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**Affine Deligne–Lusztig varieties
with finite Coxeter parts**

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We study affine Deligne–Lusztig varieties $X_{w(b)}$ when the finite part of the element w in the Iwahori–Weyl group is a partial σ -Coxeter element. We show that such w is a cordial element and $X_{w(b)} \neq \emptyset$ if and only if b satisfies a certain Hodge–Newton indecomposability condition. Our main result is that for such w and b , $X_{w(b)}$ has a simple geometric structure: the σ -centralizer of b acts transitively on the set of irreducible components of $X_{w(b)}$; and each irreducible component is an iterated fibration over a classical Deligne–Lusztig variety of Coxeter type, and the iterated fibers are either \mathbb{A}^1 or \mathbb{G}_m .

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1. Introduction

1A. Classical/affine Deligne–Lusztig varieties. The classical Deligne–Lusztig varieties were introduced by Deligne and Lusztig in [3]. They play a crucial role in the representation theory of finite reductive groups. They are defined for a connected reductive group \mathbf{G} over a finite field \mathbb{F}_q . For any w in the (finite) Weyl group of $\mathbf{G}(\bar{\mathbb{F}}_q)$, the corresponding Deligne–Lusztig variety X_w is a certain locally closed subvariety of the flag variety of $\mathbf{G}(\bar{\mathbb{F}}_q)$, and it admits a natural action of the finite reductive group $\mathbf{G}(\mathbb{F}_q)$. It is known that

- (a) the classical Deligne–Lusztig variety X_w is smooth and of dimension equal to the length of w , and the finite reductive group $\mathbf{G}(\mathbb{F}_q)$ acts transitively on the set of irreducible components of X_w .

Affine Deligne–Lusztig varieties were introduced by Rapoport in [27] as the affine analog of classical Deligne–Lusztig varieties. They serve as group-theoretic models for Shimura varieties and shtukas. They are defined for a connected reductive group \mathbf{G} over a nonarchimedean local field F . Let \check{F} be the

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completion of the maximal unramified extension of F . For any element w in the Iwahori–Weyl group \tilde{W} of $\mathbf{G}(\check{F})$ and any element $b \in \mathbf{G}(\check{F})$, the corresponding affine Deligne–Lusztig variety $X_w(b)$ is a certain locally closed subscheme of finite type in the affine flag variety of $\mathbf{G}(\check{F})$, and it admits a natural action of the σ -centralizer $\mathbf{J}_b(F)$ of b . Unlike classical Deligne–Lusztig varieties, which have the nice geometric structure described in (a), the geometric structures of affine Deligne–Lusztig varieties are very complicated:

- For many pairs (w, b) , $X_w(b)$ are empty.
- Even if $X_w(b)$ is nonempty, it is not equidimensional in general, and it is very difficult to determine its dimension.
- In general, the group $\mathbf{J}_b(F)$ does not act transitively on the set of irreducible components of $X_w(b)$, and very little is known about this $\mathbf{J}_b(F)$ -action.
- The irreducible components of $X_w(b)$, in general, have a very complicated geometric structure.

We refer to the survey article [15] and [16] for recent developments regarding the nonemptiness pattern and the dimension formula for $X_w(b)$.

1B. Main result. Milićević and Viehmann [25] introduced the notion of cordial elements. The geometry of the affine Deligne–Lusztig varieties associated with cordial elements is “well-behaved” in the following sense. If w is a cordial element, then the elements b with $X_w(b) \neq \emptyset$ form a saturated set in the sense of [25, Theorem 1.1], and for any such b , there is a simple dimension formula for $X_w(b)$. Moreover, $X_w(b)$ is equidimensional. Schremmer gave a classification of the cordial elements in [28].

However, even for a cordial element w , very little is known about the $\mathbf{J}_b(F)$ -orbits on the set of irreducible components of $X_w(b)$ or about the geometric structure of the irreducible components of $X_w(b)$.

On the other hand, by [12], there is a family of elements in the Iwahori–Weyl group whose associated affine Deligne–Lusztig varieties have very simple geometric structures. We denote by σ the Frobenius morphism on $\mathbf{G}(\check{F})$ and the induced group automorphism on the Iwahori–Weyl group \tilde{W} . Suppose that w is a minimal length element in its σ -conjugacy class of \tilde{W} ; then there exists a unique σ -conjugacy class $[b]$ with $X_w(b) \neq \emptyset$. In this case, there exist a parahoric subgroup P of $\mathbf{J}_b(F)$ and a classical Deligne–Lusztig variety X (associated with the reductive quotient of P) such that $X_w(b) \cong \mathbf{J}_b(F) \times^P X$. Such a simple geometric structure has been used in the study of certain Shimura varieties with simple geometric structure (see [4; 7; 8]). However, these minimal length elements w form only a tiny fraction of the whole Iwahori–Weyl group, and such a simple geometric structure only occurs in a few cases.

We will focus on another family of elements in \tilde{W} . For any $w \in \tilde{W}$, we define its finite part to be the image of w under the map $\eta_\sigma : \tilde{W} \rightarrow W$ (see Section 2E). Our main result is that if the finite part of w is a σ -Coxeter element of W , then the associated affine Deligne–Lusztig variety $X_w(b)$ for any b has a simple geometric structure.

Theorem 1.1 (see Theorem 2.6). *Let $w \in \tilde{W}$ such that $\eta_\sigma(w)$ is a σ -Coxeter element of W . Then*

- (1) w is a cordial element;

- (2) $X_w(b) \neq \emptyset$ if and only if b satisfies a certain Hodge–Newton indecomposability condition;
- (3) for any b with $X_w(b) \neq \emptyset$, there exists a parahoric subgroup P of $J_b(F)$ and a classical Deligne–Lusztig variety X of Coxeter type, and an iterated fibration $Y \rightarrow X$ whose iterated fibers are either \mathbb{A}^1 or \mathbb{G}_m such that $X_w(b) \cong J_b(F) \times^P Y$.

We refer to Section 2 for the precise statement and definitions of the notions used here. The special case of part (3) where $G = GL_n$, b is basic and w is a certain element with finite Coxeter part was studied by Shimada [29].

1C. Strategy. One major tool used in the study of affine Deligne–Lusztig varieties is the Deligne–Lusztig reduction method [12]. Based on the Deligne–Lusztig reduction, a close relationship between affine Deligne–Lusztig varieties and the class polynomials of affine Hecke algebras was established in [12]. One remarkable property of these class polynomials is that they are polynomials in $(q - 1)$ with nonnegative integral coefficients. With each element $w \in \tilde{W}$, we may associate a reduction tree, which encodes the information on the reduction steps and determines the class polynomials associated with w . However, obtaining an explicit description of the reduction trees is quite challenging.

Another key ingredient is the Chen–Zhu conjecture. This conjecture predicts the $J_b(F)$ -action on the top-dimensional irreducible components of affine Deligne–Lusztig varieties in the affine Grassmannian. This conjecture was verified recently in [20; 26; 33]. Part of the Chen–Zhu conjecture predicts the isotropy group for the $J_b(F)$ -action, which gives some information about the end points of the reduction trees.

Combining the above two ingredients, in Section 5, we show that the end points of each reduction tree for w with finite Coxeter part must be certain σ -Coxeter elements, and each path, which corresponds to a σ -conjugacy class $[b]$ of b , in a reduction tree provides a $J_b(F)$ -orbit of irreducible components of $X_w(b)$. It remains to show that for any b , there is at most one path in the reduction tree that corresponds to $[b]$ (i.e., the “multiplicity one” result). For the (unique) maximal σ -conjugacy class $[b]$ with $X_w(b) \neq \emptyset$, this “multiplicity one” result is obvious. For the basic σ -conjugacy class $[b]$, one may deduce the “multiplicity one” result by showing that any path corresponding to $[b]$ is of a unique type. See Section 5F.

It is more challenging to determine the numbers of reduction paths for other σ -conjugacy classes in a reduction tree. We use the following indirect approach to establish the “multiplicity one” result. We first interpret the class polynomials as the number of rational points for certain admissible subsets. We then use the positivity property of the class polynomials to show that the “multiplicity one” result for all b is equivalent to the single combinatorial identity

$$\sum_{[b] \in B(G, \mu)_{\text{indec}}} (q - 1)^? q^{-??} = 1. \tag{*}$$

Here $B(G, \mu)_{\text{indec}}$ is the set of all Hodge–Newton indecomposable σ -conjugacy classes (see Section 2C), and the powers “?” and “??” are certain nonnegative integers determined by w and b (see Section 6A).

Verifying the combinatorial identity (*) is the most technical part of this paper and is done in Section 6. We first establish natural bijections between the sets $B(G, \mu)_{\text{indec}}$ for various pairs (G, μ) , which is of

independent interest. In combination with other techniques, we reduce the verification of (*) to the case for simply laced, \check{F} -simple and split groups and for fundamental coweights.

For classical groups, we may further reduce to the case where μ is minuscule. In this case, the “multiplicity one” result follows from the Chen–Zhu conjecture. For exceptional groups, we use a computer to verify (*). The most complicated case for the exceptional group is (E_8, ω_4^\vee) . In this case, the left-hand side of (*) involves a summation of 729 terms. It is also worth mentioning that in the case (A_{n-1}, ω_i^\vee) , we may write (*) explicitly as

$$\sum_{\substack{k \geq 1 \\ 1 > a_1/b_1 > \dots > a_k/b_k > 0 \\ a_1 + \dots + a_k = i \\ b_1 + \dots + b_k = n}} (q-1)^{k-1} q^{1-k + (\sum_{1 \leq l_1 < l_2 \leq k} (a_{l_1} b_{l_2} - a_{l_2} b_{l_1}) + \sum_{1 \leq l \leq k} \gcd(a_l, b_l))/2} = q^{(i(n-i)-n)/2+1}.$$

We do not know if there is a purely combinatorial proof of this identity.

2. Preliminaries

2A. Reductive groups. Let F be a nonarchimedean local field with residue field \mathbb{F}_q and let \check{F} be the completion of the maximal unramified extension of F . We write Γ for $\text{Gal}(\bar{F}/F)$, and Γ_0 for the inertia subgroup of Γ .

Let G be a quasisplit connected reductive group over F . We set $\check{G} = G(\check{F})$. Let σ be the Frobenius morphism of \check{F} over F . We use the same symbol σ for the induced Frobenius morphism on \check{G} . Let S be a maximal \check{F} -split torus of G defined over F , which contains a maximal F -split torus. Let T be the centralizer of S in G . Then T is a maximal torus. We denote by N the normalizer of T in G . Let $W = N(\check{F})/T(\check{F})$ be the *relative Weyl group*. We fix a σ -stable Iwahori subgroup \check{I} of \check{G} . Let $\tilde{W} = N(\check{F})/T(\check{F}) \cap \check{I}$ be the *Iwahori–Weyl group*. The action σ on \check{G} induces a natural action on \tilde{W} and W , which we still denote by σ . For any $w \in \tilde{W}$, we choose a representative \dot{w} in $N(\check{F})$. We have the splitting

$$\tilde{W} = X_*(T)_{\Gamma_0} \rtimes W = \{t^\lambda w \mid \lambda \in X_*(T)_{\Gamma_0}, w \in W\}.$$

Here $X_*(T)_{\Gamma_0}$ denotes the Γ_0 -coinvariants of $X_*(T)$.

Since G is quasisplit over F , σ acts naturally on $X_*(T)_{\Gamma_0}$ and on W . We denote by ℓ the length function on \tilde{W} and on W , and by \leq the Bruhat order on \tilde{W} and on W . Let $\tilde{\mathbb{S}}$ be the index set of simple reflections in \tilde{W} and let $\mathbb{S} \subset \tilde{\mathbb{S}}$ be the index set of simple reflections in W . In other words, the set of simple reflections in \tilde{W} is $\{s_i \mid i \in \tilde{\mathbb{S}}\}$.

For any $w \in W$, we denote by $\text{supp}(w)$ the set of i such that s_i occurs in some (or, equivalently, any) reduced expressions of w , and we set $\text{supp}_\sigma(w) = \bigcup_{l \in \mathbb{N}} \sigma^l(\text{supp}(w))$.

An element $c \in W$ is called a (*full*) σ -Coxeter element if it is a product of simple reflections, one from each σ -orbit of \mathbb{S} . An element $c \in W$ is called a *partial* σ -Coxeter element if it is a product of simple reflections, at most one from each σ -orbit of \mathbb{S} .

Let Φ be the reduced root system underlying the relative root system of \mathbf{G} over \check{F} (the *échelonnage* root system). For any $i \in \mathbb{S}$, we denote by α_i and α_i^\vee the corresponding (relative) simple root and simple coroot, respectively.

2B. The σ -conjugacy classes of \check{G} . The σ -conjugation action on \check{G} is defined by $g \cdot_\sigma g' = gg'\sigma(g)^{-1}$. For $b \in \check{G}$, we denote by $[b]$ the σ -conjugacy class of b . Let $B(\mathbf{G})$ be the set of σ -conjugacy classes on \check{G} . The classification of the σ -conjugacy classes is due to Kottwitz in [21; 22]. Any σ -conjugacy class $[b]$ is determined by two invariants:

- the element $\kappa([b]) \in \pi_1(\mathbf{G})_\sigma$;
- the Newton point $\nu_b \in ((X_*(T)_{\Gamma_0, \mathbb{Q}})^+)^{(\sigma)}$.

Here $\pi_1(\mathbf{G})_\sigma$ denotes the σ -coinvariants of $\pi_1(\mathbf{G})$, and $(X_*(T)_{\Gamma_0, \mathbb{Q}})^+$ denotes the intersection of $X_*(T)_{\Gamma_0} \otimes \mathbb{Q} = X_*(T)^{\Gamma_0} \otimes \mathbb{Q}$ with the set $X_*(T)_{\mathbb{Q}}^+$ of dominant elements in $X_*(T)_{\mathbb{Q}}$. Define

$$V = X_*(T)_{\Gamma_0} \otimes \mathbb{R}.$$

For any $v \in V$, define

$$I(v) = \{i \in \mathbb{S} \mid \langle v, \alpha_i \rangle = 0\}.$$

Here $\langle \cdot, \cdot \rangle : V \times \mathbb{R}\Phi \rightarrow \mathbb{R}$ is the natural pairing. Let V^+ be the set of dominant vectors $v \in V$, that is, $\langle v, \alpha_i \rangle \geq 0$ for $i \in \mathbb{S}$.

The set $B(\mathbf{G})$ is equipped with a natural partial order: $[b] \leq [b']$ if and only if $\kappa([b]) = \kappa([b'])$ and $\nu_b \leq \nu_{b'}$. Here \leq is the dominance order on the set of dominant elements in $X_*(T)_{\mathbb{Q}}$, that is, $\nu \leq \nu'$ if $\nu' - \nu$ is a nonnegative rational linear combination of positive relative coroots. It is proved in [2, Theorem 7.4] that the poset $B(\mathbf{G})$ is ranked. For any $[b] \leq [b']$ in $B(\mathbf{G})$, we denote by $\text{length}([b], [b'])$ the length of any maximal chain between $[b]$ and $[b']$.

Let μ be a dominant coweight. Let μ^\diamond be the average of the σ -orbit of μ . The set of *neutrally acceptable* σ -conjugacy classes is defined by

$$B(\mathbf{G}, \mu) = \{[b] \in B(\mathbf{G}) \mid \kappa([b]) = \kappa(\mu), \nu_b \leq \mu^\diamond\}.$$

For any $i \in \mathbb{S}$, let $\omega_i \in \mathbb{R}\Phi$ be the corresponding fundamental weight. For any σ -orbit \mathcal{O} of \mathbb{S} , let $\omega_{\mathcal{O}} = \sum_{i \in \mathcal{O}} \omega_i$. The following length formula is due to Chai (see [2, Theorem 7.4; 31, Theorem 3.4]):

- (a) For $[b] \in B(\mathbf{G}, \mu)$, $\text{length}([b], [t^\mu]) = \sum_{\mathcal{O} \in \mathbb{S}/\langle \sigma \rangle} \lceil \langle \mu - \nu_b, \omega_{\mathcal{O}} \rangle \rceil$.

2C. Hodge–Newton indecomposable/irreducible set. For any σ -stable subset J of \mathbb{S} , we denote by M_J the standard Levi subgroup of $\mathbf{G}_{\check{F}}$ associated with J . Let $W_J \subseteq W$ be the parabolic subgroup generated by the simple reflections in J . Then W_J is the Weyl group of M_J . Let $b \in \check{G}$. We say that (μ, b) is *Hodge–Newton decomposable* with respect to M_J if $I(\nu_b) \subseteq J$ and $\mu^\diamond - \nu_b \in \sum_{j \in J} \mathbb{R}_{\geq 0} \alpha_j^\vee$. If (μ, b) is not Hodge–Newton decomposable with respect to any proper σ -stable standard Levi subgroup of $\mathbf{G}_{\check{F}}$, then we say that $[b]$ is *Hodge–Newton indecomposable*. Set

$$B(\mathbf{G}, \mu)_{\text{indec}} = \{[b] \in B(\mathbf{G}, \mu) \mid [b] \text{ is Hodge–Newton indecomposable}\}.$$

We say that (μ, b) is *Hodge–Newton J -irreducible* if $\mu^\diamond - \nu_b \in \sum_{j \in J} \mathbb{R}_{>0} \alpha_j^\vee$. Set

$$B(\mathbf{G}, \mu)_{J\text{-irr}} = \{[b] \in B(\mathbf{G}, \mu) \mid [b] \text{ is Hodge–Newton } J\text{-irreducible}\}.$$

We simply write $B(\mathbf{G}, \mu)_{\text{irr}} = B(\mathbf{G}, \mu)_{\mathbb{S}\text{-irr}}$.

