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To the memory of my mother

Let $p \in \mathbb{Z}$ be a prime, $\overline{\mathbb{Q}_p}$ a fixed algebraic closure of the field of p-adic numbers and $\overline{\mathbb{Z}_p}$ the absolute integral closure of the ring of p-adic integers. Given a residually algebraic torsion extension W of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}(X)$, by Kaplansky's characterization of immediate extensions of valued fields, there exists a pseudo-convergent sequence of transcendental type $E = \{s_n\}_{n \in \mathbb{N}} \subset \overline{\mathbb{Q}_p}$ such that

$$W = \mathbb{Z}_{(p),E} = \{\phi \in \mathbb{Q}(X) \mid \phi(s_n) \in \overline{\mathbb{Z}_p} \text{ for all sufficiently large } n \in \mathbb{N}\}.$$

We show here that we may assume that E is stacked, in the sense that, for each $n \in \mathbb{N}$, the residue field (resp. the value group) of $\overline{\mathbb{Z}_p} \cap \mathbb{Q}_p(s_n)$ is contained in the residue field (resp. the value group) of $\overline{\mathbb{Z}_p} \cap \mathbb{Q}_p(s_{n+1})$; this property of E allows us to describe the residue field and value group of W. In particular, if W is a DVR, then there exists α in the completion $\mathbb{C}_p$ of $\overline{\mathbb{Q}_p}$, α transcendental over $\mathbb{Q}$, such that $W = \mathbb{Z}_{(p),\alpha} = \{\phi \in \mathbb{Q}(X) \mid \phi(\alpha) \in \mathbb{O}_p\}$, where $\mathbb{O}_p$ is the unique local ring of $\mathbb{C}_p$; α belongs to $\overline{\mathbb{Q}_p}$ if and only if the residue field extension $W/M \supseteq \mathbb{Z}/p\mathbb{Z}$ is finite. As an application, we provide a full characterization of the Dedekind domains between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$.

Introduction

The problem of characterizing the set of the extensions of a valuation domain V with quotient field K to the field of rational functions $K(X)$ has a long and rich tradition (for example, see [Alexandru and Popescu 1988; Alexandru et al. 1988; 1990a; 1990b; Kaplansky 1942; Matignon and Ohm 1988; Peruginelli 2017; Peruginelli and Spirito 2020; 2021]). One recent direction of research is to describe these extensions by means of pseudo-monotone sequences of K [Peruginelli and Spirito 2021] in the original spirit of Ostrowski [1935a; 1935b], who introduced the well-known notion of pseudo-convergent sequence, later expanded by Kaplansky [1942] to study immediate extensions of valued fields.

Here, given a prime $p \in \mathbb{Z}$ and the DVR $\mathbb{Z}_{(p)}$ of $\mathbb{Q}$, we are interested in describing residually algebraic torsion extensions of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}(X)$, that is, valuation domains W of $\mathbb{Q}(X)$ lying above $\mathbb{Z}_{(p)}$ such that the residue field extension $W/M \supseteq \mathbb{Z}/p\mathbb{Z}$ is algebraic and the value group Γ_w of the associated valuation w to W is contained in the divisible hull of the value group of $\mathbb{Z}_{(p)}$ (i.e., the rationals). These valuation

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domains arise naturally as overrings of rings of integer-valued polynomials and Dedekind domains between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ [Eakin and Heinzer 1973; Peruginelli 2023] and also in the description of closed subfields of $\mathbb{C}_p$ [Ioviţă and Zaharescu 1995], the completion of an algebraic closure $\overline{\mathbb{Q}_p}$ of the field of p-adic numbers $\mathbb{Q}_p$. In the case when W is a DVR and the residue field extension is finite, by [Peruginelli 2017, Theorem 2.5 & Proposition 2.2], there exists an element α in $\overline{\mathbb{Q}_p}$, transcendental over $\mathbb{Q}$, such that $W = \mathbb{Z}_{(p),\alpha} = \{\phi \in \mathbb{Q}(X) \mid \phi(\alpha) \in \overline{\mathbb{Z}_p}\}$, where $\overline{\mathbb{Z}_p}$ is the absolute integral closure of $\mathbb{Z}_p$ (i.e., the integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}_p}$; note that $\overline{\mathbb{Z}_p}$ is the valuation domain of the unique extension of v_p to $\overline{\mathbb{Q}_p}$). In general, given a residually algebraic torsion extension W of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}(X)$, there exists a pseudo-convergent sequence $E = \{s_n\}_{n\in\mathbb{N}}$ in $\overline{\mathbb{Q}_p}$ such that

$$W = \mathbb{Z}_{(p),E} = \{\phi \in \mathbb{Q}(X) \mid \phi(s_n) \in \overline{\mathbb{Z}_p} \text{ for all sufficiently large } n \in \mathbb{N}\}$$

(Proposition 2.24). One of the main results of this paper is to show that we may assume that E is stacked (in a sense we make clear in Section 2; see Theorem 2.5). In particular, if W is a DVR of $\mathbb{Q}(X)$ extending $\mathbb{Z}_{(p)}$ such that the extension of the residue fields is infinite algebraic, then there exists α in $\mathbb{C}_p \setminus \overline{\mathbb{Q}_p}$ such that $W = \mathbb{Z}_{(p),\alpha} = \{\phi \in \mathbb{Q}(X) \mid \phi(\alpha) \in \mathbb{O}_p\}$, where $\mathbb{O}_p$ is the completion of $\overline{\mathbb{Z}_p}$ (equivalently, $\mathbb{O}_p$ is the valuation domain of the unique extension of v_p to $\mathbb{C}_p$; see Corollary 2.28). Necessarily, the (transcendental) extension $\mathbb{Q}_p(\alpha)/\mathbb{Q}_p$ has finite ramification.

It is worth recalling that in [Alexandru et al. 1990a, §5.1, & Theorem 5.1] a residually algebraic torsion extension W of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}(X)$ is realized as the limit of a sequence of residually transcendental extensions W_n of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}(X)$ (i.e., the residue field extension of W_n over $\mathbb{Z}_{(p)}$ is transcendental); moreover, for each $n \in \mathbb{N}$, W_n is defined by a minimal pair (s_n, δ_n) (as explained in [Alexandru et al. 1990a, p. 282]; for the definition of minimal pair see Section 1.2). Here, W is realized as the valuation domain $\mathbb{Z}_{(p),E}$, where, for each $n \in \mathbb{N}$, $(s_n, \delta_n = v_p(s_{n+1} - s_n))$ is a minimal pair.

The motivations behind these results are based on [Alexandru et al. 1998], in which the authors study closed subfields of $\mathbb{C}_p$ and show that any transcendental element of $\mathbb{C}_p$ is the limit of a particular kind of Cauchy sequence in $\overline{\mathbb{Q}_p}$ called distinguished [Alexandru et al. 1998, Proposition 2.2], which allows them to associate to such an element a set of invariants [Alexandru et al. 1998, Remark 2.4]. The notion of a stacked sequence that we introduce in this paper is a generalization of the notion of a distinguished sequence and falls into the well-known class of pseudo-convergent sequences. It allows us to describe the whole class of residually algebraic torsion extensions of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}(X)$, which strictly comprise the valuation domains $\mathbb{Z}_{(p),\alpha}$ arising from elements $\alpha \in \mathbb{C}_p \setminus \overline{\mathbb{Q}_p}$.

As an application of the above results, we are able to complete the classification of the family of Dedekind domains R between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ started in [Peruginelli 2023]. In that paper we described the Dedekind domains of this family whose residue fields of prime characteristic are finite fields [Peruginelli 2023, Theorem 2.17]; the description is obtained by means of the notion of rings of integer-valued polynomials over algebras. We also showed that, given a group G which is the direct sum of a countable family of finitely generated abelian groups, there exists a Dedekind domain R with finite residue fields of prime characteristic, $\mathbb{Z}[X] \subset R \subseteq \mathbb{Q}[X]$, with class group G [Peruginelli 2023, Theorem 3.1].

The paper is organized as follows. In Section 1 we recall the relevant notions we need in our paper: First, we review the definition of a pseudo-convergent sequence of a valued field K and the valuation domain of $K(X)$ associated to such a sequence in the spirit of Ostrowski [1935a; 1935b], as developed recently in [Peruginelli and Spirito 2020; 2021]. Then, we recall the notion of a distinguished pair introduced in [Popescu and Zaharescu 1995], which later was used in [Alexandru et al. 1998] to describe closed subfields of $\mathbb{C}_p$ in terms of a specific kind of pseudo-convergent Cauchy sequence called distinguished.

In Section 2, we introduce the notion of a stacked sequence $E = \{s_n\}_{n\in\mathbb{N}}$ in $\overline{\mathbb{Q}_p}$, which turns out to be a pseudo-convergent sequence of transcendental type such that, for each $n \in \mathbb{N}$, the value group (resp. the residue field) of $\overline{\mathbb{Z}_p} \cap \mathbb{Q}_p(s_n)$ is contained in the value group (resp. the residue field) of $\overline{\mathbb{Z}_p} \cap \mathbb{Q}_p(s_{n+1})$. By Theorem 2.5, every residually algebraic extension W of $\mathbb{Z}_p$ to $\mathbb{Q}_p(X)$ can be realized by means of a stacked sequence $E \subset \overline{\mathbb{Q}_p}$, that is,

$$W = \mathbb{Z}_{p,E} = \{\phi \in \mathbb{Q}_p(X) \mid \phi(s_n) \in \overline{\mathbb{Z}_p} \text{ for all sufficiently large } n \in \mathbb{N}\}.$$

Moreover, the above specific property of stacked sequences is crucial for the description of the residue field and value group of W as the union of the ascending chain of residue fields and value groups of $\overline{\mathbb{Z}_p} \cap \mathbb{Q}_p(s_n)$, respectively (Proposition 2.7). We mentioned above that the elements $\alpha \in \mathbb{C}_p \setminus \overline{\mathbb{Q}_p}$ such that the extension $\mathbb{Q}_p(\alpha)/\mathbb{Q}_p$ has finite ramification give raise to DVRs of $\mathbb{Q}(X)$; we characterize such elements as the limits of sequences contained in the maximal unramified extension of a finite extension of $\mathbb{Q}_p$ (Proposition 2.20). We close this section by pointing out an incorrect statement in [Ioviţă and Zaharescu 1995], namely, that the completion of $\mathbb{Q}_p(X)$ with respect to a residually algebraic torsion extension W of $\mathbb{Z}_p$ is a subfield of $\mathbb{C}_p$; this is not true in general and it depends on whether the above sequence E is Cauchy or not. In Section 2.3, we use the result of Section 2.1 about residually algebraic torsion extensions of $\mathbb{Z}_p$ to $\mathbb{Q}_p(X)$ to characterize the analogous extensions of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}(X)$ (Proposition 2.24). In Theorem 2.26, we show that, for any prescribed algebraic extension k of $\mathbb{F}_p$ and value group Γ, $\mathbb{Z} \subseteq \Gamma \subseteq \mathbb{Q}$, there exists $\alpha \in \mathbb{C}_p$, transcendental over $\mathbb{Q}$, such that $\mathbb{Z}_{(p),\alpha}$ has residue field k and value group Γ.

Finally, in Section 3 we provide the aforementioned classification of the Dedekind domains between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ by means of the notion of the ring of integer-valued polynomials over an algebra: given such a domain R, we show that, for each prime $p \in \mathbb{Z}$, there exists a finite set $E_p \subset \mathbb{C}_p$ of transcendental elements over $\mathbb{Q}$ such that $R = \{f \in \mathbb{Q}[X] \mid f(E_p) \subseteq \mathbb{O}_p, \forall p \in \mathbb{P}\}$ (Theorem 3.4).

1. Preliminaries

We refer to [Bourbaki 1985a; Engler and Prestel 2005; Ribenboim 1968; Zariski and Samuel 1960] for generalities about valuation theory. A valuation domain W of the field of rational functions $K(X)$ is an extension of a valuation domain V of K if $W \cap K = V$. We denote by w a valuation associated to W, by Γ_w the value group of w and by k_w the residue field of W. We recall that an extension W of V to $K(X)$ is called residually algebraic if the residue field extension is algebraic, and it is called torsion if Γ_w is contained in the divisible hull of the value group Γ_v of V; see [Alexandru et al. 1990a]. Given a valuation domain W with quotient field F, a subfield K of F and the valuation domain $V = W \cap K$, we say that W is an immediate extension of V (or simply immediate over V) if the value groups (resp. the residue fields) of V and W are the same. Given a field K with a valuation domain V, we denote by $\widehat{K}$ (resp. $\widehat{V}$) the completion of K (resp. V) with respect to V-adic topology.

1.1. ***Pseudo-convergent sequences.*** The following basic material about pseudo-convergent sequences can be found for example in [Kaplansky 1942; Peruginelli and Spirito 2020; 2021].

Given a valued field (K, v), a sequence $E = \{s_n\}_{n\in\mathbb{N}} \subset K$ is said to be *pseudo-convergent* if, for all $n < m < k$, we have

$$v(s_n - s_m) < v(s_m - s_k).$$

In particular, for all n and $m > n$, we have $v(s_n - s_m) = v(s_n - s_{n+1})$. For each $n \in \mathbb{N}$, we set $\delta_n = v(s_n - s_{n+1})$. The strictly increasing sequence $\{\delta_n\}_{n\in\mathbb{N}}$ of the value group Γ_v of v is called the *gauge* of E. The sequence E is a classical Cauchy sequence in K if and only if the gauge of E is cofinal in Γ_v. In this case, E converges to a unique limit $\alpha \in \widehat{K}$. In general, if $E = \{s_n\}_{n\in\mathbb{N}} \subset K$ is a pseudo-convergent sequence, we say that an element $\alpha \in K$ is a *pseudo-limit* of E if $v(s_n - \alpha)$ is a strictly increasing sequence. Equivalently, $v(s_n - \alpha) = \delta_n$ for each $n \in \mathbb{N}$. The set of pseudo-limits $\mathcal{L}_E$ in K of a pseudo-convergent sequence E is equal to $\mathcal{L}_E = \alpha + \mathrm{Br}(E)$ [Kaplansky 1942, Lemma 3], where

$$\mathrm{Br}(E) = \{x \in K \mid v(x) > \delta_n, \forall n \in \mathbb{N}\}$$

is a fractional ideal, called the *breadth ideal* of E. Clearly, E is a Cauchy sequence if and only if $\mathrm{Br}(E) = \{0\}$.

As in [Kaplansky 1942, Definitions, p. 306], a pseudo-convergent sequence $E = \{s_n\}_{n\in\mathbb{N}} \subset K$ is of *transcendental type* if, for all $f \in K[X]$, $v(f(s_n))$ is eventually constant. Otherwise, E is said to be of *algebraic type* if $v(f(s_n))$ is eventually strictly increasing for some $f \in K[X]$. The sequence E is of algebraic type if and only if, for some extension u of v to the algebraic closure $\overline{K}$ of K, there exists $\alpha \in \overline{K}$ which is a pseudo-limit of E with respect to u. If F is a subfield of K, then we say that E is of *transcendental type over* F if, for all $f \in F[X]$, $v(f(s_n))$ is eventually constant. Almost all the pseudo-convergent sequences considered in this paper in order to describe residually algebraic torsion extensions to the field of rational functions are of transcendental type.

Given a pseudo-convergent sequence $E = \{s_n\}_{n\in\mathbb{N}} \subset K$, the following is a valuation domain of $K(X)$ extending V associated to E [Peruginelli and Spirito 2020, Theorem 3.8]:

$$V_E = \{\phi \in K(X) \mid \phi(s_n) \in V \text{ for all sufficiently large } n \in \mathbb{N}\}.$$

Moreover, by the same theorem, X is a pseudo-limit of E with respect to the valuation v_E associated to V_E, so, in particular, $v_E(X - s_n) = \delta_n$ for every $n \in \mathbb{N}$. Also, if E is of transcendental type, then, for all $f \in K[X]$, we have $v_E(f) = v(f(s_n))$ for all n sufficiently large; see [Kaplansky 1942, Theorem 2] or [Peruginelli and Spirito 2020, Theorem 4.9, (a)].

