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The strong maximal rank conjecture and moduli spaces of curves

Fu Liu, Brian Osserman, Montserrat Teixidor i Bigas and Naizhen Zhang

Building on recent work of the authors, we use degenerations to chains of elliptic curves to prove two cases of the Aprodu–Farkas strong maximal rank conjecture, in genus 22 and 23. This constitutes a major step forward in Farkas' program to prove that the moduli spaces of curves of genus 22 and 23 are of general type. Our techniques involve a combination of the Eisenbud–Harris theory of limit linear series, and the notion of linked linear series developed by Osserman.

1. Introduction

The moduli space $\mathcal{M}_g$ of curves of fixed genus g is one of the most classically studied in algebraic geometry. Going back to Severi and based on examples in low genus there was a general expectation that these moduli spaces ought to be unirational. However, groundbreaking work of Harris, Mumford and Eisenbud [Harris and Mumford 1982; Harris 1984; Eisenbud and Harris 1987] in the 1980s showed that not only is $\mathcal{M}_g$ not unirational for large g, but it is in fact of general type for $g \geq 24$. Their fundamental technique was to compute the classes of certain explicit effective divisors on $\overline{\mathcal{M}}_g$ arising from Brill–Noether theory, and use this to show that the canonical class of $\overline{\mathcal{M}}_g$ can be written as the sum of an ample and an effective divisor. The particular families of divisors they considered were computable in all applicable genera, but did not suffice to prove that $\mathcal{M}_g$ is of general type for $g \leq 23$. For the last thirty

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years, no new cases have been proved of $\mathcal{M}_g$ being of general type. Over a decade ago, Farkas [2009a, §7; 2009b, §4; 2009c] proposed new families of expected divisors on $\mathcal{M}_g$ as an approach to showing that $\mathcal{M}_{22}$ and $\mathcal{M}_{23}$ are of general type. Let $\mathcal{D}_g \subseteq \mathcal{M}_g$ consist of curves X of genus g which admit a $\mathfrak{g}^6_d$ such that the resulting image of X in $\mathbb{P}^6$ lies on a quadric hypersurface. Farkas computed "virtual classes" for these expected divisors $\mathcal{D}_g$ in [Farkas 2009a] for genus 22 and in [Farkas 2018] for genus 23, and in both cases found that the classes satisfy the necessary inequalities to conclude that $\mathcal{M}_{22}$ and $\mathcal{M}_{23}$ are of general type, provided that they are indeed represented by effective divisors.

In order to conclude that $\mathcal{M}_g$ is of general type for $g = 22$ or 23, one has to check two statements: first, that $\mathcal{D}_g$ yields an effective divisor, or equivalently, that $\mathcal{D}_g \subsetneq \mathcal{M}_g$; and second, that the class induced by $\mathcal{D}_g$ agrees with the class previously computed by Farkas, or equivalently, that the subset of $\mathcal{D}_g$ consisting of curves carrying *infinitely* many $\mathfrak{g}^6_d$s whose image lie on a quadric occurs in codimension strictly higher than 1.

In this paper, we prove the first of these two statements, for both $g = 22$ and $g = 23$. An independent proof of this result has been obtained by Jensen and Payne [2018] using a tropical approach. Their tropical proof has now been merged in [Farkas et al. 2020] with the prior results of Farkas and with the missing piece that the map from the space of linear series to the moduli space of curves does not have infinite fibers over a divisorial component of the image. This completes the proof that $\mathcal{M}_{22}$, $\mathcal{M}_{23}$ are of general type.

Our main theorem is thus the following:

Theorem 1.1. *In characteristic* 0, *the loci $\mathcal{D}_{22}$ and $\mathcal{D}_{23}$ are proper subsets of $\mathcal{M}_{22}$ and $\mathcal{M}_{23}$ respectively.*

Our proof goes through unmodified for characteristic $p \geq 29$, and our techniques can in principle be applied to lower characteristics as well, but due to characteristic restrictions on the application to the geometry of $\mathcal{M}_g$, we have not pursued this. See Remark 1.3 below.

The divisor $\mathcal{D}_g \subsetneq \mathcal{M}_g$ can be presented as a particular case of a more general (conjectural) subsets of $\mathcal{M}_g$: With applications to moduli,spaces of curves in mind, Aprodu and Farkas [2011, Conjecture 5.4] proposed a "strong maximal rank conjecture", about ranks of multiplication maps of line bundles on curves. Specifically, given a linear series $(\mathscr{L}, V)$ on a curve X, we have the multiplication map

$$\operatorname{Sym}^2 V \to \Gamma(X, \mathscr{L}^{\otimes 2}). \tag{1-1}$$

Note that the source has dimension $\binom{r+2}{2}$, and assuming X is Petri-general, the target has dimension $2d+1-g$. The image of X under the linear series lies on a quadric if and only if (1-1) has a nonzero kernel. The classical maximal rank conjecture asserts that if $r \geq 3$, for a general X and a general $\mathfrak{g}^r_d$ on X, the map (1-1) should always be injective or surjective (and similarly for the higher-order multiplication maps). Many special cases of this were proved by various people; we omit discussion of most of these, but mention that the case of quadrics was first proved by Ballico [2012]. Subsequent proofs were given by Jensen and Payne [2016] using a tropical approach, and by the present authors [Liu et al. 2021] using a degeneration to a chain of genus-1 curves. Recently, Larson [2017; 2020] has proved the full classical maximal rank conjecture.

Since the failure of (1-1) to have maximal rank is a determinantal condition, the strong maximal rank conjecture of Aprodu and Farkas is the following:

Conjecture 1.2 [Aprodu and Farkas 2011, Conjecture 5.4]. *Set* $\rho := g-(r+1)(g+r-d)$.

On a general curve of genus g, *if* $\rho < r-2$, *the locus of* $\mathfrak{g}^r_d$*s for which* (1-1) *fails to have maximal rank is equal to the expected determinantal codimension, which is* $1+\left|\binom{r+2}{2}-(2d+1-g)\right|$. *In particular, when this expected codimension exceeds* ρ, *every linear series on* X *should have maximal rank.*[1]

The strong maximal rank conjecture remains wide open, even in the case of quadrics. The only cases solved (to our knowledge) are for $k=2,\ d \le g+1$ (see [Teixidor i Bigas 2003]) or for Brill–Noether number $\rho=0$, because it is equivalent to the (weak) maximal rank conjecture.

For the divisors $\mathcal{D}_g$, we compute that $\rho = g-21 = (2d+1-g)-\binom{r+2}{2}$, so in this case Conjecture 1.2 predicts that every linear series on the generic curve should yield (1-1) of maximal rank, and more specifically, should have injective multiplication map, just as we prove in Theorem 1.1 for the cases $g=22, 23$.

Our proof builds on the ideas introduced in [Liu et al. 2021], which combine the Eisenbud–Harris theory of limit linear series with ideas from the theory of linked linear series introduced by the second author [Osserman 2006; 2014]. We start with a limit linear series on a chain X_0 of genus-1 curves, and describe a collection of global sections living in different multidegrees on X_0. We then take tensors of these sections and consider their image in a carefully chosen multidegree, showing that they have the correct-dimensional span. The first major difficulty is that while we can choose the curve, we have to consider all possible limit linear series. As a consequence, we cannot ignore degenerate limit linear series (which occur already in codimension 1). We systematically use ideas from linked linear series to prove that when $\rho=1$ or $\rho=2$ we can always produce global sections of certain prescribed forms which must lie in the specialization of the family of linear series.

The structure of the paper is as follows. In Section 2 we consider certain maps from genus-1 curves to projective spaces which arise naturally from tensor squares of linear series, and show that these are nondegenerate morphisms in cases of interest. In Section 3, we review the Eisenbud–Harris theory of limit linear series, and the related theory of linked linear series introduced by the second author. In Section 4, we describe the possible structures of linked linear series lying over a given limit linear series in the cases that appear when $\rho \le 2$ (see also Proposition 5.3). In Sections 5 and 6, we give a criterion for certain collection of sections in the tensor square of a limit linear series to be linearly independent. In Section 7, we improve and apply this criterion to a family of examples with $r=6$, which include the genus-22 and genus-23 cases of interest for the proof of Theorem 1.1. In Section 8, we focus on the behavior of degenerate sections under tensor product. For $\rho=2$ (that is, genus 23), the situation is quite a bit more complicated than for $\rho=1$ (i.e., genus 22). To handle the degenerate cases, we consider in Section 8 variant multidegrees which depend more tightly on the limit linear series in question, and (partially inspired by the earlier work of Jensen and Payne [2016] on a tropical approach to the classical

[1] In fact, Aprodu and Farkas also include higher-degree multiplication maps in their conjecture. Farkas and Ortega [2011] subsequently relax the $\rho < r-2$ hypothesis in cases such as ours, where ρ is less than the expected codimension.

maximal rank conjecture), we also consider families of curves with highly specialized directions of approach, which gives us further control over the behavior of the global sections in different multidegrees. Finally, in Section 9, we complete the proof of Theorem 1.1.

We expect that the tools we develop here will lead to proofs of specific cases of the strong maximal rank conjecture of geometric relevance. In the tropical setting, this has been carried out in the case $g = 13$ (see [Farkas et al. 2024]). We have written the different parts of the argument to be independent of r and/or ρ wherever this does not lead to unnecessary complication. In particular, Theorem 9.1 has been stated in greater generality than what we need to prove the results for $g = 22$, $g = 23$. The nature of our approach also allows for proving cases of the maximal rank conjecture where the expected codimension does not exceed ρ, so that the locus of linear series which do not have maximal rank is nonempty. Our approach should also be useful in other questions involving multiplication maps for linear series, such as the conjecture of Bakker and Farkas [2018, Remark 14], which was motivated by connections to higher-rank Brill–Noether theory. Their conjecture treats a certain specific family of cases, but with products of distinct linear series in place of symmetric squares of a fixed one. In addition, our work in Section 2 on nondegeneracy of certain morphisms from genus-1 curves to projective spaces and in Section 4 on the structure of exact linked linear series is likely to be useful in other settings as well.

In a different direction, the ideas and approach of this paper can be used in the context of vector bundles (see [Teixidor i Bigas 2023]). Brill–Noether related vector bundle problems cannot at the moment be treated with tropical techniques as there is no satisfactory theory of sections of vector bundles in the tropical settings.

Remark 1.3. We mention that although we impose characteristic-0 hypotheses in our main theorem, these do not appear to be essential. Nearly everything we do is characteristic-independent, but we use a characteristic-dependent result (Theorem 3.4 below) of Eisenbud and Harris to simplify the situation slightly by restricting our attention to "refined" limit linear series (Definition 3.1 below). In fact, the only characteristic dependence in Theorem 3.4 is the use of the Plücker inequality, which still holds in characteristic p and degree d when $p > d$; see for instance Proposition 2.4 and Corollary 2.5 of [Osserman 2006]. Thus, our proof of Theorem 1.1 extends as written to characteristic $p > 25$ for $g = 22$ and $p > 26$ for $g = 23$.

Moreover, since our key specialization result (Proposition 3.10 below) on linked linear series applies in arbitrary characteristic, there is no visible obstruction to extending our proof to lower characteristics as well. However, key portions of the argument for the implications for the geometry of $\mathcal{M}_g$ were written using characteristic 0, and as far as we are aware no one has carefully analyzed which positive characteristics they may apply to, so it seems preferable to work in characteristic zero and we will assume this is the case from now on.

2. Nondegeneracy calculations

In this section, we study maps from elliptic curves to projective space determined by comparing values of tensor products of certain tuples of sections at two points P and Q. We will need two distinct results in this direction: first, we consider the situation that we let the point Q vary. This is already considered in [Liu et al. 2021], where we showed that these maps are morphisms, described them explicitly, and gave partial criteria for nondegeneracy. Here we extend the nondegeneracy criterion to a sharp statement for the case of tensor pairs. This is used to show that if we vary the location of the nodes on individual components, we can get possible linear dependencies to vary sufficiently nontrivially. Next, we will consider a new case, where Q is fixed, but we have a separate varying parameter. This situation was not considered in [Liu et al. 2021], but will be important to us in dealing with situations where the discrete data of the limit linear series does not fix the underlying line bundle in some components.

First, given a genus-1 curve C and distinct P, Q on C, and $c, d \geq 0$, let $\mathscr{L} = \mathscr{O}_C(cP + (d-c)Q)$. Then for any $a, b \geq 0$ with $a+b = d-1$, there is a unique section (up to scaling by $k^\times$) of $\mathscr{L}$ vanishing to order at least a at P and at least b at Q. Thus, we have a uniquely determined point R such that the divisor of the aforementioned section is $aP + bQ + R$; explicitly, R is determined by $aP + bQ + R \sim cP + (d-c)Q$, or

$$\begin{aligned} R \sim (c-a)P + (d-c-b)Q &= (c-a)P + (1+a-c)Q \\ &= P + (a+1-c)(Q-P) = Q + (a-c)(Q-P). \end{aligned} \tag{2-1}$$

We see that $R = P$ if and only if $Q - P$ is $|a+1-c|$-torsion, and $R = Q$ if and only if $Q - P$ is $|a-c|$-torsion. Note that (2-1) makes sense even when $Q = P$ (in which case $R = Q = P$), so we will use the formula for all P, Q, understanding that it has the initial interpretation as long as $Q \neq P$. To avoid trivial cases, we will assume that $a \neq c-1$, and $b \neq d-c-1$, or equivalently, $a+1-c \neq 0$, and $a - c \neq 0$.

Notation 2.1. Fix $P \in C$, $\ell \geq 1$, and for $j = 0, \dots, \ell$, set numbers $a_1^j, a_2^j, b_1^j, b_2^j$ satisfying for $i = 1, 2$, $j, j' \in \{0, \dots, \ell\}$,

$$a_i^j + b_i^j = d-1, \quad a_i^j - c \neq 0, -1, \quad a_1^j + a_2^j = a_1^{j'} + a_2^{j'}.$$

Let U be the open subset of C consisting of all Q such that $Q - P$ is not $|a_i^j - c|$-torsion or $|a_i^j + 1 - c|$-torsion for any i, j. For $Q \in U$, choose sections s_i^j with divisors

$$a_i^j P + b_i^j Q + R_i^{j,Q}.$$

Define $s^j = s_1^j \otimes s_2^j \in \Gamma(C, \mathscr{L}^{\otimes 2})$, and normalize the s^j, uniquely up to simultaneous scalar, so that their values at P are all the same. Considering $(s^0(Q), \dots, s^\ell(Q))$ gives a well-defined point of $\mathbb{P}^\ell$, denote by f_Q the point of $\mathbb{P}^\ell$ determined by $(s^0(Q), \dots, s^\ell(Q))$.

In [Liu et al. 2021] we showed that the map $U \to \mathbb{P}^\ell$ given by $Q \mapsto f_Q$ extends to a morphism $f : C \to \mathbb{P}^\ell$.

Extending Corollary 2.5 of [Liu et al. 2021], we have:

Proposition 2.2. *If all the a_i^j are distinct, and $a_1^j + a_2^j \neq 2c - 1$, then f is nondegenerate.*

The proof relies on reduction to a good understanding of the $\ell = 1$ case. Indeed, we can view our map as being given by $(1, f_1, \dots, f_\ell)$, where f_j is the rational function constructed from the quotient of sections s^j, s^0. Thus, nondegeneracy is equivalent to linear independence of the rational functions $1, f_1, \dots, f_\ell$, whose zeroes and poles are described explicitly by the following result, which combines Lemma 2.2 and Corollary 2.3 of [Liu et al. 2021]. Recall the following notation introduced in [Liu et al. 2021]. For k an integer, X a point in C and $L_1, \dots, L_{k^2}$ the line bundles in $\mathrm{Pic}^0(C)$ of order a divisor of $|k|$, $\mathcal{O}_C(X) \otimes L_i = \mathcal{O}_C(Y_i)$ for a unique $Y_i \in C$. Then, $X + T[k] := \sum_i Y_i$.

Lemma 2.3. *In the $\ell = 1$ case of Notation 2.1, the function $f : U \to k^\times$ given by $Q \mapsto (s^0/s^1)(Q)$ determines a rational function on C. We then have*

$$\operatorname{div} f = \sum_{i=1}^{2} \big((P + T[|a_i^0 - c|]) - (P + T[|a_i^1 - c|]) - (P + T[|a_i^0 + 1 - c|]) + (P + T[|a_i^1 + 1 - c|]) \big),$$

Moreover, f is nonconstant if and only if $a_1^j + a_2^j \neq 2c - 1$.

Proof of Proposition 2.2. By Lemma 2.3, $f_1, \dots, f_\ell$ are all nonconstant. By reindexing the pairs we may further assume

$$a_1^\ell < a_1^{\ell-1} < \cdots < a_1^0 \leq a_2^0 < a_2^1 < \cdots < a_2^\ell.$$

Let $n_i^j := |a_i^j - c + 1|$, $m_i^j := |a_i^j - c|$, and $n^j = \max\{n_1^j, n_2^j\}$.

A first observation is that $n^j > n^{j-1}$ for all j: if $a_1^{j-1} < c$ (respectively, $a_1^{j-1} > c$), then $n_1^{j-1} < n_1^j$ (respectively, $n_1^{j-1} < n_2^j$), and thus $n_1^{j-1} < n^j$; by a similar calculation, $n_2^{j-1} < n^j$; thus, $n^{j-1} < n^j$.

A second observation is that $n^j \geq \max\{m_1^0, m_2^0\}$ for all $j \geq 1$, and if equality is attained, j must be 1. Indeed, when $c < a_2^0$, we have $m_2^0 < m_2^j < n_2^j$ for all $j \geq 1$; meanwhile, either $m_1^0 \leq m_2^0$ (if $c < a_1^0$) or $m_1^0 \leq n_1^1 < n_1^j$ (if $c > a_1^0$) for all $j > 1$; thus, $\max\{m_1^0, m_2^0\} \leq n^j$ for all $j \geq 1$. When $c > a_2^0$, $m_2^0 \leq m_1^0 \leq n_1^1 < n_1^j$ for all $j > 1$, and hence the same conclusion holds.

Now, we claim that f_j has poles at the strict n_j-torsion points. Recalling from Lemma 2.3 that the poles of f_j are supported among the m_i^0- and n_i^j-torsion points for $i = 1, 2$, the above two observations show that $1, \dots, f_{j-1}$ cannot have any poles at strict n^j-torsion points, which immediately implies that $1, f_1, \dots, f_\ell$ are k-linearly independent. Thus, it suffices to prove the claim. Since the potential zeroes of f_j are supported among the m_i^j- and n_i^0-torsion points, we just need to show that n^j does not divide m_i^j or n_i^0 for $i = 1, 2$ and any $j \geq 1$. Moreover, we already know that $n^j > n^0 \geq n_i^0$, so it is enough to consider the m_i^j. We consider two cases.

Case 1: $c < a_2^0$, so that also $c < a_2^j$ for all j. In this case,

$$m_2^0 < n_2^0 \leq m_2^1 < n_2^1 \leq \cdots \leq m_2^\ell < n_2^\ell.$$

In particular, we have $n^j > m_2^j$, so it remains to compare n^j against m_1^j. If $n^j = n_1^j$, since n_1^j is always coprime to m_1^j, the claim follows instantly. If $n^j = n_2^j > n_1^j$, since $|n_1^j - m_1^j| = 1$, we have $n^j \geq m_1^j$. But equality cannot hold as it would imply that $a_1^j + a_2^j = 2c - 1$, which is ruled out by our assumption. So we conclude the claim in this case.

Case 2: $c > a_1^0$, so that $c > a_1^j$ for all j. If $a_2^j > c$, $n_2^j = m_2^j + 1$ and hence $n^j > m_2^j$. Meanwhile, $n_1^j = m_1^j - 1$. Similarly to the previous case, either $n^j > m_1^j$ or n^j is coprime to m_1^j, and the claim follows. If $a_2^j < c$, $n_2^j = m_2^j - 1$. Under our assumption, $n^j = n_1^j$ so is coprime to m_1^j. But because $j \geq 1$, we have $n_1^j \geq n_2^j + 2 = m_2^j + 1$, so $n_1^j > m_2^j$ and the claim follows. □

Notation 2.4. We now consider the point Q fixed, but the line bundle $\mathscr{L}$ varies (in particular, we do not have a c). As before, for $\ell \geq 1$, and $j = 0, \ldots, \ell$, set nonnegative integers $a_1^j, a_2^j, b_1^j, b_2^j$ satisfying

$$a_i^j + b_i^j = d - 1 \quad \text{for all } i, j; \qquad a_1^j + a_2^j \text{ is independent of } j.$$

Choose a point $R = R_1^0$. Then, for every $i = 1, 2,\ j = 0, \ldots, l$, there is a well determined R_i^j such that $\mathcal{O}(a_i^j P + b_i^j Q + R_i^j) = \mathcal{O}(a_1^0 P + b_1^0 Q + R) = \mathscr{L}$. Note that, using that $a_i^j + b_i^j = d - 1$, the last equation is equivalent to

$$R_i^j = R_1^0 + (a_1^0 - a_i^j)(P - Q).$$

We have sections s_i^j of $\mathscr{L}$ with divisors $a_i^j P + b_i^j Q + R_i^j$. We can take tensor products to obtain $s^j = s_1^j \otimes s_2^j$ and obtain sections of the line bundle $\mathscr{L}^{\otimes 2}$ having divisors $(a_1^j + a_2^j)P + (b_1^j + b_2^j)Q + R_1^j + R_2^j$. Note that the condition that $a_1^j + a_2^j$ is independent of j implies that the divisors $R_1^j + R_2^j$ will all be linearly equivalent to one another. If the a_i^j, b_i^j are generic, none of the R_i^j are equal to P and we can normalize the s^j to have the same value at P. We obtain a well-defined point $(s^0(Q), \ldots, s^\ell(Q)) \in \mathbb{P}^\ell$. But because we have said that $\mathscr{L}$ is uniquely determined by R_1^0, we can view this procedure as giving a rational map from C to $\mathbb{P}^\ell$, which we will now study. The argument will be similar to that of Lemma 2.2 and Corollary 2.5 of [Liu et al. 2021], but a bit simpler.

Proposition 2.5. *Suppose that $P - Q$ is not m-torsion for any $m \leq d$. Let $U \subseteq C$ be the open subset of points $R = R_1^0$ on which the map $\varphi : U \to \mathbb{P}^\ell$ that sends $R \in U$ to $(s^0(Q), \ldots, s^\ell(Q))$ is well defined. Then φ extends to a nondegenerate morphism $C \to \mathbb{P}^\ell$.*

Proof. We first consider the case $\ell = 1$, proving that we obtain a nonconstant rational function, and showing further that the divisor of this function is equal to

$$\begin{aligned} Q + (Q - (a_1^0 - a_2^0)(P - Q)) + (P - (a_1^0 - a_1^1)(P - Q)) + (P - (a_1^0 - a_2^1)(P - Q)) \\ - (Q - (a_1^0 - a_1^1)(P - Q)) - (Q - (a_1^0 - a_2^1)(P - Q)) - P - (P - (a_1^0 - a_2^0)(P - Q)). \end{aligned}$$

Let D_i^j be the divisor on $C \times C$ obtained as the graph of the morphism

$$R \mapsto R + (a_1^0 - a_i^j)(P - Q)$$

so that D_1^0 is simply the diagonal, and $(R_1^0, R_i^j) \in D_i^j$. Set

$$D^j = D_1^j + D_2^j + (P - (a_1^0 - a_1^{1-j})(P - Q)) \times C + (P - (a_1^0 - a_2^{1-j})(P - Q)) \times C$$

for $j = 0, 1$. Then we claim that D^0 and D^1 are linearly equivalent. By construction, if we restrict to $\{R\} \times C$ for any R not among the $P - (a_1^0 - a_i^j)(P - Q)$, we get that D^0 and D^1 are linearly equivalent, so $D^0 - D^1 \sim D \times C$ for some divisor D on C. But if we restrict to $C \times \{P\}$, we see that $D_1^j + D_2^j$ restricts to $(P - (a_1^0 - a_1^j)(P - Q)) + (P - (a_1^0 - a_2^j)(P - Q))$, so the restrictions of D^0 and D^1 are linearly equivalent on $C \times \{P\}$, and hence on $C \times C$, as desired. Moreover, this shows that if t_0 and t_1 are sections of the resulting line bundle having D^0 and D^1 as divisors, then $t_0|_{C\times\{P\}}$ has the same divisor as $t_1|_{C\times\{P\}}$, so we can scale so that t_0 and t_1 are equal on $C \times \{P\}$. We then see that our map $U \to \mathbb{P}^1$ is given by composing $R \mapsto (R, Q)$ with the rational function induced by our normalized choice of (t_0, t_1). Thus, it is a rational function, as desired. We compute its divisor simply by looking at the restrictions of D^0 and D^1 to $C \times \{Q\}$, which gives the claimed formula.

Now, for the case of arbitrary ℓ, we can consider the map to $\mathbb{P}^\ell$ to be given by a tuple of rational functions induced from the $\ell = 1$ case, specifically by $(f_0, \dots, f_{\ell-1}, 1)$, where f_j comes from looking at s^j and s^ℓ. To show nondegeneracy, it suffices to show that the f_j are linearly independent, which we do by showing that each of them (other than $f_\ell = 1$) has a pole which none of the others have. If we order so that

$$a_1^0 < a_1^1 < \cdots < a_1^\ell \le a_2^\ell < a_2^{\ell-1} < \cdots < a_2^0,$$

we see that $P - (a_1^j - a_2^j)(P - Q)$ occurs among the poles of f_j: indeed, given our nontorsion hypothesis on $P - Q$, the only positive term in the divisor which could possibly cancel it is Q, which would require $a_1^j - a_2^j = 1$, which is not possible with our above ordering. But again using our nontorsion hypothesis, and the fact that $a_2^j - a_1^j$ strictly decreases as j increases, we see that we obtain the desired distinct poles. □

3. Background on limit linear series and linked linear series

In this section we review background on limit linear series, as introduced by Eisenbud and Harris [1986], and on linked linear series, introduced by the second author [Osserman 2006] for two-component curves and generalized to arbitrary curves of compact type in [Osserman 2014].[2] Recall that a curve of *compact type* is a projective nodal curve such that every node is disconnecting, or equivalently, the dual graph is a tree. To streamline our presentation, we will largely restrict our attention to the situation of curves of compact type together with one-parameter smoothings.

Definition 3.1. Let X_0 be a curve of compact type, with dual graph Γ. Given $r, d \ge 0$, a *limit linear series* on X_0 of dimension r and degree d is a tuple $(\mathscr{L}^v, V^v)_{v \in V(\Gamma)}$, where each $(\mathscr{L}^v, V^v)$ is a linear series of dimension r and degree d on the component Z_v of X_0 corresponding to v. Write $a_\bullet^{(v,e)} = (a_0^{(v,e)}, \dots, a_r^{(v,e)})$ with $a_0^{(v,e)} < a_1^{(v,e)} < \cdots < a_r^{(v,e)}$ for the vanishing sequence (the $r+1$ different orders of vanishing of the sections in V^v) at P_e. The following condition must be satisfied: if Z_v and $Z_{v'}$ meet at a node P_e, then

$$a_j^{(v,e)} + a_{r-j}^{(v',e)} \ge d \quad \text{for } j = 0, \dots, r.$$

A limit linear series is said to be *refined* if the above inequalities are equalities for all e and j.

[2] In [Osserman 2006], linked linear series were called "limit linear series", but the name was changed subsequently to reduce confusion.

We now consider a one-parameter smoothing of X_0, as follows.

Remark 3.2. Suppose B is the spectrum of a discrete valuation ring with algebraically closed residue field, and $\pi : X \to B$ is flat and proper, with special fiber X_0 a curve of compact type, and smooth generic fiber X_η. Suppose further that the total space X is regular, that π admits a section.

Now, suppose we have a line bundle $\mathscr{L}_\eta$ generically — more precisely, we allow for the possibility that $\mathscr{L}_\eta$ is only defined after a finite extension of the base field of X_η. We can then take a finite base change $B' \to B$ so that $\mathscr{L}_\eta$ is defined over X'_η, and then X' may not be regular, but the line bundle $\mathscr{L}_\eta$ will still extend over X_0 because X_0 is of compact type. Moreover, there is a unique extension of $\mathscr{L}_\eta$ having any specified *multidegree* (i.e., tuple of degrees one for each component) adding up to d: because X was regular each component Z_v of X_0 is a Cartier divisor in X, and twisting by the $\mathscr{O}_X(Z_v)$ (or more precisely, their pullbacks to X') will increase the degree by 1 on each component meeting Z_v, and decrease the degree on Z_v correspondingly. For a multidegree ω, we denote this unique extension by $\widetilde{\mathscr{L}}_{\boldsymbol{\omega}}$. In particular, for each Z_v, we can consider the multidegree $\boldsymbol{\omega}^v$ which concentrates degree d on Z_v, and has degree 0 elsewhere.

Proposition 3.3 [Eisenbud and Harris 1986, Proposition 2.1]. *Given a linear series $(\mathscr{L}_\eta, V^v)$ on X'_η of dimension r and degree d, if we set $\mathscr{L}^v := (\widetilde{\mathscr{L}}_{\omega^v})|_{Z_v}$, and $V^v := (V_\eta \cap \Gamma(X', \widetilde{\mathscr{L}}_{\omega^v}))|_{Z_v}$, then the resulting tuple $(\mathscr{L}^v, V^v)_v$ is a limit linear series on X_0.*

Theorem 3.4 [Eisenbud and Harris 1986, Theorem 2.6]. *In characteristic* 0, *after finite base change and blowing up nodes in the special fiber, we may assume that the specialized limit linear series constructed by Proposition 3.3 is refined.*

Note that the only effect on X_0 of the base change and blowup is that chains of genus-0 curves are introduced at the nodes. Assuming we blow up to fully resolve the singularities resulting from the base change, these chain of curves have length equal to one less than the ramification index of the base change, so in particular they are the same at every node.

We now move on to linked linear series. The first observation is that if we have two multidegrees ω and ω', then there is a unique collection of nonnegative coefficients $c_v \in \mathbb{Z}$, not all positive, such that $\widetilde{\mathscr{L}}_\omega \cong \widetilde{\mathscr{L}}_{\omega'}(-\sum_v c_v Z_v)$. In this way, we obtain an inclusion $\widetilde{\mathscr{L}}_\omega \hookrightarrow \widetilde{\mathscr{L}}_{\omega'}$ which is defined uniquely up to scaling. If we define $\mathscr{L}_\omega := \widetilde{\mathscr{L}}_\omega|_{X_0}$, we get induced maps $\mathscr{L}_\omega \to \mathscr{L}_{\omega'}$ which are no longer injective, as they vanish identically on the components Z_v with $c_v > 0$. However, they are injective on the remaining components. Passing to global sections we obtain maps

$$f_{\omega,\omega'} : \Gamma(X_0, \mathscr{L}_\omega) \to \Gamma(X_0, \mathscr{L}_{\omega'}).$$

From the construction we see that $f_{\omega,\omega'} \circ f_{\omega',\omega}$ always vanishes identically. Although the twisted line bundles $\mathscr{L}_\omega$ can be described intrinsically on the special fiber, the maps $f_{\omega,\omega'}$ depend on the smoothing of X_0 whenever the locus on which they are nonvanishing. is disconnected.

