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# Sym-Noetherianity for powers of GL-varieties

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Much recent literature concerns finiteness properties of infinite-dimensional algebraic varieties equipped with an action of the infinite symmetric group, or of the infinite general linear group. In this paper, we study a common generalisation in which the product of both groups acts on infinite-dimensional spaces, and we show that these spaces are topologically Noetherian with respect to this action.

## 1. Introduction

**1.1. Sym-*Noetherianity and* GL-*Noetherianity.*** It has been well-established since the 1980s that if $Z$ is finite-dimensional variety, then the topological space $Z^{\mathbb{N}}$, equipped with the inverse-limit topology of the Zariski topologies, has the property that if

$$X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$$

is a descending chain of closed subvarieties, each stable under the infinite symmetric group $\mathrm{Sym} = \bigcup_n \mathrm{Sym}([n])$ permuting the copies of $Z$, then $X_n = X_{n+1}$ for all $n \gg 0$. We say that $Z^{\mathbb{N}}$ is Sym-*Noetherian*; see [Cohen 1967; 1987; Aschenbrenner and Hillar 2007; Hillar and Sullivant 2012] for the relevant literature.

On the other hand, the third author proved that if $Z$ is a GL-*variety*: a (typically infinite-dimensional) affine variety equipped with a suitable action of the infinite general linear group $\mathrm{GL} = \bigcup_n \mathrm{GL}_n$ — see below for precise definitions — then $Z$ is topologically GL-Noetherian. See [Draisma 2019] for Noetherianity, and see [Bik et al. 2023a; 2023b] for the structure theory of GL-varieties.

**1.2. *Our result:* Sym × GL-*Noetherianity.*** Given a GL-variety $Z$, the group $\mathrm{Sym} \times \mathrm{GL}$ acts naturally on $Z^{\mathbb{N}}$, and our main goal in this paper is to prove the following theorem.

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**Theorem 1.1** (main theorem). *Let $Z$ be a* GL*-variety over a field of characteristic zero. Then $Z^{\mathbb{N}}$ is topologically* $\mathrm{Sym} \times \mathrm{GL}$*-Noetherian. In other words, every descending chain*

$$X_1 \supseteq X_2 \supseteq \cdots$$

*of* $\mathrm{Sym} \times \mathrm{GL}$*-stable closed subvarieties of $Z^{\mathbb{N}}$ stabilises eventually. Equivalently, any* $\mathrm{Sym} \times \mathrm{GL}$*-stable closed subvariety of $Z^{\mathbb{N}}$ is defined by finitely many* $\mathrm{Sym} \times \mathrm{GL}$*-orbits of polynomial equations.*

Below we give two examples of $\mathrm{Sym} \times \mathrm{GL}$-varieties; these illustrate that even when $Z$ is a rather simple GL-variety, $Z^{\mathbb{N}}$ will have many $\mathrm{Sym} \times \mathrm{GL}$-stable closed subvarieties.

**Example 1.2.** Consider the space of $\mathbb{N} \times \mathbb{N}$-matrices where Sym permutes the rows, and GL acts simultaneously on all rows. We can think about this space as $Z^{\mathbb{N}}$, where $Z$ is the space $\mathbb{A}^{\mathbb{N}}$ with the obvious GL-action. We write $x_{i,j}$ $(i, j \in \mathbb{N})$ for the coordinates on this space. Let $X$ be a $\mathrm{Sym} \times \mathrm{GL}$-stable proper closed subvariety of this space. Let $f$ be a nonzero polynomial vanishing identically on $X$ involving only the $x_{ij}$ with $1 \leq i, j \leq n$, chosen such that $n$ is minimal among all defining equations of $X$. We claim that $X$ is contained in the variety of matrices with rank at most $n - 1$. Indeed, suppose that a matrix $A$ in $X$ has rank at least $n$. Then by basic linear algebra the $\mathrm{Sym} \times \mathrm{GL}$-orbit of $A$ projects dominantly in the affine space $\mathbb{A}^{n \times n}$ corresponding to the upper left $n \times n$-block. This implies that $f$ is the zero polynomial; a contradiction.

Also, by the minimality of $n$, there must exist a matrix in $X$ whose rank is $n-1$. However, it is not easy to completely classify the $\mathrm{Sym} \times \mathrm{GL}$-stable closed subvarieties of $Z^{\mathbb{N}}$ containing a matrix of rank $n-1$ and no matrices of rank $n$. For instance, fix any matroid $M$ of rank $n - 1$ on the ground set $[m] := \{1, \ldots, m\}$ and let $R \subseteq \mathbb{A}^{m \times (n-1)}$ be the variety defined by the determinants of the $(n - 1) \times (n - 1)$-submatrices whose rows correspond to nonbases of $M$. Regard $R$ as a subvariety of $\mathbb{A}^{\mathbb{N} \times \mathbb{N}}$ by extending with zeros and set $X_M := \overline{(\mathrm{Sym} \times \mathrm{GL})R}$. This $\mathrm{Sym} \times \mathrm{GL}$-stable subvariety of $\mathbb{A}^{\mathbb{N} \times \mathbb{N}}$ is the common zero set of two classes of polynomials: all monomials containing variables from at least $m + 1$ distinct rows, and the Sym-orbits of all products of the form

$$\prod_{\pi \in \mathrm{Sym}([m])} \det(x[\pi(I_\pi), J_\pi]),$$

where each $I_\pi \subseteq [m]$ is an $(n-1)$-element set that is not a basis of $M$, each $J_\pi$ is an arbitrary $(n-1)$-element subset of $\mathbb{N}$, and $x[\pi(I_\pi), J_\pi]$ stands for the matrix of variables $x_{ij}$ with $i \in \pi(I_\pi)$ and $j \in J_\pi$.

Now suppose that $M, M'$ are loopless matroids on ground sets $[m], [m']$, both realisable over the algebraic closure of the ground field. We then claim that $X_M = X_M$ holds (if and) only if $M, M'$ are isomorphic. Indeed, if $X_M = X_{M'}$, then let $p \in \mathbb{A}^{m \times (n-1)} \subseteq \mathbb{A}^{\mathbb{N} \times \mathbb{N}}$ realise $M$, so that $p \in X_M = X_{M'}$. This means that $p$ satisfies all equations for $X_{M'}$. Since $M$ is loopless, all rows of $p$ are nonzero, and the monomial equations for $X_{M'}$ imply that $m' \geq m$. The converse follows by taking a realisation of $M'$. That the determinantal equations for $X_{M'}$ vanish on $p$ imply that, after a permutation, all nonbases of $M'$ are also nonbases of $M$. Again, the converse holds by taking a realisation of $M'$. Hence $M$ and $M'$ are isomorphic.

We conclude that the considerable combinatorial complexity of the class of realisable matroids is contained in the classification problem for $\mathrm{Sym} \times \mathrm{GL}$-subvarieties of $Z^{\mathbb{N}}$.

**Remark 1.3.** Already the classification of Sym-stable closed subvarieties of $(A^1)^{\mathbb{N}}$ is nontrivial [Nagpal and Snowden 2020], so it is not so surprising that also the classification of $\mathrm{Sym} \times \mathrm{GL}$-stable closed subvarieties of $Z^{\mathbb{N}}$ in Example 1.2 is difficult.

**Example 1.4.** Let $Z$ be the space of symmetric $\mathbb{N} \times \mathbb{N}$ matrices, acted upon by GL via $(g, A) \mapsto gAg^T$. It is not hard to classify the GL-stable closed subvarieties of $Z$: they are the empty set and the varieties of matrices whose rank is bounded by some $k \in \{0, \dots, \infty\}$.

Now let $X$ be a $\mathrm{Sym} \times \mathrm{GL}$-stable proper closed subvariety of $Z^{\mathbb{N}}$, and let $n$ be minimal such that there exists a nonzero polynomial that vanishes on $X$ and involves only coordinates on the first $n$ copies of $Z$. Then it follows from [Eggermont 2015, Proposition 3.3] that $X$ is contained in the variety $X_{n,r}$ of $\mathbb{N}$-tuples in which every $n$-tuple has a nontrivial linear combination whose rank is at most some integer $r$. However, completely classifying all $\mathrm{Sym} \times \mathrm{GL}$-stable closed subvarieties of $X_{n,r}$ seems completely out of reach.

## 1.3. *A generalisation:* $\mathbf{Sym}^k \times \mathbf{GL}$*-Noetherianity.*

We prove the main theorem by establishing first the following more general result.

**Theorem 1.5.** *Let $Z_1, \dots, Z_k$ be GL-varieties over a field of characteristic zero. Then the variety $Z_1^{\mathbb{N}} \times \cdots \times Z_k^{\mathbb{N}}$ is $\mathrm{Sym}^k \times \mathrm{GL}$-Noetherian.*

Here there is one copy of GL that acts diagonally, and there are $k$ copies of Sym that act on separate copies of $\mathbb{N}$. We believe it is impossible to prove the main theorem without considering multiple copies of Sym. Indeed, covering a proper closed $\mathrm{Sym} \times \mathrm{GL}$-stable subvariety of $Z^{\mathbb{N}}$ requires partitioning $\mathbb{N}$ into finitely many parts such that the points in $Z$ labelled by the indices in one same part behave in a similar fashion. The following example illustrates this point.

**Example 1.6.** Let $Z$ be the space of $\mathbb{N} \times \mathbb{N}$-matrices over a field of characteristic zero, equipped with the GL-action given by $(g, A) \mapsto gAg^T$. Let $X$ be the closed $\mathrm{Sym} \times \mathrm{GL}$-stable subvariety of $Z^{\mathbb{N}}$ consisting of all infinite matrix tuples $(A_1, A_2, \dots)$ such that each $A_i$ is either symmetric or skew-symmetric. It is easy to see that $X$ is defined by the $\mathrm{Sym} \times \mathrm{GL}$-orbit of the equation $(x_{112} + x_{121})(x_{112} - x_{121})$, where $x_{ijk}$ is the $(j, k)$-entry of the $i$-th matrix. We will see that the $\mathrm{Sym} \times \mathrm{GL}$-Noetherianity of $X$ follows from the $\mathrm{Sym}^2 \times \mathrm{GL}$-Noetherianity of the *"smaller"* variety $Z_1^{\mathbb{N}} \times Z_2^{\mathbb{N}}$, where $Z_1 \subseteq Z$ is the GL-subvariety of symmetric matrices, and $Z_2 \subseteq Z$ is the GL-subvariety of skew-symmetric matrices. Here the term "smaller" refers to the fact that both $Z_1$ and $Z_2$ are quotients of $Z$. The exact meaning of smaller varieties is given in Section 2.7.2.

## 1.4. *Relation to existing literature.*

The main theorem generalises the results mentioned in Section 1.1: taking for $Z$ a finite-dimensional affine variety with trivial GL-action, one recovers the Sym-Noetherianity of $Z^{\mathbb{N}}$; and on the other hand, if $Z$ is a GL-variety, then considering chains $X_1 \supseteq X_2 \supseteq \cdots$ in which each $X_i$ is of the form $Z_i^{\mathbb{N}}$ with $Z_i \subseteq Z$ a GL-subvariety, one recovers the GL-Noetherianity of $Z$.

The proof of the main theorem will reflect these two special cases. We will use the proof method from [Draisma 2019] for the GL-Noetherianity of $Z$, and similarly, we will use methods for Sym-varieties from [Draisma et al. 2022]. In fact, we do not explicitly use Higman's lemma in our proofs as is classically done [Aschenbrenner and Hillar 2007; Hillar and Sullivant 2012; Draisma 2014], and in passing we give a new proof of the Sym-Noetherianity of $Z^{\mathbb{N}}$ for a finite-dimensional variety $Z$. However, our proof only yields a *set-theoretic* Noetherianity result, while in the pure Sym-setting (much) stronger results are known: increasing chains of Sym-stable ideals in the coordinate ring of $Z^{\mathbb{N}}$ with $Z$ a finite-dimensional variety stabilise [Cohen 1967; 1987; Aschenbrenner and Hillar 2007; Hillar and Sullivant 2012], and even finitely generated modules over such rings with a compatible Sym-action are Noetherian [Nagel and Römer 2019]. In the pure GL-setting, however, such stronger Noetherianity results are known only for very few classes of GL-varieties: over a field of characteristic zero ring-theoretic Noetherianity holds for a direct sum of copies of the first symmetric power $S^1$ [Sam and Snowden 2016; 2019], for the second symmetric power $S^2$, for $\bigwedge^2$ [Nagpal et al. 2016], for $S^1 \oplus S^2$ and for $S^1 \oplus \bigwedge^2$ [Sam and Snowden 2022].

Partitions of $\mathbb{N}$ into finitely many subsets also feature in the classification of symmetric subvarieties of infinite affine space $(\mathbb{A}^1)^{\mathbb{N}}$ [Nagpal and Snowden 2020], and while our proofs do not logically depend on this classification, that paper did serve as an inspiration.

**1.5.** ***Organisation of this paper.*** This paper is organised as follows. In Sections 2.1 and 2.2 we introduce polynomial functors and affine varieties over the categories **Vec**, $\mathbf{FI^{op}} \times \mathbf{Vec}$ and $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$. This language happens to be more convenient than a purely infinite-dimensional approach, as shown in Remark 2.9. In Section 2.3 we introduce the category **PM** with morphisms between $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties, in which, for the reasons explained in Example 1.6 and above it, $k$ varies. In Section 2.4 we describe $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties of *product type*. The simplest ones among these are of the form

$$Z : (V; S_1, \ldots, S_k) \mapsto \prod_{i=1}^{k} Z_i(V)^{S_i},$$

which are the ones of interest in our Theorems 1.1 and 1.5. Reformulations of our main theorem and its generalisation Theorem 1.5 in this language are in Remark 2.19.

Our proofs rely on induction on the "complexity" of product-type $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties. The several well-founded orders used in this induction are the topic of Section 2.7, which builds on **FI**-techniques developed in Sections 2.5 and 2.6. We introduce orders on

(1) polynomial functors (Section 2.7.1),

(2) **Vec**-varieties with a specified closed embedding in $B \times Q$ where $B$ is a finite-dimensional algebraic variety and $Q$ is a suitable polynomial functor (Section 2.7.2),

(3) $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties of product type in the category **PM** (Section 2.7.3),

(4) closed subvarieties of $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties of product type (Section 2.7.4).

Then in Section 3 we formulate and prove the parameterisation theorem, Theorem 3.1, the core technical result of this paper. The statement roughly says that if $X$ is a proper closed $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-subvariety of a variety $Z$ of product type, then $X$ is covered by finitely many morphisms in **PM** from $(\mathbf{FI^{op}})^l \times \mathbf{Vec}$-varieties of product form that are smaller than $Z$ in the sense of Section 2.7.3. We prove this theorem by induction on closed subvarieties mentioned in (4). The description of these smaller $(\mathbf{FI}^{\mathbf{op}})^l \times \mathbf{Vec}$-varieties of product type relies on Proposition 3.3. This proposition allows to partition points according to their common behaviour with respect to a specific defining equation, similarly to what happens in [Draisma 2019]. Indeed, our Lemma 3.8 is proven as an iteration of the argument for the embedding theorem in [Bik et al. 2023b], which in turn uses a technique developed in [Draisma 2019]. Essential for applying Proposition 3.3 is the operation of shifting over a tuple of finite sets, described in Section 2.6. In the final Section 4 we use all the above to prove that all $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-varieties of product type are Noetherian via an induction on their order of Section 2.7.3. Theorem 1.5 and the main theorem follow as corollaries.

**1.6.** ***Notation and conventions.***

- For a nonnegative integer $k$, we set $[k] := \{1, \ldots, k\}$; so in particular $[0] = \varnothing$.
- Let $S$ be a finite set. We denote by $|S|$ the cardinality of $S$.
- Throughout this paper, we work over a field $K$ of characteristic zero.
- Sym denotes the infinite symmetric group. It is defined as the direct limit over $\mathrm{Sym}(n)$, the symmetric group on the set $[n]$, with the obvious inclusion maps.
- GL denotes the infinite general linear group. It is defined as the direct limit of $\mathrm{GL}_n$, the general linear group on $K^n$, with inclusion maps $\mathrm{GL}_n \to \mathrm{GL}_{n+1}$ given by

$$g \mapsto \left(\begin{array}{c|c} g & 0 \\ \hline 0 & 1 \end{array}\right).$$

- The category of schemes over $K$ is denoted by $\mathbf{Sch}_K$. A product $X \times Y$ of two schemes will always mean a product in this category.
- A *variety* $X$ here is a reduced affine scheme of finite type over $K$. By $K[X]$ we denote its coordinate ring, so $X = \mathrm{Spec}\, K[X]$. If $Y$ is a subvariety of $X$, then we write $\mathcal{I}(Y) \subseteq K[X]$ for the (radical) ideal of functions on $X$ vanishing on $Y$.
- If $f \in K[X]$ then we write $X[1/f] := \mathrm{Spec}(K[X]_f)$.
- Let $\varphi : X \to Y$ be a morphism of varieties. We denote by $\varphi^\# : K[Y] \to K[X]$ the induced morphism on coordinate rings.
- By a point $x$ of a variety $X$ we always mean a closed point of $X$, i.e., an element of $X(\overline{K})$.

## 2. The categories of $(\mathbf{FI}^{\mathbf{op}})^k$ × Vec-varieties

**2.1.** **Vec-*varieties.*** Let $K$ be a field of characteristic zero, and let **Vec** be the category of finite-dimensional vector spaces over $K$ with $K$-linear morphisms. We will be working with **Vec**-varieties, a functorial finite-dimensional counterpart of GL-varieties. Below, we quickly recap the theory of polynomial functors: definitions, relevant properties; and we define the notion of **Vec**-variety. See Remark 2.6 for the connection with GL-varieties.

**Definition 2.1.** A *polynomial functor* is a functor $P : \mathbf{Vec} \to \mathbf{Vec}$ such that for each $U, V \in \mathbf{Vec}$ the map $P : \mathrm{Hom}_{\mathbf{Vec}}(U, V) \to \mathrm{Hom}_{\mathbf{Vec}}(P(U), P(V))$ is polynomial, and such that the degree of this polynomial map is upper-bounded independently of $U, V$. The minimal such bound is called the *degree* of $P$.

We will also regard a polynomial functor $P$ as a functor $\mathbf{Vec} \to \mathbf{Sch}_K$ by composing with the embedding $\mathbf{Vec} \to \mathbf{Sch}_K$ given by $V \mapsto \mathrm{Spec}(\mathrm{Sym}_K(V^*))$, the spectrum of the symmetric algebra on the dual space $V^*$ of $V$. Every polynomial functor $P$ equals $P_0 \oplus \cdots \oplus P_d$, where $d$ is the degree of $P$ and $P_i$ is defined as

$$P_i(V) := \{v \in P(V) \mid \forall t \in K : P(t\,\mathrm{id}_V)v = t^i v\}.$$

Considering $P$ as a functor $\mathbf{Vec} \to \mathbf{Sch}_K$ we have $P(V) = P_0(V) \times \cdots \times P_d(V)$. We note that $P_0$ is a constant polynomial functor, which assigns a fixed vector space $P(0) \in \mathbf{Vec}$ to all $V \in \mathbf{Vec}$ and the identity map to each linear map. We call $P$ *pure* if $P_0 = \{0\}$.

Let $X, Y : \mathbf{Vec} \to \mathbf{Sch}_K$ be functors. A *closed immersion* $\iota : X \to Y$ is a natural transformation such that $\iota(V) : X(V) \to Y(V)$ is a closed immersion for all $V \in \mathbf{Vec}$. In particular, $X$ is then a subfunctor of $Y$.

**Definition 2.2.** An *affine* **Vec**-*scheme* is a functor $X : \mathbf{Vec} \to \mathbf{Sch}_K$ that admits a closed immersion $X \to P$ with $P : \mathbf{Vec} \to \mathbf{Sch}_K$ a polynomial functor. A **Vec**-*variety* is an affine **Vec**-scheme $X$ such that $X(V)$ is reduced for all $V \in \mathbf{Vec}$. The *category of affine* **Vec**-*schemes* is the full subcategory of the functor category $\mathbf{Sch}_K^{\mathbf{Vec}}$ whose objects are affine **Vec**-schemes.

Spelled out explicitly, a **Vec**-variety $X$ can be described by the data of a polynomial functor $P$ and a subvariety $X(V) \subseteq P(V)$ for each $V \in \mathbf{Vec}$ such that, for each $\varphi \in \mathrm{Hom}_{\mathbf{Vec}}(U, V)$, the linear map $P(\varphi)$ maps $X(U)$ into $X(V)$. A morphism of **Vec**-varieties $\tau : X \to Y$ consists of a morphism of varieties $\tau(V) : X(V) \to Y(V)$ for each $V \in \mathbf{Vec}$ such that, for each $\varphi \in \mathrm{Hom}_{\mathbf{Vec}}(U, V)$, we have $\tau(V) \circ X(\varphi) = Y(\varphi) \circ \tau(U)$.

**Remark 2.3.** The subcategory of **Vec**-varieties is closed under taking closed immersions and finite products. To see the latter, note that the product of $X, Y : \mathbf{Vec} \to \mathbf{Sch}_K$ in $\mathbf{Sch}_K^{\mathbf{Vec}}$ is given by $V \mapsto X(V) \times Y(V)$; and furthermore, given closed immersions $X \hookrightarrow P$ and $Y \hookrightarrow Q$, the assignment

$$X(V) \times Y(V) \to P(V) \times Q(V)$$

defines a closed immersion of the product $X \times Y$ into the polynomial functor $P \oplus Q$.

**Lemma 2.4.** *The category of affine* **Vec**-*schemes admits fibre products.*

*Proof.* First note that for morphisms of affine **Vec**-schemes $X \to Y$ and $Z \to Y$ the fibre product $X \times_Y Z$ of $X$ and $Z$ over $Y$ exists in the functor category $\mathbf{Sch}_K^{\mathbf{Vec}}$ and is given by

$$(X \times_Y Z)(V) := X(V) \times_{Y(V)} Z(V).$$

Moreover, since $Y(V)$ is affine (or more generally since $Y(V)$ is separated; see [Stacks, Tag 01KR]) the natural morphism $X(V) \times_{Y(V)} Z(V) \to X(V) \times Z(V)$ is a closed immersion. The statement then follows by Remark 2.3. $\square$

The main result of [Draisma 2019] says that **Vec**-varieties are topologically Noetherian.

**Theorem 2.5** [Draisma 2019, Theorem 1]. *Let $X$ be a* **Vec**-*variety. Then every descending chain of* **Vec**-*subvarieties*

$$X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$$

*stabilises, that is, there exists $N \geq 0$ such that for each $n \geq N$ we have $X_n = X_{n+1}$.*

**Remark 2.6.** If $X$ is a **Vec**-variety, then $X_\infty := \lim_{\leftarrow n} X(K^n)$ is a GL-variety in the sense of [Bik et al. 2023b]. This yields an equivalence of categories between **Vec**-varieties and GL-varieties. Most of our reasoning will be in the former terminology, but could be rephrased in the latter.

### 2.2. $(\mathbf{FI}^{\mathbf{op}})^k \times$ Vec-*varieties*.

Let **FI** be the category of finite sets with injections.

**Definition 2.7.** Let $k \in \mathbb{Z}_{\geq 0}$. An $(\mathbf{FI}^{\mathbf{op}})^k \times$ **Vec**-*variety* is a covariant functor $X$ from $(\mathbf{FI}^{\mathbf{op}})^k$ to the category of **Vec**-varieties.

Explicitly, an $(\mathbf{FI}^{\mathbf{op}})^k \times$ **Vec**-variety is given by the following data: for any $k$-tuple $(S_1, \ldots, S_k)$ we have a **Vec**-variety $X(S_1, \ldots, S_k)$, and for any $k$-tuple of injective maps $\iota = (\iota_1 : S_1 \to T_1, \ldots, \iota_k : S_k \to T_k)$, we have a corresponding morphism $X(\iota) : X(T_1, \ldots, T_k) \to X(S_1, \ldots, S_k)$ of **Vec**-varieties and the usual requirements that $X(\tau \circ \iota) = X(\iota) \circ X(\tau)$ and $X(\mathrm{id}_{S_1}, \ldots, \mathrm{id}_{S_k}) = \mathrm{id}_{X(S_1,\ldots,S_k)}$.

Again, there are natural notions of morphism and closed immersion of $(\mathbf{FI}^{\mathbf{op}})^k \times$ **Vec**-varieties, and we call an $(\mathbf{FI}^{\mathbf{op}})^k \times$ **Vec**-variety Noetherian if every descending chain of closed $(\mathbf{FI}^{\mathbf{op}})^k \times$ **Vec**-subvarieties stabilises.

**Remark 2.8.** In particular, any contravariant functor from **FI** to finite-dimensional affine varieties, i.e., an $\mathbf{FI}^{\mathbf{op}}$-variety, is trivially an $\mathbf{FI}^{\mathbf{op}} \times$ **Vec**-variety. In this generality, $\mathbf{FI}^{\mathbf{op}}$-varieties are certainly not Noetherian; see [Hillar and Sullivant 2012, Example 3.8].

However, we will be largely concerned with $(\mathbf{FI}^{\mathbf{op}})^k \times$ **Vec**-varieties defined as follows. Let $Z_1, \ldots, Z_k$ be **Vec**-varieties, define

$$X(S_1, \ldots, S_k) := Z_1^{S_1} \times \cdots \times Z_k^{S_k} \tag{1}$$

and for $\iota = (\iota_1, \ldots, \iota_k) : (S_1, \ldots, S_k) \to (T_1, \ldots, T_k)$ define $X(\iota)$ as the product of the natural projections $Z^{T_i} \to Z^{S_i}$ associated to $\iota_i$. We will prove that $(\mathbf{FI}^{\mathbf{op}})^k \times$ **Vec**-varieties of this form are, indeed, Noetherian.

Note that we may also regard a $(\mathbf{FI}^{\mathbf{op}})^k \times$ **Vec**-variety as a functor $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec} \to \mathbf{Sch}_K$. For fixed $k$, the $(\mathbf{FI}^{\mathbf{op}})^k \times$ **Vec**-varieties thus form a category by considering it as the full subcategory in the corresponding functor category.

**Remark 2.9.** If $X$ is an $(\mathbf{FI}^{\mathbf{op}})^k \times$ **Vec**-variety, then the group $\mathrm{Sym}^k \times \mathrm{GL}$ acts on the inverse limit

$$\varprojlim_{n_1,\ldots,n_k,n} X([n_1], \ldots, [n_k])(K^n).$$

This gives a functor from $(\mathbf{FI}^{\mathbf{op}})^k \times$ **Vec**-varieties to (infinite-dimensional) schemes equipped with a $\mathrm{Sym}^k \times \mathrm{GL}$-action. Unlike in Remark 2.6, this is not quite an equivalence of categories (even under reasonable restrictions on the $\mathrm{Sym}^k \times \mathrm{GL}$-action). For example, $X([n_1], \ldots, [n_k])$ could be empty for large $n_i$ and a fixed nontrivial GL-variety for smaller $n_i$. We will consider an explicit example of this type later in Example 2.35. In that case, the inverse limit is empty but the $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety is not trivial. Our theorems will be formulated in the richer category of $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-varieties.

**2.3.** ***Partition morphisms and the category* PM.** Suppose we are given a point $p$ in some $X(S_1, \dots, S_k)(V)$, where $X$ is as in (1). Then the components of $p$ labelled by one of the finite sets $S_i$ may exhibit different behaviours, which prompts us to further partition $S_i$ into subsets labelling components where the behaviour is similar. See Example 1.6 for an instance of this phenomenon. In that case, $p$ will be in the image of some partition morphism; for Example 1.6 this is further explained in Example 2.15. Partition morphisms are defined below, after another motivating example.

**Example 2.10.** We revisit a step in the classification of Sym-invariant subvarieties of infinite affine space from [Nagpal and Snowden 2020]. We do so in the **FI**-framework, where this corresponds to closed $\mathbf{FI}^{\mathbf{op}}$-subvarieties of the $\mathbf{FI}^{\mathbf{op}}$-variety $X(S) := (\mathbb{A}^1)^S = \mathbb{A}^S$, where, for an injection $\pi : S \to T$, the map $X(\pi)$ is the corresponding projection $\mathbb{A}^T \to \mathbb{A}^S$. Let $Z$ be a proper closed $\mathbf{FI}^{\mathbf{op}}$-subvariety of $X$. By (the **FI**-analogue of) [Nagpal and Snowden 2020, Proposition 2.6], the number of distinct coordinates of points in $Z(S)$ is bounded by some natural number $l$, independently of $S$. This means that for every $S \in \mathbf{FI}$, $Z(S)$ is contained in the union over all partitions of $S$ into subsets $T_1, \dots, T_l$ of the morphism $\varphi(T_1, \dots, T_l) : \mathbb{A}^l \to X(S)$ that maps $(p_1, \dots, p_l)$ to the tuple $(q_i)_{i \in S}$ with $q_i = p_j$ for the unique $j \in [l]$ with $i \in S_j$. The morphisms $\varphi(T_1, \dots, T_l)$ for varying $(T_1, \dots, T_l) \in \mathbf{FI}^l$ form a partition morphism into $X$ from the constant $(\mathbf{FI}^{\mathbf{op}})^l$-variety $Y : (T_1, \dots, T_l) \mapsto \mathbb{A}^l$, an object that is arguably simpler than $X$. In the definition that follows, we generalise this notion to the setting where $X$ is an arbitrary $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety.

**Definition 2.11.** Let $X$ be an $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety and let $Y$ be an $(\mathbf{FI}^{\mathbf{op}})^l \times \mathbf{Vec}$-variety. A *partition morphism* $Y \to X$ consists of

(1) a map $\pi : [l] \to [k]$; and

(2) for each $l$-tuple of finite sets $(T_1, \dots, T_l)$ a morphism

$$\varphi(T_1, \dots, T_l) : Y(T_1, \dots, T_l) \to X\Big(\bigsqcup_{j \in \pi^{-1}(1)} T_j, \dots, \bigsqcup_{j \in \pi^{-1}(k)} T_j\Big)$$

of **Vec**-varieties in such a manner that for any $l$-tuple $\iota = (\iota_j)_j \in \mathrm{Hom}_{\mathbf{FI}}(S_j, T_j)^l$ the following diagram of **Vec**-variety morphisms commutes:

$$\begin{array}{ccc} Y(T_1, \dots, T_l) & \xrightarrow{\varphi(T_1, \dots, T_l)} & X\Big(\bigsqcup_{j \in \pi^{-1}(1)} T_j, \dots, \bigsqcup_{j \in \pi^{-1}(k)} T_j\Big) \\ \downarrow{\scriptstyle Y(\iota_1, \dots, \iota_l)} & & \downarrow{\scriptstyle X\left(\bigsqcup_{j \in \pi^{-1}(1)} \iota_j, \dots, \bigsqcup_{j \in \pi^{-1}(k)} \iota_j\right)} \\ Y(S_1, \dots, S_l) & \xrightarrow[\varphi(S_1, \dots, S_l)]{} & X\Big(\bigsqcup_{j \in \pi^{-1}(1)} S_j, \dots, \bigsqcup_{j \in \pi^{-1}(k)} S_j\Big) \end{array}$$

**Remark 2.12.** Note that if we take $k = l$ and $\pi = \mathrm{id}_{[k]}$, then a partition morphism is just a morphism of $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties.

There is a natural way to compose partition morphisms: if $(\pi, \varphi)$ is a partition morphism $Y \to X$ as above and $(\rho, \psi)$ is a partition morphism $Z \to Y$, where $Z$ is an $(\mathbf{FI^{op}})^m \times \mathbf{Vec}$-variety, then $(\pi, \varphi) \circ (\rho, \psi)$ is the partition morphism given by the data $\pi \circ \rho : [m] \to [k]$ and the morphisms

$$\varphi\Biggl(\bigsqcup_{n\in\rho^{-1}(1)} R_n, \dots, \bigsqcup_{n\in\rho^{-1}(l)} R_n\Biggr) \circ \psi(R_1, \dots, R_m) : Z(R_1, \dots, R_m) \to X\Biggl(\bigsqcup_{n\in(\pi\circ\rho)^{-1}(1)} R_n, \dots, \bigsqcup_{n\in(\pi\circ\rho)^{-1}(k)} R_n\Biggr).$$

A tedious but straightforward computation shows that partition morphisms turn the class of $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties, with varying $k$, into a category. We call this category **PM**.

**Definition 2.13.** Let $X$ be an $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety, let $Y$ be an $(\mathbf{FI^{op}})^l \times \mathbf{Vec}$-variety, and let $(\pi, \varphi) : Y \to X$ be a partition morphism. Let $S_1, \dots, S_k \in \mathbf{FI}$ and $V \in \mathbf{Vec}$. The (set-theoretic) *image* of $(\pi, \varphi)$ in $X(S_1, \dots, S_k)(V)$ is defined as the set of all points of the form $(X(\iota_1, \dots, \iota_k)(V) \circ \varphi(T_1, \dots, T_l)(V))(q)$, where $T_1, \dots, T_l$ are finite sets, $q$ is a point in $Y(T_1, \dots, T_l)(V)$, and each $\iota_i$ is a bijection from $S_i$ to $\bigsqcup_{j\in\pi^{-1}(i)} T_j$. The partition morphism $(\pi, \varphi)$ is called *surjective* if its image in $X(S_1, \dots, S_k)(V)$ equals $X(S_1, \dots, S_k)(V)$ for all choices of $S_1, \dots, S_k$ and $V$.

**Remark 2.14.** In the previous definition, each bijection $\iota_i$ induces a partition of the set $S_i$. Furthermore, if a partition morphism $(\pi, \varphi)$ is surjective and for every $i$ the **Vec**-variety

$$X(\varnothing, \dots, \varnothing, \{*\}, \varnothing, \dots, \varnothing),$$

where $\{*\}$ is a singleton in the $i$-th position, is nonempty, then the map $\pi$ is automatically surjective, so that $\pi$ induces a partition of $[l]$ into $k$ labelled, nonempty parts. This is our reason for calling the morphisms in **PM** partition morphisms.

The following example rephrases Example 1.6 in the current terminology.

**Example 2.15.** Let $Z$ be the **Vec**-variety that maps $V$ to $V \otimes V$, and let $Z_1, Z_2$ be the closed **Vec**-subvarieties consisting of symmetric and skew-symmetric tensors, respectively. Consider the $\mathbf{FI^{op}} \times \mathbf{Vec}$-variety defined by $S \mapsto Z^S$, and for every finite set $S$ let $X(S)$ be the closed **Vec**-subvariety given by the points $x = (x_s)_{s\in S} \in Z(V)^S$ such that each component $x_s$ is either symmetric or skew-symmetric. Note that $X$ is a closed $\mathbf{FI^{op}} \times \mathbf{Vec}$-subvariety. Let $Y$ be the $(\mathbf{FI^{op}})^2 \times \mathbf{Vec}$-variety defined by

$$Y(S_1, S_2) = Z_1^{S_1} \times Z_2^{S_2}.$$

We now construct a partition morphism $\varphi : Y \to X$ as follows. The map $\pi : [2] \to [1]$ is the only possible map, and for every $V \in \mathbf{Vec}$ and $(S_1, S_2) \in \mathbf{FI^{op}}^2$, the map

$$\varphi(S_1, S_2)(V) : Y(S_1, S_2)(V) = Z_1(V)^{S_1} \times Z_2(V)^{S_2} \to X(S_1 \sqcup S_2)(V)$$

is defined by

$$((x_{s_1})_{s_1\in S_1}, (x_{s_2})_{s_2\in S_2}) \mapsto (x_s)_{s\in S_1\sqcup S_2}.$$

Note that the partition morphism $\varphi$ is surjective. In particular, we say that $X$ *is covered* by $Y$, and, as we have already hinted in Example 1.6, $Y$ is in some sense smaller than the assignment $S \mapsto Z^S$. The fact that we can do this in general is the content of the *parameterisation theorem* (Theorem 3.1).

The following lemma is immediate.

**Lemma 2.16.** *Let $X$ be an $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety, let $X'$ be a closed $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-subvariety of $X$, and let $(\pi, \varphi)$ be a partition morphism from an $(\mathbf{FI}^{\mathbf{op}})^l \times \mathbf{Vec}$-variety $Y$ to $X$. Then $Y' := (\pi, \varphi)^{-1}(X')$ defined by*

$$Y'(T_1, \dots, T_l) := \varphi(T_1, \dots, T_l)^{-1}\Big(X'\Big(\bigsqcup_{j \in \pi^{-1}(1)} T_j, \dots, \bigsqcup_{j \in \pi^{-1}(k)} T_j\Big)\Big)$$

*is a closed $(\mathbf{FI}^{\mathbf{op}})^l \times \mathbf{Vec}$-subvariety of $Y$, and the data of $\pi$ together with the restrictions of the morphisms $\varphi(T_1, \dots, T_l)$ gives a partition morphism from $Y'$ to $X$. Moreover, if $(\pi, \varphi)$ is surjective, then so is its restriction to $Y' \to X'$.*

The following easy proposition is crucial in our approach to the main theorem.

**Proposition 2.17.** *If $(\pi, \varphi)$ is a surjective partition morphism from $Y$ to $X$, and $Y$ is a Noetherian $(\mathbf{FI}^{\mathbf{op}})^l \times \mathbf{Vec}$-variety, then $X$ is a Noetherian $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety.*

*Proof.* Let $X_1 \supseteq X_2 \supseteq \cdots$ be a descending chain of closed $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-subvarieties. By Lemma 2.16, the preimages $Y_i := (\pi, \varphi)^{-1}(X_i)$ are closed $(\mathbf{FI}^{\mathbf{op}})^l \times \mathbf{Vec}$-subvarieties of $Y$. Hence the chain $Y_1 \supseteq Y_2 \supseteq \cdots$ stabilises by assumption. The surjectivity of $(\pi, \varphi)$ implies the surjectivity of its restriction to $Y_i \to X_i$. This implies that $X_i$ is uniquely determined by $Y_i$, and hence the chain $X_1 \supseteq X_2 \supseteq \cdots$ stabilises at the same point. □

**2.4.** ***Product type.*** We now introduce the $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$ varieties of product type. Essentially, these are the varieties from Remark 2.8, but for our proofs we will need a finer control over these products. Therefore, we will work over a general base $\mathbf{Vec}$-variety $Y$, and keep track of the "constant parts" $B_i$ of the $\mathbf{Vec}$-varieties whose products we consider.

