call  $\mathcal{B}$  the set of *bases* of  $M$ , and define the *matroid polytope*  $P(M)$  of  $M$  to be the polytope  $P(\mathcal{B})$ . When  $\mathcal{B}$  indexes the nonvanishing Plücker coordinates of an element  $A$  of the Grassmannian  $\text{Gr}_{d,n}(\mathbb{C})$ , we say that  $A$  *realizes*  $M$ , and it is well-known that  $P(\mathcal{B})$  is the moment map image of the closure of the torus orbit of  $A$  in the Grassmannian [Gelfand et al. 1987]. We assume familiarity with the fundamentals of matroid theory as in [Oxley 2011] and [Borovik et al. 2003].

The above definition of matroid in terms of its polytope is due to [Gelfand et al. 1987]. Flag matroids are natural generalizations of matroids that admit the following polytopal definition.

**Definition 1.1** [Borovik et al. 2003, Corollary 1.13.5 and Theorem 1.13.6]. Let  $\mathbf{r} = (r_1, \dots, r_k)$  be a sequence of increasing integers in  $[n]$ . A *flag matroid* of ranks  $\mathbf{r}$  on  $[n]$  is a sequence  $\mathbf{M} = (M_1, \dots, M_k)$  of matroids of ranks  $(r_1, \dots, r_k)$  on  $[n]$  such that all vertices of the polytope

$$P(\mathbf{M}) = P(M_1) + \dots + P(M_k), \text{ the Minkowski sum of matroid polytopes,}$$

are equidistant from the origin. The polytope  $P(\mathbf{M})$  is called the *flag matroid polytope* of  $\mathbf{M}$ ; we sometimes say it is a flag matroid polytope of *rank*  $\mathbf{r}$ .

Flag matroids are exactly the type  $A$  objects in the theory of Coxeter matroids [Gelfand and Serganova 1987; Borovik et al. 2003]. Just as a realization of a matroid is a point in a Grassmannian, a realization of a flag matroid is a point in a flag variety. More concretely, a *realization* of a flag matroid of ranks  $(r_1, \dots, r_k)$  is an  $r_k \times n$  matrix  $A$  over a field such that for each  $1 \leq i \leq k$ , the  $r_i \times n$  submatrix of  $A$  formed by the first  $r_i$  rows of  $A$  is a realization of  $M_i$ . For an equivalent definition of flag matroids in terms of Plücker relations on partial flag varieties; see [Jarra and Lorscheid 2024, Proposition A].

There is a notion of moment map for any flag variety (indeed for any generalized partial flag variety  $G/P$ ) [Gelfand and Serganova 1987; Borovik et al. 2003]. When a flag matroid  $\mathbf{M}$  can be realized by a point  $A$  in the flag variety, then its matroid polytope  $P(\mathbf{M})$  is the moment map image of the closure of the torus orbit of  $A$  in the flag variety [Gelfand and Serganova 1987; Borovik et al. 2003, Corollary 1.13.5].

There are natural “positive” analogues of matroids, flag matroids, and their polytopes.

**Definition 1.2.** Let  $\mathbf{r} = (r_1, \dots, r_k)$  be a sequence of increasing integers in  $[n]$ . We say that a flag matroid  $(M_1, \dots, M_k)$  of ranks  $\mathbf{r}$  on  $[n]$  is a *flag positroid* if it has a realization by a real matrix  $A$  such that the  $r_i \times n$  submatrix of  $A$  formed by the first  $r_i$  rows of  $A$  has all nonnegative minors for each  $1 \leq i \leq k$ .

We refer to the flag matroid polytope of a flag positroid as a *flag positroid polytope*. It follows from our definition above that flag positroids are realizable.

Setting  $k = 1$  in Definition 1.2 gives the well-studied notion of *positroids* and *positroid polytopes* [Postnikov 2007; Oh 2008; Ardila et al. 2016]. Therefore each flag positroid is a sequence of positroids.

In recent years it has been gradually understood that the tropical geometry of the Grassmannian and flag variety, and in particular, the *Dressian* and *flag Dressian*, are intimately connected to (flag) matroid polytopes and their subdivisions [Speyer 2008; Herrmann et al. 2009; Brandt et al. 2021]; see also [Maclagan and Sturmfels 2015, Section 4]. A particularly attractive point of view, which sheds light on the above connections, is the theory of (*flag*) *matroids over hyperfields* [Baker and Bowler 2019; Jarra and Lorscheid 2024]. In this framework, the Dressian and flag Dressian are the Grassmannian and flag variety over the *tropical hyperfield*, while matroids and flag matroids are the points of the Grassmannian and flag variety over the *Krasner hyperfield*.

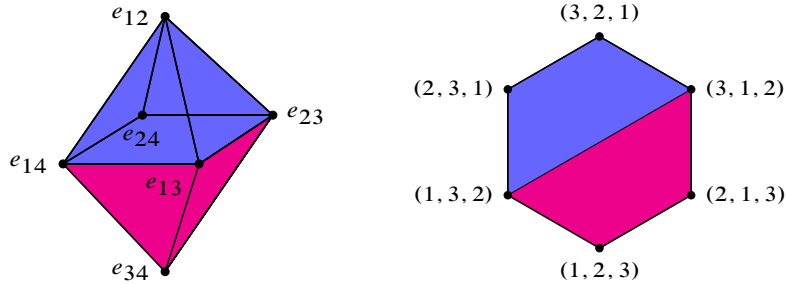
The tropical geometry of the *positive* Grassmannian and flag variety are particularly nice: the positive tropical Grassmannian equals the positive Dressian, whose cones in turn parametrize subdivisions of the hypersimplex into positroid polytopes [Speyer and Williams 2005; 2021; Lukowski et al. 2023; Arkani-Hamed et al. 2021b]. And the positive tropical complete flag variety equals the positive complete flag Dressian, whose cones parametrize subdivisions of the permutohedron into *Bruhat interval polytopes* [Boretsky 2022; Joswig et al. 2023]. Theorem A below unifies and generalizes the above results.

**Definition 1.3.** Let  $\mathbb{T} = \mathbb{R} \cup \{\infty\}$  be the set underlying the tropical hyperfield, endowed with the topology such that  $-\log : \mathbb{R}_{\geq 0} \rightarrow \mathbb{T}$  is a homeomorphism. Given a point  $w \in \mathbb{T}^{\binom{[n]}{r}}$ , we define the *support* of  $w$  to be  $\underline{w} = \{S \in \binom{[n]}{r} : w_S \neq \infty\}$ . When  $\underline{w}$  is the set of bases of a matroid, we identify  $\underline{w}$  with that matroid. Let  $\mathbb{P}(\mathbb{T}^{\binom{[n]}{r}})$  be the tropical projective space of  $\mathbb{T}^{\binom{[n]}{r}}$ , which is defined as  $(\mathbb{T}^{\binom{[n]}{r}} \setminus \{(\infty, \dots, \infty)\}) / \sim$ , where  $w \sim w'$  if  $w = w' + (c, \dots, c)$  for some  $c \in \mathbb{R}$ .

Our main result is the following.

**Theorem A.** Suppose  $\mathbf{r}$  is a sequence of consecutive integers  $(a, \dots, b)$  for some  $1 \leq a \leq b \leq n$ . Then, for  $\underline{\mu} = (\mu_a, \dots, \mu_b) \in \prod_{i=a}^b \mathbb{P}(\mathbb{T}^{\binom{[n]}{i}})$ , the following statements are equivalent:

- (a)  $\underline{\mu} \in \text{TrFI}_{\mathbf{r},n}^{\geq 0}$ , the nonnegative tropicalization of the flag variety, i.e., the closure of the coordinate-wise valuation of points in  $\text{Fl}_{\mathbf{r},n}(\mathbb{C}_{\geq 0})$ .
- (b)  $\underline{\mu} \in \text{FIDr}_{\mathbf{r},n}^{\geq 0}$ , the nonnegative flag Dressian, i.e., the “solutions” to the positive-tropical Grassmann–Plücker and incidence–Plücker relations.
- (c) Every face in the coherent subdivision  $\mathcal{D}_{\underline{\mu}}$  of the polytope  $P(\underline{\mu}) = P(\underline{\mu}_1) + \dots + P(\underline{\mu}_k)$  induced by  $\underline{\mu}$  is a flag positroid polytope (of rank  $\mathbf{r}$ ).



**Figure 1.** Left-hand side: the coherent subdivision of the hypersimplex into positroid polytopes induced by a point  $\underline{\mu} \in \text{Dr}_{2,4}^{>0}$  such that  $\mu_{13} + \mu_{24} = \mu_{23} + \mu_{14} < \mu_{12} + \mu_{34}$ . Right-hand side: the coherent subdivision of the permutohedron into Bruhat interval polytopes induced by a point  $\underline{\mu} \in \text{FIDr}_{(1,2,3),3}^{>0}$  such that  $\mu_2 + \mu_{13} = \mu_1 + \mu_{23} < \mu_3 + \mu_{12}$ .

- (d) Every face of dimension at most 2 in the subdivision  $\mathcal{D}_{\underline{\mu}}$  of  $P(\underline{\mu})$  is a flag positroid polytope (of rank  $r$ ).
- (e) The support  $\underline{\mu}$  of  $\underline{\mu}$  is a flag matroid, and  $\underline{\mu}$  satisfies every three-term positive-tropical incidence relation (respectively, every three-term positive-tropical Grassmann–Plücker relation) when  $a < b$  (respectively,  $a = b$ ).

For the definitions of the objects in Theorem A, see Proposition 3.6 for (a), Definition 3.3 for (b), Definition 5.1 for (c), and Definition 3.8 for (e).

We note that if  $r = (d)$  is a single integer, Theorem A describes the relationship between the nonnegative tropical Grassmannian, the nonnegative Dressian, and subdivisions of positroid polytopes (e.g., the hypersimplex, if  $\underline{\mu}$  has no coordinates equal to  $\infty$ ) into positroid polytopes. And when  $r = (1, 2, \dots, n)$ , Theorem A describes the relationship between the nonnegative tropical complete flag variety, the nonnegative complete flag Dressian, and subdivisions of Bruhat interval polytopes (e.g., the permutohedron, if  $\underline{\mu}$  has no coordinates equal to  $\infty$ ) into Bruhat interval polytopes. We illustrate this relationship in the case where  $\underline{\mu}$  has no coordinates equal to  $\infty$  in Figure 1.

We prove the equivalence (a) $\iff$ (b) in Section 3.2, the implications (b) $\implies$ (c) $\implies$ (d) $\implies$ (e) in Section 5.2, and the implication (e) $\implies$ (b) in Section 6.1.

Theorem A has applications to flag positroid polytopes.

**Corollary 1.4.** *For a flag matroid  $\mathbf{M} = (M_a, M_{a+1}, \dots, M_b)$  of consecutive ranks  $\mathbf{r} = (a, a + 1, \dots, b)$ , its flag matroid polytope  $P(\mathbf{M})$  is a flag positroid polytope if and only if its ( $\leq 2$ )-dimensional faces are flag positroid polytopes (of rank  $\mathbf{r}$ ).*

*Proof.* Let  $\underline{\mu} = (\mu_a, \dots, \mu_b)$ , with  $\mu_i \in \{0, \infty\}^{\binom{[n]}{i}}$ , where the coordinates of each  $\mu_i$  are either 0 or  $\infty$  based on whether we have a basis or nonbasis of  $M_i$ . This gives rise to the trivial subdivision of the corresponding flag matroid polytope  $P(\underline{\mu}) = P(\mathbf{M})$ . The result now follows from the equivalence of (c) and (d) in Theorem A. □

In the Grassmannian case, that is, the case that  $\mathbf{r} = (d)$  is a single integer, the flag positroid polytopes of rank  $\mathbf{r}$  are precisely the positroid polytopes, and in that case the above corollary appeared as [Lukowski et al. 2023, Theorem 3.9].

Also in the Grassmannian case, the objects discussed in Theorem A are closely related to questions of realizability. Note that by definition, every positroid has a realization by a matrix whose Plücker coordinates are nonnegative, so it naturally defines a *positively oriented matroid*, that is, an oriented matroid defined by a chirotope whose values are all 0 and 1. Conversely, every positively oriented matroid can be realized by a positroid: this was first proved in [Ardila et al. 2017] using positroid polytopes, and subsequently in [Speyer and Williams 2021], using the positive tropical Grassmannian. It is natural then to ask if there is an analogous realizability statement in the setting of flag matroids, and if one can characterize when a sequence of positroids forms a flag positroid; indeed, this was part of the motivation for [Benedetti et al. 2022], which studied quotients of uniform positroids. Note however that questions of realizability for flag matroids are rather subtle: for example, a sequence of positroids that form a realizable flag matroid can still fail to be a flag positroid (see Example 4.4). By working with *oriented* flag matroids, we give an answer to this realizability question in Corollary 1.5, in the case of consecutive ranks.

**Corollary 1.5.** *Suppose  $(M_1, \dots, M_k)$  is a sequence of positroids on  $[n]$  of consecutive ranks  $\mathbf{r} = (r_1, \dots, r_k)$ . Then, when considered as a sequence of positively oriented matroids,  $(M_1, \dots, M_k)$  is a flag positroid if and only if it is an oriented flag matroid.*

We define a *positively oriented flag matroid* to be a sequence of positively oriented matroids  $(\chi_1, \dots, \chi_k)$  which is also an oriented flag matroid. Corollary 1.5 then says that every positively oriented flag matroid of consecutive ranks  $(r_1, \dots, r_k)$  is realizable.

See Section 4.1 for a review of oriented matroids and oriented flag matroids. Note that because a positroid by definition has a realization over  $\mathbb{R}$  with all nonnegative minors, it defines a positively oriented matroid. In Section 4.2, we deduce Corollary 1.5 from the equivalence of (a) and (b) in Theorem A. Another proof using ideas from discrete convex analysis is sketched in Remark 4.7. In both proofs, the consecutive ranks condition is indispensable. We do not know whether the corollary holds if  $\mathbf{r} = (r_1, \dots, r_k)$  fails to satisfy the consecutive rank condition.

**Question 1.6.** Suppose  $M$  and  $M'$  are positroids on  $[n]$  such that, when considered as positively oriented matroids, they form an oriented flag matroid  $(M, M')$ . Is  $(M, M')$  then a flag positroid?

One may attempt to answer the question by appealing to the fact [Kung 1986, Exercise 8.14] that for a flag matroid  $(M, M')$ , one can always find a flag matroid  $(M_1, \dots, M_k)$  of consecutive ranks such that  $M_1 = M$  and  $M_k = M'$ . However, the analogous statement fails for flag positroids; see Example 4.6 for an example of a flag positroid  $(M, M')$  on  $[4]$  of ranks  $(1, 3)$  such that there is no flag positroid  $(M, M_2, M')$  with rank of  $M_2$  equal to 2.

The consecutive rank condition has recently shown up in [Bloch and Karp 2023], which studied the relation between two notions of total positivity for partial flag varieties, “Lusztig positivity” and “Plücker

positivity” (see Section 2.1). In particular, the Plücker positive subset of a partial flag variety agrees with the Lusztig positive subset of the partial flag variety precisely when the flag variety consists of linear subspaces of consecutive ranks [Bloch and Karp 2023, Theorem 1.1].

A *generalized Bruhat interval polytope* [Tsukerman and Williams 2015, Definition 7.8 and Lemma 7.9] can be defined as the moment map image of the closure of the torus orbit of a point  $A$  in the nonnegative part  $(G/P)^{\geq 0}$  (in the sense of Lusztig) of a flag variety  $G/P$ . When  $\mathbf{r}$  is a sequence of consecutive integers, it then follows from [Bloch and Karp 2023] that generalized Bruhat interval polytopes for  $\text{Fl}_{\mathbf{r};n}^{\geq 0}$  are precisely the flag positroid polytopes of ranks  $\mathbf{r}$ . In the complete flag case, a generalized Bruhat interval polytope is just a *Bruhat interval polytope* [Kodama and Williams 2015], that is, the convex hull of the permutation vectors  $(z(1), \dots, z(n))$  for all permutations  $z$  lying in some Bruhat interval  $[u, v]$ .

We can now restate Corollary 1.4 as follows.

**Corollary 1.7.** *For a flag matroid on  $[n]$  of consecutive ranks  $\mathbf{r}$ , its flag matroid polytope is a generalized Bruhat interval polytope if and only if its  $(\leq 2)$ -dimensional faces are generalized Bruhat interval polytopes. In particular, for a complete flag matroid on  $[n]$ , its flag matroid polytope is a Bruhat interval polytope if and only if its  $(\leq 2)$ -dimensional faces are Bruhat interval polytopes.*

The structure of this paper is as follows. In Section 2, we give background on total positivity and Bruhat interval polytopes. In Section 3, we introduce the tropical flag variety, the flag Dressian, and nonnegative analogues of these objects; we also prove the equivalence of (a) and (b) in Theorem A. In Section 4 we discuss positively oriented flag matroids and prove Corollary 1.5. In Section 5 we explain the relation between the flag Dressian and subdivisions of flag matroid polytopes, then prove that (b) $\implies$ (c) $\implies$ (d) $\implies$ (e) in Theorem A. We prove some key results about three-term incidence and Grassmann–Plücker relations in Section 6, which allow us to prove (e) $\implies$ (b) in Theorem A. Section 7 concerns projections of positive Richardson’s to positroids: we characterize the positroid constituents of complete flag positroids, and we characterize when two adjacent-rank positroids form an oriented matroid quotient, or equivalently, can appear as constituents of a complete flag positroid. In Section 8, we make some remarks about the various fan structures for  $\text{TrFl}_{\mathbf{r};n}^{>0}$ ; we then discuss fan structures and coherent subdivisions in the case of the Grassmannian and complete flag variety, including a detailed look at the case of  $\text{TrFl}_4^{>0}$ .

## 2. Background on total positivity and Bruhat interval polytopes

**2.1. Background on total positivity.** Let  $n \in \mathbb{Z}_+$  and let  $\mathbf{r} = \{r_1 < \dots < r_k\} \subseteq [n]$ . For a field  $\mathbb{k}$ , let  $G = \text{GL}_n(\mathbb{k})$ , and let  $P_{\mathbf{r};n}(\mathbb{k})$  denote the parabolic subgroup of  $G$  of block upper-triangular matrices with diagonal blocks of sizes  $r_1, r_2 - r_1, \dots, r_k - r_{k-1}, n - r_k$ . We define the *partial flag variety*

$$\text{Fl}_{\mathbf{r};n}(\mathbb{k}) := \text{GL}_n(\mathbb{k}) / P_{\mathbf{r};n}(\mathbb{k}).$$

As usual, we identify  $\text{Fl}_{\mathbf{r};n}(\mathbb{k})$  with the variety of partial flags of subspaces in  $\mathbb{k}^n$ :

$$\text{Fl}_{\mathbf{r};n}(\mathbb{k}) = \{(V_1 \subset \dots \subset V_k) : V_i \text{ a linear subspace of } \mathbb{k}^n \text{ of dimension } r_i \text{ for } i = 1, \dots, k\}.$$

We write  $\text{Fl}_n(\mathbb{k})$  for the complete flag variety  $\text{Fl}_{1,2,\dots,n;n}(\mathbb{k})$ . Note that  $\text{Fl}_n(\mathbb{k})$  can be identified with  $\text{GL}_n(\mathbb{k})/B(\mathbb{k})$ , where  $B(\mathbb{k})$  is the subgroup of upper-triangular matrices. There is a natural projection  $\pi$  from  $\text{Fl}_n(\mathbb{k})$  to any partial flag variety by simply forgetting some of the subspaces.

If  $A$  is an  $r_k \times n$  matrix such that  $V_{r_i}$  is the span of the first  $r_i$  rows, we say that  $A$  is a *realization* of  $V := (V_1 \subset \dots \subset V_k) \in \text{Fl}_{r;n}$ . Given any realization  $A$  of  $V$  and any  $1 \leq i \leq k$ , we have the *Plücker coordinates* or *flag minors*  $p_I(A)$  where  $I \in \binom{[n]}{r_i}$ ; concretely,  $p_I(A)$  is the determinant of the submatrix of  $A$  occupying the first  $r_i$  rows and columns  $I$ . This gives the Plücker embedding of  $\text{Fl}_{r;n}(\mathbb{k})$  into  $\mathbb{P}^{\binom{[n]}{r_1}-1} \times \dots \times \mathbb{P}^{\binom{[n]}{r_k}-1}$  taking  $V$  to  $((p_I(A))_{I \in \binom{[n]}{r_1}}, \dots, (p_I(A))_{I \in \binom{[n]}{r_k}})$ .

We now let  $\mathbb{k}$  be the field  $\mathbb{R}$  of real numbers. With this understanding, we will often drop the  $\mathbb{R}$  from our notation.

**Definition 2.1.** We say that a real matrix is *totally positive* if all of its minors are positive. We let  $\text{GL}_n^{>0}$  denote the subset of  $\text{GL}_n$  of totally positive matrices.

