Semistable models for some unitary Shimura varieties over ramified primes

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We consider Shimura varieties associated to a unitary group of signature $(n-2, 2)$. We give regular p-adic integral models for these varieties over odd primes p which ramify in the imaginary quadratic field with level subgroup at p given by the stabilizer of a selfdual lattice in the hermitian space. Our construction is given by an explicit resolution of a corresponding local model.

1. Introduction

1A. This paper is a contribution to the problem of constructing regular integral models for Shimura varieties over places of bad reduction. There are several implicit examples of constructions of such regular integral models in special cases; see, for example, work of de Jong [1993], Genestier [2000], Pappas [2000b], Faltings [1997] and the very recent work of Pappas and Zachos [2022]. Here, we consider Shimura varieties associated to unitary groups of signature (r, s) over an imaginary quadratic field F_0. These Shimura varieties are of PEL type, so they can be written as a moduli space of abelian varieties with polarization, endomorphisms and level structure. Shimura varieties have canonical models over the "reflex" number field E. In the cases we consider here the reflex field is the field of rational numbers $\mathbb{Q}$ if $r = s$ and $E = F_0$ otherwise.

Constructing such well-behaved integral models is an interesting and hard problem whose solution has many applications to number theory. The behavior of these depends very much on the "level subgroup". Here, the level subgroup is the stabilizer of a selfdual lattice in the hermitian space. This stabilizer, by what follows below, is not connected when n is even, so not parahoric. However, by using work of

Rapoport and Zink [1996] and Pappas [2000b] we construct p-adic integral models, which have simple and explicit moduli descriptions, and are étale locally around each point isomorphic to certain simpler schemes the *naive local models*. Inspired by the work of Pappas and Rapoport [2005] and Krämer [2003], we consider a variation of the above moduli problem where we add in the moduli problem an additional subspace in the deRham filtration $\mathrm{Fil}^0(A) \subset H^1_{\mathrm{dR}}(A)$ of the universal abelian variety A, which satisfies certain conditions. This is essentially an instance of the notion of a "linear modification" introduced in [Pappas 2000b]. We then show that the blow-up of this model along a smooth (non-Cartier) divisor produces a semistable integral model of the corresponding Shimura variety, i.e., it is regular and the irreducible components of the special fiber are smooth divisors crossing normally. We expect that our construction will find applications to the study of arithmetic intersections of special cycles and Kudla's program; see [Zhang 2021; Bruinier et al. 2020; He et al. 2023] for important applications of integral models of unitary Shimura varieties to number theory.

1B. To explain our results, we need to introduce some notation. We consider the group G of unitary similitudes for a hermitian vector space (W, ϕ) of dimension $n > 3$ over an imaginary quadratic field $F_0 \subset \mathbb{C}$, and fix a conjugacy class of homomorphisms $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$ corresponding to a Shimura datum (G, X_h) of signature $(r, s) = (n - 2, 2)$ (see Section 6). Let us mention here that the case $(r, s) = (1, 2)$, when $n = 3$, was studied in [Pappas 2000b, 4.5, 4.15]; see also [Pappas and Rapoport 2009, Section 6].

We assume that $F_0/\mathbb{Q}$ is ramified over p, where p is an odd prime number. Let $F_1 = F_0 \otimes \mathbb{Q}_p$ and $V = W \otimes_{\mathbb{Q}} \mathbb{Q}_p$. We fix a square root π of p and we set $k = \bar{\mathbb{F}}_p$. We assume that the hermitian form ϕ on V is split, i.e., that there is a basis $e_1, \ldots, e_n$ such that $\phi(e_i, e_{n+1-j}) = \delta_{ij}$ for $i, j \in \{1, \ldots n\}$.

In addition, we denote by Λ the standard lattice $O_{F_1}^n$ in V and we let $\mathcal{L}$ be the self-dual multichain consisting of $\{\pi^k \Lambda\}_{k \in \mathbb{Z}}$. Denote by K the stabilizer of Λ in $G(\mathbb{Q}_p)$ and let $\mathcal{G}$ be the (smooth) group scheme of automorphisms of the polarized chain $\mathcal{L}$ over $\mathbb{Z}_p$; see [Pappas and Rapoport 2009, Section 1.5]. Then $\mathcal{G}(\mathbb{Z}_p) = K$ and the group scheme $\mathcal{G}$ has $G \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ as its generic fiber. It turns out that when n is odd the stabilizer K is a parahoric subgroup. When n is even, K is not a parahoric subgroup since it contains a parahoric subgroup with index 2 and the corresponding parahoric group scheme is its connected component K°; see [Pappas and Rapoport 2009, Section 1.2] for more details.

Choose also a sufficiently small compact open subgroup K^p of the prime-to-p finite adelic points $G(\mathbb{A}_f^p)$ of G and set $\boldsymbol{K} = K^p K$. The Shimura variety $\mathrm{Sh}_{\boldsymbol{K}}(G, X)$ with complex points

$$\mathrm{Sh}_{\boldsymbol{K}}(G, X)(\mathbb{C}) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/\boldsymbol{K}$$

is of PEL type. We set $\mathcal{O} = O_{E_v}$ where v the unique prime ideal of E above (p).

Next, we follow [Rapoport and Zink 1996, Definition 6.9] to define the moduli scheme $\mathcal{A}_{\boldsymbol{K}}^{\mathrm{naive}}$ over $\mathcal{O}$ whose generic fiber agrees with $\mathrm{Sh}_{\boldsymbol{K}}(G, X)$ (see also Section 6). A point of $\mathcal{A}_{\boldsymbol{K}}^{\mathrm{naive}}$ with values in the $\mathcal{O}$-scheme S is the isomorphism class of the following set of data $(A, \bar{\lambda}, \bar{\eta})$:

(1) An $\mathcal{L}$-set of abelian varieties $A = \{A_\Lambda\}$.

(2) A $\mathbb{Q}$-homogeneous principal polarization $\bar{\lambda}$ of the $\mathcal{L}$-set A.

(3) A K^p-level structure

$$\bar{\eta} : H_1(A, \mathbb{A}_f^p) \simeq W \otimes \mathbb{A}_f^p \bmod K^p$$

which respects the bilinear forms on both sides up to a constant in $(\mathbb{A}_f^p)^\times$; see [loc. cit.] for details. The set A should satisfy the determinant condition (i) of [loc. cit.]

For the definitions of the terms employed here we refer to [loc. cit., 6.3–6.8] and [Pappas 2000b, Section 3]. The functor $\mathcal{A}_K^{\text{naive}}$ is representable by a quasiprojective scheme over $\mathcal{O}$. The moduli scheme $\mathcal{A}_K^{\text{naive}}$ is connected to the *naive local model* M^{naive}, see Section 2 for the explicit definition of M^{naive}, via the local model diagram

$$\begin{array}{ccc} & \widetilde{\mathcal{A}}_K^{\text{naive}} & \\ {\scriptstyle \pi_K}\swarrow & & \searrow{\scriptstyle q_K} \\ \mathcal{A}_K^{\text{naive}} & & M^{\text{naive}} \end{array} \qquad (1\text{B}.1)$$

where the morphism π_K is a $\mathcal{G}$-torsor and q_K is a smooth and $\mathcal{G}$-equivariant morphism (see Section 6). Equivalently, using the language of algebraic stacks, there is a relatively representable smooth morphism

$$\mathcal{A}_K^{\text{naive}} \to [\mathcal{G}\backslash M^{\text{naive}}]$$

where the target is the quotient algebraic stack. In particular, since $\mathcal{G}$ is smooth, the above imply that $\mathcal{A}_K^{\text{naive}}$ is étale locally isomorphic to M^{naive}.

One can now consider a variation of the moduli of abelian schemes $\mathcal{A}_K^{\text{spl}}$ over Spec O_{F_1} where we add in the moduli problem an additional subspace in the Hodge filtration $\text{Fil}^0(A) \subset H_{\text{dR}}^1(A)$ of the universal abelian variety A with certain conditions to imitate the definition of the splitting local model $\mathcal{M}$; see Section 6B for the explicit definition of $\mathcal{A}_K^{\text{spl}}$ and Section 2 where we define $\mathcal{M}$ for general signature (r, s). (Actually, $\mathcal{M}$ is a generalization of Krämer's local models [Krämer 2003, Definition 4.1]). There is a forgetful morphism

$$\tau : \mathcal{A}_K^{\text{spl}} \to \mathcal{A}_K^{\text{naive}} \otimes_{\mathcal{O}} O_{F_1}$$

defined by forgetting the extra subspace. Moreover, $\mathcal{A}_K^{\text{spl}}$ has the same étale local structure as $\mathcal{M}$ and is a linear modification of $\mathcal{A}_K^{\text{naive}} \otimes_{\mathcal{O}} O_{F_1}$ in the sense of [Pappas 2000b, Section 2] (see also [Pappas and Rapoport 2005, Section 15]). Therefore, there is a local model diagram for $\mathcal{A}_K^{\text{spl}}$ similar to (1B.1) but with M^{naive} replaced by $\mathcal{M}$. Note, that there is also a corresponding forgetful morphism

$$\tau_1 : \mathcal{M} \to M^{\text{naive}} \otimes_{\mathcal{O}} O_{F_1}.$$

In Section 2, we show that $\tau_1^{-1}(*)$ is isomorphic to the Grassmannian $\text{Gr}(2, n)_k$. Here, $*$ is the "worst point" of M^{naive}, i.e., the unique closed $\mathcal{G}$-orbit supported in the special fiber; see [Pappas 2000b, Section 4] for more details. Under the local model diagram, (see Section 6), $\tau_1^{-1}(*)$ corresponds to the

locus where the Hodge filtration $\mathrm{Fil}^0(A)$ of the universal abelian scheme A is annihilated by the action of the uniformizer π. Consider the blow-up $\mathcal{A}_K^{\mathrm{bl}}$ of $\mathcal{A}_K^{\mathrm{spl}}$ along this locus.

1C. The main result of the paper is the following theorem.

Theorem 1.1. $\mathcal{A}_K^{\mathrm{bl}}$ *is a semistable integral model for the Shimura variety* $\mathrm{Sh}_K(G, X)$.