We say that μ is *essentially noncentral* with respect to \mathbf{M}_J if it is noncentral on every σ -orbit of connected components of J . It is easy to see that $B(\mathbf{G}, \mu)_{J\text{-irr}} \neq \emptyset$ if and only if μ is essentially noncentral with respect to \mathbf{M}_J . We may simply say that μ is essentially noncentral if it is essentially noncentral with respect to \mathbf{G} . If μ is essentially noncentral, then $B(\mathbf{G}, \mu)_{\text{irr}} = B(\mathbf{G}, \mu)_{\text{indec}}$.

Let \mathfrak{M}_μ be the set of σ -stable subsets $J \subseteq \mathbb{S}$ such that μ is essentially noncentral in J . Note that if $B(\mathbf{G}, \mu)_{J\text{-irr}} \neq \emptyset$, then $J \in \mathfrak{M}_\mu$. By definition, any $[b] \in B(\mathbf{G}, \mu)$ lies in some $B(\mathbf{G}, \mu)_{J\text{-irr}}$. Then we have

$$B(\mathbf{G}, \mu) = \bigsqcup_{J \in \mathfrak{M}_\mu} B(\mathbf{G}, \mu)_{J\text{-irr}}.$$

Let $J = \sigma(J) \subseteq \mathbb{S}$. For $b \in \mathbf{M}_J(\check{F})$, we denote by $[b]_{\mathbf{M}_J}$ the σ -conjugacy class of b in $\mathbf{M}_J(\check{F})$, and denote by $\nu_b^{\mathbf{M}_J}$ its \mathbf{M}_J -dominant Newton point.

Lemma 2.1. *Let μ be a dominant coweight and J be a σ -stable subset of \mathbb{S} . Then*

- (1) *the map $\phi_J : B(\mathbf{M}_J, \mu) \rightarrow B(\mathbf{G}, \mu)$, $[b]_{\mathbf{M}_J} \mapsto [b]$, is injective;*
- (2) *the image of ϕ_J consists of $[b] \in B(\mathbf{G}, \mu)$ with $\mu^\diamond - \nu_b \in \sum_{i \in J} \mathbb{R}_{\geq 0} \alpha_i^\vee$;*
- (3) *for $[b]_{\mathbf{M}_J} \in B(\mathbf{M}_J, \mu)$, $\text{length}_{\mathbf{G}}([b], [t^\mu]) = \text{length}_{\mathbf{M}_J}([b]_{\mathbf{M}_J}, [t^\mu]_{\mathbf{M}_J})$.*

Proof. Let $[b]_{\mathbf{M}_J} \in B(\mathbf{M}_J, \mu)$. Then $\mu^\diamond - \nu_b^{\mathbf{M}_J} \in \sum_{i \in J} \mathbb{R}_{\geq 0} \alpha_i^\vee$, which implies that $\nu_b^{\mathbf{M}_J}$ is dominant with respect to \mathbf{G} , and hence $\nu_b^{\mathbf{M}_J} = \nu_b$. Now the Newton point and the Kottwitz point of $[b]_{\mathbf{M}_J}$ are determined by $[b]$ and μ , respectively. Hence ϕ_J is injective.

Part (2) follows from [2, §7.1; 19, Lemma 3.5]. Part (3) follows from part (2) and Chai’s length formula Section 2B(a). □

As a consequence, we have the following.

Corollary 2.2. *Let $J \in \mathfrak{M}_\mu$. Then the map ϕ_J in Lemma 2.1 induces a bijection $B(\mathbf{M}_J, \mu)_{\text{irr}} \cong B(\mathbf{G}, \mu)_{J\text{-irr}}$.*

2D. Affine Deligne–Lusztig varieties. Let $\text{Fl} = \check{G}/\check{I}$ be an affine flag variety. For any $b \in \check{G}$ and $w \in \check{W}$, we define the corresponding affine Deligne–Lusztig variety in the affine flag variety

$$X_w(b) = \{g\check{I} \in \check{G}/\check{I} \mid g^{-1}b\sigma(g) \in \check{I}w\check{I}\} \subset \text{Fl}.$$

Let \mathbf{k} be the residue field of \check{F} . In the equal characteristic setting, the affine Deligne–Lusztig variety $X_w(b)$ is the set of \mathbf{k} -valued points of a locally closed subscheme of the affine flag variety, equipped with the reduced scheme structure. In the mixed characteristic setting, we consider $X_w(b)$ as the \mathbf{k} -valued points of a perfect scheme in the sense of Zhu [34] and Bhatt and Scholze [1], a locally closed perfect subscheme of the p -adic partial flag variety.

We denote by $\Sigma^{\text{top}}(X_w(b))$ the set of top-dimensional irreducible components of $X_w(b)$. Let \mathbf{J}_b be the σ -centralizer of b and let $\mathbf{J}_b(F) = \{g \in \check{G} \mid gb\sigma(g)^{-1} = b\}$ be the group of F -points of \mathbf{J}_b . The left action of $\mathbf{J}_b(F)$ on $X_w(b)$ induces an action of $\mathbf{J}_b(F)$ on $\Sigma^{\text{top}}(X_w(b))$.

We denote by $\mathbf{J}_b(F) \backslash \Sigma^{\text{top}}(X_w(b))$ the set of $\mathbf{J}_b(F)$ -orbits on $\Sigma^{\text{top}}(X_w(b))$.

If b and b' are σ -conjugate in \check{G} , then $X_w(b)$ and $X_w(b')$ are isomorphic. Thus the affine Deligne–Lusztig variety $X_w(b)$ (up to isomorphism) depends only on the element w in the Iwahori–Weyl group \tilde{W} and the σ -conjugacy class $[b]$ in $B(\mathbf{G})$. We set

$$B(\mathbf{G})_w = \{[b] \in B(\mathbf{G}) \mid X_w(b) \neq \emptyset\}.$$

There is a unique maximal σ -conjugacy class in $B(\mathbf{G})_w$, which we denote by $[b_w]$. By [25, Lemma 3.2], $\dim X_w(b_w) = \ell(w) - \langle \nu_{b_w}, 2\rho \rangle$. Here ρ is the half sum of the positive roots in Φ .

2E. Cordial elements. For $b \in \check{G}$, the *defect* of b is defined to be

$$\text{def}(b) = \text{rank}_F \mathbf{G} - \text{rank}_F \mathbf{J}_b,$$

where rank_F denotes the F -rank of a reductive group over F .

By [31, Theorem 3.4], we have the following length formula:

(a) For $[b] \in B(\mathbf{G}, \mu)$, $\text{length}([b], [t^\mu]) = \langle \mu - \nu_b, \rho \rangle + \frac{1}{2}\text{def}(b)$.

Let ${}^{\mathbb{S}}\tilde{W}$ be the set of minimal length representatives for the cosets in $W \backslash \tilde{W}$. Any element $w \in \tilde{W}$ can be written in a unique way as $w = xt^\mu y$ with μ dominant, $x, y \in W$ such that $t^\mu y \in {}^{\mathbb{S}}\tilde{W}$. We have $\ell(w) = \ell(x) + \langle \mu, 2\rho \rangle - \ell(y)$. In this case, we set $\eta_\sigma(w) = \sigma^{-1}(y)x$. The *virtual dimension* is defined to be

$$d_w(b) = \frac{1}{2}(\ell(w) + \ell(\eta_\sigma(w)) - \text{def}(b) - \langle \nu_b, 2\rho \rangle).$$

For $w = t^\mu y \in {}^{\mathbb{S}}\tilde{W}$, it is easy to see that $d_w(b) = \langle \mu - \nu_b, \rho \rangle - \frac{1}{2}\text{def}(b)$.

By [12, Theorem 10.3; 13, Theorem 2.30], we have

(b) for $w \in \tilde{W}$ and $b \in \check{G}$, $\dim X_w(b) \leq d_w(b)$.

In the special case $[b] = [b_w]$, (b) implies that

$$\ell(w) - \ell(\eta_\sigma(w)) \leq \langle \nu_{b_w}, 2\rho \rangle - \text{def}(b_w).$$

Cordial elements were introduced by Milićević and Viehmann in [25]. By definition, an element $w \in \tilde{W}$ is *cordial* if $\dim X_w(b_w) = d_w(b_w)$. This condition is equivalent to the condition that $\ell(w) - \ell(\eta_\sigma(w)) = \langle \nu_{b_w}, 2\rho \rangle - \text{def}(b_w)$. The following nice properties of the cordial elements are established in [25].

Theorem 2.3. *Let $w \in \tilde{W}$ be a cordial element. Then*

- (1) $B(\mathbf{G})_w$ is saturated, that is, if $[b_1] \leq [b_2] \leq [b_3]$ in $B(\mathbf{G})$ and $[b_1], [b_3] \in B(\mathbf{G})_w$, then $[b_2] \in B(\mathbf{G})_w$;
- (2) for each $[b] \in B(\mathbf{G})_w$, $X_w(b)$ is equidimensional of dimension equal to $d_w(b)$.

2F. Minimal length elements. We consider the σ -conjugation action on \tilde{W} defined by $w \cdot_{\sigma} w' = ww'\sigma(w)^{-1}$. Let $B(\tilde{W}, \sigma)$ be the set of σ -conjugacy classes of \tilde{W} . For any σ -conjugacy class \mathcal{O} of \tilde{W} , we let \mathcal{O}_{\min} be the set of minimal length elements in \mathcal{O} , and we write $\ell(\mathcal{O}) = \ell(w)$ for any $w \in \mathcal{O}_{\min}$.

For $w, w' \in \tilde{W}$ and $i \in \tilde{S}$, we write $w \xrightarrow{s_i}_{\sigma} w'$ if $w' = s_i w \sigma(s_i)$ and $\ell(w') \leq \ell(w)$. We write $w \rightarrow_{\sigma} w'$ if there is a sequence $w = w_0, w_1, \dots, w_n = w'$ of elements in \tilde{W} such that for any $k, w_{k-1} \xrightarrow{s_i}_{\sigma} w_k$ for some $s_i \in \tilde{S}$. We write $w \approx_{\sigma} w'$ if $w \rightarrow_{\sigma} w'$ and $w' \rightarrow_{\sigma} w$.

We call $w, w' \in \tilde{W}$ *elementarily strongly σ -conjugate* if $\ell(w) = \ell(w')$ and there exists $x \in \tilde{W}$ such that $w' = xw\sigma(x)^{-1}$ and $\ell(xw) = \ell(x) + \ell(w)$ or $\ell(w\sigma(x)^{-1}) = \ell(x) + \ell(w)$. We call w, w' *strongly σ -conjugate* if there is a sequence $w = w_0, w_1, \dots, w_n = w'$ such that for each i, w_{i-1} is elementarily strongly σ -conjugate to w_i . We write $w \sim_{\sigma} w'$ if w and w' are strongly σ -conjugate.

The following result is proved in [17, Theorem 2.10].

Theorem 2.4. *Let \mathcal{O} be a σ -conjugacy class in \tilde{W} . Then the following hold:*

- (1) *For each element $w \in \mathcal{O}$, there exists $w' \in \mathcal{O}_{\min}$ such that $w \rightarrow_{\sigma} w'$.*
- (2) *Let $w, w' \in \mathcal{O}_{\min}$. Then $w \sim_{\sigma} w'$.*

2G. Decompositions of affine Deligne–Lusztig varieties. We recall the Deligne–Lusztig reduction method on affine Deligne–Lusztig varieties.

Proposition 2.5 [20, Proposition 3.3.1]. *Let $w \in \tilde{W}, i \in \tilde{S}$, and $b \in \check{G}$. If $\text{char}(F) > 0$, then the following two statements hold:*

- (1) *If $\ell(s_i w \sigma(s_i)) = \ell(w)$, then there exists a $\mathbf{J}_b(F)$ -equivariant morphism $X_w(b) \rightarrow X_{s_i w \sigma(s_i)}(b)$ which is a universal homeomorphism.*
- (2) *If $\ell(s_i w \sigma(s_i)) = \ell(w) - 2$, then $X_w(b) = X_1 \sqcup X_2$, where X_1 is a $\mathbf{J}_b(F)$ -stable open subscheme X of $X_w(b)$ and X_2 is its closed complement satisfying the following conditions:*
 - X_1 is $\mathbf{J}_b(F)$ -equivariant universally homeomorphic to a Zariski-locally trivial \mathbb{G}_m -bundle over $X_{s_i w \sigma(s_i)}(b)$.
 - X_2 is $\mathbf{J}_b(F)$ -equivariant universally homeomorphic to a Zariski-locally trivial \mathbb{A}^1 -bundle over $X_{s_i w \sigma(s_i)}(b)$.

If $\text{char}(F) = 0$, then the above two statements still hold, but with \mathbb{A}^1 and \mathbb{G}_m replaced by the perfection $\mathbb{A}^{1, \text{perf}}$ and $\mathbb{G}_m^{\text{perf}}$, respectively.

For any $a_1, a_2 \in \mathbb{N}$, we say that a scheme X is an *iterated fibration* of type (a_1, a_2) over a scheme Y if there exist morphisms

$$X = Y_0 \rightarrow Y_1 \rightarrow \dots \rightarrow Y_{a_1+a_2} = Y$$

such that for any i with $0 \leq i < a_1 + a_2$, Y_i is a Zariski-locally trivial $\mathbb{A}^{1, (\text{perf})}$ -bundle or $\mathbb{G}_m^{(\text{perf})}$ -bundle over Y_{i+1} , and there are exactly a_1 locally trivial $\mathbb{G}_m^{(\text{perf})}$ -bundles in the sequence. In this case, there are exactly a_2 locally trivial $\mathbb{A}^{1, (\text{perf})}$ -bundles in the sequence.

2H. Classical Deligne–Lusztig varieties. Let \mathbb{F}_q be a finite field and let $\bar{\mathbb{F}}_q$ an algebraic closure of \mathbb{F}_q . Let H be a connected reductive group over \mathbb{F}_q and $H = H(\bar{\mathbb{F}}_q)$. Let σ_H be the Frobenius morphism on H . Let B be a σ_H -stable Borel subgroup of H . Let W_H be the Weyl group of H and let \mathbb{S}_H be the set of simple reflections. Then σ_H induces a group automorphism on W_H preserving \mathbb{S}_H . (Classical) Deligne–Lusztig varieties were introduced in [3]. They are defined as follows. For $x \in W_H$, we set

$$X_x^H = \{hB \in H/B \mid h^{-1}\sigma(h) \in Bx B\}.$$

If x is a σ_H -Coxeter element of W_H , then we say that X_x^H is a *classical Deligne–Lusztig variety of Coxeter type* for H . It is well known that classical Deligne–Lusztig varieties of Coxeter type are irreducible.

It is proved in [12, Theorem 4.8] that if $w \in \tilde{W}$ is a minimal length element in its σ -conjugacy class, then $X_w(b) \neq \emptyset$ if and only if b and \check{w} are in the same σ -conjugacy class of \check{G} . In this case,

$$X_w(b) \cong J_b(F) \times^P X. \tag{2-1}$$

Here \cong means a $J_b(F)$ -equivariant universal homeomorphism, P is a parahoric subgroup of $J_b(F)$, and X is (the perfection of) a classical Deligne–Lusztig variety for some connected reductive group H over \mathbb{F}_q with $H(\mathbb{F}_q)$ isomorphic to the reductive quotient of P . The group P acts on $J_b(F) \times X$ by $p \cdot (g, z) = (gp^{-1}, p \cdot z)$, and $J_b(F) \times^P X$ is the quotient space.

2I. Very special parahoric subgroups. We follow [20, §2] for the definition of very special parahoric subgroups.

Let G_1 be a (not necessarily quasisplit) connected reductive group over F and let σ_1 be its Frobenius morphism. Let \tilde{W}_1 be the Iwahori–Weyl group G_1 . We still denote by σ_1 the action on \tilde{W}_1 induced from the Frobenius morphism on G_1 . Let $\tilde{\mathbb{S}}_1$ be the set of simple reflections in \tilde{W}_1 .

A parahoric subgroup P of $G_1(F)$ is called *very special* if it is of maximal volume among all the parahoric subgroups of $G_1(F)$. A σ_1 -stable subset $\check{K} \subseteq \tilde{\mathbb{S}}_1$ is called *very special* with respect to σ_1 if the parabolic subgroup $W_{\check{K}}$ generated by \check{K} is finite and the longest element of $W_{\check{K}}$ is of maximal length among all such σ_1 -stable subsets of $\tilde{\mathbb{S}}_1$. By [20, Proposition 2.2.5],

- (a) a σ_1 -stable subset \check{K} of $\tilde{\mathbb{S}}_1$ is very special if and only if $\check{P}_{\check{K}} \cap G_1(F)$ is a very special parahoric subgroup of $G_1(F)$, where $\check{P}_{\check{K}}$ is the parahoric subgroup of $G_1(\check{F})$ corresponding to \check{K} .

2J. Statement of the main result. Let $w \in \tilde{W}$. We say that w has *finite σ -Coxeter part* if $\eta_\sigma(w)$ is a partial σ -Coxeter element. In this case, we set $J(w) = \text{supp}_\sigma(\eta_\sigma(w))$. For any σ -stable subset J of $J(w)$, denote by J'_μ (resp. J''_μ) the union of all σ -orbits of connected components of J in which μ is noncentral (resp. central). Then μ is essentially noncentral in J'_μ . By Section 2C, $B(G, \mu)_{J'_\mu\text{-irr}} \neq \emptyset$, and we have a natural bijection $B(M_{J'_\mu}, \mu)_{\text{irr}} \cong B(G, \mu)_{J'_\mu\text{-irr}}$.

