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Given a finite group G and a prime p , let $\mathcal{A}_p(G)$ be the poset of nontrivial elementary abelian p -subgroups of G . The group G satisfies the Quillen dimension property at p if $\mathcal{A}_p(G)$ has nonzero homology in the maximal possible degree, which is the p -rank of G minus 1. For example, D. Quillen showed that solvable groups with trivial p -core satisfy this property, and later, M. Aschbacher and S. D. Smith provided a list of all p -extensions of simple groups that may fail this property if p is odd. In particular, a group G with this property satisfies Quillen's conjecture: G has trivial p -core and the poset $\mathcal{A}_p(G)$ is not contractible.

In this article, we focus on the prime $p = 2$ and prove that the 2-extensions of finite simple groups of exceptional Lie type in odd characteristic satisfy the Quillen dimension property, with only finitely many exceptions. We achieve these conclusions by studying maximal subgroups and usually reducing the problem to the same question in small linear groups, where we establish this property via counting arguments. As a corollary, we reduce the list of possible components in a minimal counterexample to Quillen's conjecture at $p = 2$.

1. Introduction

Since the early 70s, there has been a growing interest in the p -subgroup posets and their connections with finite group theory, the classification of the finite simple groups, finite geometries, group cohomology and representation theory. The poset $\mathcal{S}_p(G)$ of nontrivial p -subgroups of a group G was introduced by Kenneth Brown [1975]. In that paper, Brown worked with the Euler characteristic $\chi(G)$ of groups G satisfying certain finiteness conditions and established connections between the p -fractional part of $\chi(G)$ and the p -subgroup structure of G . One of the consequences of his results is the commonly known "Homological Sylow theorem", which states that the Euler characteristic of $\mathcal{S}_p(G)$ is 1 modulo $|G|_p$, the order of a Sylow p -subgroup of G .

Some years later, Daniel Quillen [1978] introduced the poset $\mathcal{A}_p(G)$ of nontrivial elementary abelian p -subgroups of a finite group G and exhibited several applications of the topological properties of these posets. Indeed, the study of elementary abelian p -subgroups goes back to Quillen's earlier work on the Bredon cohomology of G -spaces and his proof of the Atiyah–Swan conjecture, that relates the Krull dimension of a ring to the dimension of $\mathcal{A}_p(G)$ (see [Quillen 1971]).

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Quillen [1978] showed that $\mathcal{S}_p(G)$ and $\mathcal{A}_p(G)$ are (G -equivariantly) homotopy equivalent, and provided a new proof of Brown's result. In fact, when G is the set of rational points of a semisimple algebraic group over a finite field of characteristic p , these posets are homotopy equivalent to the building of G and, hence, they have the homotopy type of a wedge of spheres of dimension $l - 1$, where l is the rank of the underlying algebraic group. Furthermore, in that case, the homology $\tilde{H}_*(\mathcal{A}_p(G))$ affords the classical Steinberg module for G .

Quillen also exhibited other connections between intrinsic algebraic properties of G and the topology of these posets. For instance, he showed that $\mathcal{A}_p(G)$ is disconnected if and only if G contains a strongly p -embedded subgroup. Recall that the classification of the groups with this property is indeed one of the many important steps towards the classification of the finite simple groups (see, for example, Section 7.6 of [Gorenstein et al. 1998]).

On the other hand, Quillen proved that if G has a fixed point on $\mathcal{A}_p(G)$ (or, equivalently on $\mathcal{S}_p(G)$), then these posets are contractible. Note that G has a fixed point if and only if its p -core $O_p(G)$ is nontrivial. In view of this and further evidence, Quillen conjectured that the reciprocal to this statement should hold. That is, if $\mathcal{A}_p(G)$ is contractible then there is a fixed point, or, equivalently, $O_p(G) \neq 1$ (see Conjecture 2.9 of [Quillen 1978]). In other words, Quillen's conjecture asserts that $\mathcal{A}_p(G)$ is contractible if and only if $O_p(G) \neq 1$.

A significant part of Quillen's article is devoted to proving the solvable case of this conjecture. In [Quillen 1978] it is shown that for a p -nilpotent group G with abelian Sylow p -subgroups and $O_p(G) = 1$, $\mathcal{A}_p(G)$ is homotopy equivalent to a nontrivial wedge of spheres of the maximal possible dimension, which is $m_p(G) - 1$, the p -rank of G minus 1. Then, if G is any solvable group with $O_p(G) = 1$, G contains a p -nilpotent subgroup $O_{p'}(G)A$, with $A \in \mathcal{A}_p(G)$ of maximal p -rank and $O_p(O_{p'}(G)A) = 1$, and thus $\tilde{H}_{m_p(G)-1}(\mathcal{A}_p(G)) \neq 0$.

Later, Michael Aschbacher and Stephen D. Smith [1993] formalised this property and gave a name to it: an arbitrary group G with $\tilde{H}_{m_p(G)-1}(\mathcal{A}_p(G)) \neq 0$ is said to satisfy the *Quillen dimension property at p* , or $(\mathbb{Q}\mathcal{D})_p$ for short. Therefore, a solvable group G with $O_p(G) = 1$ satisfies $(\mathbb{Q}\mathcal{D})_p$ and thus Quillen's conjecture. Furthermore, it was shown that p -solvable groups satisfy this property by using Quillen's techniques and, in addition, the CFSG (see [Díaz Ramos 2018; Smith 2011]). These results also suggest that a stronger statement of the conjecture may hold: if $O_p(G) = 1$ then $\tilde{H}_*(\mathcal{A}_p(G); \mathbb{Q}) \neq 0$. Therefore, from now on, by Quillen's conjecture we will be referring to this stronger version.

It is not hard to see that not every group G with $O_p(G) = 1$ satisfies $(\mathbb{Q}\mathcal{D})_p$. For example, we mentioned that finite groups of Lie type in characteristic p satisfy the conjecture, but since the Lie rank is usually strictly smaller than the p -rank, they fail $(\mathbb{Q}\mathcal{D})_p$. This has led to the development of new methods to prove Quillen's conjecture. One of the most notorious advances in the conjecture was achieved by Aschbacher and Smith [1993]. They established Quillen's conjecture for a group G if $p > 5$ and in addition, roughly, all the p -extensions of finite unitary groups $\text{PSU}_n(q)$, with q odd and $p \mid q+1$, satisfy $(\mathbb{Q}\mathcal{D})_p$ (see Main Theorem of [Aschbacher and Smith 1993] for the precise statement). Here, a p -extension of a group L is a split extension of L by an elementary abelian p -subgroup of $\text{Out}(L)$. In [Aschbacher and Smith 1993]

it is not shown that the group G satisfies $(\mathbb{Q}\mathcal{D})_p$. Instead, they proved that if every p -extension of a fixed component of G satisfies $(\mathbb{Q}\mathcal{D})_p$, then

$$\tilde{H}_*(\mathcal{A}_p(G); \mathbb{Q}) \neq 0 \quad \text{if } O_p(G) = 1$$

(under suitable inductive hypotheses). This result restricts the possibilities of the components of a minimal counterexample to Quillen's conjecture: every component has a p -extension failing $(\mathbb{Q}\mathcal{D})_p$. In view of this result and the classification of the finite simple groups, Aschbacher and Smith described for $p \geq 3$, all the possible p -extensions of simple groups which may potentially fail $(\mathbb{Q}\mathcal{D})_p$. This is the $(\mathbb{Q}\mathcal{D})$ -List, Theorem 3.1, of [Aschbacher and Smith 1993]. Moreover, it is conjectured in [Aschbacher and Smith 1993] that the unitary groups $\text{PSU}_n(q)$ with q odd and $p \mid q+1$ should not appear in this list, and so the extra hypothesis on the unitary groups in the main result of [Aschbacher and Smith 1993] could be omitted. Nevertheless, this problem remains open (see [Piterman and Welker 2022] for recent results in this direction).

In the last few years, there have been further developments in the Quillen conjecture [Piterman 2021; Piterman et al. 2021; Piterman and Smith 2022a; 2022b]. Recently, in [Piterman and Smith 2022b], new tools for the study of the conjecture have been provided. For example, it is shown that the Aschbacher–Smith general approach to the conjecture can be extended to *every* prime p by reducing reliance on results of [Aschbacher and Smith 1993] stated only for odd primes and invoking the classification. In particular, Theorem 1.1 of [Piterman and Smith 2022b] shows that Main Theorem of [Aschbacher and Smith 1993] extends to $p \geq 3$, keeping the additional constraint on the unitary groups. On the other hand, for $p = 2$, one important obstruction for this extension is the lack of a $(\mathbb{Q}\mathcal{D})$ -List for this prime. Roughly, Corollary 1.8 of [Piterman and Smith 2022b] concludes that a minimal counterexample to Quillen's conjecture contains a component of Lie type in characteristic $r \neq 3$, and every component of G has a 2-extension failing $(\mathbb{Q}\mathcal{D})_2$.

In view of these results on Quillen's conjecture, in this article, we focus on showing that the 2-extensions of the finite simple groups of exceptional Lie type in odd characteristic satisfy $(\mathbb{Q}\mathcal{D})_2$, with a small number of exceptions. This improves the conclusions of [Piterman and Smith 2022b] on Quillen's conjecture for $p = 2$, and allows us to conclude then that exceptional groups of Lie type in odd characteristic different from 3 cannot be components of a minimal counterexample to the conjecture (see Corollary 1.2 below).

The main result of this article is the following theorem, whose proof is given through different propositions in Section 5.

Theorem 1.1. *Let L be a finite simple group of exceptional Lie type in odd characteristic. That is, $L = {}^3D_4(q)$, $F_4(q)$, $G_2(q)$, ${}^2G_2(q)'$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$ or $E_8(q)$, with q odd. Then every 2-extension of L satisfies the Quillen dimension property at $p = 2$, except possibly in the following cases:*

- ${}^3D_4(9)$ extended with field automorphisms;
- $F_4(3)$, $F_4(9)$ extended with field automorphisms;
- 2-extensions of $G_2(3)$, $G_2(9)$ extended with field automorphisms;
- ${}^2G_2(3)'$, $E_8(3)$, $E_8(9)$ extended with field automorphisms.

Indeed, the extensions of $G_2(3)$, $G_2(9)$ and ${}^2G_2(3)'$ mentioned above do fail $(\mathbb{Q}\mathcal{D})_2$ by Example 5.3 and Proposition 5.1.

To achieve the conclusions of Theorem 1.1, in most cases we exhibit a maximal subgroup M of a 2-extension LB of L such that $m_2(M) = m_2(LB)$ and M satisfies $(\mathbb{Q}\mathcal{D})_2$. Since there is an inclusion $\tilde{H}_{m_2(LB)-1}(\mathcal{A}_2(M)) \hookrightarrow \tilde{H}_{m_2(LB)-1}(\mathcal{A}_2(LB))$ in the top-degree homology groups, this establishes $(\mathbb{Q}\mathcal{D})_2$ for LB (see Lemma 3.3). In some cases, the subgroup M arises from suitable parabolic subgroups. More concretely, when it is possible, we pick P to be a maximal parabolic subgroup of L which is stabilised by B and such that $M := PB$ realises the 2-rank of LB . Then we get a 2-nilpotent configuration UA , where U is the unipotent radical of P , A is an elementary abelian 2-subgroup realising the 2-rank of PB , and $O_2(UA) = C_A(U) = 1$ by one of the corollaries of the Borel–Tits theorem. Hence, by Quillen’s results on the solvable case, UA satisfies $(\mathbb{Q}\mathcal{D})_2$, and thus also M and LB .

When the choice of such parabolic P is not possible, we pick one of the maximal rank subgroups of L . Here, the components of the maximal subgroup M are usually smaller exceptional groups, low-dimensional linear group $A_1(q)$ and $A_2(q)$ or unitary groups ${}^2A_2(q)$. Therefore, we first prove that the 2-extensions of simple linear and unitary groups in dimensions 2 and 3 satisfy $(\mathbb{Q}\mathcal{D})_2$.

Although there is a large literature on maximal subgroups of exceptional groups of Lie type, we will only need the results from [Cohen et al. 1992; Kleidman 1988; Liebeck et al. 1992; Liebeck and Seitz 1990; 2004].

Finally, from Theorem 1.1 and the results of [Piterman and Smith 2022b] for $p = 2$, we can conclude:

Corollary 1.2. *Let G be a minimal counterexample to Quillen’s conjecture for $p = 2$. Then G contains a component of Lie type in characteristic $r \neq 3$. Moreover, every such component fails $(\mathbb{Q}\mathcal{D})_2$ in some 2-extension and belongs to one of the following families:*

$$\begin{aligned} & \text{PSL}_n(2^a) \ (n \geq 3), \quad D_n(2^a) \ (n \geq 4), \quad E_6(2^a), \\ & \text{PSL}_n^\pm(q) \ (n \geq 4), \quad B_n(q) \ (n \geq 2), \quad C_n(q) \ (n \geq 3), \quad D_n^\pm(q) \ (n \geq 4), \end{aligned}$$

where $q = r^a$ and $r > 3$.

