A unipotent realization
of the chromatic quasisymmetric function

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We realize two families of combinatorial symmetric functions via the complex character theory of the finite general linear group $GL_n(\mathbb{F}_q)$: chromatic quasisymmetric functions and vertical strip LLT polynomials. The associated $GL_n(\mathbb{F}_q)$ characters are elementary in nature and can be obtained by induction from certain well-behaved characters of the unipotent upper triangular groups $UT_n(\mathbb{F}_q)$. The proof of these results also gives a general Hopf algebraic approach to computing the induction map. Additional results include a connection between the relevant $GL_n(\mathbb{F}_q)$ characters and Hessenberg varieties and a reinterpretation of known theorems and conjectures about the relevant symmetric functions in terms of $GL_n(\mathbb{F}_q)$.

1. Introduction

The chromatic symmetric function sits at a nexus of disparate areas of mathematics. At face value, this symmetric function encodes the coloring problem of a graph as an analogue of the chromatic polynomial [40]. However, through a well-known equivalence between the ring of symmetric functions and the representation theory of the symmetric groups (see, e.g., [35]), some chromatic symmetric functions are also complex characters of the symmetric group [21]. By way of a t-analogue known as the chromatic quasisymmetric function, Brosnan and Chow [10] and Guay-Paquet [25] independently proved that the characters corresponding to indifference graphs are afforded by symmetric group representations on the cohomology rings of regular semisimple Hessenberg varieties, as predicted by a conjecture of Shareshian and Wachs [39]. Thus, certain questions about graphs, representation theory, and algebraic geometry coincide in the combinatorics of these symmetric functions, and vice versa.

At about the same time, a sequence of superficially unrelated developments occurred in the character theory of the group of unipotent upper triangular matrices UT_n over a finite field $\mathbb{F}_q$. Unlike the symmetric group, the conjugacy classes and irreducible characters of UT_n are exceptionally complicated and cannot be described with modern combinatorial tools [26]. However, beginning with the work of André [8], a theory of well-behaved reducible characters — known a supercharacters — has developed, leading to a combinatorial representation theory of UT_n without irreducible characters, as in [3; 4]. A recent example given by Aliniaeifard and Thiem [7] constructs supercharacters which are imbued with Catalan combinatorics coming from a family of normal subgroups of UT_n. These same subgroups and

supercharacters will appear in this paper, where they will be indexed by indifference graphs using a canonical bijection between Catalan-enumerated objects.

This paper uses the representation theory of the general linear group GL_n over $\mathbb{F}_q$ to establish a connection between the supercharacter theory of UT_n and the chromatic (quasi)symmetric function. Both UT_n and its subgroups are contained in GL_n. The main result, Theorem 3.1, shows that up to a factor of $(q-1)^n$, inducing the trivial character from each of these subgroups gives a map

$$\left\{\begin{array}{l}\text{indifference graph}\\\text{indexed subgroups}\end{array}\right\} \xrightarrow{\;\mathrm{Ind}^{\mathrm{GL}_n}_{(-)}(\mathbb{1})\;} \left\{\begin{array}{l}\text{chromatic quasisymmetric functions for}\\\text{indifference graphs evaluated at } t=q\end{array}\right\},$$

using an implicit identification between characters of GL_n with unipotent support and symmetric functions coming from the Hall algebra; more details can be found in Section 3. This result is a $\mathrm{GL}_n(\mathbb{F}_q)$-analogue of the Brosnan–Chow–Guay-Paquet theorem, in which cohomology rings are replaced by a permutation representation on the cosets of certain unipotent subgroups.

The remaining sections of the paper explore the implications of the main result for the theory of chromatic quasisymmetric functions. Many of these consequences are reminiscent of consequences of the Brosnan–Chow–Guay-Paquet theorem. Along with Theorem 3.1 itself, these similarities come as a surprise, especially since the association between characters of GL_n and symmetric functions used above is markedly different from the classical association for the symmetric groups. Intuition notwithstanding, each result appears to be straightforward, or even inevitable once the right perspective is achieved.

Section 4 relates Theorem 3.1 to the study of Hessenberg varieties, but not the ones appearing in the Brosnan–Chow–Guay-Paquet theorem. Instead, the values of the GL_n characters in Theorem 3.1 count the points of a nilpotent Hessenberg variety over $\mathbb{F}_q$ associated to an ad-nilpotent ideal. The analogous complex Hessenberg varieties have been studied by Precup and Sommers [38], who found an independent connection to the chromatic quasisymmetric function via Poincaré polynomials. Corollary 4.5 links these results by showing that the Poincaré polynomials for certain complex Hessenberg varieties also count the points of the corresponding Hessenberg variety over $\mathbb{F}_q$.

The chromatic quasisymmetric functions of indifference graphs are also closely related to another family of symmetric functions known as unicellular LLT polynomials [11] (see also [28]), and Section 5 reframes this relationship as a GL_n representation theoretic one. There is a second, more standard realization of symmetric functions as *unipotent* characters of GL_n, and up to a twist by the involution ω, Theorem 5.1 gives a map

$$\left\{\begin{array}{l}\text{indifference graph}\\\text{indexed subgroups}\end{array}\right\} \xrightarrow{\;\mathrm{proj}_{\mathrm{unipotent}} \circ \mathrm{Ind}^{\mathrm{GL}_n}_{(-)}(\mathbb{1})\;} \left\{\begin{array}{l}\text{unicellular LLT polynomials}\\\text{evaluated at } t=q\end{array}\right\},$$

where $\mathrm{proj}_{\mathrm{unipotent}}$ is the operation which replaces a character of GL_n with the sum of its irreducible unipotent constituents. In fact, by applying the composite map to additional characters of UT_n — including supercharacters — Theorem 5.1 finds the larger family of vertical strip LLT polynomials as unipotent characters of GL_n. These symmetric functions are known to appear in the representation theory of

quantum groups [34], affine Hecke algebras [23], and the symmetric groups [25; 28], but this is their first appearance in the representation theory of GL_n.

Finally, both chromatic quasisymmetric functions and LLT polynomials are the subject of "positivity conjectures" which are at least partially open. Such a conjecture postulates that when a particular symmetric function is expressed in a chosen basis, the coefficient of each basis element will be a polynomial in t with nonnegative coefficients. For chromatic quasisymmetric functions, the modified Stanley–Stembridge conjecture [39, Conjecture 1.3] (see also [42]) concerns the elementary basis, and is almost entirely open. For LLT polynomials, positivity in the Schur basis has been established by Grojnowski and Haiman [23], but no "positive" combinatorial formula is known in general [27]. Section 6 describes the meaning of these conjectures — and one more, recently resolved by D'Adderio [13] and Alexandersson and Sulzgruber [6] — in GL_n representation theory. This does not lead to immediate progress on any conjecture, but it may be a useful guide for future work.

The method of proof for Theorems 3.1 and 5.1 may also be of independent interest. At a high level, I am able to translate Guay-Paquet's proof in [25] into the (super)character theory of UT_n and GL_n in such a way that both results follow immediately. However, this translation also gives a more general Hopf algebraic conduit from the combinatorial representation theory of UT_n to that of GL_n. Since matters of UT_n character theory are usually very difficult, the tractability of this approach alone is a significant development. These results begin to answer lingering questions from [3] about the Hopf algebraic enumerative invariants of certain supercharacters of UT_n.

A short summary of the aforementioned framework and the machinery of [25] is given in this paragraph. In [2], Aguiar, Bergeron, and Sottile constructively classify all Hopf algebra homomorphisms from an arbitrary Hopf algebra to the Hopf algebra of symmetric functions Sym using linear functionals of the domain. This generalizes Zelevinsky's theory of PSH algebras, which completely describes the character theory of GL_n by constructing a collection of homomorphisms from a Hopf algebra $cf(GL_\bullet)$ of GL_n-class functions to Sym. In [19], I construct an analogous Hopf algebra $cf(UT_\bullet)$ on the class functions of UT_n, and show that induction $Ind_{UT_n}^{GL_n}$ induces a Hopf algebra homomorphism to $cf(GL_\bullet)$. By composing induction with any of Zelevinsky's maps to Sym, the classification of [2] can be used to describe the induction map itself, and Theorems 3.8 and 5.11 do so. The classification of [2] was also used in [25] to construct the chromatic quasisymmetric function using a Hopf algebra structure on Hessenberg varieties, and I show that this coincides with induction of Catalan supercharacters and related objects.

This Hopf algebraic approach builds on the previously understood relationship between the combinatorics of unipotent subgroups and of finite groups of Lie type, including GL_n [9; 22; 32; 45]. Future work should continue to push this connection: it may be possible to transplant some of the framework in this paper and [19] into other Lie types. In doing so, one might find the generalized LLT polynomials defined in [23], yet-to-be-discovered variants of the chromatic quasisymmetric function, and more nilpotent Hessenberg varieties.

The remainder of the paper is organized as follows. Section 2 describes the general background material for the paper. Section 3 concerns Theorem 3.1 and the chromatic quasisymmetric function, and Section 4

relates these results to Hessenberg varieties. Section 5 concerns Theorem 5.1 and the vertical strip LLT polynomial, and is essentially independent of Sections 3 and 4. Finally, Section 6 connects my results to various positivity conjectures.

2. Preliminaries

This section gives the shared preliminary material for Sections 3 and 5. This includes definitions of each of the relevant Hopf algebras, background material from representation theory and combinatorics, and a short review of the theory of combinatorial Hopf algebras.

2A. *Hopf algebras and (quasi)symmetric functions.* This section will describe the Hopf algebras of quasisymmetric and symmetric functions, and their role as universal objects in the theory of combinatorial Hopf algebras. Throughout this paper, the term "Hopf algebra" will refer to a graded connected Hopf algebra over the field of complex numbers $\mathbb{C}$, and all homomorphisms and sub-Hopf algebras are graded.

A *composition* of $n \in \mathbb{Z}_{\geq 0}$ is a finite (possibly empty) sequence of positive integers $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $\alpha_1 + \cdots + \alpha_k = n$. Call each α_i a *part* of α, and write $\ell(\alpha) = k$ for the number of parts of α. The *monomial quasisymmetric function* associated to the composition α is

$$M_\alpha = \sum_{i_1 < \cdots < i_{\ell(\alpha)}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_{\ell(\alpha)}}^{\alpha_{\ell(\alpha)}} \in \mathbb{C}[\![x]\!].$$

where $x = \{x_1, x_2, \ldots\}$ is an infinite, totally ordered set of commuting indeterminates. The *Hopf algebra of quasisymmetric functions* is the graded commutative, noncocommutative Hopf algebra

$$\mathcal{Q}\text{Sym} = \mathbb{C}\text{-span}\{M_\alpha \mid \alpha \text{ is a composition}\}.$$

The product of $\mathcal{Q}$Sym is inherited from $\mathbb{C}[\![x]\!]$ and the coproduct is given by deconcatenation:

$$\Delta(M_\alpha) = \sum_{\ell(\alpha) \geq k \geq 0} M_{(\alpha_1, \ldots, \alpha_k)} \otimes M_{(\alpha_k, \ldots, \alpha_{\ell(\alpha)})}.$$

A *partition of n* is a composition of n is with nonincreasing parts. Let

$$\mathcal{P} = \bigsqcup_{n \geq 0} \mathcal{P}(n) \quad \text{with } \mathcal{P}(n) = \{\text{partitions of } n\}.$$

The *Hopf algebra of symmetric functions* is the cocommutative sub-Hopf algebra

$$\text{Sym} = \mathbb{C}\text{-span}\{m_\lambda \mid \lambda \in \mathcal{P}\} \subseteq \mathcal{Q}\text{Sym} \quad \text{with } m_\lambda = \sum_{\text{sort}(\alpha) = \lambda} M_\alpha,$$

where sort(α) denotes the partition obtained by listing the parts of α in nonincreasing order.

Three additional bases of Sym will be used in later sections. The first basis consists of the *elementary symmetric functions* $\{e_\lambda \mid \lambda \in \mathcal{P}\}$ defined by

$$e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell} \quad \text{where } e_k = m_{(1^k)}.$$

The second basis comprises the *Schur functions* $\{s_\lambda \mid \lambda \in \mathcal{P}\}$, which I will not define; see [35, I.3]. The final basis consists of the Hall–Littlewood elements $P_\lambda(\boldsymbol{x}; t)$ [35, III.2], which are discussed further in Section 3A.

The antipode of Sym acts as $(-1)^n \omega$ on the n-th graded component, where ω is the involutive automorphism of Sym defined in [35, I.4], given by $\omega(s_\lambda) = s_{\lambda'}$, where λ' denotes the transpose partition of λ: $(\lambda')_i = \#\{j \in [\ell(\lambda)] \mid \lambda_j \geq i\}$ for $1 \leq i \leq \lambda_1$.

2A1. *Combinatorial Hopf algebras.* This section will give an abridged description of the framework for classifying Hopf algebra homomorphisms to $\mathcal{Q}$Sym established in [2]. The original result also includes an explicit formula for any such map, which is omitted from this paper as the relevant maps are already known.

A *combinatorial Hopf algebra* (*CHA*) is a pair (H, ζ) where H is a Hopf algebra and $\zeta : H \to \mathbb{C}$ is an algebra homomorphism, which will be called a *zeta function* in order to avoid confusion with group characters. An important example of a CHA is $\mathcal{Q}$Sym with the *first principal specialization*,

$$(\mathcal{Q}\text{Sym}, \text{ps}_1) \qquad \text{with} \quad \text{ps}_1 : \mathcal{Q}\text{Sym} \to \mathbb{C}, \quad M_\alpha \mapsto \begin{cases} 1 & \text{if } \ell(\alpha) \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.1. The name "first principal specialization" comes from the fact that ps_1 is equivalent to specializing $x_1 = 1$ and $x_i = 0$ for $i > 1$ in any quasisymmetric function.