To minimize notation, we will define linked linear series only in the above specialization context.

Definition 3.5. Given $\mathscr{L}_\eta$ of degree d and the induced tuple $(\mathscr{L}_\omega)_\omega$ of line bundles, a *linked linear series* of dimension r (and degree d) on the $\mathscr{L}_\omega$ is a tuple $(V_\omega)_\omega$ for all multidegrees of total degree d where each $V_\omega \subseteq \Gamma(X_0, \mathscr{L}_\omega)$ is an $(r+1)$-dimensional space of global sections, and for every ω, ω', we have

$$f_{\omega,\omega'}(V_\omega) \subseteq V_{\omega'}.$$

From the definitions and using Remark 3.2, we have:

Proposition 3.6. *Given $(\mathscr{L}_\eta, V_\eta)$, for all ω set $V_\omega = (V_\eta \cap \Gamma(X', \widetilde{\mathscr{L}}_\omega))|_{X_0}$. We obtain a linked linear series.*

This process is compatible with the Eisenbud–Harris specialization process, and the forgetful map commutes with specialization. The definition of linked series includes a linear series for every meaningful multidegree. In particular, there are linear series for the degrees ω^v which concentrate all the degree on Z_v. Ignoring the other multidegrees, we obtain a forgetful functor:

Theorem 3.7. *If $(V_\omega)_\omega$ is a linked linear series on $\mathscr{L}_\omega$, and we set $\mathscr{L}^v = \mathscr{L}_{\omega^v}|_{Z_v}$ and $V^v = V_{\omega^v}|_{Z_v}$ for all $v \in V(\Gamma)$, then $(\mathscr{L}^v, V^v)$ is a limit linear series. We will say that the linked linear series* ***lies over*** *the limit linear series*

This is explicitly stated (in the generality of higher-rank vector bundles) as part of Theorem 4.3.4 of [Osserman 2014], but is primarily a consequence of Lemma 4.1.6 of [loc. cit.].

In [Osserman 2014], the following notion is introduced:

Definition 3.8. A linked linear series is *simple* if there exist multidegrees $\omega_0, \dots, \omega_r$ and sections $s_j \in \Gamma(X_0, \mathscr{L}_{\omega_j})$ such that for every ω, the $f_{\omega_j,\omega}(s_j)$ form a basis of V_ω.

The simple linked linear series form an open subset, and are particularly easy to understand (hence the name). However, we will be forced to consider more general linked linear series arising under specialization. We therefore introduce the following open subset, originally introduced in [Osserman 2006] in the two-component case.

Definition 3.9. A linked linear series is *exact* if for every multidegree ω, and every proper subset $S \subseteq V(\Gamma)$, if $\mathscr{L}_{\omega'} \cong \mathscr{L}_\omega(-\sum_{v \in S} Z_v)$, then

$$f_{\omega,\omega'}(V_\omega) = V_{\omega'} \cap \ker f_{\omega',\omega}.$$

An important special case in the definition, and the only one which we will use in the present paper, is that ω' is obtained from ω by decreasing the degree by 1 on a single component and increasing it correspondingly on an adjacent component.

While we cannot always ensure our linked linear series are simple, we can ensure they are exact:

Proposition 3.10. *If $(\mathscr{L}_\eta, V_\eta)$ is defined over X_η itself, then the resulting linked linear series is exact.*

The proof is exactly the same as in the two-component case, which is explained immediately before the statement of Theorem 5.2 of [Esteves and Osserman 2013]. Thus, even if $(\mathscr{L}_\eta, V_\eta)$ is not defined over X_η, we can take a finite base change to make it defined, and blow up the resulting singularities of the total space to put ourselves into position to apply Proposition 3.10.

4. Degenerate linked linear series

The purpose of this section is to analyze the structures of the possible exact linked linear series lying over limit linear series mostly when $\rho \le 2$. We will restrict our attention to the case that the reducible curve X_0 is a chain.

Definition 4.1. Let $Z_1, \dots, Z_N$ be smooth curves with distinct marked points P_i, Q_i on each Z_i. Construct X_0 by gluing Q_i to P_{i+1} for each $i = 1, \dots, N-1$. Fix a total degree which in this section, we will denote with d. Given $w = (c_2, \dots, c_N)$, define the *multidegree of a line bundle on X_0 associated to w*,

$$\mathrm{md}_d(w) = (d_1, \dots, d_N), \qquad \text{by} \qquad d_1 = c_2, \quad d_i = c_{i+1} - c_i, \quad i = 2, \dots, N-1, \quad d_N = d - c_N.$$

Note that conversely, given a multidegree ω with total degree d, there is a unique w such that $\omega = \mathrm{md}(w)$. We will assume that $0 \le c_i \le c_{i+1} \le d$ for all i.

The notation c_i is very helpful in connection with the way in which we encode the combinatorial data of a limit linear series. In order to avoid treating the end points separately, it will be convenient to use the convention that $c_1 = 0$ and $c_{N+1} = d$. To avoid notational clutter, we will frequently write simply $\mathrm{md}(w)$ when the total degree is clear, and we will abbreviate $\mathscr{L}_{\mathrm{md}(w)}$ by $\mathscr{L}_w$, $f_{\mathrm{md}(w),\mathrm{md}(w')}$ by $f_{w,w'}$, and so fourth. The assumption that $0 \le c_i \le d$ for all i guarantees that the map f_{w,w^i} is injective on the component Z_i (see Proposition 3.6 of [Liu et al. 2021]), so we can understand sections in multidegree w as being glued from the Z_i-parts of sections in the multidegrees w^i.

From Remark 3.2, given a limit linear series $(\mathscr{L}^i, \mathscr{V}^i)_{i=1,\dots,N}$ and the multidegree associated to a w, then $\mathscr{L}_w|_{Z_i}$ is obtained from $\mathscr{L}^i$ by twisting down by $c_i P_i + (d - c_{i+1}) Q_i$, leaving degree $d - c_i - (d - c_{i+1}) = c_{i+1} - c_i$. Therefore, the condition $c_i \le c_{i+1}$ is equivalent to the d_i being positive.

Notation 4.2. By construction, the components of X_0 are ordered from 1 to N. We will think of a horizontal representation of the curve, numbering the components from left to right. For example, when we talk of the curve "strictly left of i", we mean $\bigcup_{j<i} Z_j$.

We first describe the behavior of the maps $f_{w,w'}$ under the above encoding (See Proposition 3.6 of [Liu et al. 2021] for a proof):

Proposition 4.3. *Given $w = (c_2, \dots, c_N)$, $w' = (c'_2, \dots, c'_N)$ and total degree d, the map $\mathscr{L}_{w'} \to \mathscr{L}_w$ vanishes identically on the component Z_i if and only if*

$$\sum_{j=i+1}^{N} (c'_j - c_j) > \min_{1 \le i' \le N} \sum_{j=i'+1}^{N} (c'_j - c_j).$$

In particular, if $c'_i < c_i$ or $c'_{i+1} > c_{i+1}$ then the map vanishes identically on Z_i, and if $c'_i = c_i$ for $i > 1$, then the map vanishes identically on Z_i if and only if it vanishes identically on Z_{i-1}.

Proposition 4.4. *Let Z be a smooth projective curve, and $P, Q \in Z$ distinct. Let $(\mathscr{L}, V)$ be a $\mathfrak{g}^r_d$ on Z. Then there is a unique (unordered) set of pairs $(a_0, b_0), \dots, (a_r, b_r)$ with all a_j distinct and all b_j distinct such that there exists a basis $s_0, \dots, s_r$ of V with* $\operatorname{ord}_P s_j = a_j$ *and* $\operatorname{ord}_Q s_j = b_j$ *for $j = 0, \dots, r$.*

Proof. Start with a basis $s_0, \dots, s_r$ with vanishing $a_0 < \dots < a_r$ at P. Add multiples of the s_i to s_0 to maximize vanishing at Q. Then repeat the process replacing s_1, by adding multiples of the s_j, $j \geq 2$.

Note that the s_j themselves are not unique, although a given s_j can be modified only by adding multiples of $s_{j'}$ which simultaneously satisfy $\operatorname{ord}_P s_{j'} > \operatorname{ord}_P s_j$ and $\operatorname{ord}_Q s_{j'} > \operatorname{ord}_Q s_j$. □

To each refined limit linear series, we can associate a table of numbers as follows:

Definition 4.5. Let $(\mathscr{L}^i, V^i)$ be a refined limit $\mathfrak{g}^r_d$ on X_0, and for each i let $(a^i_j, b^i_j)_j$ be the set of pairs given by Proposition 4.4. Construct the $(r+1) \times N$ *table* T' from left to right, with the i-th column of T' consisting of the pairs (a^i_j, b^i_j) for $j = 0, \dots, r$, and the ordering of each column determined as follows: a^1_j should be strictly increasing, and for $i > 1$ and each j, we require $a^i_j = d - b^{i-1}_j$. For fixed i, we refer to the a^i_j and the b^i_j as making up the *subcolumns* of the i-th column of T'. For each j, let $w_j = (a^2_j, \dots, a^N_j)$, and set $\omega_j = \operatorname{md}_d(w_j)$.

Example 4.6. Let X_0 be a chain of 5 elliptic curves. Construct a limit linear series on X_0 of degree 4 and dimension 1 with the following line bundles on the components:

$$L_1 = \mathcal{O}(4Q_1), \quad L_2 = \mathcal{O}(2P_2 + 2Q_2), \quad L_3 = \mathcal{O}(P_3 + 3Q_3), \quad L_4 \text{ generic}, \quad L_5 = \mathcal{O}(4P_5).$$

and sections with vanishing associated to the table

$$\begin{array}{cc|cc|cc|cc|cc} 0 & 4 & 0 & 3 & 1 & 3 & 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 2 & 2 & 1 & 3 & 0 & 4 & 0 \end{array}$$

The table has two rows corresponding to the two sections. The five columns correspond to the 5 elliptic curves with the left and right semicolumns corresponding to the vanishing at P_i and Q_i, respectively. There are two w_i one for each of the two sections and left semicolumns, starting with the second one and corresponding multidegrees ω_j as follows:

$$w_0 = (0, 1, 1, 2), \quad \omega_0 = (0, 1, 0, 1, 2), \quad w_1 = (2, 2, 3, 4), \quad \omega_1 = (2, 0, 1, 1, 0).$$

For instance $\mathscr{L}_{w_0}$ is a line bundle on the chain with restrictions to the 5 components

$$L^0_1 = \mathcal{O}, \quad L^0_2 = \mathcal{O}(2P_2 - Q_2), \quad L^0_3 = \mathcal{O}, \quad L^0_4 = L_4(-P_4 - 2Q_4), \quad L^0_5 = \mathcal{O}(2P_5),$$

while $\mathscr{L}_{w_1}$ has restrictions to the 5 components

$$L^0_1 = \mathcal{O}(2Q_1), \quad L^0_2 = \mathcal{O}, \quad L^0_3 = \mathcal{O}(2Q_3 - P_3), \quad L^0_4 = L_4(-3P_4), \quad L^0_5 = \mathcal{O}.$$

Note that the set of pairs of Proposition 4.4 is giving a relative ordering of the vanishing sequences at P and Q, so the condition that the limit linear series is refined means that we can always impose that $a^i_j = d - b^{i-1}_j$. Arranging our table ordering in this way, we can always choose sections $s^i_j \in V^i$ such that

$\mathrm{ord}_{P_i} s^i_j = a^i_j$ and $\mathrm{ord}_{Q_i} s^i_j = b^i_j$. Then in multidegree ω_j there is a unique section s_j obtained from gluing together the s^i_j (although as noted above, the choices of s^i_j are not unique in general).

Definition 4.7. We say that a *swap* occurs in column i between rows j, j' if $a^i_j < a^i_{j'}$ and $b^i_j < b^i_{j'}$ or if $a^i_j > a^i_{j'}$ and $b^i_j > b^i_{j'}$. A swap is *minimal* if further $|a^i_j - a^i_{j'}| = |b^i_j - b^i_{j'}| = 1$ and either $a^i_j + b^i_j = d$ or $a^i_{j'} + b^i_{j'} = d$.

Example 4.8. If X_0 is again a chain of 5 elliptic curves, construct a limit linear series on X_0 of degree 4 and dimension 1 with line bundles and table of vanishing

$$L_1 = \mathcal{O}(4Q_1), \quad L_2 = \mathcal{O}(2P_2+2Q_2), \quad L_3 = \mathcal{O}(2P_3+2Q_3), \quad L_4 = \mathcal{O}(2P_4+2Q_4), \quad L_5 = \mathcal{O}(4P_5),$$

$$\begin{array}{cc|cc|cc|cc|cc} 0 & 4 & 0 & 3 & 1 & 1 & 3 & 0 & 4 & 0 \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \end{array}$$

A swap appears on C_3 between the only two sections on the linear series s_0, s_1. This swap is minimal as $|a^3_1 - a^3_0| = 2 - 1 = |b^3_1 - b^3_0|$ is 1 and $a^3_1 + b^3_1 = 2 + 2 = 4 = d$.

A limit linear series is *chain-adaptable* in the sense of [Osserman 2014] if there are no swaps in the table T'. For a chain-adaptable limit linear series, there is only one linked linear series lying over it that is simple, generated by the s_j described above. In the nonchain-adaptable case, the linked linear series is not necessarily unique.

A nonempty open subset of the set of possible linked linear series will always be simple, generated by sections similar to the s_j described above. However, even some exact linked linear series are not simple. We can nonetheless use exactness to obtain fairly good control over what these linked linear series look like. We address all the cases that can arise for $\rho \le 2$ below.

We will use the following observation: Fix a refined limit linear series and a choice of all the s^i_j. For any $w = (c_2, \dots, c_N)$ (assumed bounded), the linkage condition implies that the sections in the $(r+1)$-dimensional space V_w in the linked linear series are linear combinations of sections obtained by gluing, for a fixed j, the sections s^i_j to one another as i varies, where each s^i_j that appears must satisfy $a^i_j \ge c_i$ and $b^i_j \ge d - c_{i+1}$, and if the first (respectively, second) inequality is an equality, we must also have s^{i-1}_j (respectively, s^{i+1}_j) included in the gluing. Indeed, a section in V_w must be a linear combination of such s^i_j, and since the a^i_j and b^i_j are all distinct for fixed i, at most one can have equality on each side, leading to the desired form for the gluing.

Proposition 4.9. *Suppose that the j_0-th row of T' has the property that for all $j < j_0$ we have $b^i_j > b^i_{j_0}$ for $i = 1, \dots, N-1$, and for all $j > j_0$ we have $a^i_j > a^i_{j_0}$ for $i = 2, \dots, N$. Then any linked linear series lying over the given limit linear series* (*in the sense of Theorem 3.7*) *contains the expected section s_{j_0}.*

Proof. It suffices to see that the space of global sections in multidegree ω_{j_0} obtained from all possible gluings of the s^i_j has dimension exactly $r+1$, so that any linked linear series must contain the whole space, including s_{j_0}. But for $j < j_0$ since $b^i_j > b^i_{j_0}$ for $i < N$, we have $a^{i+1}_j < a^{i+1}_{j_0}$, so s^{i+1}_j cannot appear at all in multidegree ω_{j_0}. Thus, only s^1_j can appear, glued to the zero section on every other component. Similarly, for $j > j_0$ only s^N_j can appear. And since each $s^i_{j_0}$ has precisely the desired vanishing at the nodes, s_{j_0} is the unique way to glue them together, so we obtain an $(r+1)$-dimensional space in total, as desired. □

In this paper we consider mostly spaces of linear series with Brill–Noether number $\rho = 1, 2$. We will see in Proposition 5.3 that on a generic chain of elliptic curves, the number of swaps is bounded by ρ. So, we now systematically consider all cases where the limit linear series has only one or two swaps.

Proposition 4.10. *Suppose that a limit linear series has a single swap occurring in the i_0-th column between the (j_0-1)-st and j_0-th rows, Then any linked linear series lying over that limit linear series (in the sense of Theorem 3.7) contains the expected section s_{j_0-1}. The spaces of sections of the linear series with multidegrees associated to*

$$(a^2_{j_0-1}, \dots, a^{i_0}_{j_0-1}, a^{i_0+1}_{j_0}, \dots, a^N_{j_0}) \quad \textit{and} \quad (a^2_{j_0}, \dots, a^{i_0}_{j_0}, a^{i_0+1}_{j_0-1}, \dots, a^N_{j_0-1})$$

contain the respective images of the section s_{j_0}. These images consist of 0 on the first $i_0 - 1$ components and $s^i_{j_0}$ for $i = i_0, \dots, N$, and of $s^i_{j_0}$ for $i = 1, \dots, i_0$ and 0 on the last $N - i_0$ components, respectively.

Given $w = (c_2, \dots, c_N)$. If $c_i < a^i_{j_0-1}, a^i_{j_0}$ for all i, the linked linear series contains $s^1_{j_0}$ in multidegree $\mathrm{md}(w)$, and if $c_i > a^i_{j_0-1}, a^i_{j_0}$ for all i, the linked linear series contains $s^N_{j_0}$ in multidegree $\mathrm{md}(w)$ (in both cases, glued to 0 on the other components).

Proof. Note that our assumptions imply that

$$b^i_j > b^i_{j_0-1}, b^i_{j_0}, \quad \text{for all } j < j_0-1,\ i = 1, \dots, N-1, \qquad a^i_j > a^i_{j_0}, a^i_{j_0-1} \quad \text{for all } j > j_0,\ i = 2, \dots, N.$$

In multidegree ω_{j_0-1}, as in the proof of Proposition 4.9, the s^i_j for $j \neq j_0 - 1, j_0$ can only contribute for $i = 1$ (if $j < j_0 - 1$) or $i = N$ (if $j > j_0$), and the $s^i_{j_0-1}$ glue uniquely to give s_{j_0-1}. Finally, the $s^i_{j_0}$ can only contribute at $i = i_0$, so we find that the space obtained from all the s^i_j is $(r + 1)$-dimensional, and s_{j_0-1} must be in the linked linear series, as desired.

Next, consider $w' = (a^2_{j_0-1}, \dots, a^{i_0}_{j_0-1}, a^{i_0+1}_{j_0}, \dots, a^N_{j_0})$. Note that $f_{w_{j_0},w'}(s_{j_0})$ is equal to s_{j_0} from i_0 to N (inclusive), and 0 strictly before i_0. We claim that the space of possible sections from the s^i_j in multidegree $\mathrm{md}(w')$ is precisely $(r+1)$-dimensional, so the linked linear series is uniquely determined in this multidegree. By hypothesis, the s^i_j for $j < j_0 - 1$ can only contribute for $i = 1$, and the s^i_j for $j > j_0$ can only contribute for $i = N$. The $s^i_{j_0-1}$ could in principle contribute for $i < i_0$ and $i = N$, but if the $s^i_{j_0-1}$ gave rise to nonzero sections for $i < i_0$, they all would be nonvanishing at the relevant nodes, and they would have to glue to something nonvanishing in the i_0-th column. This would have to be $s^{i_0}_{j_0-1}$, which does not have enough vanishing on the right to appear in multidegree $\mathrm{md}(w')$. Thus, we conclude that the $s^i_{j_0-1}$ can only appear for $i = N$ (where it is glued to the zero section on all other columns). Finally, the $s^i_{j_0}$ can only appear for $i \geq i_0$, where they are nonzero at all interior nodes, and therefore have a unique gluing, which must yield $f_{w_{j_0},w'}(s_{j_0})$. Thus we get the claimed dimension $r + 1$, and conclude that $f_{w_{j_0},w'}(s_{j_0})$ is contained in the linked linear series.

Similarly, if $w'' = (a^2_{j_0}, \dots, a^{i_0}_{j_0}, a^{i_0+1}_{j_0-1}, \dots, a^N_{j_0-1})$, we find that space of possible sections is $(r+1)$-dimensional, and contains $f_{w_{j_0},w''}(s_{j_0})$.

Now, suppose we are given w with $c_i < a^i_{j_0-1}, a^i_{j_0}$ for all i. Then Proposition 4.3 implies that $f_{w'',w}$ is nonzero precisely on the first component, so $f_{w'',w}(f_{w_{j_0},w''}(s_{j_0}))$ is equal to $s^1_{j_0}$ glued to 0, as desired. The situation with $c_i > a^i_{j_0-1}, a^i_{j_0}$ is similar, but with w' in place of w''. □

Example 4.11. In Example 4.8, $(a^2_{j_0-1}, \dots, a^{i_0}_{j_0-1}, a^{i_0+1}_{j_0}, \dots, a^N_{j_0}) = (0, 1, 2, 2)$. This numbers give the required vanishing at P_i, $i = 2, \dots, 5$, while the vanishing at $Q_i = d - c_{i+1} = 4 - c_{i+1}$. This means that the required vanishing is

$$4 \text{ at } Q_1, \quad 0 \text{ at } P_2, \quad 3 \text{ at } Q_2, \quad 1 \text{ at } P_3, \quad 2 \text{ at } Q_3, \quad 2 \text{ at } P_4, \quad 2 \text{ at } Q_4, \quad 2 \text{ at } P_5.$$

The order of vanishing of s_1 at the nodes is 2 at P_3, 2 at Q_3, 2 at P_4, 2 at Q_4, 2 at P_5. As the order at P_3 is 2, strictly greater than the required 1, it can be glued to the zero section in the first two components to give rise to a section of the linked linear series. Similarly, $(a^2_{j_0}, \dots, a^{i_0}_{j_0}, a^{i_0+1}_{j_0-1}, \dots, a^N_{j_0-1}) = (2, 2, 3, 1)$. The required vanishing is then

$$2 \text{ at } Q_1, \quad 2 \text{ at } P_2, \quad 2 \text{ at } Q_2, \quad 2 \text{ at } P_3, \quad 1 \text{ at } Q_3, \quad 3 \text{ at } P_4, \quad 0 \text{ at } Q_4, \quad 4 \text{ at } P_5.$$

The order of vanishing of s_1 at the nodes is 2 at Q_1, 2 at P_2, 2 at Q_2, 2 at P_3, 2 at Q_3. As the order at Q_3 is 1, strictly greater than the required 0, it can be glued to the zero section in the last two components to give rise to a section of the linked linear series.

When the hypotheses of Proposition 4.9 are not satisfied for every j_0, there may be linked linear series — even exact ones — which do not contain all of the s_{j_0}. They may occur as specializations of linear series on the generic fiber. This leads us to introduce the notion of mixed sections below. We will then show that there are mixed sections of rather precise forms, which can in some sense take the place of the missing s_j.

Definition 4.12. For $\ell > 1$, let $\vec{S} = (S_1, \dots, S_\ell)$ be a tuple of subsets of $\{1, \dots, N\}$ (some of which may be empty) such that for all pairs $i < i'$, every element of S_i is less than or equal to every element of $S_{i'}$ and such that every element of $\{1, \dots, g\}$ is contained in some S_i. Let $\vec{j} = (j_1, \dots, j_\ell)$ be a tuple of elements of $\{0, \dots, r\}$, possibly with repetitions. Then given a fixed limit linear series and corresponding choices of the s^i_j, a *mixed section* of type $(\vec{S}, \vec{j})$ is a w and a section s in multidegree $\text{md}(w)$ which is a sum from $i = 1$ to ℓ of sections obtained by gluing $s^{i'}_{j_i}$ for all $i' \in S_i$ to the zero section on other components.

Proposition 4.13. *Suppose that a limit linear series has precisely one swap, between the j_0-th and (j_0-1)-st rows occurring in the i_0-th column. Then any linked linear series lying over the given limit linear series contains the expected sections s_j for all $j \neq j_0$. If the linked linear series is exact, then it must contain mixed sections s'_{j_0} of type $((S'_1, S'_2), (j_0 - 1, j_0))$ with S'_1 supported on $\bigcup_{i<i_0} Z_i$ and s''_{j_0} of type $((S''_1, S''_2), (j_0, j_0 - 1))$ with S''_2 supported on $\bigcup_{i>i_0} Z_i$.*

Note that this allows for the linked linear series to contain the section s_{j_0} itself when S'_1 and S''_2 are both empty.

Proof. Start with the $w' = (a^2_{j_0-1}, \dots, a^{i_0}_{j_0-1}, a^{i_0+1}_{j_0}, \dots, a^N_{j_0})$ as in Proposition 4.10. Note that if $i_0 = 1$, then the proposition says that s_{j_0} itself is in our linked linear series, consistent with the stated form for s'_{j_0}.

If $i_0 > 1$, we consider iteratively changing w' by increasing the twists by 1 for $i' \leq i_0$ (starting at i_0) until they each agree with $a_{j_0}^{i'}$. We note that every such modified w' has an $(r+2)$-dimensional space of global sections obtained from the s_j^i, described explicitly as follows: s_j^1 for $j < j_0 - 1$; s_j^N for $j > j_0$; $s_{j_0-1}^N$; a section obtained by gluing the $s_{j_0-1}^i$ for i from 1 to $i'-1$ (which is the last column in which w' agrees with $a_{j_0-1}^i$); and a section obtained by gluing the $s_{j_0}^i$ from either $i'-1$ or i' to N, beginning with the last column in which w' has coefficient strictly less than $a_{j_0}^i$. For each $j \neq j_0 - 1$, since there is a unique section constructed from the s_j^i, it is necessarily equal to $f_{w_j,w'}(s_j)$. In addition, since we know s_j is in the linked linear series for $j \neq j_0$, then $f_{w_j,w'}(s_j)$ is necessarily contained in the linked linear series for $j \neq j_0 - 1, j_0$.

Now, suppose that the linked linear series contained $f_{w_{j_0},w'}(s_{j_0})$ for the old w'; we claim that it either also contains it for the new w', or contains a section of the form desired for s'_{j_0}. Indeed, increasing the twist in the i-th column corresponding to twisting once by every component from i to N., we observe that $f_{w_{j_0},w'}(s_{j_0})$ is in the kernel of the map from the old w' to the new one. By definition of exactness, the linked linear series must contain some s in the new multidegree mapping to $f_{w_{j_0},w'}(s_{j_0})$ in the old one. Using the above description of the space of global sections, this is necessarily a combination of the $f_{w_j,w'}(s_j)$ for $j < j_0 - 1$ and $j = j_0$, together with the section from the $s_{j_0-1}^i$ for $i = 1$ to $i'-1$. Moreover, since we observed above that $f_{w_j,w'}(s_j)$ is contained in the linked linear series for $j < j_0 - 1$, we can subtract these off to obtain a combination of the sections from the $j_0 - 1$ and j_0 rows. If the $j_0 - 1$ term vanishes, we have that $f_{w_{j_0},w'}(s_{j_0})$ is contained in our linked linear series for the new w', and if the $j_0 - 1$ term is nonzero, we have something of the desired form for s'_{j_0} (with the minimal element of S'_2 being either i' or $i'-1$ according to where $f_{w_{j_0},w'}(s_{j_0})$ begins), as claimed. Iterating this process, we either obtain the desired s'_{j_0}, or we eventually reach $w' = w_{j_0}$ and find that the linked linear series actually contains s_{j_0} itself.

As the situation is completely symmetric, the construction of s''_{j_0} is similar, starting from the multidegree w'' from the proof of Proposition 4.10. □

When $\rho = 2$, there are four additional types of swap (see Definition 4.7), which we consider one by one. They all involve having exactly two swaps, occurring in distinct columns. The first case is when the swaps occur in disjoint pairs of rows.

Proposition 4.14 ("disjoint swap"). *Suppose that a limit linear series contains precisely two swaps, and these occur in disjoint pairs of rows, say $j_0 - 1$, j_0 and $j_1 - 1$, j_1. Then any linked linear series lying over the given limit linear series contains the expected sections s_j for all $j \neq j_0, j_1$. If the linked linear series is exact, then for $\ell = 0, 1$ it must contain mixed sections s'_{j_ℓ} of type $((S'_{1+2\ell}, S'_{2+2\ell}), (j_\ell - 1, j_\ell))$ with $S'_{1+2\ell}$ supported strictly left of i_ℓ and s''_{j_ℓ} of type $((S''_{1+2\ell}, S''_{2+2\ell}), (j_\ell, j_\ell - 1))$ with $S''_{2+2\ell}$ supported strictly right of i_ℓ.*

Proof. This is essentially identical to the proof of Proposition 4.13. The only new point which arises is that in constructing the sections s'_{j_0}, s''_{j_0}, we need to know that we can always subtract off any s_{j_1} part which arises in the iterative procedure, and similarly with j_0 and j_1 switched. But this follows from the last assertion of Proposition 4.10. □

Example 4.15. Assume that X_0 is a chain of 6 elliptic curves. One can construct a limit linear series on X_0 of degree 8 and dimension 3 by choosing the following line bundles on each component:

$$\mathcal{O}(8Q_1),\quad \mathcal{O}(2P_2+6Q_2),\quad \mathcal{O}(2P_3+6Q_3),\quad \mathcal{O}(6P_4+2Q_4),\quad \mathcal{O}(6P_5+2Q_5), \mathcal{O}(8P_6),$$

and sections with associated table

0	8	0	7	1	5	3	4	4	3	5	2
1	6	2	6	2	6	2	5	3	4	4	3
2	5	3	4	4	3	5	1	7	0	8	0
3	4	4	3	5	2	6	2	6	2	6	1

A swap appears on C_3 between s_0, s_1 and another at C_4 between s_2, s_3. Both swaps are minimal as required as there are the maximum number possible (two) of swaps for $\rho = 2$. In general there may be larger gaps between the rows where the swap occurs and between the columns where it occurs as well.

The next case is that a single pair of rows can undergo two swaps in different columns.

Proposition 4.16 ("repeated swap"). *Suppose that the limit linear series has precisely two swaps, both between the j_0-th and (j_0-1)-st rows, with the first occurring in the i_0-th column, and the second in the i_1-st column for some $i_1 > i_0$. Then any exact linked linear series lying over the given limit linear series contains mixed sections s'_{j_0-1} of type $((S'_1, S'_2, S'_3), (j_0-1, j_0, j_0-1))$ with S'_2 supported strictly left of i_1 and s''_{j_0-1} of type $((S''_1, S''_2), (j_0-1, j_0))$ with S''_2 supported strictly right of i_1, and it contains mixed sections s'_{j_0} of type $((S'_4, S'_5), (j_0-1, j_0))$ with S'_4 supported strictly left of i_0 and s''_{j_0} of type $((S''_3, S''_4, S''_5), (j_0, j_0-1, j_0))$ with S''_4 supported strictly right of i_0.*

Proof. The proof is similar to the proof of Proposition 4.13. For s'_{j_0-1}, we first consider

$$w' = (a^2_{j_0-1}, \ldots, a^{i_0}_{j_0-1}, a^{i_0+1}_{j_0}, \ldots, a^{i_1}_{j_0}, a^{i_1+1}_{j_0-1}, \ldots, a^N_{j_0-1}).$$

Note that $f_{w_{j_0-1},w'}(s_{j_0-1})$ is equal to s_{j_0-1} from i_1 to N (inclusive), and 0 elsewhere. Indeed, these are the only columns in which the $s^i_{j_0-1}$ can be supported, since they do not satisfy the correct inequalities from i_0 to i_1-1, and for $i < i_0$ they satisfy them with equality, so would have to be glued to a nonzero element in the i_0-th column. As in the proof of Proposition 4.9, we check that we have dimension exactly $r+1$ in multidegree $\mathrm{md}(w')$, with the unique contribution from the j_0 row coming from $s^N_{j_0}$. Thus, we find that $f_{w_{j_0-1},w'}(s_{j_0-1})$ is necessarily contained in multidegree $\mathrm{md}(w')$.