**Definition 2.18.** Let $Y$ be a $\mathbf{Vec}$-variety and $k, n_1, \dots, n_k \in \mathbb{Z}_{\geq 0}$. For each $i \in [k]$, let $B_i$ be a $\mathbf{Vec}$-subvariety of $Y \times \mathbb{A}^{n_i}$, and $Q_i$ be a pure polynomial functor. By construction each $\mathbf{Vec}$-variety $B_i \times Q_i$ has a morphism to $Y$ induced by the projection $Y \times \mathbb{A}^{n_i} \to Y$. We define the $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety $Z = [Y; B_1 \times Q_1, \dots, B_k \times Q_k]$ via

$$Z(S_1, \dots, S_k) := \underbrace{(B_1 \times Q_1) \times_Y \cdots \times_Y (B_1 \times Q_1)}_{\text{cardinality-of-}S_1\text{ times}} \times_Y (B_2 \times Q_2) \times_Y \cdots \times_Y (B_k \times Q_k),$$

where for every index $i \in [k]$ the fibre product over $Y$ of $B_i \times Q_i$ with itself is taken $|S_i|$ times, and these copies are labelled by the elements of $S_i$; and where the morphism $Z(T_1, \dots, T_k) \to Z(S_1, \dots, S_k)$ corresponding to $\iota : S \to T$ is the projection as in Remark 2.8. We also write the above product in a more compact notation as

$$(B_1 \times Q_1)_Y^{S_1} \times_Y \cdots \times_Y (B_k \times Q_k)_Y^{S_k}.$$

We say that $Z$ is an $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety of *product type* (over $Y$).

Note that $Z(S_1, \dots, S_k)$ is naturally a closed **Vec**-subvariety of

$$Y \times \prod_{i=1}^{k} (\mathbb{A}^{n_i} \times Q_i)^{S_i},$$

where the product is over $K$. Moreover, if $k = 0$, then by definition $Z = Y$.

When we talk of $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties of product type, we will always specify each $B_i$ together with its closed embedding in $Y \times \mathbb{A}^{n_i}$; the reason being that, in the proof of the main theorem, we aim to argue by induction on both $Y$ and $n_i$.

**Remark 2.19.** The settings of Theorems 1.1 and 1.5 can be rephrased in our current terminology as follows. Consider **Vec**-varieties $Z_1, \dots, Z_k$. Then for every $i \in [k]$ there exist $n_i \in \mathbb{Z}_{\geq 0}$, a closed subvariety $A_i \subseteq \mathbb{A}^{n_i}$, and a pure polynomial functor $Q_i$ such that $Z_i \subseteq A_i \times Q_i$. Define $Y$ to be a point, and $B_i := Y \times A_i$. Then the variety $Z_1^{\mathbb{N}} \times \cdots \times Z_k^{\mathbb{N}}$ of Theorem 1.5 is a subvariety of the product-type $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety

$$[Y; B_1 \times Q_1, \dots, B_k \times Q_k],$$

with $k = 1$ being the special case addressed in Theorem 1.1.

**Remark 2.20.** In [Draisma et al. 2022], for $\mathbf{FI^{op}}$-varieties (no dependence on **Vec**), the notion of product type is more restrictive. Essentially, there the last three authors considered a single finite-dimensional affine variety $Z$ with a morphism to a finite-dimensional, irreducible, affine variety $Y$, with the additional requirement that $K[Z]$ is a free $K[Y]$-module. This then ensures that each irreducible component of $Z^S$ maps dominantly to $Y$. In [Draisma et al. 2022] this is used to count the orbits of $\mathrm{Sym}(S)$ on these irreducible components.

The following example describes the partition morphisms between product-type varieties. It is particularly relevant as this is the shape of the partition morphisms we will be dealing with in our proof of the parameterisation theorem (Theorem 3.1).

**Example 2.21.** Let $Z' := [Y'; B'_1 \times Q'_1, \dots, B'_l \times Q'_l]$ and $Z := [Y; B_1 \times Q_1, \dots, B_k \times Q_k]$ be an $(\mathbf{FI^{op}})^l \times \mathbf{Vec}$-variety and an $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety of product type over $Y'$ and $Y$, respectively. We want to construct a partition morphism $(\pi, \varphi) : Z' \to Z$. Consider the following data:

- Let $\pi : [l] \to [k]$ be any map.
- Let $\alpha : Y' \to Y$ be a morphism of **Vec**-varieties.
- For each $j \in [l]$ let $\beta_j : B'_j \times Q'_j \to B_{\pi(j)} \times Q_{\pi(j)}$ be a morphism of **Vec**-varieties such that the following diagram commutes:

$$\begin{array}{ccc} B'_j \times Q'_j & \xrightarrow{\beta_j} & B_{\pi(j)} \times Q_{\pi(j)} \\ \downarrow & & \downarrow \\ Y' & \xrightarrow[\alpha]{} & Y \end{array} \tag{2}$$

For each $(T_1, \dots, T_l) \in \mathbf{FI}^l$ we define the morphism of **Vec**-varieties

$$\varphi(T_1, \dots, T_l) : Z'(T_1, \dots, T_l) \to Z\Big( \bigsqcup_{j \in \pi^{-1}(1)} T_j, \dots, \bigsqcup_{j \in \pi^{-1}(k)} T_j \Big)$$

as follows. Let $S_i := \bigsqcup_{j \in \pi^{-1}(i)} T_j$. Then for any $V \in \mathbf{Vec}$ the element

$$((b'_{j,t}, q'_{j,t})_{t \in T_j})_{j \in [l]} \in (B'_1 \times Q'_1)^{T_1}_{Y'}(V) \times_{Y'} \cdots \times_{Y'} (B'_l \times Q'_l)^{T_l}_{Y'}(V)$$

is mapped to the element

$$(((\beta_j(V)(b'_{j,t}, q'_{j,t}))_{t \in T_j})_{j \in \pi^{-1}(i)})_{i \in [k]} \in (B_1 \times Q_1)^{S_1}_Y(V) \times_Y \cdots \times_Y (B_k \times Q_k)^{S_k}_Y(V).$$

By construction, the pair $(\pi, \varphi)$ is a partition morphism $Z' \to Z$. Conversely, every partition morphism $Z' \to Z$ is of this form. Indeed, from a general partition morphism $Z' \to Z$, $\alpha$ is recovered by taking all $T_j$ empty and $\beta_j$ is recovered by taking $T_j$ a singleton and all $T_{j'}$ with $j' \neq j$ empty. That (2) commutes then follows by applying the commuting diagram from the definition of a partition morphism to the morphism $(\varnothing, \dots, \varnothing, \dots, \varnothing) \to (\varnothing, \dots, \{*\}, \dots, \varnothing)$ in $\mathbf{FI}^l$.

**2.5.** ***The leading monomial ideal.*** The following definition gives a size measure for a closed subvariety $B \subseteq Y \times \mathbb{A}^n$.

**Definition 2.22.** Let $Y$ be a **Vec**-variety, $n \in \mathbb{Z}_{\geq 0}$ and $B$ a closed **Vec**-subvariety of $Y \times \mathbb{A}^n$. For $V \in \mathbf{Vec}$ consider the ideal $\mathcal{I}(B(V))$ of $K[Y(V)][x_1, \dots, x_n]$ defining $B(V)$. We fix the lexicographic order on monomials in $x_1, \dots, x_n$, and denote by $\mathrm{LM}(B)$ the set of those monomials that appear as leading monomials of *monic* polynomials in $\mathcal{I}(B(V))$, i.e., those with leading coefficient $1 \in K[Y(V)]$.

Indeed, $LM(B)$ is well-defined:

**Lemma 2.23.** *The set* $\mathrm{LM}(B)$ *does not depend on the choice of* $V$.

*Proof.* Let $V \in \mathbf{Vec}$ and consider the linear maps $\iota : 0 \to V$ and $\pi : V \to 0$. If $f \in \mathcal{I}(B(V))$ is monic with leading monomial $x^u$, then applying $Y(\iota)^\#$ to all coefficients of $f$ yields a polynomial in $\mathcal{I}(B(0))$ which is monic with leading monomial $x^u$. This shows that the leading monomials of monic polynomials in $\mathcal{I}(B(V))$ remain leading monomials of monic elements in $\mathcal{I}(B(0))$. One obtains the converse inclusion by applying $Y(\pi)^\#$. □

The following lemma monitors the size of LM of the constant parts after a base change in product-type varieties. It is used in Proposition 2.28.

**Lemma 2.24.** *Let* $Y' \to Y$ *be a morphism of* **Vec***-varieties, let* $B$ *be a closed* **Vec***-subvariety of* $Y \times \mathbb{A}^n$, *and define* $B' := Y' \times_Y B \subseteq Y' \times \mathbb{A}^n$. *Then* $\mathrm{LM}(B') \supseteq \mathrm{LM}(B)$.

*Proof.* Pulling back a monic equation for $B(V)$ along $Y'(V) \times \mathbb{A}^n \to Y(V) \times \mathbb{A}^n$ yields a monic equation for $B'(V)$ with the same leading monomial. □

**2.6.** ***Shifting over tuples of finite sets.*** Shifting over a finite set is a standard technique in the theory of **FI**-modules [Church et al. 2015], and was also used by the last three authors in [Draisma et al. 2022] to turn certain $\mathbf{FI^{op}}$-varieties into products. The third author used the operation of shifting over a vector space in [Draisma 2019] to prove what became “the embedding theorem” for GL-varieties in [Bik et al. 2023b, Theorems 4.1, 4.2]. Here we describe this operation in the context of $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties.

**Definition 2.25.** Let $X$ be an $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety and let $S = (S_1, \dots, S_k) \in \mathbf{FI}^k$. Then the *shift* $\mathrm{Sh}_S X$ *of* $X$ *over* $S$ is the $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety defined by

$$(\mathrm{Sh}_S X)(T_1, \dots, T_k) := X(S_1 \sqcup T_1, \dots, S_k \sqcup T_k)$$

and, for injections $\iota_i : T_i \to T_i'$,

$$(\mathrm{Sh}_S X)(\iota_1, \dots, \iota_k) := X(\mathrm{id}_{S_1} \sqcup \iota_1, \dots, \mathrm{id}_{S_k} \sqcup \iota_k).$$

**Remark 2.26.** Consider a tuple $S = (S_1, \dots, S_k)$ in $(\mathbf{FI^{op}})^k$ and define the covariant functor

$$\mathrm{Sh}_S : (\mathbf{FI^{op}})^k \times \mathbf{Vec} \to (\mathbf{FI^{op}})^k \times \mathbf{Vec}$$

by assigning to each tuple $(T_1, \dots, T_k)$ the tuple $(S_1 \sqcup T_1, \dots, S_k \sqcup T_k)$ and assigning to each morphism $\iota : (\iota_1, \dots, \iota_k) : (T_1, \dots, T_k) \to (T_1', \dots, T_k')$ the morphism $\iota \sqcup \mathrm{id}_S$. In particular $\mathrm{Sh}_S X$ is the composition $X \circ \mathrm{Sh}_S$.

**Remark 2.27.** Let $V \in \mathbf{Vec}$. While $\mathrm{Sh}_S X(T_1, \dots, T_k)(V)$ and $X(S_1 \sqcup T_1, \dots, S_k \sqcup T_k)(V)$ coincide as sets, the action induced by functoriality of the $k$ copies of the symmetric group on them is different. Indeed, the groups $\mathrm{Sym}(T_1) \times \cdots \times \mathrm{Sym}(T_k)$ and $\mathrm{Sym}(S_1 \sqcup T_1) \times \cdots \times \mathrm{Sym}(S_k \sqcup T_k)$ act, respectively, on the former and on the latter.

The following proposition describes the shift operation on product-type varieties.

**Proposition 2.28.** *The shift* $\mathrm{Sh}_S Z$ *over* $S = (S_1, \dots, S_k)$ *of an* $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$*-variety*

$$Z := [Y; B_1 \times Q_1, \dots, B_k \times Q_k]$$

*of product type is itself isomorphic to a variety of product type*:

$$\mathrm{Sh}_S Z \cong [Y'; B_1' \times Q_1, \dots, B_k' \times Q_k]$$

*with*

$$Y' := (B_1 \times Q_1)_Y^{S_1} \times_Y \cdots \times_Y (B_k \times Q_k)_Y^{S_k} \quad \textit{and} \quad B_i' := Y' \times_Y B_i.$$

*Furthermore, each* $B_i'$ *is naturally a* $\mathbf{Vec}$*-subvariety of* $Y' \times \mathbb{A}^{n_i}$, *and we have* $\mathrm{LM}(B_i') \supseteq \mathrm{LM}(B_i)$.

*Proof.* Straightforward. The last statement follows from Lemma 2.24. □

**2.7.** ***Well-founded orders.*** In this paper a *preorder* $\preceq$ on a class is a reflexive and transitive relation. We also write $B \succeq A$ for $A \preceq B$. Furthermore, write $A \prec B$ or $B \succ A$ to mean that $A \preceq B$ but not $B \preceq A$. The preorder is well-founded if it admits no infinite strictly decreasing chains $A_1 \succ A_2 \succ \cdots$.

In this section we first recall a well-founded preorder on polynomial functors. Building on it, we define well-founded preorders

- on varieties appearing in the definition of $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-varieties of product type,
- on product-type varieties, and
- on closed subvarieties of a fixed product-type variety.

**2.7.1.** *Order on polynomial functors.*

**Definition 2.29.** For polynomial functors $P$, $Q$, we write $P \preceq Q$ if $P \cong Q$ or else, for the largest $e$ with $P_e \ncong Q_e$, $P_e$ is a quotient of $Q_e$.

This is a well-founded partial order on polynomial functors; see [Draisma 2019, Lemma 12].

**2.7.2.** *Order on* **Vec***-varieties of type* $B \times Q$. Consider **Vec**-varieties $Y$, $Y'$, integers $n$, $n'$, pure polynomial functors $Q$, $Q'$, and **Vec**-subvarieties $B \subset Y \times \mathbb{A}^n$, $B' \subset Y' \times \mathbb{A}^{n'}$. We say that $B' \times Q' \preceq B \times Q$ if

(1) $Q' \prec Q$ in the order of Definition 2.29; or

(2) $Q' \cong Q$, $n' = n$ and $\mathrm{LM}(B') \supseteq \mathrm{LM}(B)$.

This is a preorder on **Vec**-varieties of this type.

**Remark 2.30.** We remark that $\preceq$ is defined on **Vec**-varieties *with a specified product decomposition* $B \times Q$ where $B$ is a **Vec**-variety *with a specified closed embedding into a specified product* $Y \times \mathbb{A}^n$ of a **Vec**-variety $Y$ and some $n$. It is not a preorder on **Vec**-varieties without further data.

**Lemma 2.31.** *The preorder on* **Vec***-varieties defined as above is well-founded.*

*Proof.* Suppose we have an infinite strictly decreasing chain

$$B_1 \times Q_1 \succ B_2 \times Q_2 \succ \cdots$$

with $B_i \subseteq Y_i \times \mathbb{A}^{n_i}$. Then we have $Q_1 \succeq Q_2 \succeq \cdots$. By the well-foundedness of $\succeq$ on polynomial functors, there exists a $j \geq 1$ such that both $Q_i$ and $n_i$ are constant for $i \geq j$. But then

$$\mathrm{LM}(B_i) \subsetneq \mathrm{LM}(B_{i+1}) \subsetneq \cdots,$$

which contradicts Dickson's lemma. □

**2.7.3.** *Order on product-type varieties.* Consider an $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety $Z := [Y;\, B_1 \times Q_1, \ldots, B_k \times Q_k]$, and an $(\mathbf{FI}^{\mathbf{op}})^l \times \mathbf{Vec}$-variety $Z' := [Y';\, B'_1 \times Q'_1, \ldots, B'_l \times Q'_l]$. We say that $Z' \preceq Z$ if there exists a map $\pi : [l] \to [k]$ with the following properties:

(1) $B'_j \times Q'_j \preceq B_{\pi(j)} \times Q_{\pi(j)}$ holds for all $j \in [l]$.

(2) For all $j$ whose $\pi$-fibre $\pi^{-1}(\pi(j))$ has cardinality at least 2 we have $B'_j \times Q'_j \prec B_{\pi(j)} \times Q_{\pi(j)}$.

(3) If $\pi$ is a bijection, then either at least one of the inequalities in (1) is strict, or else $Y'$ is a closed **Vec**-subvariety of $Y$.

**Lemma 2.32.** *Suppose $Z' \preceq Z$ is witnessed by $\pi : [l] \to [k]$ and suppose that at least one of the following holds*:

- $l \neq k$, *or*
- *at least one of the inequalities in* (1) *is strict.*

*Then we have $Z' \prec Z$.*

*Proof.* Assume, on the contrary, that $\sigma : [k] \to [l]$ witnesses $Z \preceq Z'$. Construct a directed graph $\Gamma$ with vertex set $[l] \sqcup [k]$ and an arrow from each $j \in [l]$ to $\pi(j)$ and an arrow from each $i \in [k]$ to $\sigma(i)$. Like any digraph in which each vertex has out-degree 1, $\Gamma$ is a union of disjoint directed cycles (here of even length) plus a number of trees rooted at vertices in those cycles and directed towards those roots. Moreover, those cycles have the same number of vertices in $[l]$ as in $[k]$.

The assumptions imply that at least one of the vertices of $\Gamma$ does not lie on a directed cycle. Without loss of generality, there exists an $i \in [k]$ not in any cycle such that $j := \sigma(i)$ lies on a cycle. Let $n$ be half the length of that cycle, so that $(\sigma\pi)^n(j) = j$. Then we have

$$B'_j \times Q'_j \preceq B_{\pi(j)} \times Q_{\pi(j)} \preceq \cdots \preceq B_{\pi(\sigma\pi)^{n-1}(j)} \times Q_{\pi(\sigma\pi)^{n-1}(j)} \prec B'_{(\sigma\pi)^n(j)} \times Q'_{(\sigma\pi)^n(j)} = B'_j \times Q'_j,$$

where the strict inequality holds because $\sigma^{-1}(j)$ has at least two elements: $i$ and $\pi(\sigma\pi)^{n-1}(j)$. By transitivity of the preorder from Section 2.7.2, we find $B'_j \times Q'_j \prec B'_j \times Q'_j$, which however contradicts the reflexivity of that preorder. □

**Lemma 2.33.** *The relation $\preceq$ is a well-founded preorder on varieties in* **PM** *of product type.*

*Proof.* For reflexivity we may take $\pi$ equal to the identity. For transitivity, if $\pi : [l] \to [k]$ witnesses $Z' \preceq Z$ and $\sigma : [k] \to [m]$ witnesses $Z \preceq Z''$, then $\tau := \sigma \circ \pi$ witnesses $Z' \preceq Z''$ — here we note that if $|\tau^{-1}(\tau(j))| > 1$ for some $j \in [l]$, then either $|\pi^{-1}(\pi(j))| > 1$ or else $|\sigma^{-1}(\sigma(\pi(j)))| > 1$; in both cases we find that $B'_j \times Q'_j \prec B''_{\tau(j)} \times Q''_{\tau(j)}$.

For well-foundedness, suppose that we had a sequence $Z_1 \succ Z_2 \succ Z_3 \succ \cdots$, where

$$Z_i = [Y_i; B_{i,1} \times Q_{i,1}, \ldots, B_{i,k_i} \times Q_{i,k_i}],$$

and where $\pi_i : [k_{i+1}] \to [k_i]$ is a witness to $Z_i \succ Z_{i+1}$. We note that $k_i > 0$ for all $i$. Otherwise $0 = k_i = k_{i+1} = \cdots$ and then $Z_i = Y_i \succ Z_{i+1} = Y_{i+1} \succ \cdots$ implies that $Y_i \supsetneq Y_{i+1} \supsetneq \cdots$, which contradicts the Noetherianity of the **Vec**-variety $Y_i$; see Theorem 2.5.

From the chain, we construct an infinite rooted forest with vertex set $[k_1] \sqcup [k_2] \sqcup \cdots$ as follows: $[k_1]$ is the set of roots, and we attach each $j \in [k_{i+1}]$ via an edge with $\pi_i(j)$; the latter is called the *parent* of the former. We further label each vertex $j \in [k_i]$ with the product $B_{i,j} \times Q_{i,j}$.

We claim that $\pi_i$ is an injection for all $i \gg 0$, i.e., that there are only finitely many vertices with more than one child. Indeed, if not, then by König's lemma the forest would have an infinite path starting at a root in $[k_1]$ and passing through infinitely many vertices with at least two children. By construction, the labels $B \times Q$ decrease weakly along such a path and strictly whenever going from a vertex to one of its more than one children, a contradiction to Lemma 2.31.

For even larger $i$, the $k_i$ are constant, say equal to $k$, and hence the $\pi_i$ are bijections. After reordering, we may assume that the $\pi_i$ all equal the identity on $[k]$. Moreover, for all such $i$ we still have $B_{i,j} \times Q_{i,j} \succeq B_{i+1,j} \times Q_{i+1,j} \succeq \cdots$ for all $j \in [k]$, and all these chains stabilise. When they do, we have $Y_i \supsetneq Y_{i+1} \supsetneq \cdots$, which is a strictly decreasing chain of **Vec**-varieties — but this again contradicts the Noetherianity of **Vec**-varieties. □

**2.7.4.** *Order on closed subvarieties of product-type varieties in* **PM**. Consider the $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety $Z = [Y; B_1 \times Q_1, \ldots, B_k \times Q_k]$ and let $X$ be a closed $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-subvariety of $Z$; $X$ is not required to be of product type. We define

$$\delta_X := \min_{(S_1,\ldots,S_k)\in\mathbf{FI}^k} \left\{ \sum_{i=1}^{k} |S_i| : X(S_1, \ldots, S_k) \neq Z(S_1, \ldots, S_k) \right\}$$

Let $X$ and $X'$ be closed $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-subvarieties of $Z$. Then we say $X' \preceq X$ if $\delta_{X'} \leq \delta_X$. This is a well-founded preorder on the $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-subvarieties of $Z$.

**Remark 2.34.** If $f$ is a nonzero equation for $X(S_1, \ldots, S_k)(V)$ with $\sum_i |S_i| = \delta_X$, then $f$ may still "come from smaller sets". More specifically, there might exist a $k$-tuple $(S_1', \ldots, S_k')$ with $|S_i'| \leq |S_i|$ for all $i \in [k]$ and with strict inequality for at least one $i$, an $\mathbf{FI}^k$-morphism $\iota := (\iota_1, \ldots, \iota_k) : (S_1', \ldots, S_k') \to (S_1, \ldots, S_k)$, and an element $f' \in K[Z(S_1', \ldots, S_k')(V)]$ such that $Z(\iota)(V)^{\#}(f') = f$. This is related to Remark 2.9. The following example demonstrates this phenomenon.

**Example 2.35.** Consider the $\mathbf{FI}^{\mathbf{op}} \times \mathbf{Vec}$-variety $Z := [\mathrm{Spec}(K); \mathbb{A}^1]$. The coordinate ring $K[Z(S)]$ is isomorphic to the polynomial ring over $K$ in $|S|$ variables. Let $n \in \mathbb{Z}_{>0}$ and define the proper closed variety $X$ of $Z$ by

$$X(S) := \begin{cases} Z(S) & \text{for } |S| < n, \\ \varnothing & \text{otherwise.} \end{cases}$$

Then $\delta_X$ is equal to $n$ and computed by the element $1 \in K[Z([n])]$, which is the image of $1 \in K[Z(\varnothing)]$ under the natural map $K[Z(\varnothing)] \to K[Z([n])]$.

## 3. Covering $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-varieties by smaller ones

**3.1.** ***The parameterisation theorem.*** The goal of this section is to prove the following core result, which says that any proper closed subvariety of an $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety of product type is covered by finitely many smaller such varieties.

**Theorem 3.1** (parameterisation theorem). *Consider an* $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$*-variety $Z$ of product type and let $X \subsetneq Z$ be a proper closed* $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$*-subvariety. Then there exist a finite number of quadruples consisting of*

- *an $l \in \mathbb{Z}_{\geq 0}$;*
- *an* $(\mathbf{FI}^{\mathbf{op}})^l \times \mathbf{Vec}$*-variety $Z'$ of product type with $Z' \prec Z$;*
- *a $k$-tuple $S = (S_1, \ldots, S_k) \in \mathbf{FI}^k$; and*
- *a partition morphism $(\pi, \varphi) : Z' \to \mathrm{Sh}_S Z$;*

*such that for any* $T_1, \dots, T_k \in \mathbf{FI}^k$, *any* $V \in \mathbf{Vec}$, *and any* $p \in X(T_1, \dots, T_k)(V)$ *there exist*: *one of these finitely many quadruples*; *finite sets* $U_1, \dots, U_k$; *and bijections* $\sigma_i : T_i \to S_i \sqcup U_i$; *such that* $p$ *lies in the image under* $Z(\sigma_1, \dots, \sigma_k)(V)$ *of the image of* $(\pi, \varphi)$ *in* $\mathrm{Sh}_S(Z)(U_1, \dots, U_k)(V) = Z(S_1 \sqcup U_1, \dots, S_k \sqcup U_k)(V)$.

**Remark 3.2.** Recall Definition 2.13 of the image of a partition morphism. Explicitly, the conclusion above means that there exist finite sets $U'_1, \dots, U'_l$ and, for each $i \in [k]$, a bijection $\iota_i : U_i \to \bigsqcup_{j \in \pi^{-1}(i)} U'_j$, and a point $q \in Z'(U'_1, \dots, U'_l)(V)$ such that

$$(Z(\sigma_1, \dots, \sigma_k)(V) \circ (\mathrm{Sh}_S Z)(\iota_1, \dots, \iota_l)(V) \circ \varphi(U'_1, \dots, U'_l)(V))(q) = p.$$

Informally, we will say that all points in $X$ are *hit* by finitely many partition morphisms from varieties $Z'$ in **PM** of product type with $Z' \prec Z$.

**3.2.** ***A key proposition.*** The proof of Theorem 3.1 uses a key proposition that we establish first. The reader may prefer to read only the statement of this proposition and postpone its proof until after reading the proof of Theorem 3.1 in Section 3.5.

**Proposition 3.3.** *Let* $Y$ *be a* **Vec**-*variety*; $n \in \mathbb{Z}_{\geq 0}$; $B$ *a closed* **Vec**-*subvariety of* $Y \times \mathbb{A}^n$; $Q$ *a pure polynomial functor*; *and* $X$ *a proper closed* **Vec**-*subvariety of* $B \times Q \subseteq Y \times \mathbb{A}^n \times Q$. *Then there exist*

- *a proper closed* **Vec**-*subvariety* $Y_0$ *of* $Y$;
- *a* **Vec**-*variety* $Y'$ *together with a morphism* $\alpha : Y' \to Y$;
- $k \in \mathbb{Z}_{\geq 0}$;

*and*, *for each* $l = 0, \dots, k$,

- *an integer* $n_l \in \mathbb{Z}_{\geq 0}$;
- *a closed* **Vec**-*subvariety* $B_l \subseteq Y' \times \mathbb{A}^{n_l}$;
- *a pure polynomial functor* $Q_l$;
- *and a morphism* $\beta_l : B_l \times Q_l \to B \times Q$,

*such that the following properties hold*:

(1) *For each* $l = 0, \dots, k$, *we have that* $B_l \times Q_l \prec B \times Q$ *in the preorder from Section 2.7.2, and the following diagram commutes*:

$$\begin{array}{ccc} B_l \times Q_l & \xrightarrow{\beta_l} & B \times Q \\ \downarrow & & \downarrow \\ Y' & \xrightarrow[\alpha]{} & Y \end{array}$$

(2) *Let* $m \in \mathbb{Z}_{\geq 0}$, *let* $V \in \mathbf{Vec}$, *and let* $p_1, \dots, p_m \in X(V) \subseteq Y(V) \times \mathbb{A}^n \times Q(V)$ *be points whose images in* $Y(V)$ *are all equal to the same point* $y \in Y(V) \setminus Y_0(V)$. *Then there exist indices* $l_j \in \{0, \dots, k\}$ *for* $j \in [m]$ *and points* $p'_j \in B_{l_j}(V) \times Q_{l_j}(V)$ *whose images in* $Y'(V)$ *are all equal to the same point* $y'$ *and such that* $\beta_{l_j}(V)(p'_j) = p_j$ *for all* $j \in [m]$.

**Remark 3.4.** The condition $\beta_{l_j}(V)(p'_j) = p_j$, together with the commuting diagram in (1), implies $\alpha(y') = y$.

**Remark 3.5.** The labelling by $l \in \{0, \dots, k\}$ rather than by $l \in [k]$ is chosen because in the proof of Proposition 3.3 the data for $l = 0$ are chosen in a slightly different manner than those for $l > 0$. However, in the statement of that proposition, all $l$ play equivalent roles.

To apply Proposition 3.3 in the proof of Theorem 3.1 we will do a shift over an appropriate $k$-tuple of finite sets. After this shift, we deal with the points of $X$ lying over $Y_0$ by induction, while we cover those in the complement by a partition morphism constructed with the morphisms $\alpha$ and $\beta_j$'s, and whose domain is a product-type variety strictly smaller than $Z$. Before proving Proposition 3.3 in Section 3.4, we demonstrate its statement in two special cases.

**Example 3.6.** Consider the case where $Y = \operatorname{Spec} K$ and $n = 0$; then $B \subseteq Y \times \mathbb{A}^n$ is also isomorphic to $\operatorname{Spec} K$. Let $Q$ be an arbitrary polynomial functor. In this case, $X$ is a proper closed **Vec**-subvariety of $Q$ and by [Bik et al. 2023b] there exist $k \in \mathbb{Z}_{\geq 0}$, (finite-dimensional) varieties $B_0, \dots, B_k$, pure polynomial functors $Q_0, \dots, Q_k \prec Q$ and morphisms $\beta_l : B_l \times Q_l \to Q$ such that $X$ is the union of the images of the $\beta_l$. This is an instance of Proposition 3.3 with $Y_0 = \varnothing$, $Y' = Y$, and $\alpha = \mathrm{id}_Y$. Note that then $B_l \times Q_l \prec Q$ since $Q_l \prec Q$, so the specific choice of embedding $B_l \subseteq \mathbb{A}^{n_l}$ is not relevant.

**Example 3.7.** Consider the case where $Y$ is constant, that is, just given by a (finite-dimensional) variety, and $Q = 0$. Since $X$ is a proper closed subvariety of $B \subseteq Y \times \mathbb{A}^n$, there exist a $V \in$ **Vec** and a nonzero function $f \in K[B(V)]$ that vanishes identically on $X(V)$.

Then $f$ is represented by a polynomial in $K[Y(V)][x_1, \dots, x_n]$, also denoted by $f$. We may reduce $f$ modulo $\mathcal{I}(B(V))$ in such a manner that its leading term $c \cdot x^u$ has the property that $c \in K[Y(V)]$ is nonzero and $x^u \notin \mathrm{LM}(B)$. Then we take for $Y_0$ the closed subvariety of $Y$ defined by the vanishing of $c$ and for $Y'$ the complement $Y \setminus Y_0$, with $\alpha : Y' \to Y$ being the inclusion. Furthermore, we take $k = 0$, and $B_0$ to be the intersection of $B$ with $Y' \times \mathbb{A}^n$ and with the vanishing locus of $f$ in $Y \times \mathbb{A}^n$. Then $\mathrm{LM}(B_0) \supseteq \mathrm{LM}(B)$ and since $c$ is invertible on $Y'$ and $f$ vanishes on $B_0$, $x^u \in \mathrm{LM}(B_0) \setminus \mathrm{LM}(B)$. To verify (2) of Proposition 3.3, we observe that the $p_j$ all map to the same point in $Y' = Y \setminus Y_0$, i.e., $p_j$ lies in the set $B_0 \subseteq B$, and we can just take $p'_j := p_j$ for all $j$.

**3.3.** ***Iterated partial derivatives.*** The main technical result for proving Proposition 3.3 is Lemma 3.8 below. This is essentially an iteration of the argument used to establish the embedding theorem in [Bik et al. 2023b], which involves directional derivatives of a function defining a **Vec**-variety along a direction lying in an irreducible subobject of the top-degree part of the ambient polynomial functor.

**Lemma 3.8.** *Let $B$ be a* **Vec***-variety and $Q$ a pure polynomial functor. Decompose*

$$Q = R_1 \oplus \cdots \oplus R_t,$$

*where the $R_i$ are irreducible objects in the abelian category of polynomial functors, arranged in weakly increasing degrees. Denote with $R_{\leq s}$ the functor $\bigoplus_{i=1}^{s} R_i$, so that $R_{\leq 0} = 0$. Let $X$ be a proper closed* **Vec***-subvariety of $B \times Q$.*

*Then there exist*

- *a* $k \in \mathbb{Z}_{\geq 0}$;
- $U_0, \dots, U_k \in \mathbf{Vec}$ *with partial sums* $U_{\leq s} := \bigoplus_{i=0}^{s} U_i$ *for* $s \geq 0$;
- *indices* $0 = s_0 < s_1 \leq \cdots \leq s_k \leq t$;
- *for each* $l \in \{0, \dots, k\}$ *a nonzero function* $h_l \in K[B(U_{\leq l}) \times R_{\leq s_l}(U_{\leq l})]$ (*so that* $h_0 \in K[B(U_0)]$); *and*
- *for each* $l \in \{1, \dots, k\}$, *a nonzero coordinate* $x_l \in R_{s_l}(U_l)^*$ *and a function* $r_l$ *in* $K[B(U_{\leq l}) \times (R_{\leq s_l}(U_{\leq l})/R_{s_l}(U_l))]$ *such that*
$$h_l = x_l \cdot h_{l-1} + r_l;$$

*and such that*, *moreover*, *the function* $h_k$ *vanishes on* $X(U_{\leq k})$.

**Remark 3.9.** It is here that we use the fact that $K$ has characteristic zero, in at least two different ways: the fact that an arbitrary polynomial functor is a direct sum of irreducible ones, and the fact that, by acting with the Lie algebra of $\mathrm{GL}_n$, we can go from an equation to an equation of weight $(1, \dots, 1)$. We think that our main theorem may be true in positive characteristic as well, but the proof would be more technical and involve techniques from [Bik et al. 2024], where the theory of GL-varieties in positive characteristic is developed.

*Proof.* Let $U$ be a finite-dimensional vector space for which there exists a nonzero $f \in K[B(U) \times Q(U)]$ that vanishes identically on $X(U)$. Without loss of generality, $U = K^n$ for some $n$. Since the vanishing ideal of $X(U)$ is a $\mathrm{GL}(U)$-module, we may assume that $f$ is a weight vector with respect to the standard maximal torus in $\mathrm{GL}(U) = \mathrm{GL}_n$. Furthermore, by enlarging $U$ if necessary ($n = \deg(f)$ suffices) we may assume that the weight of $f$ is $(1, \dots, 1)$ (see [Snowden 2021, Lemma 3.2]; strictly speaking, our $\mathrm{GL}(U)$-action is contragredient to the action there, and writing $(-1, \dots, -1)$ would be more consistent).

Choose $s_k$ as the maximal index in $[t]$ such that $f$ involves coordinates in $R_{s_k}(U)^*$; if no such index exists, then $k$ is set to zero, and we may take $U_0 = U$ and $h_0 = f \in K[B(U_0)]$ and we are done.

After acting with the symmetric group $\mathrm{Sym}([n])$ if necessary, we may assume that $f$ contains at least one coordinate in $R_{s_k}(U)^*$ of weight $(0, \dots, 0, 1, \dots, 1) =: (0^{n'}, 1^{n_k})$, where there are $n'$ zeroes and $n_k$ ones, with $n' + n_k = n$. We set $U' := K^{n'}$ and $U_k := K^{n_k}$, so that $U = U' \oplus U_k$. Since $f$ has weight $(1, \dots, 1)$, we can decompose
$$f = \left( \sum_{i=1}^{N} f_i \cdot y_i \right) + r$$
where $N \geq 1$, the $f_i$ have weight $(1^{n'}, 0^{n_k})$; the $y_i$ are elements in $R_{s_k}(U)^*$ of weight $(0^{n'}, 1^{n_k})$ and hence lie in $R_{s_k}(U_k)^*$; and $r$ does not contain elements in $R_{s_k}(U_k)^*$. This implies that the $f_i$ are elements of $K[B(U') \times R_{\leq s_k}(U')]$ and $r$ is an element of $K[B(U' \oplus U_k) \times (R_{\leq s_k}(U' \oplus U_k)/R_{s_k}(U_k))]$. Furthermore, we may assume that the $f_i$ are linearly independent over $K$.