The 2-extensions of $\text{PSL}_2(q)$, $\text{PSL}_3(q)$ and $\text{PSU}_3(q)$ satisfy $(\mathbb{Q}\mathcal{D})_2$ by Propositions 4.2, 4.5 and 4.6, respectively, with exceptions when $q = 3, 5, 9$. Nevertheless, the results of [Piterman and Smith 2022b] eliminate these possibilities from a minimal counterexample.

Further results on the Quillen dimension property at $p = 2$ for the classical groups could be pursued by combining the methods presented in this article with the results of [Díaz Ramos 2018; Díaz Ramos and Mazza 2022].

The paper is organised as follows. In Section 2 we set the notation and conventions that we will need to work with the finite groups of Lie type. We also provide some useful properties to work out the p -extensions and compute p -ranks. In Section 3 we gather previous results on the Quillen dimension property and related tools that will help us establish this property. Then in Section 4 we establish $(\mathbb{Q}\mathcal{D})_2$

for some 2-extensions of linear groups and recall the structure of the centralisers of graph automorphisms, following Table 4.5.1 of [Gorenstein et al. 1998]. In Section 5 we prove each case of Theorem 1.1.

All groups considered in this article are finite. We suppress the notation for the homology coefficients, and we assume that they are always taken over \mathbb{Q} . The interested reader may note that our results can be extended to homology with coefficients in other rings. Finally, we emphasise that we adopt the language and conventions of [Gorenstein et al. 1998]. This is particularly important when we name the different types of automorphisms of groups of Lie type. Computer calculations were performed with [GAP].

2. Preliminaries

We assume that the reader is familiar with the construction of the finite groups of Lie type as fixed points of Steinberg endomorphisms, and the basic properties concerning root systems of reductive algebraic groups. We will follow the language of [Gorenstein et al. 1998], which also contains the required background on finite groups of Lie type. In this section, we will only recall some notation and names, and state results that will be used later.

We denote by C_n , D_n , Sym_n and Alt_n the cyclic group of order n , the dihedral group of order n , the symmetric group on n points and the alternating group on n points.

If G is a group, then $\text{Aut}(G)$, $\text{Inn}(G)$ and $\text{Out}(G)$ denote the automorphism group, the group of inner automorphisms and the outer automorphism group of G respectively. We denote by $Z(G)$ the centre of G . We usually write $G : H$, or simply GH , for a split extension of G by H . When an extension of G by H may not split, we denote it by $G.H$. By an element g (resp. a subgroup B) of G inducing outer automorphisms on $L \leq G$ we mean that $g \in N_G(L)$ embeds into $\text{Aut}(L) \setminus \text{Inn}(L)$ (resp. $B \leq N_G(L)$ embeds in $\text{Aut}(L)$ with $B \cap \text{Inn}(L) = 1$). Finally, $H \circ_m K$ denotes a central product of H and K by a central cyclic subgroup of order m . That is, $H \circ_m K = (H \times K) / C_m$, where C_m embeds into both $Z(H)$ and $Z(K)$.

We will usually use the notation n in a group extension to denote a cyclic group of order n , and n^m a direct product of m copies of cyclic groups of order n . A number between brackets $[n]$ in the structure description of an extension means some group of order n .

In this article, we are mainly interested in extensions by elementary abelian groups. Below we recall the definition of p -extension given in the introduction and introduce some useful notation.

Definition 2.1. Let L be a finite group and p a prime number. A p -extension of L is a split extension LB of L by an elementary abelian p -group B inducing outer automorphisms on L .

If $L \leq G$, we denote by $\mathcal{O}_G(L)$ the poset of elements $B \in \mathcal{A}_p(N_G(L))$ such that $B \cap LC_G(L) = 1$ (that is, B induces outer automorphisms on L). We write $\mathcal{O}_2(L)$ for $\mathcal{O}_{\text{Aut}(L)}(L)$ at $p = 2$. We also let $\hat{\mathcal{O}}_G(L) = \mathcal{O}_G(L) \cup \{1\}$ and $\hat{\mathcal{O}}_2(L) = \mathcal{O}_2(L) \cup \{1\}$.

Definition 2.2. For a prime number p , we say that a group G satisfies the *Quillen dimension property* at p if $\mathcal{A}_p(G)$ has nonzero homology in dimension $m_p(G) - 1$, where $m_p(G)$ denotes the p -rank of G :

$$\tilde{H}_{m_p(G)-1}(\mathcal{A}_p(G)) \neq 0. \tag{QD}_p$$

A remarkable study of the Quillen dimension property for odd primes p was carried out in Theorem 3.1 of [Aschbacher and Smith 1993]. This theorem provides a list with the p -extensions of simple groups that might fail $(\mathbb{Q}\mathcal{D})_p$, for $p \geq 3$. In particular, this list contains the p -extensions of unitary groups $\text{PSU}_n(q)$ with q odd and $p|q+1$. However, Conjecture 4.1 of [Aschbacher and Smith 1993] basically claims that these groups should not belong to this list. In fact, it is shown there that if $n < q(q - 1)$ then these p -extensions satisfy $(\mathbb{Q}\mathcal{D})_p$. Nevertheless, this problem remains open.

The aim of this article is to achieve some progress on a similar list for the prime $p = 2$. Therefore, we will focus on showing that 2-extensions of certain simple groups satisfy $(\mathbb{Q}\mathcal{D})_2$. To that end, we introduce the following convenient definition.

Definition 2.3. A group L satisfies (E- $(\mathbb{Q}\mathcal{D})$) if every 2-extension of L satisfies $(\mathbb{Q}\mathcal{D})_2$:

$$\text{For every } B \in \hat{\mathcal{O}}_2(L), LB \text{ satisfies } (\mathbb{Q}\mathcal{D})_2. \tag{E- $(\mathbb{Q}\mathcal{D})$ }$$

In order to establish $(\mathbb{Q}\mathcal{D})_p$ for p -extensions, it is crucial to be able to compute p -ranks of extensions. The following result, extracted from Lemma 4.2 in [Piterman et al. 2021], will be a useful tool to compute p -ranks of extensions.

Lemma 2.4 (p -rank of extensions). *Let $G = N.K$ be an extension of finite groups, and let p be a prime number. Then*

$$m_p(G) = \max_{A \in \mathcal{S}} (m_p(C_N(A)) + m_p(A)),$$

where $\mathcal{S} = \{A \in \mathcal{A}_p(G) \cup \{1\} : A \cap N = 1\}$. In particular, $m_p(G) \leq m_p(N) + m_p(K)$, and if K has order prime to p then $\mathcal{A}_p(G) = \mathcal{A}_p(N)$ and $m_p(G) = m_p(N)$.

We will implicitly use this result at many points of the proofs. Note that, in order to apply this lemma, we should be able to compute centralisers of elementary abelian 2-subgroups, usually inducing outer automorphisms. We will often proceed as follows: if LB is a 2-extension of L , then take a suitable decomposition $B = B_0 \oplus B_1$, with $|B_1| = 2$. Suppose that we can inductively compute the 2-rank of LB_0 . Then, by Lemma 2.4, we have

$$m_2(LB) = \max\{m_2(LB_0), 1 + m_2(C_{LB_0}(t)) : t \in LB \setminus LB_0 \text{ is an involution}\}. \tag{2-1}$$

Moreover, this computation depends only on the conjugacy classes of the involutions t , and, in most of the cases that we are interested in, such classes are completely classified.

Now we recall, rather informally, the names of the different *types* of automorphisms of a simple group of Lie type K defined over a field of odd characteristic, following Definition 2.5.13 of [Gorenstein et al. 1998]. We refer to [Gorenstein et al. 1998] for the full details. Let $t \in \text{Aut}(K)$ be an involution and $K^* = \text{Inndiag}(K)$. Then we have the following names for t :

- (1) inner-diagonal if $t \in K^*$;
- (2) inner if $t \in \text{Inn}(K)$;

- (3) diagonal if $t \in K^* \setminus \text{Inn}(K)$;
- (4) field automorphism if $t \in \text{Aut}(K) \setminus K^*$ is $\text{Aut}(K)$ -conjugate to a field automorphism of the ground field and K is not ${}^2A_n(q)$, ${}^2D_n(q)$ or ${}^2E_6(q)$;
- (5) graph if $t \in \text{Aut}(K) \setminus K^*$, roughly, is $\text{Aut}(K)$ -conjugate to an involution arising as an automorphism of the underlying Dynkin diagram (except for $K = G_2(q)$), or else from a field automorphism in cases ${}^2A_n(q)$, ${}^2D_n(q)$ and ${}^2E_6(q)$; and
- (6) graph-field automorphism if it can be expressed as a product gf of a graph involution g and a field automorphism f , or else $K = G_2(q)$ and t arises from an $\text{Aut}(K)$ -conjugate of an involution automorphism of the underlying Coxeter diagram.

It follows from Proposition 4.9.1 of [Gorenstein et al. 1998] that the centralisers of field involutions t verify that $m_2(C_K(t)) = m_2(K)$ and $m_2(C_{K^*}(t)) = m_2(K^*)$. By (2-1), we see that $m_2(K \langle t \rangle) = m_2(K) + 1$. Below we reproduce a simplified version of this proposition.

Proposition 2.5. *Let $K = {}^d\Sigma(q)$ be a group of Lie type in adjoint version in characteristic r , and let x be a field or graph-field automorphism of prime order p . Set $K_x = O^{r'}(C_K(x))$. Then the following hold:*

- (1) *If x is a field automorphism then $K_x \cong {}^d\Sigma(q^{1/p})$.*
- (2) *If x is a graph-field automorphism then $d = 1$, $p = 2$ or 3 , and $K_x \cong {}^p\Sigma(q^{1/p})$.*
- (3) *K_x is adjoint and $C_{\text{Inndiag}(K)}(x) \cong \text{Inndiag}(K_x)$.*
- (4) *Field (resp. graph-field) automorphisms are all $\text{Inndiag}(K)$ -conjugate, except for graph-fields for $K = D_4(q)$ and $p = 3$.*

The previous proposition does not determine, a priori, the structure of $C_K(x)$, but just of the centraliser taken over the inner-diagonal automorphism group. Since we are interested in computing $m_2(C_K(x))$, it will be crucial for us to decide when a diagonal involution can centralise a field or graph-field automorphism x . We recall below Lemma 12.8 of [Gorenstein et al. 2018, Chapter 17], which provides a partial solution to this problem.

Lemma 2.6. *Let $K \cong \text{PSL}_2(q)$, $\text{P}\Omega_{2n+1}(q)$, $\text{PSp}_{2n}(q)$ or $E_7(q)$, where q is a power of an odd prime r . Let ϕ be a field automorphism of order 2, and let $K_\phi = O^{r'}(C_K(\phi))$. Then*

$$\text{Inndiag}(K_\phi) = C_{\text{Inndiag}(K)}(\phi) = C_{\text{Inn}(K)}(\phi).$$

In particular, ϕ does not commute with diagonal involutions of $\text{Inndiag}(K)$.

We will mainly work with Table 4.5.1 of [Gorenstein et al. 1998] to compute the 2-ranks of extensions by diagonal and graph involutions, mostly for the groups of type $A_m^\pm(q)$ and the exceptional groups. In the next paragraph, we briefly and informally describe how to read such a table. See [Gorenstein et al. 1998, pp. 171–182] for a complete and accurate description of Table 4.5.1.

This table records the K^* -conjugacy classes of inner-diagonal and graph involutions t of a finite group of Lie type K in adjoint version, and the structure of their centralisers when taken over $K^* = \text{Inndiag}(K)$.

The centraliser of an involution t is denoted by $C^* = C_{K^*}(t)$. The first column of Table 4.5.1 denotes the family for which the involutions are listed (A_n, B_n, C_n , etc.) The second column indicates the restrictions for these classes to exist, while the third column is a label for the conjugacy class of that involution. For the purposes of this article, we will not need to interpret the fourth column. In the fifth column, it is indicated when such classes are of inner type (denoted by 1), diagonal type (denoted by d) or graph type (several notation like g, g'). The notation $1/d$ indicates that it is inner if the condition inside the parentheses at the right holds, and it is diagonal otherwise. From the sixth column to the end, the structure of the centraliser C^* is described. Roughly, C^* is an extension of a central product of groups of Lie type $L^* = O^{r'}(C^*)$ (column six), whose versions are specified in the column “version” and whose centres can be recovered from the column $Z(L^*)$. An extra part centralising this product can be computed from the column $C_{C^{*\circ}}(L^*)$. Here $C^{*\circ} = L^*T^*$ is the connected-centraliser, where T^* is a certain r' -subgroup arising from a torus T normalised by t and inducing inner-diagonal automorphisms on L^* . From the columns L^* , version, $Z(L^*)$ and $C_{C^{*\circ}}(L^*)$, one can compute the “inner-part” of $C^{*\circ}$. Finally, from the last two columns we can recover the outer automorphisms of L^* arising in $C^{*\circ}$ (in general of diagonal type) and the remaining part of $C^*/C^{*\circ}$, which is often an involution acting on the components of L^* (as field or graph automorphism, or by switching two components) and on the central part $C_{C^*}(L^*)$ (which is usually cyclic and the involution acts by inversion). To recover the action of the last column, the symbols $i, \leftrightarrow, \phi, \gamma, 1$ mean, respectively, an action by inversion, a swap of two components, a field automorphism of order 2, a graph automorphism of order 2, and an inner action on a component or trivial action on $C_{C^{*\circ}}(L^*)$.