Let $J_0(w)$ be the subset of \mathbb{S} with $\mu - \nu_{b_w} \in \sum_{i \in J_0(w)} \mathbb{R}_{>0} \alpha_i^\vee$. Then $J_0(w) = J_0(w)'_\mu$. By definition, $[\check{w}] \leq [b_w]$. Thus $\nu_{\check{w}} \leq \nu_{b_w}$ and $J_0(w) \subseteq J(w)$.

For any σ -stable $J_1, J_2 \subseteq \mathbb{S}$, denote by $[J_1, J_2]$ the set of σ -stable subsets $J \subseteq \mathbb{S}$ such that $J_1 \subseteq J \subseteq J_2$ and denote by $[J_1, J_2]_\mu$ the set of $J \in [J_1, J_2]$ such that μ is essentially noncentral in J (or equivalently, $J = J'_\mu$).

Now we state our main result.

Theorem 2.6. *Let $w \in Wt^\mu W$ such that $\eta_\sigma(w)$ is a partial σ -Coxeter element. Then*

- (1) w is a cordial element;
- (2) $B(\mathbf{G})_w = \bigsqcup_{J \in [J_0(w), J(w)]_\mu} B(\mathbf{G}, \mu)_{J\text{-irr}}$;
- (3) for any $[b] \in B(\mathbf{G})_w$, we have

$$X_w(b) \cong \mathbf{J}_b(F) \times^P Y,$$

where P is a parahoric subgroup of $\mathbf{J}_b(F)$, and Y is an iterated fibration, whose iterated fibers are either \mathbb{A}^1 or \mathbb{G}_m , over (the perfection of) a classical Deligne–Lusztig variety of Coxeter type for some connected reductive group \mathbf{H} over \mathbb{F}_q with $\mathbf{H}(\mathbb{F}_q)$ isomorphic to the reductive quotient of P .

- Remark 2.7.** (1) In particular, $\mathbf{J}_b(F)$ acts transitively on the set of irreducible components of $X_w(b)$.
 (2) Combining Remark 3.10 with Theorem 7.1, we have a detailed description of the classical Deligne–Lusztig variety in Theorem 2.6(3). If $w \in {}^S\tilde{W}$ and $\eta_\sigma(w)$ is a (full) σ -Coxeter element, then the parahoric subgroup P in Theorem 2.6(3) is very special; see Section 5E.
 (3) In Theorem 7.1, we explicitly compute the numbers of \mathbb{G}_m -bundles and \mathbb{A}^1 -bundles appearing in the iterated fibration.

Parts (1) and (2) will be proved in Section 4. Part (3) is the most difficult part of this paper and will be proved in Sections 6 and 7. The proof is based on a deep analysis of the reduction tree of w , which will be introduced in Section 3.

3. Class polynomials and reduction trees

We recall the class polynomials of Hecke algebras and the connection with affine Deligne–Lusztig varieties discovered in [12]. We then introduce the reduction tree, which encodes more information than the class polynomials.

3A. Hecke algebras and their cocenters. Let q be an indeterminate. Let H be the Hecke algebra associated with \tilde{W} , that is, it is the $\mathbb{Z}[q^{\pm 1}]$ -algebra generated by T_w for $w \in \tilde{W}$ subject to the relations

- $T_w T_{w'} = T_{ww'}$ if $\ell(ww') = \ell(w) + \ell(w')$;
- $(T_{s_i} + 1)(T_{s_i} - q) = 0$ for $i \in \tilde{S}$.

The action of σ on \tilde{W} induces an action on H , which we still denote by σ . The σ -commutator $[H, H]_\sigma$ is the $\mathbb{Z}[q^{\pm 1}]$ -submodule generated by $hh' - h'\sigma(h)$ for $h, h' \in H$. The σ -cocenter of H is defined to be

$$\bar{H}_\sigma = H/[H, H]_\sigma.$$

By Theorem 2.4(2), for any σ -conjugacy class \mathcal{O} of \tilde{W} and $w, w' \in \mathcal{O}_{\min}$, we have $T_w + [H, H]_\sigma = T_{w'} + [H, H]_\sigma$. We write $T_{\mathcal{O}} = T_w + [H, H]_\sigma \in \tilde{H}_\sigma$ for any $w \in \mathcal{O}_{\min}$.

Theorem 3.1 [17, Theorem 6.7]. \tilde{H}_σ is a free $\mathbb{Z}[q^{\pm 1}]$ -module with basis $\{T_{\mathcal{O}}\}_{\mathcal{O} \in B(\tilde{W}, \sigma)}$.

By [12, §2.3; 13, §2.8.2], for any $w \in \tilde{W}$ and $\mathcal{O} \in B(\tilde{W}, \sigma)$, there exists a unique polynomial $F_{w, \mathcal{O}} \in \mathbb{N}[q - 1]$ such that

$$T_w + [H, H]_\sigma = \sum_{\mathcal{O} \in B(\tilde{W}, \sigma)} F_{w, \mathcal{O}} T_{\mathcal{O}} \in \tilde{H}_\sigma.$$

The polynomials $F_{w, \mathcal{O}}$ are called *class polynomials*.¹

3B. Class polynomials and affine Deligne–Lusztig varieties. The class polynomials encode a lot of information about affine Deligne–Lusztig varieties.

Let $\tilde{W} \rightarrow B(\mathbf{G})$ be the map sending $w \in \tilde{W}$ to the σ -conjugacy class $[\dot{w}]$ of \check{G} . It is known that this map is independent of the choice of the representative \dot{w} of w , and it induces a map

$$\Psi : B(\tilde{W}, \sigma) \rightarrow B(\mathbf{G}).$$

By [12, Theorem 3.7], the map Ψ is surjective.

Let $w \in \tilde{W}$ and $[b] \in B(\mathbf{G})$. We set

$$F_{w, [b]} = \sum_{\substack{\mathcal{O} \in B(\tilde{W}, \sigma) \\ \Psi(\mathcal{O}) = [b]}} q^{\ell(\mathcal{O})} F_{w, \mathcal{O}} \in \mathbb{N}[q - 1].$$

Here $\ell(\mathcal{O}) = \ell(x)$ for any $x \in \mathcal{O}_{\min}$.

The following “dimension = degree” theorem is established in [12, Theorem 6.1].

Theorem 3.2. Let $w \in \tilde{W}$ and $b \in \check{G}$. Then $\dim X_w(b) = \deg F_{w, [b]} - \langle \nu_b, 2\rho \rangle$.

Remark 3.3. Here, by convention, $\dim \emptyset = \deg 0 = -\infty$.

We have the following “leading coefficients = irreducible components” theorem. This is established in [13, Theorem 2.19]. See also [20, Theorem 3.3.9 and Corollary 3.3.11].

Theorem 3.4. For $w \in \tilde{W}$ and $b \in \check{G}$, the cardinality of $\mathbf{J}_b(F) \setminus \Sigma^{\text{top}}(X_w(b))$ equals the leading coefficient of $F_{w, [b]}$.

Although not needed in this paper, it is also worth mentioning that in the superbasic case, the class polynomial gives the number of rational points of an affine Deligne–Lusztig variety. This is established in [12, Proposition 8.3].

¹The polynomials we use here coincide with those in [13] and differ from the polynomials used in [12] by a certain monomial. See [13, footnote on p. 106].

Proposition 3.5. *Suppose that the residue field of F is \mathbb{F}_q . Assume that $\mathbf{G} = \mathrm{PGL}_n$ split over F and $b \in \mathbf{G}(F)$ is a superbasic element in \check{G} . Then*

$$\sharp X_w(b)^\sigma = nF_{w,[b]}|_{q=q}.$$

3C. An identity on the class polynomials. We have the following identity on the class polynomials. We first give a proof using representations of Hecke algebras. Then we provide a geometric interpretation of this identity.

Proposition 3.6. *Let $w \in \check{W}$. Then*

$$\mathbf{q}^{\ell(w)} = \sum_{\mathcal{O} \in B(\check{W}, \sigma)} \mathbf{q}^{\ell(\mathcal{O})} F_{w, \mathcal{O}} = \sum_{[b] \in B(\mathbf{G})} F_{w, [b]}.$$

Proof. We prove the first equality. The second follows from the definition.

Let $\pi : H \rightarrow \mathbb{Z}[\mathbf{q}^{\pm 1}]$ be the homomorphism of $\mathbb{Z}[\mathbf{q}^{\pm 1}]$ -algebras sending T_{s_i} to \mathbf{q} for any $i \in \check{S}$. As $\pi \circ \sigma = \pi$, $\pi([H, H]_\sigma) = 0$ and thus π induces a homomorphism of the $\mathbb{Z}[\mathbf{q}^{\pm 1}]$ -modules $\bar{H}_\sigma \rightarrow \mathbb{Z}[\mathbf{q}^{\pm 1}]$, which we still denote by π .

We have $T_w + [H, H]_\sigma = \sum_{\mathcal{O} \in B(\check{W}, \sigma)} F_{w, \mathcal{O}} T_{\mathcal{O}}$. Applying π to both sides, we obtain $\mathbf{q}^{\ell(w)} = \sum_{\mathcal{O} \in B(\check{W}, \sigma)} \mathbf{q}^{\ell(\mathcal{O})} F_{w, \mathcal{O}}$. □

In the rest of this subsection, we assume that $F = \mathbb{F}_q((\epsilon))$ and that \mathbf{G} is split over F . We give a geometric interpretation of the above identity.

For any $n \in \mathbb{N}$, let \check{I}_n be the n -th congruence subgroup of \check{I} . Following [5, §2.10], we call a subset X of \check{G} *admissible* if for any $w \in \check{W}$, there exists $n \in \mathbb{N}$ such that $X \cap \check{I} \dot{w} \check{I}$ is stable under right multiplication of \check{I}_n . In this case, the action of \check{I}_n on $X \cap \check{I} \dot{w} \check{I}$ is free, and $\frac{\sharp((X \cap \check{I} \dot{w} \check{I}) / \check{I}_n)^\sigma}{\sharp(\check{I} / \check{I}_n)^\sigma}$ is independent of the choice of such n . We set

$$\sharp_{\check{I}}(X \cap \check{I} \dot{w} \check{I}) = \frac{\sharp((X \cap \check{I} \dot{w} \check{I}) / \check{I}_n)^\sigma}{\sharp(\check{I} / \check{I}_n)^\sigma}.$$

An admissible subset X is called *bounded* if $X \cap \check{I} \dot{w} \check{I} = \emptyset$ for all but finitely many $w \in \check{W}$. For any bounded admissible subset X , we set

$$\sharp_{\check{I}} X = \sum_{w \in \check{W}} \sharp_{\check{I}}(X \cap \check{I} \dot{w} \check{I})$$

and call it the *normalized cardinality of the rational points* of X . By [14, Theorem A.1], each σ -conjugacy class of \check{G} is admissible. We have the following geometric interpretation of the class polynomials.

Proposition 3.7. *Let $w \in \check{W}$ and $[b] \in B(\mathbf{G})$. Then*

$$\sharp_{\check{I}}([b] \cap \check{I} \dot{w} \check{I}) = F_{w, [b]}|_{q=q}.$$

Proof. We argue by induction on $\ell(w)$.

If w is a minimal length element in its σ -conjugacy class in \tilde{W} , then

$$[b] \cap \check{I}\check{w}\check{I} = \begin{cases} \check{I}\check{w}\check{I} & \text{if } [b] = [\check{w}], \\ \emptyset & \text{otherwise.} \end{cases}$$

On the other hand, by [13, §2.8.2],

$$F_{w,[b]} = \begin{cases} q^{\ell(w)} & \text{if } [b] = [\check{w}], \\ 0 & \text{otherwise.} \end{cases}$$

Thus the proposition holds in this case.

Now we assume that w is not a minimal length element in its σ -conjugacy class. By Theorem 2.4(1), there exist $w' \in \tilde{W}$ and $i \in \tilde{S}$ such that $w \approx_\sigma w'$ and $s_i w' \sigma(s_i) < w'$. Set $w_1 = s_i w'$ and $w_2 = s_i w' \sigma(s_i)$. Then $\ell(w_1), \ell(w_2) < \ell(w)$. By the proof of [13, Theorem 2.16],

$$\sharp_{\check{I}}([b] \cap \check{I}\check{w}\check{I}) = \sharp_{\check{I}}([b] \cap \check{I}\check{w}'\check{I}) = (q - 1)\sharp_{\check{I}}([b] \cap \check{I}\check{w}_1\check{I}) + q\sharp_{\check{I}}([b] \cap \check{I}\check{w}_2\check{I}).$$

On the other hand, by [13, §2.8.2], $F_{w,[b]} = (q - 1)F_{w_1,[b]} + qF_{w_2,[b]}$. Now the statement for w follows from the inductive hypothesis on w_1 and w_2 . □

We have the decomposition $\check{I}\check{w}\check{I} = \bigsqcup_{[b] \in B(\mathcal{G})} [b] \cap \check{I}\check{w}\check{I}$. Thus

$$q^{\ell(w)} = \sharp_{\check{I}}(\check{I}\check{w}\check{I}) = \sum_{[b] \in B(\mathcal{G})} \sharp_{\check{I}}([b] \cap \check{I}\check{w}\check{I}) = \sum_{[b] \in B(\mathcal{G})} F_{w,[b]}.$$

This gives an alternative proof of Proposition 3.6 and a geometric interpretation of Proposition 3.6 in the case where the σ -action on \tilde{W} is trivial via counting (the normalized cardinality of) the rational points of $\check{I}\check{w}\check{I}$.

3D. Reduction trees. Let $w \in \tilde{W}$. We construct the reduction tree for w , which encodes the Deligne–Lusztig reduction for the affine Deligne–Lusztig varieties associated with w (and with all $b \in \check{G}$).

The vertices of the graphs are the elements of \tilde{W} , and the (oriented) edges are of the form $x \rightarrow y$, where $x, y \in \tilde{W}$, and there exist $x' \in \tilde{W}$ and $i \in \tilde{S}$ with $x \approx_\sigma x'$, $s_i x' \sigma(s_i) < x'$ and $y \in \{s_i x', s_i x' \sigma(s_i)\}$. Some elements of \tilde{W} may occur more than once in a reduction tree.

The reduction trees are constructed inductively.

Suppose that w is of minimal length in its σ -conjugacy class of \tilde{W} . Then the reduction tree of w consists of a single vertex w and no edges.

Suppose that w is not of minimal length in its σ -conjugacy class of \tilde{W} and that a reduction tree is given for any $z \in \tilde{W}$ with $\ell(z) < \ell(w)$. By Theorem 2.4(1), there exist $w' \in \tilde{W}$ and $i \in \tilde{S}$ with $w \approx_\sigma w'$ and $s_i w' \sigma(s_i) < w'$. The reduction tree of w is the graph containing the given reduction tree for $s_i w'$ and the reduction tree for $s_i w' \sigma(s_i)$, and the edges $w \rightarrow s_i w'$ and $w \rightarrow s_i w' \sigma(s_i)$.

The reduction trees of w are not unique. They depend on the choices of w' and s_i in the construction. We will see in the rest of this section that the reduction trees encode more information than the class polynomials.

3E. Reduction path. Let \mathcal{T} be a reduction tree of w . An *end point* of the tree \mathcal{T} is a vertex x of \mathcal{T} such that there is no edge of the form $x \rightarrow x'$ in \mathcal{T} . By [Theorem 2.4](#), each end point is of minimal length in its σ -conjugacy class. A *reduction path* in \mathcal{T} is a path $\underline{p} : w \rightarrow w_1 \rightarrow \cdots \rightarrow w_n$, where w_n is an end point of \mathcal{T} . The *length* $\ell(\underline{p})$ of the reduction path \underline{p} is the number of edges in \underline{p} . We also write $\text{end}(\underline{p}) = w_n$ and $[b]_{\underline{p}} = \Psi(\text{end}(\underline{p})) \in B(G)$.

If $x \rightarrow y$, then $\ell(x) - \ell(y) \in \{1, 2\}$. We say that the edge $x \rightarrow y$ is of type I if $\ell(x) - \ell(y) = 1$ and of type II if $\ell(x) - \ell(y) = 2$. For any reduction path \underline{p} , we denote by $\ell_I(\underline{p})$ the number of type-I edges in \underline{p} and by $\ell_{II}(\underline{p})$ the number of type-II edges in \underline{p} . Then $\ell(\underline{p}) = \ell_I(\underline{p}) + \ell_{II}(\underline{p})$.

The following relation between class polynomials and reduction trees follows easily from the inductive construction, and we omit the details of its proof.

Lemma 3.8. *Let $w \in \tilde{W}$ and let \mathcal{T} be a reduction tree of w . Then, for any σ -conjugacy class \mathcal{O} of \tilde{W} ,*

$$F_{w, \mathcal{O}} = \sum_{\underline{p}} (q - 1)^{\ell_I(\underline{p})} q^{\ell_{II}(\underline{p})},$$

where \underline{p} runs over all the reduction paths in \mathcal{T} with $\Psi(\text{end}(\underline{p})) = \mathcal{O}$.

Combining [Proposition 2.5](#) with the construction of the reduction trees, we obtain the following result.