There are two natural ways to define positivity for partial flag varieties. The first notion comes from work of Lusztig [1994]. The second notion uses Plücker coordinates, and was initiated in work of Postnikov [2007].

**Definition 2.2.** We define the (*Lusztig*) *positive part* of  $\text{Fl}_{r;n}$ , denoted by  $\text{Fl}_{r;n}^{>0}$ , as the image of  $\text{GL}_n^{>0}$  inside  $\text{Fl}_{r;n} = \text{GL}_n / \text{P}_{r;n}$ . We define the (*Lusztig*) *nonnegative part* of  $\text{Fl}_{r;n}$ , denoted by  $\text{Fl}_{r;n}^{\geq 0}$ , as the closure of  $\text{Fl}_{r;n}^{>0}$  in the Euclidean topology.

We define the *Plücker positive part* (respectively, *Plücker nonnegative part*) of  $\text{Fl}_{r;n}$  to be the subset of  $\text{Fl}_{r;n}$  where all Plücker coordinates are positive (respectively, nonnegative).<sup>1</sup>

It is well-known that the Lusztig positive part of  $\text{Fl}_{r;n}$  is a subset of the Plücker positive part of  $\text{Fl}_{r;n}$ , and that the two notions agree in the case of the Grassmannian [Talaska and Williams 2013, Corollary 1.2]. The two notions also agree in the case of the complete flag variety [Boretsky 2022, Theorem 5.21]. More generally, we have the following.

**Theorem 2.3** [Bloch and Karp 2023, Theorem 1.1]. *The Lusztig positive (respectively, Lusztig nonnegative) part of  $\text{Fl}_{r;n}$  equals the Plücker positive (respectively, Plücker nonnegative) part of  $\text{Fl}_{r;n}$  if and only if the set  $\mathbf{r}$  consists of consecutive integers.*

See [Bloch and Karp 2023, Section 1.4] for more references and a nice discussion of the history. Since in this paper we will be mainly studying the case where  $\mathbf{r}$  consists of consecutive integers, we will use the two notions interchangeably when there is no ambiguity.

Let  $B$  and  $B^-$  be the opposite Borel subgroups consisting of upper-triangular and lower-triangular matrices. Let  $W = S_n$  be the Weyl group of  $\text{GL}_n$ . Given  $u, v \in W$ , the *Richardson variety* is the intersection of opposite Bruhat cells

$$\mathcal{R}_{u,v} := (B \dot{v} B / B) \cap (B^- \dot{u} B / B),$$

---

<sup>1</sup>The reader who is concerned about the fact that we are working with projective coordinates can replace “all Plücker coordinates are positive” by “all Plücker coordinates are nonzero and have the same sign”.

where  $\dot{v}$  and  $\dot{u}$  denote permutation matrices in  $GL_n$  representing  $v$  and  $u$ . It is well-known that  $\mathcal{R}_{u,v}$  is nonempty precisely when  $u \leq v$  in Bruhat order, and in that case is irreducible of dimension  $\ell(v) - \ell(u)$ .

For  $u, v \in W$  with  $u \leq v$ , let  $\mathcal{R}_{u,v}^{>0} := \mathcal{R}_{u,v} \cap \text{Fl}_n^{\geq 0}$  be the positive part of the Richardson variety. Lusztig conjectured and Rietsch [1998] proved that

$$\text{Fl}_n^{\geq 0} = \bigsqcup_{u \leq v} \mathcal{R}_{u,v}^{>0} \tag{1}$$

is a cell decomposition of  $\text{Fl}_n^{\geq 0}$ . Moreover, Rietsch showed that one obtains a cell decomposition of the nonnegative partial flag variety  $\text{Fl}_{r;n}^{\geq 0}$  by projecting the cell decomposition of  $\text{Fl}_n^{\geq 0}$  [Rietsch 1998; 2006, Section 6]. Specifically, if we let  $W_r$  be the parabolic subgroup of  $W$  generated by the simple reflections  $\{s_i \mid 1 \leq i \leq n - 1 \text{ and } i \notin \{r_1, \dots, r_k\}\}$ , then one obtains a cell decomposition by using the projections  $\pi(\mathcal{R}_{u,v}^{>0})$  of the cells  $\mathcal{R}_{u,v}^{>0}$  where  $u \leq v$  and  $v$  is a minimal-length coset representative of  $W/W_r$ . (We note moreover that Rietsch’s results hold for  $G$  a semisimple, simply connected linear algebraic group over  $\mathbb{C}$  split over  $\mathbb{R}$ ).

In the case of the Grassmannian, Postnikov [2007] studied the Plücker nonnegative part  $\text{Gr}_{d,n}^{\geq 0}$  of the Grassmannian, and gave a decomposition of it into *positroid cells*  $S_{\mathcal{B}}^{>0}$  by intersecting  $\text{Gr}_{d,n}^{\geq 0}$  with the matroid strata. Concretely, if  $\mathcal{B}$  is the collection of bases of an element of  $\text{Gr}_{d,n}^{\geq 0}$ , then  $S_{\mathcal{B}}^{>0} = \{A \in \text{Gr}_{d,n}^{\geq 0} \mid p_I(A) \neq 0 \text{ if and only if } I \in \mathcal{B}\}$ . This cell decomposition of  $\text{Gr}_{d,n}^{\geq 0}$  agrees with Rietsch’s cell decomposition [Talaska and Williams 2013, Corollary 1.2].

**2.2. Background on (generalized) Bruhat interval polytopes.** Bruhat interval polytopes were defined in [Kodama and Williams 2015], motivated by the connections to the full Kostant–Toda hierarchy.

**Definition 2.4** [Kodama and Williams 2015]. Given two permutations  $u$  and  $v$  in  $S_n$  with  $u \leq v$  in Bruhat order, the *Bruhat interval polytope*  $P_{u,v}$  is defined as

$$P_{u,v} = \text{Conv}\{(x(1), x(2), \dots, x(n)) \mid u \leq x \leq v\} \subset \mathbb{R}^n. \tag{2}$$

We also define the (*twisted*) *Bruhat interval polytope*  $\tilde{P}_{u,v}$  by

$$\tilde{P}_{u,v} = \text{Conv}\{(n + 1 - x^{-1}(1), n + 1 - x^{-1}(2), \dots, n + 1 - x^{-1}(n)) \mid u \leq x \leq v\} \subset \mathbb{R}^n. \tag{3}$$

While the definition of Bruhat interval polytope in (2) is more natural from a combinatorial point of view, as we’ll see shortly, the definition in (3) is more natural from the point of view of the moment map. Note that the set of Bruhat interval polytopes is the same as the set of twisted Bruhat interval polytopes; it is just a difference in labeling.

**Remark 2.5.** If we choose any point  $A$  in the cell  $\mathcal{R}_{u,v}^{>0} \subset \text{Fl}_n^{\geq 0}$  (thought of as an  $n \times n$  matrix), and let  $M_i$  be the matroid represented by the first  $i$  rows of  $A$ , then  $\tilde{P}_{u,v}$  is the Minkowski sum of the matroid polytopes  $P(M_1), \dots, P(M_n)$  [Kodama and Williams 2015, Corollary 6.11]. In particular,  $\tilde{P}_{u,v}$  is the matroid polytope of the flag matroid  $M_1, \dots, M_n$ .

Following [Tsukerman and Williams 2015], we can generalize the notion of Bruhat interval polytope as follows; see [loc. cit., Section 7.2] for notation.

**Definition 2.6.** Choose a generalized partial flag variety  $G/P = G/P_J$ , let  $W_J$  be the associated parabolic subgroup of the Weyl group  $W$ , and let  $u, v \in W$  with  $u \leq v$  in Bruhat order and  $v$  a minimal-length coset representative of  $W/W_J$ . Let  $\pi$  denote the projection from  $G/B$  to  $G/P$ , and let  $A$  be an element of the cell  $\pi(\mathcal{R}_{u,v}^{>0})$  of (Lusztig’s definition of)  $(G/P)^{\geq 0}$ .

A *generalized Bruhat interval polytope*  $\tilde{P}_{u,v}^J$  can be defined in any of the following equivalent ways [Tsukerman and Williams 2015, Definition 7.8, Lemma 7.9, Proposition 7.10, Remark 7.11] and [Borovik et al. 2003, Preface]:

- The moment map image of the closure of the torus orbit of  $A$  in  $G/P$  (which is a Coxeter matroid polytope).
- The moment map image of the closure of the cell  $\pi(\mathcal{R}_{u,v}^{>0})$ .
- The moment map image of the closure of the projected Richardson variety  $\pi(\mathcal{R}_{u,v})$ .
- The convex hull  $\text{Conv}\{z \cdot \rho_J \mid u \leq z \leq v\} \subset \mathfrak{t}_{\mathbb{R}}^*$ , where  $\rho_J$  is the sum of fundamental weights  $\sum_{j \in J} \omega_j$ , and  $\mathfrak{t}_{\mathbb{R}}^*$  is the dual of the real part of the Lie algebra  $\mathfrak{t}$  of the torus  $T \subset G$ .

**Remark 2.7.** When  $G = \text{GL}_n$  with fundamental weights  $e_1, e_1 + e_2, \dots, e_1 + \dots + e_{n-1}$ , each generalized Bruhat interval polytope  $\tilde{P}_{u,v}^J$  is the flag positroid polytope associated to a matrix  $A$  representing a point of  $\text{Fl}_{r;n}^{\geq 0}$ , with  $r = (r_1, \dots, r_k)$ . In this case the generalized Bruhat interval polytope is precisely the Minkowski sum  $P(M_1) + \dots + P(M_k)$  of the matroid polytopes  $P(M_i)$ , where  $M_i$  is the matroid realized by the first  $r_i$  rows of  $A$ . In particular, the generalized Bruhat interval polytope  $\tilde{P}_{u,v}^{\emptyset} = \tilde{P}_{u,v}$  is the Minkowski sum  $P(M_1) + \dots + P(M_n)$ , where  $M_i$  is the positroid realized by the first  $i$  rows of any matrix representing a point of  $A \in \mathcal{R}_{v,w}^{>0}$ . We will discuss how to read off the matroids  $M_i$  from  $(u, v)$  in Section 7.2.

As mentioned in the introduction, when  $r$  is a sequence of consecutive ranks, the generalized Bruhat interval polytopes for  $\text{Fl}_{r;n}^{\geq 0}$  are precisely the flag positroid polytopes of ranks  $r$ . When  $r = (1, 2, \dots, n)$ , we recover the notion of Bruhat interval polytope, and when  $r$  is a single integer, we recover the notion of positroid polytope.

### 3. The nonnegative tropicalization

**3.1. Background on tropical geometry.** We define the main objects in (a) and (b) of Theorem A, and record some basic properties. For a more comprehensive treatment of tropicalizations and positive-tropicalizations, we refer to [Maclagan and Sturmfels 2015, Chapter 6] and [Speyer and Williams 2005], respectively.

For a point  $w = (w_1, \dots, w_m) \in \mathbb{T}^m$ , we write  $\bar{w}$  for its image in the tropical projective space  $\mathbb{P}(\mathbb{T}^m)$ . For  $a = (a_1, \dots, a_n) \in \mathbb{Z}^m$ , write  $a \bullet w = a_1 w_1 + \dots + a_m w_m$ .

**Definition 3.1.** For a real homogeneous polynomial

$$f = \sum_{a \in \mathcal{A}} c_a x^a \in \mathbb{R}[x_1, \dots, x_m], \quad \text{where } \mathcal{A} \text{ is a finite subset of } \mathbb{Z}_{\geq 0}^m \text{ and } 0 \neq c_a \in \mathbb{R},$$

the *extended tropical hypersurface*  $V_{\text{trop}}(f)$  and the *nonnegative tropical hypersurface*  $V_{\text{trop}}^{\geq 0}(f)$  are subsets of the tropical projective space  $\mathbb{P}(\mathbb{T}^m)$  defined by

$$V_{\text{trop}}(f) = \{\bar{w} \in \mathbb{P}(\mathbb{T}^m) \mid \text{the minimum in } \min_{a \in \mathcal{A}}(a \bullet w), \text{ if finite, is achieved at least twice}\},$$

and

$$V_{\text{trop}}^{\geq 0}(f) = \left\{ \bar{w} \in \mathbb{P}(\mathbb{T}^m) \mid \begin{array}{l} \text{the minimum in } \min_{a \in \mathcal{A}}(a \bullet w), \text{ if finite, is achieved at least twice,} \\ \text{including at some } a, a' \in \mathcal{A} \text{ such that } c_a \text{ and } c_{a'} \text{ have opposite signs} \end{array} \right\}.$$

We say that a point *satisfies the tropical relation* of  $f$  if it is in  $V_{\text{trop}}(f)$ , and that it *satisfies the positive-tropical relation* of  $f$  if it is in  $V_{\text{trop}}^{\geq 0}(f)$ .

When  $f$  is a multihomogeneous real polynomial, we define  $V_{\text{trop}}(f)$  and  $V_{\text{trop}}^{\geq 0}(f)$  similarly as subsets of a product of tropical projective spaces. We will consider tropical hypersurfaces of polynomials that define the Plücker embedding of a partial flag variety.

**Definition 3.2.** For integers  $0 < r \leq s < n$ , the (*single-exchange*) *Plücker relations of type*  $(r, s; n)$  are polynomials in variables  $\{x_I : I \in \binom{[n]}{r} \cup \binom{[n]}{s}\}$  defined as

$$\mathcal{P}_{r,s;n} = \left\{ \sum_{j \in J \setminus I} \text{sign}(j, I, J) x_{I \cup j} x_{J \setminus j} \mid I \in \binom{[n]}{r-1}, J \in \binom{[n]}{s+1} \right\},$$

where  $\text{sign}(j, I, J) = (-1)^{|k \in J|k < j| + |i \in I|j < i|}$ . When  $r = s$ , the elements of  $\mathcal{P}_{r,r;n}$  are called the *Grassmann–Plücker relations* (of type  $(r; n)$ ), and when  $r < s$ , the elements of  $\mathcal{P}_{r,s;n}$  are called the *incidence-Plücker relations* (of type  $(r, s; n)$ ).

As in the introduction, let  $\mathbf{r} = (r_1 < \dots < r_k)$  be a sequence of increasing integers in  $[n]$ . We let  $\mathcal{P}_{\mathbf{r};n} = \bigcup_{r \leq s, r, s \in \mathbf{r}} \mathcal{P}_{r,s;n}$ , and let  $\langle \mathcal{P}_{\mathbf{r};n} \rangle$  be the ideal generated by the elements of  $\mathcal{P}_{\mathbf{r};n}$ . It is well-known that for any field  $\mathbb{k}$  the ideal  $\langle \mathcal{P}_{\mathbf{r};n} \rangle$  set-theoretically carves out the partial flag variety  $\text{Fl}_{\mathbf{r};n}(\mathbb{k})$  embedded in  $\prod_{i=1}^k \mathbb{P}(\mathbb{k}^{\binom{[n]}{r_i}})$  via the standard Plücker embedding [Fulton 1997, Section 9]. Similarly, the Plücker relations define the tropical analogues of partial flag varieties as follows.

**Definition 3.3.** The *tropicalization*  $\text{TrFl}_{\mathbf{r};n}$  of  $\text{Fl}_{\mathbf{r};n}$ , *nonnegative tropicalization*  $\text{TrFl}_{\mathbf{r};n}^{\geq 0}$  of  $\text{Fl}_{\mathbf{r};n}$ , *flag Dressian*  $\text{FIDr}_{\mathbf{r};n}$ , and *nonnegative flag Dressian*  $\text{FIDr}_{\mathbf{r};n}^{\geq 0}$  are subsets of  $\prod_{i=1}^k \mathbb{P}(\mathbb{T}^{\binom{[n]}{r_i}})$  defined as

$$\begin{aligned} \text{TrFl}_{\mathbf{r};n} &= \bigcap_{f \in \langle \mathcal{P}_{\mathbf{r};n} \rangle} V_{\text{trop}}(f) & \text{and} & & \text{TrFl}_{\mathbf{r};n}^{\geq 0} &= \bigcap_{f \in \langle \mathcal{P}_{\mathbf{r};n} \rangle} V_{\text{trop}}^{\geq 0}(f), \\ \text{FIDr}_{\mathbf{r};n} &= \bigcap_{f \in \mathcal{P}_{\mathbf{r};n}} V_{\text{trop}}(f) & \text{and} & & \text{FIDr}_{\mathbf{r};n}^{\geq 0} &= \bigcap_{f \in \mathcal{P}_{\mathbf{r};n}} V_{\text{trop}}^{\geq 0}(f). \end{aligned}$$

When  $k = 1$ , i.e., when  $\mathbf{r}$  consists of one integer  $d$ , one obtains the (nonnegative) tropicalization of the Grassmannian  $\text{TrGr}_{d;n}^{(\geq 0)}$  and the (nonnegative) Dressian  $\text{Dr}_{d;n}^{(\geq 0)}$  studied in [Speyer and Sturmfels 2004; Speyer and Williams 2005; 2021; Arkani-Hamed et al. 2021b]. Like  $\text{Fl}_n$ , we write only  $n$  in the subscript when  $\mathbf{r} = (1, 2, \dots, n)$ .

**Remark 3.4.** In [Joswig et al. 2023, Section 6], the authors define the “positive flag Dressian” to consist of the elements  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k) \in \text{FlDr}_{\mathbf{r};n}$  whose constituents  $\mu_i$  are each in the strictly positive Dressian. In our language, this is equal to considering the points of

$$\bigcap_{f \in \bigcup_{i=1}^k \mathcal{P}_{r_i, r_i; n}} V_{\text{trop}}^{\geq 0}(f) \cap \bigcap_{f \in \bigcup_{r_i < r_j} \mathcal{P}_{r_i, r_j; n}} V_{\text{trop}}(f)$$

that have no  $\infty$  coordinates. In a similar vein, we could consider defining the “nonnegative flag Dressian” to be the elements of the flag Dressian whose constituents are in the nonnegative Dressian. This gives a strictly larger set than our definition of the nonnegative flag Dressian, and has the shortcoming that the equivalence of (a) and (b) in Theorem A would no longer hold; see Example 4.4.

We record a useful equivalent description of the (nonnegative) tropicalization of a partial flag variety using Puiseux series. Recall the notion of the tropical semifield from Definition 1.3.

**Definition 3.5.** Let  $\mathcal{C} = \mathbb{C}\{\{t\}\}$  be the field of Puiseux series with coefficients in  $\mathbb{C}$ , with the usual valuation map  $\text{val} : \mathcal{C} \rightarrow \mathbb{T}$ . Concretely, for  $f \neq 0$ ,  $\text{val}(f)$  is the exponent of the initial term of  $f$ , and  $\text{val}(0) = \infty$ . Let

$$\mathcal{C}_{>0} = \{f \in \mathcal{C} \setminus \{0\} : \text{the initial coefficient of } f \text{ is real and positive}\} \quad \text{and} \quad \mathcal{C}_{\geq 0} = \mathcal{C}_{>0} \cup \{0\}.$$

For a point  $p \in \text{Fl}_{\mathbf{r};n}(\mathcal{C}) \subseteq \prod_{i=1}^k \mathbb{P}(\mathcal{C}^{\binom{[n]}{r_i}})$ , applying the valuation  $\text{val} : \mathcal{C} \rightarrow \mathbb{T}$  coordinate-wise to the Plücker coordinates gives a point  $\text{val}(p) \in \prod_{i=1}^k \mathbb{P}(\mathbb{T}^{\binom{[n]}{r_i}})$ . Noting that  $\text{val}(\mathcal{C}) = \mathbb{Q} \cup \{\infty\} \subset \mathbb{T}$ , we say that a point in  $\prod_{i=1}^k \mathbb{P}(\mathbb{T}^{\binom{[n]}{r_i}})$  has *rational coordinates* if it is a point in  $\prod_{i=1}^k \mathbb{P}((\mathbb{Q} \cup \{\infty\})^{\binom{[n]}{r_i}})$ . Let  $\text{Fl}_{\mathbf{r};n}(\mathcal{C}_{\geq 0})$  be the subset of  $\text{Fl}_{\mathbf{r};n}(\mathcal{C})$  consisting of points with all coordinates in  $\mathcal{C}_{\geq 0}$ , i.e., the points  $p \in \text{Fl}_{\mathbf{r};n}(\mathcal{C}) \subseteq \prod_{i=1}^k \mathbb{P}(\mathcal{C}^{\binom{[n]}{r_i}})$  that have a representative in  $\prod_{i=1}^k \mathcal{C}_{\geq 0}^{\binom{[n]}{r_i}}$ .