Now act on $f$ with upper triangular elements of $\mathfrak{gl}(U_k)$. With respect to this action, the $f_i$ are constants, the $y_i$ are replaced by higher-weight vectors in $R_{s_k}(U_k)^*$, and $r$ remains an element of

$K[B(U'\oplus U_k)\times(R_{\leq s_k}(U'\oplus U_k)/R_{s_k}(U_k))]$. We can choose a sequence of such upper triangular elements that takes $y_1$ to a nonzero highest weight vector $v$ in $R_{s_k}(U_k)^*$, and the same sequence will take each $y_i$ to a scalar multiple of $v$. Since the $f_i$ are linearly independent, the term $f_1\cdot v$ in the result is not cancelled. Hence after this action, $f$ has been transformed to the desired shape

$$f = h\cdot x_k + r$$

with $h\in K[B(U')\times R_{\leq s_k}(U')]$, $x_k$ a nonzero highest weight vector in $R_{s_k}(U_k)^*$ and $r$ lies in the ring

$$K[B(U'\oplus U_k)\times(R_{\leq s_k}(U'\oplus U_k)/R_{s_k}(U_k))].$$

Now we treat the pair $(U', h)$ in exactly the same manner as we treated the pair $(U, f)$, dragging $r$ along in the process: pick $s_{k-1}$ maximal such that $h$ contains elements from $R_{s_{k-1}}(U')^*$. By acting with the symmetric group $\mathrm{Sym}([n'])$ on $f$ we may assume that $h$ contains an element from $R_{s_{k-1}}(U')^*$ of weight $(0^{n''}, 1^{n_{k-1}})$, with $n''+n_{k-1}=n'$. Then set $U''=K^{n''}$ and $U_{k-1}=K^{n_{k-1}}$, so that $U'=U''\oplus U_{k-1}$. By acting on $f$ with upper triangular elements of $\mathfrak{gl}(U_{k-1})$ we transform it into the shape

$$f = (\tilde{h}\cdot x_{k-1}+\tilde{r})\cdot x_k + r,$$

where $x_k$ has not changed, $r$ has changed within the space $K[B(U'\oplus U_k)\times(R_{\leq s_k}(U'\oplus U_k)/R_{s_k}(U_k))]$, $x_{k-1}$ is a highest weight vector in $R_{s_{k-1}}(U_{k-1})^*$, $\tilde{h}$ lies in $K[B(U'')\times R_{\leq s_{k-1}}(U'')]$, and $\tilde{r}$ lies in the ring

$$K[B(U''\oplus U_{k-1})\times(R_{\leq s_{k-1}}(U''\oplus U_{k-1})/R_{s_{k-1}}(U_{k-1}))].$$

Continuing in this fashion, we eventually put $f$ in the form

$$f = x_k(x_{k-1}(\cdots(x_2(x_1h_0+r_1)+r_2)\cdots)+r_{k-1})+r_k$$

where $h_0\in K[B(U_0)]$ and $U_0$ is the space left over from $U$ after splitting off all the $U_i$ with $i>0$. Now set

$$h_l := x_l(x_{l-1}(\cdots(x_2(x_1h_0+r_1)+r_2)\cdots)+r_{l-1})+r_l$$

and we are done. □

**3.4.** ***Proof of Proposition 3.3.*** This section contains the proof of the Proposition 3.3, and, for clarity's sake, we spell it out in a concrete example at the end.

**Remark 3.10.** We recall that, for any **Vec**-variety $Z$ and any $U\in$ **Vec**, the shift $\mathrm{Sh}_U\, Z$ of $Z$ over $U$ is the **Vec**-variety defined by $(\mathrm{Sh}_U\, Z)(V)=Z(U\oplus V)$. There is a *natural morphism* $\mathrm{Sh}_U\, Z\to Z$ of **Vec**-varieties: for $V\in$ **Vec**, this morphism $(\mathrm{Sh}_U\, Z)(V)=Z(U\oplus V)\to Z(V)$ is just $Z(\pi_V)$, where $\pi_V$ is the projection $U\oplus V\to V$.

**Lemma 3.11.** *Let $Y$ be a* **Vec**-*variety, $n\in\mathbb{Z}_{\geq 0}$, and $B$ a closed* **Vec**-*subvariety of $Y\times\mathbb{A}^n$. Then for any $U\in$* **Vec**, *$\mathrm{Sh}_U\, B$ is a closed* **Vec**-*subvariety of $(\mathrm{Sh}_U\, Y)\times\mathbb{A}^n$, and $\mathrm{LM}(B)=\mathrm{LM}(\mathrm{Sh}_U(B))$.*

*Proof.* This follows from Lemma 2.24. □

**Remark 3.12.** Let $X$ be a **Vec**-variety, $U \in$ **Vec** and $f \in K[X(U)]$. We define $(\mathrm{Sh}_U X)[1/f]$ to be the **Vec**-variety given by $V \mapsto X(U \oplus V)[1/f]$, where we identify $f$ with its image under the natural map $K[X(U)] \to K[X(U \oplus V)]$. Note that the action of the group GL on the coordinate ring of $\mathrm{Sh}_U X$ is the identity on the element $f$. In particular, for every $V \in$ **Vec**, $(\mathrm{Sh}_U X[1/f])(V) \subseteq \mathrm{Sh}_U X(V)$ is the distinguished open set of points not vanishing on the single $f$.

*Proof of Proposition 3.3.* Since $X$ is a proper closed subvariety of $B \times Q$, we apply the machinery of Lemma 3.8. Decompose $Q$ as $R_1 \oplus \cdots \oplus R_t$, where the $R_s$ are irreducible polynomial functors and $\deg(R_s) \le \deg(R_{s+1})$ for all $s = 1, \dots, t-1$. Write $R_{\le s} := R_1 \oplus \cdots \oplus R_s$ and $R_{>s} := R_{s+1} \oplus \cdots \oplus R_t$, so that $R_{\le 0} = \{0\}$ and $R_{>t} = \{0\}$.

By Lemma 3.8, we can construct a sequence of vector spaces $U_0, U_1, \dots, U_k$ with partial sums $U_{\le l} := \bigoplus_{i=0}^{l} U_i$, indices $0 = s_0 < s_1 \le \cdots \le s_k \le t$, nonzero coordinates $x_l \in R_{s_l}(U_l)^*$ for $l \in [k]$, nonzero functions $h_l \in K[B(U_{\le l}) \times R_{\le s_l}(U_{\le l})]$ for $l = 0, \dots, k$ and functions $r_l \in K[B(U_{\le l}) \times (R_{\le s_l}(U_{\le l})/R_{s_l}(U_l))]$ for $l \in [k]$ such that

$$h_l = x_l \cdot h_{l-1} + r_l \tag{A}$$

for each $l = 1, \dots, k$ and such that $h_k$ that vanishes on $X(U_{\le k})$.

Now $h_0 \in K[B(U_0)]$ is represented by a polynomial in $K[Y(U_0)][x_1, \dots, x_n]$, and after reducing modulo $\mathcal{I}(B(U_0))$, we may assume that its leading term equals $c \cdot x^u$ where $c \in K[Y(U_0)]$ is nonzero and $x^u \notin \mathrm{LM}(B)$.

Now set $U := U_{\le k} = U_0 \oplus \cdots \oplus U_k$. Then we construct the relevant data as follows.

(1) Define $Y_0$ as the closed **Vec**-subvariety of $Y$ defined by the vanishing of $c$, so that

$$Y_0(V) := \{y \in Y(V) \mid \forall \varphi \in \mathrm{Hom}_{\mathbf{Vec}}(V, U_0) : c(Y(\varphi)y) = 0\}.$$

(2) Set $Y' := (\mathrm{Sh}_U Y)[1/c]$ with $\alpha : Y' \to Y$ the restriction to $Y'$ of the natural morphism $\mathrm{Sh}_U Y \to Y$.

(3) Let $B_0$ be the closed **Vec**-subvariety of $(\mathrm{Sh}_U B)[1/c]$ defined by the vanishing of the single equation $h_0$. Note that $B_0$ is a closed **Vec**-subvariety of $Y' \times \mathbb{A}^{n_0}$ with $n_0 := n$. Define $Q_0 := Q$ and $\beta_0 : B_0 \times Q_0 \to B \times Q$ as the identity on $Q$ and equal to the restriction to $B_0$ of the natural morphism $\mathrm{Sh}_U B \to B$ on $B_0$. Note that $\mathrm{LM}(B_0) \supseteq \mathrm{LM}(B)$ by virtue of Lemma 3.11, and since $h_0 \in \mathcal{I}(B_0(U_0))$ has leading term $c \cdot x^u$ and $c$ is invertible on $Y'$, we have $x^u \in \mathrm{LM}(B_0) \setminus \mathrm{LM}(B)$. Thus $B_0 \times Q_0 \prec B \times Q$.

(4) For $l \in [k]$, set

$$Q_l := ((\mathrm{Sh}_U R_{\le s_l})/(R_{\le s_l}(U) \oplus R_{s_l})) \oplus R_{>s_l}.$$

Here we recall that, for any pure polynomial functor $R$, the top-degree part of $\mathrm{Sh}_U R$ is naturally isomorphic to that of $R$, and its constant part is isomorphic to $R(U)$ (see [Draisma 2019, Lemma 14] for the first statement; the second is proved in a similar fashion). So, since we ordered the irreducible factors $R_s$ by ascending degrees, $R_{s_l}$ is naturally a subobject of the top-degree part of $\mathrm{Sh}_U R_{\le s_l}$; and the constant polynomial functor $R_{\le s_l}(U)$ is the constant part of $\mathrm{Sh}_U R_{\le s_l}$. Both are modded out, and we have $Q_l \prec Q$.

(5) For $l \in [k]$, we define $B_l$ as

$$B_l := (\mathrm{Sh}_U\, B)[1/c] \times R_{\leq s_l}(U) \times \mathbb{A}^1 \subseteq Y' \times \mathbb{A}^n \times R_{\leq s_l}(U) \times \mathbb{A}^1 \cong Y' \times \mathbb{A}^{n_l}.$$

where $n_l := n + \dim(R_{\leq s_l}(U)) + 1$. Note that the factor $R_{\leq s_l}(U)$ is precisely the constant term modded out in the definition of $Q_l$; the role of the factor $\mathbb{A}^1$ will become clear below.

(6) To construct $\beta_l : B_l \times Q_l \to B \times Q$ we proceed as follows. Let $X_l$ be the closed **Vec**-subvariety of $B \times R_{\leq s_l}$ defined by the vanishing of $h_l$. Then (A) shows that, on the distinguished open subvariety $(\mathrm{Sh}_{U_{\leq l-1}}\, X_l)[1/h_{l-1}]$, the coordinate $x_l$ can be expressed as a function on $\mathrm{Sh}_{U_{\leq l-1}}\, B \times ((\mathrm{Sh}_{U_{\leq l-1}}\, R_{\leq s_l})/R_{s_l})$ evaluated at $U_l$. Since $R_{s_l}$ is irreducible, *each* coordinate on it can be thus expressed; this is a crucial point in the proof of [Draisma 2019, Lemma 25]. This implies that the projection

$$\mathrm{Sh}_{U_{\leq l-1}}\, B \times \mathrm{Sh}_{U_{\leq l-1}}\, R_{\leq s_l} \to (\mathrm{Sh}_{U_{\leq l-1}}\, B) \times (\mathrm{Sh}_{U_{\leq l-1}}\, R_{\leq s_l})/R_{s_l}$$

restricts to a closed immersion of $(\mathrm{Sh}_{U_{\leq l-1}}\, X_l)[1/h_{l-1}]$ into the open subvariety of the right-hand side where $h_{l-1}$ is nonzero. This statement remains true when we replace $U_{\leq l-1}$ everywhere by the larger space $U$. After also inverting $c$, we find a closed immersion

$$(\mathrm{Sh}_U\, X_l)[1/h_{l-1}][1/c] \to (\mathrm{Sh}_U\, B)[1/c] \times (\mathrm{Sh}_U\, R_{\leq s_l})/R_{s_l} \times \mathbb{A}^1,$$

where the map to the last factor is given by $1/h_{l-1}$. By [Bik 2020, Proposition 1.3.22] the inverse morphism from the image of this closed immersion lifts to a morphism of ambient **Vec**-varieties

$$\iota : B_l \times (\mathrm{Sh}_U\, R_{\leq s_l})/(R_{\leq s_l}(U) \oplus R_{s_l}) \cong (\mathrm{Sh}_U\, B)[1/c] \times (\mathrm{Sh}_U\, R_{\leq s_l})/R_{s_l} \times \mathbb{A}^1 \to \mathrm{Sh}_U(B \times R_{\leq s_l})$$

that hits all the points in $(\mathrm{Sh}_U\, X_l)[1/h_{l-1}][1/c]$. Finally, we define $\beta_l := \beta_l' \times \mathrm{id}_{R_{>s_l}}$ where $\beta_l'$ is the composition of $\iota$ and the natural morphism $\mathrm{Sh}_U(B \times R_{\leq s_l}) \to B \times R_{\leq s_l}$.

Property (1) in the proposition holds by construction. We now verify property (2). Thus let $V \in$ **Vec**, $m \in \mathbb{Z}_{\geq 0}$, and let $p_1, \ldots, p_m \in X(V) \subseteq Y(V) \times \mathbb{A}^n \times Q(V)$. Assume that the images of $p_1, \ldots, p_m$ in $Y(V)$ are all equal to $y$, and that $y \notin Y_0(V)$. By definition of $Y_0$, this means that there exists a $\varphi \in \mathrm{Hom}_{\mathbf{Vec}}(V, U)$ such that $c(Y(\varphi)(y)) \neq 0$.

On the other hand, we have $h_k(X(\psi)(p_j)) = 0$ for all $j$ and all $\psi : V \to U$, because $h_k$ vanishes identically on $X$. For $j \in [k]$ define

$$l_j := \min\{l \mid \forall \psi \in \mathrm{Hom}_{\mathbf{Vec}}(V, U) : h_l(X(\psi)(p_j)) = 0\}.$$

Put differently, $l_j$ is the smallest index $l$ such that the projection of $p_j$ in $B \times R_{\leq s_l}$ lies in $X_l \subseteq B \times R_{\leq s_l}$. Note that, if $l_j > 0$, then there exists a linear map $\psi : V \to U$ such that $h_{l_j - 1}(X(\psi)(p_j)) \neq 0$.

Since $\mathrm{Hom}_{\mathbf{Vec}}(V, U)$ is irreducible, there exists a linear map $\varphi : V \to U$ such that first, $c(Y(\varphi)(y)) \neq 0$; and second, $h_{l_j - 1}(X(\varphi)(p_j)) \neq 0$ for all $j$ with $l_j > 0$.

We now define the $p_j'$ as follows. First, we decompose $p_j = (p_{j,1}, p_{j,2})$, where $p_{j,1} \in B(V) \times R_{\leq s_{l_j}}(V)$ and $p_{j,2} \in R_{>s_{l_j}}(V)$. Similarly, we decompose the point $p_j' = (p_{j,1}', p_{j,2}')$ to be constructed.

(1) Set $p'_{j,2} := p_{j,2}$ for all $j$. Recall that we had defined $s_0 := 0$, so that this implies that if $l_j = 0$, then the component $p'_{j,2}$ of $p'_j$ in $Q$ equals the component $p_{j,2}$ of $p_j$ in $Q$.

(2) If $l_j = 0$, then $p_{j,1} \in B(V)$, and $p'_{j,1} \in B_0(V) \subseteq (\mathrm{Sh}_U\, B)[1/c](V)$ is defined as $B(\varphi \oplus \mathrm{id}_V)(p_{j,1})$. Note that $p'_{j,1}$ does indeed lie in $B_0(V)$; this follows from the fact $l_j = 0$, so that $h_0(B(\psi)(p_{j,1})) = 0$ for all $\psi : V \to U_0$, and hence also for all $\psi$ that decompose as $\psi' \circ (\varphi \oplus \mathrm{id}_V)$.

Furthermore, note that $\beta_0(V)(p'_j) = p_j$; this follows from the equality $\pi_V \circ (\varphi \oplus \mathrm{id}_V) = \mathrm{id}_V$. Also, the image of $p'_j$ in $Y'(V)$ equals $Y(\varphi \oplus \mathrm{id}_V)(y) =: y'$.

(3) If $l := l_j > 0$, then $p_{j,1} \in B(V) \times R_{\leq s_l}(V)$ with $s_l \geq 1$, and $p'_{j,1}$ is constructed as follows. First apply $(B \times R_{\leq s_l})(\varphi \oplus \mathrm{id}_V)$ to $p_{j,1}$ and then forget the component in $R_{s_l}(V)$. The morphism $\beta'_l$ was constructed in such a manner that $\beta'_l(V)(p'_{j,1}) = p_{j,1}$ and therefore $\beta_l(V)(p'_j) = p_j$. Note that also the image of $p'_j$ in $Y'(V)$ equals $y'$. □

**Example 3.13.** Write $Y$ for the polynomial functor $V \to V \oplus V$ and write $K[x_i, y_i \mid i \in [n]]$ for the coordinate ring of $Y(K^n)$. Consider the **Vec**-subvariety $B$ of $Y \times \mathbb{A}^1$ defined by $y_1 - t \cdot x_1$, where $t$ is the coordinate of $\mathbb{A}^1$. Then $\mathrm{LM}(B) = \varnothing$ and $B(V)$ is the set of triples $(v, \lambda v, \lambda)$ with $v \in V$ and $\lambda \in K$. Set $Q(V) := S^2 V$, and choose coordinates $z_{ij}$, $i \leq j$ on $Q(K^n)$ by writing an arbitrary element of $Q(K^n)$ as

$$\sum_{i=1}^{n} z_{ii} e_i^2 + \sum_{1 \leq i < j \leq n} 2 z_{ij} e_i e_j.$$

Note that $Q$ is an irreducible polynomial functor, so, in the notation of Proposition 3.3, we have $R = R_1 = Q$. Define the **Vec**-subvariety

$$X \subset B \times Q \subset Y \times \mathbb{A}^1 \times Q$$

by

$$X(V) := \{(v, w, \lambda, q) \mid (v, w, \lambda) \in B(V) \text{ and } w^2, q \text{ are linearly dependent}\}.$$

An equation for $X(K^2)$ is the determinant

$$f := z_{12} y_1^2 - z_{11} y_1 y_2 = t^2 (z_{12} x_1^2 - z_{11} x_1 x_2) \in K[B(U_0) \times Q(U_0)]$$

with $U_0 := K^2$. Define $U_1 := \langle e_3, e_4 \rangle \cong K^2$, so that $U_0 \oplus U_1 = K^4$. Acting on $f$ equation with the (upper triangular) elements $E_{1,3}$ and $E_{2,4}$ of the Lie algebra $\mathfrak{gl}(U_0 \oplus U_1)$ gives the equation

$$h_1 := z_{34}(x_1^2 t^2) + (2 z_{14} x_1 x_3 - 2 z_{13} x_1 x_4 - z_{11} x_3 x_4) t^2$$

that, by construction, vanishes on $X(U_0 \oplus U_1)$. Note that $z_{34} \in Q(U_1)^*$, $h_0 := x_1^2 t^2 \in K[B(U_0)]$ (and we let $c$ be the leading coefficient: $c := x_1^2$), and the rest belongs to $K[B(U_0 \oplus U_1) \times Q(U_0 \oplus U_1)/Q(U_1)]$.

By acting with permutations $(3, i)$ and $(4, j)$ with $i < j$ on $h_1$ we find that, where $h_0$ is nonzero, on $X$ we have

$$z_{ij} = -\frac{1}{h_0} \cdot (2 z_{1j} x_1 x_i - 2 z_{1i} x_1 x_j - z_{11} x_i x_j) t^2. \tag{3}$$

A similar expression can be found for $z_{ii}$, with the same denominator $h_0$.

In this case, $Y_0$ from the proposition is the **Vec**-subvariety of $Y$ defined by $c = x_1^2$. This consists of all pairs $(0, w) \in V \oplus V$. The preimage in $X$ consists of all quadruples $(0, 0, \lambda, q)$ with $q$ arbitrary.

Set $U := U_0 \oplus U_1$, $Y' := \mathrm{Sh}_U\, Y[1/c]$, and let $B_0$ be the vanishing locus of $h_0$ in $\mathrm{Sh}_U\, B[1/c] \subset Y' \times \mathbb{A}^1$. Note that we have $t^2 \in \mathrm{LM}(B_0)$ — indeed, $t$ even vanishes identically on $B_0$. With $Q_0 := Q$ we find $B_0 \times Q_0 \prec B \times Q$, and we define the map

$$\beta_0 : B_0 \times Q_0 \to B \times Q$$

as $B(\pi_V)|_{B_0} \times \mathrm{id}_{Q(V)}$ for every $V \in \mathbf{Vec}$. This covers all the points in $X(V)$ of the form $(v, 0, 0, q)$ with $v, q$ arbitrary.

Finally, consider the map

$$\begin{aligned}\mathrm{Sh}_U(B \times Q)[1/h_0][1/c] \to \mathrm{Sh}_U(B \times Q)/Q \times \mathbb{A}^1 &\cong (\mathrm{Sh}_U\, B \times Q(U) \times \mathbb{A}^1) \times (\mathrm{Sh}_U\, Q/(Q(U) \oplus Q)) \\ &=: B_1 \times Q_1\end{aligned}$$

where the coordinate on $\mathbb{A}^1$ is given by $1/h_0$. This is a closed immersion, because where $h_0$ is nonzero, coordinates on $Q(V)$ with can be recovered from the coordinates on the right-hand side via (3). We use this to construct the map

$$\beta_1 : B_1 \times Q_1 = \mathrm{Sh}_U(B \times Q)/Q \times \mathbb{A}^1 \to \mathrm{Sh}_U(B \times Q) \to B \times Q.$$

The first arrow is given by the identity on the coordinates not in $Q(V)$, while the coordinates on $Q(V)$ are computed via (3). The second arrow projects into $B(V) \times Q(V)$. This map hits points in $X(V)$ of the form $(v, \lambda v, \lambda, \mu(\lambda v)^2)$ with $v, \lambda$ nonzero.

**3.5.** ***Proof of Theorem 3.1.*** The $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety $Z$ is of product type; hence by Definition 2.18 it can be written as

$$Z = [Y; B_1 \times Q_1, \dots, B_k \times Q_k]$$

for some **Vec**-subvarieties $B_i$ of $Y \times \mathbb{A}^{n_i}$ and pure polynomial functors $Q_i$. Furthermore, $X$ is a proper closed $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-subvariety of $Z$.

We prove, by induction on the quantity $\delta_X$, that all points in $X$ can be hit by partition morphisms from finitely many $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-varieties $Z'$ of product type with $Z' \prec Z$. So in the proof we may assume that this is true for all proper closed $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-subvarieties $X' \subsetneq Z$ with $\delta_{X'} < \delta_X$.

Let $(S_1, \dots, S_k) \in \mathbf{FI}^k$ be such that $\sum_i |S_i| = \delta_X$ and $X(S_1, \dots, S_k) \neq Z(S_1, \dots, S_k)$. If all $S_i$ are empty, then set $Y' := X(\varnothing, \dots, \varnothing)$, a proper closed **Vec**-subvariety of $Y$, $B_i' := Y' \times_Y B_i$, and $Z := [Y'; B_1' \times Q_1, \dots, B_k' \times Q_k]$. The partition morphism $(\mathrm{id}_{[k]}, \varphi)$ with $\varphi(T_1, \dots, T_k)$ the inclusion $\prod_i (B_i' \times Q_i)^{T_i} \to \prod_i (B_i \times Q_i)^{T_i}$ has $X$ in its image, and we have $Z' \prec Z$ because the $Q_i$ remain the same, $\mathrm{LM}(B_i') \supseteq \mathrm{LM}(B_i)$ by Lemma 2.24, and $Y'$ is a proper closed **Vec**-subvariety of $Y$. In this case, no shift of $Z$ is necessary.

Next assume that not all $S_i$ are empty. First we argue that the points of $X(T_1, \dots, T_k)$, where, for some $i$, $|T_i|$ is strictly smaller than $|S_i|$, are hit by partition morphisms from finitely many $Z' \prec Z$. We give the argument for $i=k$. Define the $k$-tuple $S$ to be shifted over as $S:=(\varnothing,\dots,\varnothing,T_k)\in\mathbf{FI}^k$, and define the $(\mathbf{FI}^{\mathbf{op}})^{k-1}\times\mathbf{Vec}$- variety $Z'$ of product type

$$Z':=[(B_k\times Q_k)^{T_k};\,B_1'\times Q_1,\dots,B_{k-1}'\times Q_{k-1}]$$

with $B_i'=(B_k\times Q_k)^{T_k}\times_Y B_i$. Consider the partition morphism $(\pi,\varphi):Z'\to\mathrm{Sh}_S Z$, where $\pi:[k-1]\to[k]$ is the inclusion and $\varphi(T_1,\dots,T_{k-1})$ is the natural isomorphism of **Vec**-varieties

$$Z'(T_1,\dots,T_{k-1})\to(\mathrm{Sh}_S Z)(T_1,\dots,T_{k-1},\varnothing)=Z(T_1,\dots,T_{k-1},T_k).$$

Note that $\pi$ witnesses $Z'\preceq Z$ since the $Q_i$ with $i\le k-1$ remain the same and $\mathrm{LM}(B_i')\supseteq\mathrm{LM}(B_i)$ by Lemma 2.24. Furthermore, since $k-1<k$, we have $Z'\prec Z$ by Lemma 2.32. All points in $X$ where the last index set has cardinality $|T_k|$ are hit by this partition morphism. Since there are only finitely many values of $|T_k|$ that are strictly smaller than $|S_k|$, we are done.

So it remains to hit points in $X(T_1,\dots,T_k)$ where $|T_i|\ge|S_i|$ for all $i$. In this phase we will apply Proposition 3.3.

As by assumption not all $S_i$ are empty, after a permutation of $[k]$ we may assume that $S_k\ne\varnothing$. Let $*$ be an element of $S_k$ and define $\widetilde{S_k}:=S_k\setminus\{*\}$. Consider the **Vec**-varieties

$$Z(S_1,\dots,S_k)=(B_1\times Q_1)^{S_1}_Y\times_Y\cdots\times_Y(B_k\times Q_k)^{\widetilde{S_k}}_Y\times_Y(B_k\times Q_k)^{\{*\}},$$
$$\widetilde{Y}:=Z(S_1,\dots,S_{k-1},\widetilde{S_k})=(B_1\times Q_1)^{S_1}_Y\times_Y\cdots\times_Y(B_k\times Q_k)^{\widetilde{S_k}}_Y.$$

Set $\widetilde{B_k}:=\widetilde{Y}\times_Y B_k\subseteq\widetilde{Y}\times\mathbb{A}^{n_k}$, and note that $X(S_1,\dots,S_k)$ is a proper closed **Vec**-subvariety of $\widetilde{B_k}\times Q_k$. We may therefore apply Proposition 3.3 to $\widetilde{Y},n_k,\widetilde{B_k},Q_k$ and $X(S_1,\dots,S_k)$.

First consider the proper closed **Vec**-subvariety $Y_0$ of $\widetilde{Y}$ promised by Proposition 3.3, and let $X'$ be the largest closed $(\mathbf{FI}^{\mathbf{op}})^k\times\mathbf{Vec}$-subvariety of $Z$ that intersects $Z(S_1,\dots,S_{k-1},\widetilde{S_k})$ in $Y_0$. Then $X'(S_1,\dots,\widetilde{S_k})\ne Z(S_1,\dots,\widetilde{S_k})$, and thus $\delta_{X'}\le\delta_X-1<\delta_X$. Hence, by the induction hypothesis, all points in $X'(T_1,\dots,T_k)$ can be hit by finitely many partition morphisms from varieties $Z'\prec Z$ of product type.

Next we consider the remaining pieces of data from Proposition 3.3. First, we have the **Vec**-variety $Y'$ with a morphism $\alpha:Y'\to\widetilde{Y}$. Further, we have an integer $s\in\mathbb{Z}_{\ge0}$ and for each $i=0,\dots,s$ we have integers $n'_{k+i}$; **Vec**-varieties $B'_{k+i}\subseteq Y'\times\mathbb{A}^{n'_{k+i}}$; pure polynomial functors $Q'_{k+i}$; and morphisms $\beta_{k+i}:B'_{k+i}\times Q'_{k+i}\to\widetilde{B_k}\times Q_k$ satisfying the conditions (1) and (2).

Define $B_i':=Y'\times_Y B_i$ for $i=1,\dots,k-1$ and the $(\mathbf{FI}^{\mathbf{op}})^{k+s}\times\mathbf{Vec}$-variety

$$Z':=[Y';\,B_1'\times Q_1,\dots,B_{k-1}'\times Q_{k-1},B_k'\times Q_k',\dots,B_{k+s}'\times Q_{k+s}'].$$

Now the map $\pi:[k+s]\to[k]$ that is the identity on $[k-1]$ and maps $[k+s]\setminus[k-1]$ to $\{k\}$ witnesses that $Z'\preceq Z$; here we use that $B'_{k+j}\times Q'_{k+j}\prec B_k\times Q_k$ for $j\in\{0,\dots,s\}$ by the conclusion of Proposition 3.3, and also Lemma 2.24 to show that $B_i'\times Q_i\preceq B_i\times Q_i$ for $i\in[k-1]$. In fact, we have $Z'\prec Z$ by Lemma 2.32.

Now the base variety $Y'$ of $Z'$ comes with a morphism $\alpha$ to the base variety $\widetilde{Y}$ of $\mathrm{Sh}_S Z$; we have morphisms $\beta_i:B_i'\times Q_i\to\widetilde{B_i}\times Q_i$ for $i=1,\dots,k-1$ (the natural map $B_i'\to\widetilde{B_i}$ times the identity on $Q_i$) and the morphisms $\beta_{k+j}:B'_{k+j}\times Q'_{k+j}\to\widetilde{B_k}\times Q_k$ defined earlier. By Example 2.21, these data yield a partition morphism $(\pi,\varphi): Z' \to \mathrm{Sh}_S\, Z$. We have to show that this partition morphism hits all points in $X$ that are not in $X'$.

First we show, for a $V \in \mathbf{Vec}$, that a point $p \in \mathrm{Sh}_S\, X(\widetilde{T}_1,\dots,\widetilde{T}_k)(V)$ whose projection to $\widetilde{Y}(V)$ is not in $Y_0(V)$ lies in the image of $\varphi(\widetilde{T}_1,\dots,\widetilde{T}_k)(V)$. To this end, we write

$$p = ((p_{i,t})_{t\in\widetilde{T}_i})_{i\in[k]}$$

with

$$p_{i,t} \in \mathrm{Sh}_S\, X(\varnothing,\dots,\varnothing,\{t\},\varnothing,\dots,\varnothing)(V) = \widetilde{Y}(V)\times_{Y(V)} B_i(V)\times Q_i(V) \subset \widetilde{Y}(V)\times \mathbb{A}^{n_i}\times Q_i(V),$$

where the singleton $\{t\}$ is in the $i$-th position. We write $p_{i,t} = (\widetilde{y}, a_{i,t}, b_{i,t})$ with $\widetilde{y}\in\widetilde{Y}(V)$, $a_{i,t}\in\mathbb{A}^{n_i}$, and $b_{i,t}\in Q_i(V)$.

By definition of a fibre product, the $p_{i,t}$ all have the same projection $\widetilde{y}$ in $\widetilde{Y}(V)\setminus Y_0(V)$, and hence we can apply (2) of Proposition 3.3 to the points $p_{k,t}$ with $t\in\widetilde{T}_k$. This yields integers $l_t\in\{0,\dots,s\}$ and points $p'_{k,t}\in B'_{k+l_t}(V)\times Q'_{k+l_t}(V)$ for $t\in\widetilde{T}_k$ whose images in $Y'(V)$ are all equal, say to $y'\in Y'(V)$, and which satisfy $\beta_{k+l_t}(V)(p'_{k,t}) = p_{k,t}$ for all $t$. This implies that $\alpha(y') = \widetilde{y}$.

Define

$$T'_{k+j} := \{t\in\widetilde{T}_k \mid l_t = j\},$$

$j = 0,\dots,s$, and set $T'_i := \widetilde{T}_i$ for $i = 1,\dots,k-1$. In $Z'(T'_1,\dots,T'_{k+s})$ we define the point $q = ((q_{i,t})_{t\in T'_i})_{i\in[k+s]}$ as follows. We set $q_{i,t}$ to be $(y', a_{i,t}, b_{i,t})$ for $i = 1,\dots,k-1$ and $t\in T'_i$, and $q_{i,t} = p'_{k,t}$ for $i = k,\dots,k+s$ and $t\in T'_i$. Then

$$\varphi(T'_1,\dots,T'_{k+s})(q) = p,$$

as desired.

Now, more generally, consider a point $p$ in $X(T_1,\dots,T_k)(V)\setminus X'(T_1,\dots,T_k)(V)$, where the cardinalities satisfy $|T_i|\ge|S_i|$. Then there exists an $\mathbf{FI}^k$-morphism $\iota = (\iota_1,\dots,\iota_k): S\to(T_1,\dots,T_k)$ such that $X(\iota)(p)\notin Y_0(V)$. Define $\widetilde{T}_i := T_i\setminus \mathrm{Im}(\iota_i)$ and extend $\iota$ to an isomorphism $\iota^e: S\sqcup(\widetilde{T}_1,\dots,\widetilde{T}_k)\to(T_1,\dots,T_k)$ by defining $\iota_i$ on $\widetilde{T}_i$ to be the inclusion. Consider $X(\iota^e)(p)\in X(S\sqcup(\widetilde{T}_1,\dots,\widetilde{T}_k))(V)$. This is also a point in $\mathrm{Sh}_S\, X(\widetilde{T}_1,\dots,\widetilde{T}_k)(V)$ whose projection to $\widetilde{Y}(V)$ does not lie in $Y_0(V)$. We can therefore find a point $q$ as described above showing that $X(\iota^e)(p)$ is in the image of $(\pi,\varphi): Z'\to\mathrm{Sh}_S\, Z$; by Definition 2.13, then so is $p$. □

## 4. Proof of the main theorem

The most general version of our Noetherianity result is the following.

**Theorem 4.1.** *Any $(\mathbf{FI}^{\mathbf{op}})^k\times\mathbf{Vec}$-variety of product type is Noetherian.*

*Proof.* We proceed by induction along the well-founded order on objects of product type in **PM** from Section 2.7.3.

Let $Z$ be an $(\mathbf{FI}^{\mathbf{op}})^k\times\mathbf{Vec}$-variety of product type and let $X_1\supseteq X_2\supseteq\cdots$ be a descending chain of closed $(\mathbf{FI}^{\mathbf{op}})^k\times\mathbf{Vec}$-subvarieties. Then either all $X_i$ are equal to $Z$, or there exists an $i_0$ such that $X := X_{i_0}$ is a proper closed $(\mathbf{FI}^{\mathbf{op}})^k\times\mathbf{Vec}$-subvariety of $Z$. In the latter case, by Theorem 3.1, there exist a finite number of objects $Z_1, \dots, Z_N$ in **PM** of product type, along with $k$-tuples $S_1, \dots, S_N \in \mathbf{FI}^k$ and partition morphisms $(\pi_j, \varphi_j) : Z_j \to \mathrm{Sh}_{S_j} Z$ such that every point of $X$ is hit by one of these. By the induction hypothesis, all $Z_j$s are Noetherian. For each $j$, by Lemma 2.16, the preimage in $Z_j$ of the chain $(\mathrm{Sh}_{S_j} X_i)_{i \geq i_0}$ is a chain of closed subvarieties, which therefore stabilises. As soon as these $N$ chains have all stabilised, then so has the chain $(X_i)_i$ — here we have used a version of Proposition 2.17. □

To deduce from this Theorems 1.1 and 1.5, we consider GL-varieties $Z_1, \dots, Z_k$ as well as the product $Z := Z_1^{\mathbb{N}} \times \cdots \times Z_k^{\mathbb{N}}$. Recall Remark 2.9.

*Proof of Theorem 1.5.* We need to prove that any descending chain $Z \supseteq X_1 \supseteq \dots$ of $\mathrm{Sym}^k \times \mathrm{GL}$-stable closed subvarieties of $Z$ stabilises.

To each $Z_i$ is associated a **Vec**-variety, which by abuse of notation we also denote $Z_i$; see Remark 2.6. Furthermore, $Z_i$ is a closed subvariety of $B_i \times Q_i$ for some finite-dimensional variety $B_i$ and some pure polynomial functor $Q_i$, and hence $Z$ is a closed subvariety of

$$(B_1 \times Q_1)^{\mathbb{N}} \times \cdots \times (B_k \times Q_k)^{\mathbb{N}}.$$

Now each $X_i$ defines a closed $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-subvariety $\widetilde{X}_i$ of

$$\widetilde{Z} := [Y; B_1 \times Q_1, \dots, B_k \times Q_k],$$

where $Y$ is a point. By Theorem 4.1, the $\widetilde{X}_i$ stabilise. As soon as they do, so do the $X_i$. □

*Proof of the main theorem.* Apply Theorem 1.5 with $k = 1$. □

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# On the boundedness of canonical models

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It is conjectured that the canonical models of varieties (not of general type) are bounded when the Iitaka volume is fixed. We confirm this conjecture when a general fiber of the corresponding Iitaka fibration is in a fixed bounded family of polarized log Calabi–Yau pairs.

## 1. Introduction

Throughout this paper, we work over the complex number field $\mathbb{C}$.

By analogy with the definition of volumes of divisors, the Iitaka volume of a $\mathbb{Q}$-divisor is defined as follows: Let $X$ be a normal projective variety and $D$ be a $\mathbb{Q}$-Cartier divisor. When the Iitaka dimension $\kappa(D)$ of $D$ is nonnegative, the Iitaka volume of $D$ is defined to be

$$\mathrm{Ivol}(D) := \limsup_{m\to\infty} \frac{\kappa(D)!\, h^0(X, \mathcal{O}_X(\lfloor mD \rfloor))}{m^{\kappa(D)}}.$$

For the definition of the Iitaka dimension, see [Lazarsfeld 2004, Definition 2.1.3].

For a pair $(X, \Delta)$, if the Iitaka dimension of the log canonical divisor $K_X + \Delta$ is nonnegative, it is conjectured that a general fiber of the Iitaka fibration of $K_X + \Delta$ is birationally equivalent to a log Calabi–Yau pair, according to the abundance conjecture. The main theorem states that, when a general fiber of $K_X + \Delta$ belongs to a fixed bounded family with bounded polarization, the Iitaka volume of the log canonical divisor lies in a set satisfying the descending chain condition (DCC). Furthermore, if the Iitaka volume is fixed, then the canonical model is in a bounded family.

**Theorem 1.1.** *Fix $\mathscr{C}$ a log bounded class of polarized log Calabi–Yau pairs, $\mathcal{I} \subset [0, 1] \cap \mathbb{Q}$ a DCC set of rational numbers, $n$ a positive integer and $v$ a positive rational number. Suppose $(X, \Delta)$ is a projective klt pair of dimension $n$, $L$ is a divisor on $X$, and $f : X \to Z$ is a contraction which is birationally equivalent to the Iitaka fibration of $K_X + \Delta$.*

*If a general fiber $(X_g, \Delta_g, L_g)$ of $f$ is in $\mathscr{C}$ and $\mathrm{coeff}(\Delta) \subset \mathcal{I}$, then*

(1) $\mathrm{Ivol}(K_X + \Delta)$ *is in a DCC set, and*

(2) *if* $\mathrm{Ivol}(K_X + \Delta) = v$ *is a constant, then*

$$\mathrm{Proj} \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(mK_X + \lfloor m\Delta \rfloor))$$

*is in a bounded family.*

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*Keywords:* canonical models, Iitaka fibration, polarized log Calabi–Yau pairs, boundedness.