Proposition 3.9. *Let $w \in \tilde{W}$ and \mathcal{T} be a reduction tree of w . Then, for any $b \in \check{G}$, there exists a decomposition*

$$X_w(b) = \bigsqcup_{\substack{\underline{p} \text{ is a reduction path of } \mathcal{T} \\ [b]_{\underline{p}} = [b]}} X_{\underline{p}},$$

where $X_{\underline{p}}$ is a locally closed subscheme of $X_w(b)$ and is $\mathbf{J}_b(F)$ -equivariant universally homeomorphic to an iterated fibration of type $(\ell_I(\underline{p}), \ell_{II}(\underline{p}))$ over $X_{\text{end}(\underline{p})}(b)$.

Remark 3.10. Since $\text{end}(\underline{p})$ is a minimal length element in its σ -conjugacy class, by [\(2-1\)](#) we have $X_{\text{end}(\underline{p})}(b) \cong \mathbf{J}_b(F) \times^P X$, where P is a parahoric subgroup of $\mathbf{J}_b(F)$ and X is (the perfection of) an irreducible component of a classical Deligne–Lusztig variety. Thus each irreducible component Y of $X_{\underline{p}}$ is universally homeomorphic to an iterated fibration of type $(\ell_I(\underline{p}), \ell_{II}(\underline{p}))$ over X . We have a natural action of $\mathbf{J}_b(F)$ on $X_w(b)$, and $X_{\underline{p}}$ is stable under this action. In this case, $X_{\underline{p}} \cong \mathbf{J}_b(F) \times^P Y$.

4. Cordiality and the set $B(G)_w$

4A. Maximal Hodge–Newton irreducible elements. Recall that $[b_{t^\mu}]$ is the unique maximal element of $B(G, \mu)$. By [\[32, Corollary 7.6\]](#), there is a unique maximal element $[b_{\mu, G\text{-indec}}]$ in $B(G, \mu)_{\text{indec}}$, whose Newton point is denoted by $v_{b_{\mu, G\text{-indec}}}$. We give an explicit description of $v_{b_{\mu, G\text{-indec}}}$ below.

Following [\[2\]](#), for any subset $E \subseteq (V^+)^\sigma$, we set

$$C_{\geq E} = \{v \in (V^+)^\sigma \mid v \geq v', \forall v' \in E\}.$$

By [\[2, Theorem 6.5\]](#), $C_{\geq E}$ has a unique minimal element, which we denote by $\min C_{\geq E}$.

Proposition 4.1. *We have $v_{b_{\mu, G\text{-indec}}} = \min C_{\geq E_0}$, where $E_0 = \{e_i \mid i \in \mathbb{S}\}$ with $e_i \in \mathbb{R}\omega_i^\vee$ such that $\langle e_i, \omega_i \rangle = \frac{1}{\sharp \mathcal{O}_i} \max\{0, \langle \mu, \omega_{\mathcal{O}_i} \rangle - 1\}$.*

Proof. By [2, Theorem 6.5], there exists a unique σ -conjugacy class $[b] \in B(\mathbf{G}, \mu)$ such that $v_b = \min C_{\geq E_0}$ and $\langle \mu - v_b, \omega_{\mathcal{O}_k} \rangle = 1$ for any $k \in \mathbb{S} - I(v)$.

Suppose that $(\mu, [b])$ is Hodge–Newton decomposable with respect to some standard Levi subgroup \mathbf{M}_J with $J = \sigma(J) \subsetneq \mathbb{S}$. Let $j \in \mathbb{S} - J$. By definition, $\langle \mu - v_b, \omega_{\mathcal{O}_j} \rangle = 0$. On the other hand, as $I(v_b) \subseteq J$ we have $\langle v_b, \alpha_j \rangle \neq 0$. Thus $\langle \mu - v_b, \omega_{\mathcal{O}_j} \rangle = 1$, which is a contradiction. Therefore $[b] \in B(\mathbf{G}, \mu)_{\text{indec}}$.

On the other hand, let $[b'] \in B(\mathbf{G}, \mu)_{\text{indec}}$. For any $i \in \mathbb{S} - I(v_{b'})$, we denote by $\text{pr}_{(i)} : V = \mathbb{R}\omega_i^\vee \oplus \sum_{j \neq i} \mathbb{R}\alpha_j^\vee \rightarrow \mathbb{R}\omega_i^\vee$ the natural projection. Set $e'_i = \text{pr}_{(i)}(v_{b'}) \in \mathbb{R}\omega_i^\vee$. Let $E' = \{e'_i \mid i \in \mathbb{S} - I(v_{b'})\}$. Again by [2, Theorem 6.5], we have $v_{b'} = \min C_{\geq E'}$. By Section 2B(a), we have $\langle \mu - v_{b'}, \omega_{\mathcal{O}_i} \rangle \in \mathbb{Z}_{\geq 1}$ for $i \in \mathbb{S} - I(v')$. This means that $e'_i \leq e_i$ for $i \in \mathbb{S} - I(v')$. So $v_b \in C_{\geq E'}$ and $v_b \geq \min C_{\geq E'} = v_{b'}$. Hence $[b]$ is the unique maximal element of $B(\mathbf{G}, \mu)_{\text{indec}}$. □

Corollary 4.2. *Suppose that μ is essentially noncentral. Then*

$$\text{length}([b_{\mu, G\text{-indec}}], [t^\mu]) = \sharp(\mathbb{S}/\langle \sigma \rangle).$$

Proof. As μ is essentially noncentral, we have $B(\mathbf{G}, \mu)_{\text{indec}} = B(\mathbf{G}, \mu)_{\text{irr}}$ and hence $0 < \langle \mu - v_{b_{\mu, G\text{-indec}}}, \omega_{\mathcal{O}_i} \rangle$ for any $i \in \mathbb{S}$. On the other hand, we have $\langle \mu - v_{b_{\mu, G\text{-indec}}}, \omega_{\mathcal{O}_i} \rangle \leq 1$ by Proposition 4.1. Then the statement follows directly from Section 2B(a). □

4B. Proof of Theorem 2.6(1) and (2) for $w = t^\mu c$. Now we prove Theorem 2.6(1) and (2) in the case $w = t^\mu c \in {}^{\mathbb{S}}\tilde{W}$, where c is a partial σ -Coxeter element. If μ is central over some connected components of $\text{supp}_\sigma(c)$, then the element $t^\mu c$ would not be in ${}^{\mathbb{S}}\tilde{W}$. Thus μ is essentially noncentral in $\text{supp}_\sigma(c)$. Set $J = \text{supp}_\sigma(c)$. Reviewing the definition of $J(w)$ in Section 2J, we have $J(w) = J(w)'_\mu = J_0(w) = J$. We need to show that w is cordial and $B(\mathbf{G})_w = B(\mathbf{G}, \mu)_{J\text{-irr}}$.

By Section 2C, we have a natural bijection $B(\mathbf{M}_J, \mu)_{\text{irr}} \cong B(\mathbf{G}, \mu)_{J\text{-irr}}$. By [6, Theorem 3.3.1], we have $B(\mathbf{G})_w = B(\mathbf{M}_J)_w$. Note that $\mu^\diamond - v_{b_w} \in \sum_{j \in J} \mathbb{R}\alpha_j^\vee$. Hence $\langle \mu - v_{b_w}, \rho \rangle = \langle \mu - v_{b_w}, \rho_J \rangle$, where ρ_J is the half sum of positive roots of \mathbf{M}_J . By definition, w is cordial in \mathbf{G} if and only if it is cordial in \mathbf{M}_J . Hence we may assume that $J = \mathbb{S}$ and that c is a σ -Coxeter element.

We first show that

(a) $B(\mathbf{G})_w \subseteq B(\mathbf{G}, \mu)_{\text{indec}}$.

Suppose that $X_w(b) \neq \emptyset$ and (μ, b) is Hodge–Newton decomposable with respect to some proper standard Levi subgroup \mathbf{M} . By [7, Theorem 1.11], there is some $u \in W$ such that $u^{-1}w\sigma(u)$ lies in $\tilde{W}_{\mathbf{M}}$, which contradicts the fact that c is σ -Coxeter. Thus (a) is proved.

Next we show that

(b) $[b_w] = [b_{\mu, G\text{-indec}}]$ and w is a cordial element.

By Section 2E(b), we have

$$\langle \mu, 2\rho \rangle - 2\ell(c) = \ell(w) - \ell(\eta_\sigma(w)) \leq \langle \nu_{b_w}, 2\rho \rangle - \text{def}(b_w).$$

Combined with Section 2E(a), we get $\text{length}([b_w], [t^\mu]) \leq \ell(c) = \sharp(\mathbb{S}/\langle \sigma \rangle)$. However, by Corollary 4.2, we have $\text{length}([b_w], [b_{t^\mu}]) \geq \text{length}([b_{\mu, G\text{-indec}}], [b_{t^\mu}]) = \sharp(\mathbb{S}/\langle \sigma \rangle)$. Thus we must have $[b_w] = [b_{\mu, G\text{-indec}}]$ and $\dim X_w(b_w) = d_w(b_w)$. Thus (b) is proved.

Finally, we show that

$$(c) \quad B(\mathbf{G})_w = B(\mathbf{G}, \mu)_{\text{indec}} = B(\mathbf{G}, \mu)_{\text{irr}}.$$

Let $[b_{\min}]$ be the unique basic σ -conjugacy class in $B(\mathbf{G})$ with $\kappa(b_{\min}) = \kappa(\mu)$. Then $[b_{\min}]$ is the unique minimal element in $B(\mathbf{G}, \mu)_{\text{indec}}$. Note that c is a σ -Coxeter element of W . Thus $\nu_{\tilde{w}}$ is central and $[\tilde{w}] = [b_{\min}]$. In particular, $[b_{\min}] \in B(\mathbf{G})_w$. By Theorem 2.3, $B(\mathbf{G})_w$ is saturated and hence must be equal to $B(\mathbf{G}, \mu)_{\text{indec}}$.

This completes the proof of the $t^\mu c$ case.

4C. Partial conjugation. To handle the general case, we use the partial conjugation method introduced in [11].

By partial conjugation, we mean conjugating by elements in the finite Weyl group W . For any $x \in {}^{\mathbb{S}}\tilde{W}$, set

$$I(x) = \max\{J \subseteq \mathbb{S} \mid \text{Ad}(x)\sigma(J) = J\}.$$

This is well-defined. Indeed, if $\text{Ad}(x)\sigma(J_i) = J_i$ for $i = 1, 2$, then $\text{Ad}(x)\sigma(J_1 \cup J_2) = J_1 \cup J_2$. Let $W_{I(x)}$ be the subgroup of W generated by the simple reflections in $I(x)$. Then $\text{Ad}(x) \circ \sigma$ gives a length-preserving group automorphism on $W_{I(x)}$. By [11, Proposition 2.4], we have

$$\tilde{W} = \bigsqcup_{x \in {}^{\mathbb{S}}\tilde{W}} W \cdot_\sigma (W_{I(x)}x) = \bigsqcup_{x \in {}^{\mathbb{S}}\tilde{W}} W \cdot_\sigma (xW_{\sigma(I(x))}).$$

Moreover, by [11, Proposition 3.4], we have the following:

- (a) For any $w \in \tilde{W}$, there exist $x \in {}^{\mathbb{S}}\tilde{W}$ and $u \in W_{I(x)}$ such that $w \rightarrow_\sigma ux$ and all the simple reflections involved in the conjugations are in \mathbb{S} .

By [12, Proposition 4.9], we have the following:

- (b) Let $x \in {}^{\mathbb{S}}\tilde{W}$ and $u \in W_{I(x)}$. Then $B(G)_{ux} = B(G)_x$ and $\dim X_{ux}(b) = \dim X_x(b) + \ell(u)$ for any $[b] \in B(\mathbf{G})_x$.

Similar to Section 3D, we may consider partial reductions. By partial reduction, we mean reduction $w \rightarrow s_i w$ or $w \rightarrow s_i w \sigma(s_i)$ with $i \in \mathbb{S}$. We show that partial reduction preserves elements with finite Coxeter parts.

Lemma 4.3. *Let $w \in \tilde{W}$ with $\eta_\sigma(w)$ a partial σ -Coxeter element of W . Let $i \in \mathbb{S}$ with $s_i w < w$. Then:*

- (1) $\eta_\sigma(s_i w \sigma(s_i))$ is a partial σ -Coxeter element of W and $\text{supp}_\sigma(\eta_\sigma(s_i w \sigma(s_i))) = \text{supp}_\sigma(\eta_\sigma(w))$.

(2) If, moreover, $s_i w \sigma(s_i) < w$, then $\eta_\sigma(s_i w)$ is a partial σ -Coxeter element of W and $\text{supp}_\sigma(\eta_\sigma(s_i w)) = \text{supp}_\sigma(\eta_\sigma(w)) - \{\sigma^l(i') \mid l \in \mathbb{Z}\}$ for some $i' \in \text{supp}_\sigma(\eta_\sigma(w))$.

Proof. We prove part (1). The proof of part (2) is similar, and we skip the details.

Write $w = xt^\mu y$ with $t^\mu y \in {}^{\mathbb{S}}\tilde{W}$. Set $c = \eta_\sigma(w) = \sigma^{-1}(y)x$. If $y\sigma(s_i) \in {}^{\mathbb{S}}\tilde{W}$, then $\eta_\sigma(s_i w \sigma(s_i)) = \eta_\sigma(w)$, and the statement is obvious. Now assume that $y\sigma(s_i) = s_{i'}y$ for some $i' \in I(\mu)$. Then we have $x^{-1}(\alpha_i) < 0$ and $\sigma^{-1}(y)(\alpha_i) > 0$. Thus $(\sigma^{-1}(y)x)(-x^{-1}(\alpha_i)) < 0$. It follows that $\sigma^{-1}(y)s_i x = (\sigma^{-1}(y)x)(x^{-1}s_i x) < \sigma^{-1}(y)x$. By the cancellation property of Coxeter groups, we conclude that $\sigma^{-1}(y)s_i x$ is a partial σ -Coxeter element and $\ell(\sigma^{-1}(y)s_i x) = \ell(\sigma^{-1}(y)x) - 1$. Write $c' = \sigma^{-1}(y)s_i x$. Notice that $\sigma^{-1}(s_{i'})c = \sigma^{-1}(s_{i'}y)x = \sigma^{-1}(y)s_i x = c'$. It follows that $\sigma^{-1}(i')$ is not in the σ -support of c' . Hence $\eta_\sigma(s_i w \sigma(s_i)) = \sigma^{-1}(y)s_i x s_{i'} = c' s_{i'}$ is a partial σ -Coxeter element, and

$$\text{supp}_\sigma(\eta_\sigma(s_i w \sigma(s_i))) = \text{supp}_\sigma(\eta_\sigma(w)). \quad \square$$

4D. Proof of Theorem 2.6(1) and (2): general case. Let $w = xt^\mu y \in \tilde{W}$ with $t^\mu y \in {}^{\mathbb{S}}\tilde{W}$. We assume that $\eta_\sigma(w)$ a partial σ -Coxeter element. We prove Theorem 2.6(1) and (2) by induction on $\ell(x)$.

The case $\ell(x) = 0$ has already been proved in Section 4B. Assume that $\ell(x) > 0$. Let $i \in \mathbb{S}$ such that $s_i x < x$. There are three different cases.

Case (1): $\ell(s_i w \sigma(s_i)) < \ell(w)$. Write $w_1 = s_i w$ and $w_2 = s_i w \sigma(s_i)$. By Lemma 4.3, $\eta_\sigma(w_1)$ and $\eta_\sigma(w_2)$ are both partial σ -Coxeter elements. The inductive hypothesis applies for w_1 and w_2 . Since $w_1 > w_2$, we have $[b_w] = \max([b_{w_1}], [b_{w_2}]) = [b_{w_1}]$. Then $d_w(b_w) = d_{w_1}(b_w) + 1$ and $\dim X_{w_1}(b_w) + 1 = \dim X_w(b_w)$. Thus w is cordial.

Observe that

$$B(\mathbf{G})_w = B(\mathbf{G})_{w_1} \cup B(\mathbf{G})_{w_2} = \bigsqcup_{J \in [J_0(w_1), J(w_1)]_\mu \cup [J_0(w_2), J(w_2)]_\mu} B(\mathbf{G}, \mu)_{J\text{-irr}}.$$

Note that $J(w_2) = J(w)$, $J_0(w_1) = J_0(w)$ and $J(w_1) \subset J(w)$. By Section 2E(b), $B(\mathbf{G})_w$ is saturated. Hence we must have

$$[J_0(w_1), J(w_1)]_\mu \cup [J_0(w_2), J(w_2)]_\mu = [J_0(w), J(w)]_\mu.$$

This proves part (2) of Theorem 2.6.

Case (2): $y\sigma(s_i) < y$. Write $w' = s_i w \sigma(s_i) = s_i x t^\mu y \sigma(s_i)$. The inductive hypothesis applies for w' . Note that $w \approx_\sigma w'$, in particular, $B(\mathbf{G})_w = B(\mathbf{G})_{w'}$. Also $J(w) = J(w')$. Hence the statements hold for w .

Case (3): $y\sigma(s_i) = s_{i'}y$ for some $i' \in I(\mu)$ and $\ell(s_i x s_{i'}) = \ell(x)$. Write $w' = s_i w \sigma(s_i) = s_i x s_{i'} t^\mu y$. Then $w \approx_\sigma w'$. By Lemma 4.3, $\eta_\sigma(w') = \sigma^{-1}(y)s_i x s_{i'}$ is a partial σ -Coxeter element with length equal to $\ell(\sigma^{-1}(y)x)$. Hence the statements hold for w if and only if they hold for w' . We continue the procedure until case (1) or (2) happens. If case (3) happens all the time and the procedure does not end, then $x \in W_{I(t^\mu y)}$, and both x and y are partial σ -Coxeter elements. Then the statements follow from Section 4C(b).

