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Functorial embedded resolution via weighted blowings up

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We provide a simple procedure for resolving, in characteristic 0, singularities of a variety X embedded in a smooth variety Y by repeatedly blowing up the worst singularities, in the sense of stack-theoretic weighted blowings up. No history, no exceptional divisors, and no logarithmic structures are necessary to carry this out; the steps are explicit geometric operations requiring no choices; and the resulting algorithm is efficient.

A similar result was discovered independently by McQuillan (2020).

1. Introduction

1.1. ***Classical embedded resolution.*** All known methods to canonically (or functorially) resolve singularities of a variety X are embedded: first, one locally embeds X into a smooth *ambient variety* Y, and then gradually improves the transforms (either proper or weak) X_i of X by a series of *basic modifications* $\cdots Y_2 \to Y_1 \to Y_0 = Y$ such that each Y_i is smooth. In fact, the embedded framework was already used by Hironaka [1964a; 1964b], then, based on Hironaka's and Giraud's works, canonical methods were introduced by Bierstone-Milman [1997] and Villamayor [1989], and the full functoriality with respect to smooth morphisms was achieved by Schwartz [1992] and Włodarczyk [2005]. Note that it suffices to construct a functorial resolution étale-locally as globalization follows from the reembedding principle

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(recalled in Section 8.1) and functoriality. We refer to the methods developed in these papers, as well as [Encinas and Hauser 2002; Encinas and Villamayor 2003; Kollár 2007], etc., the *classical methods*.

Basic modifications in the classical methods are blowings up with smooth centers $V_i \subset Y_i$, in particular, this is the way to guarantee that Y_{i+1} is also smooth. Most naturally, one would like to choose V_i to be the worst singularity locus for a natural singularity invariant $\mathrm{inv}_{(Y,X)} : X \to I$ with values in a well-ordered set so that each blowing up improves the invariant. This would lead to a simplest resolution method controlled by a geometrically meaningful singularity measure. But it was common knowledge for decades that this dream is unrealizable; see, for instance, [Kollár 2007, Example 3.6.1] and Section 1.8.

Starting with Hironaka's work, the classical methods use history and the choice of V_i, as well as the invariant $\mathrm{inv}_{(Y_i,X_i)}$, depends on the whole earlier resolution process rather than just on (Y_i, X_i). In particular, only the center $V_0 \subset Y_0$ of the first blowup of the process, sometimes known as the "year-zero center", and invariant $\mathrm{inv}_{(X_0,Y_0)}$, possess a clear geometric meaning.

Remark 1.1.1. (i) The composed sequence $Y_n \to Y$ can be canonically realized as a single blowing up along a highly nonreduced center V, but this is a rather useless presentation, no clear connection between the geometry of V and the singularities of X is known, and it is even unclear how to show that $\mathrm{Bl}_V(Y)$ is smooth if not by a direct computation.

(ii) In classical methods, a basic embedded resolution operates with weak (or principal) transforms, so the intermediate X_i may have new components contained in the exceptional divisor, the center V_i does not have to lie in the i-th proper (or strict) transform X_i^{st} and though $X_{i+1}^{\mathrm{st}} \to X_i^{\mathrm{st}}$ is a blowing up, its center $V_i \cap X_i^{\mathrm{st}}$ may be singular. Using basic resolution and Hilbert–Samuel function one can develop a much more technical method, usually called strong resolution, which operates with proper transforms and hence satisfies $V_i \subset X_i$ and $X_{i+1} = \mathrm{Bl}_{V_i}(X_i)$. It outputs a desingularization blowing up sequence $X_n \to \cdots X_0$ whose centers are smooth.

As noted in Section 1.3, our desingularization Theorem 1.2.2 works directly with proper transforms, and thus achieves strong resolution without the need for further reductions.

1.2. *Statement of main results.* In this paper we show that the unrealizable dream becomes possible (probably, even the most natural solution) once one enlarges the pool of basic modifications to the class of weighted blowings up along smooth centers. In fact, just the classical year-zero blowings up with correct weights, which are encoded in the classical year zero invariant, does the job! A similar result was obtained independently by McQuillan [2020].

The stumbling block all these years was the fact that weighted blowings up were not a legitimate tool in embedded resolution because the output ambient variety may be singular. Recently, it was discovered that such blowings up possess a smooth stack-theoretic refinement, and this makes them an absolutely kosher embedded resolution tool at the price of working with Deligne–Mumford stacks instead of varieties. Since the construction is étale-local and the coarse moduli space can be easily resolved (using simple combinatorial tools going under the name "destackification"), this is not a real burden; see Section 1.6 and Theorem 8.1.3.

Definition 1.2.1. By a *DM pair* (X, Y) we mean a quasicompact Deligne–Mumford stack Y smooth *over a field of characteristic zero* and a closed substack $X \subset Y$.

To extend the pool of blowings up we introduce in Section 2 *valuative* $\mathbb{Q}$*-ideals*, providing a convenient formalism to work with Hironaka's idealistic exponents. The basic examples are called *centers*, locally they are of the form $J = (x_1^{a_1}, \ldots, x_k^{a_k})$, where $a_i \in \mathbb{Q}_{>0}$ and $x_1, \ldots, x_n$ is a regular system of parameters. A center is called *reduced* if $w_i = 1/a_i$ are natural numbers with $\gcd(w_1, \ldots, w_k) = 1$. Note that a usual ideal and its normalization give rise to the same valuative ideal, and $J^l = (x_1^{la_1}, \ldots, x_k^{la_k})$ as valuative ideals for $l \in \mathbb{N}$. In particular, there is a unique reduced center $\bar{J}$ such that $J = \bar{J}^l$ with $l \in \mathbb{Q}$. In Section 3 we associate to any center J a blowing up $\mathrm{Bl}\,\bar{J}(Y)$, which is a smooth stack-theoretic enhancement of the classical weighted blowing up along $x_1, \ldots, x_k$ with weights $w_1, \ldots, w_k$. Such blowings up are compatible with smooth morphisms $f : Y' \to Y$, that is, $\mathrm{Bl}_{f^{-1}\bar{J}}(Y') = \mathrm{Bl}\,\bar{J}(Y) \times_Y Y'$. In particular, $\mathrm{Bl}\,\bar{J}(Y) \to (Y)$ is an isomorphism outside of $V(J) := V(x_1, \ldots, x_k)$, so the proper transform of closed subschemes is defined as usual.

Theorem 1.2.2 (a step towards resolution). *There is a construction which associates to each DM pair* (X, Y), *with* X *nonempty*, *a semicontinuous function* $\mathrm{inv}_{(X,Y)} : X \to \Xi_m$ *with values in a well-ordered set* Ξ_m *and a reduced center* $\bar{J} = \bar{J}(X, Y)$ *with the associated blowing up* $F_1(X, Y) : Y_1 \to Y$ *and proper transform* $X_1 \subset Y_1$ *such that the following conditions hold*:

(1) *The vanishing locus*: $V(\bar{J})$ *is precisely the locus where* $\mathrm{inv}_{(X,Y)}$ *attains its maximal value* $\mathrm{maxinv}(X, Y)$.

(2) *The invariant drops*: $\mathrm{maxinv}(X_1, Y_1) < \mathrm{maxinv}(X, Y)$.

(3) *Functoriality*: *for any smooth morphism* $f : Y' \to Y$ *with* $X' = X \times_Y Y'$, *one has that* $\mathrm{inv}_{(X',Y')} = \mathrm{inv}_{(X,Y)} \circ f$. *Furthermore*, *either* $f^{-1}\bar{J}(X, Y) = (1)$, *or* $\bar{J}(X', Y') = f^{-1}\bar{J}(X, Y)$ *and hence* $(X_1', Y_1') = (X_1, Y_1) \times_Y Y_1$.

The set Ξ_m does not depend on X or Y, only on $m := \dim Y$. It is a well-ordered subset $\Xi_m \subset \mathbb{Q}^{\leq m}$ of the set of sequences of length at most m, described in the context of Theorem 1.2.5 and in Section 5.1. Moreover, as m varies these sets are nested: $\Xi_m \subset \Xi_{m+1}$ allowing for the necessary comparison in (3).

The index 1 of $F_1(X, Y)$ indicates that it is a one-step operation on the way to a final product; the final product is achieved when X is empty so F_1 does not exist.

Since Ξ_m is well-ordered, composing the one-step partial resolution blowings up $F_1(X_i, Y_i) : Y_{i+1} \to Y_i$ one obtains a sequence $(X_l, Y_l) \to \cdots \to (X_0, Y_0) = (X, Y)$ with $X_l = \varnothing$.

The full weighted embedded resolution is obtained by stopping this process once a center containing an irreducible component of X is chosen, and here an equicodimensionality condition has to be imposed. From the description of the invariant below one sees that the minimal invariant locus is precisely the largest codimension component of the smooth locus of X, hence the theorem immediately implies

Corollary 1.2.3 (weighted resolution). *For a DM pair* (X, Y) *let* $F(X, Y) : (X_n, Y_n) \to \cdots \to (X_0, Y_0) = (X, Y)$ *denote the maximal sequence of blowings up* $F_1(X_i, Y_i)$ *whose centers are nowhere dense in* X_i. *In particular*, $X_n \to X$ *is proper and birational*:

(1) *If X is generically reduced and of constant codimension in Y, then X_n is smooth.*

(2) *If, in addition, $Y' \to Y$ is a smooth morphism and $X' = X \times_Y Y'$, then the sequence $F(X', Y')$ is obtained from $F(X, Y) \times_Y Y'$ by removing all blowings up with empty centers. In particular, $(X'_{n'}, Y'_{n'}) = (X_n, Y_n) \times_Y Y'$.*

Remark 1.2.4 (functorial formulation). One can spell out the results in terms of functors on categories. This is not used in the paper, so we only outline the formulation: F_1 can be viewed as a partial resolution endofunctor on the category of DM pairs with smooth surjective morphisms. Its birational stabilization $F = F_1^{\circ n}$ gives rise to a resolution endofunctor on the category of generically reduced DM pairs of constant codimension with arbitrary smooth morphisms:

- (nonembedded resolution) Using standard arguments, one deduces nonembedded resolution — see Theorem 8.1.1.
- (principalization) Theorem 1.2.2 relies on principalization of ideals on Deligne–Mumford stacks. See Theorem 6.3.1, where strict transforms in Theorem 1.2.2 and Corollary 1.2.3 are replaced by weak transforms.
- (coarse resolution) The reader may wonder about the coarse moduli spaces when Y is a variety. As we note in Section 8.2, the stacks Y_i and X_n have finite abelian stabilizers, hence their coarse moduli spaces $\underline{Y}_i$ and $\underline{X}_n$ have finite abelian quotient singularities. These are eminently resolvable, see Section 1.6 and Theorem 8.1.3. The transformations $\underline{Y}_{i+1} \to \underline{Y}_i$ are best described as the coarse transformations of the weighted blowings up $Y_{i+1} \to Y_i$.

Finally, we provide a very simple and geometric characterization of the invariant $\mathrm{inv}_{(X,Y)}$ and center $\bar{J}(X, Y)$, and we view this as a part of our main results. We will always order local parameters at a point p giving a center $J = (x_1^{a_1}, \dots, x_k^{a_k})$ so that $a_1 \le a_2 \le \cdots$ and set $\mathrm{inv}_J(p) = (a_1, \dots, a_k) \in \mathbb{Q}^{\le m} = \bigsqcup_{k=0}^m \mathbb{Q}^k$. We provide the set of invariants with the natural lexicographic order, where shorter sequences are declared to be of larger order.

Theorem 1.2.5. *Let (X, Y) be a variety pair, $I \subset \mathcal{O}_Y$ the ideal of X and $p \in X$ a point*:

(1) *There exists a neighborhood $p \in U$ and a center J on U such that $p \in V(J)$, $I|_U \subseteq J$ and $\mathrm{inv}_J(p)$ is maximal possible among such pairs (U, J). Moreover for all $p' \in V(J)$ we have $\mathrm{inv}_J(p') = \mathrm{inv}_J(p)$ and is locally maximal at p'. In particular, the invariant $\mathrm{inv}_{(X,Y)}(p) = \mathrm{inv}_J(p)$ is well defined and upper semicontinuous.*

(2) *The localization J_p is unique and does not depend on the choice of (U, J).*

(3) *If $\mathrm{inv}_J(p) = (a_1, \dots, a_k)$, then the numbers $b_1 = a_1$ and $b_l = a_l \prod_{i=1}^{l-1} b_i!$ for $2 \le l \le k$ are integers.*

The theorem is stated for varieties as it is local in nature. The theorem immediately implies that the set Ξ_m of actual invariants is well ordered, and there exists a unique center $J = J(X, Y)$ whose invariant is $\mathrm{maxinv}(X, Y)$, whose vanishing locus is the maximality locus of $\mathrm{inv}_{(X,Y)}$ and such that $V(I_X) \supseteq V(J)$. The center $\bar{J}(X, Y)$ is simply the reduction of J. So, what the algorithm really does — it blows up the unique center J such that $V(I_X) \supseteq V(J)$ and inv_J is maximal possible. Loosely speaking, this is just the center of maximal invariant contained in X.

1.3. ***Our quest for the present algorithm.*** Several times during our study, we were positively surprised by the properties of the algorithm presented here.

In [Abramovich et al. 2020a] we extended the pool of smooth blowings up in the logarithmic setting, and this allowed to produce algorithms with better efficiency and functoriality properties. That work required to consider stack-theoretic blowings up with nontrivial weights for monomial parameters, so our next project was to study what is the natural resolution algorithm that uses weighted blowings up with arbitrary weights. Our expectation was that the algorithm will be more efficient than the classical ones, but we did not expect at all that it would not require the history of prior operations, a "memoryless" algorithm visibly improving singularities by each weighted blowing up, as it turned out to be. The paradigm that such things do not exist was too strong. We do not know if there exists a simpler algorithm, or a faster one, or a more geometrically informative one, but to the best of our knowledge currently there is not even a conjecture in that direction.

