

Algebra & Number Theory

Volume 19
2025
No. 1

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Let E be a field of characteristic p . In a previous paper of ours, we defined and studied super-Hölder vectors in certain E -linear representations of \mathbb{Z}_p . In the present paper, we define and study super-Hölder vectors in certain E -linear representations of a general p -adic Lie group. We then consider certain p -adic Lie extensions K_∞/K of a p -adic field K , and compute the super-Hölder vectors in the tilt of K_∞ . We show that these super-Hölder vectors are the perfection of the field of norms of K_∞/K . By specializing to the case of a Lubin–Tate extension, we are able to recover $E((Y))$ inside the Y -adic completion of its perfection, seen as a valued E -vector space endowed with the action of \mathcal{O}_K^\times given by the endomorphisms of the corresponding Lubin–Tate group.

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Introduction

Let E be a field of characteristic p , for example a finite field. In our paper [Berger and Rozensztajn 2022], we defined and studied super-Hölder vectors in certain E -linear representations of the p -adic Lie group \mathbb{Z}_p . These vectors are a characteristic p analogue of locally analytic vectors. They allowed us to recover $E((X))$ inside the X -adic completion of its perfection, seen as a valued E -vector space endowed with the action of \mathbb{Z}_p^\times given by $a \cdot f(X) = f((1 + X)^a - 1)$.

In the present paper, we define and study super-Hölder vectors in certain E -linear representations of a general p -adic Lie group. We then consider certain p -adic Lie extensions K_∞/K of a p -adic field K , and compute the super-Hölder vectors in the tilt of K_∞ . We show that these super-Hölder vectors are the perfection of the field of norms of K_∞/K . By specializing to the case of a Lubin–Tate extension, we are able to recover $E((Y))$ inside the Y -adic completion of its perfection, seen as a valued E -vector space endowed with the action of \mathcal{O}_K^\times given by the endomorphisms of the corresponding Lubin–Tate group.

MSC2020: primary 11S15, 11S20, 11S80; secondary 11S31, 12J25, 13J05, 22E35.

Keywords: Super-Hölder function, locally analytic function, field of norms, pro- p -group, perfectoid ring, decompletion, Lubin–Tate group, Mahler basis, p -adic wavelet.

We now give more details about the contents of our paper. Let Γ be a p -adic Lie group. It is known that Γ always has a uniform open pro- p subgroup G . Let G be such a subgroup, and let $G_i = G^{p^i}$ for $i \geq 0$. Let M be an E -vector space, endowed with a valuation val_M such that $\text{val}_M(xm) = \text{val}_M(m)$ if $x \in E^\times$. We assume that M is separated and complete for the val_M -adic topology. We say that a function $f : G \rightarrow M$ is super-Hölder if there exist constants $e > 0$ and $\lambda, \mu \in \mathbb{R}$ such that $\text{val}_M(f(g) - f(h)) \geq p^\lambda \cdot p^{ei} + \mu$ whenever $gh^{-1} \in G_i$, for all $g, h \in G$ and $i \geq 0$. If M is now endowed with an action of G by isometries, and $m \in M$, we say that m is a super-Hölder vector if the orbit map $g \mapsto g \cdot m$ is a super-Hölder function $G \rightarrow M$. We let $M^{G-e\text{-sh}, \lambda}$ denote the space of super-Hölder vectors for given constants e and λ as in the definition above. The space of vectors of M that are super-Hölder for a given e is independent of the choice of the uniform subgroup G , and denoted by $M^{e\text{-sh}}$. When $G = \mathbb{Z}_p$ and $e = 1$, we recover the definitions of [Berger and Rozensztajn 2022]. If Γ is a p -adic Lie group and $e = 1$, we get an analogue of locally \mathbb{Q}_p -analytic vectors. If K is a finite extension of \mathbb{Q}_p , Γ is the Galois group of a Lubin–Tate extension of K , and $e = [K : \mathbb{Q}_p]$, we seem to get an analogue of locally K -analytic vectors.

From now on, assume that $p \neq 2$. Let K be a p -adic field and let K_∞/K be an almost totally ramified p -adic Lie extension, with Galois group Γ of dimension $d \geq 1$. The tilt of K_∞ is the fraction field $\tilde{\mathbb{E}}_{K_\infty}$ of $\varprojlim_{x \mapsto x^p} \mathcal{O}_{K_\infty}/p$. It is a perfect complete valued field of characteristic p , endowed with an action of Γ by isometries. The field $\tilde{\mathbb{E}}_{K_\infty}$ naturally contains the field of norms $X_K(K_\infty)$ of the extension K_∞/K , and it is known that $\tilde{\mathbb{E}}_{K_\infty}$ is the completion of the perfection of $X_K(K_\infty)$. We have the following result (Theorem 2.2.3).

Theorem A. *We have $\tilde{\mathbb{E}}_{K_\infty}^{d\text{-sh}} = \bigcup_{n \geq 0} \varphi^{-n}(X_K(K_\infty))$.*

Assume now that K is a finite extension of \mathbb{Q}_p , with residue field k , and let LT be a Lubin–Tate formal group attached to K . Let K_∞ be the extension of K generated by the torsion points of LT, so that $\text{Gal}(K_\infty/K)$ is isomorphic to \mathcal{O}_K^\times . The field of norms $X_K(K_\infty)$ is isomorphic to $k((Y))$, and \mathcal{O}_K^\times acts on this field by the endomorphisms of the Lubin–Tate group: $a \cdot f(Y) = f([a](Y))$. Let $d = [K : \mathbb{Q}_p]$. The following (Theorem 3.2.1) is a more precise version of Theorem A in this situation.

Theorem B. *If $j \geq 1$, then $\tilde{\mathbb{E}}_{K_\infty}^{1+p^j \mathcal{O}_K\text{-}d\text{-sh}, dj} = k((Y))$.*

If $K = \mathbb{Q}_p$ and K_∞/K is the cyclotomic extension, Theorem B was proved in [Berger and Rozensztajn 2022]. A crucial ingredient of the proof of this theorem was Colmez’ analogue of Tate traces for $\tilde{\mathbb{E}}_{K_\infty}$. If the Lubin–Tate group is of height ≥ 2 , there are no such traces (we state and prove a precise version of this assertion in Section 3.2). Instead of Tate traces, we use a theorem of Ax and a precise characterization of the field of norms $X_K(K_\infty)$ inside $\tilde{\mathbb{E}}_{K_\infty}$ in order to prove Theorem A.

As an application of Theorem B, we compute the perfectoid commutant of $\text{Aut}(\text{LT})$. If $b \in \mathcal{O}_K^\times$ and $n \in \mathbb{Z}$, then $u(Y) = [b](Y^{q^n})$ is an element of $\tilde{\mathbb{E}}_{K_\infty}^+$ that satisfies the functional equation $u \circ [g](Y) = [g] \circ u(Y)$ for all $g \in \mathcal{O}_K^\times$. Conversely, we prove the following (Theorem 3.3.1).

Theorem C. *If $u \in \tilde{\mathbb{E}}_{K_\infty}^+$ is such that $\text{val}_Y(u) > 0$ and $u \circ [g] = [g] \circ u$ for all $g \in \mathcal{O}_K^\times$, there exists $b \in \mathcal{O}_K^\times$ and $n \in \mathbb{Z}$ such that $u(Y) = [b](Y^{q^n})$.*

In the last section, we give a characterization of super-Hölder functions on a uniform pro- p group in terms of their Mahler expansions (Theorem 4.3.4). In order to do so, we prove some results of independent interest on the space of continuous functions on \mathcal{O}_K^d with values in a valued E -vector space M as above.

At the end of [Berger and Rozensztajn 2022], we suggested an application of super-Hölder vectors for the action of \mathbb{Z}_p to the p -adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. We hope that this general theory of super-Hölder vectors, especially in the Lubin–Tate case, will have applications to the p -adic local Langlands correspondence for other fields than \mathbb{Q}_p .

1. Super-Hölder functions and vectors

In this section, we define super-Hölder vectors inside a valued E -vector space M endowed with an action of a p -adic Lie group Γ . The definition is very similar to the one that we gave for $\Gamma = \mathbb{Z}_p$ in [Berger and Rozensztajn 2022]. The main new technical tool is the existence of uniform open subgroups of Γ . These uniform subgroups look very much like \mathbb{Z}_p^d in a sense that we make precise.

1.1. Uniform pro- p groups. Uniform pro- p groups are defined at the beginning of Section 4 of [Dixon et al. 1991]. We do not recall the definition, nor the notion of rank of a uniform pro- p group, but rather point out the following properties of uniform pro- p groups. A coordinate (below) is simply a homeomorphism.

Proposition 1.1.1. *If G is a uniform pro- p group of rank d , then:*

- (1) $G_i = \{g^{p^i}, g \in G\}$ is an open normal (and uniform) subgroup of G for $i \geq 0$.
- (2) We have $[G_i : G_{i+1}] = p^d$ for $i \geq 0$.
- (3) There is a coordinate $c : G \rightarrow \mathbb{Z}_p^d$ such that $c(G_i) = (p^i \mathbb{Z}_p)^d$ for $i \geq 0$.
- (4) If $g, h \in G$, then $gh^{-1} \in G_i$ if and only if $c(g) - c(h) \in (p^i \mathbb{Z}_p)^d$.

Proof. Properties (1)–(4) are proved in Section 4 of [Dixon et al. 1991]. Alternatively, a uniform pro- p group G has a natural integer valued p -valuation ω such that (G, ω) is saturated [Klopsch 2005, Remark 2.1]. Properties (1)–(4) are then proved in Section 26 of [Schneider 2011]. \square

For example, the pro- p group \mathbb{Z}_p^d is uniform for all $d \geq 1$.

Lemma 1.1.2. *If G is a uniform pro- p group, and H is a uniform open subgroup of G , there exists $j \geq 0$ such that $G_{i+j} \subset H_i$ for all $i \geq 0$.*

Proof. This follows from the fact that $\{G_i\}_{i \geq 0}$ forms a basis of neighborhoods of the identity in G . \square

A p -adic Lie group is a p -adic manifold that has a compatible group structure. For example, $\mathrm{GL}_n(\mathbb{Z}_p)$ and its closed subgroups are p -adic Lie groups. We refer to [Schneider 2011] for a comprehensive treatment of the theory. Every uniform pro- p group is a p -adic Lie group. Conversely, we have the following.