We then iterate changing w' by 1, increasing the twist by 1 in the i'-th column for $i' \le i_1$ to change them from $a^{i'}_{j_0}$ to $a^{i'}_{j_0-1}$. Using exactness, at each stage we either find the linked linear series still contains $f_{w_{j_0-1},w'}(s_{j_0-1})$ for the new value of w', or it contains the sum of $f_{w_{j_0-1},w'}(s_{j_0-1})$ with a section obtained by gluing the $s^i_{j_0}$ for $i = i_0, \ldots, i'-1$. In the first case, we continue to iterate the process of changing w', and if we do not ever get the second case, we end up with s_{j_0-1} itself in our linked linear series. On the other hand, once the second case occurs, we begin to iteratively change w' by increasing the twist by 1 in the i'-th column for $i' \le i_0$ to change them from $a^{i'}_{j_0-1}$ to $a^{i'}_{j_0}$. Each time the twist increases above $a^{i'}_{j_0-1}$, we could obtain a contribution obtained from gluing $s^i_{j_0-1}$ from $i = 1$ to $i'-1$, and if this occurs, we get our desired s'_{j_0-1}. Otherwise, we keep iterating, and each time the twist at i' reaches $a^{i'}_{j_0}$, the portion of the section obtained from the $s^i_{j_0}$ extends to include $i'-1$. Again, if we never get a contribution from the $s^i_{j_0-1}$ for $i \leq i'$, we will end up with a section as required for s'_{j_0-1}, having $S'_1 = \varnothing$.

The construction of s''_{j_0-1} is similar, but simpler: we set our initial $w'' = (a^2_{j_0-1}, \dots, a^{i_1}_{j_0-1}, a^{i_1+1}_{j_0}, \dots, a^N_{j_0})$, and then we iteratively decrease the twists for $i' > i_1$ by 1 to change them from $a^{i'}_{j_0}$ to $a^{i'}_{j_0-1}$, until we obtain the desired result.

The construction of s'_{j_0} and s''_{j_0} follows the same process. For s'_{j_0}, we start with

$$w' = (a^2_{j_0-1}, \dots, a^{i_0}_{j_0-1}, a^{i_0+1}_{j_0}, \dots, a^N_{j_0}),$$

and we iteratively increase the twists for $i' \leq i_0$ by 1 to change them from $a^{i'}_{j_0-1}$ to $a^{i'}_{j_0}$. Finally, for s''_{j_0}, we start with $w'' = (a^2_{j_0}, \dots, a^{i_0}_{j_0}, a^{i_0+1}_{j_0-1}, \dots, a^{i_1}_{j_0-1}, a^{i_1+1}_{j_0}, \dots, a^N_{j_0})$, obtaining a section glued from the $s^i_{j_0}$ for $i \leq i_0$. We iteratively decrease the twists for $i' > i_0$ by 1 to change them from $a^{i'}_{j_0-1}$ to $a^{i'}_{j_0}$, until we obtain a contribution from the $a^i_{j_0-1}$ (necessarily ending at i_1), and then we iteratively decrease the twists for $i' > i_1$ by 1 to change them from $a^{i'}_{j_0}$ to $a^{i'}_{j_0-1}$, eventually obtaining either s_{j_0} itself, or the desired s''_{j_0}. □

Example 4.17. Assume that X_0 is a chain of 6 elliptic curves. One can construct a limit linear series on X_0 of degree 5 and dimension 1 by choosing the following line bundles on each component,

$$\mathcal{O}(5Q_1), \quad \mathcal{O}(2P_2+3Q_2), \quad \mathcal{O}(2P_3+3Q_3), \quad \mathcal{O}(3P_4+2Q_4), \quad \mathcal{O}(3P_5+2Q_5), \quad \mathcal{O}(5P_6);$$

a table associated to this limit linear series is

$$\begin{array}{cc|cc|cc|cc|cc|cc} 0 & 5 & 0 & 4 & 1 & 2 & 3 & 2 & 3 & 2 & 3 & 1 \\ 1 & 3 & 2 & 3 & 2 & 3 & 2 & 1 & 4 & 0 & 5 & 0 \end{array}$$

A swap appears on C_3 and again at C_4 between the only two sections on the linear series s_0, s_1. Both swap are minimal as required as there are the maximum number possible (two) of swaps for $\rho = 2$

Finally, three consecutive rows can undergo two swaps. This can happen in two different ways.

Proposition 4.18 ("first 3-cycle"). *Suppose that the limit linear series has one swap between the j_0-th and (j_0+1)-st rows occurring in the i_0-th column, and a second swap between the (j_0-1)-st and (j_0+1)-st rows in the i_1-st column for some $i_1 > i_0$, and no other swaps. Then any linked linear series lying over the given limit linear series contains s_{j_0-1} and s_{j_0}. If further the linked linear series is exact, then it contains mixed sections s'_{j_0+1} of type $((S'_1, S'_2, S'_3), (j_0-1, j_0, j_0+1))$ with S'_1 supported strictly left of i_1, S'_2 supported strictly left of i_0, s''_{j_0+1} of type $((S''_1, S''_2, S''_3), (j_0+1, j_0-1, j_0))$ with S''_2 supported strictly right of i_1, S''_3 supported strictly right of i_0, and s'''_{j_0+1} of type $((S'''_1, S'''_2, S'''_3), (j_0, j_0+1, j_0-1))$ with S'''_1 supported strictly left of i_0, and S'''_3 supported strictly right of i_1.*

Note that if $S'_2 = \varnothing$, then S'_1 may contain elements greater than i_0, and similarly if $S''_2 = \varnothing$, then S''_3 may contain elements less than i_1.

Proof. First, check that the multidegrees ω_{j_0-1} and ω_{j_0} both have only $(r+1)$-dimensional spaces of possible sections, so that s_{j_0-1} and s_{j_0} must both lie in any linked linear series. Indeed, for the former, the

$s^i_{j_0}$ can contribute only for $i = N$, while the $s^i_{j_0+1}$ can contribute only for $i = i_1$, while for the latter, the $s^i_{j_0-1}$ can contribute only for $i = 1$, and the $s^i_{j_0+1}$ can contribute only for $i = i_0$.

Now, to construct the sections s'_{j_0+1}, s''_{j_0+1} and s'''_{j_0+1}, we proceed as in the previous propositions. For s'_{j_0+1}, we start with $w' = (a^2_{j_0-1}, \ldots, a^{i_1}_{j_0-1}, a^{i_1+1}_{j_0+1}, \ldots, a^N_{j_0+1})$, and then iteratively increase the twist by 1 at a time for $i' \leq i_1$, initially increasing it from $a^{i'}_{j_0-1}$ to $a^{i'}_{j_0+1}$. For $i' > i_0$, this process behaves as before, either extending the contribution from the $a^i_{j_0+1}$ iteratively to the left without introducing any other nonzero contributions, or producing a section s'_{j_0+1} as desired, having $S_2 = \varnothing$. Once $i' \leq i_0$, we still iteratively increase the twist from $a^{i'}_{j_0-1}$ to $a^{i'}_{j_0+1}$, but we are required to pass $a^{i'}_{j_0}$ in the process. This introduces a third possibility: once the twist at i' is strictly greater than $a^{i'}_{j_0}$, we could obtain a contribution from $s^{i'-1}_{j_0}$. Also, for $i' < i_0$, once the twist at i' is equal to $a^{i'}_{j_0}$, we could obtain a contribution from both $s^{i'-1}_{j_0}$ and $s^{i'}_{j_0}$. If either of these occurs, we move to the next i', and for the remaining i', instead of increasing the twist from $a^{i'}_{j_0-1}$ to $a^{i'}_{j_0+1}$, we only increase to $a^{i'}_{j_0}$. Note that we may obtain contributions from the $s^i_{j_0}$ (for $i = i' - 1$ or $i = i' - 1, i'$) and $s^i_{j_0-1}$ (for $i = 1, \ldots, i' - 1$) simultaneously at some point, which still gives an s'_{j_0+1} of the desired form. On the other hand, if we never obtain a contribution from the $s^i_{j_0}$, then the resulting s'_{j_0+1} simply has $S'_2 = \varnothing$.

For s''_{j_0+1}, we start with $w'' = (a^2_{j_0+1}, \ldots, a^{i_0}_{j_0+1}, a^{i_0+1}_{j_0}, \ldots, a^N_{j_0})$, and then follow the same procedure as for s'_{j_0+1}, iteratively decreasing the twist at $i' > i_0$ from $a^{i'}_{j_0}$ to $a^{i'}_{j_0+1}$, with the possibility of a contribution from the $s^i_{j_0-1}$ once i' passes i_1.

Finally, for s'''_{j_0+1} set $w''' = (a^2_{j_0}, \ldots, a^{i_0}_{j_0}, a^{i_0+1}_{j_0+1}, \ldots, a^{i_1}_{j_0+1}, a^{i_1+1}_{j_0-1}, \ldots, a^N_{j_0-1})$ initially. We then iteratively increase the twist at $i' \leq i_0$ from $a^{i'}_{j_0}$ to $a^{i'}_{j_0+1}$, and iteratively decrease the twist at $i' > i_1$ from $a^{i'}_{j_0-1}$ to $a^{i'}_{j_0+1}$ to construct s'''_{j_0+1}. □

Example 4.19. Assume that X_0 is a chain of 8 elliptic curves. One can construct a limit linear series on X_0 of degree 8 and dimension 2 by choosing the following line bundles on each component

$$\mathcal{O}(8Q_1), \quad \mathcal{O}(2P_2+6Q_2), \quad \mathcal{O}(4P_3+4Q_3), \quad \mathcal{O}(4P_4+4Q_4),$$
$$\mathcal{O}(4P_5+4Q_5), \quad \mathcal{O}(4P_6+4Q_6), \quad \mathcal{O}(6P_7+2Q_7), \quad \mathcal{O}(8P_8)$$

and sections with associated table

0	8	0	7	1	6	2	5	3	3	5	2	6	2	6	1
1	6	2	6	2	5	3	3	5	2	6	1	7	0	8	0
2	5	3	4	4	4	4	4	4	4	4	4	4	3	5	2

A swap appears on C_4 between s_1, s_2 and another at C_5 between s_0, s_2. In general there may be larger gaps between the columns where the swap occurs.

Proposition 4.20 ("second 3-cycle"). *Suppose that our limit linear series has one swap between the (j_0-1)-st and j_0-th rows occurring in the i_0-th column, and a second swap between the (j_0-1)-st and (j_0+1)-st rows in the i_1-st column for some $i_1 > i_0$, and no other swaps.*

Then any linked linear series lying over the given limit linear series contains s_{j_0-1}. If further the linked linear series is exact, then it contains mixed sections s'_{j_0} and s''_{j_0} of type $((S'_1, S'_2), (j_0-1, j_0))$ and

$((S''_1, S''_2, S''_3), (j_0, j_0+1, j_0-1))$ *respectively, with* S'_1 *supported strictly left of* i_0, S''_2 *supported at or right of* i_1, *and* S''_3 *supported strictly right of* i_0. *Similarly, it contains mixed sections* s'_{j_0+1} *and* s''_{j_0+1} *of type* $((S'_3, S'_4, S'_5), (j_0-1, j_0, j_0+1))$ *and* $((S''_4, S''_5), (j_0+1, j_0-1))$ *respectively, with*S'_3 *supported strictly left of* i_1, S'_4 *supported at or left of* i_0, *and* S''_5 *supported strictly right of* i_1. *Moreover, if* $i_1 \in S''_2$ *then also* $i_1 \in S''_1$, *and if* $i_0 \in S'_4$, *then also* $i_0 \in S'_5$. *Finally, either we can have* $S'_2 = S''_4 = \{1, \dots, N\}$, *or it also contains a mixed section* s''' *of type* $((S'''_1, S'''_2, S'''_3), (j_0, j_0-1, j_0+1))$, *where every element of* S'''_2 *is strictly between* i_0 *and* i_1.

Proof. For the most part, this is straightforward and similar to the previous propositions, but there is one new subtlety to address, and the idea for the construction of s''' is new. We first construct s'_{j_0}, starting with $w' = (a^2_{j_0-1}, \dots, a^{i_0}_{j_0-1}, a^{i_0+1}_{j_0}, \dots, a^N_{j_0})$. We then iteratively increase the twist from $a^{i'}_{j_0-1}$ to $a^{i'}_{j_0}$ for $i' \leq i_0$, and obtain s'_{j_0} as before. We then do the same procedure for s''_{j_0+1}, starting with $w'' = (a^2_{j_0+1}, \dots, a^{i_1}_{j_0+1}, a^{i_1+1}_{j_0-1}, \dots, a^N_{j_0-1})$.

Next, we construct s''_{j_0}, starting with $w'' = (a^2_{j_0}, \dots, a^{i_0}_{j_0}, a^{i_0+1}_{j_0-1}, \dots, a^N_{j_0-1})$. We then iteratively decrease the twist at $i' > i_0$ from $a^{i'}_{j_0-1}$ to $a^{i'}_{j_0}$. For $i' \leq i_1$, this behaves as in the previous propositions, with one new subtlety: for each intermediate value of w', the $s^i_{j_0+1}$ can contribute only in the i_1 column, but because we do not know that s_{j_0+1} is contained in the linked linear series, we also do not know *a priori* that this contribution from $s^{i_1}_{j_0+1}$ in multidegree $\mathrm{md}(w')$ is contained in our linked linear series. However, since we have already constructed s''_{j_0+1}, we can use its image in $\mathrm{md}(w')$. One checks that its only possible support in $\mathrm{md}(w')$ is in the i_1 column, so that in fact the multidegree-$\mathrm{md}(w')$ part of the linked linear series necessarily contains the section given by $s^{i_1}_{j_0+1}$, and we can subtract it off as necessary from the section we are constructing. Thus, for $i' \leq i_1$, we can iterate as before, and will either obtain an s''_{j_0} as desired (with $S''_2 = \varnothing$), or we will obtain a section made up of the $s^i_{j_0}$ for $i \leq i_1$, and vanishing identically for $i > i_1$. In the latter case, we continue to iteratively decrease the twists defining w' for $i > i_1$, but as in the construction of s'_{j_0+1} in the proof of Proposition 4.18, to get from $a^{i'}_{j_0-1}$ to $a^{i'}_{j_0}$ we need to pass $a^{i'}_{j_0+1}$, which is where the possible contribution from the j_0+1 may occur.

The construction of s'_{j_0+1} follows the same pattern as that of s''_{j_0}, but starting with

$$w' = (a^2_{j_0-1}, \dots, a^{i_1}_{j_0-1}, a^{i_1+1}_{j_0+1}, \dots, a^N_{j_0+1}).$$

Here we use the image of s'_{j_0} in order to subtract off any contributions of $s^{i_0}_{j_0}$ which occur.

Finally, for s''', we start with $w' = w_{j_0}$. We observe that there is an $(r+2)$-dimensional space of potential sections in multidegree ω_{j_0}, with the s^i_j for $j < j_0 - 1$ contributing only for $i = 1$, the s^i_j for $j \geq j_0 + 1$ contributing only for $i = N$, the $s^i_{j_0}$ contributing only with s_{j_0} itself, and the $s^i_{j_0-1}$ contributing separately for $i = 1$ and $i = N$.

We must therefore have a three-dimensional space of combinations of the four sections $s^1_{j_0-1}$, $s^N_{j_0-1}$, $s^N_{j_0+1}$, and s_{j_0}. It follows by elimination that this space must contain (at least) one of the following: s_{j_0} plus a (possibly zero) multiple of $s^1_{j_0-1}$; s_{j_0} plus a (possibly zero) multiple of $s^N_{j_0+1}$; $s^1_{j_0-1}$ and $s^N_{j_0+1}$. The first case means that we can take $S'_2 = \{1, \dots, N\}$, while in the second we get a valid choice of s'''. In the third case, we begin with $s^N_{j_0+1}$, and iteratively twist the multidegree as before. For $i' > i_1$, we change w' from twisting down by $a^{i'}_{j_0}$ to $a^{i'}_{j_0+1}$, and at each stage, we must either obtain the desired s''', or a section made up purely of the $s^i_{j_0+1}$, in which case we continue to iterate. Note that in these multidegrees, we continue to have that the only possible contributions of the s^i_j (for $j \neq j_0$) supported strictly left of i' come for $j \leq j_0 - 1$, and we can take the image of $s^1_{j_0-1}$ from multidegree ω_{j_0}, so all these can be subtracted off as necessary. When $i' \leq i_1$, we will have $a^{i'}_{j_0-1}$ between $a^{i'}_{j_0}$ and $a^{i'}_{j_0+1}$; we still iteratively increase the twist, but a new possibility occurs: once we are twisting down by strictly more than $a^{i'}_{j_0-1}$, we could obtain a contribution from $a^{i'-1}_{j_0-1}$. If this occurs, we will continue to iterate, but stopping after increasing the twist from $a^{i'}_{j_0}$ to $a^{i'}_{j_0-1}$ for each smaller i'.

If we have continued with contributions from $s^{i'}_{j_0+1}$ for each i', then once we reach i_0, we will again have no other a^i_j between $a^i_{j_0}$ and $a^i_{j_0+1}$, so we will ultimately obtain an s''' of the desired form, with $S'''_2 = \varnothing$. On the other hand, if we have switched from the $s^{i'}_{j_0+1}$ to the $s^{i'}_{j_0-1}$, then we see that this must terminate (necessarily with an s''' of the desired form) before we reach $i' = i_0$, because there is no section in column i_0 which can glue to $s^{i_0+1}_{j_0-1}$.

Now, if the above construction did not give s''' because we had $S'_2 = \{1, \dots, N\}$, we apply precisely the same process starting in multidegree ω_{j_0+1}, and we find that unless we also have $S''_4 = \{1, \dots, N\}$, we end up with the desired s'''. □

Example 4.21. Assume that X_0 is a chain of 8 elliptic curves. One can construct a limit linear series on X_0 of degree 8 and dimension 2 by choosing the following line bundles on each component

$$\mathcal{O}(8Q_1), \quad \mathcal{O}(2P_2+6Q_2), \quad \mathcal{O}(2P_3+6Q_3), \quad \mathcal{O}(5P_4+3Q_4),$$
$$\mathcal{O}(5P_5+3Q_5), \quad \mathcal{O}(4P_6+4Q_6), \quad \mathcal{O}(6P_7+2Q_7), \quad \mathcal{O}(8P_8)$$

and sections with associated table

0	8	0	7	1	5	3	4	4	2	6	1	7	0	8	0
1	6	2	6	2	6	2	5	3	4	4	4	4	3	5	2
2	5	3	4	4	3	5	3	5	3	5	2	6	2	6	1

A swap appears on C_3 between s_0, s_1 and another at C_5 between s_0, s_2. In general there may be larger gaps between the columns where the swap occurs.

Up until now, everything we have done has been insensitive to insertion of genus-0 components. However, to handle the genus-23 case, we will need to impose restrictions on direction of approach; more precisely, we will require that the genus-1 components be separated by exponentially increasing numbers of genus-0 components (going from right to left). The reason for doing this is that, if our limit linear series has all changes to the λ_i occurring in the genus-1 components, the pattern of the genus-0 components will force the support of every s_j in every multidegree to be precisely the leftmost segment of potential support (see Proposition 8.8). So, we obtain better control over the situation when the potential support is disconnected. That this sort of restriction could potentially be useful is already pointed out in

Remark 4.12 of [Liu et al. 2021], and is also influenced by the earlier work of Jensen and Payne [2016] on their tropical approach to the classical maximal rank conjecture.

Definition 4.22. Assume that we have a chain of curves of genus zero and one with the first and last components of the chain being of genus 1. We denote by $\boldsymbol{\ell}_i$ the number of nodes between the i-th and $(i+1)$-st components of genus-1. We say that X_0 is *left-weighted* if

$$\ell_i \geq 4d \sum_{i'=i+1}^{g-1} \ell_{i'}.$$

In our notation, $\ell_i - 1$ is the number of genus-0 components between two genus one components. If we take a ramified base change with ramification index e, and then blow up to resolve the resulting singularities, we will insert e new genus-0 components at every node, which has the effect of multiplying all the ℓ_i by e. Thus, the ratios of the ℓ_i (and therefore the left-weightedness) are invariant under this operation.

Definition 4.23. Given $\vec{S} = (S_1, \dots, S_\ell)$ and $\vec{j} = (j_1, \dots, j_\ell)$, a mixed section of type $(\vec{S}, \vec{j})$ is said to be *controlled* if for every $i = 2, \dots, \ell$ with $S_i \neq \varnothing$, the minimal element of S_i is either a genus-1 component or strictly closer to the next genus-1 component to the right than to the previous one on the left.

Proposition 4.24. *Suppose that X_0 is left-weighted. Then*:

1. *In the situation of Proposition 4.14, if we assume further that i_0 and i_1 have genus* 1, *then we may require that s'_{j_0} and s'_{j_1} are controlled, that S'_2 does not contain any $i < i_0$ which has genus* 1, *and that S'_4 does not contain any $i < i_1$ which has genus* 1.
2. *In the situation of Proposition 4.20, if we assume further that i_0 and i_1 have genus* 1, *then we may require that s'_{j_0} is controlled, and that S'_2 does not contain any $i < i_0$ which has genus* 1.

Proof. (1) With the notations of the proof of Proposition 4.14, for $s = 0, 1$ and every value of w' arising in the iterative procedure, the potential support of the $s^i_{j_s-1}$ in multidegree $\mathrm{md}(w')$ has two connected components: one extending from $i = 1$ to $i = i' - 1$, and the other supported at $i = N$. We cannot continue our iterative procedure indefinitely if $f_{w_{j_s-1},w'}(s_{j_s-1})$ is supported (partially or entirely) at $i = N$. If we write $w' = (c'_2, \dots, c'_N)$, then $a^i_{j_s-1} - c'_i > 0$ for $i > i_s$, $a^i_{j_s-1} - c'_i < 0$ for $i' \leq i \leq i_s$, and $a^i_{j_s-1} - c'_i = 0$ for $i < i'$. Suppose i_s is the m_s-th genus-1 component. Note that $a^i_{j_s-1} - c'_i \leq a^i_{j_s-1} \leq d$. Then in the notation used in Definition 4.22 above, we can say (extremely conservatively) that

$$\sum_{i=i_s+1}^{N} (a^i_{j_s-1} - c'_i) \leq d \sum_{i=m_s}^{g-1} \ell_i \leq \tfrac{1}{4}\ell_{m_s-1}. \tag{4-1}$$

Thus, if $i_s - i' > \frac{1}{4}\ell_{m_s-1}$, then $\sum_{i=i'}^{N}(a^i_{j_s-1} - c'_i) < 0$, so $f_{w_{j_s-1},w'}(s_{j_s-1})$ is supported entirely on the left. We can then subtract off any contribution from the $s^i_{j_s-1}$ and continue our iterative procedure. The desired conditions on s'_{j_s} follow.

(2) This is essentially the same as (1) (the analogous statement for s'_{j_0+1} is a bit more complicated, but we don't need it). ☐

row	col = 1	2	3	4	5	6	7	8	9	10	11
0	0 25	0 24	1 23	2 22	3 21	4 20	5 19	6 19	6 18	7 17	8 16
1	1 23	2 23	2 22	3 21	4 20	5 19	6 18	7 17	8 16	9 16	9 15
2	2 22	3 21	4 21	4 20	5 19	6 18	7 17	8 16	9 14	11 13	12 12
3	3 21	4 20	5 19	6 19	6 18	7 17	8 16	9 15	10 15	10 14	11 14
4	4 20	5 19	6 18	7 17	8 17	8 16	9 15	10 14	11 13	12 12	13 11
5	5 19	6 18	7 17	8 16	9 15	10 15	10 14	11 13	12 12	13 11	14 10
6	6 18	7 17	8 16	9 15	10 14	11 13	12 13	12 12	13 11	14 10	15 9

row	col = 12	13	14	15	16	17	18	19	20	21	22
0	9 15	10 14	11 13	12 12	13 12	13 11	14 10	15 9	16 8	17 7	18 6
1	10 14	11 13	12 12	13 11	14 10	15 10	15 9	16 8	17 7	18 6	19 5
2	13 12	13 11	14 10	15 9	16 8	17 7	18 6	19 6	19 5	20 4	21 3
3	11 13	12 12	13 11	14 10	15 9	16 8	17 8	17 7	18 6	19 5	20 4
4	14 10	15 10	15 9	16 8	17 7	18 6	19 5	20 4	21 4	21 3	22 2
5	15 9	16 8	17 8	17 7	18 6	19 5	20 4	21 3	22 2	23 2	23 1
6	16 8	17 7	18 6	19 6	19 5	20 4	21 3	22 2	23 1	24 0	25 0

Table 1. Example 4.25. A possible table T' associated to a limit linear series for $r = 6$, $g = 22$, $d = 25$ with all component of genus 1. The table is split in two. Top: left part. Bottom: right part.

An example in a case we are ultimately interested in is presented now.

Example 4.25. Table 1 shows a possible table T' associated to a limit linear series for $r = 6$, $g = 22$, $d = 25$ with all component of genus 1.

Since there is no ramification at P_1, the first entries of the table agree with the row labels. Note that we have a single swap, occurring in the ninth column between the $j = 2$ and $j = 3$ rows. This leads to having an extra dimension of possibilities in the multidegree obtained from the $j = 3$ row, as the $j = 2$ row can appear either in the first or last columns. Consequently, it is possible that an exact linked linear series lying over this limit linear series might not contain s_3, but might only contain mixed sections s_3' and s_3'', as in Proposition 4.13, with s_3' agreeing with s_3 for $i \geq 9$, but switching to s_2 at some $i < 9$, and s_3'' agreeing with s_3 for $i \leq 9$, but switching to s_2 at some $i > 9$. In both cases, the switch occurs in a column mixing s_2^i and s_3^i unless, the column in question has a gap of at least 2 between the $j = 2$ and $j = 3$ rows. Since this doesn't occur for $i < 9$, we see that s_3' always has a mixed column, while s_3'' may not.

5. General setup

We are working with chains of N curves. While we imagine starting from a chain of genus-1 curves, we allow for inserting any number of rational components at nodes so that all components Z_i are of genus at most one, and exactly g have genus 1 (including the first and last components). Given i between 1 and N, we denote by $g(i)$ the number of genus-1 components between 1 and i, inclusive. In particular, $g(0) = 0$ by convention.

We will suppose also that all the P_i and Q_i are general (and in particular each $P_i - Q_i$ is not ℓ-torsion for any $\ell \leq d$). We need generality conditions that go beyond what can be imposed component by component, as it also involves interaction between components. This will be needed in the proof of Lemma 6.3.

No matter the genus of Z_i, for all j, $a^i_j + b^i_j \leq d$. If Z_i has genus 0, there are no further restrictions. If the genus of Z_i is 1, the genericity hypothesis implies that there is at most one value of j such that $a^i_j + b^i_j = d$. In this case the underlying line bundle is uniquely determined as $\mathcal{O}(a^i_j P_i + b^i_j Q_i)$. The generic situation is that $a^i_j + b^i_j = d - 1$ for all other j, but in positive codimension we can have strictly smaller sums as well

Definition 5.1. We say that the j-th row is *exceptional* in column i if $a^i_j + b^i_j < d - 1$ when Z_i has genus 1, or if $a^i_j + b^i_j < d$ when Z_i has genus 0. For $i = 1, \ldots, N$, we write $\boldsymbol{g}(i)$ for the number of genus one components between 1 and i inclusive. For $j = 0, \ldots, r$ and $i = 0, \ldots, N$, define $\boldsymbol{\lambda}_{i,j}$ by $a^{i+1}_j = g(i) + j - \lambda_{i,j}$. For a given i, if there is a j such that $\lambda_{i,j} > \lambda_{i-1,j}$, we say that *there is a* $\boldsymbol{\delta}_i$. and more specifically, that $\boldsymbol{\delta}_i = j$. Otherwise, we say that *there is no* δ_i.

In this way, we obtain a sequence $\lambda_i = (\lambda_{i,0}, \ldots, \lambda_{i,r})$. For a fixed i, we think of the $\lambda_{i,j}$, $j = 0, \ldots, r$ as rows with $\lambda_{i,j}$, squares, where a negative number of squares appears as "holes" to the left of the level 0 line. In this ways, we get a collection of "shapes", $i = 0, \ldots, N$ (not necessarily skew, or connected) generalizing the Young Tableaux usually associated to limit linear series on chains of elliptic curves. They behave as follows: $\lambda_{0,j} \leq 0$ for all j, λ_0 is the empty shape if $a^1_j = j$ and in general the λ_0 shape is entirely made of holes. Going from i to $i+1$, any number of "squares" can be removed from the right of any row (and then the row is exceptional). At most one "square" is added (and then $\delta_i = j$), with the possibility of adding a "square" only in the genus-1 components.