Another surprise is that the algorithm shares common features with the strong resolution methods and uses proper transforms. In particular, the centers (with an appropriate formalism) are contained in X itself, so it can even be interpreted as a nonembedded algorithm and described without using an ambient manifold. In fact, while proving the principalization we show that already the weak transform reduces the invariant on each blowing up, hence the same is true for the proper transform. Since the algorithm is "memoryless", independent of the history of prior operations, this allows to work with proper transforms as well. No need to use Hilbert–Samuel function and much of the usual classical machinery.

Remark 1.3.1. In fact, our method produces a sequence of stack-theoretic modifications $X_n \to \cdots \to X_0 = X$ with a smooth source $\mathcal{F}_{\mathrm{ner}}(X) = X_n$ such that each $f_i : X_{i+1} \to X_i$ satisfies the following two properties:

(i) The method is "memoryless": f_i depends only on X_i.

(ii) The resolution is strong in the extended (stack-theoretic) meaning: each f_i is a stack-theoretic blowing up of a weighted smooth center.

Since we only introduce a narrow class of weighted blowings up — blowings up of *smooth* varieties along *smooth* weighted centers, our interpretation of the second property is the following naive one: (locally) X_i embeds into a smooth stack Y and f_i is the proper transform of a weighted blowing up $g : Y' \to Y$ along a weighted smooth center $\mathcal{J}$ which contains the ideal $\mathcal{I}_{X_i}$ of $\mathcal{O}_Y$. However, Quek and Rydh [2021] define weighted blowings up of arbitrary schemes along arbitrary Rees algebras, not necessarily smooth, and establish their basic properties. In particular, in the formalism of [Quek and Rydh 2021], f_i is indeed the strict transform of g and it is the weighted blowing up of the restriction of $\mathcal{J}$ onto X_i.

Finally, the choice of the center fits and clarifies very well the classical constructions, see Section 1.5. Loosely speaking, we just take the year-zero center with correct weights predicted by the year-zero invariant.

1.4. ***Weighted blowings up, stacks, and resolutions.*** Weighted blowings up in a scheme theoretic sense have been used in birational geometry (as well as many other subjects in mathematics) for a long time. Varchenko used them to characterize the log canonical threshold of a surface; see [Varčenko 1976; Kollár et al. 2004, Theorem 6.40]. Reid [1980; 2002] employs them in the foundation of canonical singularities and in the geometry of surfaces. Kawamata [1992] used them to relate discrepancies to indices. Martín-Morales [2013; 2014] uses them to efficiently study monodromy zeta functions as well as explicit $\mathbb{Q}$-desingularizations of certain singularities. Artal Bartolo, Martín-Morales, and Ortigas-Galindo [Artal Bartolo et al. 2012; 2014] further study the geometry of surfaces. All this on top of the enormous literature on weighted projective spaces.

All these authors show that weighted blowings up are remarkably efficient in computing invariants of singularities. In [Martín-Morales 2013; 2014], they are shown, in a wide class of examples, to be remarkably efficient in finding $\mathbb{Q}$-resolutions, namely modifications with at most quotient singularities.

Most relevant to the present paper, Panazzolo [2006] used scheme theoretic weighted blowings up to simplify foliations in dimension three, and McQuillan and Panazzolo [2013] revisited the problem using stack theoretic blowings up. In particular it is shown there that weighted blowings up are unavoidable for their goals. The paper [McQuillan and Panazzolo 2013] led to the paper [McQuillan 2020] concurrent to ours.

In our work, stack theoretic modification appeared in [Abramovich et al. 2020a] and shown to be unavoidable for functoriality of logarithmic resolution, leading us to investigate weighted blowings up in general.

1.5. ***Invariants and parameters.*** The notation for the present invariant $\mathrm{inv}_{\mathcal{I}}(p)$ in [Abramovich et al. 2020a] was $a_1 \cdot \mathrm{inv}_{\mathcal{I}_X, a_1}(p)$, and extends to arbitrary ideal sheaves on logarithmic orbifolds. Here it is applied solely when Y is smooth with trivial logarithmic structure.

Both this invariant and our center of blowing up are present in earlier work:

This invariant $(a_1, \ldots, a_k)$ is closely related to invariants developed in earlier papers on resolution of singularities, in particular [Bierstone and Milman 1997] and [Włodarczyk 2005]. In fact $(a_1, \ldots, a_k)$ is determined by a sequence $(b_1, \ldots, b_k)$ of integers, which is "interspersed" in Bierstone and Milman's richer invariant $(H_1, s_1, b_2, \ldots, b_k, s_k)$. Here b_1 is determined by the Hilbert–Samuel function H_1 and the $s_i = 0$ since no divisors are present — our invariant is in essence the classical "year zero invariant". Invariants of similar nature are already introduced in [Hironaka 1964b].

The center $J = (x_1^{a_1}, \ldots, x_k^{a_k})$ can be interpreted in terms of Newton polyhedra, and as such it appears in [Youssin 1990, Section 1], with a closely related precedent in [Hironaka 1967]. The local parameters $x_1, \ldots, x_k$ in the definition of J were already introduced in [Bierstone and Milman 1997; Encinas and Villamayor 2003; Włodarczyk 2005; Abramovich et al. 2020a] as a sequence of iterated hypersurfaces of maximal contact for appropriate coefficient ideals, see Section 5.1. In this paper we prove the necessary properties of the invariant $\mathrm{inv}_{\mathcal{I}}(p)$ and the center J, but many of these properties are directly implied by these cited works.

In earlier work the ideal $(x_1, \dots, x_k)$ was used to locally define the unique center of blowing up satisfying appropriate admissibility and functoriality properties for resolution using smooth blowings up. A central observation here is that the center $(x_1^{a_1}, \dots, x_k^{a_k})$ is uniquely defined as a valuative $\mathbb{Q}$-ideal, see Theorem 5.3.1(3).

As recalled below, in general, after blowing up the reduced ideal $(x_1, \dots, x_k)$, the invariant does not drop, and may increase; Earlier work enhanced this invariant by including data of exceptional divisors and their history, or more recently, logarithmic structures. Another central observation here is that, with the use of weighted blowings up, no history, no exceptional divisors, and no logarithmic structures are necessary.

1.6. ***Tools and methods.*** The present treatment requires the theory of Deligne–Mumford stacks. The reader is assumed to be comfortable with their basic notions, such as coherent sheaves and coarse moduli spaces, though there is little harm in viewing a stack as "locally the quotient of a variety by the action of a finite group", in which case coherent sheaves are represented by equivariant sheaves on the variety, and the coarse moduli space is the schematic (or algebraic space) quotient.

An application of Bergh's destackification theorem [Bergh 2017, Theorem 1.2] or its generalization [Bergh and Rydh 2019, Theorem B] allows one to replace $X_n \subset Y_n$ by a smooth embedded scheme $X'_n \subset Y'_n$ projective over $X \subset Y$, giving a resolution in the schematic sense, see Theorem 8.1.3. Alternatively the coarse moduli space admits only abelian quotient singularities (see Section 8.2) and can be resolved directly by combinatorial methods; see [Bogomolov 1992; Abramovich and de Jong 1997; Abramovich et al. 2002; 2020c; Włodarczyk 2003; Illusie and Temkin 2014; Włodarczyk 2022]. Both destackification and this resolution process apply in arbitrary characteristics, as the stabilizer group-schemes involved are tame.[1]

Our center J can be identified as an *idealistic exponent*, see [Hironaka 1977], which we present here through the slightly more flexible formalism of *valuative* $\mathbb{Q}$*-ideals*, see Section 2.2, or equivalently equivariant ideals in the h topology, see Section 2.5. This formalism allows us to show with little effort that centers are unique and functorial. We believe the formalism, which is inspired by existing work on $\mathbb{Q}$-ideals, graded families of ideals, and B-divisors, is the correct formalism to consider ideals with rational multiplicities up to blowings up, a topic permeating birational geometry.

We provide a proof of the theorem based on existing theory of resolution of singularities, using concepts and methods from [Hironaka 1964a; 1964b; Villamayor 1989; Bierstone and Milman 1997;

[1]We remind the reader that, by a theorem of de Jong [1997, Corollary 5.15], as stated in [Bergh and Rydh 2019, Theorem 1.4], any variety X over a field of any characteristic admits a purely inseparable alteration $X' \to X$ with X' the coarse moduli space of a smooth Deligne–Mumford stack $\mathcal{X}'$. Thus, if the field is perfect, resolution of X is reduced to the combination of destackification of a possibly wild Deligne–Mumford stack $\mathcal{X}'$ and the resolution of a purely inseparable cover of a smooth scheme X' — using Frobenius we can realize a modification of X as a purely inseparable alteration of X'.

2008; Encinas and Villamayor 2003; 2007; Włodarczyk 2005; Kollár 2007], among others. The reader is assumed to be familiar with the introductory material in [Kollár 2007, 3.1–3.2]. We explicitly use [loc. cit., Theorem 3.67] (or [Bierstone and Milman 2008, Lemma 3.3]), [Kollár 2007, Theorem 3.92], and [loc. cit., Proposition 3.99], and our terminology (maximal contact, coefficient ideals) is consistent with Kollár's (and others') treatment.

1.7. ***Concurrent and future work.*** As indicated before, Theorem 1.2.2 was discovered independently by McQuillan [2020].

The present paper is a beginning for several other works, all requiring additional techniques.

The present treatment does not address logarithmic resolutions, a critical requirement of birational geometry. As Section 8.3 shows this does not follow by accident. The necessary modifications were worked out by Quek [2022]. This requires, in addition to the present methods, bringing in the theory of logarithmic structures as in [Abramovich et al. 2020a]. A variant of Quek's work using smooth Artin stacks is provided in [Abramovich and Quek 2021]. A variant using only smooth Deligne–Mumford stacks is provided in [Włodarczyk 2023]. The work [Włodarczyk 2023] provides an alternative view on the current work, representing the stacks as global quotients of varieties with torus actions.

The present results were discovered along the way of our work [Abramovich et al. 2020b], addressing resolution of singularities in families and semistable reduction, again using the logarithmic theory of [Abramovich et al. 2020a]. The chapter [Temkin 2023] indicates how the present methods should be introduced into that project, and carried out in the appropriate generality of *quasiexcellent schemes*, to deduce results in other geometric categories of interest, as is done in [Temkin 2012; Abramovich and Temkin 2019]. McQuillan's method [2020] is developed in the generality of quasiexcellent schemes.

Further discussion of these and other aspects is included in the volume [Abramovich et al. 2023].

1.8. ***Examples: comparing smooth and weighted blowings up.***

1.8.1. *Blowing up without weights.* It is well-known that there exists no classical "memoryless algorithm" which blows up smooth centers and is compatible with smooth morphisms in the sense of Theorem 1.2.2(3); for example, see [Kollár 2007, Claim 3.6.3]. We give here slightly different examples.

Consider first the 3-dimensional singularity

$$x^2 = y_1 y_2 y_3.$$

The singular locus consists of the three lines $x = y_i = y_j = 0$, for $i \neq j$, meeting at the origin. Due to the group of permutations acting on the singularity the only possible *invariant* smooth center is the origin: $\{x = y_1 = y_2 = y_3 = 0\}$, but its blowing up leads to the three points with singularities identical to the original one, occurring on the three y_i-charts. Writing

$$x = x' y_3', \quad y_1 = y_1' y_3', \quad y_2 = y_2' y_3', \quad \text{and} \quad y_3 = y_3'$$

we get, after clearing out y_3^2, the equation

$$x'^2 = y_1' y_2' y_3'$$

in the new coordinates.

Thus functorial embedded desingularization by smooth blowings up, using no additional structure — called "history" by some authors — is simply impossible, as it may lead to an infinite cycle.[2]

This paucity of functorial centers leads to choices which are far from optimal, and resulting in *worse* singularities.

Consider the equation

$$x^2 = y_1^a y_2^a y_3^a,$$

with $a \geq 2$ instead. The origin is again the unique possible functorial center, and leads to a singularity of the form $x^2 = y_1^a y_2^a y_3^{3a-2}$ in the y_3-chart. This visibly is a worse singularity.

1.8.2. *Weighted blowing up.* The main reason for working with smooth centers in Hironaka's approach is that we want to keep the ambient space Y smooth.

A birational geometer knows that the singularity $x^2 = y_1 y_2 y_3$ asks for the blowing up of $J = (x^2, y_1^3, y_2^3, y_3^3)$. This is the observation used by the authors mentioned in Section 1.4 above. But a weighted blowing up in the *schematic* sense gives rise to a singular ambient space Y, with abelian quotient singularities. For the classical algorithm this is a nonstarter.

As explained in Section 3, we use instead the *stack theoretic* weighted blowing up of the associated *reduced* center — in the example $J^{1/6} = (x^{1/3}, y_1^{1/2}, y_2^{1/2}, y_3^{1/2})$. The chart corresponding to y_3 is of the form

$$[\operatorname{Spec} \mathbb{C}[x', y_1', y_2', u]/\boldsymbol{\mu}_2],$$

evidently smooth, where

$$y_3 = u^2, \quad x = x'u^3, \quad y_1 = y_1' u^2, \quad y_2 = y_2' u^2,$$

and $\boldsymbol{\mu}_2 = \pm 1$ acts by $(x', y_1', y_2', u) \mapsto (-x', y_1', y_2', -u)$. The general equations, and their derivation, are given in Section 3.