Proposition 1.1.3. *Every p -adic Lie group Γ has a uniform open subgroup G , and the rank of G is the dimension of Γ .*

Proof. See Interlude A of [Dixon et al. 1991, pages 97–98]. \square

Proposition 1.1.4. *Let G be a pro- p group of finite rank, and N a closed normal subgroup of G . There exists an open subgroup G' of G such that G' , $G' \cap N$ and $G'/G' \cap N$ are all uniform.*

Proof. This is stated and proved on page 64 of [Dixon et al. 1991] (their H is our G'). \square

1.2. Super-Hölder functions and vectors. Let M be an E -vector space, endowed with a valuation val_M such that $\text{val}_M(xm) = \text{val}_M(m)$ if $x \in E^\times$. We assume that M is separated and complete for the val_M -adic topology. Throughout this section, G denotes a uniform pro- p group.

Definition 1.2.1. We say that $f : G \rightarrow M$ is super-Hölder if there exist constants $\lambda, \mu \in \mathbb{R}$ and $e > 0$ such that $\text{val}_M(f(g) - f(h)) \geq p^\lambda \cdot p^{ei} + \mu$ whenever $gh^{-1} \in G_i$, for all $g, h \in G$ and $i \geq 0$.

Remark 1.2.2. If $G = \mathbb{Z}_p$ and $e = 1$, we recover the functions defined in [Berger and Rozensztajn 2022, Section 1.1]; see also Remark 1.12 of [loc. cit.].

In the above definition, e will usually be equal to either 1 or $\dim(G)$.

We let $\mathcal{H}_e^{\lambda, \mu}(G, M)$ denote the space of functions such that $\text{val}_M(f(g) - f(h)) \geq p^\lambda \cdot p^{ei} + \mu$ whenever $gh^{-1} \in G_i$, for all $g, h \in G$ and $i \geq 0$, and $\mathcal{H}_e^\lambda(G, M) = \bigcup_{\mu \in \mathbb{R}} \mathcal{H}_e^{\lambda, \mu}(G, M)$ and $\mathcal{H}_e(G, M) = \bigcup_{\lambda \in \mathbb{R}} \mathcal{H}_e^\lambda(G, M)$.

If M, N are two valued E -vector spaces, and $f : M \rightarrow N$ is an E -linear map, we say that f is Hölder-continuous if there exists $c > 0, d \in \mathbb{R}$ such that $\text{val}_N(f(x)) \geq c \cdot \text{val}_M(x) + d$ for all $x \in M$.

Proposition 1.2.3. *If $\pi : M \rightarrow N$ is a Hölder-continuous linear map, we get a map $\mathcal{H}_e(G, M) \rightarrow \mathcal{H}_e(G, N)$.*

Proof. Take $c, d \in \mathbb{R}$ of Hölder continuity for π , $f \in \mathcal{H}_e^{\lambda, \mu}(G, M)$, and $g, h \in G$ with $gh^{-1} \in G_i$. We have $\text{val}_N(\pi(f(g)) - \pi(f(h))) \geq c \cdot \text{val}_M(f(g) - f(h)) + d \geq cp^\lambda \cdot p^{ei} + (\mu + d)$, so that $\pi \circ f \in \mathcal{H}_e^{\lambda', \mu'}(G, N)$ with $p^{\lambda'} = cp^\lambda$, and $\mu' = \mu + d$. \square

Proposition 1.2.4. *If $\alpha : G \rightarrow H$ is a group homomorphism, we get a map $\alpha^* : \mathcal{H}_e(H, M) \rightarrow \mathcal{H}_e(G, M)$.*

Proof. By definition of the subgroups G_i and H_i , we have $\alpha(G_i) \subset H_i$ for all i . Take $f \in \mathcal{H}_e^{\lambda, \mu}(H, M)$, and $g, h \in G$ with $gh^{-1} \in G_i$. We have $\text{val}_M(f(\alpha(g)) - f(\alpha(h))) \geq p^\lambda \cdot p^{ei} + \mu$ as $\alpha(g)\alpha(h)^{-1} \in H_i$, so that $\alpha^*(f) = f \circ \alpha \in \mathcal{H}_e^{\lambda, \mu}(G, M)$. \square

Proposition 1.2.5. *Suppose that M is a ring, and that $\text{val}_M(mm') \geq \text{val}_M(m) + \text{val}_M(m')$ for all $m, m' \in M$. If $c \in \mathbb{R}$, let $M_c = M^{\text{val}_M \geq c}$:*

- (1) *If $f \in \mathcal{H}_e^{\lambda, \mu}(G, M_c)$ and $g \in \mathcal{H}_e^{\lambda, \nu}(G, M_d)$, and $\xi = \min(\mu + d, \nu + c)$, then $fg \in \mathcal{H}_e^{\lambda, \xi}(G, M_{c+d})$.*
- (2) *If $\lambda, \mu \in \mathbb{R}$, then $\mathcal{H}_e^{\lambda, \mu}(G, M_0)$ is a subring of $C^0(G, M)$.*
- (3) *If $\lambda \in \mathbb{R}$, then $\mathcal{H}_e^\lambda(G, M)$ is a subring of $C^0(G, M)$.*

Proof. Items (2) and (3) follow from item (1), which we now prove. If $x, y \in G$, then

$$(fg)(x) - (fg)(y) = (f(x) - f(y))g(x) + (g(x) - g(y))f(y),$$

which implies the claim. \square

We now assume that M is endowed with an E -linear action by isometries of G . If $m \in M$, let $\text{orb}_m : G \rightarrow M$ denote the function defined by $\text{orb}_m(g) = g \cdot m$.

Definition 1.2.6. Let $M^{G-e\text{-sh},\lambda,\mu}$ be those $m \in M$ such that $\text{orb}_m \in \mathcal{H}_e^{\lambda,\mu}(G, M)$, and let $M^{G-e\text{-sh},\lambda}$ and $M^{G-e\text{-sh}}$ be the corresponding sub- E -vector spaces of M .

Remark 1.2.7. We assume that G acts by isometries on M , but not that G acts continuously on M , namely that $G \times M \rightarrow M$ is continuous. However, let M^{cont} denote the set of $m \in M$ such that $\text{orb}_m : G \rightarrow M$ is continuous. It is easy to see that M^{cont} is a closed sub- E -vector space of M , and that $G \times M^{\text{cont}} \rightarrow M^{\text{cont}}$ is continuous; compare with Section 3 of [Emerton 2017]. We then have $M^{\text{sh}} \subset M^{\text{cont}}$.

Lemma 1.2.8. *If $m \in M$, then $m \in M^{G-e\text{-sh},\lambda,\mu}$ if and only if for all $i \geq 0$, we have $\text{val}_M(g \cdot m - m) \geq p^\lambda \cdot p^{ei} + \mu$ for all $g \in G_i$.*

Proof. If $m \in M$, then $m \in M^{G-e\text{-sh},\lambda,\mu}$ if and only if the function orb_m is in $\mathcal{H}_e^{\lambda,\mu}(G, M)$, that is, for all g, h with $gh^{-1} \in G_i$, we have $\text{val}_M(g \cdot m - h \cdot m) \geq p^\lambda \cdot p^{ei} + \mu$. As G acts by isometries, we have $\text{val}_M(g \cdot m - h \cdot m) = \text{val}_M(h^{-1}g \cdot m - m)$. The result follows, as $h^{-1}g = h^{-1} \cdot gh^{-1} \cdot h \in G_i$. \square

Lemma 1.2.9. *The space $M^{G-e\text{-sh},\lambda,\mu}$ is a closed sub- E -vector space of M .*

Lemma 1.2.10. *If $i_0 \geq 0$, and $m \in M$ is such that $\text{val}_M(g \cdot m - m) \geq p^\lambda \cdot p^{ei} + \mu$ for all $g \in G_i$ with $i \geq i_0$, then $m \in M^{G-e\text{-sh},\lambda}$.*

Proof. Take $i < i_0$, and let R_i be a set of representatives of $G_{i_0} \setminus G_i$. This is a finite set, so there exists $\mu_i \in \mathbb{R}$ such that $\text{val}_M(r \cdot m - m) \geq p^\lambda \cdot p^{ei} + \mu_i$ for all $r \in R_i$. If $g \in G_i$, it can be written as $g = hr$ for some $h \in G_{i_0}$ and $r \in R_i$. We then have $g \cdot m - m = hr \cdot m - h \cdot m + h \cdot m - m$, so that $\text{val}_M(g \cdot m - m) \geq \min(\text{val}_M(r \cdot m - m), \text{val}_M(h \cdot m - m))$ (recall that G acts by isometries), so $\text{val}_M(g \cdot m - m) \geq \min(p^\lambda \cdot p^{ei} + \mu_i, p^\lambda \cdot p^{ei_0} + \mu) \geq p^\lambda \cdot p^{ei} + \min(\mu, \mu_i)$ as $i_0 > i$. If μ' is the min of μ and the μ_i for $0 \leq i < i_0$, then $m \in M^{G-e\text{-sh},\lambda,\mu'}$. \square

Recall that if $k \geq 0$, then G_k is also a uniform pro- p group.

Lemma 1.2.11. *If $k \geq 0$ then $M^{G-e\text{-sh},\lambda} = M^{G_k-e\text{-sh},\lambda+k}$.*

Proof. Note that $(G_k)_i = G_{i+k}$. The inclusion $M^{G-e\text{-sh},\lambda} \subset M^{G_k-e\text{-sh},\lambda+k}$ is obvious, and the reverse inclusion follows from Lemma 1.2.10. \square

Proposition 1.2.12. *The space $M^{H-e\text{-sh}}$ does not depend on the choice of a uniform open subgroup $H \subset G$.*

Proof. Let H and H' be uniform open subgroups of G . The group $H \cap H'$ contains an open uniform subgroup by Proposition 1.1.3, so to prove the proposition, we can further assume that $H' \subset H$. We then have $H'_i \subset H_i$ for all i , so that if $m \in M^{H-e\text{-sh},\lambda,\mu}$, then $m \in M^{H'-e\text{-sh},\lambda,\mu}$. This implies that

$M^{H-e\text{-sh},\lambda} \subset M^{H'-e\text{-sh},\lambda}$. Conversely, by Lemma 1.1.2, there exists j such that $H_j \subset H'$. The previous reasoning implies that $M^{H'-e\text{-sh},\lambda} \subset M^{H_j-e\text{-sh},\lambda}$. Lemma 1.2.11 now implies that $M^{H_j-e\text{-sh},\lambda} = M^{H-e\text{-sh},\lambda-j}$.

