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**Paucity of rational points
on fibrations with multiple fibres**

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Given a family of varieties over the projective line, we study the density of fibres that are everywhere locally soluble in the case that components of higher multiplicity are allowed. We use log geometry to formulate a new sparsity criterion for the existence of everywhere locally soluble fibres and formulate new conjectures that generalise previous work of Loughran and Smeets. These conjectures involve geometric invariants of the associated multiplicity orbifolds on the base of the fibration in the spirit of Campana. We give evidence for the conjectures by providing an assortment of bounds using Chebotarev's theorem and sieve methods, with most of the evidence involving upper bounds.

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1. Introduction

Let X be a smooth, proper, geometrically irreducible variety over \mathbb{Q} , which is equipped with a dominant morphism $\pi : X \rightarrow \mathbb{P}^1$ with geometrically integral generic fibre. We shall refer to such fibrations as *standard*. The focus of this article is on situations where multiple fibres are present. Work of Colliot-Thélène, Skorobogatov and Swinnerton-Dyer [Colliot-Thélène et al. 1997] shows that the set $X(\mathbb{Q})$ of \mathbb{Q} -rational points on X is not Zariski dense when there are at least 5 geometric double fibres. Our goal is to put this kind of result on a quantitative footing by analysing the simpler question of solubility over the ring of adèles $A_{\mathbb{Q}}$. Let

$$N_{\text{loc}}(\pi, H, B) = \#\{x \in \mathbb{P}^1(\mathbb{Q}) \cap \pi(X(A_{\mathbb{Q}})) : H(x) \leq B\}, \quad (1-1)$$

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where H is a height function on $\mathbb{P}^1(\mathbb{Q})$. In general, we will need to allow the height H to be any adelic height on a line bundle $\mathcal{O}(d)$. However, most of the time we shall use an $\mathcal{O}(1)$ -height. In this case we will simply write $N_{\text{loc}}(\pi, H, B) = N_{\text{loc}}(\pi, B)$. Usually we will take the naive height $H(x) = \max\{|x_0|, |x_1|\}$ if $x \in \mathbb{P}^1(\mathbb{Q})$ is represented by a vector $\mathbf{x} = (x_0, x_1) \in \mathbb{Z}_{\text{prim}}^2$, in which case it is easy to prove that

$$\#\{x \in \mathbb{P}^1(\mathbb{Q}) : H(x) \leq B\} \sim \frac{2}{\zeta(2)} B^2 \quad \text{as } B \rightarrow \infty.$$

Loughran and Smeets [2016] have shown that

$$N_{\text{loc}}(\pi, B) \ll \frac{B^2}{(\log B)^{\Delta(\pi)}} \quad (1-2)$$

for a certain exponent $\Delta(\pi) \geq 0$ that is defined in terms of the data of the fibration. (Here, as throughout our work, all implied constants are allowed to depend on the fibration π .) Roughly speaking, the size of $\Delta(\pi)$ is determined by the number of nonsplit fibres of π , thereby lending credence to a philosophy put forward by Serre [1990] and further developed by Loughran [2018]. In [Loughran and Smeets 2016, Conjecture 1.6] it is conjectured that the upper bound (1-2) is sharp provided that the fibre of π over every closed point of \mathbb{P}^1 has an irreducible component of multiplicity 1. (In fact, [Loughran and Smeets 2016] works over arbitrary number fields k and concerns fibrations $X \rightarrow \mathbb{P}^n$ over projective space of arbitrary dimension, but we shall restrict to $k = \mathbb{Q}$ and $n = 1$ in our work.) Our goal is to explore what happens to $N_{\text{loc}}(\pi, B)$ when the assumption about components of multiplicity 1 is violated.

There are relatively few examples in the number theory literature that feature standard fibrations with multiple fibres. When the generic fibre of π is rationally connected, it follows from work of Graber, Harris and Starr [Graber et al. 2003] that every fibre contains a geometrically integral component of multiplicity 1. In particular, when $\dim X = 2$, we must look to fibrations over \mathbb{P}^1 into curves of positive genus to find examples with multiple fibres. Let $c, d, f \in \mathbb{Q}[t]$ be nonzero polynomials such that f is square-free of even degree and such that f and $c - d$ are coprime. Let $\pi : X \rightarrow \mathbb{P}^1$ be a smooth, proper model of the affine variety cut out by the pair of equations

$$x^2 - c(t) = f(t)y^2, \quad x^2 - d(t) = f(t)z^2. \quad (1-3)$$

Then it follows from [Colliot-Thélène et al. 1997, Proposition 4.1] that all the fibres of π over the zeros of f are double fibres and that the generic fibre is a geometrically integral curve whose projective model is isomorphic to a curve of genus 1. When $\deg(f) \geq 6$, as pointed out in [Loughran and Matthiesen 2024, Theorem 1.4], the argument of [Colliot-Thélène et al. 1997, Corollary 2.2] implies that $N_{\text{loc}}(\pi, B) = O(1)$. Further examples involving genus-2 fibrations over \mathbb{P}^1 have been worked out in [Stoppino 2011].

In the spirit of [Campana 2005], our approach to this problem comes from relating the arithmetic of $\pi : X \rightarrow \mathbb{P}^1$ to the arithmetic of the orbifold base $(\mathbb{P}^1, \partial_\pi)$ for a certain \mathbb{Q} -divisor ∂_π , in the sense of Definition 4.6. For each closed point $D \in (\mathbb{P}^1)^{(1)}$, we let $m_D \geq 1$ denote the minimum multiplicity of the irreducible components of $\pi^{-1}(D)$. We will call the fibre over D *multiple* if $m_D > 1$. We emphasise that we have not defined the multiplicity of a fibre as the greatest common divisor of the multiplicities of its

components, as is common in some applications, but rather as the minimum. Then we may define

$$\partial_\pi = \sum_{D \in (\mathbb{P}^1)^{(1)}} \left(1 - \frac{1}{m_D}\right) [D]. \tag{1-4}$$

With this notation, we make the following conjecture.

Conjecture 1.1. *Let $\pi : X \rightarrow \mathbb{P}^1$ be a standard fibration such that the \mathbb{Q} -divisor $-(K_{\mathbb{P}^1} + \partial_\pi)$ is ample. Then*

$$N_{\text{loc}}(\pi, B) = O_\varepsilon(B^{2-\text{deg } \partial_\pi + \varepsilon})$$

for any $\varepsilon > 0$.

Note that $-\text{deg}(K_{\mathbb{P}^1} + \partial_\pi) = 2 - \text{deg } \partial_\pi$. Hence $-(K_{\mathbb{P}^1} + \partial_\pi)$ is ample if and only if $\text{deg } \partial_\pi < 2$. The main feature of Conjecture 1.1 is that we expect $N_{\text{loc}}(\pi, B)$ to be much smaller in the presence of multiple fibres. Our remaining results give evidence towards this, as well as a proposal about the replacement of B^ε by an explicit *nonpositive* power of $\log B$.

In the case that $\text{deg } \partial_\pi > 2$, the Mordell orbifold conjecture shows that the rational points of X can only lie in finitely many fibres of π . This conjecture follows from the *abc*-conjecture, as shown by Smeets [2017]. Examples where the conclusion can be proven unconditionally are found in [Colliot-Thélène et al. 1997]. It remains unclear what can be said generally about the number of everywhere locally soluble fibres in this situation. In the intermediate case $\text{deg } \partial_\pi = 2$, very little is known about the number of soluble or everywhere locally soluble fibres.

1.1. Upper bounds. For each closed point $D \in (\mathbb{P}^1)^{(1)}$, let S_D be the set of geometrically irreducible components of $\pi^{-1}(D)$ of multiplicity m_D , and let $\kappa(D)$ be the residue field. For any number field N/\mathbb{Q} , we write

$$\delta_{D,N}(\pi) = \frac{\#\{\sigma \in \Gamma_{D,N} : \sigma \text{ acts with a fixed point on } S_D\}}{\#\Gamma_{D,N}}, \tag{1-5}$$

where $\Gamma_{D,N}$ is a finite group through which the action of $\text{Gal}(\bar{N}/N)$ on S_D factors. (We take $\delta_{D,N}(\pi) = 0$ when no such components exist.) Note that

$$0 \leq \delta_{D,N}(\pi) \leq 1. \tag{1-6}$$

Moreover, we shall write $\delta_D(\pi) = \delta_{D,\kappa(D)}(\pi)$. When $\pi^{-1}(D)$ has components of multiplicity 1, this agrees with the definition given in [Loughran and Smeets 2016, (1.4)]. A natural analogue of the exponent appearing in [Loughran and Smeets 2016, Theorem 1.2] is then

$$\Delta(\pi) = \sum_{D \in (\mathbb{P}^1)^{(1)}} (1 - \delta_D(\pi)), \tag{1-7}$$

which agrees with the exponent appearing in (1-2) whenever $\pi^{-1}(D)$ contains a multiplicity-1 component for every $D \in (\mathbb{P}^1)^{(1)}$.

The following upper bound treats the case of one multiple fibre above a degree-1 point of \mathbb{P}^1 and is consistent with Conjecture 1.1.

Theorem 1.2. *Let $\pi : X \rightarrow \mathbb{P}^1$ be a standard fibration with a unique multiple fibre at 0. Then*

$$N_{\text{loc}}(\pi, B) \ll \frac{B^{2-\text{deg } \partial\pi}}{(\log B)^{\Delta(\pi)}},$$

where $\Delta(\pi)$ is given by (1-7).

It is tempting to suppose that the same estimate continues to hold when there is more than one closed point of \mathbb{P}^1 above which multiple fibres exist. However, in Theorem 7.1, we shall illustrate that a smaller exponent than $\Delta(\pi)$ is sometimes necessary.

Let $\pi : X \rightarrow \mathbb{P}^1$ be a standard fibration, and let $D \in (\mathbb{P}^1)^{(1)}$, which we suppose is defined by an irreducible binary form $g \in \mathbb{Q}[x, y]$. Assume first that $g(1, 0) \neq 0$. Then the residue field is $\kappa(D) = \mathbb{Q}[x]/(g(x, 1))$. Moreover, for any $d \in \mathbb{N}$ and any $v \in \mathbb{Q}$, let $h(x) = h^{(1)}(x)^{e_1} \dots h^{(s_D)}(x)^{e_s}$ be the factorisation of $h(x) = g(x^d, v)$ into distinct irreducible polynomial $h^{(i)}(x)$. We define $N_{D,d,v}^{(i)} = \mathbb{Q}[x]/(h^{(i)}(x))$ and

$$N_{D,d,v} = N_{D,d,v}^{(1)} \times \dots \times N_{D,d,v}^{(s_D)}. \tag{1-8}$$

For typical v this forms a number field of degree $\text{deg}(g) + d$, but in general an étale algebra of possibly lower degree is formed since h need not be irreducible nor separable. It still remains to deal with the case $g(1, 0) = 0$. But then $D = \infty$ and we apply the same construction to the polynomial $g(1, vy^d) \in \mathbb{Q}[y]$.

We may now define

$$\Theta_v(\pi) = \sum_{D \in (\mathbb{P}^1)^{(1)}} \sum_{k=1}^{s_D} (1 - \delta_{D, N_{D,d,v}^{(k)}}(\pi)) \tag{1-9}$$

in the notation of (1-5). Our main upper bound is as follows.

Theorem 1.3. *Let $\pi : X \rightarrow \mathbb{P}^1$ be a standard fibration with multiple fibres at 0 and ∞ and nowhere else. Let $d = \text{gcd}(m_0, m_\infty)$. Then*

$$N_{\text{loc}}(\pi, B) \ll \frac{B^{2-\text{deg } \partial\pi}}{(\log B)^{\min_{v \in \mathbb{Q}^\times/\mathbb{Q}^{\times,d}} \Theta_v(\pi)}}.$$

It will be convenient to put

$$\Theta(\pi) = \min_{v \in \mathbb{Q}^\times/\mathbb{Q}^{\times,d}} \Theta_v(\pi). \tag{1-10}$$

Let us first note that $\Theta(\pi) \geq 0$ by (1-6). Secondly, $\Delta(\pi)$ and $\Theta(\pi)$ can be different; in Theorem 7.1 we will see an example with $\Theta(\pi) = 0$ but $\Delta(\pi) = 1$. However, we will see that

$$\Theta(\pi) = \Delta(\pi) \quad \text{if } \text{gcd}(m_0, m_\infty) = 1. \tag{1-11}$$

The following result shows that there are only finitely many values that $\Theta_v(\pi)$ can take.

Proposition 1.4. *Let $\pi : X \rightarrow \mathbb{P}^1$ be a standard fibration, and let $D \in (\mathbb{P}^1)^{(1)}$. Let E be the field of definition of the elements of S_D , and let N/\mathbb{Q} be a number field. Then $\delta_{D,N}(\pi) = \delta_{D, N \cap E^{\text{normal}}}(\pi)$, where E^{normal} is the normal closure of E .*

As we have seen, our understanding of $N_{\text{loc}}(\pi, B)$ is inexorably linked to the arithmetic of the orbifold base $(\mathbb{P}^1, \partial_\pi)$. The study of rational points on orbifolds is the focus of work by Pieropan, Smeets, Tanimoto and Várilly-Alvarado [Pieropan et al. 2021], which offers a far-reaching conjectural asymptotic formula for any orbifold (Y, ∂) with \mathbb{Q} -ample divisor $-(K_Y + \partial)$. Pieropan and Schindler [2024] have verified many cases of the conjecture when Y is a split toric variety over \mathbb{Q} . Their work covers the orbifolds that arise in the proof of Theorems 1.2 and 1.3 and would yield the upper bound $N_{\text{loc}}(\pi, B) = O(B^{2-\text{deg } \partial_\pi})$. In order to achieve the desired nonpositive powers of $\log B$, we need to incorporate extra Chebotarev-type conditions that arise when counting locally soluble fibres.

The proofs of Theorems 1.2 and 1.3 are based on the large sieve and will be carried out in Section 6. A crucial ingredient will be a *sparcity criterion*, which gives explicit control over which fibres are everywhere locally soluble. This criterion will be proved in Section 5 using log geometry and may be of independent interest.

The condition $\text{deg } \partial_\pi < 2$ restricts us to only considering fibrations over \mathbb{P}^1 with at most three multiple fibres, and the multiplicities of these fibres cannot be too large. Extending Theorem 1.3 to three multiple fibres represents a formidable challenge. The easiest such case corresponds to the \mathbb{Q} -divisor

$$\partial_\pi = \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\infty].$$

Conjecture 1.1 would predict that $N_{\text{loc}}(\pi, B) = O_\varepsilon(B^{1/2+\varepsilon})$ for any $\varepsilon > 0$. However, the best upper bound we have is due to [Browning and Van Valckenborgh 2012], which only yields the exponent $\frac{3}{5} + \varepsilon$.

1.2. A new conjecture. We are now ready to reveal a new conjecture for the density of locally soluble fibres for standard fibrations in which multiple fibres are allowed. Let $\pi : X \rightarrow \mathbb{P}^1$ be a standard fibration, and let $\theta : \mathbb{P}^1 \rightarrow (\mathbb{P}^1, \partial_\pi)$ be a finite étale orbifold morphism, as defined in Definition 4.2.

We assume that $(\mathbb{P}^1, \partial_\pi)$ does not admit a finite étale orbifold morphism which factors through θ , and θ is a G -torsor under a finite étale group scheme G . Let $\theta_v : C_v \rightarrow \mathbb{P}^1$ denote the twist of θ by any $v \in H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), G)$, which is a torsor under the inner twist G_v of G [Skorobogatov 2001, p. 20]. Finally, let $\pi_v : X_v \rightarrow C_v$ denote the normalisation of the pullback of π along θ_v . We will only consider v for which $C_v(\mathbb{Q}) \neq \emptyset$, in which case we identify $C_v \cong \mathbb{P}^1$.

Conjecture 1.5. *Let $\pi : X \rightarrow \mathbb{P}^1$ be a standard fibration such that the \mathbb{Q} -divisor $-(K_{\mathbb{P}^1} + \partial_\pi)$ is ample and $X(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$. Then there exists a constant $c_\pi > 0$ such that*

$$N_{\text{loc}}(\pi, B) \sim c_\pi \frac{B^{2-\text{deg } \partial_\pi}}{(\log B)^{\min_{v \in H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), G)} \Delta(\pi_v)}},$$

where $\Delta(\pi_v)$ is given by (1-7).

Note that it follows from Proposition 1.4 that $\Delta(\pi_v)$ takes only finitely many values. In the special case that the orbifold base is simply connected as an orbifold, which in the setting of Theorem 1.3 covers the case $\text{gcd}(m_0, m_\infty) = 1$, the exponent will simply equal $\Delta(\pi)$. Thus Conjecture 1.5 implies that

$$N_{\text{loc}}(\pi, B) \sim c_\pi \frac{B^{2-\text{deg } \partial_\pi}}{(\log B)^{\Delta(\pi)}}$$

in this case, which is consistent with the upper bound in Theorem 1.2. In Corollary 4.9 we take $G = \mu_d$ and prove that $\Theta_v(\pi) = \Delta(\pi_v)$ in (1-9). Hence the upper bound in Theorem 7.1 is also consistent with Conjecture 1.5. In Section 7 we provide further evidence for the conjecture by establishing a range of estimates for the variant $N_{\text{loc},S}(\pi, B)$ of $N_{\text{loc}}(\pi, B)$, in which local solubility is only required away from a finite set S of primes. In Theorem 7.2, for example, we establish a precise lower bound for $N_{\text{loc},S}(\pi, B)$ in the case that $\pi : X \rightarrow \mathbb{P}^1$ is a standard fibration for which the only nonsplit fibres lie over 0 and ∞ .

One further source of examples that can be used to illustrate our conjectures is the class of *Halphen surfaces*. These were introduced by Halphen [1882] and correspond to standard fibrations admitting a unique multiple fibre. In Theorems 7.3–7.8 we provide several estimates for $N_{\text{loc},S}(\pi, B)$ that are consistent with Conjecture 1.5 for appropriate surfaces of Halphen type. In the proof of Theorem 7.8 we are led to a concrete problem in analytic number theory that was solved by Friedlander and Iwaniec [2010, Theorem 11.31]. Indeed, we need matching upper and lower bounds for the number of positive integers a, b satisfying $a^6 + b^2 \leq x$, as $x \rightarrow \infty$, such that the only prime divisors of $a^6 + b^2$ are those that split in a given cubic Galois extension K/\mathbb{Q} . It would be useful to have a similar result for non-Galois extensions, but this appears to be difficult.