A *CHA morphism* between combinatorial Hopf algebras (H, ζ) and (H', ζ') is a graded Hopf algebra homomorphism $\Psi : H \to H'$ for which $\zeta = \zeta' \circ \Psi$. For example, the inclusion of Sym into $\mathcal{Q}$Sym gives a CHA morphism $(\text{Sym}, \text{ps}_1) \to (\mathcal{Q}\text{Sym}, \text{ps}_1)$.

Theorem 2.2 [2, Theorem 4.1]. *Let (H, ζ) be a combinatorial Hopf algebra. There is a unique CHA morphism*

$$\text{cano} : (H, \zeta) \to (\mathcal{Q}\text{Sym}, \text{ps}_1).$$

A consequence of Theorem 2.2 is that for every Hopf algebra H, there is a bijection

$$\{\text{homomorphisms } H \to \mathcal{Q}\text{Sym}\} \longleftrightarrow \{\text{combinatorial Hopf algebras } (H, \zeta)\},$$

$$\Psi \mapsto (H, \text{ps}_1 \circ \Psi), \quad \text{cano} \leftarrow\!\!\shortmid (H, \zeta),$$

where cano refers to the Hopf algebra homomorphisms underlying the CHA morphism in Theorem 2.2. This paper will frequently appeal to this bijective interpretation.

2B. *Dyck paths and related objects.* The results of this paper build on the combinatorics of Dyck paths, indifference graphs, and Schröder paths, each of which are described in this section.

A *Dyck path* of size $n \geq 0$ is a lattice path consisting of $2n$ steps east $E = (1, 0)$ and south $S = (0, -1)$ from $(0, 0)$ to $(n, -n)$ which does not go below the main diagonal $y = -x$. Let

$$\mathcal{D} = \bigsqcup_{n \geq 0} \mathcal{D}_n \quad \text{with} \quad \mathcal{D}_n = \{\text{Dyck paths of size } n\}.$$

For example,

$$\text{(figure)} = (EESESS) \in \mathcal{D}_3. \tag{2.3}$$

It is well known that the size of $\mathcal{D}_n$ is the n-th Catalan number, $\frac{1}{n+1}\binom{2n}{n}$; see, for instance, [41].

An *indifference graph* of size $n \geq 0$ is a simple, undirected graph γ with vertex set $[n] = \{1, \ldots, n\}$ and edge set $E(\gamma)$ satisfying

$$\text{for each } \{i < l\} \in E(\gamma), \quad \{\{j, k\} \mid i \leq j < k \leq l\} \subseteq E(\gamma).$$

The empty graph on $\varnothing$ is the unique indifference graph of size zero. Let

$$\mathcal{IG} = \bigsqcup_{n \geq 0} \mathcal{IG}_n \quad \text{with } \mathcal{IG}_n = \{\text{indifference graphs on } [n]\}.$$

For example,

$$\gamma = \underset{1 \quad 2 \quad 3 \quad 4}{\overset{\frown}{\bullet \quad \bullet \quad \bullet \quad \bullet}} \in \mathcal{IG}_4 \qquad \text{but} \qquad \sigma = \underset{1 \quad 2 \quad 3 \quad 4}{\overset{\frown}{\bullet \quad \bullet \quad \bullet \quad \bullet}} \notin \mathcal{IG}_4,$$

as $\{1, 4\} \in E(\sigma)$ but $\{3, 4\} \notin E(\sigma)$.

There is a size-preserving bijection between Dyck paths and indifference graphs. Label the unit squares above $y = -x$ in the fourth quadrant of $\mathbb{Z} \times \mathbb{Z}$ by edges so that the square with lower right corner $(j, -i)$ is labeled by $\{i, j\}$; for example,

shows the first three of these unit squares with their labels. For any Dyck path π, let

$$\text{Area}(\pi) = \{\{i, j\} \mid \text{the unit square } \{i, j\} \text{ is below } \pi\}$$

and if π has size n, define the *graph of π* to be

$$\text{Graph}(\pi) = ([n], \text{Area}(\pi)).$$

For example, taking the Dyck path in (2.3),

$$\text{Area}\left(\text{(figure)} \right) = \{\{1, 2\}, \{2, 3\}\} \qquad \text{and} \qquad \text{Graph}\left(\text{(figure)} \right) = \underset{1 \quad 2 \quad 3}{\bullet \!-\! \bullet \!-\! \bullet}. \tag{2.4}$$

Proposition 2.5 [41, Solution 187]. *For $n \geq 0$, the map $\pi \mapsto \text{Graph}(\pi)$ is a bijection from $\mathcal{D}_n$ to $\mathcal{IG}_n$.*

Remark 2.6. Both $\mathcal{D}_n$ and $\mathcal{IG}_n$ also correspond to the family of integer partitions bounded termwise by $(n - 1, \ldots, 2, 1)$ [41, Item 167]. Reflecting a Dyck path π across $y = -x$ gives the Ferrer's shape (in

French notation) of such a partition, and the edges of $\text{Graph}(\pi)$ are the excluded squares. For example, the objects in (2.4) correspond to $\lambda = (1)$.

A common generalization of Dyck paths will appear in Sections 5 and 6. A *Schröder path* of size $n \geq 0$ is a lattice path from $(0, 0)$ to $(n, -n)$ consisting of steps E, S, and $D = (1, -1)$ that never goes below the main diagonal. Thus, every Dyck path is a Schröder path, but there are more Schröder paths, for example

$$\begin{array}{c}\text{[figure]}\end{array} = (EEDSS). \tag{2.7}$$

Say that a Schröder path σ is *tall* if σ has no D steps along the main diagonal. Let

$$\mathcal{TS} = \bigsqcup_{n \geq 0} \mathcal{TS}_n \quad \text{with } \mathcal{TS}_n = \{\text{tall Schröder paths of size } n\}.$$

The Schröder path in (2.7) above is tall, as is any Dyck path, taken as a Schröder path. The number of tall Schröder paths by size is given by the small Schröder numbers, [37, A001003].

Finally, for any tall Schröder path $\sigma \in \mathcal{TS}$, define

$$\text{Area}(\sigma) = \{\{i, j\} \mid \text{the unit square } \{i, j\} \text{ is completely below } \sigma\}$$

and

$$\text{Diag}(\sigma) = \{\{i, j\} \mid \sigma \text{ has a diagonal step through the unit square } \{i, j\}\},$$

so that taking σ as in (2.7) gives $\text{Area}(\sigma) = \{\{1, 2\}, \{2, 3\}\}$ and $\text{Diag}(\sigma) = \{\{1, 3\}\}$.

2C. *Supercharacter theory.* Let G be a finite group, let $\text{Irr}(G)$ denote the irreducible complex characters of G, and let $\text{cf}(G)$ denote the space of complex-valued class functions on G. The set $\text{Irr}(G)$ is an orthonormal basis for $\text{cf}(G)$ under the inner product $\langle \cdot, \cdot \rangle : \text{cf}(G) \otimes \text{cf}(G) \to \mathbb{C}$ defined by

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)},$$

where $\overline{\psi(g)}$ denotes the complex conjugate of $\psi(g)$.

Following Diaconis and Isaacs [15], a *supercharacter theory* (Cl, Ch) of G comprises a set partition Cl of G and a basis of orthogonal characters Ch for the space

$$\text{scf}(G) = \{\phi : G \to \mathbb{C} \mid \phi \text{ is constant on each part of Cl}\},$$

such that $\text{scf}(G)$ contains the regular character reg_G. Since

$$\text{reg}_G(g) = \begin{cases} |G| & \text{if } g = 1_G, \\ 0 & \text{otherwise,} \end{cases}$$

the final condition above is equivalent to $\{1_G\} \in \text{Cl}$.

The elements of Cl and Ch are respectively called *superclasses* and *supercharacters*. Every group has at least one supercharacter theory, with superclasses given by conjugacy classes and supercharacters given by irreducible characters, and in this case $\mathrm{scf}(G) = \mathrm{cf}(G)$.

Each supercharacter theory of G comes with two canonical bases: the supercharacters in Ch and the set of *superclass identifier functions*

$$\{\delta_K \mid K \in \mathtt{Cl}\} \quad \text{with } \delta_K(g) = \begin{cases} 1 & \text{if } g \in K, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

These bases are each orthogonal. For any $\chi \in \mathrm{scf}(G)$, define an element $\chi\rangle \in \mathrm{scf}(G)^*$ by

$$\chi\rangle : \mathrm{scf}(G) \to \mathbb{C}, \quad \psi \mapsto \langle \psi, \chi \rangle, \tag{2.9}$$

so that $\mathrm{scf}(G)^* = \{\chi\rangle \mid \chi \in \mathrm{scf}(G)\}$.

The rest of the section describes a particular collection of supercharacter theories originating in the work of Aliniaeifard and Thiem [7]. Fix a prime power q, let $\mathbb{F}_q$ denote the field with q elements, and let $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{F}_q)$. The *unipotent upper triangular group* is the subgroup

$$\mathrm{UT}_n = \{g \in \mathrm{GL}_n \mid (g - 1_n)_{i,j} \neq 0 \text{ only if } i < j\}$$

where 1_n denotes the $n \times n$ identity matrix. This group has a family of normal subgroups — called *normal pattern subgroups* — indexed by indifference graphs [36, Lemma 4.1]: for $\gamma \in \mathcal{IG}_n$, let

$$\mathrm{UT}_\gamma = \{g \in \mathrm{UT}_n \mid g_{i,j} = 0 \text{ if } \{i, j\} \in E(\gamma)\}$$

where $E(\gamma)$ denotes the edge set of γ. If $\pi \in \mathcal{D}_n$ is the Dyck path for which $\gamma = \mathrm{Graph}(\pi)$, UT_γ can be visualized in terms of π: UT_γ is the subset of elements of UT_n with nonzero entries occurring only on the diagonal or above the path π. For example, using the graph and Dyck path from (2.4),

$$\mathrm{UT}_{\underset{1 \quad 2 \quad 3}{\bullet\text{—}\bullet\text{—}\bullet}} = \begin{Vmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}.$$

The paper [7] also shows that the set $\{\mathrm{UT}_\gamma \mid \gamma \in \mathcal{IG}_n\}$ is a lattice under containment. This order is dual to the spanning subgraph relation on $\mathcal{IG}_n$, in that the containment $\mathrm{UT}_\gamma \subseteq \mathrm{UT}_\sigma$ holds if and only if σ is a spanning subgraph of γ. The top element of this lattice is UT_n, corresponding to the edgeless graph $([n], \varnothing)$, and $|\mathrm{UT}_n : \mathrm{UT}_\gamma| = q^{|E(\gamma)|}$ for all $\gamma \in \mathcal{IG}_n$.

The lattice structure on normal pattern subgroups partitions the set UT_n into parts

$$\mathrm{UT}_\gamma^\circ = \{g \in \mathrm{UT}_\gamma \mid g \notin \mathrm{UT}_\sigma \text{ for any } \sigma \supsetneq \gamma\}$$

for each $\gamma \in \mathcal{IG}_n$. Similarly, $\mathcal{IG}_n$ indexes the parts of a partition of the set of irreducible characters $\mathrm{Irr}(\mathrm{UT}_n)$ of UT_n: let

$$\widehat{\mathrm{UT}}_\gamma^\circ = \{\psi \in \mathrm{Irr}(\mathrm{UT}_n) \mid \mathrm{UT}_\gamma \subseteq \ker(\psi) \text{ and } \mathrm{UT}_\sigma \not\subseteq \ker(\psi) \text{ for each } \sigma \supsetneq \gamma\}.$$

for each $\gamma \in \mathcal{IG}_n$, and further define

$$\chi^\gamma = \sum_{\psi \in \widehat{\mathrm{UT}}^\circ_\gamma} \psi(1)\psi.$$

Proposition 2.10 [7, Section 3.2]. *With*

$$\mathtt{Ch} = \{\mathrm{UT}^\circ_\gamma \mid \gamma \in \mathcal{IG}_n\} \quad and \quad \mathtt{Cl} = \{\chi^\gamma \mid \gamma \in \mathcal{IG}_n\},$$

the pair $(\mathtt{Cl}, \mathtt{Ch})$ *is a supercharacter theory of* UT_n.

For the remainder of the paper, write $\delta_\gamma = \delta_{\mathrm{UT}^\circ_\gamma}$ for the superclass identifier functions in this supercharacter theory. In addition to these functions and the supercharacters, the space $\mathrm{scf}(\mathrm{UT}_n)$ has two interesting bases: $\{\bar{\delta}_\gamma \mid \gamma \in \mathcal{IG}_n\}$ and $\{\bar{\chi}^\gamma \mid \gamma \in \mathcal{IG}_n\}$, with

$$\bar{\delta}_\gamma = \sum_{\sigma \geq \gamma} \delta_\sigma \quad \text{and} \quad \bar{\chi}^\gamma = \sum_{\sigma \subseteq \gamma} \chi^\sigma.$$

Remarkably, if $\mathbb{1} \in \mathrm{cf}(\mathrm{UT}_\gamma)$ denotes the character of the trivial representation then

$$\bar{\chi}^\gamma = \mathrm{Ind}^{\mathrm{UT}_n}_{\mathrm{UT}_\gamma}(\mathbb{1}) = q^{|E(\gamma)|}\bar{\delta}_\gamma, \tag{2.11}$$

the character of the UT_n-module $\mathbb{C}[\mathrm{UT}_n/\mathrm{UT}_\gamma]$.