These inclusions imply the proposition. \square

Definition 1.2.13. If Γ is a p -adic Lie group that acts by isometries on M , we let $M^{e\text{-sh}} = M^{G-e\text{-sh}}$ where G is any uniform open subgroup of Γ .

Remark 1.2.14. If $e \leq f$, then $M^{f\text{-sh}} \subset M^{e\text{-sh}}$.

Recall that G is a uniform pro- p group. If a closed normal subgroup N of G acts trivially on M , then G/N acts on M .

Proposition 1.2.15. *If a closed normal subgroup N of G acts trivially on M , then $M^{G-e\text{-sh}} = M^{G/N-e\text{-sh}}$.*

Proof. By Proposition 1.1.4, G has an open subgroup G' such that G' and G'/N' are uniform (where $N' = G' \cap N$). By Proposition 1.2.12, we have $M^{G-e\text{-sh}} = M^{G'-e\text{-sh}}$ and $M^{G/N-e\text{-sh}} = M^{G'/N'-e\text{-sh}}$. Let $\pi : G' \rightarrow G'/N'$ denote the projection. We have $\pi(G'_i) = (G'/N')_i$ for all i . Hence if $m \in M$, then $\text{val}_M(g \cdot m - m) \geq p^\lambda \cdot p^{ei} + \mu$ for all $g \in G'_i$ if and only if $\text{val}_M(\pi(g) \cdot m - m) \geq p^\lambda \cdot p^{ei} + \mu$ for all $\pi(g) \in (G'/N')_i$. \square

Proposition 1.2.16. *Suppose that M is a ring, and that $g(mm') = g(m)g(m')$ and $\text{val}_M(mm') \geq \text{val}_M(m) + \text{val}_M(m')$ for all $m, m' \in M$ and $g \in G$:*

- (1) *If $v \in \mathbb{R}$ and $m, m' \in M^{G-e\text{-sh},\lambda,\mu} \cap M^{\text{val}_M \geq v}$, then $m \cdot m' \in M^{G-e\text{-sh},\lambda,\mu+v}$.*
- (2) *If $m \in M^{G-e\text{-sh},\lambda,\mu} \cap M^\times$, then $1/m \in M^{G-e\text{-sh},\lambda,\mu-2\text{val}_M(m)}$.*

Proof. Item (1) follows from Proposition 1.2.5 and Lemma 1.2.8. Item (2) follows from

$$g\left(\frac{1}{m}\right) - \frac{1}{m} = \frac{m - g(m)}{g(m)m}. \quad \square$$

2. The field of norms

Let K be a p -adic field, and let K_∞ be an algebraic Galois extension of K , whose Galois group G is a p -adic Lie group of dimension ≥ 1 . We assume that K_∞/K is almost totally ramified, namely that the inertia subgroup of G is open in G . Let $d = \dim(G)$ and let $\ell = p^d$. Let $\tilde{\mathbb{E}}_{K_\infty}^+$ denote the ring $\varprojlim_{x \mapsto x^\ell} \mathcal{O}_{K_\infty}/p$. This is a perfect domain of characteristic p , which has a natural action of G . The map $(y_j)_{j \geq 0} \mapsto (y_{di})_{i \geq 0}$ gives an isomorphism between $\varprojlim_{x \mapsto x^p} \mathcal{O}_{K_\infty}/p$ and $\tilde{\mathbb{E}}_{K_\infty}^+$, so that $\tilde{\mathbb{E}}_{K_\infty}^+$ is the ring of integers of the tilt of \hat{K}_∞ ; see Section 3 of [Scholze 2012].

If $x = (x_i)_{i \geq 0}$, and \hat{x}_i is a lift of x_i to \mathcal{O}_{K_∞} , then $\ell^i \text{val}_p(\hat{x}_i)$ is independent of $i \geq 0$ such that $x_i \neq 0$. We define a valuation on $\tilde{\mathbb{E}}_{K_\infty}^+$ by $\text{val}_E(x) = \lim_{i \rightarrow +\infty} \ell^i \text{val}_p(\hat{x}_i)$.

The aim of this section is to compute $(\tilde{\mathbb{E}}_{K_\infty}^+)^{d\text{-sh}}$. Given Definition 1.2.13, we assume from now on (replacing K by a finite subextension if necessary) that G is uniform and that K_∞/K is totally ramified. Let k denote the common residue field of K and K_∞ .

2.1. The field of norms. Let $\mathcal{E}(K_\infty)$ denote the set of finite extensions E of K such that $E \subset K_\infty$. Let $X_K(K_\infty)$ denote the set of sequences $(x_E)_{E \in \mathcal{E}(K_\infty)}$ such that $x_E \in E$ for all $E \in \mathcal{E}(K_\infty)$, and $N_{F/E}(x_F) = x_E$ whenever $E \subset F$ with $E, F \in \mathcal{E}(K_\infty)$.

If $n \geq 0$, let $K_n = K_\infty^{G_n}$ so that $[K_{n+1} : K_n] = \ell$, $\{K_n\}_{n \geq 0}$ is a cofinal subset of $\mathcal{E}(K_\infty)$, and $X_K(K_\infty) = \varprojlim_{N_{K_n/K_{n-1}}} K_n$. If $x = (x_n)_{n \geq 0} \in X_K(K_\infty)$, let $\text{val}_E(x) = \text{val}_p(x_0)$.

Theorem 2.1.1. *Let K and K_∞ be as above:*

- (1) *If $x, y \in X_K(K_\infty)$, then $\{N_{K_{n+j}/K_n}(x_{n+j} + y_{n+j})\}_{j \geq 0}$ converges for all $n \geq 0$.*
- (2) *If we set $(x + y)_n = \lim_{j \rightarrow +\infty} N_{K_{n+j}/K_n}(x_{n+j} + y_{n+j})$, then $x + y \in X_K(K_\infty)$, and the set $X_K(K_\infty)$ with this addition law, and componentwise multiplication, is a field of characteristic p .*
- (3) *The function val_E is a valuation on $X_K(K_\infty)$, for which it is complete.*
- (4) *If $\varpi = (\varpi_n)_{n \geq 0}$ is a norm compatible sequence of uniformizers of \mathcal{O}_{K_n} , the valued field $X_K(K_\infty)$ is isomorphic to $k((\varpi))$ (with $\text{val}(\varpi) = \text{val}_p(\varpi_0)$).*

Proof. By a result of Sen [1972], K_∞/K is strictly APF in the terminology of Section 1.2 of [Wintenberger 1983]; see 1.2.2 of [loc. cit.]. The theorem is then proved in Section 2 of [loc. cit.]. \square

Let $X_K^+(K_\infty) = \varprojlim_{N_{K_n/K_{n-1}}} \mathcal{O}_{K_n}$ be the ring of integers of the valued field $X_K(K_\infty)$.

If $c > 0$, let $I_n^c = \{x \in \mathcal{O}_{K_n} \text{ such that } \text{val}_p(x) \geq c\}$. If $m, n \geq 0$, the map $\mathcal{O}_{K_n}/I_n^c \rightarrow \mathcal{O}_{K_{m+n}}/I_{m+n}^c$ is well-defined and injective.

Proposition 2.1.2. *There exists $c(K_\infty/K) \leq 1$ such that if $0 < c \leq c(K_\infty/K)$, then*

$$\text{val}_p(N_{K_{n+k}/K_n}(x)/x^{[K_{n+k}:K_n]} - 1) \geq c$$

for all $n, k \geq 0$ and $x \in \mathcal{O}_{K_{n+k}}$.

Proof. See [Wintenberger 1983] as well as Section 4 of [Cais and Davis 2015]. The result follows from the fact (see 1.2.2 of [Wintenberger 1983]) that the extension K_∞/K is strictly APF. One can then apply 1.2.1, 4.2.2 and 1.2.3 of [Wintenberger 1983]. \square

Using Proposition 2.1.2, we get a map $\iota : X_K^+(K_\infty) \rightarrow \varprojlim_{x \mapsto x^\ell} \mathcal{O}_{K_\infty}/I_\infty^c$ given by

$$(x_n)_{n \geq 0} \in \varprojlim_{N_{K_n/K_{n-1}}} \mathcal{O}_{K_n} \mapsto (\bar{x}_n)_{n \geq 0}.$$

Let $\varprojlim_{x \mapsto x^\ell} \mathcal{O}_{K_n}/I_n^c$ denote the set of $(x_n)_{n \geq 0} \in \varprojlim_{x \mapsto x^\ell} \mathcal{O}_{K_\infty}/I_\infty^c$ such that $x_n \in \mathcal{O}_{K_n}/I_n^c$ for all $n \geq 0$.

Proposition 2.1.3. *Let $0 < c \leq c(K_\infty/K)$ be as in Proposition 2.1.2:*

- (1) *The natural map $\widetilde{\mathbb{E}}_{K_\infty}^+ \rightarrow \varprojlim_{x \mapsto x^\ell} \mathcal{O}_{K_\infty}/I_\infty^c$ is a bijection.*
- (2) *The map $\iota : X_K^+(K_\infty) \rightarrow \varprojlim_{x \mapsto x^\ell} \mathcal{O}_{K_\infty}/I_\infty^c = \widetilde{\mathbb{E}}_{K_\infty}^+$ is injective and isometric.*
- (3) *The image of ι is $\varprojlim_{x \mapsto x^\ell} \mathcal{O}_{K_n}/I_n^c$.*

Proof. See [Wintenberger 1983] and Section 4 of [Cais and Davis 2015]. We give a few more details for the convenience of the reader. Item (1) is classical; see for instance Proposition 4.2 of [Cais and Davis 2015]. The map ι is obviously injective and isometric. For (3), choose $x = (x_n)_{n \geq 0} \in \varprojlim_{x \mapsto x^\ell} \mathcal{O}_{K_n}/I_n^c$, and choose a lift $\hat{x}_n \in \mathcal{O}_{K_n}$ of x_n . One proves that $\{N_{K_{n+j}/K_n}(\hat{x}_{n+j})\}_{j \geq 0}$ converges to some $y_n \in \mathcal{O}_{K_n}$, and that $(y_n)_{n \geq 0} \in X_K^+(K_\infty)$ is a lift of $(x_n)_{n \geq 0}$. See Section 4 of [loc. cit.] for details, for instance the proof of Lemma 4.1. \square

Proposition 2.1.3 allows us to see $X_K^+(K_\infty)$, and hence $\varphi^{-n}(X_K^+(K_\infty))$ for all $n \geq 0$, as a subring of $\tilde{\mathbb{E}}_{K_\infty}^+$.