**Proposition 3.6.** *The set  $\{\text{val}(p) : p \in \text{Fl}_{\mathbf{r};n}(\mathcal{C})\}$  equals the set of points in  $\text{TrFl}_{\mathbf{r};n}$  with rational coordinates. Likewise, the set  $\{\text{val}(p) : p \in \text{Fl}_{\mathbf{r};n}(\mathcal{C}_{\geq 0})\}$  equals the set of points in  $\text{TrFl}_{\mathbf{r};n}^{\geq 0}$  with rational coordinates. Moreover, we have*

$$\begin{aligned} \text{TrFl}_{\mathbf{r};n} &= \text{the closure of } \{\text{val}(p) : p \in \text{Fl}_{\mathbf{r};n}(\mathcal{C})\} \text{ in } \prod_{i=1}^k \mathbb{P}(\mathbb{T}^{\binom{[n]}{r_i}}) \quad \text{and} \\ \text{TrFl}_{\mathbf{r};n}^{\geq 0} &= \text{the closure of } \{\text{val}(p) : p \in \text{Fl}_{\mathbf{r};n}(\mathcal{C}_{\geq 0})\} \text{ in } \prod_{i=1}^k \mathbb{P}(\mathbb{T}^{\binom{[n]}{r_i}}). \end{aligned}$$

*Proof.* The first equality is known as the (extended) *fundamental theorem of tropical geometry* [Maclagan and Sturmfels 2015, Theorems 3.2.3 and 6.2.15]. The second equality is the analogue for nonnegative tropicalizations, established in [Speyer and Williams 2005, Proposition 2.2].  $\square$

**Remark 3.7.** The need to restrict to rational coordinates and the need to take the closure in Proposition 3.6 can be removed if we let  $\mathcal{C}$  be the Maltsev–Neumann ring  $\mathbb{C}((\mathbb{R}))$  (see [Poonen 1993, Section 3]) which satisfies  $\text{val}(\mathcal{C}) = \mathbb{T}$ ; see also [Markwig 2010].

Let us also record an equivalent description of the (nonnegative) flag Dressian when  $\mathbf{r}$  is a sequence of consecutive integers. We need the following definition. As is customary in matroid theory, we write  $Sij$  for the union  $S \cup \{i, j\}$  of subsets  $S$  and  $\{i, j\}$  of  $[n]$ .

**Definition 3.8.** The set  $\mathcal{P}_{r,r;n}^{(3)}$  of *three-term Grassmann–Plücker relations* (of type  $(r; n)$ ) is the subset of  $\mathcal{P}_{r,r;n}$  consisting of polynomials of the form

$$x_{Sij}x_{Sk\ell} - x_{Sik}x_{Sj\ell} + x_{Si\ell}x_{Sjk}$$

for a subset  $S \subseteq [n]$  of cardinality  $r - 2$  and a subset  $\{i < j < k < \ell\} \subseteq [n]$  disjoint from  $S$ . Similarly, the set  $\mathcal{P}_{r,r+1;n}^{(3)}$  of *three-term incidence–Plücker relations* (of type  $(r, r + 1)$ ) is the subset of  $\mathcal{P}_{r,r;n}$  consisting of polynomials of the form

$$x_{Si}x_{Sjk} - x_{Sj}x_{Sik} + x_{Sk}x_{Sij}$$

for a subset  $S \subseteq [n]$  of cardinality  $r - 1$  and a subset  $\{i < j < k\} \subseteq [n]$  disjoint from  $S$ .

Let  $\mathcal{P}_{r;n}^{(3)}$  be the union of the three-term Grassmann–Plücker and three-term incidence-Plücker relations, which we refer to as the *three-term Plücker relations*.

**Proposition 3.9.** *Suppose  $\mathbf{r} = (r_1 < \dots < r_k)$  consists of consecutive integers. Then a point  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k) \in \prod_{i=1}^k \mathbb{P}(\mathbb{T}^{\binom{[n]}{r_i}})$  is in the (nonnegative) flag Dressian if and only if its support  $\underline{\boldsymbol{\mu}} = (\underline{\mu}_1, \dots, \underline{\mu}_k)$  is a flag matroid and  $\boldsymbol{\mu}$  satisfies the (nonnegative-)tropical three-term Plücker relations. More explicitly, we have*

$$\begin{aligned} \text{FIDr}_{r;n} &= \left\{ \boldsymbol{\mu} \in \prod_{i=1}^k \mathbb{P}(\mathbb{T}^{\binom{[n]}{r_i}}) \mid \underline{\boldsymbol{\mu}} \text{ is a flag matroid and } \boldsymbol{\mu} \in \bigcap_{f \in \mathcal{P}_{r;n}^{(3)}} V_{\text{trop}}(f) \right\}, \text{ and} \\ \text{FIDr}_{r;n}^{\geq 0} &= \left\{ \boldsymbol{\mu} \in \prod_{i=1}^k \mathbb{P}(\mathbb{T}^{\binom{[n]}{r_i}}) \mid \underline{\boldsymbol{\mu}} \text{ is a flag matroid and } \boldsymbol{\mu} \in \bigcap_{f \in \mathcal{P}_{r;n}^{(3)}} V_{\text{trop}}^{\geq 0}(f) \right\}. \end{aligned}$$

*Proof.* We will use the language and results from the study of matroids over hyperfields. See [Baker and Bowler 2019] for hyperfields and relation to matroid theory, and see [Gunn 2019, Section 2.3] for a description of the signed tropical hyperfield  $\mathbb{T}\mathbb{R}$ , for which we note the following fact: The underlying set of  $\mathbb{T}\mathbb{R}$  is  $(\mathbb{R} \times \{+, -\}) \cup \{\infty\}$ , so given  $c \in \mathbb{T}$ , one can identify it with the element  $(c, +) \in \mathbb{R} \times \{+, -\}$  of  $\mathbb{T}\mathbb{R}$  if  $c < \infty$  and  $\infty$  otherwise.

In the language of hyperfields, for a homogeneous polynomial  $f$  in  $m$  variables and a hyperfield  $\mathbb{F}$ , one has the notion of the “hypersurface of  $f$  over  $\mathbb{F}$ ,” which is a subset  $V_{\mathbb{F}}(f)$  of  $\mathbb{P}(\mathbb{F}^m)$ . When  $\mathbb{F}$  is the tropical hyperfield  $\mathbb{T}$ , this coincides with  $V_{\text{trop}}(f)$  in Definition 3.1. When  $\mathbb{F}$  is the signed tropical hyperfield  $\mathbb{TR}$ , a point  $w \in \mathbb{T}^m$ , when considered as a point of  $\mathbb{TR}^m$ , is in  $V_{\mathbb{TR}}(f)$  if and only if it is in  $V_{\text{trop}}^{\geq 0}(f)$ . Thus, in the language of flag matroids over hyperfields [Jarra and Lorscheid 2024], the flag Dressian is the partial flag variety  $\text{Fl}_{r;n}(\mathbb{T})$  over  $\mathbb{T}$ , and the nonnegative flag Dressian is the subset of the partial flag variety  $\text{Fl}_{r;n}(\mathbb{TR})$  over  $\mathbb{TR}$  consisting of points that come from  $\mathbb{T}$ .

Now, both the tropical hyperfield and the signed tropical hyperfield are perfect hyperfields because they are doubly distributive [Baker and Bowler 2019, Corollary 3.45]. Our proposition then follows from [Jarra and Lorscheid 2024, Theorem 2.16 and Corollary 2.24], which together state the following: When  $r$  consists of consecutive integers, for a perfect hyperfield  $\mathbb{F}$ , a point  $p \in \prod_{i=1}^k \mathbb{P}(\mathbb{F}^{\binom{m}{r_i}})$  is in the partial flag variety  $\text{Fl}_{r;n}(\mathbb{F})$  over  $\mathbb{F}$  if and only if the support of  $p$  is a flag matroid and  $p$  satisfies the three-term Plücker relations over  $\mathbb{F}$ . □

For completeness, we include the proof of the following fact.

**Lemma 3.10.** *The signed tropical hyperfield  $\mathbb{TR}$  is doubly distributive. That is, for any  $x, y, z, w \in \mathbb{TR}$ , one has an equality of sets  $(x \boxplus y) \cdot (z \boxplus w) = xz \boxplus xw \boxplus yz \boxplus yw$ .*

*Proof.* If any one of the four  $x, y, z, w$  is  $\infty$ , then the desired equality is the usual distributivity of the signed tropical hyperfield. Thus, we now assume that all four elements are in  $\mathbb{R} \times \{+, -\}$ , and write  $x = (x_{\mathbb{R}}, x_{\mathbb{S}}) \in \mathbb{R} \times \{+, -\}$  and similarly for  $y, z, w$ . If  $x_{\mathbb{R}} > y_{\mathbb{R}}$ , then  $xz_{\mathbb{R}} > yz_{\mathbb{R}}$  and  $xw_{\mathbb{R}} > yw_{\mathbb{R}}$ , so the equality follows again from the usual distributivity. So we now assume that all four elements have the same value in  $\mathbb{R}$ , and the equality then follows from the fact that the signed hyperfield  $\mathbb{S}$  is doubly distributive. □

**Remark 3.11.** Even when  $r$  does not consist of consecutive integers, [Jarra and Lorscheid 2024, Theorem 2.16] implies that the flag Dressian and the nonnegative flag Dressian are carved out by fewer polynomials than  $\mathcal{P}_{r;n}$  in the following way: Denoting by

$$\mathcal{P}_{r;n}^{adj} = \bigcup_{i=1}^k \mathcal{P}_{r_i,r_i;n} \cup \bigcup_{i=1}^{k-1} \mathcal{P}_{r_i,r_{i+1};n},$$

one has

$$\text{FIDr}_{r;n} = \bigcap_{f \in \mathcal{P}_{r;n}^{adj}} V_{\text{trop}}(f) \quad \text{and} \quad \text{FIDr}_{r;n}^{\geq 0} = \bigcap_{f \in \mathcal{P}_{r;n}^{adj}} V_{\text{trop}}^{\geq 0}(f).$$

This generalizes the fact that a sequence of matroids  $(M_1, \dots, M_k)$  is a flag matroid if and only if  $(M_i, M_{i+1})$  is a flag matroid for all  $i = 1, \dots, k - 1$  [Borovik et al. 2003, Theorems 1.7.1 and 1.11.1].

The following corollary of Proposition 3.9 is often useful in computation. It states that the nonnegative tropical flag Dressian is in some sense “convex” inside the tropical flag Dressian.

**Corollary 3.12.** *Suppose that  $\mathbf{r} = (r_1 < \dots < r_k)$  consists of consecutive integers, and suppose we have points  $\mu_1, \dots, \mu_\ell \in \prod_i^k \mathbb{T}^{\binom{[n]}{r_i}}$  that are in  $\text{FIDr}_{\mathbf{r};n}^{\geq 0}$ . Then, if a nonnegative linear combination  $c_1\mu_1 + \dots + c_\ell\mu_\ell$  is in  $\text{FIDr}_{\mathbf{r};n}$ , it is in  $\text{FIDr}_{\mathbf{r};n}^{\geq 0}$ .*

*Proof.* We make the following general observation: Suppose  $f = c_\alpha x^\alpha - c_\beta x^\beta + c_\gamma x^\gamma$  is a three-term polynomial in  $\mathbb{R}[x_1, \dots, x_m]$  with  $c_\alpha, c_\beta, c_\gamma$  positive. Then an element  $u \in \mathbb{T}^m$  satisfies the positive-tropical relation of  $f$  if and only if  $\beta \bullet u = \min\{\alpha \bullet u, \gamma \bullet u\}$ . Hence, if  $u_1, \dots, u_\ell \in \mathbb{T}^m$  each satisfy this relation, then a nonnegative linear combination of them can satisfy the tropical relation of  $f$  only if the term at  $\beta$  achieves the minimum, that is, only if the positive-tropical relation is satisfied. The corollary now follows from this general observation and Proposition 3.9.  $\square$

**3.2. Equivalence of (a) and (b) in Theorem A.** Let  $\mathbf{r}$  be a sequence of consecutive integers  $(a, \dots, b)$  for some  $1 \leq a \leq b \leq n$ . We will show that  $\text{TrFl}_{\mathbf{r};n}^{\geq 0} = \text{FIDr}_{\mathbf{r};n}^{\geq 0}$ . The inclusion  $\text{TrFl}_{\mathbf{r};n}^{\geq 0} \subseteq \text{FIDr}_{\mathbf{r};n}^{\geq 0}$  is immediate from Definition 3.3. We will deduce  $\text{TrFl}_{\mathbf{r};n}^{\geq 0} \supseteq \text{FIDr}_{\mathbf{r};n}^{\geq 0}$  by utilizing the two known cases of the equality  $\text{TrFl}_{\mathbf{r};n}^{\geq 0} = \text{FIDr}_{\mathbf{r};n}^{\geq 0}$  — when  $\mathbf{r} = (r)$  and when  $\mathbf{r} = (1, 2, \dots, n)$ .

We start by recalling that tropicalization behaves well on subtraction-free rational maps.

**Definition 3.13.** Let  $f = \sum_{a \in \mathcal{A}} c_a x^a \in \mathbb{R}[x_1, \dots, x_m]$  be a real polynomial, where  $\mathcal{A}$  is a finite subset of  $\mathbb{Z}_{\geq 0}^m$  and  $0 \neq c_a \in \mathbb{R}$ . We define the *tropicalization*  $\text{Trop}(f) : \mathbb{R}^m \rightarrow \mathbb{R}$  to be the piecewise-linear map  $w \mapsto \min_{a \in \mathcal{A}} (a \bullet w)$ , where as before,  $a \bullet w = a_1 w_1 + \dots + a_m w_m$ .

Note that  $\text{Trop}(f_1 f_2) = \text{Trop}(f_1) + \text{Trop}(f_2)$ . Moreover, if  $f_1$  and  $f_2$  are two polynomials with positive coefficients, and  $a_1, a_2 \in \mathbb{R}_{>0}$ , then  $\text{Trop}(a_1 f_1 + a_2 f_2) = \min(\text{Trop}(f_1), \text{Trop}(f_2))$ . These facts imply the following simple lemma, which appears as [Rietsch and Williams 2019, Lemma 11.5]; see [Speyer and Williams 2005, Proposition 2.5] and [Pachter and Sturmfels 2004] for closely related statements.

**Lemma 3.14.** *Let  $f = (f_1, \dots, f_n) : \mathcal{C}^m \rightarrow \mathcal{C}^n$  be a rational map defined by polynomials  $f_1, \dots, f_n$  with positive coefficients (or more generally by subtraction-free rational expressions). Let  $(x_1, \dots, x_m) \in (\mathcal{C}_{\geq 0})^m$ , such that  $f(x_1, \dots, x_m) = (y_1, \dots, y_n)$ . Then*

$$(\text{Trop}(f))(\text{val}(x_1), \dots, \text{val}(x_m)) = (\text{val}(y_1), \dots, \text{val}(y_n)).$$

The next result states that we can extend points in the nonnegative Dressian to points in the nonnegative two-step flag Dressian.

**Proposition 3.15.** *Given  $\mu_d \in \text{Dr}_{d;n}^{\geq 0}$  with rational coordinates, there exists  $\mu_{d+1} \in \text{Dr}_{d+1;n}^{\geq 0}$  such that  $(\mu_d, \mu_{d+1}) \in \text{FIDr}_{d,d+1;n}^{\geq 0}$ . Similarly, there exists  $\mu_{d-1} \in \text{Dr}_{d-1;n}^{\geq 0}$  such that  $(\mu_{d-1}, \mu_d) \in \text{FIDr}_{d-1,d;n}^{\geq 0}$ .*

The proof of Proposition 3.15 requires the following refined results about Rietsch’s cell decomposition of the nonnegative flag variety.

**Theorem 3.16.** *The nonnegative flag variety has a cell decomposition into positive Richardsons*

$$\mathrm{Fl}_n(\mathcal{C}_{\geq 0}) = \bigsqcup_{v \leq w} \mathcal{R}_{v,w}(\mathcal{C}_{> 0})$$

where each cell  $\mathcal{R}_{v,w}(\mathcal{C}_{> 0})$  can be parametrized using a map

$$\phi_{v,w} : (\mathcal{C}_{> 0})^{\ell(w) - \ell(v)} \rightarrow \mathcal{R}_{v,w}(\mathcal{C}_{> 0}).$$

Moreover, this parametrization can be expressed as an embedding into projective space (e.g., using the flag minors) using polynomials in the parameters with positive coefficients.

*Proof.* The first statement comes from [Marsh and Rietsch 2004, Theorem 11.3]; Marsh and Rietsch were working over  $\mathbb{R}$  and  $\mathbb{R}_{> 0}$  but the same proof holds over Puiseux series. The statement that the parametrization can be expressed as an embedding into projective space using positive polynomials comes from [Rietsch and Williams 2008, Proposition 5.1].  $\square$

**Corollary 3.17.** *Each  $m$ -dimensional positroid cell  $S_{\mathcal{B}}(\mathcal{C}_{> 0})$  in the nonnegative Grassmannian  $\mathrm{Gr}_{d,n}(\mathcal{C}_{\geq 0})$  is the projection  $\pi_d(\mathcal{R}_{v,w}(\mathcal{C}_{> 0}))$  of some positive Richardson of dimension  $m = \ell(w) - \ell(v)$  in  $\mathrm{Fl}_n(\mathcal{C}_{\geq 0})$ , so we get a subtraction-free rational map*

$$\pi_d \circ \phi_{v,w} : (\mathcal{C}_{> 0})^m \rightarrow \mathcal{R}_{v,w}(\mathcal{C}_{> 0}) \rightarrow S_{\mathcal{B}}(\mathcal{C}_{> 0}).$$

*Proof.* That fact that each positroid cell is the projection of a positive Richardson was discussed in Section 2.1. The result now follows from Theorem 3.16.  $\square$

*Proof of Proposition 3.15.* Using [Arkani-Hamed et al. 2021b, Theorem 9.2], the fact that  $\mu_d \in \mathrm{Dr}_{d;n}^{\geq 0}$  with rational coordinates implies that  $\mu_d = \mathrm{val}(\{\Delta_I(V_d)\})$  for some subspace  $V_d \in \mathrm{Gr}_{d,n}(\mathcal{C}_{\geq 0})$ , and hence  $V_d$  lies in some positroid cell  $S_{\mathcal{B}}(\mathcal{C}_{> 0})$  over Puiseux series.

By Corollary 3.17,  $V_d$  is the projection of a point  $(V_1, \dots, V_n)$  of  $\mathrm{Fl}_n(\mathcal{C}_{\geq 0})$ , which in turn is the image of a point  $(x_1, \dots, x_m) \in (\mathcal{C}_{> 0})^m$ , and the Plücker coordinates  $\Delta_I(V_j)$  of each  $V_j$  are expressed as positive polynomials  $\Delta_I(x_1, \dots, x_m)$  in the parameters  $x_1, \dots, x_m$ .

In particular, we have subtraction-free maps

$$\pi_d \circ \phi_{v,w} : (\mathcal{C}_{> 0})^m \rightarrow \mathrm{Fl}_n(\mathcal{C}_{\geq 0}) \rightarrow \mathrm{Gr}_{d,n}(\mathcal{C}_{\geq 0})$$

taking

$$(x_1, \dots, x_m) \mapsto \{\Delta_I(x_1, \dots, x_m) \mid I \subset [n]\} \mapsto \left\{ \Delta_I(x_1, \dots, x_m) \mid I \in \binom{[n]}{d} \right\}.$$

The fact that the maps  $\phi_{v,w}$  and  $\pi_d$  are subtraction-free implies by Lemma 3.14 that we can tropicalize them, obtaining maps

$$\mathrm{Trop}(\pi_d \circ \phi_{v,w}) : \mathbb{R}^m \rightarrow \mathrm{TrFl}_n^{\geq 0} \rightarrow \mathrm{TrGr}_{d,n}^{\geq 0}$$

taking

$$(\mathrm{val}(x_1), \dots, \mathrm{val}(x_m)) \mapsto \{\mathrm{val}(\Delta_I(x_1, \dots, x_m)) \mid I \subset [n]\} \mapsto \left\{ \mathrm{val}(\Delta_I(x_1, \dots, x_m)) \mid I \in \binom{[n]}{d} \right\}.$$

We now let  $\mu_{d+1} = \{\text{val}(\Delta_I(V_{d+1})) \mid I \in \binom{[n]}{d+1}\}$  and  $\mu_{d-1} = \{\text{val}(\Delta_I(V_{d-1})) \mid I \in \binom{[n]}{d-1}\}$ . By construction we have that all the three-term (incidence) Plücker relations hold for  $(\mu_d, \mu_{d+1})$ , and similarly for  $(\mu_{d-1}, \mu_d)$ . Therefore  $(\mu_d, \mu_{d+1}) \in \text{FIDr}_{d,d+1;n}^{\geq 0}$  and  $(\mu_{d-1}, \mu_d) \in \text{FIDr}_{d-1,d;n}^{\geq 0}$ .  $\square$

The following consequence of Proposition 3.15 is very useful.