3. Tools to achieve $(\mathbb{Q}\mathcal{D})_p$

In this section, we provide tools and collect results that will help us to establish $(\mathbb{Q}\mathcal{D})_2$ on certain 2-extensions. Many of these tools were introduced and exploited by Aschbacher–Smith to determine the $(\mathbb{Q}\mathcal{D})$ -list in [Aschbacher and Smith 1993].

The following proposition is an easy consequence of the Künneth formula for the join of spaces and the fact that $\mathcal{A}_p(H \times K) \simeq \mathcal{A}_p(H) * \mathcal{A}_p(K)$ (see [Quillen 1978, Proposition 2.6]).

Proposition 3.1. *If p is a prime and H, K satisfy $(\mathbb{Q}\mathcal{D})_p$, then $H \times K$ satisfies $(\mathbb{Q}\mathcal{D})_p$.*

The following lemma corresponds to Lemmas 0.11 and 0.12 of [Aschbacher and Smith 1993].

Lemma 3.2. *Let $N \trianglelefteq G$ be such that $N \leq O_{p'}(G)$. Then there is an inclusion*

$$\tilde{H}_*(\mathcal{A}_p(G/N)) \subseteq \tilde{H}_*(\mathcal{A}_p(G)).$$

In particular, $m_2(G) = m_2(G/N)$, and if G/N satisfies $(\mathbb{Q}\mathcal{D})_p$ then so does G .

If $N \leq Z(G)$, then the quotient map induces a poset isomorphism $\mathcal{A}_p(G) \cong \mathcal{A}_p(G/N)$.

The following observation is an easy consequence of the inclusion between the homology groups of top-degree.

Lemma 3.3. *Let $H \leq G$ be such that $m_p(H) = m_p(G)$. If H satisfies $(\mathbb{Q}\mathcal{D})_p$, then so does G .*

Next, we recall one of the essential results on the Quillen dimension property.

Theorem 3.4 (Quillen). *If G is a solvable group with $O_p(G) = 1$, then G satisfies $(\mathbb{Q}\mathcal{D})_p$.*

This theorem settles the solvable case of Quillen's conjecture (see [Quillen 1978, Theorem 12.1]). Later, it was extended to the family of p -solvable groups by using the CFSG if p is odd. We refer to Chapter 8 of [Smith 2011] for further details on Quillen's conjecture and the Quillen dimension property.

In view of Theorem 3.4 and the inclusion lemma (Lemma 3.3), it is convenient to look for solvable subgroups of G with maximal p -rank. Some standard solvable subgroups in a group of Lie type L arise by taking extensions of unipotent radicals by elementary abelian subgroups of their normalisers. These extensions lie then inside parabolic subgroups. The following result on parabolic subgroups will help us to achieve (E- $(\mathbb{Q}\mathcal{D})$) for arbitrary groups of Lie type (see [Aschbacher and Smith 1993, Step v on p. 506]).

Lemma 3.5. *Let L be a simple group of Lie type, and p a prime not dividing the characteristic of L . Suppose that LB is a p -extension of L and that there exists a B -invariant proper parabolic subgroup $P \leq L$ such that $m_p(LB) = m_p(PB)$. Then LB satisfies $(\mathbb{Q}\mathcal{D})_p$.*

Proof. Let $R := O_r(P)$, where r is the characteristic of the ground field. Then, as a consequence of the Borel–Tits theorem, $C_{\text{Aut}(L)}(R) \leq R$ (see Corollary 3.1.4 of [Gorenstein et al. 1998]). In particular, if $T \leq PB$ realises the p -rank of PB , then T normalises R , and $C_T(R) \leq R \cap T = 1$. This means that T is faithful on R , i.e., $O_p(RT) = 1$, and $m_p(RT) = m_p(PB) = m_p(LB)$. Then RT is a solvable group with trivial p -core, and by Theorem 3.4, RT satisfies $(\mathbb{Q}\mathcal{D})_p$. By Lemma 3.3, LB satisfies $(\mathbb{Q}\mathcal{D})_p$. \square

Lemma 3.6. *Let L be a simple group of Lie type defined in odd characteristic. Suppose that P is a proper parabolic subgroup of L containing a Sylow 2-subgroup of L (that is, $|L : P|$ is odd). Then L and the extension of L by a field automorphism of order 2 satisfy $(\mathbb{Q}\mathcal{D})_2$.*

Proof. Let L and P be as in the hypotheses of the lemma. Since P has odd index in L , it contains a Sylow 2-subgroup of L . Therefore, $m_2(P) = m_2(L)$ and by Lemma 3.5, L satisfies $(\mathbb{Q}\mathcal{D})_2$.

Next, let $B \in \widehat{\mathcal{O}}_2(L)$ be cyclic inducing field automorphisms. By passing through algebraic groups and root systems, it can be shown that B normalises some conjugate of P , which we may assume is P itself. Thus, after conjugation, we suppose that $B \leq N_{\text{Aut}(L)}(P)$. Note that a Sylow 2-subgroup of PB is a Sylow 2-subgroup of LB , so $m_2(PB) = m_2(LB)$. By Lemma 3.5, LB satisfies $(\mathbb{Q}\mathcal{D})_2$. \square

We close this section with a few more results on low p -ranks. The following lemma follows from the p -rank 2 case of Quillen's conjecture. See [Quillen 1978, Proposition 2.10].

Lemma 3.7. *If $\mathcal{A}_p(G)$ is connected, $m_p(G) = 2$ and $O_p(G) = 1$, then G satisfies $(\mathbb{Q}\mathcal{D})_p$.*

It will be convenient to recall the classification of groups with a strongly 2-embedded subgroup, that is, those groups with disconnected 2-subgroup poset. See [Gorenstein et al. 1998, Theorem 7.6.1] and [Quillen 1978, Sec. 5].

Theorem 3.8. *Let $p = 2$ and G be a finite group. Then $\mathcal{A}_2(G)$ is disconnected if and only if $O_2(G) = 1$ and one of the following holds:*

- (1) $m_2(G) = 1$;
- (2) $\Omega_1(G)/O_{p'}(\Omega_1(G)) \cong \text{PSL}_2(2^n)$, $\text{PSU}_3(2^n)$ or $\text{Sz}(2^{2n-1})$ for some $n \geq 2$.

In particular, from the isomorphisms among the simple groups, we see that

$$\text{Alt}_5 \cong \text{PSL}_2(5) \cong \text{PSL}_2(2^2), \quad {}^2G_2(3)' \cong \text{PSL}_2(2^3),$$

are included in the list of item (2).

Indeed, sometimes in low dimensions, we will be able to conclude $(\mathbb{Q}\mathcal{D})_p$ by computing the sign of the Euler characteristic of $\mathcal{A}_p(G)$. Therefore, we will use the following well-known expression of this invariant. We write \bar{E} for the conjugacy class of a subgroup E of G .

Proposition 3.9. *The reduced Euler characteristic of $\mathcal{A}_p(G)$ is*

$$\tilde{\chi}(\mathcal{A}_p(G)) = \sum_{\bar{E} \in \mathcal{A}_p(G)/G \cup \{1\}} (-1)^{m_p(E)-1} p^{\binom{m_p(E)}{2}} |G : N_G(E)|.$$

Proof. This follows from the results of [Jacobsen and Møller 2012], as we briefly explain below. By [Jacobsen and Møller 2012, Example 2.10], we have

$$\tilde{\chi}(\mathcal{A}_p(G)) = - \sum_{E \in \mathcal{A}_p(G) \cup \{1\}} \tilde{\chi}(\mathcal{A}_p(E) \setminus \{E\}).$$

Since $\mathcal{A}_p(E) \setminus \{E\}$ is the poset of proper nonzero subspaces of the vector space E of dimension $m_p(E)$ over the finite field of p elements, we see that

$$\tilde{\chi}(\mathcal{A}_p(E) \setminus \{E\}) = (-1)^{m_p(E)-2} p^{\binom{m_p(E)}{2}}.$$

Grouping by conjugacy classes yields the formula given in the statement of the proposition. □

Finally, the next lemma will help us to produce nonzero homology by inductively looking into the homology of the Quillen poset of a certain normal subgroup and centralisers of outer elements acting on it. The main reference for this lemma is [Segev and Webb 1994].

Lemma 3.10. *Let G be a finite group and p a prime number. Suppose that $L \trianglelefteq G$ is a normal subgroup such that $\mathcal{O}_G(L)$ consists only of cyclic subgroups. Then we have a long exact sequence*

$$\cdots \rightarrow \tilde{H}_{m+1}(\mathcal{A}_p(G)) \rightarrow \bigoplus_{B \in \mathcal{O}_G(L)} \tilde{H}_m(\mathcal{A}_p(C_L(B))) \xrightarrow{i_*} \tilde{H}_m(\mathcal{A}_p(L)) \xrightarrow{j_*} \tilde{H}_m(\mathcal{A}_p(G)) \rightarrow \cdots$$

where i_* and j_* are the natural maps induced by the inclusions $\mathcal{A}_p(C_L(B)) \subseteq \mathcal{A}_p(L)$ and $\mathcal{A}_p(L) \subseteq \mathcal{A}_p(G)$, respectively.

In particular, the following hold:

(1) Let X be the union of the subposets $\mathcal{A}_p(C_L(B))$ for $B \in \mathcal{O}_G(L)$. We have indeed a factorisation

$$\begin{array}{ccc} \bigoplus_{B \in \mathcal{O}_G(L)} \tilde{H}_m(\mathcal{A}_p(C_L(B))) & \xrightarrow{i_*} & \tilde{H}_m(\mathcal{A}_p(L)) \\ & \searrow i'_* & \nearrow k_* \\ & \tilde{H}_m(X) & \end{array} \tag{3-1}$$

where also i'_* and k_* are induced by the inclusions $\mathcal{A}_p(C_L(B)) \subseteq X$ and $X \subseteq \mathcal{A}_p(L)$, respectively.

(2) $m_p(G) \leq m_p(L) + 1$.

(3) If $\tilde{H}_{m_p(L)-1}(\mathcal{A}_p(C_L(B))) = 0$ for all $B \in \mathcal{O}_G(L)$, then $H_{m_p(L)}(\mathcal{A}_p(G)) = 0$.

(4) We have a bound

$$\begin{aligned} \dim H_{m_p(L)}(\mathcal{A}_p(G)) &\geq \sum_{B \in \mathcal{O}_G(L)} \dim \tilde{H}_{m_p(L)-1}(\mathcal{A}_p(C_L(B))) - \dim \tilde{H}_{m_p(L)-1}(X) \\ &\geq \sum_{B \in \mathcal{O}_G(L)} \dim \tilde{H}_{m_p(L)-1}(\mathcal{A}_p(C_L(B))) - \dim \tilde{H}_{m_p(L)-1}(\mathcal{A}_p(L)). \end{aligned}$$

(5) If $m_p(G) = m_p(L) + 1$ and G fails $(\mathbb{Q}\mathcal{D})_p$, then, for $m = m_p(L) - 1$, we get inclusions

$$\bigoplus_{B \in \mathcal{O}_G(L)} \tilde{H}_m(\mathcal{A}_p(C_L(B))) \hookrightarrow \tilde{H}_m(X) \hookrightarrow \tilde{H}_m(\mathcal{A}_p(L)).$$

Proof. The long exact sequence arises from the main result of [Segev and Webb 1994]. Then equation (3-1) in item (1) is an immediate consequence of this sequence. Item (2) holds by Lemma 2.4. Items (3)–(5) follow by looking into the last terms of the long exact sequence, at $m = m_p(L)$. \square

4. Some linear groups satisfy $(\mathbb{Q}\mathcal{D})_2$

In this section, we prove that the linear groups $\text{PSL}_2(q)$ and $\text{PSL}_3(q)$ satisfy $(E-(\mathbb{Q}\mathcal{D}))$ for every q , with a few exceptions for $q = 3, 5, 9$. These cases will serve as basic cases for the exceptional groups, where we will occasionally find linear groups as direct factors in some of their maximal subgroups.

From [Gorenstein et al. 1998, Proposition 4.10.5], we recall the 2-ranks of the small dimensional linear groups:

Proposition 4.1. *If q is a power of an odd prime and $n = 2, 3$, then $\text{PSL}_n^\pm(q)$ and $\text{PGL}_n^\pm(q)$ have 2-rank 2.*

We begin by studying the linear group of dimension 2.

Proposition 4.2. *Let $L \cong \text{PSL}_2(q)$ with q odd and $q \neq 3$. Then every 2-extension LB of L satisfies $(\mathbb{Q}\mathcal{D})_2$, with the following exceptions:*

- (1) $L \cong \text{PSL}_2(5)$, $B = 1$;
- (2) $L \cong \text{PSL}_2(9)$, B induces field automorphisms of order 2.