In the case that E is a Cauchy sequence converging to $\alpha \in \widehat{K}$, we have

$$V_E = V_\alpha = \{\phi \in K(X) \mid \phi(\alpha) \in \widehat{V}\};$$

see [Peruginelli and Spirito 2020, Remark 3.10].

Given two pseudo-convergent sequences $E = \{s_n\}_{n\in\mathbb{N}}$, $E' = \{s'_n\}_{n\in\mathbb{N}} \subset K$, we say that E and E' are equivalent if $\mathrm{Br}(E) = \mathrm{Br}(E')$ and, for each $k \in \mathbb{N}$, there exist $i_0, j_0 \in \mathbb{N}$ such that $v(s_i - s'_j) > v(s_{k+1} - s_k)$ for each $i \geq i_0$ and $j \geq j_0$; see [Peruginelli and Spirito 2020, §5]. By Proposition 5.3 in that work, E and E' are equivalent if and only if $V_E = V_{E'}$.

1.2. ***Distinguished pairs.*** We suppose in this section that (K, v) is a complete valued field, where v is a rank-1 discrete valuation (so, in particular, (K, v) is Henselian). Let $\overline{K}$ be a fixed algebraic closure, and let v denote the unique extension of v to $\overline{K}$. Let also $\Gamma_{\bar{v}} = \Gamma_v \otimes \mathbb{Q}$ be the divisible hull of Γ_v. Given an element $a \in \overline{K}$, let O_a, k_a and Γ_a be the valuation domain of the restriction of v to $K(a)$, the residue field of O_a and the value group of O_a, respectively.

As in [Khanduja and Saha 1999], given $a \in \overline{K} \setminus K$, we set

$$\delta_K(a) = \sup\{v(a - c) \mid c \in \overline{K}, [K(c) : K] < [K(a) : K]\},$$
$$\omega_K(a) = \sup\{v(a - a') \mid a' \neq a \text{ runs over the } K\text{-conjugates of } a\}.$$

The following is the well-known Krasner's lemma. Essentially, given a separable element $a \in \overline{K}$, if another element $b \in \overline{K}$ is closer to a than to any of its other conjugates, then $K(a)$ is a subfield of $K(b)$.

Lemma 1.1 (Krasner). *If $a \in \overline{K}^{\mathrm{sep}}$ and $b \in \overline{K}$ are such that $v(a - b) > \omega_K(a)$, then $K(a) \subseteq K(b)$.*

In particular, for every $a \in \overline{K}^{\mathrm{sep}}$, we have $\delta_K(a) \leq \omega_K(a)$. Moreover, it follows also that $\delta_K(a)$ is a maximum, since v is supposed to be discrete. This is known (see, for example, [Popescu and Zaharescu 1995, p. 105]), but for the sake of the reader we give a short proof.

Lemma 1.2. *In the above setting,*

$$\delta_K(a) = \max\{v(a - c) \mid c \in \overline{K}, [K(c) : K] < [K(a) : K]\}.$$

Proof. By Krasner's lemma, for each of the relevant c we have $v(a-c) \leq \omega_K(a)$. Note that the ramification index of $K(a, c)$ over K is (strictly) bounded by $[K(a) : K]^2$. In particular, since the value $v(a - c)$ belongs to Γ_{a-c}, it follows that there exists $N \in \mathbb{N}$, independent from each of the above c, such that $N v(a - c) \in \Gamma_v \cong \mathbb{Z}$. Hence the set

$$\{v(a - c) \mid c \in \overline{K}, [K(c) : K] < [K(a) : K]\}$$

(which is a subset of $\Gamma_{\bar{v}}$) is bounded from above and its elements have bounded torsion. It follows that this set has a maximum, which is equal to $\delta_K(a)$ by its very definition. □

Similar to Krasner's lemma, we have the following fundamental principle (see [Khanduja and Saha 1999, Theorem 1.1]), first discovered in [Popescu and Zaharescu 1995].

Theorem 1.3. *Suppose that $a, b \in \overline{K}$ are such that $v(a-b) > \delta_K(b)$. Then*:

(i) $\Gamma_b \subseteq \Gamma_a$.

(ii) $k_b \subseteq k_a$.

(iii) $[K(b):K] \mid [K(a):K]$.

Next, we recall the definition of a distinguished pair introduced in [Popescu and Zaharescu 1995, p. 105].

Definition 1.4. A pair of elements $(b, a) \in \overline{K}^2$ is said to be *distinguished* if the following hold:

(i) $[K(b):K] < [K(a):K]$.

(ii) For all $c \in \overline{K}$ such that $[K(c):K] < [K(a):K]$, we have $v(a-c) \leq v(a-b)$.

(iii) For all $c \in \overline{K}$ such that $[K(c):K] < [K(b):K]$, we have $v(a-c) < v(a-b)$.

Part of the definition of a distinguished pair is related to the notion of a minimal pair, which we now recall (see, for example, [Alexandru et al. 1988; 1990a; 1990b]).

Definition 1.5. Let $(a, \delta) \in \overline{K} \times \Gamma_{\bar{v}}$. We say that (a, δ) is a *minimal pair* if, for every $c \in \overline{K}$ such that $[K(c):K] < [K(a):K]$, we have $v(a-c) < \delta$.

In other words, (a, δ) is a minimal pair if, for every $b \in B(a, \delta) = \{x \in \overline{K} \mid v(a-x) \geq \delta\}$, we have $[K(b):K] \geq [K(a):K]$ (i.e., a is a "center" of the ball $B(a, \delta)$ of minimal degree). By Lemma 1.2, (a, δ) is a minimal pair if and only if $\delta > \delta_K(a)$. In particular, if $\delta > \omega_K(a)$, then (a, δ) is a minimal pair.

Remarks 1.6. Let (b, a) be a distinguished pair.

(1) Note that conditions (i) and (ii) above imply that $v(a-b) = \delta_K(a)$. In fact, by (i) and (ii), it immediately follows that the inequality "$\leq$" holds. Conversely, by (ii) we also have that $v(a-b) \geq v(a-c)$ for all c such that $[K(c):K] < [K(a):K]$; that is, $v(a-b) \geq \delta_K(a)$.

(2) Note that (iii) is equivalent to the following:

(iii′) For all $c \in \overline{K}$ such that $[K(c):K] < [K(b):K]$, we have $v(b-c) < v(a-b)$.

This precisely says that $(b, v(a-b))$ is a minimal pair with respect to K. In fact, if (iii) holds and $c \in \overline{K}$ is such that $[K(c):K] < [K(b):K]$, then $v(b-c) = v(b-a+a-c) = v(a-c) < v(a-b)$. Similarly, (iii′) implies (iii). Note also that (iii′) is equivalent to

$$v(a-b) > \delta_K(b).$$

In particular, by the above theorem, $\Gamma_b \subseteq \Gamma_a$, $k_b \subseteq k_a$ and $[K(b):K] \mid [K(a):K]$.

(3) Finally, note also that $\delta_K(b) < \delta_K(a)$.

2. Stacked pseudo-convergent sequences of $\overline{\mathbb{Q}_p}$

Let $\mathbb{P} \subset \mathbb{Z}$ be the set of prime numbers, and let $p \in \mathbb{P}$ be a fixed prime. We let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$, $\mathbb{Z}_p$ the ring of p-adic integers and $\mathbb{Q}_p$ its field of fractions, the field of p-adic numbers. If v_p denotes the usual p-adic valuation, then $\mathbb{Z}_p$ (resp. $\mathbb{Q}_p$) is the completion of $\mathbb{Z}$ (resp. $\mathbb{Q}$) with respect to the p-adic valuation. We denote by $\overline{\mathbb{Q}_p}$ a fixed algebraic closure of $\mathbb{Q}_p$ and still denote the unique extension of v_p to $\overline{\mathbb{Q}_p}$ by v_p. Note that $\overline{\mathbb{Q}_p}$ is a rank-1 nondiscrete valued field with valuation domain denoted by $\overline{\mathbb{Z}_p}$, the integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}_p}$. We will use the well-known fact that $\mathbb{Q}_p$ has only finitely many extensions of a given degree; see, for example, [Narkiewicz 2004, Corollary 2, Chapter V, p. 202].

Finally, we let $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}_p}$ with respect to the p-adic valuation, and we denote by $\mathbb{O}_p$ the completion of $\overline{\mathbb{Z}_p}$; v_p still denotes the unique extension of v_p to $\mathbb{C}_p$. For $\alpha \in \overline{\mathbb{Q}_p} \setminus \mathbb{Q}_p$, we write the abbreviations $\delta_{\mathbb{Q}_p}(\alpha) = \delta(\alpha)$ and $\omega_{\mathbb{Q}_p}(\alpha) = \omega(\alpha)$. For $\alpha \in \mathbb{C}_p$, we denote by e_α (resp. f_α) the ramification index (resp. the residue field degree) of $\mathbb{Q}_p(\alpha)$ over $\mathbb{Q}_p$. Clearly, if $\alpha \in \overline{\mathbb{Q}_p}$, then $e_\alpha \cdot f_\alpha < \infty$; we show that the converse holds in Remark 2.15. Note that each element of $\mathbb{C}_p \setminus \overline{\mathbb{Q}_p}$ is transcendental over $\mathbb{Q}_p$; we call such elements simply transcendental. For a transcendental element $\alpha \in \mathbb{C}_p$, even if $e_\alpha \cdot f_\alpha = \infty$, we will show in Theorem 2.21 that either one of e_α or f_α can be finite.

2.1. ***Residually algebraic torsion extensions of $\mathbb{Z}_p$.*** In this section we describe residually algebraic torsion extensions of $\mathbb{Z}_p$ to $\mathbb{Q}_p(X)$ by means of a suitable class of pseudo-convergent sequences of transcendental type contained in $\overline{\mathbb{Q}_p}$, called a stacked sequence, which we now introduce. This definition is a generalization of [Alexandru et al. 1998, p. 135].[1]

Definition 2.1. Let $E = \{s_n\}_{n\geq 0} \subset \overline{\mathbb{Q}_p}$ be a sequence with $s_0 \in \mathbb{Q}_p$. For every $n \geq 0$, we consider the following properties:

(i) $[\mathbb{Q}_p(s_n) : \mathbb{Q}_p] < [\mathbb{Q}_p(s_{n+1}) : \mathbb{Q}_p]$.

(ii) For every $c \in \overline{\mathbb{Q}_p}$ such that $[\mathbb{Q}_p(c) : \mathbb{Q}_p] < [\mathbb{Q}_p(s_{n+1}) : \mathbb{Q}_p]$, we have $v(s_{n+1} - c) \leq v(s_{n+1} - s_n)$.

(iii) For every $c \in \overline{\mathbb{Q}_p}$ such that $[\mathbb{Q}_p(c) : \mathbb{Q}_p] < [\mathbb{Q}_p(s_n) : \mathbb{Q}_p]$, we have $v(s_n - c) < v(s_{n+1} - s_n)$.

We say that E is *unbounded* if (i) holds for every n, *stacked* if (i) and (iii) hold for every n, and *strongly stacked* if (i), (ii), (iii) hold for every n. Equivalently, E is stacked if (i) holds and $(s_n, \delta_n = v(s_{n+1} - s_n))$ is a minimal pair for every $n \geq 0$, and E is strongly stacked if (s_n, s_{n+1}) is distinguished for every $n \geq 0$.

Remark 2.2. Let $E = \{s_n\}_{n\in\mathbb{N}} \subset \overline{\mathbb{Q}_p}$ be a stacked sequence. Note that the sequence $\{v(s_{n+1} - s_n) = \delta_n\}_{n\in\mathbb{N}}$ is strictly increasing since $[\mathbb{Q}_p(s_{n-1}) : \mathbb{Q}_p] < [\mathbb{Q}_p(s_n) : \mathbb{Q}_p]$ and (s_n, δ_n) is a minimal pair. In the original definition of a distinguished sequence E in [Alexandru et al. 1998], the sequence $\{\delta_n\}_{n\in\mathbb{N}}$ is unbounded; thus, in this case E is a Cauchy sequence. In our setting we are not imposing that restriction; we show in Lemma 2.3 below that a stacked sequence is a pseudo-convergent sequence of transcendental type of $\overline{\mathbb{Q}_p}$.

[1]The notion of a distinguished sequence was introduced in [Alexandru et al. 1998]. We cannot borrow that term here for our sequences for the following reason: by Lemma 2.3, a stacked sequence is pseudo-convergent, and distinguished pseudo-convergent sequences have already been defined by P. Ribenboim [1958, p. 474] to denote pseudo-convergent sequences of a valued field whose breadth ideal is a nonmaximal prime ideal.

The motivation for the terminology of these kind of sequences is due to the following fact. For each $n \in \mathbb{N}$, we abbreviate $\Gamma_n = \Gamma_{s_n}$ and $k_n = k_{s_n}$ (i.e., the value group and the residue field of the valuation domain O_{s_n} of $\mathbb{Q}_p(s_n)$, respectively). By Remarks 1.6, $v(s_{n+1} - s_n) > \delta(s_n)$. Hence, by Theorem 1.3, we have $\Gamma_n \subseteq \Gamma_{n+1}$ and $k_n \subseteq k_{n+1}$. For each $n \in \mathbb{N}$, we set $e_n = e(\mathbb{Q}_p(s_n)\,|\,\mathbb{Q}_p)$ and $f_n = f(\mathbb{Q}_p(s_n)\,|\,\mathbb{Q}_p)$, the ramification index and the residue field degree of O_{s_n} over $\mathbb{Z}_p$, respectively; we remark that $[\mathbb{Q}_p(s_n) : \mathbb{Q}_p] = e_n f_n = d_n$ for each $n \in \mathbb{N}$, and since $\{d_n\}_{n\in\mathbb{N}}$ is unbounded by assumption, either $\{e_n\}_{n\in\mathbb{N}}$ is unbounded or $\{f_n\}_{n\in\mathbb{N}}$ is unbounded. Since $e_n\,|\,e_{n+1}$ for each $n \in \mathbb{N}$, $\{e_n\}_{n\in\mathbb{N}}$ is bounded if and only if $e_n = e$ for all $n \in \mathbb{N}$ sufficiently large. Similarly for $\{f_n\}_{n\in\mathbb{N}}$.

By Remarks 1.6, condition (ii) is equivalent to $\delta_n = v(s_n - s_{n+1}) = \delta(s_{n+1})$ (note that in general the inequality $\delta_n \le \delta(s_{n+1})$ holds). In other words, among all the elements $c \in \overline{\mathbb{Q}_p}$ such that

$$[\mathbb{Q}_p(s_n) : \mathbb{Q}_p] \le [\mathbb{Q}_p(c) : \mathbb{Q}_p] < [\mathbb{Q}_p(s_{n+1}) : \mathbb{Q}_p],$$

s_n is one of those which is closest to s_{n+1}.

Let $E = \{t_n\}_{n\in\mathbb{N}} \subset \overline{\mathbb{Q}_p}$ be a pseudo-convergent sequence. If $\{[\mathbb{Q}_p(t_n) : \mathbb{Q}_p] \mid n \in \mathbb{N}\}$ is bounded, then E is contained in a finite extension K of $\mathbb{Q}_p$, and hence E is Cauchy and therefore converges to an element $\alpha \in K$. In particular, if E is of transcendental type, then the set $\{[\mathbb{Q}_p(t_n) : \mathbb{Q}_p]\}_{n\in\mathbb{N}}$ is necessarily unbounded. Stacked sequences are of this kind, as the next lemma shows.

Lemma 2.3. *Let $E \subset \overline{\mathbb{Q}_p}$ be a stacked sequence. Then E is a pseudo-convergent sequence of transcendental type.*

Proof. Let $E = \{s_n\}_{n\in\mathbb{N}}$, and set $\delta_n = v(s_{n+1} - s_n)$ for each $n \in \mathbb{N}$. We have already observed in Remark 2.2 that $\{\delta_n\}_{n\in\mathbb{N}}$ is a strictly increasing sequence. Moreover, for every $m > n$, we have $v(s_n - s_m) > v(s_n - s_{n-1})$. In particular, $v(s_{n-1} - s_m) = v(s_{n-1} - s_n)$ for every $m \ge n$. Let now $n < m < k$. Then

$$v(s_n - s_m) = v(s_n - s_{n+1}) < v(s_m - s_{m+1}) = v(s_m - s_k),$$

which shows that E is a pseudo-convergent sequence.