Remark 1.6. Returning to the example (1-3), we see that the associated \mathbb{Q} -divisor ∂_π has degree $\frac{1}{2} \deg(f)$. Since f is assumed to have even degree, it follows that $-(K_{\mathbb{P}^1} + \partial_\pi)$ is ample only when $\deg(f) = 2$. When f is a quadratic polynomial, Conjecture 1.1 implies that $N_{\text{loc}}(\pi, B) = O_\varepsilon(B^{1+\varepsilon})$ for any $\varepsilon > 0$. The orbifold base $(\mathbb{P}^1, \partial_\pi)$ admits μ_2 -covers, and it is possible to apply Conjecture 1.5 to predict an explicit power of $\log B$. The outcome will depend on the Galois action on the geometric components of the fibres.

1.3. Further questions. We expect similar conjectures to hold when looking at fibrations $\pi : X \rightarrow Y$ over other bases for which $-(K_Y + \partial_\pi)$ is \mathbb{Q} -ample. However, when $\dim(Y) > 1$ the sparsity criterion we work out in Section 5 will be significantly more complicated. Moreover, care also needs to be taken around the effect of thin subsets of $Y(\mathbb{Q})$ on the counting problem. A counter-example to the most naive expectation has recently been provided [Browning et al. 2023] in the case that Y is a split quadric in \mathbb{P}^3 .

In a different direction, when $Y = \mathbb{P}^1$, we can extend the definition (1-1) by defining $N_{\text{loc}}(\pi, B; Z)$ to be the number of $x \in (\mathbb{P}^1(\mathbb{Q}) \setminus Z) \cap \pi(X(A_{\mathbb{Q}}))$ for which $H(x) \leq B$ for any *thin subset* $Z \subseteq \mathbb{P}^1(\mathbb{Q})$. It is then very natural to ask whether or not we should expect a bound of the shape

$$N_{\text{loc}}(\pi, B; Z) \ll \frac{B^{1/m_0+1/m_\infty}}{(\log B)^{\Delta(\pi)}},$$

where $\Delta(\pi)$ is given by (1-7), if we have the freedom to remove any thin set Z . Of course, as pointed out by the anonymous referee, it is not completely clear whether anything is left if we are able to remove arbitrary thin sets from $\mathbb{P}^1(\mathbb{Q})$.

1.4. Summary of the paper. The main sparsity criterion for locally soluble fibres is Theorem 5.5. It is proved using log geometry in Section 5 and leads to Chebotarev-type conditions about the splitting

behaviour of primes. In Section 2 we shall collect together some basic group-theoretic results that allow us to interpret the output from Chebotarev’s theorem. Section 3 uses recent work of Arango-Piñeros, Keliher and Keyes [Arango-Piñeros et al. 2022] to count pairs of power-full integers which lie in the multiplicative span of Frobenian sets of primes. In Section 4 we shall introduce the necessary background on orbifolds that is required to interpret the exponent of $\log B$ in Conjecture 1.5. Section 6 contains the proof of Theorems 1.2 and 1.3 and is based on an application of the large sieve. Finally, Section 7 builds on the work in Section 3 and contains new evidence for Conjecture 1.5, including estimates for $N_{\text{loc},S}(\pi, B)$ in the case of Halphen surfaces and other families admitting multiple fibres.

2. Group-theoretic results

We will need some preliminary results on the density of primes with a prescribed splitting behaviour. Using Chebotarev’s theorem, we will be able to translate it into statements about groups and group actions. We begin by proving some results in elementary group theory.

2.1. Group theory lemmas. Let G be a finite group, and let $H \subseteq G$ be a subgroup. For an element $g \in G$, we will write $\text{Fix}_g(G/H)$ for the set of fixed points of g under the natural action of G on G/H .

Lemma 2.1. *Let $C \subseteq G$ be a conjugacy class. Then we have*

$$\sum_{g \in C} \#\text{Fix}_g(G/H) = \frac{\#G}{\#H} \#(C \cap H).$$

Proof. First note that, for conjugate elements $g, y \in C$, there is an element $u \in G$ such that $u^{-1}yu = g$. Hence

$$\{x \in G : x^{-1}gx = y\} = \{x \in G : (ux)^{-1}y(ux) = y\} = u^{-1} \text{Stab}_y,$$

whose cardinality is $\#G/\#C$ by the orbit-stabiliser theorem since C is the orbit of y under conjugation.

We now see that

$$\begin{aligned} \sum_{g \in C} \#\text{Fix}_g(G/H) &= \#\{(g, xH) \in C \times G/H : gxH = xH\} \\ &= \#\{(g, xH) \in C \times G/H : x^{-1}gx \in H\}. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{g \in C} \#\text{Fix}_g(G/H) &= \frac{1}{\#H} \#\{(g, x) \in C \times G : x^{-1}gx \in H \cap C\} \\ &= \frac{1}{\#H} \#\{(g, x, y) \in C \times G \times (H \cap C) : x^{-1}gx = y\} \\ &= \frac{1}{\#H} \#C \cdot \frac{\#G}{\#C} \cdot \#(C \cap H), \end{aligned}$$

which proves the lemma. □

Lemma 2.2. *Let S and T be subgroups of G . Then*

$$\#S\#T = \#(S \cap T)\#(ST).$$

Proof. Consider the action $S \times T$ on G by $(s, t)g = sgt^{-1}$. The stabiliser of e_G equals the image of diagonal map $S \cap T \hookrightarrow S \times T$, and the set ST is the orbit of e_G . The result now follows from the orbit-stabiliser formula. \square

2.2. Density of primes. Let F/\mathbb{Q} be a number field with ring of integers \mathcal{O}_F . Define $\mathcal{P}_{F,m}$ to be the set of rational primes p unramified in F which are divisible by exactly m primes $\mathfrak{p}_i \subseteq \mathcal{O}_F$ of degree 1. Let

$$\mathcal{P}_F = \bigcup_{m \geq 1} \mathcal{P}_{F,m}.$$

We define

$$\delta(\mathcal{E}, K) = 1 - \sum_{m=1}^d m \operatorname{dens} \left(\mathcal{P}_{K,m} \cap \bigcup_{E \in \mathcal{E}} \mathcal{P}_E \right)$$

for a finite set \mathcal{E} of number fields and any number field $K \subseteq \overline{\mathbb{Q}}$ with $d = [K : \mathbb{Q}]$. If $\mathcal{E} = \{E\}$ consists of a single number field $E \subseteq \overline{\mathbb{Q}}$, we will write $\delta(E, K) = \delta(\mathcal{E}, K)$. The main result of this section is the following result.

Theorem 2.3. *Let \mathcal{E} be a finite set of number fields and $K \subseteq \overline{\mathbb{Q}}$ a number field with $d = [K : \mathbb{Q}]$. Define*

$$\delta(\mathcal{E}, K) = 1 - \sum_{m=1}^d m \operatorname{dens} \left(\mathcal{P}_{K,m} \cap \bigcup_{E \in \mathcal{E}} \mathcal{P}_E \right).$$

Let $L \subseteq \overline{\mathbb{Q}}$ be a Galois extension of \mathbb{Q} which contains both K and all $E \in \mathcal{E}$. Then

$$\delta(\mathcal{E}, K) = 1 - \frac{\#\{\sigma \in \operatorname{Gal}(L/K) : \sigma \text{ fixes a conjugate of some } E \in \mathcal{E}\}}{\#\operatorname{Gal}(L/K)}.$$

The quantity $\delta(\mathcal{E}, K)$ generalises a quantity that is implicit in [Loughran and Smeets 2016, (1.4)]. Let $\pi : X \rightarrow \mathbb{P}^1$ be a standard fibration, and let D be a closed point of \mathbb{P}^1 with residue field $\kappa(D)$. Let $I_D(\pi)$ be the set of geometrically irreducible components of $\pi^{-1}(D)$ of multiplicity 1, and let \mathcal{E}_D be the set of number fields obtained from taking the algebraic closure of \mathbb{Q} in the function field of each irreducible component of $\pi^{-1}(D)$, i.e., the minimal finite extensions of \mathbb{Q} over which the irreducible components of $\pi^{-1}(D)$ split into their geometrically irreducible components. Then

$$\delta_D(\pi) = 1 - \delta(\mathcal{E}_D, \kappa(D))$$

in [Loughran and Smeets 2016, (1.4)]. Moreover, if we take S_D to be the set of geometrically irreducible components of $\pi^{-1}(D)$ of minimal multiplicity m_D and we let \mathcal{E}_D be the set of fields of definition of the elements of S_D , then we also have

$$\delta_{D,N}(\pi) = 1 - \delta(\mathcal{E}_D, N) \tag{2-1}$$

in (1-5) for any number field N/\mathbb{Q} .

Proof of Theorem 2.3. Write $G = \operatorname{Gal}(L/\mathbb{Q})$, and let K be the fixed field of the subgroup H_1 of G . Similarly, let $E \in \mathcal{E}$ be the fixed fields of the subgroups $H \in \mathcal{H}$ of G . Let $\mathcal{P}'_{K,m}$ denote the set of primes

in $\mathcal{P}_{K,m}$ that are unramified in L , and similarly for each \mathcal{P}'_E . Then we have

$$\mathcal{P}'_{K,m} = \{\text{primes } p \in \mathbb{Z} \text{ unramified in } L \text{ for which } \#\text{Fix}_{\text{Frob}_p}(G/H_1) = m\}$$

and

$$\mathcal{P}'_E = \{\text{primes } p \in \mathbb{Z} \text{ unramified in } L \text{ for which } \#\text{Fix}_{\text{Frob}_p}(G/H) \geq 1\}.$$

Note that

$$C_m = \left\{ g \in G : \#\text{Fix}_g(G/H_1) = m, \sum_{H \in \mathcal{H}} \#\text{Fix}_g(G/H) \geq 1 \right\}$$

is closed under conjugation, since conjugate elements have the same number of fixed points. By Chebotarev’s theorem, in the form presented in [Serre 2012, Theorem 3.4], for example, we therefore obtain

$$\text{dens}\left(\mathcal{P}_{K,m} \cap \bigcup_{E \in \mathcal{E}} \mathcal{P}_E\right) = \text{dens}\left(\mathcal{P}'_{K,m} \cap \bigcup_{E \in \mathcal{E}} \mathcal{P}'_E\right) = \frac{\#C_m}{\#G}.$$

Let $T = \bigcup_{t \in G, H \in \mathcal{H}} tHt^{-1}$, which we note is closed under conjugation. Since $g \in G$ has at least a fixed point on one of the G/H if and only if $g \in T$, we arrive at

$$\sum_{m=1}^d m \text{dens}\left(\mathcal{P}_{K,m} \cap \bigcup_{E \in \mathcal{E}} \mathcal{P}_E\right) = \frac{1}{\#G} \sum_{m=1}^d m \#C_m = \frac{1}{\#G} \sum_{g \in T} \#\text{Fix}_g(G/H_1).$$

We may now conclude from Lemma 2.1 that

$$\sum_{m=1}^d m \text{dens}\left(\mathcal{P}_{K,m} \cap \bigcup_{E \in \mathcal{E}} \mathcal{P}_E\right) = \frac{\#(T \cap H_1)}{\#H_1}. \tag{2-2}$$

The statement of the theorem follows on noting that $H_1 = \text{Gal}(L/K)$ and

$$T = \{\sigma \in G : \sigma \text{ fixes a conjugate of some } E \in \mathcal{E}\}. \quad \square$$

Note that we could not have applied the Chebotarev theorem to $\#(T \cap H_1)$, since $T \cap H_1$ is not necessarily fixed under conjugation in G . It is however closed under conjugation in H_1 .

2.3. Computation of δ in specific cases. Theorem 2.3 allows us to compute the density $\delta(\mathcal{E}, K)$ in the common Galois closure L of both K and each $E \in \mathcal{E}$. The following theorem says that this can be reduced to a computation in a Galois closure of the fields $E \in \mathcal{E}$.

Proposition 2.4. *Let E^{normal} be the normal closure of the compositum of the $E \in \mathcal{E}$ in $\overline{\mathbb{Q}}$. Then*

$$\delta(\mathcal{E}, K) = \delta(\mathcal{E}, E^{\text{normal}} \cap K).$$

Proof. We adopt the notation from the proof of Theorem 2.3. Let $A^{\{j\}}$ be the subgroups of G indexed by a set J , which are of the form tHt^{-1} for $t \in G$ and $H \in \mathcal{H}$. For a set $I \subseteq J$, we write $A^I = \bigcap_{i \in I} A^{\{i\}}$.

The field $E^{\text{normal}} \cap K$ corresponds to the subgroup $\langle H_1, A^J \rangle \subseteq G$ generated by H_1 and A^J . (Since A^J is normal, one can actually show that $\langle H_1, A^J \rangle = H_1 A^J$.) It follows from Lemma 2.2 that

$$\frac{\#(S \cap H_1)}{\#H_1} = \frac{\#(S \cap \langle H_1, A^J \rangle)}{\#\langle H_1, A^J \rangle}$$

when S is equal to A^I for any $I \subseteq J$. Since both sides are additive in S , the statement extends to $S = T = \bigcup_{j \in J} A^{\{j\}}$ by the principle of inclusion and exclusion. □

Proof of Proposition 1.4. Combine Proposition 2.4 with (2-1). □

Our remaining results summarise some special situations in which we can use Theorem 2.3 and Proposition 2.4 to calculate the densities $\delta(\mathcal{E}, K)$ easily.

Lemma 2.5. *If $E \subseteq K$ for some $E \in \mathcal{E}$, then $\delta(\mathcal{E}, K) = 0$.*

Proof. Since K, E are the fixed fields of the subgroups $H_1, H' \subseteq \text{Gal}(L/\mathbb{Q})$, we have $E \subseteq K$ if and only if $H' \supseteq H_1$. But then $H_1 \subseteq H' \subseteq T = \bigcup_{t \in G, H \in \mathcal{H}} t H t^{-1}$, whence $\#(T \cap H_1)/\#H_1 = 1$ in (2-2). □

Let us now consider some cases in which \mathcal{E} contains a single element.

Lemma 2.6. *If E/\mathbb{Q} is Galois, then $\delta(E, K) = 1 - \text{deg}(E \cap K)/\text{deg } E$.*

Proof. Since E/\mathbb{Q} is Galois, E is also Galois over $E^{\text{normal}} \cap K = E \cap K$. Thus we conclude $\delta(E, K) = \delta(E, E \cap K) = 1 - 1/[E : E \cap K]$. □

Lemma 2.7. *If K/\mathbb{Q} is Galois and $KE = E^{\text{normal}}$, then $\delta(E, K) = 1 - \text{deg}(E \cap K)/\text{deg } E$.*

Proof. Since $KE = E^{\text{normal}}$ and K/\mathbb{Q} is Galois, we have $H_1 \cap A^{\{j\}} = A^j$ for all $j \in J$. Thus

$$\frac{\#(T \cap H_1)}{\#H_1} = \frac{\#A^J}{\#H_1} = \frac{\text{deg } K}{\text{deg } E^{\text{normal}}} = \frac{\text{deg } K}{\text{deg } KE}$$

in (2-2). Since K is Galois, we have $[KE : K] = [E : E \cap K]$, from which the lemma follows. □

3. Pairs of integers with Frobenian conditions

We say that a set \mathcal{P} of rational primes is *Frobenian* if there is a finite Galois extension K/\mathbb{Q} and a union of conjugacy classes H in $\text{Gal}(K/\mathbb{Q})$ such that \mathcal{P} is equal to the set of primes p that are unramified in K and for which the Frobenius conjugacy class of p in $\text{Gal}(K/\mathbb{Q})$ lies in H . In this section we produce an asymptotic formula for the density of coprime integers a_0, a_1 which are both power-full and lie in the multiplicative span of a Frobenian set of primes.

It will be convenient to introduce the notation

$$c_S(\alpha) = \prod_{p \in S} \left(1 - \frac{1}{p^\alpha}\right) \tag{3-1}$$

for any $\alpha > 0$ and any finite set of primes S . We shall prove the following result.

Proposition 3.1. *For $i \in \{0, 1\}$, let $m_i \in \mathbb{N}$ and let \mathcal{P}_i be a Frobenian set of rational primes of density ∂_i . Then, for any finite set of primes S , we have*

$$\#\{(a_0, a_1) \in \mathbb{Z}_{\text{prim}}^2 : |a_i| \leq B, p \notin S \Rightarrow [m_i \mid v_p(a_i) \text{ and } (p \mid a_i \Rightarrow p \in \mathcal{P}_i)]\} \sim c_{m_i, \mathcal{P}_i, S} \frac{B^{1/m_0+1/m_1}}{(\log B)^{2-\partial_0-\partial_1}}$$

as $B \rightarrow \infty$, where

$$\begin{aligned} c_{m_i, \mathcal{P}_i, S} &= \frac{4m_0^{1-\partial_0}m_1^{1-\partial_1}}{\Gamma(\partial_0)\Gamma(\partial_1)} \cdot \frac{c_S(\frac{1}{m_0} + \frac{1}{m_1})}{c_S(\frac{1}{m_0})c_S(\frac{1}{m_1})} \prod_{\substack{p \in \mathcal{P}_0 \cap \mathcal{P}_1 \\ p \notin S}} \left(1 - \frac{1}{p^2}\right) \\ &\times \prod_{p \in \mathcal{P}_0 \cap S} \left(1 - \frac{1}{p}\right) \prod_{p \in \mathcal{P}_0} \left(1 - \frac{1}{p}\right)^{-1+\partial_0} \prod_{p \notin \mathcal{P}_0} \left(1 - \frac{1}{p}\right)^{\partial_0} \\ &\times \prod_{p \in \mathcal{P}_1 \cap S} \left(1 - \frac{1}{p}\right) \prod_{p \in \mathcal{P}_1} \left(1 - \frac{1}{p}\right)^{-1+\partial_1} \prod_{p \notin \mathcal{P}_1} \left(1 - \frac{1}{p}\right)^{\partial_1}. \end{aligned}$$

There are only $O(1)$ elements with $a_0a_1 = 0$ that contribute to the counting function. Let $M(B) = M(m_i, \mathcal{P}_i, B, S)$ denote the overall contribution with $a_0a_1 \neq 0$. Hence, on accounting for signs, we have

$$M(B) = 4\#\left\{(a_0, a_1) \in \mathbb{N}^2 : \begin{array}{l} a_0, a_1 \leq B, \gcd(a_0, a_1) = 1, \\ p \notin S \Rightarrow [m_i \mid v_p(a_i) \text{ and } (p \mid a_i \Rightarrow p \in \mathcal{P}_i)] \end{array}\right\}.$$

For (a_0, a_1) appearing in the counting function, we may clearly write

$$a_0 = b_0u_0^{m_0} \quad \text{and} \quad a_1 = b_1u_1^{m_1},$$

where $p \mid b_0b_1 \Rightarrow p \in S$, $\gcd(u_0u_1, \prod_{p \in S} p) = 1$, and $p \mid u_i \Rightarrow p \in \mathcal{P}_i$. Moreover, we have $\gcd(b_0, b_1) = \gcd(u_0, u_1) = 1$. Let $\mathcal{Q} = \mathcal{P}_0 \cap \mathcal{P}_1$.