**Corollary 3.18.** *Let  $a' \leq a \leq b \leq b'$  be positive integers, and let  $\mathbf{r} = (a, a + 1, \dots, b)$  and  $\mathbf{r}' = (a', a' + 1, \dots, b')$  be sequences of consecutive integers. Then any point  $(\mu_a, \dots, \mu_b) \in \text{FIDr}_{\mathbf{r};n}^{\geq 0}$  with rational coordinates can be extended to a point  $(\mu_{a'}, \mu_{a'+1}, \dots, \mu_a, \dots, \mu_b, \dots, \mu_{b'}) \in \text{FIDr}_{\mathbf{r}';n}^{\geq 0}$ .*

*Proof.* We start with  $\boldsymbol{\mu} = (\mu_a, \mu_{a+1}, \dots, \mu_b) \in \text{FIDr}_{\mathbf{r};n}^{\geq 0}$ . We take  $\mu_b$  and repeatedly use Proposition 3.15 to construct  $\mu_{b+1}$ , then  $\mu_{b+2}, \dots, \mu_{b'}$ . Similarly we take  $\mu_a$  and use Proposition 3.15 to construct  $\mu_{a-1}, \mu_{a-2}, \dots, \mu_{a'}$ . Now by construction  $(\mu_{a'}, \mu_{a'+1}, \dots, \mu_a, \dots, \mu_b, \dots, \mu_{b'})$  satisfies:

- $\mu_i \in \text{Dr}_{i;n}^{\geq 0}$  for  $i = a', a' + 1, \dots, b'$ .
- All three-term incidence-Plücker relations hold (because the three-term incidence-Plücker relations occur only in consecutive ranks).

Therefore  $(\mu_{a'}, \mu_{a'+1}, \dots, \mu_{b'}) \in \text{FIDr}_{\mathbf{r}';n}^{\geq 0}$  by Proposition 3.9.  $\square$

**Theorem 3.19.** *Let  $\mathbf{r} = (a, a + 1, \dots, b)$  be a sequence of consecutive integers, and let  $\boldsymbol{\mu} \in \text{FIDr}_{\mathbf{r};n}^{\geq 0}$  with rational coordinates. Then  $\boldsymbol{\mu} \in \text{TrFI}_{\mathbf{r};n}^{\geq 0}$ .*

*Proof.* We start with  $\boldsymbol{\mu} = (\mu_a, \mu_{a+1}, \dots, \mu_b) \in \text{FIDr}_{\mathbf{r};n}^{\geq 0}$  and use Corollary 3.18 to construct  $(\mu_1, \dots, \mu_n) \in \text{FIDr}_n^{\geq 0}$ . Now [Boretsky 2022, Theorem 5.21<sup>trop</sup>] states that  $\text{FIDr}_n^{\geq 0} = \text{TrFI}_n^{\geq 0}$ . Hence, we have  $(\mu_1, \dots, \mu_n) \in \text{TrFI}_n^{\geq 0}$ , so  $(\mu_a, \mu_{a+1}, \dots, \mu_b) \in \text{TrFI}_{\mathbf{r};n}^{\geq 0}$ .  $\square$

*Proof of (a)  $\iff$  (b) in Theorem A.* We only need show that (b) $\implies$ (a), i.e., that  $\text{TrFI}_{\mathbf{r};n}^{\geq 0} \supseteq \text{FIDr}_{\mathbf{r};n}^{\geq 0}$ , since the other direction is trivial. But this follows from Theorem 3.19 because the points in  $\text{FIDr}_{\mathbf{r};n}^{\geq 0}$  with rational coordinates are dense in  $\text{FIDr}_{\mathbf{r};n}^{\geq 0}$ , and  $\text{TrFI}_{\mathbf{r};n}^{\geq 0}$  is closed.  $\square$

**Remark 3.20.** Note that our method of proof crucially used the fact that  $\mathbf{r}$  is a sequence of consecutive integers: we used Proposition 3.15 to fill in the ranks from  $b$  through  $n$  and from  $a$  down to 1. But if say we were considering  $\mathbf{r} = \{a, b\}$  with  $b - a > 1$  and  $\boldsymbol{\mu} = (\mu_a, \mu_b)$ , we could not guarantee using Proposition 3.15 that we could construct  $\mu_{b-1}, \mu_{b-2}, \dots, \mu_{a+1}$  in a way that is consistent with  $\mu_a$ .

**Remark 3.21.** Recall from Theorem 2.3 that if  $\mathbf{r}$  is a sequence of consecutive integers, the two notions of the positive/nonnegative part of the flag variety (see Definition 2.2) coincide. The method used to prove the equivalence of (a) and (b) in Theorem A can be applied in a nontropical context to prove Theorem 2.3 in an alternate way. We start by noting that the result holds when  $\mathbf{r} = (a)$ , which is to say, for the nonnegative Grassmannian [Talaska and Williams 2013, Corollary 1.2] and also when  $\mathbf{r} = (1, 2, \dots, n)$ , which is to say, for the nonnegative complete flag variety [Boretsky 2022, Theorem 5.21]. To prove the result for  $\mathbf{r} = (a, a + 1, \dots, b)$ , we start with a flag  $V_\bullet = (V_a, \dots, V_b)$  in ranks  $\mathbf{r}$  whose Plücker coordinates are all nonnegative, so that  $V_\bullet$  is Plücker nonnegative. As in Proposition 3.15, we can use the  $\mathbf{r} = (a)$  case to argue that the flag can be extended to lower ranks in such a way that all the Plücker

coordinates are nonnegative. Dually, we can extend to higher ranks from the  $r = (b)$  case. This yields a complete flag  $(V_1, \dots, V_n)$  with all nonnegative Plücker coordinates. We can then apply the result in the complete flag case to conclude that  $(V_1, \dots, V_n)$  lies in  $\text{Fl}_n^{\geq 0}$ . Thus,  $V_\bullet$  is a projection of the nonnegative complete flag  $(V_1, \dots, V_n)$  and itself lies in  $\text{Fl}_{r;n}^{\geq 0}$ , which is to say,  $V_\bullet$  is Lusztig nonnegative.

The *strictly positive tropicalization* of a partial flag variety  $\text{TrFl}_{r;n}^{>0}$  is the subset of  $\text{TrFl}_{r;n}^{\geq 0}$  consisting of points whose coordinates are never  $\infty$ . Define similarly the *strictly positive flag Dressian*  $\text{FIDr}_{r;n}^{>0}$ . The weaker version of Theorem 3.19 stating that  $\text{TrFl}_n^{>0} = \text{FIDr}_n^{>0}$  was established in [Joswig et al. 2023, Lemma 19] as follows. One starts by noting that if  $\mu \in \text{Dr}_{r+m;n+m}^{\geq 0}$ , then the sequence of minors  $(\mu_r, \dots, \mu_{r+m})$  where  $\mu_{r+i} = \mu \setminus \{n+1, \dots, n+i\} / \{n+i+1, \dots, n+m\}$  is a point in  $\text{FIDr}_{r,\dots,r+m;n}^{\geq 0}$ . Then, the crucial step is a construction in discrete convex analysis [Murota and Shioura 2018, Proposition 2] that shows that every element of  $\text{FIDr}_n^{>0}$  arises from an element of  $\text{Dr}_{n;2n}^{>0}$  in this way. One then appeals to  $\text{Gr}_{r;n}^{>0} = \text{Dr}_{r;n}^{>0}$  established in [Speyer and Williams 2021].

Example 3.22 shows that the above argument does not work if one replaces “strictly positive” with “nonnegative.” In particular, the crucial step fails: that is, not every element of  $\text{FIDr}_n^{\geq 0}$  arises from an element of  $\text{Dr}_{n;2n}^{\geq 0}$  in such a way.

**Example 3.22.** Let  $(M_1, M_2, M_3)$  be matroids on  $[3]$  whose sets of bases are  $(\{1, 3\}, \{13\}, \{123\})$ . The matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

shows that it is a flag positroid. However, we claim that there is no positroid  $M$  of rank 3 on  $[6]$  such that  $M_1 = M \setminus 4/56$ ,  $M_2 = M \setminus 45/6$ , and  $M_3 = M \setminus 456$ . Since all three cases involve deletion by 4, if we replace  $M \setminus 4$  by  $M'$ , and decrease each of 5, 6 by 1, then we are claiming that there is no positroid  $M'$  of rank 3 on  $[5]$  such that

$$M_1 = M'/45, \quad M_2 = M' \setminus 4/5, \quad \text{and} \quad M_3 = M' \setminus 45. \tag{4}$$

From  $M_1 = M'/45$  and  $M_2 = M' \setminus 4/5$ , we have that  $M'/5$  has bases  $\{14, 34, 13\}$ , and similarly, we have  $M' \setminus 4$  has bases  $\{135, 123\}$ . Hence, the set of bases of  $M'$  contains  $\{123, 135, 145, 345\}$ , and does not contain  $\{125, 235, 245\}$ . By considering the Plücker relation

$$p_{134}p_{235} = p_{123}p_{345} + p_{135}p_{234},$$

we see that no positroid satisfies these properties.

#### 4. Positively oriented flag matroids

In this section we explain the relationship between the nonnegative flag Dressian and positively oriented flag matroids, and we apply our previous results to flag matroids. In particular, we prove Corollary 1.5, which says that every positively oriented flag matroid of consecutive ranks is realizable. We also prove

Corollary 4.8, which says that a positively oriented flag matroid of consecutive ranks  $a, \dots, b$  can be extended to ranks  $a', \dots, b'$  (for  $a' \leq a \leq a \leq b$ ).

**4.1. Oriented matroids and flag matroids.** We give here a brief review of oriented matroids in terms of Plücker relations. Let  $\mathbb{S} = \{-1, 0, 1\}$  be the *hyperfield of signs*. For a polynomial  $f = \sum_{a \in \mathcal{A}} c_a x^a \in \mathbb{R}[x_1, \dots, x_m]$ , we say that an element  $\chi \in \mathbb{S}^m$  is *in the null set* of  $f$  if the set  $\{\text{sign}(c_a)\chi^a\}_{a \in \mathcal{A}}$  is either  $\{0\}$  or contains  $\{-1, 1\}$ .

**Definition 4.1.** An *oriented matroid* of rank  $r$  on  $[n]$  is a point  $\chi \in \mathbb{S}^{\binom{[n]}{r}}$ , called a *chirotope*, such that  $\chi$  is in the null set of  $f$  for every  $f \in \mathcal{P}_{r,r;n}$ . Similarly, an *oriented flag matroid* of ranks  $\mathbf{r}$  is a point  $\chi = (\chi_1, \dots, \chi_k) \in \prod_{i=1}^k \mathbb{S}^{\binom{[n]}{r_i}}$  such that  $\chi$  is in the null set of  $f$  for every  $f \in \mathcal{P}_{\mathbf{r};n}$ .

While these definitions may seem different from those in the standard reference [Björner et al. 1999] on oriented matroids, Definition 4.1 is equivalent to [Björner et al. 1999, Definition 3.5.3] by [Baker and Bowler 2019, Example 3.33]. The definition of oriented flag matroid here is equivalent to the definition of a sequence of oriented matroid quotients [Björner et al. 1999, Definition 7.7.2] by [Jarra and Lorscheid 2024, Example above Theorem D].

**Definition 4.2.** A *positively oriented matroid* is an oriented matroid  $\chi$  such that  $\chi$  only takes values 0 or 1. Similarly, we define a *positively oriented flag matroid* to be an oriented flag matroid  $\chi$  such that  $\chi$  only takes values 0 or 1.

A positroid  $M$  defines a positively oriented matroid  $\chi = \chi_M$  where  $\chi$  takes value 1 on its bases and 0 otherwise. da Silva [1987] conjectured that every positively oriented matroid arises in this way; this conjecture was subsequently proved in [Ardila et al. 2017] and then [Speyer and Williams 2021].

**Theorem 4.3** [Ardila et al. 2017]. *Every positively oriented matroid  $\chi$  is realizable, i.e.,  $\chi$  has the form  $\chi_M$  for some positroid  $M$ .*

By Theorem 4.3, each positively oriented flag matroid is a sequence of positroids which is also an oriented flag matroid.

In this section we will prove Corollary 1.5, which generalizes Theorem 4.3, and says that every positively oriented flag matroid  $(\chi_1, \dots, \chi_k)$  of consecutive ranks  $r_1 < \dots < r_k$  can be realized by a flag positroid. But before we prove it, let us give an example that shows that imposing the oriented flag matroid condition is stronger than imposing that we have a realizable flag matroid whose consistent matroids are positroids.

**Example 4.4.** We give an example of a realizable flag matroid that has positroids as its constituent matroids but is not a flag positroid. This example also appeared in [Joswig et al. 2023, Example 5] and [Bloch and Karp 2023, Example 6]. Let  $(M, M')$  be matroids of ranks 1 and 2 on  $[3]$  whose sets of bases are  $\{1, 3\}$  and  $\{12, 13, 23\}$ , respectively. Both are positroids. We can realize  $(M, M')$  as a flag matroid using the matrix

$$\begin{bmatrix} a & 0 & b \\ c & d & e \end{bmatrix},$$

where the nonvanishing minors  $a, b, ad, -bd, ae - bc$  are nonzero. In order to realize  $(M, M')$  as a flag positroid, we need to choose real numbers  $a, b, c, d, e$  such that all these minors are strictly positive. However,  $a > 0$  and  $ad > 0$  implies  $d > 0$ , while  $b > 0$  and  $-bd > 0$  implies  $d < 0$ .

This example is consistent with Corollary 1.5 because  $(M, M')$ , when considered as a sequence of positively oriented matroids, is not an oriented flag matroid.

**4.2. From the nonnegative flag Dressian to positively oriented flag matroids.** We start with the following simple observation. While the proof is very simple, we label it a “theorem” to emphasize its importance.

**Theorem 4.5.** *The set of positively oriented flag matroids of ranks  $\mathbf{r}$  can be identified with the set of points of the nonnegative flag Dressian  $\text{FIDr}_{\mathbf{r};n}^{\geq 0}$  whose coordinates are all either 0 or  $\infty$ .*

*Proof.* Given a point  $\chi = (\chi_1, \dots, \chi_m) \in \{0, 1\}^m \subset \mathbb{S}^m$ ,<sup>2</sup> we define  $t(\chi) = (t_1, \dots, t_m) \in \mathbb{T}^m$  by setting  $t_i = 0$  if  $\chi_i = 1$  and  $t_i = \infty$  if  $\chi_i = 0$ . Then, we observe that  $\chi$  is in the null set of a polynomial  $f \in \mathbb{R}[x_1, \dots, x_m]$  if and only if the image of  $t(\chi)$  in  $\mathbb{P}(\mathbb{T}^m)$  is a point in  $V_{\text{trop}}^{\geq 0}(f)$ . Therefore, each positively oriented flag matroid  $\chi$  can be identified with the element  $t(\chi)$  in the nonnegative flag Dressian  $\text{FIDr}_{\mathbf{r};n}^{\geq 0}$ . □

We now prove that every positively oriented flag matroid  $\chi = (\chi_1, \dots, \chi_k)$  of consecutive ranks  $r_1 < \dots < r_k$  is realizable.

*Proof of Corollary 1.5.* By the lemma, we may identify a positively oriented flag matroid  $\chi$  as an element  $t(\chi)$  of the nonnegative flag Dressian. Because the ranks  $\mathbf{r}$  are consecutive integers, the equivalence (a) $\iff$ (b) of Theorem A implies that  $t(\chi)$  is thus a point in  $\text{TrFl}_{\mathbf{r};n}^{\geq 0}$ . Because  $t(\chi)$  has rational coordinates (all non- $\infty$  coordinates are 0), Proposition 3.6 implies that  $t(\chi) = \text{val}(p)$  for some  $p \in \prod_{i=1}^k \mathbb{P}(\mathcal{C}_{\geq 0}^{\binom{[n]}{r_i}})$ . Setting the parameter  $t$  in each Puiseux series of  $p$  to 0 now gives the realization of  $\chi$  as a flag positroid. □

As in Question 1.6, we do not know whether the corollary holds when  $\mathbf{r}$  does not consist of consecutive integers. The following example shows that one cannot reduce to the consecutive ranks case.

**Example 4.6.** We give an example of a flag positroid  $(M, M')$  on [4] of ranks (1, 3) such that there is no flag positroid  $(M, M_2, M')$  with rank of  $M_2$  equal to 2. Let the sets of bases of  $M$  and  $M'$  be  $\{1, 2, 3, 4\}$  and  $\{123, 234\}$ , respectively. The matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

for example shows that  $(M, M')$  is a flag positroid. However, this flag positroid cannot be extended to a flag positroid with consecutive ranks. To see this, note that any realization of  $(M, M')$  as a flag positroid,

<sup>2</sup>Note that  $(\chi_1, \dots, \chi_m)$  is not a sequence of chirotopes in this proof, instead each  $\chi_i \in \mathbb{S}$ .

after row-reducing by the first row, is of the form

$$\begin{bmatrix} 1 & a & b & c \\ 0 & x & y & 0 \\ 0 & z & w & 0 \end{bmatrix}$$

where  $a, b, c > 0$  and  $xw - yz > 0$ . The minors of the matrix formed by the first two rows include  $x, y, -cx, -cy$ , which cannot be all nonnegative since  $c > 0$  and not both of  $x$  and  $y$  are zero.

**Remark 4.7.** Let us sketch an alternate proof of Corollary 1.5 that relies only on the weaker version of (a) $\iff$ (b) in Theorem A that the strictly positive parts agree, i.e., that  $\text{TrFI}_{r;n}^{>0} = \text{FIDr}_{r;n}^{>0}$ . For a matroid  $M$  of rank  $d$ , define  $\rho_M \in \mathbb{R}^{\binom{[n]}{d}}$  by  $\rho_M(S) = d - \text{rk}_M(S)$  for  $S \in \binom{[n]}{d}$ , where  $\text{rk}_M$  is the rank function of  $M$ . If  $M$  is a positively oriented matroid, then  $\rho_M$  is a point in the positive Dressian  $\text{Dr}_{d,n}^{>0}$  [Speyer and Williams 2021, proof of Theorem 5.1]. One can use this to show that if  $\mathbf{M} = (M_1, \dots, M_k)$  is a positively oriented flag matroid of consecutive ranks  $\mathbf{r}$ , then the sequence  $\boldsymbol{\rho} = (\rho_{M_1}, \dots, \rho_{M_k})$  is a point in  $\text{FIDr}_{r;n}^{>0}$ . Since  $\text{TrFI}_{r;n}^{>0} = \text{FIDr}_{r;n}^{>0}$  and  $\boldsymbol{\rho}$  has rational coordinates, Proposition 3.6 implies that there is a point  $p \in \text{Fl}_{r;n}(\mathcal{C}_{\geq 0})$  with  $\text{val}(p) = \boldsymbol{\rho}$ . Consider the coordinate  $p(S) \in \mathcal{C}$  of  $p$  at a subset  $S \in \binom{[n]}{r_i}$ . By construction, the initial term of  $p(S)$  is  $ct^q$  for some positive real  $c$  and a nonnegative integer  $q$ , where  $q$  is zero exactly when  $S$  is a basis of  $M_i$ . Thus, setting the parameter  $t$  to 0 in the Puiseux series of  $p$  gives a realization of  $\mathbf{M}$  as a flag positroid.

We now use Theorem 4.5 to give a matroidal analogue of Corollary 3.18.

**Corollary 4.8.** *Let  $a' \leq a \leq b \leq b'$  be positive integers, and let  $(M_a, M_{a+1}, \dots, M_b)$  be a positively oriented flag matroid on  $[n]$  of consecutive ranks  $a, a + 1, \dots, b$ , that is, a sequence of positroids  $M_a, \dots, M_b$  which is also an oriented flag matroid. Then we can extend it to a positively oriented flag matroid  $(M_{a'}, M_{a'+1}, \dots, M_a, \dots, M_b, \dots, M_{b'})$  of consecutive ranks  $a', a' + 1, \dots, b'$ .*

*Proof.* As in Theorem 4.5, we view the positively oriented flag matroid  $(M_a, \dots, M_b)$  as a point of the nonnegative flag Dressian  $(\mu_a, \dots, \mu_b) \in \text{FIDr}_{r;n}^{\geq 0}$  whose coordinates are all either 0 or  $\infty$ . The desired statement *almost* follows from Proposition 3.15: we just need to check that we can extend  $(\mu_a, \dots, \mu_b)$  in a way which preserves the fact that coordinates are all either 0 or  $\infty$ . This is true, and we prove it by following the proof of Proposition 3.15 and replacing all instances of the positive Puiseux series  $\mathcal{C}_{>0}$  by the positive Puiseux series with *constant coefficients*, that is, by  $\mathbb{R}_{>0}$ . Alternatively, we can use our result that  $(M_a, \dots, M_b)$  is realizable by a flag positroid, and then argue as in Remark 3.21.  $\square$

### 5. Subdivisions of flag matroid polytopes

**5.1. Flag Dressian and flag matroidal subdivisions.** Consider a point  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k) \in \prod_{i=1}^k \mathbb{P}(\mathbb{T}^{\binom{[n]}{r_i}})$  such that its support  $\underline{\boldsymbol{\mu}}$  is a flag matroid. By construction, the vertices of the flag matroid polytope  $P(\underline{\boldsymbol{\mu}})$  have the form  $\mathbf{e}_{B_1} + \dots + \mathbf{e}_{B_k}$  where  $B_i$  is a basis of the matroid  $\underline{\mu}_i$  for each  $i = 1, \dots, k$ .