Since blowing-up commutes with étale localization and the étale local structure of the moduli scheme $\mathcal{A}_K^{\mathrm{spl}}$ is controlled by the local structure of the local model $\mathcal{M}$, it is enough to show the above statement for the corresponding local models. In particular, it suffices to prove:

Theorem 1.2. *The blow-up* M^{bl} *of* $\mathcal{M}$ *along the smooth irreducible component* $\tau_1^{-1}(*)$ *of its special fiber is regular and has special fiber a divisor with normal crossings.*

To show the above theorem, we explicitly calculate an affine chart $\mathcal{U}$ of $\mathcal{M}$ in a neighborhood of $\tau_1^{-1}(*)$. In fact, we consider a more general situation where we calculate $\mathcal{U}$ for a general signature (r, s) and we show that $\mathcal{G}$-translates of $\mathcal{U}$ cover $\mathcal{M}$.

Proposition 1.3. *An affine chart* $\mathcal{U} \subset \mathcal{M}$ *containing a preimage of the worst point is isomorphic to*

$$\mathrm{Spec}\, O_{F_1}[X, Y]/(X - X^t, X \cdot (I_s + Y^t \cdot Y) - 2\pi I_s)$$

where X, Y *are of sizes* $s \times s$ *and* $(n - s) \times s$ *respectively.*

When $(r, s) = (n - 1, 1)$, Krämer [2003] shows that $\mathcal{U}$, and so $\mathcal{M}$, has semistable reduction. Therefore, she obtains a semistable integral model for the corresponding Shimura variety.

When $(r, s) = (n - 2, 2)$, $\mathcal{U}$ does not have semistable reduction anymore and so $\mathcal{M}$ does not give us a resolution. However, we use the explicit description of $\mathcal{U}$ above to calculate the blow-up of $\mathcal{M}$ along the $\mathcal{G}$-invariant smooth subscheme $\tau_1^{-1}(*)$. The blow-up gives a $\mathcal{G}$-birational projective morphism

$$r^{\mathrm{bl}} : \mathrm{M}^{\mathrm{bl}} \to \mathcal{M}$$

such that M^{bl} is regular and has special fiber a reduced divisor with normal crossings. We quickly see that the corresponding blow-up $\mathcal{A}_K^{\mathrm{bl}}$ of the integral model $\mathcal{A}_K^{\mathrm{spl}}$ inherits the same nice properties as M^{bl}. In fact, there is a local model diagram for $\mathcal{A}_K^{\mathrm{bl}}$ similar to (1B.1) but with $\mathrm{M}^{\mathrm{naive}}$ replaced by M^{bl}. See Theorem 6.1 for the precise statement about the model $\mathcal{A}_K^{\mathrm{bl}}$.

Let us mention here that we can obtain similar results for the Shimura varieties $\mathrm{Sh}_{K'}(G, X)$ where $K' = K^p K^\circ$ (see Section 6). (Recall that K° is the parahoric connected component of the stabilizer K.) Also, we can apply these results to obtain regular (formal) models of the corresponding Rapoport–Zink spaces.

Let us now explain the lay-out of the paper. In Section 2, we recall the definitions of certain variants of local models for ramified unitary groups. In Section 3, we give explicit equations that describe the affine chart $\mathcal{U}$ of the *splitting model* $\mathcal{M}$ for a general signature (r, s) and we also show that $\mathcal{G}$-translates of $\mathcal{U}$ cover $\mathcal{M}$. For the rest of the paper we assume $(r, s) = (n - 2, 2)$. In Section 4, we construct the semistable resolution $\rho : \mathcal{U}^{\mathrm{bl}} \to \mathcal{U}$ of the affine chart $\mathcal{U}$. In Section 5, we show that M^{bl} has semistable

reduction by using the results of Section 4 and the structure of local models. In Section 6, we apply the above results to construct regular integral models for the corresponding Shimura varieties.

2. Preliminaries: local models and variants

We use the notation of [Pappas 2000b]. We take $F = \mathbb{Q}_p[t]/(t^2 - pu)$ and $O_F = \mathbb{Z}_p[t]/(t^2 - pu)$, where p is an odd prime and u is a unit in $\mathbb{Z}_p$. For $n > 3$, we set $V = F^n$ and denote by e_i, $1 \le i \le n$, the standard O_F-generators of the standard lattice $\Lambda = O_F^n \subset V$. Fix a uniformizer π of O_F with $\pi^2 = p\delta$. Also, since $p \ne 2$, $\delta = \pi^2/p$ has a square root in a finite étale extension of $\mathrm{Spec}(\mathbb{Z}_p)$. After such a base extension there is a uniformizer π such that $\pi^2 = p$. We will assume that we have such a uniformizer and suppress the notation of the étale base extension.

Set $k = \bar{\mathbb{F}}_p$. The uniformizing element π induces a $\mathbb{Z}_p$-linear mapping on Λ which we denote by t. We define a nondegenerate alternating $\mathbb{Q}_p$-bilinear form $\langle \, , \rangle : V \times V \to \mathbb{Q}_p$ given by

$$\langle e_i, t e_j \rangle = \delta_{i,j}, \quad \langle e_i, e_j \rangle = 0, \quad \langle t e_i, t e_j \rangle = 0.$$

The restriction $\langle \, , \rangle : O_F^n \times O_F^n \to \mathbb{Z}_p$ is a perfect $\mathbb{Z}_p$-bilinear form. Using the duality isomorphism $\mathrm{Hom}_F(V, F) \cong \mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p)$ given by composing with the trace $\mathrm{Tr}_{F/\mathbb{Q}_p} : F \to \mathbb{Q}_p$ we see, as in [Pappas 2000b, Section 3], that there exists a unique nondegenerate hermitian form $\phi : V \times V \to F$ such that

$$\langle x, y \rangle = \mathrm{Tr}_{F/\mathbb{Q}_p}(\pi^{-1}\phi(x, y)), \quad x, y \in V.$$

We take $G := GU_n := GU(\phi)$ and we choose a partition $n = r + s$; we refer to the pair (r, s) as the signature. By replacing ϕ by $-\phi$ if needed, we can make sure that $s \le r$ and so we assume that $s \le r$ (see [Pappas and Rapoport 2009, Section 1.1] for more details). Identifying $G \otimes F \simeq \mathrm{GL}_{n,F} \times \mathbb{G}_{m,F}$, we define the cocharacter $\mu_{r,s}$ as $(1^{(s)}, 0^{(r)}, 1)$ of $D \times \mathbb{G}_m$, where D is the standard maximal torus of diagonal matrices in GL_n; for more details we refer the reader to [Smithling 2011, Section 3.2]. We denote by E the reflex field of $\{\mu_{r,s}\}$; then $E = \mathbb{Q}_p$ if $r = s$ and $E = F$ otherwise; see [Pappas and Rapoport 2009, Section 1.1]. We set $O := O_E$.

Next, we denote by K the stabilizer of Λ in $G(\mathbb{Q}_p)$. We also let $\mathcal{L}$ be the self-dual multichain consisting of $\{\pi^k \Lambda\}_{k \in \mathbb{Z}}$. Here $\mathcal{G} = \underline{\mathrm{Aut}}(\mathcal{L})$ is the group scheme over $\mathbb{Z}_p$ with $K = \mathcal{G}(\mathbb{Z}_p)$ the subgroup of $G(\mathbb{Q}_p)$ fixing the lattice chain $\mathcal{L}$. When n is odd, the stabilizer K is a parahoric subgroup but when n is even, K is not a parahoric subgroup since it contains a parahoric subgroup with index 2. The corresponding parahoric group scheme is its connected component K°; see [Pappas and Rapoport 2009, Section 1.2] for more details.

We briefly recall the definition of certain variants of local models for ramified unitary groups.

2A. *Rapoport–Zink local models and variants.* Let $\mathrm{M}^{\mathrm{naive}}$ be the functor which associates to each scheme S over $\mathrm{Spec}\, O$ the set of subsheaves $\mathcal{F}$ of $O \otimes \mathcal{O}_S$-modules of $\Lambda \otimes \mathcal{O}_S$ such that:

(1) $\mathcal{F}$ as an $\mathcal{O}_S$-module is Zariski locally on S a direct summand of rank n.

(2) $\mathcal{F}$ is totally isotropic for $\langle \, , \rangle \otimes \mathcal{O}_S$. item (Kottwitz condition) $\mathrm{char}_{t|\mathcal{F}}(X) = (X + \pi)^r (X - \pi)^s$.

The functor M^{naive} is represented by a closed subscheme, which we again denote M^{naive}, of $\text{Gr}(n, 2n) \otimes$ Spec O; hence M^{naive} is a projective O-scheme. (Here we denote by $\text{Gr}(n, d)$ the Grassmannian scheme parametrizing locally direct summands of rank n of a free module of rank d.) M^{naive} is the *naive local model* of Rapoport and Zink [1996]. Also, M^{naive} supports an action of $\mathcal{G}$.

Proposition 2.1. (a) *We have*

$$M^{\text{naive}} \otimes_O E \cong \text{Gr}(s, n) \otimes E.$$

In particular, the generic fiber of M^{naive} is smooth and geometrically irreducible of dimension rs.

(b) *We have*

$$\dim M^{\text{naive}} \otimes_O k \geq \begin{cases} n^2/4 & \text{if } n \text{ is even,} \\ (n^2 - 1)/4 & \text{if } n \text{ is odd.} \end{cases}$$

In particular, M^{naive} is not flat if $|r - s| > 1$.

Proof. See [Pappas 2000b, Proposition 3.8; Krämer 2003, Proposition 2.2; 2003, Corollary 2.3]. □

The flat closure of $M^{\text{naive}} \otimes_O E$ in M^{naive} is by definition the *local model* M^{loc}.