5. Analyzing the reduction paths

5A. σ -Conjugacy classes of \tilde{W} . We first recall the definition of elliptic conjugacy classes. Let W_1 be a Coxeter group and let S_1 be the index set of simple reflections in W_1 . Let δ be a length-preserving group automorphism on W_1 . A δ -conjugacy class C of W_1 is called *elliptic* if it contains no elements in any proper δ -stable standard parabolic subgroup of W_1 .

Let $x \in \tilde{W}$. We regard $x\sigma$ as an element in $\tilde{W} \rtimes \langle \sigma \rangle$. There exists a positive integer such that $(x\sigma)^n = t^\lambda$ for some $\lambda \in X_*(T)_{\Gamma_0}$. Then we set $v_x = \lambda/n \in V$. It is easy to see that v_x is independent of the choice of n . Moreover, the unique dominant element in the W -orbit of v_x equals the (dominant) Newton point $v_{\check{x}}$ for $\check{x} \in \check{G}$.

We follow [13, §1.8.3]. Let $J \subseteq \mathbb{S}$. Let $\tilde{W}_J = X_*(T)_{\Gamma_0} \rtimes W_J$ be the Iwahori–Weyl group of the standard Levi subgroup M_J of G . Let ${}^J\tilde{W}$ be the set of minimal length representatives for cosets in $W_J \backslash \tilde{W}$. Let $\tilde{J} \supseteq J$ be the set of simple reflections for the Iwahori–Weyl group \tilde{W}_J .

We say that (J, x, \check{K}, C) is a *standard quadruple* if

- (1) $\sigma(J) = J$;
- (2) $x \in {}^J\tilde{W}$ such that v_x is dominant, $J = I(v_x)$, and $\text{Ad}(x) \circ \sigma$ preserves \tilde{J} ;
- (3) $\check{K} \subseteq \tilde{J}$ with $W_{\check{K}}$ finite and $\text{Ad}(x)(\sigma(\check{K})) = \check{K}$;
- (4) C is an elliptic $(\text{Ad}(x) \circ \sigma)$ -conjugacy class of $W_{\check{K}}$.

We say that the standard quadruples (J, x, \check{K}, C) and (J', x', \check{K}', C') are *equivalent* in \tilde{W} if $J = J'$, there exists a length-zero element τ of \tilde{W}_J with $x' = \tau x \sigma(\tau)^{-1}$, and there exists $w \in \tilde{W}_J$ with $x' \sigma(w)(x')^{-1} = w$ and $C' = w \tau C (w \tau)^{-1}$.

By [13, Theorem 1.19], we have the following:

- (a) The map $(J, x, \check{K}, C) \mapsto \tilde{W} \cdot \sigma C x$ induces a bijection between the equivalence classes of standard quadruples and the set of σ -conjugacy classes of \tilde{W} .

Let $\mathcal{O} \in B(\tilde{W}, \sigma)$ and let (J, x, \check{K}, C) be a standard quadruple associated with \mathcal{O} . We say that \mathcal{O} is *Coxeter* (resp. *elliptic*) associated with $[b] \in B(G)$ if $\Psi(\mathcal{O}) = [b]$, $\check{K} \subset \tilde{J}$ is very special with respect to $\text{Ad}(\check{x}) \circ \sigma$, and C is an $(\text{Ad}(x) \circ \sigma)$ -Coxeter (resp. elliptic) conjugacy class of $W_{\check{K}}$. Namely, C contains an $(\text{Ad}(x) \circ \sigma)$ -Coxeter (resp. elliptic) element of the finite Coxeter group $W_{\check{K}}$. We say that $w \in \tilde{W}$ is a σ -Coxeter element associated with $[b]$ if it is a minimal length element in a Coxeter σ -conjugacy class associated with $[b]$.

5B. Description of reduction trees. For any $[b] \in B(G, \mu)$, set

$$\begin{aligned} \ell_I(\mu, [b]) &= \sharp(\mathbb{S}/\langle \sigma \rangle) - \sharp(I(v_b)/\langle \sigma \rangle), \\ \ell_{II}(\mu, [b]) &= \text{length}([b], [t^\mu]) - \sharp(\mathbb{S}/\langle \sigma \rangle). \end{aligned}$$

We have the following description of reduction trees.

Theorem 5.1. *Let c be a σ -Coxeter element of W such that $t^\mu c \in {}^{\mathbb{S}}\tilde{W}$. Let \mathcal{T} be a reduction tree of $t^\mu c$. Then, for any reduction path \underline{p} in \mathcal{T} , we have*

- (1) $\ell_{\text{I}}(\underline{p}) = \ell_{\text{I}}(\mu, [b]_{\underline{p}})$ and $\ell_{\text{II}}(\underline{p}) = \ell_{\text{II}}(\mu, [b]_{\underline{p}})$;
- (2) $\text{end}(\underline{p})$ is a σ -Coxeter element associated with $[b]_{\underline{p}}$.

Moreover, for any $[b] \in B(\mathbf{G}, \mu)_{\text{indec}}$, there exists a unique reduction path \underline{p} in \mathcal{T} with $[b]_{\underline{p}} = [b]$.

Combining [Theorem 5.1](#) with [Proposition 3.9](#) and [Remark 3.10](#), we obtain part (3) of [Theorem 2.6](#) for $w = t^\mu c$. We will describe the reduction trees of the elements with finite partial σ -Coxeter part in [Section 7](#) and deduce [Theorem 2.6\(3\)](#) for such elements.

In the rest of this section, we shall prove parts (1) and (2) of [Theorem 5.1](#). The “moreover” part (i.e., the multiplicity-one result) is the most difficult part and will be proved in [Section 6](#).

5C. Estimate ℓ_{I} . Let $\text{Aff}(V)$ be the group of affine transformations on V . For any $g \in \text{Aff}(V)$, define $V^g = \{v \in V \mid g(v) = v\}$. We have a natural projection map $p : \tilde{W} \times \langle \sigma \rangle \rightarrow \text{Aff}(V) \rightarrow \text{GL}(V)$. For any $w \in \tilde{W}$, define $V_w = \{v \in V \mid w\sigma(v) = v + v_w\}$. We have $\dim V_w = \dim V^{p(w\sigma)}$.

It is easy to see that for any $g \in \text{GL}(V)$ and any reflection $r \in \text{GL}(V)$, we have

$$|\dim V^{rg} - \dim V^g| \leq 1.$$

In particular, for any $w \in \tilde{W}$ and $i \in \tilde{\mathbb{S}}$, we have

$$|\dim V_{s_i w} - \dim V_w| \leq 1.$$

Now let $w = t^\mu c$ be as in [Theorem 5.1](#). Let \mathcal{T} be a reduction tree of w and let \underline{p} be a reduction path in \mathcal{T} . Set $e = \text{end}(\underline{p})$ and $[b] = \Psi(e) \in B(\mathbf{G})$. Let (J, x, \check{K}, C) be a standard quadruple associated with the σ -conjugacy class of e .

Consider the variation of $\dim V_{\gamma}$ along the reduction path \underline{p} , where γ stands for any element in \tilde{W} . Type-II edges do not change $\dim V_{\gamma}$, and type-I edges change $\dim V_{\gamma}$ by at most 1. Therefore

$$\ell_{\text{I}}(\underline{p}) \geq |\dim V_e - \dim V_w|.$$

Since $p(w) = c$ is a σ -Coxeter element, $V^{p(w\sigma)} = V^{p(\tilde{W} \times \langle \sigma \rangle)}$. Since C is $\text{Ad}(x) \circ \sigma$ -elliptic in \check{K} , we have

$$\dim V_e = \dim V_x - \sharp(\check{K} / \langle \text{Ad}(x) \circ \sigma \rangle).$$

Hence

$$\ell_{\text{I}}(\underline{p}) \geq \dim V_x - \sharp(\check{K} / \langle \text{Ad}(x) \circ \sigma \rangle). \tag{5-1}$$

By [\[23, §1.9\]](#), $\text{def}(b) = \sharp(\mathbb{S} / \langle \sigma \rangle) - \dim V_x$. Note that $\ell_{\text{I}}(\underline{p}) + 2\ell_{\text{II}}(\underline{p}) = \ell(t^\mu c) - \ell(e)$. Moreover,

$$\ell(e) \geq \langle v_b, 2\rho \rangle + \sharp(\check{K} / \langle \text{Ad}(x) \circ \sigma \rangle), \tag{5-2}$$

with equality holding if and only if C is an $(\text{Ad}(x) \circ \sigma)$ -Coxeter conjugacy class in \check{K} . We have

$$\begin{aligned} \dim X_p &= \ell_I(\underline{p}) + \ell_{II}(\underline{p}) + \ell(e) - \langle \nu_b, 2\rho \rangle \\ &= \frac{1}{2}(\ell_I(\underline{p}) + \langle \mu, 2\rho \rangle - \sharp(\mathbb{S}/\langle \sigma \rangle) + \ell(e)) - \langle \nu_b, 2\rho \rangle \\ &\geq \langle \mu - \nu_b, \rho \rangle + \frac{1}{2}(\dim V_x - \sharp(\mathbb{S}/\langle \sigma \rangle)) \\ &= \langle \mu - \nu_b, \rho \rangle - \frac{1}{2}\text{def}(b) = d_w(b). \end{aligned}$$

By Section 2E(b), we have $\dim X_p \leq \dim X_w(b) \leq \dim d_w(b)$. Thus the inequalities in (a) and (b) are equalities, and $\dim X_p = \dim X_w(b)$. In particular, C is an $(\text{Ad}(x) \circ \sigma)$ -Coxeter conjugacy class in \check{K} .

5D. Affine Deligne–Lusztig varieties in the affine Grassmannian. It remains to show that \check{K} occurring in Section 5C is very special. To do this, we need some information on affine Deligne–Lusztig varieties in the affine Grassmannian.

Let $\check{P} \subseteq \check{G}$ be a special parahoric subgroup containing \check{I} . The affine Deligne–Lusztig variety in the affine Grassmannian \check{G}/\check{P} is defined by

$$X_\mu(b) = \{g \in \check{G}/\check{P} \mid g^{-1}b\sigma(g) \in \check{P}t^\mu\check{P}\}.$$

The following dimension formula is proved for split groups [5; 30], for unramified groups [9; 34], and in general [13, Theorem 2.29].

Theorem 5.2. *Suppose that $[b] \in B(\mathbf{G}, \mu)$. Then $\dim X_\mu(b) = \langle \mu - \nu_b, \rho \rangle - \frac{1}{2}\text{def}(b)$.*

Let $\Sigma^{\text{top}}(X_\mu(b))$ be the set of top-dimensional irreducible components of $X_\mu(b)$.

Let $\widehat{\mathbf{G}}$ be the Langlands dual of \mathbf{G} over the complex number field \mathbb{C} . Let \widehat{T} be the maximal torus dual to T . Then σ acts on \widehat{T} in a natural way, and we denote by \widehat{T}^σ the σ -fixed points of \widehat{T} . Let $\lambda_b \in X^*(\widehat{T}^\sigma)$ be the “best integral approximation” of the Newton point of b in the sense of [10, Definition 2.1]. Let V_μ be the irreducible representation of $\widehat{\mathbf{G}}$ with highest weight μ . Write $V_\mu(\lambda_b)$ for the corresponding λ_b -weight subspace of \widehat{T}^σ . The following result was conjectured by M. Chen and X. Zhu, and is proved in [20; 26; 33].

Theorem 5.3. *The number of $\mathbf{J}_b(F)$ -orbits on $\Sigma^{\text{top}}(X_\mu(b))$ equals $\dim V_\mu(\lambda_b)$. The stabilizer of each element in $\Sigma^{\text{top}}(X_\mu(b))$ is a very special parahoric subgroup of $\mathbf{J}_b(F)$.*

We also need the following result that connects affine Deligne–Lusztig varieties in the affine flag and in the affine Grassmannian.

Lemma 5.4. *The $\mathbf{J}_b(F)$ -equivariant projection map $X_{t^\mu c}(b) \rightarrow X_\mu(b)$ is injective.*

Proof. Let $g\check{I}, g'\check{I} \in \check{G}/\check{I}$ be in the same fiber of the natural projection $\text{map } X_{t^\mu c}(b) \rightarrow X_\mu(b)$. Then $g'^{-1}g \in \check{P}$. We have $g'^{-1}g \in \check{I}\check{x}\check{I}$ for some $x \in W$. Since $(g'^{-1}g)(g^{-1}b\sigma(g)) = (g'^{-1}b\sigma(g'))\sigma(g'^{-1}g)$, $(\check{I}\check{x}\check{I})(\check{I}t^\mu\check{c}\check{I}) \cap (\check{I}t^\mu\check{c}\check{I})\sigma(\check{I}\check{x}\check{I}) \neq \emptyset$. Since $t^\mu c \in {}^{\mathbb{S}}\check{W}$, $(\check{I}\check{x}\check{I})(\check{I}t^\mu\check{c}\check{I}) = \check{I}\check{x}t^\mu\check{c}\check{I}$. Thus we have $xt^\mu c = t^\mu c\sigma(x)$ and $\text{supp}_\sigma(x) \subset I(t^\mu c)$. As c is σ -elliptic, we conclude that $\text{supp}_\sigma(x) = \emptyset$ and hence $g'g^{-1} \in \check{I}$ as desired. □

5E. Proof of Theorem 5.1(1) and (2). We continue our analysis of reduction paths. All the notation is the same as in Section 5C.

Equalities (5-1) and (5-2) hold and $\text{def}(b) = \sharp(I(v_b)/\langle\sigma\rangle) - \sharp(\check{K}/\langle\text{Ad}(x) \circ \sigma\rangle)$. It follows that

$$\ell_I(\underline{p}) = \sharp(\mathbb{S}/\langle\sigma\rangle) - \sharp(I(v_b)/\langle\sigma\rangle) = \ell_I(\mu, [b]).$$

Using Section 2E(a) and the simple fact that $\ell_I(\underline{p}) + 2\ell_{II}(\underline{p}) = \ell(w) - \ell(e)$, one can prove that $\ell_{II}(\underline{p}) = \ell_{II}(\mu, [b])$. This proves part (1) of Theorem 5.1.

By (2-1), the stabilizer in $\mathbf{J}_b(F)$ of any irreducible component of $X_{\underline{p}}$ is isomorphic to the parahoric subgroup $\check{P}_{\check{K}} \cap \mathbf{J}_b(F) \subseteq \mathbf{J}_b(F)$. By Section 5C, we have $\dim X_{\underline{p}} = d_w(b) = \dim X_{\mu}(b)$. By Lemma 5.4, the image of each irreducible component Z of $X_{\underline{p}}$ in $X_{\mu}(b)$ is an open dense subset of some top-dimensional irreducible component Y of $X_{\mu}(b)$. Thus the stabilizers of Z and Y coincide. By Theorem 5.3, $\check{P}_{\check{K}} \cap \mathbf{J}_b(F)$ is a very special parahoric subgroup of $\mathbf{J}_b(F)$. Hence, by Section 2I(a), $\check{K} \subset \check{J}$ is very special with respect to $\text{Ad}(\dot{x}) \circ \sigma$. By Section 5C, C is an $(\text{Ad}(x) \circ \sigma)$ -Coxeter conjugacy class of $W_{\check{K}}$. This proves part (2) of Theorem 5.1.

5F. The extreme cases. Let \mathcal{T} be a reduction tree of $t^\mu c$. Let $[b] \in B(\mathbf{G})_{t^\mu c}$ and let \underline{p} be a path in \mathcal{T} such that $[b]_{\underline{p}} = [b]$.

If $[b] = [b_{\mu, \mathbf{G}\text{-indec}}]$, then

$$\ell_{II}(\underline{p}) = \ell_{II}(\mu, [b]) = 0.$$

Therefore \underline{p} consists only of type-I edges and is unique.

If $[b]$ is basic, then

$$I(v_b) = \mathbb{S} \quad \text{and} \quad \ell_I(\underline{p}) = \ell_I(\mu, [b]) = 0.$$

Therefore \underline{p} consists only of type-II edges and is unique.

This proves the “moreover” part of Theorem 2.6 for these two extreme cases.

6. Some combinatorial identities

6A. Reduction to combinatorial identities. In this section, we assume that μ is essentially noncentral. Let c be a σ -Coxeter element of W such that $t^\mu c \in {}^{\mathbb{S}}\check{W}$. Let \mathcal{T} be a reduction tree of $t^\mu c$. For any $[b] \in B(\mathbf{G}, \mu)_{\text{indec}}$, let $n_{[b]}$ be the number of reduction paths \underline{p} in \mathcal{T} with $[b]_{\underline{p}} = [b]$. By Theorem 2.6(2), $n_{[b]} \geq 1$ for all $[b] \in B(\mathbf{G}, \mu)_{\text{indec}}$. By Section 5F, $n_{[b]} = 1$ if $[b]$ is either the minimal or the maximal element in $B(\mathbf{G}, \mu)_{\text{indec}}$.

Combining Theorem 5.1(1) and (2) with Proposition 3.6 and Lemma 3.8, we have

$$q^{\langle\mu, 2\rho\rangle - \sharp(\mathbb{S}/\langle\sigma\rangle)} = \sum_{[b] \in B(\mathbf{G}, \mu)_{\text{indec}}} n_{[b]} (q - 1)^{\ell_I(\mu, [b])} q^{\ell_{II}(\mu, [b]) + \ell_{[b]}}$$

where $\ell_{[b]} = \langle v_b, 2\rho \rangle + \sharp(I(v_b)/\langle\sigma\rangle) - \text{def}(b)$ and equals $\ell(\mathcal{O})$ for any σ -Coxeter class \mathcal{O} associated with $[b]$.