Proposition 2.1.4. *The ring $\bigcup_{n \geq 0} \varphi^{-n}(X_K^+(K_\infty))$ is dense in $\tilde{\mathbb{E}}_{K_\infty}^+$.*

Proof. See Section 4.3 of [Wintenberger 1983]. \square

2.2. Decompleting the tilt. We now compute $(\tilde{\mathbb{E}}_{K_\infty}^+)^{d\text{-sh}}$. Since Proposition 2.2.1 below is vacuous if $p = 2$, we assume in this section that $p \neq 2$.

Proposition 2.2.1. *If $0 < c \leq 1 - 1/(p - 1)$, and $x \in \mathcal{O}_{K_\infty}$ is such that $\text{val}_p(g(x) - x) \geq 1$ for all $g \in G_n$, then the image of x in $\mathcal{O}_{K_\infty}/I_\infty^c$ belongs to \mathcal{O}_{K_n}/I_n^c .*

Proof. If $\text{val}_p(g(x) - x) \geq 1$ for all $g \in \text{Gal}(K^{\text{alg}}/K_n)$, then by Theorem 1.7 of [Le Borgne 2010] (an optimal version of a theorem of Ax), there exists $y \in K_n$ such that $\text{val}_p(x - y) \geq 1 - 1/(p - 1)$. This implies the proposition. \square

Proposition 2.2.2. *If $c = p^\gamma$ is as above, then $X_K^+(K_\infty) \subset (\tilde{\mathbb{E}}_{K_\infty}^+)^{G\text{-}d\text{-sh}, \gamma, 0}$.*

Proof. Take $x = (x_n)_{n \geq 0} \in \varprojlim_{x \mapsto x^\ell} \mathcal{O}_{K_n}/I_n^c$. If $g \in G_i$, then $g(x_n) = x_n$ for $n \leq i$, so that $\text{val}_E(gx - x) \geq p^{di} p^\gamma$. \square

Theorem 2.2.3. *We have:*

- (1) $(\tilde{\mathbb{E}}_{K_\infty}^+)^{G\text{-}d\text{-sh}, 0, 0} \subset X_K^+(K_\infty)$.
- (2) $(\tilde{\mathbb{E}}_{K_\infty}^+)^{d\text{-sh}} = \bigcup_{n \geq 0} \varphi^{-n}(X_K^+(K_\infty))$ and $\tilde{\mathbb{E}}_{K_\infty}^{d\text{-sh}} = \bigcup_{n \geq 0} \varphi^{-n}(X_K(K_\infty))$.

Proof. Take $c \leq \min(c(K_\infty/K), 1 - 1/(p - 1))$. Take $x = (x_n)_{n \geq 0} \in \varprojlim_{x \mapsto x^\ell} \mathcal{O}_{K_\infty}/p$. If $n \geq 0$ and $x \in (\tilde{\mathbb{E}}_{K_\infty}^+)^{G\text{-}d\text{-sh}, 0, 0}$, then $\text{val}_E(g(x) - x) \geq p^{dn}$ if $g \in G_n$. This implies that $\text{val}_p(g(x_n) - x_n) \geq 1$ if $g \in G_n$. By Proposition 2.2.1, the image of x_n in $\mathcal{O}_{K_\infty}/I_\infty^c$ belongs to \mathcal{O}_{K_n}/I_n^c . Hence the image of x in $\varprojlim_{x \mapsto x^\ell} \mathcal{O}_{K_\infty}/I_\infty^c$ belongs to $\varprojlim_{x \mapsto x^\ell} \mathcal{O}_{K_n}/I_n^c$. By Proposition 2.1.3, x belongs to $X_K^+(K_\infty)$. This proves (1).

Since $\text{val}_E(\varphi(x)) = p \cdot \text{val}_E(x)$, item (2) follows from (1) and Propositions 2.2.2 and 1.2.16. \square

Remark 2.2.4. We have $\tilde{\mathbb{E}}_{K_\infty}^{d\text{-sh}} \subset \tilde{\mathbb{E}}_{K_\infty}^{1\text{-sh}}$. The field $\tilde{\mathbb{E}}_{K_\infty}^{1\text{-sh}}$ contains the field of norms $X_K(L_\infty)$ of any p -adic Lie extension L_∞/K contained in K_∞ . Indeed, $\tilde{\mathbb{E}}_{L_\infty} \subset \tilde{\mathbb{E}}_{K_\infty}$ and if $e = \dim \text{Gal}(L_\infty/K)$, then $X_K(L_\infty) \subset \tilde{\mathbb{E}}_{L_\infty}^{e\text{-sh}} \subset \tilde{\mathbb{E}}_{K_\infty}^{1\text{-sh}}$ (see Proposition 1.2.15).

Can one give a description of $\tilde{\mathbb{E}}_{K_\infty}^{1\text{-sh}}$, for example along the lines of Section 5 of [Berger 2016]?

3. The Lubin–Tate case

We now specialize the constructions of the previous section to the case when K_∞ is generated over K by the torsion points of a Lubin–Tate formal group.

3.1. Lubin–Tate formal groups. Let K be a finite extension of \mathbb{Q}_p of degree d , with ring of integers \mathcal{O}_K , inertia index f , ramification index e , and residue field k . Let $q = p^f = \text{Card}(k)$ and let π be a uniformizer of \mathcal{O}_K . Let LT be the Lubin–Tate formal \mathcal{O}_K -module attached to π ; see [Lubin and Tate 1965]. We choose a coordinate Y on LT. For each $a \in \mathcal{O}_K$ we get a power series $[a](Y) \in \mathcal{O}_K[[Y]]$, that we now see as an element of $k[[Y]]$. In particular, $[\pi](Y) = Y^q$. Let $S(T, U) \in k[[T, U]]$ denote the reduction mod π of the power series giving the addition law in LT in that coordinate. Recall that $S(T, 0) = T$ and $S(0, U) = U$.

Lemma 3.1.1. *If $a, b \in \mathcal{O}_K$ and $i \geq 0$, then $\text{val}_Y([a + p^i b](Y) - [a](Y)) \geq p^{di}$.*

Furthermore, $[1 + \pi^i](Y) = Y + Y^{q^i} + \mathcal{O}(Y^{q^{i+1}})$.

Proof. We have $[\pi](Y) = Y^q$, so $\text{val}_Y([\pi](Y)) \geq p^f$. Writing $p = u\pi^e$ for a unit u , we see that $\text{val}_Y([p^i b](Y)) \geq p^{di}$ if $b \in \mathcal{O}_K$. If $a, b \in \mathcal{O}_K$ and $i \geq 0$, then $[a + bp^i](Y) = S([a](Y), [bp^i](Y))$. We have $S(T, U) = T + U + TU \cdot R(T, U)$, so that $[a + bp^i](Y) - [a](Y) = S([a](Y), [bp^i](Y)) - [a](Y) \in [bp^i](Y) \cdot k[[Y]]$. This implies the first result.

The second claim follows likewise from the fact that $[1 + \pi^i](Y) = S(Y, [\pi^i](Y)) = Y + [\pi^i](Y) + Y \cdot [\pi^i](Y) \cdot R(Y, [\pi^i](Y))$. \square

Let $\mathbb{E} = k((Y))$. Let $\mathbb{E}_n = k((Y^{1/q^n}))$ and let $\mathbb{E}_\infty = \bigcup_{n \geq 0} \mathbb{E}_n$. These fields are endowed with the Y -adic valuation val_Y , and we let \mathbb{E}_\star^+ denote the ring of integers of \mathbb{E}_\star . The group \mathcal{O}_K^\times acts on \mathbb{E}_n by $a \cdot f(Y^{1/q^n}) = f([a](Y^{1/q^n}))$.

Lemma 3.1.2. *If $j \geq 1$ ($j \geq 2$ if $p = 2$), then $1 + p^j \mathcal{O}_K$ is uniform, and $(1 + p^j \mathcal{O}_K)_i = 1 + p^{i+j} \mathcal{O}_K$.*

Proof. The map $1 + p^j \mathcal{O}_K \rightarrow \mathcal{O}_K$, given by $x \mapsto p^{-j} \cdot \log_p(x - 1)$, is an isomorphism of pro- p groups taking $1 + p^{i+j} \mathcal{O}_K$ to $p^i \mathcal{O}_K$. \square

Recall that $d = [K : \mathbb{Q}_p]$, that $f = [k : \mathbb{F}_p]$, and that $q = p^f$.

Proposition 3.1.3. *We have $\mathbb{E}_n^+ = (\mathbb{E}_n^+)^{1+p^j \mathcal{O}_K - d \cdot \text{sh}, dj - fn, 0}$.*

Proof. If $b \in \mathcal{O}_K$ and $i, j \geq 0$, then by Lemma 3.1.1, we have

$$\text{val}_Y([1 + p^{i+j} b](Y^{1/q^n}) - Y^{1/q^n}) \geq 1/q^n \cdot p^{d(i+j)} = p^{dj - fn} \cdot p^{di}.$$

Lemma 3.1.2 then implies that $Y^{1/q^n} \in (\mathbb{E}_n^+)^{1+p^j \mathcal{O}_K - d \cdot \text{sh}, dj - fn, 0}$. The lemma follows from Proposition 1.2.16 and Lemma 1.2.9. \square

Corollary 3.1.4. *We have $\mathbb{E} = \mathbb{E}^{1+p^j \mathcal{O}_K - d \cdot \text{sh}, dj}$.*

Proof. This follows from Proposition 3.1.3 with $n = 0$, and Proposition 1.2.16. \square

Proposition 3.1.5. *If $\varepsilon > 0$, then $k[[Y]]^{1+p^j \mathcal{O}_K - d \cdot \text{sh}, dj + \varepsilon} \subset k[[Y^p]]$.*

Proof. Take $f(Y) \in k[[Y]]$. There is a power series $h(T, U) \in k[[T, U]]$ such that

$$f(T + U) = f(T) + U \cdot f'(T) + U^2 \cdot h(T, U).$$

If $m \geq 0$, Lemma 3.1.1 implies that $[1 + \pi^m](Y) = Y + Y^{q^m} + \mathcal{O}(Y^{q^m+1})$. Therefore,

$$f([1 + \pi^m](Y)) = f(Y) + (Y^{q^m} + \mathcal{O}(Y^{q^m+1})) \cdot f'(Y) + \mathcal{O}(Y^{2q^m}).$$

If $f(Y) \notin k[[Y^p]]$, then $f'(Y) \neq 0$. Let $\mu = \text{val}_Y(f'(Y))$. The above computations imply that $\text{val}_Y(f([1 + \pi^{ei+ej}](Y)) - f(Y)) = p^{dj} \cdot p^{di} + \mu$ for $i \gg 0$.