In [Pappas 2000b, Section 4], Pappas introduces the *wedge local model* $M^{\wedge}$, in order to correct the nonflatness problem, by imposing the following additional condition:

$$\wedge^{r+1}(t - \pi \mid \mathcal{F}) = (0) \quad \text{and} \quad \wedge^{s+1}(t + \pi \mid \mathcal{F}) = (0).$$

More precisely, $M^{\wedge}$ is the closed subscheme of M^{naive} that classifies points given by $\mathcal{F}$ which satisfy the wedge condition. The scheme $M^{\wedge}$ supports an action of $\mathcal{G}$ and the immersion $i : M^{\wedge} \to M^{\text{naive}}$ is $\mathcal{G}$-equivariant. It is easy to see that:

Proposition 2.2. *The generic fibers of M^{naive} and $M^{\wedge}$ coincide, in particular the generic fiber of $M^{\wedge}$ is a smooth, geometrically irreducible variety of dimension rs.*

Proof. See [Krämer 2003, Proposition 3.4] and [Arzdorf 2009, Lemma 1.1]. □

Also, $M^{\text{loc}} \subset M^{\wedge}$ and $M^{\text{loc}} \otimes E = M^{\wedge} \otimes E$. As in [Pappas 2000b, Section 4] and [Pappas and Rapoport 2009, Section 2.4.2, Section 5.5], the worst point of $M^{\wedge}$, i.e., the unique closed $\mathcal{G}$-orbit which lies in the closure of any other orbit, is given by the k-valued point $\mathcal{F} = t\Lambda \subset \Lambda \otimes k \cong (k[t]/(t^2))^n$.

It is conjectured in [Pappas 2000b] that $M^{\wedge}$ is flat for $n \geq 2$ and any signature and so $M^{\text{loc}} = M^{\wedge}$. It has been shown, see [Pappas 2000b, Theorem 4.5], that this is true for the signature $(n - 1, 1)$. For more general lattice chains, the wedge condition turns out to be insufficient, see [Pappas and Rapoport 2009, Remarks 5.3, 7.4]. In [loc. cit.], the authors introduced a further refinement of the moduli problem by also adding the so-called "spin condition" (for more information we refer the reader to [loc. cit.]); this will play no role in this paper.

Next, we consider the moduli scheme $\mathcal{M}$ over O_F, the *splitting (or Krämer) local model* as in [Pappas and Rapoport 2005, Remark 14.2] and [Krämer 2003, Definition 4.1], whose points for an O_F-scheme S are Zariski locally $\mathcal{O}_S$-direct summands $\mathcal{F}_0, \mathcal{F}_1$ of ranks s, n respectively, such that:

(1) $(0) \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \Lambda \otimes \mathcal{O}_S$.

(2) $\mathcal{F}_1 = \mathcal{F}_1^\perp$, i.e., $\mathcal{F}_1$ is totally isotropic for $\langle\,,\,\rangle \otimes \mathcal{O}_S$.

(3) $(t + \pi)\mathcal{F}_1 \subset \mathcal{F}_0$.

(4) $(t - \pi)\mathcal{F}_0 = (0)$.

The functor is represented by a projective O_F-scheme $\mathcal{M}$. The scheme $\mathcal{M}$ supports an action of $\mathcal{G}$ and there is a $\mathcal{G}$-equivariant morphism

$$\tau : \mathcal{M} \to \mathrm{M}^\wedge \otimes_O O_F$$

which is given by $(\mathcal{F}_0, \mathcal{F}_1) \mapsto \mathcal{F}_1$ on S-valued points. (Indeed, we can easily see, as in [Krämer 2003, Definition 4.1], that τ is well defined.)

Proposition 2.3. *The morphism $\tau : \mathcal{M} \to \mathrm{M}^\wedge \otimes_O O_F$ induces an isomorphism on the generic fibers.*

Proof. It follows by [Krämer 2003, Remark 4.2] and the proof of [Pappas 2000b, Proposition 3.8]. $\square$

The following discussion appears in [Pappas 2000a]. Over the special fiber, the condition (4) amounts to $t\mathcal{F}_0 = (0)$. Thus, we have

$$(0) \subset \mathcal{F}_0 \subset t\Lambda \otimes k \subset \mathcal{F}_0^\perp \subset \Lambda \otimes k.$$

Also, we have

$$(0) \subset (t^{-1}(\mathcal{F}_0))^\perp \subset t\Lambda \otimes k \subset t^{-1}(\mathcal{F}_0) \subset \Lambda \otimes k.$$

The spaces $t^{-1}(\mathcal{F}_0)$, $\mathcal{F}_0^\perp$ have rank $n + s$, $2n - s = n + r$ respectively. Fixing $\mathcal{F}_0$, the rank of $t^{-1}(\mathcal{F}_0) \cap \mathcal{F}_0^\perp$ influences the dimension of the space of allowable $\mathcal{F}_1$ since

$$\mathcal{F}_0 + (t^{-1}(\mathcal{F}_0))^\perp \subset \mathcal{F}_1 \subset t^{-1}(\mathcal{F}_0) \cap \mathcal{F}_0^\perp.$$

Note that $\mathcal{F}_0 \subset t\Lambda \otimes k \simeq k^n \otimes \mathcal{O}_S$. Hence, we consider the morphism

$$\pi : \mathcal{M} \otimes k \to \mathrm{Gr}(s, n) \otimes k$$

given by $(\mathcal{F}_0, \mathcal{F}_1) \mapsto \mathcal{F}_0$. This has a section

$$\phi : \mathrm{Gr}(s, n) \otimes k \to \mathcal{M} \otimes k$$

given by $\mathcal{F}_0 \mapsto (\mathcal{F}_0, \mathcal{F}_1)$ with $\mathcal{F}_1 = t\Lambda \otimes k$. The image of the section ϕ is an irreducible component of $\mathcal{M} \otimes_O k$ which is the fiber $\tau^{-1}(t\Lambda)$ over the worst point. Hence, $\tau^{-1}(t\Lambda)$ is isomorphic to the Grassmannian $\mathrm{Gr}(s, n) \otimes k$ of dimension rs. Also, observe that $\{(\mathcal{F}_0, \mathcal{F}_1) \mid \mathrm{rank}(t^{-1}(\mathcal{F}_0) \cap \mathcal{F}_0^\perp) = n\} \subset \tau^{-1}(t\Lambda)$ since when $t^{-1}(\mathcal{F}_0) \cap \mathcal{F}_0^\perp$ has rank n then $t^{-1}(\mathcal{F}_0) \cap \mathcal{F}_0^\perp = t\Lambda$ which gives $\mathcal{F}_1 = t\Lambda$.

However, the morphism π has fibers of positive dimension over points of $\mathrm{Gr}(s, n) \otimes k$ which correspond to subspaces of $\mathrm{Gr}(s, n) \otimes k$ on which the dimension $t^{-1}(\mathcal{F}_0) \cap \mathcal{F}_0^\perp$ is more than n. Actually, $t^{-1}(\mathcal{F}_0) \cap \mathcal{F}_0^\perp$ has maximal dimension, i.e., $t^{-1}(\mathcal{F}_0) \subset \mathcal{F}_0^\perp$, if and only if $\mathcal{F}_0 \subset t\Lambda \otimes k \simeq k^n \otimes \mathcal{O}_S$ is totally isotropic for the (nondegenerate) symmetric form on $t\Lambda \otimes k$ which is defined as $\{tv, tw\} := \langle tv, w \rangle$; see the proof of [Krämer 2003, Theorem 4.5] for more details. Denote by $\mathcal{Q}(s, n)$ the smooth closed subscheme of

$\mathrm{Gr}(s, n) \otimes k$ of dimension $s(2n - 3s - 1)/2$ which parametrizes all these isotropic s-subspaces $\mathcal{F}_0$ in the n-space $k^n \otimes \mathcal{O}_S$. For such $\mathcal{F}_0 \in \mathcal{Q}(s, n)$ we have that $t^{-1}(\mathcal{F}_0) \subset \mathcal{F}_0^{\perp}$ and thus the fiber $\pi^{-1}(\mathcal{F}_0)$ is given by $\mathcal{F}_1$ with $\mathcal{F}_1 = \mathcal{F}_1^{\perp}$ such that

$$\mathcal{F}_0 \subset (t^{-1}(\mathcal{F}_0))^{\perp} \subset \mathcal{F}_1 \subset t^{-1}(\mathcal{F}_0).$$

We can see that these $\{\mathcal{F}_1\}$ correspond to Lagrangian subspaces in $t^{-1}(\mathcal{F}_0)/(t^{-1}(\mathcal{F}_0))^{\perp}$ which have dimension $2s$. This is a smooth $s(s + 1)/2$-dimensional scheme which we denote by $L(s, 2s)$. From the above discussion we see that $\pi^{-1}(\mathcal{Q}(s, n))$ is a $L(s, 2s)$-bundle over $\mathcal{Q}(s, n)$ with dimension rs. Thus, $\pi^{-1}(\mathcal{Q}(s, n))$ is an irreducible component of $\mathcal{M} \otimes k$ which intersects with $\tau^{-1}(t\Lambda)$ over $\mathcal{Q}(s, n)$.

Krämer [2003] shows that τ defines a resolution of $\mathrm{M}^{\wedge}$ in the case that the signature is $(n - 1, 1)$. In particular, she proves that $\mathcal{M}$ is regular with special fiber a reduced divisor with simple normal crossings. Also she shows that the special fiber of $\mathcal{M}$ consists of two smooth irreducible components of dimension $n - 1$ — one of which being isomorphic to $\mathbb{P}_k^{n-1}$ (this corresponds to $\tau^{-1}(t\Lambda)$), and the other one being a $\mathbb{P}_k^1$-bundle over a smooth quadric (this corresponds to $\pi^{-1}(\mathcal{Q}(1, n))$) — which intersect transversely in a smooth irreducible variety of dimension $n - 2$ (this corresponds to $\mathcal{Q}(1, n)$).

3. An affine chart

The goal of this section is to write down the equations that define $\mathcal{M}$ in a neighborhood $\mathcal{U}$ of $(\mathcal{F}_0, t\Lambda)$ for a general signature (r, s); see Proposition 3.1. We also deduce, see Proposition 3.7, that $\mathcal{G}$-translates of $\mathcal{U}$ cover $\mathcal{M}$. (Recall from Section 2 that $(\mathcal{F}_0, t\Lambda)$ is a point in the fiber of $\tau : \mathcal{M} \to \mathrm{M}^{\wedge} \otimes_O O_F$ over the worst point.) In order to find the explicit equations that describe $\mathcal{U}$, we use similar arguments as in the proof of [Krämer 2003, Theorem 4.5]. In our case we consider:

$$\mathcal{F}_1 = \begin{bmatrix} A \\ I_n \end{bmatrix}, \quad \mathcal{F}_0 = X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where A is of size $n \times n$, X is of size $2n \times s$ and X_1, X_2 are of size $n \times s$; with the additional condition that $\mathcal{F}_0$ has rank s and so X has a nonvanishing $s \times s$-minor. We also ask that $(\mathcal{F}_0, \mathcal{F}_1)$ satisfy the following four conditions:

(1) $\mathcal{F}_1^{\perp} = \mathcal{F}_1$.