We prove now that E is of transcendental type. Let $\alpha \in \overline{\mathbb{Q}_p}$. Then there exists $n \in \mathbb{N}$ such that

$$[\mathbb{Q}_p(\alpha) : \mathbb{Q}_p] < [\mathbb{Q}_p(s_n) : \mathbb{Q}_p].$$

Since (s_n, δ_n) is a minimal pair, $v(s_n - \alpha) < \delta_n$, so, in particular, α cannot be a pseudo-limit of E. This shows that E has no pseudo-limits in $\overline{\mathbb{Q}_p}$, and thus E is of transcendental type. □

Let $E = \{s_n\}_{n\in\mathbb{N}} \subset \overline{\mathbb{Q}_p}$ be a stacked sequence. In particular, by Lemma 2.3, the sequence

$$\{\delta_n = v(s_{n+1} - s_n)\}_{n\in\mathbb{N}}$$

is the gauge of the pseudo-convergent sequence E. Moreover, by the same lemma, if E is Cauchy, then E converges to a transcendental element $\alpha \in \mathbb{C}_p$.

The next proposition shows that any residually algebraic torsion extension of $\mathbb{Z}_p$ to $\mathbb{Q}_p(X)$ is obtained by means of a pseudo-convergent sequence of transcendental type of $\overline{\mathbb{Q}_p}$. We recall that if $E \subset \overline{\mathbb{Q}_p}$ is a pseudo-convergent sequence of transcendental type, then $\overline{\mathbb{Z}_p}_E$, the associated valuation domain of $\overline{\mathbb{Q}_p}(X)$, is an immediate extension of $\overline{\mathbb{Z}_p}$ and conversely every immediate extension of $\overline{\mathbb{Z}_p}$ to $\overline{\mathbb{Q}_p}(X)$ can be realized in this way; see, for example, [Kaplansky 1942; Peruginelli and Spirito 2021]. If $\mathbb{Z}_{p,E} = \overline{\mathbb{Z}_p}_E \cap \mathbb{Q}_p(X)$, then $\mathbb{Z}_{p,E}$ is a residually algebraic torsion extension of $\mathbb{Z}_p$ to $\mathbb{Q}_p(X)$.

Proposition 2.4. *Let W be a residually algebraic torsion extension of $\mathbb{Z}_p$ to $\mathbb{Q}_p(X)$. Then there exists a pseudo-convergent sequence $E \subset \overline{\mathbb{Q}_p}$ of transcendental type such that*

$$W = \mathbb{Z}_{p,E} = \{\phi \in \mathbb{Q}_p(X) \mid \phi(s_n) \in \overline{\mathbb{Z}_p} \textit{ for all sufficiently large } n \in \mathbb{N}\}.$$

Proof. Let $\overline{W}$ be an extension of W to $\overline{\mathbb{Q}_p}(X)$. Then $\overline{W}$ is an immediate extension of $\overline{\mathbb{Z}_p}$ to $\overline{\mathbb{Q}_p}(X)$ (and, in particular, is a residually algebraic torsion extension of $\overline{\mathbb{Z}_p}$). By [Kaplansky 1942, Theorems 1 and 2] or [Peruginelli and Spirito 2021, Theorem 6.2 (a)], there exists a pseudo-convergent sequence $E \subset \overline{\mathbb{Q}_p}$ of transcendental type such that $\overline{W} = \overline{\mathbb{Z}_p}_E$. The claim follows by contracting down to $\mathbb{Q}_p(X)$. □

Clearly, not every pseudo-convergent sequence of transcendental type in $\overline{\mathbb{Q}_p}$ is stacked. However, the next theorem is the converse of Lemma 2.3: it shows that any pseudo-convergent sequence of transcendental type is equivalent to a strongly stacked sequence. In particular, every stacked sequence is equivalent to a strongly stacked sequence. Moreover, given a valuation domain $\mathbb{Z}_{p,E}$ of $\mathbb{Q}_p(X)$ associated to a pseudo-convergent sequence $E \subset \overline{\mathbb{Q}_p}$ of transcendental type, without loss of generality, we may also assume that E is strongly stacked.

By [Alexandru et al. 1998, Proposition 2.2], every transcendental element $t \in \mathbb{C}_p$ is the limit of a strongly stacked sequence E of $\overline{\mathbb{Q}_p}$. The next theorem is the analog of that result for residually algebraic extensions W of $\mathbb{Z}_p$ to $\mathbb{Q}_p(X)$: for such a valuation W, there exists a strongly stacked sequence $E \subset \overline{\mathbb{Q}_p}$ such that $W = \mathbb{Z}_{p,E}$; it is not difficult to show that, for a transcendental element $t \in \mathbb{C}_p$, the valuation domain

$$\mathbb{Z}_{p,t} = \{\phi \in \mathbb{Q}_p(X) \mid \phi(t) \in \mathbb{O}_p\}$$

is a residually algebraic torsion extension of $\mathbb{Z}_p$.

Theorem 2.5. *Let $E \subset \overline{\mathbb{Q}_p}$ be a pseudo-convergent sequence of transcendental type. Then there exists a strongly stacked sequence $E' \subset \overline{\mathbb{Q}_p}$ which is equivalent to E. In particular, given a residually algebraic torsion extension W of $\mathbb{Z}_p$ to $\mathbb{Q}_p(X)$, there exists a strongly stacked sequence $E' \subset \overline{\mathbb{Q}_p}$ such that $W = \mathbb{Z}_{p,E'}$.*

Proof. Let $E = \{t_n\}_{n\in\mathbb{N}}$, and let $\bar{v}_E$ be a valuation associated to $\overline{\mathbb{Z}_p}_E \subset \overline{\mathbb{Q}_p}(X)$.

First, we consider the following subset of $\Gamma_{v_E} \subseteq \mathbb{Q}$:

$$M_E(X, \mathbb{Q}_p) = \{v_E(X - s) \mid s \in \mathbb{Q}_p\}.$$

If $M_E(X, \mathbb{Q}_p)$ is not bounded, then there exists a sequence $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{Q}_p$ such that $v_E(X - s_n)$ tends to ∞. Necessarily, the sequence $\{s_n\}_{n\in\mathbb{N}}$ is Cauchy and so converges to an element s of $\mathbb{Q}_p$. Now, for every n, $v_E(X - s_n) = \bar{v}_E(X - s_n) = v(t_m - s_n)$ for all m sufficiently large since E is of transcendental type (see Section 1.1). Hence E would be a Cauchy sequence equivalent to $\{s_n\}_{n\in\mathbb{N}}$ and E would converge to s, too, which is not possible. Let then $\delta_0 = \sup M_E(X, \mathbb{Q}_p) \in \mathbb{R}$. We claim that $\delta_0 \in M_E(X, \mathbb{Q}_p)$; that is, δ_0 is a maximum. Suppose otherwise: there exists a sequence $\{r_k\}_{k\in\mathbb{N}} \subset \mathbb{Q}_p$ such that $v_E(X - r_k) \nearrow \delta_0$. Then $\{r_k\}_{k\in\mathbb{N}} \subset \mathbb{Q}_p$ would be a pseudo-convergent sequence which is not Cauchy, which is not possible, since $\mathbb{Q}_p$ is a complete valued field. Hence there exists $s_0 \in \mathbb{Q}_p$ such that $v_E(X - s_0) = \delta_0$.

For $n > 0$, we now choose $s_n \in \overline{\mathbb{Q}_p}$ so that (s_{n-1}, s_n) is distinguished. Let B_n be the subset of the α in $\overline{\mathbb{Q}_p}$ satisfying the following properties:

(i) $[\mathbb{Q}_p(\alpha) : \mathbb{Q}_p] > [\mathbb{Q}_p(s_{n-1}) : \mathbb{Q}_p]$.

(ii) $\bar{v}_E(X - \alpha) > \bar{v}_E(X - s_{n-1})$.

(iii) The positive integer $[\mathbb{Q}_p(\alpha) : \mathbb{Q}_p] - [\mathbb{Q}_p(s_{n-1}) : \mathbb{Q}_p]$ is minimal.

Note that since $\mathbb{N}$ is well-ordered, condition (iii) can be satisfied (that is, among the $\alpha \in \overline{\mathbb{Q}_p}$ satisfying (i) and (ii), we can find one which also satisfies (iii)). Since $\bar{v}_E(X - s_{n-1}) = v(t_m - s_{n-1})$ for all m sufficiently large, for all such m we also have $\bar{v}_E(X - s_{n-1}) < \bar{v}_E(X - t_m)$. Moreover, without loss of generality, we may also assume that $[\mathbb{Q}_p(t_m) : \mathbb{Q}_p] > [\mathbb{Q}_p(s_{n-1}) : \mathbb{Q}_p]$ since $\{[\mathbb{Q}_p(t_m) : \mathbb{Q}_p]\}_{m\in\mathbb{N}}$ is unbounded. This shows that the set B_n is nonempty. Let

$$M_E(X, B_n) = \{\bar{v}_E(X - \alpha) \mid \alpha \in B_n\},$$

which is a subset of $\mathbb{Q}$. Let $\delta_n = \sup M_E(X, B_n)$. Since each element of B_n has the same degree over $\mathbb{Q}_p$, it follows that B_n is contained in a finite extension K of $\mathbb{Q}_p$. In particular, it follows as above that $M_E(X, B_n)$ is bounded above. Let $\delta_n = \sup M_E(X, B_n) \in \mathbb{R}$. Next, we show that $M_E(X, B_n)$ contains its upper bound (which is, in particular, a rational number). Suppose otherwise: then there exists a sequence $\{\alpha_k\}_{k\in\mathbb{N}} \subset B_n$ such that $\bar{v}_E(X - \alpha_k) \nearrow \delta_n$. In particular, $\{\alpha_k\}_{k\in\mathbb{N}}$ would be a pseudo-convergent sequence of a finite extension of $\mathbb{Q}_p$ which is not Cauchy, which is impossible. Let $s_n \in B_n$ be such that $\bar{v}_E(X - s_n) = \delta_n$. Note that

$$v_p(s_n - s_{n-1}) = \bar{v}_E(s_n - X + X - s_{n-1}) = \bar{v}_E(X - s_{n-1}) = \delta_{n-1}.$$

We now show that (s_{n-1}, s_n) is distinguished. Clearly, $[\mathbb{Q}_p(s_{n-1}) : \mathbb{Q}_p] < [\mathbb{Q}_p(s_n) : \mathbb{Q}_p]$.

Let $c \in \overline{\mathbb{Q}_p}$ be such that $[\mathbb{Q}_p(c) : \mathbb{Q}_p] < [\mathbb{Q}_p(s_n) : \mathbb{Q}_p]$.

If $[\mathbb{Q}_p(c) : \mathbb{Q}_p] > [\mathbb{Q}_p(s_{n-1}) : \mathbb{Q}_p]$, then, by the minimality of the degree of s_n, we have

$$\bar{v}_E(X - c) \leq \bar{v}_E(X - s_{n-1}) = \delta_{n-1},$$

so

$$v_p(s_n - c) = \bar{v}_E(s_n - X + X - c) = \bar{v}_E(X - c) \leq \delta_{n-1} = v_p(s_n - s_{n-1}).$$

Suppose now that $[\mathbb{Q}_p(c) : \mathbb{Q}_p] = [\mathbb{Q}_p(s_{n-1}) : \mathbb{Q}_p]$.

If $\bar{v}_E(X - c) \leq \bar{v}_E(X - s_{n-2})$, then $\bar{v}_E(X - c) < \delta_{n-1}$.

If $\bar{v}_E(X - c) > \bar{v}_E(X - s_{n-2})$, then $c \in B_{n-1}$, so

$$\bar{v}_E(X - c) \leq \delta_{n-1} = \bar{v}_E(X - s_{n-1}).$$

In either case,

$$v_p(s_n - c) = \bar{v}_E(s_n - X + X - c) = \bar{v}_E(X - c) \leq \delta_{n-1} = v_p(s_n - s_{n-1}).$$

Note that, in particular, for $n = 1$, we have that (s_0, s_1) is distinguished since condition (iii) of Definition 2.1 is empty, since $s_0 \in \mathbb{Q}_p$.

Suppose now that $n \geq 2$, and assume by induction that (s_{n-2}, s_{n-1}) is distinguished. Let $c \in \overline{\mathbb{Q}_p}$ be such that $[\mathbb{Q}_p(c) : \mathbb{Q}_p] < [\mathbb{Q}_p(s_{n-1}) : \mathbb{Q}_p]$. Since (s_{n-2}, s_{n-1}) is distinguished, we have

$$v_p(s_{n-1} - c) \leq v_p(s_{n-1} - s_{n-2}) = \delta_{n-2} < \delta_{n-1}.$$

Hence

$$v_p(s_n - c) = v_p(s_n - s_{n-1} + s_{n-1} - c) = v_p(s_{n-1} - c) < v_p(s_n - s_{n-1}).$$

We now show that $E' = \{s_n\}_{n\in\mathbb{N}}$ is equivalent to $E = \{t_n\}_{n\in\mathbb{N}}$. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\delta_n\}_{n\in\mathbb{N}}$ be the gauges of E and E', respectively. We need to show that, for each $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $\lambda_k \leq \delta_n$. Since E' is unbounded, there exists $n \in \mathbb{N}$ such that $[\mathbb{Q}_p(t_k) : \mathbb{Q}_p] < [\mathbb{Q}_p(s_n) : \mathbb{Q}_p]$. Since (s_n, δ_n) is a minimal pair, we have $v_p(s_n - t_k) < \delta_n$, so that

$$\lambda_k = \bar{v}_E(X - t_k) = \bar{v}_E(X - s_n + s_n - t_k) < \bar{v}_E(X - s_n) = \delta_n. \tag{2.6}$$

Conversely, let $n \in \mathbb{N}$. We need to show that there exists $k \in \mathbb{N}$ such that $\delta_n \leq \lambda_k$. For all m sufficiently large, we have

$$\bar{v}_E(X - s_n) = v_p(t_m - s_n) = \bar{v}_E(t_m - X + X - s_n),$$

and since n is fixed and $\bar{v}_E(t_m - X) = \lambda_m$ is strictly increasing, it follows that

$$\bar{v}_E(t_m - X) = \lambda_m > \bar{v}_E(X - s_n)$$

for all such m.

Hence $\mathrm{Br}(E) = \mathrm{Br}(E')$.

Finally, we need to show that, if $k \in \mathbb{N}$, then there exist $n_0, m_0 \in \mathbb{N}$ such that, for each $n \geq n_0$ and $m \geq m_0$, we have $v_p(t_n - s_m) > \lambda_k$. Let n_0 be the smallest integer such that $[\mathbb{Q}_p(t_k) : \mathbb{Q}_p] < [\mathbb{Q}_p(s_{n_0}) : \mathbb{Q}_p]$. As in (2.6) above, $\lambda_k < v_E(X - s_{n_0}) = \delta_{n_0}$. Let now $m > k$ and $n \geq n_0$. Then,

$$v(t_m - s_n) = \bar{v}_E(t_m - X + X - s_n) > \lambda_k$$

since

$$\bar{v}_E(t_m - X) = \lambda_m > \lambda_k \quad \text{and} \quad \bar{v}_E(X - s_n) \geq \bar{v}_E(X - s_{n_0}) = \delta_{n_0} > \lambda_k.$$

Hence E and E' are equivalent.

By [Peruginelli and Spirito 2021, Proposition 5.3], $\overline{\mathbb{Z}_p}_E = \overline{\mathbb{Z}_p}_{E'}$, so, in particular, $\mathbb{Z}_{p,E} = \mathbb{Z}_{p,E'}$. The final claim follows by Proposition 2.4. □

The following proposition describes the value group and the residue field of a residually algebraic torsion extension W of $\mathbb{Z}_p$ to $\mathbb{Q}_p(X)$. By Theorem 2.5, W is equal to $\mathbb{Z}_{p,E}$ for some strongly stacked sequence $E \subset \overline{\mathbb{Q}_p}$. We keep the notation of Remark 2.2.