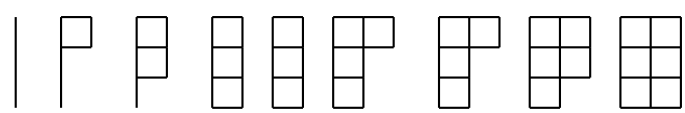
Example 5.2. Let X_0 be a generic chain of 8 elliptic curves. Construct a limit linear series on X_0 of degree 8 and dimension 2 by choosing generic line bundles L_4, L_6 on C_4, C_6 and the rest as follows:

$$\mathcal{O}(8Q_1), \quad \mathcal{O}(2P_2+6Q_2), \quad \mathcal{O}(4P_3+4Q_3), \quad L_4, \quad \mathcal{O}(3P_5+5Q_5), \quad L_6, \quad \mathcal{O}(6P_7+2Q_7), \quad \mathcal{O}(8P_8)$$

and sections with associated table

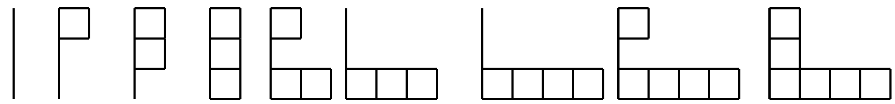
0	8	0	7	1	6	2	5	3	5	3	4	4	3	5	2
1	6	2	6	2	5	3	4	4	3	5	2	6	2	6	1
2	5	3	4	4	4	4	3	5	2	6	1	7	0	8	0

The corresponding λ shapes are



λ_0 λ_1 λ_2 λ_3 λ_4 λ_5 λ_6 λ_7 λ_8

By contrast, the λ shapes corresponding to Example 4.19 which has a swap on C_4 between rows 1 and 2 and another swap on C_5 between rows 0 and 2 are



λ_0 λ_1 λ_2 λ_3 λ_4 λ_5 λ_6 λ_7 λ_8

Proposition 5.3. *Assume that the curve X_0 is a chain of elliptic and rational curves. Choose a limit linear series and a linked series lying over it, then the number of swaps is bounded by ρ. Moreover, if there are ρ swaps, then the swaps must all be minimal, and occur in genus-1 columns, and there cannot be any exceptional behavior (as in Definition 5.1) other than what is needed for the swaps.*

Proof. The b_j^N are nonnegative (and distinct) for all j. Equivalently the a_j^{N+1} are bounded by

$$d-r, d-r+1, \dots, d.$$

In particular, $\sum_{j=0}^{r} a_j^{N+1} \leq (r+1)d - \binom{r+1}{2}$, so

$$\sum_{j=0}^{r} \lambda_{N,j} \geq (r+1)g + \binom{r+1}{2} - (r+1)d + \binom{r+1}{2} = (r+1)(g+r-d) = g-\rho.$$

Since $\sum_j \lambda_{i,j}$ can increase by at most 1 as i increases (and only on genus-1 components), and $\lambda_{0,j} \leq 0$ for all j, we see that for $\rho = 0$, we must have $\lambda_{0,j} = 0$ for all j (i.e., minimal initial vanishing sequence at P_1), no places where $\lambda_{i,j}$ decreases (i.e., no exceptional columns for any row), and a δ_i for every genus-1 column i. When $\rho > 0$, the total amount of initial ramification, exceptional columns, and genus-1 columns without δ_i is bounded by ρ.

A swap occurs when the vanishings of two of the sections at P_i, Q_i are of the form $(a, d-a-l)$ and $(a+k, d-a-k-l')$, respectively, with $k > 0$, $k + l' < l$. Hence, a swap is necessarily a case of an exceptional column, and can contribute exactly 1 to ρ precisely when it is minimal and occurs in a genus-1 column. □

We now describe the tensor square of a limit linear series considering images in a fixed multidegree of total degree $2d$. Essentially the discrete data from the base limit linear series is extended to its tensor square.

Notation 5.4. In the situation of Definition 4.5, let T be the $\binom{r+2}{2} \times N$ table with rows indexed by unordered pairs (j, j') with $j, j' \in \{0, \dots, r\}$, having entries $(a^i_{(j,j')}, b^i_{(j,j')})$ defined by

$$a^i_{(j,j')} = a^i_j + a^i_{j'}, \quad \text{and} \quad b^i_{(j,j')} = b^i_j + b^i_{j'}.$$

We update a definition from [Liu et al. 2021] to allow for genus-0 components:

Definition 5.5. We say a multidegree of total degree $2d$ is *unimaginative* if it assigns degree 0 to every genus-0 component, and degree 2 or 3 to every genus-1 component. By extension, we will say that w is unimaginative if $\mathrm{md}_{2d}(w)$ is. Given a fixed unimaginative multidegree, we will let $\boldsymbol{\gamma}_i$ be the number of 3s in the first i columns.

We will work throughout only with unimaginative multidegrees. Thus, the multidegree is encoded by twisting down by $2d-2g(i)-\gamma_i$ on the right-hand of the i-th column, and by twisting down by $2g(i)+\gamma_i$ on the left-hand side of the $(i+1)$-st column, for all $i < N$. We introduce some notation that we will use:

Definition 5.6. In the situation of Notation 5.4, fix total degree $2d$, and $w = (c_2, \dots, c_N)$. We say that the (j, j') row is *potentially present* (respectively *potentially starting*, respectively *potentially ending*) in column i and multidegree $\mathrm{md}_{2d}(w)$ if $a^i_{(j,j')} \geq c_i$ and $b^i_{(j,j')} \geq 2d - c_{i+1}$ (respectively $a^i_{(j,j')} > c_i$ and $b^i_{(j,j')} \geq 2d - c_{i+1}$, respectively $a^i_{(j,j')} \geq c_i$ and $b^i_{(j,j')} > 2d - c_{i+1}$).

The next proposition is an immediate consequence of the definitions.

Proposition 5.7. *If a row (j_1, j_2) is potentially present in the i-th column, then*

$$j_1 + j_2 - \lambda_{i-1,j_1} - \lambda_{i-1,j_2} \geq \gamma_{i-1} \quad \textit{and} \quad j_1 + j_2 - \lambda_{i,j_1} - \lambda_{i,j_2} \leq \gamma_i .$$

If a row (j_1, j_2) is potentially starting (respectively ending) in the i-th column, then the first (respectively second) inequality is strict.

If a row (j_1, j_2) is potentially present in the i-th and $(i+1)$-st columns, then

$$j_1 + j_2 - \lambda_{i,j_1} - \lambda_{i,j_2} = \gamma_i .$$

Note that the sequence $j_1 + j_2 - \lambda_{i,j_1} - \lambda_{i,j_2}$ decreases by at most 1 each time i increases, unless $j_1 = j_2 = \delta_i$, when it can decrease by 2. Similarly, from our assumptions on degrees and the definition of γ_i, γ_i is nondecreasing, and increases by at most 1 each time i increases.

Corollary 5.8. *Assume the multidegree is 2 on the i-th column. There can be a row potentially starting in the i-th column only if δ_i exists and either there exists j such that*

$$\delta_i + j - \lambda_{i,\delta_i} - \lambda_{i,j} = \gamma_i \quad \textit{or} \quad 2\delta_i - 2\lambda_{i,\delta_i} = \gamma_i - 1.$$

In these cases, the potentially starting rows are (δ_i, j) or (δ_i, δ_i), respectively.

There can be a row potentially ending in the i-th column only if δ_i exists and either there exists j such that

$$\delta_i + j - \lambda_{i,\delta_i} - \lambda_{i,j} = \gamma_i - 1 \quad \textit{or} \quad 2\delta_i - 2\lambda_{i,\delta_i} = \gamma_i - 2.$$

In these cases, the potentially ending rows are (δ_i, j) or (δ_i, δ_i), respectively.

In any case, there can be at most one row potentially starting on the i-th column, and at most one row potentially ending in it.

Proof. Since in this case $\gamma_i = \gamma_{i-1}$, Proposition 5.7 implies that the (j_1, j_2) row can be potentially starting in the i-th column only if $\lambda_{i,j_1} > \lambda_{i-1,j_1}$ or $\lambda_{i,j_2} > \lambda_{i-1,j_2}$, which is to say if δ_i exists and j_1 or j_2 is equal to δ_i. Moreover, in this case $\lambda_{i,\delta_i} = \lambda_{i-1,\delta_i} + 1$, so we conclude that the two stated cases are the only possibilities for having

$$j_1 + j_2 - \lambda_{i-1,j_1} - \lambda_{i-1,j_2} > \gamma_{i-1} = \gamma_i \geq j_1 + j_2 - \lambda_{i,j_1} - \lambda_{i,j_2},$$

and that moreover in the first case we must also have $\lambda_{i-1,j} = \lambda_{i,j}$ unless $j = \delta_i$.

Now, there is at most one j satisfying the first identity of the corollary, since the $j - \lambda_{i,j}$ are all distinct. Moreover, if there is some j satisfying the first, then the second one cannot hold, since this would force

$$\delta_i - \lambda_{i-1,\delta_i} = \delta_i - \lambda_{i,\delta_i} + 1 = j - \lambda_{i,j} = j - \lambda_{i-1,j},$$

which is not allowed. This completes the proof of the assertions on rows potentially starting in the i-th column. The assertion on rows potentially ending in the i-th column is proved similarly. □

Next corollary has a similar proof, which we omit.

Corollary 5.9. *If the multidegree has a* 3 *in the i-th column, then there can be at most one row potentially starting and ending in the i-th column, and this occurs only if δ_i exists and either there exists j such that*

$$\delta_i + j - \lambda_{i,\delta_i} - \lambda_{i,j} = \gamma_i - 1 \quad or \quad 2\delta_i - 2\lambda_{i,\delta_i} = \gamma_i - 2.$$

In addition, for a fixed $j \neq \delta_i$, there is at most one value of j' such that the (j, j') row is potentially starting in column i, and at most one value of j' such that the (j, j') row is potentially ending in column i.

Proposition 5.10. *For a fixed limit linear series, w, and column i, if $a^i_{(j,j')} > c_i$, then (j, j') has a component of potential support $\bigcup_{j\geq i} Z_j$. If $a^i_{(j,j')} < c_i$, then (j, j') has a component of potential support $\bigcup_{j<i} Z_j$.*

Conversely, suppose further that w is unimaginative. If (j, j') has a component of potential support $\bigcup_{j\geq i} Z_j$, and if neither j nor j' is exceptional in any column strictly right of $i-1$, then $a^i_{(j,j')} > c_i$. Similarly, if (j, j') has a component of potential support $\bigcup_{j<i} Z_j$., and if neither j nor j' is exceptional in any column $j < i$, then $a^i_{(j,j')} < c_i$.

In particular, in the unimaginative case, the potential support of (j, j') is connected unless the sum of the number of swaps for which the j-th row is exceptional and the number of swaps for which the j'-th row is exceptional is at least 2.

Proof. The first part is straightforward, and we omit the proof. For the second part, the point is that the unimaginative hypothesis together with the nonexceptional hypothesis imply that the relevant portion of the sequence $a^{i'}_{(j,j')} - c_{i'}$ is nondecreasing in the relevant range as i' decreases, so in the first case if its positivity for some $i' \geq i$ implies it remains positive at $i' = i$, while in the second case its negativity for some $i' \leq i$ implies it remains negative at $i' = i$.

For the last assertion, we can have disconnected potential support in the (j, j') row only if the sequence $a^i_{(j,j')} - c_i$ goes from positive to negative as i decreases, possibly over multiple columns. But we observe that if only one of j and j' are exceptional at a swap, which is moreover minimal and in a genus-1 column, then the sequence $a^i_{(j,j')} - c_i$ can decrease only by 1 as i decreases. Thus, if this occurs only once, it cannot go from positive to negative, and it cannot have disconnected potential support. □

Definition 5.11. Given a refined limit linear series, we construct a second table $\overline{\boldsymbol{T}}$ of vanishing numbers obtained from the first by reordering each subcolumn into strictly increasing (respectively, decreasing) order. We denote the λ sequence obtained from $\overline{T}$ by $\bar{\boldsymbol{\lambda}}_i$, and the entries of the table $\overline{T}$ by $(\bar{\boldsymbol{a}}^i_j, \bar{\boldsymbol{b}}^i_j)$. For $\ell \geq 1$, we denote by $\bar{\boldsymbol{\lambda}}^\ell_i$ the number of j such that $\bar{\lambda}_{i,j} \geq \ell$.

Put differently, $\overline{T}$ is obtained from the limit linear series simply by taking vanishing sequences at each point, and ignoring the interplay between the pair of points. If we picture skewing the rows of the $\bar{\lambda}_i$ according to the initial ramification sequence $a^1_j - j$, the sequence $\bar{\lambda}_i$ will give a genuine sequence of skew shapes, terminating with a skew shape containing the one obtained by starting from the usual $(r+1) \times (r+g-d)$ center rectangle, and adding squares on the left determined by the initial ramification sequence.

The following lemma is the key to our analysis, showing in particular that if we place multidegree 3 in the correct places, we can obtain fine control over what happens with the rows involving δ_{i+1}.

Lemma 5.12. *Given $1 \leq \ell_1 < \ell_2$ and $n > 0$, suppose that $\bar{\lambda}^{\ell_1}_i + \bar{\lambda}^{\ell_2}_i = n$ and $\bar{\lambda}^{\ell_1}_{i-1} + \bar{\lambda}^{\ell_2}_{i-1} < n$. Then*

$$\text{for all } j, \quad \delta_i + j - \lambda_{i,\delta_i} - \lambda_{i,j} \neq n - 1 - \ell_1 - \ell_2, \tag{5-1}$$

$$2\delta_i - 2\lambda_{i,\delta_i} \neq n - 2 - \ell_1 - \ell_2. \tag{5-2}$$

Moreover, if for some j, $\lambda_{i,j} < \lambda_{i-1,j}$, then

$$\delta_i + j - \lambda_{i,\delta_i} - \lambda_{i,j} \neq n - \ell_1 - \ell_2, \tag{5-3}$$

$$\delta_i + j - \lambda_{i-1,\delta_i} - \lambda_{i-1,j} \neq n - 1 - \ell_1 - \ell_2. \tag{5-4}$$

Proof. We first prove the case that $\bar{\lambda}_{i'} = \lambda_{i'}$ for all i'. Note that the assumption that $\bar{\lambda}^{\ell_1}_i + \bar{\lambda}^{\ell_2}_i > \bar{\lambda}^{\ell_1}_{i-1} + \bar{\lambda}^{\ell_2}_{i-1}$ implies that there is a δ_i and it is the row of the last square in either the ℓ_1-th or ℓ_2-th columns of λ_i.

Case when $\lambda^{\ell_1}_i, \lambda^{\ell_2}_i$ are distinct and positive. Set $j_s = \lambda^{\ell_s}_i - 1$, $s = 1, 2$. In particular, $\delta_i = j_1$ or j_2 and $(j_1 + 1) + (j_2 + 1) = n$. For $s = 1, 2$ write $m_s = \lambda_{i,j_s} - \ell_s$, so that $m_s \geq 0$ for $s = 1, 2$, with equality for at least one s. Then,

$$\begin{aligned} j_1 + j_2 - \lambda_{i,j_1} - \lambda_{i,j_2} &= (j_1 + 1) + (j_2 + 1) - 2 - (m_1 + \ell_1) - (m_2 + \ell_2) \\ &= n - 2 - \ell_1 - \ell_2 - m_1 - m_2 \\ &< n - 1 - \ell_1 - \ell_2. \end{aligned}$$

Thus, the only way to get equality in (5-1) would be to set j to be strictly greater than whichever j_s is not equal to δ_i. Now, because j_s was determined as the lowest row with a square in the ℓ_s-th column, we have

$$\lambda_{i,j_s+1} < \ell_s = \lambda_{i,j_s} - m_s,$$

so if we use $j > j_s$ in place of j_s, the value of the above expression jumps by at least $2 + m_s$. Moreover, we can only use j in place of j_1 if $\delta_i = j_2$, in which case we must have $m_2 = 0$, and similarly if we use j in place of j_2. We conclude that (5-1) is satisfied.

Similarly, if we had equality in (5-3), then necessarily $j = j_s + 1$ and $\lambda_{i,j} = \ell_s - 1$. On the other hand, the assumption that $\bar{\lambda}_i^{\ell_1} + \bar{\lambda}_i^{\ell_2} > \bar{\lambda}_{i-1}^{\ell_1} + \bar{\lambda}_{i-1}^{\ell_2}$ implies that if $\lambda_{i,j} < \lambda_{i-1,j}$ for some j, then $\lambda_{i,j} \neq \ell_1 - 1, \ell_2 - 1$. Therefore, (5-3) is an inequality as stated.

By the same reasoning, if $\delta_i = j_1 > j_2$, then equality in (5-2) is also impossible: because $m_1 = 0$ replacing j_2 by δ_i increases the left-hand side by at least $2 + m_2$. On the other hand, if $\delta_i = j_2$ then replacing j_1 by δ_i decreases the left-hand side, making it too small to satisfy (5-2).

Finally, suppose we have some j such that $\lambda_{i,j} < \lambda_{i-1,j}$; say $\lambda_{i,j} = \lambda_{i-1,j} - p$ for some $p > 0$. Then equality in (5-4) is equivalent to

$$\delta_i + j - \lambda_{i,\delta_i} - \lambda_{i,j} = n - 2 - \ell_1 - \ell_2 + p.$$

If as above $j_s \neq \delta_i$, then necessarily $j > j_s$. Then, by definition of j_s we must have $\lambda_{i,j} < \ell_s$. On the other hand, since we have assumed that $\lambda_{i-1}^{\ell_1} + \lambda_{i-1}^{\ell_2} < n$, we must have that $\lambda_{i,j}, \ldots, \lambda_{i,j} + p$ does not contain ℓ_s. It follows that $\lambda_{i,j} + p < \ell_s = \lambda_{i,j_s} - m_s$. We conclude that $j - \lambda_{i,j} > 1 + j_s - \lambda_{i,j_s} + p + m_s$, so

$$\delta_i + j - \lambda_{i,\delta_i} - \lambda_{i,j} > (n - 2 - \ell_1 - \ell_2 - m_s) + 1 + p + m_s = n - 1 - \ell_1 - \ell_2 + p,$$

proving (5-4).

Case when λ_i has no entries in the ℓ_2-th column. Then, $\bar{\lambda}_i^{\ell_2} = 0$, δ_i is the row of the last square in the ℓ_1-th column and $\lambda_{i,\delta_i} = \ell_1$. As the rows are numbered starting at 0, the condition $\bar{\lambda}_i^{\ell_1} + \bar{\lambda}_i^{\ell_2} = n$ becomes $\delta_i + 1 = n$. Hence,

$$\delta_i - \lambda_{i,\delta_i} = (\delta_i + 1) - 1 - \ell_1 = n - 1 - \ell_1.$$

But since the ℓ_2-th column is empty for all j, we have $\lambda_{i,j} < \ell_2$, so we find that

$$\delta_i + j - \lambda_{i,\delta_i} - \lambda_{i,j} > n - 1 - \ell_1 + j - \ell_2 \geq n - 1 - \ell_1 - \ell_2,$$

showing that (5-1) holds, and that equality in (5-3) occurs only if $\lambda_{i,j} = \ell_2 - 1$. But if we assume $\lambda_{i,j} < \lambda_{i-1,j}$, then $\lambda_{i-1,j} \geq \ell_2$, contradicting the assumption that $\bar{\lambda}_{i-1}^{\ell_1} + \bar{\lambda}_{i-1}^{\ell_2} < n$. Therefore, (5-3) holds.

We also see that

$$2\delta_i - 2\lambda_{i,\delta_i} = 2n - 2 - 2\ell_1 > 2n - 2 - \ell_1 - \ell_2 > n - 2 - \ell_1 - \ell_2,$$

so (5-2) holds. Finally, as $\lambda_{i-1}^{\ell_1} + \lambda_{i-1}^{\ell_2} < n$, then $\lambda_{i-1,j} < \ell_2$, so

$$\delta_i + j - \lambda_{i-1,\delta_i} - \lambda_{i-1,j} = \delta_i + j - \lambda_{i,\delta_i} - \lambda_{i-1,j} + 1 > n - 1 - \ell_1 - \ell_2 + 1 = n - \ell_1 - \ell_2,$$

proving (5-4).

Case when λ_i has the same number of entries in the ℓ_1-th and ℓ_2-th columns. So we must have n even, with $\delta_i + 1 = n/2$, and also $\lambda_{i,\delta_i} = \ell_2$. In this case, we have

$$\delta_i - \lambda_{i,\delta_i} = (\delta_i + 1) - 1 - \ell_2 = n/2 - 1 - \ell_2.$$

Therefore

$$2\delta_i - 2\lambda_{i,\delta_i} = n - 2 - 2\ell_2 < n - 2 - \ell_1 - \ell_2,$$

proving (5-2).

For $j_1 < \delta_i$, $\lambda_{i,j_1} \geq \lambda_{i,\delta_i}$. Therefore

$$j_1 + \delta_i - \lambda_{i,j_1} - \lambda_{i,\delta_i} < 2\delta_i - 2\lambda_{i,\delta_i} < n - 2 - \ell_1 - \ell_2,$$

proving (5-1) in this case. For $j_1 > \delta_i$, as the ℓ_1-th column has exactly $\delta_i + 1$ entries,

$$\lambda_{i,j_1} \leq \ell_1 - 1 = \ell_2 - (\ell_2 - \ell_1) - 1 = \lambda_{i,\delta_i} - (\ell_2 - \ell_1) - 1,$$

$$j_1 + \delta_i - \lambda_{i,j_1} - \lambda_{i,\delta_i} \geq \delta_i + 1 + \delta_i - 2\lambda_{i,\delta_i} + \ell_2 - \ell_1 + 1 = n - 2\ell_2 + \ell_2 - \ell_1 = n - \ell_2 - \ell_1,$$

completing the proof of (5-1). Moreover, we can have equality in (5-3) only if $j = \delta_i + 1$ and $\lambda_{i,j} = \ell_1 - 1$, so (5-3) holds under the stated condition $\lambda_{i,j} < \lambda_{i-1,j}$. Finally, if $\lambda_{i,j} = \lambda_{i-1,j} - p$ for $p > 0$, then in order to have equality in (5-4) we would need to have $j > \delta_i$, which implies that $\lambda_{i,j} + p < \ell_1 = \lambda_{i,\delta_i} - \ell_2 + \ell_1$, so

$$\delta_i + j - \lambda_{i,\delta_i} - \lambda_{i,j} > (n - 2 - 2\ell_2) + 1 + (\ell_2 - \ell_1 + p) = n - 1 - \ell_1 - \ell_2 + p,$$

again yielding (5-4).

This completes the proof of the lemma in the case that $\bar{\lambda}_{i'} = \lambda_{i'}$ for all i'. We will see that the general case follows.

General case ($\bar{\lambda}_{i'}$ not necessarily equal to $\lambda_{i'}$). The main observation is the following: if $\bar{\lambda}_{i,j} = \bar{\lambda}_{i-1,j} + 1$, and we let j' be such that $\bar{a}^{i+1}_j = a^{i+1}_{j'}$, then we necessarily have $\lambda_{i,j'} = \lambda_{i-1,j'} + 1$, and we cannot have any swaps in the i-th column involving the j'-th row. Indeed, the identity $\bar{\lambda}_{i,j} = \bar{\lambda}_{i-1,j} + 1$ means that we have $\bar{a}^i_j = \bar{a}^{i+1}_j$, which means that $a^{i+1}_{j'} = a^i_{j''}$ for the j'' such that exactly j values of a^i_m are less than $a^i_{j''}$. We also have exactly j values of a^{i+1}_m less than $a^{i+1}_{j'}$. It then follows that we must have $j'' = j'$: we cannot have $a^i_{j'} > a^i_{j''}$, since then we would have $a^i_{j'} > a^{i+1}_{j'}$. But if $a^i_{j'} < a^i_{j''}$, then j' occurs among the values of m with $a^i_m < a^i_{j''}$, so there is necessarily some m with $a^{i+1}_m < a^{i+1}_{j'}$ but $a^i_m \geq a^i_{j''}$, again leading to a contradiction. This proves the observation, noting that the fact that $j' = j''$ rules out any swaps involving the j'-th row.

Note that equations (5-1) and (5-2) can be phrased in terms of the values of $j - \lambda_{i',j} = a^{i'+1}_j - g(i')$. Using $\bar{\delta}_i$ to denote the values of δ coming from $\overline{T}$, our above observation implies that we have $a^i_{\delta_i} = a^{i+1}_{\delta_i} = \bar{a}^{i+1}_{\bar{\delta}_i} = \bar{a}^i_{\bar{\delta}_i}$. Therefore, the proof of these two equations follows from the case $\bar{\lambda}_{i'} = \lambda_{i'}$.

Next, suppose that we have some j with $\lambda_{i,j} < \lambda_{i-1,j}$. We claim that if j' is such that $a^{i+1}_j = \bar{a}^{i+1}_{j'}$, and j'' is such that $a^i_j = \bar{a}^i_{j''}$, then we necessarily also have that $\bar{\lambda}_{i,j'} < \bar{\lambda}_{i-1,j'}$ and $\bar{\lambda}_{i,j''} < \bar{\lambda}_{i-1,j''}$. Given this claim, (5-3) and (5-4) follow from the case that $\bar{\lambda}_{i'} = \lambda_{i'}$ for all i'. By our above observations on the case $\bar{\lambda}_{i,j} = \bar{\lambda}_{i-1,j} + 1$, it suffices to prove that $\bar{\lambda}_{i,j'} \neq \bar{\lambda}_{i-1,j'}$ and $\bar{\lambda}_{i,j''} \neq \bar{\lambda}_{i-1,j''}$, Equivalently, $\bar{a}^{i+1}_{j'} \neq \bar{a}^i_{j'} + 1$ and $\bar{a}^{i+1}_{j''} \neq \bar{a}^i_{j''} + 1$. In order to have $a^{i+1}_j = \bar{a}^{i+1}_{j'} = \bar{a}^i_{j'} + 1$, we would need to have $a^{i+1}_j - 1$ occurring among the $a^i_\bullet$, with precisely j' strictly smaller values also occurring. By definition we have j' values strictly smaller than a^{i+1}_j occurring in $a^{i+1}_\bullet$, and using our observation on lack of swaps when $\bar{\lambda}_{i,j} = \bar{\lambda}_{i-1,j} + 1$, we see that every one of these also must yield a value of $a^i_\bullet$ strictly smaller than $a^{i+1}_j - 1$. But we have in addition that $a^i_j < a^{i+1}_j - 1$, so we conclude that there are at least $j'+1$ values in $a^i_\bullet$ strictly less than $a^{i+1}_j - 1$, proving the desired inequality by contradiction.

Similarly, in order to have $\bar{a}^{i+1}_{j''} = \bar{a}^i_{j''} + 1 = a^i_j + 1$, we would need to have $a^i_j + 1$ occurring among the $a^{i+1}_\bullet$, with precisely j'' strictly smaller values also occurring. By definition, we have only j'' values among the $a^i_\bullet$ strictly smaller than a^i_j, and every value of $a^{i+1}_\bullet$ which is strictly smaller than $a^i_j + 1$ must come from one of these. But again using our observation on the lack of swaps when $\bar{\lambda}_{i,j} = \bar{\lambda}_{i-1,j} + 1$, we see that the value $a^i_j + 1$ in $a^{i+1}_\bullet$ must itself come from a row in $a^i_\bullet$ with value strictly smaller than a^i_j, so we conclude that if $a^i_j + 1$ occurs in $a^{i+1}_\bullet$, there must be strictly fewer than j'' entries in $a^{i+1}_\bullet$ which are strictly smaller than it. This proves the claim, and the lemma. □

6. An independence criterion

Suppose we have a limit linear series, and fix choices of sections s^i_j matching the vanishing orders in our table. We make the following definition:

Definition 6.1. Given an unimaginative multidegree ω, for all (j_1, j_2), let $n_{(j_1,j_2)}$ be the number of places (i.e., collections of contiguous columns) where the (j_1, j_2) row is potentially present in the multidegree ω. Let $s_{(j_1,j_2),i}$ for $i = 1, \dots, n_{(j_1,j_2)}$ be the induced sections in multidegree ω with precisely the given support. Then the full collection of $s_{(j_1,j_2),i}$ are the *potentially present* sections in multidegree ω, and their span in $\Gamma(X_0, (\mathscr{L}^{\otimes 2})_\omega)$ is the *potential ambient space*.

Note that in the above, we require that each $s_{(j_1,j_2),i}$ be potentially starting in its first column of support and potentially ending in its last column of support. Thus, there may be individual columns in which the (j, j') row satisfies the inequalities to be potentially present in that column, but which does not occur in any of the $s_{(j_1,j_2),i}$ because it fails inequalities in other columns.

The $s_{(j_1,j_2),i}$ are each unique up to scaling given a choice of the s^i_j. The s^i_j are not unique, but they can differ only by multiples of $s^i_{j'}$ with strictly higher vanishing at both points. Then if s^i_j has potential support (in the i-th column), necessarily $s^i_{j'}$ has a connected component of potential support consisting precisely of the i-th column. We conclude that the potential ambient space is independent of the choice of the s^i_j. Consequently, the dimension of the span — and in particular the linear independence — of the potentially appearing sections is likewise independent of choices.

Fix a multidegree ω and consider the images of $s_j \otimes s_{j'}$ focusing attention on potentially present sections. Assume we have a linear combination of images of sections equal to 0. As in [Liu et al. 2021],, we prove successively that the coefficient of particular sections must be zero. When this happens, we say that we "*drop*" that section and talk about *"remaining" sections* (i.e., those which have not yet been dropped).

Definition 6.2. We say that the i-th column of T is *semicritical* in multidegree ω if it satisfies the following conditions:

- it has a value of δ_i (see Definition 5.1), in particular, it has genus 1;

- on the potentially present sections remaining, the two subcolumns of column i exhibiting the minimal values add to at least $2d-2$;
- if the (j, δ_i) row remains in the i-th column for some $j \neq \delta_i$, then the j-th row is not exceptional.

If further the minimal values among the remaining potentially present sections are not both one less than the values in the (δ_i, δ_i) row, we say that the i-th column is *critical*.

We start with the following independence criterion:

Lemma 6.3. *For a given limit linear series, and given unimaginative multidegree ω, we can drop potentially present sections if they satisfy the following rules*:

(i) *If in some column i, there is a unique remaining potentially present section supported in that column having minimal $a^i_{(j,j')}$ value, or a unique one having minimal $b^i_{(j,j')}$, then the one achieving the minimum may be dropped.*

(ii) *If there are at most two remaining potentially present sections with support in some genus-*1 *column i, and neither of them involves an exceptional row,* (*see Definition 5.1*), *then they can both be dropped.*

(iii) *If there are $i < i'$ such that the block of columns from i to i' has the following properties, then all the remaining potentially present sections supported in this block can be dropped*:

- *There are at most* 3 *remaining potentially present sections supported in each of the i-th and i'-th columns.*
- *Within the block, there are at most three potentially present sections continuing from any column to the next.*
- *Every column strictly between i and i' has degree* 2.
- *Both the i-th column and the i'-th column are semicritical and either i is critical with no remaining potentially present section ending in the i-th column, or i' is critical with no remaining potentially present section starting in the i'-th column.*

Proof. Suppose we had a hypothetical linear dependence among the potentially present sections. We claim that in each case (i), (ii), (iii), the coefficients of the relevant potentially present sections would be forced to vanish.

In case (i), the uniqueness of the minimal value of $a^i_{(j,j')}$ (or of $b^i_{(j,j')}$) means that $s^i_{(j,j')}$ vanishes to strictly smaller order at P_i than any other remaining potentially present section, which forces us to drop that section.

In case (ii), we need to see that for a fixed column i, any two $s^i_{(j,j')}$ have to be linearly independent provided that they do not involve any exceptional rows. If either of them involves δ_i, this is automatic, since either the $a^i_{(j,j')}$ or $b^i_{(j,j')}$ values are forced to be distinct. On the other hand, if neither involves δ_i, we claim that the sections in question must have distinct zeroes on Z_i away from P_i and Q_i. Indeed, if we have a, b, a', b' with $a+b=d-1=a'+b'$, then the unique sections s, s' of the given line bundle vanishing to order at least a at P_i and b at Q_i (respectively, a' at P_i and b' at Q_i) have

$$\operatorname{div} s = aP_i + bQ_i + R \quad \text{and} \quad \operatorname{div} s' = a'P_i + b'Q_i + R'$$

for some R, R'. We see that we have a linear equivalence $R - R' \sim (a'-a)P_i + (b'-b)Q_i$, and if $0 \le a, a' \le d$, we see that $R \neq R'$ because of the generality hypothesis on P_i, Q_i. Thus, tensors of different sections of this form always have zeroes in distinct places on Z_i, and must be linearly independent.