Theorem 1.1 is a special case of the following conjecture.

**Conjecture 1.2.** Let $n$ be a positive integer, $v$ a nonnegative rational number, and $\mathcal{I} \subset [0, 1] \cap \mathbb{Q}$ a DCC set of rational numbers. Let $\mathcal{D}(n, v, \mathcal{I})$ be the set of varieties $Z$ such that

- $(X, \Delta)$ is a projective klt pair of dimension $n$,
- $\operatorname{coeff}(\Delta) \subset \mathcal{I}$,
- $\operatorname{Ivol}(K_X + \Delta) = v$ is a constant, and
- $f : X \dashrightarrow Z$ is the Iitaka fibration associated with $K_X + \Delta$, where

$$Z = \operatorname{Proj} \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(m(K_X + \Delta))).$$

Then $\mathcal{D}(n, v, \mathcal{I})$ is in a bounded family.

An interesting application of Theorem 1.1 is when $X \to Z$ is a Fano-type fibration whose general fibers are $\epsilon$-lc. In this case, a general fiber of $f$ is bounded according to the Birkar-BAB theorem, see [Birkar 2021b], and $-K_X$ will induce a natural polarization on a general fiber. We have the following corollary.

**Corollary 1.3.** *Let $n$ be a positive integer, $v$ a positive rational number and $\mathcal{I} \subset [0, 1] \cap \mathbb{Q}$ a DCC set of rational numbers. Suppose $(X, \Delta)$ is a projective klt pair of dimension $n$ and $f : X \to Z$ is a contraction such that*

- $\operatorname{coeff}(\Delta) \subset \mathcal{I}$,
- $K_X + \Delta \sim_{\mathbb{Q},Z} 0$, *and*
- $\Delta$ *is big over* $Z$.

*Then*

- $\operatorname{Ivol}(K_X + \Delta)$ *is in a DCC set, and*
- *if* $\operatorname{Ivol}(K_X + \Delta) = v$ *is a constant, then*

$$\operatorname{Proj} \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(mK_X + \lfloor m\Delta \rfloor))$$

  *is in a bounded family.*

According to [Birkar et al. 2010], the canonical ring $R(X, K_X+\Delta) := \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(m(K_X+\Delta)))$ is finitely generated, which implies that $Z = \operatorname{Proj} \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$ is well-defined and $v = \operatorname{Ivol}(K_X + \Delta)$ is a positive rational number. The validity of Conjecture 1.2 has been established in different scenarios: when $K_X + \Delta$ is big, it was proved in [Hacon et al. 2014]; for the case where a general fiber of $f$ is $\epsilon$-lc Fano-type, it was demonstrated in [Li 2024]; and when $f$ is an elliptic curve, [Filipazzi 2024] shows that $X$ is actually bounded in codimension one. Notably, around the same time this paper was completed, [Birkar 2021a] provided a proof of Conjecture 1.2 for the situation where a general fiber of $f$ belongs to a bounded family.

It is shown in [Hacon et al. 2013] that the boundedness of varieties of general type is connected with the DCC of volumes of the log canonical divisors. We think that the following conjecture is closely related to Conjecture 1.2.

**Conjecture 1.4.** Let $n \in \mathbb{N}$, and consider a DCC set $\mathcal{I} \subset [0, 1] \cap \mathbb{Q}$. Then the set of Iitaka volumes

$$\{\mathrm{Ivol}(K_X + \Delta) \mid X \text{ is projective, } (X, \Delta) \text{ is klt, } \dim X = n \text{ and } \operatorname{coeff}(\Delta) \subset \mathcal{I}\}$$

is a DCC set.

The main idea is to prove the DCC of Iitaka volumes and the boundedness of the canonical models when the locus of singular fibers of the Iitaka fibrations is "bounded". We show that, in this case, we can choose a uniform base such that the moduli part (see Theorem 2.11) descends.

To be precise, we are interested in the following set of pairs and the corresponding Iitaka fibrations.

**Definition 1.5.** Fix a DCC set $\mathcal{I} \subset [0, 1] \cap \mathbb{Q}$ and positive integers $n, r, l$. Let $\mathcal{D}(n, \mathcal{I}, l, r)$ be the set of pairs $(X, \Delta)$ satisfying the following conditions:

- $(X, \Delta)$ is a projective klt pair of dimension $n$.
- $\operatorname{coeff}(\Delta) \subset \mathcal{I}$.
- $f : X \to Z$ is the canonical model of $(X, \Delta)$.
- A general fiber $(X_g, \Delta_g)$ of $f$ has a good minimal model.
- Let $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$ be the generalized pair defined in Theorem 2.12; then $l\boldsymbol{M}_{Z'}$ is nef and Cartier.
- There is a $\mathbb{Q}$-Cartier integral divisor $D$ and a $\mathbb{Q}$-divisor $F \in |K_X + \Delta|_{\mathbb{Q}/Z}$ such that $(X, \operatorname{Supp}(\Delta - F))$ is log smooth over $Z \setminus D$ and

$$\mathrm{Ivol}(K_X + \Delta + f^*D) \le r\,\mathrm{Ivol}(K_X + \Delta).$$

**Theorem 1.6.** *Fix a DCC set $\mathcal{I} \subset [0, 1] \cap \mathbb{Q}$ and positive integers $n, r, l$. Then the set*

$$\{\mathrm{Ivol}(K_X + \Delta) \mid (X, \Delta) \in \mathcal{D}(n, \mathcal{I}, l, r)\}$$

*satisfies the DCC.*

As an application, we prove the following boundedness result.

**Theorem 1.7.** *Fix a DCC set $\mathcal{I} \subset [0, 1] \cap \mathbb{Q}$, positive integers $n, r, l$ and a positive rational number $v$. Then the set*

$$\{\operatorname{Proj} R(X, K_X + \Delta) \mid (X, \Delta) \in \mathcal{D}(n, \mathcal{I}, l, r),\ \mathrm{Ivol}(K_X + \Delta) = v\}$$

*is bounded.*

The idea is to prove that we can choose an snc model (see Definition 2.10) of $(X, \Delta - F) \to Z$ to be in a bounded family: this is why we need the last condition in Definition 1.5. We believe that the existence of $D$ and the integer $r$ naturally comes from a suitable moduli space of a general fiber of $f$. Theorem 1.1 is an application of Theorems 1.6 and 1.7 based on this idea.

## 2. Preliminaries

***Notation and conventions.*** Let $\mathcal{I} \subset \mathbb{Q}$ be a subset. We say $\mathcal{I}$ satisfies the DCC if there is no strictly decreasing subsequence in $\mathcal{I}$. For a birational morphism $f : Y \to X$ and a divisor $B$ on $X$, $f_*^{-1}(B)$ denotes the strict transform of $B$ on $Y$, and $\operatorname{Exc}(f)$ denotes the sum of the reduced exceptional divisors of $f$. For a $\mathbb{Q}$-divisor $D$, a map defined by the linear system $|D|$ means a map defined by $|\lfloor D \rfloor|$. Given two $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors $A$ and $B$, $A \sim_{\mathbb{Q}} B$ means that there is an integer $m > 0$ such that $m(A - B) \sim 0$. For a $\mathbb{Q}$-divisor $D$, we write $D = D_{\geq 0} - D_{\leq 0}$ as the difference of its positive and negative parts.

A subpair $(X, \Delta)$ consists of a normal variety $X$ and a $\mathbb{Q}$-divisor $\Delta$ on $X$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We call $(X, \Delta)$ a pair if, in addition, $\Delta$ is effective. If $g : Y \to X$ is a birational morphism and $E$ is a divisor on $Y$, the discrepancy $a(E, X, \Delta)$ is $-\operatorname{coeff}_E(\Delta_Y)$, where $K_Y + \Delta_Y := g^*(K_X + \Delta)$. A subpair $(X, \Delta)$ is called sub-klt (resp. sub-lc) if, for every birational morphism $Y \to X$ as above, $a(E, X, \Delta) > -1$ (resp. $\geq -1$) for every divisor $E$ on $Y$. A pair $(X, \Delta)$ is called klt (resp. lc) if $(X, \Delta)$ is sub-klt (resp. sub-lc) and $(X, \Delta)$ is a pair.

Let $(X, \Delta)$ and $(Y, \Delta_Y)$ be two subpairs, and let $h : Y \to X$ be a birational morphism. We say that $(Y, \Delta_Y) \to (X, \Delta)$ is a crepant birational morphism if $K_Y + \Delta_Y \sim_{\mathbb{Q}} h^*(K_X + \Delta)$ and $h_*\Delta_Y = \Delta$. Two pairs $(X_i, \Delta_i)$, $i = 1, 2$, are crepant birationally equivalent if there is a subpair $(Y, \Delta_Y)$ and two crepant birational morphisms $(Y, \Delta_Y) \to (X_i, \Delta_i)$, $i = 1, 2$.

A generalized pair $(X, \Delta + \boldsymbol{M}_X)$ consists of a normal variety $X$ equipped with a projective morphism $X \to U$, a birational morphism $f : X' \to X$ where $X$ is normal, a $\mathbb{Q}$-boundary $\Delta$, and a $\mathbb{Q}$-Cartier divisor $\boldsymbol{M}_{X'}$ on $X'$ such that $K_X + \Delta + \boldsymbol{M}_X$ is $\mathbb{Q}$-Cartier, $\boldsymbol{M}_{X'}$ is nef over $U$, and $\boldsymbol{M}_X = f_*\boldsymbol{M}_{X'}$. Let $\Delta'$ be the $\mathbb{Q}$-divisor such that

$$K_{X'} + \Delta' + \boldsymbol{M}_{X'} = f^*(K_X + \Delta + \boldsymbol{M}_X).$$

We call $(X, \Delta + \boldsymbol{M}_X)$ a generalized klt (resp. lc) pair if $(X', \Delta')$ is sub-klt (resp. sub-lc). When $U$ is a point we drop it by saying $X$ is projective.

A contraction is a projective morphism $f : X \to Z$ with $f_*\mathcal{O}_X = \mathcal{O}_Z$; hence it is surjective with connected fibers. A fibration means a contraction $X \to Z$ such that $\dim X > \dim Z$. Let $X \to Z$ be a fibration and $R$ a $\mathbb{Q}$-divisor on $X$. We write $R = R_v + R_h$, where $R_v$ is the vertical part and $R_h$ is the horizontal part.

For a scheme $X$, a stratification of $X$ is a disjoint union $\coprod_i X_i$ of finitely many locally closed subschemes $X_i \hookrightarrow X$ such that the morphism $\coprod_i X_i \to X$ is both a monomorphism and surjective.

The language of the $\boldsymbol{b}$-divisor was introduced by Shokurov.

**Definition 2.1.** Let $X$ be a projective scheme. We say a formal sum $\boldsymbol{B} = \sum a_\nu \nu$, $a_\nu \in \mathbb{Q}$, where the sum ranges over all divisorial valuations of $X$, is a $\boldsymbol{b}$-divisor if the set

$$F_X = \{\nu \mid a_\nu \neq 0 \text{ and the center } \nu \text{ on } X \text{ is a divisor}\}$$

is finite. The trace $\boldsymbol{B}_Y$ of $\boldsymbol{B}$ is the sum $\sum a_\nu B_\nu$, where the sum now ranges over the elements of $F_Y$.

Notice that, by definition, a generalized pair $(X, \Delta + \boldsymbol{M}_X)$ defines a $\boldsymbol{b}$-divisor $\boldsymbol{M}$.

**Definition 2.2.** For a klt pair $(X, \Delta)$ with a projective morphism $\mu : X \to U$, by [Birkar et al. 2010], the canonical ring

$$R(X/U, K_X + \Delta) := \bigoplus_{m \geq 0} \mu_* \mathcal{O}_X(m(K_X + \Delta))$$

is a finitely generated $\mathcal{O}_U$-algebra. We define the canonical model of $(X, \Delta)$ over $U$ to be

$$\operatorname{Proj} R(X/U, K_X + \Delta).$$

When $U$ is a point we drop it by saying $X$ is projective.

Next, we state some results that we will use in what follows.

**Theorem 2.3** [Hacon et al. 2013, Theorem 2.12]. *Let $f : X \to U$ be a surjective projective morphism and $(X, \Delta)$ a dlt pair such that*

- *for a very general point $u \in U$, the fiber $(X_u, \Delta_u)$ has a good minimal model, and*
- *the ring $R(X/U, K_X + \Delta)$ is finitely generated.*

*Then $(X, \Delta)$ has a good minimal model over $U$.*

**Theorem 2.4** [Birkar and Zhang 2016, Theorem 1.3]. *Let $d$ and $l$ be two positive integers and $\mathcal{I} \subset [0, 1]$ a DCC set of real numbers. Then there is a positive number $m_0$ depending only on $d$, $l$ and $\mathcal{I}$ satisfying the following. Assume that*

- *$(Z, B)$ is a projective lc pair of dimension $d$,*
- $\operatorname{coeff}(B) \in \mathcal{I}$,
- *$lM$ is a nef Cartier divisor, and*
- *$K_Z + B + M$ is big,*

*then the linear system $|m(K_Z + B + M)|$ defines a birational map for every positive integer $m$ such that $m_0 \mid m$.*

**Theorem 2.5** [Birkar and Zhang 2016, Theorem 8.1]. *Let $\mathcal{I}$ be a DCC set of nonnegative real numbers and $d$ a natural number. Then there is a real number $e \in (0, 1)$ depending only on $\mathcal{I}$ and $d$ such that, if*

- *$(Z, B)$ is projective lc of dimension $d$,*
- *$M = \sum \mu_j M_j$, where $M_j$ are nef Cartier divisors,*
- *the coefficients of $B$ and the $\mu_j$ are in $\mathcal{I}$, and*
- *$K_Z + B + M$ is a big divisor,*

*then $K_Z + eB + eM$ is a big divisor.*

**Theorem 2.6** [Filipazzi 2018, Theorem 1.10]. *Let $\mathcal{I} \subset [0, 1] \cap \mathbb{Q}$ be a DCC set, $(W, D)$ a log smooth pair with $D$ reduced, and $M$ a fixed $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $W$. Suppose $\mathcal{D}$ is the set of all projective simple normal crossing pairs $(Z, B)$ such that* $\operatorname{coeff}(B) \subset \mathcal{I}$, *there exists a birational morphism $f : Z \to W$ and $f_*B \leq D$. Then, the set*

$$\{\operatorname{vol}(K_Z + B + f^*M) \mid (Z, B) \in \mathcal{D}\}$$

*satisfies the DCC.*

**Theorem 2.7** [Filipazzi 2018, Theorem 1.12]. *Let $(\mathcal{Z}, \operatorname{Supp}(\mathcal{B})) \to T$ be a projective log smooth morphism and $\{x_i\}_{i \geq 1} \subset T$ a set of closed points. Denote by $(Z_i, B_i)$ the pair given by the fiber product $(\mathcal{Z}, \mathcal{B}) \times_T x_i$. Assume that*

- $0 \leq \mathcal{B} \leq \operatorname{red}(\mathcal{B})$, *and*
- *there is a $\mathbb{Q}$-divisor $\mathcal{M}$ on $\mathcal{Z}$ such that $M_i = \mathcal{M}|_{Z_i}$ is nef for every $i$.*

*Then, we have* $\operatorname{vol}(K_{Z_i} + B_i + M_i) = \operatorname{vol}(K_{Z_j} + B_j + M_j)$ *for every $i, j \in \mathbb{N}$.*

**Definition 2.8.** Let $X$ and $Z$ be normal quasiprojective varieties and $f : X \to Z$ a contraction. Let $R$ be a $\mathbb{Q}$-divisor on $X$ such that $K_X + R$ is $\mathbb{Q}$-Cartier. We call $(X, R) \to Z$ an lc-trivial fibration if

- $(X, R)$ is sub-klt over the generic point of $Z$,
- $K_X + R \sim_{\mathbb{Q},Z} 0$, and
- $h^0(X_\eta, \mathcal{O}_{X_\eta}(\lceil R_{\leq 0} \rceil)) = 1$, where $X_\eta$ is the generic fiber of $f$.

**Definition 2.9** [Kollár 2007, Definition 8.3.6]. Let $f : X \to Z$ be a projective morphism between normal projective varieties, $R$ be a $\mathbb{Q}$-divisor on $X$ and $B$ be a divisor on $Z$. We say that $f : X \to Z$ and the divisors $R$ and $B$ satisfy the standard normal crossing assumption if the following hold:

- $X$ and $Z$ are smooth.
- $\operatorname{Supp}(R) + \operatorname{Supp}(f^*B)$ and $B$ are snc divisors.
- $(X, \operatorname{Supp}(R))$ is log smooth over $Z \setminus B$.

In practice, the assumptions on $X$ and the divisors $R$ and $B$ are completely harmless. By contrast, it takes some work to reduce the problems on $Z$ to problems on the following "nice" birational model of $Z$.

**Definition 2.10.** An snc model of $f : (X, R) \to Z$ is a birational model $Z' \to Z$ such that there is a reduced divisor $D'$ on $Z'$, a $\mathbb{Q}$-divisor $B$ on $Z$, and a crepant birational morphism $\phi : (X', R') \to (X, R + f^*B)$, such that the morphism $X' \to Z'$ and $R'$, $D'$ satisfy the standard normal crossing assumption.

The following is a general version of the canonical bundle formula given in [Kollár 2007].

**Theorem 2.11** (the canonical bundle formula). *Let $X$, $Z$ be normal projective varieties and $f : (X, R) \to Z$ an lc-trivial fibration with generic fiber $X_\eta$. Suppose $B$ is a reduced divisor on $Z$ such that $f$ has slc fibers in codimension* 1 *over $Z \setminus B$; that is, if $D$ is a prime divisor not contained in $B$, then*

- *no component of $R_v$ dominates $D$, and*
- *$(X, R + f^*D)$ is sub-lc over the generic point of $D$.*

*Then one can write*

$$K_X + R \sim_{\mathbb{Q}} f^*(K_Z + B_Z + \boldsymbol{M}_Z),$$

*where the following hold*:

(a) $\boldsymbol{M}_Z = M(X/Z, R)$ *is the moduli part. It is a* $\boldsymbol{b}$*-divisor depending only on the crepant birational equivalence class of* $(X_\eta, R|_{X_\eta})$ *and* $Z$ *such that the following hold*:
- *There is a birational morphism* $Z' \to Z$ *such that* $\boldsymbol{M}_Z$ *is the pushforward of* $\boldsymbol{M}_{Z'} := M(X'/Z', R')$ *and* $\boldsymbol{M}_{Z''} = M(X''/Z'', R'') = \pi^* \boldsymbol{M}_{Z'}$ *for any birational morphism* $Z'' \to Z'$, *where* $X'$ *is the normalization of the main component of* $X \times_Z Z'$ *and* $(X', R') \to (X, R)$ *is a crepant birational morphism. In this case, we say* $\boldsymbol{M}$ *descends on* $Z'$.
- *If* $X \to Z$ *and* $R$, $B$ *satisfy the standard normal crossing assumption, see Definition 2.9, then* $\boldsymbol{M}$ *descends on* $Z$.

(b) $B_Z$ *is the unique* $\mathbb{Q}$*-divisor supported on* $B$ *for which there is a codimension* $\geq 2$ *closed subset* $W \subset Z$ *such that the following hold*:
- $(X \setminus f^{-1}(W), R + f^*(B - B_Z))$ *is lc.*
- *Every irreducible component of* $B$ *is dominated by an lc center of* $(X, R + f^*(B - B_Z))$.

(c) *If the morphism* $X \to Z$ *and the divisors* $R$ *and* $B$ *satisfy the standard normal crossing assumption, see Definition 2.9, then* $B_Z$ *is also the unique smallest* $\mathbb{Q}$*-divisor such that* $R_v + f^*(B - B_Z) \leq \mathrm{red}(f^*B)$.

*Proof.* Items (a) and (b) follow from [Kollár 2007, Theorem 8.5.1].

For (c): when $R_h \geq 0$, item (c) is [Kollár 2007, Theorem 8.3.7]. We use the idea in this result to tackle the general case. If the morphism $X \to Z$ and the divisors $R$ and $B$ satisfy the standard normal crossing assumption, then $(Z, \mathrm{Supp}(B))$ is log smooth. We replace $R$ with $R + f^*(B - B_Z)$ and $B_Z$ with $B_Z + (B - B_Z) = B$; then

- there is a codimension $\geq 2$ closed subset $W \subset Z$ such that $(X \setminus f^{-1}(W), R)$ is sub-lc, and
- every irreducible component of $B$ is dominated by an lc center of $(X, R)$.

It is easy to see that, to prove (c), we only need to prove that $W$ can be chosen to be the empty set, which is equal to saying that $(X, R)$ is lc.

Suppose $(X, R)$ is not lc, and consider the diagram

$$\begin{array}{ccc} (X', R') & \xrightarrow{\pi_X} & (X, R) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow[\pi]{} & Z \end{array}$$

where $\pi$ is birational, $\pi_X$ is crepant birational, $f' : X' \to Z'$ is equidimensional and $\pi_X$ extracts a non-lc place of $(X, R)$, which is denoted by $E$. Thus we have that $\mathrm{coeff}_E(R') > 1$. Applying (a) and (b) for the lc-trivial fibration $f' : (X', R') \to Z'$, we have

$$K_{X'} + R' \sim_{\mathbb{Q}} f'^*(K_{Z'} + B' + \boldsymbol{M}_{Z'}).$$

By assumption, $X \to Z$ and the divisors $R$ and $B$ satisfy the standard normal crossing assumption. Then $\boldsymbol{M}$ descends on $Z$, $\pi^*\boldsymbol{M}_Z = \boldsymbol{M}_{Z'}$ and $K_{Z'} + B' \sim_{\mathbb{Q}} \pi^*(K_Z + B)$. Because $(Z, B)$ is lc, $(Z', B')$ is sub-lc.

Let $\widetilde{B}$ be a reduced divisor on $Z'$ such that $f'$ has slc fibers in codimension 1 over $Z' \setminus \widetilde{B}$. By (b), $B'$ is the unique $\mathbb{Q}$-divisor for which there is a codimension $\geq 2$ closed subset $W' \subset Z'$ such that

- $(X' \setminus f'^{-1}(W'), R' + f'^*(\widetilde{B} - B'))$ is sub-lc, and
- every irreducible component of $B'$ is dominated by an lc center of $(X', R' + f'^*(\widetilde{B} - B'))$.

Because $f'$ is equidimensional, $\mathrm{coeff}_E(R' + f'^*(\widetilde{B} - B')) \leq 1$ and $\mathrm{coeff}_E(R') > 1$, we then have that $\mathrm{coeff}_E(f'^*(\widetilde{B} - B')) < 0$. Since $\widetilde{B}$ is reduced and $E$ is vertical, we have $\mathrm{coeff}_{f'(E)}(B') > 1$, which contradicts the fact that $(Z', B')$ is sub-lc. □

The next theorem says that the canonical bundle formula works on the Iitaka fibration of a klt pair.

**Theorem 2.12.** *Let $(X, \Delta)$ be an $n$-dimensional projective klt pair, let $f : X \to Z$ be a contraction such that $\kappa(X_\eta, K_{X_\eta} + \Delta|_{X_\eta}) = 0$, where $X_\eta$ is the generic fiber of $f$, and let $g : W \to Z$ be a birational morphism. Then there is a commutative diagram*

$$\begin{array}{ccc} X & \xleftarrow{\ h_X\ } & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Z & \xleftarrow[\ h\ ]{} & Z' \end{array}$$

*such that the following hold*:

(1) *$h$ and $h_X$ are birational, $h$ factors through $g$, and $f'$ is equidimensional.*

(2) *$Z'$ is smooth and $X'$ has only quotient singularities.*

(3) *There are a klt pair $K_{X'} + \Delta'$, an effective $\mathbb{Q}$-divisor $F'$ on $X'$, and $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$ a generalized klt pair such that*
- *$\boldsymbol{M}$ descends on $Z'$,*
- *$K_{X'} + \Delta' \sim_{\mathbb{Q}} f'^*(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'}) + F'$,*
- *$f'_*\mathcal{O}_{X'}(mF') \cong \mathcal{O}_{Z'}$ for any $m \geq 0$,*
- *$h_{X*}\mathcal{O}_{X'}(m(K_{X'} + \Delta')) \cong \mathcal{O}_X(m(K_X + \Delta))$ for all $m \geq 0$,*
- *$(X, \Delta)$, $(X', \Delta')$ and $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$ have the same canonical models, and*
- $\mathrm{Ivol}(K_X + \Delta) = \mathrm{Ivol}(K'_X + \Delta') = \mathrm{vol}(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'})$.

(4) *If* $\mathrm{coeff}(\Delta)$ *is in a DCC set and a general fiber $(X_g, \Delta_g)$ of $f$ has a good minimal model, then* $\mathrm{coeff}(B_{Z'})$ *and* $\mathrm{coeff}(B_Z)$ *are in a DCC set, where $B_Z := h_*B_{Z'}$.*

*Proof.* Fix $R$ with $R_h \leq 0$ such that $f$ is an lc-trivial fibration for the subpair $(X, \Delta + R)$. Notice that such an $R$ exists by the assumption $\kappa(X_\eta, K_{X_\eta} + \Delta|_{X_\eta}) = 0$, and the choice of $R_h$ is unique. Let $\boldsymbol{M}$ be the moduli $\boldsymbol{b}$-divisor of this lc-trivial fibration. By the weak semi-stable reduction theorem by Abramovich and Karu [2000], we can construct $X'$ and $Z'$ satisfying (1) and (2) such that $\boldsymbol{M}$ descends on $Z'$.

For (3), because $(X, \Delta)$ is klt, we can choose a sufficiently large integer $k$ such that if we define $\Delta' := (h_X)_*^{-1}\Delta + (1-1/k)E$, where $E$ is the reduced exceptional divisor, then $K_{X'}+\Delta' \geq h_X^*(K_X+\Delta)$. Also by the semistable reduction, $X'$ has a toroidal structure $(X' \setminus \mathrm{Supp}(\Delta')) \subset X'$, and we have that $(X', \Delta')$ is klt. It is easy to see that

$$(h_X)_*\mathcal{O}_{X'}(m(K_{X'}+\Delta')) \cong \mathcal{O}_X(m(K_X+\Delta))$$

for all $m \geq 0$; hence $\kappa(X, K_X+\Delta) = \kappa(X', K_{X'}+\Delta')$.

If $\kappa(X, K_X+\Delta) < 0$, choose $a \gg 0$ such that $a(K_{X'}+\Delta')$ is Cartier. Because $\kappa(X_\eta, K_{X_\eta}+\Delta|_{X_\eta}) = 0$, we may also assume that $h^0(X'_\eta, \mathcal{O}_{X'_\eta}(a(K_{X'_\eta}+\Delta'|_{X'_\eta}))) = 1$. Since $Z'$ is smooth and $f'$ is equidimensional, by [Hartshorne 1980, Corollary 1.7], $f'_*\mathcal{O}_{X'}(a(K_{X'}+\Delta'))$ is a reflexive sheaf of rank 1. Moreover, since $Z'$ is smooth, $f'_*\mathcal{O}_{X'}(a(K_{X'}+\Delta'))$ is a line bundle on $Z'$; denote it by $\mathcal{O}_{Z'}(D)$. Choose a general sufficiently ample divisor $A'$ on $Z'$ such that $\mathcal{O}_{Z'}(A'+D)$ is big. Let $A := (1/a)A'$; then $f'_*\mathcal{O}_{X'}(a(K_{X'}+\Delta'+f'^*A))$ is a big line bundle and $\kappa(X', K_{X'}+\Delta'+f'^*A) = \dim Z' \geq 0$. Because $A'$ is general, $(X', \Delta'+f'^*A)$ is klt. It is easy to see that, to prove (3), we may replace $(X', \Delta')$ with $(X', \Delta'+f'^*A)$ and assume $\kappa(X, K_X+\Delta) \geq 0$.

Suppose $\kappa(X, K_X+\Delta) \geq 0$ and choose $a \gg 0$ such that $H^0(X', \mathcal{O}_{X'}(a(K_{X'}+\Delta'))) > 0$; then we can choose $L \in |a(K_{X'}+\Delta')|$. Define

$$G := \max\{N \mid N \text{ is an effective } \mathbb{Q}\text{-divisor such that } L \geq f'^*N\}$$

and

$$D := \frac{1}{a}G \quad \text{and} \quad F' := \frac{1}{a}(L - f'^*G).$$

Then we have $K_{X'}+\Delta' \sim_{\mathbb{Q}} f'^*D + F'$. It is easy to see that $h^0(X'_\eta, \mathcal{O}_{X'_\eta}(mF'|_{X'_\eta})) = 1$ for all $m \geq 0$. Because $f'$ is equidimensional, $f'_*\mathcal{O}_{X'}(mF')$ is a reflexive sheaf of rank 1 for every $m \geq 0$. Moreover, since $Z'$ is smooth, $f'_*\mathcal{O}_{X'}(mF')$ is an invertible sheaf for every $m \geq 0$. Since $\mathrm{Supp}(F')$ does not contain the whole fiber over any codimension 1 point on $Z'$, it is easy to see that $f'_*\mathcal{O}_{X'}(mF') \cong \mathcal{O}_{Z'}$ for all $m \geq 0$.

Let $X'_\eta$ be the generic fiber of $f'$; then $(K_{X'}+\Delta')|_{X'_\eta} \sim_{\mathbb{Q}} F'|_{X'_\eta}$. Because $f'_*\mathcal{O}_{X'}(mF') \cong \mathcal{O}_{Z'}$, we have $H^0(X'_\eta, \mathcal{O}_{X'_\eta}(\lceil(\Delta'-F')_{\leq 0}\rceil)) = 1$ and $f' : (X', \Delta'-F') \to Z'$ is an lc-trivial fibration. By the canonical bundle formula, there is a generalized pair $(Z', B_{Z'}+\boldsymbol{M}_{Z'})$ such that

$$K_{X'}+\Delta'-F' \sim_{\mathbb{Q}} f'^*(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'}).$$

Also because $f'_*\mathcal{O}_{X'}(mF') \cong \mathcal{O}_{Z'}$, there is an integer $l > 0$ such that

$$H^0(X', \mathcal{O}_{X'}(ml(K_{X'}+\Delta'))) \cong H^0(Z', \mathcal{O}_{Z'}(ml(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'})))$$

for all $m \geq 0$. Then $(X, \Delta)$, $(X', \Delta')$ and $(Z', B_{Z'}+\boldsymbol{M}_{Z'})$ all have the same canonical models, and $\mathrm{Ivol}(K_X+\Delta) = \mathrm{Ivol}(K'_X+\Delta') = \mathrm{vol}(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'})$.

For (4), if $\mathrm{coeff}(\Delta)$ is in a DCC set, by the construction of $\Delta'$, $\mathrm{coeff}(\Delta')$ is also in a DCC set. Because $(X', \Delta')$ is a klt pair, by the main theorem of [Birkar et al. 2010], $R(X'/Z, K_{X'}+\Delta')$ is finitely generated. Because a general fiber $(X_g, \Delta_g)$ has a good minimal model and $K_{X'}+\Delta'-h^*(K_X+\Delta)$ is effective and exceptional over $X$, we have that a general fiber $(X'_g, \Delta'_g)$ of $(X', \Delta') \to Z'$ has a good minimal model. By Theorem 2.3, $(X', \Delta')$ has a good minimal model over $Z'$; we denote it by $h_Y : (X', \Delta') \dashrightarrow (Y, \Delta_Y)$. By (2), $K_{X'} + \Delta' \sim_{\mathbb{Q},Z'} F'$ and $f'_*\mathcal{O}_{X'}(mF') \cong \mathcal{O}_{Z'}$ for all $m \geq 0$; therefore $Z'$ is the canonical model of $(X', \Delta')$ over $Z'$. By the definition of good minimal models, we have $K_Y + \Delta_Y \sim_{\mathbb{Q},Z'} 0$ and $(h_Y)_*F' = 0$.

Since $\mathrm{coeff}(\Delta')$ is in a DCC set and $\Delta_Y$ is the pushforward of $\Delta'$, $\mathrm{coeff}(\Delta_Y)$ is also in a DCC set. Let $B'$ be the unique smallest reduced divisor on $Z'$ such that $f'$ has slc fibers in codimension 1 over $Z' \setminus B'$. By Theorem 2.11, there is a codimension $\geq 2$ closed subset $W \subset Z'$ such that $B_{Z'}$ is the smallest $\mathbb{Q}$-divisor supported on $B'$ such that $(X' \setminus f'^{-1}(W), \Delta' - F' + f'^*(B' - B_{Z'}))$ is sub-lc.

Because $K_{X'} + \Delta' - F' \sim_{\mathbb{Q},Z'} 0$, $K_Y + \Delta_Y \sim_{\mathbb{Q},Z'} 0$, and $(h_Y)_*(\Delta' - F') = 0$, we have that $B_{Z'}$ is also the smallest $\mathbb{Q}$-divisor supported on $B'$ such that $(Y \setminus f_Y^{-1}(W), \Delta_Y + h_{Y*}f'^*(B' - B_{Z'}))$ is lc. Because $\mathrm{coeff}(\Delta_Y)$ is in a DCC set, by [Hacon et al. 2014, Theorem 1.1], $\mathrm{coeff}(B_{Z'})$ is in a DCC set. □

**Remark 2.13.** Suppose $(X, \Delta)$ is a projective klt pair and $f : X \dashrightarrow Z$ is the canonical model of $(X, \Delta)$. Let $g : Y \to X$ be a resolution of the indeterminacy of $f$. Choose a sufficiently large integer $k$ such that if we define $\Delta_Y := g_*^{-1}\Delta + (1 - 1/k)E$, where $E$ is the reduced exceptional divisor, then $K_Y + \Delta_Y \geq g^*(K_X + \Delta)$.

Because $K_Y + \Delta_Y - g^*(K_X + \Delta)$ is effective and exceptional over $X$, $Z$ is also the canonical model of $(Y, \Delta_Y)$ and $\kappa(Y_\eta, K_{Y_\eta} + \Delta_Y|_{Y_\eta}) = 0$. By Theorem 2.12, the contraction $Y \to Z$ defines a moduli $\boldsymbol{b}$-divisor $\boldsymbol{M}$ and a generalized pair $(Z', B' + \boldsymbol{M}_{Z'})$ with a birational morphism $Z' \to Z$, $Z$ is also the canonical model of $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$, and $\mathrm{Ivol}(K_X + \Delta) = \mathrm{Ivol}(K_Y + \Delta_Y) = \mathrm{vol}(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'})$.

Furthermore, if $\mathrm{coeff}(\Delta)$ is in a DCC set, then $\mathrm{coeff}(\Delta_Y)$ is also in a DCC set. If a general fiber $(X_g, \Delta_g)$ of $f$ has a good minimal model, then a general fiber $(Y_g, \Delta_{Y_g})$ of $f_Y$ has a good minimal model, and therefore $\mathrm{coeff}(B')$ is in a DCC set by Theorem 2.12.

## 3. DCC of Iitaka volumes

**Lemma 3.1.** *Fix a positive integer $C$ and a finite set $\mathcal{I} \subset [0, 1] \cap \mathbb{Q}$. Suppose $\mathcal{Z} \to T$ is a family of projective smooth varieties, where $T$ is of finite type. Let $\mathcal{A}$ be a relative very ample divisor on $\mathcal{Z}$ over $T$. Let $\mathcal{S}$ be a set of generalized pairs such that, for every $(Z, B_Z + \boldsymbol{M}_Z) \in \mathcal{S}$, there is a closed point $t \in T$ such that*

- *there is a birational morphism $\phi : Z \to \mathcal{Z}_t$, and*
- *$\phi_*\boldsymbol{M}_Z \sim_{\mathbb{Q}} D_1 - D_2$ for two effective $\mathbb{Q}$-divisors $D_k$ with* $\mathrm{coeff}(D_k) \subset \mathcal{I}$ *and* $\deg_{\mathcal{A}_t}(D_k) \leq C$, $k = 1, 2$.

*Then there is a smooth projective morphism $\mathcal{Z}' \to T'$, where $T'$ is of finite type, and finitely many $\mathbb{Q}$-divisors $\mathcal{M}_k$ on $\mathcal{Z}'$ over $T'$ such that, for any $(Z, B_Z + \boldsymbol{M}_Z) \in \mathcal{S}$, there is a closed point $t' \in T'$ and an isomorphism $\psi : \mathcal{Z}_t \to \mathcal{Z}'_{t'}$ such that $\psi_*\phi_*\boldsymbol{M}_Z \sim_{\mathbb{Q}} \mathcal{M}_k|_{\mathcal{Z}'_{t'}}$.*

*Proof.* Since the coefficients of $D_k$, $k = 1, 2$, are in a finite set $\mathcal{I}$, there is a positive number $\delta$ such that $\mathrm{coeff}(D_k) > \delta$, which implies $(1/\delta)D_k \geq \lfloor D_k \rfloor$, $k = 1, 2$. Because $\deg_{\mathcal{A}_t}(D_k) \leq C$, we have that $\deg_{\mathcal{A}_t}(\mathrm{Supp}(D_k)) \leq C/\delta$. By boundedness of the Chow variety, see [Kollár 1996, §1.3], there is a morphism $\mathcal{Z}' \to T'$ and a divisor $\mathcal{D}$ on $\mathcal{Z}'$ such that, for every closed point $t \in T$, there is a closed point $t' \in T'$ and an isomorphism $\psi : \mathcal{Z}_{t'} \to \mathcal{Z}'_t$ such that $\operatorname{Supp}(\psi_* D_k) \subset \mathcal{D}|_{\mathcal{Z}'_{t'}}$, $k = 1, 2$.

Let $R$ be a component of $\mathcal{D}$, let $S \to T'$ be the normalization of the Stein factorization of $R \to T'$ such that $S \to T'$ is finite and $S$ is normal, and consider the diagram

$$\begin{array}{ccc} \mathcal{Z}'' & \longrightarrow & \mathcal{Z}' \\ \downarrow & & \downarrow \\ S & \longrightarrow & T' \end{array}$$

Because $S \to T'$ is finite, $S$ is irreducible and $\mathcal{Z}'' \to S$ is flat, we have that $\mathcal{Z}''$ is a quasiprojective variety. $\mathcal{Z}''$ is normal by [EGA IV$_3$ 1966, 5.12.7]. Replacing $\mathcal{Z}' \to T'$ by $\mathcal{Z}'' \to S$ finitely many times, we may assume that the fibers of $R \to T'$ are irreducible for every component $R$ of $\mathcal{D}$.