Plugging this into the original equation $x^2 = y_1 y_2 y_3$ we get $u^6 x'^2 = u^6 y_1' y_2'$, where the factor u^6 is exceptional, with proper transform

$$x'^2 = y_1' y_2'.$$

[2]To resolve this, in Hironaka's classical algorithm one must encode $y_3' = 0$ as an exceptional divisor — this is quite natural and useful. One must then note that upon restriction to the first maximal contact $x = 0$ the ideal $y_1' y_1' y_3'$ factors an exceptional "monomial" part y_3'. Unfortunately in general the monomial part makes it impossible to proceed with transverse maximal contact. One must then separate it from the order-2 locus with a resolution subroutine sometimes called "the monomial stage". Only then one can find further maximal contact elements and proceed.

In other words, the vector of degrees (2, 3, 3, 3) is reduced to (2, 2, 2), an immediate and visible improvement. One more blowing up resolves the singularities (in the category of Deligne–Mumford stacks).[3]

Similarly, our general algorithm, which requires no knowledge of prior steps taken, assigns to a singularity a canonical weighted blowing up which improves actual singularities, rather than an intricate additional auxiliary structure. Consequently the natural centers and resulting valuations are much better suited for computations of various birational invariants, such as log canonical thresholds, as recalled in Section 1.4.

1.9. *Efficiency.* As most algorithms in algebraic geometry, our algorithm is woefully expensive computationally, and can only be carried out in low dimension and degree. One source for computational costs here is the use of the iterated factorial in the construction of invariant and centers. Still, empirically the improvements are significant. Already in [Abramovich et al. 2020a] we showed how more limited use of stack-theoretic blowings up leads to a vast improvement in efficiency. In examples the present algorithm is remarkably efficient, with great improvements even on [loc. cit.]. For instance, in the example above, two weighted blowings up suffice. Cases of interest which were out of reach for computer calculation are now computed. This adds to the evidence recalled in Section 1.4. Our process is explicitly computable, and an implementation in SINGULAR [Decker et al. 2019] is available in [Lee et al. 2020].

2. Valuative ideals, idealistic exponents, and centers

To simplify the exposition we will mainly work with schemes. All intermediate constructions can be extended to Deligne–Mumford stacks, étale topology and geometric points via étale descent, but we will use this only in the main statements and constructions, including centers, weighted blowings up and resolution invariants. For completeness, we provide in remarks and complementary sections some additional material with only outlined arguments; it will not be used and can be safely ignored if the reader prefers.

2.1. *Zariski–Riemann spaces.* Given an integral noetherian scheme Y we are interested in understanding ideals, and more generally $\mathbb{Q}$-ideals, as they behave after arbitrary blowing up. For instance the ideals (x^2, y^2) and (x^2, xy, y^2) coincide after blowing up the origin, and a formalism in which they are the same object is desirable. We propose to work with the Zariski–Riemann space $\boldsymbol{ZR}(Y)$ of Y, the projective limit of all projective birational transformations of Y, whose points consist of all valuation rings R of $K(Y)$ extending to a morphism $\operatorname{Spec} R \to Y$.

The space $\boldsymbol{ZR}(Y)$ carries a constant sheaf $K = K(Y)$, a subsheaf of rings $\mathcal{O}$ with stalk at v consisting of the valuation ring R_v, and a sheaf of ordered groups $\Gamma = K^*/\mathcal{O}^*$ such that $v : K^* \to \Gamma$ is the valuation. The image $v(\mathcal{O} \setminus \{0\}) =: \Gamma_+ \subset \Gamma$ is the valuation monoid consisting of nonnegative sections of Γ.

The space $\boldsymbol{ZR}(Y)$ is quasicompact; see [Temkin 2010, Proposition 3.2.1]. If $Y = \bigcup Y_i$ is reduced but possibly reducible with irreducible components Y_i, we define $\boldsymbol{ZR}(Y) := \bigsqcup \boldsymbol{ZR}(Y_i)$.

[3]Hironaka's classical algorithm requires many more blowings up, and, as indicated in the previous note, is quite technically involved.

Remark 2.1.1. While Theorem 1.2.2 is applied to Deligne–Mumford stacks $X \subset Y$, functoriality means that we can always work on an étale cover by a scheme $\tilde{X} \subset \tilde{Y}$: the resolution step $F_1(X \subset Y)$ is obtained by étale descent from $F_1(\tilde{X} \subset \tilde{Y})$. In particular we need not introduce $\boldsymbol{ZR}(Y)$ for a stack. Nevertheless we note that such $\boldsymbol{ZR}(Y)$ can be constructed as well, be it by étale descent, or directly as a limit, or as a suitably normalized fibered product of Y with the Zariski–Riemann space of the coarse moduli space.

2.2. *Valuative $\mathbb{Q}$-ideals.*

Definition 2.2.1. (1) By a *valuative ideal* on Y we mean a section $\gamma \in H^0(\boldsymbol{ZR}(Y), \Gamma_+)$. Every ideal $\mathcal{I}$ on every birational model $Y' \to Y$, proper over Y, defines a valuative ideal that we denote $v(\mathcal{I})$ by taking the minimal element of the image of $\mathcal{I}$ in Γ_+.

(2) The group $\Gamma_{\mathbb{Q}} = \Gamma \otimes \mathbb{Q}$ is also ordered. We denote the monoid of nonnegative elements by $\Gamma_{\mathbb{Q}+}$. By a *valuative $\mathbb{Q}$-ideal* we mean a section $\gamma \in H^0(\boldsymbol{ZR}(Y), \Gamma_{\mathbb{Q}+})$.

(3) Any dominant morphism $f : Z \to Y$ induces a map $\boldsymbol{ZR}(Z) \to \boldsymbol{ZR}(Y)$. For a valuative $\mathbb{Q}$-ideal γ on Y its image under the induced map $\Gamma_Y \otimes \mathbb{Q} \to \Gamma_Z \otimes \mathbb{Q}$ will be denoted $f^{-1}(\gamma)$ and called the *preimage of γ on Z*.

Ideals with the same integral closure have the same valuative ideal. Every valuative ideal γ defines an ideal sheaf $\mathcal{I}'_\gamma$ on every modification Y' of Y by taking $\mathcal{I}'_\gamma := \{f \in \mathcal{O}_{Y'} \mid v(f) \geq \gamma_v \forall v\}$, which is automatically integrally closed. We will use only the ideal $\mathcal{I}_\gamma$ thus defined on Y itself.

The definition of $\mathcal{I}_\gamma$ extends to valuative $\mathbb{Q}$-ideals. Conversely, there is a convenient way to consider $\mathbb{Q}$-ideals, extending the definition of $v(\mathcal{I})$: given a finite collection $f_i \in \mathcal{O}_Y$ and $a_i \in \mathbb{Q}_{>0}$ we write

$$(f_1^{a_1}, \ldots, f_k^{a_k}) := (\min\{a_i \cdot v(f_i)\})_v \in H^0(\boldsymbol{ZR}(Y), \Gamma_{\mathbb{Q}+}) \tag{1}$$

for the naturally associated valuative $\mathbb{Q}$-ideal. When a_i are integers this coincides with $v(f_1^{a_1}, \ldots, f_k^{a_k})$.

Remark 2.2.2. As was pointed out by D. Rydh, valuative $\mathbb{Q}$-ideals are equivalent to effective $\mathbb{Q}$-Cartier divisors on $\boldsymbol{ZR}(X)$. Indeed, any section γ of Γ_+ is locally the image of an element of $\mathcal{O}$, and since $\boldsymbol{ZR}(X)$ is quasicompact, finitely many such representatives suffice. Moreover, taking a common birational model $Y' \to Y$ over which all the representative sections are regular, we find that γ is an invertible ideal on Y'. Allowing denominators, any valuative $\mathbb{Q}$-ideal γ is written, using the notation of (1), locally on the model Y' as $\gamma = (f^a)$.

2.3. *Complements: idealistic exponents.* A valuative $\mathbb{Q}$-ideal which is represented locally on Y itself as $(f_1^{a_1}, \ldots, f_k^{a_k})$ is an *idealistic exponent*. This notion coincides with Hironaka's [1977, Definition 3] by [loc. cit., Remark (2.2)]. Hironaka's notation $(\mathcal{J}, b)$, with $\mathcal{J} \subset \mathcal{O}_Y$, $b \in \mathbb{N}$ translates to the valuative $\mathbb{Q}$-ideal $\mathcal{J}^{1/b}$. Hironaka's definition of pullback of an idealistic exponent under a dominant morphism $Y' \to Y$ extends to an arbitrary valuative $\mathbb{Q}$-ideal.

As indicated in the next section, these are related to Rees algebras [Encinas and Villamayor 2007] or graded families of ideals [Lazarsfeld 2004, Section 2.4.B]. This relationship was pursued in greater depth by Quek [2022].

2.4. *Centers and admissibility.*

Definition 2.4.1. (1) By a *center* J on a regular scheme Y we mean a valuative $\mathbb{Q}$-ideal for which there is an affine covering $Y = \cup U_i$ and regular systems of parameters $(x_1^{(i)}, \ldots, x_k^{(i)}) = (x_1, \ldots, x_k)$ on U_i such that $J_{U_i} = (x_1^{a_1}, \ldots, x_k^{a_k})$ for some $a_j \in \mathbb{Q}_{>0}$ independent of i.

(2) A center J is *admissible* for a valuative $\mathbb{Q}$-ideal β if $J_v \le \beta_v$ for all v. A center is *admissible* for an ideal $\mathcal{I}$ if it is admissible for the associated valuative $\mathbb{Q}$-ideal $v(\mathcal{I})$, in which case we use the suggestive notation $\mathcal{I} \subseteq J$.

(3) The center J is *reduced* if $w_i = 1/a_i$ are positive integers with $\gcd(w_1, \ldots, w_k) = 1$. For any center J we write $\bar{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$ for the unique reduced center such that $\bar{J}^\ell = J$ for some $\ell \in \mathbb{Q}_{>0}$.

In Section 3 below we define the blowing up of $(x_1^{1/w_1}, \ldots, x_k^{1/w_k})$. In Section 5.2 we show how admissibility is manifested in terms of this blowing up, and becomes very much analogous to the notion used in earlier resolution algorithms.

Remark 2.4.2. Using the coordinates as in (1), the center J corresponds to a unique monomial valuation associated to the cocharacter

$$(a_1^{-1}, \ldots, a_k^{-1}, 0, \ldots, 0),$$

where $v\bigl(\prod x_i^{c_i}\bigr) = \sum_{i=1}^k c_i/a_i$.

The definition of centers extends to stacks similarly to usual ideals.

Definition 2.4.3. Let Y be a Deligne–Mumford stack:

(1) By $\mathrm{Cov}(Y)$ we denote the category of étale covers $Y' \to Y$ with Y' a scheme and Y-morphisms between the covers.

(2) A *center* J on Y is a compatible family of centers J' on the elements Y' of $\mathrm{Cov}(Y)$: for any morphism $f : Y'' \to Y'$ in $\mathrm{Cov}(Y)$ one has $f^{-1}J' = J''$.

Partial regular families of parameters are preserved by preimages under smooth and, more generally, regular morphisms (see [Stacks 2005–, 07R6]), hence we have:

Lemma 2.4.4. *If $f : Y' \to Y$ is a regular morphism of regular schemes and J is a center on Y, then $f^{-1}J$ is a center on Y'.*

Remark 2.4.5. In fact, the inverse is also true: is f is surjective, γ is a valuative $\mathbb{Q}$-ideal and $f^{-1}\gamma$ is a center, then γ is a center. As a corollary one obtains an extension of these claims to stacks and the claim that a center on a stack can be defined using a single presentation rather than the whole category $\mathrm{Cov}(Y)$. However, we will not need these natural but not completely trivial results.

2.5. ***Complements: relation with the h topology.*** The following observation is not used in the paper, so we just outline it without proof. Valuative $\mathbb{Q}$-ideals are closely related to what we call equivariant ideals in the h topology, where Zariski open coverings and alterations generate a cofinal collection of coverings; see [Voevodsky 1996, Definition 3.1.5 and Theorem 3.1.9]. The structure sheaf $\mathcal{O}_{X_h}$ is the sheafification of the presheaf $U \mapsto \Gamma(\mathcal{O}_U)$. In fact, $\mathcal{O}_{X_h}(U) = \Gamma(\mathcal{O}_{U^{sn}})$, where U^{sn} is the seminormalized reduction of U; see [Huber and Jörder 2014, Proposition 4.5]. Any finitely generated ideal $\mathcal{J} \subseteq \mathcal{O}_{X_h}$ is generated by ideals $J_i \subseteq \mathcal{O}_{Y_i}$ on a Zariski cover $Y' = \cup Y_i$ of an alteration $Y' \to Y$. Refining the alteration we can achieve that pullbacks of J_i agree on the intersections, so $\mathcal{J}$ comes from an ideal J' on Y' and hence yields a valuative ideal γ' on Y'. Refining Y' further we can achieve that $Y' \to Y$ is a Galois alteration, namely it splits into a composition of a Galois cover $Y' \to Y''$, with Galois group G, and a generically radicial alteration $Y'' \to Y$. On the level of sets $\boldsymbol{ZR}(Y')/G = \boldsymbol{ZR}(Y'') = \boldsymbol{ZR}(Y)$, hence γ' comes from a valuative $\mathbb{Q}$-ideal γ if and only if γ' is G-equivariant. In fact, the latter happens if and only if one can choose Y' and J' so that already J' is G-equivariant.