(2) $\mathcal{F}_0 \subset \mathcal{F}_1$.

(3) $(t - \pi)\mathcal{F}_0 = (0)$.

(4) $(t + \pi)\mathcal{F}_1 \subset \mathcal{F}_0$.

Observe that

$$M_t = \left[\begin{array}{c|c} 0_n & pI_n \\ \hline I_n & 0_n \end{array} \right]$$

of size $2n \times 2n$ is the matrix giving multiplication by t:

(1) The condition that $\mathcal{F}_1$ is isotropic translates to

$$A^t = A.$$

(2) The condition $\mathcal{F}_0 \subset \mathcal{F}_1$ translates to

$$\exists Y : X = \left[\frac{A}{I_n} \right] \cdot Y$$

where Y is of size $n \times s$. Thus, we have

$$\left[\frac{X_1}{X_2} \right] = \left[\frac{A}{I_n} \right] \cdot Y = \left[\frac{AY}{I_n Y} \right]$$

and so $X_1 = AY$ and $X_2 = Y$.

(3) The condition $(t - \pi)\mathcal{F}_0 = (0)$ is equivalent to

$$M_t \cdot X = \left[\frac{\pi X_1}{\pi X_2} \right],$$

which amounts to

$$\left[\frac{p X_2}{X_1} \right] = \left[\frac{\pi X_1}{\pi X_2} \right].$$

Thus, $X_1 = \pi X_2$ which translates to $AY = \pi Y$ by condition (2).

(4) The last condition $(t + \pi)\mathcal{F}_1 \subset \mathcal{F}_0$ translates to

$$\exists Z : M_t \cdot \left[\frac{A}{I_n} \right] + \left[\frac{\pi A}{\pi I_n Y} \right] = X \cdot Z^t$$

where Z is of size $n \times s$. This amounts to

$$\left[\frac{p I_n + \pi A}{A + \pi I_n} \right] = \left[\frac{X_1 Z^t}{X_2 Z^t} \right].$$

From the above we get $A + \pi I_n = X_2 Z^t$ which by condition (2) translates to $A = Y Z^t - \pi I_n$. Thus, A can be expressed in terms of Y, Z. In addition, by condition (2) and in particular by the relations $X_1 = AY$ and $X_2 = Y$ we deduce that the matrix X is given in terms of Y, Z. Also from $Y = X_2$ we get that the matrix Y is given in terms of A, X. (Below we will also show that Z can be expressed in terms of A, X.)

For later use, we break up the matrices Y, Z into blocks as follows. We write

$$Y = \left[\frac{Y_1}{Y_2} \right], \quad Z = \left[\frac{Z_1}{Z_2} \right]$$

where Y_1, Z_1 are of size $s \times s$ and Y_2, Z_2 are of size $(n - s) \times s$. Observe from $X_1 = \pi X_2$ that all the entries of X_1 are in the maximal ideal and thus a minor involving entries of X_1 cannot be a unit. Recall that the matrix X has a nonvanishing $s \times s$-minor and $X_2 = Y$ from condition (2). Therefore, we can assume that $Y_1 = I_s$ up to a change of basis.

We replace A by $YZ^t - \pi I_n$. Hence, conditions (1) and (3) are equivalent to

$$ZY^t = YZ^t, \tag{3.1}$$

$$YZ^tY = 2\pi Y. \tag{3.2}$$

Here, we want to mention how we can express Z in terms of A, X. From the above we have $YZ^t = A + \pi I_n$ and $Y = \left[\frac{I_s}{Y_2}\right]$ which gives $\left[\frac{Z^t}{Y_2 Z^t}\right] = A + \pi I_n$. Next, we break the matrices A, I_n into blocks: $A = \left[\frac{A_1}{A_2}\right]$, $I_n = \left[\frac{I_1}{I_2}\right]$ where A_1, I_1 are of size $s \times n$ and A_2, I_2 are of size $(n - s) \times n$. Hence, from $\left[\frac{Z^t}{Y_2 Z^t}\right] = A + \pi I_n$ we obtain $Z^t = A_1 + \pi I_1$ and thus $Z = A_1^t + \pi I_1^t$.

From the above we deduce that an affine neighborhod of $\mathcal{M}$ around $(\mathcal{F}_0, t\Lambda)$ is given by $\mathcal{U} = \operatorname{Spec} O_F[Y, Z]/(Y_1 - I_s, ZY^t - YZ^t, YZ^tY - 2\pi Y)$.

Our goal in this section is to prove the simplification of equations given by the following proposition.

Proposition 3.1. *The affine chart $\mathcal{U} \subset \mathcal{M}$ is isomorphic to*

$$\operatorname{Spec} O_F[Y_2, Z_1]/(Z_1 - Z_1^t, Z_1(I_s + Y_2^t Y_2) - 2\pi I_s).$$

Proof. From (3.1) we get

$$\left[\frac{Z_1}{Z_2}\right] \cdot [I_s \mid Y_2^t] = \left[\frac{I_s}{Y_2^t}\right] \cdot [Z_1^t \mid Z_2^t],$$

which gives

$$\left[\begin{array}{c|c} Z_1 & Z_1 Y_2^t \\ \hline Z_2 & Z_2 Y_2^t \end{array}\right] = \left[\begin{array}{c|c} Z_1^t & Z_2^t \\ \hline Y_2 Z_1 & Y_2 Z_2^t \end{array}\right].$$

From the above relation we obtain the relations $Z_1 = Z_1^t$ and $Z_2 = Y_2 Z_1^t$. Thus, Z_1 is symmetric and Z_2 can be expressed in terms of Y_2, Z_1.

Next, the relation (3.2) amounts to

$$\left[\frac{I_s}{Y_2}\right] \cdot [Z_1^t \mid Z_2^t] \cdot \left[\frac{I_s}{Y_2}\right] = \left[\frac{2\pi I_s}{2\pi Y_2}\right]$$

which is equivalent to

$$\left[\frac{Z_1^t + Z_2^t Y_2}{Y_2 Z_1^t + Y_2 Z_2^t Y_2}\right] = \left[\frac{2\pi I_s}{2\pi Y_2}\right].$$

From this we get $Z_1^t + Z_2^t Y_2 = 2\pi I_s$ which translates to $Z_1(I_s + Y_2^t Y_2) = 2\pi I_s$ as $Z_1^t = Z_1$ and $Z_2 = Y_2 Z_1^t$. From all the above the proof of the proposition follows. $\qquad\square$

As corollaries of the above result we have:

Corollary 3.2. *For $(r, s) = (n - 1, 1)$, the corresponding affine chart $\mathcal{U}$ will be isomorphic to*:

$$\mathcal{U} \cong \operatorname{Spec}\left(O_F[(y_i)_{1 \le i \le n}, a] \Big/ \left(a \sum_{c=1}^n y_c^2 - 2\pi, y_1 - 1\right)\right). \qquad\square$$

Remark 3.3. For the above signature, Krämer [2003] shows that $\mathcal{U}$ is regular with special fiber a divisor with simple normal crossings.

Corollary 3.4. *For* $(r, s) = (n - 2, 2)$ *the corresponding affine chart* $\mathcal{U}$ *will be isomorphic to*

$$\mathcal{U} \cong \mathrm{Spec}(O_F[(x_i)_{3 \leq i \leq n}, (y_i)_{3 \leq i \leq n}, a, b, c]/(Z_1 N - 2\pi I_2))$$

where

$$Z_1 = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad N = \begin{pmatrix} 1 + \sum_{i=3}^{n} x_i^2 & \sum_{i=3}^{n} x_i y_i \\ \sum_{i=3}^{n} x_i y_i & 1 + \sum_{i=3}^{n} y_i^2 \end{pmatrix}. \qquad \square$$

Remark 3.5. In this case $s = 2$, $\mathcal{U}$ does not have semistable reduction as one of the irreducible components of the special fiber of $\mathcal{U}$ is not smooth. More precisely, over the special fiber $(\pi = 0)$ $\mathcal{U}$ has three irreducible components given by

$$T_i = \{(Z_1, N) \mid Z_1 N = 0, \text{rank } Z_1 \leq i, \text{rank } N \leq 2 - i\}, \quad \text{for } i = 0, 1, 2.$$

We can easily see that T_0, T_2 are smooth but T_1 is singular. We resolve the singularities of $\mathcal{U}$ in Section 3 by blowing up the irreducible component T_0 or in other words by blowing up the ideal (Z_1) generated by the entries of Z_1. Observe from the above and the proof of Proposition 3.1 that $A = Y \cdot \left[\frac{Z_1}{Z_2} \right]^t$ and $Z_2 = Y_2 Z_1^t$ over the special fiber. Hence, $Z_1 = 0$, i.e., $a = b = c = 0$, gives $A = 0$ which corresponds to $\mathcal{F}_1 = t\Lambda$. Thus $T_0 = \mathcal{U} \cap \tau^{-1}(t\Lambda)$ where $\tau : \mathcal{M} \to \mathrm{M}^{\wedge} \otimes_O O_F$; see Section 2A.

Remark 3.6. For a general signature (r, s), over the special fiber of $\mathcal{U}$ we have that $Z_1 = Z_1^t$ and $Z_1(I_s + Y_2^t Y_2) = 0$. As in Remark 3.5, $Z_1 = 0$ gives $A = 0$ which corresponds to $\mathcal{F}_1 = t\Lambda$.

Moreover, from the above and the definition of the (nondegenerate) symmetric form $\{ , \}$ on $t\Lambda \otimes k$ (see Section 2A) we have $\{\mathcal{F}_0, \mathcal{F}_0\} = Y^t \cdot Y = I_s + Y_2^t Y_2$ since $\mathcal{F}_0 = \left[\frac{X_1}{X_2} \right]$ where $X_1 = \pi X_2$, $X_2 = Y$ and $Y = \left[\frac{I_s}{Y_2} \right]$. Thus, from the rank of $I_s + Y_2^t Y_2$ we read how isotropic $\mathcal{F}_0$ is with respect to $\{ , \}$. When the rank of the matrix $I_s + Y_2^t Y_2$ is zero, which actually occurs, we have $\{\mathcal{F}_0, \mathcal{F}_0\} = 0$.