This implies the claim, since $\pi^e \mathcal{O}_K = p \mathcal{O}_K$. \square

Corollary 3.1.6. *We have $\mathbb{E}_\infty^{1+p^j \mathcal{O}_K\text{-d-sh}, dj-fn} = \mathbb{E}_n$.*

Proof. We prove that, more generally,

$$\mathbb{E}_\infty^{1+p^j \mathcal{O}_K\text{-d-sh}, dj-\ell} = k((Y^{1/p^\ell})).$$

Take $f(Y^{1/p^m}) \in (\mathbb{E}_\infty^+)^{1+p^j \mathcal{O}_K\text{-d-sh}, dj-\ell}$ where $f(Y) \in k[[Y]]$. Since $\text{val}_Y(h^p) = p \cdot \text{val}_Y(h)$ for all $h \in \tilde{\mathbb{E}}^+$, we have $f^{p^m}(Y) \in (\mathbb{E}_\infty^+)^{1+p^j \mathcal{O}_K\text{-d-sh}, dj-\ell+m}$, where $f^{p^m}(Y) \in E[[Y]]$ is $f^{p^m}(Y) = f(Y^{1/p^m})^{p^m}$. If $m \geq \ell+1$, then Proposition 3.1.5 implies that $f^{p^m}(Y) \in E[[Y^p]]$, so that $f(Y) = g(Y^p)$, and $f(Y^{1/p^m}) = g(Y^{1/p^{m-1}})$. This implies the claim. \square

3.2. Decompletion of $\tilde{\mathbb{E}}$. Since we use the results of Section 2.2, we once more assume that $p \neq 2$. Let $\tilde{\mathbb{E}}$ denote the Y -adic completion of \mathbb{E}_∞ .

Theorem 3.2.1. *We have $\tilde{\mathbb{E}}^{1+p^j \mathcal{O}_K\text{-d-sh}, dj} = \mathbb{E}$, and $\tilde{\mathbb{E}}^{\text{d-sh}} = \mathbb{E}_\infty$.*

Proof. Let $K_\infty = K(\text{LT}[\pi^\infty])$ denote the extension of K generated by the torsion points of LT , and let $\Gamma = \text{Gal}(K_\infty/K)$. The Lubin–Tate character χ_π gives rise to an isomorphism $\chi_\pi : \Gamma \rightarrow \mathcal{O}_K^\times$. For $n \geq 1$, let $K_n = K(\text{LT}[\pi^n])$. If $(\pi_n)_{n \geq 1}$ is a compatible sequence of primitive π^n -torsion points of LT , then π_n is a uniformizer of \mathcal{O}_{K_n} , $\varpi = (\pi_n)_{n \geq 0}$ belongs to $\varprojlim_{N_{K_n/K_{n-1}}} \mathcal{O}_{K_n}$, and $X_K(K_\infty) = k((\varpi))$ by Theorem 2.1.1. If $g \in \Gamma$, then $g(\varpi) = [\chi_\pi(g)](\varpi)$, so that if we identify Γ and \mathcal{O}_K^\times , then $X_K(K_\infty) = \mathbb{E}$ with its action of \mathcal{O}_K^\times . Proposition 2.1.4 implies that $\tilde{\mathbb{E}} = \tilde{\mathbb{E}}_{K_\infty}$ as valued fields with an action of (an open subgroup of) \mathcal{O}_K^\times . We can therefore apply Theorem 2.2.3, and get $(\tilde{\mathbb{E}}^+)^{\text{d-sh}} = \mathbb{E}_\infty^+$. This implies the second statement. The first one then follows from Corollary 3.1.6. \square

Remark 3.2.2. In the above proof, note that $K_\infty^{1+p^n \mathcal{O}_K} = K_{ne}$, so that the numbering is not the same as in Section 2.1.

Remark 3.2.3. We can define Lubin–Tate Γ -modules over \mathbb{E} as in Section 3.2 of [Berger and Rozensztajn 2022]. The results proved in that section carry over to the Lubin–Tate setting without difficulty.

In Theorem 2.9 of [Berger and Rozensztajn 2022], we proved Theorem 3.2.1 above in the cyclotomic case, using Tate traces. There are no such Tate traces in the Lubin–Tate case if $K \neq \mathbb{Q}_p$. We now explain why this is so. More precisely, we prove that there is no Γ -equivariant k -linear projector $\tilde{\mathbb{E}} \rightarrow \mathbb{E}$ if $K \neq \mathbb{Q}_p$.

Choose a coordinate T on LT such that $\log_{\text{LT}}(T) = \sum_{n \geq 0} T^{q^n} / \pi^n$, so that $\log'_{\text{LT}}(T) \equiv 1 \pmod{\pi}$. Let $\partial = 1 / \log'_{\text{LT}}(T) \cdot d/dT$ be the invariant derivative on LT . Let $\varphi_q = \varphi^f$ where $q = p^f$.

Lemma 3.2.4. *We have $d\gamma(Y)/dY \equiv \chi_\pi(\gamma)$ in \mathbb{E} for all $\gamma \in \Gamma$.*

Proof. Since $\log'_{\text{LT}} \equiv 1 \pmod{\pi}$, we have $\partial = d/dY$ in \mathbb{E} . Applying $\partial \circ \gamma = \chi_\pi(\gamma)\gamma \circ \partial$ to Y , we get the claim. \square

Lemma 3.2.5. *If $\gamma \in \Gamma$ is nontorsion, then $\mathbb{E}^{\gamma=1} = k$.*

Proposition 3.2.6. *If $K \neq \mathbb{Q}_p$, there is no Γ -equivariant map $R : \mathbb{E} \rightarrow \mathbb{E}$ such that $R(\varphi_q(f)) = f$ for all $f \in \mathbb{E}$.*

Proof. Suppose that such a map exists, and take $\gamma \in \Gamma$ nontorsion and such that $\chi_\pi(\gamma) \equiv 1 \pmod{\pi}$. We first show that if $f \in \mathbb{E}$ is such that $(1 - \gamma)f \in \varphi_q(\mathbb{E})$, then $f \in \varphi_q(\mathbb{E})$. Write $f = f_0 + \varphi_q(R(f))$ where $f_0 = f - \varphi_q(R(f))$, so that $R(f_0) = 0$ and $(1 - \gamma)f_0 = \varphi_q(g) \in \varphi_q(\mathbb{E})$. Applying R , we get $0 = (1 - \gamma)R(f_0) = g$. Hence $g = 0$ so that $(1 - \gamma)f_0 = 0$. Since $\mathbb{E}^{\gamma=1} = k$ by Lemma 3.2.5, this implies $f_0 \in k$, so that $f \in \varphi_q(\mathbb{E})$.

However, Lemma 3.2.4 and the fact that $\chi_\pi(\gamma) \equiv 1 \pmod{\pi}$ imply that $\gamma(Y) = Y + f_\gamma(Y^p)$ for some $f_\gamma \in \mathbb{E}$, so that $\gamma(Y^{q/p}) = Y^{q/p} + \varphi_q(g_\gamma)$. Hence $(1 - \gamma)(Y^{q/p}) \in \varphi_q(\mathbb{E})$ even though $Y^{q/p}$ does not belong to $\varphi_q(\mathbb{E})$. Therefore, no such map R can exist. \square

Corollary 3.2.7. *If $K \neq \mathbb{Q}_p$, there is no Γ -equivariant k -linear projector $\varphi_q^{-1}(\mathbb{E}) \rightarrow \mathbb{E}$. A fortiori, there is no Γ -equivariant k -linear projector $\tilde{\mathbb{E}} \rightarrow \mathbb{E}$.*

Proof. Given such a projector Π , we could define R as in Proposition 3.2.6 by $R = \Pi \circ \varphi_q^{-1}$. \square

3.3. The perfectoid commutant of $\text{Aut}(\text{LT})$. In Section 3.1 of [Berger and Rozensztajn 2022], we computed the perfectoid commutant of $\text{Aut}(\mathbb{G}_m)$. We now use Theorem 3.2.1 to do the same for $\text{Aut}(\text{LT})$. We still assume that $p \neq 2$.

Theorem 3.3.1. *If $u \in \tilde{\mathbb{E}}^+$ is such that $\text{val}_Y(u) > 0$ and $u \circ [g] = [g] \circ u$ for all $g \in \mathcal{O}_K^\times$, there exists $b \in \mathcal{O}_K^\times$ and $n \in \mathbb{Z}$ such that $u(Y) = [b](Y^{q^n})$.*

Recall that a power series $f(Y) \in k[[Y]]$ is separable if $f'(Y) \neq 0$. If $f(Y) \in Y \cdot k[[Y]]$, we say that f is invertible if $f'(0) \in k^\times$, which is equivalent to f being invertible for composition (denoted by \circ). We say that $w(Y) \in Y \cdot k[[Y]]$ is nontorsion if $w^{on}(Y) \neq Y$ for all $n \geq 1$. If $w(Y) = \sum_{i \geq 0} w_i Y^i \in k[[Y]]$ and $m \in \mathbb{Z}$, let $w^{(m)}(Y) = \sum_{i \geq 0} w_i^{p^m} Y^i$. Note that $(w \circ v)^{(m)} = w^{(m)} \circ v^{(m)}$.