3. Weighted blowings up

Stack theoretic projective spectra were considered informally by Miles Reid, introduced officially in [Abramovich and Hassett 2011] to study moduli spaces of varieties, and treated in Olsson's book [2016, Section 10.2.7].

The manuscript by Quek and Rydh [2021] provides foundations for stack-theoretic blowings up. The presentation here is rather terse as complete details already appear there. The local equations we present here can be found in [Kollár et al. 2004, page 167], where they are developed for the study of log canonical thresholds. The graded algebras we present below are special cases of the graded families of ideals discussed in [Lazarsfeld 2004, Section 2.4.B], especially Example 2.4.8.

From now on Y is a smooth Deligne–Mumford stack over a field k of characteristic zero. In Sections 3–5, if not said to the contrary, Y is also assumed to be a variety.

3.1. ***Graded algebras and their $\mathcal{P}$roj.*** Given a quasicoherent graded algebra $\mathcal{A} = \bigoplus_{m \geq 0} \mathcal{A}_m$ on Y with associated $\mathbb{G}_m$-action defined by $(t, s) \mapsto t^m s$ for $s \in \mathcal{A}_m$ we define its stack-theoretic projective spectrum to be

$$\mathcal{P}\mathrm{roj}_Y \mathcal{A} := [(\mathrm{Spec}_{\mathcal{O}_Y} \mathcal{A} \setminus S_0)/\mathbb{G}_m],$$

where the vertex S_0 is the zero scheme of the ideal $\bigoplus_{m>0} \mathcal{A}_m$; see [Quek and Rydh 2021, Section 1.2]. When $\mathcal{A}_1$ is coherent and generates $\mathcal{A}$ over $\mathcal{A}_0$ this agrees with the construction in [Hartshorne 1977, II.7, page 160]; see [Quek and Rydh 2021, Corollary 1.6.2]. As usual $\mathcal{P}\mathrm{roj}_Y \mathcal{A}$ carries an invertible sheaf $\mathcal{O}_{\mathcal{P}\mathrm{roj}_Y \mathcal{A}}(1)$ corresponding to the graded module $\mathcal{A}(1)$. When $\mathcal{A}$ is finitely generated over $\mathcal{O}_Y$ with coherent graded components the resulting morphism $\mathcal{P}\mathrm{roj}_Y \mathcal{A} \to Y$ is proper; see [Quek and Rydh 2021, Proposition 1.6.1(ii)].

3.2. *Rees algebras of ideals.* If $\mathcal{I}$ is an ideal on Y, its Rees algebra is $\mathcal{A}_{\mathcal{I}} := \bigoplus_{m\geq 0}\mathcal{I}^m$, and the blowing up of $\mathcal{I}$ is $Y' = \mathrm{Bl}_Y(\mathcal{I}) := \mathcal{P}\mathrm{roj}_Y(\mathcal{A}_{\mathcal{I}})$. It is the universal birational map making $\mathcal{I}\mathcal{O}_{Y'}$ invertible, in this case $Y' \to Y$ projective; see definition [Hartshorne 1977, II.7, page 163].

3.3. *Rees algebras of valuative $\mathbb{Q}$-ideals.*

Definition 3.3.1. (1) Given a valuative $\mathbb{Q}$-ideal γ we define its *Rees algebra* to be

$$\mathcal{A}_\gamma := \bigoplus_{m\in\mathbb{N}} \mathcal{I}_{m\gamma}.$$

(2) The *blowing up* of γ is defined to be $Y' = \mathrm{Bl}_Y(\gamma) := \mathcal{P}\mathrm{roj}_Y\,\mathcal{A}_\gamma$.

At least when $\gamma = (f_1^{a_1}, \ldots, f_k^{a_k})$ is an idealistic exponent, $Y' \to Y$ satisfies a corresponding universal property; see [Quek and Rydh 2021, Proposition 3.5.3]. Since we will not use this property in this paper, we just mention that the valuative $\mathbb{Q}$-ideal $E = \gamma\mathcal{O}_{Y'}$, in a suitable sense of Zariski–Riemann spaces of stacks, or as an h-ideal, becomes an *invertible ideal sheaf* on Y'. We only show this below for the blowing up of a center.

Note that if $Y_1 \to Y$ is flat and $Y_1' = \mathrm{Bl}_Y(\gamma\mathcal{O}_{Y_1})$ then $Y_1' = Y' \times_Y Y_1$.

3.4. *Weighted blowings up: local equations.* Now consider the situation where γ is a center of the special form $J = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$, with $w_i \in \mathbb{N}$. In this case the algebra $\mathcal{A}_\gamma = \bigoplus_{m\in\mathbb{N}}\mathcal{I}_{m\gamma}$, with $\mathcal{I}_{m\gamma} = \left(x_i^{b_1}\cdots x_n^{b_n} \mid \sum w_i b_i \geq m\right)$ is finitely generated. It is the integral closure inside $\mathcal{O}_Y[T, T^{-1}]$ of the simpler algebra with generators $(x_i)T^{w_i}$. We can therefore describe $\mathrm{Bl}_Y(J) = \mathrm{Bl}_Y(\gamma)$, which deserves to be called a stack-theoretic weighted blowing up, explicitly in local coordinates, as follows [Quek and Rydh 2021, Corollary 4.4.4]:

The chart associated to x_1 has local variables $u, x_2', \ldots, x_n'$, where

- $x_1 = u^{w_1}$,
- $x_i' = x_i/u^{w_i}$ for $2 \leq i \leq k$, and
- $x_j' = x_j$ for $j > k$.

The group $\boldsymbol{\mu}_{w_1}$ acts through

$$(u, x_2', \ldots, x_k') \mapsto (\zeta_{w_1}u, \zeta_{w_1}^{-w_2}x_2', \ldots, \zeta_{w_1}^{-w_k}x_k')$$

and trivially on x_j', $j > k$, giving an étale local isomorphism of the chart with

$$[\mathrm{Spec}\,k[u, x_2', \ldots, x_n']/\boldsymbol{\mu}_{w_1}].$$

It is easy to see that these charts glue to a stack-theoretic modification $Y' \to Y$ with a smooth Y' and its coarse space is the classical (singular) weighted blowing up.

Write $E = (u)$ for the exceptional ideal. Then $v(E) = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$, and this persists on all charts, in other words the center $(x_1^{1/w_1}, \ldots, x_k^{1/w_k})$ becomes an invertible ideal sheaf on Y'.

We sometimes, but not always, insist on $\gcd(w_1, \dots, w_k) = 1$, in which case the center is *reduced*. We will however need to consider the proper transform of the locus $H = \{x_1 = 0\}$, where it may happen that $\gcd(w_2, \dots, w_k) \neq 1$. The relationships are summarized by the following lemma, which uses the construction of the root stack $Y(E^{1/c})$ along a divisor E (for a treatment on a stack see [Abramovich and Fantechi 2016, Section 1.1]) and follows from [Quek and Rydh 2021, Corollary 3.2.1] or by considering the charts:

Lemma 3.4.1. *If $J' = (x_1^{1/w_1}, \dots, x_k^{1/w_k})$ and $J'' = (x_1^{1/cw_1}, \dots, x_k^{1/cw_k})$ with w_i, c positive integers, and if $Y', Y'' \to Y$ are the corresponding blowings up, with E', E'' the exceptional divisors, then $Y'' = Y'(\sqrt[c]{E'})$ is the root stack of Y' along E'.*

Write $H = \{x_1 = 0\}$, and $H' \to H$ the blowing up of the ***reduced*** *center $\bar{J}'_H$ associated to $J'_H := (x_2^{1/w_2}, \dots, x_k^{1/w_k})$, with exceptional E_H. Then the proper transform $\tilde{H}' \to H$ of H via the blowing up of J'' is the root stack $H'(\sqrt[(cc')]{E_H})$ of H' along $E_H \subset H'$, where $c' = \gcd(w_2, \dots, w_k)$. Therefore $\tilde{H}'$ is the blowing up of ${\bar{J}'_H}{}^{1/(cc')}$.*

3.5. ***Derivation of equations.*** Let us derive the description in Section 3.4 above, in a manner similar to [Quek and Rydh 2021, Lemma 1.3.1]. Write $y_i = x_i T^{w_i}$. The x_1-chart is the stack $[\operatorname{Spec} \mathcal{A}[y_1^{-1}]/\mathbb{G}_m]$. The slice $W_1 := \operatorname{Spec} \mathcal{A}[y_1^{-1}]/(y_1 - 1)$ is stabilized by $\boldsymbol{\mu}_{w_1}$, so the embedding $W_1 \subset \operatorname{Spec} \mathcal{A}[y_1^{-1}]$ gives rise to a morphism $\phi : [W_1/\boldsymbol{\mu}_{w_1}] \to [\operatorname{Spec} \mathcal{A}[y_1^{-1}]/\mathbb{G}_m]$. This is an isomorphism: the equation $u^{w_1} = x_1$ describes a $\boldsymbol{\mu}_{w_1}$-torsor on $\operatorname{Spec} \mathcal{A}[y_1^{-1}]$ mapping to W_1 equivariantly via $T \mapsto u^{-1}$. The resulting morphism $\operatorname{Spec} \mathcal{A}[y_1^{-1}] \to [W_1/\boldsymbol{\mu}_{w_1}]$ descends to $[\operatorname{Spec} \mathcal{A}[y_1^{-1}]/\mathbb{G}_m] \to [W_1/\boldsymbol{\mu}_{w_1}]$ which is an inverse to ϕ.

It thus remains to show that $[W_1/\boldsymbol{\mu}_{w_1}]$ has the local description above. Since $T^{-w_1} = y_1^{-1} x_1 \in \mathcal{A}[y_1^{-1}]$ and $\mathcal{A}$ is integrally closed in $\mathcal{O}_Y[T, T^{-1}]$ we have $u := T^{-1} \in \mathcal{A}[y_1^{-1}]$, and its restriction to W_1 satisfies $u^{w_1} = x_1$. For $i = 2, \dots, k$ we write x'_i for the restriction of y_i, obtaining $x'_i = x_i/u^{w_i}$. Now W_1 is normal and finite birational over $\operatorname{Spec} k[u, x'_2, \dots, x'_n]$, hence they are isomorphic.

3.6. ***Complements: local toric description of weighted blowings up*** [Quek and Rydh 2021, Section 4.5.5]. Again working locally, assume that $Y = \operatorname{Spec} k[x_1, \dots, x_n]$. It is the affine toric variety associated to the monoid $\mathbb{N}^n \subset \sigma = \mathbb{R}^n_{\geq 0}$. Here the generator e_i of $\mathbb{N}^n$ corresponds to the monomial valuation v_i associated to the divisor $x_i = 0$, namely $v_i(x_j) = \delta_{ij}$.

The monomial x_i^{1/w_i} defines the linear function on σ whose value on $(b_1, \dots, b_n)$ is its valuation b_i/w_i. The ideal $(x_1^{1/w_1}, \dots, x_k^{1/w_k})$ thus defines the piecewise linear function $\min_i\{b_i/w_i\}$, which becomes linear precisely on the star subdivision $\Sigma = v_{\bar{J}} \star \sigma$ with

$$v_{\bar{J}} = (w_1, \dots, w_k, 0, \dots, 0).$$

This defines the scheme theoretic weighted blowing up $\bar{Y}'$; see [Reid 1980, Section 4]. Note that this cocharacter $v_{\bar{J}}$ is a multiple of the valuation associated to the exceptional divisor of the center.

Since $v_{\bar{J}}$ is assumed integral, we can apply the theory of toric stacks [Borisov et al. 2005; Fantechi et al. 2010; Geraschenko and Satriano 2015a; 2015b; Gillam and Molcho 2015]. We have a smooth toric stack $Y' \to \overline{Y}'$ associated to the same fan Σ with the cone $\sigma_i = \langle v_{\bar{J}}, e_1, \ldots, \hat{e}_i, \ldots, e_n \rangle$ endowed with the sublattice $N_i \subset N$ generated by the elements $v_{\bar{J}}, e_1, \ldots, \hat{e}_i, \ldots, e_n$, for all $i = 1, \ldots, k$. This toric stack is precisely the stack theoretic weighted blowing up $Y' \to Y$. One can derive the equations in Section 3.4 from this toric picture.

4. Coefficient ideals

In this section we recall some notions from the classical embedded resolution. By Y we denote a smooth k-variety.

4.1. ***Graded algebra and coefficient ideals.*** Fix an ideal $\mathcal{I} \subset \mathcal{O}_Y$ and an integer $a > 0$. We use the notation of [Abramovich et al. 2020a], except that we use the saturated coefficient ideal as in [Kollár 2007; Abramovich et al. 2020b], which is consistent with the Rees algebra approach of [Encinas and Villamayor 2007]:

Definition 4.1.1. (1) Consider the graded subalgebra $\mathcal{G} = \mathcal{G}(\mathcal{I}, a) \subseteq \mathcal{O}_Y[T]$ generated by placing $\mathcal{D}^{\leq a-i}\mathcal{I}$ in degree i. Its graded pieces are

$$\mathcal{G}_j = \sum_{\sum_{i=0}^{a-1}(a-i)\cdot b_i \geq j} \mathcal{I}^{b_0} \cdot (\mathcal{D}^{\leq 1}\mathcal{I})^{b_1} \cdots (\mathcal{D}^{\leq a-1}\mathcal{I})^{b_{a-1}},$$

where the sum runs over all monomials in the ideals $\mathcal{I}, \ldots, \mathcal{D}^{\leq a-1}\mathcal{I}$ of weighted degree

$$\sum_{i=0}^{a-1}(a-i) \cdot b_i \geq j.$$

(2) Let $\mathcal{I} \subset \mathcal{O}_Y$ and $a \geq 1$ an integer. Define the *coefficient ideal*

$$C(\mathcal{I}, a) := \mathcal{G}_{a!}.$$

The product rule, and the trivial inclusion $\mathcal{D}^{\leq 1}\mathcal{D}^{\leq a-1}\mathcal{I} \subset (1)$, imply that $\mathcal{D}\mathcal{G}_{k+1} \subset \mathcal{G}_k$ for $k \geq 0$. The formation of $\mathcal{G}$ and $C(\mathcal{I}, a)$ is functorial for smooth morphisms: if $Y_1 \to Y$ is smooth then $C(\mathcal{I}, a)\mathcal{O}_{Y_1} = C(\mathcal{I}\mathcal{O}_{Y_1}, a)$. This follows since the formation of $\mathcal{D}^{\leq 1}\mathcal{I}$, ideal product, and ideal sum are all functorial.