For case (iii), note that the condition that the degree is 2 on every column between i and i' means by Corollary 5.8 that there is at most one potentially starting and at most one potentially ending section in each of these columns. Noting that the situation is fully symmetric, suppose without loss of generality that i' is critical, with no remaining potentially present sections starting in it. If i or i' has fewer than three remaining potentially present sections, we may use (ii) to drop these, and then move iteratively through the rest of the block, using that at most one section starts or ends in each column together with (i) to drop the remaining sections. Thus, suppose that i and i' both have three remaining potentially present sections. Note also that if any column i'' has only one potentially present section spanning i'' and $i''+1$, then the minimal value in the right subcolumn of i'' is necessarily unique, so we can use case (i) to drop the section in question. Moreover, there can be at most one other potentially present section supported in column i'' (the one ending there), so we can drop this one as well, and then we can move iteratively left and right to drop the entire block. Thus, we may further suppose that every column in the block has at least two potentially present sections spanning it and the next column.

Next, normalize the sections as follows: scale all sections spanning the $i'-1$ and i' column so they agree at $Q_{i'-1}$, and then go back one column at a time, scaling any newly appearing section so that its value at the previous node agrees with the value of a section which has already been fixed. In this way, we will fix a normalization of all the sections except for those which are supported in only one column. Although the normalization depends on some choices, they are of a discrete nature, and can be fixed based purely on the discrete data of the limit linear series.

Now, consider a hypothetical nonzero linear dependence involving the rows in our block. First, the coefficients of the linear dependence cannot vanish identically in the remaining potential sections of any column, since otherwise the condition that at most one potentially present section ends in each column would imply that there was a column with exactly one nonzero coefficient among its remaining sections. Next, we see that the coefficients are unique up to simultaneous scaling for the three potentially present sections in column i. Indeed, since we have assumed that i is semicritical, its three potentially present sections must be pairwise independent.

Since we have at most one new potentially present section in each column, we find that the coefficient for any new one is always uniquely determined by the previous ones. Since there are no new potentially present sections in column i', we find that even before considering this column, we have already uniquely determined all of the coefficients (up to simultaneous scaling) of all of the potentially present sections remaining in the block. Moreover, we claim that these coefficients (excluding the ones for potentially present sections supported only in a single column) are uniquely determined up to finite indeterminacy by the marked curves $Z_i, \dots, Z_{i'-1}$ together with the discrete data of the limit linear series. Indeed, there are only two ways in which nontrivial moduli can enter the picture: if there are columns i'' between i and $i'-1$ either having no $\delta_{i''}$, or having some sections $s_j^{i''}$ which are not uniquely determined up to scalar. This becomes slightly delicate, since in both these cases, varying the moduli could affect both the normalization we have chosen and the linear dependence. However, we will show that in both cases, there will in fact be only finitely many possibilities which still preserve the linear dependence. Note that by hypothesis, neither of these nontrivial moduli occurs in the i-th column. Note also that we cannot have both occurring at once, as the $s_j^{i''}$ can only fail to be determined up to scalar if they involve an exceptional row, and since we have assumed we have degree 2 between i and i', these can only appear if paired with the $\delta_{i''}$ row.

First consider the case that we have no $\delta_{i''}$. Then since we have degree 2, every potentially present section in column i'' must extend to both the preceding and subsequent columns. By assumption, there are then at most three such sections. If there are fewer than three, they cannot be independent, leading to an immediate contradiction. As a line bundle of degree two on an elliptic curve has at most two independent sections, if there are three, say $s_0^{i''}, s_1^{i''}, s_2^{i''}$, they are necessarily dependent, with a unique dependence $c_0 s_0^{i''} + c_1 s_1^{i''} + c_2 s_2^{i''} = 0$ which can be determined by requiring that it holds at both $P_{i''}$ and $Q_{i''}$. We claim that for any fixed choice of c_0, c_1, c_2 (not all zero), there are only finitely many choices of the line bundle $\mathscr{L}^{i''}$ such that the resulting cancellation holds at both points. For this claim, we can renormalize our sections so that the values of the $s_j^{i''}$ agree at $P_{i''}$. We want to see that the values at $Q_{i''}$ move nondegenerately in $\mathbb{P}^2$ as $\mathscr{L}^{i''}$ varies. But this is precisely the content of Proposition 2.5.

Next, suppose that we have an exceptional row j involved in column i'', necessarily paired with the $\delta_{i''}$ row. As before, a linear dependence in the i'' necessarily has to give cancellation at both $P_{i''}$ and $Q_{i''}$. Suppose that the j-th row and the $\delta_{i''}$-th row have entries a, b and a', b' respectively, so that $a+b=d-2$ and $a'+b'=d$. There are two cases: if $a=a'-1$, so that also $b=b'-1$ (and i'' has a swap in it), then the moduli for the section $s_j^{i''}$ consists simply of adding multiples of the section $s_{\delta_{i''}}^{i''}$, which doesn't affect the value at either $P_{i''}$ or $Q_{i''}$, and only affects the coefficient of the $(\delta_{i''}, \delta_{i''})$ row, which in this case is supported purely in the i'' column. On the other hand, if $a \neq a'-1$, observe that since the degree is 2 in this column, we cannot have any other sections involving $\delta_{i''}$ starting or ending in the column, and therefore we have no sections starting or ending in the column. Thus, there are at most three potentially present sections in column i'', and the other ones can't involve any exceptional row and must therefore be linearly independent. It follows that in our linear dependence, the coefficient of $s_{(j,\delta_{i''})}$ must be nonzero. Now, varying s_j will change the relationship between the values at $P_{i''}$ and $Q_{i''}$ (we can view the moduli for s_j as adding multiples of a section vanishing to order $a+1$ at $P_{i''}$ and order b at $Q_{i''}$). Since this variation of moduli affects only a single potentially present section, and we know it must have nonzero coefficient in our linear dependence, there is only one choice of $s_j^{i''}$ compatible with the previously determined linear dependence, and we have no nontrivial moduli in this case.

Finally, note that although our normalization was not determined for potentially present sections supported in a single column, scaling these does not affect the coefficients of any of the sections spanning the $i'-1$ and i' column, so we have that the possible coefficients of these sections are determined up to finitely many possibilities. It thus suffices to show that if we vary the gluing points on the component corresponding to the final column, the (unique, if it exists) linear independence on the three potentially present sections varies nontrivially.

As there are no remaining potentially present section starting in the i'-th column, the three rows in its left subcolumn necessarily have the same a value. Let b be the minimal value for the right subcolumn. By criticality, $a+b=2d-2$. Using (i), there are two cases to consider, either b is attained twice, or in all three rows. The last condition in the definition of criticality implies that none of the (a,b) rows are obtained by adding the $\delta_{i'}$ row to an exceptional row. Now, if all three rows are (a,b) rows, we can directly apply Proposition 2.2 to conclude that the linear dependence in the i'-th column varies nontrivially with $P_{i'}$, $Q_{i'}$, as desired. On the other hand, if two rows are (a,b) rows, we again apply Proposition 2.2 to these two rows, and since we have normalized all three rows so that the values at $P_{i'}$ agree, we again see that the linear dependence among the three has to vary nontrivially with $P_{i'}$, $Q_{i'}$, as desired. □

7. The case $r=6$

We now specialize to $r=6$, $g=21+\epsilon$ and $d=24+\epsilon$ for some $\epsilon\geq 0$ (so that $\rho=\epsilon$). As the total degree is $2d=2g+6=3\times 6+2\times(g-6)$, a multidegree can be determined by placing threes in six columns, and twos in the rest.

Definition 7.1. For a limit linear series and with the $\bar{\lambda}_i$ of Definition 5.11, the *default multidegree* ω_{def} is determined by placing a 3

(1) in the first column;

(2) in the first column with $\bar{\lambda}_i^1+\bar{\lambda}_i^2=5$;

(3) in the first column with $\bar{\lambda}_i^1+\bar{\lambda}_i^3=7$;

(4) in the column immediately after the last column with $\bar{\lambda}_i^1+\bar{\lambda}_i^3=7$;

(5) in the column immediately after the last column with $\bar{\lambda}_i^2+\bar{\lambda}_i^3=9$;

(6) in the last column.

As we are assuming that the first and last component in the chain have genus one and $\bar{\lambda}_i^\ell$ can only increase in a genus-1 column, the default multidegree is unimaginative. Also, as the goal is to fill an $(r+1)(g-d+r)=7\times 3$ rectangle, conditions (2)–(5), (3)–(4) and of course (1)–(6) are symmetric with respect to flipping the start and end of the chain. Recall that an unimaginative multidegree has degrees 2 or 3 on every elliptic component and that γ_i denotes the number of 3s in the first i components(see Definition 5.5).

Proposition 7.2. *Fix an unimaginative multidegree. Then for a column i, there can be at most three rows spanning columns i and $i+1$ except in the following circumstances*:

(i) $\gamma_i=0$ *and* $\bar{\lambda}_i^1+\bar{\lambda}_i^3\geq 8$;

(ii) $\gamma_i = 2$ *and* $\bar{\lambda}_i^1 + \bar{\lambda}_i^3 \geq 7$;

(iii) $\gamma_i = 4$ *and* $\bar{\lambda}_i^1 + \bar{\lambda}_i^3 \leq 7$;

(iv) $\gamma_i = 6$ *and* $\bar{\lambda}_i^1 + \bar{\lambda}_i^3 \leq 6$.

In particular, in the default multidegree there are never more than three rows spanning a given pair of columns.

Proof. We will use the criterion from Proposition 5.7. Since this only involves the values of $j - \lambda_{i,j} = a_j^{i+1} - g(i)$, the general case reduces immediately to the notationally simpler situation that $\bar{\lambda}_i = \lambda_i$ for all i. We thus assume that we are in this situation. Then, because the sequence $j - \lambda_{i,j}$ is strictly increasing in j, we see that pairs (j_1, j_2) satisfying the identity for appearing in the i-th and $(i+1)$-st columns from Proposition 5.7 must be strictly nested, so we can have at most $r/2+1 = 4$ of them, and we only have all of these if $\lambda_{i,j} + \lambda_{i,r-j}$ is constant for all j, in particular $2\lambda_{i,r/2} = \lambda_{i,0} + \lambda_{i,r}$. Moreover,

$$\gamma_i = r - 2\lambda_{i,r/2} = 6 - 2\lambda_{i,r/2}, \qquad \gamma_i = r - \lambda_{i,j} - \lambda_{i,r-j}, \quad j = 0, \dots, r;$$

in particular γ_i is even. Adding these identities, we find that

$$\sum_{j=0}^{r} \lambda_{i,j} = \frac{(r+1)(r-\gamma_i)}{2} = 7\big(3 - \tfrac{1}{2}\gamma_i\big),$$

so $\lambda_{i,r/2} = 3 - \gamma_i/2$.

If $\gamma_i = 0$, then $\lambda_{i,r/2} = 3$. As $r/2 = 3$ and we start numbering the sections at 0, there are at least 4 values of λ that contribute to $\bar{\lambda}_i^3$ (and therefore to $\bar{\lambda}_i^1$). We conclude that $\bar{\lambda}_i^1 + \bar{\lambda}_i^3 \geq 8$.

If $\gamma_i = 2$, then $\lambda_{i,r/2} = 2$. Let n be the number of values of j with $\lambda_{i,j} \leq 0$. Then also $\lambda_{i,r-j} \geq 4$ for the same n values of j. So $\lambda_i^1 + \lambda_i^3 \geq (r+1-n) + n = 7$, as desired.

Similarly, if $\gamma_i = 4$ then $\lambda_{i,r/2} = 1$. If there are n values of j with $\lambda_{i,j} \geq 3$, then also $\lambda_{i,r-j} \leq -1$. As before we find $\lambda_i^1 + \lambda_i^3 \leq (r+1-n) + n = 7$.

Finally, if $\gamma_i = 6$ then $\lambda_{i,r/2} = 0$. Therefore, $\bar{\lambda}_i^1 + \bar{\lambda}_i^3 \leq 6$, as claimed. □

We can now prove the following theorem, which will in particular prove the desired maximal rank statement in all sufficiently nondegenerate cases for all ϵ in our family of cases. It will also suffice to prove the genus-22 case of our main theorem.

Theorem 7.3. *In the default multidegree, we can always drop all potentially present sections using the rules from Lemma 6.3, so the potentially present sections are all linearly independent.*

Proof. The vanishing of sections of a line bundle of degree d at Q_1 is at most either d or $d-1$(but not both), $d-2, d-3 \dots$. So, at most the rows (0, 0), (0, 1) and (0, 2) can be among the potentially present if there is no swap and at most the rows (0, 1) and (1, 1) can be potentially present if there is a swap. In both cases, these sections have distinct orders of vanishing at Q_1, so they can be dropped.

According to Corollary 5.8, we will have at most one new row with a potentially present section in each column until we get to the next column of degree 3, so these can all be dropped.

Now, suppose that i is minimal such that $\bar{\lambda}_i^1 + \bar{\lambda}_i^2 = 5$. Take $\ell_1 = 1$ and $\ell_2 = 2$ in Lemma 5.12, so $\gamma_i - 1 = 1 = 5 - 1 - \ell_1 - \ell_2$. From Lemma 5.12 and Corollary 5.9, we have no potentially present sections supported entirely in the i-th column. Any other new potentially present sections would have to be supported in the i-th and $(i+1)$-st columns, so by Proposition 7.2, we have at most three of these. Choose i' minimal so that $\bar{\lambda}_{i'}^1 + \bar{\lambda}_{i'}^2 = 6$, then with $\ell_1 = 1$ and $\ell_2 = 2$, $\gamma_{i'} = 2 = 6 - 1 - \ell_1 - \ell_2$. According to Corollary 5.8 and Lemma 5.12, there is no row starting in the i'-th column. Then the i-th (respectively, i'-th) columns are critical: if a, b are the minimum values in the subcolumns, they have to add to at least $2d - 2$ or the rows would not be potentially starting in the i-th column (respectively, potentially supported in the i'-th column). The last condition of semicriticality and the condition for criticality then follow from the second and first parts of Lemma 5.12, respectively. It follows that the hypotheses of Lemma 6.3(iii) are satisfied, so we can drop all rows occurring in this block. We can then again handle any additional columns before the next degree-3 one.

The setup being symmetric, we can also start at the end of the chain and in the same manner, eliminating all potentially present sections occurring in any columns outside the middle two degree-3 columns. For these columns, we are considering $\ell_1 = 1$ and $\ell_2 = 3$, so we have

$$\gamma_{i+1} - 1 = 2 = 7 - 1 - \ell_1 - \ell_2 \quad \text{and} \quad \gamma_{i+1} - 1 = 3 = 8 - 1 - \ell_1 - \ell_2,$$

respectively, and according to Corollary 5.9 and Lemma 5.12, neither column has any potentially present section supported entirely in it. As before, we find we must have a block satisfying the hypotheses of Lemma 6.3(iii), which we can then eliminate. □

If the specialization of our linear series contains the "expected" sections s_j for every $j = 0, \dots, r$ in the expected multidegrees ω_j (as in Proposition 4.9), then Theorem 7.3 implies that the images of each $s_j \otimes s_{j'}$ in the default multidegree are linearly independent, so the multiplication map has the desired rank $\binom{r+2}{2} = 28$. However, some linear series may have more degenerate specializations. The remainder of the paper will be devoted to applying Theorem 7.3 (and variants thereof) to handle these situations as well. For this, the statement in terms of potentially present sections (as opposed to the separate rows considered in [Liu et al. 2021]) is crucial. In interesting cases, we can have strictly more than 28 potentially present sections. This does not contradict the fact that the multiplication map can have rank at most 28, because these do not occur separately in the linked linear series coming as the specialization of any fixed family of linear series on the smooth fibers. In most limits, for every (j_1, j_2) we will have a unique linear combination of the potentially present sections in the (j_1, j_2) row which actually arise in the specialization. What makes the degenerate cases more interesting is that in these cases, we may have more than one linear combination occurring from a given row, precisely in situations where the specialization fails to contain any potentially present sections from some other row — see Example 8.3.

Ultimately, the default multidegree used in Theorem 7.3 will be sufficient to handle the genus-22 case, and most of the genus-23 cases. However, for certain degenerate cases we will need to consider other multidegrees instead. The following results allows for some flexibility in the choice of multidegree while maintaining linear independence.

Proposition 7.4. *Suppose ω is an unimaginative multidegree determined by placing degree 3 in genus-1 columns as follows*:

(1) *In one column which is either the first, or a column with no exceptional rows and satisfying $\bar{\lambda}_i^1+\bar{\lambda}_i^2 \leq 4$ and $\bar{\lambda}_{i,0} \leq 2$.*

(2) *In one column with $\bar{\lambda}_i^1+\bar{\lambda}_i^2 = 5$ but $\bar{\lambda}_{i-1}^1+\bar{\lambda}_{i-1}^2 = 4$.*

(3) *In one column between the first column with $\bar{\lambda}_i^1+\bar{\lambda}_i^2 = 6$ and the first column with $\bar{\lambda}_i^1+\bar{\lambda}_i^3 = 7$ (inclusive).*

(4) *In one column between the column immediately after the last column with $\bar{\lambda}_i^1+\bar{\lambda}_i^3 = 7$ and the column immediately after the last column with $\bar{\lambda}_i^2+\bar{\lambda}_i^3 = 8$ (inclusive).*

(5) *In one column with $\bar{\lambda}_i^2+\bar{\lambda}_i^3 = 10$ but $\bar{\lambda}_{i-1}^2+\bar{\lambda}_{i-1}^3 = 9$.*

(6) *In one column which is either the last, or a column with no exceptional rows and satisfying*

$$\bar{\lambda}_{i-1}^2+\bar{\lambda}_{i-1}^3 \geq 10 \quad \text{and} \quad \bar{\lambda}_{i-1,6} \geq 1.$$

Then the potentially present sections in multidegree ω are still linearly independent.

Proof. The main new ingredient is verifying that if we place the first degree 3 in a (genus-1) column after the first, but still satisfying $\bar{\lambda}_i^1+\bar{\lambda}_i^2 \leq 4$ and $\bar{\lambda}_{i,0} \leq 2$, then provided we also have no exceptional rows (and therefore no swaps), we will in fact obtain at most two potentially present sections starting in the i-th column. By Proposition 5.7, for the (j, j') row to have a potentially present section starting in the i-th column, we will need $j + j' - \lambda_{i-1,j} - \lambda_{i-1,j'} > \gamma_{i-1} = 0$ and $j + j' - \lambda_{i,j} - \lambda_{i,j'} \leq \gamma_i = 1$, or equivalently

$$\lambda_{i-1,j} + \lambda_{i-1,j'} < j + j' \leq 1 + \lambda_{i,j} + \lambda_{i,j'}. \tag{7-1}$$

As there are no swaps in the i-th column, it suffices to check this assertion with $\bar{\lambda}_i = \lambda_i$ for all i. Then, $\bar{\lambda}_i^1+\bar{\lambda}_i^2 \leq 4$ implies $\lambda_{i,j} \leq 0$ for $j \geq 4$ and $\lambda_{i,j} \leq 1$ for $j = 2, 3$. It follows that to satisfy the right-hand inequality above, we must have at least one of j, j' equal to 0 or 1. Moreover, by Corollary 5.9 for $j = 0, 1$, if $j \neq \delta_i$, then there is at most one value of j' satisfying the above inequalities. In particular, we conclude that if $\delta_i \neq 0, 1$, there are at most two potentially present sections, as claimed.

Assume $\delta_i = 0$, and show that at most two rows of the form $(0, j')$ are present in the i-th column, and if two are present, then none of the form $(1, j')$ is for $j' > 0$. Suppose first $(0, 0)$ is potentially starting in the i-th column. By (7-1) this could only happen if $\lambda_{i-1,0} < 0$, so $\lambda_{i,j'} = \lambda_{i-1,j'} < 0$ for all $j' > 0$, and then $(0, j')$ cannot satisfy the right-hand side of (7-1) for any $j' > 0$. On the other hand, if $j'' > j' > 0$ are such that $(0, j')$ and $(0, j'')$ are both present, then

$$\lambda_{i-1,0} + \lambda_{i-1,j'} < j' < j'' \leq 1 + \lambda_{i,0} + \lambda_{i,j''} \leq 2 + \lambda_{i-1,0} + \lambda_{i-1,j'},$$

so the only possibility is that $j' = 1 + \lambda_{i-1,0} + \lambda_{i-1,j'}$ and $j'' = j' + 1$, with $\lambda_{i,j''} = \lambda_{i,j'}$. It follows that no other $(0, j''')$ is present for $j''' \neq 0, j', j''$. Moreover, $(1, j''')$ cannot be potentially present for any $j''' > 0$ in this situation: If the $(1, 1)$ row were present, (7-1) implies $\lambda_{i-1,1} \leq 0$, $\lambda_{i,1} \geq 1$ against the assumption $\delta_i = 0$. As $1 + j'''$ will be too large if $j''' \geq j'$, in order to have $(1, j''')$, $j''' \geq 2$ present we would need $j'' \geq 4$. But the original assumptions imply $1 + \lambda_{i,0} + \lambda_{i,j''} \leq 3$, contradicting (7-1).

Finally, consider the case that $\delta_i = 1$. If the $(1, 1)$ row is potentially starting in the i-th column, by parity we have $1 = \lambda_{i,1}$. So for all $j > 1$ $\lambda_{i,j} = \lambda_{i-1,j} \leq \lambda_{i-1,1} = 0$. Then we cannot have $(1, j')$ potentially starting for any $j' > 1$, so we have at most two rows potentially starting. On the other hand, if we have $j'' > j' > 1$ potentially starting in the i-th column, we are just as above forced to have $j' = \lambda_{i-1,1} + \lambda_{i-1,j'}$ and $j'' = j' + 1$, with $\lambda_{i,j''} = \lambda_{i,j'}$, and we claim we cannot have $(0, j''')$ potentially starting for any j'''. Indeed, if $j''' \leq j''$, then we have $j''' - \lambda_{i-1,j'''} \leq j'' - \lambda_{i-1,j''}$, so

$$\begin{aligned} j''' &\leq j'' - \lambda_{i-1,j''} + \lambda_{i-1,j'''} \\ &= 1 + \lambda_{i-1,1} + \lambda_{i-1,j''} - \lambda_{i-1,j''} + \lambda_{i-1,j'''} \\ &\leq \lambda_{i-1,0} + \lambda_{i-1,j'''}, \end{aligned}$$

violating (7-1). But $j'' \geq 3$, so if $j''' \geq 4$ we cannot satisfy (7-1) without violating our hypothesis that $\lambda_{i-1,0} \leq 2$. We thus conclude the desired statement on the number of potentially present sections starting in column i.

Now, since we have assumed that our first column with degree 3 has no exceptional rows, the fact that it has at most two potentially present sections starting in it means that we can still eliminate sections starting at the beginning of the chain until we reach the second column of degree 3, just as in the proof of Theorem 7.3 and the second column of degree 3 will still be critical, with at most three potentially present sections starting in it. The next step depends on the location of the third column of degree 3. If the first column with $\bar{\lambda}_i^1 + \bar{\lambda}_i^2 = 6$ still has degree 2, we will eliminate this block in increasing order, as before. On the other hand, if the first column with $\bar{\lambda}_i^1 + \bar{\lambda}_i^3 = 7$ has degree 2, we do not need to have eliminated everything in components with smaller indices in order to eliminate the central block, since the potentially supported rows in multidegree ω will be precisely the same as the potentially starting rows in ω_{def}. Thus, if the third column of degree 3 is strictly between these, we can eliminate both adjacent blocks first, and then eliminate all potentially present sections one by one from both sides until we reach this final column, which can have at most one remaining potentially present section by Corollary 5.9. However, if the third column of degree 3 is the first column with $\bar{\lambda}_i^1 + \bar{\lambda}_i^2 = 6$, we see that this will be critical with at most three potentially present sections ending in it, and we will instead eliminate the central block first, and then eliminate the block between the second and third columns of degree 3 last.

The situation is symmetric on the right, so we see that in all cases we will be able to eliminate all potentially present sections in a suitable order. □

It will also be important to consider moving degree 3 into a column with a swap, which we analyze below:

Lemma 7.5. *Choose a multidegree* $\mathrm{md}_{2d}(w)$ *for* $w = (c_2, \dots, c_g)$ *that assigns degree* 3 *to the* i*-th column having a* δ_i. *If* $a^i_{\delta_i} = a^i_{j_0} + 1,\ a^{i+1}_{\delta_i} = a^{i+1}_{j_0} - 1$ *for some* j_0, *then there are at most four potentially starting* *sections on the i-th column. Moreover, if there are actually four either*

$$2\bar{a}^i_3 = c_i + 1, \tag{7-2}$$

or (δ_i, δ_i) is potentially starting, and one of the following three possibilities holds:

(1) $a^i_{(\delta_i,\delta_i)} = c_i + 1$;

(2) $a^i_{(\delta_i,\delta_i)} = c_i + 2$, *and the (j_0, δ_i) row is potentially starting*;

(3) $a^i_{(\delta_i,\delta_i)} = c_i + 3$, *with* $a^i_{\delta_i} = \bar{a}^i_4$.

At most four potentially present sections end in the i-th column, with four of them ending only if either

$$2\bar{a}^i_3 = c_i, \tag{7-3}$$

or if (δ_i, δ_i) is potentially ending, and one the following three possibilities holds:

(1) $a^i_{(\delta_i,\delta_i)} = c_i$, *with* $a^i_{\delta_i} = \bar{a}^i_3$;

(2) $a^i_{(\delta_i,\delta_i)} = c_i + 1$, *and the (j_0, δ_i) row is potentially ending*;

(3) $a^i_{(\delta_i,\delta_i)} = c_i + 2$;

We have written the above to allow for swaps having occurred prior to the i-th column. If no swaps have occurred, the j_0 in the lemma statement is necessarily $\delta_i - 1$, and the third exceptional case would require $\delta_i = 4$ (respectively, $\delta_i = 2$) in the statement on potential support starting (respectively, ending).

Proof. In order to have a potentially starting section in the (j, j') row, one needs $a^i_{(j,j')} > c_i$ and $a^{i+1}_{(j,j')} \le c_{i+1} = c_i + 3$. It follows that if $j, j' \neq \delta_i$, then $a^i_{(j,j')} = c_i + 1$, and neither j nor j' equal to j_0. If $j' = \delta_i,\ j \neq \delta_i$, it is possible that $a^i_{(j,\delta_i)} = c_i + 2$, provided that $j \neq j_0$. And (j_0, δ_i) is potentially starting only if $a^i_{(j_0,\delta_i)} = c_i + 1$, or equivalently if $a^i_{(\delta_i,\delta_i)} = c_i + 2$. Recall that Corollary 5.9 says that if (j, δ_i) is potentially starting for some $j \neq \delta_i$, then there is no $j' \neq \delta_i$ with (j, j') also potentially starting. Next, we note that we can have at most two rows of the form (j, δ_i) potentially starting. Indeed, if $a^i_{(\delta_i,\delta_i)} = c_i + 3$, then $a^i_{(j,\delta_i)} = c_i + 2$ only for $j = j_0$, so the (j_0, δ_i) row does not occur, and we can have at most one additional row, having $a^i_{(j,\delta_i)} = c_i + 1$. On the other hand, if $a^i_{(\delta_i,\delta_i)} \neq c_i + 3$, then we have at most two rows, because they have to satisfy $a^i_{(j,\delta_i)} = c_i + 1$ or $c_i + 2$. We also observe that we can have a row of the form (j, j) for $j \neq \delta_i$ only for a unique choice of j, necessarily with $2a^i_j = c_i + 1$ and $j \neq j_0$, and then we cannot have (δ_i, δ_i) occurring, since $a^i_j \neq a^i_{\delta_i} - 1$ for $j \neq j_0$.

If no (j, δ_i) is potentially starting, then in particular (j_0, δ_i) also cannot be potentially starting. We can obtain at most three pairs (allowing one of them to have repeated entries) with fixed sum of vanishing. Similarly, if exactly one (j, δ_i) is potentially starting, then necessarily $j \neq j_0$, or we would be in the 2nd exceptional case with also (δ_i, δ_i) potentially starting, so for the remaining pairs we must choose from values not equal to δ_i, j_0, j, leaving four values, and at most two pairs. We therefore see that in order to have four rows potentially starting, two of them need to involve δ_i.

If (j_1, δ_i) and (j_2, δ_i) are potentially starting, with neither j_1, j_2 equal to δ_i (and hence also neither equal to j_0), then any remaining rows have to be chosen as distinct pairs from the remaining $(r+1)-4 = 3$

indices, with at most one pair having repeated value. We thus obtain at most four rows, with four occurring only if $2a^i_j = a^i_{j_3} + a^i_{j_4} = c_i + 1$ for some $j, j_3, j_4 \neq \delta_i, j_0, j_1, j_2$. Moreover, we see that there must be exactly three values of j' with $a^i_{j'} < a^i_j$ in this case: if $a^i_{\delta_i} < a^i_j$, then these are δ_i, j_0, and exactly one of j_3, j_4, with necessarily j_1, j_2 and the other of j_3, j_4 having $a^i_{j'} > a^i_j$. If $a^i_{\delta_i} > a^i_j$, then $a^i_{j_0}$ must also be greater than a^i_j, so we similarly find exactly three values are smaller. Thus (7-2) must hold.

It remains to consider the case the (δ_i, δ_i) row is potentially starting, and the only thing left to prove is the description of case (3), where $a^i_{(\delta_i,\delta_i)} = c_i + 3$. Here, we must also have a j with $a^i_{(j,\delta_i)} = c_i + 1$, and if we have two additional rows appearing, these must come from two additional pairs nested around $a^i_{(j,\delta_i)}$, so since $a^i_j < a^i_{j_0} < a^i_{\delta_i}$ in this case, we obtain the desired statement.

The statement on rows ending is symmetric. □

We are now ready to deal with the two possible cases of 3-cycles in the next two corollaries.