**Definition 5.1.** We define  $\mathcal{D}_\mu$  to be the coherent subdivision of  $P(\underline{\mu})$  induced by assigning each vertex  $\mathbf{e}_{B_1} + \dots + \mathbf{e}_{B_k}$  of  $P(\underline{\mu})$  the weight  $\mu_1(B_1) + \dots + \mu_k(B_k)$ . That is, the faces of  $\mathcal{D}_\mu$  correspond to the faces of the lower convex hull of the set of points

$$\{(\mathbf{e}_{B_1} + \dots + \mathbf{e}_{B_k}, \mu_1(B_1) + \dots + \mu_k(B_k)) \in \mathbb{R}^n \times \mathbb{R} : \mathbf{e}_{B_1} + \dots + \mathbf{e}_{B_k} \text{ a vertex of } P(\underline{\mu})\}.$$

The points of the flag Dressians are exactly the ones for which the subdivision  $\mathcal{D}_\mu$  consists of flag matroid polytopes.

**Theorem 5.2** [Brandt et al. 2021, Theorem A(a) and (c)]. *A point  $\mu \in \prod_{i=1}^k \mathbb{P}(\mathbb{T}^{\binom{[n]}{r_i}})$  is in the flag Dressian  $\text{FIDr}_{r,n}$  if and only if the all faces of the subdivision  $\mathcal{D}_\mu$  are flag matroid polytopes.*

When  $r$  consists of consecutive integers  $(a, a + 1, \dots, b)$ , the nonnegative analogue of this theorem is the equivalence of (b) and (c) in Theorem A, which states that a point  $\mu \in \prod_{i=a}^b \mathbb{P}(\mathbb{T}^{\binom{[n]}{i}})$  is in the nonnegative flag Dressian  $\text{FIDr}_{r,n}^{\geq 0}$  if and only if all faces of the subdivision  $\mathcal{D}_\mu$  are flag positroid polytopes. A different nonnegative analogue of Theorem 5.2 that holds for  $r$  not necessarily consecutive, but loses the flag positroid property, can be found in Remark 5.6.

**5.2. The proof of (b)  $\implies$  (c)  $\implies$  (d)  $\implies$  (e) in Theorem A.** We start by recording two observations. The first is a well-known consequence of the greedy algorithm for matroids; see for instance [Ardila and Klivans 2006, Proposition 4.3]. For a matroid  $M$  on  $[n]$  and a vector  $\mathbf{v} \in \mathbb{R}^n$ , let  $\text{face}(P(M), \mathbf{v})$  be the face of the matroid polytope  $P(M)$  that maximizes the standard pairing with  $\mathbf{v}$ .

**Proposition 5.3.** *Let  $M$  be a matroid on  $[n]$  and let  $\mathcal{S} = (\emptyset \subsetneq S_1 \subsetneq \dots \subsetneq S_\ell \subsetneq [n])$  be a chain of nonempty proper subsets of  $[n]$ . For a vector  $\mathbf{v}_\mathcal{S}$  in the relative interior of the cone  $\mathbb{R}_{\geq 0}\{\mathbf{e}_{S_1}, \dots, \mathbf{e}_{S_\ell}\}$ , we have*

$$\text{face}(P(M), \mathbf{v}_\mathcal{S}) = P(M^\mathcal{S}),$$

where  $M^\mathcal{S} = M|_{S_1} \oplus M|_{S_2/S_1} \oplus M|_{S_3/S_2} \oplus \dots \oplus M|_{S_\ell}$  is the direct sum of minors of  $M$ .

For  $\mathbf{M} = (M_1, \dots, M_k)$  a flag matroid, since  $P(\mathbf{M})$  is the Minkowski sum  $P(M_1) + \dots + P(M_k)$ , we likewise have that  $\text{face}(P(\mathbf{M}), \mathbf{v}_\mathcal{S}) = P(\mathbf{M}^\mathcal{S}) = P(M_1^\mathcal{S}) + \dots + P(M_k^\mathcal{S})$ , where  $\mathbf{M}^\mathcal{S} = (M_1^\mathcal{S}, \dots, M_k^\mathcal{S})$ . In particular, the face of a flag matroid polytope is a flag matroid polytope.

The second observation concerns the following operations that we will show preserve the nonnegative flag Dressian. Recall that for  $w \in \mathbb{T}^{\binom{[n]}{r}}$ , its support  $\underline{w}$  is  $\{S \in \binom{[n]}{r} : w_S \neq \infty\}$ :

- We consider a point  $w \in \mathbb{T}^{\binom{[n]}{r}}$  as a set of weights on the vertices  $\{\mathbf{e}_S : S \in \underline{w}\}$  of  $P(\underline{w}) \subset \mathbb{R}^n$ . Given an affine-linear function  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  and an element  $w \in \mathbb{T}^{\binom{[n]}{r}}$ , we define

$$\varphi w \in \mathbb{T}^{\binom{[n]}{r}} \quad \text{by} \quad (\varphi w)(S) = \varphi(\mathbf{e}_S) + w(S) \text{ for } S \in \binom{[n]}{r}.$$

- For a point  $w \in \mathbb{T}^{\binom{[n]}{r}}$ , denote by  $w^{\text{in}} \in \mathbb{T}^{\binom{[n]}{r}}$  its *initial part*, i.e.,

$$w^{\text{in}}(S) = \begin{cases} 0 & \text{if } w(S) = \min\{w(S') : S' \in \binom{[n]}{r}\}, \\ \infty & \text{otherwise.} \end{cases}$$

**Proposition 5.4.** *Let  $\mathbf{r} = (r_1, \dots, r_k)$  be a sequence of increasing integers in  $[n]$ . Suppose  $\mathbf{w} = (w_1, \dots, w_k) \in \text{FIDr}_{\mathbf{r};n}^{\geq 0}$ . Then, the following hold:*

- (1) *The support  $\underline{\mathbf{w}}$  is a positively oriented flag matroid. In particular, it is a flag positroid when  $\mathbf{r} = (r_1, \dots, r_k)$  consists of consecutive integers.*
- (2) *We have  $\varphi\mathbf{w} = (\varphi w_1, \dots, \varphi w_k) \in \text{FIDr}_{\mathbf{r};n}^{\geq 0}$  for any affine-linear functional  $\varphi$  on  $\mathbb{R}^n$ .*
- (3) *We have  $\mathbf{w}^{\text{in}} = (w_1^{\text{in}}, \dots, w_k^{\text{in}}) \in \text{FIDr}_{\mathbf{r};n}^{\geq 0}$ .*

*Proof.* We may consider  $\underline{\mathbf{w}}$  as an element  $\prod_{i=1}^k \mathbb{P}(\mathbb{T}^{\binom{[n]}{r_i}})$  by assigning the value 0 to a subset  $S$  if it is in the support of  $\mathbf{w}$  and  $\infty$  otherwise. Then, we have  $\underline{\mathbf{w}} \in \text{FIDr}_{\mathbf{r};n}^{\geq 0}$  because the terms in each of the tropical Plücker relations that achieve the minimum when evaluated at  $\mathbf{w}$  continue to do so when evaluated at  $\underline{\mathbf{w}}$ . The statement (1) follows from Theorem 4.5 and Corollary 1.5

The support is unchanged by  $\varphi$ , so  $\varphi\mathbf{w}$  is a flag matroid. The statement (2) now follows because for each of the positive-tropical Plücker relations, the operation  $\varphi$  preserves the terms at which the minimum is achieved.

The support  $\underline{\mathbf{w}}^{\text{in}}$  is a flag matroid by Theorem 5.2 and because  $P(\underline{\mathbf{w}}^{\text{in}})$  is a face in the subdivision  $\mathcal{D}_{\mathbf{w}}$  of  $P(\mathbf{w})$ . The statement (3) now follows because for each of the positive-tropical Plücker relations, the operation  $^{\text{in}}$  either preserves the terms at which the minimum is achieved or changes all the terms involved to  $\infty$ . □

**Remark 5.5.** While it’s not needed here, we note that Proposition 5.4 is the “positive” analogue of the following statement, which is proved similarly: If  $\mathbf{w} \in \text{FIDr}_{\mathbf{r};n}$ , then (1)  $\underline{\mathbf{w}}$  is a flag matroid, (2)  $\varphi\mathbf{w} \in \text{FIDr}_{\mathbf{r};n}$ , and (3)  $\mathbf{w}^{\text{in}} \in \text{FIDr}_{\mathbf{r};n}$ ; see also [Brandt et al. 2021, Corollary 4.3.2] for related statements.

*Proof of (b)  $\implies$  (c).* Every face in the coherent subdivision is the initial one after an affine-linear transformation. Hence, the implication follows from Proposition 5.4. □

**Remark 5.6.** One may modify the statement (c) to the following:

- (c’) Every face in the coherent subdivision  $\mathcal{D}_{\underline{\boldsymbol{\mu}}}$  of  $P(\underline{\boldsymbol{\mu}})$  is the flag matroid polytope of a positively oriented flag matroid.

Similar argument as above shows that (b) $\implies$ (c’) even when  $\mathbf{r}$  doesn’t consist of consecutive integers. One can also verify the converse (c’) $\implies$ (b) in this more general case as follows:

Suppose for contradiction (c’) but not (b) for some  $\underline{\boldsymbol{\mu}}$ . Then Theorem 5.2 implies that  $\underline{\boldsymbol{\mu}}$  is in the flag Dressian, and thus the failure of (b) implies that there is a Plücker relation where the minimum occurs at least twice but at the terms whose coefficients have the same sign. Proposition 5.4 implies that, replacing  $\underline{\boldsymbol{\mu}}$  by  $\varphi\underline{\boldsymbol{\mu}}$  for some  $\varphi$  if necessary, we may conclude that the same is true for that Plücker relation evaluated at  $\underline{\boldsymbol{\mu}}^{\text{in}}$ . But then  $\underline{\boldsymbol{\mu}}^{\text{in}}$ , which arise as a face in the subdivision, is not a positively oriented flag matroid by Theorem 4.5, contradicting (c’).

There is no equivalence of (c’) and (e) since three-term incidence relations exist only for consecutive ranks.

The implication (c)⇒(d) is immediate.

*Proof of (d) ⇒ (e).* First, (d) implies that every edge of the subdivision  $\mathcal{D}_\mu$  of  $P(\underline{\mu})$  is a flag matroid polytope, i.e., it is parallel to  $\mathbf{e}_i - \mathbf{e}_j$  for some  $i \neq j \in [n]$ . Hence the edges of  $P(\underline{\mu})$  have the same property, so  $\underline{\mu}$  is a flag matroid.

We start with the case  $a = b$ , where  $\underline{\mu}$  is just  $(\mu)$ . We need check the validity of the three-term positive-tropical Grassmann–Plücker relations, say for an arbitrary choice of  $S \subset [n]$  and  $\{i < j < k < \ell\} \subseteq [n] \setminus S$ . Let  $\mathcal{S}$  be a maximal chain  $S_1 \subsetneq \dots \subsetneq S_m$  of subsets of  $[n]$  with the property that  $S_a = S$  and  $S_{a+1} = S \cup \{ijkl\}$  for some  $a \in [m]$ . Then, Proposition 5.3 implies that for a vector  $\mathbf{v}_\mathcal{S}$  in the relative interior of the cone  $\mathbb{R}_{\geq 0}\{\mathbf{e}_{S_1}, \dots, \mathbf{e}_{S_m}\}$ , we have

$$\text{face}(P(\underline{\mu}), \mathbf{v}_\mathcal{S}) = P(\underline{\mu}^\mathcal{S}) \simeq P(\underline{\mu}|S \cup ijkl/S).$$

For the second identification, we have used that

- the matroid polytope of a direct sum of matroids is the product of the matroid polytopes;
- with the exception of  $(S_a, S_{a+1}) = (S, S \cup ijkl)$ , all other minors of the matroid  $\underline{\mu}$  corresponding to  $(S_b, S_{b+1})$  in the chain have their polytopes being a point because  $|S_{b+1} \setminus S_b| = 1$ .

Let  $\widehat{r}$  be the rank of the matroid minor  $\underline{\mu}|S \cup ijkl/S$ . For a basis  $\widehat{B}$  of  $\underline{\mu}|S \cup ijkl/S$ , let  $B$  be the basis of  $\underline{\mu}$  such that the vertex  $\mathbf{e}_B$  of  $P(\underline{\mu})$  corresponds to the vertex  $\mathbf{e}_{\widehat{B}}$  of  $P(\underline{\mu}|S \cup ijkl/S)$  under the identification above. Identifying  $[4] = \{1 < 2 < 3 < 4\}$  with  $\{i < j < k < \ell\}$ , we may thus consider “restricting”  $\mu$  to the face  $P(\underline{\mu}|S \cup ijkl/S)$  to obtain an element  $\widehat{\mu} = \mu|S \cup ijkl/S \in \text{Dr}_{\widehat{r};4}$  defined by

$$\widehat{\mu}(\widetilde{B}) = \begin{cases} \mu(B) & \text{if } \widehat{B} \text{ a basis of } \underline{\mu}|S \cup ijkl/S, \\ \infty & \text{otherwise,} \end{cases} \quad \text{for } \widehat{B} \in \binom{[4]}{\widehat{r}}.$$

It is straightforward to check that for points in a Dressian on four elements, the three-term positive-tropical Grassmann–Plücker relations are satisfied if and only if every 2-dimensional faces in the corresponding subdivision are positroid polytopes. Since the faces of the subdivision  $\mathcal{D}_{\widehat{\mu}}$  of  $P(\underline{\mu}|S \cup ijkl/S)$  are a subset of the faces of the subdivision  $\mathcal{D}_\mu$ , we have that  $\mu$  satisfies the three-term tropical-positive Grassmann–Plücker relation involving  $ijkl$  and  $S$ .

Let us now treat the case  $a < b$ , whose proof is similar. We check the validity of the three-term positive-tropical incidence–Plücker relations, say for an arbitrary choice of  $S \subset [n]$  and  $\{i < j < k\} \subseteq [n] \setminus S$ . Let  $\mathcal{S}$  be a maximal chain  $S_1 \subsetneq \dots \subsetneq S_m$  of subsets of  $[n]$  with the property that  $S_a = S$  and  $S_{a+1} = S \cup ijk$  for some  $a \in [m]$ . Then, Proposition 5.3 implies that for a vector  $\mathbf{v}_\mathcal{S}$  in the relative interior of the cone  $\mathbb{R}_{\geq 0}\{\mathbf{e}_{S_1}, \dots, \mathbf{e}_{S_m}\}$ , we have

$$\text{face}(P(\underline{\mu}), \mathbf{v}_\mathcal{S}) = P(\underline{\mu}^\mathcal{S}) \simeq P(\underline{\mu}|S \cup ijk/S).$$

For the second identification, we have used that

- the matroid polytope of a direct sum of matroids is the product of the matroid polytopes;
- with the exception of  $(S_a, S_{a+1}) = (S, S \cup ijk)$ , all other minors of the constituent matroids of  $\underline{\mu}$  corresponding to  $(S_b, S_{b+1})$  in the chain have their polytopes being a point because  $|S_{b+1} \setminus S_b| = 1$ .

Note that the polytope  $P(\underline{\mu}|S \cup ijk/S)$  is at most 2-dimensional since it is a flag matroid polytope on 3 elements. Similarly to the  $a = b$  case, we may “restrict”  $\mu$  to the face  $P(\mu|S \cup ijk/S)$  to obtain an element  $\hat{\mu} = \mu|S \cup ijk/S \in \text{FIDr}_3$ , it is straightforward to verify that the unique three-term positive-tropical incidence relation involving  $S$  and  $ijk$  is satisfied if and only if the subdivision  $\mathcal{D}_{\hat{\mu}}$  consists only of flag positroid polytopes. Since the faces of the subdivision  $\mathcal{D}_{\hat{\mu}}$  are a subset of the faces of the subdivision  $\mathcal{D}_{\mu}$ , we have that  $\mu$  satisfies the three-term incidence relation involving  $S$  and  $\{i, j, k\}$ .  $\square$

### 6. Three-term incidence relations

**6.1. The proof of (e)  $\implies$  (b) in Theorem A.** In the case that  $a = b$  in Theorem A, the implication (e)  $\implies$  (b) is the content of Proposition 3.9.

To prove the implication when  $a < b$ , we will show the following key theorem.

**Theorem 6.1.** *Suppose  $\mu = (\mu_1, \mu_2) \in \mathbb{P}(\mathbb{T}^{\binom{[n]}{r}}) \times \mathbb{P}(\mathbb{T}^{\binom{[n]}{r+1}})$  satisfies every three-term positive-tropical incidence relation. If the support  $\underline{\mu}$  is a flag matroid, then  $\mu \in \text{FIDr}_{r,r+1;n}^{\geq 0}$ .*

*Proof of (e)  $\implies$  (b).* Since  $r$  consists of consecutive integers, Theorem 6.1 implies that if  $\mu$  is a flag matroid and  $\mu$  satisfies every three-term positive-tropical incidence relation, then  $\mu$  also satisfies every three-term positive-tropical Grassmann–Plücker relation. Hence  $\mu$  is an element of  $\text{FIDr}_{r;n}^{\geq 0}$  by Proposition 3.9.  $\square$

The proof of Theorem 6.1 relies on the following technical lemma.

**Lemma 6.2.** *Suppose  $w \in \mathbb{T}^{\binom{[5]}{2}}$  satisfies all three-term positive-tropical Grassmann–Plücker relations involving the element 5. Suppose moreover that  $w_{i5} < \infty$  for some  $i = 1, 2, 3, 4$ . Then  $w \in \text{Dr}_{2,5}^{\geq 0}$ , i.e.,  $w$  also satisfies the three-term positive-tropical Grassmann–Plücker relation not involving 5.*

*Proof.* The idea of the proof of Lemma 6.2 is that in the usual Grassmannian  $\text{Gr}_{2,5}$ , if we can invert certain Plücker coordinates, then we can write the three-term Grassmann–Plücker relation not involving 5 as a linear combination of three of the other three-term Grassmann–Plücker relations. In particular, we have the following identity, which is easy to verify.

**Lemma 6.3.** *If  $p_{25} \neq 0$  (respectively,  $p_{35} \neq 0$ ) then  $p_{13}p_{24} - p_{12}p_{34} - p_{14}p_{23}$  can be written in the following ways.*

$$\begin{aligned} & p_{13}p_{24} - p_{12}p_{34} - p_{14}p_{23} \\ &= (p_{13}p_{25} - p_{12}p_{35} - p_{15}p_{23}) \frac{p_{24}}{p_{25}} - (p_{14}p_{25} - p_{12}p_{45} - p_{15}p_{24}) \frac{p_{23}}{p_{25}} + (p_{24}p_{35} - p_{23}p_{45} - p_{25}p_{34}) \frac{p_{12}}{p_{25}} \\ &= (p_{13}p_{25} - p_{12}p_{35} - p_{15}p_{23}) \frac{p_{34}}{p_{35}} - (p_{14}p_{35} - p_{13}p_{45} - p_{15}p_{34}) \frac{p_{23}}{p_{35}} + (p_{24}p_{35} - p_{23}p_{45} - p_{25}p_{34}) \frac{p_{13}}{p_{35}}. \end{aligned}$$

We next note that we can interpret the first (respectively, second) expression in Lemma 6.3 tropically as long as  $w_{25} < \infty$  (respectively,  $w_{35} < \infty$ ).

**Case 1:**  $w_{25} < \infty$ . Then we can make sense of the terms on the right hand side of the first expression of Lemma 6.3 tropically. Since the three-term positive tropical Plücker relations involving 5 hold, and

$w_{25} < \infty$ , we have

$$w_{13} + w_{25} + w_{24} - w_{25} = \min(w_{12} + w_{35} + w_{24} - w_{25}, w_{15} + w_{23} + w_{24} - w_{25}),$$

$$w_{14} + w_{25} + w_{23} - w_{25} = \min(w_{12} + w_{45} + w_{23} - w_{25}, w_{15} + w_{24} + w_{23} - w_{25}),$$

$$w_{24} + w_{35} + w_{12} - w_{25} = \min(w_{23} + w_{45} + w_{12} - w_{25}, w_{25} + w_{34} + w_{12} - w_{25}).$$

We now simplify these expressions and underline terms that agree, obtaining

$$w_{13} + w_{24} = \min(\underline{w_{12} + w_{35} + w_{24} - w_{25}}, \underline{w_{15} + w_{23} + w_{24} - w_{25}}), \tag{5}$$

$$w_{14} + w_{23} = \min(\underline{w_{12} + w_{45} + w_{23} - w_{25}}, \underline{w_{15} + w_{24} + w_{23} - w_{25}}), \tag{6}$$

$$\underline{w_{24} + w_{35} + w_{12} - w_{25}} = \min(\underline{w_{23} + w_{45} + w_{12} - w_{25}}, w_{34} + w_{12}). \tag{7}$$

There are now eight cases to consider, based on whether the minimum is achieved by the first or second term in each of (5), (6), (7). All cases are straightforward. If the minimum is achieved by the first term in (5) and the second term in (7), then we find that  $w_{13} + w_{24} = w_{12} + w_{34} \leq w_{14} + w_{23}$ . In the other six cases, we find that  $w_{13} + w_{24} = w_{14} + w_{23} \leq w_{12} + w_{34}$ . Therefore the positive tropical Plücker relation involving 1, 2, 3, 4 is satisfied.