4.2. ***Maximal contact.*** For the rest of the section we assume that $\mathcal{I} \subset \mathcal{O}_Y$ has maximal order $\leq a$. Recall that an element $x \in \mathcal{D}^{\leq a-1}\mathcal{I}$ which is a regular parameter at $p \in Y$ is called *a maximal contact element* at p, and its vanishing locus *a maximal contact hypersurface* at p. In general, maximal contact only exists locally. For completeness, any parameter is a maximal contact element for the unit ideal.

The coefficient ideal combines sufficient information from derivatives of $\mathcal{I}$ so that when one restricts $C(\mathcal{I}, a)$ to a hypersurface of maximal contact H no information necessary for resolution is lost. For example, this is manifested in the equivalence (in the sense of [Bierstone and Milman 1997]) of $(\mathcal{I}, a)$ and $C(\mathcal{I}, a)|_H$.

4.3. ***Invariance.*** Now consider $\mathcal{I} \subset \mathcal{O}_Y$ and assume that $x_1 \in \mathcal{D}^{\leq a-1}\mathcal{I}$ is a maximal contact element at $p \in Y$. The ideals $\mathcal{G}_i$ enjoy a strong invariance property summarized in the following theorem:[4]

Theorem 4.3.1. *Let x_1 and x_1' be maximal contact elements at p, and $x_2, \dots, x_n \in \mathcal{O}_{Y,p}$ such that $(x_1, x_2, \dots, x_n)$ and $(x_1', x_2, \dots, x_n)$ are both regular sequences of parameters. There is a scheme $\tilde{Y}$ with point $\tilde{p} \in \tilde{Y}$ and two morphisms $\phi, \phi' : \tilde{Y} \to Y$ with $\phi(\tilde{p}) = \phi'(\tilde{p}) = p$, both étale at p, satisfying*

(1) $\phi^* x_1 = \phi'^* x_1'$,

(2) $\phi^* x_i = \phi'^* x_i$ *for* $i = 2, \dots, n$, *and*

(3) $\phi^* \mathcal{G}_i = \phi'^* \mathcal{G}_i$.

This is [Kollár 2007, Theorem 3.92], generalizing [Włodarczyk 2005, Lemma 3.5.5].[5]

4.4. ***Formal decomposition.*** We now pass to formal completions. Fixing a field of coefficients $k_p = k(p) \hookrightarrow \hat{\mathcal{O}}_{Y,p}$ and extending to a regular sequence of parameters we have $\hat{\mathcal{O}}_{Y,p} = k_p[\![x_1, \dots, x_n]\!]$. We use the reduction homomorphism $k_p[\![x_1, \dots, x_n]\!] \to k_p[\![x_2, \dots, x_n]\!]$ and the inclusion $k_p[\![x_2, \dots, x_n]\!] \to k_p[\![x_1, \dots, x_n]\!]$.

We have $\mathcal{G}_j = (x_1^j) + (x_1^{j-1})\mathcal{G}_1 + \cdots + (x_1)\mathcal{G}_{j-1} + \mathcal{G}_j$ since the ideal on the left contains every term on the right. Write $\bar{\mathcal{C}}_j = \mathcal{G}_j k_p[\![x_2, \dots, x_n]\!] \subset k_p[\![x_2, \dots, x_n]\!]$ via the reduction homomorphism sending x_1 to 0, and $\widetilde{\mathcal{C}}_j = \bar{\mathcal{C}}_j k_p[\![x_1, \dots, x_n]\!] \subset k_p[\![x_1, \dots, x_n]\!]$ its image via inclusion. We hope the reader can distinguish the notation $\bar{\mathcal{C}}_j$ from $\widetilde{\mathcal{C}}_j$.

Proposition 4.4.1. *Denoting the completions $\hat{\mathcal{G}}_j = \mathcal{G}_j \hat{\mathcal{O}}_{Y,p}$ and $\hat{C}(\mathcal{I}, a) = C(\mathcal{I}, a)\hat{\mathcal{O}}_{Y,p}$, we have*

$$\hat{\mathcal{G}}_j = (x_1^j) + (x_1^{j-1})\widetilde{\mathcal{C}}_1 + \cdots + (x_1)\widetilde{\mathcal{C}}_{j-1} + \widetilde{\mathcal{C}}_j,$$

in particular

$$\hat{C}(\mathcal{I}, a) = (x_1^{a!}) + (x_1^{a!-1}\widetilde{\mathcal{C}}_1) + \cdots + (x_1\widetilde{\mathcal{C}}_{a!-1}) + \widetilde{\mathcal{C}}_{a!}.$$

Proof. We write $x = x_1$. Apply induction on j, noting that $\hat{\mathcal{G}}_0 = (1)$ so that we may start with $(1) = \widetilde{\mathcal{C}}_0$ and inductively assume the equality holds up to $j - 1$.

For an integer $M > j$ the ideals $\hat{\mathcal{G}}_j \supset (x^M)$ are stable under the linear operator $x\partial/\partial x$. Hence the quotient $\hat{\mathcal{G}}_j/(x^M)$ inherits a linear action, with m-eigenspaces we denote $x^m \cdot \hat{\mathcal{G}}_j^{(m)} \subset x^m k_p[\![x_2, \dots, x_n]\!]$, giving

$$\hat{\mathcal{G}}_j/(x^{M+1}) = \hat{\mathcal{G}}_j^{(0)} \oplus x \cdot \hat{\mathcal{G}}_j^{(1)} \oplus \cdots \oplus x^m \cdot \hat{\mathcal{G}}_j^{(m)} \oplus \cdots \oplus x^M \cdot \hat{\mathcal{G}}_j^{(M)},$$

with $\hat{\mathcal{G}}_j^{(m)} \subset k_p[\![x_2, \dots, x_n]\!]$ and equality holding for $m \geq j$. Note that $\hat{\mathcal{G}}_j^{(0)} = \bar{\mathcal{C}}_j$.

[4]The reader familiar with [Kollár 2007, Section 3.53] will recognize that $\mathcal{G}_i$ are all *MC-invariant*: $\mathcal{G}_1 \cdot \mathcal{D}^{\leq 1}\mathcal{G}_i \subset \mathcal{G}_i$, hence they are *homogeneous* in the sense of [Włodarczyk 2005].

[5]These are the easier properties of coefficient ideals. We emphasize that we do not require the harder part (4) of [Włodarczyk 2005, Lemma 3.5.5] or [Kollár 2007, Theorem 3.97] describing the behavior after a sequence of blowings up.

The subspaces $\hat{\mathcal{G}}_j^{(m)} \subset k_p[\![x_2, \dots, x_n]\!]$ are independent of the choice of $M \geq m$. Moreover $x^j \cdot \hat{\mathcal{G}}_j^{(m)} \subset \hat{\mathcal{G}}_j \cap x^j \cdot k_p[\![x_2, \dots, x_n]\!]$, so that

$$\hat{\mathcal{G}}_j^{(m)} = \frac{\partial^j}{\partial x^j}(x^j \cdot \hat{\mathcal{G}}_j^{(m)}) \subset \hat{\mathcal{G}}_{j-m} \cap k_p[\![x_2, \dots, x_n]\!] \subset \bar{\mathcal{C}}_{j-m}.$$

Taking ideals we obtain

$$\hat{\mathcal{G}}_j \subset \hat{\mathcal{G}}_j^{(0)} + (x)\widetilde{\mathcal{C}}_{j-1} + \dots + (x^{j-1})\widetilde{\mathcal{C}}_1 + (x^j).$$

Induction gives

$$(x)\widetilde{\mathcal{C}}_{j-1} + \dots + (x^{j-1})\widetilde{\mathcal{C}}_1 + (x^j) = (x)\hat{\mathcal{G}}_{j-1} \subset \hat{\mathcal{G}}_j.$$

Together with $\bar{\mathcal{C}}_j = \hat{\mathcal{G}}_j^{(0)} \subset \hat{\mathcal{G}}_j$ the equality follows. □

By [Kollár 2007, Proposition 3.99] we have $(\mathcal{D}^{\leq j} C(\mathcal{I}, a))^{a!} \subset C(\mathcal{I}, a)^{a!-j}$. This implies:

Corollary 4.4.2. $$(\widetilde{\mathcal{C}}_{a!-j})^{a!} \subset \widetilde{\mathcal{C}}_{a!}^{a!-j}.$$

5. Invariants, local centers, and admissibility

In this section we continue to work on a smooth variety Y and fix an ideal $\mathcal{I} \subseteq \mathcal{O}_Y$. All definitions and results will be local at a point p, and to simplify notation we will use the same letter Y after passing to a neighborhood, where a maximal contact at p is defined (a pedantic reader can simply work with the localization $Y_p = \operatorname{Spec}(\mathcal{O}_{Y,p})$ instead).

5.1. *Existence of invariants and centers.*

Definition 5.1.1. (1) For an ideal $\mathcal{I} \subset \mathcal{O}_Y$ and sequence of parameters $x_1, \dots, x_k$ at p one defines $\mathcal{I}[1] = \mathcal{I}$ and recursively ideals $\mathcal{I}[i]$ and integers b_i by setting $b_i = \operatorname{ord}_p(\mathcal{I}[i])$ and $\mathcal{I}[i+1] = C(\mathcal{I}[i], b_i)|_{V(x_1,\dots,x_i)}$, ending with either $k = 1, \mathcal{I} = (1)$ or $\mathcal{I}[k+1] = 0$. The sequence of parameters $x_1, \dots, x_k$ at p is called a *maximal contact sequence* if each x_i is a maximal contact for $(\mathcal{I}[i], b_i)$ at p.

(2) To a maximal contact sequence we associate the invariant $\operatorname{inv}_{\mathcal{I}}(p) = (a_1, \dots, a_k)$, where $a_i = b_i / \prod_{j=1}^{i-1} b_j!$ and the center $J = J_p(\mathcal{I}) = (x_1^{a_1}, \dots, x_k^{a_k})$.

Obviously, a maximal contact sequence exists, and it is empty if and only if $\mathcal{I} = 0$ at p, in which case we also have that $J = 0$ and $\operatorname{inv} = ()$ is empty. The other extreme occurs when $\mathcal{I} = (1)$, in which case $J = (1)$ and $\operatorname{inv} = (0)$. Note also that $\operatorname{inv}_{\mathcal{I}[1]}(p) = (a_1, \operatorname{inv}_{\mathcal{I}[2]}(p)/(a_1 - 1)!)$ the concatenation, and $x_2, \dots, x_k$ are lifts of the parameters for $\mathcal{I}[2]$. In the notation of Section 4.4, $\mathcal{I}[2] = \bar{\mathcal{C}}_{a_1!}$.

The invariant and center in Definition 5.1.1(2) require the choice of a maximal contact sequence. The goal of Section 5 is to prove that the invariant and the center (as a valuative $\mathbb{Q}$-ideal) are independent of the choice of maximal contacts. This is at once a consequence and a generalization of Theorem 4.3.1.

A posteriori, this will also imply that $\operatorname{inv}_{\mathcal{I}}(p)$ is the maximal invariant of a center admissible for $\mathcal{I}$ at p and J is the unique center of maximal invariant admissible for $\mathcal{I}$ at p — a characterization which can be used as a choice-free definition.

Remark 5.1.2. The string $(b_1, \dots, b_k)$ was used as a singularity invariant in [Abramovich et al. 2020a], but it is its rescaling $(a_1, \dots, a_k)$ which gives a natural definition of the canonical center J independent of choices.

We order the set of invariants lexicographically, with truncated sequences considered larger, for instance

$$(1, 1, 1) < (1, 1, 2) < (1, 2, 1) < (1, 2) < (2, 2, 1).$$

The invariant takes values in a well-ordered subset $\Xi_n, n = \dim Y$, since it is order-equivalent to $(b_1, \dots, b_k)$. Explicitly write $\Xi_1 = \mathbb{N}^{\geq 1}$ and

$$\Xi_n = \Xi_1 \sqcup \bigsqcup_{a \geq 1} \{a\} \times \frac{\Xi_{n-1}}{(a-1)!}.$$

In particular, the denominators are bounded in terms of the previous entries of the invariant.

Theorem 5.1.3 [Abramovich et al. 2020a]. *Keep the above notation, then:*

(1) *The invariant* $\mathrm{inv}_{\mathcal{I}}(p)$ *is independent of the choices.*

(2) *The invariant function* $\mathrm{inv}_{\mathcal{I}} : Y \to \Xi_n$ *is constructible and upper-semicontinuous.*

(3) *The invariant is functorial for smooth morphisms: if* $f : Y' \to Y$ *is smooth and* $\mathcal{I}' = f^{-1}\mathcal{I}$, *then* $\mathrm{inv}_{\mathcal{I}'} = \mathrm{inv}_{\mathcal{I}} \circ f$.