From all the above, we can easily see that $\mathcal{U}$ contains points $(\mathcal{F}_0, \mathcal{F}_1)$ where $\mathcal{F}_0 \in \mathcal{Q}(s, n)$ and $\mathcal{F}_1 = t\Lambda$. (Recall from Section 2A that $\mathcal{Q}(s, n)$ is the closed subscheme of $\mathrm{Gr}(s, n) \otimes k$ which contains all the totally isotropic s-subspaces $\mathcal{F}_0$ with respect to the symmetric form $\{ , \}$.)

Proposition 3.7. *When* $s \geq 1$, $\mathcal{G}$-*translates of* $\mathcal{U}$ *cover* $\mathcal{M}$.

Proof. From Section 2A, we have the forgetful $\mathcal{G}$-equivariant morphism $\tau : \mathcal{M} \to \mathrm{M}^{\wedge} \otimes_O O_F$ given by $(\mathcal{F}_0, \mathcal{F}_1) \mapsto \mathcal{F}_1$. As in [Pappas 2000b, Section 4] and [Pappas and Rapoport 2009, Sections 2.4.2, 5.5], the worst point of $\mathrm{M}^{\wedge} \otimes_O O_F$ is given by the k-valued point $t\Lambda$. The reason for this terminology is that the geometric special fiber of $\mathrm{M}^{\wedge} \otimes_O O_F$ embeds into an appropriate affine flag variety, where it decomposes into unions of finitely many Schubert cells, and the worst point is the unique closed Schubert cell. This one point stratum lies in the closure of any other stratum and the inclusion relations between the Schubert varieties are given by the Bruhat order. From the construction of splitting local models and the above, in order to show that $\mathcal{G}$-translates of $\mathcal{U}$ cover $\mathcal{M}$ it is enough to prove that $\mathcal{G}$-translates of $\mathcal{U}$ cover $\tau^{-1}(t\Lambda)$.

Recall that K is the stabilizer of Λ in $G(\mathbb{Q}_p)$ and K° is the neutral component of K, i.e., the parahoric stabilizer of Λ. When n is odd $K = K^{\circ}$ and when n is even $K/K^{\circ} \simeq \mathbb{Z}/2\mathbb{Z}$; see §2. Also, $\mathcal{G}$ is the smooth group scheme of automorphisms of the polarized chain $\mathcal{L}$ over $\mathbb{Z}_p$ with $\mathcal{G}(\mathbb{Z}_p) = K$ and $\mathcal{G}^{\circ}$ is the neutral component of $\mathcal{G}$ with $\mathcal{G}^{\circ}(\mathbb{Z}_p) = K^{\circ}$.

From Section 2A, we have that $\tau^{-1}(t\Lambda) \cong \mathrm{Gr}(s,n) \otimes k$ and $\mathcal{G}_k$ acts via its action by reduction to $t\Lambda/t^2\Lambda$. This action factors through the orthogonal group $O(n)_k$ of the symmetric form $\{\,,\}$ on $t\Lambda \otimes k$ and gives the map $\mathcal{G}_k \to O(n)_k$. As in [Pappas and Rapoport 2008, Section 4] (see also [Tits 1979, Section 3.11]), $\mathcal{G}_k^\circ$ has $SO(n)_k$ as its maximal reductive quotient if n is even and $O(n)_k$ if n is odd via its action by reduction to $t\Lambda/t^2\Lambda$. The maps $\mathcal{G}_k^\circ \to O(n)_k$ and $\mathcal{G}_k^\circ \to SO(n)_k$ are surjective when n is odd and even respectively. Therefore, the map $\mathcal{G}_k \to O(n)_k$ is always surjective.

Next, the $O(n)$-action on $\mathrm{Gr}(s,n)$ has a finite number of orbits. More precisely, there are $s+1$ orbits $Q(0), Q(1), \ldots, Q(s)$ where $Q(i) = \{\mathcal{F}_0 \in \mathrm{Gr}(s,n) \mid \dim(\mathrm{rad}(\mathcal{F}_0)) = i\}$; see [Barbasch and Evens 1994, Section 4]. For example in the case $s = 2$ there are three $O(n)$-orbits: $\mathcal{F}_0$ can either contain no isotropic vectors at all or one isotropic vector or be totally isotropic. Observe that $Q(j)$ is contained in the (Zariski) closure of $Q(i)$ if and only if $j \geq i$ and $Q(s) = \mathcal{Q}(s,n)$ is the unique closed orbit; see for example [Barbasch and Evens 1994, Section 3.1] and [Arbarello et al. 1985]. Thus, $\mathcal{Q}(s,n)$ is contained in the closure of each orbit $Q(i)$.

Lastly, from Remark 3.6 we have that $\mathcal{U}$ contains points $(\mathcal{F}_0, t\Lambda)$ with $\mathcal{F}_0 \in \mathcal{Q}(s,n)$ and so $\mathcal{U}$ contains points from all the orbits. Therefore, from all the above we deduce that the $\mathcal{G}$-translates of $\mathcal{U}$ cover $\tau^{-1}(t\Lambda)$. $\square$

Conjecture 3.8. *When $s \geq 1$, the scheme $\mathcal{M}$ is flat over* $\mathrm{Spec}\, O_F$, *Cohen–Macaulay and normal. It's special fiber is reduced.*

Remark 3.9.   (a) By Proposition 3.7, to prove the conjecture, it is enough to show that the affine chart $\mathcal{U}$ has the above properties. More precisely, the hard part of the conjecture is to prove that the special fiber of $\mathcal{U}$ is reduced and Cohen–Macaulay.

  (b) For $(r,s) = (n-1, 1)$, the conjecture is true as we can see from Remark 3.3.

  (c) The above conjecture is supported by some computer calculations that we obtained with the help of Macaulay 2. In particular, we verified the conjecture when $(r,s) = (n-2, 2)$ where $n = 5, 6, 7, 8, 9, 10$ for various primes $p > 2$.

4. A blow-up

In what follows, we assume $(r,s) = (n-2, 2)$. The goal of this section is to find a semistable resolution of the affine chart $\mathcal{U}$ (see Corollary 4.2).

From Corollary 3.4 we have that $\mathcal{U} \cong \mathrm{Spec}\, B$ where B is the quotient ring

$$B = O_F[(x_i)_{3 \leq i \leq n}, (y_i)_{3 \leq i \leq n}, a, b, c]/J$$

and J is the ideal generated by the entries of the relation:

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} Q(\underline{x}) & P(\underline{x}, \underline{y}) \\ P(\underline{x}, \underline{y}) & Q(\underline{y}) \end{pmatrix} = 2\pi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $Q(\underline{x}) = 1 + \sum_{i=3}^{n} x_i^2$, $Q(\underline{y}) = 1 + \sum_{i=3}^{n} y_i^2$ and $P(\underline{x}, \underline{y}) = \sum_{i=3}^{n} x_i y_i$.

Now, we will explicitly calculate the blow-up $\mathcal{U}^{\mathrm{bl}}$ of $\mathrm{Spec}(B)$ along the ideal (a, b, c). By Remark 3.5, $\mathcal{U}^{\mathrm{bl}}$ is the blow-up of $\mathcal{U}$ along the smooth subscheme $\mathcal{U} \cap \tau^{-1}(t\Lambda)$. Let $\rho : \mathcal{U}^{\mathrm{bl}} \to \mathcal{U}$ be the blow-up morphism. Define

$$\mathcal{U}' := \mathrm{Proj}(B[t_1, t_2, t_3]/J')$$

where

$$J' = (t_1 Q(\underline{x}) - t_3 Q(\underline{y}), t_2 Q(\underline{y}) + t_1 P(\underline{x}, \underline{y}), t_2 Q(\underline{x}) + t_3 P(\underline{x}, \underline{y}), at_2 - bt_1, at_3 - ct_1, bt_3 - ct_2).$$

By definition, $\mathcal{U}^{\mathrm{bl}}$ is a closed subscheme of the projective $\mathrm{Spec}(B)$-scheme $\mathcal{U}'$ (as the blow-up $\mathcal{U}^{\mathrm{bl}}$ may be, a priori, cut out by more equations). In fact, as a result of our analysis we will see that $\mathcal{U}^{\mathrm{bl}} = \mathcal{U}'$.

Proposition 4.1. (a) $\mathcal{U}'$ *has semistable reduction over* O_F.

 (b) *The closed immersion* $\mathcal{U}^{\mathrm{bl}} \to \mathcal{U}'$ *is an isomorphism.*

Proof. There are three affine patches that cover $\mathcal{U}'$: For $t_1 = 1$ the affine open chart is given by $V(J_1) = \mathrm{Spec}\, R_1/J_1$ where $R_1 = O_F[(x_i)_{3 \le i \le n}, (y_i)_{3 \le i \le n}, a, t_2, t_3]$ and

$$J_1 = (t_2 Q(\underline{y}) + P(\underline{x}, \underline{y}), Q(\underline{x}) - t_3 Q(\underline{y}), a(Q(\underline{x}) + t_2 P(\underline{x}, \underline{y})) - 2\pi).$$

We will show that the scheme $V(J_1)$ has semistable reduction over O_F. It suffices to prove that $V(J_1)$ is regular and its special fiber is reduced with smooth irreducible components that have smooth intersections with correct dimensions. First we observe that

$$J_1 = (t_2 Q(\underline{y}) + P(\underline{x}, \underline{y}), Q(\underline{x}) - t_3 Q(\underline{y}), a(t_3 - t_2^2)Q(\underline{y}) - 2\pi).$$