Corollary 7.6. *Suppose that $\rho = 2$ and $r = 6$ and we are in the "first 3-cycle" situation of Proposition 4.18. Then, there exists an unimaginative multidegree ω' such that the $(j_0 - 1, j_0)$ row has a unique potentially present section in multidegree ω', whose support does not contain i_0 or i_1, and such that all the potentially present sections are linearly independent.*

Proof. Consider the default multidegree ω_{def}. If all (j_1, j_2) have connected potential support, we are done. With the notation of Proposition 4.18, the only rows that have a semicolumn adding to $d-2$ are $j_0, j_0 - 1$. From Proposition 5.10, the only row which could have disconnected potential support in some unimaginative multidegree is $(j_0 - 1, j_0)$. More specifically, the potential support of $(j_0 - 1, j_0)$ can be disconnected only if $a^{i_0}_{j_0-1} + a^{i_0}_{j_0} = c_{i_0} - 1$, $a^{i_1+1}_{j_0-1} + a^{i_1+1}_{j_0} = c_{i_1+1} + 1$, there is degree 2 in every column from i_0 to i_1 inclusive, and no δ_i equals $j_0 - 1$ or j_0 for any i between i_0 and i_1. It then follows in particular that $a^{i_0}_{(j_0+1, j_0+1)} \geq c_{i_0} + 2$ and $a^{i_1+1}_{(j_0+1, j_0+1)} \leq c_{i_1+1} - 2$, or equivalently, $a^{i_1}_{(j_0+1, j_0+1)} \leq c_{i_1}$. If the $(j_0 - 1, j_0)$ row has disconnected potential support, then we will use Lemma 7.5 to verify that we can move one degree 3 into either the i_0 or i_1 column to achieve connected potential support while maintaining the independence conclusion of Theorem 7.3. If the 3 was moved to the i_0 column, then the $(j_0 - 1, j_0)$ still cannot have any potential support at i_1. If the 3 was moved from the right, we still have $a^{i_0}_{j_0-1} + a^{i_0}_{j_0} = c_{i_0} - 1$, ruling out potential support at i_0. But if it was moved from the left, then this will decrease c_{i_0} by 1, and we will then have $a^i_{j_0-1} + a^i_{j_0} = c_i$ for $i_0 \leq i \leq i_1$, meaning that any potential support at i_0 would have to continue right to i_1, but we will still have $a^{i_1+1}_{j_0-1} + a^{i_1+1}_{j_0} = c_{i_1} + 1$, so there cannot be any potential support at i_1. A similar analysis holds if we moved the 3 to i_1, proving the desired result.

To prove that we can always move a 3 as desired, we first make some general observations regarding when we will be able to move degree 3 from the left or right onto i_0 or i_1. Recall that, from the assumption of having a 3-cycle, $\delta_{i_0} = \delta_{i_1} = j_0 + 1$. Since $a^{i_1}_{(j_0+1, j_0+1)} \leq c_{i_1}$, moving a degree 3 to i_1 from the right will always lead to at most 3 rows starting in the i_1 column, unless $2\bar{a}^{i_1}_3 = c_{i_1} + 1$, or equivalently,

$$5 - \gamma_{i_1-1} = 2\bar{\lambda}_{i_1-1,3}. \tag{7-4}$$

In addition, $a^{i_1}_{(j_0-1, j_0+1)} < c_{i_1}$, so the $(j_0 - 1, j_0 + 1)$ row will not be among the potentially present rows.

We next consider what happens if we move a degree 3 to i_0 from the left. This will decrease c_{i_0} by 1, so we have to rule out that in multidegree ω_{def} we have $2\bar{a}_3^{i_0} = c_{i_0}$, or equivalently,

$$6 - \gamma_{i_0-1} = 2\bar{\lambda}_{i_0-1,3}. \tag{7-5}$$

Additionally, if $a^{i_0}_{(j_0+1,\,j_0+1)} \geq c_{i_0} + 3$ in ω_{def}, then after moving the degree 3 to i_0, none of the other exceptional cases of Lemma 7.5 can occur, so as long as we do not have (7-5), we will have at most three rows with potential support starting at i_0. The only other possibility is that $a^{i_0}_{(j_0+1,\,j_0+1)} = c_{i_0} + 2$, which is equivalent to $2j_0 - \gamma_{i_0-1} = 2\lambda_{i_0-1,\,j_0+1}$; moreover, after moving a 3 from the left to i_0 we will have $a^{i_0}_{(j_0+1,\,j_0+1)} = c_{i_0} + 3$, so we could potentially be only in the third exceptional case in Lemma 7.5. Thus, the only case for concern is that $j_0 + 1 = 4$, so we simply need to check that in cases where we wish to move a 3 from the left, we never have

$$6 - \gamma_{i_0-1} = 2\lambda_{i_0-1,4}. \tag{7-6}$$

Finally, in either case after the move we will have $a^{i_0}_{(j_0,\,j_0+1)} = a^{i_0}_{(j_0+1,\,j_0+1)} - 1 \geq c_{i_0} + 2$, so the $(j_0,\, j_0+1)$ row cannot be among the rows starting at i_0.

We now describe how to modify our default multidegree, depending on the location of i_0 and i_1. If we have $\gamma_{i_0} = \gamma_{i_1} = 1$, then we will move the next 3 from the right to column i_1, and we will obtain at most three rows with potential support starting in i_1: by the above observation, it suffices to rule out (7-4), but we have $5 - \gamma_{i_1-1} = 4$. To have equality we would need $\bar{\lambda}_{i_1-1,3} = 2$, which would imply $\bar{\lambda}^1_{i_1-1} + \bar{\lambda}^2_{i_1-1} \geq 8$, in which case we would not have had $\gamma_{i_1} = 1$ in ω_{def}.

Next, suppose $\gamma_{i_0} = \gamma_{i_1} = 2$, and we have $\bar{\lambda}^1_{i_1} + \bar{\lambda}^2_{i_1} < 6$. In this case, we will move the 3 to i_0 from the left, and $\gamma_{i_0-1} = 2$ in ω_{def}, so if either (7-6) or (7-5) is satisfied, we must have $\lambda_{i_0-1,3} \geq 2$. But this would force

$$\bar{\lambda}^1_{i_1} + \bar{\lambda}^2_{i_1} \geq \bar{\lambda}^1_{i_0-1} + \bar{\lambda}^2_{i_0-1} \geq 8,$$

contradicting the hypothesis for the case in question. We again conclude that there are at most 3 rows starting, and again the $(j_0,\, j_0+1)$ row is not among them.

On the other hand, if $\gamma_{i_0} = \gamma_{i_1} = 2$, and $\bar{\lambda}^1_{i_1} + \bar{\lambda}^2_{i_1} \geq 6$, then we will move a 3 to i_1 from the right, and (7-4) is not satisfied for parity reasons, so we will have at most three new rows starting. Finally, if $\gamma_{i_0} = \gamma_{i_1} = 3$, neither (7-6) nor (7-5) can be satisfied for parity reasons, so we can move a 3 from the left to i_0, and have at most three starting rows.

The remaining cases are treated symmetrically, with rows starting replaced by rows ending. In each case, we see that the basic structure of the proof of Theorem 7.3 is preserved by our change of multidegree, so our linear independence is likewise preserved, yielding the desired statement. □

Corollary 7.7. *Assume that $\rho = 2$, $r = 6$ and we are in the "second 3-cycle" situation of Proposition 4.20. Suppose that in the default multidegree ω_{def}, we have the inequalities*

$$2a^{i_0}_{j_0-1} \leq c_{i_0} - 1, \quad \textit{and} \quad 2a^{i_1+1}_{j_0-1} \geq c_{i_1+1} + 1,$$

with exactly one of the two inequalities satisfied with equality. Then there exists an unimaginative multidegree ω' such that the (j_0-1, j_0-1) row does not have potentially present sections both left of i_0 and right of i_1 in multidegree ω', and such that all the potentially present sections are linearly independent.

Proof. The only row that has a semicolumn adding to $d-2$ is j_0-1 and it has two of them. From Proposition 5.10, the only row which could have disconnected potential support in some unimaginative multidegree is (j_0-1, j_0-1).

Suppose that in multidegree ω_{def}, we have

$$2a^{i_0}_{j_0-1} = c_{i_0} - 1, \quad \text{but} \quad 2a^{i_1+1}_{j_0-1} > c_{i_1+1} + 1.$$

We will show that we can always move a 3 from the left to a genus-1 column on or right of i_0, while preserving linear independence. This will eliminate potential support in the (j_0-1, j_0-1) row left of i_0, as desired. From the definition of $\lambda_{i,j}$ in Definition 5.1, this condition can be written as

$$2(j_0-1) + 1 - \gamma_{i_0-1} = 2\lambda_{i_0-1, j_0-1},$$

so in particular γ_{i_0-1} must be odd.

Case $\gamma_{i_0-1} = 1$. Then $j_0 - 1 = \lambda_{i_0-1, j_0-1}$. From the definition of default multidegree $\bar{\lambda}^1_{i_0-1} + \bar{\lambda}^2_{i_0-1} < 5$, which forces $j_0 - 1 = 1$, so $\bar{\lambda}_{i_0,1} = \lambda_{i_0-1,1} = 1$.

First, if i_1 is the genus-1 column immediately following i_0, we observe that if we move the first 3 to i_0, considering only the inequalities at i_0, there can be at most three rows with potential support starting at i_0: $(1,2)$, $(2,2)$ and $(0, j)$ for a unique $j > 2$: For the row (j, j') to be present, we need

$$\lambda_{i_0-1,j} + \lambda_{i_0-1,j'} < j + j' \le 1 + \lambda_{i_0,j} + \lambda_{i_0,j'}.$$

From $j_0 - 1 = 1$, $\lambda_{i_0-1,1} = 1$ and the 3-cycle situation, $\lambda_{i_0,1} = 0$, $\lambda_{i_0-1,2} = 1$, $\lambda_{i_0,2} = 2$. So $(1,2)$, $(2,2)$ are potentially present and one pair $(0, j)$, $j > 2$. But in this case the actual potential support of $(1,2)$ is connected and supported strictly to the right of i_1. Thus, there are in fact at most two rows with potential support starting at i_0, and neither of them involves the exceptional row (specifically, $j = 1$). So moving the first 3 to i_0 we will still be able to eliminate potentially present sections from left to right as before.

Next, suppose that i_1 is not the genus-1 column immediately following i_0, and denote this column by i. Suppose also that there is no degree-3 column between i_0 and i_1, so that in particular $\bar{\lambda}^1_i + \bar{\lambda}^2_i \le 4$. We observe that we must also have $\bar{\lambda}_{i,0} = 1$, since we have $\bar{\lambda}_{i_0,1} = \bar{\lambda}_{i_0,2} = 1$, and we must have $\bar{\lambda}_{i_1-1,2} = \bar{\lambda}_{i_1-1,3} \ge 1$. So the only way we can avoid having a column of degree 3 before i_1 is if also $\bar{\lambda}_{i_1-1,0} = 1$. We can then apply Proposition 7.4 to move the first 3 to column i, and we will still obtain linear independence.

Finally, if we have a column of degree 3 between i_0 and i_1, say in column i, so that $\bar{\lambda}^1_i + \bar{\lambda}^2_i = 5$, then we claim that if we move the first 3 from the left to i_0, we will have at most two potentially present sections ending in column i, and at most two potentially present sections supported in the first column with $\bar{\lambda}^1_{i'} + \bar{\lambda}^2_{i'} = 6$. This will prove the desired statement, since we can then eliminate the potentially present sections starting from i' and moving both left and right from there. Checking the possible inequalities in column i', moving the 3 from the left to i_0 won't affect anything, so the argument for Theorem 7.3 implies *a priori* that there are at most three rows satisfying the inequalities at i' for potentially present sections to be supported there. We will check that there is always one such row which satisfies the inequalities at i', but does not in fact have potential support there. Because we have a 3 between i_0 and i_1, we must have $2a^{i_1+1}_{j_0-1} = c_{i_1+1} + 2$. If $i' < i_1$, the row in question is $(1, 1) = (j_0 - 1, j_0 - 1)$: indeed, in this situation we will have $a^{i''}_{(j_0-1, j_0-1)} = c_{i''}$ for all i'' with $i < i'' \leq i_1$, so $(j_0 - 1, j_0 - 1)$ does satisfy the necessary inequalities at i', but its actual potential support (after moving the 3 to i_0) is strictly to the right of i_1. On the other hand, if $i' > i_1$, the row in question will be $(1, 2) = (j_0 - 1, j_0)$:

$$\text{we have}\quad a^{i_1+1}_{j_0-1} + a^{i_1+1}_{j_0} \leq c_{i_1+1}, \quad \text{so} \quad a^{i_1}_{j_0-1} + a^{i_1}_{j_0} < c_{i_1},$$

and because the potential support is connected, it must be strictly left of i_1. However, we claim that we must have $a^{i_1+1}_{j_0-1} + a^{i_1+1}_{j_0} = c_{i_1+1}$, and that this must extend through the column i', so that the inequalities for potential support are satisfied at i'. Indeed, the only way this could fail is if $\delta_{i''} = j_0$ for some i'' with $i_0 < i'' < i'$. But we know that $\lambda_{i_0-1, j_0-1} = \lambda_{i_0-1, j_0} = 1$. So if $\delta_{i''} = j_0$ anywhere after i_0, it increases $\bar{\lambda}^1_{i''} + \bar{\lambda}^2_{i''}$ to at least 5. Thus, this could only happen for $i'' < i'$ if $i'' = i$, which then forces us to have $\bar{\lambda}^1_{i_0} = 3$ and $\bar{\lambda}^2_{i_0} = 1$. However, in this case, because we cannot have a gap between the $j_0 - 1$ and $j_0 + 1$ column at i_1, this would force us to also increase $\bar{\lambda}^1_{i''}$ to 4 before i_1, which violates our hypothesis that $i' > i_1$. Thus, in either situation we have shown that the column i' has at most two potentially present sections supported on it, and it remains to check that the column i has at most two potentially present sections ending on it. But we either have $\bar{\lambda}^1_{i-1} = 4$ and $\bar{\lambda}^2_{i-1} = 0$ or $\bar{\lambda}^1_{i-1} = 3$ and $\bar{\lambda}^2_{i-1} = 1$, and one can calculate directly that because we cannot have $\delta_i = 0$ or 4 in the second case, $\delta_i = 1$ in either case (recalling that by column i we have had a swap between rows 1 and 2), or $\delta_i = 3$ in the first case, the only rows with potential support ending in column i are $(1, 2)$ and $(0, j)$ for a unique value of j, yielding the desired statement.

Case $\gamma_{i_0-1} = 3$. Then either $j_0 - 1 = 2$ and $\lambda_{i_0-1, j_0-1} = 1$, or $j_0 - 1 = 3$ and $\lambda_{i_0-1, j_0-1} = 2$. First, suppose that $(j_0 - 1, j_0)$ has potential support strictly to the right of i_1, or equivalently, that there are no columns between i_0 and i_1 having degree 3, or with $\delta_i = j_0 - 1$ or j_0. In this case, if we move a 3 from the left to i_0, by Lemma 7.5 at most four rows satisfy the inequalities at i_0 to have potentially starting sections at i_0, and we see that these include $(j_0 - 1, j_0)$. But $(j_0 - 1, j_0)$ does not actually have potential support at i_0, so in this case we have at most three rows starting at i_0, and none of them involve the exceptional row (specifically, $j_0 - 1$), so we can eliminate this central block just as in Theorem 7.3, and we conclude we still have linear independence.

Now, the possibility that we have $\delta_i = j_0 - 1$ in between i_0 and i_1 is ruled out by the inequality $2a^{i_1+1}_{j_0-1} > c_{i_1+1} + 1$. If there is a column with $\delta_i = j_0$, but no column having degree 3 between i_0 and i_1, we will move the third degree-3 from the left to i_1, and the $(j_0 + 1, j_0 + 1) = (\delta_{i_1}, \delta_{i_1})$ row is supported strictly to the right of i_1. In addition (7-2) is ruled out by parity reasons, so by Lemma 7.5 we have at most three rows starting at i_1, and we also see that $(j_0 - 1, j_0 + 1)$ is not among them, as it will have potential support strictly to the right of i_1. Thus, no row involving j_0-1 (the exceptional row) has potential support starting at i_1, and in this case we can eliminate all potentially present sections just as in Theorem 7.3.

Next, suppose there is some column with degree 3 between i_0 and i_1, but no column with $\delta_i = j_0$. In this case, we will move the fourth 3 to the first column i with $\bar{\lambda}^2_i + \bar{\lambda}^3_i = 9$, and the third 3 to i_0. If $\bar{\lambda}^2_{i_1} + \bar{\lambda}^3_{i_1} < 9$, then according to Proposition 7.4, moving the fourth 3 doesn't disrupt linear independence, and then we are in exactly the same situation as the first case considered above, with (j_0-1, j_0) having potential support strictly to the right of i_1. On the other hand, if $\bar{\lambda}^2_{i_1} + \bar{\lambda}^3_{i_1} = 9$, we will still maintain linear independence, but for different reasons: we claim that will have at most three rows ending in the i-th column, no row ending in the first column i' with $\bar{\lambda}^2_{i'} + \bar{\lambda}^3_{i'} = 8$, and only two rows ending in the first column i'' with $\bar{\lambda}^2_{i''} + \bar{\lambda}^3_{i''} = 10$. Thus, we will be able to eliminate potentially present sections from the right, treating the columns from i' to i as a block to which to apply Lemma 6.3(3), and we will in this way eliminate all potentially present sections supported on either side of i_0. This leaves at most one potentially present section, which can then be eliminated. Thus, it suffices to prove the above claim. By the argument for Proposition 7.4, we have no potentially present section supported only in the i-th column, and at most three continuing from the previous column. So, there are at most three ending in the i-th column, as claimed. The fact that there are no rows ending in the i'-th column is immediate from Corollary 5.8 and Lemma 5.12. Finally, we know from the proof of Theorem 7.3 that there at most three rows satisfying the inequalities in column i'' to have potential support ending there. Moreover, we see that (j_0-1, j_0) is necessarily one of them. Indeed, since we have one column with degree 3 and none with $\delta_i = j_0-1$ or j_0 between i_0 and i_1, we see that $a^{i_1+1}_{(j_0-1,j_0)} = c_{i_1+1}$ even after changing the multidegree. But after i_1, any column with $\delta_i = j_0-1$ or j_0 will increase $\bar{\lambda}^2_i + \bar{\lambda}^3_i$, so this cannot occur strictly between i_1 and i'', and we conclude that $a^{i''}_{(j_0-1,j_0)} = c_{i''}$ as well. Since column i'' has degree 3, this means that (j_0-1, j_0) satisfies the inequalities to have potential support ending at i''. But again using that the fourth 3 is still left of i_1, the actual potential support of (j_0-1, j_0) is contained to the left of i_1, so we conclude that column i'' has at most two rows with potential support ending there, completing the proof of the claim.

It remains to analyze the possibility that we have a column of degree 3 and a column with $\delta_i = j_0$ in between i_0 and i_1. Recall that we have either

$$j_0-1=2 \quad \text{and} \quad \lambda_{i_0-1,j_0-1}=1, \qquad \text{or} \qquad j_0-1=3 \quad \text{and} \quad \lambda_{i_0-1,j_0-1}=2.$$

We first claim that in the latter case, we cannot have $\delta_i = j_0$ in between i_0 and i_1 without forcing there to be two columns of degree 3 in between, or equivalently, forcing $\bar{\lambda}^2_{i_1} + \bar{\lambda}^3_{i_1} \geq 10$. Indeed, since we cannot have a gap between $a^{i_0}_{j_0-1}$ and $a^{i_0}_{j_0}$ for the swap, we must have $\bar{\lambda}^2_{i_0} \geq 5$, and then for the same reason at i_1 we must have $\bar{\lambda}^2_{i_1} \geq 6$. But having some $\delta_i = j_0$ also requires $\bar{\lambda}^3_i \geq 4$, so we conclude that we would necessarily have $\bar{\lambda}^2_{i_1} + \bar{\lambda}^3_{i_1} \geq 10$, as claimed. Thus, it suffices to treat the situation that $\lambda_{i_0-1,j_0-1} = 1$. In this situation, we have $\bar{\lambda}^2_{i_1} + \bar{\lambda}^3_{i_1} \leq 6$, and we will move the third 3 to column i_0 and the fourth 3 to column i_1. We claim that we will have at most two rows with potentially present sections ending in i_1, and neither involves the exceptional row (specifically, j_0-1, which is 2). Thus, we will be able to eliminate all potentially present sections from the left and from the right of i_0, and finally eliminate the at most one potentially present section supported only at i_0. To verify the claim, we see that we necessarily have

$$5 \leq \bar{\lambda}^1_{i_1} \leq 7, \quad \bar{\lambda}^2_{i_1} = 3, \quad \text{and} \quad 1 \leq \bar{\lambda}^3_{i_1} \leq 3.$$

We compute that the only rows satisfying the inequalities to potentially end at i_1 are $(3,4)$, $(0,6)$, $(1,5)$, $(1,6)$ and $(3,5)$, but by the uniqueness part of Corollary 5.9, we see that the only way we can have three of these occurring at once is if we have $(3,4)$, $(0,6)$ and $(1,5)$. However, we also have that $(3,4)$ can only end if $\bar{\lambda}^3_{i_1} \leq 2$, $(0,6)$ can only end if $\bar{\lambda}^1_{i_1} \leq 6$, and $(1,5)$ can only end if one of the preceding two inequalities is strict. But together these imply that $\bar{\lambda}^1_{i_1} + \bar{\lambda}^3_{i_1} \leq 7$, meaning that we cannot have all the rows ending at i_1 under our hypothesis that the fourth 3 comes before i_1.

This concludes the case $\gamma_{i_0-1} = 3$.

Case $\gamma_{i_0-1} = 5$. In this case $j_0 - 1 = \lambda_{i_0-1, j_0-1} + 2$. From the definition of default multidegree, $\bar{\lambda}^2_{i_0-1} + \bar{\lambda}^3_{i_0-1} \geq 10$, so $j_0 - 1 \geq 4$. But, to allow for the double swap (there is a j_0+1 row), $j_0 - 1 \leq 4$, so $j_0 - 1 = 4$. With an argument as in Lemma 7.5, if we move the fifth 3 to i_0, even if we obtain two rows involving $\delta_{i_0} = 5$ with potential support ending at i_0, there can be at most one more (necessarily of the form $(j,6)$ for some j). Moreover, the $(4,5)$ row is not one of these, as it will have potential support starting, not ending, at i_0. We can therefore still eliminate the block spanning from the first column with $\bar{\lambda}^2_i + \bar{\lambda}^3_i = 9$ to column i_0 just as before.

The case that $2a^{i_1+1}_{j_0-1} = c_{i_1+1} + 1$ but $2a^{i_0}_{j_0-1} < c_{i_0} - 1$ is handled completely symmetrically, completing the proof. □

8. An analysis of the degenerate case

To conclude the proof of the main theorem, we need multidegrees such that on the one hand, the potentially present sections are still linearly independent, and on the other hand, tensors coming from any exact linked linear series generate at least $\binom{r+2}{2} = 28$ linearly independent combinations of the potentially present sections. The key point is that even though there are cases where some row may not have any potentially present section occurring in the linked linear series in the chosen multidegree, in those cases there is more than one combination of sections from some other row. We first look at the behavior of mixed sections under tensor product

Lemma 8.1. *Suppose s, s' are mixed sections of multidegrees* $\mathrm{md}_d(w)$ *and* $\mathrm{md}_d(w')$, *and let* $\mathrm{md}_{2d}(w'')$ *be another multidegree. Then $f_{w+w',w''}(s \otimes s')$ lies in the potential ambient space in multidegree* $\mathrm{md}(w'')$.

Proof. By definition of mixed sections as sums, it suffices to treat the case that s is obtained purely from gluing together s^i_j for fixed j, and s' is obtained from gluing together $s^i_{j'}$ for fixed j'. But in this case the result is clear, since $f_{w+w',w''}(s \otimes s')$ must be a combination of potentially present sections from the (j, j') row. □

The following lemma is convenient for cutting down the number of possibilities to consider.

Lemma 8.2. *Let s, s' be mixed sections of multidegrees* $\mathrm{md}(w)$ *and* $\mathrm{md}(w')$ *and types* $(\vec{S}, \vec{j})$ *and* $(\vec{S}', \vec{j}')$ *respectively. Suppose that for some i with $1 < i < N$, we have*

$$\ell_1 \neq \ell_2 \quad \textit{and} \quad \ell'_1 \neq \ell'_2 \quad \textit{such that } i \in S_{\ell_1} \cap S_{\ell_2} \cap S'_{\ell'_1} \cap S'_{\ell'_2}.$$

Then for any unimaginative w'', the map $f_{w+w',w''}$ vanishes identically on Z_i.

If further either $\{j_{\ell_1}, j_{\ell_2}\} = \{j'_{\ell'_1}, j'_{\ell'_2}\}$ or $\{j_{\ell_1}, j_{\ell_2}\} \cap \{j'_{\ell'_1}, j'_{\ell'_2}\} = \varnothing$, then the same conclusion holds when $i = 1$ or $i = N$.

Proof. First consider the case $1 < i < N$, and write $w = (c_2, \ldots, c_N)$ and $w' = (c'_2, \ldots, c'_N)$. The hypotheses mean that w allows for support of both s^i_j and $s^i_{j'}$ for some distinct j, j', so $a^i_j, a^i_{j'} \geq c_i$ and $b^i_j, b^i_{j'} \geq d - c_{i+1}$. Without loss of generality, suppose $a^i_j < a^i_{j'}$. Then, either $a^i_j + b^i_j < d$ or $a^i_{j'} + b^i_{j'} < d$. If $b^i_j > b^i_{j'}$, then either $c_{i+1} \geq d - b^i_{j'} > d - b^i_j > a^i_j \geq c_i$ or $c_{i+1} \geq d - b^i_{j'} > a^i_{j'} > a^i_j \geq c_i$, so in either case we have $c_{i+1} \geq c_i + 2$. On the other hand, if $b^i_j < b^i_{j'}$, then $c_{i+1} \geq d - b^i_j > d - b^i_{j'} \geq a^i_{j'} > a^i_j \geq c_i$, so again $c_{i+1} \geq c_i + 2$. The same argument holds for w', so we conclude that $c_{i+1} + c'_{i+1} \geq c_i + c'_i + 4$, which implies that $f_{w+w',w''}$ vanishes on Z_i, since if we write $w'' = (c''_2, \ldots, c''_N)$, the unimaginative hypothesis means that $c''_{i+1} \leq c''_i + 3$.

Next, if $i = 1$, the unimaginative hypothesis means that c_2 is equal to 2 or 3. It follows (see the proof of Theorem 7.3) that only the rows (0, 0), (0, 1), (1, 1) and (0, 2) can have potential support in the column, with not both (0, 0) and (1, 1) occurring. If $f_{w+w',w''}$ is nonzero on Z_1, then $f_{w+w',w''}(s \otimes s')$ must have $(j_{\ell_u}, j'_{\ell'_v})$ parts with potential support at $i = 1$ for $u = 1, 2$ and $v = 1, 2$, and this isn't possible if either $\{j_{\ell_1}, j_{\ell_2}\} = \{j'_{\ell'_1}, j'_{\ell'_2}\}$ or $\{j_{\ell_1}, j_{\ell_2}\} \cap \{j'_{\ell'_1}, j'_{\ell'_2}\} = \varnothing$. The case $i = N$ is symmetric. □

The cases of mixed sections appearing after each type of swap will require individual analysis. We treat the case of a single swap in the next proposition but we first give an example:

Example 8.3. Tables 2a and 2b together show the table T obtained from the tensor square of the limit linear series considered in Example 4.25, which has $r = 6$, $g = 22$, and $d = 25$.

We have highlighted the potentially present sections; note that the (2, 2) row contains two (the first in Table 2a, the second in Table 2b), while the rest all have a unique one. These two potentially present sections are thus treated separately in Theorem 7.3; the first appears as part of a block in the fifth and sixth columns which is eliminated using rule (iii) of Lemma 6.3, while the second occurs as the only new potentially present sections in the twelfth column, which is part of another block, extending from the seventh column to the sixteenth column, which is again eliminated using rule (iii), after all other potentially present sections have been eliminated on both the left and right. The only other block that requires rule (iii) contains the seventeenth and eighteenth columns, and is eliminated after the potentially present sections appearing to the right have all been dropped. Following the proof of Theorem 7.3, we see that we can eliminate all sections outside the aforementioned three blocks going inward from both the left and right ends, using only iterated applications of rule (i).