Proof. (3) Since both $\mathrm{ord}_p(\mathcal{I})$ and the formation of coefficient ideals are functorial for smooth morphisms, the invariant is functorial for smooth morphisms, once parameters are chosen.

(1) We now show that the choices of maximal contacts do not change the invariant. The integer $a_1 = \mathrm{ord}_p(\mathcal{I}) = \max\{a : \mathcal{I}_p \subseteq \mathfrak{m}_p^a\}$ requires no choices. Given a regular sequence of parameters $(x_1, \dots, x_n)$ extending $(x_1, \dots, x_k)$, and given another maximal contact element x_1', we may choose constants t_i, and replace $x_2, \dots, x_n$ by $x_2 + t_2 x_1, \dots, x_n + t_n x_1$ so that also $(x_1', x_2, \dots, x_n)$ is a regular sequence of parameters.

Taking étale $\phi, \phi' : \tilde{Y} \to Y$ as in Theorem 4.3.1, we have $\phi^*\mathcal{I}[2] = \phi'^*\mathcal{I}[2]'$, where $\mathcal{I}[2]'$ is defined using x_1'. By induction $a_2, \dots, a_k$ are independent of choices. Hence $(a_1, \dots, a_k)$ is independent of choices.

(2) Since the closed subscheme $V(\mathcal{D}^{\leq a-1}\mathcal{I})$ is the locus where $\mathrm{ord}_p(\mathcal{I}) \geq a$, the order is constructible and upper-semicontinuous. The subscheme $V(\mathcal{D}^{\leq a_1 - 1}\mathcal{I})$ is contained in $V(x_1)$ on which $\mathrm{inv}_p(\mathcal{I}[2])$ is constructible and upper-semicontinuous by induction, hence $\mathrm{inv}_p(\mathcal{I})$ is constructible and upper-semicontinuous. □

Remark 5.1.4. Theorem 5.1.3(3) allows to extend the definition of inv to the case of smooth stacks Y. Indeed, if $\mathcal{I}$ is an ideal, choose a smooth presentation $p_{1,2} : Y_1 \rightrightarrows Y_0$ of Y and let $\mathcal{I}_i \subseteq \mathcal{O}_{Y_i}$ be the pullbacks of $\mathcal{I}$. Then $\mathrm{inv}_{\mathcal{I}_1} = \mathrm{inv}_{\mathcal{I}_0} \circ p_i$ for $i = 1, 2$, hence $\mathrm{inv}_{\mathcal{I}_0}$ factors through $Y_0 \to |Y|$ uniquely. A similar argument shows that the induced map $\mathrm{inv}_{\mathcal{I}} : |Y| \to \Xi_n$ is independent of the presentation.

Concerning the independence of J, we note the following consequence of Theorem 4.3.1:

Lemma 5.1.5. *If x_1' is another maximal contact element such that $(x_1', x_2, \dots, x_n)$ is a regular sequence of parameters at p, then $J' = (x_1'^{a_1}, x_2^{a_2}, \dots, x_k^{a_k})$ is also a center associated to $\mathcal{I}$ at p.*

Proof. As above, this follows since $\phi^*\mathcal{I}[2] = \phi'^*\mathcal{I}[2]'$, where $\mathcal{I}[2]'$ is defined using x_1'. □

5.2. ***Admissibility of centers.*** As in earlier work on resolution of singularities, *admissibility* allows flexibility in studying the behavior of ideals under blowings up of centers. This becomes important when an ideal is related to the sum of ideals with different invariants of their own, but all admitting a common admissible center.

In this section we assume that a_1 is a positive integer and $a_i \le a_{i+1}$. We deliberately do not assume $(a_1, \dots, a_k)$ is $\mathrm{inv}_p(\mathcal{I})$ — see Remark 5.3.2.

5.2.1. *Admissibility and blowing up.* Recall that by Definition 2.4.1(2), a center $J = (x_1^{a_1}, \dots, x_k^{a_k})$ is $\mathcal{I}$-admissible at p if the inequality $(x_1^{a_1}, \dots, x_k^{a_k}) \le v(\mathcal{I})$ of valuative $\mathbb{Q}$-ideals is satisfied on a neighborhood of p.

Very much in analogy to the notion used in earlier resolution algorithms, this can be described in terms of the associated weighted blowing up $Y' = \mathrm{Bl}_{\bar{J}}(Y) \to Y$ along $\bar{J} := (x_1^{1/w_1}, \dots, x_k^{1/w_k})$ as follows: let $E = \bar{J}\mathcal{O}_{Y'}$, which is an invertible ideal sheaf. Note that since $a_1 w_1$ is an integer also $J\mathcal{O}_{Y'} = E^{a_1 w_1}$ is an invertible ideal sheaf. Therefore $J = (x_1^{a_1}, \dots, x_k^{a_k})$ is $\mathcal{I}$-admissible if and only if $E^{a_1 w_1}$ is $\mathcal{I}\mathcal{O}_{Y'}$ admissible, if and only if $\mathcal{I}\mathcal{O}_{Y'} = E^{a_1 w_1}\mathcal{I}'$, with $\mathcal{I}'$ an ideal.

Definition 5.2.2. In the situation as above, $\mathcal{I}'$ is called the *weak transform* of $\mathcal{I}$ under the weighted blowing up.

We will only use this operation when J is the center associated to $\mathcal{I}$, which is shown to be $\mathcal{I}$-admissible below.

Remark 5.2.3. In terms of its monomial valuation, J is admissible for $\mathcal{I}$ if and only if $v_J(f) \ge 1$ for all $f \in \mathcal{I}$. This means that if $f = \sum c_{\bar{\alpha}} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ then $\sum_{i=1}^k \alpha_i / a_i \ge 1$ whenever $c_{\bar{\alpha}} \ne 0$. This is convenient for testing admissibility, as long as one remembers that $v_{J^m} = v_J / m$.

If $Y_1 \to Y$ is smooth and J is $\mathcal{I}$-admissible then $J\mathcal{O}_{Y_1}$ is $\mathcal{I}\mathcal{O}_{Y_1}$-admissible, with the converse holding when $Y_1 \to Y$ is surjective.

5.2.4. *Working with rescaled centers.* For induction to work in the arguments below, it is worthwhile to consider blowings up of centers of the form

$$\bar{J}^{1/c} := (x_1^{1/(w_1 c)}, \dots, x_k^{1/(w_k c)})$$

for a positive integer c. We also use the notation $J^\alpha := (x_1^{a_1\alpha}, \dots, x_k^{a_k\alpha})$ throughout — this being an equality of valuative $\mathbb{Q}$-ideals.

5.2.5. *Basic properties.* The description in Section 5.2.1 of the monomial valuation of J immediately provides the following lemmas:

Lemma 5.2.6. *If J is both $\mathcal{I}_1$-admissible and $\mathcal{I}_2$-admissible then J is $\mathcal{I}_1+\mathcal{I}_2$-admissible. If J is $\mathcal{I}$-admissible then J^k is $\mathcal{I}^k$-admissible. More generally if J^{c_j} is $\mathcal{I}_j$-admissible then $J^{\sum c_j}$ is $\prod \mathcal{I}_j$-admissible.*

Indeed if $v_J(f)\geq 1$ and $v_J(g)\geq 1$ then $v_J(f+g)\geq 1$ and $v_J(f^{c_1}\cdot g^{c_2})\geq c_1+c_2$, etc.

Lemma 5.2.7. *If J is $\mathcal{I}$-admissible then $J'=J^{(a_1-1)/a_1}$ is $\mathcal{D}(\mathcal{I})$-admissible. If $a_1>1$ and $J^{(a_1-1)/a_1}$ is $\mathcal{I}$-admissible then J is $x_1\mathcal{I}$-admissible.*

Proof. For the first statement note that if $\sum_{i=1}^k \alpha_i/a_i\geq 1$ and $\alpha_j\geq 1$ then

$$v_J\left(\frac{\partial(x_1^{\alpha_1}\cdots x_n^{\alpha_n})}{\partial x_j}\right)=\sum_{i=1}^k \alpha_i/a_i-1/a_j\geq 1-1/a_1,$$

so

$$v_{J'}\left(\frac{\partial(x_1^{\alpha_1}\cdots x_n^{\alpha_n})}{\partial x_j}\right)\geq 1,$$

as needed. The other statement is similar. □

As in Section 4.4 by $k_p=k(p)$ we denote a fixed field of coefficients.

Lemma 5.2.8. *For $\mathcal{I}_0\subset k_p[\![x_2,\dots,x_n]\!]$ write $\tilde{\mathcal{I}}_0=\mathcal{I}_0 k_p[\![x_1,\dots,x_n]\!]$. Assume $(x_2^{a_2},\dots,x_k^{a_k})$ is $\mathcal{I}_0$-admissible. Then $(x_1^{a_1},\dots,x_k^{a_k})$ is $\tilde{\mathcal{I}}_0$-admissible.*

Proof. Here for generators of $\tilde{\mathcal{I}}_0$ we have $\sum_{i=1}^k \alpha_i/a_i=\sum_{i=2}^k \alpha_i/a_i$. □

Lemma 5.2.9. *J is $\mathcal{I}$-admissible if and only if $J^{(a_1-1)!}$ is $C(\mathcal{I},a_1)$-admissible.*

Proof. When $\mathcal{I}$ has order $<a_1$ then J is not admissible for $\mathcal{I}$ and $J^{(a_1-1)!}$ is not admissible for $\mathcal{C}(\mathcal{I},a_1)=(1)$. When $\mathcal{I}$ has order $\geq a_1$ this combines Lemmas 5.2.6 and 5.2.7 for the terms defining $C(\mathcal{I},a_1)$. □

This statement is only relevant, and will only be used, when $\mathcal{I}$ has order a_1. If $a_1<a:=\operatorname{ord}(\mathcal{I})$ then $J^{(a_1-1)!}$ is in general not $C(\mathcal{I},a)$-admissible. For instance $J=(x_1)$ is admissible for $\mathcal{I}=(x_1x_2)$ but not for $C(\mathcal{I},2)=(x_1^2,x_1x_2,x_2^2)$.

Lemma 5.2.10. *Assume $(x_1,x_2,\dots,x_n)$ and $(x_1',x_2,\dots,x_n)$ are both regular sequences of parameters, and suppose $(x_1^{a_1},x_2^{a_2},\dots,x_k^{a_k})\leq v(x_1'^{a_1})$. Then $(x_1^{a_1},x_2^{a_2},\dots,x_k^{a_k})=(x_1'^{a_1},x_2^{a_2},\dots,x_k^{a_k})$ as centers.*

Proof. We may rescale a_i and assume they are all integers. The inequality $(x_1^{a_1},x_2^{a_2},\dots,x_k^{a_k})\leq v(x_1'^{a_1})$ implies that $x_1'^{a_1}$ lies in the integral closure $(x_1^{a_1},x_2^{a_2},\dots,x_k^{a_k})^{\mathrm{int}}$, hence

$$(x_1'^{a_1},x_2^{a_2},\dots,x_k^{a_k})^{\mathrm{int}}\subset(x_1^{a_1},x_2^{a_2},\dots,x_k^{a_k})^{\mathrm{int}}.$$

Since these two ideals have the same Hilbert–Samuel functions they coincide. □

5.3. ***Unique admissibility of $J_p(\mathcal{I})$.*** Finally, we prove the second main result of Section 5 in addition to Theorem 5.1.3.

Theorem 5.3.1. *Let Y be a smooth variety, $p \in Y$ a point, and $\mathcal{I} \subseteq \mathcal{O}_Y$ an ideal with $\mathrm{inv}_{\mathcal{I}}(p) = (a_1, \dots, a_k)$:*

(1) *If $x_1, \dots, x_k$ is a maximal contact sequence at p and $J = (x_1^{a_1}, \dots, x_k^{a_k})$ the corresponding center, then J is $\mathcal{I}$-admissible at p.*

(2) $$\mathrm{inv}_{\mathcal{I}}(p) = \max_{(x_1'^{b_1}, \dots, x_k'^{b_k}) \le v(\mathcal{I})} (b_1, \dots, b_k),$$
in other words, $\mathrm{inv}_{\mathcal{I}}(p)$ is the maximal invariant of a center admissible for $\mathcal{I}$.

(3) *Locally at p, J is the unique admissible center with invariant $\mathrm{inv}_{\mathcal{I}}(p)$. In particular, it is in fact independent of the maximal contact sequence $(x_1, \dots, x_k)$.*

(4) *Locally at p, any point p' with $\mathrm{inv}_{\mathcal{I}}(p') = \mathrm{inv}_{\mathcal{I}}(p)$ lies in $V(J)$.*

Proof. We first prove (1). We can work on formal completions as the usual admissibility is equivalent to the formal one: J is dominated by $\mathcal{I}$ at p if and only if the completion $\hat{J} = J\hat{\mathcal{O}}_{Y,p}$ is dominated by $\hat{\mathcal{I}} = \mathcal{I}\hat{\mathcal{O}}_{Y,p}$. Applying Lemma 5.2.9, we replace $\mathcal{I}$ by $\mathcal{C} = C(\mathcal{I}, a_1)$ and rescale the invariant up to $a_1!$. Recall that by Proposition 4.4.1

$$\hat{\mathcal{C}} = (x_1^{a_1!}) + (x_1^{a_1!-1}\widetilde{\mathcal{C}}_1) + \cdots + (x_1\widetilde{\mathcal{C}}_{a_1!-1}) + \widetilde{\mathcal{C}}_{a_1!}.$$

The inductive hypothesis implies that $\hat{J}^{(a_1-1)!}$ is $\overline{\mathcal{C}}_{a_1!}$-admissible. By Lemma 5.2.8 $\hat{J}^{(a_1-1)!}$ is $\widetilde{\mathcal{C}}_{a_1!}$-admissible. By Corollary 4.4.2 and Lemma 5.2.7 $\hat{J}^{(a_1-1)!}$ is $(x_1^{a_1!-j}\widetilde{\mathcal{C}}_j)$-admissible, so by Lemma 5.2.6 $\hat{J}^{(a_1-1)!}$ is $\hat{\mathcal{C}}$-admissible, as needed.