We proceed by introducing the counting functions

$$M_i(x) = \#\{v \leq x : p \mid v \Rightarrow p \in \mathcal{P}_{i,S}\}$$

for $i = 0, 1$, where $\mathcal{P}_{i,S} = \mathcal{P}_i \setminus (S \cap \mathcal{P}_i)$. On using the Möbius function to detect the condition $\gcd(u_0, u_1) = 1$, we may now write

$$M(B) = 4 \sum_{\substack{b_0, b_1 \in \mathbb{N} \\ \gcd(b_0, b_1) = 1 \\ p \mid b_0b_1 \Rightarrow p \in S}} \sum_{\substack{k \in \mathbb{N} \\ p \mid k \Rightarrow p \in \mathcal{Q}_S}} \mu(k) M_0(k^{-1}(B/b_0)^{1/m_0}) M_1(k^{-1}(B/b_1)^{1/m_1}),$$

where $\mathcal{Q}_S = \mathcal{Q} \setminus (S \cap \mathcal{Q})$. The treatment of $M_i(x)$ is handled by the following result.

Lemma 3.2. *Let $i \in \{0, 1\}$. Then*

$$M_i(x) \sim \frac{\kappa_{i,S}}{\Gamma(\partial_i)} \frac{x}{(\log x)^{1-\partial_i}}$$

as $x \rightarrow \infty$, where

$$\kappa_{i,S} = \prod_{p \in \mathcal{P}_i \cap S} \left(1 - \frac{1}{p}\right) \prod_{p \in \mathcal{P}_i} \left(1 - \frac{1}{p}\right)^{-1+\partial_i} \prod_{p \notin \mathcal{P}_i} \left(1 - \frac{1}{p}\right)^{\partial_i}. \tag{3-2}$$

Proof. Let $i \in \{0, 1\}$. There are several approaches to estimating $M_i(x)$, but the one we shall adopt is via a general result of [Wirsing 1967] on mean values of multiplicative arithmetic functions $g : \mathbb{N} \rightarrow [0, 1]$. (In fact, this result applies to general nonnegative multiplicative arithmetic functions under further assumptions on the behaviour of g at prime powers.) Suppose that

$$\sum_{p \leq x} g(p) \log p \sim \tau x$$

for some $\tau > 0$. Then it follows that

$$\sum_{n \leq x} g(n) \sim \frac{e^{-\gamma\tau}}{\Gamma(\tau)} \frac{x}{\log x} \prod_{p \leq x} \left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \dots\right),$$

where γ is Euler’s constant.

In our case we take

$$g(n) = \begin{cases} 1 & \text{if } p \mid n \Rightarrow p \in \mathcal{P}_{i,S}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, since \mathcal{P}_i is a Frobenian set of primes of density ∂_i , it follows from the Chebotarev density theorem that

$$\sum_{p \leq x} g(p) \log p = \sum_{\substack{p \leq x \\ p \in \mathcal{P}_{i,S}}} \log p \sim \partial_i \log x$$

as $x \rightarrow \infty$. Hence $\tau = \partial_i$ and we obtain

$$M(x) \sim \frac{e^{-\gamma\partial_i}}{\Gamma(\partial_i)} \frac{x}{\log x} \prod_{\substack{p \leq x \\ p \in \mathcal{P}_{i,S}}} \left(1 - \frac{1}{p}\right)^{-1}$$

as $x \rightarrow \infty$. It remains to study

$$\prod_{\substack{p \leq x \\ p \in \mathcal{P}_{i,S}}} \left(1 - \frac{1}{p}\right)^{-1} = \prod_{p \in \mathcal{P}_i \cap S} \left(1 - \frac{1}{p}\right) \prod_{\substack{p \leq x \\ p \in \mathcal{P}_i}} \left(1 - \frac{1}{p}\right)^{-1}.$$

However, on appealing to [Arango-Piñeros et al. 2022, Theorem A], we quickly arrive at the expression

$$\prod_{\substack{p \leq x \\ p \in \mathcal{P}_i}} \left(1 - \frac{1}{p}\right)^{-1} \sim \left(\frac{\log x}{e^{-\gamma\kappa}}\right)^{\partial_i}$$

as $x \rightarrow \infty$, where

$$e^{-\gamma\kappa} = e^{-\gamma} \prod_{p \in \mathcal{P}_i} \left(1 - \frac{1}{p}\right)^{\partial_i^{-1}-1} \prod_{p \notin \mathcal{P}_i} \left(1 - \frac{1}{p}\right)^{-1}.$$

It now follows that

$$\prod_{\substack{p \leq x \\ p \in \mathcal{P}_{i,S}}} \left(1 - \frac{1}{p}\right)^{-1} \sim \kappa_{i,S} (\log x)^{\partial_i} e^{\gamma\partial_i}$$

in the notation of lemma. Inserting this into our previous asymptotic formula for $M_i(x)$, we finally arrive at the statement of the lemma. □

We clearly have

$$(\log(k^{-1}(B/b_i)^{1/m_i}))^{-(1-\partial_i)} = m_i^{1-\partial_i} (\log B)^{-(1-\partial_i)} \left(1 + O\left(\frac{\log kb_i}{\log B}\right)\right)$$

for $i = 0, 1$. Hence, on substituting Lemma 3.2 into our previous expression for $M(B)$, we thereby obtain

$$M(B) = 4 \sum_{\substack{b_0, b_1 \in \mathbb{N} \\ \gcd(b_0, b_1) = 1 \\ p|b_0 b_1 \Rightarrow p \in S}} \sum_{\substack{k \in \mathbb{N} \\ p|k \Rightarrow p \in \mathcal{Q}_S}} A_{b_0, b_1, k}(B) + o\left(\frac{B^{1/m_0+1/m_1}}{(\log B)^{2-\partial_0-\partial_1}}\right),$$

with

$$\begin{aligned} A_{b_0, b_1, k}(B) &= \frac{\kappa_{0,S} \kappa_{1,S}}{\Gamma(\partial_0) \Gamma(\partial_1)} \cdot \frac{\mu(k) m_0^{1-\partial_0} m_1^{1-\partial_1} (k^{-1}(B/b_0)^{1/m_0}) (k^{-1}(B/b_1)^{1/m_1})}{(\log B)^{2-\partial_0-\partial_1}} \\ &= \frac{\kappa_{0,S} \kappa_{1,S}}{\Gamma(\partial_0) \Gamma(\partial_1)} \cdot m_0^{1-\partial_0} m_1^{1-\partial_1} \cdot \frac{B^{1/m_0+1/m_1}}{(\log B)^{2-\partial_0-\partial_1}} \cdot \frac{\mu(k)}{k^2} \cdot \frac{1}{b_0^{1/m_0} b_1^{1/m_1}} \end{aligned}$$

and where $\kappa_{0,S}, \kappa_{1,S}$ are given by (3-2)

Next, on recalling the notation of (3-1), a simple calculation furnishes the identities

$$\sum_{\substack{b_0, b_1 \in \mathbb{N} \\ \gcd(b_0, b_1) = 1 \\ p|b_0 b_1 \Rightarrow p \in S}} \frac{1}{b_0^{1/m_0} b_1^{1/m_1}} = \frac{c_S\left(\frac{1}{m_0} + \frac{1}{m_1}\right)}{c_S\left(\frac{1}{m_0}\right) c_S\left(\frac{1}{m_1}\right)} \quad \text{and} \quad \sum_{\substack{k \in \mathbb{N} \\ p|k \Rightarrow p \in \mathcal{Q}_S}} \frac{\mu(k)}{k^2} = \prod_{\substack{p \in \mathcal{P}_0 \cap \mathcal{P}_1 \\ p \notin S}} \left(1 - \frac{1}{p^2}\right).$$

Hence it follows that the asymptotic formula in Proposition 3.1 holds with the leading constant

$$c_{m_i, \mathcal{P}_i, S} = 4 \cdot \frac{\kappa_{0,S} \kappa_{1,S}}{\Gamma(\partial_0) \Gamma(\partial_1)} \cdot m_0^{1-\partial_0} m_1^{1-\partial_1} \cdot \frac{c_S\left(\frac{1}{m_0} + \frac{1}{m_1}\right)}{c_S\left(\frac{1}{m_0}\right) c_S\left(\frac{1}{m_1}\right)} \cdot \prod_{\substack{p \in \mathcal{P}_0 \cap \mathcal{P}_1 \\ p \notin S}} \left(1 - \frac{1}{p^2}\right),$$

where $\kappa_{0,S}, \kappa_{1,S}$ are given by (3-2). This therefore completes the proof of Proposition 3.1.

4. Orbifolds and étale orbifold morphisms

Campana [2004] related the study of fibrations $\pi : X \rightarrow Y$ of varieties over a fixed field k to orbifolds on the base. He studied *multiplicity orbifolds*, but since these are the only orbifolds in this paper we will simply call them *orbifolds*. In this section we summarise the construction of the most important invariant of orbifolds.

4.1. Orbifold pairs. Throughout this section, let k be an arbitrary field of characteristic 0.

Definition 4.1. An *orbifold* is a pair (B, Δ) , where B is a normal, proper k -scheme and Δ is a \mathbb{Q} -divisor

$$\Delta = \sum_D \left(1 - \frac{1}{m_D}\right) [D]$$

for positive integers m_D associated to prime divisors D on B . We call m_D the *multiplicity* of the orbifold over D .

Definition 4.2. Let (B, Δ) be an orbifold on a normal and proper k -variety B . A *finite étale (orbifold) morphism* is a morphism $\theta : C \rightarrow B$, with C normal, which is

- (i) finite,
- (ii) étale away from Δ ,
- (iii) has the property $e(D'/D) \mid m_D$ for any prime divisor $D' \mid D$ (meaning any prime divisor $D' \subseteq C$ above $D \subseteq B$), where $e(D'/D)$ is the ramification index.

Let us explain the use of the word étale. Consider a finite dominant morphism $\theta : C \rightarrow B$ between integral, normal, proper k -varieties. Then we can always endow B with an orbifold structure such that θ becomes a finite étale orbifold morphism by assigning $m_D = \text{lcm}\{e(D'/D) : D' \mid D\}$. If B has an orbifold divisor Δ under which θ is a finite étale orbifold morphism, then we can endow C with the \mathbb{Q} -divisor

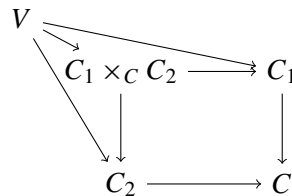
$$\Delta_C = \sum_{D'} \left(1 - \frac{1}{m_{D'}}\right) [D'], \quad \text{where } m_{D'} = \frac{m_D}{e(D'/D)}.$$

This is the unique orbifold structure on C such that the orbifold morphism $(C, \Delta_C) \rightarrow (B, \Delta)$ is étale in codimension 1, in the sense of [Campana 2011, Definition 2.21]. In the latter case, the Riemann–Hurwitz formula yields

$$K_{C, \Delta_C} = \theta^* K_{B, \Delta},$$

where $K_{B, \Delta} = K_B + \Delta$ is the *canonical divisor class* on an orbifold (B, Δ) . (This statement can be proven along similar lines to the proof of Proposition 4.7 (c).)

Proposition 4.3. Let $C_1, C_2 \rightarrow C$ be morphisms of normal k -varieties. Let $V = \overline{C_1 \times_C C_2}$ be the normalisation of the product $C_1 \times_C C_2$:



Let $D_V \subseteq V$ be a prime divisor lying above prime divisors $D_i \subseteq C_i$ and $D \subseteq C$. Then

$$e(D_V/D_1) = \frac{e_2}{\text{gcd}(e_1, e_2)},$$

where $e_i = e(D_i/D)$ for $i = 1, 2$.

Proof. Replacing the prime divisors with their generic points we can compute the normalisation étale locally over D . Hence we assume k is algebraically closed and consider the normalisation of the tensor product of the two homomorphisms $\varrho_i : k[[t]] \rightarrow k[[t_i]]$ given by $t \mapsto t_i^{e_i}$. The tensor product is

$R = k[[t_1, t_2]]/(t_1^{e_1} - t_2^{e_2})$ generated by the images of the t_i . Let us define $d = \gcd(e_1, e_2)$, and let $\zeta \in k$ be a primitive d -th root of unity. We can write R as the product

$$R = \prod_i k[[t_1, t_2]]/(t_1^{e_1/d} - \zeta^i t_2^{e_2/d}).$$

All factors are principal ideal domains, since polynomials $X^a - \lambda Y^b$ with $\lambda \in k^\times$ and $\gcd(a, b) = 1$ are irreducible over an algebraically closed field. We will compute the integral closure of each component separately. Let us write $\alpha_1 e_1 + \alpha_2 e_2 = d$ for $\alpha_i \in \mathbb{Z}$. Then $T = t_1^{\alpha_2} t_2^{\alpha_1}$ is integral in each factor since $T^{e_1/d} = t_2 \cdot (t_1^{e_1/d} / t_2^{e_2/d})^{\alpha_2}$ and $T^{e_2/d} = t_1 \cdot (t_2^{e_2/d} / t_1^{e_1/d})^{\alpha_1}$. It follows that

$$k[[t_1, t_2]]/(t_1^{e_1/d} - \zeta^i t_2^{e_2/d}) \hookrightarrow k[[T]]$$

is the integral closure. Finally, to compute $e(D_V/D_1)$, we look at the image of t_1 under the map

$$k[[t_1]] \rightarrow k[[T]],$$

which has valuation e_2/d . □

Remark 4.4. Campana [2011, Definition 11.1] defines the orbifold fundamental group $\pi_1(X|\Delta)$ for a complex orbifold $(X|\Delta)$ and relates it to covers unramified away from Δ . Likewise, we can define the (algebraic) orbifold fundamental group and relate it to the structure of all finite étale orbifold morphisms over a fixed base (B, Δ) of dimension 1. (Note that we could do this in arbitrary dimension if we allow finite étale morphisms to be defined away from a codimension-2 locus.) Consider the category $\mathbf{FEt}_{(B, \Delta)}$ of all finite étale orbifold morphisms to (B, Δ) , where the morphisms are given by B -morphisms. Given a point $\bar{x} \in B(\bar{k}) \setminus \text{supp}(\Delta)$, we have the fibre functor

$$F : \mathbf{FEt}_{(B, \Delta)} \rightarrow \mathbf{Sets}$$

given by $C \mapsto C_{\bar{x}}$, and one can show that $(\mathbf{FEt}_{(B, \Delta)}, F)$ is a Galois category. The only nontrivial part is to show that $\mathbf{FEt}_{(B, \Delta)}$ has products, but this follows from Proposition 4.3. In particular, this implies that, for any two finite étale covers of (C, ∂) , there is another cover mapping to both. We define the (algebraic) orbifold fundamental group $\pi_1^{\text{orb}}(B, \Delta)$ to be the automorphism group of the fibre functor F . Many relations between the topological and algebraic fundamental group can be directly translated to fundamental groups of orbifolds. For example, if $k \subseteq \mathbb{C}$ then

$$\pi_1^{\text{orb}}(B, \Delta) = \overline{\pi_1(B(\mathbb{C})|\Delta)}.$$

Campana [2011, Sections 11 and 12] studied the complex orbifold fundamental group and provided several results and conjectures about their structure.

For our application we will need the following definition.

Definition 4.5. Let G/k be a finite étale group and (B, Δ) an orbifold. Let $\theta : C \rightarrow B$ be a finite étale orbifold morphism endowed with a G -action on C , which is compatible with θ . We say that θ is a G -torsor (of orbifolds) if the restriction of θ away from the support of Δ is a G -torsor.

Since we are dealing with curves, it makes sense to talk about torsors. The natural morphism $G \times C \rightarrow C \times_B C$ is not necessarily an isomorphism over B , but it is so over $B \setminus \Delta$ by definition. Since $G \times C$ is a smooth curve over k , this morphism factors through the normalisation $G \times C \rightarrow \overline{C \times_B C} \rightarrow C \times_B C$. Now $G \times C \rightarrow \overline{C \times_B C}$ is a morphism between normal curves, which is an isomorphism on a dense open subset. Note that this agrees with the observation that $\overline{C \times_B C} \rightarrow C$ is unramified by Proposition 4.3; $\overline{C \times_B C}$ is just a union of copies of C .

4.2. Orbifold base of a fibration. As we saw in Section 1, we can associate a natural orbifold to any fibration. In this section we discuss this further before passing to our reasoning behind Conjecture 1.5.

Definition 4.6. Consider a fibration $\pi : X \rightarrow Y$, which we assume is a morphism between integral, normal, proper k -schemes such that the generic fibre is geometrically irreducible. For a prime divisor $D \subseteq Y$ with generic point η_D , we define m_D as the minimum multiplicity of the components of X_{η_D} as a divisor on X . The orbifold base of π is (Y, ∂_π) , where

$$\partial_\pi = \sum_D \left(1 - \frac{1}{m_D}\right) [D].$$

Possibly up to thin sets, we expect the geometry of the base orbifold (Y, ∂_π) to govern the arithmetic properties of the fibration. We henceforth focus our attention on standard fibrations $\pi : X \rightarrow \mathbb{P}^1$ defined over \mathbb{Q} , with the aim of interpreting the growth of the counting function $N_{\text{loc}}(\pi, B)$ that was defined in (1-1). Occasionally we will write $N_{\text{loc}}^\circ(\pi, B)$ for the same counting function but excluding the finitely many points in the orbifold divisors ∂_π .

Let us begin by discussing the conjectured power of B in Conjecture 1.5, which is equal to

$$2 - \deg \partial_\pi = -\deg(K_{\mathbb{P}^1, \partial_\pi}), \tag{4-1}$$

where $K_{\mathbb{P}^1, \partial_\pi} = K_{\mathbb{P}^1} + \partial_\pi$. The following result relates the geometry of π to the geometry of a normalisation of the fibre product of π with a finite cover.

Proposition 4.7. *Let $\pi : X \rightarrow \mathbb{P}^1$ be a standard fibration, and let*

$$\theta : \mathbb{P}^1 \rightarrow \mathbb{P}^1$$

be a (possibly ramified) finite cover of degree d . We define $\pi_\theta : X_\theta \rightarrow \mathbb{P}^1$ to be the normalisation of the fibre product of θ and π . Then we have the following properties.

- (a) $\pi_\theta : X_\theta \rightarrow \mathbb{P}^1$ is a standard fibration.
- (b) The orbifold multiplicities $m_{P'}$ for π_θ satisfy

$$m_{P'} \geq \frac{m_P}{e(P'/P)}$$

for any prime divisor P' of \mathbb{P}^1 , where $P = \theta(P')$. We have equality precisely when condition (iii) in Definition 4.2 is satisfied at P' .