Moreover, every 2-extension of $\text{Inndiag}(L) \cong \text{PGL}_2(q)$ satisfies $(\mathbb{Q}\mathcal{D})_2$, except in case (2).

2-extension LB	$C_L(B)$	$m_2(LB)$
$B = 1$	L	2
$B = \langle \phi \rangle$	$\text{PGL}_2(q^{1/2})$	3
$B = \langle d \rangle$	$D_{q+\epsilon}$	2

Table 1. 2-extensions of $\text{PSL}_2(q)$, $q \geq 5$ odd. Here $q \equiv \epsilon \pmod{4}$, $\epsilon \in \{1, -1\}$.

Proof. We consider the possible 2-extensions of L . In any case, we know that L is simple and that $\text{Out}(L) = C_2 \times C_a$, where $C_2 \cong \text{Outdiag}(L)$ and C_a is the group of field automorphisms of \mathbb{F}_q . Suppose that ϕ is an order 2-field automorphisms of \mathbb{F}_q (if it exists), and that $d \in \text{Inndiag}(L) \setminus L$ is a diagonal involution. Then the 2-extensions of L are given in Table 1.

This table follows since every involution of $\text{Aut}(\text{PSL}_2(q)) \setminus \text{PGL}_2(q)$ is a field automorphism. Recall also that field and diagonal automorphisms of order 2 do not commute by Lemma 2.6. The structure of the centraliser for d follows from the first row of Table 4.5.1 of [Gorenstein et al. 1998]. Finally, observe that $L\langle d \rangle = \text{Inndiag}(L)$ and $m_2(\text{Inndiag}(L)\langle \phi \rangle) = 3$ since $m_2(L) = m_2(\text{Inndiag}(L)) = 2$ by Proposition 4.1.

By computing the Euler characteristic, we prove that each 2-extension of L satisfies $(\mathbb{Q}\mathcal{D})_2$. First, 2-extensions LB and $\text{Inndiag}(L)\langle \phi \rangle$ have connected \mathcal{A}_2 -poset by Theorem 3.8, except for $L = \text{PSL}_2(5)$, $B = 1$. Therefore, by Lemma 3.7, L and $\text{Inndiag}(L)$ satisfy $(\mathbb{Q}\mathcal{D})_2$, except for $L = \text{PSL}_2(5)$. Note that $\mathcal{A}_2(\text{PSL}_2(5)) = \mathcal{A}_2(\text{Alt}_5) = \mathcal{A}_2(\text{PSL}_2(4))$ is homotopically discrete with 5 points, and the 2-extension $\text{PGL}_2(5) \cong \text{Sym}_5$ does satisfy $(\mathbb{Q}\mathcal{D})_2$. This yields the conclusions of the statement for the case $q = 5$.

Next we show $(\mathbb{Q}\mathcal{D})_2$ for the 2-extensions $L\langle \phi \rangle$ and $\text{Inndiag}(L)\langle \phi \rangle$. Since both have 2-rank 3, by Lemma 3.3 it is enough to show that $L\langle \phi \rangle$ satisfies $(\mathbb{Q}\mathcal{D})_2$. In order to do this, we compute the dimensions of $H_1(\mathcal{A}_2(L))$ and $H_1(\mathcal{A}_2(C_L(\phi)))$.

Since in this situation, q is a square, $q \neq 5$. Second, if $q = 25$, $C_L(\phi) = \text{PGL}_2(5)$. Hence, in any case, the dimension of these degree 1 homology groups can be computed from the reduced Euler characteristic of the underlying \mathcal{A}_2 -poset. Here we use the formula given in Proposition 3.9. Thus, for $K = L$ or $C_L(\phi)$,

$$\dim H_1(\mathcal{A}_2(K)) = -\tilde{\chi}(\mathcal{A}_2(K)) = 1 - \# \text{ of involutions in } K + 2 \cdot \# \text{ of 4-subgroups of } K. \tag{4-1}$$

In Table 2 we describe these numbers.

Proof of Table 2. The number of involutions and 4-subgroups of $\text{PSL}_2(q)$ follows from Dickson’s classification of the subgroups of $\text{PSL}_2(q)$ (see also Theorem 6.5.1 of [Gorenstein et al. 1998]).

group	number of involutions	number of 4-subgroups
$\text{PSL}_2(q)$	$\frac{1}{2}q(q + \epsilon)$	$\frac{1}{24}q(q^2 - 1)$
$\text{PGL}_2(q)$	q^2	$\frac{1}{6}q(q^2 - 1)$

Table 2. Here $q \equiv \epsilon \pmod{4}$, $\epsilon \in \{1, -1\}$.

The number of involutions of $\text{PGL}_2(q)$ follows since there is a unique conjugacy class of diagonal involutions d by Table 4.5.1 of [Gorenstein et al. 1998]. Thus, the number of elements in such conjugacy class is equal to $\frac{1}{2}q(q - \epsilon)$, which gives q^2 after adding the number of involutions in $\text{PSL}_2(q)$.

Finally, to compute the number of four-subgroups of $\text{PGL}_2(q)$ we proceed as follows: each four-subgroup of $\text{PGL}_2(q)$ is either contained in $\text{PSL}_2(q)$ or else it contains a unique involution of $\text{PSL}_2(q)$ and 2 diagonal involutions. Therefore, for a given diagonal involution d , there is a one-to-one correspondence between 4-subgroups containing d and involutions in $C_L(d) \cong D_{q+\epsilon}$. This shows that each diagonal involution is contained in $(q + \epsilon)/2$ 4-subgroups. Since we have $\frac{1}{2}q(q - \epsilon)$ diagonal involutions, the total number of 4-subgroups in $\text{PGL}_2(q)$ containing diagonal involutions is

$$\frac{q(q - \epsilon)}{2} \cdot \frac{(q + \epsilon)}{2} \cdot \frac{1}{2} = \frac{q(q^2 - 1)}{8}.$$

Thus the total number of 4-subgroups in $\text{PGL}_2(q)$ is

$$\frac{q(q^2 - 1)}{24} + \frac{q(q^2 - 1)}{8} = \frac{q(q^2 - 1)}{6}.$$

This completes the proof of Table 2. □

Indeed, by Table 2, we get concrete values for the dimensions of the degree 1 homology groups of $\mathcal{A}_2(\text{PSL}_2(q))$ and $\mathcal{A}_2(\text{PGL}_2(q))$:

$$\dim H_1(\mathcal{A}_2(\text{PSL}_2(q))) = -\tilde{\chi}(\mathcal{A}_2(\text{PSL}_2(q))) = \frac{1}{12}(q - \epsilon)(q^2 - (6 - \epsilon)q - \epsilon 12), \tag{4-2}$$

$$\dim H_1(\mathcal{A}_2(\text{PGL}_2(q))) = -\tilde{\chi}(\mathcal{A}_2(\text{PGL}_2(q))) = \frac{1}{3}(q - 3)(q^2 - 1). \tag{4-3}$$

Now we need to describe the number of field automorphisms in $\text{PSL}_2(q)\langle\phi\rangle$ and in $\text{PGL}_2(q)\langle\phi\rangle$.

Recall that the field automorphisms of $\text{PSL}_2(q)\langle\phi\rangle$ are all $\text{PGL}_2(q)$ -conjugate, with centraliser $C_{\text{PGL}_2(q)}(\phi) = C_{\text{PSL}_2(q)}(\phi)$. Thus, the number of field automorphisms of order 2 in $\text{PSL}_2(q)\langle\phi\rangle$ is exactly

$$\frac{|\text{PGL}_2(q)|}{|C_{\text{PSL}_2(q)}(q)|} = \frac{q(q^2 - 1)}{q^{1/2}(q - 1)} = q^{1/2}(q + 1).$$

This gives $q^{1/2}(q + 1)$ involutions in $\text{PSL}_2(q)\langle\phi\rangle \setminus \text{PSL}_2(q)$. Let $L = \text{PSL}_2(q)$, $B = \langle\phi\rangle$. By Lemma 3.10, the values in Table 2 and formula (4-1), we conclude that

$$\begin{aligned} \dim H_2(\mathcal{A}_2(LB)) &\geq q^{1/2}(q + 1) \dim H_1(\mathcal{A}_2(\text{PGL}_2(q^{1/2}))) - \dim H_1(\mathcal{A}_2(\text{PSL}_2(q))) \\ &= q^{1/2}(q + 1) \frac{1}{3}(q^{1/2} - 3)(q - 1) - \frac{1}{12}(q - 1)(q^2 - 5q - 12) \\ &= \frac{1}{4}(q^{1/2} - 1)(q - 1)(q^{3/2} - 3q - 4). \end{aligned}$$

Note that $q \equiv 1 \pmod{4}$. The above number is positive for all $q \geq 13$, which is our case since q is an even power of an odd prime and $q \neq 9$ by hypothesis. We conclude that $LB = \text{PSL}_2(q)\langle\phi\rangle$ satisfies

$(\mathbb{Q}\mathcal{D})_2$. Then also $\mathrm{PGL}_2(q)\langle\phi\rangle$ satisfies $(\mathbb{Q}\mathcal{D})_2$. Moreover,

$$\begin{aligned} \dim H_2(\mathcal{A}_2(\mathrm{PGL}_2(q)\langle\phi\rangle)) &\geq \dim H_2(\mathcal{A}_2(\mathrm{PSL}_2(q)\langle\phi\rangle)) \\ &\geq \frac{1}{4}(q^{1/2} - 1)(q - 1)(q^{3/2} - 3q - 4). \end{aligned} \tag{4-4}$$

We have shown that every possible 2-extension of $\mathrm{PSL}_2(q)$ and $\mathrm{PGL}_2(q)$ satisfies $(\mathbb{Q}\mathcal{D})_2$, except for the cases described in the statement of the theorem. □

We note that the excluded cases in Proposition 4.2 actually fail $(\mathbb{Q}\mathcal{D})_2$. Indeed, $\mathrm{PSL}_2(5)$ fails $(\mathbb{Q}\mathcal{D})_2$ since it has 2-rank 2 and $\mathcal{A}_2(\mathrm{PSL}_2(5)) = \mathcal{A}_2(\mathrm{PSL}_2(4))$ is homotopically discrete. The following example provides the details that show that $\mathrm{PSL}_2(9)\langle\phi\rangle$ and $\mathrm{PGL}_2(9)\langle\phi\rangle$ fail $(\mathbb{Q}\mathcal{D})_2$, where ϕ is a field automorphism of order 2.

Example 4.3. Let $L = \mathrm{PSL}_2(9)$ and let $A = \mathrm{Aut}(L)$. Then $A/L \cong C_2 \times C_2$, so every 2-extension of L is a nontrivial normal subgroup of A . This gives 3 possible 2-extensions of L , but not 4. Let ϕ be a field automorphism of L and d a diagonal automorphism of L , both of order 2. Then the possible 2-extensions of L are

- (1) L , with 2-rank 2, satisfies $(\mathbb{Q}\mathcal{D})_2$ with $H_1(\mathcal{A}_2(L))$ of rank 16;
- (2) $L\langle\phi\rangle$, with 2-rank 3, fails $(\mathbb{Q}\mathcal{D})_2$ since $C_L(\phi) \cong \mathrm{Sym}_4$, which has nontrivial 2-core

$$O_2(C_L(\phi)) \cong C_2 \times C_2 \neq 1;$$

- (3) $L\langle d\rangle = \mathrm{PGL}_2(9)$, with 2-rank 2, satisfies $(\mathbb{Q}\mathcal{D})_2$ with $H_1(\mathcal{A}_2(L)\langle d\rangle)$ of rank 160 and $C_L(d) \cong D_{10}$.

Note that $\mathrm{Aut}(L)$ has 2-rank 3 and does not satisfy $(\mathbb{Q}\mathcal{D})_2$, and it is not a 2-extension of L since diagonal and field automorphisms do not commute in $\mathrm{Aut}(L)$. Also $\mathrm{PGL}_2(9)\langle\phi\rangle$ fails $(\mathbb{Q}\mathcal{D})_2$ since $C_{\mathrm{PGL}_2(9)}(\phi) = C_L(\phi)$ has nontrivial 2-core.

There is also a remaining almost simple group N with $L < N < \mathrm{Aut}(L)$, not contained in the previous cases. This is the extension $N = \mathrm{PSL}_2(9).2 \cong \mathrm{Alt}_6.2$, and it satisfies that $\mathcal{A}_2(N) = \mathcal{A}_2(L)$. Therefore, although this group N is not a 2-extension of L , it is a “nonsplit 2-extension”, and it does satisfy $(\mathbb{Q}\mathcal{D})_2$.

Finally, these computations show that $\mathcal{A}_2(L) \hookrightarrow \mathcal{A}_2(\mathrm{Aut}(L))$ induces an inclusion in homology, and hence a nonzero map. By the main result of [Piterman and Smith 2022a], $\mathrm{PSL}_2(9)$ is not a component of a minimal counterexample to Quillen’s conjecture.