2D. *Homomorphisms between Hopf algebras of class functions.* In [45, III], Zelevinsky defines a graded connected Hopf algebra on the space

$$\mathrm{cf}(\mathrm{GL}_\bullet) = \bigoplus_{n \geq 0} \mathrm{cf}(\mathrm{GL}_n),$$

with structure maps coming from the parabolic induction and restriction functors. The paper [19] defines a similar Hopf structure on the spaces

$$\mathrm{scf}(\mathrm{UT}_\bullet) = \bigoplus_{n \geq 0} \mathrm{scf}(\mathrm{UT}_n) \quad \text{and} \quad \mathrm{cf}(\mathrm{UT}_\bullet) = \bigoplus_{n \geq 0} \mathrm{cf}(\mathrm{UT}_n),$$

in which $\mathrm{scf}(\mathrm{UT}_n)$ is the subspace of class functions defined in Section 2C, with $\mathrm{scf}(\mathrm{UT}_\bullet)$ a sub-Hopf algebra of $\mathrm{cf}(\mathrm{UT}_\bullet)$. This section will describe several homomorphisms involving these Hopf algebras.

In [25, Section 6], Guay-Paquet defines a $\mathbb{C}[t]$-Hopf algebra on the free $\mathbb{C}[t]$-module $\mathbb{C}[t][\mathcal{IG}]$, and specializing $t \mapsto q^{-1}$ gives a Hopf algebra over $\mathbb{C}$; see [19, Section 7]. Recall the basis $\{\bar{\delta}_\gamma \mid \gamma \in \mathcal{IG}\}$ of $\mathrm{scf}(\mathrm{UT}_\bullet)$ defined in Section 2C.

Theorem 2.12 [19, Corollary 7.2]. *The map $\gamma \mapsto \bar{\delta}_\gamma$ is an isomorphism from Guay-Paquet's specialized Hopf algebra to* $\mathrm{scf}(\mathrm{UT}_\bullet)$.

A second map comes from the induction functors $\mathrm{Ind}^{\mathrm{GL}_n}_{\mathrm{UT}_n} : \mathrm{cf}(\mathrm{UT}_n) \to \mathrm{cf}(\mathrm{GL}_n)$: let

$$\mathrm{Ind}^{\mathrm{GL}}_{\mathrm{UT}} = \bigoplus_{n \geq 0} \mathrm{Ind}^{\mathrm{GL}_n}_{\mathrm{UT}_n} : \mathrm{cf}(\mathrm{UT}_\bullet) \to \mathrm{cf}(\mathrm{GL}_\bullet). \tag{2.13}$$

Theorem 2.14 [19, Theorem 6.1]. *The map* $\mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}}$ *is a Hopf algebra homomorphism.*

The homomorphism $\mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}}$ also induces a linear map on dual spaces. Using the identification in (2.8), the dual of the direct sum $\mathrm{cf}(\mathrm{GL}_\bullet)$ becomes a product

$$\mathrm{cf}(\mathrm{GL}_\bullet)^* = \prod_{n \geq 0} \mathrm{cf}(\mathrm{GL}_n)^* = \big\{ (\chi_n) \rangle_{n \geq 0} \mid \chi_n \in \mathrm{cf}(\mathrm{GL}_n) \big\}.$$

Making the analogous identification for $\mathrm{cf}(\mathrm{UT}_\bullet)^*$ and $\mathrm{scf}(\mathrm{UT}_\bullet)^*$, Frobenius reciprocity gives that

$$(\chi_n) \rangle_{n \geq 0} \circ \mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}} = \big(\mathrm{Res}_{\mathrm{UT}_n}^{\mathrm{GL}_n}(\chi_n) \rangle \big)_{n \geq 0}.$$

If $\mathrm{Res}_{\mathrm{UT}_n}^{\mathrm{GL}_n}(\chi_n) \in \mathrm{scf}(\mathrm{UT}_\bullet)$ for each $n \geq 0$, the same equation applies when considering each side as an element of $\mathrm{scf}(\mathrm{UT}_\bullet)^*$.

3. The chromatic quasisymmetric function as a GL_n character

This section will state and prove Theorem 3.1, following some initial context. Recall the Hopf algebras $\mathrm{scf}(\mathrm{UT}_\bullet)$ and $\mathrm{cf}(\mathrm{UT}_\bullet)$ from Section 2D. The Hopf algebra of GL-*class functions with unipotent support* is the image

$$\mathrm{cf}_{\mathrm{supp}}^{\mathrm{uni}}(\mathrm{GL}_\bullet) = \mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}}(\mathrm{cf}(\mathrm{UT}_\bullet)) \subseteq \mathrm{cf}(\mathrm{GL}_\bullet).$$

Zelevinsky [45] has defined a Hopf algebra isomorphism $\boldsymbol{p}_{\{1\}} : \mathrm{cf}_{\mathrm{supp}}^{\mathrm{uni}}(\mathrm{GL}_\bullet) \to \mathrm{Sym}$ which will be used in the theorem; see Section 3A. Finally, for each indifference graph γ, recall the subgroup UT_γ defined in Section 2C, and let $X_\gamma(\boldsymbol{x}; t)$ denote the chromatic quasisymmetric function of γ in an indeterminate 't', which will be formally defined in Section 3B.

Theorem 3.1. *For $n \geq 0$ and $\gamma \in \mathcal{IG}_n$,*

$$\mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{GL}_n}(\mathbb{1}) = (q-1)^n \, \boldsymbol{p}_{\{1\}}^{-1}(X_\gamma(\boldsymbol{x}; q)).$$

I will describe briefly how the results in this section prove Theorem 3.1. Define a Hopf algebra homomorphism $\boldsymbol{c}_{\{1\}} : \mathrm{scf}(\mathrm{UT}_\bullet) \to \mathcal{QSym}$ as the composite map in the diagram

$$\tag{3.2}$$

of Hopf algebra homomorphisms. By the transitivity of induction, the theorem is equivalent to computing the image of the character $\overline{\chi}^\gamma = \mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{UT}_n}(\mathbb{1}) \in \mathrm{scf}(\mathrm{UT}_\bullet)$ under $\boldsymbol{c}_{\{1\}}$.

Recalling the theory of combinatorial Hopf algebras from Section 2A1, there is a unique combinatorial Hopf algebra structure on $\mathrm{scf}(\mathrm{UT}_\bullet)$ for which $\boldsymbol{c}_{\{1\}}$ is a CHA morphism to $(\mathcal{QSym}, \mathrm{ps}_1)$, and this structure is given by a zeta function of the Hopf algebra $\mathrm{scf}(\mathrm{UT}_\bullet)$. Theorem 3.8 computes this zeta function, and

Proposition 3.13 shows that it is essentially the same as one defined by Guay-Paquet [25]. This leads to a formula for $c_{\{1\}}$ on the basis $\{\bar{\delta}_\gamma \mid \gamma \in \mathcal{IG}\}$ of $\mathrm{scf}(\mathrm{UT}_\bullet)$ from Section 2C, stated formally in Corollary 3.14:

$$c_{\{1\}}(\bar{\delta}_\gamma) = (q-1)^n X_\gamma(\boldsymbol{x}; q^{-1}) \quad \text{for } \gamma \in \mathcal{IG}_n. \tag{3.3}$$

From here, the theorem follows from an identity of Shareshian–Wachs [39]. Recalling from Section 2C that $\bar{\chi}^\gamma = q^{|E(\gamma)|}\bar{\delta}_\gamma$, [39, Proposition 2.6] reformulates (3.3) as

$$c_{\{1\}}(\bar{\chi}^\gamma) = (q-1)^n q^{|E(\gamma)|} X_\gamma(\boldsymbol{x}; q^{-1}) = (q-1)^n X_\gamma(\boldsymbol{x}; q).$$

The results used in the proof are given in the remainder of this section, which comprises two subsections. Section 3A describes the zeta functions of the Hopf algebras $\mathrm{cf}^{\mathrm{uni}}_{\mathrm{supp}}(\mathrm{GL}_\bullet)$ and $\mathrm{scf}(\mathrm{UT}_\bullet)$ needed to make Diagram (3.2) a diagram of combinatorial Hopf algebras. Then, Section 3B uses results from [25] and Section 2 to describe the chromatic quasisymmetric function as the image of a CHA morphism from $\mathrm{scf}(\mathrm{UT}_\bullet)$ and subsequently shows that up to a power of $(q-1)$ this map coincides with $c_{\{1\}}$.

3A. *Factoring $c_{\{1\}}$ through $\mathrm{cf}^{\mathrm{uni}}_{\mathrm{supp}}(\mathrm{GL}_\bullet)$.* This section describes the Hopf algebra $\mathrm{cf}^{\mathrm{uni}}_{\mathrm{supp}}(\mathrm{GL}_\bullet)$ and its isomorphism with Sym.

An element $g \in \mathrm{GL}_n$ is called *unipotent* if $g - 1_n$ is nilpotent. There is a canonical indexing of the unipotent GL_n-conjugacy classes by partitions; this is stated without proof in [45, 10.1] so a bit more detail has been included here. The Jordan canonical form of an element $g \in \mathrm{GL}_n$ is defined over any field that contains every root of the characteristic polynomial of g. Assuming that g is unipotent, the characteristic polynomial is $(t-1)^n$, so the Jordan canonical form of g is defined over $\mathbb{F}_q$. The Jordan matrices corresponding to $(t-1)^n$ are naturally indexed by partitions of n: $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ corresponds to

$$J_\lambda = \begin{bmatrix} J_{\lambda_1} & 0 & \cdots & 0 \\ 0 & J_{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{\lambda_\ell} \end{bmatrix} \quad \text{with} \quad J_k = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Thus, if we write O_λ for the GL_n conjugacy class of J_λ, the set of unipotent elements of GL_n is partitioned by the conjugacy classes $\{O_\lambda \mid \lambda \in \mathcal{P}_n\}$.

This shows that an element of GL_n is unipotent if and only if it is conjugate to an element of $\mathrm{UT}_n(\mathbb{F}_q)$, so that the sub-Hopf algebra $\mathrm{cf}^{\mathrm{uni}}_{\mathrm{supp}}(\mathrm{GL}_\bullet)$ from Section 3 is exactly

$$\mathrm{cf}^{\mathrm{uni}}_{\mathrm{supp}}(\mathrm{GL}_\bullet) = \bigoplus_{n \geq 0} \{\psi \in \mathrm{cf}(\mathrm{GL}_n) \mid \psi(h) = 0 \text{ for } h \in \mathrm{GL}_n \text{ not unipotent}\},$$

the space of GL-class functions with support only on unipotent elements. This fact is the source of the notational choice "$\mathrm{cf}^{\mathrm{uni}}_{\mathrm{supp}}$".

The preceding paragraphs demonstrate that $\mathrm{cf}^{\mathrm{uni}}_{\mathrm{supp}}(\mathrm{GL}_\bullet)$ has a $\mathcal{P}$-indexed basis of identifier functions $\delta_\lambda = \delta_{O_\lambda}$ for unipotent conjugacy classes,

$$\mathrm{cf}^{\mathrm{uni}}_{\mathrm{supp}}(\mathrm{GL}_\bullet) = \mathbb{C}\text{-span}\{\delta_\lambda \mid \lambda \in \mathcal{P}\}.$$

Zelevinsky [45, 10.13] (see also [35, IV.4.1]) constructs a graded Hopf algebra isomorphism

$$\boldsymbol{p}_{\{1\}} : \mathsf{cf}^{\mathrm{uni}}_{\mathrm{supp}}(\mathrm{GL}_\bullet) \to \mathrm{Sym}, \quad \delta_\lambda \mapsto \widetilde{P_\lambda}(\boldsymbol{x}; q) = q^{-n(\lambda)} P_\lambda(\boldsymbol{x}; q^{-1}), \tag{3.4}$$

where $n(\lambda) = \sum_{i=1}^{\lambda_1} \binom{\lambda_i'}{2}$, $P_\lambda(\boldsymbol{x}; t)$ is an element of the Hall–Littlewood 'P' basis of $\mathrm{Sym}[t]$ defined in [35, III.2], and we have specialized $t = q^{-1}$.

In the framework of Theorem 2.2, the isomorphism $\boldsymbol{p}_{\{1\}}$ is equivalent to a zeta function of $\mathsf{cf}^{\mathrm{uni}}_{\mathrm{supp}}(\mathrm{GL}_\bullet)$. This datum was also determined by Zelevinsky in [45]. The *regular* unipotent elements of GL_n are the members of the conjugacy class $O_{(n)}$. Using the notation of Section 2D, define a linear functional

$$\delta^*_{(\bullet)}\rangle = (\delta^*_{(n)}\rangle)_{n \geq 0} \in \mathsf{cf}(\mathrm{GL}_\bullet)^* \quad \text{with} \quad \delta^*_{(n)}\rangle = \frac{\delta_{(n)}\rangle}{\langle \delta_{(n)}, \delta_{(n)} \rangle},$$

so that for $\psi \in \mathsf{cf}(\mathrm{GL}_n)$, the value of $\delta^*_{(\bullet)}\rangle(\psi)$ is the value of ψ at any regular unipotent element, $\psi(J_{(n)})$. By embedding $\mathsf{cf}^{\mathrm{uni}}_{\mathrm{supp}}(\mathrm{GL}_\bullet)$ into $\mathsf{cf}(\mathrm{GL}_\bullet)$, $\delta^*_{(\bullet)}\rangle$ is also a linear functional on $\mathsf{cf}^{\mathrm{uni}}_{\mathrm{supp}}(\mathrm{GL}_\bullet)$.

Proposition 3.5 [45, 10.8]. *The map* $\delta^*_{(\bullet)}\rangle$ *is a zeta function of the Hopf algebra* $\mathsf{cf}^{\mathrm{uni}}_{\mathrm{supp}}(\mathrm{GL}_\bullet)$ *and* $\boldsymbol{p}_{\{1\}}$ *is the unique CHA morphism* $(\mathsf{cf}^{\mathrm{uni}}_{\mathrm{supp}}(\mathrm{GL}_\bullet), \delta^*_{(\bullet)}\rangle) \to (\mathcal{Q}\mathrm{Sym}, \mathrm{ps}_1)$.