Note that $(q - 1)^a q^{a'} \in \mathbb{N}[q - 1]$ for all $a, a' \in \mathbb{N}$. Thus, to show that $n_{[b]} = 1$ for all $[b]$, it suffices to show that

$$\sum_{[b] \in B(\mathbf{G}, \mu)_{\text{indec}}} (q - 1)^{\ell_1(\mu, [b])} q^{\ell_{\Pi}(\mu, [b]) + \ell_{[b]}} = q^{\langle \mu, 2\rho \rangle - \sharp(\mathbb{S}/\langle \sigma \rangle)}. \quad (\spadesuit)$$

(In fact, it is enough to prove the inequality \geq .)

Using Section 2E(a), one computes that

$$\ell_{\Pi}(\mu, [b]) + \ell_{[b]} - (\langle \mu, 2\rho \rangle - \sharp(\mathbb{S}/\langle \sigma \rangle)) = \sharp(I(v_b)/\langle \sigma \rangle) - \text{length}([b], [t^\mu]).$$

Thus, (\spadesuit) is equivalent to

$$\sum_{[b] \in B(\mathbf{G}, \mu)_{\text{indec}}} (q - 1)^{\sharp(\mathbb{S}/\langle \sigma \rangle) - \sharp(I(v_b)/\langle \sigma \rangle)} q^{\sharp(I(v_b)/\langle \sigma \rangle) - \text{length}([b], [t^\mu])} = 1. \quad (\spadesuit')$$

6B. Reduction to unramified adjoint groups. Let \mathbf{G}_{ad} be the adjoint group of \mathbf{G} and let T_{ad} be the image of T in \mathbf{G}_{ad} . We denote by μ_{ad} the image of μ in $X_*(T_{\text{ad}})_{\Gamma_0}$. For any $b \in \check{G}$, we denote by b_{ad} its image in \check{G}_{ad} . By [22, Proposition 4.10], the map $\mathbf{G} \rightarrow \mathbf{G}_{\text{ad}}$ identifies the reduced root system of \mathbf{G}_{ad} with that of \mathbf{G} and induces an isomorphism of posets

$$B(\mathbf{G}, \mu)_{\text{indec}} \cong B(\mathbf{G}_{\text{ad}}, \mu_{\text{ad}})_{\text{indec}}, \quad [b] \mapsto [b_{\text{ad}}].$$

Therefore, (\spadesuit') for \mathbf{G} is equivalent to that for \mathbf{G}_{ad} . We can therefore assume that \mathbf{G} is adjoint. In this case, it is convenient to work with the reduced root system Φ of \mathbf{G} . Define

$$B(\Phi, \sigma, \mu) = \{v \in (V^+)^\sigma \mid \langle \mu^\diamond - v, \omega_{\mathcal{O}_i} \rangle \in \mathbb{Z}_{\geq 0} \text{ for any } i \in \mathbb{S} - I(v)\},$$

where \mathcal{O}_i denotes the σ -orbit of i . By [19, Lemma 3.5], the map $[b] \mapsto v_b$ identifies $B(\mathbf{G}, \mu)$ with $B(\Phi, \sigma, \mu)$ as posets. For any $v \in B(\Phi, \sigma, \mu)$, we set $\text{length}(v, \mu^\diamond) = \sum_{\mathcal{O} \in \mathbb{S}/\langle \sigma \rangle} [\langle \mu^\diamond - v, \omega_{\mathcal{O}} \rangle]$. Then by Section 2B(a),

$$\text{length}([b], [t^\mu]) = \text{length}(v_b, \mu^\diamond) \quad \text{for any } [b] \in B(\mathbf{G}, \mu).$$

We set

$$f_{\Phi, \sigma, \mu}(v) = (q - 1)^{\sharp(\mathbb{S}/\langle \sigma \rangle) - \sharp(I(v)/\langle \sigma \rangle)} q^{\sharp(I(v)/\langle \sigma \rangle) - \text{length}(v, \mu^\diamond)}.$$

Now (\spadesuit') can be reformulated in a purely combinatorial way as

$$\sum_{v \in B(\Phi, \sigma, \mu)_{\text{indec}}} f_{\Phi, \sigma, \mu}(v) = 1. \quad (\spadesuit'')$$

As any triple (Φ, σ, μ) arises from an unramified group, it suffices to prove (\spadesuit'') for the triples (Φ, σ, μ) arising from unramified adjoint groups. In the rest of this section, we assume that \mathbf{G} is an unramified adjoint group.

6C. Reduction to F -simple groups. Write $G = G_1 \times \cdots \times G_l$, where G_i are F -simple adjoint groups. Write $\mu = (\mu_1, \dots, \mu_l)$, where μ_i is a dominant coweight of G_i . Also we have $\Phi = \Phi_1 \sqcup \cdots \sqcup \Phi_l$, where Φ_i is the root system of G_i . It is easy to see that

$$B(\Phi, \sigma, \mu)_{\text{indec}} = B(\Phi_1, \sigma_1, \mu_1)_{\text{indec}} \times \cdots \times B(\Phi_l, \sigma_l, \mu_l)_{\text{indec}},$$

where σ_i is the restriction of σ on Φ_i . It is clear that $f_{\Phi, \sigma, \mu} = \prod_{i=1}^l f_{\Phi_i, \sigma_i, \mu_i}$. In the rest of this section, we assume that G is an F -simple unramified adjoint group.

6D. Reduction to \check{F} -simple groups. We have $\Phi = \Phi_1 \sqcup \cdots \sqcup \Phi_l$, where $\Phi_1 \cong \cdots \cong \Phi_l$ are irreducible root systems and σ induces an isomorphism from Φ_i to Φ_{i+1} . Here, by convention, we set $\Phi_{l+1} = \Phi_1$. Then the map

$$v = (v_1, \dots, v_l) \mapsto |v| = v_l + \sigma(v_{l-1}) + \cdots + \sigma^{l-1}(v_1)$$

induces an isomorphism of posets

$$B(\Phi, \sigma, \mu)_{\text{indec}} \xrightarrow{\sim} B(\Phi_l, \sigma^l, |\mu|)_{\text{indec}}.$$

There is a natural bijection $\mathbb{S}/\langle \sigma \rangle \cong \mathbb{S}_l/\langle \sigma^l \rangle$, where \mathbb{S}_l is the set of simple reflections for Φ_l . Thus $f_{\Phi, \sigma, \mu} = f_{\Phi_l, \sigma^l, |\mu|}$. In the rest of this section, we assume that G is an \check{F} -simple unramified adjoint group.

6E. Reduction to split groups. Let \mathcal{O} be a σ -orbit of \mathbb{S} . If all the simple roots in \mathcal{O} commute with each other, we define $\alpha'_{\mathcal{O}} = \sum_{i \in \mathcal{O}} \alpha_i$. If $\mathcal{O} = \{i_0, j_0\}$ with $\langle \alpha_{j_0}^\vee, \alpha_{i_0} \rangle = \langle \alpha_{i_0}^\vee, \alpha_{j_0} \rangle = -1$, we define $\alpha'_{\mathcal{O}} = 2(\alpha_{i_0} + \alpha_{j_0})$. Let $\mathbb{S}' = \mathbb{S}/\langle \sigma \rangle$ and let Φ' be the root system generated by $\alpha'_{\mathcal{O}}$ for all $\mathcal{O} \in \mathbb{S}'$. The coroot corresponding to \mathcal{O} is given by $\alpha_{\mathcal{O}}^\vee = \frac{1}{\#\mathcal{O}} \sum_{i \in \mathcal{O}} \alpha_i^\vee$, and the fundamental weight corresponding to \mathcal{O} is given by $\omega'_{\mathcal{O}} = \sum_{i \in \mathcal{O}} \omega_i$. For any $v \in (V^+)^\sigma$, $v \in B(\Phi, \sigma, \mu)$ if and only if $\langle \mu^\diamond - v, \omega'_{\mathcal{O}} \rangle \in \mathbb{Z}_{\geq 0}$ for any $\mathcal{O} \in \mathbb{S}/\langle \sigma \rangle$ such that $\langle v, \alpha'_{\mathcal{O}} \rangle \neq 0$, which is also equivalent to $v \in B(\Phi', \text{id}, \mu^\diamond)$. Hence we have the following:

(a) The natural identification $(\mathbb{R}\Phi^\vee)^\sigma = \mathbb{R}\Phi'^\vee$ induces a bijection of posets

$$B(\Phi, \sigma, \mu)_{\text{indec}} \xrightarrow{\sim} B(\Phi', \text{id}, \mu^\diamond)_{\text{indec}}.$$

It follows from (a) that $f_{\Phi, \sigma, \mu} = f_{\Phi', \text{id}, \mu^\diamond}$. Therefore, it suffices to prove (\spadesuit'') for the triples (Φ, σ, μ) arising from split groups. In the rest of this section, we assume that G is a split \check{F} -simple adjoint group. We identify $B(G, \mu)$ with $B(\Phi, \sigma, \mu)$ and write $f_{G, \mu}(v) = f_{\Phi, \sigma, \mu}(v)$.

6F. Reduction to simply laced groups. By Section 6, (\spadesuit) is equivalent to the condition that in some (or, equivalently, any) reduction tree of $t^\mu c$, there exists only one reduction path whose end point is associated with a given $[b] \in B(G, \mu)_{\text{indec}}$.

There exists an irreducible, simply laced, extended affine Weyl group (\tilde{W}', \tilde{S}') of adjoint type and a length-preserving automorphism δ on \tilde{W}' such that $\tilde{W} = (\tilde{W}')^\delta$. We have a natural bijection between $\tilde{\mathbb{S}}$ and $\tilde{S}'/\langle \delta \rangle$. We may assume that the simple reflections in each δ -orbit in \tilde{S}' commute. More explicitly,

- if \tilde{W} is of type \tilde{B}_n , then we take \tilde{W}' to be of type \tilde{D}_{n+1} and δ is of order 2;

- if \tilde{W} is of type \tilde{C}_n , then we take \tilde{W}' to be of type \tilde{A}_{2n-1} and δ is of order 2;
- if \tilde{W} is of type \tilde{F}_4 , then we take \tilde{W}' to be of type \tilde{E}_6 and δ is of order 2;
- if \tilde{W} is of type \tilde{G}_2 , then we take \tilde{W}' to be of type \tilde{D}_4 and δ is of order 3.

Let $\iota : \tilde{W} \rightarrow \tilde{W}'$ be the natural embedding. For each $i \in \tilde{S}$, we have $\iota(s_i) = s_{i'_1} \cdots s_{i'_k}$, where i'_1, \dots, i'_k are the δ -orbits of i in \tilde{S}' . Let $w \in \tilde{W}$ and $w \rightarrow s_i w$ be a type-I reduction edge (see Section 3D). Then one can construct a k -step reduction path

$$\iota(w) \rightarrow s_{i'_k} \iota(w) \rightarrow s_{i'_{k-1}} s_{i'_k} \iota(w) \rightarrow \cdots \rightarrow s_{i'_1} \cdots s_{i'_k} \iota(w) = \iota(s_i w)$$

in \tilde{W}' . Similarly, a type-II reduction edge $w \rightarrow s_i w s_i$ corresponds to a k -step reduction path from $\iota(w)$ to $\iota(s w s)$, whose edges are all of type II. Now considering a reduction tree Γ of w , we can construct a reduction tree \mathcal{T}' of $\iota(w)$ such that \mathcal{T} can be viewed as a subtree of \mathcal{T}' in the above way. Hence the multiplicity-one result of $\iota(t^\mu c) \in \tilde{W}'$ implies the multiplicity-one result of $t^\mu c \in \tilde{W}$.

In the rest of this section, we assume that G is a split \check{F} -simple simply laced adjoint group. We will then reduce to the case where μ is a fundamental coweight. We first need a combinatorial identity on finite graphs.

6G. A combinatorial identity on graphs. Let X be a finite graph and $Y \subseteq X$. Denote by $\mathcal{A}(Y, X)$ the set of subsets $J \subseteq X$ such that none of the connected components of $X - J$ lies in Y . Define

$$f_{Y,X} = \sum_{J \in \mathcal{A}(Y,X)} (q-1)^{\sharp(J-Y \cap J^\circ)} q^{\sharp(Y \cap J^\circ)} \in \mathbb{Z}[q],$$

where $J^\circ = \{i \in J \mid i \text{ has no neighbors in } X - J\}$ is the interior of J .

Lemma 6.1. *We have $f_{Y,X} = q^{\sharp X}$ for any $Y \subseteq X$.*

Proof. Define

$$\alpha : \{(J, K) \mid J \in \mathcal{A}(Y, X), K \subseteq Y \cap J^\circ\} \rightarrow \{\text{subsets of } X\}, \quad (J, K) \mapsto J - K.$$

We construct the inverse map β of α as follows. Let $H \subseteq X$. Let C be the union of connected components of $X - H$ that are contained in Y . Then we define $\beta(H) = (H \sqcup C, C)$. By definition, $\alpha \circ \beta = \text{id}$. On the other hand, for any $J \in \mathcal{A}(Y, X)$ and $K \subseteq Y \cap J^\circ$, $\alpha((J, K)) = J - K$. Moreover, $X - (J - K) = (X - J) \sqcup K$. As $K \subseteq Y \cap J^\circ$, K and $X - J$ are not connected with each other. Hence K is the union of connected components of $X - (J - K)$ contained in Y . Therefore $\beta \circ \alpha((J, K)) = \beta(J - K) = ((J - K) \sqcup K, K) = (J, K)$ and hence $\beta \circ \alpha = \text{id}$. Therefore α is a bijection. Using the binomial expansion, we get

$$\begin{aligned} f_{Y,X} &= \sum_{J \in \mathcal{A}(Y,X)} (q-1)^{\sharp(J-Y \cap J^\circ)} q^{\sharp(Y \cap J^\circ)} = \sum_{J \in \mathcal{A}(Y,X)} (q-1)^{\sharp(J-Y \cap J^\circ)} \sum_{K \subseteq Y \cap J^\circ} (q-1)^{\sharp(Y \cap J^\circ - K)} \\ &= \sum_{J \in \mathcal{A}(X,Y)} \sum_{K \subseteq Y \cap J^\circ} (q-1)^{\sharp(J-K)} = \sum_{H \subseteq X} (q-1)^{\sharp H} = q^{\sharp X}. \end{aligned} \quad \square$$

6H. Reduction to fundamental coweights. Assume that μ is not a fundamental coweight. Then there exist $i, j \in \mathbb{S}$ (here i and j are not necessarily distinct) such that $\mu - \omega_i^\vee - \omega_j^\vee$ is also dominant. Let X be the (unique) shortest path in the Dynkin diagram of \mathbb{S} with end points i, j .

Let $\lambda = \mu - \sum_{k \in X} \alpha_k^\vee$. Set

$$Y = \{i \in X \cap I(\lambda) \mid i \text{ has no neighbors in } \mathbb{S} - X\}.$$

Let \mathcal{A} be the set of subsets $J \subseteq X$ such that λ is noncentral on each connected component of $\mathbb{S} - J$.

Lemma 6.2. *We have*

- (1) λ is dominant;
- (2) $B(\mathbf{G}, \mu)_{\text{indec}} = \bigsqcup_{J \in \mathcal{A}} B(\mathbf{G}, \lambda)_{(\mathbb{S}-J)\text{-irr}}$;
- (3) $\mathcal{A} = \mathcal{A}(Y, X)$, where $\mathcal{A}(Y, X)$ is defined as in [Section 6G](#).

Proof. Let $l \in \mathbb{S}$. If $l \in \mathbb{S} - X$; then $\langle \lambda, \alpha_l \rangle \geq -\langle \sum_{k \in X} \alpha_k^\vee, \alpha_l \rangle \geq 0$. If $l \in X - \{i, j\}$, then $\langle \lambda, \alpha_l \rangle = -\langle \sum_{k \in X} \alpha_k^\vee, \alpha_l \rangle = -\langle \alpha_k^\vee + \alpha_l^\vee + \alpha_{k'}^\vee, \alpha_l \rangle \geq -(-1 + 2 - 1) = 0$, where k and k' are the neighbors of l in X . If $l \in \{i, j\}$ and $i \neq j$, then $\langle \lambda, \alpha_l \rangle = \langle \mu, \alpha_l \rangle - \langle \sum_{k \in X} \alpha_k^\vee, \alpha_l \rangle \geq 1 - 1 = 0$. If $l = i = j$, then $\langle \lambda, \alpha_l \rangle = \langle \mu, \alpha_l \rangle - \langle \sum_{k \in X} \alpha_k^\vee, \alpha_l \rangle \geq 2 - 2 = 0$. This proves part (1).

By definition, $B(\mathbf{G}, \lambda)_{(\mathbb{S}-J)\text{-irr}} \subseteq B(\mathbf{G}, \mu)_{\text{indec}}$ for any $J \subseteq X$. Now we prove the other direction. Let $E_0 = \{e_i; i \in \mathbb{S}\}$ be as in [Proposition 4.1](#). Then, for each $i \in \mathbb{S}$, by the definition of λ we have

$$\langle \lambda, \omega_i \rangle \geq \max\{0, \langle \mu, \omega_i \rangle - 1\} = \langle e_i, \omega_i \rangle.$$

Hence $\lambda \in C_{\geq E_0}$, and it follows from [Proposition 4.1](#) that $\lambda \geq \min C_{\geq E_0} = [b]_{\mu, \mathbf{G}\text{-indec}}$.