Our next aim is to show that 2-extensions of $\mathrm{PSL}_3(q)$ satisfy $(\mathbb{Q}\mathcal{D})_2$, with only a few exceptions. We will need the following lemma which records the values of the Euler characteristic of the Quillen poset of some linear groups and the unitary groups in dimension 3.

Lemma 4.4. *For $L = \mathrm{PSL}_n(q)$ and n odd, we have*

$$\tilde{\chi}(\mathcal{A}_2(L)) = \tilde{\chi}(\mathcal{A}_2(\mathrm{PGL}_n(q))) = \frac{(-1)^n}{n} \prod_{i=1}^{n-1} (q^i - 1) f_n(q),$$

where $f_n(q)$ denotes a polynomial as described in [Welker 1995]. For instance, $f_3(q) = q^3 + 3q^2 + 3q + 3$. Moreover, since $\mathcal{A}_2(L)$ is Cohen–Macaulay of dimension $n - 2$, the above Euler characteristic computes the dimension of $H_{n-2}(\mathcal{A}_2(L))$.

If $L = \text{PSU}_3(q)$, then

$$\tilde{\chi}(\mathcal{A}_2(L)) = \tilde{\chi}(\mathcal{A}_2(\text{PGU}_3(q))) = -\frac{1}{3}(q^6 - 2q^5 - q^4 + 2q^3 - 3q^2 + 3).$$

Proof. The value of the Euler characteristic for $\text{PGL}_n(q)$ follows from Proposition 4.1 and Theorem 4.4 of [Welker 1995] (note that there is a typo in the formula of Theorem 4.4, and the product over i should be up to $r - 1$). Also, since n is odd, by Proposition 7.5 of [Pitman and Welker 2022], $\mathcal{A}_2(\text{PSL}_n(q)) = \mathcal{A}_2(\text{PGL}_n(q)) = \mathcal{A}_2(\text{GL}_n(q))_{>Z}$ where Z is the cyclic subgroup of order 2 of $Z(\text{GL}_n(q))$. By [Quillen 1978] (see also [Welker 1995]), $\mathcal{A}_2(\text{PSL}_n(q))$ is Cohen–Macaulay of dimension $n - 2$.

The formula for $\text{PGU}_3(q)$ follows from Example 7.6 of [Pitman and Welker 2022]. \square

Next, we show that the 2-extensions of $\text{PSU}_3(q)$ satisfy $(\mathbb{Q}\mathcal{D})_2$, except for $q = 3$. These cases will be important during our analysis for $\text{PSL}_3(q)$, especially when working with 2-extensions by graph-field automorphisms.

Proposition 4.5. *Let $L = \text{PSU}_3(q)$ with q odd. Then L satisfies $(E-(\mathbb{Q}\mathcal{D}))$ if $q \neq 3$. Moreover, let ϕ be a graph automorphism of order 2 of L . Then we have*

$$\begin{aligned} \dim H_2(\mathcal{A}_2(\text{PGU}_3(q)\langle\phi\rangle)) &\geq \dim H_2(\mathcal{A}_2(\text{PSU}_3(q)\langle\phi\rangle)) \\ &\geq \frac{1}{3}(q^2 - 1)(q + 1) \left(\frac{q^2(q^2 - q + 1)}{(3, q + 1)}(q - 3) - (q^3 - 3q^2 + 3q - 3) \right), \end{aligned}$$

which is a positive polynomial for $q > 3$. Finally, for $q = 3$, $\text{PSU}_3(3)$ satisfies $(\mathbb{Q}\mathcal{D})_2$ but $\text{PSU}_3(3)\langle\phi\rangle$ fails $(\mathbb{Q}\mathcal{D})_2$.

Proof. We have that $\mathcal{A}_2(L)$ is connected by Theorem 3.8, and $m_2(L) = 2$ by Proposition 4.1. Thus L satisfies $(\mathbb{Q}\mathcal{D})_2$ by Lemma 3.7. Moreover, by Lemma 4.4,

$$\dim H_1(\mathcal{A}_2(L)) = -\tilde{\chi}(\mathcal{A}_2(L)) = \frac{1}{3}(q^6 - 2q^5 - q^4 + 2q^3 - 3q^2 + 3). \quad (4-5)$$

Next, the only possible nontrivial 2-extension of L is by a graph automorphism ϕ of order 2 (which indeed arises from the field automorphism $x \mapsto x^q$). Let $L_1 = L\langle\phi\rangle$ be such extension. By Table 4.5.1 of [Gorenstein et al. 1998],

$$C_{\text{PGU}_3(q)}(\phi) \cong \text{Inndiag}(\Omega_3(q)) = \text{PGL}_2(q).$$

This implies that $C_L(\phi) = \text{PGL}_2(q)$. Moreover, there is a unique $\text{PGU}_3(q)$ -conjugacy class of graph automorphisms, and such elements act by inversion on $\text{Outdiag}(L) = (3, q + 1)$. Thus the conjugacy class of ϕ in $\text{Out}(L)$ has size $(3, q + 1)$, and this gives rise to exactly $(3, q + 1)$ extensions $L\langle\phi'\rangle \leq \text{Aut}(L)$ of L by a conjugate ϕ' of ϕ , and these extensions are $\text{Aut}(L)$ -conjugate. We conclude then that the number

of graph automorphisms contained in L_1 is

$$n_g := \frac{|\mathrm{PGU}_3(q)|}{|\mathrm{PGL}_2(q)|(3, q + 1)} = \frac{q^2(q^3 + 1)}{(3, q + 1)}.$$

Finally, by Lemma 3.10, we conclude that

$$\begin{aligned} \dim H_2(\mathcal{A}_2(\mathrm{PGU}_3(q)\langle\phi\rangle)) &\geq \dim H_2(\mathcal{A}_2(\mathrm{PSU}_3(q)\langle\phi\rangle)) \\ &\geq n_g \dim H_1(\mathcal{A}_2(\mathrm{PGL}_2(q))) - \dim H_1(\mathcal{A}_2(\mathrm{PSU}_3(q))) \\ &= \frac{q^2(q^3 + 1)}{(3, q + 1)} \frac{1}{3}(q - 3)(q^2 - 1) - \frac{1}{3}(q^6 - 2q^5 - q^4 + 2q^3 - 3q^2 + 3) \\ &= \frac{1}{3}(q^2 - 1)(q + 1) \left(\frac{q^2(q^2 - q + 1)}{(3, q + 1)}(q - 3) - (q^3 - 3q^2 + 3q - 3) \right). \end{aligned}$$

This polynomial is positive for all $q > 3$. Therefore, L_1 satisfies $(\mathbb{Q}\mathcal{D})_2$ if $q \neq 3$.

When $q = 3$, $C_L(\phi) = \mathrm{PGL}_2(3)$ has nontrivial 2-core, so $H_1(\mathcal{A}_2(C_L(\phi))) = 0$, and by Lemma 3.10(3), $H_2(\mathcal{A}_2(L_1)) = 0$. □

Now we have the necessary background to prove that $\mathrm{PSL}_3(q)$ satisfies $(\mathbb{E}(\mathbb{Q}\mathcal{D}))$, except for a small number of cases.

Proposition 4.6. *Let $L = \mathrm{PSL}_n(q)$ with n, q odd. The following assertions hold:*

- (1) L , and L extended by a field involution, satisfy $(\mathbb{Q}\mathcal{D})_2$.
- (2) If $n = 3$, then every 2-extension of L satisfies $(\mathbb{Q}\mathcal{D})_2$, with the following exceptions that fail $(\mathbb{Q}\mathcal{D})_2$:
 - $L = \mathrm{PSL}_3(3)$ extended by a graph automorphism, and
 - $L = \mathrm{PSL}_3(9)$ extended by a group generated by a field involution and a graph automorphism.

Proof. Let $L = \mathrm{PSL}_n(q)$, with n odd, and consider the stabiliser P of a 1-dimensional subspace of the underlying module $V = \mathbb{F}_q^n$. Then P is a parabolic subgroup with structure $P \cong [q^{n-1}]L_P$, where L_P , a Levi complement for P , has structure $\mathrm{SL}_{n-1}(q) \circ_{(n, q-1)} C_{q-1}$. Thus $|L_P| = |\mathrm{GL}_{n-1}(q)|/(n, q - 1)$ and the index of P in L is

$$|L : P| = \frac{q^{n(n-1)/2} \prod_{i=2}^n (q^i - 1)}{q^{n-1} \cdot q^{(n-1)(n-2)/2} \prod_{i=1}^{n-1} (q^i - 1)} = \frac{q^n - 1}{q - 1} = q^{n-1} + q^{n-2} + \dots + q + 1.$$

Since n is odd, the index of P in $\mathrm{PSL}_n(q)$ is odd. By Lemma 3.6, $L = \mathrm{PSL}_n(q)$ and L extended by a field involution satisfy $(\mathbb{Q}\mathcal{D})_2$. This proves item (1).

Before moving to the case $n = 3$, we list all the possible 2-extensions of L . Denote by ϕ, γ and δ a field automorphism of order 2, a graph automorphism and a graph-field automorphism of L , respectively, such that $[\phi, \gamma] = 1$ and $\delta = \phi\gamma$. Let also $L^* = \mathrm{PGL}_n(q)$. Then the 2-extensions of L are

- (i) L ;
- (ii) $L\langle\phi\rangle$, with $C_{L^*}(\phi) \cong \mathrm{PGL}_n(q^{1/2})$ by Proposition 2.5;
- (iii) $L\langle\gamma\rangle$, with $C_L(\gamma) \cong \mathrm{Inndiag}(\Omega_n(q))$ by Table 4.5.1 of [Gorenstein et al. 1998];

- (iv) $L\langle\delta\rangle$, with $C_{L^*}(\delta) \cong \text{PGU}_n(q^{1/2})$ by Proposition 2.5;
- (v) $L\langle\phi, \gamma\rangle$, with $C_L(\phi, \gamma) \cong \text{Inndiag}(\Omega_n(q^{1/2}))$ by (iii) and Proposition 2.5.

Now suppose that $n = 3$, that is $L = \text{PSL}_3(q)$. We know that the extensions of cases (i) and (ii) above satisfy $(\mathbb{Q}\mathcal{D})_2$ by the parabolic argument. So it remains to show that the 2-extensions by graph, graph-field and both graph and field automorphisms, satisfy $(\mathbb{Q}\mathcal{D})_2$. To that end, we compute the dimensions of the top-degree homology groups, similar to what we did for $\text{PSL}_2(q)$ in the proof of Proposition 4.2.

First, recall that we have the following number of involutions of each type. Let $B = \langle\phi, \gamma\rangle$.

$$\begin{aligned} n_f &:= \# \text{ field involutions in } L\langle\phi\rangle = \# \text{ field involutions in } LB \\ &= \frac{|\text{PGL}_3(q)|}{|\text{PGL}_3(q^{1/2})|(3, q^{1/2} + 1)}, \\ n_g &:= \# \text{ graph involutions in } L\langle\gamma\rangle = \# \text{ graph involutions in } LB \\ &= \frac{|\text{PGL}_3(q)|}{|\text{PGL}_2(q)|(3, q - 1)}, \\ n_{gf} &:= \# \text{ graph-field involutions in } L\langle\delta\rangle = \# \text{ graph-field involutions in } LB \\ &= \frac{|\text{PGL}_3(q)|}{|\text{PGU}_3(q^{1/2})|(3, q^{1/2} - 1)}. \end{aligned}$$

To compute these numbers, we have used the structure of the centraliser in each case, the fact that there is a unique L^* -conjugacy class for each type of involution, and the structure of $\text{Out}(L) = (3, q - 1) : \langle\phi, \gamma\rangle$ (see Theorem 2.5.12 of [Gorenstein et al. 1998]).

Let t be a field, graph or graph-field involution of L , and let $L_1 = L\langle t\rangle$. Then the number n_t of involutions in $L_1 \setminus L$ is n_f, n_g or n_{gf} , accordingly to the type of t . Note also that $m_2(L_1) = m_2(L) + 1 = 3$.