Remark 3.6. In [45], this result is stated in terms of symmetric functions, since the language of CHAs was not yet available. However, the underlying theory naturally extends to this context, essentially because the inclusion $(\mathrm{Sym}, \mathrm{ps}_1) \hookrightarrow (\mathcal{Q}\mathrm{Sym}, \mathrm{ps}_1)$ is a CHA morphism.

Now consider the Hopf algebra $\mathsf{scf}(\mathrm{UT}_\bullet)$. Recall that $([n], \varnothing)$ is the minimal indifference graph on n vertices and define a linear functional

$$(q-1)^\bullet \delta^*_{([\bullet], \varnothing)}\rangle = ((q-1)^n \delta^*_{([n], \varnothing)}\rangle)_{n \geq 0} \in \mathsf{scf}(\mathrm{UT}_\bullet)^* \quad \text{with} \quad \delta^*_{([n], \varnothing)}\rangle = \frac{\delta_{([n], \varnothing)}\rangle}{\langle \delta_{([n], \varnothing)}, \delta_{([n], \varnothing)} \rangle}.$$

Remark 3.7. There is an unfortunate coincidence of notation between the class functions $\delta^*_{(n)}$ and $\delta^*_{([n], \varnothing)}$, and care should be taken to distinguish between the two: up to normalization $\delta^*_{(n)}$ is the GL_n-class function which identifies the conjugacy class $O_{(n)}$ of regular unipotent elements, and $\delta^*_{([n], \varnothing)}$ is the UT_n-class function which identifies the superclass

$$\mathrm{UT}^\circ_{([n], \varnothing)} = \{X \in \mathrm{UT}_n \mid X_{i,i+1} \neq 0 \text{ for } 1 \leq i < n\}.$$

However, the two are closely related, as described in the proof of Theorem 3.8 below.

Theorem 3.8. *The linear functional* $(q-1)^\bullet \delta^*_{([\bullet], \varnothing)}\rangle$ *is a zeta function of* $\mathsf{scf}(\mathrm{UT}_\bullet)$ *and*

$$(q-1)^\bullet \delta^*_{([\bullet], \varnothing)}\rangle = \delta^*_{(\bullet)}\rangle \circ \mathrm{Ind}^{\mathrm{GL}}_{\mathrm{UT}},$$

so $\mathrm{Ind}^{\mathrm{GL}}_{\mathrm{UT}}$ *is a CHA morphism*

$$(\mathsf{scf}(\mathrm{UT}_\bullet), (q-1)^\bullet \delta^*_{([\bullet], \varnothing)}\rangle) \xrightarrow{\mathrm{Ind}^{\mathrm{GL}}_{\mathrm{UT}}} (\mathsf{cf}(\mathrm{GL}_\bullet), \delta^*_{(\bullet)}\rangle).$$

Proof. The first and third assertions follow from the second. The proof of the second assertion will make use of the fact that the superclass $\mathrm{UT}^\circ_{([n],\varnothing)}$ is also the set of all regular unipotent elements in UT_n, so that $\delta_{([n],\varnothing)} = \mathrm{Res}^{\mathrm{GL}_n}_{\mathrm{UT}_n}(\delta_{(n)})$.

For $\gamma \in \mathcal{IG}_n$, Frobenius reciprocity (as described in Section 2D) gives

$$\langle \delta^*_{(\bullet)}\rangle \circ \mathrm{Ind}^{\mathrm{GL}}_{\mathrm{UT}}(\bar{\delta}_\gamma) = \frac{\mathrm{Res}^{\mathrm{GL}_n}_{\mathrm{UT}_n}(\delta_{(n)})\rangle(\bar{\delta}_\gamma)}{\langle \delta_{(n)}, \delta_{(n)}\rangle} = \frac{\langle \bar{\delta}_\gamma, \delta_{([n],\varnothing)}\rangle}{\langle \delta_{(n)}, \delta_{(n)}\rangle} = \begin{cases} \dfrac{\langle \delta_{([n],\varnothing)}, \delta_{([n],\varnothing)}\rangle}{\langle \delta_{(n)}, \delta_{(n)}\rangle} & \text{if } \gamma = ([n], \varnothing), \\ 0 & \text{otherwise,} \end{cases}$$

with the last equation following from the definition of $\bar{\delta}_\gamma$, the minimality of $([n], \varnothing)$, and the orthogonality of the superclass identifiers; see Section 2C. Direct computation then gives that

$$\frac{\langle \delta_{([n],\varnothing)}, \delta_{([n],\varnothing)}\rangle}{\langle \delta_{(n)}, \delta_{(n)}\rangle} = \frac{|\mathrm{GL}_n|}{|O_{(n)}|} \frac{|\mathrm{UT}^\circ_{([n],\varnothing)}|}{|\mathrm{UT}_n|} = (q-1)^n,$$

where the final equality comes from the order formulas

$$O_{(n)} = \frac{|\mathrm{GL}_n|}{q^{n-1}(q-1)} \quad \text{and} \quad \mathrm{UT}^\circ_{([n],\varnothing)} = (q-1)^{n-1} \frac{|\mathrm{UT}_n|}{q^{n-1}}. \qquad \square$$

Now recall the map $c_{\{1\}}$ defined in Diagram (3.2). Theorem 3.8 and Proposition 3.5 give the following.

Corollary 3.9. *The map $c_{\{1\}}$ is the unique CHA morphism*

$$c_{\{1\}} : (\mathrm{scf}(\mathrm{UT}_\bullet), (q-1)^\bullet \delta^*_{([\bullet],\varnothing)}\rangle) \to (\mathcal{Q}\mathrm{Sym}, \mathrm{ps}_1).$$

Remark 3.10. Theorem 3.8 actually establishes the stronger result that $(q-1)^\bullet \delta^*_{([\bullet],\varnothing)}\rangle$ is a zeta function of $\mathrm{cf}(\mathrm{UT}_\bullet)$, and that we may extend the domain of the CHA morphisms $\mathrm{Ind}^{\mathrm{GL}}_{\mathrm{UT}}$ and $c_{\{1\}}$ to the combinatorial Hopf algebra $\big(\mathrm{cf}(\mathrm{UT}_\bullet), (q-1)^\bullet \delta^*_{([\bullet],\varnothing)}\rangle\big)$. While this level of generality is unnecessary for the scope of this work, it may be of general interest.

3B. *The chromatic quasisymmetric function.* This section defines the chromatic quasisymmetric function of a graph and describes how it can be realized as the image of a character of $\mathrm{GL}_n(\mathbb{F}_q)$ under a particular a CHA morphism, leading to a proof of Theorem 3.1.

Let γ be a simple, undirected graph with vertex set $[n]$ and edge set $E(\gamma)$. A *coloring* of γ is a function $\kappa : [n] \to \mathbb{Z}_{>0}$. A coloring κ of γ is *proper* if $\kappa(i) \neq \kappa(j)$ for all $\{i, j\} \in E(\gamma)$. The *γ-ascent number* of a coloring κ is

$$\mathrm{asc}_\gamma(\kappa) = \big|\{\{i, j\} \in E(\gamma) \mid i < j \text{ and } \kappa(i) < \kappa(j)\}\big|. \tag{3.11}$$

For example, if $\kappa : [5] \to \mathbb{Z}_{>0}$ is given by $\kappa(1) = 2$, $\kappa(2) = 5$, $\kappa(3) = 1$, and $\kappa(4) = 5$, then

$$\mathrm{asc}_{\underset{1 \ \ 2 \ \ 3 \ \ 4}{\frown\quad\frown}}(\kappa) = \big|\{\{1, 2\}, \{3, 4\}\}\big| = 2.$$

In this example, κ is a proper coloring of the given graph.

The *chromatic quasisymmetric function* of γ is

$$X_\gamma(\boldsymbol{x}; t) = \sum_{\substack{\kappa:[n]\to\mathbb{Z}_{>0} \\ \text{proper}}} t^{\mathrm{asc}_\gamma(\kappa)} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)} \in \mathcal{Q}\mathrm{Sym}[t],$$

so that $X_\gamma(\boldsymbol{x}; t)$ is a polynomial in an indeterminate t whose coefficients — by properties of the ascent statistic — are quasisymmetric functions. For an indifference graph $\gamma \in \mathcal{IG}_n$, it is known that these coefficients are elements of Sym [39, Theorem 4.5]. For example,

$$X_{\underset{1 \quad 2 \quad 3}{\longleftrightarrow}}(\boldsymbol{x}; t) = t m_{(2,1)} + (t^2 + 4t + 1) m_{(1^3)}.$$

However, this property is not used, and a novel proof of it follows from Corollary 3.14 below; see Remarks 3.15 (R1).

Evaluating the indeterminate t in $X_\gamma(\boldsymbol{x}; t)$ at a complex number gives an actual (quasi)symmetric function. For example, $X_\gamma(\boldsymbol{x}; 1)$ is the ordinary chromatic symmetric function of the graph γ, as defined by Stanley in [40]. In Theorem 3.1 the chromatic quasisymmetric functions are evaluated at q, the order of the finite field $\mathbb{F}_q$.

In [25], Guay-Paquet constructs the chromatic quasisymmetric by way of a homomorphism of $\mathbb{C}[t]$-Hopf algebras. By evaluating at $t = q^{-1}$ as in Theorem 2.12, this result descends to a Hopf algebra homomorphism $\mathrm{scf}(\mathrm{UT}_\bullet) \to \mathcal{Q}\mathrm{Sym}$. Define a linear functional

$$\zeta_0 : \mathrm{scf}(\mathrm{UT}_\bullet) \to \mathbb{C}, \quad \bar{\delta}_\gamma \mapsto \begin{cases} 1 & \text{if } \gamma = ([n], \varnothing), \\ 0 & \text{otherwise.} \end{cases}$$

The following theorem is translated from its original context in [25] to that of the Hopf algebra $\mathrm{scf}(\mathrm{UT}_\bullet)$ using the Hopf algebra isomorphism in Theorem 2.12.

Theorem 3.12 [25, Theorem 57]. *The map ζ_0 is a zeta function of* $\mathrm{scf}(\mathrm{UT}_\bullet)$, *and the unique CHA morphism*

$$(\mathrm{scf}(\mathrm{UT}_\bullet), \zeta_0) \to (\mathcal{Q}\mathrm{Sym}, \mathrm{ps}_1)$$

is given by

$$\bar{\delta}_\gamma \mapsto X_\gamma(\boldsymbol{x}; q^{-1}).$$

Along with Theorem 2.2, this result is the key to compute the image of the map $\boldsymbol{c}_{\{1\}}$ defined at the outset of Section 3. Recall the zeta function $(q-1)^\bullet \delta^*_{([\bullet], \varnothing)}\rangle$ of the Hopf algebra $\mathrm{scf}(\mathrm{UT}_\bullet)$ defined in Section 3A.

Proposition 3.13. *Let γ be an indifference graph of size $n \geq 0$. Then*

$$(q-1)^\bullet \delta^*_{([\bullet], \varnothing)}\rangle(\bar{\delta}_\gamma) = \begin{cases} (q-1)^n & \text{if } \gamma = ([n], \varnothing), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By definition, $\bar{\delta}_\gamma = \sum_{\sigma \geq \gamma} \delta_\sigma$. Explicit computation then gives

$$\frac{\langle (q-1)^n \delta_{([n], \varnothing)}, \bar{\delta}_\gamma \rangle}{\langle \delta_{([n], \varnothing)}, \delta_{([n], \varnothing)} \rangle} = (q-1)^n \frac{\sum_{\sigma \geq \gamma} \langle \delta_{([n], \varnothing)}, \delta_\sigma \rangle}{\langle \delta_{([n], \varnothing)}, \delta_{([n], \varnothing)} \rangle}.$$

Using the orthogonality of the basis $\{\delta_\gamma \mid \gamma \in \mathcal{IG}_n\}$ and the minimality of $([n], \varnothing)$ under the spanning subgraph order on $\mathcal{IG}_n$, the above expression reduces to the desired formula. $\square$

Thus, on homogeneous elements of degree n, the zeta functions $(q-1)^\bullet \delta^*_{([\bullet], \varnothing)}\rangle$ and ζ_0 only differ by a factor of $(q-1)^n$. This leads to the following result, which is a restatement of (3.3) and accordingly the last step in the proof of Theorem 3.1.

Corollary 3.14. *Let γ be an indifference graph of size $n \geq 0$. Then*

$$c_{\{1\}}(\bar{\delta}_\zeta) = (q-1)^n X_\gamma(\boldsymbol{x}; q^{-1}).$$

Proof. By comparison with the Hopf algebra homomorphism in Theorem 3.12, it is clear that the given map is a graded Hopf algebra homomorphism, and further, that

$$\mathrm{ps}_1((q-1)^n X_\gamma(\boldsymbol{x}; q^{-1})) = (q-1)^n \zeta_0(\bar{\delta}_\gamma) = (q-1)^\bullet \delta^*_{([\bullet], \varnothing)}\rangle(\bar{\delta}_\gamma).$$

Thus, the given map is a CHA morphism

$$(\mathrm{scf}(\mathrm{UT}_\bullet), (q-1)^\bullet \delta^*_{([\bullet], \varnothing)}\rangle) \to (\mathcal{Q}\mathrm{Sym}, \mathrm{ps}_1).$$

By Theorem 2.2, the above map must be equal to $c_{\{1\}}$. $\square$

Remarks 3.15. (R1) As the image of $c_{\{1\}}$ is $\mathrm{Sym} \subseteq \mathcal{Q}\mathrm{Sym}$, Corollary 3.14 gives a novel proof that the coefficients of $X_\gamma(\boldsymbol{x}; t)$ are symmetric functions.