Over the special fiber ($\pi = 0$) we have $V(\bar{J}_1) = \mathrm{Spec}\, \bar{R}_1/\bar{J}_1$ where

$$\bar{J}_1 = (t_2 Q(\underline{y}) + P(\underline{x}, \underline{y}), \quad Q(\underline{x}) - t_3 Q(\underline{y}), \quad a(t_3 - t_2^2)Q(\underline{y}))$$

and $\bar{R}_1 = k[(x_i)_{3 \le i \le n}, (y_i)_{3 \le i \le n}, a, t_2, t_3]$. Let $V(I_i) = \mathrm{Spec}\, \bar{R}_1/I_i$ of dimension $2(n-2)$, where

$$I_1 = (a, t_2 Q(\underline{y}) + P(\underline{x}, \underline{y}), Q(\underline{x}) - t_3 Q(\underline{y})),$$
$$I_2 = (t_3 - t_2^2, t_2 Q(\underline{y}) + P(\underline{x}, \underline{y}), Q(\underline{x}) - t_2^2 Q(\underline{y})),$$
$$I_3 = (Q(\underline{y}), P(\underline{x}, \underline{y}), Q(\underline{x})).$$

We can easily see that

$$V(\bar{J}_1) = V(I_1) \cup V(I_2) \cup V(I_3).$$

Using the Jacobi criterion we see that $V(I_i)$ are smooth and that their intersections

$$I_1 + I_2 = (a, t_3 - t_2^2, t_2 Q(\underline{y}) + P(\underline{x}, \underline{y}), Q(\underline{x}) - t_2^2 Q(\underline{y})),$$
$$I_1 + I_3 = (a, Q(\underline{y}), P(\underline{x}, \underline{y}), Q(\underline{x})),$$
$$I_2 + I_3 = (t_3 - t_2^2, Q(\underline{y}), P(\underline{x}, \underline{y}), Q(\underline{x})),$$
$$I_1 + I_2 + I_3 = (a, t_3 - t_2^2, Q(\underline{y}), P(\underline{x}, \underline{y}), Q(\underline{x})),$$

are also smooth and with the correct dimensions. By the above, we get that $V(I_i)$ are the smooth irreducible components of $V(\bar{J}_1)$.

Now, we prove that the special fiber of $V(J_1)$ is reduced by showing that

$$\bar{J}_1 = I_1 \cap I_2 \cap I_3.$$

Recall that $\bar{J}_1 = (m_1, m_2, a(t_3 - t_2^2)Q(\underline{y}))$ where $m_1 := t_2 Q(\underline{y}) + P(\underline{x}, \underline{y})$ and $m_2 := Q(\underline{x}) - t_3 Q(\underline{y})$. Clearly, $\bar{J}_1 \subset I_1 \cap I_2 \cap I_3$. Let $g \in I_1 \cap I_2 \cap I_3$. Thus, $g \in I_1$ and

$$g = f_1 a + f_2 m_1 + f_3 m_2 \equiv f_1 a \bmod \bar{J}_1$$

for $f_i \in \bar{R}_1$. Also, $g \in I_2$ and so $f_1 a \in I_2$. I_2 is a prime ideal and $a \notin I_2$. Thus, $f_1 \in I_2$ and

$$f_1 = h_1(t_3 - t_2^2) + h_2 m_1 + h_3 m_2 \equiv h_1(t_3 - t_2^2) \bmod \bar{J}_1$$

for $h_i \in \bar{R}_1$. Lastly, $g \in I_3$ and from the above we obtain $h_1 \in I_3$ as $a, (t_3 - t_2^2) \notin I_3$. Thus,

$$h_1 = k_1 Q(\underline{y}) + k_2 P(\underline{x}, \underline{y}) + k_3 Q(\underline{x}) \equiv Q(\underline{y})(k_1 - k_2 t_2 + k_3 t_3) \bmod \bar{J}_1$$

for $k_i \in \bar{R}_1$. Therefore, $g \equiv a(t_3 - t_2^2)Q(\underline{y})(k_1 - k_2 t_2 + k_3 t_3) \equiv 0 \bmod \bar{J}_1$ and so $g \in \bar{J}_1$. Hence, $\bar{J}_1 = I_1 \cap I_2 \cap I_3$.

Next, we can easily see that the ideals I_1, I_2, I_3 are principal over $V(J_1)$. In particular, for I_1 we have $I_1 = (a)$, for I_2 we have $I_2 = (t_3 - t^2)$ and for I_3 we get $I_3 = (Q(\underline{y}))$. From the above we deduce that $V(J_1)$ is regular; see [Hartl 2001, Remark 1.1.1].

From all the above discussion we deduce that $V(J_1)$ has semistable reduction over O. By symmetry, we get similar results for $t_3 = 1$.

For $t_2 = 1$, the affine open chart is given by $V(J_2) = \operatorname{Spec} R_2/J_2$ where

$$R_2 = O_F[(x_i)_{3 \le i \le n}, (y_i)_{3 \le i \le n}, b, t_1, t_3]$$

and

$$J_2 = (Q(\underline{y}) + t_1 P(\underline{x}, \underline{y}), \; Q(\underline{x}) + t_3 P(\underline{x}, \underline{y}), \; b(1 - t_1 t_3)P(\underline{x}, \underline{y}) - 2\pi).$$

To show that $V(J_2)$ has semistable reduction one proceeds exactly as above. In this case, the special fiber of $V(J_2)$ is isomorphic to $\operatorname{Spec} \bar{R}_2/\bar{J}_2$ where

$$\bar{J}_2 = (Q(\underline{y}) + t_1 P(\underline{x}, \underline{y}), \quad Q(\underline{x}) + t_3 P(\underline{x}, \underline{y}), \quad b(1 - t_1 t_3)P(\underline{x}, \underline{y}))$$

and $\bar{R}_2 = k[(x_i)_{3 \le i \le n}, (y_i)_{3 \le i \le n}, b, t_1, t_3]$. Let $V(I_i') = \operatorname{Spec} \bar{R}_2/I_i'$ of dimension $2(n-2)$, where

$$I_1' = (b, Q(\underline{y}) + t_1 P(\underline{x}, \underline{y}), Q(\underline{x}) + t_3 P(\underline{x}, \underline{y})),$$
$$I_2' = (1 - t_1 t_3, Q(\underline{y}) + t_1 P(\underline{x}, \underline{y}), Q(\underline{x}) + t_3 P(\underline{x}, \underline{y})),$$
$$I_3' = (P(\underline{x}, \underline{y}), Q(\underline{y}), Q(\underline{x})).$$

and their intersections

$$I_1' + I_2' = (b, 1 - t_1 t_3, Q(\underline{y}) + t_1 P(\underline{x}, \underline{y}), Q(\underline{x}) + t_3 P(\underline{x}, \underline{y})),$$
$$I_1' + I_3' = (b, Q(\underline{y}), P(\underline{x}, \underline{y}), Q(\underline{x})),$$
$$I_2' + I_3' = (1 - t_1 t_3, Q(\underline{y}), P(\underline{x}, \underline{y}), Q(\underline{x})),$$
$$I_1' + I_2' + I_3' = (b, 1 - t_1 t_3, Q(\underline{y}), P(\underline{x}, \underline{y}), Q(\underline{x})).$$

As in the case $t_1 = 1$, by using the Jacobi criterion we see that the irreducible components $V(I_i')$ are smooth and they intersect transversely. Also, by a similar argument as above we can easily see that $V(J_2)$ is regular and its special fiber is reduced. Now, the semistability of $V(J_2)$ follows.

By the above, we conclude that $\mathcal{U}'$ is regular, of relative dimension $2(n-2)$, that $\mathcal{U}'$ is O_F-flat and that its special fiber is a reduced divisor with normal crossings. This shows part (a). Let us show part (b). The blow-up $\mathcal{U}^{\mathrm{bl}}$ is a closed subscheme of $\mathcal{U}'$. By the above, $\mathcal{U}'$ is integral of dimension $2(n-2)$. However, the dimension of the blow-up $\mathcal{U}^{\mathrm{bl}}$ is also $2(n-2)$. Indeed, on one hand $\mathcal{U}^{\mathrm{bl}}$ is a closed subscheme of $\mathcal{U}'$ while on the other hand it is birational to $\mathrm{Spec}(B)$. We deduce that $\mathcal{U}^{\mathrm{bl}} = \mathcal{U}'$ which is the claim in (b). $\square$

As a consequence of the above proposition we obtain:

Corollary 4.2. *The morphism $\rho : \mathcal{U}^{\mathrm{bl}} \to \mathcal{U}$ is a semistable resolution, i.e., $\mathcal{U}^{\mathrm{bl}}$ has semistable reduction over O_F.*

Proof. It follows from part (a) and (b) of Proposition 4.1. $\square$

Remark 4.3. From the proof of Proposition 4.1 we obtain that the special fiber of $\mathcal{U}^{\mathrm{bl}}$ has three irreducible components. In fact, we explicitly describe the equations defining these irreducible components over the three affine patches that cover $\mathcal{U}^{\mathrm{bl}}$. It is then easy to see that the exceptional locus of $\rho : \mathcal{U}^{\mathrm{bl}} \to \mathcal{U}$ is the irreducible component of the special fiber of $\mathcal{U}^{\mathrm{bl}}$

$$\mathrm{Proj}\left(\frac{k[(x_i)_{3 \le i \le n}, (y_i)_{3 \le i \le n}][t_1, t_2, t_3]}{(t_1 Q(\underline{x}) - t_3 Q(\underline{y}), t_2 Q(\underline{y}) + t_1 P(\underline{x}, \underline{y}), t_2 Q(\underline{x}) + t_3 P(\underline{x}, \underline{y}))}\right)$$

that corresponds to $V(I_1)$ and $V(I_1')$ for the affine patches $t_1 = 1$ and $t_2 = 1$ respectively.

5. A resolution for the local model

We use the notation from Section 2. In particular, recall the morphism

$$\tau : \mathcal{M} \to \mathrm{M}^{\wedge} \otimes_O O_F$$

and the following isomorphisms over the generic fiber

$$\mathcal{M} \otimes F \cong \mathrm{M}^{\wedge} \otimes F \cong \mathrm{M}^{\mathrm{loc}} \otimes F. \tag{5.1}$$

Let $\mathcal{Z} = \tau^{-1}(t\Lambda)$ be the smooth $\mathcal{G}$-invariant subscheme of dimension $2(n-2)$, which is supported in the special fiber. (Recall from Section 2 that $t\Lambda$ is the worst point of $\mathrm{M}^{\wedge}$ and $\tau^{-1}(t\Lambda) \cong \mathrm{Gr}(2, n) \otimes k$.)