By Lemma 3.10,

$$\dim H_2(\mathcal{A}_2(L_1)) \geq n_t \cdot \dim H_1(\mathcal{A}_2(C_L(t))) - \dim H_1(\mathcal{A}_2(L)). \tag{4-6}$$

We compute $d(t) := \dim H_1(\mathcal{A}_2(C_L(t)))$ in each case, by using Lemma 4.4 and (4-3). Note that $\Omega_1(C_L(\phi)) = \text{PSL}_3(q^{1/2})$ by item (ii) above. Also $C_L(\gamma) = \text{PGL}_2(q)$ by the classical isomorphism $\text{Inndiag}(\Omega_3(q)) \cong \text{PGL}_2(q)$. By Lemma 4.4, we have

$$\begin{aligned} d(\phi) &= \dim H_1(\mathcal{A}_2(\text{PSL}_3(q^{1/2}))) = \frac{1}{3}(q^{1/2} - 1)(q - 1)(q^{3/2} + 3q + 3q^{1/2} + 3), \\ d(\gamma) &= \dim H_1(\mathcal{A}_2(\text{PGL}_2(q))) = \frac{1}{3}(q - 3)(q^2 - 1), \\ d(\delta) &= \dim H_1(\mathcal{A}_2(\text{PGU}_3(q^{1/2}))) = \frac{1}{3}(q^3 - 2q^{5/2} - q^2 + 2q^{3/2} - 3q + 3). \end{aligned}$$

Let $d := \dim H_1(\mathcal{A}_2(L))$. Since $\mathcal{A}_2(L)$ is connected and $m_2(L) = 2$, by Lemma 4.4 we have

$$d = -\tilde{\chi}(\mathcal{A}_2(L)) = \frac{1}{3}(q - 1)(q^2 - 1)(q^3 + 3q^2 + 3q + 3).$$

Now it is routine to verify that $n_t d(t) > d$ if $t = \gamma$ or $t = \delta$, if and only if $(t, q) \neq (\gamma, 3)$. Indeed, for $q = 3$, $C_L(\gamma) = \text{PGL}_2(3) \cong \text{Sym}_4$ has nontrivial 2-core, so $d(\gamma) = 0$ and in consequence, $H_2(L\langle\gamma\rangle) = 0$.

This shows that $L\langle\gamma\rangle$ fails $(\mathbb{Q}\mathcal{D})_2$ if $q = 3$. Therefore, a 2-extension of L by a field, graph or graph-field involution satisfies $(\mathbb{Q}\mathcal{D})_2$ if and only if $q \neq 3$ when L is extended by a graph involution.

It remains to show that $LB = L\langle\phi, \gamma\rangle$ verifies $(\mathbb{Q}\mathcal{D})_2$. For this case, we take $L_f = L\langle\phi\rangle$, $L_2 = LB$ and consider the long exact sequence of Lemma 3.10 at $m = 2$ there (since $m_2(L_2) = 4$). That is, we need to show that $H_3(\mathcal{A}_2(L_2)) \neq 0$.

Note that the set of involutions $t \in L_2 \setminus L_1$ is exactly the set of all graph and graph-field automorphisms of the extension $L_2 = LB$. Let $d_g := \dim H_2(\mathcal{A}_2(\text{PGL}_2(q)\langle\phi\rangle))$, $d_{gf} := \dim H_2(\mathcal{A}_2(\text{PGU}_3(q^{1/2})\langle\phi\rangle))$ and $d_f := \dim H_2(\mathcal{A}_2(L_f))$. Therefore, by Lemma 3.10,

$$\dim H_3(\mathcal{A}_2(L_2)) \geq n_g d_g + n_{gf} d_{gf} - d_f. \tag{4-7}$$

We show that the right-hand side of this equation is positive if $q \neq 9$ by providing proper bounds of the dimensions d_g , d_{gf} and d_f .

By (4-4),

$$d_g = \dim H_2(\mathcal{A}_2(\text{PGL}_2(q)\langle\phi\rangle)) \geq \frac{1}{4}(q^{1/2} - 1)(q - 1)(q^{3/2} - 3q - 4). \tag{4-8}$$

Next, by Proposition 4.5,

$$d_{gf} \geq \frac{1}{3}(q - 1)(q^{1/2} + 1) \left(\frac{q(q - q^{1/2} + 1)}{(3, q^{1/2} + 1)}(q^{1/2} - 3) - (q^{3/2} - 3q + 3q^{1/2} - 3) \right), \tag{4-9}$$

which is positive for all $q > 9$.

Finally, we need to bound d_f from above. Indeed, by Lemma 3.10 at $m = 2$, we have

$$\begin{aligned} d_f &= \dim H_2(\mathcal{A}_2(L_f)) = \dim H_2(\mathcal{A}_2(\text{PSL}_3(q)\langle\phi\rangle)) \\ &\leq n_f \dim H_1(\mathcal{A}_2(\text{PSL}_3(q^{1/2}))) \\ &= \frac{q^{3/2}(q + 1)(q^{3/2} + 1)}{(3, q^{1/2} + 1)} \frac{1}{3}(q^{1/2} - 1)(q - 1)(q^{3/2} + 3q + 3q^{1/2} + 3). \end{aligned}$$

Now we check with the given bounds that $n_g d_g + n_{gf} d_{gf} - d_f$ is positive if and only if $q > 9$. In fact, if $q = 9$, similar arguments show $H_3(\mathcal{A}_2(LB)) = 0$ since $d_g = 0$ by Example 4.3 and $d_{gf} = 0$ by Proposition 4.5.

We conclude that every 2-extension of $\text{PSL}_3(q)$ satisfies $(\mathbb{Q}\mathcal{D})_2$, except for $\text{PSL}_3(3)$ extended by a graph automorphism and for $\text{PSL}_3(9)$ extended by field and graph automorphisms, which actually fail $(\mathbb{Q}\mathcal{D})_2$. □

5. The Quillen dimension property on exceptional groups of Lie type

We use the results of the preceding sections to show that, with only finitely many exceptions, the 2-extensions of the finite simple groups of exceptional Lie type satisfy $(\mathbb{Q}\mathcal{D})_2$. For that purpose, it will be convenient to recall first which 2-extensions can arise in each case. Table 3 records the 2-ranks of the exceptional groups of Lie type in adjoint version and the structure of the outer automorphism group. The

group	2-rank	Outdiag	Out / Outdiag
${}^3D_4(q)$	3	1	3Φ
$G_2(q)$	3	1	$\Phi\Gamma$, where $ \Phi\Gamma : \Gamma = 2$ if $q = 3^a$, and $\Gamma = 1$ otherwise
${}^2G_2(q)$	3	1	Φ (odd order)
$F_4(q)$	5	1	Φ
$E_6(q)$	6	$(3, q - 1)$	$\Phi \times \Gamma, \Gamma \cong C_2$
${}^2E_6(q)$	6	$(3, q + 1)$	2Φ
$E_7(q)$	8	2	Φ
$E_8(q)$	9	1	Φ

Table 3. Out / Outdiag is cyclic unless specified; $\Phi = \text{Aut}(\mathbb{F}_q) \cong C_a$, where $q = r^a$, r is an odd prime, and the usual conventions for the twisted types hold. Also, Γ is a set of graph automorphisms.

2-ranks were extracted from [Cohen and Seitz 1987; Gorenstein et al. 1998, Proposition 4.10.5]. From Table 3, we will compute the possible 2-extensions in each case.

Recall that we follow the terminology of [Gorenstein et al. 1998]. In particular, by a group of Lie type K we mean the finite group $O'_{\sigma}(C_{\sigma}(\bar{K}))$, where \bar{K} is a simple $\bar{\mathbb{F}}_r$ -algebraic group and σ a Steinberg endomorphism of \bar{K} . Also, K is the adjoint version if $Z(K) = 1$. Unless we specify the version, we will always work with the adjoint versions of the exceptional groups of Lie type.

Cases $G_2(q)$ and ${}^2G_2(q)$. We start by proving that the Ree groups ${}^2G_2(q)$ satisfy $(\mathbb{Q}\mathcal{D})_2$ if and only if $q \neq 3$. Note that, by Table 3 for example, ${}^2G_2(q)$ has no nontrivial 2-extension.

Proposition 5.1. *Let L be the Ree group ${}^2G_2(q)$, where q is a power of 3 by an odd positive integer. Then the following hold:*

- (1) L has no nontrivial 2-extensions.
- (2) A Sylow 2-subgroup of L is an elementary abelian group of order 8, so $m_2(L) = 3$.
- (3) 2-subgroups of equal order of L are conjugate.
- (4) L satisfies $(\mathbb{Q}\mathcal{D})_2$ if and only if $q \neq 3$. Moreover, if $q > 3$ then

$$\dim H_2(\mathcal{A}_2(L)) \geq \tilde{\chi}(\mathcal{A}_2(L)) = \frac{1}{21}(q^2 - 1)(q^5 - 8q^4 + 15q^3 + 21) > 0. \tag{5-1}$$

- (5) For $q = 3$, $\mathcal{A}_2(L) = \mathcal{A}_2(\text{PSL}_2(8))$ is homotopy equivalent to a discrete space of 8 points.

Proof. Items (1)–(3) are well-known facts about the Ree groups and can be found in [Ward 1966].

If $L = {}^2G_2(3)$, then $L' = \text{PSL}_2(8)$ has index 3 in L , and $\mathcal{A}_2(L) \cong \mathcal{A}_2(\text{PSL}_2(8))$ is homotopy equivalent to a discrete space with 8 points. Since $m_2(L) = 3$, we conclude that L fails $(\mathbb{Q}\mathcal{D})_2$ for $q = 3$. This proves item (5) and the “only if” part of item (4).

Now suppose that $q \neq 3$ and $L = {}^2G_2(q)$. Since $\mathcal{A}_2(L)$ has dimension 2 by item (2), we show that its second homology group is nonzero. To that end, it is enough to see that its Euler characteristic is positive

since $\mathcal{A}_2(L)$ is connected for $q \neq 3$ by Theorem 3.8. Indeed,

$$\tilde{\chi}(\mathcal{A}_2(L)) = \dim H_2(\mathcal{A}_2(L)) - \dim H_1(\mathcal{A}_2(L)) \leq \dim H_2(\mathcal{A}_2(L)).$$

We invoke Theorem C of [Kleidman 1988] to describe the normalisers of 2-subgroups: the centraliser of an involution is $2 \times \text{PSL}_2(q)$, the normaliser of a four-subgroup is $(2^2 \times D_{(q+1)/2}) : 3$, and the normaliser of a Sylow 2-subgroup is $2^3 : 7 : 3$. From this information, items (2), (3) and Proposition 3.9, we can compute the Euler characteristic of $\mathcal{A}_2(L)$:

$$\begin{aligned} \tilde{\chi}(\mathcal{A}_2(L)) &= -1 + \frac{|L|}{2|\text{PSL}_2(q)|} - 2\frac{|L|}{6(q+1)} + 8\frac{|L|}{168} \\ &= -1 + q^3(q^3 + 1)(q - 1)\left(\frac{1}{q(q^2 - 1)} - \frac{1}{3(q+1)} + \frac{1}{21}\right) \\ &= \frac{1}{21}(q^2 - 1)(q^5 - 8q^4 + 15q^3 + 21). \end{aligned}$$

Since the polynomial $q^5 - 8q^4 + 15q^3 + 21 = q^4(q - 8) + 15q^3 + 21$ is positive for every prime power $q \neq 4$, we conclude that $H_2(\mathcal{A}_2(L)) \neq 0$. In consequence, L satisfies $(\mathbb{Q}\mathcal{D})_2$ if $q \neq 3$. This completes the proof of item (4), and hence of this proposition. \square

For the case $G_2(q)$, we refer the reader to the classification of maximal subgroups of $G_2(q)$ by P. Kleidman [1988]. We will follow the terminology of that article.

Proposition 5.2. *Let $L = G_2(q)$, with q odd. Then every 2-extension of L satisfies $(\mathbb{Q}\mathcal{D})_2$, except possibly for the 2-extensions of $G_2(3)$ and the 2-extension of $G_2(9)$ by a field involution.*

Proof. Let $L = G_2(q)$. We prove first that $G_2(q)$ and its extension by a field automorphism of order 2 satisfy $(\mathbb{Q}\mathcal{D})_2$, by exhibiting a maximal subgroup of the same rank that satisfies $(\mathbb{Q}\mathcal{D})_2$.

By Theorem A in [Kleidman 1988], $G_2(q)$ contains a subgroup $K_+ = \text{SL}_3(q) : 2$. Let $L_+ = F^*(K_+) \cong \text{SL}_3(q)$ and $Z = Z(L_+)$. Then $L_0 := L_+/Z = \text{PSL}_3(q)$ and $H_0 := K_+/Z = L_0 \langle \gamma \rangle$, where γ induces a graph automorphism on L_0 (see Proposition 2.2 and its proof in [Kleidman 1988]). By Proposition 4.6, L_0 satisfies $(\mathbb{Q}\mathcal{D})_2$ if $q \neq 3$, so H_0 satisfies $(\mathbb{Q}\mathcal{D})_2$.

On the other hand, $m_2(L) = 3$ by Table 3, and also $m_2(L_0) = 3$ by the proof of Proposition 4.6. Recall from Lemma 3.2 that

$$\tilde{H}_*(\mathcal{A}_2(H_0)) = \tilde{H}_*(\mathcal{A}_2(K_+/Z)) \subseteq \tilde{H}_*(\mathcal{A}_2(K_+)).$$

In particular, we get the following inclusions between the top-degree homology groups

$$\tilde{H}_2(\mathcal{A}_2(H_0)) \subseteq \tilde{H}_2(\mathcal{A}_2(K_+)) \subseteq \tilde{H}_2(\mathcal{A}_2(L)),$$

which show that L satisfies $(\mathbb{Q}\mathcal{D})_2$ if $q \neq 3$.