(R2) Proposition 3.13 also shows that $\zeta_0 = (\delta^*_{([n], \varnothing)}\rangle)_{n \geq 0}$; however this fact seems not to have any representation theoretic significance beyond its relation to the proof above.

4. Connections to Hessenberg varieties

This section will describe the relationship between the characters $\mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{GL}_n}(\mathbb{1})$ in Theorem 3.1, certain Hessenberg varieties over $\mathbb{F}_q$, and the analogous Hessenberg varieties over $\mathbb{C}$. These results follow a short overview of Hessenberg varieties. Throughout, the algebraic groups defined over $\mathbb{F}_q$ in Section 2C and their analogues over $\mathbb{C}$ are used, so the underling field will be explicitly written for each such group to avoid confusion.

Take a field $\mathbb{K} \in \{\mathbb{F}_q, \mathbb{C}\}$, and for $n \geq 0$ let $B_n(\mathbb{K})$ denote the subgroup of upper triangular matrices in $\mathrm{GL}_n(\mathbb{K})$. For each subspace $M \subseteq \mathrm{Mat}_n(\mathbb{K})$ which is stable under conjugation by elements of $B_n(\mathbb{K})$ and each matrix $A \in \mathrm{Mat}_n(\mathbb{K})$, the *Hessenberg variety* associated to A and M is

$$\mathcal{B}_A^M = \{g B_n(\mathbb{K}) \in \mathrm{GL}_n(\mathbb{K}) / B_n(\mathbb{K}) \mid g^{-1} A g \in M\}.$$

This is a slight variation — apparently due to [44] — of the original definition in [14], which requires that M contain all upper triangular matrices. The generalization is crucial, since the following results exclusively concern Hessenberg varieties associated to strictly upper triangular subspaces known as

ad-nilpotent ideals. For $\gamma \in \mathcal{IG}_n$, let

$$\mathfrak{ut}_\gamma(\mathbb{K}) = \{A \in \mathrm{Mat}_n(\mathbb{K}) \mid A_{i,j} \neq 0 \text{ only if } i < j \text{ and } (i,j) \notin \gamma\}$$
$$= \mathrm{UT}_\gamma(\mathbb{K}) - 1.$$

These sets are in fact ideals in the algebra (and Lie algebra) of upper triangular matrices. Key examples of the Hessenberg varieties of the form $\mathcal{B}_A^{\mathfrak{ut}_\gamma(\mathbb{K})}$ have been known for some time, but a specific study of these varieties is quite recent; see [31; 38].

Proposition 4.1. *Let $n \geq 0$ and $\gamma \in \mathcal{IG}_n$. For any $A \in \mathrm{Mat}_n(\mathbb{F}_q)$ with $1 + A \in \mathrm{GL}_n(\mathbb{F}_q)$,*

$$\mathrm{Ind}_{\mathrm{UT}_\gamma(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)}(\mathbb{1})(1+A) = (q-1)^n q^{|E(\gamma)|} |\mathcal{B}_A^{\mathfrak{ut}_\gamma(\mathbb{F}_q)}|.$$

Proof. The proof will compute the left side of the equation directly. Equation (2.11) and the standard formula for induced character values give

$$\mathrm{Ind}_{\mathrm{UT}_\gamma(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)}(\mathbb{1})(1+A) = \left|\{h\mathrm{UT}_\gamma(\mathbb{F}_q) \in \mathrm{GL}_n(\mathbb{F}_q)/\mathrm{UT}_\gamma(\mathbb{F}_q) \mid h^{-1}(1+A)h \in \mathrm{UT}_\gamma(\mathbb{F}_q)\}\right|.$$

Each left $B_n(\mathbb{F}_q)$ coset in $\mathrm{GL}_n(\mathbb{F}_q)$ comprises $q^{|E(\gamma)|}(q-1)^n$ left $\mathrm{UT}_\gamma(\mathbb{F}_q)$ cosets, and for each $h\mathrm{UT}_\gamma(\mathbb{F}_q) \subseteq gB_n(\mathbb{F}_q)$, it is the case that $h^{-1}(1+A)h \in \mathrm{UT}_\gamma(\mathbb{F}_q)$ if and only if $g^{-1}(1+A)g \in \mathrm{UT}_\gamma(\mathbb{F}_q)$, because $\mathrm{UT}_\gamma(\mathbb{F}_q)$ is normalized by $B_n(\mathbb{F}_q)$. Finally, $g^{-1}(1+A)g \in \mathrm{UT}_\gamma(\mathbb{F}_q)$ if and only if $g^{-1}Ag \in \mathfrak{ut}_\gamma(\mathbb{F}_q)$, in which case $gB_n(\mathbb{F}_q)$ belongs to $\mathcal{B}_A^{\mathfrak{ut}_\gamma(\mathbb{F}_q)}$. $\square$

This result reveals a relationship between the chromatic quasisymmetric function and Hessenberg varieties for ad-nilpotent ideals over $\mathbb{F}_q$. To state this relationship precisely, recall the degree-shifted Hall–Littlewood elements $\widetilde{P}_\lambda(\boldsymbol{x}; t)$ from Section 3A, and define Laurent polynomials $d_\lambda^\gamma(t)$ by

$$X_\gamma(\boldsymbol{x}; t) = \sum_{\lambda \in \mathcal{P}_n} d_\lambda^\gamma(t) \widetilde{P}_\lambda(\boldsymbol{x}; t). \tag{4.2}$$

Each $\widetilde{P}_\lambda(\boldsymbol{x}; t)$ is a polynomial in t^{-1} rather than t, so there is some subtlety to this definition: one must first express $t^{-|E(\gamma)|}X_\gamma(\boldsymbol{x}; t)$ in the basis $P_\lambda(\boldsymbol{x}; t^{-1})$ of $\mathrm{Sym}[t^{-1}]$ and then multiply each term by appropriate powers of t to obtain (4.2).

Evaluating both sides of (4.2) at $t = q$ and applying the map $\boldsymbol{p}_{\{1\}}^{-1}$ defined in Section 3A gives

$$\frac{1}{(q-1)^n} \mathrm{Ind}_{\mathrm{UT}_\gamma(\mathbb{F}_q)}^{\mathrm{GL}n(\mathbb{F}_q)}(\mathbb{1}) = \sum_{\lambda \in \mathcal{P}_n} d_\lambda^\gamma(q) \delta_\lambda,$$

where Theorem 3.1 is used to evaluate the left side. Each side of the above equation is a class function, so for any partition $\lambda \in \mathcal{P}_n$ we can evaluate both sides at an element $1 + A \in O_\lambda$ for some fixed partition $\lambda \in \mathcal{P}_n$, Proposition 4.1 gives

$$q^{|E(\gamma)|} |\mathcal{B}_A^{\mathfrak{ut}_\gamma(\mathbb{F}_q)}| = d_\lambda^\gamma(q). \tag{4.3}$$

The coefficients $d_\lambda^\gamma(t)$ also appear in the complex geometry of Hessenberg varieties for ad-nilpotent subspaces in a manner discovered by Precup and Sommers in [38]. For the following theorem, note that the discussion of Jordan canonical form in Section 3A shows that the similarity classes of nilpotent matrices over any field are indexed by partitions of n: the class indexed by $\lambda \in \mathcal{P}_n$ consists of all matrices similar to $J_\lambda - 1$.

Theorem 4.4 [38, Equation (4.7)]. *For $n \geq 0$, take $\gamma \in \mathcal{IG}_n$ and $\lambda \in \mathcal{P}_n$. Then*

$$\sum_{k \geq 0} \beta_k^\lambda t^{k/2} = t^{-|E(\gamma)|} d_\lambda^\gamma(t),$$

where β_k^λ denotes the k-th Betti number of $\mathcal{B}_A^{\mathfrak{ut}_\gamma(\mathbb{C})}$ for any nilpotent matrix $A \in \mathrm{Mat}_n(\mathbb{C})$ in the similarity class indexed by λ.

Thus, [38] shows that the $d_\lambda^\gamma(t)$ are in fact polynomials. Combining this result with (4.3) leads to the following corollary.

Corollary 4.5. *For $n \geq 0$, take $\gamma \in \mathcal{IG}_n$ and $\lambda \in \mathcal{P}_n$. Let $A \in \mathrm{Mat}_n(\mathbb{F}_q)$ be a nilpotent elements in similarity class indexed by λ. Then*

$$\sum_{k \geq 0} \beta_k^\lambda q^{k/2} = |\mathcal{B}_A^{\mathfrak{ut}_\gamma(\mathbb{F}_q)}|,$$

where the numbers β_k^λ are as in Theorem 4.4.

Remarks 4.6. (R1) Aside from this paper, I am aware of two works about Hessenberg varieties over $\mathbb{F}_q$. The preprint [17] concerns the Hessenberg variety associated to a split regular element of $\mathrm{GL}_n(\mathbb{F}_q)$ and a subspace containing all upper triangular matrices; under some nontrivial assumptions on q a result similar to Corollary 4.5 is established. This generalizes Fulman's use of Weil conjecture machinery on a subset of smooth Hessenberg varieties in order prove some identities on q-Eulerian numbers [18].

(R2) In [31], Ji and Precup give a combinatorial formula for the polynomials $d_\lambda^\gamma(t)$ by constructing an affine paving of $\mathcal{B}_A^{\mathfrak{ut}_\gamma(\mathbb{C})}$. Precup has also suggested that a second proof of Corollary 4.5 could be obtained from a careful study of this paving, which would independently reprove Theorem 3.1 (private communication, 2022).

5. The vertical strip LLT polynomial as a GL_n character

This section gives a second result of the same type as Theorem 3.1, in that it interprets a family of t-graded symmetric functions as the images of certain GL_n characters obtained by induction from UT_n under a particular isomorphism; see Table 1. Here, the initial UT_n characters come from a larger set $\{\psi^\sigma \mid \sigma \in \mathcal{TS}\}$ indexed by the set of tall Schröder paths $\mathcal{TS}$ from Section 2B, the map to Sym is a homomorphism $\boldsymbol{p}_\mathbb{1} : \mathrm{cf}(\mathrm{GL}_\bullet) \to \mathrm{Sym}$ which records the unipotent constituent of a character, and the symmetric functions are the vertical strip LLT polynomials $G_\sigma(\boldsymbol{x}; t)$, also indexed by the set $\mathcal{TS}$. Each object mentioned will be defined in this section.

	Theorem 3.1	Theorem 5.1
indexing set	indifference graphs $\gamma \in \mathcal{IG}_n$	tall Schröder paths $\sigma \in \mathcal{TS}_n$
UT_n-characters	permutation characters $\bar{\chi}^\gamma$	pseudosupercharacters ψ^σ
symmetric functions	chromatic quasisymmetric functions $X_\gamma(\boldsymbol{x}; t)$	vertical strip LLT polynomials $G_\sigma(\boldsymbol{x}; t)$
map to Sym	$\boldsymbol{p}_{\{1\}} : \mathrm{cf}^{\mathrm{uni}}_{\mathrm{supp}}(\mathrm{GL}_\bullet) \to \mathrm{Sym}$	$\boldsymbol{p}_{\mathbb{1}} : \mathrm{cf}(\mathrm{GL}_\bullet) \to \mathrm{Sym}$
meaning of map to Sym	records unipotently supported GL_n-class functions	records the irreducible unipotent constituents

Table 1. A comparison of the results of Theorems 3.1 and 5.1 in degree n.

Theorem 5.1. *Let σ be a tall Schröder path. Then*

$$\boldsymbol{p}_{\mathbb{1}} \circ \mathrm{Ind}^{\mathrm{GL}}_{\mathrm{UT}}(\psi^\sigma) = (q-1)^{|\mathrm{Diag}(\sigma)|} \omega G_\sigma(\boldsymbol{x}; q),$$

where $\mathrm{Diag}(\sigma)$ is the set of diagonal steps in σ.

I will now describe the meaning of this result in greater depth and outline its proof. In the study of finite groups of Lie type, including GL_n, Deligne–Lusztig theory identifies an exemplary set of irreducible characters known as *unipotent characters*. For GL_n, the unipotent characters are relatively well understood and will be described in Section 5B. Here, the relevant fact is that Zelevinsky [45] has shown that the subspace

$$\mathrm{cf}^{\mathrm{uni}}_{\mathrm{char}}(\mathrm{GL}_\bullet) = \mathbb{C}\text{-span}\{\text{irreducible unipotent characters of } \mathrm{GL}_n, n \geq 0\}$$

is a sub-Hopf algebra of $\mathrm{cf}(\mathrm{GL}_\bullet)$, and that $\mathrm{cf}^{\mathrm{uni}}_{\mathrm{char}}(\mathrm{GL}_\bullet)$ is isomorphic to Sym. Furthermore, [45] shows that the orthogonal projection from $\mathrm{cf}(\mathrm{GL}_\bullet)$ to $\mathrm{cf}^{\mathrm{uni}}_{\mathrm{char}}(\mathrm{GL}_\bullet)$ (with respect to the inner product $\langle \cdot, \cdot \rangle$ in Section 2C) is a Hopf algebra homomorphism. Consequently, there is a homomorphism $\boldsymbol{p}_{\mathbb{1}} : \mathrm{cf}(\mathrm{GL}_\bullet) \to \mathrm{Sym}$ obtained by projecting onto $\mathrm{cf}^{\mathrm{uni}}_{\mathrm{char}}(\mathrm{GL}_\bullet)$ and then applying the aforementioned isomorphism, as in the diagram

$$
\begin{array}{ccc}
\mathrm{cf}(\mathrm{GL}_\bullet) & \xrightarrow{\boldsymbol{p}_{\mathbb{1}}} & \mathrm{Sym} \\
& & \\
& \mathrm{cf}^{\mathrm{uni}}_{\mathrm{char}}(\mathrm{GL}_\bullet) & \cong
\end{array}
\tag{5.2}
$$

of Hopf algebra homomorphisms. The map $\boldsymbol{p}_{\mathbb{1}}$ faithfully records the irreducible unipotent constituents of any class function of GL_n, which can be recovered by reversing the isomorphism $\mathrm{cf}^{\mathrm{uni}}_{\mathrm{char}}(\mathrm{GL}_\bullet) \cong \mathrm{Sym}$. Thus, Theorem 5.1 states that the vertical strip LLT polynomial $G_\sigma(\boldsymbol{x}; q)$ determines the irreducible unipotent constituents of the character $\mathrm{Ind}^{\mathrm{GL}}_{\mathrm{UT}}(\psi^\sigma)$.