(c) We have

$$\deg(K_{\mathbb{P}^1, \partial_{\pi_\theta}}) \geq d \deg(K_{\mathbb{P}^1, \partial_\pi}),$$

with equality precisely when θ is a finite étale orbifold morphism.

Proof. (a) This is clear from the definition.

(b) Consider a component Z' of the fibre of π_θ over a prime divisor P' of \mathbb{P}^1 . Suppose that Z' lies over $Z \subseteq X$ and P' lies over P . Let $m_{P'}(Z')$ and $m_P(Z)$ denote the multiplicities of these components in their respective fibres. We wish to apply Proposition 4.3 with $C_1 \rightarrow C$ being the morphism $\theta : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ and $C_2 \rightarrow C$ being the morphism $\pi : X \rightarrow \mathbb{P}^1$. Then $V \rightarrow C_1$ is the morphism $\pi_\theta : X_\theta \rightarrow \mathbb{P}^1$. It follows that

$$e(Z'/P') = \frac{e(Z/P)}{\gcd(e(Z/P), e(P'/P))}.$$

Hence, since the ramification indices over a codimension-1 point are precisely the multiplicities of the different components of the fibre, we obtain

$$m_{P'}(Z') = \frac{m_P(Z)}{\gcd(m_P(Z), e(P'/P))}.$$

Since $m_P(Z) \geq m_P$ and $\gcd(m_P(Z), e(P'/P)) \leq e(P'/P)$, we conclude

$$m_{P'}(Z') \geq m_P/e(P'/P)$$

for all components Z' in the fibre over P' .

Clearly, if $m_{P'} = m_P/e(P'/P)$, we have $e(P'/P) \mid m_P$. Now suppose that $e(P'/P) \nmid m_P$. To prove the statement we must show that there is a component Z' over P' with $m_{P'}(Z') = m_P/e(P'/P)$. By the definition of m_P , there exists a component Z over P with $m_P = m_P(Z)$. Now let Z' be any component over P' which lies over P . Then

$$m_{P'}(Z') = \frac{m_P(Z)}{\gcd(m_P(Z), e(P'/P))} = \frac{m_P}{\gcd(m_P, e(P'/P))} = \frac{m_P}{e(P'/P)}.$$

This concludes the proof of part (b).

(c) We will prove the result for orbifolds equipped with a degree- d morphism $(C', \partial') \rightarrow (C, \partial)$ for general smooth curves C and C' , in order to distinguish between the two copies of \mathbb{P}^1 . The statement is invariant under base change, so we can assume we are working over an algebraically closed field $k = \bar{k}$. We begin by noting that

$$\deg K_{C, \partial} = 2g(C) - 2 + \sum_{P \in C^{(1)}} \left(1 - \frac{1}{m_P}\right)$$

and

$$\deg K_{C', \partial'} = 2g(C') - 2 + \sum_{P' \in C'^{(1)}} \left(1 - \frac{1}{m_{P'}}\right).$$

The Riemann–Hurwitz formula yields

$$2g(C') - 2 = d(2g(C) - 2) + \sum_{P' \in C^{(1)}} (e(P'/P) - 1),$$

where $P = \theta(P')$. Hence

$$\deg K_{C', \partial'} = d(2g(C) - 2) + \sum_{P' \in C^{(1)}} \left(e(P'/P) - \frac{1}{m_{P'}} \right).$$

It now follows that

$$\begin{aligned} \deg K_{C', \partial'} - d \deg K_{C, \partial} &= \sum_{P' \in C^{(1)}} \left(e(P'/P) - \frac{1}{m_{P'}} \right) - d \sum_{P \in C^{(1)}} \left(1 - \frac{1}{m_P} \right) \\ &= \sum_{P \in C^{(1)}} \left[\left(\sum_{P' | P} e(P'/P) - d \right) + \left(\frac{d}{m_P} - \sum_{P' | P} \frac{1}{m_{P'}} \right) \right]. \end{aligned}$$

Using $\sum_{P' | P} e(P'/P) = d$, we see that the first terms all vanish and so

$$\deg K_{C', \partial'} - d \deg K_{C, \partial} = \sum_{P \in C^{(1)}} \sum_{P' | P} \left(\frac{e(P'/P)}{m_P} - \frac{1}{m_{P'}} \right).$$

This is clearly nonnegative by (b), and we have equality if and only if condition (iii) of Definition 4.2 is satisfied at all P' . □

In the setting of this result, it follows that the points in $N_{\text{loc}}(\pi, B)$ that are counted by $N_{\text{loc}}(\pi_\theta, H_\theta, B)$ are expected to contribute at most to the same order of B , where H_θ is the pullback height along θ . Indeed, in Conjecture 1.1, we have

$$N_{\text{loc}}(\pi_\theta, H_\theta, B) = O_\varepsilon((B^{1/d})^{\deg(-K_{\mathbb{P}^1, \partial\pi_\theta}) + \varepsilon})$$

for any $\varepsilon > 0$, where we use $B^{1/d}$ since H_θ is an $\mathcal{O}(d)$ -height on \mathbb{P}^1 . Hence, in the light of Proposition 4.7 (c), we should expect no higher-order contribution from $N_{\text{loc}}(\pi_\theta, H_\theta, B)$ to $N_{\text{loc}}(\pi, B)$. Moreover, we should obtain the same exponent of B when θ is a finite étale orbifold morphism.

We are now ready to address the possible power of $\log B$. Let $\pi : X \rightarrow \mathbb{P}^1$ be a standard fibration, and suppose that $\theta : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is a G -torsor of orbifolds under a finite étale group scheme G of degree d , as presented in Definition 4.5. We write $\theta_v : C_v \rightarrow \mathbb{P}^1$ for the twists of θ by $v \in H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), G)$. Finally, we shall write $\pi_v : X_v \rightarrow C_v$ for the normalisation of the pullback of π along θ_v , which is a torsor under the inner twists G_v of G [Skorobogatov 2001, p. 20]. We usually restrict to the v for which $C_v(\mathbb{Q}) \neq \emptyset$ and identify $C_v \cong \mathbb{P}^1$. For our applications, G will be abelian and we will have $G_v \cong G$.

We are now ready to compare the counting function $N_{\text{loc}}^\circ(\pi, B)$ with the counting functions

$$N_{\text{loc}}^\circ(\pi_v, H_v, B)$$

for various $v \in H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), G)$, where H_v is the pullback height along θ_v . (Note that this is an $\mathcal{O}(d)$ -height on the domain $C_v \cong \mathbb{P}^1$ of θ_v when $C_v(\mathbb{Q}) \neq \emptyset$.)

Proposition 4.8. *In the setting above we have the following:*

(a) *A point $x \in \mathbb{P}^1(\mathbb{Q})$ is counted by $N_{\text{loc}}^\circ(\pi, B)$ if and only if there exists $v \in H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), G)$ and $y \in C_v(\mathbb{Q})$ such that $\theta_v(y) = x$ and such that y is counted by $N_{\text{loc}}^\circ(\pi_v, H_v, B)$.*

(b) *We have*

$$N_{\text{loc}}^\circ(\pi, B) = \sum_{v \in H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), G)} \frac{1}{\#G_v(\mathbb{Q})} N_{\text{loc}}^\circ(\pi_v, H_v, B).$$

(c) *Let $\theta_v^{-1}(D) = \bigcup_{1 \leq i \leq s_D} E_v^{(i)}$ be a decomposition into irreducible components, and write $N_v^{(i)} = \kappa(E_v^{(i)})$ for their function fields. Then*

$$\Delta(\pi_v) = \sum_{D \in (\mathbb{P}^1)^{(1)}} \sum_{i=1}^{s_D} (1 - \delta_{D, N_{D,v}^{(i)}}(\pi)),$$

where $\delta_{D, N_{D,v}^{(i)}}$ is given by (1-5).

(d) *The expression $\Delta(\pi_v)$ only assumes finitely many values.*

Proof. (a) Let $U \subseteq \mathbb{P}^1$ be the image of the étale locus of θ . The restrictions $\theta_v : U_v \rightarrow U$ are G_v -torsors, and so we have a partition

$$U(\mathbb{Q}) = \bigsqcup_{v \in H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), G)} \theta_v(U_v(\mathbb{Q})).$$

Furthermore, the fibre of π_v over $y \in U_v(\mathbb{Q})$ is isomorphic to the fibre of π over $x = \theta_v(y)$. Hence one of these fibres is locally soluble precisely when the other is. Finally, since $\theta : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ has degree d , the pullback of the $\mathcal{O}(1)$ -height pulls back to an $\mathcal{O}(d)$ -height.

(b) This follows from the partition in (a) and the fact that each fibre has $\#G_v(\mathbb{Q})$ points.

(c) This directly follows from the definition of $\delta_{D,N}$ and π_v .

(d) This follows from Proposition 1.4. □

In the setting of Theorem 1.3, we consider μ_d -covers parametrised by $\mathbb{Q}^\times/\mathbb{Q}^{\times,d}$. The following result therefore follows from part (c) of Proposition 4.8.

Corollary 4.9. *We have $\Delta(\pi_v) = \Theta_v(\pi)$ in (1-9).*

In principle there might be infinitely many twists π_v for which $\Delta(\pi_v)$ differs from the expected exponent $\Delta(\pi)$ defined in (1-7). The following example illustrates an instance where the points counted by the covers for which $\Delta(\pi_v) = \Delta(\pi)$ can form a nontrivial cothin set in $\mathbb{P}^1(\mathbb{Q})$.

Example 4.10. Consider the fibration $\pi : X \rightarrow \mathbb{P}^1$ with three double fibres over 0, -1 and ∞ , together with precisely one other nonsplit fibre over 1 which has multiplicity 1 and is split by a quadratic extension K/\mathbb{Q} . Let C_v be the conic

$$v_1x_1^2 + v_2x_2^2 = x_0^2$$

in \mathbb{P}^2 defined by $v = (v_1, v_2) \in \mathbb{Q}^\times / \mathbb{Q}^{\times,2} \times \mathbb{Q}^\times / \mathbb{Q}^{\times,2}$. We apply the partition in part (b) of Proposition 4.8 with the full family of twists

$$\theta_v : C_v \rightarrow \mathbb{P}^1, \quad [x_0 : x_1 : x_2] \mapsto [v_1 x_0^2 : v_2 x_1^2].$$

This is the finest partition in the sense of Remark 4.4 since we have $\pi_1^{\text{orb}}(\mathbb{P}^1, \partial\pi) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and any θ_v is geometrically a universal orbifold cover. Consider the fibres $\theta_v^{-1}(1)$ as v varies, which on algebras are biquadratic étale \mathbb{Q} -algebras $\prod_i N_{1,v}^{(i)}$. Infinitely many of these contain the splitting field K of the fibre, and for such v we have

$$1 - \delta_{1,\mathbb{Q}} < 1 = 1 - \delta_{1,N_{1,v}^{(\alpha)}}(\pi) < \sum_i (1 - \delta_{1,N_{1,v}^{(i)}}(\pi)),$$

where α is such that $K \subseteq N_{1,v}^{(\alpha)}$. However, each of these infinitely many $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ -covers factors through only two $\mathbb{Z}/2\mathbb{Z}$ -covers. Hence the set of points counted through the v for which

$$1 - \delta_{1,\mathbb{Q}} \neq \sum_i (1 - \delta_{1,N_{1,v}^{(i)}}(\pi))$$

is a thin set. In the case of a nontrivial Galois action on the components of the multiple fibres, we will need to deal with them in a similar manner to conclude that the points counted in the covers θ_v for $\Delta(\pi_v) \neq \Delta(\pi)$ form a thin set.

5. A sparsity criterion

In this section we derive a sparsity condition for the fibres of a morphism $f : X \rightarrow Y$ to have a \mathbb{Q}_p -point under geometric conditions on X , Y and f . After fixing a model $f : \mathcal{X} \rightarrow \mathcal{Y}$ over \mathbb{Z}_S away from finitely many primes S , and after possibly enlarging S for $p \notin S$ and $\tilde{t} \in \mathcal{Y}(\mathbb{F}_p)$, we will give an exact criterion for which lifts $t \in \mathcal{Y}(\mathbb{Z}_p)$ of \tilde{t} we can lift an \mathbb{F}_p -point on the \mathbb{F}_p -scheme $\mathcal{X}_{\tilde{t}}$ to a \mathbb{Z}_p -point on \mathcal{X}_t .

The exact criterion, Theorem 5.5, is based on a version of Hensel’s lemma which takes into account the intersection multiplicities of a component of a fibre of f with the point in $\mathcal{X}_t(\mathbb{Z}_p)$, and the lifting is done in such a way that these required relations are preserved at each step. In order to include this information naturally, we use a logarithmic structure. For a basic introduction to log geometry with a view towards arithmetic applications, the reader is referred to [Loughran et al. 2020, §5].

5.1. Logarithmic lifting in families. Let k be a number field. Let X and Y be smooth, proper varieties over k , and let D and E be strict normal crossing divisors on X and Y , respectively, where $f^{-1}(E) \subseteq D$. Assume that the induced morphism $f : (X, D) \rightarrow (Y, E)$ is a *toroidal* morphism, i.e., a toroidal morphism between toroidal embeddings, or equivalently, a log smooth morphism of (Zariski) log regular schemes. Fix $Q \in Y(k)$. We want to understand when $f^{-1}(Q)$ is everywhere locally soluble.

Let S be a finite set of places including all *places of bad reduction for f* . This means that we have a good model $\tilde{f} : (\mathcal{X}, \mathcal{D}) \rightarrow (\mathcal{Y}, \mathcal{E})$ for f over $\mathcal{O}_{k,S}$ with the property that $\tilde{f}^{-1}(\mathcal{E}) \subseteq \mathcal{D}$ such that $(\mathcal{X}, \mathcal{D})$ and $(\mathcal{Y}, \mathcal{E})$ are still log regular, and such that \tilde{f} is still log smooth with respect to the divisorial log structures induced by \mathcal{D} and \mathcal{E} .

Let $v \notin S$ be a finite place of k . Let $\mathcal{Q}_v \in \mathcal{Y}(\mathcal{O}_v)$ be the unique lift of $Q \in Y(k)$ to an \mathcal{O}_v -point. We will give necessary and sufficient conditions for the existence of an \mathcal{O}_v -point \mathcal{P}_v on \mathcal{X} such that $f(\mathcal{P})_v = \mathcal{Q}_v$ for $v \gg 0$.

If $\mathcal{Q}_v \notin \mathcal{E}$, then the \mathcal{O}_v -point \mathcal{Q}_v can be seen as a morphism

$$\mathcal{Q}_v : (\text{Spec } \mathcal{O}_v)^\dagger \rightarrow (\mathcal{Y}, \mathcal{E})$$

of log schemes, where $(\text{Spec } \mathcal{O}_v)^\dagger$ is the scheme $\text{Spec } \mathcal{O}_v$ equipped with the divisorial log structure induced by the closed point. This morphism induces a morphism of associated *Kato fans*

$$F(\mathcal{Q}_v) : \text{Spec } \mathbb{N} \cong F((\text{Spec } \mathcal{O}_v)^\dagger) \rightarrow F(\mathcal{Y}, \mathcal{E}).$$

In other words, we get an \mathbb{N} -valued point $F(\mathcal{Q}_v) \in F(\mathcal{Y}, \mathcal{E})(\mathbb{N})$.

If \mathcal{Q}_v is the image of $\mathcal{P}_v \in \mathcal{X}(\mathcal{O}_v)$, then clearly $F(\mathcal{Q}_v)$ cannot lie anywhere in $F(\mathcal{Y}, \mathcal{E})(\mathbb{N})$; it needs to be an element of the potentially smaller set

$$\text{image}(F(\mathcal{X}, \mathcal{D})(\mathbb{N}) \rightarrow F(\mathcal{Y}, \mathcal{E})(\mathbb{N})).$$

This means that if $F(\mathcal{Q}_v)$ does *not* lie in the image of $F(\mathcal{X}, \mathcal{D})(\mathbb{N})$, then surely \mathcal{Q}_v cannot lift to an \mathcal{O}_v -point on \mathcal{X} . This is a *sparsity criterion* in the sense of [Loughran and Smeets 2016, §2] but still a rather naive one, since it does not take important arithmetic information into account.

Definition 5.1. Let $\overline{\mathcal{P}}_v$ be an \mathbb{F}_p -point on \mathcal{X} . With the notation above, we define $F(\mathcal{X}, \mathcal{D})(\mathbb{N})_{\overline{\mathcal{P}}_v}$ as the subset of $F(\mathcal{X}, \mathcal{D})(\mathbb{N})$ with the property that $\overline{\mathcal{P}}_v$ lies in the logarithmic stratum associated to the image of the closed point $\mathbb{N}_{>0}$ of $\text{Spec } \mathbb{N}$.

Proposition 5.2. *With notation as above, let $\overline{\mathcal{P}}_v$ be an \mathbb{F}_p -point on $\mathcal{X}_{\mathcal{Q}_v}$, and assume that $F(\mathcal{Q}_v)$ does not lie in*

$$\text{image}(F(\mathcal{X}, \mathcal{D})(\mathbb{N})_{\overline{\mathcal{P}}_v} \rightarrow F(\mathcal{Y}, \mathcal{E})(\mathbb{N})).$$

Then $\overline{\mathcal{P}}_v \in \mathcal{X}_{\mathcal{Q}_v}(\mathbb{F}_p)$ does not lift to $\mathcal{P}_v \in \mathcal{X}_{\mathcal{Q}_v}(\mathcal{O}_v)$.