We consider the blow-up of $\mathcal{M}$ along the subscheme $\mathcal{Z}$. This gives a $\mathcal{G}$-birational projective morphism

$$r^{\mathrm{bl}} : \mathrm{M}^{\mathrm{bl}} \to \mathcal{M}$$

which induces an isomorphism on the generic fibers.

Theorem 5.1. *The scheme* M^{bl} *is regular and has special fiber a reduced divisor with normal crossings.*

Proof. From Proposition 3.7 we have that the $\mathcal{G}$-translates of $\mathcal{U}$ cover $\mathcal{M}$ and since r^{bl} is $\mathcal{G}$-equivariant we obtain that the $\mathcal{G}$-translates of the open $\mathcal{U}^{\mathrm{bl}} = (r^{\mathrm{bl}})^{-1}(\mathcal{U}) \subset \mathrm{M}^{\mathrm{bl}}$ cover M^{bl}. Therefore, it is enough to show the conclusion of the theorem for the blow-up $\mathcal{U}^{\mathrm{bl}}$ of $\mathcal{U}$ at the ideal (a, b, c) and by Corollary 4.2 the proof of the theorem follows. $\qquad\square$

Remark 5.2. It would be useful to have a simple moduli-theoretic description of the blow-up M^{bl} similar in spirit to the description of $\mathcal{M}$ given in Section 2.

We just proved that M^{bl} has semistable reduction, and is therefore flat over O_F. Combining all the above we have

$$\mathrm{M}^{\mathrm{bl}} \xrightarrow{r^{\mathrm{bl}}} \mathcal{M} \xrightarrow{\tau} \mathrm{M}^{\wedge} \otimes_O O_F$$

which factors through $\mathrm{M}^{\mathrm{loc}} \otimes_O O_F \subset \mathrm{M}^{\wedge} \otimes_O O_F$ because of flatness; the generic fiber of all of these is the same as we can see from (5.1). Then, we obtain that $\mathrm{M}^{\mathrm{bl}} \to \mathrm{M}^{\mathrm{loc}} \otimes_O O_F$ is a $\mathcal{G}$-equivariant birational projective morphism.

6. Application to Shimura varieties

6A. *Unitary Shimura data.* We now discuss some Shimura varieties to which we can apply these results. We follow [Pappas and Rapoport 2009, Section 1.1] for the description of the unitary Shimura varieties; see also [Pappas 2000b, Section 3].

Let F_0 be an imaginary quadratic field and fix an embedding $\epsilon : F_0 \hookrightarrow \mathbb{C}$. Let O be the ring of integers of F_0 and denote by $a \mapsto \bar{a}$ the nontrivial automorphism of F_0. Assuming $n > 3$, we let $W = F_0^n$ be an n-dimensional F_0-vector space, and we suppose that $\phi : W \times W \to F_0$ is a nondegenerate hermitian form. Set $W_{\mathbb{C}} = W \otimes_{F_0,\epsilon} \mathbb{C}$. Choosing a suitable isomorphism $W_{\mathbb{C}} \cong \mathbb{C}^n$ we may write ϕ on $W_{\mathbb{C}}$ in a normal form $\phi(w_1, w_2) = {}^t\bar{w}_1 H w_2$ where

$$H = \mathrm{diag}(-1, \ldots, -1, 1, \ldots, 1).$$

We denote by s (resp. r) the number of places, where -1, (resp. 1) appears in H. We will say that ϕ has signature (r, s). By replacing ϕ by $-\phi$ if needed, we can make sure that $s \le r$ and so we assume that $s \le r$. Let $J : W_{\mathbb{C}} \to W_{\mathbb{C}}$ be the endomorphism given by the matrix $-\sqrt{-1}H$. We have $J^2 = -\,\mathrm{id}$ and so the endomorphism J gives an $\mathbb{R}$-algebra homomorphism $h_0 : \mathbb{C} \to \mathrm{End}_{\mathbb{R}}(W \otimes_{\mathbb{Q}} \mathbb{R})$ with $h_0(\sqrt{-1}) = J$ and hence a complex structure on $W \otimes_{\mathbb{Q}} \mathbb{R} = W_{\mathbb{C}}$. For this complex structure we have

$$\mathrm{Tr}_{\mathbb{C}}(a; W \otimes_{\mathbb{Q}} \mathbb{R}) = s \cdot \epsilon(a) + r \cdot \bar{\epsilon}(a), \quad a \in F_0.$$

Denote by E the subfield of $\mathbb{C}$ which is generated by the traces above (the "reflex field"). We have that $E = \mathbb{Q}$ if $r = s$ and $E = F_0$ otherwise. The representation of F_0 on $W \otimes_{\mathbb{Q}} \mathbb{R}$ with the above trace is defined over E, i.e., there is an n-dimensional E-vector space W_0 on which F_0 acts such that

$$\text{Tr}_E(a; W_0) = s \cdot a + r \cdot \bar{a}$$

and such that $W_0 \otimes_E \mathbb{C}$ together with the above F_0-action is isomorphic to $W \otimes_{\mathbb{Q}} \mathbb{R}$ with the F_0-action induced by $\epsilon : F_0 \hookrightarrow \mathbb{C}$ and the above complex structure.

Next, fix a nonzero element $a \in F_0$ with $\bar{a} = -a$ and set

$$\psi(x, y) = \text{Tr}_{F_0/\mathbb{Q}}(a^{-1}\phi(x, y))$$

which is a nondegenerate alternating form $W \otimes_{\mathbb{Q}} W \to \mathbb{Q}$. This satisfies

$$\psi(av, w) = \psi(v, \bar{a}w), \quad \text{for all } a \in F_0, \ v, w \in W.$$

By replacing a by $-a$, we can make sure that the symmetric $\mathbb{R}$-bilinear form on $W_{\mathbb{C}}$ given by $\psi(x, Jy)$ for $x, y \in W_{\mathbb{C}}$ is positive definite. Let G be the reductive group over $\mathbb{Q}$ which is given by

$$G(\mathbb{Q}) = \{g \in \text{GL}_{F_0}(W) \mid \psi(gv, gw) = c(g)\psi(v, w), \ c(g) \in \mathbb{Q}^{\times}\}.$$

The group G can be identified with the unitary similitude group of the form ϕ. Set

$$GU(r, s) := \{A \in \text{GL}_n(\mathbb{C}) \mid {}^t\bar{A}HA^- = c(A)H, c(A) \in \mathbb{R}^{\times}\}.$$

By the above, the embedding $\epsilon : F_0 \hookrightarrow \mathbb{C}$ induces an isomorphism $G(\mathbb{R}) \cong GU(r, s)$. We define a homomorphism $h : \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{R}}$ by restricting h_0 to $\mathbb{C}^{\times}$. Then $h(a)$ for $a \in \mathbb{R}^{\times}$ acts on $W \otimes_{\mathbb{Q}} \mathbb{R}$ by multiplication by a and $h(\sqrt{-1})$ acts as J. Consider $h_{\mathbb{C}}(z, 1) : \mathbb{C}^{\times} \to G(\mathbb{C}) \cong \text{GL}_n(\mathbb{C}) \times \mathbb{C}^{\times}$. Up to conjugation $h_{\mathbb{C}}(z, 1)$ is given by

$$\mu_{r,s}(z) = (\text{diag}(z^{(s)}, 1^{(r)}), z);$$

this is a cocharacter of G defined over the number field E. Denote by X_h the conjugation orbit of $h(i)$ under $G(\mathbb{R})$. The pair (G, h) gives rise to a Shimura variety $\text{Sh}(G, h)$ which is defined over the reflex field E.

6B. *Unitary integral models.* We continue with the notations and assumptions of the previous paragraph. In particular, we take $G = GU_n$ and $X = X_h$ above that define the unitary similitude Shimura datum (G, X). Assume that $(r, s) = (n - 2, 2)$.

Assume that p is an odd prime number and is ramified in F_0. Let $F_1 = F_0 \otimes \mathbb{Q}_p$ and $V = W \otimes_{\mathbb{Q}} \mathbb{Q}_p$. We fix a square root π of p and we set $k = \bar{\mathbb{F}}_p$. In addition, we assume that the hermitian form ϕ on V is split. This means that there exists a basis $e_1, \ldots, e_n$ of V such that $\phi(e_i, e_{n+1-j}) = \delta_{ij}$ for $i, j \in \{1, \ldots, n\}$. We denote by Λ the standard lattice $O^n \otimes_{\mathbb{Z}} \mathbb{Z}_p$ in V. Denote by K the stabilizer of Λ in $G(\mathbb{Q}_p)$.

We let $\mathcal{L}$ be the self-dual multichain consisting of $\{\pi^k \Lambda\}_{k \in \mathbb{Z}}$. Here $\mathcal{G} = \underline{\mathrm{Aut}}(\mathcal{L})$ is the group scheme over $\mathbb{Z}_p$ with $K = \mathcal{G}(\mathbb{Z}_p)$ the subgroup of $G(\mathbb{Q}_p)$ fixing the lattice chain $\mathcal{L}$. Denote by K° the neutral component of K. As in Section 2, when n is odd $K = K^\circ$ and when n is even $K/K^\circ \simeq \mathbb{Z}/2\mathbb{Z}$.

Choose also a sufficiently small compact open subgroup K^p of the prime-to-p finite adelic points $G(\mathbb{A}_f^p)$ of G and set $\boldsymbol{K} = K^p K$ and $\boldsymbol{K}' = K^p K^\circ$. As was observed in [Pappas and Rapoport 2009, Section 1.3], the Shimura varieties $\mathrm{Sh}_{\boldsymbol{K}'}(G, X)$ and $\mathrm{Sh}_{\boldsymbol{K}}(G, X)$ have isomorphic geometric connected components. Therefore, from the point of view of constructing reasonable integral models, we may restrict our attention to $\mathrm{Sh}_{\boldsymbol{K}}(G, X)$; since K corresponds to a lattice set stabilizer, this Shimura variety is given by a simpler moduli problem. The Shimura variety $\mathrm{Sh}_{\boldsymbol{K}}(G, X)$ with complex points

$$\mathrm{Sh}_{\boldsymbol{K}}(G, X)(\mathbb{C}) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/\boldsymbol{K}$$

is of PEL type and has a canonical model over the reflex field E. We set $\mathcal{O} = O_{E_v}$ where v the unique prime ideal of E above (p).