Next, a nontrivial 2-extension of $L = G_2(q)$ can only be given by field automorphisms of order 2 if q is not a power of 3. Moreover, by the construction of the subgroup K_+ given in [Kleidman 1988], field automorphisms of $G_2(q)$ induce field automorphisms on (a suitable conjugate of) K_+ , and hence on the quotient H_0 . Thus, for $B \in \mathcal{O}_2(L)$ inducing field automorphisms, we may take

K_+ fixed by B , and then $K_+B \cong \text{SL}_3(q) : (2 \times B)$ after a suitable choice of conjugates (recall that $\text{Out}(\text{SL}_3(q)) = (3, q - 1) : (\text{Aut}(\mathbb{F}_q) \times \Gamma)$, where $\Gamma = 2$ is a group of graph automorphisms). Similar as before, we have a split extension $K_+B/Z = L_0B'$, where $B' = \langle \gamma \rangle \times B \in \mathcal{O}(L_0)$. By Proposition 4.6, L_0B' satisfies $(\mathcal{QD})_2$ if $q \neq 9$. Analogously to the previous case, $m_2(L_0B') = 4 = m_2(L) = m_2(K_+B)$, and we get an inclusion in the degree 3 homology groups, showing that K_+B and LB satisfy $(\mathcal{QD})_2$. Therefore, an extension of L by a field automorphism of order 2 satisfies $(\mathcal{QD})_2$ if $q \neq 9$.

It remains to analyse the case $q = 3^a$. By Table 4.5.1 of [Gorenstein et al. 1998] (see also Theorem 2.5.12 of [Gorenstein et al. 1998]), only field or graph-field automorphisms can arise in $\text{Aut}(L)$. We have shown above that the extension of L by a field automorphism of order 2 satisfies $(\mathcal{QD})_2$ if $q \neq 9$. Thus we need to prove that if t is a graph-field automorphism of L , then $L\langle t \rangle$ satisfies $(\mathcal{QD})_2$. In that case, $q = 3^{2a+1}$ and by Proposition 2.5, $C_L(t) = {}^2G_2(q)$, which has 2-rank 3. Therefore $m_2(L\langle t \rangle) = 4$. However, by Theorem B of [Kleidman 1988], every maximal subgroup of $L\langle t \rangle$ containing t is either 2-local or has 2-rank at most 3. This shows that we cannot proceed as before via maximal subgroups. In view of this, we will proceed by using the long exact sequence of Lemma 3.10.

We have subgroups $M_0 := C_L(t) = {}^2G_2(q)$, $M_1 := G_2(3)\langle t \rangle \leq L\langle t \rangle$ and $M_2 := {}^2G_2(3)$ such that $M_2 \leq M_1 \cap M_0$. Fix A a Sylow 2-subgroup of M_2 . By Proposition 5.1(2) and [Kleidman 1988, Lemma 2.4], A is also a Sylow 2-subgroup of M_0 and it is self-centralising in L , i.e., $C_L(A) = A$. A direct computation also shows that $N_{M_1}(A) = A.\text{PSL}_3(2)$, which immediately implies $N_L(A) = A.\text{PSL}_3(2)$.

Now, suppose by the way of contradiction that $L\langle t \rangle$ fails $(\mathcal{QD})_2$, that is, the homology group $H_3(\mathcal{A}_2(L\langle t \rangle))$ vanishes. Recall that $C_L(t) = {}^2G_2(q)$ and there is a unique L -conjugacy class of involutions $t' \in L\langle t \rangle - L$ by Proposition 2.5(4). Let $X = \bigcup_{C_L(t)x \in L/C_L(t)} \mathcal{A}_2(C_L(t^x))$. By Lemma 3.10, we get inclusions

$$\bigoplus_{L/C_L(t)} H_2(\mathcal{A}_2(C_L(t))) \hookrightarrow H_2(X) \hookrightarrow H_2(\mathcal{A}_2(L)). \tag{5-2}$$

Set

$$d := \dim H_2(X), \quad d' := \dim \bigoplus_{L/C_L(t)} H_2(\mathcal{A}_2(C_L(t))) = |L : C_L(t)| \dim H_2(\mathcal{A}_2({}^2G_2(q))).$$

Equation (5-2) shows that $d' \leq d$. However, we will prove that $d < d'$, arriving then at a contradiction.

On one hand, we have that X is a union of \mathcal{A}_2 -posets. Therefore, below each point, we have a wedge of spheres of maximal possible dimension. This means that the homology of X can be obtained from the chain complex that in degree i is freely generated by the spheres below each point of X of height i . In particular, for $i = 2$, the points of height 2 correspond to the conjugates of A , the fixed Sylow 2-subgroup of $M_0 = C_L(t)$ and M_2 . Thus,

$$d = \dim H_2(X) \leq |L : N_L(A)| \cdot \#(\text{spheres below } A) = \frac{q^6(q^6 - 1)(q^2 - 1)}{168}.$$

On the other hand, by Proposition 5.1(4),

$$d' \geq |L : C_L(t)| \cdot \tilde{\chi}(\mathcal{A}_2({}^2G_2(q))) = q^3(q^3 - 1)(q + 1) \frac{1}{21}(q^2 - 1)(q^5 - 8q^4 + 15q^3 + 21).$$

Finally, from these bounds, we prove that $d' > d$ if $q \geq 7$. We can bound

$$\begin{aligned} d' - d &\geq q^3(q^3 - 1)(q + 1)\frac{1}{21}(q^2 - 1)(q^5 - 8q^4 + 15q^3 + 21) - \frac{q^6(q^6 - 1)(q^2 - 1)}{168} \\ &= q^3(q^3 - 1)(q + 1)\frac{1}{168}(q^2 - 1)(8(q^5 - 8q^4 + 15q^3 + 21) - q^3(q^2 - q + 1)) \\ &> 8(q^5 - 8q^4 + 15q^3 + 21) - q^3(q^2 - q + 1) \\ &= 7(q^4(q - 9) + 17q^3 + 24). \end{aligned}$$

The latter polynomial is clearly positive for $q \geq 9$, and also for $q = 7$ by direct computation. Since $q \geq 7$ by hypothesis, we conclude that $d' > d$. This gives a contradiction to equation (5-2), and thus shows that $H_3(\mathcal{A}_2(L(t))) \neq 0$; that is, $L(t)$ satisfies $(\mathbb{Q}\mathcal{D})_2$. This finishes the proof of the proposition. \square

Example 5.3. Let $L = G_2(3)$. We show that $\mathcal{A}_2(L)$ is homotopy equivalent to a wedge of spheres of dimension 1. In particular, since $m_2(L) = 3$, L fails $(\mathbb{Q}\mathcal{D})_2$. Moreover, by Lemma 3.10, also the unique nontrivial 2-extension of L (by a graph-field automorphism) fails $(\mathbb{Q}\mathcal{D})_2$.

We construct a subposet of $\mathcal{A}_2(L)$ of dimension 1 and homotopy equivalent to $\mathcal{A}_2(L)$. First, take the subposet $i(\mathcal{A}_2(L)) = \{A \in \mathcal{A}_2(L) : A = \Omega_1(Z(\Omega_1(C_L(A))))\}$, which is homotopy equivalent to $\mathcal{A}_2(L)$ (see [Piterman 2019, Remark 4.5]). Next, there are two conjugacy classes of elementary abelian 2-subgroups of order 8, and both are contained in $i(\mathcal{A}_2(L))$. For one of these classes, say represented by A , the normaliser $N_L(A)$ has order 192. Then it can be shown that $i(\mathcal{A}_2(L))_{<A}$ is contractible. Therefore, if we remove the L -conjugates of A from $i(\mathcal{A}_2(L))$ we get a subposet $si(\mathcal{A}_2(L))$ homotopy equivalent to $i(\mathcal{A}_2(L))$. Now, there is a unique conjugacy class of four-subgroups in this new subposet $si(\mathcal{A}_2(L))$, and each such subgroup is contained in a unique element of order 8 of $si(\mathcal{A}_2(L))$. Again, we can remove all the four-subgroups from $si(\mathcal{A}_2(L))$ and obtain a new subposet Y homotopy equivalent to $\mathcal{A}_2(L)$. Since Y consists only of elements of order 2 and 8, we conclude that Y has dimension 1. Finally, an extra computation shows that $\tilde{\chi}(\mathcal{A}_2(L)) = -11584$. Therefore $\mathcal{A}_2(L)$ is homotopy equivalent to a wedge of 11584 spheres of dimension 1. In particular, L fails $(\mathbb{Q}\mathcal{D})_2$.

This also shows that $L = G_2(9)$ extended by a field automorphism of order 2 fails $(\mathbb{Q}\mathcal{D})_2$: if ϕ is a field involution, then $C_L(\phi) = G_2(3)$, and thus $H_2(\mathcal{A}_2(C_L(\phi))) = 0$ by the previous computation. Then by Lemma 3.10, we conclude that $H_3(\mathcal{A}_2(L)) = 0$.

Cases 3D_4 and $F_4(q)$.

Proposition 5.4. *The group $L = {}^3D_4(q)$ satisfies $(E-(\mathbb{Q}\mathcal{D}))$ if $q \neq 9$ is odd. Also ${}^3D_4(9)$ satisfies $(\mathbb{Q}\mathcal{D})_2$.*

Proof. Recall that $m_2(L) = 3$ by Table 3. Then a graph automorphism of order 3 of ${}^3D_4(q)$ centralises a subgroup $K = G_2(q)$. Also, if ϕ denotes a field automorphism of order 2 of L , then, after choosing a suitable conjugate, we may assume that ϕ induces a field automorphism on K . By Proposition 5.2 and its proof, $m_2(K) = 3 = m_2(L)$, $m_2(K\langle\phi\rangle) = 4 = m_2(L\langle\phi\rangle)$, and both K and $K\langle\phi\rangle$ satisfy $(\mathbb{Q}\mathcal{D})_2$ for $q \neq 3, 9$ respectively. Also note that $G_2(9)$ satisfies $(\mathbb{Q}\mathcal{D})_2$. By Lemma 3.3, L and $L\langle\phi\rangle$ satisfy $(\mathbb{Q}\mathcal{D})_2$ if $q \neq 3, 9$, respectively. Since these are the only possible 2-extensions of L by Table 3, this concluded with the proof of our proposition for $q \neq 3$.

If $q = 3$, a computation of the Euler characteristic of $\mathcal{A}_2(L)$ in GAP with the Posets package [Fernández et al. 2019] shows that $\tilde{\chi}(\mathcal{A}_2(L)) = 882634225472$. Since $\mathcal{A}_2(L)$ is connected by Theorem 3.8, we see that $H_2(\mathcal{A}_2(L)) \neq 0$, that is, L satisfies $(\mathbb{Q}\mathcal{D})_2$. \square

Proposition 5.5. *If $L = F_4(q)$, with $q \neq 3, 9$ odd, then L satisfies $(E-(\mathbb{Q}\mathcal{D}))$. Also $F_4(9)$ satisfies $(\mathbb{Q}\mathcal{D})_2$.*

Proof. Suppose that $q \neq 3, 9$ is an odd prime power. Then L contains a subgroup $H := \text{PGL}_2(q) \times G_2(q)$ (see the main result of [Liebeck and Seitz 2004]). Note that H satisfies $(\mathbb{Q}\mathcal{D})_2$ by Propositions 3.1, 4.2 and 5.2. Since both L and H have 2-rank 5 by Table 3, we conclude that L satisfies $(\mathbb{Q}\mathcal{D})_2$.

Let $B \in \mathcal{O}_2(L)$, so B is generated by a field automorphism of order 2. Thus it acts by field automorphisms in a direct product subgroup isomorphic to H , which we may assume without loss of generality that it is our H . Then $\tilde{H} = \text{PGL}_2(q)B \times G_2(q^{1/2})$, which is a subgroup of HB , satisfies $(\mathbb{Q}\mathcal{D})_2$ by Propositions 3.1, 4.2 and 5.2. Since $m_2(\tilde{H}) = 6 = m_2(LB)$, we conclude that LB also satisfies $(\mathbb{Q}\mathcal{D})_2$.

We have shown that every possible 2-extension of L satisfies $(\mathbb{Q}\mathcal{D})_2$, so L satisfies $(E-(\mathbb{Q}\mathcal{D}))$.

If $q = 9$, then $\text{PGL}_2(9) \times G_2(9)$ satisfies $(\mathbb{Q}\mathcal{D})_2$ by Propositions 3.1, 4.2 and 5.2. Therefore, $F_4(9)$ satisfies $(\mathbb{Q}\mathcal{D})_2$. \square

Cases $E_6(q)$ and ${}^2E_6(q)$.

Proposition 5.6. *Let $L = E_6^\epsilon(q)$ (any version), $\epsilon \in \{\pm 1\}$, and q odd. Then L satisfies $(E-(\mathbb{Q}\mathcal{D}))$.*

Proof. Let $L = E_6^\epsilon(q)$ in adjoint version (i.e., simple), where $\epsilon \in \{\pm 1\}$. For a 2-extension LB of the adjoint version L , we see that $m_2(LB) = m_2(L_u \tilde{B})$, where L_u is the universal version of $E_6^\epsilon(q)$ and \tilde{B} , isomorphic to B , is just a lift of the action of B on L_u (this is possible since $Z(L_u) = (3, q - \epsilon)$ is odd). Thus $LB = L_u \tilde{B} / Z(L_u)$, and by Lemma 3.2, $\tilde{H}_*(\mathcal{A}_2(LB)) \subseteq \tilde{H}_*(\mathcal{A}_2(L_u B))$. Therefore, if L satisfies $(E-(\mathbb{Q}\mathcal{D}))$, then so does the universal version of $E_6^\epsilon(q)$.