Since, for every component $R$ of $\mathcal{D}$, the coefficients of $R$ in $D_1$ and $D_2$ are in a finite set, there are only finitely many possibilities for $D_1$, $D_2$ and $D_1 - D_2$. Then there are only finitely many $\mathbb{Q}$-divisors $\mathcal{M}_k$ on $\mathcal{Z}'$ over $T'$ such that $\psi_* \phi_* \boldsymbol{M}_Z \sim_{\mathbb{Q}} \mathcal{M}_k|_{\mathcal{Z}_t}$ for some $k$. □

The next theorem says that if we bound the Iitaka volume of $(X, \Delta) \in \mathcal{D}(n, \mathcal{I}, l, r)$, then we can choose the snc model of $X \to \operatorname{Proj} R(X, K_X + \Delta)$ to be in a bounded family depending only on $n, \mathcal{I}, l$ and $r$.

**Theorem 3.2.** *Fix a DCC set $\mathcal{I} \subset [0, 1] \cap \mathbb{Q}$, positive integers $n, l, r, v > 0$, and a positive number $\delta > 0$. Define $\mathcal{D}'(n, \mathcal{I}, l, r, v, \delta)$ to be the set of $n$-dimensional projective pairs $(X, \Delta)$ such that*

- $(X, \Delta) \in \mathcal{D}(n, \mathcal{I}, l, r)$,
- $\operatorname{Ivol}(K_X + \Delta) \leq v$, *and*
- *if $Z$ is the canonical model of $(X, \Delta)$, then there is an effective ample $\mathbb{Q}$-divisor $H$ on $Z$ with* $\operatorname{coeff}(H) > \delta$ *such that*

$$\operatorname{Ivol}(K_X + \Delta + f^* H) \leq r \operatorname{Ivol}(K_X + \Delta).$$

*Then there is a family of projective log smooth pairs $(\mathcal{Z}, \mathcal{P}) \to T$, where $T$ is a scheme of finite type, and finitely many $\mathbb{Q}$-divisors $\mathcal{M}_k$, $k \in \Lambda$, on $\mathcal{Z}$, where $\Lambda$ is a finite index set, such that, for every $(X, \Delta) \in \mathcal{D}'(n, \mathcal{I}, l, r, v, \delta)$, there is a closed point $t \in T$ such that the following hold:*

- *$\mathcal{Z}_t$ is birationally equivalent to the canonical model of $(X, \Delta)$,*
- *If $\boldsymbol{M}$ is the moduli $\boldsymbol{b}$-divisor corresponding to $(X, \Delta) \dashrightarrow Z$ defined in Remark 2.13, then $\boldsymbol{M}_{\mathcal{Z}_t} \sim_{\mathbb{Q}} \mathcal{M}_k|_{\mathcal{Z}_t}$ for some $k \in \Lambda$,*
- *There is a birational morphism $X' \to X$ and a $\mathbb{Q}$-divisor $F'$ on $X'$ such that the morphism $X' \to \mathcal{Z}_t$ and $\Delta' - F'$, $\mathcal{P}_t$ satisfy the standard normal crossing assumption, where $\Delta'$ is the strict transform of $\Delta$ plus the exceptional divisor and $F' \in |K_{X'} + \Delta'|_{\mathbb{Q}/\mathcal{Z}_t}$. In particular, $\boldsymbol{M}$ descends on $\mathcal{Z}_t$.*
- *If $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$ is the generalized pair defined in Remark 2.13 such that there is a birational morphism $\phi_t : Z' \to \mathcal{Z}_t$, then $B := \phi_{t*} B_{Z'} \leq \mathcal{P}_t$.*

*Proof.* We replace $\mathcal{I}$ by $\mathcal{I} \cup \{1-1/k,\, k \in \mathbb{N}\}$; note that $\mathcal{I}$ is still a DCC set.

Step 1: We prove that $Z$ is birationally bounded.

Suppose $\kappa(X, K_X+\Delta) = d \le n$; then $\dim Z = d$. Let $Z' \to Z$ be a projective birational morphism and $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$ be the generalized klt pair defined in Remark 2.13, and denote the morphism $Z' \to Z$ by $h$. By Theorem 2.12 (4), $\mathrm{coeff}(B_{Z'})$ is in a DCC set $\mathcal{I}'$ depending only on $\mathcal{I}, d, n$. By assumption, $l\boldsymbol{M}_{Z'}$ is nef and Cartier. We may assume that $\{1-1/k,\, k \in \mathbb{N}\} \subset \mathcal{I}'$.

By Theorem 2.4, there is an integer $r'$ such that $|r'(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'})|$ defines a birational map $\phi : Z' \dashrightarrow W$. Let $h' : Z'' \to Z'$ be a birational morphism such that $\phi$ extends to a morphism $\phi' : Z'' \to W$. Because $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$ is generalized klt, we can choose an integer $k \gg 0$ such that if we define

$$B_{Z''} := h'^{-1}_{*} B_{Z'} + \Big(1 - \frac{1}{k}\Big) E,$$

where $E$ is the reduced $h'$-exceptional divisor, then $K_{Z''} + B_{Z''} + \boldsymbol{M}_{Z''} - h'^{*}(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'})$ is effective and exceptional over $Z'$. Then we replace $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$ by $(Z'', B_{Z''} + \boldsymbol{M}_{Z''})$; we may assume that $|r'(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'})|$ defines a birational morphism $\phi : Z' \to W$. Note that we keep $\mathrm{coeff}(B_{Z'}) \subset \mathcal{I}'$, $Z$ is still the canonical model of $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$ and $\mathrm{vol}(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'}) = \mathrm{Ivol}(K_X + \Delta) \le v$.

Because $|r'(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'})|$ defines a birational morphism $\phi : Z' \to W$, there is a very ample divisor $A$ on $W$ such that $r'(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'}) \sim \phi^* A + F_1$, where $F_1$ is a $\phi$-exceptional $\mathbb{Q}$-divisor. Because

$$A^d \le \mathrm{vol}(r'(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'})) = r'^d\, \mathrm{Ivol}(K_X + \Delta) < r'^d C,$$

by boundedness of the Chow variety, see [Kollár 1996, §1.3], $W$ is in a bounded family. Then there exists a projective morphism $\mathcal{W}' \to T$ over a scheme $T$ of finite type and a relative very ample divisor $\mathcal{A}'$ depending only on $n, \mathcal{I}, l, v$, such that there is a closed point $t \in T$ and an isomorphism $\chi : W \to \mathcal{W}'_t$ such that $\chi^* \mathcal{A}'_t = A$. Because $r'$ is fixed and the coefficients of $B_{Z'}$ are in a DCC set $\mathcal{I}'$, it is easy to see that the coefficients of $F_1$ are also in a DCC set $\widetilde{\mathcal{I}}$.

Passing to a stratification of $T$ and a log resolution of the generic fiber of $\mathcal{W}' \to T$, we may assume that there is a birational morphism $\xi : \mathcal{W} \to \mathcal{W}'$, and $\mathcal{W} \to T$ is a smooth morphism. Let $\mathcal{A}$ be a very ample divisor on $\mathcal{W}$ over $T$. Then there is an integer $r''$ such that $r''\xi^*\mathcal{A}' - \mathcal{A}$ is big over $T$. After increasing $r'$, replacing $Z'$ by a birational model and $(W, A)$ by $(\mathcal{W}_t, \mathcal{A}_t)$, we may assume $W$ is smooth and there is a very ample divisor $A$ on $W$ such that

$$A^d \le \mathrm{vol}(r'(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'})).$$

Step 2: We construct a birational map $Z' \dashrightarrow Z^!$, two morphisms $h^! : Z^! \to Z$, $\phi^! : Z^! \to W$ and an ample $\mathbb{Q}$-divisor $L^!$ on $Z^!$.

Let $m$ be the Cartier index of $H$, and define

$$\begin{aligned} L' &:= \frac{1}{r'}(\phi^* A + F_1) + (2d+1)\phi^* A + (2d+1) m h^* H \\ &\sim_{\mathbb{Q}} K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'} + (2d+1)\phi^* A + (2d+1) m h^* H. \end{aligned} \tag{3-1}$$

Because $H$ is an effective ample $\mathbb{Q}$-divisor on $Z$, by [Birkar et al. 2010], the canonical model of $K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'}+(2d+1)\phi^*A+(2d+1)mh^*H$ exists; denote it by $h' : Z' \dashrightarrow Z^!$. Then

$$h'_*(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'}+(2d+1)\phi^*A+(2d+1)mh^*H)\sim_{\mathbb{Q}} h'_*L'$$

is ample, and we write $L^! := h'_*L'$. Because $\phi^*A$ and $mh^*H$ are nef Cartier divisors, by [Birkar and Zhang 2016, Lemma 4.4], both $\phi^*A$ and $h^*H$ are $h'$-trivial, so there are two birational morphisms $\phi^! : Z^! \to W$ and $h^! : Z^! \to Z$ as in the following diagram:

$$\begin{array}{ccccc} & & Z' & & \\ & \overset{h}{\swarrow} & \downarrow{\scriptstyle h'} & \overset{\phi}{\searrow} & \\ Z & \overset{h^!}{\longleftarrow} & Z^! & \overset{\phi^!}{\longrightarrow} & W \end{array}$$

Because $L^!$ is ample and effective and $W$ is smooth, by the negativity lemma, $L^! = \phi^{!*}\phi^!_*L^! - F_W$, where $F_W$ is effective and has the same support as $\operatorname{Exc}(\phi^!)$. Then we have

$$\operatorname{Supp}(\phi^{!*}\phi^!_*L^!)\supset \operatorname{Exc}(\phi^!) \quad \text{and} \quad Z^!\setminus \operatorname{Supp}(L^!)\supseteq W\setminus \operatorname{Supp}(\phi^!_*L^!).$$

<u>Step 3</u>: We use the two birational morphisms $Z^!\to Z$, $Z^!\to W$ and ampleness of $L^!$ to show that if there is a $\mathbb{Q}$-Cartier integral divisor $D$ and a $\mathbb{Q}$-divisor $F\in |K_X+\Delta|_{\mathbb{Q}/Z}$ on $X$ such that $(X, \operatorname{Supp}(\Delta-F))$ is log smooth over $Z\setminus D$, then $(X, \operatorname{Supp}(\Delta-F))$ is log smooth over $W\setminus \operatorname{Supp}(\phi^!_*L^!+\phi^!_*h^{!*}D)$.

Consider the diagram

$$\begin{array}{ccc} X & \overset{g}{\longleftarrow} & X^! \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^!} \\ Z & \underset{h^!}{\longleftarrow} & Z^! \end{array}$$

where $X^!$ is the normalization of the main component of $X\times_Z Z^!$. Because $\operatorname{Supp}(h^{!*}D)=h^{!-1}(\operatorname{Supp}(D))$ and $X\to Z$ is smooth over $Z\setminus\operatorname{Supp}(D)$, we have that $X^!\to Z^!$ is smooth over $Z^!\setminus\operatorname{Supp}(h^{!*}D)$. Because $Z^!$ is normal, $f^{!-1}(Z^!\setminus\operatorname{Supp}(h^{!*}D))$ is normal and

$$f^{!-1}(Z^!\setminus\operatorname{Supp}(h^{!*}D))\cong f^{-1}(Z\setminus\operatorname{Supp}(D))\times_{Z\setminus\operatorname{Supp}(D)} Z^!\setminus\operatorname{Supp}(h^{!*}D).$$

Define $K_{X^!}+\Delta^{!'}-F^{!'} := g^*(K_X+\Delta-F)$, where $\Delta^{!'}$ and $F^{!'}$ are two effective $\mathbb{Q}$-divisors with no common component. Suppose $\Delta^{!'}=\Delta^{!''}+\Delta_v$ and $F^{!'}=F^{!''}+F_v$, where $\Delta_v$ and $F_v$ are $f^!$-vertical and not supported on $f^{!-1}(\operatorname{Supp}(h^{!*}D))$, and the prime components of $\Delta^{!''}$ and $F^{!''}$ are either $f^!$-horizontal or supported on $f^{!-1}(\operatorname{Supp}(h^{!*}D))$. Because $f^!$ is smooth over $Z^!\setminus\operatorname{Supp}(h^{!*}D)$, we have that $\Delta_v|_{f^{!-1}(Z^!\setminus\operatorname{Supp}(h^{!*}D))}$ and $F_v|_{f^{!-1}(Z^!\setminus\operatorname{Supp}(h^{!*}D))}$ are the pullback of two divisors on $Z^!\setminus\operatorname{Supp}(h^{!*}D)$. It is easy to see that there is a $\mathbb{Q}$-divisor $R^!$ on $X^!$ such that $\operatorname{Supp}(R^!)\subset f^{!-1}(\operatorname{Supp}(h^{!*}D))$ and $\Delta_v-F_v-R^!\sim_{\mathbb{Q},f^!}0$. Let $\Delta^!$ and $F^!$ be two effective $\mathbb{Q}$-divisors without common component such that $\Delta^!-F^!=\Delta^{!''}-F^{!''}+R^!$; then $K_{X^!}+\Delta^!-F^!\sim_{\mathbb{Q},f^!}0$.

If $P$ is a component of $\operatorname{Supp}(\Delta^!-F^!)$, then it is either supported on $f^{!-1}(h^{!*}D)$ or is $f^!$-horizontal. Then $\operatorname{Supp}(\Delta^!-F^!)$ does not contain any irreducible component of the fiber over any prime divisor on $Z^!$

that is not contained in $\mathrm{Supp}(h^{!*}D)$, and we have

$$\mathrm{Supp}(\Delta^! - F^!)|_{(Z^! \setminus \mathrm{Supp}(h^{!*}D))} = \mathrm{Supp}(\Delta - F) \times_Z (Z^! \setminus \mathrm{Supp}(h^{!*}D)).$$

Because $(X, \Delta) \subset \mathcal{D}(n, \mathcal{I}, l, r)$, by definition of $\mathcal{D}(n, \mathcal{I}, l, r)$, we know that $(X, \mathrm{Supp}(\Delta - F))$ is log smooth over $Z \setminus D$, and hence $f^! : (X^!, \mathrm{Supp}(\Delta^! - F^!)) \to Z^!$ is log smooth over $Z^! \setminus \mathrm{Supp}(h^{!*}D)$.

Recall that $Z^! \setminus \mathrm{Supp}(L^!) \supseteq W \setminus \mathrm{Supp}({\phi^!}_* L^!)$. Then

$$Z^! \setminus \mathrm{Supp}(h^{!*}D) \supseteq Z^! \setminus \{\mathrm{Supp}(L^! + h^{!*}D)\} \supseteq W \setminus \{\mathrm{Supp}({\phi^!}_* L^! + {\phi^!}_* h^{!*}D)\}$$

and $X^! \to W$ is isomorphic to $X^! \to Z^!$ over $W \setminus \{\mathrm{Supp}({\phi^!}_* L^! + {\phi^!}_* h^{!*}D)\}$. Then $(X^!, \mathrm{Supp}(\Delta^! - F^!)) \to W$ is log smooth over $W \setminus \{\mathrm{Supp}({\phi^!}_* L^! + {\phi^!}_* h^{!*}D)\}$.

Step 4: We prove that $(W, \mathrm{Supp}({\phi^!}_* L^! + \phi_* B_{Z'} + {\phi^!}_* h^{!*}D))$ is log bounded.

Because $\mathrm{Supp}({\phi^!}_* L^! + B_W + {\phi^!}_* h^{!*}D) = \mathrm{Supp}(\phi_*(\phi^* A + F_1 + B_{Z'} + h^*(D+H)))$, we only need to prove that $(W, \mathrm{Supp}(\phi_*(\phi^* A + F_1 + B_{Z'} + h^*(D+H))))$ is log bounded. Recall that $W$ is bounded by $A$ by construction; we only need to work on the boundary.

Recall that the coefficients of $F_1$ and $B_{Z'}$ are in a DCC set and $\mathrm{coeff}(H) \geq \delta$ by assumption. Then there is a positive number $\delta' < 1$ such that $(F_1 + B_{Z'})/\delta' \geq \mathrm{red}(F_1 + B_{Z'})$. By assumption, $A$ and $D$ are two effective integral divisors, so we only need to prove that there exists a constant $v' > 0$ such that

$$A^{d-1}.\phi_*(\mathrm{red}(\phi^* A + F_1 + B_{Z'} + h^* D) + h^* H) < v'.$$

By the projection formula, this is equivalent to proving

$$(\phi^* A)^{d-1}.(\mathrm{red}(\phi^* A + F_1 + B_{Z'} + h^* D) + h^* H) < v'.$$

Let $G = 2((2d+1)+1)\phi^* A$. By [Hacon et al. 2013, Lemma 3.2], we have

$$G^{d-1}.(\mathrm{red}(\phi^* A + F_1 + B_{Z'} + h^* D)) \leq 2^d \operatorname{vol}\Big(K_{Z'} + \frac{1}{\delta'} B_{Z'} + \phi^* A + \frac{1}{\delta'} F_1 + h^* D + G\Big). \tag{3-2}$$

Recall that the coefficients of $B_{Z'}$ are in a DCC set $\mathcal{I}'$ and the Cartier index of the $\boldsymbol{b}$-divisor $\boldsymbol{M}$ is $l$, according to the assumption that $(X, \Delta) \subset \mathcal{D}(n, \mathcal{I}, l, r)$. By Theorem 2.5, there is a positive number $e < 1$ depending only on $\mathcal{I}'$ and $l$ such that $K_{Z'} + eB_{Z'} + \boldsymbol{M}_{Z'}$ is big. Because $\boldsymbol{M}_{Z'}$ is pseudo-effective and

$$K_{Z'} + \frac{1}{\delta'} B_{Z'} + \frac{\frac{1}{\delta'} - 1}{1-e}(K_{Z'} + eB_{Z'} + \boldsymbol{M}_{Z'}) + \boldsymbol{M}_{Z'} \sim_{\mathbb{Q}} \frac{\frac{1}{\delta'} - e}{1-e}(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'}),$$

for any divisor $E$, we have that

$$\operatorname{vol}\Big(E + K_{Z'} + \frac{1}{\delta'} B_{Z'}\Big) \leq \operatorname{vol}\Big(E + \frac{\frac{1}{\delta'} - e}{1-e}(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'})\Big).$$

Because $\phi^* A$ and $F_1$ are effective, it is easy to see that

$$\phi^* A + \frac{1}{\delta'} F_1 + G \leq \Big(1 + 2((2d+1)+1) + \frac{1}{\delta'}\Big)(\phi^* A + F_1) \sim_{\mathbb{Q}} \Big(1 + 2((2d+1)+1) + \frac{1}{\delta'}\Big) r'(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'}).$$

Then we have

$$\begin{aligned}\operatorname{vol}\Big(K_{Z'}+\frac{1}{\delta'}B_{Z'}+\phi^*A+\frac{1}{\delta'}F_1+h^*D+G\Big)&\\ &\le \operatorname{vol}\bigg(\frac{\frac{1}{\delta'}-e}{1-e}(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'})+\phi^*A+\frac{1}{\delta'}F_1+h^*D+G\bigg)\\ &\le \operatorname{vol}\bigg(\Big(\frac{\frac{1}{\delta'}-e}{1-e}+r'+2r'((2d+1)+1)+\frac{r'}{\delta'}\Big)(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'})+h^*D\bigg)\\ &\le v''^d\operatorname{vol}(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'}+h^*D),\end{aligned}\tag{3-3}$$

where

$$v'':=\bigg(\frac{\frac{1}{\delta'}-e}{1-e}+r'+2r'((2d+1)+1)+\frac{r'}{\delta'}\bigg)^d.$$

Recall that, by construction, $H^0(X,m(K_X+\Delta+f^*D))\cong H^0(Z',m(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'}+h^*D))$ for all $m\gg 0$ sufficiently divisible. Then we have

$$\operatorname{vol}(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'}+h^*D)=\operatorname{Ivol}(K_X+\Delta+f^*D)\le r\operatorname{Ivol}(K_X+\Delta)\le rC,$$

where the second inequality is from the definition of $\mathcal{D}(n,\mathcal{I},l,r)$. Then we have

$$G^{d-1}.(\operatorname{red}(\phi^*A+F_1+B_{Z'}+h^*D))\le 2^d rC''v.$$

Because $\phi^*A$ and $h^*H$ are nef, we have that

$$\begin{aligned}(\phi^*A)^{d-1}.h^*H\le(\phi^*A+h^*H)^d&\le r'^d\operatorname{vol}(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'}+H)\\ &\le r'^d\operatorname{Ivol}(K_X+\Delta+f^*H)\le r'^d rC.\end{aligned}\tag{3-4}$$

Let $v':=2^dC''v/(2(2d+1)+1)^d+r'^d rC$. Then $(\phi^*A)^{d-1}.(\operatorname{red}(\phi^*A+F_1+B_{Z'}+h^*D)+h^*H)<v'$. By boundedness of the Chow varieties, see [Kollár 1996, §1.3], $(W,\operatorname{Supp}(\phi_*(\phi^*A+F_1+B_{Z'}+h^*D+h^*H)))$ is log bounded, and therefore $(W,\operatorname{Supp}(\phi_*(\phi^*A+F_1+h^*D)))$ is log bounded.

Step 5: We take a log resolution of $(W,\phi_*(\phi^*A+F_1+h^*D))$ to get a log bounded family $(\mathcal{Z},\mathcal{P})\to T$, then show the moduli part $\boldsymbol{M}$ descends on $\mathcal{Z}_t$ by using the standard normal crossing assumptions.

By the definition of log boundedness, there is a flat morphism $(\mathcal{Z},\mathcal{P})\to T$ such that, for a closed point $t\in T$, we have $(W,\phi_*(\phi^*A+F_1+h^*D))\cong(\mathcal{Z}_t,\mathcal{P}_t)$. Because $f^!:(X^!,\operatorname{Supp}(\Delta^!-F^!))\to W$ is log smooth over $W\setminus\operatorname{Supp}({\phi^!}_*L^!+{\phi^!}_*h^{!*}D)$ and $\operatorname{Supp}({\phi^!}_*L^!+{\phi^!}_*h^{!*}D)=\operatorname{Supp}(\phi_*(\phi^*A+F_1+h^*D))$, there is a rational contraction $f_t:X^!\dashrightarrow\mathcal{Z}_t$ such that $(X^!,\operatorname{Supp}(\Delta^!-F^!))$ is log smooth over $\mathcal{Z}_t\setminus\mathcal{P}_t$.

After passing to a stratification of $T$ and log resolution of the generic fiber of $\mathcal{Z}\to T$, we can assume $(\mathcal{Z}_t,\mathcal{P}_t)$ is log smooth for every closed point $t\in T$. We choose a birational model $Z'$ of $Z$ as in Theorem 2.12 such that $\phi:Z'\to\mathcal{Z}_t$ is still a birational morphism.

We replace $X$ by a higher birational model which resolves the indeterminacy of $X^!\dashrightarrow\mathcal{Z}_t$, replace $\Delta$ by its strict transform plus the exceptional divisor, and choose $F\in|K_X+\Delta|_{\mathbb{Q}/\mathcal{Z}_t}$. Because $(X^!,\operatorname{Supp}(\Delta^!-F^!))$ is log smooth over $\mathcal{Z}_t\setminus\mathcal{P}_t$, we may assume that the morphism $X\to\mathcal{Z}_t$ and divisors $\Delta-F$, $\mathcal{P}_t$ satisfy the standard normal crossing assumptions. Hence the corresponding moduli $\boldsymbol{b}$-divisor descends on $\mathcal{Z}_t$.

Because $(X, \Delta) \to Z$ has the same generic fiber $(X_\eta, \Delta_\eta)$ as $f : (X, \Delta) \to \mathcal{Z}_t$ and $\kappa(X_\eta, K_{X_\eta} + \Delta_\eta) = 0$, the moduli $\boldsymbol{b}$-divisor $\boldsymbol{M}$ of $(X, \Delta) \to Z$ descends on $\mathcal{Z}_t$. Also because $\phi : Z' \to \mathcal{Z}_t$ is a birational morphism, we have $\boldsymbol{M}_{Z'} = \phi^* \boldsymbol{M}_{\mathcal{Z}_t}$.

Step 6: We show that the boundary part is $\mathbb{Q}$-linearly equivalent to the difference of two $\mathbb{Q}$-divisors on $\mathcal{Z}_t$ both with bounded degrees. Therefore, the boundary part is bounded up to $\mathbb{Q}$-linear equivalence.

By Theorem 2.5, there is a rational number $e < 1$ depending only on $\mathcal{I}$, $d$, and $l$ such that both $K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'}$ and $K_{Z'} + B_{Z'} + e\boldsymbol{M}_{Z'}$ are big divisors. By Theorem 2.4, there is an integer $\tilde{r}$ depending only on $\mathcal{I}$, $d$, $l$ and $e$ such that both $|m\tilde{r}(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'})|$ and $|m\tilde{r}(K_{Z'} + B_{Z'} + e\boldsymbol{M}_{Z'})|$ define birational maps for all integers $m \geq 1$. By assumption, $l\boldsymbol{M}_{\mathcal{Z}_t}$ is Cartier, so we may choose $\tilde{r} = r'l$ for some integer $r' \gg 0$ such that both $\tilde{r}\boldsymbol{M}_{Z'}$ and $\tilde{r}e\boldsymbol{M}_{Z'}$ are Cartier divisors and both $|\tilde{r}(K_{Z'} + \lfloor \tilde{r} B_{Z'} \rfloor / \tilde{r} + \boldsymbol{M}_{Z'})|$ and $|\tilde{r}(K_{Z'} + \lfloor \tilde{r} B_{Z'} \rfloor / \tilde{r} + e\boldsymbol{M}_{Z'})|$ define birational maps. Let

$$D_1'' \in \left| \tilde{r}\left( K_{Z'} + \frac{\lfloor \tilde{r} B_{Z'} \rfloor}{\tilde{r}} + \boldsymbol{M}_{Z'} \right) \right|, \quad D_2'' \in \left| \tilde{r}\left( K_{Z'} + \frac{\lfloor \tilde{r} B_{Z'} \rfloor}{\tilde{r}} + e\boldsymbol{M}_{Z'} \right) \right|$$

be general members. Define two effective $\mathbb{Q}$-divisors

$$D_1' \sim_{\mathbb{Q}} \frac{K_{Z'} + \lfloor \tilde{r} B_{Z'} \rfloor / \tilde{r} + \boldsymbol{M}_{Z'}}{1-e}, \quad D_2' \sim_{\mathbb{Q}} \frac{K_{Z'} + \lfloor \tilde{r} B_{Z'} \rfloor / \tilde{r} + e\boldsymbol{M}_{Z'}}{1-e}.$$

It is easy to see that the coefficients of $D_1'$ and $D_2'$ are in a discrete set that depends only on $r, \tilde{r}, e, \mathcal{I}$. Let $D_1 = \phi_* D_1'$ and $D_2 = \phi_* D_2'$. It is easy to see the degrees of $D_1$ and $D_2$ with respect to $A$ in $W$ are bounded. Because the coefficients of $D_1$ and $D_2$ are in a finite set and $\boldsymbol{M}_Z = D_1 - D_2$, by Lemma 3.1, up to replacing the family, there are finitely many divisors $\mathcal{M}_k$, $k \in \Lambda$, on $\mathcal{Z}$ such that $\boldsymbol{M}_{\mathcal{Z}_t} = \mathcal{M}_k|_{\mathcal{Z}_t}$ for some $k \in \Lambda$. □

**Theorem 3.3.** *Suppose $(\mathcal{Z}, \mathcal{P}) \to T$ is a family of projective log smooth pairs, where $T$ is of finite type, and let $\mathcal{M}$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $\mathcal{Z}$. Fix an integer $l > 0$ and a DCC set $\mathcal{I} \subset [0, 1] \cap \mathbb{Q}$. For a closed point $t \in T$, let $\mathcal{S}(\mathcal{I}, \mathcal{Z}, \mathcal{P}, \mathcal{M}, T, t)$ denote the set of generalized pairs $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$ such that*

- $(Z', \mathrm{Supp}(B_{Z'}))$ *is log smooth,*
- $\mathrm{coeff}(B_{Z'}) \in \mathcal{I}$,
- *there is a birational morphism $\phi : Z' \to \mathcal{Z}_t$,*
- $\phi_* B_{Z'} \leq \mathcal{P}|_{\mathcal{Z}_t}$,
- $\boldsymbol{M}$ *descends on $\mathcal{Z}_t$, and*
- $\boldsymbol{M}_{Z'} = \phi^*(\mathcal{M}|_{\mathcal{Z}_t})$.

*Let $\mathcal{S}(\mathcal{I}, \mathcal{Z}, \mathcal{P}, \mathcal{M}, T) := \bigcup_{t \in T} \mathcal{S}(\mathcal{I}, \mathcal{Z}, \mathcal{P}, \mathcal{M}, T, t)$. Then the set*

$$\{\mathrm{vol}(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'}) \mid (Z', B_{Z'} + \boldsymbol{M}_{Z'}) \in \mathcal{S}(\mathcal{I}, \mathcal{Z}, \mathcal{P}, \mathcal{M}, T)\}$$

*satisfies the DCC.*

*Proof.* Let $T' \subset T$ be the subset such that $\boldsymbol{M}|_{\mathcal{Z}_t}$ is nef for every $t \in T'$. Fix a closed point $0 \in T'$. For any closed point $t \in T'$ and $(Z', B_{Z'} + \boldsymbol{M}_{Z'}) \in \mathcal{S}(\mathcal{I}, \mathcal{Z}, \mathcal{P}, \mathcal{M}, T, t)$, because $(\mathcal{Z}_t, \mathcal{P}_t)$ is log smooth, by the proof of [Filipazzi 2018, Theorem 1.10], we may assume that $\phi : Z' \to \mathcal{Z}_t$ only blows up strata of $\mathcal{P}_t$. On the other hand, by the proof of Lemma 3.1, after replacing $T$ by an étale cover, we may assume every stratum of $(\mathcal{Z}, \mathcal{P})$ has irreducible fibers over $T$. Therefore, we may find a sequence of blowups $g : \mathcal{Z}' \to \mathcal{Z}$ such that $Z' = \mathcal{Z}'_t$. It is easy to see that there is a unique divisor $\mathcal{B}_{Z'}$ supported on the strict transform of $\mathcal{P}$ and the exceptional locus of $g$ such that $B_{Z'} = \mathcal{B}_{Z'}|_{\mathcal{Z}'_t}$. Let $Y = \mathcal{Z}'_0$ be the fiber over 0 of $\mathcal{Z}' \to T$ and $B_Y := \mathcal{B}_{Z'}|_{\mathcal{Z}'_0}$. By Theorem 2.7, we have that

$$\mathrm{vol}(K_{Z'} + B_{Z'} + \phi^*\mathcal{M}|_{\mathcal{Z}_t}) = \mathrm{vol}(K_Y + B_Y + (g^*\mathcal{M})|_{\mathcal{Z}'_0}).$$

Then the set

$$\{\mathrm{vol}(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'}) \mid (Z', B_{Z'} + \boldsymbol{M}_{Z'}) \in \mathcal{S}(\mathcal{I}, \mathcal{Z}, \mathcal{P}, \mathcal{M}, T, t)\}$$

is independent of $t \in T$. Now apply Theorem 2.6. □

*Proof of Theorem 1.6.* Fix an arbitrary constant $v > 0$, let

$$\mathcal{D}(n, \mathcal{I}, l, r, v^-) := \{(X, \Delta) \in \mathcal{D}(n, \mathcal{I}, l, r), \mathrm{Ivol}(K_X + \Delta) \le v\}.$$

We only need to prove $\{\mathrm{Ivol}(K_X + \Delta) \mid (X, \Delta) \in \mathcal{D}(n, \mathcal{I}, l, r, v^-)\}$ is a DCC set.

Fix $(X, \Delta) \in \mathcal{D}(n, \mathcal{I}, l, r, v^-)$. Because $Z$ is the canonical model of $K_X + \Delta$, by Theorem 2.12, there is a generalized pair $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$ and a birational morphism $h : Z' \to Z$ such that $Z$ is the canonical model of $K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'}$. Let $B_Z$ be the pushforward of $B_{Z'}$; then $K_Z + B_Z + \boldsymbol{M}_Z$ is ample.

By Theorem 2.4, there is an integer $r' > 0$ which only depends on $\mathcal{I}$ and $l$ such that $|r'(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'})|$ defines a birational map. Choose a general member $H' \in |r'(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'})|$, and let $H := h_*H'$. Then $H$ is ample and the coefficients of $H$ are bounded below by a positive number $\delta'$. By definition of the canonical model, $h^*H \le H'$, by Theorem 2.12 (3),

$$H^0(X, \mathcal{O}_X(ml(K_X + \Delta))) \cong H^0(Z', \mathcal{O}_{Z'}(ml(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'}))),$$

and we have that

$$\mathrm{Ivol}(K_X + \Delta + f^*H) \le (1 + r)^d \, \mathrm{Ivol}(K_X + \Delta). \tag{3-5}$$

Then $(X, \Delta)$ and $H$ satisfy the conditions in Theorem 3.2.

Let $(\mathcal{Z}, \mathcal{P}) \to T$ be the bounded family, let $\mathcal{M}_k$, $k \in \Lambda$, be the $\mathbb{Q}$-divisors defined in Theorem 3.2, and let $\mathcal{D}'$ be the set of generalized klt pairs $(W', B_{W'} + \boldsymbol{M}_{W'})$ such that

- $(W', \mathrm{Supp}(B_{W'}))$ is log smooth,
- there is a morphism $\phi : W' \to \mathcal{Z}_t$ for a closed point $t \in T$,
- $\boldsymbol{M}_{W'} = \phi^*(\mathcal{M}_k|_{\mathcal{Z}_t})$ for some $k \in \Lambda$, and
- $\mathrm{coeff}(B_{W'})$ is in a fixed DCC set and $\phi_*(B_{W'}) \le \mathcal{P}_t$.

Since $Z$ is the canonical model of $(X, \Delta)$, we have $\operatorname{Ivol}(K_X+\Delta) = \operatorname{vol}(K_Z+B_Z+\boldsymbol{M}_Z)$. Let $(Z', B_{Z'}+\boldsymbol{M}_{Z'})$ be a generalized pair as in Theorem 2.12 such that there is a birational morphism $\psi_t : Z' \to \mathcal{Z}_t$ for a closed point $t \in T$. By Theorem 2.12 (3), $\operatorname{vol}(K_Z+B_Z+\boldsymbol{M}_Z) = \operatorname{vol}(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'})$, and, by Theorem 3.2, $\psi_{t*}B_{Z'} \leq \mathcal{P}_t$ and $\boldsymbol{M}_{Z'} = \psi_t^*\mathcal{M}_k$. Then $(Z', B_{Z'}+\boldsymbol{M}_{Z'}) \in \mathcal{D}'$ and

$$\{\operatorname{Ivol}(K_X+\Delta) \mid (X, \Delta) \in \mathcal{D}(n, \mathcal{I}, l, r, v^-)\} \subset \{\operatorname{vol}(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'}) \mid (Z', B_{Z'}+\boldsymbol{M}_{Z'}) \in \mathcal{D}'\}.$$

Because $\Lambda$ is a finite set, by Theorem 3.3, the set $\{\operatorname{vol}(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'}) \mid (Z', B_{Z'}+\boldsymbol{M}_{Z'}) \in \mathcal{D}'\}$ satisfies the DCC, and hence $\{\operatorname{Ivol}(K_X+\Delta) \mid (X, \Delta) \in \mathcal{D}(n, \mathcal{I}, l, r, v^-)\}$ satisfies the DCC. □

## 4. Boundedness of canonical models

In this section, we follow the method of [Hacon et al. 2018, Chapter 7].

**Definition 4.1.** Let $(Z, B)$ be a pair. Define a $\boldsymbol{b}$-divisor $\boldsymbol{M}_B$ by assigning to any divisorial valuation $\mu$

$$\boldsymbol{M}_B(\mu) = \begin{cases} \operatorname{mult}_\Gamma(B) & \text{if the center of } \mu \text{ is a divisor } \Gamma \text{ on } Z, \\ 1 & \text{otherwise.} \end{cases} \tag{4-1}$$

**Theorem 4.2.** *Let $v$ be a positive rational number, and let $\mathcal{I} \subset [0, 1]$ be a DCC set of positive rational numbers. Suppose $(\mathcal{Z}, \mathcal{P}) \to T$ is a family of projective log smooth pairs, where $T$ is of finite type, and $\mathcal{M}$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $\mathcal{Z}$. Let $\mathcal{S}(v, \mathcal{I}, \mathcal{Z}, \mathcal{P}, \mathcal{M}, T)$ be the set of generalized pairs $(Z', B_{Z'}+\boldsymbol{M}_{Z'})$ such that*

- *$(Z', B_{Z'}+\boldsymbol{M}_{Z'})$ is generalized klt,*
- $\operatorname{vol}(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'}) = v$,
- $\operatorname{coeff}(B_{Z'}) \subset \mathcal{I}$,
- *there is a closed point $t \in T$ and a birational morphism $\phi : Z' \to \mathcal{Z}_t$,*
- $\phi_* B_{Z'} \leq \mathcal{P}_t$,
- *$\boldsymbol{M}$ descends on $\mathcal{Z}_t$, and*
- $\boldsymbol{M}_{Z'} = \phi^*(\mathcal{M}|_{\mathcal{Z}_t})$.

*Let $(Z, B_Z+\boldsymbol{M}_Z)$ be the canonical model of $(Z', B_{Z'}+\boldsymbol{M}_{Z'})$. Then $Z$ is in a bounded family depending only on $v$, $\mathcal{I}$, $(\mathcal{Z}, \mathcal{P}) \to T$ and $\mathcal{M}$.*

*Proof.* It suffices to show that, for any generalized pair $(Z', B_{Z'}+\boldsymbol{M}_{Z'}) \subset \mathcal{S}(v, \mathcal{I}, \mathcal{Z}, \mathcal{P}, \mathcal{M}, T)$, there is an integer $N > 0$ such that if $(Z, B_Z+\boldsymbol{M}_Z)$ is the canonical model of $(Z', B_{Z'}+\boldsymbol{M}_{Z'})$, then $N(K_Z+B_Z+\boldsymbol{M}_Z)$ is Cartier and very ample.