Proposition 2.7. *Let $E = \{s_n\}_{n\in\mathbb{N}} \subset \overline{\mathbb{Q}_p}$ be a stacked sequence and $W = \mathbb{Z}_{p,E}$. Then we have*

$$\bigcup_{n\in\mathbb{N}} \Gamma_n = \Gamma_w, \quad \bigcup_{n\in\mathbb{N}} k_n = k_w.$$

Proof. Let $w = v_E$ be the valuation associated to $\mathbb{Z}_{p,E}$ and $\bar{v}_E$ the valuation associated to $\overline{\mathbb{Z}_p}_E$.

Since E is of transcendental type, for each $f \in \mathbb{Q}_p[X]$, we have $v_E(f) = v(f(s_n))$ for all n sufficiently large (see Section 1.1). It follows that, for each $\phi \in \mathbb{Q}_p(X)$ with $\phi = f/g$, for some $f, g \in \mathbb{Q}_p[X]$, we have that $v_E(\phi) = v_E(f) - v_E(g)$ is in Γ_n for all n sufficiently large. Hence $\Gamma_w \subseteq \bigcup_n \Gamma_n$. Conversely, let $n \in \mathbb{N}$ and $f \in \mathbb{Q}_p[X]$ be of degree smaller than $[\mathbb{Q}_p(s_n) : \mathbb{Q}_p]$. Then, each root α_i of $f(X)$ in $\overline{\mathbb{Q}_p}$ has degree smaller than $[\mathbb{Q}_p(s_n) : \mathbb{Q}_p]$ and so, since (s_n, δ_n) is a minimal pair, we have

$$v_p(s_n - \alpha_i) < \delta_n, \tag{2.8}$$

which implies that

$$\bar{v}_E(X - \alpha_i) = \bar{v}_E(X - s_n + s_n - \alpha_i) = v_p(s_n - \alpha_i), \tag{2.9}$$

and so

$$v_E(f(X)) = \sum_i \bar{v}_E(X - \alpha_i) = \sum_i v_p(s_n - \alpha_i) = v_p(f(s_n)), \tag{2.10}$$

which shows that $\Gamma_n \subseteq \Gamma_w$. Note that $\bar{v}_E(X - \alpha_i) = v(s_m - \alpha_i)$ for each $m \geq n$, and so $v_E(f(X)) = v(f(s_m))$ for each $m \geq n$.

Let now $n \in \mathbb{N}$ and $\bar{c} = \overline{f(s_n)} \in k_n$ for some $f(s_n) \in O_n^*$, where $f \in \mathbb{Q}_p[X]$ has degree strictly smaller than $[\mathbb{Q}_p(s_n) : \mathbb{Q}_p]$. In particular, $\bar{c} \neq 0$. As in (2.10), $v_E(f(X)) = v(f(s_m)) = 0$ for each $m \geq n$. Let α_i be a root of $f(X)$ in $\overline{\mathbb{Q}_p}$. Then, by (2.9), $\bar{v}_E(X - \alpha_i) = v_p(s_n - \alpha_i) = v_p(d_i)$ for some $d_i \in \overline{\mathbb{Q}_p}$. Then

$$\bar{v}_E\left(\frac{(X - \alpha_i)/d_i}{(s_n - \alpha_i)/d_i} - 1\right) = \bar{v}_E\left(\frac{X - s_n}{s_n - \alpha_i}\right) = \delta_n - v_p(s_n - \alpha_i) > 0,$$

where the last inequality holds by (2.8). Therefore, $(X - \alpha_i)/d_i$ and $(s_n - \alpha_i)/d_i$ coincide over the residue field of W. In particular,

$$\frac{f(X)}{f(s_n)} = \prod_i \frac{(X - \alpha_i)}{(s_n - \alpha_i)} = \prod_i \frac{(X - \alpha_i)/d_i}{(s_n - \alpha_i)/d_i}, \tag{2.11}$$

and since each factor of the last product has residue $\bar{1}$ in W, it follows that $f(X)$ and $f(s_n)$ coincide over the residue field of W (which contains both $f(X)$ and $f(s_n)$). Since $f \in \mathbb{Z}_{p,E} = W$, this shows that k_n is contained in the residue field k_w of W.

Conversely, let $\phi = f/g \in W \subset \mathbb{Q}_p(X)$ for some $f, g \in \mathbb{Q}_p[X]$. Let α_i and β_j be the roots in $\overline{\mathbb{Q}_p}$ of f and g, respectively. There exists $n \in \mathbb{N}$ such that $[\mathbb{Q}_p(\alpha_i) : \mathbb{Q}_p] < [\mathbb{Q}_p(s_n) : \mathbb{Q}_p]$ and $[\mathbb{Q}_p(\beta_j) : \mathbb{Q}_p] < [\mathbb{Q}_p(s_n) : \mathbb{Q}_p]$ for all i and j. Hence, as in (2.9), we have

$$\bar{v}_E(X - \alpha_i) = v_p(s_n - \alpha_i), \quad \bar{v}_E(X - \beta_j) = v(s_n - \beta_j) \quad \text{for all } i, j,$$

which again, as in (2.10), shows that

$$v_E(\phi(X)) = v(\phi(s_n)).$$

Moreover, this last equation holds if we replace s_n by s_m for all $m \geq n$. If $v_E(\phi(X)) = 0$, then, as in (2.11), one can show that $\phi(X)$ and $\phi(s_n)$ coincide over the residue field of W, so that $k_w \subseteq k_n$. □

The following corollary gives a further characterization of the residue field and the value group of a residually algebraic torsion extension W of $\mathbb{Z}_p$: either the residue field of W is an infinite algebraic extension of $\mathbb{F}_p$, or the value group Γ_w is nondiscrete.

Corollary 2.12. *Let W be a residually algebraic torsion extension of $\mathbb{Z}_p$ to $\mathbb{Q}_p(X)$, and let $e = e(W|\mathbb{Z}_p)$ and $f = f(W|\mathbb{Z}_p)$ be the ramification index and the residue field degree, respectively. Then $e \cdot f = \infty$.*

Proof. By Theorem 2.5, there exists a stacked sequence $E = \{s_n\}_{n\in\mathbb{N}} \subset \overline{\mathbb{Q}_p}$ such that $W = \mathbb{Z}_{p,E}$. By Proposition 2.7, $\Gamma_w = \bigcup_n \Gamma_n$ and $k_w = \bigcup_n k_n$. Remark 2.2 shows that either the sequence $\{e_n = [\Gamma_n : \mathbb{Z}]\}_{n\in\mathbb{N}}$ or $\{f_n = [k_n : \mathbb{F}_p]\}_{n\in\mathbb{N}}$ is unbounded; therefore, either $e = e(W|\mathbb{Z}_p)$ or $f = f(W|\mathbb{Z}_p)$ is infinite. □

The following proposition is analogous to [Alexandru et al. 1998, Proposition 2.3]. It shows that the sequence of ramification indexes, residue field degrees and gauges attached to a residually algebraic torsion extension W of $\mathbb{Z}_p$ do not depend on the strongly stacked sequence $E \subset \overline{\mathbb{Q}_p}$ such that $W = \mathbb{Z}_{p,E}$ (Theorem 2.5).

Proposition 2.13. *Let $W \subset \mathbb{Q}_p(X)$ be a residually algebraic torsion extension of $\mathbb{Z}_p$. Let $E = \{s_n\}_{n\in\mathbb{N}}$, $E' = \{t_n\}_{n\in\mathbb{N}} \subset \overline{\mathbb{Q}_p}$ be strongly stacked sequences with gauges $\{\delta_n\}_{n\in\mathbb{N}}$, $\{\delta'_n\}_{n\in\mathbb{N}}$, respectively, such that $W = \mathbb{Z}_{p,E} = \mathbb{Z}_{p,E'}$. Then, for each $n \in \mathbb{N}$, we have*

(i) $[\mathbb{Q}_p(s_n) : \mathbb{Q}_p] = [\mathbb{Q}_p(t_n) : \mathbb{Q}_p]$ *and* $\delta_n = \delta'_n$,

(ii) $e_{s_n} = e_{t_n}$ *and* $f_{s_n} = f_{t_n}$.

Proof. Without loss of generality, we may assume that in $\overline{\mathbb{Q}_p}(X)$ we have $\overline{\mathbb{Z}_p}_E = \overline{\mathbb{Z}_p}_{E'}$; we will let $\overline{W} = \overline{\mathbb{Z}_p}_E = \overline{\mathbb{Z}_p}_{E'}$ and denote by w a valuation associated to $\overline{W}$.

(i) We have $s_0, t_0 \in \mathbb{Q}_p$. There exists $n \in \mathbb{N}$, $n \geq 1$, such that

$$w(X - s_{n-1}) \leq w(X - t_0) < w(X - s_n),$$

otherwise t_0 would be a pseudo-limit of E, which is not possible. In particular,

$$v_p(s_n - t_0) = w(s_n - X + X - t_0) = w(X - t_0) \geq w(X - s_{n-1}) = \delta_{n-1}.$$

If $n > 1$, we have $[\mathbb{Q}_p(t_0) : \mathbb{Q}_p] < [\mathbb{Q}_p(s_{n-1}) : \mathbb{Q}_p]$, so by (iii) of Definition 2.1 we have that

$$v_p(s_n - t_0) = v_p(s_{n-1} - t_0) < v_p(s_n - s_{n-1}) = \delta_{n-1},$$

which is impossible. Hence $n = 1$, so $v_p(s_1 - t_0) = w(X - t_0) \geq w(X - s_0)$. Reversing the roles of s_0 and t_0, we get the other inequality, so $w(X - s_0) = w(X - t_0) = \delta_0 = \delta'_0$.

Let $n \in \mathbb{N}$, and suppose that, for each $m \leq n$, we have $[\mathbb{Q}_p(s_m) : \mathbb{Q}_p] = [\mathbb{Q}_p(t_m) : \mathbb{Q}_p]$ and $\delta_m = \delta'_m$.

Since $[\mathbb{Q}_p(s_n) : \mathbb{Q}_p] = [\mathbb{Q}_p(t_n) : \mathbb{Q}_p] < [\mathbb{Q}_p(t_{n+1}) : \mathbb{Q}_p]$, by (ii) of Definition 2.1 we have

$$v_p(t_{n+1} - s_n) \leq v_p(t_{n+1} - t_n) = \delta'_n = \delta_n.$$

Now,

$$v_p(t_{n+1} - s_n) = v_p(t_{n+1} - t_n + t_n - s_n) \geq \delta_n$$

since $v_p(t_n - s_n) = w(t_n - X + X - s_n) \geq \delta_n = \delta'_n$. This implies that $v_p(t_{n+1} - s_n) = \delta_n$, and so (s_n, t_{n+1}) is distinguished. Moreover, we have

$$v_p(t_{n+1} - s_{n+1}) = w(t_{n+1} - X + X - s_{n+1}) > \delta_n = \delta'_n = v_p(t_{n+1} - s_n).$$

Now, if $[\mathbb{Q}_p(s_{n+1}) : \mathbb{Q}_p] < [\mathbb{Q}_p(t_{n+1}) : \mathbb{Q}_p]$, then, since (s_n, t_{n+1}) is distinguished, we would have $v_p(s_{n+1} - t_{n+1}) \leq v_p(t_{n+1} - s_n)$, which is impossible. Hence $[\mathbb{Q}_p(s_{n+1}) : \mathbb{Q}_p] \geq [\mathbb{Q}_p(t_{n+1}) : \mathbb{Q}_p]$. The other inequality is proved in a symmetrical way, so $[\mathbb{Q}_p(s_{n+1}) : \mathbb{Q}_p] = [\mathbb{Q}_p(t_{n+1}) : \mathbb{Q}_p]$.

Suppose now that $w(X - s_{n+1}) < w(X - t_{n+1})$. Then

$$v_p(s_{n+2} - t_{n+1}) = w(s_{n+2} - X + X - t_{n+1}) > w(X - s_{n+1}) = v_p(s_{n+2} - s_{n+1}),$$

which is not possible since (s_{n+1}, s_{n+2}) is distinguished. Hence $w(X - s_{n+1}) \geq w(X - t_{n+1})$. The other inequality is proved similarly, so $\delta_{n+1} = \delta'_{n+1}$ as claimed.

(ii) For each $n \in \mathbb{N}$, let Γ_n and Γ'_n and k_n and k'_n be the value groups and residue fields, respectively, of $\mathbb{Q}_p(s_n)$ and $\mathbb{Q}_p(t_n)$. Let $e_n = e_{s_n}$, $e'_n = e_{t_n}$, $f_n = f_{s_n}$, $f'_n = f_{t_n}$.

Clearly, $e_0 = e'_0$ and $f_0 = f'_0$ since $s_0, t_0 \in \mathbb{Q}_p$.

Let $n \geq 1$. If $f \in \mathbb{Q}_p[X]$ has degree strictly smaller than $[\mathbb{Q}_p(s_n) : \mathbb{Q}_p] = [\mathbb{Q}_p(t_n) : \mathbb{Q}_p]$, then by (2.10) we have $w(f(X)) = v_p(f(s_n))$ and also $w(f(X)) = v_p(f(t_n))$, so $v_p(f(s_n)) = v_p(f(t_n))$. This proves that $\Gamma_n = \Gamma'_n$, and so $e_n = e'_n$.

Suppose now that $f \in \mathbb{Q}_p[X]$ of degree strictly smaller than $[\mathbb{Q}_p(s_n) : \mathbb{Q}_p] = [\mathbb{Q}_p(t_n) : \mathbb{Q}_p]$ is such that $v_p(f(s_n)) = v_p(f(t_n)) = 0$. In particular, $w(f(X)) = 0$ by (2.10). By (2.11) and the analogous equation where s_n is replaced by t_n, we get that $f(s_n)$ and $f(t_n)$ have the same residue as $f(X)$, so, in particular, $k_n = k'_n$. Therefore, $f_n = f'_n$. □

2.2. *Residually algebraic extensions of $\mathbb{Z}_p$ which are DVRs.* In this section we characterize DVRs of $\mathbb{Q}_p(X)$ extending $\mathbb{Z}_p$ such that the residue field extension is algebraic, necessarily of infinite degree by Corollary 2.12; this fact has already been noted in a different way in [Peruginelli 2017, p. 4217]. We will see in Section 2.3 that there is no such restriction on the residue field degree for DVRs of $\mathbb{Q}(X)$ which are residually algebraic extensions of $\mathbb{Z}_{(p)}$ (see Corollary 2.28).

Given $\alpha \in \mathbb{C}_p$, we denote by $\mathbb{O}_{p,\alpha}$ the unique valuation domain of $\mathbb{Q}_p(\alpha)$ lying over $\mathbb{Z}_p$ (i.e., $\mathbb{O}_{p,\alpha} = \mathbb{O}_p \cap \mathbb{Q}_p(\alpha)$). We also set

$$\mathbb{Z}_{p,\alpha} = \{\phi \in \mathbb{Q}_p(X) \mid \phi(\alpha) \in \mathbb{O}_p\},$$

which is a valuation domain of $\mathbb{Q}_p(X)$ and coincides with the previous definition if $\alpha \in \overline{\mathbb{Q}_p}$.

Proposition 2.14. *Let $\alpha \in \mathbb{C}_p$ be a transcendental element. Then there exists a Cauchy stacked sequence $E \subseteq \overline{\mathbb{Q}_p}$ converging to α. Moreover, the valued fields $(\mathbb{Q}_p(X), \mathbb{Z}_{p,\alpha})$ and $(\mathbb{Q}_p(\alpha), \mathbb{O}_{p,\alpha})$ are isomorphic. In particular, the ramification index $e(\mathbb{Z}_{p,\alpha} | \mathbb{Z}_p)$ is equal to e_α, the residue field degree $f(\mathbb{Z}_{p,\alpha} | \mathbb{Z}_p)$ is equal to f_α and $e_\alpha \cdot f_\alpha = \infty$.*

Note that the last condition implies that either e_α or f_α is infinite. It can happen that exactly one of these two quantities is finite (see Theorem 2.21).

Proof. The proof of the first claim follows also by [Alexandru et al. 1998, Proposition 2.2], but we give here a different proof based on the previous results.