We consider the moduli functor $\mathcal{A}_{\boldsymbol{K}}^{\mathrm{naive}}$ over $\mathrm{Spec}\,\mathcal{O}$ given in [Rapoport and Zink 1996, Definition 6.9]: A point of $\mathcal{A}_{\boldsymbol{K}}^{\mathrm{naive}}$ with values in the $\mathrm{Spec}\,\mathcal{O}$-scheme S is the isomorphism class of the following set of data $(A, \bar{\lambda}, \bar{\eta})$:

(1) An $\mathcal{L}$-set of abelian varieties $A = \{A_\Lambda\}$.

(2) A $\mathbb{Q}$-homogeneous principal polarization $\bar{\lambda}$ of the $\mathcal{L}$-set A.

(3) A K^p-level structure

$$\bar{\eta} : H_1(A, \mathbb{A}_f^p) \simeq W \otimes \mathbb{A}_f^p \bmod K^p$$

which respects the bilinear forms on both sides up to a constant in $(\mathbb{A}_f^p)^\times$; see [loc. cit.] for details.

The set A should satisfy the determinant condition (i) of [loc. cit.].

For the definitions of the terms employed here we refer to [6.3–6.8] and [Pappas 2000b, Section 3]. The functor $\mathcal{A}_{\boldsymbol{K}}^{\mathrm{naive}}$ is representable by a quasiprojective scheme over $\mathcal{O}$. Since the Hasse principle is satisfied for the unitary group, we can see as in [loc. cit.] that there is a natural isomorphism

$$\mathcal{A}_{\boldsymbol{K}}^{\mathrm{naive}} \otimes_\mathcal{O} E_v = \mathrm{Sh}_{\boldsymbol{K}}(G, X) \otimes_E E_v.$$

As is explained in [Rapoport and Zink 1996] and [Pappas 2000b] the naive local model $\mathrm{M}^{\mathrm{naive}}$ is connected to the moduli scheme $\mathcal{A}_{\boldsymbol{K}}^{\mathrm{naive}}$ via the local model diagram

$$\mathcal{A}_{\boldsymbol{K}}^{\mathrm{naive}} \xleftarrow{\psi_1} \tilde{\mathcal{A}}_{\boldsymbol{K}}^{\mathrm{naive}} \xrightarrow{\psi_2} \mathrm{M}^{\mathrm{naive}}$$

where the morphism ψ_1 is a $\mathcal{G}$-torsor and ψ_2 is a smooth and $\mathcal{G}$-equivariant morphism. Therefore, there is a relatively representable smooth morphism

$$\mathcal{A}_{\boldsymbol{K}}^{\mathrm{naive}} \to [\mathcal{G} \backslash \mathrm{M}^{\mathrm{naive}}]$$

where the target is the quotient algebraic stack.

As we mentioned in Section 2, the scheme M^{naive} is never flat and by the above, the same is true for $\mathcal{A}_K^{naive}$. Denote by $\mathcal{A}_K^{flat}$ the flat closure of $\mathrm{Sh}_K(G, X) \otimes_E E_v$ in $\mathcal{A}_K^{naive}$. Recall from Section 2 that the flat closure of $M^{naive} \otimes_{\mathcal{O}} E_v$ in M^{naive} is by definition the local model M^{loc}. By the above we can see, as in [Pappas and Rapoport 2009], that there is a relatively representable smooth morphism of relative dimension $\dim(G)$,

$$\mathcal{A}_K^{flat} \to [\mathcal{G} \backslash M^{loc}].$$

This of course implies imply that $\mathcal{A}_K^{flat}$ is étale locally isomorphic to the local model M^{loc}.

One can now consider a variation of the moduli of abelian schemes $\mathcal{A}_K^{spl}$ where we add in the moduli problem an additional subspace in the Hodge filtration $\mathrm{Fil}^0(A) \subset H_{dR}^1(A)$ of the universal abelian variety A (see [Haines 2005, Section 6.3] for more details) with certain conditions to imitate the definition of the splitting local model $\mathcal{M}$. $\mathcal{A}_K^{spl}$ associates to an O_{F_1}-scheme S the set of isomorphism classes of objects $(A, \bar{\lambda}, \bar{\eta}, \mathscr{F}_0)$. Here $(A, \bar{\lambda}, \bar{\eta})$ is an object of $\mathcal{A}_K^{naive}(S)$. Set $\mathscr{F}_1 := \mathrm{Fil}^0(A)$. The final ingredient $\mathscr{F}_0$ of an object of $\mathcal{A}_K^{spl}$ is the subspace $\mathscr{F}_0 \subset \mathscr{F}_1 \subset H_{dR}^1(A)$ of rank s which satisfies the following conditions:

$$(t + \pi)\mathscr{F}_1 \subset \mathscr{F}_0, \quad (t - \pi)\mathscr{F}_0 = (0).$$

There is a forgetful morphism

$$\tau : \mathcal{A}_K^{spl} \to \mathcal{A}_K^{naive} \otimes_{\mathcal{O}} O_{F_1}$$

defined by $(A, \bar{\lambda}, \bar{\eta}, \mathscr{F}_0) \mapsto (A, \bar{\lambda}, \bar{\eta})$. Moreover, $\mathcal{A}_K^{spl}$ has the same étale local structure as $\mathcal{M}$; it is a "linear modification" of $\mathcal{A}_K^{naive} \otimes_{\mathcal{O}} O_{F_1}$ in the sense of [Pappas 2000b, Section 2]; see also [Pappas and Rapoport 2005, Section 15]. Also we want to mention that under the local model diagram the subspace $\mathscr{F}_1$ corresponds to $\mathcal{F}_1$ of $(\mathcal{F}_0, \mathcal{F}_1) \in \mathcal{M}$.

Theorem 6.1. *For every K^p as above, there is a scheme $\mathcal{A}_K^{bl}$, flat over $\mathrm{Spec}(O_{F_1})$, with*

$$\mathcal{A}_K^{bl} \otimes_{O_{F_1}} F_1 = \mathrm{Sh}_K(G, X) \otimes_E F_1,$$

and which supports a local model diagram

$$\begin{array}{ccc}
 & \widetilde{\mathcal{A}}_K^{bl}(G, X) & \\
{\scriptstyle \pi_K^{reg}} \swarrow & & \searrow {\scriptstyle q_K^{reg}} \\
\mathcal{A}_K^{bl} & & M^{bl}
\end{array} \qquad (6B.1)$$

such that:

(a) π_K^{reg} *is a $\mathcal{G}$-torsor for the parahoric group scheme $\mathcal{G}$ that corresponds to K_p.*

(b) q_K^{reg} *is smooth and $\mathcal{G}$-equivariant.*

(c) $\mathcal{A}_K^{bl}$ *is regular and has special fiber which is a reduced divisor with normal crossings.*

Proof. By the above, we have

$$
\begin{array}{ccc}
 & \widetilde{\mathcal{A}}_K^{\mathrm{spl}} & \\
\pi_K \swarrow & & \searrow q_K \\
\mathcal{A}_K^{\mathrm{spl}} & & \mathcal{M}
\end{array}
\tag{6B.2}
$$

with π_K a $\mathcal{G}$-torsor and q_K smooth and $\mathcal{G}$-equivariant. We set

$$
\widetilde{\mathcal{A}}_K^{\mathrm{bl}} = \widetilde{\mathcal{A}}_K^{\mathrm{spl}} \times_{\mathcal{M}} \mathrm{M}^{\mathrm{bl}}
$$

which carries a diagonal $\mathcal{G}$-action. Since $\mathrm{M}^{\mathrm{bl}} \to \mathcal{M}$ is given by a blow-up, is projective, and we can see [Pappas 2000b, Section 2] that the quotient

$$
\pi_K^{\mathrm{reg}} : \widetilde{\mathcal{A}}_K^{\mathrm{bl}} \to \mathcal{A}_K^{\mathrm{bl}} := \mathcal{G} \backslash \widetilde{\mathcal{A}}_K^{\mathrm{bl}} (G, X)
$$

is represented by a scheme and gives a $\mathcal{G}$-torsor. (This is an example of a linear modification, see [Pappas 2000b, Section 2].) In fact, since blowing-up commutes with étale localization, $\mathcal{A}_K^{\mathrm{bl}}$ is the blow-up of $\mathcal{A}_K^{\mathrm{spl}}$ along the locus of its special fiber where $t\mathscr{F}_1 = 0$. The projection gives a smooth $\mathcal{G}$-morphism

$$
q_K^{\mathrm{reg}} : \widetilde{\mathcal{A}}_K^{\mathrm{bl}} \to \mathrm{M}^{\mathrm{bl}}
$$

which completes the local model diagram. Property (c) follows from Theorem 5.1 and properties (a) and (b) which imply that $\mathcal{A}_K^{\mathrm{bl}}$ and M^{bl} are locally isomorphic for the étale topology. $\square$

Corollary 6.2. *$\mathcal{A}_K^{\mathrm{bl}}$ is the blow-up of $\mathcal{A}_K^{\mathrm{spl}}$ along the locus of its special fiber where the deRham filtration $\mathscr{F}_1 = \mathrm{Fil}^0(A)$ is annihilated by the action of the uniformizer π.*

Proof. It follows from the proof of the above theorem. $\square$

Remarks 6.3. (1) From the above discussion, we can obtain a semistable integral model for the Shimura variety $\mathrm{Sh}_{K'}(G, X)$ where $K' = K^p K^\circ$. In this case, the corresponding local models $\mathrm{M}^{\mathrm{loc}}$ of $\mathrm{Sh}_{K'}(G, X)$ agree with the Pappas–Zhu local models $\mathbb{M}_{K^\circ}(G, \mu_{r,s})$ for the local model triples $(G, \{\mu_{r,s}\}, K^\circ)$; see [Pappas and Zhu 2013, Theorem 1.2] and [Pappas and Zhu 2013, Section 8] for more details.

(2) Similar results can be obtained for corresponding Rapoport–Zink formal schemes; see [He et al. 2020, Section 4] for an example of this parallel treatment.

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