Proof. Assume that $\overline{\mathcal{P}}_v$ lifts; i.e., $\mathcal{Q}_v = \tilde{f}(\mathcal{P}_v)$ for some $\mathcal{P}_v \in \mathcal{X}(\mathcal{O}_v)$ with $\overline{\mathcal{P}}_v = \mathcal{P}_v \bmod v$ (which is the image of $\text{Spec } \mathbb{F}_v$ under \mathcal{P}_v). Therefore the image of $F(\mathcal{P}_v) \in F(\mathcal{X}, \mathcal{D})(\mathbb{N})$ under the map $F(\mathcal{X}, \mathcal{D})(\mathbb{N}) \rightarrow F(\mathcal{Y}, \mathcal{E})(\mathbb{N})$ comes from $F(\mathcal{X}, \mathcal{D})(\mathbb{N})_{\overline{\mathcal{P}}_v}$, as desired. \square

Remark 5.3. In fact, the above sparsity condition can often be phrased in a more classical way. Let $\pi : X \rightarrow \mathbb{P}^1$ be a standard fibration, and suppose we have a projective model $\pi : \mathcal{X} \rightarrow \mathbb{P}_{\mathcal{O}_S}^1$ with \mathcal{X} smooth over \mathcal{O}_S . Let $h \in \mathcal{O}_S[x, y]$ be an irreducible binary form; in practice we will only need to consider the finitely many h for which the fibre of π over $V(h) \subseteq \mathbb{P}^1$ is nonsplit. After possibly enlarging S , we may argue as follows. Suppose that we have a point \mathcal{Q}_v in the fibre $\mathcal{X}_{\mathcal{P}_v}$ over a point $\mathcal{P}_v \in \mathbb{P}^1(\mathcal{O}_v)$ for $v \notin S$. Let \mathcal{Z}_i be the geometrically irreducible components of $\mathcal{X}_{V(h)}$ which contain $\overline{\mathcal{Q}}_v \in \mathcal{X}_{\overline{\mathcal{P}}_v}$. Then we may conclude that $v(h(\mathcal{P}_v))$ lies in the positive linear span of the multiplicities m_i of \mathcal{Z}_i . Indeed, in the

local ring of $\overline{\mathcal{Q}}_v$, we have the factorisation $\pi^*h = \prod_i h_i^{m_i}$. Hence

$$v(h(\mathcal{P}_v)) = v(\pi^*h(\mathcal{Q}_v)) = \sum m_i v(h_i(\mathcal{Q}_v)).$$

Moreover, since we are only considering the components containing $\overline{\mathcal{Q}}_v$, we clearly have $v(h_i(\mathcal{Q}_v)) > 0$ for each i . In this way one can use Proposition 5.2 to give a sparsity criterion for general fibrations that generalises [Loughran and Smeets 2016, Theorem 2.8] after excluding a subscheme of codimension at least 2 on the base.

We are considering the log smooth setting since then we can provide a converse result to Proposition 5.2 using the logarithmic Hensel lemma [Loughran et al. 2020, Proposition 5.13].

Proposition 5.4. *If $\overline{\mathcal{P}}_v$ is an \mathbb{F}_v -point on $\mathcal{X}_{\overline{\mathcal{Q}}_v}$, the following are equivalent:*

- (a) $\overline{\mathcal{P}}_v$ lifts to an \mathcal{O}_v -point on $\mathcal{X}_{\mathcal{Q}_v}$;
- (b) $F(\mathcal{Q}_v) \in \text{image}(F(\mathcal{X}, \mathcal{D})(\mathbb{N})_{\overline{\mathcal{Q}}_v} \rightarrow F(\mathcal{Y}, \mathcal{E})(\mathbb{N}))$.

Proof. Since we have already shown that (a) implies (b), it remains to prove the reverse implication. This is an application of [Loughran et al. 2020, Proposition 5.13]. Indeed, let $s^\dagger = \text{Spec } \mathbb{F}_v$, with the standard log structure of rank 1, and $S^\dagger = \text{Spec } \mathcal{O}_v$. Let $j : s^\dagger \rightarrow S^\dagger$ be the canonical closed immersion.

By assumption there is an element $p_v \in F(\mathcal{X}, \mathcal{D})(\mathbb{N})_{\overline{\mathcal{Q}}_v}$ which maps to $F(\mathcal{Q}_v) \in F(\mathcal{Y}, \mathcal{E})(\mathbb{N})$, and there is an \mathbb{F}_v -point $u : \text{Spec } \mathbb{F}_v \rightarrow X$ on the associated stratum of $(\mathcal{X}, \mathcal{D})$. We can uniquely make u into a morphism of log schemes $s^\dagger \rightarrow (\mathcal{X}, \mathcal{D})$ such that $F(u) = p_v$ under the identification $F(\mathbb{N}) = F(s^\dagger)$, similar to the proof of Proposition 6.1 in [Loughran et al. 2020].

Since $F(f)$ maps $F(u)$ to $F(\mathcal{Q}_v)$, we have a commutative diagram

$$\begin{array}{ccc} s^\dagger & \xrightarrow{u} & (\mathcal{X}, \mathcal{D}) \\ j \downarrow & & \downarrow \bar{f} \\ S^\dagger & \xrightarrow{\quad} & (\mathcal{Y}, \mathcal{E}) \\ & & \mathcal{Q}_v \end{array}$$

Now [Loughran et al. 2020, Proposition 5.13] provides a lift $S^\dagger \rightarrow (\mathcal{X}, \mathcal{D})$ of \mathcal{Q}_v . The morphism of schemes which underlies this lift is the \mathcal{O}_v -point \mathcal{P}_v we are looking for. □

5.2. Sparsity criteria. Using Proposition 5.4, we can give precise conditions for locally solubility. We allow ourselves to work over a general number field k/\mathbb{Q} and so define a *standard fibration* to be a dominant morphism $\pi : X \rightarrow \mathbb{P}^1$ with geometrically integral generic fibre such that X is a smooth, proper, geometrically irreducible k -variety.

Let E be the reduced divisor of \mathbb{P}^1 of the nonsplit fibres of π . Let D be the reduced divisor underlying $\pi^{-1}(E)$. By embedded resolutions of singularities, there exists a birational morphism $X' \rightarrow X$ such that the pullback D' of D has strict normal crossings. Since $X \setminus D \cong X' \setminus D'$ over \mathbb{P}^1 , we see that $N_{\text{loc}}(\pi', B)$ differs by a constant from $N_{\text{loc}}(\pi, B)$, where $\pi' : X' \rightarrow X \rightarrow Y$ is the composition. Thus, for the purposes of upper and lower bounds, we can assume without loss of generality that the reduced subschemes of the nonsplit fibres of π have strict normal crossings.

Theorem 5.5. *Let $X \rightarrow \mathbb{P}^1$ be a standard fibration whose nonsplit fibres in their reduced subscheme structure are sncd. There exists a finite set of primes S and a model $\mathcal{X} \rightarrow \mathbb{P}_{\mathcal{O}_S}^1$ such that the following holds for $v \notin S$. Fix a point $\mathcal{Q} \in \mathbb{P}^1(\mathcal{O}_S)$ for which the fibre $X_{\mathcal{Q}}$ is split. Then any \mathbb{F}_v -point $\overline{\mathcal{P}}_v \in \mathcal{X}_{\mathcal{Q}}(\mathbb{F}_v)$ lifts to a point $\mathcal{P}_v \in \mathcal{X}_{\mathcal{Q}}(\mathcal{O}_v)$ precisely if, for every closed point $V(h) \in (\mathbb{P}^1)^{(1)}$, we have that $v(h(\mathcal{Q}))$ lies in the positive linear span of the multiplicities m_i of the components of $\mathcal{X}_{V(h)}$ that contain $\overline{\mathcal{P}}_v$.*

From now on, when we have fixed a place $v \notin S$, we will assume that a closed point $V(h) \in (\mathbb{P}^1)^{(1)}$ is given by an \mathcal{O}_v -primitive irreducible form $h \in \mathcal{O}_v[x, y]$.

Note that the last condition in Theorem 5.5 is trivially satisfied for all closed points $V(h)$ for which $v(h(\mathcal{Q})) = 0$ and also for those for which $X_{V(h)}$ is split. By restricting S further, we can assume that there is at most one nonsplit fibre $X_{V(h)}$ for which we have to check this condition.

Proof of Theorem 5.5. By the definition of D on X and E on \mathbb{P}^1 , we see that $(X, D) \rightarrow (\mathbb{P}^1, E)$ is log smooth. For a suitable finite set of primes S , this extends to \mathcal{O}_S -schemes and divisors such that $\mathcal{D} \subseteq \mathcal{X}$ and $\mathcal{E} \subseteq \mathbb{P}_{\mathcal{O}_S}^1$ still have strict normal crossings and $(\mathcal{X}, \mathcal{D}) \rightarrow (\mathbb{P}_{\mathcal{O}_S}^1, \mathcal{E})$ is also log smooth. We will check that this model satisfies the condition.

Consider $\overline{\mathcal{P}}_v \in \mathcal{X}_{\mathcal{Q}}(\mathbb{F}_v)$, and let $V(h) \subseteq \mathbb{P}_{\mathcal{O}_S}^1$ be the unique nonsplit fibre containing $\overline{\mathcal{Q}}_v = \pi(\overline{\mathcal{P}}_v)$. Suppose that we can write $v(h(\mathcal{Q})) = \sum_i a_i m_i$, with $a_i > 0$ integers and m_i the multiplicities of the r components of $X_{v(h)}$ which contain $\overline{\mathcal{P}}_v$. Around $\overline{\mathcal{P}}_v$ and $\overline{\mathcal{Q}}_v$ the Kato fans have affine charts \mathbb{N}^r and \mathbb{N} . Under this identification, we have $F(\mathcal{Q}_v) = v(h(\mathcal{Q})) \in \mathbb{N}$, and $F(\mathcal{X}, \mathcal{D})(\mathbb{N}) \rightarrow F(\mathbb{P}_{\mathcal{O}_S}^1, \mathcal{E})(\mathbb{N})$ is given by $(u_i) \mapsto \sum m_i u_i$. Hence the result follows from Proposition 5.4. \square

Remark 5.6. In [Loughran and Smeets 2016, §2] the following was proven: if $v(h(\mathcal{Q})) = 1$, then $\mathcal{X}_{\mathcal{Q}}$ is a regular scheme. This implies that any \mathbb{F}_v -point on $\mathcal{X}_{\mathcal{Q}}$ which lies on the intersection of at least two components of the reduction $\mathcal{X}_{\mathcal{Q},v}$ does not lift to a \mathbb{Q}_v -point on $\mathcal{X}_{\mathcal{Q}}$. This last statement directly follows from our criterion above since then the valuation $v(h(\mathcal{Q})) = 1$ cannot possibly lie in the positive linear span of two positive integers.

The above conditions make it easy to check if an \mathbb{F}_v -point lifts. However, one cannot deduce the existence of \mathbb{F}_v -points purely from valuations and multiplicities, as explained by Loughran and Matthiesen [2024, Lemma 6.2]. In general, this only allows us to give necessary conditions for local solubility.

Corollary 5.7. *Let $X \rightarrow \mathbb{P}^1$ be a standard fibration, and let $Q \in \mathbb{P}^1(k)$. There exists a finite set of places S such that, for each $v \notin S$ with $X_Q(k_v) \neq \emptyset$, the following condition is satisfied: for every closed point $D = V(h) \in (\mathbb{P}^1)^{(1)}$, we have either $v(h(Q)) > m_D$, or $v(h(Q)) = m_D$ and v belongs to*

$$T_D = \{v \notin S : \text{Frob}_v \text{ fixes an element of } S_D\}.$$

(Recall that S_D is the set of geometric components of X_D of minimum multiplicity m_D .)

In the special case that the nonsplit fibres all lie above k -rational points in \mathbb{P}^1 , we can (after possibly extending the set S again) make this even more precise, as follows.

Corollary 5.8. *Let $X \rightarrow \mathbb{P}^1$ be a standard fibration, and let $Q \in \mathbb{P}^1(k)$. Assume that the nonsplit fibres of $X \rightarrow \mathbb{P}^1$ all lie above k -rational points. Then $X_Q(k_v) \neq \emptyset$ precisely if, for every $V(h) \in (\mathbb{P}^1)^{(1)}$, the fibre $X_{V(h)}$ has a stratum fixed by Frob_v which lies on the intersection of components of multiplicity m_i such that $v(h(Q))$ lies in the positive linear span of the m_i .*

Proof. We will start with S and $\mathcal{X} \rightarrow \mathbb{P}^1_{\mathcal{O}_S}$ as above. By the results above we have that $P_v \in X_Q(k_v)$ reduces to an \mathbb{F}_v -point on \mathcal{X} . Since this \mathbb{F}_v -point lifts we get the result.

For the inverse implication we will need to enlarge S , as follows. Firstly we do so to assume that all fibres of $\mathcal{X} \setminus \mathcal{D} \rightarrow \mathbb{P}^1_{\mathcal{O}_S} \setminus \mathcal{E}$ are geometrically integral. Consider an \mathbb{F}_v -point on $\mathbb{P}^1_{\mathcal{O}_S} \setminus \mathcal{E}$ for $v \notin S$. Its fibre in \mathcal{X} lies in the open stratum $\mathcal{X} \setminus \mathcal{D}$ and contains a smooth \mathbb{F}_v -point by Lang–Weil. Now let W be a geometric component of a nonopen stratum of (X, D) , which is defined over k'/k . The closure \mathcal{W} of W will have geometrically irreducible fibres over all but finitely many places of k' . Hence after enlarging S we see that \mathcal{W} has an $\mathbb{F}_{v'}$ -point for all $v' | v$ for $v \notin S$. Since there are only finitely many strata and each has again finitely many geometric components, we can enlarge S to make this true for all possible W .

Suppose now that Frob_v fixes a geometric component W of a stratum which has k'/k as its field of definition. Consider the multiplicities m_i of the components that contain W . Since Frob_v fixes W , we conclude that there is a place $v' | v$ of k' of residue degree 1. For this v' , we see that W contains an $\mathbb{F}_{v'} = \mathbb{F}_v$ -point. We can lift this point under the conditions in Theorem 5.5. □

6. Multiple fibres via the large sieve

We place ourselves in the setting of Theorems 1.2 and 1.3. Let $\pi : X \rightarrow \mathbb{P}^1$ be a standard fibration with orbifold divisor

$$\partial_\pi = \left(1 - \frac{1}{m_0}\right)[0] + \left(1 - \frac{1}{m_\infty}\right)[\infty],$$

in the notation of (1-4) for $m_0, m_\infty \in \mathbb{N}$. Note that $2 - \text{deg } \partial_\pi = 1/m_0 + 1/m_\infty$. We define $d = \text{gcd}(m_0, m_\infty)$. We shall apply the theory from Section 4 to the family of μ_d -torsors

$$\theta_v : \mathbb{P}^1 \rightarrow \mathbb{P}^1, \quad [x_0 : x_1] \rightarrow [v_0 x_0^d : v_1 x_1^d],$$

which are parametrised by $v = v_1/v_0 \in \mathbb{Q}^\times/\mathbb{Q}^{\times,d} = \text{H}^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mu_d)$. Let $\pi_v : X_v \rightarrow \mathbb{P}^1$ be the normalisation of the pullback of π along θ_v .

The main result of this section is the following, which pertains to the density of locally soluble fibres on the standard fibration $\pi_v : X_v \rightarrow \mathbb{P}^1$ relative to the pullback height H_v along θ_v . We denote by $\text{rad}(n) = \prod_{p|n} p$ the square-free radical of any $n \in \mathbb{N}$.

Proposition 6.1. *Let $\varepsilon > 0$, and let $v = v_1/v_0 \in \mathbb{Q}^\times/\mathbb{Q}^{\times,d}$. Then*

$$N_{\text{loc}}(\pi_v, H_v, B) \ll_\varepsilon c_{v,\varepsilon} B^{1/m_0+1/m_\infty},$$

where

$$c_{v,\varepsilon} = \frac{|v_0 v_1|^\varepsilon}{\text{rad}(v_0)|v_0|^{1/m_0} \text{rad}(v_1)|v_1|^{1/m_\infty}}. \tag{6-1}$$

Furthermore, if $|v_0 v_1| \leq B^\varepsilon$, then

$$N_{\text{loc}}(\pi_v, H_v, B) \ll_\varepsilon c_{v,\varepsilon} \frac{B^{1/m_0+1/m_\infty}}{(\log B)^{\Theta_v(\pi)}},$$

where $\Theta_v(\pi)$ is given by (1-9).

We shall begin the proof of this result in Section 6.2. Our argument is based on the large sieve, which is recalled in Section 6.1. Taking the result on faith for the moment, we proceed to show how it can be used to establish Theorems 1.2 and 1.3.

Remark 6.2. Proposition 6.1 is consistent with Conjecture 1.5 for a fixed choice of $v \in \mathbb{Q}^\times / \mathbb{Q}^{\times,d}$. Indeed, we have $H_v(x) = H(x)^d$, where $H(x)$ is an $\mathcal{O}(1)$ -height on \mathbb{P}^1 . It follows that

$$N_{\text{loc}}(\pi_v, B) = N_{\text{loc}}(\pi_v, H, B) = N_{\text{loc}}(\pi_v, H_v, B^d) \ll_v \frac{B^{d/m_0+d/m_\infty}}{(\log B)^{\Theta_v(\pi)}}.$$

The orbifold base of π_v is (X_v, ∂_{π_v}) , with

$$\partial_{\pi_v} = \left(1 - \frac{d}{m_0}\right)[0] + \left(1 - \frac{d}{m_\infty}\right)[\infty]$$

by part (b) of Proposition 4.7. It follows from (4-1) and part (c) of Proposition 4.7 that

$$\frac{d}{m_0} + \frac{d}{m_\infty} = -d \deg K_{\pi, \partial_{\pi}} = 2 - \deg \partial_{\pi_v}.$$

Moreover, $\Theta_v(\pi) = \Delta(\pi_v)$ by part (c) of Proposition 4.8.

Proof of Theorem 1.2. In this case there is only one multiple fibre above 0, and so $m_\infty = 1$ and $d = 1$. Thus $H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mu_d)$ is the trivial group, and it follows directly from Proposition 6.1 that

$$N_{\text{loc}}(\pi, B) \ll \frac{B^{1/m_0+1}}{(\log B)^{\Theta_1(\pi)}}.$$

We have already seen that $1/m_0 + 1 = 2 - \deg \partial_{\pi}$. Moreover, we saw that $\Theta_1(\pi) = \Delta(\pi)$ in (1-11). \square

Proof of Theorem 1.3. We appeal to the decomposition in part (b) of Proposition 4.8. This gives

$$N_{\text{loc}}(\pi, B) \ll \sum_{v=v_1/v_0 \in \mathbb{Q}^\times / \mathbb{Q}^{\times,d}} N_{\text{loc}}(\pi_v, H_v, B).$$

For any $\delta > 0$, we clearly have

$$\begin{aligned} \sum_{n>x} \frac{1}{\text{rad}(n)n^\delta} &< \sum_{n=1}^\infty \frac{(n/x)^{\delta/2}}{\text{rad}(n)n^\delta} = x^{-\delta/2} \prod_p \left(1 + \sum_{k=1}^\infty \frac{1}{p^{1+k\delta/2}}\right) \\ &\ll_\delta x^{-\delta/2}. \end{aligned}$$

Let $\varepsilon > 0$. In the light of the latter bound, it follows from the first part of Proposition 6.1 that there exists $\delta(\varepsilon) > 0$ such that the terms with $|v_0 v_1| > B^\varepsilon$ make an overall contribution $\mathcal{O}_\varepsilon(B^{1/m_0+1/m_\infty-\delta(\varepsilon)})$ to $N_{\text{loc}}(\pi, B)$. For the terms with $|v_0 v_1| \leq B^\varepsilon$, we apply the second part of Proposition 6.1.