By Theorem 2.5, there exists a stacked sequence $E \subset \overline{\mathbb{Q}_p}$ such that $\mathbb{Z}_{p,\alpha} = \mathbb{Z}_{p,E}$. Since the valuation domains $\overline{\mathbb{Z}_p}_E, \overline{\mathbb{Z}_p}_\alpha \subset \overline{\mathbb{Q}_p}(X)$ contract down to $\mathbb{Q}_p(X)$ to the same valuation domain, there exists $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ such that $\sigma(\overline{\mathbb{Z}_p}_\alpha) = \overline{\mathbb{Z}_p}_{\sigma(\alpha)} = \overline{\mathbb{Z}_p}_E$. By [Peruginelli and Spirito 2021, Proposition 5.3], E is then a Cauchy sequence converging to $\sigma(\alpha)$. Since $\mathbb{Z}_{p,\alpha} = \mathbb{Z}_{p,\sigma(\alpha)}$, without loss of generality, we may assume that E converges to α.

Since α is transcendental over $\mathbb{Q}_p$, the evaluation homomorphism $\mathrm{ev}_\alpha : \mathbb{Q}_p(X) \to \mathbb{Q}_p(\alpha)$, $\phi(X) \mapsto \phi(\alpha)$, is an isomorphism. It is easy to see that $\mathrm{ev}_\alpha(\mathbb{Z}_{p,\alpha}) = \mathbb{O}_{p,\alpha}$. Hence $\mathbb{Z}_{p,\alpha}$ and $\mathbb{O}_{p,\alpha}$ have the same ramification indexes and residue field degrees over $\mathbb{Z}_p$.

Finally, the last claim follows by Corollary 2.12. □

Remark 2.15. By Proposition 2.14, we may conclude that, in general,

$$\text{for } \alpha \in \mathbb{C}_p, \text{ we have } e_\alpha \cdot f_\alpha < \infty \text{ if and only if } \alpha \in \overline{\mathbb{Q}_p}.$$

The next lemma may be well known, but lacking a reference we give a short proof.

Lemma 2.16. *Let $p \in \mathbb{Z}$ be a prime, K_1 and K_2 finite extensions of $\mathbb{Q}_p$ and $L = K_1K_2$ the compositum. Let e_1 be the ramification index of K_1 over $\mathbb{Q}_p$ and e the ramification index of L over K_2. Then $e \le e_1$.*

Proof. If K_1 is a tame extension of $\mathbb{Q}_p$, then the ramification index of L over $\mathbb{Q}_p$ is equal to

$$\mathrm{lcm}\{e(K_1 \,|\, \mathbb{Q}_p), e(K_2 \,|\, \mathbb{Q}_p)\}$$

(see, for example, [Chabert and Halberstadt 2018]), so e divides e_1 and the claim is true.

We give a self-contained proof which works in general. Let L' be the normal closure of L over $\mathbb{Q}_p$ and I the inertia group of the maximal ideal $M_{L'}$ of $O_{L'}$ over $\mathbb{Z}_p$. Let G_i be the Galois group $\mathrm{Gal}(L' \,|\, K_i)$ for $i = 1, 2$ and G the Galois group $\mathrm{Gal}(L' \,|\, L)$. Since $L = K_1K_2$, we have $G = G_1 \cap G_2$. The inertia group of $M_{L'}$ over M_{K_1} is equal to $I \cap G_1$, and the inertia group of $M_{L'}$ over M_L is equal to $I \cap G$. We have

$$e = \frac{e(L' \,|\, K_2)}{e(L' \,|\, L)} = \frac{\#(I \cap G_2)}{\#(I \cap G)}, \qquad e_1 = \frac{e(L' \,|\, \mathbb{Q}_p)}{e(L' \,|\, K_1)} = \frac{\# I}{\#(I \cap G_1)}.$$

Note that $I \cap G = (I \cap G_1) \cap (I \cap G_2)$. Therefore, the claim follows by the following general fact for finite groups: given a finite group G with two subgroups H_1 and H_2, we have

$$\frac{\# H_2}{\#(H_1 \cap H_2)} = [H_2 : H_1 \cap H_2] \le \frac{\# G}{\# H_1} = [G : H_1],$$

which follows immediately since the map $h_2 H_1 \cap H_2 \mapsto h_2 H_1$ from the set $\{h_2(H_1 \cap H_2) \mid h_2 \in H_2\}$ of left cosets of $H_1 \cap H_2$ in H_2 to the set $\{gH_1 \mid g \in G\}$ of left cosets of H_1 in G is injective. □

The following result is analogous to [Peruginelli 2017, Theorem 2.5].

Theorem 2.17. *Let W be a DVR of $\mathbb{Q}_p(X)$ which is a residually algebraic extension of $\mathbb{Z}_p$. Then there exists a transcendental element $\alpha \in \mathbb{C}_p$ such that $W = \mathbb{Z}_{p,\alpha}$.*

Proof. Note that, by Corollary 2.12, the residue field of W is an infinite algebraic extension of $\mathbb{F}_p$.

By Theorem 2.5, there exists a stacked sequence $E = \{s_n\}_{n\in\mathbb{N}} \subset \overline{\mathbb{Q}_p}$ such that $W = \mathbb{Z}_{p,E}$. By assumption, the ramification index $e(W\,|\,\mathbb{Z}_p) = e$ is finite. By Remark 2.2 and Proposition 2.7, there exists $n_0 \in \mathbb{N}$ such that $\Gamma_w = \Gamma_n = \Gamma_{n_0}$ for each $n \geq n_0$. Equivalently, $e_n = e_{n_0} = e$ for each $n \geq n_0$. Let $n \geq n_0$. Note that $\delta_n = v_p(s_{n+1} - s_n) \in \Gamma_{O_{K_n}}$, where $K_n = \mathbb{Q}_p(s_n, s_{n+1})$. Note that the ramification index of $\mathbb{Q}_p(s_i)$ over $\mathbb{Q}_p$ is equal to e for $i = n, n+1$. By Lemma 2.16, the ramification index of K_n over $\mathbb{Q}_p$ is bounded by e^2. If $d = \prod_{i=1}^{e^2} i$, then $d\delta_n \in \mathbb{Z}$ for each $n \geq n_0$. This shows that the gauge $\{\delta_n\}_{n\in\mathbb{N}}$ of E has bounded denominator, so $\delta_n \nearrow \infty$, and thus E is Cauchy and converges to a (unique) element α of $\mathbb{C}_p \setminus \overline{\mathbb{Q}_p}$ since E is of transcendental type by Lemma 2.3. In particular, $W = \mathbb{Z}_{p,\alpha}$. ☐

Remark 2.18. We say an element $\alpha \in \mathbb{C}_p$ has *bounded ramification* if the extension $\mathbb{Q}_p(\alpha) \supseteq \mathbb{Q}_p$ has finite ramification. We denote by $\mathbb{C}_p^{\mathrm{br}}$ the set of all elements of $\mathbb{C}_p$ of bounded ramification; clearly, $\overline{\mathbb{Q}_p} \subset \mathbb{C}_p^{\mathrm{br}}$. A transcendental element $\alpha \in \mathbb{C}_p$ has bounded ramification if and only if the set of ramification indexes $\{e_n\}_{n\in\mathbb{N}}$ attached to a stacked sequence $E \subset \overline{\mathbb{Q}_p}$ converging to α is bounded; in fact, by Theorem 2.17, the integer e such that $e = e_n$ for all n sufficiently large is equal to $e(\mathbb{Q}_p(\alpha)\,|\,\mathbb{Q}_p)$.

We remark that not all the transcendental elements $\alpha \in \mathbb{C}_p$ have bounded ramification. For example, according to [Ioviţă and Zaharescu 1995], there exist *generic* transcendental elements $t \in \mathbb{C}_p$ for $\mathbb{C}_p$; that is, the completion of $\mathbb{Q}_p(t)$ is equal to $\mathbb{C}_p$. In particular, the value group of the unique valuation of $\mathbb{O}_{p,t}$ is equal to $\mathbb{Q}$, so the corresponding ramification index is ∞. Hence, by Proposition 2.7, $\mathbb{Z}_{p,t}$ has value group equal to $\mathbb{Q}$ and therefore the set of ramification indexes $\{e_n\}_{n\in\mathbb{N}}$ is unbounded.

We show in Theorem 2.21 that given any algebraic extension k of $\mathbb{F}_p$ and group Γ such that $\mathbb{Z} \subseteq \Gamma \subseteq \mathbb{Q}$, there exists a transcendental element $\alpha \in \mathbb{C}_p$ such that $\mathbb{Z}_{p,\alpha}$ has residue field k and value group Γ, provided that either $[k : \mathbb{F}_p]$ is infinite or Γ is not discrete (this condition being necessary by Corollary 2.12).

Lemma 2.19. *Let l be an infinite algebraic extension of $\mathbb{Q}_p$ such that $e(l\,|\,\mathbb{Q}_p)$ is finite. Then l is contained in the maximal unramified extension K^{unr} of a finite extension K of $\mathbb{Q}_p$.*

Proof. For each $n \in \mathbb{N}$, let $\mathbb{Q}_p^{(n)}$ be the compositum of all the extensions of $\mathbb{Q}_p$ of degree bounded by n. Clearly, $\overline{\mathbb{Q}_p} = \bigcup_{n\in\mathbb{N}} \mathbb{Q}_p^{(n)}$ and $\mathbb{Q}_p^{(n)} \subset \mathbb{Q}_p^{(n+1)}$ for each $n \in \mathbb{N}$. Since $\mathbb{Q}_p$ has only finitely many extensions of bounded degree, $\mathbb{Q}_p^{(n)} = \mathbb{Q}_p(t_n)$ for some $t_n \in \overline{\mathbb{Q}_p}$. Now, for each $n \in \mathbb{N}$, we let $\mathbb{Q}_p(t_n) \cap l = \mathbb{Q}_p(s_n)$ for some $s_n \in l$. Clearly, $l = \bigcup_{n\in\mathbb{N}} \mathbb{Q}_p(s_n)$ and $\mathbb{Q}_p(s_n) \subset \mathbb{Q}_p(s_{n+1})$ for each $n \in \mathbb{N}$. Since $\Gamma_{s_n} \subseteq \Gamma_{s_{n+1}} \subseteq \Gamma_l$ for each $n \in \mathbb{N}$ and Γ_l is discrete by assumption, there exists $n_0 \in \mathbb{N}$ such that $\Gamma_{s_n} = \Gamma_{s_{n_0}}$ for each $n \geq n_0$. Therefore, if $K = \mathbb{Q}_p(s_{n_0})$, then $s_n \in K^{\mathrm{unr}}$ for each $n \geq n_0$, so that $l \subseteq K^{\mathrm{unr}}$. ☐

The next proposition shows that a transcendental element t of $\mathbb{C}_p$ with bounded ramification arise as the limit of sequences contained in the maximal unramified extension K^{unr} of a finite extension K of $\mathbb{Q}_p$. We don't know whether there exists a stacked sequence in K^{unr} which converges to t.

Proposition 2.20. *Let $t \in \mathbb{C}_p^{\mathrm{br}}$. Then t is the limit of a sequence contained in the maximal unramified extension of a finite extension of $\mathbb{Q}_p$.*

Proof. By [Ioviţă and Zaharescu 1995, Theorem 1], the completion of $\widehat{\mathbb{Q}_p(t)} \cap \overline{\mathbb{Q}_p}$ is equal to $\widehat{\mathbb{Q}_p(t)}$. In particular, there exists a Cauchy sequence $E = \{s_n\}_{n\in\mathbb{N}} \subset \widehat{\mathbb{Q}_p(t)} \cap \overline{\mathbb{Q}_p}$ converging to t. Now, since $\mathbb{Q}_p(t) \subset \widehat{\mathbb{Q}_p(t)}$ and $\widehat{\mathbb{Q}_p(t)} \cap \overline{\mathbb{Q}_p} \subset \widehat{\mathbb{Q}_p(t)}$ are immediate extensions, it follows that $\widehat{\mathbb{Q}_p(t)} \cap \overline{\mathbb{Q}_p}$ has value group Γ_t and residue field k_t. By Lemma 2.19, $\widehat{\mathbb{Q}_p(t)} \cap \overline{\mathbb{Q}_p}$ is contained in the maximal unramified extension of a finite extension of $\mathbb{Q}_p$. The statement follows. □

The following result is not new; see for example [Lampert 1986, Lemma 2]. The present proof is different because it employs the notion of stacked sequence.

Theorem 2.21. *Let k be an algebraic extension of $\mathbb{F}_p$ and Γ a totally ordered group with $\mathbb{Z} \subseteq \Gamma \subseteq \mathbb{Q}$ such that either $[k : \mathbb{F}_p]$ or $[\Gamma : \mathbb{Z}]$ is infinite (the last condition is equivalent to Γ being not discrete). Then there exists a transcendental element $\alpha \in \mathbb{C}_p$ such that $k_\alpha = k$ and $\Gamma_\alpha = \Gamma$. In particular, $\mathbb{Z}_{p,\alpha}$ has residue field k and value group Γ.*

Note that, by Corollary 2.12, the last claim shows that $[k : \mathbb{F}_p] \cdot [\Gamma : \mathbb{Z}] = \infty$ is necessary.

Proof. Since $\overline{\mathbb{F}_p}$ is countable, we may suppose that $k = \bigcup_{n\in\mathbb{N}} k_n$, where k_n is a finite extension of $\mathbb{F}_p$, $k_n \subseteq k_{n+1}$ and $k_0 = \mathbb{F}_p$. Similarly, $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$, where Γ_n is a discrete group, $\Gamma_n \subseteq \Gamma_{n+1}$ and $\Gamma_0 = \mathbb{Z}$. Let $f = [k : \mathbb{F}_p]$ and $e = [\Gamma : \mathbb{Z}]$; then, either e or f is infinite. Without loss of generality, we may assume that, for each n, $[k_{n+1} : k_n][\Gamma_{n+1} : \Gamma_n] > 1$.

For each $n \in \mathbb{N}$, there exists a local field $K_n = \mathbb{Q}_p(s_n)$ with residue field k_n and value group Γ_n. By induction, we may also assume that $K_n \subset K_{n+1}$. Let $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{Q}$ be a strictly increasing sequence in $\mathbb{Q}$ which is unbounded and $\lambda_0 < \delta_0 = v(s_1 - s_0)$.

We define now a sequence $E = \{t_n\}_{n\in\mathbb{N}} \subset \overline{\mathbb{Q}_p}$ such that, for each $n \in \mathbb{N}$, $n \geq 1$, we have

(i) $\mathbb{Q}_p(t_n) = \mathbb{Q}_p(s_n)$,

(ii) $(t_{n-1}, \delta_{n-1} = v_p(t_n - t_{n-1}))$ is a minimal pair,

(iii) $\delta_{n-1} > \lambda_{n-1}$.

In particular, E is a stacked sequence by conditions (i) and (ii) and Cauchy by condition (iii) and the assumption on $\{\lambda_n\}_{n\in\mathbb{N}}$.

We set $t_0 = s_0 \in \mathbb{Q}_p$, $t_1 = s_1 \notin \mathbb{Q}_p$ and $\delta_0 = v_p(t_1 - t_0)$. Note that (t_0, δ_0) is a minimal pair. We proceed by induction on n. We assume that, for all $m < n$, we have chosen $t_m \in \overline{\mathbb{Q}_p}$ such that conditions (i), (ii) and (iii) above are satisfied.

We now show how to choose t_n. We choose $a_n \in \mathbb{Q}_p$, $a_n \neq 0$, such that

$$v_p(a_n) > \max\{\omega(t_{n-1}) - v_p(s_n), \lambda_{n-1} - v_p(s_n)\}.$$

We then set

$$t_n = a_n s_n + t_{n-1}.$$

Note that $\mathbb{Q}_p(t_n) \subseteq \mathbb{Q}_p(s_n)$ since by induction $\mathbb{Q}_p(t_{n-1}) = \mathbb{Q}_p(s_{n-1})$ and the last field is contained in $\mathbb{Q}_p(s_n)$. Now, since $\delta_{n-1} = v_p(t_n - t_{n-1}) > \omega(t_{n-1})$, it follows by Krasner's lemma that $\mathbb{Q}_p(t_{n-1}) \subseteq \mathbb{Q}_p(t_n)$. This containment and the fact that $s_n = (t_n - t_{n-1})/a_n$ show that s_n is in $\mathbb{Q}_p(t_n)$, so that $\mathbb{Q}_p(t_n) = \mathbb{Q}_p(s_n)$.