Observe that in the default multidegree, the (unique) potentially present section in row (2, 3) extends from the seventh column to the eleventh column. This means that if s'_3 and s''_3 have the smallest possible

col =	1		2		3		4		5		6		7		8		9		10		11	
(0, 0)	0	50	0	48	2	46	4	44	6	42	8	40	10	38	12	38	12	36	14	34	16	32
(0, 1)	1	48	2	47	3	45	5	43	7	41	9	39	11	37	13	36	14	34	16	33	17	31
(0, 2)	2	47	3	45	5	44	6	42	8	40	10	38	12	36	14	35	15	32	18	30	20	28
(1, 1)	2	46	4	46	4	44	6	42	8	40	10	38	12	36	14	34	16	32	18	32	18	30
(0, 3)	3	46	4	44	6	42	8	41	9	39	11	37	13	35	15	34	16	33	17	31	19	30
(1, 2)	3	45	5	44	6	43	7	41	9	39	11	37	13	35	15	33	17	30	20	29	21	27
(0, 4)	4	45	5	43	7	41	9	39	11	38	12	36	14	34	16	33	17	31	19	29	21	27
(1, 3)	4	44	6	43	7	41	9	40	10	38	12	36	14	34	16	32	18	31	19	30	20	29
(2, 2)	4	44	6	42	8	42	8	40	10	38	12	36	14	34	16	32	18	28	22	26	24	24
(0, 5)	5	44	6	42	8	40	10	38	12	36	14	35	15	33	17	32	18	30	20	28	22	26
(1, 4)	5	43	7	42	8	40	10	38	12	37	13	35	15	33	17	31	19	29	21	28	22	26
(2, 3)	5	43	7	41	9	40	10	39	11	37	13	35	15	33	17	31	19	29	21	27	23	26
(0, 6)	6	43	7	41	9	39	11	37	13	35	15	33	17	32	18	31	19	29	21	27	23	25
(1, 5)	6	42	8	41	9	39	11	37	13	35	15	34	16	32	18	30	20	28	22	27	23	25
(2, 4)	6	42	8	40	10	39	11	37	13	36	14	34	16	32	18	30	20	27	23	25	25	23
(3, 3)	6	42	8	40	10	38	12	38	12	36	14	34	16	32	18	30	20	30	20	28	22	28
(1, 6)	7	41	9	40	10	38	12	36	14	34	16	32	18	31	19	29	21	27	23	26	24	24
(2, 5)	7	41	9	39	11	38	12	36	14	34	16	33	17	31	19	29	21	26	24	24	26	22
(3, 4)	7	41	9	39	11	37	13	36	14	35	15	33	17	31	19	29	21	28	22	26	24	25
(2, 6)	8	40	10	38	12	37	13	35	15	33	17	31	19	30	20	28	22	25	25	23	27	21
(3, 5)	8	40	10	38	12	36	14	35	15	33	17	32	18	30	20	28	22	27	23	25	25	24
(4, 4)	8	40	10	38	12	36	14	34	16	34	16	32	18	30	20	28	22	26	24	24	26	22
(3, 6)	9	39	11	37	13	35	15	34	16	32	18	30	20	29	21	27	23	26	24	24	26	23
(4, 5)	9	39	11	37	13	35	15	33	17	32	18	31	19	29	21	27	23	25	25	23	27	21
(4, 6)	10	38	12	36	14	34	16	32	18	31	19	29	21	28	22	26	24	24	26	22	28	20
(5, 5)	10	38	12	36	14	34	16	32	18	30	20	30	20	28	22	26	24	24	26	22	28	20
(5, 6)	11	37	13	35	15	33	17	31	19	29	21	28	22	27	23	25	25	23	27	21	29	19
(6, 6)	12	36	14	34	16	32	18	30	20	28	22	26	24	26	24	24	26	22	28	20	30	18
(6, 6)	32	16	34	14	36	12	38	12	38	10	40	8	42	6	44	4	46	2	48	0	50	0
		47	3	45	5	43	7	41	9	38	12	36	14	33	17	31	19	29	21	27	23	25

Table 2a. This is the left side of the table T obtained from the tensor square of the limit linear series considered in Example 4.25, which has $r = 6$, $g = 22$, and $d = 25$. The right side is shown in Table 2b. We have also included the w corresponding to the default multidegree ω_{def} at the bottom of the table, and include not only the values c_i for $i = 2, \dots, 22$, but also $2d - c_i$ in the preceding subcolumns.

portions coming from the $j = 3$ row, so that s_3' only has nonzero s_3^i parts for $i \geq 8$ and s_3'' for $i \leq 10$, then the potentially present section for the (2, 3) row cannot come from either $s_2 \otimes s_3'$ or $s_2 \otimes s_3''$. This means that these sections (or more precisely, their images in multidegree ω_{def}) are forced to yield potentially present sections from the (2, 2) row, with $s_2 \otimes s_3'$ necessarily yielding the one supported from columns 5 through 7, and $s_2 \otimes s_3''$ necessarily yielding the one supported in column 12. Thus, we explicitly see the lack of a (2, 3) section being offset by the inclusion of two independent (2, 2) sections.

col =	12		13		14		15		16		17		18		19		20		21		22	
(0, 0)	18	30	20	28	22	26	24	24	26	24	26	22	28	20	30	18	32	16	34	14	36	12
(0, 1)	19	29	21	27	23	25	25	23	27	22	28	21	29	19	31	17	33	15	35	13	37	11
(0, 2)	22	27	23	25	25	23	27	21	29	20	30	18	32	16	34	15	35	13	37	11	39	9
(1, 1)	20	28	22	26	24	24	26	22	28	20	30	20	30	18	32	16	34	14	36	12	38	10
(0, 3)	20	28	22	26	24	24	26	22	28	21	29	19	31	18	32	16	34	14	36	12	38	10
(1, 2)	23	26	24	24	26	22	28	20	30	18	32	17	33	15	35	14	36	12	38	10	40	8
(0, 4)	23	25	25	24	26	22	28	20	30	19	31	17	33	15	35	13	37	12	38	10	40	8
(1, 3)	21	27	23	25	25	23	27	21	29	19	31	18	32	17	33	15	35	13	37	11	39	9
(2, 2)	26	24	26	22	28	20	30	18	32	16	34	14	36	12	38	12	38	10	40	8	42	6
(0, 5)	24	24	26	22	28	21	29	19	31	18	32	16	34	14	36	12	38	10	40	9	41	7
(1, 4)	24	24	26	23	27	21	29	19	31	17	33	16	34	14	36	12	38	11	39	9	41	7
(2, 3)	24	25	25	23	27	21	29	19	31	17	33	15	35	14	36	13	37	11	39	9	41	7
(0, 6)	25	23	27	21	29	19	31	18	32	17	33	15	35	13	37	11	39	9	41	7	43	6
(1, 5)	25	23	27	21	29	20	30	18	32	16	34	15	35	13	37	11	39	9	41	8	42	6
(2, 4)	27	22	28	21	29	19	31	17	33	15	35	13	37	11	39	10	40	9	41	7	43	5
(3, 3)	22	26	24	24	26	22	28	20	30	18	32	16	34	16	34	14	36	12	38	10	40	8
(1, 6)	26	22	28	20	30	18	32	17	33	15	35	14	36	12	38	10	40	8	42	6	44	5
(2, 5)	28	21	29	19	31	18	32	16	34	14	36	12	38	10	40	9	41	7	43	6	44	4
(3, 4)	25	23	27	22	28	20	30	18	32	16	34	14	36	13	37	11	39	10	40	8	42	6
(2, 6)	29	20	30	18	32	16	34	15	35	13	37	11	39	9	41	8	42	6	44	4	46	3
(3, 5)	26	22	28	20	30	19	31	17	33	15	35	13	37	12	38	10	40	8	42	7	43	5
(4, 4)	28	20	30	20	30	18	32	16	34	14	36	12	38	10	40	8	42	8	42	6	44	4
(3, 6)	27	21	29	19	31	17	33	16	34	14	36	12	38	11	39	9	41	7	43	5	45	4
(4, 5)	29	19	31	18	32	17	33	15	35	13	37	11	39	9	41	7	43	6	44	5	45	3
(4, 6)	30	18	32	17	33	15	35	14	36	12	38	10	40	8	42	6	44	5	45	3	47	2
(5, 5)	30	18	32	16	34	16	34	14	36	12	38	10	40	8	42	6	44	4	46	4	46	2
(5, 6)	31	17	33	15	35	14	36	13	37	11	39	9	41	7	43	5	45	3	47	2	48	1
(6, 6)	32	16	34	14	36	12	38	12	38	10	40	8	42	6	44	4	46	2	48	0	50	0
	25	23	27	21	29	19	31	17	33	14	36	12	38	9	41	7	43	5	45	3	47	

Table 2b. This is the right side of the table T obtained from the tensor square of the limit linear series considered in Example 4.25, which has $r = 6$, $g = 22$, and $d = 25$. The left side is shown in Table 2a. We have also included the w corresponding to the default multidegree ω_{def} at the bottom of the table, and include not only the values c_i for $i = 2, \dots, 22$, but also $2d - c_i$ in the preceding subcolumns.

Proposition 8.4. *Suppose a limit linear series contains precisely one swap, occurring between the rows j_0, $j_0 - 1$ in column i_0. With notation as in Proposition 4.13, for any multidegree ω, the tensors pairs of the s_j for $j \neq j_0$, and s'_{j_0}, s''_{j_0} contain $\binom{r+2}{2}$ independent linear combinations of the potentially present sections.*

Proof. If $j \le j'$ are both different from j_0, then s_j and $s_{j'}$ are in the linked linear series and contribute an $s_{(j,j'),i}$. This gives rise to $\binom{r}{2}$ potentially present sections, necessarily independent because they are supported in distinct rows. If $j \neq j_0, j_0 - 1$, there are three global sections $s_j \otimes s'_{j_0}$, $s_j \otimes s''_{j_0}$ and $s_j \otimes s_{j_0 - 1}$, each of which has nonzero image in multidegree ω. We claim that these three images must contain at least two distinct linear combinations of the $s_{(j,j_0),i}$ and $s_{(j,j_0-1),i}$. If $s_j \otimes s'_{j_0}$ has support in any columns greater than or equal to i_0, this necessarily includes a nonzero combination of the $s_{(j,j_0),i}$, which is distinct from the image of $s_j \otimes s_{j_0-1}$, and we are done. The same holds if $s_j \otimes s''_{j_0}$ has support in any columns less than or equal to i_0. The final case is that $s_j \otimes s'_{j_0}$ has support only in columns strictly less than i_0, and $s_j \otimes s''_{j_0}$ has support only in columns strictly greater than i_0. In this case, both may be linear combinations of the $s_{(j,j_0-1),i}$, but since their support is disjoint, they must be two distinct combinations, as desired.

Thus, we have produced $\binom{r}{2} + 2(r-1) = \binom{r+2}{2} - 3$ independent combinations of potentially present sections, supported among the rows (j, j') with $j \neq j_0 - 1, j_0$. Finally, we consider the tensors of $s_{j_0-1}, s'_{j_0}, s''_{j_0}$, and claim we obtain three distinct linear combinations, necessarily supported among the rows $(j_0-1, j_0-1), (j_0-1, j_0), (j_0, j_0)$. Consider the images of $s'_{j_0} \otimes s'_{j_0}$, $s'_{j_0} \otimes s''_{j_0}$, and $s''_{j_0} \otimes s''_{j_0}$. If any of their images contain any portion of the (j_0, j_0) row, then considering $s_{j_0-1} \otimes s_{j_0-1}, s_{j_0-1} \otimes s'_{j_0}, s_{j_0-1} \otimes s'_{j_0}$. The same argument as above shows we obtain two distinct combinations of type (j_0-1, j_0-1) and/or (j_0-1, j_0), so we are done. But the only alternative is that the first three tensors come from the (j_0-1, j_0-1), (j_0-1, j_0) and (j_0-1, j_0-1) rows respectively, with the first and last having disjoint support. Thus, in this case these three are all linearly independent, and we again obtain the desired conclusion. □

When $\rho = 2$, there can be up to two swaps (Proposition 5.3) occurring on distinct columns $i_0 < i_1$ corresponding to genus 1 components. We will show that in each of the four possible configurations of the two swaps there are enough independent linear combinations of the potentially present sections (see Propositions 8.5, 8.9, 8.10, 8.12).

We find it convenient to introduce shorthand notation as follows: we will write for instance

$$s'_{j_0} \otimes s''_{j_0+1} = (j_0-1, j_0+1)_{\mathrm{L}} + (j_0-1, j_0)_{\mathrm{R}} + (j_0, j_0+1)$$

to indicate that the image of $s'_{j_0} \otimes s''_{j_0+1}$ in the relevant multidegree is a combination of potentially present sections from the (j_0-1, j_0+1), (j_0-1, j_0) and (j_0, j_0+1) rows, where the first is supported strictly left of i_0, the second strictly right of i_1 and the third has no restrictions on its support. We will also use subscripts C to denote support strictly between i_0 and i_1, LC to denote support strictly left of i_1 and CR to denote support strictly right of i_0.

Propositions 8.6, 8.7, 8.8 help us control the potential support of mixed sections using a left-weighted X_0.

Proposition 8.5. *Suppose that the limit linear series contains precisely two swaps, and both occur in the same pair of rows, say j_0, j_0-1 in columns i_0, i_1 ("repeated swap", see Proposition 4.16). Then for any unimaginative multidegree ω, the images in multidegree ω of the tensors of pairs of the s_j for $j \neq j_0, j_0-1$, and $s'_{j_0-1}, s''_{j_0-1}, s'_{j_0}, s''_{j_0}$ contain $\binom{r+2}{2}$ independent linear combinations of the potentially present sections.*

Proof. Just as in the proof of Proposition 8.4, for $j, j' \neq j_0, j_0-1$, the linked linear series contains s_j and $s_{j'}$, so the image of $s_j \otimes s_{j'}$ always gives a potentially present section from row (j, j').

Now consider $j \neq j_0, j_0-1$; we claim that $s_j \otimes s'_{j_0-1}, s_j \otimes s''_{j_0-1}, s_j \otimes s'_{j_0}, s_j \otimes s''_{j_0}$ cannot all coincide. Hence they have a two-dimensional span. Indeed, if $s_j \otimes s''_{j_0-1}$ coincides with $s_j \otimes s'_{j_0}$, they must be of the form $(j, j_0-1)_{\mathrm{L}} + (j, j_0)_{\mathrm{R}}$. But the former cannot occur in $s_j \otimes s''_{j_0}$, and the latter cannot occur in $s_j \otimes s'_{j_0-1}$, so one of the two will provide a second section.

It remains to show that we have at least three independent sections among all tensors of the s'_{j_0-1}, s''_{j_0-1}, s'_{j_0}, s''_{j_0}. We first consider the tensor squares of each of the four sections. According to Lemma 8.2, these can only contain types (j_0-1, j_0-1) and (j_0, j_0), with no type (j_0-1, j_0) appearing. Now, the possible (j_0, j_0) parts of $s'^{\otimes 2}_{j_0-1}$ and $s''^{\otimes 2}_{j_0-1}$ are disjoint, so we conclude that either these two are distinct, or they are of pure type (j_0-1, j_0-1). Similarly, the sections $s'^{\otimes 2}_{j_0}$ and $s''^{\otimes 2}_{j_0}$ are either distinct or of pure type (j_0, j_0). Thus, it suffices to show that we cannot have all of our tensors in the span of a single pair of sections, each of pure type (j_0-1, j_0-1) or (j_0, j_0). Now, $s'_{j_0} \otimes s''_{j_0}$ cannot have a (j_0-1, j_0-1) part, and $s'_{j_0-1} \otimes s''_{j_0-1}$ cannot have a (j_0, j_0) part, so the only possibility to consider is that one of our sections is purely of type (j_0-1, j_0-1), and the other is purely of type (j_0, j_0).

If the (j_0-1, j_0-1) part occurs in $s''_{j_0-1} \otimes s'_{j_0}$, it must be supported strictly to the left of i_0. Then $s''_{j_0-1} \otimes s''_{j_0}$ cannot have a (j_0-1, j_0-1) part, so must be of type (j_0, j_0), and the support must be strictly to the right of i_1. On the other hand, if the (j_0, j_0) part occurs in $s''_{j_0-1} \otimes s'_{j_0}$, it must again be supported strictly to the right of i_1, and then $s'_{j_0-1} \otimes s'_{j_0}$ cannot have a (j_0, j_0) part, so must be of type (j_0-1, j_0-1), again supported to the left of i_0. But in either case, $s'_{j_0-1} \otimes s''_{j_0}$ cannot be a linear combination of these two sections, as desired. □

Proposition 8.6. *Suppose that $\rho = 2$ and there are two swaps between rows j_0-1, j_0 and j_1-1, j_1 in columns $i_0 < i_1$ ("disjoint swap" of Proposition 4.14), Then, in an unimaginative multidegree, the potential support of every (j, j') is connected except possibly for (j_0-1, j_0-1), (j_1-1, j_1-1), and (j_0-1, j_1-1). Moreover, if (j_0-1, j_1-1) has disconnected potential support in multidegree ω, the potential support must be made up of two components, one contained strictly to the right of i_1, and one contained strictly to the left of i_0, and the potential support of (j_0-1, j_1) is contained strictly right of i_1-1, and the potential support of (j_0, j_1-1) is contained strictly left of i_0+1. Finally, if the potential support of (j_0-1, j_1) is contained strictly left of i_0, then (j_0-1, j_1-1) must also have a component of potential support contained strictly left of i_0, and if the potential support of (j_0, j_1-1) is contained strictly right of i_1, then (j_0-1, j_1-1) must also have a component of potential support contained strictly right of i_1.*

Proof. We write as usual $\omega = \mathrm{md}(w)$ with $w = (c_2, \ldots, c_g)$.

From Proposition 5.3, no rows are exceptional except row j_0-1 at i_0 and row j_1-1 at i_1. From Proposition 5.10, in multidegree ω, the potential support of every (j, j') is connected except possibly for (j_0-1, j_0-1), (j_1-1, j_1-1), and (j_0-1, j_1-1). and following the proof we see further that in order for (j_0-1, j_1-1) to have disconnected support, the support must be split between strictly right of i_1 and strictly left of i_0, as claimed. Next, if the potential support of (j_0-1, j_1-1) has a component lying strictly right of i_1, then

$$a^{i_1}_{(j_0-1, j_1)} = a^{i_1+1}_{(j_0-1, j_1)} - 1 = a^{i_1+1}_{(j_0-1, j_1-1)} - 2 > c_{i_1+1} - 2 \geq c_{i_1},$$

and the connectedness statement implies that the potential support of (j_0-1, j_1) is supported strictly to the right of i_1-1, as desired. The corresponding statement on support left of i_0 and i_0+1 follows similarly. Finally, if the potential support of (j_0-1, j_1) is contained strictly left of i_0, then

$$a^{i_0}_{(j_0-1,j_1-1)} < a^{i_0}_{(j_0-1,j_1)} < c_{i_0},$$

so (j_0-1, j_1-1) also has a component of potential support strictly left of i_0, as desired. The last statement on support strictly right of i_1 follows similarly. □

Proposition 8.7. *Suppose that $\rho = 2$ and there are two swaps in the "first 3-cycle" situation of Proposition 4.18. In an unimaginative multidegree, the potential support of every (j, j') is connected except possibly for (j_0-1, j_0-1), (j_0-1, j_0), and (j_0, j_0). Moreover, if for some j, the potential support of (j, j_0) has a component strictly to the left of i_0, then the potential support of (j, j_0-1) is entirely contained strictly to the left of i_0, and if the potential support of (j, j_0-1) has a component strictly to the right of i_1, then the potential support of (j, j_0) is entirely contained strictly to the right of i_1.*

Finally, if (j_0-1, j_0) has potential support contained entirely strictly to the left of i_1, then the potential support of (j_0-1, j_0+1) cannot be contained to the right of i_1; if it has potential support contained entirely strictly to the right of i_0, then the potential support of (j_0, j_0+1) cannot be contained to the left of i_0; and if it has potential support contained entirely strictly between i_0 and i_1, then (j_0-1, j_0-1) has potential support contained entirely strictly to the left of i_1, and (j_0, j_0) has potential support contained entirely strictly to the right of i_0.

Proof. Most of the argument is similar to Proposition 8.6. For the support of (j, j_0) to have a component strictly to the left of i_0 we must have $a^{i_0}_{(j,j_0)} \leq c_{i_0}-1$, and then $a^{i_0}_{(j,j_0-1)} < c_{i_0}-1$. Arguing as in Proposition 8.6, we conclude that (even if $j = j_0-1$ or j_0) the support of (j, j_0-1) is connected and strictly to the left of i_0. The statement on support to the right of i_1 is proved in exactly the same way. For the last assertion, note that the (j_0-1, j_0-1) row has no support at i_1, and the (j_0, j_0) has no support at i_0, since both sum to $2d-4$ in the relevant columns. □

Proposition 8.8. *Suppose that X_0 is left-weighted, and that the rows j, j' have no exceptional behavior in any genus-0 columns. Then the image of $s_j \otimes s_{j'}$ in any unimaginative multidegree ω is equal to the leftmost potentially appearing section in the (j, j') row.*

Proof. The lack of exceptional behavior away from genus-1 components means that the $a^i_{(j,j')}$ are constant on the genus-0 components. The idea is then that the left-weighting means that the leftmost negative value of $a^i_{(j,j')} - c_i$ is repeated so many times that it must lead to a strict minimum of the partial sums. Compare the proof of Proposition 4.24, where in (4-1) we now replace d by $2d$ due to having passed to the tensor square. □

Proposition 8.9. *Suppose that $\rho = 2$, X_0 is left-weighted and we are in the "disjoint swap" case of Proposition 4.14, so that the limit linear series contains precisely two swaps in disjoint pairs of rows, say j_0, j_0-1 and j_1, j_1-1. Then for any unimaginative multidegree ω, choosing s'_{j_0} and s'_{j_1} as allowed* *by Proposition 4.24, the images in multidegree ω of the tensors of pairs of the s_j for $j \neq j_0, j_1$, and $s'_{j_0}, s''_{j_0}, s'_{j_1}, s''_{j_1}$ contain $\binom{r+2}{2}$ independent linear combinations of the potentially present sections.*

Proof. Without loss of generality, assume that $i_0 < i_1$. By Proposition 4.24, we may assume that s'_{j_1} is controlled, and that the j_1-part of s'_{j_1} does not contain any genus-1 component left of i_1. Every (j, j') has connected potential support unless $j, j' \in \{j_0 - 1, j_1 - 1\}$. Moreover, if $j, j' \neq j_0, j_0 - 1, j_1, j_1 - 1$, then we know that $f_{w_j + w'_j, w}(s_j \otimes s_{j'})$ is nonzero and composed of $s^i_{(j,j')}$. Now, suppose $j \neq j_0, j_0 - 1, j_1, j_1 - 1$. Then the same argument as in Proposition 8.4 shows that if we consider the images in multidegree ω of $s_j \otimes s_{j_0-1}$, $s_j \otimes s'_{j_0}$, and $s_j \otimes s''_{j_0}$, we either obtain one section of type $(j, j_0 - 1)$ and one with a contribution of type (j, j_0), or two sections of type $(j, j_0 - 1)$, but having disjoint support. The same holds with j_1 in place of j_0. Together, these produce $\binom{r-2}{2} + 4(r-3) = \binom{r+2}{2} - 10$ linearly independent combinations. It thus suffices to show that we have 10 linearly independent combinations coming from tensor products of pairs of the sections $s_{j_0-1}, s'_{j_0}, s''_{j_0}, s_{j_1-1}, s'_{j_1}, s''_{j_1}$. Just as in the proof of Proposition 8.4, tensor products of the first three sections yield three independent combinations, with contributions contained among the types $(j_0 - 1, j_0 - 1)$, $(j_0 - 1, j_0)$, and (j_0, j_0). Tensor products of the last three sections likewise yield three combinations, with j_1 replacing j_0 in the types.

It remains to consider the tensors with types contained among $(j_0 - 1, j_1 - 1)$, $(j_0 - 1, j_1)$, $(j_0, j_1 - 1)$ and (j_0, j_1). First suppose that $(j_0 - 1, j_1 - 1)$ has connected potential support in multidegree ω. Then just as in the single-swap case, at least one of $s_{j_0-1} \otimes s'_{j_1}$, $s_{j_0-1} \otimes s''_{j_1}$ must involve a $(j_0 - 1, j_1)$ part, and at least one of $s'_{j_0} \otimes s_{j_1-1}$, $s''_{j_0} \otimes s_{j_1-1}$ must involve a $(j_0, j_1 - 1)$ part. Since $s_{j_0-1} \otimes s_{j_1-1}$ is pure of type $(j_0 - 1, j_1 - 1)$, and all of these have unique potential support, we find that the span of these sections contains the (unique) pure types of each of $(j_0 - 1, j_1 - 1)$, $(j_0, j_1 - 1)$ and $(j_0 - 1, j_1)$. Thus, if we have anything with a nonzero part of type (j_0, j_1), this gives a fourth independent combination. On the other hand, if nothing has a (j_0, j_1) part, then we must have the following:

$$
\begin{aligned}
s'_{j_0} \otimes s''_{j_1} &= (j_0 - 1, j_1)_{\mathrm{L}} + (j_0, j_1 - 1)_{\mathrm{R}}, \\
s'_{j_0} \otimes s'_{j_1} &= (j_0 - 1, j_1 - 1)_{\mathrm{L}} + (j_0, j_1 - 1)_{\mathrm{LC}} \quad \text{and} \\
s''_{j_0} \otimes s''_{j_1} &= (j_0 - 1, j_1)_{\mathrm{CR}} + (j_0 - 1, j_1 - 1)_{\mathrm{R}}.
\end{aligned}
$$

First consider the possibility that the $(j_0-1, j_1)_{\mathrm{L}}$ part of $s'_{j_0} \otimes s''_{j_1}$ is nonzero. Then, by Proposition 8.6(1), we have that $(j_0 - 1, j_1 - 1)$ has support strictly left of i_0 too, which in turn means that $(j_0, j_1 - 1)$ can't have support strictly right of i_1. But this leaves no possibility for $s''_{j_0} \otimes s''_{j_1}$. On the other hand, if the $(j_0, j_1 - 1)_{\mathrm{R}}$ part of $s'_{j_0} \otimes s''_{j_1}$ is nonzero, we have that $(j_0 - 1, j_1 - 1)$ must have support strictly right of i_1, and hence that $(j_0 - 1, j_1)$ can't have support strictly left of i_0, leaving no possibility for $s'_{j_0} \otimes s'_{j_1}$. We conclude that it is not possible for these tensors not to have some (j_0, j_1) part, giving the desired four independent combinations when $(j_0 - 1, j_1 - 1)$ has connected potential support.

It remains to treat the case that $(j_0 - 1, j_1 - 1)$ has disconnected potential support in multidegree ω. Then Proposition 8.6 tells us that this potential support has two parts, contained strictly left of i_0 and right of i_1 respectively. Moreover, it says that the potential support of $(j_0 - 1, j_1)$ is contained strictly right of

i_1-1 and the potential support of (j_0, j_1-1) is contained strictly left of i_0+1. This forces $s'_{j_0}\otimes s''_{j_1}$ to be of pure (j_0, j_1) type. Now, we observe that two of the sections $s_{j_0-1}\otimes s_{j_1-1}$, $s_{j_0-1}\otimes s'_{j_1}$, $s_{j_0-1}\otimes s''_{j_1}$ must be independent, either involving a (j_0-1, j_1) part and a (j_0-1, j_1-1) part, or two (j_0-1, j_1-1) parts. Similarly, $s'_{j_0}\otimes s_{j_1-1}$ and $s''_{j_0}\otimes s_{j_1-1}$ must either involve a (j_0, j_1-1) part or two (j_0-1, j_1-1) parts. We see that the only way to avoid having four independent combinations would be if these five tensors are all of pure type (j_0-1, j_1-1), necessarily achieving support independently both on the left and right. But we note that because the potential support of (j_0, j_1-1) is contained strictly left of i_0+1, and because (in the disconnected support case) we must have $a^i_{(j_0-1,j_1-1)}=c_i$ for $i_0<i\le i_1$, the only way that $s''_{j_0}\otimes s_{j_0-1}$ can fail to have a (j_0, j_1-1) part is if s''_{j_0} is not controlled, and more specifically if its j_0 portion does not extend more than halfway to the next genus-1 component after i_0. On the other hand, s'_{j_1} is controlled and has j_1 part not containing any genus-1 component smaller than i_1, so we conclude that in this situation its j_1 part is disjoint from the j_0 part of s''_{j_0}, and then $s''_{j_0}\otimes s'_{j_1}=(j_0, j_1-1)+(j_0-1, j_1)$, and gives a fourth independent combination. This completes the proof of the proposition. □

Proposition 8.10. *Suppose that $\rho=2$ and we are in the "first 3-cycle" situation of Proposition 4.18. Choose an unimaginative multidegree ω such that the (j_0-1, j_0) row has a unique potentially present section in multidegree ω, whose support does not contain i_0 or i_1 (use Corollary 7.6,). Then the images in multidegree ω of the tensors of pairs of the s_j for $j\ne j_0+1$, and $s'_{j_0+1}, s''_{j_0+1}, s'''_{j_0+1}$ contain $\binom{r+2}{2}$ independent linear combinations of the potentially present sections.*

Proof. We are assuming that there is one swap between the j_0-th and $(j_0{+}1)$-st on the i_0-th column, and a second swap between the $(j_0{-}1)$-st and $(j_0{+}1)$-st rows on the i_1-st column for some $i_1>i_0$. We first show that for $j\ne j_0-1, j_0, j_0+1$, the sections

$$s_j\otimes s_{j_0-1},\quad s_j\otimes s_{j_0},\quad s_j\otimes s'_{j_0+1},\quad s_j\otimes s''_{j_0+1},\quad s_j\otimes s'''_{j_0+1}$$

must yield at least three independent combinations. But the first two tensors yield (j, j_0-1) and (j, j_0) parts, so if any of the last three have any (j, j_0+1) part, we obtain the desired independence. On the other hand, if not we find that

$$\begin{aligned} s_j\otimes s'_{j_0+1} &= (j, j_0-1)_{\mathrm{LC}}+(j, j_0)_{\mathrm{L}};\\ s_j\otimes s''_{j_0+1} &= (j, j_0-1)_{\mathrm{R}}+(j, j_0)_{\mathrm{CR}};\\ s_j\otimes s'''_{j_0+1} &= (j, j_0)_{\mathrm{L}}+(j, j_0-1)_{\mathrm{R}}.\end{aligned}$$

If the $(j, j_0)_{\mathrm{L}}$ part of the last tensor is nonzero, then by Proposition 8.7, the potential support of both the (j, j_0-1) and (j, j_0) rows are connected and contained strictly to the left of i_0, leaving no possibility for the second tensor. But if the $(j, j_0-1)_{\mathrm{R}}$ part of the last tensor is nonzero, then similarly the potential support of both the (j, j_0-1) and (j, j_0) rows are contained strictly to the right of i_1, leaving no possibility for the first tensor. Thus, we reach a contradiction, and conclude that we must obtain a (j, j_0+1) part, giving the desired three independent combinations.

Next, we consider the 15 tensors arising from

$$s_{j_0-1},\quad s_{j_0},\quad s'_{j_0+1},\quad s''_{j_0+1},\quad s'''_{j_0+1};$$

we need to show that these yield 6 independent linear combinations.

By hypothesis, we have that the potential support of the (j_0-1, j_0) row is connected and does not contain i_0 or i_1, so we organize cases according to its support. First suppose that the support of the (j_0-1, j_0) row is entirely to the left of i_0; then according to Proposition 8.7, the same holds for the (j_0-1, j_0-1) row, and the (j_0-1, j_0+1) row cannot have its support to the right of i_1. We then see that $s_{j_0-1}\otimes s''_{j_0+1}$ cannot have any (j_0-1, j_0-1) or (j_0-1, j_0) parts, so must be of (j_0-1, j_0+1) type. Similarly, $s''_{j_0+1}\otimes s'''_{j_0+1}$ cannot have any (j_0-1, j_0-1), (j_0-1, j_0), or (j_0-1, j_0+1) parts, so it must contain (j_0, j_0+1) or (j_0+1, j_0+1) parts. In addition, the pair $s_{j_0}\otimes s''_{j_0+1}$ and $s_{j_0}\otimes s'''_{j_0+1}$ must contain either a (j_0, j_0+1) part, or two distinct (j_0, j_0) parts, supported left and right of i_0, respectively. Given that we always have (j_0-1, j_0-1), (j_0-1, j_0) and (j_0, j_0) parts, the only way we could fail to have produced six independent combinations is if $s''_{j_0+1}\otimes s'''_{j_0+1}$ has type (j_0, j_0+1), and we have only one (j_0, j_0) part. But then considering $s''^{\otimes 2}_{j_0+1}$ and $s'''^{\otimes 2}_{j_0+1}$ and using Lemma 8.2, we must produce a (j_0+1, j_0+1) part or two distinct (j_0, j_0) parts, so we necessarily obtain the sixth combination.

Similarly, if the potential support of the (j_0-1, j_0) row is entirely to the right of i_1, then Proposition 8.7 tells us that the same holds for (j_0, j_0), and that the potential support of the (j_0, j_0+1) row cannot be to the left of i_0. Then $s_{j_0}\otimes s'_{j_0+1}$ must be of (j_0, j_0+1) type, and $s'_{j_0+1}\otimes s'''_{j_0+1}$ must have (j_0-1, j_0+1) or (j_0+1, j_0+1) parts. The pair $s_{j_0-1}\otimes s'_{j_0+1}$ and $s_{j_0-1}\otimes s'''_{j_0+1}$ must contain either a (j_0-1, j_0+1) part, or two distinct (j_0-1, j_0-1) parts, and in either case the tensors $s'^{\otimes 2}_{j_0+1}$ and $s'''^{\otimes 2}_{j_0+1}$ (together with the usual tensors of s_{j_0-1} and s_{j_0}) must complete the six independent combinations.