For any $\nu \in B(\mathbf{G}, \mu)_{\text{indec}}$, there exists a unique subset $J \subseteq X$ such that $\lambda - \nu \in \sum_{l \in \mathbb{S}-J} \mathbb{R}_{>0} \alpha_l^\vee$. By [Lemma 2.1](#), $\nu \in B(\mathbf{G}, \lambda)_{(\mathbb{S}-J)\text{-irr}}$. Part (2) is proved.

To prove (3) we first claim that

- (a) λ is noncentral on each connected component of $\mathbb{S} - X$.

Let H be a connected component of $\mathbb{S} - X$. Choose $a \in H, b \in X$ and let $a = i_0, i_1, \dots, i_n = b$ be the shortest path in the Dynkin diagram of \mathbb{S} . Then there exists $1 \leq m \leq n$ such that $i_m \in X$ and $i_0, \dots, i_{m-1} \in \mathbb{S} - X$. As H is a connected component of $\mathbb{S} - X$ containing i_0 , it follows that $i_0, i_1, \dots, i_{m-1} \in H$. Since $\langle -\alpha_m^\vee, \alpha_{m-1} \rangle > 0$, we deduce that $\lambda = \mu - \sum_{k \in X} \alpha_k^\vee$ is strictly dominant on H . The claim (a) is proved.

Let $J \subseteq X$ and A be a connected component of $\mathbb{S} - J$. Note that $\mathbb{S} - J = (\mathbb{S} - X) \sqcup (X - J)$. Hence $A = A_1 \sqcup A_2$, where A_1 (resp. A_2) is a union of connected components of $\mathbb{S} - X$ (resp. $X - J$). In view of (a), λ is central on A if and only if $A_1 = \emptyset$ and λ is central on A_2 , i.e., $A = A_2$ is a connected component of $X - J$ contained in Y . On the other hand, any connected component of $X - J$ contained in Y is also a connected component of $\mathbb{S} - J$. Therefore, $\mathcal{A} = \mathcal{A}(Y, X)$ and part (3) is proved. \square

Proposition 6.3. *Suppose that (\spadesuit'') holds for all fundamental coweights. Then it holds for all dominant coweights.*

Proof. We argue by induction on the semisimple rank of G and the number $\langle \rho, \mu \rangle$.

Suppose μ is not fundamental. Let the notation be as in Section 6H. By Corollary 2.2, we identify $B(G, \lambda)_{(\mathbb{S}-J)\text{-irr}}$ with $B(M_{\mathbb{S}-J}, \mu)_{\text{irr}}$ for $J \in \mathcal{A}$. We show that

(a) for any $J \in \mathcal{A}$ and $v \in B(M_{\mathbb{S}-J}, \mu)_{\text{irr}}$, we have

$$f_{G, \mu}(v) = q^{-\sharp X} (q - 1)^{\sharp(J - Y \cap J^\circ)} q^{\sharp(Y \cap J^\circ)} f_{M_{\mathbb{S}-J}, \lambda}(v).$$

By definition, $f_{M_K, \mu}(v) = (q - 1)^{\sharp(K - K \cap I(v))} q^{\sharp K \cap I(v) - \text{length}_{M_K}(\mu, v)}$ for any $K \subseteq \mathbb{S}$.

Note that $\text{length}_G(\lambda, \mu) = \langle \mu - \lambda, \rho \rangle = \sharp X$, and hence

$$\text{length}_G(v, \mu) = \text{length}_G(v, \lambda) + \text{length}_G(\lambda, \mu) = \text{length}_G(v, \lambda) + \sharp X.$$

By Lemma 2.1, we have $\text{length}_G(v, \mu) = \text{length}_M(v, \mu)$. To show (a), it remains to show that

$$I(v) - (\mathbb{S} - J) \cap I(v) = J \cap I(v) = Y \cap J^\circ.$$

Let $l \in J$. Write $v = \lambda - \delta$ for some $\delta \in \sum_{k \in \mathbb{S}-J} \mathbb{R}_{>0} \alpha_k^\vee$. Then $l \in J \cap I(v)$ if and only if $\langle v, \alpha \rangle = \langle \lambda, \alpha_l \rangle - \langle \delta, \alpha_l \rangle = 0$, which is equivalent to $\langle \lambda, \alpha_l \rangle = \langle \delta, \alpha_l \rangle = 0$, that is, $l \in Y \cap J^\circ$ as desired. Hence (a) is proved.

Now, by Lemma 6.1 and the inductive hypothesis on $M_{\mathbb{S}-J}$ and λ , we have

$$\begin{aligned} \sum_{v \in B(G, \mu)_{\text{indec}}} f_{G, \mu}(v) &= \sum_{J \in \mathcal{A}} \sum_{v \in B(M_{\mathbb{S}-J}, \lambda)_{\text{irr}}} f_{G, \mu}(v) \\ &= q^{-\sharp X} \sum_{J \in \mathcal{A}} \sum_{v \in B(M_{\mathbb{S}-J}, \lambda)_{\text{irr}}} (q - 1)^{\sharp(J - Y \cap J^\circ)} q^{\sharp(Y \cap J^\circ)} f_{M_{\mathbb{S}-J}, \lambda}(v) \\ &= q^{-\sharp X} \sum_{J \in \mathcal{A}} (q - 1)^{\sharp(J - Y \cap J^\circ)} q^{\sharp(Y \cap J^\circ)} = 1. \end{aligned} \quad \square$$

Now it is sufficient to deal with the fundamental coweights of G .

6I. Proof for the minuscule coweights. Let μ a be (nonzero) minuscule coweight. Then $\dim V_\mu(\lambda_b) = 1$ for any $[b] \in B(G, \mu)$. By Theorem 5.3, we conclude that $\sharp(J_b(F) \setminus \Sigma^{\text{top}}(X_\mu(b))) = 1$ for any $[b] \in B(G, \mu)$. As in Section 6A, we have

$$q^{\langle \mu, 2\rho \rangle - \sharp(\mathbb{S}/\langle \sigma \rangle)} = \sum_{[b] \in B(G, \mu)_{\text{indec}}} n_{[b]} (q - 1)^{\ell_1(\mu, [b])} q^{\ell_{\Pi}(\mu, [b]) + \ell_{[b]}},$$

where $n_{[b]}$ is the number of reduction paths \underline{p} in a given reduction tree \mathcal{T} of $t^\mu c$ with $[b]_{\underline{p}} = [b]$. At the end of Section 5C, we showed that $\dim X_{\underline{p}} = \dim X_{t^\mu c}([b]_{\underline{p}}) = \dim X_\mu([b]_{\underline{p}})$ for any reduction path \underline{p} . Using Lemma 5.4, we conclude that all $n_{[b]} = 1$. This implies the combinatorial identity (\spadesuit), and then (\spadesuit'') follows.

In particular, the combinatorial identity (♠'') holds for type A , since all the fundamental coweights are minuscule. For (A_{n-1}, ω_i^\vee) , we may write (♠) explicitly as

$$\sum_{\substack{k \geq 1, 1 > a_1/b_1 > \dots > a_k/b_k > 0; \\ a_i + \dots + a_k = i, b_1 + \dots + b_k = n}} (q-1)^{k-1} q^{1-k + (\sum_{1 \leq l_1 < l_2 \leq k} (a_{l_1} b_{l_2} - a_{l_2} b_{l_1}) + \sum_{1 \leq l \leq k} \gcd(a_l, b_l))/2} = q^{(i(n-i)-n)/2+1}.$$

We do not know if there is a purely combinatorial proof of this identity.

6J. Type- D_n case. In this subsection, we assume that G is of type D_n ($n \geq 4$). Note that the fundamental coweights $\omega_1^\vee, \omega_{n-1}^\vee, \omega_n^\vee$ are minuscule and have already been dealt with in Section 6I. Here we deal only with ω_i^\vee for $2 \leq i \leq n-2$.

For any integer k , denote by $[1, k]$ the set $\{m \in \mathbb{Z}; 1 \leq m \leq k\}$. Set $\omega_0^\vee = 0$. Since $\omega_{i-2}^\vee < \omega_i^\vee$, we have a natural embedding $B(G, \omega_{i-2}^\vee) \rightarrow B(G, \omega_i^\vee)$. For $i \geq 3$, Set

$$B_I = \bigsqcup_{i-2 \leq k \leq n-3} B(G, \omega_{i-2}^\vee)_{[1, k]\text{-irr}} \cong \bigsqcup_{i-2 \leq k \leq n-3} B(M_{[1, k]}, \omega_{i-2}^\vee)_{\text{irr}},$$

$$B_{II} = \bigsqcup_{J \subseteq \{n-1, n\}} B(G, \omega_{i-2}^\vee)_{(\mathbb{S}-J)\text{-irr}} \cong \bigsqcup_{J \subseteq \{n-1, n\}} B(M_{\mathbb{S}-J}, \omega_{i-2}^\vee)_{\text{irr}}.$$

For $i = 2$, set $B_I = \{0\}$ and $B_{II} = \emptyset$.

For $i \leq k \leq n-2$, the adjoint group of $M_{\mathbb{S}-\{k\}}$ is of type $A_{k-1} \times D_{n-k}$. Here, by convention, type D_3 is the same as type A_3 , and type D_2 is the same as type $A_1 \times A_1$. Set $\mu_k = (1^{i-1}, 0^{k-i+1}, 1, 0^{n-k-1})$. Then μ and ω_i^\vee are in the same W -orbit. The restriction of μ_k to $M_{\mathbb{S}-\{k\}}$ is (dominant) minuscule, and its projection to the adjoint group of $M_{\mathbb{S}-\{k\}}$ is the coweight $(\omega_{i-1}^\vee, \omega_1^\vee)$ if $k < n-2$ and $(\omega_{i-1}^\vee, \omega_1^\vee, \omega_1^\vee)$ if $k = n-2$. As in Section 2C, we identify $B(M_{\mathbb{S}-\{k\}}, \mu_k)$ with its natural image in $B(G, \omega_i^\vee)$. Set

$$B_{III} = \bigsqcup_{i \leq k \leq n-2} B(G, \mu_k)_{(\mathbb{S}-\{k\})\text{-irr}} \cong \bigsqcup_{i \leq k \leq n-2} B(M_{\mathbb{S}-\{k\}}, \mu_k)_{\text{irr}}.$$

By direct computation, B_I, B_{II} , and B_{III} are disjoint in $B(G, \omega_i^\vee)$.

We give an example to illustrate the subsets B_I, B_{II}, B_{III} . Let G be of type D_6 and $\mu = \omega_3^\vee$. We have

$$B_I = \left\{ \left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0\right), \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0\right) \right\};$$

$$B_{II} = \left\{ \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0\right), \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right), \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}\right), (0, 0, 0, 0, 0, 0) \right\};$$

$$B_{III} = \left\{ \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 0, 0, 0\right), \left(\frac{2}{4}, \frac{2}{4}, \frac{2}{4}, \frac{2}{4}, 0, 0\right) \right\}.$$

In this case, one may check directly that $B(G, \omega_3^\vee)_{\text{indec}} = B_I \sqcup B_{II} \sqcup B_{III}$.

Now we come back to the general situation. Intuitively, when $3 \leq i \leq n-2$, B_{III} consists of Newton vectors whose coordinates sum up to $i-1$, B_I consists of Newton vectors with the last two coordinates 0 and all the coordinates sum up to $i-2$, B_{II} consists of the rest of elements. This partition makes computation of the sum $\sum f_{G, \omega_i^\vee}(v)$ easier.

By Section 6A, we have

$$\sum_{v \in B_I \sqcup B_{II} \sqcup B_{III}} f_{\mathbf{G}, \omega_i^\vee}(v) \leq \sum_{v \in B(\mathbf{G}, \omega_i^\vee)_{\text{indec}}} f_{\mathbf{G}, \omega_i^\vee}(v) \leq 1.$$

In the rest of this section, we will show that

$$\sum_{v \in B_I \sqcup B_{II} \sqcup B_{III}} f_{\mathbf{G}, \omega_i^\vee}(v) = 1. \tag{**}$$

The equality (**) will, in particular, imply that $B_I \sqcup B_{II} \sqcup B_{III} = B(\mathbf{G}, \omega_i^\vee)_{\text{indec}}$.

It can be checked directly that (**) holds for D_4 . Using induction, we may assume that (**) holds for groups of type D with semisimple rank less than n . Note also that (♣'') for type A has already been proved in Section 6I. Therefore we have

$$(a) \sum_{v \in B_{II}} f_{\mathbf{G}, \omega_i^\vee}(v) = \mathbf{q}^{-\langle \alpha^\vee, \rho \rangle} \cdot (1 + 2(\mathbf{q} - 1)^2 + (\mathbf{q} - 1)^2) = \mathbf{q}^{-\langle \alpha^\vee, \rho \rangle} \cdot \mathbf{q}^2.$$

Next we handle B_I and B_{III} . Let

$$\alpha^\vee = \omega_i^\vee - \omega_{i-2}^\vee = \alpha_{i-1}^\vee + 2\alpha_i^\vee + \dots + 2\alpha_{n-2}^\vee + \alpha_{n-1}^\vee + \alpha_n^\vee.$$

Note that $\langle \alpha^\vee, \rho \rangle = 2n - 2i + 1$. We claim that:

(b) For $k \in [i - 2, n - 3]$ and $v \in B(\mathbf{M}_{[1, k]}, \omega_{i-2}^\vee)_{\text{irr}}$, we have

$$f_{\mathbf{G}, \omega_i^\vee}(v) = (\mathbf{q} - 1) \cdot \mathbf{q}^{-\langle \alpha^\vee, \rho \rangle + n - k - 1} \cdot f_{\mathbf{M}_{[1, k]}, \omega_{i-2}^\vee}(v).$$

(c) For $k \in [i, n - 2]$ and $v \in B(\mathbf{M}_{\mathbb{S} - \{k\}}, \mu_k)_{\text{irr}}$, we have

$$f_{\mathbf{G}, \omega_i^\vee}(v) = (\mathbf{q} - 1) \cdot \mathbf{q}^{-\langle \alpha^\vee, \rho \rangle + 2n - k - i} \cdot f_{\mathbf{M}_{\mathbb{S} - \{k\}}, \mu_k}(v).$$

We prove (c) here. The proof of (b) is similar.

By definition, we have

$$f_{\mathbf{G}, \omega_i^\vee}(v) = (\mathbf{q} - 1) \cdot \mathbf{q}^{n - k - n_0} \cdot f_{\mathbf{M}_{\mathbb{S} - \{k\}}, \mu_k}(v),$$

where $n_0 = \text{length}_{\mathbf{G}}(v, \omega_i^\vee) - \text{length}_{\mathbf{M}_{\mathbb{S} - \{k\}}}(v, \mu_k)$. It can be checked directly that

$$\langle \mu_k - v, \rho_{\mathbf{M}_{\mathbb{S} - \{k\}}} \rangle = \langle \omega_{i-1}^\vee - v, \rho \rangle + \langle (0^k, 1, 0^{n-k-1}), \rho_{\mathbf{M}_{[k+1, n]}} \rangle = \langle \omega_{i-1}^\vee - v, \rho \rangle + 1.$$

Note also that $\text{def}_{\mathbf{G}}(v) = \text{def}_{\mathbf{M}_{\mathbb{S} - \{k\}}}(v)$. Therefore, by the length formula, we have

$$\begin{aligned} n_0 &= \langle \omega_i^\vee - v, \rho \rangle - \langle \mu_k - v, \rho_{\mathbf{M}_{\mathbb{S} - \{k\}}} \rangle + \frac{1}{2} (\text{def}_{D_n}(v) - \text{def}_{\mathbf{M}_{\mathbb{S} - \{k\}}}(v)) \\ &= \langle \omega_i^\vee - \omega_{i-1}^\vee, \rho \rangle + 1 = n - i + 1 = \langle \alpha^\vee, \rho \rangle - n + i. \end{aligned}$$

The statement (c) follows.

	μ	ω_1^\vee	ω_2^\vee	ω_3^\vee	ω_4^\vee	ω_5^\vee	ω_6^\vee	ω_7^\vee	ω_8^\vee
E_6		7	15	30	15				
E_7	$\sharp B(\mathbf{G}, \mu)_{\text{indec}}$	13	26	50	125	69	32		
E_8		56	126	254	729	424	220	94	27

Table 1. The numbers of elements in $B(\mathbf{G}, \mu)_{\text{indec}}$ for all the fundamental, nonminuscule coweights in type E .

Combining (a), (b), and (c) with the combinatorial identity (♠) for type A and for type D_l with $l < n$, we have

$$\begin{aligned} \sum_{v \in B_I \sqcup B_{II} \sqcup B_{III}} f_{\mathbf{G}, \omega_i^\vee}(v) &= q^{-\langle \alpha^\vee, \rho \rangle} \cdot \left(q^2 + (q-1) \left(\sum_{k=i-2}^{n-3} q^{n-k-1} + \sum_{k=i}^{n-2} q^{2n-k-i} \right) \right) \\ &= q^{-\langle \alpha^\vee, \rho \rangle} \cdot q^{2n-2i+1} = 1. \end{aligned}$$

This completes the proof of (**).