Moreover, note also that $\delta_{n-1} > \lambda_{n-1}$. Hence $E = \{t_n\}_{n\in\mathbb{N}}$ is a stacked sequence which is Cauchy, so E converges to a transcendental element α of $\mathbb{C}_p$. By Proposition 2.7, $\mathbb{Z}_{p,E} = \mathbb{Z}_{p,\alpha}$ has residue field k and value group Γ, as desired. By Proposition 2.14, $\mathbb{Z}_{p,\alpha}$ is isomorphic to $\mathbb{O}_{p,\alpha}$, so it follows that $\Gamma_\alpha = \Gamma$ and $k_\alpha = k$. □

Remark 2.22. We remark that, without condition (iii) above in the proof of Theorem 2.21, in general we may only conclude that there exists a stacked sequence $E \subset \overline{\mathbb{Q}_p}$ (which may not be Cauchy) such that the valuation domain $\mathbb{Z}_{p,E}$ has residue field k and value group Γ. If instead Γ is discrete by assumption, condition (iii) is not necessary: in fact, there exists $n_0 \in \mathbb{N}$ such that $\Gamma_n = \Gamma_{n_0} = \Gamma$ for all $n \geq n_0$; that is, $K_n = \mathbb{Q}_p(s_n)$ is an unramified extension of K_{n_0} for all $n > n_0$. Hence $E \subset \bigcup_{n\in\mathbb{N}} K_n$ is Cauchy, and so $\mathbb{Z}_{p,E} = \mathbb{Z}_{p,\alpha}$, where $\alpha \in \mathbb{C}_p^{\mathrm{br}}$ is the transcendental limit of E.

We close this section showing that the statement of [Ioviţă and Zaharescu 1995, Proposition 1] is wrong, namely, in general the completion of $\mathbb{Q}_p(X)$ with respect to a residually algebraic torsion extension W of $\mathbb{Z}_p$ may not be a subfield of $\mathbb{C}_p$. The mistake is due to the fact that if $W = \mathbb{Z}_{p,E}$ for some pseudo-convergent sequence $E \subset \overline{\mathbb{Q}_p}$ of transcendental type, then X is a pseudo-limit of E with respect to w and may not be a limit (that is, E may not be Cauchy).

Proposition 2.23. *Let W be a residually algebraic torsion extension of $\mathbb{Z}_p$ to $\mathbb{Q}_p(X)$. Then the completion $\widehat{\mathbb{Q}_p(X)}$ with respect to W is (isomorphic to) a subfield of $\mathbb{C}_p$ if and only if there exists a transcendental element α in $\mathbb{C}_p$ such that $W = \mathbb{Z}_{p,\alpha}$.*

Proof. By Theorem 2.5, there exists a pseudo-convergent sequence $E = \{s_n\}_{n\in\mathbb{N}} \subset \overline{\mathbb{Q}_p}$ of transcendental type such that $W = \mathbb{Z}_{p,E}$.

Suppose that $\widehat{\mathbb{Q}_p(X)} \subseteq \mathbb{C}_p$. In particular, $X \in \mathbb{C}_p$, so there exists a Cauchy sequence $F = \{t_n\}_{n\in\mathbb{N}} \subset \overline{\mathbb{Q}_p}$ which tends to X. Since $\mathbb{Q}_p(X) \subset \overline{\mathbb{Q}_p}(X)$ is an algebraic extension and $\mathbb{C}_p$ is algebraically closed, then also the completion of $\overline{\mathbb{Q}_p}(X)$ with respect to $\overline{W} = \overline{\mathbb{Z}_{p}}_E$ is contained in $\mathbb{C}_p$. Without loss of generality, we may suppose that the restriction of v_p to $\overline{\mathbb{Q}_p}(X)$ is equal to $\overline{w}$. In particular, $\overline{w}(X - t_n) = v_p(X - t_n) \nearrow \infty$. Since E is of transcendental type, for each n, there exists m_0 such that $\overline{w}(X - t_n) < \overline{w}(X - s_m)$ for each $m \geq m_0$. This shows that the gauge of E tends to infinity, and thus E is Cauchy; in particular, E converges to a transcendental element $\alpha \in \mathbb{C}_p$. Therefore, $W = \mathbb{Z}_{p,\alpha}$.

Conversely, let $W = \mathbb{Z}_{p,\alpha}$ for some transcendental element $\alpha \in \mathbb{C}_p$. Then, by Proposition 2.14, the completion $\widehat{\mathbb{Q}_p(X)}$ with respect to $\mathbb{Z}_{p,\alpha}$ is isomorphic to the completion of $\mathbb{Q}_p(\alpha)$ and therefore can be identified to a subfield of $\mathbb{C}_p$. □

In particular, if $W = \mathbb{Z}_{p,E}$ for some stacked non-Cauchy sequence $E \subset \overline{\mathbb{Q}_p}$, then $\widehat{\mathbb{Q}_p(X)} \nsubseteq \mathbb{C}_p$.

2.3. *Residually algebraic torsion extensions of $\mathbb{Z}_{(p)}$.* We now characterize residually algebraic torsion extensions of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}(X)$. We remark that such a valuation domain may have an extension to $\mathbb{Q}_p(X)$ which is a residually algebraic extension of $\mathbb{Z}_p$ but is not torsion. For example, let $\alpha \in \overline{\mathbb{Q}_p}$ be transcendental over $\mathbb{Q}$. Then $\mathbb{Z}_{(p),\alpha}$ is torsion but $\mathbb{Z}_{p,\alpha}$ is not (the one dimensional valuation overring of $\mathbb{Z}_{p,\alpha}$ is $\mathbb{Q}_p[X]_{(p_\alpha(X))}$, where $p_\alpha(X)$ is the minimal polynomial of α over $\mathbb{Q}_p$).

Given $\alpha \in \mathbb{C}_p$, we consider the following valuation domain of $\mathbb{Q}(X)$:

$$\mathbb{Z}_{(p),\alpha} = \{\phi \in \mathbb{Q}(X) \mid \phi(\alpha) \in \mathbb{O}_p\},$$

which is just the contraction to $\mathbb{Q}(X)$ of $\mathbb{Z}_{p,\alpha}$ considered in Section 2.1. Similarly, if $E = \{s_n\}_{n\in\mathbb{N}} \subset \overline{\mathbb{Q}_p}$ is a pseudo-convergent sequence of transcendental type, then we set

$$\mathbb{Z}_{(p),E} = \{\phi \in \mathbb{Q}(X) \mid \phi(s_n) \in \overline{\mathbb{Z}_p} \text{ for all sufficiently large } n \in \mathbb{N}\},$$

which is equal to $\mathbb{Z}_{p,E} \cap \mathbb{Q}(X)$.

The next proposition is analogous to Proposition 2.4 and characterizes residually algebraic torsion extensions of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}(X)$ in terms of pseudo-convergent sequences of $\overline{\mathbb{Q}_p}$ which are of transcendental type over $\mathbb{Q}$; clearly, every pseudo-convergent sequence of transcendental type of $\overline{\mathbb{Q}_p}$ belongs to this class. As a particular case, we find again part of the result of [Peruginelli 2017, Theorem 2.5].

Proposition 2.24. *Let $p \in \mathbb{P}$, and let W be a residually algebraic torsion extension of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}(X)$. Then there exists a pseudo-convergent sequence $E \subset \overline{\mathbb{Q}_p}$ of transcendental type over $\mathbb{Q}$ such that $W = \mathbb{Z}_{(p),E}$. More precisely, let e and f be the ramification index and residue field degree of W over $\mathbb{Z}_{(p)}$, respectively. Let $\widehat{\mathbb{Q}(X)}$ be the completion of $\mathbb{Q}(X)$ with respect to the W-adic topology. Then the following conditions are equivalent*:

(1) *$\widehat{\mathbb{Q}(X)}$ is a finite extension of $\mathbb{Q}_p$.*

(2) *X is algebraic over $\mathbb{Q}_p$.*

(3) *$W = \mathbb{Z}_{(p),\alpha}$ for some $\alpha \in \overline{\mathbb{Q}_p}$ transcendental over $\mathbb{Q}$.*

(4) *$ef < \infty$.*

If any one of these conditions holds, then the sequence E above is Cauchy and converges to α (and E is therefore of algebraic type over $\mathbb{Q}_p$). Moreover, we have $\Gamma_w = \Gamma_\alpha$ and $k_w = k_\alpha$.

If $ef = \infty$, then $E \subset \overline{\mathbb{Q}_p}$ is of transcendental type over $\mathbb{Q}_p$ and $\mathbb{Z}_{(p),E} \subset \mathbb{Z}_{p,E}$ is an immediate extension.

Proof. Note that, since W is a torsion extension of $\mathbb{Z}_{(p)}$, the p-adic completion $\mathbb{Q}_p$ of $\mathbb{Q}$ is contained in $\widehat{\mathbb{Q}(X)}$; see for example the arguments given in the proof of [Alexandru et al. 1988, Corollary 2.6].

If $\widehat{\mathbb{Q}(X)}$ is a finite extension of $\mathbb{Q}_p$, then clearly X is algebraic over $\mathbb{Q}_p$, so (1) implies (2). If X is algebraic over $\mathbb{Q}_p$, we may identify X with some $\alpha \in \overline{\mathbb{Q}_p}$; $\mathbb{Q}_p(\alpha)$ is a finite extension of $\mathbb{Q}_p$ and is hence complete. So, $\widehat{\mathbb{Q}(X)} = \mathbb{Q}_p(\alpha)$. As in the proof of [Peruginelli 2017, Theorem 2.5] it follows easily that $W = \mathbb{Z}_{(p),\alpha}$. Therefore, (2) implies (3).

If $W = \mathbb{Z}_{(p),\alpha}$ for some $\alpha \in \overline{\mathbb{Q}_p}$ transcendental over $\mathbb{Q}$, then, by [Peruginelli 2017, Proposition 2.2], $ef < \infty$, so (3) implies (4). Finally, (4) implies (1) by [Peruginelli 2017, Lemma 2.4] because $e(\widehat{W} \mid \mathbb{Z}_p) = e$ and $f(\widehat{W} \mid \mathbb{Z}_p) = f$.

Note that if $E \subset \overline{\mathbb{Q}_p}$ is a pseudo-convergent sequence such that $\mathbb{Z}_{(p),E} = \mathbb{Z}_{(p),\alpha}$, then by Lemma 2.27 below we have $\mathbb{Z}_{p,E} = \mathbb{Z}_{p,\alpha}$, so by [Peruginelli and Spirito 2021, Proposition 5.3] we have that E is Cauchy and converges to α.

The claims about the value group and residue field of $\mathbb{Z}_{(p),\alpha}$ follow by [Peruginelli 2017, Proposition 2.2].

If $ef = \infty$, then X is transcendental over $\mathbb{Q}_p$ by the previous part of the proof; in particular, the field of rational functions $\mathbb{Q}_p(X)$ is contained in the completion $\widehat{\mathbb{Q}(X)}$. If $\widetilde{W} = \widehat{W} \cap \mathbb{Q}_p(X)$, then $\widetilde{W}$ is a residually algebraic torsion extension of $\mathbb{Z}_p$ to $\mathbb{Q}_p(X)$, so by Theorem 2.5 there exists a stacked sequence $E \subset \overline{\mathbb{Q}_p}$ such that $\widetilde{W} = \mathbb{Z}_{p,E}$ (by Lemma 2.3, E is a pseudo-convergent sequence of transcendental type, necessarily unbounded). Restricting down to $\mathbb{Q}(X)$, we get $W = \mathbb{Z}_{(p),E}$. Finally, since $W \subset \widehat{W}$ is an immediate extension, it follows that $\mathbb{Z}_{(p),E} \subset \mathbb{Z}_{p,E}$ is an immediate extension, too. Hence the value group and residue field of $\mathbb{Z}_{(p),E}$ are the same as those of $\mathbb{Z}_{p,E}$, respectively (see Proposition 2.7). □

The following statement is the analog of Proposition 2.23 for residually algebraic torsion extensions of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}(X)$.

Corollary 2.25. *Let W be a residually algebraic torsion extension of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}(X)$. Then the completion $\widehat{\mathbb{Q}(X)}$ with respect to W is (isomorphic to) a subfield of $\mathbb{C}_p$ if and only if there exists $\alpha \in \mathbb{C}_p$, transcendental over $\mathbb{Q}$, such that $W = \mathbb{Z}_{(p),\alpha}$.*

Proof. According to Proposition 2.24, when passing to the completion, either X is algebraic over $\mathbb{Q}_p$ or X is transcendental over $\mathbb{Q}_p$, and consequently either $\widehat{\mathbb{Q}(X)} \subset \overline{\mathbb{Q}_p} \subset \mathbb{C}_p$ or $\mathbb{Q}_p(X) \subset \widehat{\mathbb{Q}(X)}$, respectively. In the first case, $W = \mathbb{Z}_{(p),\alpha}$ for some $\alpha \in \overline{\mathbb{Q}_p} \subset \mathbb{C}_p$ transcendental over $\mathbb{Q}$. In the second case, $\widehat{\mathbb{Q}_p(X)} = \widehat{\mathbb{Q}(X)}$, where the completion of $\mathbb{Q}_p(X)$ is considered with respect to the valuation domain $\widetilde{W} = \widehat{W} \cap \mathbb{Q}_p(X)$. In particular, by Proposition 2.23, in this case we get that $\widehat{\mathbb{Q}(X)} \subseteq \mathbb{C}_p$ if and only if there exists a transcendental element $\alpha \in \mathbb{C}_p$ such that $W = \mathbb{Z}_{(p),\alpha}$. □

In particular, if $W = \mathbb{Z}_{(p),E}$ for some stacked non-Cauchy sequence $E \subset \overline{\mathbb{Q}_p}$, then $\widehat{\mathbb{Q}(X)}$ is not contained in $\mathbb{C}_p$.

The following result is the analog of Theorem 2.21 for building residually algebraic torsion extensions W of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}(X)$ with prescribed residue field k and value group Γ. Note that, contrary to that theorem, we are no longer assuming that $[k : \mathbb{F}_p] \cdot [\Gamma : \mathbb{Z}] = \infty$.

Theorem 2.26. *Let k be an algebraic extension of $\mathbb{F}_p$ and Γ a totally ordered group such that $\mathbb{Z} \subseteq \Gamma \subseteq \mathbb{Q}$. Then there exists $\alpha \in \mathbb{C}_p$, transcendental over $\mathbb{Q}$, such that $\mathbb{Z}_{(p),\alpha}$ has residue field k and value group Γ.*

Proof. Let $e = [\Gamma : \mathbb{Z}]$ and $f = [k : \mathbb{F}_p]$. If $ef < \infty$, then it is well known that there exists $\alpha \in \overline{\mathbb{Q}_p}$ transcendental over $\mathbb{Q}$ such that $\mathbb{O}_{p,\alpha}$ has residue field k and value group Γ. Hence, by [Peruginelli 2017, Proposition 2.2], $\mathbb{Z}_{(p),\alpha}$ is the desired extension of $\mathbb{Z}_{(p)}$.

If $ef = \infty$, then, by Theorem 2.21, there exists a transcendental element $\alpha \in \mathbb{C}_p$ such that $\mathbb{Z}_{p,\alpha}$ has residue field k and value group Γ. Clearly, $\mathbb{Z}_{p,\alpha} \cap \mathbb{Q}(X) = \mathbb{Z}_{(p),\alpha}$ is a residually algebraic torsion extension of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}(X)$. Moreover, by Proposition 2.14, $\mathbb{Z}_{p,\alpha} = \mathbb{Z}_{p,E}$ for some stacked Cauchy sequence $E \subset \overline{\mathbb{Q}_p}$ which converges to α. In particular, $\mathbb{Z}_{(p),\alpha} = \mathbb{Z}_{(p),E}$. By the last part of Proposition 2.24, $\mathbb{Z}_{(p),E} \subset \mathbb{Z}_{p,E}$ is an immediate extension, so $\mathbb{Z}_{(p),\alpha}$ has residue field k and value group Γ. □

Now we are able to describe the DVRs of $\mathbb{Q}(X)$ which are residually algebraic extensions of $\mathbb{Z}_{(p)}$ for some $p \in \mathbb{P}$. We recall that every $\sigma \in G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ extends uniquely to a continuous automorphism of $\mathbb{C}_p$; see [Alexandru et al. 1998, §3]. Given $\alpha, \beta \in \mathbb{C}_p$, we say that α and β are conjugate (over $\mathbb{Q}_p$) if there exists $\sigma \in G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ such that $\sigma(\alpha) = \beta$; the orbit of an element $\alpha \in \mathbb{C}_p$ is finite if and only if $\alpha \in \overline{\mathbb{Q}_p}$; see [Alexandru et al. 1998, Remark 3.2].