Suppose this is not the case: let $\{(Z'_i, B^i_{Z'_i}+\boldsymbol{M}_{Z'_i}), i \geq 1\} \subset \mathcal{S}(v, \mathcal{I}, \mathcal{Z}, \mathcal{P}, \mathcal{M}, T)$ be a sequence and $(Z_i, B^i_{Z_i}+\boldsymbol{M}^i_{Z_i})$ the corresponding canonical model such that $i!(K_{Z_i}+B^i_{Z_i}+\boldsymbol{M}^i_{Z_i})$ is not very ample for every $i \geq 1$. Let $\{t_i \in T, i \geq 1\}$ be the corresponding sequence of closed points, and let $\phi_i : Z'_i \to \mathcal{Z}_{t_i}$ be the corresponding morphisms. By the construction, we have $\phi_{i*}B^i_{Z'_i} \leq \mathcal{P}_{t_i}$ and $\boldsymbol{M}^i_{Z'_i} = \phi_i^*(\mathcal{M}|_{\mathcal{Z}_{t_i}})$. After replacing $T$ by a closed subset, we assume that $\{t_i \in T, i \geq 1\}$ is dense in $T$.

Step 1: We prove that there exists a birational morphism $g : \mathcal{Z}' \to \mathcal{Z}$ such that

- $g$ is obtained by blowing up the corresponding strata of $\boldsymbol{M}_{\mathcal{P}}$, and
- $\operatorname{vol}(K_{\mathcal{Z}'_{t_i}} + \Phi^i|_{\mathcal{Z}'_{t_i}} + g^*\mathcal{M}|_{\mathcal{Z}'_{t_i}}) = v$ for every $i \geq 1$, where $\Phi^i$ is the $\mathbb{Q}$-divisor supported on $\boldsymbol{M}_{\mathcal{P},\mathcal{Z}'}$ such that $\Phi^i|_{\mathcal{Z}'_{t_i}} = \boldsymbol{M}_{B^i_{Z'_i},\mathcal{Z}'_{t_i}}$.

Applying [Filipazzi 2018, Proposition 5.1] to $(\mathcal{Z}_{t_1}, \mathcal{P}_{t_1} + \mathcal{M}|_{\mathcal{Z}_{t_1}})$, we obtain a model $\mathcal{Z}'_{t_1} \to \mathcal{Z}_{t_1}$ and the morphism $g : (\mathcal{Z}', \mathcal{P}' := \boldsymbol{M}_{\mathcal{P},\mathcal{Z}'}) \to \mathcal{Z}$ obtained by blowing up the corresponding strata of $\boldsymbol{M}_{\mathcal{P}}$. We define $\Phi_{t_i} = \boldsymbol{M}_{B^i_{Z'_i},\mathcal{Z}'_{t_i}}$. Passing to a subsequence, we may also assume that, for any irreducible component $P$ of the support of $\mathcal{P}'$, the coefficients of $\Phi_{t_i}$ along $P_{t_i}$ are nondecreasing. Let $\Phi^i$ be the $\mathbb{Q}$-divisor supported on $\mathcal{P}'$ such that $\Phi^i|_{\mathcal{Z}'_{t_i}} = \Phi_{t_i}$. Then the coefficients of $\Phi^i$ are nondecreasing.

We claim that, for any $i \geq 1$, we have

$$\operatorname{vol}(K_{\mathcal{Z}'_{t_i}} + \Phi_{t_i} + g^*\mathcal{M}|_{\mathcal{Z}'_{t_i}}) = v.$$

To prove this, we may fix $i$. Applying the above cited result to $(\mathcal{Z}_{t_i}, \mathcal{P}_{t_i} + \mathcal{M}|_{\mathcal{Z}_{t_i}})$, we obtain a model $\mathcal{Z}''_{t_i} \to \mathcal{Z}'_{t_i}$ and the corresponding morphism $g' : (\mathcal{Z}'', \mathcal{P}'' := \boldsymbol{M}_{\mathcal{P},\mathcal{Z}''}) \to \mathcal{Z}'$ obtained by blowing up the corresponding strata of $\boldsymbol{M}_{\mathcal{P}}$. By the above cited result again, we have

$$\operatorname{vol}(K_{\mathcal{Z}''_{t_i}} + \Psi_{t_i} + g'^* g^*\mathcal{M}|_{\mathcal{Z}''_{t_i}}) = v,$$

where $\Psi_{t_i} := \boldsymbol{M}_{B^i_{Z'_i},\mathcal{Z}''_{t_i}}$. If $\Psi$ is the divisor supported on $\operatorname{Supp}(\boldsymbol{M}_{\mathcal{P}',\mathcal{Z}''})$ such that $\Psi|_{\mathcal{Z}''_{t_i}} = \Psi_{t_i}$, then

$$\begin{aligned} v &= \operatorname{vol}(K_{\mathcal{Z}''_{t_i}} + \Psi_{t_i} + g'^* g^*\mathcal{M}|_{\mathcal{Z}''_{t_i}}) \\ &= \operatorname{vol}(K_{\mathcal{Z}''_{t_1}} + \Psi|_{\mathcal{Z}''_{t_1}} + g'^* g^*\mathcal{M}|_{\mathcal{Z}''_{t_1}}) \\ &= \operatorname{vol}(K_{\mathcal{Z}'_{t_1}} + \Phi^i|_{\mathcal{Z}'_{t_1}} + g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}}) \\ &= \operatorname{vol}(K_{\mathcal{Z}'_{t_i}} + \Phi_{t_i} + g^*\mathcal{M}|_{\mathcal{Z}'_{t_i}}), \end{aligned} \tag{4-2}$$

where the second and the fourth equalities follow from Theorem 2.7 and the third one follows from [Filipazzi 2018, Proposition 5.1].

Step 2: We show that, after replacing $T$ by an open subset, $\mathcal{Z}'$ by a resolution and $\{t_i, i \geq 1\}$ by a subsequence, there exist effective $\mathbb{Q}$-divisors $\mathcal{A}$ and $\mathcal{E}^i$ on $\mathcal{Z}'$ such that

- $\mathcal{A}$ is ample over $T$,
- $\mathcal{E}^i := \mathcal{E}^1 + \Phi^i - \Phi^1$,
- $K_{\mathcal{Z}'} + \Phi^i + g^*\mathcal{M} \sim_{\mathbb{Q}} \mathcal{A} + \mathcal{E}^i$, and
- $(\mathcal{Z}', \operatorname{Supp}(\Phi^i + \mathcal{E}^i))$ is log smooth over $T$

for every $i \geq 1$.

Because $g^*\mathcal{M}|_{\mathcal{Z}'_{t_i}}$ is nef for every $i \in \mathbb{N}$ and $\{t_i \in T, i \geq 1\}$ is dense in $T$, we have that $g^*\mathcal{M}|_{\mathcal{Z}'_t}$ is nef for a very general point $t \in T$. Note that $K_{\mathcal{Z}'_{t_1}} + \Phi^1|_{\mathcal{Z}'_{t_1}} + g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}}$ is big. Suppose

$$\mathrm{vol}(K_{\mathcal{Z}'_{t_1}} + \Phi^1|_{\mathcal{Z}'_{t_1}} + g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}}) = v > 0.$$

Then by [Filipazzi 2018, Theorem 1.12], we have $\mathrm{vol}(K_{\mathcal{Z}'_t} + \Phi^1|_{\mathcal{Z}'_t} + g^*\mathcal{M}|_{\mathcal{Z}'_t}) = v$ for a very general point $t \in T$. Since sections on the very general fiber agree with sections on the generic fiber by semicontinuity of cohomology groups, $K_{\mathcal{Z}'} + \Phi^1 + g^*\mathcal{M}$ is big over the generic point of $T$, and we have that $K_{\mathcal{Z}'} + \Phi^1 + g^*\mathcal{M}$ is big over $T$.

Let $\mathcal{A}$ be a general relatively ample $\mathbb{Q}$-divisor on $\mathcal{Z}'$ and $\mathcal{E}^1$ be an effective $\mathbb{Q}$-divisor on $\mathcal{Z}'$ such that

$$K_{\mathcal{Z}'} + \Phi^1 + g^*\mathcal{M} \sim_{\mathbb{Q}} \mathcal{A} + \mathcal{E}^1.$$

Define $\mathcal{E}^i := \mathcal{E}^1 + \Phi^i - \Phi^1$; then $\mathcal{E}^i$ is effective and $K_{\mathcal{Z}'} + \Phi^i + g^*\mathcal{M} \sim_{\mathbb{Q}} \mathcal{A} + \mathcal{E}^i$.

After taking a log resolution of the generic fiber and replacing $T$ by an open subset, we may assume that there is a fiberwise log resolution $h : \mathcal{Z}^* \to \mathcal{Z}'$ of $\mathcal{P}' + \mathcal{E}$ over $T$. By the negativity lemma, there exists a $\mathbb{Q}$-divisor $\mathcal{F}$ on $\mathcal{Z}^*$ which is supported on the exceptional divisor over $\mathcal{Z}$ such that $\mathcal{A}^* := h^*\mathcal{A} - \mathcal{F}$ is relatively ample over $T$. Let $\Phi^{i*} := \boldsymbol{M}_{\Phi^i, \mathcal{Z}^*}$. Because $(\mathcal{Z}', \Phi^1)$ is lc, if we write

$$K_{\mathcal{Z}^*} + \Phi^{1*} + h^*g^*\mathcal{M} \sim_{\mathbb{Q}} \mathcal{A}^* + \mathcal{E}^{1*},$$

then $\mathcal{E}^{1*}$ is effective and supported on the $\mathrm{Supp}(h_*^{-1}\mathcal{E}^1)$ plus the $h$-exceptional divisors. Therefore, $(\mathcal{Z}^*, \mathrm{Supp}(\mathcal{E}^{1*}))$ is log smooth over $T$. Notice that $\Phi^i - \Phi^1$ is effective and supported on $\mathcal{P}'$. Define $\mathcal{E}^{i*} := \mathcal{E}^{1*} + h_*^{-1}(\Phi^i - \Phi^1)$; then

$$K_{\mathcal{Z}^*} + \Phi^{i*} + h^*g^*\mathcal{M} \sim_{\mathbb{Q}} \mathcal{A}^* + \mathcal{E}^{i*}$$

and $(\mathcal{Z}^*, \mathrm{Supp}(\mathcal{E}^{i*}))$ is log smooth over $T$.

Then we replace $\mathcal{Z}',\ \Phi^i,\ g,\ \mathcal{A}$ and $\mathcal{E}^i$ by $\mathcal{Z}^*,\ \Phi^{i*},\ h \circ g,\ \mathcal{A}^*$ and $\mathcal{E}^{i*}$, respectively. Suppose

$$k = \min\{i \mid t_i \in T,\ i \geq 1\}.$$

Then we pass to a subsequence of $\{t_i, i \in \mathbb{N}\}$ and replace $t_1,\ \Phi^1$ and $\mathcal{E}^1$ by $t_k,\ \Phi^k$ and $\mathcal{E}^k$, respectively.

Step 3: In this step we construct a $\mathbb{Q}$-divisor $\hat{\Phi}$ on $\mathcal{Z}'$ such that

- $\hat{\Phi} \leq \Phi^1$,
- $(\mathcal{Z}'_{t_i}, \hat{\Phi}|_{\mathcal{Z}'_{t_i}})$ is klt for every $i \geq 1$,
- $\mathrm{vol}(K_{\mathcal{Z}'_{t_1}} + \hat{\Phi}|_{\mathcal{Z}'_{t_1}} + g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}}) = v$, and
- $(Z_1, B^1_{Z_1} + \boldsymbol{M}^1_{Z_1})$ is the canonical model of $(\mathcal{Z}'_{t_1}, \hat{\Phi}|_{\mathcal{Z}'_{t_1}} + g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}})$.

Since $(Z_1, B^1_{Z_1} + \boldsymbol{M}^1_{Z_1})$ is generalized klt, we slightly decrease the coefficients of components of $\Phi^1$ corresponding to the exceptional divisor of $\mathcal{Z}'_{t_1} \dashrightarrow Z_1$ to define a $\mathbb{Q}$-divisor $\hat{\Phi}$ such that

- $\hat{\Phi} \leq \Phi^1$,
- $(\mathcal{Z}'_{t_1}, \hat{\Phi}|_{\mathcal{Z}'_{t_1}})$ is klt,

- $\mathrm{vol}(K_{\mathcal{Z}'_{t_1}}+\hat{\Phi}|_{\mathcal{Z}'_{t_1}}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}})=v$, and
- $(Z_1, B^1_{Z_1}+\boldsymbol{M}^1_{Z_1})$ is the canonical model of $(\mathcal{Z}'_{t_1}, \hat{\Phi}|_{\mathcal{Z}'_{t_1}}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}})$.

Note we have $\hat{\Phi}\le\Phi^1\le\Phi^2\le\cdots$. Because $(\mathcal{Z}', \mathrm{Supp}(\Phi^i+\mathcal{E}^i))$ is log smooth over $T$ and $(\mathcal{Z}'_{t_1}, \hat{\Phi}|_{\mathcal{Z}'_{t_1}})$ is klt, we have that $(\mathcal{Z}'_{t_i}, \hat{\Phi}|_{\mathcal{Z}'_{t_i}})$ is klt for every $i$.

Step 4: We show that there exist a sufficiently small positive number $\epsilon\in(0,1)$ and a birational contraction $\psi:\mathcal{Z}'\dashrightarrow\mathcal{W}$ over $T$ such that

- $\psi$ is the relative canonical model of $\big(\mathcal{Z}', \frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}+g^*\mathcal{M}\big)$, and
- $\psi_{t_i}:\mathcal{Z}'_{t_i}\dashrightarrow\mathcal{W}_{t_i}$ is the canonical model of $\big(\mathcal{Z}'_{t_i}, \big(\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}+g^*\mathcal{M}\big)\big|_{\mathcal{Z}'_{t_i}}\big)$ for every $i\ge1$.

Because $\Phi_{t_1}=\boldsymbol{M}_{B^i_{Z'_1},\mathcal{Z}'_{t_1}}$, for any common resolution $Y$ of $Z'_1$ and $\mathcal{Z}'_{t_1}$, we have $\boldsymbol{M}_{\Phi_{t_1},Y}\ge\boldsymbol{M}_{B^1_{Z'_1},Y}$. Also because $\mathrm{vol}(K_{Z'_1}+B'_{Z'_1}+\boldsymbol{M}^1_{Z'_1})=\mathrm{vol}(K_{\mathcal{Z}'_{t_1}}+\Phi_{t_1}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}})$, by [Filipazzi 2018, Lemma 5.2], $(Z'_1, B'_{Z'_1}+\boldsymbol{M}^1_{Z'_1})$ has the same canonical model as $(\mathcal{Z}'_{t_1}, \Phi_{t_1}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}})$, which is $(Z_1, B^1_{Z_1}+\boldsymbol{M}^1_{Z_1})$. In particular, there is a birational contraction $\mathcal{Z}'_{t_1}\dashrightarrow Z_1$.

Since $(\mathcal{Z}'_{t_1}, \hat{\Phi}|_{\mathcal{Z}'_{t_1}})$ is klt and $(\mathcal{Z}', \mathrm{Supp}(\Phi^i+\mathcal{A}+\mathcal{E}^i))$ is log smooth over $T$, we can choose $\epsilon\ll1$ such that $(\mathcal{Z}'_{t_i}, \hat{\Phi}|_{\mathcal{Z}'_{t_i}}+\epsilon\mathcal{E}^1|_{\mathcal{Z}'_{t_i}})$ is klt for every $i\ge1$ and $g^*\mathcal{M}+\epsilon\mathcal{A}$ is ample over $T$. We then have that $(\mathcal{Z}'_{t_i}, \hat{\Phi}|_{\mathcal{Z}'_{t_i}}+\epsilon\mathcal{E}^1|_{\mathcal{Z}'_{t_i}}+(g^*\mathcal{M}+\epsilon\mathcal{A})|_{\mathcal{Z}'_{t_i}})$ is generalized klt with nef part $(g^*\mathcal{M}+\epsilon\mathcal{A})|_{\mathcal{Z}'_{t_i}}$ for every $i\ge1$. Because

$$K_{\mathcal{Z}'_{t_1}}+\hat{\Phi}|_{\mathcal{Z}'_{t_1}}+\epsilon\mathcal{E}^1|_{\mathcal{Z}'_{t_1}}+(g^*\mathcal{M}+\epsilon\mathcal{A})|_{\mathcal{Z}'_{t_1}}\sim_{\mathbb{Q}}K_{\mathcal{Z}'_{t_1}}+\hat{\Phi}|_{\mathcal{Z}'_{t_1}}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}}+\epsilon(K_{\mathcal{Z}'_{t_1}}+\Phi^1|_{\mathcal{Z}'_{t_1}}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}})$$

and $Z_1$ is both the canonical model of $(\mathcal{Z}'_{t_1}, \hat{\Phi}|_{\mathcal{Z}'_{t_1}}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}})$ and $(\mathcal{Z}'_{t_1}, \Phi^1|_{\mathcal{Z}'_{t_1}}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}})$, we have that $Z_1$ is also the canonical model of $(\mathcal{Z}'_{t_1}, \hat{\Phi}|_{\mathcal{Z}'_{t_1}}+\epsilon\mathcal{E}^1|_{\mathcal{Z}'_{t_1}}+(g^*\mathcal{M}+\epsilon\mathcal{A})|_{\mathcal{Z}'_{t_1}})$.

Because $g^*\mathcal{M}+\epsilon\mathcal{A}$ is ample over $T$, we can choose a general effective $\mathbb{Q}$-divisor $\mathcal{H}\sim_{\mathbb{Q}}g^*\mathcal{M}+\epsilon\mathcal{A}$ and replace $T$ by an open neighborhood of $t_1$ such that $(\mathcal{Z}'_{t_i}, \hat{\Phi}|_{\mathcal{Z}'_{t_i}}+\epsilon\mathcal{E}^1|_{\mathcal{Z}'_{t_i}}+\mathcal{H}|_{\mathcal{Z}'_{t_i}})$ is klt for every $i\ge1$ and $(\mathcal{Z}', \mathrm{Supp}(\hat{\Phi}+\epsilon\mathcal{E}^1+\mathcal{H}))$ is log smooth over $T$. It is easy to see that $Z_1$ is also the canonical model of $(\mathcal{Z}'_{t_1}, \hat{\Phi}|_{\mathcal{Z}'_{t_1}}+\epsilon\mathcal{E}^1|_{\mathcal{Z}'_{t_1}}+\mathcal{H}|_{\mathcal{Z}'_{t_1}})$.

Because $\mathcal{H}|_{\mathcal{Z}'_{t_1}}$ is ample and $(\mathcal{Z}'_{t_1}, \hat{\Phi}|_{\mathcal{Z}'_{t_1}}+\epsilon\mathcal{E}^1|_{\mathcal{Z}'_{t_1}}+\mathcal{H}|_{\mathcal{Z}'_{t_1}})$ is klt, $(\mathcal{Z}'_{t_1}, \hat{\Phi}|_{\mathcal{Z}'_{t_1}}+\epsilon\mathcal{E}^1|_{\mathcal{Z}'_{t_1}}+\mathcal{H}|_{\mathcal{Z}'_{t_1}})$ has a good minimal model, according to [Birkar et al. 2010, Theorem 1.2] and [Kollár and Mori 1998, Theorem 3.3]. Because $(\mathcal{Z}', \mathrm{Supp}(\hat{\Phi}+\epsilon\mathcal{E}^1+\mathcal{H}))$ is log smooth over $T$, by [Hacon et al. 2018, Corollary 1.4], suppose $\psi:\mathcal{Z}'\dashrightarrow\mathcal{W}$ is the relative canonical model of $(\mathcal{Z}', \hat{\Phi}+\epsilon\mathcal{E}^1+\mathcal{H})$ over $T$. Then, fiber by fiber, $\psi_{t_i}:\mathcal{Z}'_{t_i}\dashrightarrow\mathcal{W}_{t_i}$ gives the canonical model for $(\mathcal{Z}'_{t_i}, \hat{\Phi}|_{\mathcal{Z}'_{t_i}}+\epsilon\mathcal{E}^1|_{\mathcal{Z}'_{t_i}}+\mathcal{H}|_{\mathcal{Z}'_{t_i}})$ for all $i\ge1$. In particular, $\mathcal{W}_{t_1}$ is the canonical model of $(\mathcal{Z}'_{t_1}, \hat{\Phi}|_{\mathcal{Z}'_{t_1}}+\epsilon\mathcal{E}^1|_{\mathcal{Z}'_{t_1}}+\mathcal{H}|_{\mathcal{Z}'_{t_1}})$, and it is isomorphic to $Z_1$.

By the definition of the canonical model, $K_{\mathcal{W}}+\psi_*(\hat{\Phi}+\epsilon\mathcal{E}^1+\mathcal{H})$ is ample over $T$. We recall that $K_{\mathcal{Z}'}+\hat{\Phi}+\epsilon\mathcal{E}^1+\mathcal{H}\sim_{\mathbb{Q}}K_{\mathcal{Z}'}+\hat{\Phi}+g^*\mathcal{M}+\epsilon(K_{\mathcal{Z}'}+\Phi^1+g^*\mathcal{M})$. Then

$$K_{\mathcal{W}}+\psi_*(\hat{\Phi}+\epsilon\mathcal{E}^1+\mathcal{H})\sim_{\mathbb{Q}}(1+\epsilon)\Big(K_{\mathcal{W}}+\psi_*\Big(\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}+g^*\mathcal{M}\Big)\Big)$$

and $K_{\mathcal{W}}+\psi_*\big(\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}+g^*\mathcal{M}\big)$ is ample over $T$. Thus $K_{\mathcal{W}_{t_i}}+\psi_*\big(\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}+g^*\mathcal{M}\big)\big|_{\mathcal{W}_{t_i}}$ is ample for every $i\ge1$.

Because $\mathcal{Z}' \dashrightarrow \mathcal{W}$ is $K_{\mathcal{Z}'}+\hat{\Phi}+\epsilon\mathcal{E}^1+\mathcal{H}$-nonpositive and

$$K_{\mathcal{Z}'}+\hat{\Phi}+\epsilon\mathcal{E}^1+\mathcal{H}\sim_{\mathbb{Q}}(1+\epsilon)\Big(K_{\mathcal{Z}'}+\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}+g^*\mathcal{M}\Big),$$

$\mathcal{Z}' \dashrightarrow \mathcal{W}$ is $K_{\mathcal{Z}'}+\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}+g^*\mathcal{M}$-nonpositive. Also because

$$K_{\mathcal{W}_{t_i}}+\psi_*\Big(\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}+g^*\mathcal{M}\Big)\Big|_{\mathcal{W}_{t_i}}$$

is ample, $\psi_{t_i}:\mathcal{Z}'_{t_i}\dashrightarrow\mathcal{W}_{t_i}$ is the canonical model of

$$\Big(\mathcal{Z}'_{t_i},\Big(\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}+g^*\mathcal{M}\Big)\Big|_{\mathcal{Z}'_{t_i}}\Big).$$

<u>Step 5</u>: We show that $\psi_{t_i}$ is also the canonical model of $(\mathcal{Z}'_{t_i},\Phi^k|_{\mathcal{Z}'_{t_i}}+g^*\mathcal{M}_{\mathcal{Z}'_{t_i}})$ for every $i,k\geq 1$ and finish the proof of the theorem.

Notice that, by Theorem 2.7,

$$v=\operatorname{vol}(K_{\mathcal{Z}'_{t_k}}+\Phi^k|_{\mathcal{Z}'_{t_k}}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_i}})=\operatorname{vol}(K_{\mathcal{Z}'_{t_1}}+\Phi^k|_{\mathcal{Z}'_{t_1}}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}})$$

for all $k>1$. By the construction of $\hat{\Phi}$, we have

$$\hat{\Phi}\leq\Phi^1\leq\Phi^2\leq\Phi^3\leq\cdots$$

and $\operatorname{vol}(K_{\mathcal{Z}'_{t_1}}+\hat{\Phi}|_{\mathcal{Z}'_{t_1}}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}})=v$; hence

$$\operatorname{vol}\Big(K_{\mathcal{Z}'_{t_1}}+\Big(\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}\Big)\Big|_{\mathcal{Z}'_{t_1}}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}}\Big)=v.$$

Because $(\mathcal{Z}',\operatorname{Supp}(\hat{\Phi}+\Phi^1))$ is log smooth over $T$, by [Filipazzi 2018, Theorem 1.12], we have

$$\operatorname{vol}\Big(K_{\mathcal{Z}'_{t_1}}+\Big(\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}\Big)\Big|_{\mathcal{Z}'_{t_1}}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}}\Big)=\operatorname{vol}\Big(K_{\mathcal{Z}'_{t_i}}+\Big(\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}\Big)\Big|_{\mathcal{Z}'_{t_i}}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_i}}\Big)=v$$

for every $i\geq 1$.

It follows from [Filipazzi 2018, Lemma 5.2] that $\psi_{t_i}:\mathcal{Z}'_{t_i}\dashrightarrow\mathcal{W}_{t_i}$ is also the canonical model of $(\mathcal{Z}'_{t_i},\Phi^k|_{\mathcal{Z}'_{t_i}}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_i}})$ for every $k\geq 1$,

$$\psi_{t_i*}\Phi^k|_{\mathcal{Z}'_{t_i}}=\psi_{t_i*}\Big(\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}\Big)\Big|_{\mathcal{Z}'_{t_i}},$$

and there is an isomorphism $\alpha_i:Z_i\to\mathcal{W}_{t_i}$. Let $N>0$ be an integer such that

$$N\Big(K_{\mathcal{W}}+\psi_*\Big(\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}\Big)+\psi_*g^*\mathcal{M}\Big)$$

is very ample over $T$. Then

$$N\Big(K_{\mathcal{W}_{t_i}}+\psi_*\Big(\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}\Big)\Big|_{\mathcal{Z}'_{t_i}}+\psi_*g^*\mathcal{M}|_{\mathcal{W}_{t_i}}\Big)$$

is very ample for every $i\geq 1$. Since

$$\psi_{t_i*}\Big(\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}\Big)\Big|_{\mathcal{Z}'_{t_i}}=\psi_{t_i*}\Phi^i|_{\mathcal{Z}'_{t_i}}=\alpha_{i*}B^i_{Z_i},$$

we have that $N(K_{Z_i}+B^i_{Z_i}+\boldsymbol{M}^i_{Z_i})$ is very ample for every $i\geq 1$, which is the required contradiction. □

*Proof of Theorem 1.7.* Define $\mathcal{D}(n, \mathcal{I}, l, r, v)$ to be the set

$$\{(X, \Delta) \mid (X, \Delta) \in \mathcal{D}(n, \mathcal{I}, l, r) \text{ and } \operatorname{Ivol}(K_X + \Delta) = v\}.$$

Suppose $(Z, B_Z + \boldsymbol{M}_Z)$ is the canonical model of $(X, \Delta) \in \mathcal{D}(n, \mathcal{I}, l, r, v)$. Let $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$ be the generalized pair defined in Theorem 2.12 (3). Then $(Z, B_Z + \boldsymbol{M}_Z)$ is the canonical model of $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$.

By Theorem 3.2, there is a log bounded log smooth family of projective varieties $(\mathcal{Z}, \mathcal{P}) \to T$ and finitely many $\mathbb{Q}$-divisors $\mathcal{M}_k$, $k \in \Lambda$, on $\mathcal{Z}$ such that there is a closed point $t \in T$ and a birational morphism $\phi : Z' \to \mathcal{Z}_t$ such that $\phi_* B_{Z'} \le \mathcal{P}_t$. Then we have $(Z', B_{Z'} + \boldsymbol{M}_{Z'}) \in \bigcup_{k \in \Lambda} \mathcal{S}(v, \mathcal{I}, \mathcal{Z}, \mathcal{P}, \mathcal{M}_k, T)$. Because $\Lambda$ is a finite set, $Z$ is in a bounded family according to Theorem 4.2. □

## 5. Weak boundedness

The definition of weak boundedness is introduced in [Kovács and Lieblich 2010].

**Definition 5.1.** A $(g, m)$-curve is an irreducible smooth curve $C^\circ$ whose smooth compactification $C$ has genus $g$ and which satisfies the requirement that $C \setminus C^\circ$ consists of $m$ closed points.

**Definition 5.2.** Let $W$ be a proper scheme with a line bundle $\mathcal{N}$, and let $U$ be an open subset of a proper variety. We say a morphism $\xi : U \to W$ is weakly bounded with respect to $\mathcal{N}$ if there exists a function $b_{\mathcal{N}} : \mathbb{Z}^2_{\ge 0} \to \mathbb{Z}$ such that, for every pair $(g, m)$ of nonnegative integers, for every $(g, m)$-curve $C^\circ \subseteq C$, and for every morphism $C^\circ \to U$, one has $\deg \xi_C^* \mathcal{N} \le b_{\mathcal{N}}(g, m)$, where $\xi_C : C \to W$ is the induced morphism. The function $b_{\mathcal{N}}$ will be called a weak bound, and we will say that $\xi$ is weakly bounded by $b_{\mathcal{N}}$.

We say a quasiprojective variety $U$ is weakly bounded if there exists a compactification $i : U \hookrightarrow W$ such that $i : U \hookrightarrow W$ is weakly bounded with respect to an ample line bundle $\mathcal{N}$ on $W$. The following lemma says that if a projective variety $U$ is weakly bounded with respect to an embedding $U \hookrightarrow W$, then it is also weakly bounded with respect to any other compactification $U \hookrightarrow W'$ and any ample line bundle on $W'$ (possibly by a different weak bound).

**Lemma 5.3.** *Let $U$ be a weakly bounded quasiprojective variety with a compactification $i : U \hookrightarrow W$ such that $i : U \hookrightarrow W$ is weakly bounded with respect to an ample line bundle $\mathcal{N}$ on $W$. Then, for any compactification $i' : U \hookrightarrow W'$ and any ample line bundle $\mathcal{N}'$ on $W'$, $i' : U \hookrightarrow W'$ is weakly bounded with respect to $\mathcal{N}'$.*

*Proof.* Let $g : W'' \to W$ and $h : W'' \to W'$ be a common resolution of $W$ and $W'$. Let $A''$ be an ample Cartier divisor on $W''$, $A$ a Cartier divisor on $W$ such that $\mathcal{O}_W(A) = \mathcal{N}$, and $A'$ a Cartier divisor on $W'$ such that $\mathcal{O}_{W'}(A') = \mathcal{N}'$.

Suppose $C^\circ \subset C$ is a $(g, m)$-curve for a pair of nonnegative integers $(g, m)$ and $C^\circ \to U$ is a morphism that extends to a morphism $\xi : C \to W$. By definition, there exists a function $b_{\mathcal{N}} : \mathbb{Z}^2_{\ge 0} \to \mathbb{Z}$ such that $\deg \xi^* \mathcal{N} \le b_{\mathcal{N}}(g, m)$.

Because $A$ is ample, $g^* A$ is big, and there exist an effective divisor $F$ on $W''$ and $l, n \in \mathbb{N}$ such that $l g^* A \sim n A'' + F''$. Write $\operatorname{Supp}(F'') = \bigcup_{1 \le i \le k} W''_i$, where the $W''_i$ are reduced divisors.

Suppose $C^\circ \to U$ extends to a morphism $\xi'' : C \to W''$. Then $\xi := g \circ \xi''$. We claim that there exists a positive number $c$ depending only on $U \hookrightarrow W''$ and $A''$ such that $\deg \xi''^* A'' \le c b_{\mathcal{N}}(g, m)$, which means $i'' : U \hookrightarrow W''$ is weakly bounded.

We argue by induction on the dimension of $U$. If $\dim(U) = 1$, then $W''$ is the normalization of $W$, $g^*A$ and $F''$ are ample, and

$$\deg \xi''^* A'' = \deg \xi''^* \Big( \frac{l}{n} g^* A - \frac{1}{n} F'' \Big) \le \frac{l}{n} \deg \xi^* A \le \frac{l}{n} b_{\mathcal{N}}(g, m).$$

Thus we may assume the claim is true in dimension one less.

Suppose $\dim(U) > 1$. We have the following two cases.

(1) If $\xi''(C) \not\subset \mathrm{Supp}(F'')$, then

$$\deg \xi''^*(nA'') = \deg \xi''^*(lg^*A - F) \le \deg \xi''^*(lg^*A) = \deg \xi^*(lA) \le l b_{\mathcal{N}}(g, m).$$

Let $c_0 := l/n$; then we have

$$\deg \xi''^* A'' \le c_0 b_{\mathcal{N}}(g, m).$$

(2) If $\xi''(C) \subset \mathrm{Supp}(F'')$, let $W_i''$ be the irreducible component of $\mathrm{Supp}(F)$ that contains $\xi''(C)$. Define $W_i = g(W_i'')$ and $U_i := U \cap W_i$. It is easy to see that $U_i \hookrightarrow W_i$ is naturally weakly bounded with respect to $A|_{W_i}$ by $b_{\mathcal{N}}(g, m)$. Also because $\dim(U_i) < \dim(U)$ and we assume the claim is true in lower dimensions, there exists $c_i > 0$ such that

$$\deg \xi''^* A'' \le c_i b_{\mathcal{N}}(g, m).$$

Because $\mathrm{Supp}(F) = \bigcup_{1 \le i \le k} W_i''$ has only finitely many components, let $c = \max\{c_i, 0 \le i \le k\}$. Then in both cases we have

$$\deg \xi''^* A'' \le c b_{\mathcal{N}}(g, m)$$

and $i'' : U \hookrightarrow W''$ is weakly bounded with respect to $\mathcal{O}_{X''}(A'')$.

Next we use the weak boundedness of $i' : U \hookrightarrow W''$ to show that $i' : U \hookrightarrow W'$ is weakly bounded. Because $A''$ is ample, there exist $d, r \in \mathbb{N}$ such that $dA'' \sim rg'^*A' + H''$, where $H''$ is an ample Cartier divisor. Let $\xi' := g' \circ \xi'' : C \to W'$. We have

$$\deg \xi'^*(rA') = \deg \xi''^*(rg'^*A') \le \deg \xi''^*(rg'^*A' + H'') = \deg \xi''^*(dA'') \le dc b_{\mathcal{N}}(g, m).$$

Thus

$$\deg \xi'^* A' \le \frac{dc}{r} b_{\mathcal{N}}(g, m),$$

and $i' : U \hookrightarrow W'$ is weakly bounded with respect to $\mathcal{O}_{X'}(A')$. □

**Lemma 5.4.** *Let $T$ be a quasiprojective variety. Then we can decompose $T$ into finitely many locally closed subsets, $T = \bigcup T_i$, such that each $T_i$ is weakly bounded.*

*Proof.* By the definition of weakly bounded, if a variety $U$ is weakly bounded, then any open subset $U^\circ \subset U$ is also weakly bounded. Therefore, we may replace $T$ with a stratification and assume that $T$ is smooth and projective; we only need to show that $T$ has a weakly bounded open subset.

Fix an integer $n \geq 2$, and let $A$ be a general very ample divisor on $\mathbb{P}^n_T$ such that $K_{\mathbb{P}^n_T/T} + A$ is also very ample. Then $\operatorname{Supp}(A)$ is smooth and dominates $T$ and, by the generic smoothness theorem, there is a normal open subset $T_1 \subset T$ such that $\operatorname{Supp}(A_{T_1})$ is smooth over $T_1$, where $A_{T_1} := A|_{T_1}$.

Since $K_{\mathbb{P}^n_T/T} + A$ is ample and $A$ is smooth, by the adjunction formula, $K_{A/T} = (K_{\mathbb{P}^n_T/T} + A)|_A$ is very ample, and we have that $A_{T_1} \to T_1$ is a family of canonically polarized smooth varieties. We may assume that $T_1$ is irreducible and every fiber of $A_{T_1} \to T_1$ has Hilbert polynomial $h(m) = \chi(\omega_{A_t}^{\otimes m})$.

Write $\mathcal{M}_h^\circ$ for the (Deligne–Mumford) stack of canonically polarized smooth varieties with Hilbert polynomial $h$ and $\boldsymbol{M}_h^\circ$ for its coarse moduli space. It is easy to see that $g$ maps $A_{T_1}$ to $T_1 \in \mathcal{M}_h^\circ(T_1)$. Let $\psi : T_1 \to \boldsymbol{M}_h^\circ$ be the induced moduli map.

By [Kovács and Patakfalvi 2017, Corollary 6.20], there is a diagram

$$\begin{array}{ccccc} A' & \xleftarrow{g} & A'' & \xrightarrow{h} & A_{T_1} \\ \downarrow & & \downarrow & & \downarrow \\ T' & \longleftarrow & T'' & \longrightarrow & T_1 \end{array}$$

with Cartesian squares such that

- $T'' \to T_1$ is finite surjective, and
- $A' \to T'$ is a family of canonically polarized smooth varieties for which the induced moduli map $\psi' : T' \to \boldsymbol{M}_h^\circ$ is finite.

Since the diagram is Cartesian,

$$K_{A''/T''} = g^* K_{A'/T'} = h^* K_{A_{T_1}/T_1}.$$

Because $h$ is finite and $K_{A_{T_1}/T_1}$ is ample, $K_{A''/T''}$ is ample and $T'' \to T'$ is quasifinite. It is easy to see that both

$$T'' \to T' \xrightarrow{\psi'} \boldsymbol{M}_h^\circ \quad \text{and} \quad T'' \to T_1 \xrightarrow{\psi} \boldsymbol{M}_h^\circ$$

give the moduli map $\psi'' : T'' \to \boldsymbol{M}_h^\circ$ induced by $A'' \to T''$; thus we have that $\psi : T_1 \to \boldsymbol{M}_h^\circ$ is quasifinite.