We will show that there exists a parabolic subgroup P of L such that for any 2-extension LB , a suitable conjugate of P is normalised by B (so we can suppose it is P itself), and $m_2(PB) = m_2(LB)$.

This parabolic subgroup P arises from the A_5 subdiagram in E_6 , so $P = U \text{GL}_6^\epsilon(q) / Z(L_u)$, where $\text{GL}_6^\epsilon(q) / Z(L_u)$ denotes the Levi complement. Then $m_2(P) = 6$, which realises the 2-rank of L . Furthermore, a graph, graph-field or field automorphism of order 2 of L (the last two only for $\epsilon = 1$) stabilises this subdiagram (and hence P), inducing a graph (resp. graph-field or field) automorphism on $\text{GL}_6^\epsilon(q) / Z(L_u)$. Denote by t such automorphism. Then $m_2(L\langle t \rangle) \leq m_2(L) + 1 = 7$. We claim that

$$m_2(P\langle t \rangle) = m_2(\text{GL}_6^\epsilon(q)\langle t \rangle) = 7 = m_2(L\langle t \rangle). \tag{5-3}$$

Note that $m_2(P\langle t \rangle) = m_2(\text{GL}_6^\epsilon(q)\langle t \rangle)$, for the lifted action of t on $\text{GL}_6^\epsilon(q)$. Then it is clear that (5-3) holds if t induces a field automorphism (so $\epsilon = 1$), since the stabiliser of t in $\text{GL}_6(q)$ is $\text{GL}_6(q^{1/2})$. Similarly, if t is a graph-field automorphism then $\epsilon = 1$ and $C_{\text{GL}_6(q)}(t) = \text{GU}_6(q^{1/2})$, which has 2-rank 6. Then, in these two situations, $m_2(P\langle t \rangle) = 7$.

Now assume that t is a graph involution. For $\epsilon = 1$, t acts on $\text{GL}_6(q)$, so $\text{GL}_6(q)\langle t \rangle$ contains a graph automorphism g inducing the map $x \mapsto (x')^{-1}$, where x' denotes the transpose of x . Therefore,

$C_{GL_6(q)}(g) = \{x \in GL_6(q) : xx' = 1\} = GO_6(q)$ is the subgroup of orthogonal matrices, which has 2-rank 6. This implies that $m_2(GL_6(q)\langle t \rangle) = 6$. If $\epsilon = -1$, t is a graph involution acting on $GU_6(q)$, so up to conjugation t is indeed the map $x \mapsto x^q$. Therefore, $C_{GU_6(q)}(t) = GO_6(q)$, so $m_2(GU_6(q)\langle t \rangle) = 6$. In any case, we see that $m_2(P\langle t \rangle) = 7$.

Finally, suppose that we have $B = \langle \phi, \gamma \rangle$, where ϕ is a field automorphism of order 2 and γ a graph automorphism of order 2 of $L = E_6(q)$. We can suppose that B stabilises P (and thus its unipotent radical), and its Levi complement $GL_6(q)/Z(L_u)$. Thus, γ induces a graph automorphism on the stabiliser of ϕ in $GL_6(q)$, which is isomorphic to $GL_6(q^{1/2})$. As we saw above, $m_2(GL_6(q^{1/2})\langle \gamma \rangle) = 7$. Therefore, $m_2(GL_6(q)B) = 8$. Since $m_2(B) = 2$ and $m_2(E_6(q)) = 6$, we conclude that $m_2(E_6(q)B) = 8$, so the 2-rank is realised in PB .

To conclude, note that a 2-extension of L is one of:

- (1) L , of 2-rank $6 = m_2(P)$,
- (2) $L\langle \gamma \rangle$ of 2-rank 7, with γ a graph automorphism of order 2, which also stabilises P and $m_2(P\langle \gamma \rangle) = 7$,
- (3) $L\langle \phi \rangle$ of 2-rank 7, with ϕ a field automorphism of order 2 ($\epsilon = 1$) that stabilises P , and so $m_2(P\langle \phi \rangle) = 7$,
- (4) $L\langle \gamma\phi \rangle$ of 2-rank 7, with $\gamma\phi$ a graph-field automorphism of order 2 ($\epsilon = 1$), which stabilises P , and thus $m_2(P\langle \gamma\phi \rangle) = 7$,
- (5) $L\langle \gamma, \phi \rangle$ of 2-rank 8, with ϕ a field automorphism of order 2 ($\epsilon = 1$) commuting with γ a graph automorphism of order 2, and $\langle \gamma, \phi \rangle$ also stabilises P with $m_2(P\langle \gamma, \phi \rangle) = 8$.

From this, we conclude that any 2-extension of the simple group $E_6^\epsilon(q)$ satisfies $(\mathbb{Q}\mathcal{D})_2$. By the remark at the beginning of the proof, we conclude that any version of $E_6^\epsilon(q)$ satisfies $(E-(\mathbb{Q}\mathcal{D}))$. □

Case $E_7(q)$.

Proposition 5.7. *Let $L = E_7(q)$ with q odd. Then L satisfies $(E-(\mathbb{Q}\mathcal{D}))$.*

Proof. Let $L = E_7(q)$. By Table 3, if ϕ denotes a field automorphism of order 2 of L , the 2-extensions of L are

$$L, \quad \text{Inndiag}(L), \quad L\langle \phi \rangle.$$

Note that $\text{Inndiag}(L)\langle \phi \rangle$ is not a 2-extension since field and diagonal automorphisms of order 2 do not commute in view of Lemma 2.6.

Next, we study the 2-ranks of these extensions, so we need to understand the centralisers of the outer involutions. From Table 3, $m_2(L) = 8$. We claim that $m_2(\text{Inndiag}(L)) = 8 = m_2(L)$. Indeed, consider $K = E_7(q^2)$ in adjoint version. Then $m_2(K) = 8$. Let ϕ' be a field automorphism of order 2 for K . Then, by Proposition 2.5 and Lemma 2.6,

$$K \geq C_K(\phi') = C_{\text{Inndiag}(K)}(\phi') = \text{Inndiag}(E_7(q)) \cong \text{Inndiag}(L).$$

From this we see that $m_2(\text{Inndiag}(L)) = 8 = m_2(L)$. In particular, $\text{Inndiag}(L)$ satisfies $(\mathbb{Q}\mathcal{D})_2$ if L does. Moreover, this also proves that if ϕ is a field automorphism of order 2 for L then

$$m_2(\text{Inndiag}(L)\langle\phi\rangle) = 9 = m_2(L\langle\phi\rangle).$$

From these observations, we conclude that, in order to establish $(E-(\mathbb{Q}\mathcal{D}))$ for $E_7(q)$, it is enough to show that $E_7(q)$ and $E_7(q)\langle\phi\rangle$ satisfy $(\mathbb{Q}\mathcal{D})_2$.

To this end, we exhibit a maximal parabolic subgroup of $E_7(q)$ of 2-rank 8. We see that D_6 is a subdiagram of E_7 , so we have a maximal parabolic subgroup in $E_7(q)$ of the form

$$P = U(D_6(q).(q - 1)).$$

Here U denotes the unipotent radical of P , and the subgroup $H = D_6(q)$ is a quotient of $\text{Spin}_{12}^+(q)$ by a central subgroup of order 2. Indeed, $H = \text{HSpin}_{12}^+(q)$ and it lies in the centraliser of the involution that generates the centre of a Sylow 2-subgroup T of L (see the t_1 involution of the $E_7(q)$ entry in Table 4.5.1 of [Gorenstein et al. 1998]). From this, we show that the Levi complement $L_P = D_6(q).(q - 1)$ of P has 2-rank 8. Let t be the involution in the centre of L_P . Then $C_L(t) = (\text{SL}_2(q) \circ_2 \text{HSpin}_{12}^+(q)).2$ by Table 4.5.1 of [Gorenstein et al. 1998]. Since $t \in Z(T)$, $T \leq C_L(t)$. Also, $\text{SL}_2(q)$ has a unique involution, so the 2-rank of T is realised in a subgroup of the extension $M := \text{HSpin}_{12}^+(q).2$. Here, the 2 at the end comes from diagonal automorphisms of the half-spin group, as in the Levi complement above. Therefore, if we identify M as a subgroup of L_P , we conclude that $m_2(L_P) = m_2(M) = m_2(E_7(q))$.

Moreover, after suitable choices of conjugates, a field automorphism ϕ of order 2 must normalise P and act as a field automorphism on our M . Since $C_M(\phi)$ contains a subgroup isomorphic to $\text{HSpin}_{12}^+(q^{1/2}).2$, we see that $P\langle\phi\rangle$ has 2-rank 9, which is the 2-rank of the 2-extension $E_7(q)\langle\phi\rangle$.

By Lemma 3.5, L and $L\langle\phi\rangle$ satisfy $(\mathbb{Q}\mathcal{D})_2$. Finally, by the previous discussion, we conclude that L satisfies $(E-(\mathbb{Q}\mathcal{D}))$. □

Case $E_8(q)$.

Proposition 5.8. *The simple group $E_8(q)$, $q \neq 3, 9$ odd, satisfies $(E-(\mathbb{Q}\mathcal{D}))$. Also $E_8(9)$ satisfies $(\mathbb{Q}\mathcal{D})_2$.*

Proof. Let $L = E_8(q)$. By Table 5.1 of [Liebeck et al. 1992], L contains a maximal subgroup

$$H \cong (3, q - 1).(\text{PSL}_3(q) \times E_6(q)).(3, q - 1).2.$$

Note that

$$F^*(H) = (3, q - 1).(\text{PSL}_3(q) \times E_6(q)), \quad \text{and} \quad H_+ := H/Z(F^*(H)) = (\text{PSL}_3(q) \times E_6(q)).(3, q - 1).2,$$

where $(3, q - 1)$ induces diagonal automorphism on each component of H_+ , and the 2 induces a graph involution, also acting on both components. In particular, by taking the centraliser of a graph involution on the $\text{PSL}_3(q)$ component, we see that H_0 contains a subgroup K_0 isomorphic to

$$\text{PGL}_2(q) \times \text{Inndiag}(E_6(q))\langle\gamma\rangle,$$

where γ is a graph involution of $E_6(q)$ centralising $\mathrm{PGL}_2(q)$. Now, recall that

$$m_2(L) = 9 \quad \text{and} \quad m_2(\mathrm{PGL}_2(q)) = 2.$$

Since $m_2(E_6(q)\langle\gamma\rangle) = 7$ by item (2) of the proof of Proposition 5.6, we see that

$$m_2(K_0) = m_2(\mathrm{PGL}_2(q)) + m_2(E_6(q)\langle\gamma\rangle) = 2 + 7 = 9 = m_2(L).$$

Therefore K_0 realises the 2-rank of L .

By Table 3, $E_8(q)$ extended by a field automorphism of order 2, say ϕ , is the unique nontrivial 2-extension. From the construction of the maximal subgroup H and K_0 (see [Liebeck et al. 1992]), we can pick a suitable L -conjugate of ϕ (and we suppose it is the same ϕ) such that it normalises H and, after passing to the quotient, normalises K_0 and induces a field automorphism on both factors of K_0 . In particular, we have a subgroup K_1 of $K_0\langle\phi\rangle$ of the form

$$\mathrm{PGL}_2(q^{1/2}) \times \mathrm{Inndiag}(E_6(q)\langle\gamma', \phi\rangle),$$

where we have chosen $\gamma' \in \mathrm{Inndiag}(E_6(q)\langle\gamma\rangle)$ to be a graph automorphism commuting with ϕ , and $\mathrm{PGL}_2(q^{1/2}) = C_{\mathrm{PGL}_2(q)}(\phi)$. Therefore, by item (5) in the proof of Proposition 5.6,

$$m_2(K_1) = 2 + m_2(E_6(q)\langle\gamma', \phi\rangle) = 2 + 8 = 10.$$

Since $m_2(L\langle\phi\rangle) \leq m_2(L) + 1 = 10$, we conclude that $m_2(K_1) = m_2(L\langle\phi\rangle)$.

Finally, note that K_0 and K_1 satisfy $(\mathbb{Q}\mathcal{D})_2$ if $q \neq 3, 9$ respectively, by Propositions 4.2, 5.6 and 3.1. Hence, by Lemmas 3.2 and 3.3, L and $L\langle t \rangle$ satisfy $(\mathbb{Q}\mathcal{D})_2$ if $q \neq 3, 9$, respectively.

Therefore every 2-extension of $E_8(q)$ satisfies $(\mathbb{Q}\mathcal{D})_2$, with the exceptions given in the statement. This concludes the proof of the proposition. \square

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