This easily leads to the conclusion that

$$N_{\text{loc}}(\pi, B) \ll_{\varepsilon} B^{1/m_0+1/m_{\infty}-\delta(\varepsilon)} + \sum_{|v_0 v_1| \leq B^{\varepsilon}} c_{v, \varepsilon} \frac{B^{1/m_0+1/m_{\infty}}}{(\log B)^{\Theta_v(\pi)}} \ll_{\varepsilon} \frac{B^{1/m_0+1/m_{\infty}}}{(\log B)^{\Theta(\pi)}},$$

where $\Theta(\pi)$ is given by (1-10). The statement of the theorem follows, since we have already remarked that $1/m_0 + 1/m_{\infty} = 2 - \deg \partial_{\pi}$. □

6.1. The large sieve. We begin by stating the version of the large sieve that we shall use in this paper.

Lemma 6.3. *Let $m \in \mathbb{N}$, let $B_0, B_1 \geq 1$ and let $\Omega \subseteq \mathbb{Z}^2$. For each prime p , assume that there exists $\bar{\omega}(p) \in [0, 1)$ such that the reduction modulo p^m of Ω has cardinality at most $(1 - \bar{\omega}(p))p^{2m}$. Then*

$$\#\{x \in \Omega : |x_i| \leq B_i \text{ for } i = 0, 1\} \ll \frac{(B_0 + Q^{2m})(B_1 + Q^{2m})}{L(Q)}$$

for any $Q \geq 1$, where

$$L(Q) = \sum_{q \leq Q} \mu^2(q) \prod_{p|q} \frac{\bar{\omega}(p)}{1 - \bar{\omega}(p)}.$$

Proof. When $m = 1$, this is a straightforward rephrasing of the multidimensional large sieve worked out in [Kowalski 2008, Theorem 4.1]. The extension to $m > 1$ is routine and will not be explained here. □

6.2. Preliminary steps. Recall that $d = \gcd(m_0, m_{\infty})$. Henceforth, we usually write $v = (v_0, v_1) \in \mathbb{Z}_{\text{prim}}^2$ for the point $v = v_1/v_0 \in \mathbb{Q}^{\times}/\mathbb{Q}^{\times, d}$. We may clearly proceed under the assumption that v_0 and v_1 are both free of d -th powers.

Let S be a large enough finite set of primes, as required for the arguments in Section 5 to go through. Suppose that $E_1, \dots, E_r \in (\mathbb{P}^1)^{(1)}$ are the closed points distinct from 0 and ∞ , where π_v is not smooth. For each $1 \leq j \leq r$, assume that $E_j = V(h_j)$ for a square-free binary form $h_j \in \mathbb{Z}_S[x_0, x_1]$. We may further assume that h_j is irreducible over \mathbb{Q} and coprime to the monomial $x_0 x_1$, and that the coefficients of h_j are relatively coprime.

We proceed by defining the sets

$$T_0 = \{p \notin S : \text{Frob}_p \text{ fixes an element of } S_0\},$$

$$T_{\infty} = \{p \notin S : \text{Frob}_p \text{ fixes an element of } S_{\infty}\},$$

$$U_j = \{p \notin S : \text{Frob}_p \text{ fixes an element of } S_{E_j}\}$$

for $1 \leq j \leq r$. The fibre $X_{v, y}$ of the fibration $\pi_v : X_v \rightarrow \mathbb{P}^1$ has a \mathbb{Q}_p -point precisely if $X_{\pi_v(y)}$ does, and thus we can apply the sparsity conditions in Corollary 5.7. This yields the upper bound $N_{\text{loc}}(\pi_v, B) \leq M_v(B)$, where $M_v(B)$ is defined to be the number of $y = (y_0, y_1) \in \mathbb{Z}^2$ such that $\gcd(v_0 y_0, v_1 y_1) = 1$ and $\max\{|v_0 y_0^d|, |v_1 y_1^d|\} \leq B$, with

$$p \notin S \Rightarrow \begin{cases} [v_p(x_0) = m_0 \text{ and } p \in T_0] \text{ or } v_p(x_0) > m_0, \\ [v_p(x_1) = m_{\infty} \text{ and } p \in T_{\infty}] \text{ or } v_p(x_1) > m_{\infty}, \\ p \parallel h_j(\mathbf{x}) \Rightarrow p \in U_j, \end{cases}$$

where $(x_0, x_1) = (v_0 y_0^d, v_1 y_1^d)$. We write $v_0 = a_0 w'_0$ and $y_0 = b_0 z_0$, where $w'_0 z_0$ is coprime to all the primes in S and $p \mid a_0 b_0 \Rightarrow p \in S$. Let $p \notin S$. Then $v_p(w'_0 z_0^d) = m_0$ if and only if $v_p(w'_0) = 0$ and $v_p(z_0) = m_0/d$, since w'_0 is free of d -th powers. Similarly, if $v_p(w'_0 z_0^d) > m_0$, then either $v_p(z_0) > m_0/d$, or $v_p(z_0) = m_0/d$ and $p \mid w'_0$. This suggests that we may write

$$v_0 = a_0 w_0, \quad y_0 = b_0 s_0^{m_0/d} t_0^{m_0/d} u_0,$$

where

- $p \mid a_0 b_0 \Rightarrow p \in S$;
- $p \mid s_0 w_0 u_0 \Rightarrow p \notin S$;
- s_0 and t_0 are square-free;
- $p \mid w_0 \Rightarrow p \mid s_0$;
- $p \mid t_0 \Rightarrow p \in T_0$; and
- u_0 is $(m_0/d + 1)$ -full.

Similarly, we have a factorisation

$$v_1 = a_1 w_1, \quad y_1 = b_1 s_1^{m_\infty/d} t_1^{m_\infty/d} u_1,$$

where

- $p \mid a_1 b_1 \Rightarrow p \in S$;
- $p \mid s_1 w_1 u_1 \Rightarrow p \notin S$;
- s_1 and t_1 are square-free;
- $p \mid w_1 \Rightarrow p \mid s_1$;
- $p \mid t_1 \Rightarrow p \in T_\infty$; and
- u_1 is $(m_\infty/d + 1)$ -full.

There are $O_\varepsilon(|v_0 v_1|^\varepsilon)$ choices for $a_i, s_i, w_i \in \mathbb{Z}$ for $i = 0, 1$ by the standard estimate for the divisor function. We fix a choice of b_0, b_1, u_0, u_1 and write

$$A_0 = a_0 b_0^d s_0^{m_0} u_0^d w_0 \quad \text{and} \quad A_1 = a_1 b_1^d s_1^{m_\infty} u_1^d w_1. \tag{6-2}$$

Note that we have $\gcd(A_0, A_1) = 1$. Moreover, let

$$R_0 = \left(\frac{B}{|A_0|} \right)^{1/m_0}, \quad R_1 = \left(\frac{B}{|A_1|} \right)^{1/m_\infty},$$

and

$$g_j(\mathbf{t}) = h_j(A_0 t_0^{m_0}, A_1 t_1^{m_\infty}) \quad \text{for } 1 \leq j \leq r. \tag{6-3}$$

The binary form $g_j(t_0, t_1)$ is square-free and coprime to the monomial t_0t_1 since $h_j(t_0, t_1)$ satisfies these properties. For a (possibly infinite) set T of primes, let

$$\mathbf{1}_T(n) = \begin{cases} 1 & \text{if } p \mid n \Rightarrow p \in T, \\ 0 & \text{otherwise.} \end{cases}$$

In what follows, we will also write T^c to denote the complement of T in the full set of primes.

Then, with all this notation in mind, we have

$$M_v(B) \ll \sum_{v_0=a_0w_0} \sum_{v_1=a_1w_1} \sum_{\substack{b_0, b_1 \\ p \mid b_0b_1 \Rightarrow p \in S}} \sum_{u_0, u_1 \in \mathbb{Z}} L(R_0, R_1),$$

where

$$L(R_0, R_1) = \sum_{\substack{(t_0, t_1) \in \mathbb{Z}^2 \\ |t_0| \leq R_0, |t_1| \leq R_1}} \mu^2(t_0t_1) \mathbf{1}_{T_0}(t_0) \mathbf{1}_{T_\infty}(t_1) \prod_{j=1}^r \mathbf{1}_{U_j}^\sharp(t_0, t_1) \tag{6-4}$$

and where

$$\mathbf{1}_{U_j}^\sharp(t_0, t_1) = \begin{cases} 1 & \text{if } p \parallel g_j(\mathbf{t}) \Rightarrow p \in U_j, \\ 0 & \text{otherwise.} \end{cases}$$

The trivial bound for $L(R_0, R_1)$ is

$$L(R_0, R_1) \ll \frac{B^{1/m_0+1/m_\infty}}{|A_0|^{1/m_0}|A_1|^{1/m_\infty}} \ll \frac{B^{1/m_0+1/m_\infty}}{|s_0||v_0|^{1/m_0}|s_1||v_1|^{1/m_\infty}|b_0u_0|^{d/m_0}|b_1u_1|^{d/m_\infty}}$$

by (6-2). Clearly

$$|s_i| \gg \text{rad}(v_i) \quad \text{for } i = 0, 1 \tag{6-5}$$

for a suitable implied constant depending only on S . Note that

$$\sum_{\substack{|b_0| > J \\ p \mid b_0 \Rightarrow p \in S}} |b_0|^{-d/m_0} \ll \frac{1}{J^{d/m_0}}$$

for any $J \geq 1$. Similarly,

$$\sum_{\substack{|u_0| > J \\ u_0 \text{ is } (m_0/d+1)\text{-full}}} |u_0|^{-d/m_0} \ll \frac{1}{J^{d^2/(m_0(m_0+d))}}.$$

Let $\varepsilon > 0$. In what follows it will be convenient to recall the notation (6-1) for $c_{v,\varepsilon}$ in the statement of Proposition 6.1. It now follows that the overall contribution to $M_v(B)$ from parameters b_0, u_0 in the range $\min(|b_0|, |u_0|) > B^\varepsilon$ or parameters b_1, u_1 in the range $\min(|b_1|, |u_1|) > B^\varepsilon$ is clearly

$$\ll_\varepsilon c_{v,\varepsilon} B^{1/m_0+1/m_\infty-\varepsilon/(m_0^2m_\infty^2)}$$

since we have seen that there are $O_\varepsilon(|v_0v_1|^\varepsilon)$ choices for $a_i, s_i, w_i \in \mathbb{Z}$ associated to a particular choice of \mathbf{v} . Thus we deduce that

$$M_v(B) \ll_\varepsilon \sum_{v_0=a_0w_0} \sum_{v_1=a_0w_1} \sum_{\substack{|b_0|, |b_1| \leq B^\varepsilon \\ p \mid b_0b_1 \Rightarrow p \in S}} \sum_{|u_0|, |u_1| \leq B^\varepsilon} L(R_0, R_1) + c_{v,\varepsilon} B^{1/m_0+1/m_\infty-\varepsilon/(m_0^2m_\infty^2)}. \tag{6-6}$$

6.3. Application of the large sieve. We shall now apply Lemma 6.3 to estimate (6-4), which we shall apply with $m = 2$. Let $\Omega \subseteq \mathbb{Z}^2$ be the set of vectors $\mathbf{t} \in \mathbb{N}^2$ such that $\mathbf{1}_{T_0}(t_0)\mathbf{1}_{T_\infty}(t_1) = 1$ and for which $p \in U_j$ whenever there exists an index j such that $p \parallel g_j(\mathbf{t})$. For any prime $p \notin S$, let

$$A_0(p) = \{\mathbf{t} \in (\mathbb{Z}/p^2\mathbb{Z})^2 : p \mid t_0 \text{ and } p \notin T_0\}$$

and

$$A_\infty(p) = \{\mathbf{t} \in (\mathbb{Z}/p^2\mathbb{Z})^2 : p \mid t_1 \text{ and } p \notin T_\infty\}.$$

Similarly, let

$$B_j(p) = \{\mathbf{t} \in (\mathbb{Z}/p^2\mathbb{Z})^2 : p \parallel g_j(\mathbf{t}) \text{ and } p \notin U_j\}$$

for $1 \leq j \leq r$. Then $\#\Omega \bmod p^2 \leq (1 - \bar{\omega}(p))p^4$, where

$$\bar{\omega}(p) = \frac{\#(A_0(p) \cup A_\infty(p) \cup B_1(p) \cup \dots \cup B_r(p))}{p^4}.$$

In particular, we have $\bar{\omega}(p) \in [0, 1)$. The following result is concerned with estimating this quantity.

Lemma 6.4. *Let $p \notin S$, and let $d = \gcd(m_0, m_\infty)$. Then*

$$\bar{\omega}(p) = \frac{\mathbf{1}_{T_0^c}(p)}{p} + \frac{\mathbf{1}_{T_\infty^c}(p)}{p} + \sum_{j=1}^r \frac{\mathbf{1}_{U_j^c}(p)v_j(p; \mathbf{v})}{p^2} + O\left(\frac{\gcd(p, A_0A_1)}{p^2}\right),$$

where

$$v_j(p; \mathbf{v}) = \#\{\mathbf{t} \in \mathbb{F}_p^2 : h_j(v_0t_0^d, v_1t_1^d) = 0\}.$$

Proof. Recall that $\gcd(A_0, A_1) = 1$, that $g_j(t_0, t_1)$ is defined in (6-3), and that $g_j(t_0, t_1)$ is square-free and coprime to the monomial t_0t_1 . If $p \mid A_0A_1$, we take the trivial upper bound

$$\#(A_0(p) \cup A_\infty(p) \cup B_1(p) \cup \dots \cup B_r(p)) = O(p^3),$$

whence $\bar{\omega}(p) = O(1/p)$, which is satisfactory.

Suppose henceforth that $p \nmid A_0A_1$. We proceed by noting that the intersection of any two sets in the union $A_0(p) \cup A_\infty(p) \cup B_1(p) \cup \dots \cup B_r(p)$ contains $O(p^2)$ elements of $(\mathbb{Z}/p^2\mathbb{Z})^2$. Thus

$$\bar{\omega}(p) = \frac{\mathbf{1}_{T_0^c}(p)}{p} + \frac{\mathbf{1}_{T_\infty^c}(p)}{p} + \sum_{j=1}^r \frac{\#B_j(p)}{p^4} + O\left(\frac{1}{p^2}\right).$$

Turning to $\#B_j(p)$ for $j \in \{1, \dots, r\}$, we write $\mathbf{u} = \mathbf{x} + p\mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{F}_p^2$. Thus

$$\#\{\mathbf{t} \in (\mathbb{Z}/p^2\mathbb{Z})^2 : p^2 \mid g_j(\mathbf{t})\} = \sum_{\substack{\mathbf{x} \in \mathbb{F}_p^2 \\ g_j(\mathbf{x})=0}} \#\{\mathbf{y} \in \mathbb{F}_p^2 : \mathbf{y} \cdot \nabla g_j(\mathbf{x}) = -g_j(\mathbf{x})/p\}.$$

On enlarging S , we can assume that $\nabla g_j(\mathbf{x}) \neq \mathbf{0}$ for any \mathbf{x} in the sum. Thus each of the $O(p)$ values of \mathbf{x} produces $O(p)$ choices of \mathbf{y} , giving

$$\#\{\mathbf{t} \in (\mathbb{Z}/p^2\mathbb{Z})^2 : p^2 \mid g_j(\mathbf{t})\} = O(p^2).$$

Hence

$$\#B_j(p) = \mathbf{1}_{U_j^c}(p)p^2\#\{\mathbf{t} \in \mathbb{F}_p^2 : g_j(\mathbf{t}) = 0\} + O(p^2).$$

Putting this together, we have shown that

$$\bar{\omega}(p) = \frac{\mathbf{1}_{T_0^c}(p)}{p} + \frac{\mathbf{1}_{T_\infty^c}(p)}{p} + \sum_{j=1}^r \frac{\mathbf{1}_{U_j^c}(p)\lambda_j(p; A_0, A_1)}{p^2} + O\left(\frac{1}{p^2}\right),$$

where

$$\lambda_j(p; A_0, A_1) = \#\{\mathbf{t} \in \mathbb{F}_p^2 : h_j(A_0t_0^{m_0}, A_1t_1^{m_\infty}) = 0\}$$

for $1 \leq j \leq r$. In order to complete the proof of the lemma, it will suffice to prove that

$$\lambda_j(p; A_0, A_1) = v_j(p; \mathbf{v}) + O(1) \tag{6-7}$$

for $1 \leq j \leq r$, in the notation of the lemma.

To see this, let e be the least common multiple of m_0 and m_∞ , so that $e = m_0m_\infty/d$. We pick a generator $\alpha \in \mathbb{F}_p^*$ of $\mathbb{F}_p^*/(\mathbb{F}_p^*)^e$. Then it is easily confirmed that

$$\langle \alpha^{de/m_0} \rangle = (\mathbb{F}_p^*)^d/(\mathbb{F}_p^*)^{m_0} \quad \text{and} \quad \langle \alpha^{de/m_\infty} \rangle = (\mathbb{F}_p^*)^d/(\mathbb{F}_p^*)^{m_\infty}$$

on noting that $(\mathbb{F}_p^*)^{m_0}$ and $(\mathbb{F}_p^*)^{m_\infty}$ are subgroups of $(\mathbb{F}_p^*)^d$. (Indeed, to check the first equality, for example, it suffices to confirm that α^{de/m_0} has order m_0/d in \mathbb{F}_p^* .) The group $(\mathbb{F}_p^*)^d/(\mathbb{F}_p^*)^{m_0}$ has order $N_0 = \gcd(m_0, p - 1)$ and, likewise, $(\mathbb{F}_p^*)^d/(\mathbb{F}_p^*)^{m_\infty}$ has order $N_\infty = \gcd(m_\infty, p - 1)$. It follows from this that any nonzero d -th power in \mathbb{F}_p can be represented as $u^{m_0}\alpha^{edk/m_0}$ for some $k \in \mathbb{Z}/N_0\mathbb{Z}$, and such a representation is unique up to multiplication of u by one of the m_0 -th roots of unity in \mathbb{F}_p , of which there are N_0 .

Similarly, we can represent any nonzero d -th power in \mathbb{F}_p as $u^{m_\infty}\alpha^{ed\ell/m_\infty}$ for some $\ell \in \mathbb{Z}/N_\infty\mathbb{Z}$ in exactly N_∞ ways.