By [Kovács and Lieblich 2010, Lemma 6.2], the stack $\boldsymbol{M}_h^\circ$ is weakly bounded with respect to $\boldsymbol{M}_h$ and $\lambda \in \operatorname{Pic}(\boldsymbol{M}_h)$ by a function $b(g, d)$, where $\boldsymbol{M}_h$ is a compactification of $\boldsymbol{M}_h^\circ$ and $\lambda$ is an ample line bundle according to [Kovács and Patakfalvi 2017]. Let $\hat{T}_1$ be a compactification of $T_1$ such that $\psi : T_1 \to \boldsymbol{M}_h^\circ$ extends to a morphism $\hat{T}_1 \to \boldsymbol{M}_h$. Let $T_1^c$ be the Stein factorization of $\hat{T}_1 \to \boldsymbol{M}_h$ and denote the finite morphism by $\psi^c : T_1^c \to \boldsymbol{M}_h$.

Suppose $C^\circ \subseteq C$ is a $(g, d)$-curve. Let $C^\circ \to T_1$ be a morphism, and let $\xi : C \to T_1^c$ be its closure. Then $\psi^c \circ \xi : C \to \boldsymbol{M}_h$ is the closure of $C^\circ \to T_1 \xrightarrow{\psi} \boldsymbol{M}_h^\circ$. By the definition of weakly boundedness,

$$\deg(\psi^c \circ \xi)^* \lambda \leq b(g, d),$$

and hence $T_1$ is weakly bounded with respect to $T_1^c$ and $\psi^* \lambda$. Because $\psi$ is a finite morphism, $\psi^* \lambda$ is ample, and hence $T_1$ is weakly bounded. $\square$

**Theorem 5.5** [Kovács and Lieblich 2010, Proposition 2.14]. *Let $T$ be a quasicompact quasiseparated reduced $\mathbb{C}$-scheme, and let $\mathcal{U} \to T$ be a smooth morphism. Given a projective $T$-variety and a polarization over $T$, $(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(1))$, an open subvariety $\mathcal{M}^\circ \hookrightarrow \mathcal{M}$ over $T$, and a weak bound $b$, there exists a $T$-scheme of finite type $\mathcal{W}^b_{\mathcal{M}^\circ}$ and a morphism $\Theta : \mathcal{W}^b_{\mathcal{M}^\circ} \times \mathcal{U} \to \mathcal{M}^\circ$ such that, for every geometric point $t \in T$ and for every morphism $\xi : \mathcal{U}_t \to \mathcal{M}^\circ_t \subset \mathcal{M}_t$ that is weakly bounded by $b$, there exists a point $p \in \mathcal{W}^b_{\mathcal{M}^\circ_t}$ such that $\xi = \Theta|_{\{p\}\times \mathcal{U}_t}$.*

*In particular, if $\mathcal{M}^\circ$ is weakly bounded and $\mathcal{M}$ is the compactification, by definition, every morphism $\xi : \mathcal{U}_t \to \mathcal{M}^\circ_t \subset \mathcal{M}_t$ is weakly bounded by a weak bound $b$; hence $\xi = \Theta|_{\{p\}\times \mathcal{U}_t}$ for a closed point $p \in \mathcal{W}^b_{\mathcal{M}^\circ_t}$.*

## 6. Hilbert scheme and the moduli part

**6.1.** ***Parameter space.*** A class of polarized log Calabi–Yau pairs is a set $\mathscr{C}$ consisting of triples $(X, \Delta, H)$ such that

- $(X, \Delta)$ is a pair,
- $H$ is an effective ample divisor,
- $K_X + \Delta \sim_{\mathbb{Q}} 0$, and
- $(X, \Delta + \epsilon H)$ is lc for a positive number $\epsilon \ll 1$.

A family of polarized log Calabi–Yau pairs over a normal base scheme $S$ consists of a flat, proper morphism $f : X \to S$, a $\mathbb{Q}$-divisor $\Delta$ on $X$ and a $\mathbb{Q}$-Cartier divisor $H$ such that $K_{X/S} + \Delta$ is $\mathbb{Q}$-Cartier and all fibers $(X_s, \Delta_s, H_s)$ are polarized log Calabi–Yau pairs. We denote it by $(X, \Delta, H) \to S$.

Given a class of polarized log Calabi–Yau pairs $\mathscr{C}$, we define $\mathscr{MC}(S)$ to be the set of families of polarized log Calabi–Yau pairs over $S$, $(X, \Delta, H) \to S$, such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $(X_s, \Delta_s, H_s) \in \mathscr{C}$ for every closed point $s \in S$.

Suppose $\mathscr{C}$ is a class of $n$-dimensional polarized log Calabi–Yau pairs. We say $\mathscr{C}$ is bounded if the following two equivalent conditions hold:

- There exists a positive number $C$ and a positive integer $d$ such that, for every $(Y, D, H) \in \mathscr{C}$, $dH$ is very ample without higher cohomology, $(dH)^n \le C$, and $(dH)^{n-1}.\operatorname{red}(D) \le C$.
- There is a flat projective morphism $\mathcal{Z} \to S$ over a scheme of finite type, two divisors $\mathcal{P}, \mathcal{L}$ on $\mathcal{Z}$ which are flat over $S$, and a positive integer $d$ such that, for every $(Y, D, H) \in \mathscr{C}$, there is a closed point $s \in S$ and an isomorphism $\phi : (Y, dH) \to (\mathcal{Z}_s, \mathcal{L}_s)$ such that $\phi_* D \le \mathcal{P}_s$.

If the first condition holds, then there is a (nonunique) natural choice of the scheme $S$ in the second condition. By boundedness of the Chow variety, see [Kollár 1996, §1.3], we may assume that $Y$ has a fixed Hilbert polynomial $H(t)$ with respect to $dH$. Let $\mathbb{P}$ be the projective space of dimension $H(1)-1$ with a fixed coordinate system. By the proof of [Kovács and Patakfalvi 2017, Proposition 6.11], because normality is an open condition, we may choose $\mathcal{H}'$ to be the locally closed subset of the Hilbert scheme of $\mathbb{P}$ which parametrizes all irreducible normal subvarieties of $\mathbb{P}$ with Hilbert polynomial $H(t)$, and we let $\mathscr{F} : \mathcal{X}_{\mathcal{H}'} \to \mathcal{H}'$ be the universal family.

Let $\Lambda$ be a finite set, and let $p_i(t),\ i \in \Lambda$, be $|\Lambda|$ polynomials such that $\deg p_i(t) = \deg H(t) - 1$ for every $i$. Let

$$\mathcal{H}_i := \mathrm{Hilb}_{p_i(t)}(\mathcal{X}_{\mathcal{H}'}/\mathcal{H}')$$

be the locally closed subset of the relative Hilbert scheme which parametrizes closed pure dimensional subschemes $D_i \subset \mathcal{X}_{\mathcal{H}'}$ such that $D_i \to \mathcal{H}'$ is a flat family of varieties with Hilbert polynomial $p_i(t)$. Let $\mathcal{D}_i \to \mathcal{H}_i$ be its universal family. For simplicity of notation, we define $\mathcal{H} := \mathcal{H}_1 \times_{\mathcal{H}'} \cdots \times_{\mathcal{H}'} \mathcal{H}_{|\Lambda|}$ and

$$(\mathcal{X}_{\mathcal{H}}, \mathcal{D}_{\mathcal{H}}) := \left(\mathcal{X}_{\mathcal{H}'} \times_{\mathcal{H}'} \mathcal{H}, \sum \mathcal{D}_i \times_{\mathcal{H}_i} \mathcal{H}\right).$$

**Remark 6.1.** Let $\mathscr{C}$ be a bounded class of polarized log Calabi–Yau pairs. With the same notation as above, let $(X, \Delta)$ be a klt pair and $L$ be a divisor on $X$, and suppose a general fiber of a contraction $f : (X, \Delta, L) \to Z$ is in $\mathscr{C}$; that is, there is an open subset $U \subset Z$ such that, for every closed point $u \in U$, $(X_u, \Delta|_{X_u}, L|_{X_u}) \in \mathscr{C}$.

Write $\Delta = \sum \Delta_i$ as the sum of irreducible components and define $\Delta_{i,u} := \Delta_i|_{X_u},\ \Delta_i := \Delta_i|_X$, for a closed point $u \in U$. Because the degree of $\mathrm{Supp}(\Delta_{i,u})$ is bounded from above, by boundedness of the Chow varieties, the Hilbert polynomial of $\Delta_{i,u}$ is in a finite set $\{p_i(t), i \in \Lambda\}$; see [Kollár 1996, §1.3]. Let $(\mathcal{X}_{\mathcal{H}}, \mathcal{D}_{\mathcal{H}}) \to \mathcal{H}$ be the family constructed as above. By the construction of $\mathcal{H}$, every closed point $u \in U$ corresponds to a closed point in $\mathcal{H}$, and there is a morphism $U \to \mathcal{H}$.

Notice that $\Delta_{i,u}$ may not be irreducible for every $u \in U$, and two irreducible components of $\Delta_u$ may be considered as two divisors or just one divisor, depending on the divisor $\Delta_i$ on $X$. That means, given two contractions $(X^i, \Delta^i) \to Z^i,\ i = 1, 2$, satisfying the given conditions, even if $(X^1_{u_1}, \Delta^1_{u_1}) \cong (X^2_{u_2}, \Delta^2_{u_2})$, $u_1$ and $u_2$ may corresponds to different points in $\mathcal{H}$.

Since $dL$ is very ample without higher cohomology and $f : X \to Z$ is flat over $U$, we have that $f_*\mathcal{O}_X(dL)$ is locally free over $U$. Replacing $U$ with an open subset, we may assume that $f_*\mathcal{O}_X(dL)$ is in fact free. Fixing a basis in the space of sections then gives a map $U \to \mathcal{H}'$, and $X_U \to U$ is isomorphic to the pullback of the universal family $\mathcal{X}_{\mathcal{H}'} \to \mathcal{H}'$. Similarly, each irreducible component $\Delta_i$ of $\Delta$ gives a map $U \to \mathcal{H}_i$. Hence there is a morphism $\phi : U \to \mathcal{H}$ such that $f : (X_U, \mathrm{Supp}(\Delta_U)) \to U$ is isomorphic to the pullback of $(\mathcal{X}_{\mathcal{H}}, \mathcal{D}_{\mathcal{H}}) \to \mathcal{H}$ by $\phi$.

Suppose $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a vector of rational numbers and

$$\Delta_U = \alpha\, \mathrm{Supp}(\Delta_U) := \sum \alpha_i\, \mathrm{Supp}(\Delta_{i,U}).$$

By the construction of $\mathcal{D}_{\mathcal{H}}$, $(X_U, \Delta_U)$ is isomorphic to the pullback of $(\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}}) \to \mathcal{H}$ by $\phi$. If there is a point $u \in U$ such that $(X_u, \Delta_u)$ is a log Calabi–Yau pair, then $(\mathcal{X}_{\phi(u)}, \alpha\mathcal{D}_{\phi(u)})$ is a log Calabi–Yau pair. If $\mathrm{coeff}(\Delta) \subset \mathcal{I}$ is a DCC set, then, by [Hacon et al. 2014, Theorem 1.5], $\alpha\mathcal{D}_{\mathcal{H}}$ is in a finite set and there are only finitely many $\alpha\mathcal{D}_{\mathcal{H}}$.

Moreover, by Lemma 7.4 in the first arXiv version of [Birkar 2023], after replacing $\mathcal{H}$ by a stratification of a locally closed subvariety, we may assume that $\mathcal{H}$ is smooth and $(\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}})$ is klt log Calabi–Yau over $\mathcal{H}$, and hence $(\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}}) \to \mathcal{H}$ is an lc-trivial fibration.

**6.2.** ***Moduli part.*** In this section, we deal with algebraic fibrations whose general fibers are log Calabi–Yau pairs. We claim that such a contraction naturally induces an lc-trivial fibration, and then any such fibration has a moduli $\boldsymbol{b}$-divisor by the canonical bundle formula.

**Theorem 6.2.** *Let $(X, \Delta)$ be a projective lc pair, and let $f : (X, \Delta) \to Z$ be a contraction to a projective normal $\mathbb{Q}$-factorial variety. Suppose that a general fiber $(X_g, \Delta_g)$ is a log Calabi–Yau pair. Assume there is a subpair $(X', \Delta')$, a crepant birational morphism $g : (X', \Delta') \to (X, \Delta)$ and a divisor $D$ on $Z$ such that the morphism $h := f \circ g : X' \to Z$ is smooth over $Z \setminus D$ and* $\mathrm{Supp}(\Delta')$ *is simple normal crossing over $Z \setminus D$.*

*Then there is a $\mathbb{Q}$-divisor $\Lambda'$ on $X'$ such that*

- $(X'_\eta, \Lambda'_{X_\eta}) \cong (X'_\eta, \Delta'_{X_\eta})$, *where $\eta$ is the generic point of $Z$,*
- $\mathrm{Supp}(\Lambda')$ *is log smooth over $Z \setminus D$, and*
- $(X', \Lambda') \to Z$ *is an lc-trivial fibration.*

*Proof.* Since $(X_g, \Delta_g)$ is a log Calabi–Yau pair, we have $K_{X'_\eta} + \Delta'_\eta \sim_{\mathbb{Q}} 0$, and hence there exists a vertical $\mathbb{Q}$-divisor $B'$ such that $K_{X'} + \Delta' + B' \sim_{\mathbb{Q}} 0$.

Suppose $B' = R + G$, where $\mathrm{Supp}(R) \not\subset h^{-1}(\mathrm{Supp}(D))$ and $\mathrm{Supp}(G) \subset h^{-1}(\mathrm{Supp}(D))$. Because $R$ is vertical, $h$ is smooth over the generic point of $h(\mathrm{Supp}(R))$ and $Z$ is $\mathbb{Q}$-factorial, $h(R)$ is a well-defined $\mathbb{Q}$-Cartier divisor on $Z$; denote it by $R_Z$. Also because $h$ is smooth over $Z \setminus \mathrm{Supp}(D)$, there exists a $\mathbb{Q}$-divisor $F_R$ supported on $h^{-1}(\mathrm{Supp}(D))$ such that $R + F_R = h^* R_Z$. Hence

$$K_{X'} + \Delta' + B' - (R + F_R) \sim_{\mathbb{Q},h} 0.$$

Let $\Lambda' := \Delta' + B' - (R + F_R)$; then $K_{X'} + \Lambda' \sim_{\mathbb{Q},h} 0$ and $\Lambda'_\eta = \Delta'_\eta$. Write $\Delta' = \Delta'_{\geq 0} - \Delta'_{\leq 0}$. Because $\Delta'_{\leq 0}$ is $g$-exceptional, it is easy to see that $(X', \Lambda') \to Z$ is an lc-trivial fibration. Because $\mathrm{Supp}(\Delta')$ is log smooth over $Z \setminus D$, $\mathrm{Supp}(F_R) \subset h^{-1}(D)$ and $\mathrm{Supp}(B' - R) \subset h^{-1}(D)$, we have that $\mathrm{Supp}(\Lambda')$ is log smooth over $Z \setminus D$. ☐

**Proposition 6.3.** *Let $f : (X, \Delta) \to Z$ be an lc-trivial fibration between normal projective varieties, $\rho : Z' \to Z$ a surjective morphism from a projective normal variety $Z'$ and $f' : (X', \Delta') \to Z'$ the lc-trivial fibration induced by the normalization of the main component of the base change.*

$$\begin{array}{ccc} (X, \Delta) & \xleftarrow{\ \rho_X\ } & (X', \Delta') \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ Z & \xleftarrow{\ \rho\ } & Z' \end{array}$$

*Let $\boldsymbol{M}$ and $\boldsymbol{M}'$ be the moduli $\boldsymbol{b}$-divisors of $f$ and $f'$. Then the following hold:*

(1) *If $\boldsymbol{M}$ descends on $Z$ and $\boldsymbol{M}'$ descends on $Z'$, then $\rho^* \boldsymbol{M}_Z = \boldsymbol{M}'_{Z'}$.*

(2) *If $\rho$ is finite, then $\rho^* \boldsymbol{M}_Z = \boldsymbol{M}'_{Z'}$. In particular, $\boldsymbol{M}$ descends on $Z$ if and only if $\boldsymbol{M}'$ descends on $Z'$.*

*Proof.* Result (1) is [Ambro 2005, Proposition 3.1].

For (2), let $g' : W' \to Z'$ and $g : W \to Z$ be birational morphisms such that $\boldsymbol{M}'$ descends on $W'$ and $\boldsymbol{M}$ descends on $W$ and $\rho : Z' \to Z$ lifts to a morphism $\rho_W : W' \to W$. Then $\rho_W^* \boldsymbol{M}_W = \boldsymbol{M}'_{W'}$ by (1). Because $\rho$ is finite and any $g$-exceptional divisor is only dominated by $g'$-exceptional divisors, the pushforward of $\rho_W^* \boldsymbol{M}_W = \boldsymbol{M}'_{W'}$ to $Z'$ gives $\rho^* \boldsymbol{M}_Z = \boldsymbol{M}'_{Z'}$. □

**Theorem 6.4.** *Let $(X, \Delta)$ be an lc pair, and let $f : (X, \Delta) \to Z$ be an lc-trivial fibration to a smooth projective variety $Z$. Suppose $X' \to X$ is a log resolution of $(X, \Delta)$ and $(X', \Delta')$ is a subpair such that $g : (X', \Delta') \to (X, \Delta)$ is a crepant birational morphism. Suppose $D \subset Z$ is a smooth divisor on $Z$ such that $(X', \operatorname{Supp}(\Delta'))$ is log smooth over the generic point $\eta_D$ of $D$. Let $Y$ be the normalization of the irreducible component of $f^{-1}(D)$ that dominates $D$, and let $\Delta_Y$ be the $\mathbb{Q}$-divisor on $Y$ such that*

$$K_Y + \Delta_Y = (K_X + \Delta + f^* D)|_Y.$$

*Let $\boldsymbol{M}_Z$ denote the moduli part of $(X, \Delta) \to Z$. Suppose there is a smooth divisor $B$ on $Z$ such that $B + D$ is a reduced simple normal crossing divisor and the morphism $h : X' \to Z$ and $\Delta'$, $B$ satisfy the standard normal crossing assumptions. Then $(Y, \Delta_Y) \to D$ is an lc-trivial fibration and its moduli **b**-divisor $\boldsymbol{N}$ is equal to the restriction of $\boldsymbol{M}$ up to $\mathbb{Q}$-linear equivalence.*

*Proof.* By assumption, $h$ is smooth over $Z \setminus B$, $D$ is smooth and the singular locus of $h^{-1}(D)$ is contained in $h^{-1}(B) \cap h^{-1}(D)$. After blowing up a sequence of smooth subvarieties whose centers are contained in the singular locus of $h^{-1}(D)$, we may assume that $(X', \operatorname{Supp}(\Delta' + h^*(B + D)))$ is log smooth. It is easy to see that the morphism $h : X' \to Z$ and $\Delta'$, $B$ also satisfy the standard normal crossing assumption.

Let $E'$ be the irreducible component of $h^* D$ that dominates $D$, and let $\Delta'_{E'}$ be the $\mathbb{Q}$-divisor on $E'$ such that

$$K_{E'} + \Delta'_{E'} = (K_{X'} + \Delta' + h^* D)|_{E'}.$$

It is easy to see that the generic fiber of $(E', \Delta'_{E'}) \to D$ is crepant birationally equivalent to the generic fiber of $(Y, \Delta_Y) \to D$, which means the two lc-trivial fibrations have the same moduli part. Then we only need to prove the result for $(E', \Delta'_{E'}) \to Z$.

By the canonical bundle formula, there is a divisor $B_Z$ supported on $B$ such that

$$K_{X'} + \Delta' + h^* D \sim_{\mathbb{Q}} h^*(K_Z + B_Z + \boldsymbol{M}_Z + D) \tag{6-1}$$

and

$$K_X + \Delta + f^* D \sim_{\mathbb{Q}} f^*(K_Z + B_Z + \boldsymbol{M}_Z + D). \tag{6-2}$$

Because $B + D$ is reduced and $(Z, B + D)$ is log smooth, $(Z, B + D)$ is an lc pair. By the canonical bundle formula,

$$K_{X'} + \Delta' + h^* D + h^*(B - B_Z) \sim_{\mathbb{Q}} h^*(K_Z + B + D + \boldsymbol{M}_Z).$$

Because $h : X' \to Z$ and $\Delta'$, $B$ satisfy the standard normal crossing assumptions, the moduli part $\boldsymbol{M}$ descends on $Z$ and $(Z, B + D + \boldsymbol{M}_Z)$ is generalized lc. Thus, by [Ambro 2004, Theorem 3.1],

$$(X', \Delta' + h^* D + h^*(B - B_Z))$$

is sub-lc. Because $Z$ is smooth and $B-B_Z$ is effective and $\mathbb{Q}$-Cartier, after replacing $\Delta'$ by $\Delta'+h^*(B-B_Z)$, $\Delta$ by $\Delta+f^*(B-B_Z)$ and $B_Z$ by $B_Z+(B-B_Z)=B$, we can assume that $B=B_Z$ and every irreducible component of $B$ is dominated by an irreducible component of $\Delta'$ which has coefficient 1, and $(X, \Delta)$ is still a pair. Since $K_{X'}+\Delta'+h^*D=g^*(K_X+\Delta+f^*D)$, we have that $(X, \Delta+f^*D)$ is lc.

Let $g(E')=E$, and suppose $h^*D=E'+E_1'$ and $f^*D=E+E_1$. Recall that $Y$ is the normalization of $E$. Restricting (6-1) to $E'$ and (6-2) to $E$, by the adjunction formula, there are a $\mathbb{Q}$-divisor $\Delta'_{E'}$ and an effective $\mathbb{Q}$-divisor $\Delta_Y$ such that

$$\begin{aligned}(K_{X'}+\Delta'+h^*D)|_{E'} &\sim_{\mathbb{Q}} K_{E'}+\Delta'_{E'}\\ &\sim_{\mathbb{Q}} h^*_{E'}(K_D+B|_D+\boldsymbol{M}_Z|_D),\\ (K_X+\Delta+f^*D)|_Y &\sim_{\mathbb{Q}} K_Y+\Delta_Y\\ &\sim_{\mathbb{Q}} f^*_E(K_D+B|_D+\boldsymbol{M}_Z|_D).\end{aligned}$$

It follows that $\Delta'_{E'}=\Delta'|_{E'}+E_1'|_{E'}$, $(E', \Delta'_{E'})$ is sub-lc, $\Delta_Y$ is effective and $K_{E'}+\Delta'_{E'}\sim_{\mathbb{Q}} g^*_{E'}(K_Y+\Delta_Y)$, where $g_{E'}: E'\to Y$ is the birational morphism induced by $g|_{E'}: E'\to E$. It follows that $\Delta'_{E',\le 0}$ is $g_{E'}$-exceptional, and hence $(E', \Delta'_{E'})\to D$ is an lc-trivial fibration.

$$\begin{array}{ccc} (E', \Delta'_{E'}) & \hookrightarrow & (X', \Delta') \\ \downarrow{\scriptstyle g'_{E'}} & & {\scriptstyle g}\downarrow \\ (Y, \Delta_Y) & \hookrightarrow & (X, \Delta) \\ \downarrow{\scriptstyle f_E} & & {\scriptstyle f}\downarrow \\ D & \hookrightarrow & Z \end{array} \qquad h: X'\to Z$$

By the canonical bundle formula for $(E', \Delta'_{E'})\to D$, we have

$$K_{E'}+\Delta'_{E'}\sim_{\mathbb{Q}} h^*_{E'}(K_D+B_D+\boldsymbol{N}_D). \tag{6-3}$$

To prove $\boldsymbol{N}_D\sim_{\mathbb{Q}}\boldsymbol{M}_Z|_D$, we only need to prove that $B_D=B|_D$.

Since the morphism $X'\to Z$ and $\Delta'$, $B$ satisfy the standard normal crossing assumption, $\boldsymbol{M}$ descends on $Z$. Similarly, because $B+D$ is snc, $(D, \operatorname{Supp}(B|_D))$ is log smooth and $(E', \operatorname{Supp}(\Delta'_{E'}))$ is log smooth over $D\setminus B\cap D$, we have that $\boldsymbol{N}_D$ descends on $D$. For the same reason, the morphism $E'\to D$ and $\Delta'_{E'}$, $B|_D$ satisfy the standard normal crossing assumption. By the construction of the boundary divisor, $B_D$ is the unique smallest $\mathbb{Q}$-divisor supported on $B|_D$ such that

$$\Delta'_{E',v}+h^*_{E'}(B|_D-B_D)\le \operatorname{red}(h^*_{E'}(B|_D)),$$

where $\Delta'_{E',v}$ is the vertical part of $\Delta'_{E'}$. Because every irreducible component of $B$ is dominated by an irreducible component of $\Delta'$ which has coefficient 1 and every irreducible component of $B|_D$ is dominated by an irreducible component of $\Delta'_{E'}=\Delta'|_{E'}+E_1'|_{E'}$ which has coefficient 1, we have $B|_D=B_D$ and the result follows. ∎

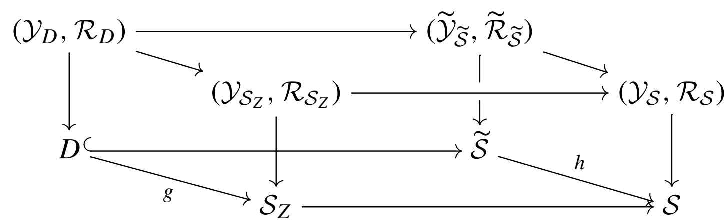
**Theorem 6.5.** *Let $\mathcal{S}$ be a normal projective variety and $\mathfrak{F} : (\mathcal{Y}_{\mathcal{S}}, \mathcal{R}_{\mathcal{S}}) \to \mathcal{S}$ be an lc-trivial fibration such that the corresponding moduli* ***b****-divisor $\boldsymbol{\mathcal{M}}$ descends on $\mathcal{S}$. Suppose there exists an open subset $\mathcal{H} \hookrightarrow \mathcal{S}$ such that $(\mathcal{Y}_{\mathcal{S}}, \mathrm{Supp}(\mathcal{R}_{\mathcal{S}}))$ is log smooth over $\mathcal{H}$. Let $Z$ be a projective normal variety and $\phi : Z \to \mathcal{S}$ be a morphism which maps the generic point of $Z$ into $\mathcal{H}$. Assume $(X, \Delta) \to Z$ is an lc-trivial fibration whose generic fiber is crepant birationally equivalent to the generic fiber of $(\mathcal{Y}_{\mathcal{S}}, \mathcal{R}_{\mathcal{S}}) \times_{\mathcal{S}} Z \to Z$. Let $\boldsymbol{M}$ be the moduli* ***b****-divisor of $f$. If $\boldsymbol{M}$ descends on $Z$, then we have*

$$\boldsymbol{M}_Z = \phi^* \boldsymbol{\mathcal{M}}_{\mathcal{S}}.$$

*Proof.* Let $(\mathcal{Y}_{\phi(Z)}, \mathcal{R}_{\phi(Z)}) \to \phi(Z)$ be the contraction induced by the restriction of $(\mathcal{Y}_{\mathcal{S}}, \mathcal{R}_{\mathcal{S}}) \to \mathcal{S}$ on $\phi(Z)$. Because $(\mathcal{Y}_{\mathcal{S}}, \mathrm{Supp}(\mathcal{R}_{\mathcal{S}}))$ is log smooth over $\mathcal{H}$ and the generic point of $\phi(Z)$ is in $\mathcal{H}$, we have that $(\mathcal{Y}_{\phi(Z)}, \mathcal{R}_{\phi(Z)}) \to \phi(Z)$ is an lc-trivial fibration over an open subset of $\phi(Z)$. We denote the corresponding moduli ***b***-divisor by $\boldsymbol{N}$. Let $\mathcal{S}_Z \to \phi(Z)$ be a birational morphism such that $\boldsymbol{N}$ descends on $\mathcal{S}_Z$. We have the following two cases:

<u>Case 1</u>: $\mathcal{S}_Z = \mathcal{S}$. Because $\phi$ is surjective, $\boldsymbol{M}$ descends on $Z$ and $\boldsymbol{\mathcal{M}}$ descends on $\mathcal{S}$, by Proposition 6.3, we have $\phi^* \boldsymbol{\mathcal{M}}_{\mathcal{S}} \sim_{\mathbb{Q}} \boldsymbol{M}_Z$.

<u>Case 2</u>: $\mathcal{S}_Z$ is a subvariety of $\mathcal{S}$ of codimension $\geq 1$. Consider the diagram




where

- $\widetilde{\mathcal{S}} \to \mathcal{S}$ is a log resolution of $(\mathcal{S}, \mathcal{S} \setminus \mathcal{H})$,
- $D$ is a divisor on $\widetilde{\mathcal{S}}$ that dominates $\mathcal{S}_Z$,
- $(\widetilde{\mathcal{S}}, D + h^{-1}(\mathcal{S} \setminus \mathcal{H}))$ is log smooth, and
- $(\mathcal{Y}_{\mathcal{S}_Z}, \mathcal{R}_{\mathcal{S}_Z}) \to \mathcal{S}_Z$, $(\mathcal{Y}_D, \mathcal{R}_D) \to D$ and $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{R}}) \to \widetilde{\mathcal{S}}$ are induced by the pullback of $(\mathcal{Y}, \mathcal{R}) \to \mathcal{S}$.

It is easy to see that $(\widetilde{\mathcal{Y}}, \mathrm{Supp}(\widetilde{\mathcal{R}})) \to \widetilde{\mathcal{S}}$ is log smooth over $\widetilde{\mathcal{S}} \setminus h^{-1}(\mathcal{S} \setminus \mathcal{H})$.

After replacing $Z$ by a higher birational model and $(X, \Delta) \to Z$ by the corresponding pullback, we may assume that $Z \to \mathcal{S}_Z$ is surjective. Because the generic fiber of $(X, \Delta) \to Z$ is crepant birationally equivalent to the generic fiber of the pullback of $(\mathcal{Y}_{\mathcal{S}}, \mathcal{R}_{\mathcal{S}}) \to \mathcal{S}$ via $\phi$, it is also crepant birationally equivalent to the generic fiber of the pullback of $(\mathcal{Y}_{\mathcal{S}_Z}, \mathcal{R}_{\mathcal{S}_Z}) \to \mathcal{S}_Z$. Then, by Proposition 6.3, we have $\boldsymbol{M}_Z = \phi^* \boldsymbol{N}_{\mathcal{S}_Z}$.

Because the generic point of $\phi(Z)$ is in $\mathcal{H}$, we have $D \not\subset h^{-1}(\mathcal{S} \setminus \mathcal{H})$. By Theorem 6.4, the induced morphism $(\mathcal{Y}_D, \mathcal{R}_D) \to D$ is an lc-trivial fibration, and the corresponding moduli divisor $\boldsymbol{M}_D$ is equal to $\boldsymbol{\mathcal{M}}_{\widetilde{\mathcal{S}}}|_D = (h^* \boldsymbol{\mathcal{M}}_{\mathcal{S}})|_D$. By Proposition 6.3, $\boldsymbol{M}_D = g^* \boldsymbol{M}_{\mathcal{S}_Z}$, and hence $\boldsymbol{M}_Z = \phi^* \boldsymbol{M}_{\mathcal{S}_Z} = \phi^* \boldsymbol{\mathcal{M}}_{\mathcal{S}}$. □

**Theorem 6.6** [Ambro 2005, Theorem 3.3]. *Let $f:(X,\Delta)\to S$ be an lc-trivial fibration over a variety $S$ such that the geometric generic fiber $X_{\bar\eta}$ is a projective variety and $\Delta_{\bar\eta}$ is effective. Then there exists a diagram*

$$\begin{array}{ccccccc} (X,\Delta) & & & & (X^!,\Delta^!) & & \\ {\scriptstyle f}\big\downarrow & & & & \big\downarrow{\scriptstyle f^!} & & \\ S & \xleftarrow{\ \tau\ } & \bar S & \xrightarrow{\ \rho\ } & S^! & \xrightarrow{\ \pi\ } & S^* \end{array} \qquad S \overset{i}{\dashleftarrow} S^!,\quad S \overset{\Phi}{\dashrightarrow} S^*$$

*such that*

1. $f^!:(X^!,\Delta^!)\to S^!$ *is an lc-trivial fibration,*
2. $\tau$ *and* $\pi$ *are generically finite and surjective morphisms and* $\rho$ *is surjective,*
3. *there exists a nonempty open subset $U\subset\bar S$ and an isomorphism*

$$\begin{array}{ccc} (X,\Delta)\times_S U & \xrightarrow{\ \cong\ } & (X^!,\Delta^!)\times_{S^!} U \\ & \searrow \quad U \quad \swarrow & \end{array}$$

4. $\Phi: S \dashrightarrow S^*$ *is an extension of the period map defined in* [Ambro 2005, Section 2], *and*
5. $i: S^! \dashrightarrow S$ *is a rational map such that the generic fiber of $f^!$ is equal to the pullback of $f$ via $i$.*

*Furthermore, if $S$ is proper, then one can choose $\bar S$, $S^!$ and $S^*$ to be proper. Let $\boldsymbol{M}$ and $\boldsymbol{M}^!$ be the corresponding moduli $\boldsymbol{b}$-divisors of $f$ and $f^!$. Then we have*

6. $\boldsymbol{M}^!$ *is $\boldsymbol{b}$-nef and big, and*
7. *if $\boldsymbol{M}$ descends on $S$ and $\boldsymbol{M}^!$ descends on $S^!$, then $\tau^*\boldsymbol{M}_S=\rho^*\boldsymbol{M}^!_{S^!}$.*

Although it is not written in [Ambro 2005], (4) and (5) are implied by its proof.

**Theorem 6.7.** *Let $(\mathcal{X}_{\mathcal{H}},\alpha\mathcal{D}_{\mathcal{H}})\to\mathcal{H}$ be the lc-trivial fibration defined in Remark 6.1. Then, after passing to a stratification of $\mathcal{H}$ and replacing $(\mathcal{X}_{\mathcal{H}},\alpha\mathcal{D}_{\mathcal{H}})\to\mathcal{H}$ by the corresponding pullback, we have the diagram*

$$\begin{array}{ccccccc} (\mathcal{X}_{\mathcal{H}},\alpha\mathcal{D}_{\mathcal{H}}) & & & & (\mathcal{X}^!_{\mathcal{H}^!},\alpha\mathcal{D}^!_{\mathcal{H}^!}) & & \\ {\scriptstyle \mathcal{F}}\big\downarrow & & & & \big\downarrow{\scriptstyle \mathcal{F}^!} & & \\ \mathcal{H} & \xleftarrow{\ \tau\ } & \overline{\mathcal{H}} & \xrightarrow{\ \rho\ } & \mathcal{H}^! & \xrightarrow{\ \pi\ } & \mathcal{H}^* \end{array} \qquad \mathcal{H} \overset{i}{\dashleftarrow} \mathcal{H}^!,\quad \mathcal{H} \overset{\Phi}{\dashrightarrow} \mathcal{H}^*$$

*where*

- $\tau$ *is finite,*
- $\pi$ *is étale,*
- $\Phi$ *is a morphism on* $\mathcal{H}$,
- $\mathcal{H}^*$ *is weakly bounded and smooth,*
- $(\mathcal{X}_{\mathcal{H}},\mathcal{D}_{\mathcal{H}})\times_{\mathcal{H}}\overline{\mathcal{H}}\cong(\mathcal{X}^!_{\mathcal{H}^!},\mathcal{D}^!_{\mathcal{H}^!})\times_{\mathcal{H}^!}\overline{\mathcal{H}}$, *and*
- $(\mathcal{X}_{\mathcal{H}},\mathcal{D}_{\mathcal{H}})\to\mathcal{H}$ *and* $(\mathcal{X}^!_{\mathcal{H}^!},\mathcal{D}^!_{\mathcal{H}^!})\to\mathcal{H}^!$ *have fiberwise log resolutions.*

*Furthermore, there exist a smooth compactification* $\mathcal{H}^* \hookrightarrow \mathcal{S}^*$ *and a positive integer* $l$ *such that if* $f : (X, \Delta) \to Z$ *is an lc trivial fibration, where*

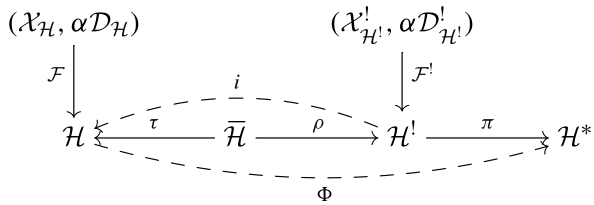
- $Z$ *is smooth and projective,*
- *there is a rational map* $\phi : Z \dashrightarrow \mathcal{H}$, *and*
- *the generic fiber of* $f$ *is isomorphic to the generic fiber of the pullback of* $(\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}}) \to \mathcal{H}$ *by* $\phi$,

*then there exists a* $\boldsymbol{b}$*-divisor* $\boldsymbol{M}^{\mathrm{fix}}$ *on birational models of* $Z$ *such that*

- $\boldsymbol{M}^{\mathrm{fix}}$ *is effective,*
- $\boldsymbol{M}^{\mathrm{fix}}_{Z'} \sim_{\mathbb{Q}} \boldsymbol{M}_{Z'}$ *for every birational map* $Z' \to Z$,
- $l\boldsymbol{M}^{\mathrm{fix}}$ *is* $\boldsymbol{b}$*-Cartier, and*
- *if* $\Phi \circ \phi$ *extends to a morphism* $Z \to \mathcal{S}^*$, *then* $\mathrm{Supp}(\boldsymbol{M}^{\mathrm{fix}}_Z) \supset Z \setminus U$, *where* $U = (\Phi \circ \phi)^{-1}\mathcal{H}^*$.

*Proof.* Step 1: We construct the stratification of $\mathcal{H}$, define $\overline{\mathcal{H}}$ and $\mathcal{H}^*$, define the lc-trivial fibration $\mathcal{F}^! : (\mathcal{X}^!_{\mathcal{H}^!}, \alpha\mathcal{D}^!_{\mathcal{H}^!}) \to \mathcal{H}^!$ and construct the diagram satisfying the requirements.

Because $\alpha\mathcal{D}_{\mathcal{H}}$ is effective, by Theorem 6.6, we have the following diagram:




We replace $\mathcal{H}$ by an open subset such that

- $\mathcal{F}$ has a fiberwise log resolution.