An interesting connection arises from the interplay of Theorems 3.1 and 5.1. Carlsson and Mellit [11, Proposition 3.5] show that for a Dyck path $\pi \in \mathcal{D}_n$, the plethystic relationship

$$(t-1)^n X_{\mathrm{Graph}(\pi)}(\boldsymbol{x}; t)\left[\frac{\boldsymbol{x}}{t-1}\right] = G_\pi(\boldsymbol{x}; t)$$

holds, where $\mathrm{Graph}(\pi)$ is the indifference graph associated to π in Section 2B. It is also known [35, IV.4] that the composite map

$$\mathrm{Sym} \xrightarrow{\ p_{\{1\}}^{-1}\ } \mathrm{cf}_{\mathrm{supp}}^{\mathrm{uni}}(\mathrm{GL}_\bullet) \hookrightarrow \mathrm{cf}(\mathrm{GL}_\bullet) \xrightarrow{\ p_{\mathbb{1}}\ } \mathrm{Sym}$$

is an isomorphism which can be expressed in plethystic notation as $f[\boldsymbol{x}] \mapsto \omega f\left[\frac{\boldsymbol{x}}{t-1}\right]\big|_{t=q}$, so my results give a GL_n-representation theoretic interpretation of Carlsson and Mellit's result; at the same time, [11, Proposition 3.5] could be used to prove Theorem 5.1 via Theorem 3.1.

The proof of Theorem 3.1 will instead use the machinery of combinatorial Hopf algebras, which has the benefit of giving a new description of the map $p_{\mathbb{1}} \circ \mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}}$. Define a Hopf algebra homomorphism $c_{\mathbb{1}} : \mathrm{scf}(\mathrm{UT}_\bullet) \to \mathcal{Q}\mathrm{Sym}$ as the composite map in the diagram

$$
\begin{array}{c}
\mathrm{scf}(\mathrm{UT}_\bullet) \\[4pt]
\Big\downarrow{\scriptstyle \mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}}} \qquad\qquad\qquad\qquad\qquad {\scriptstyle c_{\mathbb{1}}} \\[10pt]
\mathrm{cf}_{\mathrm{supp}}^{\mathrm{uni}}(\mathrm{GL}_\bullet) \hookrightarrow \mathrm{cf}(\mathrm{GL}_\bullet) \xrightarrow[\ p_{\mathbb{1}}\ ]{} \mathrm{Sym} \hookrightarrow \mathcal{Q}\mathrm{Sym}
\end{array}
\tag{5.3}
$$

of Hopf algebras, so that Theorem 5.1 describes $c_{\mathbb{1}}$ implicitly. By definition, $c_{\mathbb{1}}$ can be computed by inducing a character of UT_n to GL_n and recording its unipotent constituents as symmetric functions. However, Theorem 2.2 shows that $c_{\mathbb{1}}$ is also determined by the zeta function $\mathrm{ps}_1 \circ c_{\mathbb{1}}$ of the Hopf algebra $\mathrm{scf}(\mathrm{UT}_\bullet)$. It happens that this zeta function coincides exactly with one defined by Guay-Paquet, so that a result of [25] — restated in Corollary 5.16 — shows that

$$c_{\mathbb{1}}(\bar{\delta}_{\mathrm{Graph}(\pi)}) = G_\pi(\boldsymbol{x}; q^{-1}) \quad \text{for } \pi \in \mathcal{D}. \tag{5.4}$$

Several known identities for LLT polynomials complete the proof; these are given in Proposition 5.18.

The remainder of the section is divided into three parts. First, Section 5A describes the characters ψ^σ appearing in Theorem 5.1 and shows that this family includes both the permutation characters and supercharacters of $\mathrm{scf}(\mathrm{UT}_\bullet)$. Then, Section 5B describes the map $c_{\mathbb{1}}$ as a CHA morphism to $(\mathcal{Q}\mathrm{Sym}, \mathrm{ps}_1)$, defining the necessary combinatorial Hopf algebra structures on $\mathrm{scf}(\mathrm{UT}_\bullet)$ and $\mathrm{cf}(\mathrm{GL}_\bullet)$ along the way. Finally, Section 5C formally defines the vertical strip LLT polynomial, shows how it can be realized as the image of a CHA morphism, and concludes with a proof of Theorem 5.1.

Remark 5.5. It is possible to "remove" the factors of $q - 1$ in Theorem 5.1. With results in Section 5A, work of Andrews and Thiem [9, Remark on p. 490] and Aliniaeifard and Thiem [7, Remark (1) on p. 13]
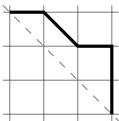
show that each ψ^σ is the sum of $(q-1)^{|\mathrm{Diag}(\sigma)|}$ distinct characters which each have the same image under $\boldsymbol{p}_\mathbb{1} \circ \mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}}$; this image must be $\omega G_\sigma(\boldsymbol{x}; q)$.

5A. *The pseudosupercharacters ψ^σ.* This section will define the characters ψ^σ appearing in Theorem 5.1. Recall the terminology used for Schröder paths in Section 2B and the characters of UT_n defined in Section 2C.

For $\sigma \in \mathcal{TS}_n$, the *pseudosupercharacter* indexed by σ is the class function

$$\psi^\sigma = \sum_{S \subseteq \mathrm{Diag}(\sigma)} (-1)^{|\mathrm{Diag}(\sigma)-S|} \overline{\chi}^{([n],\, \mathrm{Area}(\sigma) \cup S)} \in \mathrm{scf}(\mathrm{UT}_\bullet).$$

The definition of $\mathrm{Diag}(\sigma)$ ensures that each graph $([n], \mathrm{Area}(\sigma) \cup S)$ above is in fact an indifference graph. For example, with

$$\sigma = \quad\quad \text{we have} \quad \psi^\sigma = -\overline{\chi}^{\,\overset{\bullet}{1}\ \overset{\longleftrightarrow}{2\ 3}} + \overline{\chi}^{\,\overset{\longleftrightarrow}{1\ 2}\ \overset{\bullet}{3}}. \tag{5.6}$$

A noteworthy family of examples is the pseudosupercharacters indexed by Dyck paths: for $\pi \in \mathcal{D}$, $\mathrm{Diag}(\pi) = \varnothing$, from which it follows that

$$\psi^\pi = \overline{\chi}^{\mathrm{Graph}(\pi)}.$$

Proposition 5.7. *Let σ be a tall Schröder path of size $n \geq 0$. Then ψ^σ is a character, and in particular*

$$\psi^\sigma = \sum_{\substack{E(\gamma) \subseteq (\mathrm{Area}(\sigma) \cup \mathrm{Diag}(\sigma)) \\ \mathrm{Diag}(\sigma) \subseteq E(\gamma)}} \chi^\gamma,$$

where the sum is over indifference graphs $\gamma \in \mathcal{IG}_n$ satisfying the given conditions.

Proof. Using the definition of ψ^σ,

$$\psi^\sigma = \sum_{S \subseteq \mathrm{Diag}(\sigma)} (-1)^{|\mathrm{Diag}(\sigma)-S|} \sum_{E(\gamma) \subseteq \mathrm{Area}(\sigma) \cup S} \chi^\gamma,$$

where the sum is over indifference graphs γ as in the proposition. Reversing the order of summation above, we obtain

$$\psi^\sigma = \sum_{E(\gamma) \subseteq \mathrm{Area}(\sigma) \cup \mathrm{Diag}(\sigma)} \left(\sum_{\substack{T \subseteq \mathrm{Diag}(\sigma) \\ T \supseteq E(\gamma) \cap \mathrm{Diag}(\sigma)}} (-1)^{|\mathrm{Diag}(\sigma)-T|} \right) \chi^\gamma,$$

where the innermost sum is over subsets T of $\mathrm{Diag}(\sigma)$ that contain $E(\gamma) \cap \mathrm{Diag}(\sigma)$. Combining terms in this sum, the proposition follows from the binomial theorem. $\square$

As an example of Proposition 5.7, the pseudosupercharacter in (5.6) expands as the sum of supercharacters

$$\psi^\sigma = \chi^{\,\overset{\longleftrightarrow}{1\ 2}\ \overset{\bullet}{3}} + \chi^{\,\overset{\longleftrightarrow}{1\ 2}\ \overset{\bullet}{3}}.$$

The final result in this section shows that every supercharacter of $\mathrm{scf}(\mathrm{UT}_n)$ occurs as a pseudosupercharacter. Given a Dyck path π, a *peak* of π is a sequence of steps ES; say that a peak is *tall* if the first step E does not begin on the diagonal $x = -y$. For example,

$$\text{(grid figure)} = (EESESSES)$$
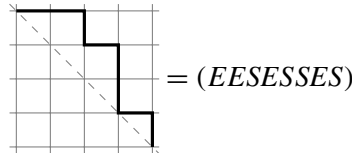
has three peaks, but only two tall peaks. Define the *Mesa path* of $\pi \in \mathcal{D}_n$ to be the tall Schröder path $\mathrm{Mesa}(\pi) \in \mathcal{TS}_n$ obtained by first constructing $\mathrm{Dyck}(\pi)$ and then replacing each tall peak ES with a diagonal step D; for example,

$$\mathrm{Mesa}\left(\text{(grid figure)} \right) = \text{(grid figure)} = (EDDSES).$$
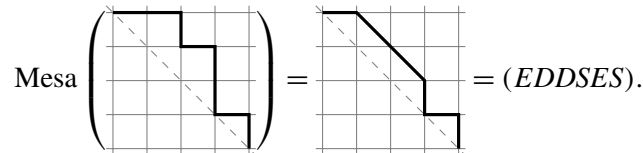
Proposition 5.8. *Let π be a Dyck path. Then $\psi^{\mathrm{Mesa}(\pi)} = \chi^{\mathrm{Graph}(\pi)}$.*

Proof. By assumption,

$$\mathrm{Area}(\pi) = \mathrm{Area}(\mathrm{Mesa}(\pi)) \cup \mathrm{Diag}(\mathrm{Mesa}(\pi)),$$

so by Proposition 5.7,

$$\psi^{\mathrm{Mesa}(\pi)} = \sum_{\substack{\gamma \subseteq \mathrm{Graph}(\pi) \\ \mathrm{Diag}(\mathrm{Mesa}(\pi)) \subseteq E(\gamma)}} \chi^{\gamma}.$$

Now suppose that an indifference graph γ is a proper spanning subgraph of $\mathrm{Graph}(\pi)$. Then γ must be missing at least one edge $\{i, j\}$ such that the unit square indexed by $\{i, j\}$ is bordered directly by a tall peak of π, so that $\{i, j\} \in \mathrm{Diag}(\mathrm{Mesa}(\pi))$, and χ^{γ} does not appear in the sum above. Thus the only summand above is $\chi^{\mathrm{Graph}(\pi)}$. $\square$

5B. *Factoring $c_{\mathbb{1}}$ through $\mathbf{cf}(\mathrm{GL}_\bullet)$.* This section will describe the unipotent characters of GL_n, and their relation to the Hopf algebra structure of $\mathrm{cf}(\mathrm{GL}_\bullet)$. As stated at the outset of Section 5, unipotent characters originate in Deligne–Lusztig theory, and are typically defined using cohomological induction. However, the unipotent characters of GL_n can also be described with much more elementary methods; see [16, Theorem 15.8 and proof] for the details. This paper will take this alternate description as a definition: an irreducible character of GL_n is *unipotent* if it is a constituent of $\mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(\mathbb{1})$, where $B_n = B_n(\mathbb{F}_q)$ is the subgroup of upper triangular matrices in GL_n.

It is also known that irreducible unipotent characters of GL_n are indexed by the partitions of n [16, Theorem 15.8]; write χ^λ for the unipotent character corresponding to $\lambda \in \mathcal{P}(n)$. This paper follows the convention of [35] in which $\chi^{(1^n)}$ is the trivial character $\mathbb{1}$ of GL_n and $\chi^{(n)}$ is the *Steinberg character* St_n; this differs from the convention of [45] and others by the transposition of each partition.

The homomorphism $\boldsymbol{p}_{\mathbb{1}}$ was constructed by Zelevinsky [45, 9.4], and is given by

$$\boldsymbol{p}_{\mathbb{1}} : \mathrm{cf}(\mathrm{GL}_\bullet) \to \mathrm{Sym}, \quad \psi \mapsto \sum_\lambda \langle \psi, \chi^\lambda \rangle s_\lambda. \tag{5.9}$$

As a linear transformation, $\boldsymbol{p}_{\mathbb{1}}$ has a right inverse $s_\lambda \mapsto \chi^\lambda$, and [45] shows that this right inverse is also a Hopf algebra homomorphism. Thus, the image

$$\mathrm{cf}^{\mathrm{uni}}_{\mathrm{char}}(\mathrm{GL}_\bullet) = \mathbb{C}\text{-span}\{\chi^\lambda \mid \lambda \in \mathcal{P}\} \subseteq \mathrm{cf}(\mathrm{GL}_\bullet)$$

is a sub-Hopf algebra of $\mathrm{cf}(\mathrm{GL}_\bullet)$ through which $\boldsymbol{p}_{\mathbb{1}}$ factors, as shown in Diagram (5.2).