We prove (2) and (3) simultaneously. Assume $(b_1, \dots, b_m) \ge (a_1, \dots, a_k)$. If $J' = (x_1'^{b_1}, \dots, x_k'^{b_k})$ is admissible for $\mathcal{I}$ then $b_1 \le a_1$. Since our chosen center J has $b_1 = a_1$ this maximum is achieved. Let $\ell = \max\{i : b_i = a_1\} \ge 1$. Evaluating $J' < v(\mathcal{I}) \le v(x^{a_1})$ at the divisorial valuation of $x_1 = 0$ we have that $x_1 \in (x_1', \dots, x_\ell') + \mathfrak{m}_p^2$, and after reordering we get that $(x_1, x_2', \dots, x_n')$ is a regular system of parameters. By Lemma 5.2.10 we may write $J' = (x_1^{a_1}, x_2'^{b_2}, \dots, x_k'^{b_k})$. Working on formal completions we may replace x_i' by a suitable $x_i' + \alpha x_1$ so we may assume $x_i' \in k_p[\![x_2, \dots, x_n]\!]$.

As in the proof of (1) above, we may replace $\mathcal{I}$ and $\mathrm{inv}_{\mathcal{I}}(p)$ by the coefficient ideal $\mathcal{C} = C(\mathcal{I}, a_1)$ and the rescaled invariant $(a_1 - 1)!(a_1, \dots, a_k)$, and for the formal completions one has

$$\hat{\mathcal{C}} = (x_1^{a_1!}) + (x_1^{a_1!-1}\widetilde{\mathcal{C}}_1) + \cdots + (x_1\widetilde{\mathcal{C}}_{a_1!-1}) + \widetilde{\mathcal{C}}_{a_1!}.$$

By induction $(a_1 - 1)!(a_2, \dots, a_k)$ is the maximal invariant for $\overline{\mathcal{C}}_{a_1!}$, with unique center $(x_2^{a_2}, \dots, x_k^{a_k})$. By functoriality, the invariant is maximal for $\widetilde{\mathcal{C}}_{a_1!}$. But $J' = (x_1^{a_1}, x_2'^{b_2}, \dots, x_k'^{b_k}) < v(\widetilde{\mathcal{C}}_{a_1!})$ is equivalent to $(x_2'^{b_2}, \dots, x_k'^{b_k}) < v(\widetilde{\mathcal{C}}_{a_1!})$. It follows that $(a_1 - 1)!(a_1, \dots, a_k)$ is the maximal invariant of a center admissible for $C(\mathcal{I}, a_1)$, with unique center J.

Finally, (4) follows from the same induction using the classical fact that a maximal contact to $(\mathcal{I}, a_1)$ contains all neighboring points p' with $\mathrm{ord}_{p'}(\mathcal{I}) = a_1$ (for example, see the proof of Theorem 5.1.3(2)). ☐

Remark 5.3.2. (1) Stated in terms of the monomial valuation v_J associated to J, the theorem says it is the unique monomial valuation with lexicographically *minimal* weights $(w_1, \dots, w_n)$ satisfying $v(\mathcal{I}) = 1$.

(2) As an example for the added flexibility provided by admissibility, the center (x_1^6, x_2^6) is $(x_1^3 x_2^3)$-admissible because this is the corresponding invariant, but also $(x_1^5, x_2^{15/2})$ is admissible. This second center becomes important when one considers instead the ideal $(x_1^5 + x_1^3 x_2^3)$, or even $(x_1^5 + x_1^3 x_2^3 + x_2^8)$, whose invariant is $\left(5, \frac{15}{2}\right)$, as described in Section 7 below.

Corollary 5.3.3. *We have* $\operatorname{inv}_{\mathcal{I}^k}(p) = k \cdot \operatorname{inv}_{\mathcal{I}}(p)$ *and* $\operatorname{inv}_{C(\mathcal{I},a_1)}(p) = (a_1 - 1)!\operatorname{inv}_{\mathcal{I}}(p)$ *when* $a_1 = \operatorname{ord}_p(\mathcal{I})$.

Proof. Indeed J^k is admissible for $\mathcal{I}^k$ if and only if J is admissible for $\mathcal{I}$, and Lemma 5.2.9 provides the analogous statement for the coefficient ideal. □

6. Principalization and resolution

6.1. ***The maximal center.*** Our local construction of centers $J_p(\mathcal{I})$ can be globalized as follows along the maximality locus of the invariant.

Theorem 6.1.1. (1) *For any smooth variety Y and an ideal $\mathcal{I} \subseteq \mathcal{O}_Y$ there exists a unique $\mathcal{I}$-admissible center $J = J(\mathcal{I})$ such that* $\operatorname{inv}_J = \max \operatorname{inv}_{\mathcal{I}}$ *and $p \in V(J)$ if and only if* $\operatorname{inv}_{\mathcal{I}}(p) = \max \operatorname{inv}_{\mathcal{I}}$.

(2) *Compatibility with smooth morphisms $f : Y' \to Y$: either $f^{-1} J(\mathcal{I}) = (1)$, or $f^{-1} J(\mathcal{I}) = J(\mathcal{I}')$, where $\mathcal{I}' = f^{-1}\mathcal{I}$.*

(3) *If Y is a smooth stack of finite type over a field of characteristic zero and $\mathcal{I}$ is an ideal on Y, then associating to each presentation $f : Y' \to Y$ the center $J(f^{-1}\mathcal{I})$ one obtains a center on Y, which will be denoted $J(\mathcal{I})$.*

Proof. Uniqueness in (1) follows from the local uniqueness in Theorem 5.3.1(3). Moreover, it implies that it suffices to establish the existence locally at p. If $\operatorname{inv}_{\mathcal{I}}(p) = \max \operatorname{inv}_{\mathcal{I}}$ then locally at p such a center is provided by Theorem 5.3.1, and otherwise the center is empty in a neighborhood of p.

Recall that the invariant is compatible with arbitrary smooth morphisms by Theorem 5.1.3(3). If $\operatorname{maxinv}(\mathcal{I}') < \operatorname{maxinv}(\mathcal{I})$, then the invariant at any $p' \in f(Y')$ is smaller than $\operatorname{maxinv}(\mathcal{I})$, and hence $V(J) \cap f(Y') = \varnothing$ and $f^{-1}(J) = (1)$. If $\operatorname{maxinv}(\mathcal{I}') = \operatorname{maxinv}(\mathcal{I})$, then the center $f^{-1}(J)$ satisfies the condition defining $J(\mathcal{I}')$. Since such a center is unique by (1), we obtain (2). Finally, (3) is a straightforward consequence of (2). □

Definition 6.1.2. The center $J = J(\mathcal{I})$ defined by Theorem 6.1.1 will be called *the maximal $\mathcal{I}$-admissible center*.

6.2. ***The invariant drops.*** The main miracle about the maximal $\mathcal{I}$-admissible center is that blowing it up one automatically reduces the invariant of the weak transform of $\mathcal{I}$ (see Definition 5.2.2). For inductive reasons we prefer to prove a slightly stronger claim:

Theorem 6.2.1. *Assume that Y is a smooth k-stack and $\mathcal{I} \neq (1)$ is a coherent ideal on Y, and $c > 0$ a natural number. Consider the blowing up $f_c : Y'_c = \mathrm{Bl}_{\bar{J}^{1/c}}(Y)$ of the rescaled reduction $\bar{J}^{1/c}$ of the maximal $\mathcal{I}$-admissible center $J = J(\mathcal{I})$, and let $\mathcal{I}' = E^{-a_1 w_1 c} f_c^{-1}\mathcal{I}$ be the weak transform of $\mathcal{I}$, where* $\operatorname{maxinv}(\mathcal{I}) = (a_1, \dots, a_k)$ *and $(w_1, \dots, w_k)$ are the corresponding weights. Then* $\operatorname{maxinv}(\mathcal{I}') < \operatorname{maxinv}(\mathcal{I})$.

Proof. All players in the assertion are compatible with surjective smooth morphisms by Theorems 5.1.3(3) and 6.1.1(2), hence we can replace Y and $\mathcal{I}$ by an étale cover and the pullback of $\mathcal{I}$. Thus, we can assume that Y is a scheme and it suffices to prove that if $p \in Y$ satisfies $\operatorname{inv}_{\mathcal{I}}(p) = (a_1, \dots, a_k)$, then any $p' \in f_c^{-1}(p)$ satisfies $\operatorname{inv}_{\mathcal{I}'}(p') < (a_1, \dots, a_k)$. In particular, working locally at p we can assume that $J = (x_1^{a_1}, \dots, x_k^{a_k})$ for a maximal contact sequence $(x_1, \dots, x_k)$, and hence $\bar{J}^{1/c} := (x_1^{1/(w_1 c)}, \dots, x_k^{1/(w_k c)})$.

If $k = 0$ the ideal is (0) and there is nothing to prove. When $k = 1$ the ideal is $(x_1^{a_1})$, which becomes exceptional with weak transform $\mathcal{I}' = (1)$. We now assume $k > 1$.

Again using Proposition 4.4.1, we choose formal coordinates, work with $\widetilde{\mathcal{C}} := \hat{C}(\mathcal{I}, a_1)$, and write

$$\widetilde{\mathcal{C}} = (x_1^{a_1!}) + (x_1^{a_1!-1}\widetilde{\mathcal{C}}_1) + \dots + (x_1\widetilde{\mathcal{C}}_1) + \widetilde{\mathcal{C}}_{a_1!}.$$

Writing $\widetilde{\mathcal{C}}\mathcal{O}_{Y'_c} = E^{a_1! w_1 c}\widetilde{\mathcal{C}}'$, we will first show that $\operatorname{inv}_{p'}(\widetilde{\mathcal{C}}') < (a_1 - 1)! \cdot (a_1, a_2, \dots, a_k)$ for all points p' over p.

Write $H = \{x_1 = 0\}$, and $H' \to H$ the blowing up of the reduced center $\bar{J}_H$ associated to $J_H := (x_2^{a_2}, \dots, x_k^{a_k})$. By Lemma 3.4.1 the proper transform $\tilde{H}' \to H$ of H via the blowing up of $\bar{J}$ is the blowing up of $\bar{J}_H^{1/(cc')}$, allowing for induction.

We now inspect the behavior on different charts. On the x_1-chart we have $x_1 = u^{w_1 c}$ so the first term becomes $(x_1^{a_1!}) = E^{a_1! w_1 c} \cdot (1)$ and $\operatorname{inv}_{p'}\widetilde{\mathcal{C}}' = \operatorname{inv}(1) = 0$.[6] This implies that on all other charts it suffices to consider $p' \in \tilde{H}' \cap E$, as all other points belong to the x_1-chart. By the inductive assumption, for such points we have

$$\operatorname{inv}_{p'}((\bar{\mathcal{C}}_{a_1!})') < (a_1 - 1)! \cdot (a_2, \dots, a_k).$$

Note that the term $(x_1^{a_1!})$ in $\widetilde{\mathcal{C}}$ is transformed, via $x_1 = u^{w_1 c} x'_1$ to the form $E^{a_1! w_1 c}(x_1'^{a_1!})$. It follows that $\operatorname{ord}_{p'}(\widetilde{\mathcal{C}}') \le a_1!$, and if $\operatorname{ord}_{p'}(\widetilde{\mathcal{C}}') < a_1!$ then a fortiori $\operatorname{inv}_{p'}(\widetilde{\mathcal{C}}') < \operatorname{inv}_p(\widetilde{\mathcal{C}})$.

If on the other hand $\operatorname{ord}_{p'}(\widetilde{\mathcal{C}}') = a_1!$ then the variable x'_1 is a maximal contact element. Using the inductive assumption we compute

$$\operatorname{inv}_{p'}((x'^{a_1!}_1) + (\widetilde{\mathcal{C}}_{a_1!})') = (a_1!, \operatorname{inv}_{p'}((\bar{\mathcal{C}}_{a_1!})')) < (a_1!, \operatorname{inv}_{p'}(\bar{\mathcal{C}}_{a_1!})) = (a_1 - 1)!(a_1, \dots, a_k).$$

Since $\widetilde{\mathcal{C}}'$ includes this ideal, we obtain again $\operatorname{inv}_{p'}(\widetilde{\mathcal{C}}') < \operatorname{inv}_p(\widetilde{\mathcal{C}})$, as claimed.

We deduce that $\operatorname{inv}_{p'}(\mathcal{I}') < \operatorname{inv}_p(\mathcal{I})$ as well. As in [Kollár 2007, Theorem 3.67; Bierstone and Milman 2008, Lemma 3.3; Abramovich et al. 2020a; 2020b], we have the inclusions $\mathcal{I}'^{(a_1-1)!} \subset \widetilde{\mathcal{C}}' \subset \hat{C}(\mathcal{I}', a_1)$,[7]

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[6] This reflects the fact that before passing to the coefficient ideal $\operatorname{ord}(\mathcal{I}') < a_1$ on this chart — it need not become a unit ideal in general!