Proposition 3.3.2. *Let $w(Y) \in Y + Y^2 \cdot k[[Y]]$ be a nontorsion series, and let $f(Y) \in Y \cdot k[[Y]]$ be a separable power series. If $w^{(m)} \circ f = f \circ w$ for some $m \in \mathbb{Z}$, then f is invertible.*

Proof. This is a slight generalization of [Lubin 1994, Lemma 6.2]. Write

$$\begin{aligned} f(Y) &= f_n Y^n + \text{O}(Y^{n+1}), \\ f'(Y) &= g_j Y^j + \text{O}(Y^{j+1}), \\ w(Y) &= Y + w_r Y^r + \text{O}(Y^{r+1}), \end{aligned}$$

with $f_n, g_j, w_r \neq 0$. Since w is nontorsion, we can replace w by $w^{\circ p^\ell}$ for $\ell \gg 0$ and assume that $r \geq j + 1$. We have

$$w^{(m)} \circ f = f(Y) + w_r^{(m)} f(Y)^r + \mathcal{O}(Y^{n(r+1)}) = f(Y) + w_r^{(m)} f_n^r Y^{nr} + \mathcal{O}(Y^{nr+1}).$$

If $j = 0$, then $n = 1$ and we are done, so assume that $j \geq 1$. We have

$$\begin{aligned} f \circ w &= f(Y + w_r Y^r + \mathcal{O}(Y^{r+1})) \\ &= f(Y) + w_r Y^r f'(Y) + \mathcal{O}(Y^{2r}) \\ &= f(Y) + w_r g_j Y^{r+j} + \mathcal{O}(Y^{r+j+1}). \end{aligned}$$

This implies that $nr = r + j$, hence $(n - 1)r = j$, which is impossible if $r > j$ unless $n = 1$. Hence $n = 1$ and f is invertible. \square

Lemma 3.3.3. *If $u \in \tilde{\mathbb{E}}^+$ is such that $\text{val}_X(u) > 0$ and $u \circ [g] = [g] \circ u$ for all $g \in \mathcal{O}_K^\times$, then $u \in (\tilde{\mathbb{E}}^+)^{d\text{-sh}}$.*

Proof. The group \mathcal{O}_K^\times acts on $\tilde{\mathbb{E}}^+$ by $g \cdot u = u \circ [g]$. By lemmas 3.1.1 and 3.1.2, the function $g \mapsto [g] \circ u$ is in $\mathcal{H}_d^\lambda(1 + p\mathcal{O}_K, \tilde{\mathbb{E}}^+)$, where $p^\lambda = \text{val}_Y(u)$. \square

Proof of Theorem 3.3.1. Take $u \in \tilde{\mathbb{E}}$ such that $\text{val}_Y(u) > 0$ and $u \circ [g] = [g] \circ u$ for all $g \in \mathcal{O}_K^\times$. By Lemma 3.3.3 and Theorem 3.2.1, there is an $m \in \mathbb{Z}$ such that $f(Y) = u(Y)^{p^m}$ belongs to $Y \cdot k[[Y]]$ and is separable. Take $g \in 1 + \pi\mathcal{O}_K$ such that g is nontorsion, and let $w(Y) = [g](Y)$ so that $u \circ w = w \circ u$. We have $f \circ w = w^{(m)} \circ f$. By Proposition 3.3.2, f is invertible. In addition, $f \circ w = w^{(m)} \circ f$ if $w(Y) = [g](Y)$ for all $g \in \mathcal{O}_K^\times$. Hence $f_0 \cdot \bar{g} = \bar{g}^{p^m} \cdot f_0$, so that $a^{p^m} = a$ for all $a = \bar{g} \in k$. This implies that $\mathbb{F}_q \subset \mathbb{F}_{p^{|m|}}$, so that $m = fn$ for some $n \in \mathbb{Z}$. Hence $w^{(m)} = w$, and $f \circ [g] = [g] \circ f$ for all $g \in \mathcal{O}_K^\times$. Theorem 6 of [Lubin and Sarkis 2007] implies that $f \in \text{Aut}(\text{LT})$. Hence there exists $b \in \mathcal{O}_K^\times$ such that $u(Y) = [b](Y^{q^n})$. \square

4. Mahler expansions and super-Hölder functions

In Section 1.3 of [Berger and Rozensztajn 2022], we proved an analogue of Mahler's theorem for continuous functions $\mathbb{Z}_p \rightarrow M$, and then gave a characterization of super-Hölder functions in terms of their Mahler expansions. We now indicate how these results generalize to functions $G \rightarrow M$ for a uniform pro- p group G . Given the definition of super-Hölder functions and the existence of a coordinate $c : G \rightarrow \mathbb{Z}_p^d$ as in Proposition 1.1.1, it is enough to study functions $\mathbb{Z}_p^d \rightarrow M$. We generalize the setting a little bit, and study functions $\mathcal{O}_K^d \rightarrow M$ where K is a finite extension of \mathbb{Q}_p . Let K be such a field, fix a uniformizer π of \mathcal{O}_K and let k be the residue field of K . Let $q = \text{Card}(k)$.

4.1. Good bases and wavelets. Let $X = \mathcal{O}_K^d$, which we endow with the valuation $\text{val}_X(x_1, \dots, x_d) = \min_i \text{val}_\pi(x_i)$. For $n \geq 0$, let $X_n = \pi^n X = \{x \in X, \text{val}_X(x) \geq n\}$.

We endow X with the val_X -adic topology. For any set Y , we denote by $\text{LC}(X, Y)$ the set of locally constant functions $X \rightarrow Y$. For $n \geq 0$ we denote by $\text{LC}_n(X, Y)$ the subset of elements of $\text{LC}(X, Y)$

that factor through X/X_n . Let $I = \bigcup_{n \geq 0} I_n$ be a set of indices, where $I_n \subset I_{n+1}$ for all $n \geq 0$, and $\text{Card}(I_n) = \text{Card}(X/X_n) = q^{nd}$. Let E be a field of characteristic p .

Definition 4.1.1. A family $\{h_i\}_{i \in I}$ is a good basis of $\text{LC}(X, E)$ if it is a basis of the E -vector space $\text{LC}(X, E)$ such that for all $n \geq 0$, $\{h_i\}_{i \in I_n}$ is a basis of $\text{LC}_n(X, E)$.

Let M be (as usual) an E -vector space with a valuation val_M , such that $\text{val}_M(ax) = \text{val}_M(x)$ for all $a \in E^\times$ and $x \in M$. We assume that M is separated and complete for the val_M -adic topology.

Proposition 4.1.2. Every $f \in \text{LC}_n(X, M)$ can be written uniquely as $\sum_{i \in I_n} h_i \cdot m_i$ for some elements $m_i \in M$. Moreover, $\inf_{x \in X} \text{val}_M(f(x)) = \inf_{i \in I_n} \text{val}_M(m_i)$.

Proof. Let $\{\delta_x\}_{x \in X/X_n}$ be the basis of $\text{LC}_n(X, E)$ defined as follows: δ_x is the characteristic function of $x + X_n$. Then $f \in \text{LC}_n(X, M)$ is equal to $\sum_{x \in X/X_n} \delta_x \cdot f(x)$.

As $\{h_i\}_{i \in I_n}$ is also a basis of $\text{LC}_n(X, E)$, we can write $\delta_x = \sum_{i \in I_n} a_{i,x} h_i$ for some elements $a_{i,x} \in E$. We now have $f = \sum_{i \in I_n} h_i \cdot m_i$ where $m_i = \sum_{x \in X/X_n} a_{i,x} f(x)$. This formula implies that $\inf_{i \in I_n} \text{val}_M(m_i) \geq \inf_{x \in X} \text{val}_M(f(x))$.

On the other hand we can also write $h_i = \sum_{x \in X/X_n} b_{x,i} \delta_x$ for some elements $b_{x,i} \in E$, so that $f(x) = \sum_{i \in I_n} b_{x,i} m_i$. This implies that $\inf_{i \in I_n} \text{val}_M(m_i) \leq \inf_{x \in X} \text{val}_M(f(x))$. \square

We now give an example of a particularly nice good basis of $\text{LC}(X, E)$, the basis of wavelets; see Section I.3 of [Colmez 2010] and Section 2.1 of [de Shalit 2016]. Let \mathcal{T} be a set of representatives of X/X_1 in X , chosen so that the representative of 0 is 0. For each $n \geq 0$, let \mathcal{R}_n be the set of representatives of X/X_n defined as follows: $\mathcal{R}_0 = \{0\}$, and for $n \geq 1$, $\mathcal{R}_n = \{\sum_{i=0}^{n-1} \pi^i x_i, x_i \in \mathcal{T} \text{ for all } i\}$. We have $\mathcal{R}_1 = \mathcal{T}$, and $\mathcal{R}_n \subset \mathcal{R}_{n+1}$ for all n . Let $\mathcal{R} = \bigcup_{n \geq 0} \mathcal{R}_n$. If $r \in \mathcal{R}$ let $\ell(r)$ be the smallest n such that $r \in \mathcal{R}_n$. For $r \in \mathcal{R}$, let χ_r be the characteristic function of the closed disc $r + X_{\ell(r)} = \{x \in X, \text{val}_X(x - r) \geq \ell(r)\}$.

Proposition 4.1.3. The set $\{\chi_r\}_{r \in \mathcal{R}}$ is a good basis of $\text{LC}(X, E)$.

Proof. We prove that for all $n \geq 0$, the set $\{\chi_r\}_{r \in \mathcal{R}_n}$ is a basis of $\text{LC}_n(X, E)$. Consider the basis $\{\delta_r\}_{r \in \mathcal{R}_n}$ of $\text{LC}_n(X, E)$, where δ_r is the characteristic function of $r + X_n$. We have

$$\chi_r = \sum_{r' \in \mathcal{R}_{n-\ell(r)}} \delta_{r+\pi^{\ell(r)}r'}.$$

This implies that if we write $\mathcal{R}_n = (\mathcal{R}_n \setminus \mathcal{R}_{n-1}) \sqcup \cdots \sqcup (\mathcal{R}_1 \setminus \mathcal{R}_0) \sqcup \mathcal{R}_0$ and we express the family $\{\chi_r\}_{r \in \mathcal{R}_n}$ in terms of the basis $\{\delta_r\}_{r \in \mathcal{R}_n}$, we get a unipotent matrix. This shows that $\{\chi_r\}_{r \in \mathcal{R}_n}$ is also a basis of $\text{LC}_n(X, E)$. \square

4.2. Expansions of continuous functions. We show that every continuous function $X \rightarrow M$ has a convergent expansion along a good basis of X , and prove some continuity estimates in terms of the coefficients of the expansion. If $\{m_i\}_{i \in I}$ is a family of M , we say that $m_i \rightarrow 0$ if $\inf_{i \notin I_n} \text{val}_M(m_i) \rightarrow +\infty$ as $n \rightarrow +\infty$.