By Theorem 2.2, the map $\boldsymbol{p}_{\mathbb{1}}$ is equivalent to a zeta function of the Hopf algebra $\mathrm{cf}(\mathrm{GL}_\bullet)$. This zeta function is also given in [45], and is

$$\mathrm{St}_\bullet \rangle = (\mathrm{St}_n \rangle)_{n \geq 0} \in \mathrm{cf}(\mathrm{GL}_\bullet)^*.$$

Proposition 5.10 [45, 9.4–5]. *The map* $\mathrm{St}_\bullet \rangle$ *is a zeta function of* $\mathrm{cf}(\mathrm{GL}_\bullet)$ *and* $\boldsymbol{p}_{\mathbb{1}}$ *is the unique CHA morphism* $(\mathrm{cf}(\mathrm{GL}_\bullet), \mathrm{St}_\bullet \rangle) \to (\mathcal{Q}\mathrm{Sym}, \mathrm{ps}_1)$.

Now, for $n \geq 0$, write $\mathrm{reg}_{\mathrm{UT}_n}$ for the regular character of UT_n. Define a linear functional

$$\mathrm{reg}_\bullet \rangle = (\mathrm{reg}_{\mathrm{UT}_n} \rangle)_{n \geq 0} \in \mathrm{scf}(\mathrm{UT}_\bullet)^*.$$

Theorem 5.11. *The function* $\mathrm{reg}_\bullet \rangle$ *is a zeta function of* $\mathrm{scf}(\mathrm{UT}_\bullet)$ *and*

$$\mathrm{reg}_\bullet \rangle = \mathrm{St}_\bullet \rangle \circ \mathrm{Ind}^{\mathrm{GL}}_{\mathrm{UT}},$$

so $\mathrm{Ind}^{\mathrm{GL}}_{\mathrm{UT}}$ *is a CHA morphism*

$$(\mathrm{scf}(\mathrm{UT}_\bullet), \mathrm{reg}_\bullet \rangle) \xrightarrow{\mathrm{Ind}^{\mathrm{GL}}_{\mathrm{UT}}} (\mathrm{cf}(\mathrm{GL}_\bullet), \mathrm{St}_\bullet \rangle).$$

Proof. It is sufficient to prove that $\mathrm{reg}_\bullet \rangle = \mathrm{St}_\bullet \rangle \circ \mathrm{Ind}^{\mathrm{GL}}_{\mathrm{UT}}$. Doing so requires the well-known fact (see, for example, [45, 10.3]) that for unipotent $X \in \mathrm{GL}_n$,

$$\mathrm{St}_n(X) = \begin{cases} |\mathrm{UT}_n| & \text{if } X = 1_n, \\ 0 & \text{for other unipotent } X. \end{cases}$$

As a consequence,

$$\mathrm{Res}^{\mathrm{GL}_n}_{\mathrm{UT}_n}(\mathrm{St}_n) = \mathrm{reg}_{\mathrm{UT}_n}.$$

With this, the claim follows from Frobenius reciprocity as described in Section 2D:

$$\mathrm{St}_\bullet \rangle \circ \mathrm{Ind}^{\mathrm{GL}}_{\mathrm{UT}} = \left(\mathrm{Res}^{\mathrm{GL}_n}_{\mathrm{UT}_n}(\mathrm{St}_n) \rangle \right)_{n \geq 0} = \mathrm{reg}_\bullet \rangle. \qquad \square$$

Remark 5.12. Like Theorem 3.8, Theorem 5.11 actually shows that $\mathrm{Ind}^{\mathrm{GL}}_{\mathrm{UT}}$ is a CHA morphism from the larger combinatorial Hopf algebra $(\mathrm{cf}(\mathrm{UT}_\bullet), \mathrm{reg}_\bullet \rangle)$ to $(\mathcal{Q}\mathrm{Sym}, \mathrm{ps}_1)$.

5C. *The vertical strip LLT polynomial.* The *vertical strip LLT polynomial* indexed by a tall Schröder path σ is

$$G_\sigma(\boldsymbol{x}; t) = \sum_{\kappa \in A(\sigma)} t^{\operatorname{asc}([n], \operatorname{Area}(\sigma))(\kappa)} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)} \in \mathbb{C}[\![\boldsymbol{x}]\!][t],$$

where the sum is over the set $A(\sigma)$ of functions $\kappa : [n] \to \mathbb{Z}_{>0}$ which satisfy $\kappa(i) < \kappa(j)$ for each $i < j$ with $\{i, j\} \in \operatorname{Diag}(\sigma)$. Viewed as a polynomial in t, the coefficients of $G_\sigma(\boldsymbol{x}; t)$ are actually symmetric functions [28, Lemma 10.2], though this is not obvious. For example,

$$G_{\boxminus}(\boldsymbol{x}; t) = tm_{(2,1)} + (t^2 + 2t)m_{(1^3)}.$$

Remark 5.13. There are several essentially equivalent definitions of LLT polynomials; the one above is due to [11] in the unicellular case and to [13] (see also [5]) in general.

If σ is a Dyck path, so that $\operatorname{Diag}(\sigma) = \varnothing$, then the sum in $G_\sigma(\boldsymbol{x}; t)$ is over all possible colorings; this special case is know as a *unicellular* LLT polynomial. In [25], Guay-Paquet realizes the unicellular LLT polynomials by way of a homomorphism of Hopf algebras over $\mathbb{C}[t]$. By evaluating at $t = q^{-1}$ as in Theorem 2.12, this result descends to a Hopf algebra homomorphism $\operatorname{scf}(\mathrm{UT}_\bullet) \to \mathcal{Q}\mathrm{Sym}$. Define a linear functional

$$\zeta_1 : \operatorname{scf}(\mathrm{UT}_\bullet) \to \mathbb{C}, \quad \bar{\delta}_\gamma \mapsto 1.$$

Theorem 5.14 [25, Theorem 57]. *The map ζ_1 is a zeta function of $\operatorname{scf}(\mathrm{UT}_\bullet)$, and the unique CHA morphism*

$$(\operatorname{scf}(\mathrm{UT}_\bullet), \zeta_1) \to (\mathcal{Q}\mathrm{Sym}, \mathrm{ps}_1)$$

is given by

$$\bar{\delta}_{\operatorname{Graph}(\pi)} \mapsto G_\pi(\boldsymbol{x}; q^{-1}) \quad \text{for } \pi \in \mathcal{D}.$$

Now recall the zeta function $\operatorname{reg}_\bullet \rangle$ defined in the previous section.

Proposition 5.15. *As a zeta function of the Hopf algebra $\operatorname{scf}(\mathrm{UT}_\bullet)$, $\operatorname{reg}_\bullet \rangle$ is equal to ζ_1; in particular*

$$\operatorname{reg}_\bullet \rangle (\bar{\delta}_\gamma) = 1 \quad \text{for } \gamma \in \mathcal{IG}.$$

Proof. This follows from direct computation: if $\gamma \in \mathcal{IG}_n$,

$$\operatorname{reg}_\bullet \rangle (\bar{\delta}_\gamma) = \langle \bar{\delta}_\gamma, \operatorname{reg}_{\mathrm{UT}_n} \rangle = \bar{\delta}_\gamma(1_n) = 1. \qquad \square$$

The uniqueness result of Theorem 2.2 now gives the following, which restates (5.4).

Corollary 5.16. *The map $\boldsymbol{c}_{\mathbb{1}}$ is the CHA morphism described in Theorem 5.14. In particular,*

$$\boldsymbol{c}_{\mathbb{1}}(\bar{\delta}_{\operatorname{Graph}(\pi)}) = G_\pi(\boldsymbol{x}; q^{-1}) \quad \text{for } \pi \in \mathcal{D}.$$

Remark 5.17 (cf. Remarks 3.15(R1)). Corollary 5.16 can be used to give a novel proof that the unicellular LLT polynomial $G_\pi(\boldsymbol{x}; t)$ has symmetric coefficients.

The proof of Theorem 5.1 is given below following two identities for LLT polynomials.

Proposition 5.18 [6, Theorem 2.1; 11, Proposition 3.4]. *Let n be a positive integer.*

(1) *For any Dyck paths $\pi \in \mathcal{D}_n$,*

$$q^{|\mathrm{Area}(\pi)|}G_{\mathrm{Dyck}(\pi)}(\boldsymbol{x}; q^{-1}) = \omega G_{\mathrm{Dyck}(\pi)}(\boldsymbol{x}; q).$$

(2) *For any tall Schröder paths $\sigma \in \mathcal{TS}_n$,*

$$(q-1)^{|\mathrm{Diag}(\sigma)|}G_\sigma(\boldsymbol{x}; q) = \sum_{S \subseteq \mathrm{Diag}(\sigma)} (-1)^{|\mathrm{Diag}(\sigma)-S|}G_{\mathrm{Area}^{-1}(\mathrm{Area}(\sigma)\cup S)}(\boldsymbol{x}; q),$$

where $\mathrm{Area}^{-1}(\mathrm{Area}(\sigma) \cup S)$ *denotes the unique Dyck path with area* $\mathrm{Area}(\sigma) \cup S$.

Proof of Theorem 5.1. For $\pi \in \mathcal{D}$, (2.11) states that $\overline{\chi}^{\mathrm{Graph}(\pi)} = q^{|\mathrm{Area}(\pi)|}\overline{\delta}_{\mathrm{Graph}(\pi)}$, so by Proposition 5.18(i),

$$\boldsymbol{c}_{\mathbb{1}}(\overline{\chi}^{\mathrm{Graph}(\pi)}) = \omega G_\pi(\boldsymbol{x}; q).$$

Combining this with Proposition 5.18(ii) and the linearity of ω,

$$\boldsymbol{c}_{\mathbb{1}}(\psi^\sigma) = \sum_{S \subseteq \mathrm{Diag}(\sigma)} (-1)^{|\mathrm{Diag}(\sigma)-S|}\omega G_{\pi+S}(\boldsymbol{x}; q) = (q-1)^{|\mathrm{Diag}(\sigma)|}\omega G_\sigma(\boldsymbol{x}; q). \qquad \square$$

6. Positivity conjectures

Recall the bases of Sym given in Section 2A. An element $f(\boldsymbol{x}; t) \in \mathrm{Sym}[t]$ is said to be *e-positive* if the coefficients $a_\lambda(t)$ in

$$f(\boldsymbol{x}; t) = \sum_{\lambda \in \mathcal{P}} a_\lambda(t)e_\lambda$$

are polynomials in t with nonnegative coefficients: $a_\lambda(t) \in \mathbb{Z}_{\geq 0}[t]$. Likewise, if the coefficients of $f(\boldsymbol{x}; t)$ in any other basis of Sym have this property — for example, the Schur basis $\{s_\lambda \mid \lambda \in \mathcal{P}\}$ — say that $f(\boldsymbol{x}; t)$ is positive in that basis. The positivity of the symmetric functions in this paper are of some interest, and this section will describe the meaning of positivity in the context of $\mathrm{GL}_n(\mathbb{F}_q)$ representation theory.

For the chromatic quasisymmetric functions in Section 3B, *e*-positivity generalizes the Stanley–Stembridge conjecture [42, Conjecture 5.5], which by [24] is the $t = 1$ case below.

Conjecture 6.1 [39, Conjecture 1.3]. *For each $\gamma \in \mathcal{IG}$, $X_\gamma(\boldsymbol{x}; t)$ is e-positive.*

Special cases of Conjecture 6.1 have explicit solutions, as in [1; 12; 29; 30].

For the vertical strip LLT polynomials in Section 5C, Schur positivity has implications for the study of Macdonald polynomials [28]. Adapting results from the case of general LLT polynomials, it is known [23, Corollary 6.9] that $G_\sigma(\boldsymbol{x}; t)$ is positive in the Schur basis for every $\sigma \in \mathcal{TS}$. However, their proof is algebraic and does not construct the Schur coefficients. In some special cases, explicit formulas are known, including the q-Kostka numbers [33] and the results of [30; 43], but in general these coefficients are a mystery.

Open Problem 6.2 [27, Open Problem 6.6]. *Find a (manifestly positive) combinatorial formula for the Schur coefficients of $G_\sigma(\boldsymbol{x}; t)$.*

The e-positivity of vertical strip LLT polynomials is also the subject of study; in this context, the paradigm is altered by considering the shifted polynomial $G_\sigma(\boldsymbol{x}; t+1)$. The e-positivity of shifted vertical strip LLT polynomials is proved in [13, Theorem 5.5], and the paper [6] gives an explicit combinatorial formula the e-coefficients, which will be restated in Section 6C. Using Theorem 5.1, this formula implies a result about the characters $\mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}}(\psi^\sigma)$, inadvertently giving some representation theoretic intuition for the $t \leftrightarrow t+1$ shift.

Returning to the general discussion of positivity, if a polynomial $f(\boldsymbol{x}; t) \in \mathrm{Sym}[t]$ is positive with respect to a certain basis, then evaluating t at any positive integer will give a symmetric function with nonnegative integer coefficients in the chosen basis. Thus, evaluating $t = q$ above gives positivity results about the GL_n characters in this paper. Conversely, polynomial equations can be verified on any infinite set — like the set of prime powers — so GL_n characters offer a novel approach to some of the open problems above.

This section reinterprets each of the positivity statements above in the context of GL_n representation theory. Section 6A will discuss the e-positivity of the chromatic quasisymmetric function, Section 6B will discuss Schur positivity of the vertical strip LLT polynomials, and Section 6C will discuss the implications of the e-positivity of vertical strip LLT polynomials.