We will use this to partition the counting function v_j and remember to divide by N_0N_∞ when collecting the parts. Define

$$\lambda_j(p; A_0, A_1; k, \ell) = \#\{\mathbf{t} \in \mathbb{F}_p^2 : h_j(A_0t_0^{m_0}\alpha^{edk/m_0}, A_1t_1^{m_\infty}\alpha^{ed\ell/m_\infty}) = 0\}$$

for any $k \in \mathbb{Z}/N_0\mathbb{Z}$ and $\ell \in \mathbb{Z}/N_\infty\mathbb{Z}$. Let $\beta = \alpha^{-edk/m_0-ed\ell/m_\infty}$. On multiplying through by $\beta^{\deg(h_j)}$ and recalling that h_j is homogeneous, we obtain

$$\begin{aligned} \lambda_j(p; A_0, A_1; k, \ell) &= \#\{\mathbf{t} \in \mathbb{F}_p^2 : h_j(A_0t_0^{m_0}\alpha^{edk/m_0}\beta, A_1t_1^{m_\infty}\alpha^{ed\ell/m_\infty}\beta) = 0\} \\ &= \#\{\mathbf{t} \in \mathbb{F}_p^2 : h_j(A_0t_0^{m_0}\alpha^{-ed\ell/m_\infty}, A_1t_1^{m_\infty}\alpha^{-edk/m_0}) = 0\}. \end{aligned}$$

But $ed/m_\infty = m_0$ and $ed/m_0 = m_\infty$. Hence a simple change of variables yields

$$\lambda_j(p; A_0, A_1; k, \ell) = \lambda_j(p; A_0, A_1; 0, 0). \tag{6-8}$$

Let $v_j^*(p; A_0, A_1)$ denote the contribution to $v_j(p; A_0, A_1)$ from $t_0 t_1 \neq 0$, and also similarly for $\lambda_j^*(p; A_0, A_1; k, \ell)$. Then we may write

$$\begin{aligned} v_j(p; A_0, A_1) &= v_j^*(p; A_0, A_1) + O(1) \\ &= \frac{1}{N_0 N_\infty} \sum_{k \in \mathbb{Z}/N_0 \mathbb{Z}} \sum_{\ell \in \mathbb{Z}/N_\infty \mathbb{Z}} \lambda_j^*(p; A_0, A_1; k, \ell) + O(1) \\ &= \lambda_j^*(p; A_0, A_1; 0, 0) + O(1) \end{aligned}$$

by (6-8). Noting that $\lambda_j^*(p; A_0, A_1; 0, 0) = \lambda_j(p; A_0, A_1) + O(1)$, we have therefore shown that

$$\lambda_j(p; A_0, A_1) = v_j(p; A_0, A_1) + O(1).$$

At this point we recall the factorisation (6-2) together with the fact that $v_i = a_i s_i w_i$ for $i = 0, 1$. Hence, since $p \nmid A_0 A_1$, a simple change of variables shows that

$$v_j(p; A_0, A_1) = \#\{t \in \mathbb{F}_p^2 : h_j(v_0(b_0 s_0^{m_0/d} t_0)^d, v_1(b_1 s_1^{m_\infty/d} t_1)^d) = 0\} = v_j(p; \mathbf{v}),$$

from which the claim (6-7) follows. □

We will need to study the average size of $\bar{\omega}(p)$ as p varies. We break this into the following results.

Lemma 6.5. *We have*

$$\sum_{\substack{p \leq x \\ p \notin T_0}} \frac{1}{p} = (1 - \delta_{0, \mathbb{Q}}(\pi)) \log \log x + O(1)$$

and

$$\sum_{\substack{p \leq x \\ p \notin T_\infty}} \frac{1}{p} = (1 - \delta_{\infty, \mathbb{Q}}(\pi)) \log \log x + O(1)$$

in the notation of (1-5).

Proof. This is a straightforward consequence of the Chebotarev density theorem in the form presented in [Serre 2012, Theorem 3.4], for example. □

Our next result concerns the average behaviour of the function $v_j(p; \mathbf{v})$ in Lemma 6.4, as we average over primes $p \notin U_j$. This is more difficult and requires the use of notation introduced at the start of Section 2.2, which we recall here. For a number field F/\mathbb{Q} , let \mathcal{P}_F denote the set of primes $p \in \mathbb{Z}$ that are unramified in F and for which there exists a prime ideal $\mathfrak{p} \mid p\mathfrak{o}_F$ of residue degree 1. For any positive integer $m \leq [F : \mathbb{Q}]$, we write $\mathcal{P}_{F,m}$ for the subset of $p \in \mathcal{P}_F$ for which there are precisely m prime ideals above p of residue degree 1.

For each $j \in \{1, \dots, r\}$, define the étale algebra

$$N_{E_j, d, v_1/v_0} = \mathbb{Q}[x]/(r_j(x)),$$

where $r_j(x) = h_j(x^d, v_1/v_0)$. As in (1-8), this has a factorisation into number fields

$$N_{E_j, d, v_1/v_0} = N^{(1)} \times \dots \times N^{(s)},$$

where $N^{(k)} = N_{E_j, d, v}^{(k)}$ for $1 \leq k \leq s$, where the dependency of s on j is suppressed for legibility.

Lemma 6.6. *For each $j \in \{1, \dots, r\}$, we have*

$$\sum_{\substack{p \leq x \\ p \notin U_j}} \frac{v_j(p; \mathbf{v})}{p^2} = \sum_{k=1}^s (1 - \delta_{D, N^{(k)}}(\pi)) \log \log x + O(1 + \omega(v_0 v_1))$$

in the notation of (1-5), where $\omega(n)$ denotes the number of distinct prime factors of $n \in \mathbb{Z}$.

Proof. We have

$$\sum_{\substack{p \leq x \\ p \notin U_j}} \frac{v_j(p; \mathbf{v})}{p^2} = \sum_{\substack{p \leq x \\ p \notin U_j \\ p \nmid v_0 v_1}} \frac{v_j(p; \mathbf{v})}{p^2} + \sum_{\substack{p \leq x \\ p \notin U_j \\ p \mid v_0 v_1}} \frac{v_j(p; \mathbf{v})}{p^2}.$$

Since $\gcd(v_0, v_1) = 1$, the second term is seen to be

$$\ll \sum_{\substack{p \leq x \\ p \mid v_0 v_1}} \frac{1}{p} \ll \omega(v_0 v_1).$$

Next, we see that

$$\sum_{\substack{p \leq x \\ p \notin U_j \\ p \nmid v_0 v_1}} \frac{v_j(p; \mathbf{v})}{p^2} = \sum_{\substack{p \leq x \\ p \notin U_j \\ p \nmid v_0 v_1}} \frac{\#\{t \in \mathbb{F}_p : h_j(t^d, v_1/v_0) = 0\}}{p} + O(1).$$

Write $r_j(t) = h_j(t^d, v_1/v_0)$, and let $r_j(t) = r_j^{(1)}(t) \cdots r_j^{(s)}(t)$ be its factorisation into irreducible factors over \mathbb{Q} . Then $N^{(k)}$ is the number field $\mathbb{Q}[t]/(r_j^{(k)})$ for $1 \leq k \leq s$. We have

$$\sum_{\substack{p \leq x \\ p \notin U_j \\ p \nmid v_0 v_1}} \frac{v_j(p; \mathbf{v})}{p^2} = \sum_{k=1}^s \sum_{\substack{p \leq x \\ p \notin U_j \\ p \nmid v_0 v_1}} \frac{\#\{t \in \mathbb{F}_p : r_j^{(k)}(t) = 0\}}{p} + O(1).$$

To begin with, it follows from the prime ideal theorem that

$$\sum_{p \leq x} \frac{\#\{t \in \mathbb{F}_p : r_j^{(k)}(t) = 0\}}{p} = \log \log x + O(1 + \omega(v_0 v_1)).$$

Next, we note that $p \in U_j$ if and only if Frob_p fixes a component of S_{E_j} . Let \mathcal{F}_j denote the set of fields of definition of the elements of S_{E_j} . Then, for any $p \notin S$, the condition $p \in U_j$ is equivalent to the condition $p \in \mathcal{P}_{\mathcal{F}_j} := \bigcup_{F \in \mathcal{F}_j} \mathcal{P}_F$. Likewise, for any positive integer $m \leq [N^{(k)} : \mathbb{Q}]$, we will have $\#\{t \in \mathbb{F}_p : r_j^{(k)}(t) = 0\} = m$ if and only if $p \in \mathcal{P}_{N^{(k)}, m}$. Hence

$$\sum_{\substack{p \leq x \\ p \notin U_j}} \frac{v_j(p; \mathbf{v})}{p^2} = \sum_{k=1}^s \left(\log \log x - \sum_{m=1}^{[N^{(k)} : \mathbb{Q}]} m \sum_{\substack{p \leq x \\ p \in \mathcal{P}_{N^{(k)}, m} \cap \mathcal{P}_{\mathcal{F}_j}}} \frac{1}{p} \right) + O(1 + \omega(v_0 v_1)).$$

The remaining sum over primes is susceptible to a further application of the Chebotarev density theorem. Once coupled with Theorem 2.3 and (2-1), this leads to the statement of the lemma. □

We may combine the previous two results to produce a lower bound for the quantity $L(Q)$ in Lemma 6.3, with the choice of $\bar{\omega}(p)$ from Lemma 6.4.

Lemma 6.7. *For any $\varepsilon > 0$, we have the lower bound*

$$L(Q) \gg_{\varepsilon} \frac{(\log Q)^{\Theta_v(\pi)}}{|A_0 A_1|^{\varepsilon}},$$

where $\Theta_v(\pi)$ is given by (1-9).

Proof. Since $1 - \bar{\omega}(p) \leq 1$, we have

$$L(Q) \geq \sum_{q \leq Q} \mu^2(q) \prod_{p|q} \bar{\omega}(p).$$

There are many results in the literature concerning mean values of nonnegative arithmetic functions. However, we can get by with the relatively crude lower bound found in [Friedlander and Iwaniec 2010, Theorem A.3], which is based on an application of Rankin’s trick. Let $\gamma : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ be a multiplicative arithmetic function that is supported on square-free integers and which satisfies

$$\sum_{y < p \leq x} \gamma(p) \log p \leq a \log(x/y) + b \tag{6-9}$$

for any $x > y > 2$ for appropriate constants $a, b > 0$. Then it follows from [Friedlander and Iwaniec 2010, Theorem A.3] that

$$\sum_{n \leq x} \gamma(n) \gg \prod_{p \leq x} (1 + \gamma(p)), \tag{6-10}$$

where the implied constant is allowed to depend on a and b . We seek to apply this with

$$\gamma(n) = \mu^2(n) \prod_{p|n} \bar{\omega}(p).$$

It is clear from Lemma 6.4 that $\bar{\omega}(p) = O(1/p)$. Hence

$$\sum_{y < p \leq x} \gamma(p) \log p \ll 1 + \sum_{y < p \leq x} \frac{\log p}{p} \ll 1 + \log(x/y)$$

uniformly in v_0 and v_1 . Hence (6-9) holds for $a, b = O(1)$, and it follows from (6-10) that

$$L(Q) \gg \prod_{p \leq Q} (1 + \bar{\omega}(p))$$

for an absolute implied constant. On appealing once more to Lemma 6.4, we find that

$$\log \left(\prod_{p \leq Q} (1 + \bar{\omega}(p)) \right) = \sum_{\substack{p \leq Q \\ p \notin T_0}} \frac{1}{p} + \sum_{\substack{p \leq Q \\ p \notin T_{\infty}}} \frac{1}{p} + \sum_{j=1}^r \sum_{\substack{p \leq Q \\ p \notin U_j}} \frac{v_j(p; \mathbf{v})}{p^2} + O(1 + \omega(A_0 A_1)).$$

These sums are estimated using Lemmas 6.5 and 6.6, leading to the conclusion that

$$\log \left(\prod_{p \leq Q} (1 + \bar{\omega}(p)) \right) = \tilde{\Theta}(\pi, v_1/v_0) \log \log Q + O(1 + \omega(A_0 A_1)),$$

where

$$\tilde{\Theta}(\pi, v_1/v_0) = 2 - \delta_{0,\mathbb{Q}}(\pi) - \delta_{\infty,\mathbb{Q}}(\pi) + \sum_{j=1}^r \sum_{k=1}^s (1 - \delta_{E_j, N_{E_j, d, v_1/v_0}^{(k)}}(\pi))$$

in the notation of (1-5). Clearly $\tilde{\Theta}(\pi, v_1/v_0) = \Theta_v(\pi)$, the latter being defined in (1-9). Hence the statement of the lemma follows on exponentiating and using the fact that $\omega(n) \ll \log |n| / \log \log |n|$ for any nonzero $n \in \mathbb{Z}$. □

6.4. Completion of the proof of Proposition 6.1. We begin by focussing on the estimation of the quantity $L(R_0, R_1)$ that was defined in (6-4). In view of (6-2), we see that

$$A_0 = v_0(b_0 s_0^{m_0/d} u_0)^d \quad \text{and} \quad A_1 = v_1(b_1 s_1^{m_\infty/d} u_1)^d.$$

Recall that $s_i \mid v_i$ for $i = 0, 1$. Taking $Q = B^\varepsilon$, we note that

$$R_0^{m_0} = \frac{B}{|A_0|} \geq \frac{B}{|v_0(s_0 b_0 u_0)^{m_0}|} \geq B^{1-(1+3m_0)\varepsilon} \geq Q^{4m_0},$$

provided that $\varepsilon \leq 1/(1+7m_0)$. Similarly, we can assume that $R_1 \geq Q^4$ if $\varepsilon > 0$ is chosen to be sufficiently small. Hence, with these choices, we have

$$(R_0 + Q^4)(R_1 + Q^4) \ll R_0 R_1 \ll \frac{B^{1/m_1+1/m_\infty}}{|A_0|^{1/m_0} |A_1|^{1/m_\infty}}.$$

We may now apply Lemma 6.7 in Lemma 6.3 to deduce that

$$L(R_0, R_1) \ll_\varepsilon \frac{B^{1/m_0+1/m_\infty}}{|A_0|^{1/m_0} |A_1|^{1/m_\infty}} \cdot \frac{|A_0 A_1|^\varepsilon}{(\log B)^{\Theta_v(\pi)}}.$$

Substituting into (6-6), recalling (6-5) and summing over b_0, b_1, u_0, u_1 , the statement of Proposition 6.1 easily follows.

7. Examples: lower bounds and asymptotics

Let $\pi : X \rightarrow \mathbb{P}^1$ be a standard fibration. It is clear from the constructions in Section 5 that we are only able to interpret local solubility conditions outside a finite set of places of S which depends on π . This set S should contain a set of places for which Corollary 5.8 holds, and such a set can be determined explicitly. With more work one might be able to incorporate local solubility at places in S , but this should not change the order of growth, which is the main interest in this paper. Accordingly, for any finite set S of primes, we introduce the counting function

$$N_{\text{loc}, S}(\pi, B) = \#\{x \in \mathbb{P}^1(\mathbb{Q}) \cap \pi(X(A_{\mathbb{Q}}^S)) : H(x) \leq B\},$$

where H is the usual height function on $\mathbb{P}^1(\mathbb{Q})$ and $A_{\mathbb{Q}}^S$ is the set of adèles away from S . We clearly have $N_{\text{loc}, S}(\pi, B) \geq N_{\text{loc}}(\pi, B)$, and we expect these two counting functions to have the same order of magnitude.

We shall prove several results about Halphen surfaces. Let $m > 1$ be an integer. A *Halphen pencil* is a geometrically irreducible pencil of plane curves of degree $3m$ with multiplicity m at nine base points P_1, \dots, P_9 . We let X be the *Halphen surface of order m* obtained by blowing up \mathbb{P}^2 at these nine points, as introduced in [Halphen 1882]. We shall assume that P_1, \dots, P_9 are globally defined over \mathbb{Q} , so that X is a smooth, proper, geometrically integral surface defined over \mathbb{Q} . In fact, X is a rational elliptic surface and we obtain a standard morphism $\pi : X \rightarrow \mathbb{P}^1$ such that there exists a unique fibre of multiplicity m . In particular, π does not admit a section.

7.1. Lower bounds. In this section we establish an array of lower bounds for $N_{\text{loc},S}(\pi, B)$. The following result demonstrates that Conjecture 1.5 would be false with the exponent $\Delta(\pi)$ and that it is indeed sometimes necessary to take a smaller exponent.

Theorem 7.1. *Let $\pi : X \rightarrow \mathbb{P}^1$ be a standard fibration. Assume it only has nonsplit fibres above 0, 1 and ∞ , comprising geometrically irreducible double fibres over 0 and ∞ , and a nonsplit fibre of multiplicity 1 above 1 that is split by a quadratic extension. Then there is a finite set of places S such that*

$$B \ll N_{\text{loc},S}(\pi, B) \ll B.$$

Proof. Suppose that $F = \mathbb{Q}(\sqrt{d})$ is the quadratic extension that splits the fibre above 1 for square-free $d \in \mathbb{Z}$. Then it is clear that

$$0 \leq \Theta(\pi) = \min_{K/\mathbb{Q} \text{ quadratic}} (1 - \delta_{1,K}(\pi)) \leq 1 - \delta_{1,F}(\pi) = 0.$$

Hence the upper bound is a direct consequence of Theorem 1.3.

For the lower bound, we compose the exact counting problem using Corollary 5.8. Thus there exists a finite set of places S , containing the prime divisors of $2d$, such that

$$N_{\text{loc},S}(\pi, B) = \frac{1}{2} \# \left\{ (a, b) \in \mathbb{Z}_{\text{prim}}^2 : \begin{array}{l} |a|, |b| \leq B, p \notin S \Rightarrow [2 \mid v_p(a) \text{ and } 2 \mid v_p(b)], \\ [p \notin S \text{ and } p \mid a - b] \Rightarrow p \in \mathcal{P}_F \end{array} \right\}.$$

The lower bound is provided by taking pairs (a, b) of the form (u^2, dv^2) . □

In this result we have $2 - \text{deg } \partial_\pi = 1$, so that the exponent of B matches the predicted exponent of B in Conjectures 1.1 and 1.5. We also have

$$\delta_{0,\mathbb{Q}}(\pi) = \delta_{\infty,\mathbb{Q}}(\pi) = 1 \quad \text{and} \quad \delta_{1,\mathbb{Q}}(\pi) = \frac{1}{2},$$

so that $\Delta(\pi) = \frac{1}{2}$. However, we saw in the proof that $\Theta(\pi) = 0$. Thus Theorem 7.1 is in agreement with Conjecture 1.5.

Let us describe what is going on geometrically. Consider the finite étale orbifold μ_2 -cover $\theta_v : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ given by $(x : y) \mapsto (x^2 : vy^2)$ and the pullback fibrations $\pi_v : X_v \rightarrow \mathbb{P}^1$ obtained from normalisation of the pullback of π along θ_v . By Proposition 4.7, we see that the two double fibres of π pull back to components of multiplicity 1 on π_v . Also, all fibres which do not lie over 1 in the composition $X_v \xrightarrow{\pi_v} \mathbb{P}^1 \xrightarrow{\theta_v} \mathbb{P}^1$ are split. We proceed by studying the fibres over 1.