We prove first the following lemma.

Lemma 2.27. *Let $p \in \mathbb{P}$ and W be a valuation domain of $\mathbb{Q}_p(X)$ such that $W \cap \mathbb{Q}(X) = \mathbb{Z}_{(p),\alpha}$ for some $\alpha \in \mathbb{C}_p$. Then $W = \mathbb{Z}_{p,\alpha}$.*

Proof. Let $n \geq 0$ be an integer such that $p^n \cdot \alpha = \alpha_0 \in \mathbb{O}_p$. The field isomorphism $X \mapsto X/p^n$ maps $\mathbb{Z}_{(p),\alpha}$ to $\mathbb{Z}_{(p),\alpha_0}$ and $\mathbb{Z}_{p,\alpha}$ to $\mathbb{Z}_{p,\alpha_0}$, respectively. Hence, in order to prove the statement, without loss of generality, we may assume that $\alpha \in \mathbb{O}_p$.

Let w be a valuation associated to W. We note first that, since $X \in \mathbb{Z}_{(p),\alpha}$, it follows that $w(X) \geq 0$. Let $f \in W \cap \mathbb{Q}_p[X]$, say $f(X) = \sum_{i=0}^{d} \alpha_i X^i$. Then, for $g(X) = \sum_{i=0}^{d} a_i X^i \in \mathbb{Q}[X]$, we have

$$w(f-g) \geq \min_{0 \leq i \leq d} \{v_p(\alpha_i - a_i) + i\,w(X)\}.$$

Therefore, if we choose $a_i \in \mathbb{Q}$ sufficiently v_p-adically close to α_i for each $i = 0, \ldots, d$, we have $w(f-g) \geq 0$. In particular, $g \in W \cap \mathbb{Q}(X) = \mathbb{Z}_{(p),\alpha}$. The polynomial $h = f - g$ is in $\mathbb{Z}_p[X]$; therefore $f(\alpha) = h(\alpha) + g(\alpha) \in \mathbb{O}_p$, so that $f \in \mathbb{Z}_{p,\alpha}$. Therefore $W \cap \mathbb{Q}_p[X] \subseteq \mathbb{Z}_{p,\alpha} \cap \mathbb{Q}_p[X]$. Similarly, one can easily show that the other containment holds, so $W \cap \mathbb{Q}_p[X] = \mathbb{Z}_{p,\alpha} \cap \mathbb{Q}_p[X]$. In the same way, $M_W \cap \mathbb{Q}_p[X] = M_{p,\alpha} \cap \mathbb{Q}_p[X]$, where M_W and $M_{p,\alpha}$ are the maximal ideals of W and $\mathbb{Z}_{p,\alpha}$, respectively.

Let now $\psi \in \mathbb{Z}_{p,\alpha}$; since $\mathbb{Z}_p[X] \subset \mathbb{Q}_p[X] \cap \mathbb{Z}_{p,\alpha}$, we may suppose that $\psi = f/g$, where $f, g \in \mathbb{Q}_p[X] \cap \mathbb{Z}_{p,\alpha}$. Clearly, $g(\alpha) \neq 0$; then, there exists $n \in \mathbb{N}$, $n \geq 1$, and $c \in \mathbb{Q}_p$, $c \neq 0$, such that $v_p(c) + v_p(g(\alpha)^n) = 0$. We consider then the rational function $\psi^n = cf^n/cg^n = f_1/g_1$, which still is in $\mathbb{Z}_{p,\alpha}$. Note that $f_1 \in \mathbb{Z}_{p,\alpha} \cap \mathbb{Q}_p[X] = W \cap \mathbb{Q}_p[X]$ and $g_1 \in \mathbb{Z}_{p,\alpha}^* \cap \mathbb{Q}_p[X] = W^* \cap \mathbb{Q}_p[X]$ since $v_{p,\alpha}(f_1) \geq v_{p,\alpha}(g_1) = 0$ (the * denotes the set of units of the valuation domains). In particular,

$$w(f_1) \geq 0 = w(g_1),$$

which proves that $\psi^n \in W$. Since W is integrally closed, it follows that $\psi \in W$. Hence $\mathbb{Z}_{p,\alpha} \subseteq W$. The equality follows because both rings are extensions of $\mathbb{Z}_p$ to $\mathbb{Q}_p(X)$, and in the case that α is algebraic over $\mathbb{Q}_p$, the one-dimensional valuation overring of $\mathbb{Z}_{p,\alpha}$ is nonunitary (i.e., $\mathbb{Q}_p[X]_{(q)}$, where $q \in \mathbb{Q}_p[X]$ is the minimal polynomial of α). □

Corollary 2.28. *Let W be a DVR of $\mathbb{Q}(X)$ which is a residually algebraic extension of $\mathbb{Z}_{(p)}$ for some $p \in \mathbb{P}$. Then there exists $\alpha \in \mathbb{C}_p^{\mathrm{br}}$, transcendental over $\mathbb{Q}$, such that $W = \mathbb{Z}_{(p),\alpha}$. The element α belongs to $\overline{\mathbb{Q}_p}$ if and only if the residue field extension $\mathbb{Z}/p\mathbb{Z} \subseteq W/M$ is finite.*

Moreover, for $\alpha, \beta \in \mathbb{C}_p$, we have $\mathbb{Z}_{(p),\alpha} = \mathbb{Z}_{(p),\beta}$ if and only if there exists $\sigma \in G_{\mathbb{Q}_p}$ such that $\sigma(\alpha) = \beta$.

Proof. Let $f = [W/M : \mathbb{Z}/p\mathbb{Z}]$. If $f < \infty$, then the claim follows by [Peruginelli 2017, Theorem 2.5] and corresponds to the first case of Proposition 2.24: $W = \mathbb{Z}_{(p),\alpha}$ for some $\alpha \in \overline{\mathbb{Q}_p}$ which is transcendental over $\mathbb{Q}$. If $f=\infty$, then we are in the last case of Proposition 2.24, so $W=\mathbb{Z}_{(p),E}$ for some pseudo-convergent sequence in $\overline{\mathbb{Q}_p}$ of transcendental type. As in the proof of Proposition 2.24, we denote by $\widehat{W}$ the completion of W; since the ramification index $e(W\,|\,\mathbb{Z}_{(p)})$ is finite, $\widetilde{W}=\widehat{W}\cap\mathbb{Q}_p(X)$ is a residually algebraic torsion extension of $\mathbb{Z}_p$ to $\mathbb{Q}_p(X)$ which is a DVR, so by Theorem 2.17, $\widetilde{W}=\mathbb{Z}_{p,\alpha}$ for some $\alpha\in\mathbb{C}_p^{\mathrm{br}}\setminus\overline{\mathbb{Q}_p}$. Hence $W=\widetilde{W}\cap\mathbb{Q}(X)=\mathbb{Z}_{(p),\alpha}$. Note that α is transcendental over $\mathbb{Q}_p$ and hence also over $\mathbb{Q}$.

We prove now the final claim. Suppose there exists $\sigma\in G_{\mathbb{Q}_p}$ such that $\sigma(\alpha)=\beta$. If $\phi\in\mathbb{Z}_{(p),\alpha}$, then $\phi(\alpha)$ is defined and belongs to $\mathbb{O}_p$. In particular, $\sigma(\phi(\alpha))=\phi(\sigma(\alpha))=\phi(\beta)\in\overline{\mathbb{Z}_p}$. Hence $\mathbb{Z}_{(p),\alpha}\subseteq\mathbb{Z}_{(p),\beta}$, and the other containment is proved in a symmetrical way.

Conversely, suppose that $\mathbb{Z}_{(p),\alpha}=\mathbb{Z}_{(p),\beta}$. By Lemma 2.27, it follows that $\mathbb{Z}_{p,\alpha}=\mathbb{Z}_{p,\beta}$. Note that the last two valuation domains are the contraction to $\mathbb{Q}_p(X)$ of the valuation domains

$$\overline{\mathbb{Z}}_{p_\alpha}=\{\phi\in\overline{\mathbb{Q}_p}(X)\mid\phi(\alpha)\in\mathbb{O}_p\}\quad\text{and}\quad\overline{\mathbb{Z}}_{p_\beta}=\{\phi\in\overline{\mathbb{Q}_p}(X)\mid\phi(\alpha)\in\mathbb{O}_p\}$$

of $\overline{\mathbb{Q}_p}(X)$, respectively. By [Bourbaki 1985b, Chapter VI, §8, 6., Corollary 1], there exists a $\mathbb{Q}_p(X)$-automorphism σ of $\overline{\mathbb{Q}_p}(X)$ such that $\sigma(\overline{\mathbb{Z}}_{p_\alpha})=\overline{\mathbb{Z}}_{p_\beta}$. It is easy to check that $\sigma(\overline{\mathbb{Z}}_{p_\alpha})=\overline{\mathbb{Z}}_{p_{\sigma(\alpha)}}$. In particular, $\overline{\mathbb{Z}}_{p_{\sigma(\alpha)}}=\overline{\mathbb{Z}}_{p_\beta}$. If $\sigma(\alpha)-\beta\neq 0$, let $c\in\overline{\mathbb{Z}_p}$ be such that $v_p(c)>v_p(\sigma(\alpha)-\beta)$. Let $a\in\overline{\mathbb{Q}_p}$ be such that $v_p(a-\sigma(\alpha))\geq v_p(c)$. Then the polynomial $(X-a)/c$ is in $\overline{\mathbb{Z}}_{p_{\sigma(\alpha)}}$ and not in $\overline{\mathbb{Z}}_{p_\beta}$, which is a contradiction. □

Note that, for a DVR W as in the statement of Corollary 2.28, there exists $\alpha\in\mathbb{O}_p\subset\mathbb{C}_p$ of bounded ramification such that $W=\mathbb{Z}_{(p),\alpha}$ if and only if $X\in W$. This last condition occurs for example if W is an overring of $\mathbb{Z}[X]$.

3. Polynomial Dedekind domains

In order to describe the family of Dedekind domains lying between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$, we briefly recall the notion of integer-valued polynomials on algebras; see [Chabert and Peruginelli 2016; Peruginelli and Werner 2017], for example. Let D be an integral domain with quotient field K and A a torsion-free D algebra. We embed K and A into the extended K-algebra $B=A\otimes_D K$, and this allows us to evaluate polynomials over K at elements of A. If $f\in K[X]$ and $a\in A$ are such that $f(a)\in A$, then we say that f is integer-valued at a. In general, given a subset S of A, we denote by

$$\mathrm{Int}_K(S,A)=\{f\in K[X]\mid f(s)\in A,\forall s\in S\}$$

the ring of integer-valued polynomials over S. We omit the subscript K if $A=D$.

In our setting, let

$$\mathcal{O}=\prod_{p\in\mathbb{P}}\mathbb{O}_p\subset\prod_{p\in\mathbb{P}}\mathbb{C}_p.$$

Given $\alpha=(\alpha_p)\in\prod_{p\in\mathbb{P}}\mathbb{C}_p$ and $f\in\mathbb{Q}[X]$, we have $f(\alpha)=(f(\alpha_p))$, which is an element of $\prod_{p\in\mathbb{P}}\mathbb{C}_p$. If $\underline{E}=\prod_{p\in\mathbb{P}}E_p$ is a subset of $\prod_p\mathbb{C}_p$, then

$$\mathrm{Int}_{\mathbb{Q}}(\underline{E},\mathcal{O})=\{f\in\mathbb{Q}[X]\mid f(\alpha)\in\mathcal{O},\forall\alpha\in\underline{E}\};$$

that is, a polynomial f is in $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \mathcal{O})$ if $f(\alpha_p) \in \mathbb{O}_p$ for each $\alpha_p \in E_p$ and $p \in \mathbb{P}$. By an argument similar to [Chabert and Peruginelli 2016, Remark 6.3], there is no loss in generality to suppose that a subset of $\prod_{p \in \mathbb{P}} \mathbb{C}_p$ is of the form $\prod_{p \in \mathbb{P}} E_p$ when dealing with such rings of integer-valued polynomials.

We remark that we have the following representation for the ring $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \mathcal{O})$ as an intersection of valuation overrings (see [Peruginelli 2023, (2.2)], for example):

$$\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \mathcal{O}) = \bigcap_{p \in \mathbb{P}} \bigcap_{\alpha_p \in E_p} \mathbb{Z}_{(p),\alpha_p} \cap \bigcap_{q \in \mathcal{P}^{\mathrm{irr}}} \mathbb{Q}[X]_{(q)}, \tag{3.1}$$

where $\mathcal{P}^{\mathrm{irr}}$ denotes the set of irreducible polynomials in $\mathbb{Q}[X]$. By [Peruginelli 2017, Proposition 2.2], the valuation domain $\mathbb{Z}_{(p),\alpha_p}$ of $\mathbb{Q}(X)$ has rank 1 if and only if α_p is transcendental over $\mathbb{Q}$ and has rank 2 otherwise (in the last case, note that necessarily $\alpha \in \overline{\mathbb{Q}_p}$).

A totally similar argument to [Peruginelli 2023, Lemma 2.5] shows that, for $p \in \mathbb{P}$, we have

$$(\mathbb{Z} \setminus p\mathbb{Z})^{-1}(\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \mathcal{O})) = \mathrm{Int}_{\mathbb{Q}}(E_p, \mathbb{O}_p) = \{f \in \mathbb{Q}[X] \mid f(E_p) \subseteq \mathbb{O}_p\}.$$

We also need to recall the following definition introduced in [Peruginelli 2023].

Definition 3.2. We say that a subset $\underline{E}$ of $\mathcal{O}$ is *polynomially factorizable* if, for each $g \in \mathbb{Z}[X]$ and $\alpha = (\alpha_p) \in \underline{E}$, there exist $n, d \in \mathbb{Z}$, $n, d \geq 1$, such that $g(\alpha)^n/d$ is a unit of $\mathcal{O}$; that is, $v_p(g(\alpha_p)^n/d) = 0$ for all $p \in \mathbb{P}$.

The next theorem characterizes which rings of integer-valued polynomials $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \mathcal{O})$ are Dedekind domains. Given $p \in \mathbb{P}$ and a subset E_p of $\mathbb{C}_p$, we say that E_p has finitely many $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$-orbits if E_p contains finitely many equivalence classes under the relation of conjugacy over $\mathbb{Q}_p$ (we stress that E_p may not necessarily contain a full $G_{\mathbb{Q}_p}$-orbit). By Corollary 2.28, this condition holds if and only if the set $\{\mathbb{Z}_{(p),\alpha_p} \mid \alpha_p \in E_p\}$ is finite. Furthermore, if $E_p \subseteq \overline{\mathbb{Q}_p}$, then the number of $G_{\mathbb{Q}_p}$-orbits is finite if and only if E_p is a finite set.

Theorem 3.3. *Let* $\underline{E} = \prod_{p \in \mathbb{P}} E_p \subset \prod_p \mathbb{C}_p$. *Then* $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \mathcal{O})$ *is a Dedekind domain if and only if, for each prime* p, E_p *is a subset of* $\mathbb{C}_p^{\mathrm{br}}$ *of transcendental elements over* $\mathbb{Q}$ *with finitely many* $G_{\mathbb{Q}_p}$*-orbits and* $\underline{E}$ *is polynomially factorizable.*

Moreover, if the above conditions hold, then the class group of $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \mathcal{O})$ *is isomorphic to the direct sum of the class groups* $\mathrm{Int}_{\mathbb{Q}}(E_p, \mathbb{O}_p)$, $p \in \mathbb{P}$, *and, if* $E_p = \{\alpha_1, \dots, \alpha_n\}$, *where the* α_i *are pairwise nonconjugate over* $\mathbb{Q}_p$, *then* $\mathrm{Cl}(\mathrm{Int}_{\mathbb{Q}}(E_p, \mathbb{O}_p)) = \mathbb{Z}/e\mathbb{Z} \oplus \mathbb{Z}^{n-1}$, *where e is the greatest common divisor of the ramifications indexes of* α_i *over* $\mathbb{Q}_p$.