**Case 2:**  $w_{35} < \infty$ . The argument for Case 2 is the same as for Case 1, except we use the tropicalization of the second identity in Lemma 6.3.

**Case 3:**  $w_{25} = w_{35} = \infty$ . In this case, since 5 is not a loop, either  $w_{15} < \infty$  or  $w_{45} < \infty$ . Suppose that  $w_{15} < \infty$ . Then the positive tropical Plücker relations

- $w_{13} + w_{25} = \min(w_{12} + w_{35}, w_{15} + w_{23}),$
- $w_{14} + w_{25} = \min(w_{12} + w_{45}, w_{15} + w_{24}),$
- $w_{14} + w_{35} = \min(w_{13} + w_{45}, w_{15} + w_{34}),$

imply that  $w_{23} = w_{24} = w_{34} = \infty$ , and hence the positive tropical Plücker relation involving 1, 2, 3, 4 is satisfied. The case where  $w_{45} < \infty$  is similar. □

For  $w \in \mathbb{T}^{\binom{[n]}{r}}$ , define its dual  $w^\perp \in \mathbb{T}^{\binom{[n]}{n-r}}$  by  $w^\perp(I) = w([n] \setminus I)$ . It is straightforward to verify that  $w$  is an element of  $\text{Dr}_{r;n}$  (resp.  $\text{Dr}_{r;n}^{\geq 0}$ ) if and only if  $w^\perp$  is an element of  $\text{Dr}_{n-r;n}$  (resp.  $\text{Dr}_{n-r;n}^{\geq 0}$ ). This matroid duality gives the following dual formulation of Lemma 6.2.

**Corollary 6.4.** *Suppose  $w \in \mathbb{T}^{\binom{[5]}{3}}$  satisfies all three-term positive-tropical Grassmann–Plücker relations that contain a variable indexed by  $S \in \binom{[5]}{3}$  with  $5 \notin S$ . If  $\underline{w}$  is a matroid such that 5 is not a coloop, then  $w \in \text{Dr}_{3;5}^{\geq 0}$ , i.e.,  $w$  also satisfies the three-term positive-tropical Grassmann–Plücker relation whose every variable contains 5 in its indexing subset.*

We are now ready to prove Theorem 6.1. We expect that the proof of Theorem 6.1 here adapts well to give an analogous statement for arbitrary perfect hyperfields.

*Proof of Theorem 6.1.* Given such  $\mu = (\mu_1, \mu_2) \in \mathbb{P}(\mathbb{T}^{\binom{[n]}{r}}) \times \mathbb{P}(\mathbb{T}^{\binom{[n]}{r+1}})$ , define  $\tilde{\mu} \in \mathbb{P}(\mathbb{T}^{\binom{[n+1]}{r+1}})$  by

$$\tilde{\mu}(S) = \begin{cases} \mu_1(S \setminus (n+1)) & \text{if } (n+1) \in S, \\ \mu_2(S) & \text{otherwise.} \end{cases}$$

Because  $\underline{\mu}$  is a flag matroid, we have that  $\tilde{\mu}$  is a matroid, with the element  $(n+1)$  that is neither a loop nor a coloop. We observe that  $\tilde{\mu} \in \text{Dr}_{r+1;n+1}^{\geq 0}$  if and only if  $\mu \in \text{FIDr}_{r,r+1;n}^{\geq 0}$  because the validity of the three-term positive-tropical Grassmann–Plücker relations for  $\tilde{\mu}$  is equivalent to the validity of both the three-term positive-tropical incidence relations and the three-term positive-tropical Grassmann–Plücker relations for  $\mu$ .

We need to check that  $\tilde{\mu}$  satisfies every three-term positive-tropical Grassmann–Plücker relation of type  $(r+1; n+1)$ . Consider the three-term relation associated to the subset  $S \subseteq [n+1]$  of cardinality  $r-1$  and  $\{i < j < k < \ell\} \subseteq [n+1]$  disjoint from  $S$ . We have three cases:

- $\ell = n+1$ . In this case, erasing the index  $n+1$  in the expression for the corresponding three-term Grassmann–Plücker relation yields a three-term incidence relation of type  $(r, r+1; n)$ , which is satisfied by our assumption on  $\mu$ .
- $(n+1) \in S$ . In this case, considering the minor  $\tilde{\mu}|_{S \cup i j k \ell / (S \setminus (n+1))}$  and then applying Corollary 6.4 implies that the three-term Grassmann–Plücker relation is satisfied.
- $(n+1) \notin S \cup i j k \ell$ . In this case, considering the minor  $\tilde{\mu}|_{S \cup i j k \ell (n+1) / S}$  and then applying Lemma 6.2 implies that the three-term Grassmann–Plücker relation is satisfied.

In every case the three-term positive-tropical Grassmann–Plücker relation is satisfied, as desired. □

### 7. Projections of positive Richardsons to positroids

One recurrent theme in our paper has been the utility of projecting a complete flag positroid (equivalently, a positive Richardson) to a positroid (or a positroid cell). This has come up in Rietsch’s cell decomposition of a nonnegative (partial) flag variety, in our proofs in Section 3.2, and in the expression of a Bruhat interval polytope as a Minkowski sum of positroid polytopes in Remark 2.7. Positive Richardsons can be indexed by pairs  $(u, v)$  of permutations with  $u \leq v$ . Meanwhile, by work of Postnikov [2007], positroid cells of  $\text{Gr}_{d,n}^{\geq 0}$  can be indexed by *Grassmann necklaces*. In this section we will give several concrete combinatorial recipes for constructing the positroids obtained by projecting a (complete) flag positroid. We will also discuss the problem of determining when a collection of positroids can be identified with a (complete) flag positroid.

**7.1. Indexing sets for cells of  $\text{Gr}_{d,n}^{\geq 0}$ .** As discussed in Section 2.1, there are two equivalent ways of thinking about the positroid cell decomposition of  $\text{Gr}_{d,n}^{\geq 0}$ :

$$\text{Gr}_{d,n}^{\geq 0} = \bigsqcup_{\mathcal{B}} S_{\mathcal{B}}^{\geq 0} = \bigsqcup_{u,v} \pi(\mathcal{R}_{u,v}^{\geq 0}).$$

In the union on the right,  $\pi$  is the projection from  $\text{Fl}_n$  to  $\text{Gr}_{d,n}$ , and  $u, v$  range over all permutations  $u \leq v$  in  $S_n$ , such that  $v$  is a minimal-length coset representative of  $W/W_d$ , and  $W_d = \langle s_1, \dots, s_{d-1}, \hat{s}_d, s_{d+1}, \dots, s_{n-1} \rangle$ . We write  $W^d$  for the set of minimal-length coset representatives of  $W/W_d$ . Recall that a *descent* of a permutation  $z$  is a position  $j$  such that  $z(j) > z(j+1)$ . We have that  $W^d$  is the subset of permutations in  $S_n$  which have at most one descent, and if it exists, that descent must be in position  $d$ .

Even if  $v \notin W^d$ , the projection of  $\mathcal{R}_{u,v}^{>0}$  to  $\text{Gr}_{d,n}^{\geq 0}$  is still a positroid, which we will characterize below. We start by defining *Grassmann necklaces* [Postnikov 2007].

**Definition 7.1.** Let  $\mathcal{I} = (I_1, \dots, I_n)$  be a sequence of subsets of  $\binom{[n]}{d}$ . We say  $\mathcal{I}$  is a *Grassmann necklace of type  $(d, n)$*  if the following holds:

- If  $i \in I_i$ , then  $I_{i+1} = (I_i \setminus i) \cup j$  for some  $j \in [n]$ .
- If  $i \notin I_i$ , then  $I_{i+1} = I_i$ .

In order to define the bijection between these Grassmann necklaces and positroids, we need to define the  *$i$ -Gale order* on  $\binom{[n]}{d}$ .

**Definition 7.2.** We write  $<_i$  for the following *shifted linear order* on  $[n]$ .

$$i <_i i + 1 <_i \dots <_i n <_i 1 <_i \dots <_i i - 1.$$

We also define the  *$i$ -Gale order* on  $d$ -element subsets by setting

$$\{a_1 <_i \dots <_i a_d\} \leq_i \{b_1 <_i \dots <_i b_d\}$$

if and only if  $a_\ell \leq_i b_\ell$  for all  $1 \leq \ell \leq d$ .

Given a positroid  $M$ , we define a sequence  $\mathcal{I}_M = (I_1, \dots, I_n)$  of subsets of  $[n]$  by letting  $I_i$  be the minimal basis of  $M$  in the  $i$ -Gale order. The following result is from [Postnikov 2007, Theorem 17.1].

**Proposition 7.3.** *For any positroid  $M$ ,  $\mathcal{I}_M$  is a Grassmann necklace. The map  $M \mapsto \mathcal{I}_M$  gives a bijection between positroids of rank  $d$  on  $[n]$  and Grassmann necklaces of type  $(d, n)$ .*

**7.2. Projecting positive Richardsons to positroids.** In this section we will give several descriptions of the constituent positroids appearing in a complete flag positroid (that is, a flag matroid represented by a positive Richardson). We start by reviewing a cryptomorphic definition of flag matroid, based on [Borovik et al. 2003, Sections 1.7–1.11].

A *flag*  $F = F_1 \subset F_2 \subset \dots \subset F_k$  on  $[n]$  is an increasing sequence of finite subsets of  $[n]$ . A *flag matroid* is a collection  $\mathcal{F}$  of flags satisfying the *maximality property*. Recall that  $e_S$  denotes the 01 indicator vector in  $\mathbb{R}^n$  associated to a subset  $S \subset [n]$ . For a flag  $F = F_1 \subset F_2 \subset \dots \subset F_k$  we let  $e_F = e_{F_1} + \dots + e_{F_k}$ . In this language, the flag matroid polytope of  $\mathcal{F}$  is  $P_{\mathcal{F}} = \text{Conv}\{e_F \mid F \in \mathcal{F}\}$ , whose vertices are precisely the points  $e_F$  for  $F \in \mathcal{F}$ .

In the complete flag case, each point  $e_F$  is a permutation vector  $(z(1), \dots, z(n))$  for some  $z \in S_n$ . Note that we can read off  $z := z(F)$  from  $F$  by setting  $z(i) = j$ , where  $j$  is the unique element of  $F_i \setminus F_{i-1}$ .

Given  $u \leq v$  in Bruhat order, we define the *Bruhat interval flag matroid*  $\mathcal{F}_{u,v}$  to be the complete flag matroid whose flags are precisely

$$\{z([1]) \subset z([2]) \subset \cdots \subset z([n])\} \text{ for } u \leq z \leq v,$$

where  $[i]$  denotes  $\{1, 2, \dots, i\}$  and  $z([i])$  denotes  $\{z(1), \dots, z(i)\}$ . Then by the above discussion, the (twisted) Bruhat interval polytope

$$\tilde{P}_{u,v} = \text{Conv}\{(n + 1 - z^{-1}(1), n + 1 - z^{-1}(2), \dots, n + 1 - z^{-1}(n)) \mid u \leq z \leq v\}$$

is the flag matroid polytope of the Bruhat interval flag matroid  $\mathcal{F}_{u,v}$ .

This observation leads naturally to the following definition.

**Definition 7.4.** Consider a complete flag matroid  $\mathcal{F}$  on  $[n]$ , which we identify with a collection  $\mathcal{S}$  of permutations on  $[n]$ . By the maximality property [Borovik et al. 2003, Section 1.7.2] and its relation to the tableau criterion for Bruhat order [Borovik et al. 2003, Theorem 5.17.3],  $\mathcal{S}$  contains a unique permutation  $u$  (respectively,  $v$ ) which is minimal (respectively, maximal) in Bruhat order among all elements of  $\mathcal{S}$ . We say that  $\mathcal{F}_{u,v}$  is the *Bruhat interval envelope* of  $\mathcal{F}$ .

It follows from Definition 7.4 that the Bruhat interval envelope of a complete flag matroid  $\mathcal{F}$  contains  $\mathcal{F}$ ; however, in general this inclusion is strict. It is an equality precisely when  $\mathcal{F}$  is a Bruhat interval flag matroid.

Recall that if  $F = (F_1, \dots, F_n)$  and  $G = (G_1, \dots, G_n)$  are flags, we say that  $F$  is *less than or equal to  $G$  in the  $\leq_j$  Gale order* (and write  $F \leq_j G$ ) if and only if  $F_i \leq_j G_i$  for all  $1 \leq i \leq n$ . (We also talk about the “usual” Gale order with respect to the total order  $1 < 2 < \dots < n$ .) The *Maximality Property* for flag matroids implies that for any flag matroid  $\mathcal{F}$ , there is always a unique element which is maximal (and a unique element which is minimal) with respect to  $\leq_j$ .

We now give a Grassmann necklace characterization of the positroid constituents of a complete flag positroid, which follows from the previous discussion plus Proposition 7.3.

**Proposition 7.5.** Consider a complete flag positroid  $\mathbf{M} = (M_1, \dots, M_n)$  on  $[n]$ , that is, the flag positroid associated to any point of  $\mathcal{R}_{u,v}^{>0}$ , for some  $u \leq v$ . For each  $1 \leq j \leq n$ , let  $z^{(j)}$  be the Gale-minimal permutation with respect to  $\leq_j$  in the interval  $[u, v]$ . Then the Grassmann necklace of the positroid  $M_j$  is  $(z^{(1)}([j]), z^{(2)}([j]), \dots, z^{(n)}([j]))$ .

**Example 7.6.** Consider the flag positroid associated to a point of  $\mathcal{R}_{u,v}^{>0}$ , where  $u = (1, 2, 4, 3)$  and  $v = (4, 2, 1, 3)$  (which we abbreviate as 1243 and 4213). The interval  $[u, v]$  consists of

$$[u, v] = \{1243, 1423, 2143, 2413, 4123, 4213\}.$$

We now use Proposition 7.5, and find that the Gale-minimal permutations of  $[u, v]$  with respect to  $\leq_1, \leq_2, \leq_3, \leq_4$  are 1243, 2413, 4123, 4123. Therefore the Grassmann necklaces for the constituent positroids  $M_1, M_2, M_3$  and  $M_4$  are  $(1, 2, 4, 4), (12, 24, 14, 14), (124, 124, 124, 124)$ , and  $(1234, 1234, 1234, 1234)$ .

Alternatively, we can read off the flags in the flag positroid from the permutations in  $[u, v]$ , obtaining the flags

$$\{1 \subset 12 \subset 124, 1 \subset 14 \subset 124, 2 \subset 12 \subset 124, 2 \subset 24 \subset 124, 4 \subset 14 \subset 124, 4 \subset 24 \subset 124\}.$$

(Note that for brevity, we have omitted the subset 1234 from the end of each flag above.) We can now read off the bases of  $M_1, M_2, M_3, M_4$  from the flags, obtaining  $\{1, 2, 4\}, \{12, 14, 24\}, \{124\}$ , and  $\{1234\}$ . We can then directly calculate the Grassmann necklaces from these sets of bases, getting the same answer as above.

If we compute the Minkowski sum of the positroids  $M_1, M_2, M_3, M_4$  above, we obtain the twisted Bruhat interval polytope  $\tilde{P}_{1243,4213} = P_{2314,4312}$ , whose vertices are

$$\{(4, 3, 1, 2), (4, 2, 1, 3), (3, 4, 1, 2), (3, 2, 1, 4), (2, 4, 1, 3), (2, 3, 1, 4)\},$$

as noted in Remark 2.7.

The following result gives an alternative description of the constituent positroids of a complete flag matroid, this time in terms of bases.

**Lemma 7.7** [Kodama and Williams 2015, Lemma 3.11; Billey and Weaver 2022, Theorem 1.4]. *Consider a complete flag positroid, that is, a flag matroid represented by a point of a positive Richardson  $\mathcal{R}_{u,v}^{>0}$ , where  $u, v \in S_n$  and  $u \leq v$  in Bruhat order. Choose  $1 \leq d \leq n$ . Let  $\pi$  denote the projection from  $\text{Fl}_n$  to  $\text{Gr}_{d,n}$ . Then the bases of the rank  $d$  positroid represented by  $\pi(\mathcal{R}_{u,v}^{>0})$  are  $\{z([d]) \mid u \leq z \leq v\}$ .*

Finally, we remark that [Bloch and Karp 2023, Remark 5.24] gives yet another description of the constituent positroids of a complete flag positroid, this time in terms of pairs of permutations.

**7.3. Characterizing when two adjacent-rank positroids form an oriented matroid quotient.** We have discussed how to compute the projection of a complete flag positroid to a positroid. Moreover, it is well-known that every positroid is the projection of a complete flag positroid. In this section we will give a criterion for determining when two positroids  $M_i$  and  $M_{i+1}$  on  $[n]$  of ranks  $i$  and  $i+1$  can be obtained as the projection of a complete flag positroid (see Theorem 7.14).

We recall the definition of oriented matroid quotient in the setting at hand.

**Definition 7.8.** We say that two positroids  $M_i$  and  $M_{i+1}$  on  $[n]$  of ranks  $i$  and  $i+1$  form an oriented matroid quotient if  $(M_i, M_{i+1})$  is an oriented flag matroid.

The following statement is a direct consequence of Corollary 4.8.

**Proposition 7.9.** *Let  $M_i$  and  $M_{i+1}$  be positroids on  $[n]$  of ranks  $i$  and  $i+1$ . Then there is a complete flag positroid with  $M_i$  and  $M_{i+1}$  as constituents if and only if  $(M_i, M_{i+1})$  form an oriented matroid quotient.*

**Proposition 7.10.** *Suppose that  $(M_1, \dots, M_n)$  is a sequence of positroids of ranks  $1, 2, \dots, n$  on  $[n]$ , such that each pair  $M_i$  and  $M_{i+1}$  forms an oriented matroid quotient. Then  $(M_1, \dots, M_n)$  is a complete flag positroid. Moreover, it is realized by a point of the positive Richardson  $\mathcal{R}_{u,v}^{>0}$ , where we can explicitly construct  $u$  and  $v$  as follows:*

- Let  $B_1^{\min}, \dots, B_n^{\min}$  (respectively,  $B_1^{\max}, \dots, B_n^{\max}$ ) be the bases of  $M_1, \dots, M_n$  which are minimal (maximal) with respect to the usual Gale ordering. Then  $u, v \in S_n$  are defined by

$$u(i) = B_i^{\min} \setminus B_{i-1}^{\min} \quad \text{and} \quad v(i) = B_i^{\max} \setminus B_{i-1}^{\max}.$$

*Proof.* As in Theorem 4.5, we identify each positroid  $M_i$  with the image  $t(\chi_i)$  of its chirotope  $\chi_i$ ; we have that  $t(\chi_i)$  lies in  $\text{Dr}_{i;n}^{\geq 0}$ . The fact that each pair  $M_i, M_{i+1}$  forms an oriented matroid quotient means that  $(t(\chi_1), \dots, t(\chi_n))$  satisfies all three-term incidence-Plücker relations, and hence  $(t(\chi_1), \dots, t(\chi_n)) \in \text{FIDr}_n^{\geq 0}$ . Since  $\text{FIDr}_n^{\geq 0} = \text{TrFl}_n^{\geq 0}$ , we have proved that  $(M_1, \dots, M_n)$  is a complete flag positroid.

To prove the characterization of  $u$  and  $v$ , we use Lemma 7.7. In particular, it follows from Lemma 7.7 and the Tableaux Criterion for Bruhat order that the Gale-minimal and Gale-maximal bases of the rank  $d$  positroid  $\pi(\mathcal{R}_{u,v}^{\geq 0})$  are  $u([d])$  and  $v([d])$ . The result now follows. □

As we’ve seen in Example 4.4 it is a subtle question to determine whether a pair of positroids  $M_1$  and  $M_2$  of ranks  $r$  and  $r + 1$  form an oriented matroid quotient. One way is to construct an  $n$  by  $r + 1$  matrix such that the minor in rows  $1, \dots, r$  and columns  $I$  is nonzero if and only if  $I$  is a basis of  $M_1$  while the maximal minor in rows  $1, \dots, r + 1$  and columns  $J$  is nonzero if and only if  $J$  is a basis of  $M_2$ . Another way is to check the three-term relations over the signed tropical hyperfield, as in Proposition 3.9. We do not have an efficient way to do either of these things. Instead, in Theorem 7.14, we will give an algorithmic, combinatorial way to verify whether  $M_1$  and  $M_2$  form an oriented matroid quotient.