Finally, if the potential support of the (j_0-1, j_0) row is between the i_0 and i_1 columns, then by Proposition 8.6, we know that the potential support of (j_0-1, j_0-1) is left of i_1 and the potential support of (j_0, j_0) is right of i_0. We then see that the tensors $s_{j_0-1}\otimes s'''_{j_0+1}$, $s_{j_0}\otimes s'''_{j_0+1}$, and $s'''^{\otimes 2}_{j_0+1}$ must be pure of types (j_0-1, j_0+1), (j_0, j_0+1), and (j_0+1, j_0+1) respectively, yielding the desired six combinations. □

Lemma 8.11. *Assume that $\rho=2$ and $r=6$ and we are in the "second 3-cycle" situation of Proposition 4.20. Then, there is an unimaginative multidegree ω_{def}, such that one of the following options is satisfied:*

(a) *the (j_0-1, j_0-1) row does not have potentially present sections both left of i_0 and right of i_1; or*

(b) $2a^{i_0}_{j_0-1}=c_{i_0}-1$, *and* $2a^{i_1+1}_{j_0-1}=c_{i_1+1}+1$; *or*

(c) $2a^{i_0}_{j_0-1}=c_{i_0}-2$, *and* $2a^{i_1+1}_{j_0-1}=c_{i_1+1}+2$, *and w has degree 2 in both i_0 and i_1.*

Proof. If (a) does not hold, the (j_0-1, j_0-1) row has support both left of i_0 and right of i_1. From Proposition 5.10, then

$$a^{i_0}_{(j_0-1, j_0-1)}<c_{i_0},\quad a^{i_1+1}_{(j_0-1, j_0-1)}>c_{i_1}.$$

Write $a^{i_0}_{(j_0-1, j_0-1)}=c_{i_0}-k_0$, $a^{i_1+1}_{(j_0-1, j_0-1)}=c_{i_1}+k_1$. Denote by t the number of elliptic components from i_0, to i_1 (inclusive). As we are assuming that $\rho=2$ and there are two swaps, there are no other exceptional columns. Therefore, between i_0 and i_1, $a^{i+1}_{j_0-1} = a^{i}_{j_0-1}$ if the component i is rational, $a^{i+1}_{j_0-1} \le a^{i}_{j_0-1} + 1$ if it is elliptic with $i \ne i_0, i_1$ and $a^{i_k+1}_{j_0-1} = a^{i_k}_{j_0-1} + 2$, $k = 0, 1$. Therefore

$$a^{i_1+1}_{(j_0-1,j_0-1)} \le a^{i_0}_{(j_0-1,j_0-1)} + 4 + 2t.$$

If ω_{def} is an unimaginative multidegree, then it has degree 0 on rational components, and 2 or 3 on elliptic components with γ_i the number of 3s in the first i components. Therefore

$$c_{i_1+1} = c_{i_0} + 2t + (\gamma_{i_1} - \gamma_{i_0-1}).$$

From these identities, it follows that

$$k_0 + k_1 \le 4 - (\gamma_{i_1} - \gamma_{i_0-1}).$$

As by assumption, both k_0, k_1 are strictly positive, the options for the pair (k_0, k_1) are

$$(1, 1), \quad (1, 2), \quad (2, 1), \quad (2, 2).$$

The case $(1, 1)$ is option (b). In case $(2, 2)$, $\gamma_{i_1} - \gamma_{i_0-1} = 0$, therefore the degree on each elliptic component from i_0 to i_1 is 2. This is option (c). From Corollary 7.7, in cases $(1, 2)$, $(2, 1)$, with a suitable choice of multidegree, we are in case (a). Therefore, the result is proved □

Proposition 8.12. *Suppose that $\rho = 2$, X_0 is left-weighted and we are in the "second 3-cycle" situation of Proposition 4.20. Choose an unimaginative multidegree $w = (c_2, \dots, c_N)$ satisfying one of the conditions of Lemma 8.11. Then the images in multidegree* $\mathrm{md}(w)$ *of the tensors of pairs of the s_j for $j \ne j_0, j_0{+}1$, and $s'_{j_0}, s''_{j_0}, s'_{j_0+1}, s''_{j_0+1}, s'''$ contain $\binom{r+2}{2}$ independent linear combinations of the potentially present sections.*

Proof. By assumption, the limit linear series contains precisely two swaps, with one swap between the $(j_0{-}1)$-st and j_0-th rows occurring in the i_0-th column, and a second swap between the $(j_0{-}1)$-st and $(j_0{+}1)$-st rows in the i_1-st column for some $i_1 > i_0$. First suppose $j \ne j_0 - 1, j_0, j_0 + 1$; we show that we can always obtain three linearly independent combinations of potentially present sections from the rows $(j, j_0 - 1)$, (j, j_0) and $(j, j_0 + 1)$. $s_j \otimes s_{j_0-1}$ always yields a pure $(j, j_0 - 1)$ part. If $S'_2 = S''_4 = \{1, \dots, N\}$, then $s_j \otimes s'_{j_0}$ has a nonzero (j, j_0) part and no $(j, j_0 + 1)$ part, while $s_j \otimes s''_{j_0+1}$ has a nonzero $(j, j_0 + 1)$ part, so we get the desired three combinations. Otherwise, we have

$$\begin{aligned} s_j \otimes s'_{j_0} &= (j, j_0 - 1)_{\mathrm{L}} + (j, j_0), \\ s_j \otimes s''_{j_0} &= (j, j_0) + (j, j_0 + 1)_{\mathrm{R}'} + (j, j_0 - 1)_{\mathrm{CR}}, \\ s_j \otimes s'_{j_0+1} &= (j, j_0 - 1)_{\mathrm{LC}} + (j, j_0)_{\mathrm{L}'} + (j, j_0 + 1), \\ s_j \otimes s''_{j_0+1} &= (j, j_0 + 1) + (j, j_0 - 1)_{\mathrm{R}}, \\ s_j \otimes s''' &= (j, j_0) + (j, j_0 - 1)_{\mathrm{C}} + (j, j_0 + 1), \end{aligned}$$

where R' and L' denote possible support at and right of i_1 and at and left of i_0, respectively, and if $s_j \otimes s''_{j_0}$ has a nonzero $(j, j_0 + 1)$ part with support containing i_1, its (j, j_0) part must be nonzero, and similarly for the (j, j_0) and $(j, j_0 + 1)$ parts of $s_j \otimes s'_{j_0+1}$. Now, suppose that $(j, j_0 - 1)$ has connected potential support which is not contained strictly right of i_0. Then (j, j_0+1) cannot have any potential support strictly right of i_1 without also forcing (j, j_0-1) to have potential support strictly right of i_1, so the (j, j_0) part of $s_j \otimes s''_{j_0}$ must be nonzero. But then adding $s_j \otimes s''_{j_0+1} = (j, j_0+1)$ and $s_j \otimes s_{j_0-1}$ yields three independent sections. Similarly, if (j, j_0-1) has connected potential support not contained strictly left of i_1, then (j, j_0) cannot have potential support strictly left of i_0, so $s_j \otimes s'_{j_0+1}$ has nonzero (j, j_0+1) part, and adding $s_j \otimes s'_{j_0} = (j, j_0)$ and $s_j \otimes s_{j_0-1}$ yields the desired combinations. For connected potential support, the only remaining possibility is that (j, j_0-1) has potential support strictly between i_0 and i_1, in which case $s_j \otimes s'_{j_0} = (j, j_0)$ and $s_j \otimes s''_{j_0+1} = (j, j_0+1)$.

Finally, since $\rho = 2$, the only remaining possibility is that (j, j_0-1) has potential support both left of i_0 and right of i_1, and in this case we must have $a^{i_0}_{(j,j_0-1)} = c_{i_0} - 1$ and $a^{i_1+1}_{(j,j_0-1)} = c_{i_1+1} + 1$. Then (j, j_0+1) cannot have potential support strictly right of i_1, and (j, j_0) cannot have potential support strictly left of i_0, so as above we find that if the (j, j_0+1) part of $s_j \otimes s''_{j_0}$ is nonzero (necessarily with support at i_1), then the (j, j_0) part must also be nonzero, and if the (j, j_0) part of $s_j \otimes s'_{j_0+1}$ is nonzero, then the (j, j_0+1) part must also be nonzero. Now, we have $s_j \otimes s'_{j_0}$ and $s_j \otimes s''_{j_0+1}$ linearly independent always, and the only way they could fail to be independent from $s_j \otimes s'''$ is if either $s_j \otimes s'_{j_0} = (j, j_0)$ or $s_j \otimes s''_{j_0+1} = (j, j_0+1)$, while the only way they could fail to be independent from $s_j \otimes s_{j_0-1}$ if is either $s_j \otimes s'_{j_0} = (j, j_0-1)_{\mathrm{L}}$ or $s_j \otimes s''_{j_0+1} = (j, j_0-1)_{\mathrm{R}}$. If $s_j \otimes s'_{j_0} = (j, j_0)$ and $s_j \otimes s''_{j_0+1} = (j, j_0-1)_{\mathrm{R}}$, we see that $s_j \otimes s'_{j_0+1}$ necessarily gives a third independent combination, while if $s_j \otimes s'_{j_0} = (j, j_0-1)_{\mathrm{L}}$ and $s_j \otimes s''_{j_0+1} = (j, j_0+1)$, we see that $s_j \otimes s''_{j_0}$ necessarily gives a third independent combination.

It remains to show that we can get six independent combinations from the rows (j_0-1, j_0-1), (j_0-1, j_0), (j_0-1, j_0+1), (j_0, j_0), (j_0, j_0+1), and (j_0+1, j_0+1). If $S'_2 = S''_4 = \{1, \dots, N\}$, then we immediately get that the six tensors coming from $s_{j_0-1}, s'_{j_0}, s''_{j_0+1}$ are linearly independent, as desired. Otherwise, we will make use of the mixed section s''' to handle certain cases. For reference, we write out the form of all the relevant tensors of $s_{j_0-1}, s'_{j_0}, s''_{j_0}, s'_{j_0+1}, s''_{j_0+1}$ (we are making use of Lemma 8.2 in the case of self-tensors):

$$\begin{aligned}
s_{j_0-1} \otimes s'_{j_0} &= (j_0-1, j_0-1)_{\mathrm{L}} + (j_0-1, j_0),\\
s_{j_0-1} \otimes s''_{j_0} &= (j_0-1, j_0) + (j_0-1, j_0+1)_{\mathrm{R}'} + (j_0-1, j_0-1)_{\mathrm{CR}},\\
s_{j_0-1} \otimes s'_{j_0+1} &= (j_0-1, j_0-1)_{\mathrm{LC}} + (j_0-1, j_0)_{\mathrm{L}'} + (j_0-1, j_0+1),\\
s_{j_0-1} \otimes s''_{j_0+1} &= (j_0-1, j_0+1) + (j_0-1, j_0-1)_{\mathrm{R}},\\
s'_{j_0} \otimes s'_{j_0} &= (j_0-1, j_0-1)_{\mathrm{L}} + (j_0, j_0),\\
s''_{j_0} \otimes s''_{j_0} &= (j_0, j_0) + (j_0+1, j_0+1)_{\mathrm{R}} + (j_0-1, j_0-1)_{\mathrm{CR}},\\
s'_{j_0} \otimes s''_{j_0} &= (j_0-1, j_0) + (j_0, j_0) + (j_0, j_0+1)_{\mathrm{R}'},\\
s'_{j_0+1} \otimes s'_{j_0+1} &= (j_0-1, j_0-1)_{\mathrm{LC}} + (j_0, j_0)_{\mathrm{L}} + (j_0+1, j_0+1),\\
s''_{j_0+1} \otimes s''_{j_0+1} &= (j_0+1, j_0+1) + (j_0-1, j_0-1)_{\mathrm{R}},\\
s'_{j_0+1} \otimes s''_{j_0+1} &= (j_0-1, j_0+1) + (j_0, j_0+1)_{\mathrm{L}'} + (j_0+1, j_0+1),
\end{aligned}$$

$$s'_{j_0} \otimes s'_{j_0+1} = (j_0-1,\, j_0-1)_{\mathrm{L}} + (j_0-1,\, j_0)_{\mathrm{LC}} + (j_0-1,\, j_0+1)_{\mathrm{L}} + (j_0,\, j_0)_{\mathrm{L}'} + (j_0,\, j_0+1),$$
$$s'_{j_0} \otimes s''_{j_0+1} = (j_0-1,\, j_0+1)_{\mathrm{L}} + (j_0-1,\, j_0)_{\mathrm{R}} + (j_0,\, j_0+1),$$
$$s''_{j_0} \otimes s''_{j_0+1} = (j_0-1,\, j_0-1)_{\mathrm{R}} + (j_0-1,\, j_0)_{\mathrm{R}} + (j_0-1,\, j_0+1)_{\mathrm{CR}} + (j_0,\, j_0+1) + (j_0+1,\, j_0+1)_{\mathrm{R}'}.$$

As above, we separate out cases by the potential support of the $(j_0-1,\, j_0-1)$ row. Note that because the entries sum to $2d-4$ in both the i_0 and i_1 columns, the $(j_0-1,\, j_0-1)$ row cannot have any potential support in either of these columns in any unimaginative multidegree. First suppose the potential support is strictly to the left of i_0. In this case none of the relevant rows can have potential support extending right of i_1, so we get $s_{j_0-1} \otimes s''_{j_0+1} = (j_0-1,\, j_0+1)$, $s''_{j_0} \otimes s''_{j_0} = (j_0,\, j_0)$, and $s''_{j_0+1} \otimes s''_{j_0+1} = (j_0+1,\, j_0+1)$, and the $(j_0-1,\, j_0)$ part of $s_{j_0-1} \otimes s''_{j_0}$ must be nonzero. We also have $s'_{j_0} \otimes s''_{j_0+1} = (j_0-1,\, j_0+1)_{\mathrm{L}} + (j_0,\, j_0+1)$ and $s''_{j_0} \otimes s''_{j_0+1} = (j_0-1,\, j_0+1)_{\mathrm{CR}} + (j_0,\, j_0+1) + (j_0+1,\, j_0+1)_{\mathrm{R}'}$, where again the latter has to have nonzero $(j_0,\, j_0+1)$ part unless it is equal to $(j_0-1,\, j_0+1)_{\mathrm{CR}}$, so these must either yield a nonzero $(j_0,\, j_0+1)$ part, or two independent $(j_0-1,\, j_0+1)$ parts (which won't happen when $\rho = 2$), and in either case together with $s_{j_0-1} \otimes s_{j_0-1}$ we get the desired six independent combinations.

Similarly, if the potential support of the $(j_0-1,\, j_0-1)$ row is strictly to the right of i_1, we will have

$$s_{j_0-1} \otimes s'_{j_0} = (j_0-1,\, j_0), \quad s'_{j_0} \otimes s'_{j_0} = (j_0,\, j_0), \quad s'_{j_0+1} \otimes s'_{j_0+1} = (j_0+1,\, j_0+1),$$

with $s_{j_0-1} \otimes s'_{j_0+1}$ having a nonzero $(j_0-1,\, j_0+1)$ part, and $s'_{j_0} \otimes s''_{j_0+1} = (j_0-1,\, j_0)_{\mathrm{R}} + (j_0,\, j_0+1)$ and $s'_{j_0} \otimes s'_{j_0+1} = (j_0-1,\, j_0)_{\mathrm{LC}} + (j_0,\, j_0+1) + (j_0,\, j_0)_{\mathrm{L}'}$, and we again obtain six independent combinations in the same manner.

If the potential support of the $(j_0-1,\, j_0-1)$ row is strictly between i_0 and i_1, then none of the relevant rows can have support either left of i_0 or right of i_1, and we get $s_{j_0-1} \otimes s'_{j_0} = (j_0-1,\, j_0)$, $s_{j_0-1} \otimes s''_{j_0+1} = (j_0-1,\, j_0+1)$, $s'_{j_0} \otimes s'_{j_0} = (j_0,\, j_0)$, $s''_{j_0+1} \otimes s''_{j_0+1} = (j_0+1,\, j_0+1)$, and $s'_{j_0} \otimes s''_{j_0+1} = (j_0,\, j_0+1)$.

If the $(j_0-1,\, j_0-1)$ row has disconnected potential support to the left of i_0 and strictly between i_0 and i_1, then once again none of the relevant rows can have potential support extending right of i_1, and because $\rho = 2$ we must have $a^{i_0}_{(j_0-1,\, j_0-1)} = c_{i_0} - 1$, so none of the other relevant rows can have their potential support contained strictly left of i_0, either. Moreover, the $(j_0-1,\, j_0)$ row must have potential support containing i_0, so $s'_{j_0} \otimes s''_{j_0}$ cannot have any $(j_0-1,\, j_0)$ part, and its $(j_0,\, j_0)$ part must be nonzero. We then find that $s_{j_0-1} \otimes s''_{j_0+1} = (j_0-1,\, j_0+1)$, $s''_{j_0+1} \otimes s''_{j_0+1} = (j_0+1,\, j_0+1)$, and $s'_{j_0} \otimes s''_{j_0+1} = (j_0,\, j_0+1)$. If the $(j_0-1,\, j_0)$ part of $s_{j_0-1} \otimes s''_{j_0}$ is nonzero, then these together with $s_{j_0-1} \otimes s_{j_0-1}$ give six independent combinations. Otherwise, we must have $s_{j_0-1} \otimes s''_{j_0} = (j_0-1,\, j_0-1)_{\mathrm{C}}$, and we see that $s_{j_0-1} \otimes s'_{j_0} = (j_0-1,\, j_0-1)_{\mathrm{L}} + (j_0-1,\, j_0)$ gives a sixth independent combination.

The situation is nearly the same if the $(j_0-1,\, j_0-1)$ row has disconnected potential support to the right of i_1 and strictly between i_0 and i_1. Here we instead obtain that $(j_0-1,\, j_0+1)$ must have potential support containing i_1, and thus that $s_{j_0-1} \otimes s'_{j_0} = (j_0-1,\, j_0)$, $s'_{j_0} \otimes s'_{j_0} = (j_0,\, j_0)$, and $s'_{j_0} \otimes s''_{j_0+1} = (j_0,\, j_0+1)$, with $s'_{j_0+1} \otimes s''_{j_0+1}$ having nonzero $(j_0+1,\, j_0+1)$ part. Then $s_{j_0-1} \otimes s'_{j_0+1}$ either has a nonzero $(j_0-1,\, j_0+1)$ part, or is equal to $(j_0-1,\, j_0-1)_{\mathrm{C}}$, and in either case we obtain a sixth combination, from $s_{j_0-1} \otimes s_{j_0-1}$ or $s_{j_0-1} \otimes s''_{j_0+1} = (j_0-1,\, j_0+1) + (j_0-1,\, j_0-1)_{\mathrm{R}}$, respectively.

If (j_0-1, j_0-1) has three components of potential support, necessarily left of i_0, strictly between i_0 and i_1, and right of i_1, then none of the relevant rows other than (j_0-1, j_0-1) can have potential support contained strictly left of i_0 or strictly right of i_1, and we also know that the potential support of the (j_0-1, j_0) (respectively, (j_0-1, j_0+1)) row contains i_0 (respectively, i_1). We then have that $s'_{j_0}\otimes s''_{j_0+1} = (j_0, j_0+1)$, and that $s'_{j_0}\otimes s''_{j_0}$ and $s'_{j_0+1}\otimes s''_{j_0+1}$ have nonzero (j_0, j_0) and (j_0+1, j_0+1) parts, respectively. We also have $s_{j_0-1}\otimes s'_{j_0} = (j_0-1, j_0-1)_{\mathrm{L}}+(j_0-1, j_0),$, $s_{j_0-1}\otimes s''_{j_0+1} = (j_0-1, j_0-1)_{\mathrm{R}}+(j_0-1, j_0+1)$, and $s_{j_0-1}\otimes s''' = (j_0-1, j_0)+(j_0-1, j_0-1)_{\mathrm{C}}+(j_0-1, j_0+1)$. To have a dependence between these, we need (at least one of) $s_{j_0-1}\otimes s'_{j_0} = (j_0-1, j_0)$ or $s_{j_0-1}\otimes s''_{j_0+1} = (j_0-1, j_0+1)$. On the other hand, to have a dependence between the first five and $s_{j_0-1}\otimes s_{j_0-1}$, we need $s_{j_0-1}\otimes s'_{j_0} = (j_0-1, j_0-1)_{\mathrm{L}}$ or $s_{j_0-1}\otimes s''_{j_0+1} = (j_0-1, j_0-1)_{\mathrm{R}}$. If $s_{j_0-1}\otimes s'_{j_0} = (j_0-1, j_0)$ and $s_{j_0-1}\otimes s''_{j_0+1} = (j_0-1, j_0-1)_{\mathrm{R}}$, we see that $s_{j_0-1}\otimes s'_{j_0+1}$ must have a nonzero $(j_0-1, j_0-1)_{\mathrm{LC}}$ or (j_0-1, j_0+1) part, and thus gives a sixth independent combination. On the other hand, if $s_{j_0-1}\otimes s'_{j_0} = (j_0-1, j_0-1)_{\mathrm{L}}$ and $s_{j_0-1}\otimes s''_{j_0+1} = (j_0-1, j_0+1)$, we see that $s_{j_0-1}\otimes s''_{j_0}$ must have a nonzero $(j_0-1, j_0-1)_{\mathrm{CR}}$ or (j_0-1, j_0) part, and again gives a sixth independent combination.

It remains to analyze the case that (j_0-1, j_0-1) has two components of potential support, one left of i_0, and the other right of i_1. By hypothesis, we only have to address the case that $a^{i_0}_{(j_0-1,j_0-1)} = c_{i_0}-2$ and $a^{i_1+1}_{(j_0-1,j_0-1)} = c_{i_1+1}+2$, and that we have degree 2 in both i_0 and i_1. In this situation, the (j_0-1, j_0) row has potential support strictly left of i_0, but none of the other relevant rows do, and the (j_0, j_0) row must have support containing i_0 and extending left to at least the previous genus-1 component. Similarly, the (j_0-1, j_0+1) row has potential support strictly right of i_1, but none of the other relevant rows do, and the (j_0+1, j_0+1) row has support containing i_1 and extending to the right to at least the next genus-1 component. We also see that the potential support of (j_0, j_0+1) must be contained between i_0 and i_1 inclusive, and cannot be equal solely to i_0 or to i_1. In particular, $s'_{j_0}\otimes s''_{j_0+1}$ cannot have a (j_0-1, j_0) or (j_0-1, j_0+1) part, so must be equal to (j_0, j_0+1).

Now, $s_{j_0-1}\otimes s_{j_0-1} = (j_0-1, j_0-1)_{\mathrm{L}}$ because X_0 is left-weighted, and we begin by considering the case that no tensor has a $(j_0-1, j_0-1)_{\mathrm{R}}$ part. Then we must have $s_{j_0-1}\otimes s''_{j_0+1} = (j_0-1, j_0+1)$, $s''_{j_0+1}\otimes s''_{j_0+1} = (j_0+1, j_0+1)$, $s''_{j_0}\otimes s''_{j_0} = (j_0, j_0)$, and we also see that $s_{j_0-1}\otimes s''_{j_0}$ must be (j_0-1, j_0), because it could only have a $(j_0-1, j_0+1)_{\mathrm{R}'}$ part if the j_0 part of s''_{j_0} extends through i_1, and in this case the fact that X_0 is left-weighted gives us that $s_{j_0-1}\otimes s''_{j_0} = (j_0-1, j_0)$ regardless. Thus, we obtain the desired six independent combinations in this case.

On the other hand, if any tensor has a $(j_0-1, j_0-1)_{\mathrm{R}}$ part, we need to produce only three more independent combinations, and we consider the four tensors,

$$s'_{j_0}\otimes s''_{j_0} = (j_0-1, j_0)+(j_0, j_0), \qquad s'_{j_0+1}\otimes s''_{j_0+1} = (j_0-1, j_0+1)+(j_0+1, j_0+1),$$
$$s_{j_0-1}\otimes s''' = (j_0-1, j_0)+(j_0-1, j_0+1), \qquad s'''\otimes s''' = (j_0, j_0)+(j_0+1, j_0+1).$$

These must have at least a three-dimensional span unless they collapse into equal pairs, and there are two possibilities for this: either $s'_{j_0}\otimes s''_{j_0} = s_{j_0-1}\otimes s''' = (j_0-1, j_0)$ and $s'_{j_0+1}\otimes s''_{j_0+1} = s'''\otimes s''' = (j_0+1, j_0+1)$, or $s'_{j_0}\otimes s''_{j_0} = s'''\otimes s''' = (j_0, j_0)$ and $s'_{j_0+1}\otimes s''_{j_0+1} = s_{j_0-1}\otimes s''' = (j_0-1, j_0+1)$. Moreover, Proposition 4.24

implies that the j_0-part of s'_{j_0} doesn't contain any genus-1 components left of i_0. Then we necessarily have $s'_{j_0} \otimes s''_{j_0} = (j_0 - 1, j_0)$, so only the first possibility above can occur. Now, in general we have $s''_{j_0} \otimes s''' = (j_0, j_0) + (j_0 - 1, j_0 + 1)_{\mathrm{CR}} + (j_0 + 1, j_0 + 1)_{\mathrm{R}'} + (j_0 - 1, j_0 - 1)_{\mathrm{C}} + (j_0 - 1, j_0)_{\mathrm{CR}} + (j_0, j_0 + 1)$, which in our case simplifies to $s''_{j_0} \otimes s''' = (j_0, j_0) + (j_0 - 1, j_0 + 1)_{\mathrm{CR}} + (j_0, j_0 + 1) + (j_0 + 1, j_0 + 1)_{\mathrm{R}'}$.

If this has nonzero (j_0, j_0) or $(j_0 - 1, j_0 + 1)$ term, we have our sixth independent combination. On the other hand, if the $(j_0 + 1, j_0 + 1)$ term is nonzero, the $(j_0, j_0 + 1)$ term must also be. Because the potential support of $(j_0, j_0 + 1)$ must end no later than i_1 and cannot be supported solely at i_1, if the $(j_0, j_0 + 1)$ term of $s''_{j_0} \otimes s'''$ is nonzero, this means that the j_0 part of s''_{j_0} must extend to cover all of $(j_0, j_0 + 1)$ (note that the proof of Lemma 8.1 indicates that a $(j_0, j_0 + 1)$ part has to come from either a j_0 part of s''_{j_0} and a $(j_0 + 1)$ part of s''' or vice versa, but not some mixture of the two). But we know that this contains at least one genus-1 component strictly right of i_0, so since the support of (j_0, j_0) ends at i_0, and X_0 is left-weighted, we conclude that we would have to have $s''_{j_0} \otimes s''_{j_0} = (j_0, j_0)$ in this case. Thus, in all cases we obtain the desired six independent combinations. $\square$

9. Proof of the main theorem

We are now ready to prove Theorem 1.1. The main point is that if we have a smoothing family $\pi : X \to B$ as in Remark 3.2, and a generic linear series $(\mathscr{L}_\eta, V_\eta)$, which after base change and blowup we may assume is rational on the generic fiber, we can apply the linked linear series construction both to $(\mathscr{L}_\eta, V_\eta)$ and to $(W_\eta, \mathscr{L}_\eta^{\otimes 2})$, where W_η is the image of the multiplication map (1-1) of sections $s' \in V_{\omega'}$ and $s'' \in V_{\omega''}$. As in the discussion following Proposition 3.12 of [Liu et al. 2021], for any multidegree ω of total degree $2d$, and any multidegrees ω', ω'' of total degree d, $f_{\omega'+\omega'',\omega}(s' \otimes s'')$ lies in W_ω. Thus, in order to give a lower bound on the rank of (1-1), we can choose many different ω', ω'' and s', s'', and show that they span a certain-dimensional subspace of $(\mathscr{L}^{\otimes 2})_\omega$.

Theorem 9.1. *We assume characteristic* 0. *Fix g, r, d with $r \geq 3$ and $\rho = 1$ or $\rho = 2$.*

If $\rho = 1$, suppose that for every generic chain of rational and elliptic curves X_0 and every refined limit $\mathfrak{g}^r_d$ on X_0, there is a multidegree ω such that the potentially present sections in multidegree ω are linearly independent.

If $\rho = 2$, suppose that for every left-weighted generic chain of elliptic and rational curves X_0 of total genus g and every refined limit $\mathfrak{g}^r_d$ on X_0, there is an unimaginative $w = (c_2, \dots, c_N)$ such that the potentially present sections in multidegree $\mathrm{md}(w)$ *are linearly independent*,

Then the strong maximal rank conjecture holds for (g, r, d), and more specifically, if we define $\mathcal{D}_{(g,r,d)} \subseteq \mathcal{M}_g$ to be the set of curves which have a $\mathfrak{g}^r_d$ for which (1-1) *is not injective, then the closure in $\overline{\mathcal{M}}_g$ of $\mathcal{D}_{g,r,d}$ does not contain a general chain of genus*-1 *curves.*

Proof. According to the above discussion together with Theorem 3.4 and Proposition 3.10, we need to show that an arbitrary exact linked linear series on X_0 lying over a refined limit linear series admits some multidegree ω such that the combined images $f_{\omega'+\omega'',\omega}(s' \otimes s'')$ span an $\binom{r+2}{2}$-dimensional space. For the ω in the statement, it then suffices to show that these sections give $\binom{r+2}{2}$ independent combinations of the potentially present sections.

When $\rho = 1$, from Proposition 5.3 , we can have at most one swap. If we have no swaps, we obtain the desired independence directly from the independence of the potentially present sections, using Proposition 4.9. On the other hand, if we have a single swap, Proposition 8.4 states that there are $\binom{r+2}{2}$-linearly independent combinations of the potentially present sections in some unimaginative multidegree. We have proved the statement for all X_0 at once, so the stronger assertion on the closure of $\mathcal{D}_{(g,r,d)}$ follows (compare with the proof of Proposition 3.13 of [Liu et al. 2021]).

If $\rho = 2$, from Proposition 5.3, there are at most two swaps. If there are no swaps or one swap, the proof above still works. If there are two swaps, we can use Propositions 8.5, 8.9, 8.10 together with Corollary 7.6 and Proposition 8.12 together with Lemma 8.11 to guarantee the existence of $\binom{r+2}{2}$- linearly independent combinations of the potentially present sections in some unimaginative multidegree.

When $\rho = 2$, we assume X_0 is left-weighted. This forces us to consider only special directions of approach to X_0 in $\overline{\mathcal{M}}_g$. Recalling that being left-weighted is preserved under the insertions of genus-0 chains which occur from base change and then blow up to resolve the resulting singularities, we conclude that for suitable smoothing families, the generic fiber cannot carry a $\mathfrak{g}^r_d$ for which (1-1) is not injective, as desired. □

Theorem 1.1 follows now from Theorem 9.1 together with Theorem 7.3 using that $\rho = 1$ when $g = 22$ and $\rho = 2$ when $g = 23$ and that in both cases, $r = 6$.

Remark 9.2. In our arguments for the $g = 23$ case, we used the $\rho = 2$ hypothesis in two distinct ways: first, to limit the number of swaps occurring to two, but then also to control the behavior of the rest of the limit linear series when two swaps did occur, for instance limiting the number of possibilities for rows having disconnected potential support. This may appear discouraging from the point of view of generalizing to cases with higher ρ, but as ρ increases, one also obtains more flexibility in choosing multidegrees while still maintaining linear independence of the potentially present sections. Indeed, we are taking advantage of this phenomenon already in the $\rho = 2$ case with Corollary 7.6.

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