6K. Type- E case. In this subsection, we assume that \mathbf{G} is of type E_n . We verify the combinatorial identity (♠'') by computer. Recall that a vector $v \in (V^+)^\sigma$ lies in $B(\mathbf{G}, \mu)$ if and only if $\langle \mu - v, \omega_i \rangle \in \mathbb{Z}_{\geq 0}$ for any $i \in \mathbb{S} - I(v)$. As a consequence, we have the following characterization of $B(\mathbf{G}, \mu)_{\text{indec}}$:

(a) The set $B(\mathbf{G}, \mu)_{\text{indec}}$ equals the set of dominant vectors of the form

$$v = \text{pr}_I \left(\mu - \sum_{i \in I} c_i \alpha_i^\vee \right),$$

where I is a σ -stable subset of \mathbb{S} , $c_i \in \mathbb{Z}$, $1 \leq c_i \leq \langle \mu, \omega_i \rangle$, and $\text{pr}_I : V \rightarrow \bigoplus_{i \in I} \mathbb{R} \omega_i^\vee$ is the natural orthogonal projection.

On the basis of (a), we can use a computer program to list all the elements in $B(\mathbf{G}, \mu)_{\text{indec}}$ and then verify the combinatorial identity (♠'') directly.

In Table 1, we provide the numbers of elements in $B(\mathbf{G}, \mu)_{\text{indec}}$ for all the fundamental, nonminuscule coweights in type E . The most complicated case is E_8 , and $\mu = \omega_4^\vee$, in which $B(\mathbf{G}, \mu)_{\text{indec}}$ contains 729 elements.

7. The general case

7A. Description of the reduction trees. Let $w \in W t^\mu W$ with finite partial σ -Coxeter part, that is, $\eta_\sigma(w)$ is a partial σ -Coxeter element. For any $J \in [J_0(w), J(w)]_\mu$ and $[b] \in B(\mathbf{G}, \mu)_{J\text{-irr}}$, we set

$$\begin{aligned} J^{b,w} &= \{i \in I(\mu^\diamond) \cap (J(w) - J) \mid i \text{ commutes with } J\}, \\ \ell_I(w, [b], J) &= \sharp(J(w)/\langle \sigma \rangle) - \sharp(J^{b,w}/\langle \sigma \rangle) - \sharp(I(v_b) \cap J/\langle \sigma \rangle), \\ \ell_{II}(w, [b], J) &= \text{length}([b], [t^\mu]) - \sharp(J_0(w)/\langle \sigma \rangle). \end{aligned}$$

By Lemma 2.1, there exists a unique σ -conjugacy class $[b]_{M_J} \in B(M_J, \mu)$ such that $[b]_{M_J} \subseteq [b]$. We similarly define $[b]_{M_{J^b, w \cup J}} \in B(M_{J^b, w \cup J}, \mu)$. In this case, a σ -Coxeter element associated with $[b]_{M_{J^b, w \cup J}}$ is equal to the product of a σ -Coxeter element of $W_{J^b, w}$ and a σ -Coxeter element associated with $[b]_{M_J}$.

The main result of this section is the following description of the reduction tree of w .

Theorem 7.1. *Let $w \in Wt^\mu W$ with $\eta_\sigma(w)$ a partial σ -Coxeter element. Let \mathcal{T} be a reduction tree of w . Then, for any $J \in [J_0(w), J(w)]_\mu$ and $[b] \in B(G, \mu)_{J\text{-irr}}$, there exists a unique reduction path \underline{p} in \mathcal{T} with $[b]_{\underline{p}} = [b]$. Moreover,*

- (1) $\ell_I(\underline{p}) = \ell_I(w, [b], J)$ and $\ell_{II}(\underline{p}) = \ell_{II}(w, [b], J)$;
- (2) $\text{end}(\underline{p})$ is a σ -Coxeter element associated with $[b]_{M_{J^b, w \cup J}}$.

Combining Theorem 7.1 with Proposition 3.9 and Remark 3.10, we obtain Theorem 2.6(3) for w .

7B. Strategy. The strategy for proving Theorem 7.1 is very different from that adopted for the proof of Theorem 5.1. In the latter case, we used the Chen–Zhu conjecture and the dimension formula to determine the end points of the reduction trees of $t^\mu c$. However, such a method, when applied to general w , cannot determine the end points.

The approach we use here is as follows. We first apply the partial reduction method and the class polynomials for $t^\mu c$ to calculate the class polynomials for w . As we mentioned earlier, class polynomials, in general, contain less information than reduction trees. Fortunately, for the elements w we consider here, by combining the information on the class polynomials and the estimates on the type-I edges, we obtain the required information for any reduction tree.

The information about the class polynomials we need is contained in the following equality on the σ -cocenter of the Iwahori–Hecke algebra H :

$$T_w + [H, H]_\sigma = \sum_{\substack{J \in [J_0(w), J(w)]_\mu \\ [b] \in B(G, \mu)_{J\text{-irr}}} } (q - 1)^{\ell_I(w, [b], J)} q^{\ell_{II}(w, [b], J)} T_{\mathcal{O}_{w, [b]}} + [H, H]_\sigma, \tag{\diamond}$$

where $\mathcal{O}_{w, [b]}$ is the σ -conjugacy class containing a σ -Coxeter element associated with $[b]_{M_{J^b, w \cup J}}$.

7C. Proof of (\diamond) .

7C1. We consider the case where $w = t^\mu c \in \mathbb{S}\tilde{W}$ for some partial σ -Coxeter element c of W . Let $J = \text{supp}_\sigma(c)$. In this case, $J_0(w) = J(w) = J$.

Suppose $J = \mathbb{S}$, that is, c is a (full) σ -Coxeter element. Let \mathcal{T} be a reduction tree of w . By the description of the reduction tree in Theorem 5.1, each end point of \mathcal{T} is a σ -Coxeter element associated with some $[b] \in B(G, \mu)_{\text{irr}}$. Then by Lemma 3.8, we get $F_{w, \mathcal{O}} = (q - 1)^{\ell_I(w, [b], J)} q^{\ell_{II}(w, [b], J)}$ if \mathcal{O} contains an end point e of \mathcal{T} with $\Phi(e) = [b]$ and $F_{w, \mathcal{O}} = 0$ otherwise. Then (\diamond) follows.

Assume $J \subsetneq \mathbb{S}$. It follows from [18, Theorem 7.3] that $F_{w, \mathcal{O}} = \sum_{\mathcal{O}' \subseteq \mathcal{O}} F_{w, \mathcal{O}'}^J$, where \mathcal{O}' denotes a σ -conjugacy class of \tilde{W}_J and $F_{w, \mathcal{O}'}^J$ denotes the corresponding class polynomial for M_J . Using the description of each $F_{w, \mathcal{O}'}^J$ we get in the $J = \mathbb{S}$ case, we conclude that $F_{w, \mathcal{O}} = (q - 1)^{\ell_I(w, [b], J)} q^{\ell_{II}(w, [b], J)}$ if \mathcal{O} contains a σ -Coxeter element associated with $[b]_{M_J}$, and $F_{w, \mathcal{O}} = 0$ otherwise. This proves (\diamond) .

7C2. We consider the case where $w = c_1 t^\mu c_2$ for some partial σ -Coxeter elements c_1, c_2 of W such that c_1 commutes with c_2 , $t^\mu c_2 \in \mathbb{S}\tilde{W}$ and $c_1 \in W_{I(t^\mu c_2)}$.

Set $J_1 = \text{supp}_\sigma(c_1)$, $J_2 = \text{supp}_\sigma(c_2)$ and $J = J_1 \sqcup J_2 = J(w)$. One can construct a reduction tree \mathcal{T} of $t^\mu c_2$ in \tilde{W}_{J_2} such that c_1 commutes with all the vertices in \mathcal{T} . In particular, if $w_1 \rightarrow w_2$ is one edge in \mathcal{T} , then we have $c_1 w_1 \rightarrow c_1 w_2$ in \tilde{W}_J . Note that c_1 is a minimal length element in its σ -conjugacy class in \tilde{W}_{J_1} , and the simple reflections in \tilde{W}_{J_1} commute with the simple reflections in \tilde{W}_{J_2} . It is easy to see that if w_1 is an end point in \mathcal{T} (and hence is a minimal length element in its σ -conjugacy class in \tilde{W}_{J_2} and commutes with c_1), then $c_1 w_1$ is a minimal length element in its σ -conjugacy class in \tilde{W}_J .

Let \mathcal{T}' be the tree with vertices $c_1 w_1$ for all vertices $w_1 \in \mathcal{T}$ and the edges $c_1 w_1 \rightarrow c_1 w_2$ for all edges $w_1 \rightarrow w_2$ in \mathcal{T} . Then, from the above discussion, \mathcal{T}' is a reduction tree of $c_1 t^\mu c_2$ in \tilde{W}_J . Note that

$$\ell_1(w, [b], J_2) = \ell_1(t^\mu c_2, [b], J_2) \quad \text{and} \quad \ell_{\text{II}}(w, [b], J_2) = \ell_{\text{II}}(t^\mu c_2, [b], J_2) \quad \text{for any } [b] \in B(\mathbf{G}, \mu)_{J_2\text{-irr}}.$$

On the other hand, we have

$$F_{w, \mathcal{O}} = \sum_{\mathcal{O}^J \subseteq \mathcal{O}} F_{w, \mathcal{O}^J}^J.$$

Hence (\diamond) for w follows from (\diamond) for $t^\mu c_2$ established in Section 7C1.

7C3. We consider the case where $w = c_1 t^\mu c_2$ for partial σ -Coxeter elements c_1 and c_2 in W such that $t^\mu c_2 \in \mathbb{S}\tilde{W}$ and $\text{supp}_\sigma(c_1) \cap \text{supp}_\sigma(c_2) = \emptyset$.

We prove (\diamond) by induction on $\ell(c_1)$. The case $\ell(c_1) = 0$ is proved in Section 7C1.

Assume that $\ell(c_1) > 0$. Let $i \in \mathbb{S}$ such that $s_i c_1 < c_1$. There are two cases as follows.

Case (1): $c_2 \sigma(s_i) \in I(\mu)W$. Write $w_1 = s_i w$ and $w_2 = s_i w \sigma(s_i) = s_i c_1 t^\mu c_2 \sigma(s_i)$. Then $T_w + [H, H]_\sigma = (q-1)T_{w_1} + qT_{w_2} + [H, H]_\sigma$. Note that $J(w_2) = J(w)$, $J_0(w_1) = J_0(w)$, $J(w_1) = J(w) - \{\sigma^\ell(s_i) \mid \ell \in \mathbb{Z}\}$, and $J_0(w_2) = J_0(w) \sqcup \{\sigma^\ell(s_i) \mid \ell \in \mathbb{Z}\}$. Then

$$[J_0(w), J(w)]_\mu = [J_0(w_1), J(w_1)]_\mu \sqcup [J_0(w_2), J(w_2)]_\mu.$$

By the induction hypothesis, it suffices to prove that

- (a) if $J \in [J_0(w_1), J(w_1)]_\mu$, then $J^{b,w} = J^{b,w_1}$; and
- (b) if $J \in [J_0(w_2), J(w_2)]_\mu$, then $J^{b,w} = J^{b,w_2}$.

Statement (b) is obvious since $J(w_2) = J(w)$. Let us prove (a). Suppose $J^{b,w} \neq J^{b,w_1}$; then $i \in I(\mu^\diamond)$ and s_i commutes with J . Since $J \supseteq J_0(w)$, s_i also commutes with $J_0(w)$. Then μ is not essentially noncentral over $J_0(w_2) = J_0(w) \sqcup \{\sigma^l(i) \mid l \in \mathbb{Z}\}$, which is a contradiction. This completes the proof.

Case (2): $\sigma(i) \in I(\mu)$, and $\sigma(s_i)$ commutes with c_2 . Then (\diamond) holds for w if and only if it holds for $s_i w \sigma(s_i)$. We continue with the procedure until case (1) happens. If case (2) happens all the time and the procedure does not stop, then $c_1 \in W_{I(t^\mu c_2)}$, and (\diamond) follows from Section 7C2.

7D. Proof of Theorem 7.1.

7D1. We consider the case $w = ct^\mu$, where c is a partial σ -Coxeter element of W . Let \mathcal{T} be a reduction tree of w . By Lemma 3.8 and (\diamond) for w , we have that, for any σ -conjugacy class \mathcal{O} of \tilde{W} ,

$$\sum_{\underline{p}; \text{end}(\underline{p}) \in \mathcal{O}} (\mathbf{q} - 1)^{\ell_1(\underline{p})} \mathbf{q}^{\ell_\Pi(\underline{p})} = \begin{cases} (\mathbf{q} - 1)^{\ell_1(w, [b], J)} \mathbf{q}^{\ell_\Pi(w, [b], J)} & \text{if } \mathcal{O} = \mathcal{O}_{w, [b]} \text{ for some } [b], \\ 0 & \text{otherwise.} \end{cases}$$

Let \underline{p} be a path in \mathcal{T} . Set $e = \text{end}(\underline{p})$ and $[b] = [b]_{\underline{p}}$. Assume that $[b] \in B(\mathbf{G}, \mu)_{J\text{-irr}}$ for some J . As $(\mathbf{q} - 1)^{\ell_1(\underline{p})} \mathbf{q}^{\ell_\Pi(\underline{p})} \in \mathbb{N}[\mathbf{q} - 1]$, there is no cancellation involved in the left-hand side of the above equality. Therefore, e must be contained in $\mathcal{O}_{w, [b]}$.

As in Section 5C, we have

$$\begin{aligned} \ell_1(\underline{p}) &\geq \dim V_e - \dim V_w = \sharp(J(w)/\langle \sigma \rangle) - \sharp(J^{b, w}/\langle \sigma \rangle) - \sharp(I(v_b) \cap J/\langle \sigma \rangle) \\ &= \ell_1(w, [b], J). \end{aligned}$$

Note that $\ell_1(\underline{p}) + 2\ell_\Pi(\underline{p}) = \ell_1(w, [b], J) + 2\ell_\Pi(w, [b], J)$. Thus

$$\deg_{\mathbf{q}} (\mathbf{q} - 1)^{\ell_1(\underline{p})} \mathbf{q}^{\ell_\Pi(\underline{p})} \geq \deg_{\mathbf{q}} (\mathbf{q} - 1)^{\ell_1(w, [b], J)} \mathbf{q}^{\ell_\Pi(w, [b], J)},$$

with equality holding if and only if $\ell_1(\underline{p}) = \ell_1(w, [b], J)$. Again, since there is no cancellation involved in $\sum_{\underline{p}; \text{end}(\underline{p}) \in \mathcal{O}} (\mathbf{q} - 1)^{\ell_1(\underline{p})} \mathbf{q}^{\ell_\Pi(\underline{p})}$, we must have $\ell_1(\underline{p}) = \ell_1(w, [b], J)$. In this case, $(\mathbf{q} - 1)^{\ell_1(\underline{p})} \mathbf{q}^{\ell_\Pi(\underline{p})} = (\mathbf{q} - 1)^{\ell_1(w, [b], J)} \mathbf{q}^{\ell_\Pi(w, [b], J)}$. This also shows that for each $\mathcal{O} = \mathcal{O}_{w, [b]}$, there is a unique reduction path \underline{p} with $\text{end}(\underline{p}) \in \mathcal{O}$.

This completes the proof of Theorem 7.1 for $w = ct^\mu$.

7D2. Now we consider the general case. Let $w = xt^\mu y$ with $t^\mu y \in \mathbb{S}\tilde{W}$. Set $c = \sigma^{-1}(y)x$ and $w' = ct^\mu$. We relate w and w' as in the proof of [12, Theorem 10.3]. Let $y = s_1 s_2 \cdots s_r$ be a reduced expression. Let $w^{(0)} = w'$, $w^{(1)} = \sigma^{-1}(s_1)w^{(0)}s_1$, $w^{(2)} = \sigma^{-1}(s_2)w^{(1)}s_2, \dots, w^{(r)} = w$. We have $w^{(0)} \rightarrow_\sigma w^{(1)} \rightarrow_\sigma \cdots \rightarrow_\sigma w^{(r)}$. This give a path $w' \rightarrow w$, consisting of $\frac{1}{2}(\ell(w') - \ell(w))$ type-II edges. We denote this path by \underline{p}_0 .

Let \mathcal{T} be a reduction tree of w . One may construct a reduction tree \mathcal{T}' of w' containing the concatenation $\underline{p}_0 \circ \mathcal{T}$ as a subgraph. In particular, for any reduction path \underline{p} in \mathcal{T} , the concatenation $\underline{p}' := \underline{p}_0 \circ \underline{p}$ is a reduction path in \mathcal{T}' . By definition, $\ell_1(\underline{p}) = \ell_1(\underline{p}')$ and $\ell_\Pi(\underline{p}) + \frac{1}{2}(\ell(w') - \ell(w)) = \ell_\Pi(\underline{p}')$. It is obvious that $J(w) = J(w')$ and $J_0(w') = \emptyset$. Hence $\ell_1(w, [b], J) = \ell_1(w', [b], J)$. On the other hand, we have $\sharp(J_0(w)/\langle \sigma \rangle) = \text{length}([b_w], [t^\mu]) = \frac{1}{2}(\ell(\eta_\sigma(w) + \ell(y) - \ell(x))) = \frac{1}{2}(\ell(w') - \ell(w))$, where the second equality follows from the definition of cordial elements. Hence $\ell_\Pi(w, [b], J) + \frac{1}{2}(\ell(w') - \ell(w)) = \ell_\Pi(w', [b], J)$. The statements for \mathcal{T} can now be deduced from the statements for \mathcal{T}' .

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On 03/08/2023, Dong Gyu Lim [24] informed us that he found a purely combinatorial proof of (*) for all quasisplit reductive groups G in the introduction. Lim’s approach is based on a probability-theoretic interpretation of a variant of (*) and is independent of our approach.

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