**Construction 1.** Given two positroids  $M_1$  and  $M_2$  on the ground set  $[n]$  of ranks  $r$  and  $r + 1$ , respectively, which form a positively oriented matroid quotient, we construct a positroid  $M := M(M_1, M_2)$  of rank  $r + 1$  on the ground set  $[n + 1]$  where  $n + 1$  is neither a loop nor a coloop. The bases of  $M$  are precisely

$$\mathcal{B}(M) = \mathcal{B}(M_2) \cup \{B \cup \{n + 1\} \mid B \in \mathcal{B}(M_1)\}.$$

**Construction 2.** Conversely, given a rank  $r$  positroid  $M$  on ground set  $[n + 1]$ , where  $(n + 1)$  is neither a loop nor coloop, we construct two positroids  $M_1 := M_1(M)$  and  $M_2 := M_2(M)$  which form a positively oriented matroid quotient, as follows. Let  $\tilde{A}$  be a matrix realizing  $M$ ; therefore its Plücker coordinates are nonnegative. We apply row operations to rewrite  $\tilde{A}$  in the form

$$A = \left[ \begin{array}{cccc|c} & & & & 0 \\ & & & & 0 \\ & & & & 0 \\ \hline * & * & * & \dots & * \\ \hline & & & & 1 \end{array} \right].$$

Let  $M_1$  denote the matroid on  $[n]$  realized by  $A'$  and let  $M_2$  denote the matroid on  $[n]$  realized by  $A'$  together with the row of  $*$ ’s below it. Then  $M_1$  and  $M_2$  are both positroids (since the Plücker coordinates of  $A'$  and  $A$  are all nonnegative), and they form a positively oriented quotient. Moreover, it is clear that  $M_1 = M \setminus (n + 1)$  and  $M_2 = M / (n + 1)$ .

The idea of our algorithm is to translate Constructions 1 and 2 into operations on Grassmann necklaces, so that Construction 1 is well-defined even if  $M_1$  and  $M_2$  fail to form a positively oriented quotient. Clearly

if we start with positroids  $M_1$  and  $M_2$  forming a positively oriented matroid quotient, then Construction 1 followed by 2 is the identity map. Conversely, if Construction 1 followed by 2 is the identity map, then since Construction 2 always outputs a positively oriented matroid quotient, we must have started with positroids forming a positively oriented matroid quotient.

We let  $\min_i\{S_1, \dots, S_k\}$  denote the minimum of the sets  $S_1, \dots, S_k$  in the  $\leq_i$  order.

**Proposition 7.11.** *Let  $M_1$  and  $M_2$  be positroids of consecutive ranks which form a positively oriented quotient. Let  $\mathcal{I}_{M_j} = (I_1^{(j)}, \dots, I_n^{(j)})$  be the Grassmann necklace of  $M_j$  for  $j = 1, 2$ . Define*

$$J_i = \begin{cases} I_1^{(2)} & \text{for } i = 1, \\ \min_i\{I_i^{(1)} \cup \{n+1\}, I_i^{(2)}\} & \text{for } 2 \leq i \leq n, \\ I_1^{(1)} \cup \{n+1\} & \text{for } i = n+1. \end{cases}$$

Then  $\mathcal{J} = (J_1, \dots, J_{n+1})$  is the Grassmann necklace of the positroid  $M = M(M_1, M_2)$  on  $[n+1]$  whose bases are precisely

$$\mathcal{B}(M) = \mathcal{B}(M_2) \cup \{B \cup \{n+1\} \mid B \in \mathcal{B}(M_1)\}.$$

*Proof.* It suffices to show that each basis of  $M$  is  $i$ -Gale greater than  $J^{(i)}$  for all  $i \in [n+1]$ . One also need to check that the  $J^{(i)}$  are in fact bases of  $M$  but this is clear by definition.

Note that the  $\leq_i$  minimal flag of a flag matroid consists of the  $\leq_i$  minimal bases of each of its constituent matroids [Borovik et al. 2003, Corollary 7.2.1]. Thus,  $I_t^{(1)} \subset I_t^{(2)}$  for each  $t \in [n]$ .

First, let  $S \subset [n]$  be a basis of  $M_2$ . For  $i \in [n]$ , we have  $S \geq_i I_i^{(2)} \geq_i J_i$ . Since neither  $S$  nor  $I_i^{(2)}$  contain  $n+1$ ,  $S \geq_{n+1} I_1^{(2)}$ . By our earlier observation,  $I_1^{(2)} = I_1^{(1)} \cup \{a\}$  for some  $a \in [n]$ . Thus,  $I_1^{(2)} \geq_{n+1} I_1^{(1)} \cup \{n+1\}$ . We conclude that  $S \geq_i J_i$  for all  $i \in [n+1]$ .

Next, consider  $S \cup \{n+1\}$  for  $S$  a basis of  $M_1$ . For  $2 \leq i \leq n$ , we have  $S \cup \{n+1\} \geq_i I_i^{(1)} \cup \{n+1\} \geq_i J_i$ . Since neither  $S$  nor  $I_i^{(1)}$  contain  $n+1$ , we have  $S \geq_1 I_1^{(1)}$  and  $S \cup \{n+1\} \geq_{n+1} I_1^{(1)} \cup \{n+1\} = J_{n+1}$ . Since  $I_1^{(2)} = I_1^{(1)} \cup \{a\}$ , we have  $I_1^{(1)} \cup \{n+1\} \geq_1 I_1^{(2)} = J_1$ . We conclude that  $S \cup \{n+1\} \geq_i J_i$  for all  $i \in [n+1]$ . □

If  $M_1$  and  $M_2$  form a positively oriented quotient, we should obtain them from the positroid  $M = M(M_1, M_2)$ , constructed as in Proposition 7.11, by deleting and contracting  $n+1$ . The following result explains how these operations affect Grassmann necklaces.

**Proposition 7.12** [Oh 2008, Proposition 7 and Lemma 9]. *Let  $M$  be a positroid on  $[n+1]$  such that  $n+1$  is neither a loop nor a coloop, with Grassmann necklace  $(J_i)_{i=1}^{n+1}$ . Then the Grassmann necklaces  $(K_1^{(1)}, \dots, K_n^{(1)})$  and  $(K_1^{(2)}, \dots, K_n^{(2)})$  of  $M_1 = M/(n+1)$  and  $M_2 = M \setminus (n+1)$ , are as follows:*

$$K_i^{(1)} = \begin{cases} J_i \setminus \{n+1\}, & n+1 \in J_i, \\ J_i \setminus \{\max_i(J_i \setminus J_{n+1})\}, & n+1 \notin J_i, \end{cases}$$

$$K_i^{(2)} = \begin{cases} (J_i \setminus \{n+1\}) \cup \{\min_i(J_{n+1} \setminus J_i)\}, & n+1 \in J_i, \\ J_i, & n+1 \notin J_i. \end{cases}$$

Taken together, the last two results yield a recipe for verifying whether two positroids, given in terms of their Grassmann necklaces, form a positively oriented quotient. First apply the construction of Proposition 7.11. If that yields a Grassmann necklace, apply Proposition 7.12 and see if that yields the original Grassmann necklaces. If so, the two Grassmann necklaces form a positively oriented quotient.

Our next goal is to streamline this recipe. Let  $\mathcal{I}^{(1)} = (I_1^{(1)}, \dots, I_n^{(1)})$  and  $\mathcal{I}^{(2)} = (I_1^{(2)}, \dots, I_n^{(2)})$  be Grassmann necklaces of positroids of ranks  $r$  and  $r + 1$ , respectively. Note that a necessary condition for the positroids corresponding to  $\mathcal{I}^{(1)}$  and  $\mathcal{I}^{(2)}$  forming a positively oriented quotient is that  $I_i^{(1)} \subset I_i^{(2)}$  for all  $i \in [n]$ . Now, we define a subset  $S$  as follows: For each  $i$ , if  $I_i^{(1)} \cup \{n + 1\} <_i I_i^{(2)}$ , let  $i \in S$ . Since  $I_i^{(2)} = I_i^{(1)} \cup a$  for some  $a \in [n]$ , this is as simple as checking whether  $a <_i n + 1$ . If the positroids corresponding to  $\mathcal{I}^{(1)}$  and  $\mathcal{I}^{(2)}$  form a positively oriented quotient, applying Proposition 7.11 and then Proposition 7.12 should leave them unchanged. It is straightforward to see that  $i \in S$  if and only if  $n + 1 \in J_i$  in Proposition 7.12. In particular, since  $\mathcal{J}$  is a Grassmann necklace,  $S$  must either be an interval of the form  $[d, n]$ , or empty.

Next we claim that, once we verify that  $S$  is an interval of the form  $[d, n]$  or is empty, then it follows automatically that  $\mathcal{J}$ , as constructed in Proposition 7.11, is a Grassmann necklace.

**Lemma 7.13.** *Let  $\mathcal{I}^{(1)} = (I_1^{(1)}, \dots, I_n^{(1)})$  and  $\mathcal{I}^{(2)} = (I_1^{(2)}, \dots, I_n^{(2)})$  be Grassmann necklaces of types  $(r, n)$  and  $(r + 1, n)$ , respectively. Construct  $\mathcal{J} = (J_1, \dots, J_{n+1})$  as in Proposition 7.11. Let  $S = \{i \in [n] \mid I_i^{(1)} \cup \{n + 1\} <_i I_i^{(2)}\}$ . If  $S = [d, n]$  for some  $d \leq n$  or  $S = \emptyset$ , then  $\mathcal{J}$  is a Grassmann necklace.*

*Proof.* It is clear from the definition that  $\mathcal{J}$  satisfies the Grassmann necklace condition for each pair of consecutive sets  $J_i$  and  $J_{i+1}$  except for when  $i = k - 1$ ,  $i = n$  and  $i = n + 1$  (where we label sets cyclically so that  $J_{n+2} = J_1$ ).

If  $S \neq \emptyset$ , then  $J_n = I_n^{(1)} \cup \{n + 1\}$ . This makes it clear that the Grassmann necklace condition holds for  $J_n$  and  $J_{n+1}$ . Also, since the fact that  $I_i^{(1)} \subset I_i^{(2)}$  for all  $i$ , it is not hard to verify the Grassmann necklace condition for  $J_{n+1}$  and  $J_1$ .

This leaves us to check the condition for  $J_{k-1}$  and  $J_k$ . In this case,  $J_{k-1} = I_{k-1}^{(2)}$  and  $J_k = I_k^{(1)} \cup \{n + 1\}$ . Our goal is to show that  $J_k = (J_{k-1} \setminus \{k - 1\}) \cup \{a\}$  for some  $a \in [n + 1]$ . It is immediately obvious that we necessarily have  $a = n + 1$ . Thus, we are left to show that  $I_k^{(1)} \cup \{n + 1\} = (I_{k-1}^{(2)} \setminus \{k - 1\}) \cup \{n + 1\}$ , or that  $I_k^{(1)} = I_{k-1}^{(2)} \setminus \{k - 1\}$ .

Let  $a_i$  be defined by  $I_i^{(1)} = (I_{i-1}^{(1)} \setminus \{i - 1\}) \cup \{a_i\}$ , let  $b_i$  be defined by  $I_i^{(2)} = (I_{i-1}^{(2)} \setminus \{i - 1\}) \cup \{b_i\}$  and let  $c_i$  be defined by  $I_i^{(2)} = I_i^{(1)} \cup \{c_i\}$ . We observe that  $I_k^{(1)} = (I_{k-1}^{(1)} \setminus \{k - 1\}) \cup \{a_k\} = (I_{k-1}^{(2)} \setminus \{c_{k-1}, k - 1\}) \cup \{a_k\}$ . Also,  $I_k^{(1)} = I_k^{(2)} \setminus \{c_k\} = (I_{k-1}^{(2)} \setminus \{c_k, k - 1\}) \cup \{b_k\}$ . Comparing these two equalities, we conclude that either  $a_k = c_{k-1}$  and  $b_k = c_k$ , or  $c_{k-1} = c_k$  and  $a_k = b_k$ . The first case is what we want to prove, so let us show by contradiction that the second case cannot occur.

Assume  $c_k = c_{k-1}$  and  $a_k = b_k$ . By assumption,  $I_{k-1}^{(1)} \cup \{c_{k-1}\} = I_{k-1}^{(2)} <_{k-1} I_{k-1}^{(1)} \cup \{n + 1\}$  and  $I_k^{(1)} \cup \{n + 1\} <_k I_k^{(2)} = I_k^{(1)} \cup \{c_k\}$ . Thus,  $c_{k-1} <_{k-1} n + 1$  and  $c_k >_k n + 1$ . Since  $c_k = c_{k-1}$ , this means they are both equal to  $k - 1$ . However, if  $c_k = k - 1$ , then  $M_2$  has  $k - 1$  as a coloop. It follows that  $b_k = k - 1$ , which means  $a_k = k - 1$  as well. Thus, in this case,  $I_k^{(1)} = I_{k-1}^{(1)} = I_{k-1}^{(2)} \setminus \{k - 1\}$ , as desired.

Finally, if  $A = \emptyset$ , we can check that the Grassmann necklace condition holds for  $J_{n+1}$  and  $J_1$  as before. The we are just left to verify this condition for  $J_n$  and  $J_{n+1}$ . We can apply the same logic but with  $J_{k-1}$  replaced by  $J_n = I_n^{(2)}$  and  $J_k$  replaced by  $J_{n+1} = I^{(1)} \cup \{n+1\}$ . Specifically, we find  $I_1^{(1)} = (I_n^{(2)} \setminus \{c_n, n\}) \cup \{a_1\} = I_n^{(2)} \setminus \{c_1, n\} \cup \{b_1\}$ . We then must show that it is impossible for  $c_1 = c_n$  and  $a_1 = b_1$ . However,  $I_n^{(1)} \cup \{c_n\} = I_n^{(2)} <_n I_n^{(1)} \cup \{n+1\}$ . Moreover, it is always true that  $I_1^{(1)} \cup \{n+1\} <_{n+1} I_1^{(1)} \cup \{c_1\} = I_2^{(1)}$ . Using  $c_1 = c_n$ , we then find  $c_n <_n (n+1)$  and  $c_n >_{n+1} (n+1)$  which means that  $c_1 = c_n = n$  and we can conclude as in the previous paragraph.  $\square$

Combining Propositions 7.11, 7.12 and Lemma 7.13, we obtain the following:

**Theorem 7.14.** *Fix positroids  $M_1$  and  $M_2$  on  $[n]$  of ranks  $r$  and  $r + 1$ , respectively. Let  $\mathcal{I} = \mathcal{I}_{M_1} = (I_1, \dots, I_n)$  and  $\mathcal{J} = \mathcal{I}_{M_2} = (J_1, \dots, J_n)$  be their Grassmann necklaces. We now set  $S = \{i \in [n] \mid I_i \cup \{n+1\} \leq_i J_i\}$ , where  $\leq_i$  denotes the  $\leq_i$  Gale order on  $[n+1]$ . Define  $a_i = \max_i(J_i \setminus I_1)$  and  $b_i = \min_i(I_1 \setminus I_i)$ . Then  $M_1$  and  $M_2$  form a positively oriented quotient if and only if the following conditions hold:*

- (1) For  $i \in [n]$ ,  $I_i \subset J_i$ .
- (2)  $S$  is an interval of the form  $[d, n]$  or  $S = \emptyset$ .
- (3) For  $i \notin S$ ,  $I_i = J_i \setminus \{a_i\}$ .
- (4) For  $i \in S$ ,  $J_i = I_i \cup \{b_i\}$ .

*Proof.* First, suppose that we have a positively oriented quotient. As explained earlier, the first two conditions always hold for positively oriented quotients. We know that applying the constructions of Propositions 7.11 and 7.12 in sequence should preserve our positively oriented quotient. Observing what conditions this imposes on the constituent Grassmann necklaces yields conditions (3) and (4).

Conversely, if the conditions in the theorem statement hold, then by Lemma 7.13, applying the construction of Proposition 7.11 to  $\mathcal{I}$  and  $\mathcal{J}$  yields another Grassmann necklace  $\mathcal{K}$  on  $[n+1]$  such that  $n+1$  is neither a loop nor a coloop of the positroid corresponding to  $\mathcal{K}$ . Then, conditions (3) and (4) guarantee that applying the construction of Proposition 7.12 to  $\mathcal{K}$  will recover  $\mathcal{I}$  and  $\mathcal{J}$ . The result of applying Proposition 7.12 to the Grassmann necklace of a positroid  $M$  with  $n+1$  neither a loop nor a coloop is the pair of Grassmann necklaces corresponding to  $M/(n+1)$  and  $M \setminus (n+1)$ , which form a positively oriented quotient.  $\square$

**Example 7.15.** Let  $\mathcal{I} = (123, 235, 356, 456, 561, 613)$  and  $\mathcal{J} = (1235, 2356, 3456, 4562, 5612, 6123)$ . Then  $A = \{4, 5, 6\}$  is an interval with upper endpoint  $n = 6$ . Note that  $a_1 = 5, a_2 = 6$  and  $a_3 = 6$ , while  $b_4 = 1, b_5 = 2$  and  $b_6 = 2$ . The positroids with these Grassmann necklaces do not form a positively oriented quotient since it is false that  $I_3 = J_3 \setminus \{a_3\}$ .

However, if we start with the Grassmann necklaces  $\mathcal{I} = (123, 235, 345, 456, 561, 613)$  and  $\mathcal{J} = (1235, 2356, 3456, 4562, 5612, 6123)$ , then the values of the  $a_i$  and  $b_i$  are unchanged. It is straightforward to verify that the conditions of Theorem 7.14 hold and so the positroids corresponding to  $\mathcal{I}$  and  $\mathcal{J}$  do in fact form a positively oriented quotient.

We now have a tool that allows us to recognize flag positroids in consecutive ranks without finding a realization or certifying the incidence relations over the signed hyperfield.

**Corollary 7.16.** *Suppose  $(M_a, M_{a+1}, \dots, M_b)$  is a sequence of positroids of ranks  $a, a+1, \dots, b$ . Then  $(M_a, M_{a+1}, \dots, M_b)$  is a flag positroid if and only if for  $a \leq i < b$ , the pair of positroids  $(M_i, M_{i+1})$  satisfy the conditions of Theorem 7.14.*

*Proof.* By Proposition 7.10, it suffices to check that each such pair forms a positively oriented quotient, which is precisely the content of Theorem 7.14.  $\square$

## 8. Fan structures for and coherent subdivisions from $\text{TrGr}_{d;n}^{>0}$ and $\text{TrFl}_n^{>0}$

In this section we make some brief remarks about the various fan structures for  $\text{TrFl}_{r;n}^{>0}$  and coherent subdivisions from points of  $\text{TrFl}_{r;n}^{>0}$ . Code written for the computations here can be found at <https://github.com/chrisweur/PosTropFlagVar>. We take a detailed look at the Grassmannian and complete flag variety, in particular the case of  $\text{TrFl}_4^{>0}$ .

**8.1. Fan structures.** There are multiple possibly different natural fan structures for  $\text{TrFl}_{r;n}^{>0}$ :

- (i) The Plücker fan (induced by the three-term tropical Plücker relations).
- (ii) The secondary fan (induced according to the coherent subdivision as in Corollary 8.3).
- (iii) The Gröbner fan (induced according to the initial ideal of the ideal  $\langle \mathcal{P}_{r;n} \rangle$ ).
- (iv) The simultaneous refinement of the fans dual to the Newton polytopes of the Plücker coordinates, when the Plücker coordinates are expressed in terms of a “positive parametrization” of  $\text{Fl}_{r;n}^{>0}$ , such as an  $\mathcal{X}$ -cluster chart.
- (v) (If the cluster algebra associated to  $\text{Fl}_{r;n}$  has finitely many cluster variables) the same fan as above but with (the larger set of) cluster variables replacing Plücker coordinates.

Note that by definition, fan (v) is always a refinement of (iv).

In the case of the positive tropical Grassmannian, the fan structures in (iv) and (v) were studied in [Speyer and Williams 2005, Definition 4.2 and Section 8], where the authors observed that for  $\text{Gr}_{2,n}$ , fan (iv) (which coincides with (v)) is isomorphic to the *cluster complex* of type  $A_{n-3}$ ;<sup>3</sup> for  $\text{Gr}_{3,6}$  and  $\text{Gr}_{3,7}$ , fan (iv) is isomorphic to a coarsening of the corresponding cluster complex, while fan (v) is isomorphic to the cluster complex (of types  $D_4$  and  $E_6$ , respectively). Conjecture 8.1 of [Speyer and Williams 2005] says that fan (v) (associated to the positive tropicalization of a full rank cluster variety of finite type) should be isomorphic to the corresponding cluster complex. This conjecture was essentially resolved in [Jahn et al. 2021; Arkani-Hamed et al. 2021a] by working with  $F$ -polynomials.

Theorem 14 of [Olarte et al. 2019] states that the Plücker fan and the secondary fan structures for Dressians coincide, and hence implies that (i) and (ii) coincide because the positive Dressian and the positive tropical Grassmannian are the same [Speyer and Williams 2021]. For  $\text{TrGr}_{2,n}$ , the results of

<sup>3</sup>See [Fomin and Zelevinsky 2003] for background on the cluster complex.