Theorem 4.2.1. *Let $\{h_i\}_{i \in I}$ be a good basis of $\text{LC}(X, E)$.*

If $\{m_i\}_{i \in I}$ is a family of M such that $m_i \rightarrow 0$, the function $f : X \rightarrow M$ given by $f = \sum_{i \in I} h_i \cdot m_i$ belongs to $C^0(X, M)$, and $\inf_{x \in X} \text{val}_M(f(x)) = \inf_{i \in I} \text{val}_M(m_i)$.

Conversely, if $f \in C^0(X, M)$, there exists a unique family $\{m_i(f)\}_{i \in I}$ of elements of M such that $m_i(f) \rightarrow 0$ and such that $f = \sum_{i \in I} h_i \cdot m_i(f)$.

Proof. Let $\{m_i\}_{i \in I}$ be a family of M such that $m_i \rightarrow 0$. If $f_n = \sum_{i \in I_n} h_i \cdot m_i$, then $f_n \in C^0(X, M)$, and f is the uniform limit of the f_n . We have $\inf_X \text{val}_M(f_n(x)) = \inf_{i \in I_n} \text{val}_M(m_i)$ by Proposition 4.1.2. Since $m_i \rightarrow 0$, we have $\inf_{i \in I} \text{val}_M(m_i) = \inf_{i \in I_n} \text{val}_M(m_i)$ for $n \gg 0$. Hence $\inf_X \text{val}_M(f_n(x)) = \inf_{i \in I} \text{val}_M(m_i)$ for $n \gg 0$. Since $\inf_{x \in X} \text{val}_M(f(x)) = \lim_n \inf_x \text{val}_M(f_n(x))$, we have $\inf_{x \in X} \text{val}_M(f(x)) = \inf_{i \in I} \text{val}_M(m_i)$.

We now prove the converse. Let $M_n = \{m \in M, \text{val}_M(m) \geq n\}$, let $\pi_n : M \rightarrow M/M_n$ be the projection, and for each n , fix a lift $\psi_n : M/M_n \rightarrow M$. Take $f \in C^0(X, M)$, and let $f_n = \psi_n \circ \pi_n \circ f$. As f and f_n coincide modulo M_n , f is the uniform limit of the f_n . On the other hand, $\pi_n \circ f$ is locally constant, and therefore so is f_n . As X is compact, there exists some $k(n) \geq 0$ such that $f_n \in \text{LC}_{k(n)}(X, M)$. By Proposition 4.1.2, we can write $f_n = \sum_{i \in I} h_i \cdot m_{i,n}$, where $m_{i,n} = 0$ if $i \notin I_{k(n)}$. We have $\text{val}_M(m_{i,n} - m_{i,n'}) \geq \min(n, n')$ by construction, so that for each i , the sequence $\{m_{i,n}\}_n$ converges to some $m_i \in M$. Moreover, if $i \notin I_{k(n)}$, then $\text{val}_M(m_i) \geq n$, so that $m_i \rightarrow 0$. The continuous function $\sum_{i \in I} h_i \cdot m_i$ is the uniform limit of the f_n , so that finally $f = \sum_{i \in I} h_i \cdot m_i$. \square

Proposition 4.2.2. *Take $f \in C^0(X, M)$ and $t \in \mathbb{Z}_{\geq 0}$. If $\{h_i\}_{i \in I}$ is a good basis of $\text{LC}(X, E)$, and we write $f = \sum_i h_i \cdot m_i$ with $m_i \rightarrow 0$, then $\inf_{i \notin I_t} \text{val}_M(m_i)$ depends only on f and not on the choice of the good basis.*

Proof. Fix two good bases $\{h_i\}_{i \in I}$ and $\{h'_i\}_{i \in I}$ of $\text{LC}(X, E)$. There exists a family $\{\lambda_{i,j}\}_{(i,j) \in I \times I}$ of elements of E such that $h_i = \sum_j \lambda_{i,j} h'_j$ for all i . Moreover, if $i \in I_c$ then $\lambda_{i,j} = 0$ for all $j \notin I_c$. Now write $f = \sum_{i \in I} h_i \cdot m_i(f) = \sum_{i \in I} h'_i \cdot m'_i(f)$. We also have

$$f = \sum_i \left(\sum_j \lambda_{i,j} h'_j \right) \cdot m_i(f) = \sum_j h'_j \cdot \left(\sum_i \lambda_{i,j} m_i(f) \right),$$

so that $m'_j(f) = \sum_i \lambda_{i,j} m_i(f)$. If $j \notin I_t$, then $m'_j(f) = \sum_{i \notin I_t} \lambda_{i,j} m_i(f)$, as $\lambda_{i,j} = 0$ if $i \in I_t$ and $j \notin I_t$. This implies that $\inf_{j \notin I_t} \text{val}_M(m'_j(f)) \geq \inf_{i \notin I_t} \text{val}_M(m_i(f))$.

By symmetry, we get that $\inf_{j \notin I_t} \text{val}_M(m'_j(f)) = \inf_{i \notin I_t} \text{val}_M(m_i(f))$. \square

Theorem 4.2.3. *Take $f \in C^0(X, M)$ and $t \in \mathbb{Z}_{\geq 0}$.*

If $\{h_i\}_{i \in I}$ is a good basis of $\text{LC}(X, E)$, and we write $f = \sum_i h_i \cdot m_i$ with $m_i \rightarrow 0$, then

$$\inf_{i \notin I_t} \text{val}_M(m_i) = \inf_{\substack{x, y \in X \\ \text{val}_X(x-y) \geq t}} \text{val}_M(f(x) - f(y)).$$

Proof. Let $C_t(f) = \inf_{x, y \in X, \text{val}_X(x-y) \geq t} \text{val}_M(f(x) - f(y))$ and $B_t(f) = \inf_{i \notin I_t} \text{val}_M(m_i)$.

If $x \in X$ and $z \in X_t$, then $f(x+z) - f(x) = \sum_{i \in I} (h_i(x+z) - h_i(z)) \cdot m_i(f)$. As $h_i \in \text{LC}_i(X, E)$ for $i \in I_t$, the above equality gives us

$$f(x+z) - f(x) = \sum_{i \notin I_t} (h_i(x+z) - h_i(z)) \cdot m_i(f).$$

This implies that $C_t(f) \geq B_t(f)$.

We now prove the converse inequality. By Proposition 4.2.2, $B_t(f)$ is independent of the choice of a good basis, and we choose the wavelet basis of Proposition 4.1.3. Write $f = \sum_{r \in \mathcal{R}} \chi_r \cdot m_r(f)$, so that we want to show that $\text{val}_M(m_r(f)) \geq C_t(f)$ for all $r \notin \mathcal{R}_t$. If $x \in X$, define $g_x : X \rightarrow M$ by $g_x(z) = f(x + \pi^t z) - f(x)$, and write $g_x = \sum_{r \in \mathcal{R}} \chi_r \cdot m_r(g_x)$. For each $r \in \mathcal{R}$, we can write uniquely $r = r_t + \pi^t s$ with $r_t \in \mathcal{R}_t$, where $s = 0$ if $r \in \mathcal{R}_t$, and $s \neq 0 \in \mathcal{R}_{\ell(r)-t}$ if $r \notin \mathcal{R}_t$. For $x \in \mathcal{R}_t$ and $r \notin \mathcal{R}_t$, the map $z \mapsto \chi_r(x + \pi^t z) - \chi_r(x)$ is the zero function if $r_t \neq x$, and is χ_s if $r_t = x$. This implies that if $x \in \mathcal{R}_t$, then

$$\begin{aligned} g_x(z) &= \sum_{r \in \mathcal{R}} (\chi_r(x + \pi^t z) - \chi_r(x)) \cdot m_r(f) \\ &= \sum_{r \notin \mathcal{R}_t} (\chi_r(x + \pi^t z) - \chi_r(x)) \cdot m_r(f) \\ &= \sum_{s \notin \mathcal{R}_0} \chi_s(z) \cdot m_{x+\pi^t s}(f). \end{aligned}$$

Therefore if $x \in \mathcal{R}_t$, then $m_0(g_x) = 0$ and $m_s(g_x) = m_{x+\pi^t s}(f)$ if $s \neq 0$. We have $\inf_{s \in \mathcal{R}} \text{val}_M(m_s(g_x)) = \inf_{z \in X} \text{val}_M(g_x(z)) \geq C_t(f)$, so that $\text{val}_M(m_s(g_x)) \geq C_t(f)$ for all $x \in X$ and $s \in \mathcal{R}$. This implies that for all $x \in \mathcal{R}_t$ and $s \neq 0$, $\text{val}_M(m_{x+\pi^t s}(f)) \geq C_t(f)$. Hence for all $r \notin \mathcal{R}_t$, we have $\text{val}_M(m_r(f)) \geq C_t(f)$. \square

4.3. Mahler bases. We now construct some other examples of good bases. For $n \geq 0$, let $\text{Int}_n(\mathcal{O}_K)$ denote the set of polynomials $f(T) \in K[T]$ such that $\deg(P) \leq n$ and $f(\mathcal{O}_K) \subset \mathcal{O}_K$. Recall (see for instance Section 1.2 of [de Shalit 2016]) that a Mahler basis for \mathcal{O}_K is a sequence $\{h_n\}_{n \geq 0}$ with $h_n(T) \in K[T]$ of degree n , and such that $\{h_0, \dots, h_n\}$ is a basis of the free \mathcal{O}_K -module $\text{Int}_n(\mathcal{O}_K)$ for all $n \geq 0$. For example, if $K = \mathbb{Q}_p$, we can take $h_n(T) = \binom{T}{n}$. Let $\{h_n\}_{n \geq 0}$ be a Mahler basis for \mathcal{O}_K . Each h_n defines a function $\mathcal{O}_K \rightarrow \mathcal{O}_K$ and hence $\mathcal{O}_K \rightarrow k$. Let $I = \mathbb{Z}_{\geq 0}$ and let $I_n = \{0, \dots, q^n - 1\}$ for $n \geq 0$.