[7] These are the "easy" inclusions — which hold even in the logarithmic situation.

hence $\operatorname{ord}_{p'}(\mathcal{I}') \leq a_1$. We may again assume x_1' is a maximal contact element and $\operatorname{ord}_{p'}(\mathcal{I}') = a_1$. By Theorem 5.3.1(2)

$$\operatorname{inv}_{p'}(\mathcal{I}'^{(a_1-1)!}) \geq \operatorname{inv}_{p'}(\widetilde{\mathcal{C}}') \geq \operatorname{inv}_{p'}(\hat{C}(\mathcal{I}', a_1)).$$

By Corollary 5.3.3 we have $\operatorname{inv}_{p'}(\mathcal{I}'^{(a_1-1)!}) = \operatorname{inv}_{p'}(\hat{C}(\mathcal{I}', a_1))$ giving equalities throughout, hence

$$\operatorname{inv}_{p'}(\mathcal{I}') = \frac{1}{(a_1-1)!}\operatorname{inv}_{p'}(\widetilde{\mathcal{C}}') < \frac{1}{(a_1-1)!}\operatorname{inv}_p(\widetilde{\mathcal{C}}) = \operatorname{inv}_p(\mathcal{I}),$$

as needed. □

6.3. *The principalization theorem.* It remains to summarize our results. First, we obtain principalization. Given a pair $(Y, \mathcal{I})$ consisting of a smooth Deligne–Mumford k-stack Y and an ideal $\mathcal{I} \subset \mathcal{O}_{Y_{\text{ét}}}$, let $J = J(\mathcal{I})$ be the maximal $\mathcal{I}$-admissible center with reduction $\bar{J}$, let $Y_1 = \operatorname{Bl}\bar{J}(Y)$ and let $\mathcal{I}_1$ be the weak transform of $\mathcal{I}$. We set $\mathcal{P}_1(Y, \mathcal{I}) = (Y_1, \mathcal{I}_1)$.

Theorem 6.3.1 (principalization). (1) *Partial principalization*:

- (a) $\mathcal{P}_1$ *reduces the invariant*: $\operatorname{maxinv}(\mathcal{I}_1) < \operatorname{maxinv}(\mathcal{I})$.
- (b) *If $f : Y' \to Y$ is smooth and $\mathcal{I}' = f^{-1}\mathcal{I}$, then either $\mathcal{P}_1(Y, \mathcal{I})$ pullbacks to the empty blowing up of Y', or $\mathcal{P}_1(Y', f^{-1}\mathcal{I}) = \mathcal{P}_1(Y, \mathcal{I}) \times_Y Y'$. In particular, $\mathcal{P}_1$ is compatible with surjective smooth morphisms.*

(2) *Full principalization*:

- (a) *The sequence $(Y_{i+1}, \mathcal{I}_{i+1}) = \mathcal{P}_1(Y_i, \mathcal{I}_i)$ starting with $(Y_0, \mathcal{I}_0) = (Y, \mathcal{I})$ stabilizes and this happens at the smallest n with $\mathcal{I}_n = (1)$.*
- (b) *The principalization blowings up sequence $\mathcal{P}(Y, \mathcal{I}) : Y_n \to \cdots \to Y$ is compatible with arbitrary smooth morphisms $f : Y' \to Y$: the sequence $\mathcal{P}(Y', f^{-1}\mathcal{I}')$ is obtained from $\mathcal{P}(Y, \mathcal{I}) \times_Y Y'$ by removing all empty blowings up.*

Proof. Claim (1) is covered by Theorems 6.2.1 and 6.1.1(2). Since the set of invariants Ξ_n is well-ordered and attains its minimum (0) on the trivial ideal (1), we obtain (2a). The functoriality of $\mathcal{P}$ follows from the functoriality of $\mathcal{P}_1$. □

As a corollary we deduce embedded resolution. This is a standard reduction, except the fact that here we are also able to replace the weak transform by the proper transform.

Proof of Theorems 1.2.2 and 1.2.5. Given a DM pair $X \subset Y$ consider the ideal $\mathcal{I} = \mathcal{I}_X$ defining X in Y, and set $\operatorname{inv}_{(X,Y)} = \operatorname{inv}_{\mathcal{I}}$ and $J(X, Y) = J(\mathcal{I})$. Thus, we take $Y_1 \to Y$ to be the same weighted blowing up as in $\mathcal{P}_1(Y, \mathcal{I})$ and take X_1 to be the proper transform of X. Since $\mathcal{I}_1 \subseteq \mathcal{I}_{X_1}$ we obtain that $\operatorname{maxinv}(X_1, Y_1) \leq \operatorname{maxinv}(\mathcal{I}_1)$. Therefore all assertions of the two theorems follow from the properties of the center $\mathcal{J}(\mathcal{I})$ and the invariant $\operatorname{inv}_{\mathcal{I}}$ proven in Theorems 6.1.1 and 6.3.1. □

Note that in the deduction of Corollary 1.2.3 we also used that if X is of codimension c at $p \in Y$, then $\operatorname{inv}_{\mathcal{I}}(p) \geq (1, \ldots, 1)$ of length c, and the equality holds if and only if X is smooth at p.

7. An example

Consider the plane curve

$$X = V(x^5 + x^3y^3 + y^k)$$

with $k \geq 5$. Its resolution depends on whether or not $k \geq 8$.

7.1. ***The case k ≥ 8.*** This curve is singular at the origin p. We have $a_1 = \operatorname{ord}_p(\mathcal{I}_X) = 5$. Since $\mathcal{D}^{\leq 4}\mathcal{I} = (x, y^2)$ we may take $x_1 = x$ and $H = V(x)$. A direct computation provides the coefficient ideal

$$C(\mathcal{I}_X, 5)|_H = (\mathcal{D}^{\leq 3}(\mathcal{I}_X)|_H)^{120/2} = (y^{180}),$$

with $b_2 = 180$ and $a_2 = \frac{180}{4!} = \frac{15}{2}$. Rescaling, we need to take the weighted blowup of $\bar{J} = (x^{1/3}, y^{1/2})$:

- In the x-chart we have $x = u^3$, $y = u^2y'$, giving

  $$Y'_x = [\operatorname{Spec} k[u, y']/\boldsymbol{\mu}_3],$$

  the action given by $(u, y') \mapsto (\zeta_3 u, \zeta_3 y')$. The equation of X becomes

  $$u^{15}(1 + y'^3 + u^{2k-15}y'^k),$$

  with proper transform $X'_x = V(1 + y'^3 + u^{2k-15}y'^k)$ smooth.
- In the y-chart we have $y = v^2$, $x = v^3x'$, giving

  $$Y'_y = [\operatorname{Spec} k[x', v]/\boldsymbol{\mu}_2],$$

  the action given by $(x', v) \mapsto (-x', -v)$. The equation of X becomes $v^{15}(x'^5 + x'^3 + v^{2k-15})$, with proper transform $X'_y = V(x'^5 + x'^3 + v^{2k-15})$.

  Note that X'_y is smooth when $k = 8$. Otherwise it is singular at the origin with invariant $(3, 2k-15)$, which is lexicographically strictly smaller than $\left(5, \frac{15}{2}\right)$; A single weighted blowing up resolves the singularity.

7.2. ***The case k ≤ 7.*** Consider now the same equation with $k = 7$ (the cases $k = 5, 6$ being similar). We still take $a_1 = 5$, $x_1 = x$ and $H = V(x)$. This time

$$C(\mathcal{I}_X)|_H = ((\mathcal{I}_X)|_H)^{120/5} = (y^{168}),$$

with $b_2 = 7 \cdot (4!)$ and $a_2 = 7$. We take the weighted blowup of $J = (x^{1/7}, y^{1/5})$:

- In the x-chart we have $x = u^7$, $y = u^5y'$, giving

  $$Y'_x = [\operatorname{Spec} k[u, y']/\boldsymbol{\mu}_7],$$

  the action given by $(u, y') \mapsto (\zeta_7 u, \zeta_7^{-5} y')$. The equation of X becomes

  $$u^{35}(1 + uy'^3 + y'^7),$$

  with proper transform $X'_x = V(1 + uy'^3 + y'^7)$ smooth.

- In the y-chart we have $y = v^5$, $x = v^7 x'$, giving

$$Y'_y = [\operatorname{Spec} k[x', v]/\boldsymbol{\mu}_5],$$

the action given by $(x', v) \mapsto (\zeta_5^{-7} x', \zeta_5 v)$. The equation of X becomes $v^{35}(x'^5 + vx'^3 + 1)$, with smooth proper transform $X'_y = V(x'^5 + vx'^3 + 1)$.

8. Further comments

8.1. ***Nonembedded resolution.*** Given two embeddings $X \subset Y_1$ and $X \subset Y_2$ such that $\dim_p(Y_1) = \dim_p(Y_2)$ for all $p \in X$, the two embeddings are étale locally equivalent. By functoriality the embedded resolutions of $X \subset Y_1$ and $X \subset Y_2$ are étale locally isomorphic, hence the resolutions $X'_1 \to X$ and $X'_2 \to X$ coincide.

Our resolutions also satisfy the reembedding principle [Abramovich et al. 2020a, proposition 2.12.3]: given an embedding $Y \subset Y_1 := Y \times \operatorname{Spec} k[x_0]$ and $\operatorname{inv}_p(\mathcal{I}_{X \subset Y}) = (a_1, \dots, a_k)$ with parameters $(x_1, \dots, x_k)$ we have $\operatorname{inv}_p(\mathcal{I}_{X \subset Y_1}) = (1, a_1, \dots, a_k)$ with parameters $(x_0, x_1, \dots, x_k)$. The proper transform X'_1 of X in Y'_1 is disjoint from the x_0-chart, and on every other chart we have $Y'_1 = Y' \times \operatorname{Spec} k[x_0]$ so that $X'_1 = X'$ and induction applies.

Since every pure-dimensional stack can be étale locally embedded in pure codimension, we deduce:

Theorem 8.1.1 (nonembedded resolution). *There is a functor F_{ner} associating to a pure-dimensional reduced stack X of finite type over a characteristic-0 field k a proper, generically representable and birational morphism $F_{\mathrm{ner}}(X) \to X$ with $F_{\mathrm{ner}}(X)$ regular. This is functorial for smooth morphisms: if $X_1 \to X$ is smooth then $F_{\mathrm{ner}}(X_1) = F_{\mathrm{ner}}(X) \times_X X_1$.*

Remark 8.1.2. Of course one can deduce functorial resolution of X which is not pure dimensional just by applying F_{ner} to the normalization of X. One can also use other operations to separate components, for example, the disjoint union of the schematic closures of the generic points of X does the job.

Carefully using Bergh's destackification theorem we also obtain:

Theorem 8.1.3 (coarse resolution). *There is a construction F_{crs} associating to a pure-dimensional reduced stack X of finite type over a characteristic-0 field k a **projective** birational morphism $F_{\mathrm{crs}}(X) \to X$ with $F_{\mathrm{crs}}(X)$ smooth. This is functorial for smooth representable morphisms $X_1 \to X$, namely, $F_{\mathrm{crs}}(X_1) = F_{\mathrm{crs}}(X) \times_X X_1$.*

Proof. We apply [Bergh and Rydh 2019, Theorem 7.1], using $F_{\mathrm{ner}}(X) \to X \to \operatorname{Spec} k$ for $X \to T \to S$ in that theorem. This provides a projective morphism $F_{\mathrm{ner}}(X)' \to F_{\mathrm{ner}}(X)$, functorial for smooth morphisms $X_1 \to X$, such that the relative coarse moduli space $F_{\mathrm{ner}}(X)' \to \underline{F_{\mathrm{ner}}(X)'} \to X$ is projective over X, and such that $F_{\mathrm{ner}}(X)'$ and $\underline{F_{\mathrm{ner}}(X)'}$ are regular. We may take $F_{\mathrm{crs}}(X) = \underline{F_{\mathrm{ner}}(X)'}$. □

Remark 8.1.4. In general, $F_{\mathrm{crs}}(X)$ is only representable (even projective) over X, but not over k. This implies that when X is an algebraic space (or projective) so is $F_{\mathrm{crs}}(X)$. Of course one can replace in the construction relative destackification by absolute destackification. In such a case, the resulting resolution of the coarse moduli space $\underline{X}$ would be an algebraic space, but the construction would not be compatible with smooth morphisms.

8.2. ***Note on stabilizers.*** Even though Bergh's destackification is known for tame stacks, one might wonder about the stabilizers occurring in our resolution. We note, however, that the stabilizers of a weighted blowing up locally embed in $I_Y \times \mathbb{G}_m$, where I_Y denotes the inertia stack of Y. We therefore have that the stabilizers of Y_n locally embed in $I_Y \times \mathbb{G}_m^n$. In particular, if Y is a scheme then Y_n has abelian inertia, and its coarse moduli space has abelian quotient singularities.

8.3. ***Note on exceptional loci.*** We show by way of an example that the exceptional loci produced in our algorithm do not necessarily have normal crossings with centers.

Consider $\mathcal{I} = (x^2yz + yz^4) \subset \mathbb{C}[x, y, z]$. Then $\operatorname{maxinv}(\mathcal{I}) = (4, 4, 4)$ is attained at the origin with center (x^4, y^4, z^4) and reduced center (x, y, z). In the z-chart one obtains the ideal $(y_3(x_3^2 + z))$. The new invariant is $(2, 2)$ with reduced center $(y_3, x_3^2 + z)$, which is tangent to the exceptional $z = 0$.

The methods of [Abramovich et al. 2020a] suggest using the logarithmic derivative in z, resulting in the invariant $(3, 3, \infty)$ with center $(y_3^3, x_3^3, z^{3/2})$ and reduced Kummer center $(y_3, x_3, z^{1/2})$. This reduces logarithmic invariants respecting logarithmic, hence exceptional, divisors. A general algorithm is worked out in [Quek 2022].

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