In particular, assuming that E_p *is formed by pairwise nonconjugate elements over* $\mathbb{Q}_p$ *for each* $p \in \mathbb{P}$, $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \mathcal{O})$ *is a PID if and only if* $\underline{E}$ *is polynomially factorizable and, for each* $p \in \mathbb{P}$, E_p *contains at most one element* $\alpha_p \in \mathbb{O}_p \cap \mathbb{C}_p^{\mathrm{br}}$ *such that* α_p *is transcendental over* $\mathbb{Q}$ *and unramified over* $\mathbb{Q}_p$.

Proof. Let $R = \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \mathcal{O})$.

Suppose that the above conditions on $\underline{E}$ are satisfied. By (3.1), R is equal to an intersection of DVRs. Moreover, R has finite character; that is, for every nonzero $f \in R$, f belongs to finitely many maximal ideals of the family of DVRs appearing in (3.1): in fact, if $f(X) = g(X)/n$ for some $g \in \mathbb{Z}[X]$ and $n \in \mathbb{Z}$, $n \neq 0$, then f is divisible only by finitely many $q \in \mathcal{P}^{\mathrm{irr}}$; since $\underline{E}$ is polynomially factorizable, by [Peruginelli 2023, Lemma 2.12], the set $\{p \in \mathbb{P} \mid \exists \alpha_p \in E_p, v_p(g(\alpha_p)) > 0\}$ is finite, so that f belongs to finitely many maximal ideals of the family $\mathbb{Z}_{(p),\alpha_p}$, $\alpha_p \in E_p$, $p \in \mathbb{P}$. Hence R is a Krull domain.

Suppose that R is not a Dedekind domain. By [Heitmann 1974, Proposition 2.2], there exists a maximal ideal $M \subset R$ of height strictly greater than one. If $M \cap \mathbb{Z} = (0)$, then, since $\mathbb{Z}[X] \subseteq R \subseteq \mathbb{Q}[X]$, it follows that $R_{\mathbb{Z}\setminus\{0\}} = \mathbb{Q}[X]$ and $2 \leq ht M = ht(M_{\mathbb{Z}\setminus\{0\}}) \leq \dim(\mathbb{Q}[X]) = 1$, a contradiction. Hence $M \cap \mathbb{Z} = p\mathbb{Z}$ for some $p \in \mathbb{P}$. If we now localize at p, we have that $(\mathbb{Z} \setminus p\mathbb{Z})^{-1}R = R_p = \mathrm{Int}_{\mathbb{Q}}(E_p, \mathbb{O}_p)$, which is a Dedekind domain by [Eakin and Heinzer 1973, Theorem]. So $(\mathbb{Z} \setminus p\mathbb{Z})^{-1}M \subset R_p$ cannot have dimension strictly greater than one, a contradiction.

Conversely, suppose that R is a Dedekind domain. In particular, for each $p \in \mathbb{P}$,

$$(\mathbb{Z} \setminus p\mathbb{Z})^{-1}R = R_p = \mathrm{Int}_{\mathbb{Q}}(E_p, \mathbb{O}_p)$$

is a Dedekind domain, so $\{\mathbb{Z}_{(p),\alpha_p} \mid \alpha_p \in E_p\}$ is a finite set of DVRs (because p is contained in only finitely many maximal ideals of these valuation overrings) which implies that E_p is a subset of $\mathbb{C}_p^{\mathrm{br}}$ of transcendental elements over $\mathbb{Q}$ and E_p has finitely many $G_{\mathbb{Q}_p}$-orbits. Since every polynomial of R is contained in only finitely many maximal ideals, it follows easily that $\underline{E}$ is polynomially factorizable.

Finally, suppose that R is a Dedekind domain. As in [Peruginelli 2023, Lemma 2.14], we have $\mathrm{Cl}(R) = \bigoplus_{p\in\mathbb{P}} \mathrm{Cl}(R_p)$, where $R_p = \mathrm{Int}_{\mathbb{Q}}(E_p, \mathbb{O}_p)$ for $p \in \mathbb{P}$. The claim about the class group of R_p follows by [Peruginelli 2023, Proposition 2.10] or by [Eakin and Heinzer 1973, Theorem], since, for each $p \in \mathbb{P}$, we are assuming that $E_p = \{\alpha_1, \ldots, \alpha_n\}$ is formed by pairwise nonconjugate elements over $\mathbb{Q}_p$.

The claim about when $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \mathbb{O})$ is a PID is now straightforward. □

Let $\overline{\overline{\mathbb{Z}}} = \prod_{p\in\mathbb{P}} \overline{\mathbb{Z}_p}$. In [Peruginelli 2023, Theorem 2.17], we show that if R is a Dedekind domain between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ such that the residue fields of prime characteristic are finite fields, then $R = \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\overline{\mathbb{Z}}})$, for some $\underline{E} = \prod_p E_p \subset \overline{\overline{\mathbb{Z}}}$ such that $\underline{E}$ is polynomially factorizable and, for each $p \in \mathbb{P}$, E_p is a finite subset of $\overline{\mathbb{Z}_p}$ of transcendental elements over $\mathbb{Q}$. Now, we are able to complete the classification of the Dedekind domains R, $\mathbb{Z}[X] \subset R \subseteq \mathbb{Q}[X]$, without any restriction on the residue fields.

Theorem 3.4. *Let R be a Dedekind domain such that $\mathbb{Z}[X] \subset R \subseteq \mathbb{Q}[X]$. Then R is equal to* $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \mathcal{O})$ *for some polynomially factorizable subset $\underline{E} = \prod_{p\in\mathbb{P}} E_p \subset \mathcal{O}$ such that, for each prime p, $E_p \subset \mathbb{O}_p \cap \mathbb{C}_p^{\mathrm{br}}$ is a finite set of transcendental elements over $\mathbb{Q}$.*

Proof. Note first that, by [Peruginelli 2018, Theorem 3.14], no valuation overring of W of R can be a residually transcendental extension of $W \cap \mathbb{Q}$ since, for such a valuation domain W, the domain $W \cap \mathbb{Q}[X]$ is not Prüfer. Hence, for each prime ideal $P \subset R$ such that $P \cap \mathbb{Z} = p\mathbb{Z}$, $p \in \mathbb{P}$, R_P is a DVR of $\mathbb{Q}(X)$ which is a residually algebraic extension of $\mathbb{Z}_{(p)}$. By Corollary 2.28, there exists $\alpha \in \mathbb{O}_p \cap \mathbb{C}_p^{\mathrm{br}}$ such that $R_p = \mathbb{Z}_{(p),\alpha}$. Let E_p be the subset of $\mathbb{C}_p^{\mathrm{br}}$ formed by all such α_p. Note that, since p is contained in only finitely many maximal ideals P of R, it follows that E_p is a finite set; moreover, each element of E_p is transcendental over $\mathbb{Q}$ since R_P is a DVR. It now follows that

$$R = \bigcap_{p\in\mathbb{P}} \bigcap_{\substack{P\subset R\\ P\cap\mathbb{Z}=p\mathbb{Z}}} R_P \cap \mathbb{Q}[X] = \bigcap_{p\in\mathbb{P}} \mathrm{Int}_{\mathbb{Q}}(E_p, \mathbb{O}_p) = \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \mathcal{O}).$$

The rest of the statement follows by Theorem 3.3. □

Finally, the next corollary describes the PIDs among the family of Dedekind domains between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$.

Corollary 3.5. *Let R be a PID such that $\mathbb{Z}[X] \subset R \subseteq \mathbb{Q}[X]$. Then R is equal to $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \mathcal{O})$ for some $\underline{E} = \prod_{p\in\mathbb{P}} E_p \subset \mathcal{O}$ such that, for each prime p, E_p contains at most one element $\alpha_p \in \mathbb{O}_p \cap \mathbb{C}_p^{\mathrm{br}}$ such that α_p is transcendental over $\mathbb{Q}$ and unramified over $\mathbb{Q}_p$ and $\underline{E} = \{\alpha = (\alpha_p)\}$ is polynomially factorizable.*

Proof. By Theorem 3.4, the ring R is equal to $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \mathcal{O})$ for some polynomially factorizable subset $\underline{E} = \prod_{p\in\mathbb{P}} E_p \subset \mathcal{O}$ such that, for each prime p, $E_p \subset \mathbb{O}_p \cap \mathbb{C}_p^{\mathrm{br}}$ is a set of transcendental elements over $\mathbb{Q}$ with finitely many $G_{\mathbb{Q}_p}$-orbits. Since by hypothesis the class group of R is trivial, it follows by Theorem 3.3 that, for each $p \in \mathbb{P}$, E_p contains at most one element, which is transcendental over $\mathbb{Q}$ and unramified over $\mathbb{Q}_p$. □

Remark 3.6. As we mentioned in the Introduction, given a group G which is the direct sum of a countable family of finitely generated abelian groups, there exists a Dedekind domain R between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ with class group G [Peruginelli 2023, Theorem 3.1]. The domain R of that construction is equal to $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \mathcal{O})$ for some polynomially factorizable subset $\underline{E} = \prod_{p\in\mathbb{P}} E_p$, where E_p is a finite subset of $\overline{\mathbb{Q}_p}$ of transcendental elements over $\mathbb{Q}$. In particular, R has finite residue fields of prime characteristic [Peruginelli 2023, Theorem 2.17]; the reason is that the valuation overrings $\mathbb{Z}_{(p),\alpha_p}$ of R in (3.1) have finite residue fields precisely because α_p is chosen in $\overline{\mathbb{Q}_p}$ for each $p \in \mathbb{P}$ (Proposition 2.24).

Now, by means of Theorem 2.26, with the same method used in [Peruginelli 2023, Theorem 3.1], we can build a Dedekind domain R, $\mathbb{Z}[X] \subset R \subseteq \mathbb{Q}[X]$, with prescribed class group G as above and prescribed residue fields of prime characteristic, which can be finite or infinite algebraic extensions of the prime field $\mathbb{F}_p$ according to whether the above elements $\alpha_p \in E_p \subset \mathbb{C}_p^{\mathrm{br}}$ transcendental over $\mathbb{Q}$ are either algebraic or transcendental over $\mathbb{Q}_p$.

References

[Alexandru and Popescu 1988] V. Alexandru and N. Popescu, "Sur une classe de prolongements à $K(X)$ d'une valuation sur un corps K", *Rev. Roumaine Math. Pures Appl.* **33**:5 (1988), 393–400. MR

[Alexandru et al. 1988] V. Alexandru, N. Popescu, and A. Zaharescu, "A theorem of characterization of residual transcendental extensions of a valuation", *J. Math. Kyoto Univ.* **28**:4 (1988), 579–592. MR

[Alexandru et al. 1990a] V. Alexandru, N. Popescu, and A. Zaharescu, "All valuations on $K(X)$", *J. Math. Kyoto Univ.* **30**:2 (1990), 281–296. MR

[Alexandru et al. 1990b] V. Alexandru, N. Popescu, and A. Zaharescu, "Minimal pairs of definition of a residual transcendental extension of a valuation", *J. Math. Kyoto Univ.* **30**:2 (1990), 207–225. MR

[Alexandru et al. 1998] V. Alexandru, N. Popescu, and A. Zaharescu, "On the closed subfields of $\mathbb{C}_p$", *J. Number Theory* **68**:2 (1998), 131–150. MR

[Bourbaki 1985a] N. Bourbaki, *Éléments de mathématique*: *Algèbre commutative*, *Chapitres* 1 *à* 4, Masson, Paris, 1985. MR

[Bourbaki 1985b] N. Bourbaki, *Éléments de mathématique*: *Algèbre commutative*, *Chapitres* 5 *à* 7, Masson, Paris, 1985. MR

[Chabert and Halberstadt 2018] J.-L. Chabert and E. Halberstadt, "On Abhyankar's lemma about ramification indices", preprint, 2018. arXiv 1805.08869

[Chabert and Peruginelli 2016] J.-L. Chabert and G. Peruginelli, "Polynomial overrings of Int($\mathbb{Z}$)", *J. Commut. Algebra* **8**:1 (2016), 1–28. MR

[Eakin and Heinzer 1973] P. Eakin and W. Heinzer, "More noneuclidian PID's and Dedekind domains with prescribed class group", *Proc. Amer. Math. Soc.* **40** (1973), 66–68. MR

[Engler and Prestel 2005] A. J. Engler and A. Prestel, *Valued fields*, Springer, 2005. MR

[Heitmann 1974] R. C. Heitmann, "PID's with specified residue fields", *Duke Math. J.* **41** (1974), 565–582. MR

[Iovită and Zaharescu 1995] A. Iovită and A. Zaharescu, "Completions of r.a.t.-valued fields of rational functions", *J. Number Theory* **50**:2 (1995), 202–205. MR

[Kaplansky 1942] I. Kaplansky, "Maximal fields with valuations", *Duke Math. J.* **9** (1942), 303–321. MR

[Khanduja and Saha 1999] S. K. Khanduja and J. Saha, "A generalized fundamental principle", *Mathematika* **46**:1 (1999), 83–92. MR

[Lampert 1986] D. Lampert, "Algebraic p-adic expansions", *J. Number Theory* **23**:3 (1986), 279–284. MR

[Matignon and Ohm 1988] M. Matignon and J. Ohm, "A structure theorem for simple transcendental extensions of valued fields", *Proc. Amer. Math. Soc.* **104**:2 (1988), 392–402. MR

[Narkiewicz 2004] W. Narkiewicz, *Elementary and analytic theory of algebraic numbers*, 3rd ed., Springer, 2004. MR

[Ostrowski 1935a] A. Ostrowski, "Untersuchungen zur arthmetischen Theorie der Körper, I", *Math. Z.* **39**:1 (1935), 269–320. MR

[Ostrowski 1935b] A. Ostrowski, "Untersuchungen zur arithmetischen Theorie der Körper, II–III", *Math. Z.* **39**:1 (1935), 321–404. MR

[Peruginelli 2017] G. Peruginelli, "Transcendental extensions of a valuation domain of rank one", *Proc. Amer. Math. Soc.* **145**:10 (2017), 4211–4226. MR

[Peruginelli 2018] G. Peruginelli, "Prüfer intersection of valuation domains of a field of rational functions", *J. Algebra* **509** (2018), 240–262. MR

[Peruginelli 2023] G. Peruginelli, "Polynomial Dedekind domains with finite residue fields of prime characteristic", *Pacific J. Math.* **324**:2 (2023), 333–351. MR

[Peruginelli and Spirito 2020] G. Peruginelli and D. Spirito, "The Zariski–Riemann space of valuation domains associated to pseudo-convergent sequences", *Trans. Amer. Math. Soc.* **373**:11 (2020), 7959–7990. MR

[Peruginelli and Spirito 2021] G. Peruginelli and D. Spirito, "Extending valuations to the field of rational functions using pseudo-monotone sequences", *J. Algebra* **586** (2021), 756–786. MR

[Peruginelli and Werner 2017] G. Peruginelli and N. J. Werner, "Non-triviality conditions for integer-valued polynomial rings on algebras", *Monatsh. Math.* **183**:1 (2017), 177–189. MR

[Popescu and Zaharescu 1995] N. Popescu and A. Zaharescu, "On the structure of the irreducible polynomials over local fields", *J. Number Theory* **52**:1 (1995), 98–118. MR

[Ribenboim 1958] P. Ribenboim, "Corps maximaux et complets par des valuations de Krull", *Math. Z.* **69** (1958), 466–479. MR

[Ribenboim 1968] P. Ribenboim, *Théorie des valuations*, Sémin. Math. Sup. **9**, Presses Univ. Montréal, 1968. MR

[Zariski and Samuel 1960] O. Zariski and P. Samuel, *Commutative algebra, II*, Van Nostrand, Princeton, NJ, 1960. MR

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