Proposition 4.3.1. *If $\{h_n\}_{n \geq 0}$ is a Mahler basis for \mathcal{O}_K , then $\{h_i\}_{i \in I}$ is a good basis of $\text{LC}(\mathcal{O}_K, k)$.*

Proof. By Theorem 1.2 of [de Shalit 2016], $\{h_0, \dots, h_{q^m-1}\}$ is a basis of the k -vector space $\text{LC}_m(\mathcal{O}_K, k)$ for all $m \geq 0$. This implies the claim. \square

We now specialize to $K = \mathbb{Q}_p$. Write \mathbb{N} for $\mathbb{Z}_{\geq 0}$ and \mathbf{n} for an element $(n_1, \dots, n_d) \in \mathbb{N}^d$. For each $\mathbf{n} \in \mathbb{N}^d$, we denote by $h_{\mathbf{n}}$ the function $\mathbb{Z}_p^d \rightarrow E$ given by $(x_1, \dots, x_d) \mapsto \binom{x_1}{n_1} \cdots \binom{x_d}{n_d}$. For $m \in \mathbb{Z}_{\geq 0}$, let $I_m = \{\mathbf{n} \in \mathbb{N}^d \text{ such that } \max(n_1, \dots, n_d) \leq p^m - 1\}$.

Proposition 4.3.2. *The functions $\{h_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{N}^d}$ form a good basis of $\text{LC}(\mathbb{Z}_p^d, \mathbb{F}_p)$.*

Proof. The claim follows from Proposition 4.3.1 for $K = \mathbb{Q}_p$, and Lemma 4.3.3 below. \square

Lemma 4.3.3. *If X and X' are as in Section 4.1, and $\{h_i\}_{i \in I}$ and $\{h'_j\}_{j \in J}$ are good bases of $\mathrm{LC}(X, E)$ and $\mathrm{LC}(X', E)$, then $\{h_i \otimes h'_j\}_{(i,j) \in I \times J}$ is a good basis of $\mathrm{LC}(X \times X', E)$, with $(I \times J)_n = I_n \times J_n$.*

Let G be a uniform pro- p group, and let $c : G \rightarrow \mathbb{Z}_p^d$ be a coordinate as in Proposition 1.1.1. The theorem below follows from Proposition 4.3.2 and Theorems 4.2.1 and 4.2.3.

Theorem 4.3.4. *If $\{m_n\}_{n \in \mathbb{N}^d}$ is a sequence of M such that $m_n \rightarrow 0$, the function $f : G \rightarrow M$ given by $f(g) = \sum_{n \in \mathbb{N}^d} \binom{c_1(g)}{n_1} \cdots \binom{c_d(g)}{n_d} m_n$ belongs to $C^0(G, M)$. We have $\inf_{g \in G} \mathrm{val}_M(f(g)) = \inf_{n \in \mathbb{N}^d} \mathrm{val}_M(m_n)$.*

Conversely, if $f \in C^0(G, M)$, there exists a unique sequence $\{m_n(f)\}_{n \in \mathbb{N}^d}$ such that $m_n(f) \rightarrow 0$ and such that $f(g) = \sum_{n \in \mathbb{N}^d} \binom{c_1(g)}{n_1} \cdots \binom{c_d(g)}{n_d} m_n(f)$.

We have $f \in \mathcal{H}_e^{\lambda, \mu}(G, M)$ if and only if for all $i \geq 0$, we have $\mathrm{val}_M(m_n(f)) \geq p^\lambda \cdot p^{ei} + \mu$ whenever $\max(n_1, \dots, n_d) \geq p^i$.

Remark 4.3.5. The first two assertions in the above theorem also follow from Theorem 1.2.4 in Section III of [Lazard 1965] (we thank Konstantin Ardakov for pointing this out).

We finish by considering the case $G = \mathcal{O}_K$ for K a finite extension of \mathbb{Q}_p , and working with a Mahler basis for \mathcal{O}_K . Let K be a finite extension of \mathbb{Q}_p as before. Assume that E is an extension of k . Let $\{h_n\}_{n \geq 0}$ be a Mahler basis for \mathcal{O}_K . If $f \in C^0(\mathcal{O}_K, M)$, write $f = \sum_{n \geq 0} h_n m_n(f)$ with $m_n(f) \rightarrow 0$. Let e denote the ramification index of K .

Proposition 4.3.6. *If $f = \sum_{n \geq 0} h_n m_n(f)$ as above, then $f \in \mathcal{H}_i^{\lambda, \mu}(\mathcal{O}_K, M)$ if and only if $\mathrm{val}_M(m_n(f)) \geq p^\lambda \cdot p^{ti} + \mu$ whenever $n \geq p^{di}$.*

Proof. This follows from Theorem 4.2.3, since $\mathrm{val}_p(x - y) \geq i$ if and only if $\mathrm{val}_\pi(x - y) \geq ei$, and since $q^e = p^d$. \square

In this situation we can also define a slightly different version of super-Hölder functions. We say that a function $f : \mathcal{O}_K \rightarrow M$ is in $\mathcal{H}_{K,t}^{\lambda, \mu}(\mathcal{O}_K, M)$ if $\mathrm{val}_M(f(x) - f(y)) \geq p^\lambda \cdot p^{ti} + \mu$ whenever $\mathrm{val}_\pi(x - y) \geq i$. We then have

$$\mathcal{H}_{te}^{\lambda+t(e-1), \mu}(\mathcal{O}_K, M) \subset \mathcal{H}_{K,t}^{\lambda, \mu}(\mathcal{O}_K, M) \subset \mathcal{H}_{te}^{\lambda, \mu}(\mathcal{O}_K, M).$$

In particular, $\mathcal{H}_{K,t}(\mathcal{O}_K, M) = \mathcal{H}_{te}(\mathcal{O}_K, M)$. If K/\mathbb{Q}_p is unramified then $\mathcal{H}_{K,t}^{\lambda, \mu}(\mathcal{O}_K, M) = \mathcal{H}_t^{\lambda, \mu}(\mathcal{O}_K, M)$. Moreover we have the following criterion:

Proposition 4.3.7. *If $f = \sum_{n \geq 0} h_n m_n(f)$ as above, then $f \in \mathcal{H}_{K,t}^{\lambda, \mu}(\mathcal{O}_K, M)$ if and only if $\mathrm{val}_M(m_n(f)) \geq p^\lambda \cdot p^{ti} + \mu$ whenever $n \geq q^i$.*

Example 4.3.8. For all $n \geq 0$, there exists $c_n(T) \in \mathrm{Int}_n(\mathcal{O}_K)$ such that $[a](Y) = \sum_{n \geq 0} c_n(a) Y^n$. This implies that $\mathrm{val}_Y(m_n(a \mapsto [a](Y))) \geq n$, so that the function $a \mapsto [a](Y)$ is in $\mathcal{H}_d^{0,0}(\mathcal{O}_K, E[[Y]])$, and in $\mathcal{H}_{K,f}^{0,0}(\mathcal{O}_K, E[[Y]])$ where $q = p^f$.

References

- [Berger 2016] L. Berger, “Multivariable (φ, Γ) -modules and locally analytic vectors”, *Duke Math. J.* **165**:18 (2016), 3567–3595. MR Zbl
- [Berger and Rozensztajn 2022] L. Berger and S. Rozensztajn, “Decompletion of cyclotomic perfectoid fields in positive characteristic”, *Ann. H. Lebesgue* **5** (2022), 1261–1276. MR Zbl
- [Cais and Davis 2015] B. Cais and C. Davis, “Canonical Cohen rings for norm fields”, *Int. Math. Res. Not.* **2015**:14 (2015), 5473–5517. MR Zbl
- [Colmez 2010] P. Colmez, “Fonctions d’une variable p -adique”, pp. 13–59 in *Représentations p -adiques de groupes p -adiques, II: Représentations de $GL_2(\mathbb{Q}_p)$ et (φ, Γ) -modules*, edited by L. Berger et al., Astérisque **330**, Soc. Math. France, Paris, 2010. MR Zbl
- [Dixon et al. 1991] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal, *Analytic pro- p groups*, Lond. Math. Soc. Lect. Note Ser. **157**, Cambridge Univ. Press, 1991. MR Zbl
- [Emerton 2017] M. Emerton, *Locally analytic vectors in representations of locally p -adic analytic groups*, Mem. Amer. Math. Soc. **1175**, Amer. Math. Soc., Providence, RI, 2017. MR Zbl
- [Klopsch 2005] B. Klopsch, “On the Lie theory of p -adic analytic groups”, *Math. Z.* **249**:4 (2005), 713–730. MR Zbl
- [Lazard 1965] M. Lazard, “Groupes analytiques p -adiques”, *Inst. Hautes Études Sci. Publ. Math.* **26** (1965), 389–603. MR Zbl
- [Le Borgne 2010] J. Le Borgne, “Optimisation du théorème d’Ax–Sen–Tate et application à un calcul de cohomologie galoisienne p -adique”, *Ann. Inst. Fourier (Grenoble)* **60**:3 (2010), 1105–1123. MR Zbl
- [Lubin 1994] J. Lubin, “Non-Archimedean dynamical systems”, *Compos. Math.* **94**:3 (1994), 321–346. MR Zbl
- [Lubin and Sarkis 2007] J. D. Lubin and G. Y. Sarkis, “Extrinsic properties of automorphism groups of formal groups”, *J. Algebra* **315**:2 (2007), 874–884. MR Zbl
- [Lubin and Tate 1965] J. Lubin and J. Tate, “Formal complex multiplication in local fields”, *Ann. of Math. (2)* **81** (1965), 380–387. MR Zbl
- [Schneider 2011] P. Schneider, *p -adic Lie groups*, Grundle Math. Wissen. **344**, Springer, 2011. MR Zbl
- [Scholze 2012] P. Scholze, “Perfectoid spaces”, *Publ. Math. Inst. Hautes Études Sci.* **116** (2012), 245–313. MR Zbl
- [Sen 1972] S. Sen, “Ramification in p -adic Lie extensions”, *Invent. Math.* **17** (1972), 44–50. MR Zbl
- [de Shalit 2016] E. de Shalit, “Mahler bases and elementary p -adic analysis”, *J. Théor. Nombres Bordeaux* **28**:3 (2016), 597–620. MR Zbl
- [Wintenberger 1983] J.-P. Wintenberger, “Le corps des normes de certaines extensions infinies de corps locaux: applications”, *Ann. Sci. École Norm. Sup. (4)* **16**:1 (1983), 59–89. MR Zbl

Communicated by Wee Teck Gan

Received 2023-03-16 Accepted 2024-02-13

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

ANT peer review and production are managed by EditFLOW[®] from MSP.

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Algebra & Number Theory

Volume 19 No. 1 2025

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