First we study the fibre of 1 in θ_v . For $v \in \mathbb{Q}^\times/\mathbb{Q}^{\times,2}$, we have $\theta^{-1}(1) = \text{Spec } A$, where A is the degree-2 étale algebra $\mathbb{Q}(\sqrt{v})$ if $v \notin \mathbb{Q}^{\times,2}$ and $\mathbb{Q} \times \mathbb{Q}$ for $v \in \mathbb{Q}^{\times,2}$. This gives

$$\Delta(\pi_v) = \sum_{D'|D} (1 - \delta_{D'}(\pi_v)) = \begin{cases} 0 & \text{if } v \equiv d, \\ \frac{1}{2} + \frac{1}{2} = 1 & \text{if } v \equiv 1, \\ \frac{1}{2} & \text{otherwise,} \end{cases}$$

where the sum ranges over all points D' lying above $D = 1 \in (\mathbb{P}^1)^{(1)}$. In the first case, the fibre over 1 (which is split by F) pulls back to an F -point and becomes split. In the second case, the fibre pulls back to two \mathbb{Q} -points. In the last case, the fibre is irreducible and its residue field is linearly disjoint from the splitting field, and we obtain $\Delta(\pi_v) = \Delta(\pi)$, in general.

Theorem 7.1 indicates that the main contribution to the point count comes from the single cover π_d . If we were to exclude the thin set of points coming from this cover, we are left with infinitely many covers π_v , with $\Delta(\pi_v) = \Delta(\pi)$ for $v \neq 1$. Proposition 4.7 (b) implies that the covers have no multiple fibres, since it gives

$$m_{P'} = \frac{m_P}{\gcd(m_P, e(P'/P))} = \frac{2}{\gcd(2, 2)} = 1$$

for each $P' \mid 0, \infty$. Hence, in the light of the original Loughran–Smeets conjecture [2016, Conjecture 1.6], we expect the remaining covers to contribute order $B/\sqrt{\log B}$ to the counting function, apart from the cover corresponding to 1, which should contribute order $B/\log B$.

Our second lower bound deals with the case of precisely two nonsplit fibres and is consistent with Conjecture 1.5 since $\deg \partial_\pi = 2 - 1/m_0 - 1/m_\infty$.

Theorem 7.2. *Let $\pi : X \rightarrow \mathbb{P}^1$ be a standard fibration for which the only nonsplit fibres lie over 0 and ∞ . Then there is a finite set of places S such that*

$$N_{\text{loc},S}(\pi, B) \gg \frac{B^{1/m_0+1/m_\infty}}{(\log B)^{\Delta(\pi)}}.$$

Proof. We begin by using Corollary 5.8 to give explicit conditions for local solubility away from S after passing to an sncd model $X' \rightarrow \mathbb{P}^1$. This leads to the conclusion that $N_{\text{loc},S}(\pi, B)$ is equal to the number of $x = (x_0 : x_1) \in \mathbb{P}^1(\mathbb{Q})$ with $H(x) \leq B$ such that, for each $i \in \{0, 1\}$ and every $p \notin S$, Frob_p fixes a collection of intersecting components Z_j of X'_{D_i} such that $v_p(x_i) \in \langle m(Z_j) \rangle_{\mathbb{N}}$, where $D_i = V(x_i)$. The following is clearly a sufficient condition for the fibre over x to have a \mathbb{Q}_p -point: for all i , the Frobenius Frob_p fixes a component of Z of minimal multiplicity in X'_{D_i} and $m(Z) \mid v_p(x_i)$. The density ∂_i of rational primes p for which Frob_p fixes an element of S_{D_i} is equal to $\delta_{D_i}(\pi) = \delta_{D_i, \kappa(D_i)}(\pi)$ in the notation of (1-5). Hence the statement of the theorem now follows from Proposition 3.1 and (1-7). □

7.2. Halphen surfaces with one nonsplit fibre. Generically, a Halphen surface has no other nonsplit fibres apart from the multiple one. Even in these cases the counting problem still depends on the Galois action on the components of the multiple fibres and how these components intersect. We record some results which illustrate this phenomenon; it will be convenient to keep in mind the notation (3-1).

We begin with the following result, which agrees with Conjecture 1.5 since $\deg \partial_\pi = 1 - 1/m$ and $\Delta(\pi) = 0$.

Theorem 7.3. *Let $X \rightarrow \mathbb{P}^1$ be a Halphen surface with a single nonsplit fibre over 0, that is the fibre of multiplicity m . Suppose that this fibre has a geometric component fixed by $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then there exists a finite set S such that*

$$N_{\text{loc},S}(\pi, B) \sim c_{\pi,S} B^{1+1/m},$$

where

$$c_{\pi,S} = \frac{2c_S(1 + \frac{1}{m})}{\zeta(2)c_S(\frac{1}{m})} \prod_{p \in S} \left(1 + \frac{1}{p}\right)^{-1}.$$

Proof. By Corollary 5.8, we see that there is a finite set of places S such that

$$N_{\text{loc},S}(\pi, B) = \frac{1}{2} \#\{(a, b) \in \mathbb{Z}_{\text{prim}}^2 : |a|, |b| \leq B, p \notin S \Rightarrow m \mid v_p(a)\}.$$

We may apply Proposition 3.1 with $m_0 = m$ and $m_1 = 1$, and with $\mathcal{P}_0 = \mathcal{P}_1$ equal to the full set of rational primes. In particular $\partial_0 = \partial_1 = 1$, and it follows that $N_{\text{loc},S}(\pi, B) \sim c_{\pi,S} B^{1+1/m}$ as $B \rightarrow \infty$, where

$$c_{\pi,S} = \frac{2c_S(1 + \frac{1}{m})}{c_S(1)c_S(\frac{1}{m})} \prod_{p \notin S} \left(1 - \frac{1}{p^2}\right) \prod_{p \in S} \left(1 - \frac{1}{p}\right)^2$$

in the notation of (3-1). The statement easily follows on simplifying the expression for the constant. \square

The following two results agree with Conjecture 1.5 since in both cases we have $\deg \partial_\pi = 1 - 1/m$ and $\Delta(\pi) = \frac{2}{3}$. Moreover, in these two examples, we have multiple fibres which do not have a geometrically integral component. This demonstrates the need to define (1-5) in terms of S_D for each divisor D , which allows us to work with the Galois action on the components of a fibre of minimum multiplicity.

Theorem 7.4. *Let $X \rightarrow \mathbb{P}^1$ be a Halphen surface with a single nonsplit fibre over 0, that is the fibre of multiplicity m . Suppose that this fibre consists of three conjugate lines split by a cubic Galois extension K/\mathbb{Q} that do **not** all meet in a point. Then there exists a finite set S such that*

$$N_{\text{loc},S}(\pi, B) \sim c_{\pi,S} \frac{B^{1+1/m}}{(\log B)^{2/3}},$$

where

$$c_{\pi,S} = \frac{2m^{2/3}c_S(1)^{1/3}c_S(1 + \frac{1}{m})}{\Gamma(\frac{1}{3})c_S(\frac{1}{m})} \prod_{\substack{p \in \mathcal{P}_K \\ p \notin S}} \left(1 + \frac{1}{p}\right) \left(1 - \frac{1}{p}\right)^{1/3} \prod_{\substack{p \notin \mathcal{P}_K \\ p \notin S}} \left(1 - \frac{1}{p}\right)^{1/3}.$$

Proof. Suppose that the three conjugate lines are split by the cubic Galois extension K/\mathbb{Q} . By Corollary 5.8, we see that there is a finite set of places S such that $N_{\text{loc},S}(\pi, B)$ is equal to

$$\frac{1}{2} \#\{(a, b) \in \mathbb{Z}_{\text{prim}}^2 : |a|, |b| \leq B, [p \notin S \text{ and } p \mid a] \Rightarrow [m \mid v_p(a) \text{ and } p \in \mathcal{P}_K]\},$$

where \mathcal{P}_K is the set of rational primes p that are unramified in K and split completely. We may apply Proposition 3.1 with $m_0 = m$ and $m_1 = 1$, and with $\mathcal{P}_0 = \mathcal{P}_K$ and \mathcal{P}_1 equal to the full set of rational primes. In particular, $\partial_0 = \frac{1}{3}$ and $\partial_1 = 1$. It follows that

$$N_{\text{loc},S}(\pi, B) \sim c_{\pi,S} \frac{B^{1+1/m}}{(\log B)^{2/3}},$$

where

$$c_{\pi,S} = \frac{2m^{2/3}}{\Gamma(\frac{1}{3})} \cdot \frac{c_S(1 + \frac{1}{m})}{c_S(\frac{1}{m})} \prod_{\substack{p \in \mathcal{P}_K \\ p \notin S}} \left(1 - \frac{1}{p^2}\right) \prod_{p \in \mathcal{P}_K \cap S} \left(1 - \frac{1}{p}\right) \prod_{p \in \mathcal{P}_K} \left(1 - \frac{1}{p}\right)^{-2/3} \prod_{p \notin \mathcal{P}_K} \left(1 - \frac{1}{p}\right)^{1/3}.$$

The statement of the proposition follows on simplifying this expression. □

The next result agrees with Conjecture 1.5 since

$$\deg \partial_\pi = 1 - \frac{1}{m} \quad \text{and} \quad \Delta(\pi) = \frac{2}{3}.$$

Theorem 7.5. *Let $X \rightarrow \mathbb{P}^1$ be a Halphen surface with a single nonsplit fibre over 0, that is the fibre of multiplicity m . Suppose that this fibre consists of three conjugate lines split by a cubic Galois extension K/\mathbb{Q} that do meet in a point. Then there exists a finite set S such that*

$$\frac{B^{1+1/m}}{(\log B)^{2/3}} \ll N_{\text{loc},S}(\pi, B) \ll \frac{B^{1+1/m}}{(\log B)^{2/3}}.$$

Proof. The upper bound follows from Theorem 1.2. The lower bound was proven in Theorem 7.2. □

Theorem 7.5 illustrates the need for the nonsplit fibres to be sncd; the counting problem for this setting will be

$$[p \notin S \text{ and } p \mid a] \Rightarrow [(3m \mid v_p(a)) \text{ or } (m \mid v_p(a) \text{ and } p \in \mathcal{P}_K)].$$

The condition $3m \mid v_p(a)$ comes from a Galois fixed component of multiplicity $3m$ on the multiple fibre of the sncd-model of X . However, no such component exists on the multiple fibre of X itself.

7.3. Halphen surfaces with two nonsplit fibres. In practice, it can be difficult to construct Halphen surfaces with more than one nonsplit fibre. We present two such examples, both of which verify Conjecture 1.5.

Theorem 7.6. *There exists a Halphen surface $X \rightarrow \mathbb{P}^1$ of degree 2 with two nonsplit fibres: the multiple fibre is geometrically irreducible and has multiplicity 2, and the other is an sncd divisor of Kodaira classification I_6 split by a cubic Galois extension K/\mathbb{Q} . Moreover, there exists a finite set of places S , and an explicit constant $c_{\pi,S} > 0$ such that*

$$N_{\text{loc},S}(\pi, B) \sim c_{\pi,S} \frac{B^{1+1/2}}{(\log B)^{2/3}}.$$

Proof. Let us first fix the cyclic cubic number field K/\mathbb{Q} . Now choose two sets of three conjugate points $P_i, Q_i \in \mathbb{P}^2(K)$, indexed by $i \in \mathbb{Z}/3\mathbb{Z}$. We let R_i be the intersecting point of the lines $P_{i+1}P_{i+2}$ and $Q_{i+1}Q_{i+2}$. For generic choices of P_i and Q_i , the R_i are well-defined and there is a unique smooth cubic through the nine points P_i, Q_i and R_i .

We will consider $X = \text{Bl}_{P_i, Q_i, R_i} \mathbb{P}^2$. The two nonsplit fibres of X come from the double cubic passing through these nine points, and the sextic curve which is geometrically the union of the six lines $P_{i+1}P_{i+2}$ and $Q_{i+1}Q_{i+2}$. Under blowup the first curve turns into a geometrically integral fibre of multiplicity 2, and the other into six lines meeting in a cycle. The three lines P_1P_2, P_2P_3 and P_3P_1 are permuted by $\text{Gal}(K/\mathbb{Q})$ and no longer meet on X . For a generic choice of P_i and Q_i , there will be no other nonsplit fibres.

Let us assume the multiple fibre lies above 0 and the other nonsplit fibre over ∞ . The fibres of $X \rightarrow \mathbb{P}^1$ are all sncd, so we can directly compose the counting problem to find that

$$N_{\text{loc}, S}(\pi, B) = \frac{1}{2} \#\{(a, b) \in \mathbb{Z}_{\text{prim}}^2 : |a|, |b| \leq B, p \notin S \Rightarrow 2 \mid v_p(a), [p \notin S \text{ and } p \mid b] \Rightarrow p \in \mathcal{P}_K\}.$$

Such a counting problem is dealt with by Proposition 3.1. □

Theorem 7.7. *There exists a Halphen surface $X \rightarrow \mathbb{P}^1$ of degree 3 with two nonsplit fibres: the multiple fibre is geometrically irreducible and has multiplicity 3, and the other is a non-sncd divisor of Kodaira classification I_3 split by a cubic Galois extension K/\mathbb{Q} . Moreover, there exists a finite set of places S such that*

$$\frac{B^{1+1/3}}{(\log B)^{2/3}} \ll N_{\text{loc}, S}(\pi, B) \ll \frac{B^{1+1/3}}{(\log B)^{2/3}}.$$

We will return to this surface in Section 7.4 to create another interesting example. There we will assume that the multiple fibre lies over 0 and the remaining nonsplit fibre lies over ∞ .

Proof of Theorem 7.7. Let E/\mathbb{Q} be an elliptic curve with $E(\mathbb{Q})_{\text{tors}} = \mathbb{Z}/9\mathbb{Z}$. Let K/\mathbb{Q} be a cyclic cubic number field K/\mathbb{Q} such that $\text{rank } E(\mathbb{Q}) < \text{rank } E(K)$. We will fix

- (i) a generator $\sigma \in \text{Gal}(K/\mathbb{Q})$,
- (ii) a generator $A \in E(\mathbb{Q})_{\text{tors}}$,
- (iii) $B \in E(K) \setminus E(\mathbb{Q})$ such that $B + \sigma(B) + \sigma^2(B) = O \in E(\mathbb{Q})$, and any
- (iv) $C \in E(K) \setminus E(\mathbb{Q})$.

With this notation in mind, consider the nine points

$$\begin{aligned} P_i &= \sigma^i(C), \\ Q_i &= \sigma^i(-2C + B + A), \\ R_i &= \sigma^i(C - 3A). \end{aligned}$$

For general choices, we find that $\text{Bl}_{P_i, Q_i, R_i} \mathbb{P}^2$ is a Halphen surface of degree 3. In particular, there is a smooth cubic through the nine points, which becomes the geometrically irreducible triple fibre on X .

Moreover, we have

$$\sum_i (P_i + Q_i + R_i) - P_j + R_j = O,$$

so that there is a cubic curve which passes through all nine points except P_j and has a singularity at R_j . The union of these three curves becomes the nonsplit I_3 -fibre split by K .

For the lower bound we may apply Theorem 7.2, and the upper bound follows from Theorem 1.2. \square

7.4. A nonsplit fibre over a point of higher degree. Our final result concerns a surface of Halphen type with a fibration over \mathbb{P}^1 that has one multiple fibre and a nonsplit fibre over a degree-2 point. Our local solubility criteria do not apply to this case in general, but we are nonetheless able to deduce explicit criteria.

Consider the Halphen surface $\pi : X \rightarrow \mathbb{P}^1$ from Theorem 7.7 with $m = 3$, with a multiple fibre over 0 and a nonsplit fibre over ∞ split by a Galois cubic extension K/\mathbb{Q} . Let $\pi' : X' \rightarrow \mathbb{P}^1$ be the normalisation of the pullback of π along the morphism $\theta : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ given by $[u : v] \mapsto [u^2 : u^2 + v^2]$. We claim that the surface X' has a unique multiple fibre over $u = 0$, whose multiplicity is 3, and that the only other nonsplit fibre lies over the degree-2 point $u^2 + v^2 = 0$ and is split by K . To see this we note that the fibres of the pullback of X are precisely the fibres of X' , and normalisation only changes the fibres over 0 and ∞ . The multiplicities of the new fibres can then be computed using Proposition 4.3. Note that $\partial_{\pi'} = \frac{2}{3}[0]$ and $\Delta(\pi') = 1 - \delta_{u^2+v^2}(\pi') = \frac{2}{3}$. We shall now prove the following result, which is easily seen to agree with the prediction in Conjecture 1.5.

Theorem 7.8. *For the surface $\pi' : X' \rightarrow \mathbb{P}^1$ as above, there exists a finite set S such that*

$$\frac{B^{4/3}}{(\log B)^{2/3}} \ll N_{\text{loc},S}(\pi', B) \ll \frac{B^{4/3}}{(\log B)^{2/3}}.$$

Proof. The upper bound follows directly from Theorem 1.2. To prove the lower bound, we note that, for all but finitely many points $x \in \mathbb{P}^1(\mathbb{Q})$, the fibre of $X' \rightarrow \mathbb{P}^1$ is isomorphic to the fibre of $X \rightarrow \mathbb{P}^1$ over $\theta(x) \in \mathbb{P}^1(\mathbb{Q})$. Hence we can apply the criterion in Corollary 5.8 to determine local solubility for X . Noting that $v_p(u^2)$ is divisible by 3 precisely if this is true for $v_p(u)$, we find that $N_{\text{loc},S}(\pi', B)$ is

$$\frac{1}{2} \# \left\{ (u, v) \in \mathbb{Z}_{\text{prim}}^2 : \begin{array}{l} |u|, |v| \leq B, p \notin S \Rightarrow 3 \mid v_p(u), \\ [p \notin S \text{ and } p \mid u^2 + v^2] \Rightarrow p \in \mathcal{P}_K \end{array} \right\} + O(1).$$

On restricting to positive coprime u and v and demanding that u is a cube, we arrive at the lower bound

$$N_{\text{loc},S}(\pi', B) \geq \frac{1}{2} M(B) + O(1),$$

where

$$M(B) = \#\{(u, v) \in \mathbb{Z}_{\text{prim}}^2 : 0 \leq u^3, v \leq B, p \mid u^6 + v^2 \Rightarrow p \in \mathcal{P}_K\}.$$

Note that $u^3, v \leq B$ whenever $u^6 + v^2 \leq B^2$. Hence

$$M(B) \geq \#\{(u, v) \in \mathbb{Z}_{\geq 0}^2 : \gcd(u, v) = 1, u^6 + v^2 \leq B^2, p \mid u^6 + v^2 \Rightarrow p \in \mathcal{P}_K\}.$$

The right-hand side is exactly the quantity estimated via the β -sieve in [Friedlander and Iwaniec 2010, Theorem 11.31], with the outcome that

$$M(B) \gg \left(\frac{B^2}{\log(B^2)} \right)^{2/3}.$$

The statement of the theorem now follows. □

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