6A. *Interpreting the e-positivity of $X_\gamma(\boldsymbol{x}; t)$.* In light of Theorem 3.1, there should be a restatement of Conjecture 6.1 involving the characters $\mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{GL}_n}(\mathbb{1})$. However, the isomorphism $\boldsymbol{p}_{\{1\}}$ in Theorem 3.1 does not associate e_λ to a character of GL_n, so some interpretation is required. My choice to use the particular restatement below is informed by ongoing work on the subject.

Recall the Steinberg character $\mathrm{St}_n \in \mathrm{cf}(\mathrm{GL}_n)$ defined in Section 5B. For any partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, define $\mathrm{St}_\lambda \in \mathrm{cf}(\mathrm{GL}_\bullet)$ to be the product

$$\mathrm{St}_\lambda = \mathrm{St}_{\lambda_1} \mathrm{St}_{\lambda_2} \cdots \mathrm{St}_{\lambda_\ell}.$$

Conjecture 6.3. *Let $n \geq 0$ and $\gamma \in \mathcal{IG}_n$. There are polynomials $a_\lambda^\gamma(t) \in \mathbb{Z}_{\geq 0}[t]$ such that for each prime power q the character*

$$\eta_\gamma = \sum_{\lambda \in \mathcal{P}_n} a_\lambda^\gamma(q) \, \mathrm{St}_\lambda$$

satisfies $(q-1)^n \eta_\gamma(u) = \mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{GL}_n}(\mathbb{1})(u)$ for every unipotent element $u \in \mathrm{GL}_n(\mathbb{F}_q)$.

Proposition 6.4. *Conjectures 6.1 and 6.3 are equivalent.*

Proof. For a class function $\psi \in \mathrm{cf}(\mathrm{GL}_n)$, write $\psi|_{\mathrm{uni}} \in \mathrm{cf}_{\mathrm{supp}}^{\mathrm{uni}}(\mathrm{GL}_\bullet)$ for the element defined by

$$\psi|_{\mathrm{uni}}(g) = \begin{cases} \psi(g) & \text{if } g \text{ is unipotent,} \\ 0 & \text{otherwise,} \end{cases}$$

so that Conjecture 6.3 states $\sum_{\lambda \in \mathcal{P}_n} a_\lambda^\gamma(q) \mathrm{St}_\lambda|_{\mathrm{uni}} = \frac{1}{(q-1)^n} \mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{GL}_n}(\mathbb{1})$.

I now claim that $\boldsymbol{p}_{\{1\}}(\mathrm{St}_\lambda|_{\mathrm{uni}}) = e_\lambda$, so that with the preceding remarks and Theorem 3.1 the proof will be complete. The claim is relatively well-known to experts, but a proof sketch is included for the sake of

completeness. Direct computation gives that $\mathrm{St}_n|_{\mathrm{uni}} = q^{\binom{n}{2}}\delta_{(1^n)}$ (see the proof of Theorem 5.11), and

$$\boldsymbol{p}_{\{1\}}(q^{\binom{n}{2}}\delta_{(1^n)}) = \tilde{P}_{(1^n)}(\boldsymbol{x}; q) = e_n,$$

with the second equality due to the definition of the Hall–Littlewood polynomial; see [35, III.2 (2.8)]. The claim then follows from the fact that the extension of $\psi \mapsto \psi_{\mathrm{uni}}$ to all of $\mathrm{cf}(\mathrm{GL}_\bullet)$ is a Hopf algebra homomorphism to $\mathrm{cf}^{\mathrm{uni}}_{\mathrm{supp}}(\mathrm{GL}_\bullet)$ [45, 10.1]. $\qquad\square$

Remarks 6.5. (R1) A direct proof of Conjecture 6.3 would probably find an organic realization of the character η_γ using the representation theory of GL_n, and in a manner which does not depend on q. Ongoing work has identified a promising candidate for the character η_γ, but has not led to any progress on the conjecture itself.

(R2) It is not clear that Conjecture 6.3 offers an easier approach to Conjecture 6.1 than other equivalent statements. However, as the clearest restatement of Conjecture 6.1 in the $\mathrm{GL}_n(\mathbb{F}_q)$ context, the wide interest in e-positivity seems to justify its inclusion.

6B. *Interpreting the Schur positivity of $G_\sigma(\boldsymbol{x}; t)$.* Let σ be a tall Schröder path and write

$$G_\sigma(\boldsymbol{x}; t) = \sum_{\lambda \in \mathcal{P}} b_\lambda^\sigma(t) s_\lambda.$$

It is immediate that each $b_\lambda^\sigma(t)$ is a polynomial in t with integral coefficients, and the content of [23, Corollary 6.9] is that the coefficients of this polynomial are nonnegative.

Recall from Section 5B that the irreducible unipotent characters of GL_n are $\{\chi^\lambda \mid \lambda \in \mathcal{P}_n\}$, and that $\boldsymbol{p}_{\mathbb{1}}(\chi^\lambda) = s_\lambda$ for each partition $\lambda \in \mathcal{P}$. Thus, for a tall Schröder path σ, Theorem 5.1 implies that

$$(q-1)^{|\mathrm{Diag}(\sigma)|} b_\lambda^\sigma(q) = \langle \chi^{\lambda'}, \mathrm{Ind}^{\mathrm{GL}}_{\mathrm{UT}}(\psi^\sigma) \rangle, \tag{6.6}$$

which is the multiplicity of the irreducible unipotent GL_n-module indexed by λ' in the GL_n-module affording $\mathrm{Ind}^{\mathrm{GL}}_{\mathrm{UT}}(\psi^\sigma)$. Thus, Theorem 5.1 implies the known fact that $b_\lambda^\sigma(q)$ is nonnegative for each prime power q, but falls short of giving a second proof of Schur positivity: a polynomial with negative coefficients can still take on infinitely many positive values. Nonetheless, progress on Open Problem 6.2 might be obtained through explicit representation theoretic formulas.

Open Problem 6.7. *For $n \geq 0$, $\sigma \in \mathcal{TS}_n$, and $\lambda \in \mathcal{P}_n$, find a combinatorial formula for $\langle \chi^{\lambda'}, \mathrm{Ind}^{\mathrm{GL}}_{\mathrm{UT}}(\psi^\sigma) \rangle$ as a function of q.*

Such a formula would almost certainly be divisible by $(q-1)^{|\mathrm{Diag}(\sigma)|}$ in a straightforward manner; see Remark 5.5. This would give an answer to Open Problem 6.2.

6C. *Interpreting the e-positivity of $G_\sigma(\boldsymbol{x}; t)$.* The final section of this paper will show how the explicit e-positivity formula for vertical strip LLT polynomials given in [6] leads to a deeper understanding of the characters $\mathrm{Ind}^{\mathrm{GL}}_{\mathrm{UT}}(\psi^\sigma)$ from Theorem 5.1; see Corollary 6.10. I will begin by recalling the main result of [6].

Fix a graph $\gamma = ([n], E(\gamma))$ on $[n]$. An *orientation* of γ is a collection of directed edges

$$\theta = \{(i, j) \mid \{i, j\} \in E(\gamma)\},$$

so that $([n], \theta)$ is a directed graph whose underlying undirected graph is γ. For example, with

$$\gamma = \underset{1 \quad 2 \quad 3 \quad 4}{\bullet\!\!\frown\!\!\bullet\!\!-\!\!\bullet\quad\bullet} \quad \text{and} \quad \theta = \{(2, 1), (1, 3), (3, 2), (3, 4)\} \tag{6.8}$$

we have

$$([n], \theta) = \underset{1 \quad 2 \quad 3 \quad 4}{\bullet\!\!\overset{\frown}{\leftarrow}\!\!\bullet\!\!\rightarrow\!\!\bullet\!\!\rightarrow\!\!\bullet} \ .$$

Write $\mathcal{O}(\gamma)$ for the set of orientations of γ. For $\theta \in \mathcal{O}(\gamma)$ and $i \in [n]$, say that the *highest reachable vertex* from i under θ is

$$\mathrm{hrv}(\theta, i) = \max\{j \in [n] \mid \text{there is an } increasing \text{ path in } ([n], \theta) \text{ from } i \text{ to } j\}.$$

For example, taking γ and θ as in (6.8)

$$\mathrm{hrv}(\theta, 1) = 4, \quad \mathrm{hrv}(\theta, 2) = 2, \quad \mathrm{hrv}(\theta, 3) = 4, \quad \text{and} \quad \mathrm{hrv}(\theta, 4) = 4.$$

Finally, for $\theta \in \mathcal{O}(\gamma)$, the *type* of θ is the partition $\mathrm{type}(\theta) \in \mathcal{P}_n$ obtained by truncating all zeros from the nonincreasing reordering of the sequence

$$\left(\left|\{i \in [n] \mid \mathrm{hrv}(\theta, i) = 1\}\right|, \ldots, \left|\{i \in [n] \mid \mathrm{hrv}(\theta, i) = n\}\right|\right).$$

For example, taking γ and θ as in (6.8), $\mathrm{type}(\theta) = (3, 1)$.

Theorem 6.9 [6, Theorem 2.9]. *For $n \geq 0$, let $\sigma \in \mathcal{T}\mathcal{S}_n$ and let γ be the natural unit interval order on $[n]$ with edge set $E(\gamma) = \mathrm{Area}(\sigma) \cup \mathrm{Diag}(\sigma)$. Then*

$$G_\sigma(\boldsymbol{x}; t) = \sum_{\substack{\mathrm{Diag}(\sigma)\text{-}ascending \\ \theta \in \mathcal{O}(\gamma)}} (t - 1)^{\left|\{\{i, j\} \in \mathrm{Area}(\sigma) \mid (i, j) \in \theta \text{ with } i < j\}\right|} e_{\mathrm{type}(\theta)},$$

where the sum is over orientations $\theta \in \mathcal{O}(\gamma)$ with $(i, j) \in \theta$ for each $i < j$ with $\{i, j\} \in \mathrm{Diag}(\sigma)$.

Evaluating the identity above at $t = q$, the expression $q - 1$ can be interpreted as $|\mathbb{F}_q^\times|$, the number of units in the field $\mathbb{F}_q$. As $|\mathbb{F}_q^\times|$ is a positive integer, it can be interpreted as the multiplicity of a submodule, as will be discussed at the end of this section.

The *Gelfand–Graev character* of GL_n [22] is the class function

$$\Gamma_n = \frac{1}{(q - 1)^{n-1}} \mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}}(\psi^{ED^{n-1}S}),$$

where $\psi^{ED^{n-1}S}$ is as defined in Section 5A; as the name suggests, Γ_n is actually a character of GL_n; see Remark 5.5. The *degenerate Gelfand–Graev character* [45, 12] indexed by a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is

$$\Gamma_\lambda = \Gamma_{\lambda_1} \cdots \Gamma_{\lambda_\ell}.$$

Corollary 6.10. *For $n \geq 0$, let $\sigma \in \mathcal{TS}_n$, and let γ be the natural unit interval order on $[n]$ with edge set $E(\gamma) = \mathrm{Area}(\sigma) \cup \mathrm{Diag}(\sigma)$. Then*

$$\mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}}(\psi^{\sigma}) = \sum_{\substack{\mathrm{Diag}(\sigma)\text{-}ascending \\ \theta \in \mathcal{O}(\gamma)}} (q-1)^{\left|\{\{i,j\} \in E(\gamma) \mid (i,j) \in \theta \text{ with } i < j\}\right|} \Gamma_{\mathrm{type}(\theta)},$$

where the sum is over orientations $\theta \in \mathcal{O}(\gamma)$ with $(i,j) \in \theta$ for each $i < j$ with $\{i,j\} \in \mathrm{Diag}(\sigma)$.

Proof. Since the map $\boldsymbol{p}_{\mathbb{1}}$ restricts to an isomorphism from $\mathrm{cf}_{\mathrm{supp}}^{\mathrm{uni}}(\mathrm{GL}_{\bullet})$ to Sym (discussed in Section 5), and the involution ω is also an isomorphism, it is sufficient to establish that the above equation holds after the application of $\omega \circ \boldsymbol{p}_{\mathbb{1}}$ to both sides. By Theorems 5.1 and 6.9, the left side becomes

$$\omega \circ \boldsymbol{p}_{\mathbb{1}} \circ \mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}}(\psi^{\sigma}) = \sum_{\substack{\mathrm{Diag}(\sigma)\text{-ascending} \\ \theta \in \mathcal{O}(\gamma)}} (q-1)^{\left|\{\{i,j\} \in E(\gamma) \mid (i,j) \in \theta \text{ with } i < j\}\right|} e_{\mathrm{type}(\theta)},$$

so the claim will follow from $\omega \circ \boldsymbol{p}_{\mathbb{1}}(\Gamma_n) = e_n$. This fact is known, but a short proof is included below for completeness.

Theorem 5.1 states that $\omega \circ \boldsymbol{p}_{\mathbb{1}}(\Gamma_n) = G_{ED^{n-1}S}(\boldsymbol{x}; q)$. With $\mathrm{Diag}(ND^{n-1}S) = \{\{i, i+1\} \mid 1 \leq i < n\}$, the definition of vertical strip LLT polynomials given in Section 5C becomes

$$G_{ED^{n-1}S}(\boldsymbol{x}; q) = \sum_{\substack{\kappa:[n] \to \mathbb{Z}_{>0} \\ \kappa(1) < \cdots < \kappa(n)}} x_{\kappa(1)} \cdots x_{\kappa(n)} = e_n. \qquad \square$$

This result implies that the GL_n-module affording $\mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}}(\psi^{\sigma})$ decomposes into a direct sum of submodules that each afford some degenerate Gelfand–Graev character. Exhibiting this decomposition explicitly would give a new proof of Corollary 6.10 and Theorem 6.9.

Open Problem 6.11. *Find a module theoretic proof of Corollary 6.10.*

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