[Speyer and Sturmfels 2004, Section 4] imply that (i), (ii), and (iii) agree, and combining this with [Speyer and Williams 2005, Section 5] implies that all five fan structures agree for  $\text{TrGr}_{2,n}^{>0}$ . For  $\text{TrGr}_{3,6}^{>0}$ , we computed that (iii) and (v) strictly refine (i), but the two fan structures are not comparable.

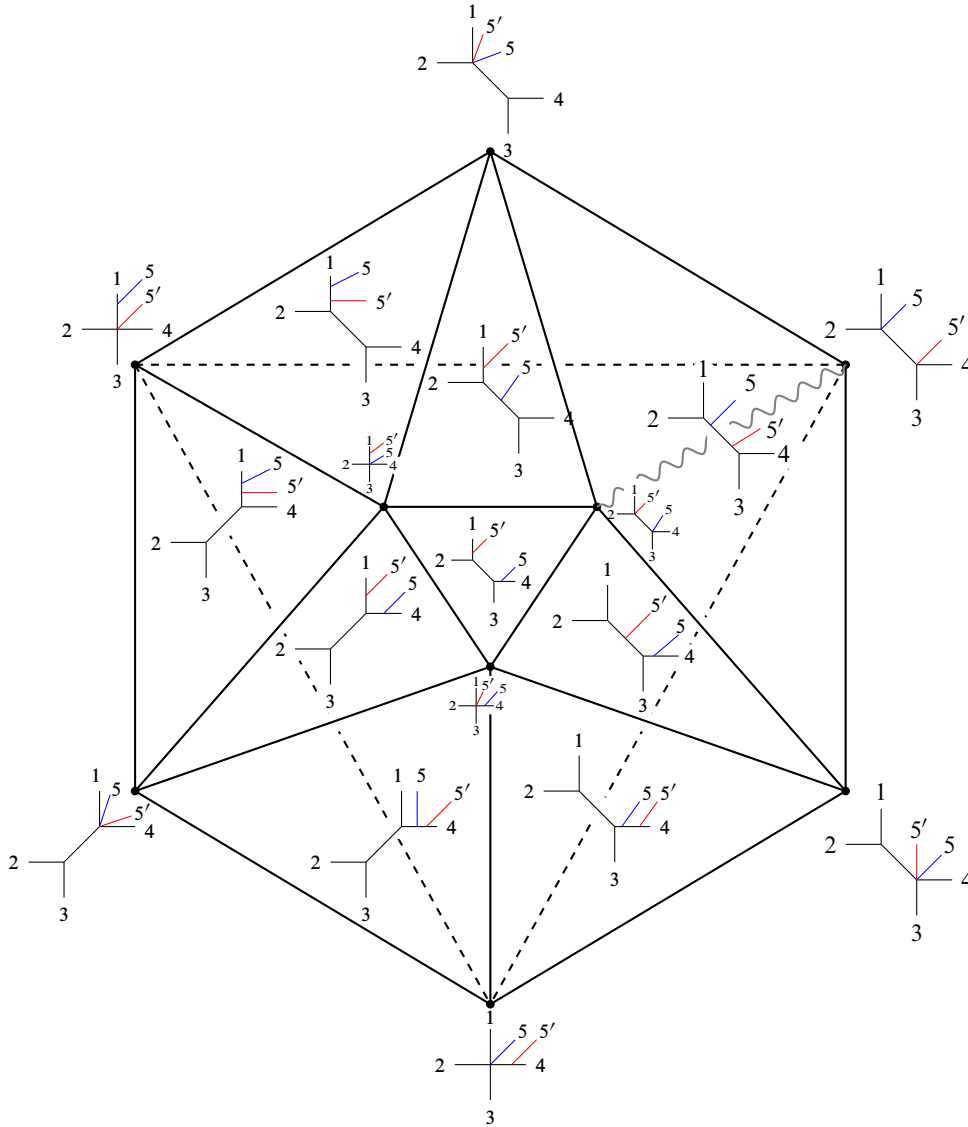
We can consider the same fan structures in the case of the positive tropical complete flag variety. When  $n = 3$ , the fan  $\text{TrFl}_n^{>0}$  modulo its lineality space is a one-dimensional fan, and all fan structures coincide. For  $\text{TrFl}_n$  (before taking the positive part), one can find computations of the fan (iii) for  $n = 4$  and  $n = 5$  in [Bossinger et al. 2017, Section 3], the fan (i) and its relation to (iii) for  $n = 4$  in [Brandt et al. 2021, Example 5.2.3], and the fan (ii) and its relation to (iii) for  $n = 4$  in [Joswig et al. 2023, Section 5]. Returning to the positive tropicalization, Bossinger [2022, Section 5.1] computed the fan structure (iii) for  $\text{TrFl}_4^{>0}$ , and found it was dual to the three-dimensional associahedron; in particular, there are 14 maximal cones and the  $f$ -vector is  $(14, 21, 9, 1)$ . Using the positive parametrization of [Boretsky 2022] (a graphical version of the parametrizations of [Marsh and Rietsch 2004]) for  $\text{TrFl}_n^{>0}$ , we computed the polyhedral complex underlying (iv) for  $n = 4$  in Macaulay2 by computing the normal fan of the Minkowski sum of the Newton polytopes of the Plücker coordinates expressed in the chosen parametrization; we obtained the  $f$ -vector  $(13, 20, 9, 1)$ . We also computed (v) after incorporating the additional non-Plücker cluster variable  $p_2 p_{134} - p_1 p_{234}$ . Combining these, we find that for  $n = 4$ , (i)=(iv) and (ii)=(iii). We also find that both (ii) and (v) strictly refine (i)=(iv) and are both isomorphic to the normal fan of the three-dimensional associahedron, but are not comparable fan structures.

The fact that the fan structure (v) of  $\text{TrFl}_4^{>0}$  is dual to the three-dimensional associahedron is consistent with [Speyer and Williams 2005, Conjecture 8.1] and the fact that  $\text{Fl}_4$  has a cluster algebra structure of finite type  $A_3$  [Geiss et al. 2008, Table 1], whose cluster complex is dual to the associahedron.

We now give a graphical way to think about the fan structure on  $\text{TrFl}_4^{>0}$ , building on the ideas of [Speyer and Williams 2005] and [Brandt et al. 2021, Example 5.2.3].

**Example 8.1.** A *planar tree* on  $[n]$  is an unrooted tree drawn in the plane with  $n$  leaves labeled by  $1, 2, \dots, n$  (in counterclockwise order). By [Speyer and Williams 2005],  $\text{TrGr}_{2;n}^{>0}$  parametrizes metric planar trees, and its cones correspond to the various combinatorial types of planar trees. In particular, if we assign real-valued lengths to the edges of a planar tree, then the negative of the distance between leaf  $i$  and  $j$  encodes the positive tropical Plücker coordinate  $w_{ij}$  of a point in the corresponding cone. In particular, it is easy to see that the negative distances  $w_{ij}$  associated to such a planar tree satisfy the positive tropical Plücker relations.

Now as in [Brandt et al. 2021, Example 5.2.3], we note that for a valuated matroid  $\mu$  whose underlying matroid is the uniform matroid  $U_{2,4}$ , the tropical linear spaces  $\text{trop}(\mu)$  and  $\text{trop}(\mu^*)$  associated to  $\mu$  and its dual  $\mu^*$  are translates of each other. This allows us to identify points  $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$  of  $\text{TrFl}_4^{>0}$  with planar trees on the vertices  $\{1, 2, 3, 4, 5, 5'\}$  such that the vertices  $\{1, 2, 3, 4, 5\}$  and separately the vertices  $\{1, 2, 3, 4, 5'\}$  appear in counterclockwise order. To see this, note that (using the same idea as Construction 1 from Section 7) we can identify  $(\mu_1, \mu_2)$ , with Plücker coordinates  $(w_1, \dots, w_4; w_{12}, \dots, w_{34})$ , with an element  $(w_{ab})$  of  $\text{TrGr}_{2,5}^{>0}$ : we simply set  $w_{a5} := w_a$  for  $1 \leq a \leq 4$ .



**Figure 2.** The fan structure (ii)=(iii) of  $\text{TrFl}_4^{>0}$ .

Similarly, we identify  $(\mu_2, \mu_3)$ , where  $\mu_3$  has Plücker coordinates  $(w_{123}, \dots, w_{234})$ , with an element of  $\text{TrGr}_{2,5}^{>0}$ : we simply set  $w_{d5'} := w_{abc}$ , where  $\{a, b, c\} := [4] \setminus \{d\}$ .

This gives us the Plücker fan structure (i)=(iv) with thirteen maximal cones, as shown in Figure 2. To get the Gröbner fan structure (iii) we subdivide one of the cones into two, along the squiggly line shown in Figure 2. This squiggly line occurs when  $\text{dist}(x_1, \text{blue}) = \text{dist}(x_2, \text{red})$ , where  $x_1$  and  $x_2$  are the two black trivalent nodes in the tree on  $[4]$ . To obtain the fan structure (v), instead of the squiggly line, the square face is subdivided along the other diagonal.

Using the computation of  $\text{TrFl}_5$  in [Bossinger et al. 2017], which can be found on github at <https://github.com/Saralamboglia/Toric-Degenerations/blob/master/Flag5.rtf> and Corollary 3.12, we further computed that  $\text{TrFl}_5^+$  with (iii) has 938 maximal cones (906 of which are simplicial) and that (iv) has 406 maximal cones. According to [Speyer and Williams 2005, Conjecture 8.1], the (v) fan structure for  $\text{TrFl}_5^+$  has 672 maximal cones.

**8.2. Coherent subdivisions.** We next discuss coherent subdivisions coming from the positive tropical Grassmannian and positive tropical complete flag variety. When  $\text{Fl}_{r,n}$  is the Grassmannian  $\text{Gr}_{d,n}$  and the support  $\underline{\mu}$  is the uniform matroid, Theorem A gives rise to the following corollary (which was first proved in [Lukowski et al. 2023] and [Arkani-Hamed et al. 2021b]).

**Corollary 8.2.** *Let  $\underline{\mu} = (\mu_d) \in \mathbb{P}(\mathbb{T}^{\binom{[n]}{d}})$ , and suppose it has no  $\infty$  coordinates. Then the following statements are equivalent:*

- $\underline{\mu} \in \text{TrGr}_{d,n}^{>0}$ , that is,  $\underline{\mu}$  lies in the strictly positive tropical Grassmannian.
- Every face in the coherent subdivision  $\mathcal{D}_{\underline{\mu}}$  of the hypersimplex  $\Delta_{d,n}$  induced by  $\underline{\mu}$  is a positroid polytope.

The coherent subdivisions above (called *positroidal subdivisions*) were further studied in [Speyer and Williams 2021], where the finest positroidal subdivisions were characterized in terms of series-parallel matroids. Furthermore, all finest positroidal subdivisions of  $\Delta_{d,n}$  achieve equality in Speyer's *f-vector theorem*; in particular, they all consist of  $\binom{n-2}{d-1}$  facets [Speyer and Williams 2021, Corollary 6.7].

When  $\text{Fl}_{r,n}$  is the complete flag variety  $\text{Fl}_n$ , and the support  $\underline{\mu}$  is the uniform flag matroid, Theorem A gives rise to the following corollary, which appeared in [Joswig et al. 2023, Theorem 20].

**Corollary 8.3.** *Let  $\underline{\mu} = (\mu_1, \dots, \mu_n) \in \prod_{i=a}^b \mathbb{P}(\mathbb{T}^{\binom{[n]}{i}})$ , and suppose it has no  $\infty$  coordinates. Then the following statements are equivalent.*

- $\underline{\mu} \in \text{TrFl}_n^{>0}$ , that is,  $\underline{\mu}$  lies in the strictly positive tropical flag variety.
- Every face in the coherent subdivision  $\mathcal{D}_{\underline{\mu}}$  of the permutohedron  $\text{Perm}_n$  induced by  $\underline{\mu}$  is a Bruhat interval polytope.

In light of the results of [Speyer and Williams 2021], it is natural to ask if one can characterize the finest coherent subdivisions of the permutohedron  $\text{Perm}_n$  into Bruhat interval polytopes. Furthermore, do they all have the same *f*-vector?

Explicit computations for  $\text{TrFl}_4$  show that the answer to the second question is no. We find that  $\text{TrFl}_4$  with the fan structure (iii) (which agrees with (ii) by [Joswig et al. 2023, Section 5]) has 78 maximal cones. We choose a point in the relative interior of each of the 78 cones to use as a height function (thinking of points in  $\text{TrFl}_4$  as weights on the vertices of  $\text{Perm}_4$  as in (c) of Theorem A), then use Sage to compute the corresponding coherent subdivision of  $\text{Perm}_4$ . As expected, precisely 14 of the 78 cones induce subdivisions of  $\text{Perm}_4$  into Bruhat interval polytopes, see Table 1.

height function $(P_1, P_2, P_3, P_4; P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}; P_{123}, P_{124}, P_{134}, P_{234})$	Bruhat interval polytopes in subdivision	$f$ -vector
$(15, -1, -7, -7; 4, -2, -2, -2, -2, 4; -7, -7, -1, 15)$	$P_{3214,4321}, P_{3124,4231}, P_{2314,3421}, P_{2134,3241}, P_{1324,2431}, P_{1234,2341}$	(24, 46, 29, 6)
$(15, 3, -9, -9; 4, -8, -8, -4, -4, 20; -1, -1, -1, 3)$	$P_{2413,4321}, P_{3124,4231}, P_{2314,4231}, P_{2134,3241}, P_{1324,2431}, P_{1234,2341}$	
$(15, -7, -1, -7; -2, 4, -2, -2, 4, -2; -7, -1, -7, 15)$	$P_{3142,4321}, P_{3124,4312}, P_{2143,3421}, P_{2134,3412}, P_{1243,2431}, P_{1234,2413}$	
$(-1, -1, -1, 3; 4, -8, -4, -8, -4, 20; 15, 3, -9, -9)$	$P_{2413,4321}, P_{1423,4231}, P_{1342,4231}, P_{1324,4213}, P_{1243,4132}, P_{1234,4123}$	
$(-7, -7, -1, 15; 4, -2, -2, -2, -2, 4; 15, -1, -7, -7)$	$P_{1432,4321}, P_{1423,4312}, P_{1342,4231}, P_{1324,4213}, P_{1243,4132}, P_{1234,4123}$	
$(-1, -7, -7, 15; -2, -2, 4, 4, -2, -2; 15, -7, -7, -1)$	$P_{3142,4321}, P_{2143,4312}, P_{2134,4213}, P_{1342,3421}, P_{1243,3412}, P_{1234,2413}$	
$(-9, -9, 3, 15; 20, -4, -8, -4, -8, 4; 3, -1, -1, -1)$	$P_{1432,4321}, P_{1423,4312}, P_{1342,4231}, P_{1324,4213}, P_{1324,4132}, P_{1234,3142}$	
$(11, -7, -7, 3; -6, -6, 4, 4, 2, 2; 11, -7, -7, 3)$	$P_{3142,4321}, P_{2143,4312}, P_{2134,4213}, P_{2143,3421}, P_{1243,2431}, P_{1234,2413}$	
$(3, 3, -3, -3; 20, -10, -10, -10, -10, 20; -3, -3, 3, 3)$	$P_{2413,4321}, P_{3124,4231}, P_{2314,4231}, P_{1324,2431}, P_{1324,3241}, P_{1234,3142}$	
$(3, -1, -1, -1; 20, -4, -4, -8, -8, 4; -9, -9, 3, 15)$	$P_{3214,4321}, P_{3124,4231}, P_{2314,3421}, P_{1324,3241}, P_{1324,2431}, P_{1234,3142}$	
$(-3, -3, 3, 3; 20, -10, -10, -10, -10, 20; 3, 3, -3, -3)$	$P_{2413,4321}, P_{1423,4231}, P_{1342,4231}, P_{1324,4132}, P_{1324,4213}, P_{1234,3142}$	
$(3, -7, -7, 11; 2, 2, 4, 4, -6, -6; 3, -7, -7, 11)$	$P_{3142,4321}, P_{3124,4312}, P_{1342,3421}, P_{2134,3412}, P_{1243,3412}, P_{1234,2413}$	
$(11, -1, -7, -3; -2, -8, -4, -4, 0, 18; 11, -1, -7, -3)$	$P_{2413,4321}, P_{2143,4231}, P_{2134,4213}, P_{1243,2431}, P_{1234,2413}$	
$(-3, -7, -1, 11; 18, 0, -4, -4, -8, -2; -3, -7, -1, 11)$	$P_{3142,4321}, P_{3124,4312}, P_{1342,3421}, P_{1324,3412}, P_{1234,3142}$	

**Table 1.** Table documenting the 14 finest coherent subdivisions of  $\text{Perm}_4$  into Bruhat interval polytopes. There are two possible  $f$ -vectors, each of which can be realized in multiple ways.

Of the 14 coherent subdivisions coming from maximal cones of  $\text{TrFl}_4^{>0}$ , 12 of them contain 6 facets, while the other 2 contain 5 facets. Table 1 lists the facets and  $f$ -vectors of each of these 14 subdivisions. Note that each Bruhat interval polytope  $P_{v,w}$  which appears as a facet satisfies  $\ell(w) - \ell(v) = 3$ . Thus, any

height function ( $P_1, P_2, P_3, P_4; P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}; P_{123}, P_{124}, P_{134}, P_{234}$ )	Bruhat interval polytopes in subdivision	$f$ -vector
$(-1, -1, -1, 0; -1, -1, 0, -1, 0, 0; 0, 0, 0, 0)$	$P_{1243,4321}, P_{1234,4213}$	$(24, 39, 18, 2)$
$(-1, -1, -1, 0; 0, 0, 0, 0, 0, 0; 0, 0, 0, 0)$	$P_{1342,4321}, P_{1234,4312}$	
$(1, 0, 0, 0; 0, 0, 0, 0, 0, 0; 0, 0, 0, 0)$	$P_{2134,4321}, P_{1234,2431}$	
$(1, 0, 0, 0; 0, 0, 0, 1, 1, 1; 0, 0, 0, 0)$	$P_{3124,4321}, P_{1234,3421}$	
$(0, 0, 0, 0; -1, -1, -1, -1, -1, 0; 0, 0, 0, 0)$	$P_{2413,4321}, P_{1234,4231}$	$(24, 40, 19, 2)$
$(0, 0, 0, 0; 1, 0, 0, 0, 0, 0; 0, 0, 0, 0)$	$P_{1324,4321}, P_{1234,3142}$	
$(-1, -1, 0, 0; -1, -1, -1, -1, -1, 0; 0, 0, 0, 0)$	$P_{1423,4321}, P_{1342,4231}, P_{1324,4213}, P_{1234,4132}$	$(24, 42, 23, 4)$
$(0, -1, -1, 0; 0, 0, 1, 0, 0, 0; 0, 0, 0, 0)$	$P_{3142,4321}, P_{1243,3421}, P_{2134,4312}, P_{1234,2413}$	
$(1, 1, 0, 0; 1, 0, 0, 0, 0, 0; 0, 0, 0, 0)$	$P_{2314,4321}, P_{1324,2431}, P_{3124,4231}, P_{1234,3241}$	

**Table 2.** Table documenting the 9 coarsest coherent subdivisions of  $\text{Perm}_4$  into Bruhat interval polytopes. There are three possible  $f$ -vectors, each of which can be realized in multiple ways.

Bruhat interval polytope  $P_{v',w}$  properly contained inside  $P_{v,w}$  would have the property that  $\ell(w') - \ell(v') \leq 2$ , and hence  $\dim(P_{v',w'}) \leq 2$ . Since  $\text{Perm}_4$  is 3-dimensional, all 14 of these subdivisions are finest subdivisions.

We note that the 12 finest subdivisions whose  $f$ -vector is  $(24, 46, 29, 6)$  are subdivisions of the permutohedron into cubes. Subdivisions of the permutohedron into Bruhat interval polytopes which are cubes have been previously studied in [Harada et al. 2019, Sections 5 and 6; Lee et al. 2021; Nadeau and Tewari 2023, Section 6]. In particular, there is a subdivision of  $\text{Perm}_n$  into  $(n-1)!$  Bruhat interval polytopes

$$\{P_{u,v} \mid u = (u_1, \dots, u_n) \text{ with } u_n = n, \text{ and } v = (v_1, \dots, v_n) \text{ with } v_i = u_i + 1 \text{ modulo } n\}.$$

The first subdivision in Table 1 has this form.

We can further study the  $f$ -vectors of subdivisions of  $\text{TrFl}_4^{>0}$  which are coarsest (without being trivial), rather than finest. In this case, we observe three different  $f$ -vectors, each of which occurs in multiple subdivisions. The detailed results of our explicit computations on coarsest subdivisions can be found in Table 2.

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jboresky@math.harvard.edu

*Department of Mathematics, Harvard University, Cambridge, MA, United States*

ceur@math.harvard.edu

*Department of Mathematics, Harvard University, Cambridge, MA, United States*

williams@math.harvard.edu

*Department of Mathematics, Harvard University, Cambridge, MA, United States*

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