Then we replace $\mathcal{H}^*$ by an open subset, and $\mathcal{H}^!$ and $\overline{\mathcal{H}}$ by the corresponding preimages such that

- $\pi$ is étale,
- $\mathcal{H}^*$ is weakly bounded and smooth, and
- $(\mathcal{X}^!_{\mathcal{H}^!}, \mathcal{D}^!_{\mathcal{H}^!}) \to \mathcal{H}^!$ has a fiberwise log resolution.

Next we replace $\mathcal{H}$ by an open subset and $\overline{\mathcal{H}}$ by the corresponding preimage such that

- $\tau$ is finite,
- $\Phi$ is a morphism on $\mathcal{H}$,
- $(\mathcal{X}_{\mathcal{H}}, \mathcal{D}_{\mathcal{H}}) \times_{\mathcal{H}} \overline{\mathcal{H}} \cong (\mathcal{X}^!_{\mathcal{H}^!}, \mathcal{D}^!_{\mathcal{H}^!}) \times_{\mathcal{H}^!} \overline{\mathcal{H}}$, and
- $(\mathcal{X}_{\mathcal{H}}, \mathcal{D}_{\mathcal{H}}) \to \mathcal{H}$ has a fiberwise log resolution.

Then we repeat this construction with the complement of $\mathcal{H}$. By Noetherian induction, we have a stratification of $\mathcal{H}$ satisfying the properties.

Step 2: We construct smooth compactifications $\mathcal{H} \hookrightarrow \mathcal{S}$, $\mathcal{H}^! \hookrightarrow \mathcal{S}^!$, and $\mathcal{H}^* \hookrightarrow \mathcal{S}^*$, a $\mathbb{Q}$-divisor $\boldsymbol{\mathcal{M}}^{\mathrm{fix}}_{\mathcal{S}}$ on $\mathcal{S}$, and a $\mathbb{Q}$-divisor $\boldsymbol{\mathcal{M}}^{!\mathrm{fix}}_{\mathcal{S}^!}$ on $\mathcal{S}^!$ such that

- $\boldsymbol{\mathcal{M}}^{\mathrm{fix}}_{\mathcal{S}} \sim_{\mathbb{Q}} \boldsymbol{\mathcal{M}}_{\mathcal{S}}$, where $\boldsymbol{\mathcal{M}}$ is the moduli $\boldsymbol{b}$-divisor of $\mathcal{F}$,
- $\boldsymbol{\mathcal{M}}^{!\mathrm{fix}}_{\mathcal{S}^!} \sim_{\mathbb{Q}} \boldsymbol{\mathcal{M}}^!_{\mathcal{S}^!}$, where $\boldsymbol{\mathcal{M}}^!$ is the moduli $\boldsymbol{b}$-divisor of $\mathcal{F}^!$,
- $\mathrm{Supp}(\boldsymbol{\mathcal{M}}^{!\mathrm{fix}}_{\mathcal{S}^!}) \supset \pi^{-1}(\mathcal{S}^* \setminus \mathcal{H}^*)$, and
- $\tau(\mathrm{Supp}(\rho^* \boldsymbol{\mathcal{M}}^{!\mathrm{fix}}_{\mathcal{S}^!})) \subset \mathrm{Supp}(\boldsymbol{\mathcal{M}}^{\mathrm{fix}}_{\mathcal{S}})$.

Let $(\mathcal{Y}_{\mathcal{H}}, \mathcal{R}_{\mathcal{H}}) \to (\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}})$ and $(\mathcal{Y}^!_{\mathcal{H}^!}, \mathcal{R}^!_{\mathcal{H}^!}) \to (\mathcal{X}^!_{\mathcal{H}^!}, \alpha\mathcal{D}^!_{\mathcal{H}^!})$ be crepant birational morphisms which are fiberwise log resolutions of $\mathcal{F}$ and $\mathcal{F}^!$. After taking smooth compactifications of the bases $\mathcal{H}$, $\bar{\mathcal{H}}$, $\mathcal{H}^!$ and $\mathcal{H}^*$, and choosing extensions of the fibrations, we have the following diagram:

$$\begin{array}{ccccccc} (\mathcal{Y}_{\mathcal{S}}, \mathcal{R}_{\mathcal{S}}) & & & & (\mathcal{Y}^!_{\mathcal{S}^!}, \mathcal{R}^!_{\mathcal{S}^!}) & & \\ \downarrow{\scriptstyle \mathcal{F}} & & & & \downarrow{\scriptstyle \mathcal{F}^!} & & \\ \mathcal{S} & \xleftarrow{\ \tau\ } & \bar{\mathcal{S}} & \xrightarrow{\ \rho\ } & \mathcal{S}^! & \xrightarrow{\ \pi\ } & \mathcal{S}^* \end{array}$$
$$i : \mathcal{S} \dashrightarrow \mathcal{S}^!, \qquad \Phi : \mathcal{S} \dashrightarrow \mathcal{S}^*$$

Recall that $\mathcal{H}^*$ is weakly bounded. By Lemma 5.3, we may assume that $\mathcal{H}^* \hookrightarrow \mathcal{S}^*$ is weakly bounded with respect to an ample divisor $H$ on $\mathcal{S}^*$.

Furthermore, by choosing the compactification appropriately, we may assume that the moduli part $\boldsymbol{\mathcal{M}}^!$ of $\mathcal{F}^!$ descends on $\mathcal{S}^!$ and the moduli part $\boldsymbol{\mathcal{M}}$ of $\mathcal{F}$ descends on $\mathcal{S}$. By Theorem 6.6, we have $\tau^* \boldsymbol{\mathcal{M}}_{\mathcal{S}} = \rho^* \boldsymbol{\mathcal{M}}^!_{\mathcal{S}^!}$.

Because $\boldsymbol{\mathcal{M}}^!_{\mathcal{S}^!}$ is big, we can fix a section of $\boldsymbol{\mathcal{M}}^{!\mathrm{fix}}_{\mathcal{S}^!} \in |\boldsymbol{\mathcal{M}}^!_{\mathcal{S}^!}|_{\mathbb{Q}}$ such that $\mathrm{Supp}(\boldsymbol{\mathcal{M}}^{!\mathrm{fix}}_{\mathcal{S}^!}) \supset \pi^{-1}(\mathcal{S}^* \setminus \mathcal{H}^*)$. Because $\tau^* \boldsymbol{\mathcal{M}}_{\mathcal{S}} = \rho^* \boldsymbol{\mathcal{M}}^!_{\mathcal{S}^!}$, we can choose $\boldsymbol{\mathcal{M}}^{\mathrm{fix}}_{\mathcal{S}}$ such that $\tau(\mathrm{Supp}(\rho^* \boldsymbol{\mathcal{M}}^{!\mathrm{fix}}_{\mathcal{S}^!})) \subset \mathrm{Supp}(\boldsymbol{\mathcal{M}}^{\mathrm{fix}}_{\mathcal{S}})$.

Step 3: We show that, to construct $\boldsymbol{M}^{\mathrm{fix}}$ satisfying the requirements, we are free to replace $Z$ by a higher birational model.

Let $h : Z' \to Z$ be a birational morphism such that $\boldsymbol{M}$ descends on $Z'$. Suppose there exists a $\boldsymbol{b}$-divisor $\boldsymbol{M}^{\mathrm{fix}}$ satisfying the requirements. Because $Z$ is smooth, $\boldsymbol{M}^{\mathrm{fix}}_Z$ is $\mathbb{Q}$-Cartier. Note $\boldsymbol{M}^{\mathrm{fix}}_{Z'} \sim_{\mathbb{Q}} \boldsymbol{M}_{Z'}$ is nef. By the negativity lemma, $\boldsymbol{M}^{\mathrm{fix}}_{Z'} \leq f^* \boldsymbol{M}^{\mathrm{fix}}_Z$, and we have

$$\mathrm{Supp}(\boldsymbol{M}^{\mathrm{fix}}_{Z'}) \subset f^{-1} \mathrm{Supp}(\boldsymbol{M}^{\mathrm{fix}}_Z).$$

Then $\mathrm{Supp}(\boldsymbol{M}^{\mathrm{fix}}_{Z'}) \supset Z' \setminus h^{-1}(U)$ implies that $\mathrm{Supp}(\boldsymbol{M}^{\mathrm{fix}}_Z) \supset Z \setminus U$, so we can replace $Z$ by a higher birational model such that $\boldsymbol{M}$ descends on $Z$.

Step 4: We construct $\boldsymbol{M}^{\mathrm{fix}}$ and finish the proof.

We have the following two cases:

Case 1: The generic point of $\phi(Z)$ is contained in $\mathrm{Supp}(\boldsymbol{\mathcal{M}}^{\mathrm{fix}}_{\mathcal{S}})$. We stratify $\mathcal{S}$ further to the disjoint union of the irreducible components of $\mathrm{Supp}(\boldsymbol{\mathcal{M}}^{\mathrm{fix}}_{\mathcal{S}})$ and its complement, then replace $(\mathcal{X}, \alpha\mathcal{D}_{\mathcal{H}}) \to \mathcal{S}$ by its restriction and repeat this process. By Noetherian reduction, this will stop.

Case 2: The generic point of $\phi(Z)$ is not contained in $\mathrm{Supp}(\mathcal{M}^{\mathrm{fix}}_{\mathcal{S}})$. Because $(\mathcal{Y}_{\mathcal{S}}, \mathrm{Supp}(\mathcal{R}_{\mathcal{S}}))$ is log smooth over $\mathcal{H}$, the generic fiber of $(X, \Delta) \to Z$ is crepant birationally equivalent to the generic fiber of the pullback of $(\mathcal{Y}_{\mathcal{S}}, \mathcal{R}_{\mathcal{S}}) \to \mathcal{S}$ via $\phi$, and the generic point of $\phi(Z)$ is contained in $\mathcal{H}$, by Theorem 6.5, we have $\phi^*\mathcal{M}_{\mathcal{S}} \sim_{\mathbb{Q}} \boldsymbol{M}_Z$. We define the $\boldsymbol{b}$-divisor $\boldsymbol{M}^{\mathrm{fix}}$ by

- $\boldsymbol{M}^{\mathrm{fix}}_Z := \phi^*\mathcal{M}^{\mathrm{fix}}_{\mathcal{S}}$, and
- $\boldsymbol{M}^{\mathrm{fix}}_{Z'} = h^*\boldsymbol{M}^{\mathrm{fix}}_Z$ for any birational morphism $h : Z' \to Z$.

Suppose $l\mathcal{M}^{\mathrm{fix}}_{\mathcal{S}}$ is Cartier and $l\boldsymbol{M}^{\mathrm{fix}}$ is $\boldsymbol{b}$-Cartier.

Because $\mathrm{Supp}(\mathcal{M}^{!\mathrm{fix}}_{\mathcal{S}^!}) \supset \pi^{-1}(\mathcal{S}^* \setminus \mathcal{H}^*)$, $\tau(\mathrm{Supp}(\rho^*\mathcal{M}^{!\mathrm{fix}}_{\mathcal{S}^!})) \subset \mathrm{Supp}(\mathcal{M}^{\mathrm{fix}}_{\mathcal{S}})$, and $\pi \circ \rho = \Phi \circ \tau$, we have $\mathrm{Supp}(\mathcal{M}^{\mathrm{fix}}_{\mathcal{S}}) \supset \tau(\mathrm{Supp}(\rho^*\mathcal{M}^{!\mathrm{fix}}_{\mathcal{S}^!})) \supset \Phi^{-1}(\mathcal{S}^* \setminus \mathcal{H}^*)$, and thus

$$\Phi \circ \phi(Z \setminus \mathrm{Supp}(\phi^*\mathcal{M}^{\mathrm{fix}}_{\mathcal{S}})) \subset \mathcal{H}^*.$$

Also because $\phi^*\mathcal{M}^{\mathrm{fix}}_{\mathcal{S}} \sim_{\mathbb{Q}} \boldsymbol{M}^{\mathrm{fix}}_Z$ and $(\Phi \circ \phi)^{-1}(\mathcal{H}^*) = U$, we have $\mathrm{Supp}(\boldsymbol{M}^{\mathrm{fix}}_Z) \supset Z \setminus U$. □

Suppose there is a family of bases $\mathcal{U} \to T$ of log Calabi–Yau fibrations whose fibers are parametrized by the Hilbert scheme defined in Remark 6.1. That is, every fiber $\mathcal{U}_t$ is the base of a log Calabi–Yau fibration whose fibers belong to the moduli defined in Remark 6.1. Then, for a closed point $t \in T$, we have a moduli map $\phi : \mathcal{U}_t \to \mathcal{H}$. Let $\phi^* : \mathcal{U}_t \to \mathcal{H}^*$ be the composition of $\phi : \mathcal{U}_t \to \mathcal{H}$ with $\Phi : \mathcal{H} \to \mathcal{H}^*$. We define $\overline{\mathcal{U}}_t := \mathcal{U}_t \times_{\mathcal{H}^*} \mathcal{H}^!$ (possibly not connected). Because $\overline{\mathcal{H}} \times_{\mathcal{H}^!} (\mathcal{X}^!_{\mathcal{H}^!}, \alpha\mathcal{D}^!_{\mathcal{H}^!}) \cong \overline{\mathcal{H}} \times_{\mathcal{H}} (\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}})$, there exists a finite cover $V \to \overline{\mathcal{U}}_t$ such that $V \times_{\mathcal{H}^!} (\mathcal{X}^!_{\mathcal{H}^!}, \alpha\mathcal{D}^!_{\mathcal{H}^!}) \cong V \times_{\mathcal{H}} (\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}})$. The next theorem says: if there exists a morphism $\Theta : \mathcal{U} \to \mathcal{H}^*$ such that $\phi^* = \Theta|_{\mathcal{U}_t}$, then we can find a relative compactification of $\mathcal{U} \hookrightarrow \mathcal{Z}$ over $T$, so that the moduli $\boldsymbol{b}$-divisor of the log Calabi–Yau fibration over $\mathcal{U}_t$ descends on $\mathcal{Z}_t$.

**Theorem 6.8.** *Consider the diagram*

$$\begin{array}{ccc}
(\mathcal{Y}^!_{\mathcal{S}^!}, \mathcal{R}^!_{\mathcal{S}^!}) & & \\
\downarrow & & \\
\mathcal{S}^! & \xrightarrow{\ \pi\ } & \mathcal{S}^* \\
\uparrow & & \uparrow \\
\mathcal{H}^! & \xrightarrow[\pi|_{\mathcal{H}^!}]{} & \mathcal{H}^*
\end{array}$$

*where*

- *$\mathcal{S}^*$ and $\mathcal{S}^!$ are smooth schemes,*
- *$\mathcal{H}^* \hookrightarrow \mathcal{S}^*$ and $\mathcal{H}^! \hookrightarrow \mathcal{S}^!$ are dense open subsets,*
- *$\pi|_{\mathcal{H}^!}$ is étale,*
- *$(\mathcal{Y}^!_{\mathcal{S}^!}, \mathrm{Supp}(\mathcal{R}^!_{\mathcal{S}^!}))$ is log smooth over $\mathcal{H}^!$, and*
- *$(\mathcal{Y}^!_{\mathcal{H}^!}, \mathcal{R}^!_{\mathcal{H}^!}) \to \mathcal{H}^!$ is an lc-trivial fibration whose moduli $\boldsymbol{b}$-divisor $\mathcal{M}^!$ descends on $\mathcal{S}^!$, where $(\mathcal{Y}^!_{\mathcal{H}^!}, \mathcal{R}^!_{\mathcal{H}^!}) := (\mathcal{Y}^!_{\mathcal{S}^!}, \mathcal{R}^!_{\mathcal{S}^!}) \times_{\mathcal{S}^!} \mathcal{H}^!$.*

*Suppose there is a family of smooth quasiprojective (possibly not proper) varieties $\mathcal{U} \to T$, where $T$ is of finite type, and a morphism $\Theta : \mathcal{U} \to \mathcal{H}^*$. Let $\overline{\mathcal{U}} := \mathcal{U} \times_{\mathcal{H}^*} \mathcal{H}^!$. Then, after passing to a stratification of $T$, there is a family of projective varieties $\mathcal{Z} \to T$ and a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\mathcal{M}$ on $\mathcal{Z}$ such that $\mathcal{Z}_s$ is a compactification of $\mathcal{U}_s$ for every closed point $s \in T$ and, for any closed point $t \in T$, if $(X, \Delta) \to Z$ is an lc-trivial fibration such that*

- *there is a closed point $t \in T$ together with a birational morphism $Z \to \mathcal{Z}_t$, and*
- *there exist a scheme $V$ and a finite cover $V \to \overline{\mathcal{U}}_t$ such that, for every irreducible component $V_i$ of $V$, the generic fiber of $(X, \Delta) \times_{\mathcal{Z}_t} V_i \to V_i$ is crepant birationally equivalent to the generic fiber of*

$$(\mathcal{Y}^!_{\mathcal{H}^!}, \mathcal{R}^!_{\mathcal{H}^!}) \times_{\mathcal{H}^!} V_i \to V_i,$$

*then the moduli part $\boldsymbol{M}$ of $(X, \Delta) \to Z$ descends on $\mathcal{Z}_t$ and $\boldsymbol{M}_{\mathcal{Z}_t} = \mathcal{M}|_{\mathcal{Z}_t}$*

*Proof.* To prove the result, we may assume $\mathcal{S}^*$ is irreducible.

After passing to a stratification of $T$, we may assume that $T$ is smooth and $\mathcal{U} \to T$ is a smooth morphism. Because $\mathcal{H}^! \to \mathcal{H}^*$ is étale, $\overline{\mathcal{U}} \to T$ is smooth and $\overline{\mathcal{U}} \to \mathcal{U}$ is étale. Let $K(\widetilde{\mathcal{U}})/K(\mathcal{U})$ be the Galois closure of $K(\overline{\mathcal{U}})/K(\mathcal{U})$ and $\widetilde{\mathcal{U}} \to \mathcal{U}$ be the Galois cover with group $G$. After replacing $\mathcal{U}$ by an open subset and passing $T$ to a stratification, we assume that $\widetilde{\mathcal{U}}_t \to \mathcal{U}_t$ is an étale morphism for every closed point $t \in T$. Note the fiber of $\widetilde{\mathcal{U}} \to T$ may not be irreducible.

The composition of $\widetilde{\mathcal{U}} \to \overline{\mathcal{U}}$ and base change of $\Theta : \mathcal{U} \to \mathcal{H}^*$ via $\overline{\mathcal{U}} \to \mathcal{U}$ defines a morphism $\widetilde{\Phi}^\circ : \widetilde{\mathcal{U}} \to \mathcal{H}^!$. Suppose $\widetilde{\mathcal{U}} \hookrightarrow \widetilde{\mathcal{Z}}'$ is a compactification over $T$ such that $\widetilde{\Phi}^\circ$ extends to a morphism $\widetilde{\mathcal{Z}}' \to \mathcal{S}^!$. Because $\widetilde{\mathcal{U}}$ is smooth, we may let $\widetilde{\mathcal{Z}} \to \widetilde{\mathcal{Z}}'$ be a $G$-equivariant log resolution of $(\widetilde{\mathcal{Z}}', \widetilde{\mathcal{Z}}' \setminus \widetilde{\mathcal{U}})$ which is an isomorphism over $\widetilde{\mathcal{U}}$. Note $\widetilde{\Phi}^\circ$ extends to a morphism $\widetilde{\Phi} : \widetilde{\mathcal{Z}} \to \mathcal{S}^!$. After replacing $T$ by a finite cover, we may assume every strata of $(\widetilde{\mathcal{Z}}, \widetilde{\mathcal{Z}} \setminus \widetilde{\mathcal{U}})$ is irreducible over $T$. By the generic smoothness theorem, after passing to a stratification of $T$, we may assume that $(\widetilde{\mathcal{Z}}_{s,j}, (\widetilde{\mathcal{Z}}_s \setminus \widetilde{\mathcal{U}}_s)|_{\widetilde{\mathcal{Z}}_{s,j}})$ is log smooth for every closed point $s \in T$ and every connected component $\widetilde{\mathcal{Z}}_{s,j}$ of $\widetilde{\mathcal{Z}}_s$.

Let $\mathcal{Z}$ be the quotient of $\widetilde{\mathcal{Z}}$ by $G$. Because $\widetilde{\mathcal{Z}}$ is a compactification of $\widetilde{\mathcal{U}}$ over $T$ and the quotient of $\widetilde{\mathcal{U}}$ by $G$ is $\mathcal{U}$, we have that $\mathcal{Z}$ is a compactification of $\mathcal{U}$ over $T$. Next, we show that $\mathcal{Z}$ satisfies the requirements.

Suppose $(X, \Delta) \to Z$ is an lc-trivial fibration that satisfies the conditions, let $Z \to \mathcal{Z}_t$ be the corresponding birational morphism and $V \to \overline{\mathcal{U}}_t$ the corresponding finite cover, and denote its moduli $\boldsymbol{b}$-divisor by $\boldsymbol{M}$. We replace $V$ by $V \times_{\overline{\mathcal{U}}_t} \widetilde{\mathcal{U}}_t$ and assume $V \to \overline{\mathcal{U}}_t$ factors through $V \to \widetilde{\mathcal{U}}_t$. Because $V \to \widetilde{\mathcal{U}}_t$ and $\widetilde{\mathcal{U}}_t \to \mathcal{U}_t$ are finite covers, we can choose a compactification $V \hookrightarrow W$ such that the induced morphisms $W \to \widetilde{\mathcal{Z}}_t$ and $W \to \mathcal{Z}_t$ are finite covers.

Write

$$(\mathcal{Y}^!_{\widetilde{\mathcal{U}}_t}, \mathcal{R}^!_{\widetilde{\mathcal{U}}_t}) := (\mathcal{Y}^!_{\mathcal{H}^!}, \mathcal{R}^!_{\mathcal{H}^!}) \times_{\mathcal{H}^!} \widetilde{\mathcal{U}}_t,$$

where the morphism $\widetilde{\mathcal{U}}_t \to \mathcal{H}^!$ is $\widetilde{\Phi}^\circ|_{\widetilde{\mathcal{U}}_t}$. Because $(\mathcal{Y}^!_{\mathcal{S}^!}, \operatorname{Supp}(\mathcal{R}^!_{\mathcal{S}^!}))$ is log smooth over $\mathcal{H}^!$, we then have that $(\mathcal{Y}^!_{\widetilde{\mathcal{U}}_t}, \operatorname{Supp}(\mathcal{R}^!_{\widetilde{\mathcal{U}}_t}))$ is log smooth over $\widetilde{\mathcal{U}}_t$. Let $\widetilde{\mathcal{Z}}_{t,i}$ be any irreducible component of $\widetilde{\mathcal{Z}}_t$, and let $\widetilde{\mathcal{U}}_{t,i} := \widetilde{\mathcal{U}}_t \cap \widetilde{\mathcal{Z}}_{t,i}$. Because $(\widetilde{\mathcal{Z}}_{t,i}, (\widetilde{\mathcal{Z}}_t \setminus \widetilde{\mathcal{U}}_t)|_{\widetilde{\mathcal{Z}}_{t,i}})$ is log smooth, the moduli $\boldsymbol{b}$-divisor $\widetilde{\boldsymbol{M}}^i$ of a compactification of $(\mathcal{Y}^!_{\mathcal{H}^!}, \mathcal{R}^!_{\mathcal{H}^!}) \times_{\mathcal{H}^!} \widetilde{\mathcal{U}}_{t,i} \to \widetilde{\mathcal{U}}_{t,i}$ descends on $\widetilde{\mathcal{Z}}_{t,i}$ according to Definition 2.9. We define $\widetilde{\boldsymbol{M}}$ to be the $\boldsymbol{b}$-divisor on $\widetilde{\mathcal{Z}}_t$ whose restriction on $\widetilde{\mathcal{Z}}_{t,i}$ is $\widetilde{\boldsymbol{M}}^i$.

Let $V_i$ be an irreducible component of $V$ which dominates $\widetilde{\mathcal{U}}_{t,i}$. By assumption, the generic fiber of $(X, \Delta) \times_{\mathcal{Z}_t} V_i \to V_i$ is crepant birationally equivalent to the generic fiber of $(\mathcal{Y}^!_{\mathcal{H}^!}, \mathcal{R}^!_{\mathcal{H}^!}) \times_{\mathcal{H}^!} V_i \to V_i$; hence the generic fiber of $(X, \Delta) \times_{\mathcal{Z}_t} W_i \to W_i$ is crepant birationally equivalent to the generic fiber of $(\mathcal{Y}^!_{\widetilde{\mathcal{U}}_t}, \mathcal{R}^!_{\widetilde{\mathcal{U}}_t}) \times_{\widetilde{\mathcal{U}}_t} V_i \to V_i$, where $W_i$ is the irreducible component of $W$ corresponding to $V_i$. Note that the moduli $\boldsymbol{b}$-divisor only depends on the crepant birational equivalence class of the generic fiber. By Proposition 6.3, because the moduli $\boldsymbol{b}$-divisor $\widetilde{\boldsymbol{M}}^i$ descends on $\widetilde{\mathcal{Z}}_{t,i}$ and $\widetilde{\mathcal{Z}}_{t,i} \to W_i$ is a finite cover, the moduli $\boldsymbol{b}$-divisor of a compactification of $(X, \Delta) \times_{\mathcal{Z}_t} V_i \to V_i$ descends on $W_i$. Also because $W_i \to \mathcal{Z}_t$ is a finite cover, $\boldsymbol{M}$ descends on $\mathcal{Z}_t$. By considering every irreducible component of $\widetilde{\mathcal{Z}}_t$, we have that $\widetilde{\boldsymbol{M}}_{\widetilde{\mathcal{Z}}_t}$ is equal to the pullback of $\boldsymbol{M}_{\mathcal{Z}_t}$.

Recall that $\widetilde{\Phi}^\circ$ extends to a morphism $\widetilde{\Phi} : \widetilde{\mathcal{Z}} \to \mathcal{S}^!$. Because $(\mathcal{Y}^!_{\mathcal{H}^!}, \operatorname{Supp}(\mathcal{R}^!_{\mathcal{H}^!}))$ is log smooth over $\mathcal{H}^!$, the generic point of $\widetilde{\mathcal{Z}}_{t,i}$ maps into $\mathcal{H}^!$ and $\widetilde{\boldsymbol{M}}^i$ descends on $\widetilde{\mathcal{Z}}_{t,i}$ for every irreducible component $\widetilde{\mathcal{Z}}_{t,i}$ of $\widetilde{\mathcal{Z}}_t$. Then, by Theorem 6.5, we have

$$\widetilde{\boldsymbol{M}}_{\widetilde{\mathcal{Z}}_t} = (\widetilde{\Phi}|_{\widetilde{\mathcal{Z}}_t})^* \boldsymbol{\mathcal{M}}^!_{\mathcal{S}^!}.$$

Let $\widetilde{\mathcal{M}} := \widetilde{\Phi}^* \boldsymbol{\mathcal{M}}^!_{\mathcal{S}^!}$. Because $\widetilde{\mathcal{Z}} \to \mathcal{Z}$ is the quotient by $G$ and $\sum_{g\in G} g^* \widetilde{\mathcal{M}}$ is $G$-invariant,

$$\frac{1}{|G|} \sum_{g\in G} g^* \widetilde{\mathcal{M}}$$

is equal to the pullback of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\mathcal{M}$ on $\mathcal{Z}$.

Because $\widetilde{\boldsymbol{M}}_{\widetilde{\mathcal{Z}}_t}$ is equal to the pullback of $\boldsymbol{M}_{\mathcal{Z}_t}$ and $\widetilde{\boldsymbol{M}}_{\widetilde{\mathcal{Z}}_t} = \widetilde{\mathcal{M}}|_{\widetilde{\mathcal{Z}}_t}$, we have that $\widetilde{\mathcal{M}}|_{\widetilde{\mathcal{Z}}_t}$ is equal to the pullback of a $\mathbb{Q}$-divisor on $\mathcal{Z}_t$. By the construction of $\mathcal{M}$, we have $\boldsymbol{M}_{\mathcal{Z}_t} = \mathcal{M}|_{\mathcal{Z}_t}$. □

## 7. Proof of Theorem 1.1

*Proof of Theorem 1.1.* We use the same notation as in Remark 6.1 and Theorem 6.7.

Let $C > v$ be any fixed number. To prove the DCC, we only need to prove that if $\operatorname{Ivol}(K_X + \Delta) \le C$, then $\operatorname{Ivol}(K_X+\Delta)$ is in a DCC set. By Theorem 2.12, we can construct a generalized pair $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$ and birational morphism $Z' \to Z$ such that

- $\operatorname{coeff}(B_{Z'})$ belongs to a DCC set $\mathcal{I}'$,
- the moduli $\boldsymbol{b}$-divisor $\boldsymbol{M}$ of $f$ descends on $Z'$,
- $\operatorname{Ivol}(K_X + \Delta) = \operatorname{vol}(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'})$, and
- $(X, \Delta)$ has the same canonical model as $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$.

After replacing $Z$ by $Z'$, and $B_{Z'}$ and $\boldsymbol{M}_{Z'}$ by $B_Z$ and $\boldsymbol{M}_Z$, respectively, we only need to prove that $\operatorname{vol}(K_Z + B_Z + \boldsymbol{M}_Z)$ belongs to a DCC set. To this end, we add $\{1 - 1/k, k \in \mathbb{N}\}$ into $\mathcal{I}'$ and assume that $\{1 - 1/k, k \in \mathbb{N}\} \subset \mathcal{I}'$.

By Remark 6.1, we have an lc-trivial fibration $(\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}}) \to \mathcal{H}$ corresponding to the class $\mathscr{C}$ of polarized log Calabi–Yau pairs. Consider the diagram

$$\begin{array}{ccccccc} (\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}}) & & & & (\mathcal{X}^!_{\mathcal{H}^!}, \alpha\mathcal{D}^!_{\mathcal{H}^!}) & & \\ \downarrow{\scriptstyle \mathcal{F}} & & \overset{i}{\dashrightarrow} & & \downarrow{\scriptstyle \mathcal{F}^!} & & \\ \mathcal{H} & \xleftarrow{\tau} & \overline{\mathcal{H}} & \xrightarrow{\rho} & \mathcal{H}^! & \xrightarrow{\pi} & \mathcal{H}^* \end{array}$$

$$\Phi : \mathcal{H} \to \mathcal{H}^*$$

constructed in Theorem 6.7. Because $\mathcal{H}$ has only finitely many irreducible components, to prove the results, we may assume $\mathcal{H}$ is irreducible. Let $\mathcal{S}^*$ be the compactification of $\mathcal{H}^*$, $l$ be the positive integer defined in Theorem 6.7, and $(\mathcal{Y}_{\mathcal{H}}, \mathcal{R}_{\mathcal{H}}) \to (\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}})$ and $(\mathcal{Y}^!_{\mathcal{H}^!}, \mathcal{R}^!_{\mathcal{H}^!}) \to (\mathcal{X}^!_{\mathcal{H}^!}, \alpha\mathcal{D}^!_{\mathcal{H}^!})$ be crepant birational morphisms which are fiberwise log resolutions of $\mathcal{F}$ and $\mathcal{F}^!$.

Since a general fiber $(X_g, \Delta_g, L_g)$ is in $\mathscr{C}$, by Remark 6.1, there is an open subset $U \hookrightarrow Z$ such that $(X_U, \Delta|_{X_U})$ is crepant birationally equivalent to the pullback of $(\mathcal{Y}_{\mathcal{H}}, \mathcal{R}_{\mathcal{H}}) \to \mathcal{H}$ by a morphism $U \to \mathcal{H}$. Let $h : Z' \to Z$ be a birational morphism such that $U \to \mathcal{H} \xrightarrow{\Phi} \mathcal{H}^*$ extends to a morphism $\phi : Z' \to \mathcal{S}^*$. Let $k$ be a sufficiently large integer such that $K_{Z'} + h_*^{-1}B_Z + (1-1/k)E + \boldsymbol{M}_{Z'} \geq h^*(K_Z + B_Z + \boldsymbol{M}_Z)$, where $E$ is the exceptional divisor of $h$. Then we replace $Z$ by $Z'$, $B_Z$ by $h_*^{-1}B_Z + (1-1/k)E$, and $\boldsymbol{M}_Z$ by $\boldsymbol{M}_{Z'}$, and assume that there is a morphism $\phi : Z \to \mathcal{S}^*$. Note that we keep the facts that $\operatorname{coeff}(B_Z)$ is in the DCC set $\mathcal{I}'$, the moduli $\boldsymbol{b}$-divisor $\boldsymbol{M}$ of $f$ descends on $Z$, $\operatorname{Ivol}(K_X + \Delta) = \operatorname{vol}(K_Z + B_Z + \boldsymbol{M}_Z)$, and $(X, \Delta)$ has the same canonical model as $(Z, B_Z + \boldsymbol{M}_Z)$.

Because $\dim Z \leq \dim X = n$, to prove the results, we may assume $\dim Z = d$ is fixed. Let $\boldsymbol{M}^{\mathrm{fix}}$ be the $\boldsymbol{b}$-divisor defined in Theorem 6.7. Then

- $\boldsymbol{M}^{\mathrm{fix}}_Z$ is effective and nef,
- $\boldsymbol{M}^{\mathrm{fix}}_{Z'} \sim_{\mathbb{Q}} \boldsymbol{M}_{Z'}$ for every birational map $Z' \dashrightarrow Z$, and
- $l\boldsymbol{M}^{\mathrm{fix}}_Z$ is Cartier.

By Step 1 of the proof of Theorem 3.2, there is a positive integer $r$ depending only on $d$, $l$, and $\mathcal{I}'$ such that, after replacing $Z$ by a birational model and $B_Z$ by the strict transform plus $(1-1/k)E$, where $E$ denotes the reduced exceptional divisor and $k$ is a sufficiently large integer, there is a birational contraction $g : Z \to W$ and a very ample divisor $A$ on $W$ such that $g^*A + F' \sim r(K_Z + B_Z + \boldsymbol{M}^{\mathrm{fix}}_Z)$ for an effective $\mathbb{Q}$-divisor $F' \geq 0$. Because

$$\operatorname{vol}(A) \leq \operatorname{vol}(r(K_Z + B_Z + \boldsymbol{M}^{\mathrm{fix}}_Z)) = r^d \operatorname{Ivol}(K_X + \Delta) \leq r^d C,$$

$W$ is in a bounded family $\mathcal{W} \to S$ and there is a relative very ample divisor $\mathcal{A}$ on $\mathcal{W}$ such that $\mathcal{A}|_{\mathcal{W}_0} \sim A$, where $0$ is a closed point of $S$ such that $W \cong \mathcal{W}_0$.

After passing to a stratification of $S$, we may assume $\mathcal{W} \to S$ has a fiberwise log resolution $\mathcal{W}' \xrightarrow{\mathscr{G}} \mathcal{W} \to S$. Because $\mathcal{A}$ is relatively very ample, we can stratify $S$ further, so that there exists a sufficiently large integer $r'$, a relative very ample divisor $\mathcal{A}'$ on $\mathcal{W}'$ and an effective divisor $\mathcal{E} \sim r'\mathscr{G}^*\mathcal{A} - \mathcal{A}'$ such that $\mathcal{E}|_{\mathcal{W}'_s}$ is effective for every closed point $s \in S$. Then we replace $W$ by $\mathcal{W}'_0$, $A$ by $\mathcal{A}'|_{\mathcal{W}'_0}$, $F'$ by $r'F' + \mathcal{E}|_{\mathcal{W}'_0}$,

$Z$ by a birational model, and $B_Z$ by the strict transform plus $(1-1/k)E$, where $E$ denotes the reduced exceptional divisor and $k$ is a sufficiently large integer. We have

- $A$ is very ample,
- $W$ is smooth, and
- $g^*A+F' \sim r(K_Z+B_Z+\boldsymbol{M}_Z^{\mathrm{fix}})$ for an effective $\mathbb{Q}$-divisor $F' \geq 0$.

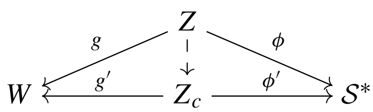
We define $F := F'+r((2d+1)l-1)\boldsymbol{M}_Z^{\mathrm{fix}}$. Then $F$ is effective, $\mathrm{Supp}(\boldsymbol{M}_Z^{\mathrm{fix}}) \subset \mathrm{Supp}(F)$, and $g^*A+F \sim r(K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}})$.

Next, we construct a birational open subset of $Z$ which maps into $\mathcal{H}^*$ via $\phi$ and belongs to a bounded family of quasiprojective varieties. This is similar to Step 2 of the proof of Theorem 3.2.

Recall that $\mathcal{H}^* \hookrightarrow \mathcal{S}^*$ is weakly bounded with respect to an ample Cartier divisor $\Lambda$ on $\mathcal{S}^*$. Let $Z \dashrightarrow Z_c$ be the canonical model of $K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}+(2d+1)\phi^*\Lambda+(2d+1)g^*A$. By [Birkar and Zhang 2016, Lemma 4.4], $Z \dashrightarrow Z_c$ is $\boldsymbol{M}_Z^{\mathrm{fix}}$-, $g^*A$- and $\phi^*\Lambda$-trivial. Then there are two morphisms $g' : Z_c \to W$ and $\phi' : Z_c \to \mathcal{S}^*$. Let $B_{Z_c}$ and $F_c$ be the pushforward of $B_Z$ and $F$ on $Z_c$. Then $K_{Z_c}+B_{Z_c}+(2d+1)l\boldsymbol{M}_{Z_c}^{\mathrm{fix}}+(2d+1)\phi'^*\Lambda+(2d+1)g'^*A$ is ample: note $l\boldsymbol{M}_{Z_c}^{\mathrm{fix}}$ is nef, effective and Cartier. Because $K_{Z_c}+B_{Z_c}+(2d+1)l\boldsymbol{M}_{Z_c}^{\mathrm{fix}} \sim_{\mathbb{Q}} (g'^*A+F_c)/r$, we have

$$\frac{1}{r}(g'^*A+F_c)+(2d+1)\phi'^*\Lambda+(2d+1)g'^*A$$

is ample. We denote it by $A'$; clearly $A'$ is effective.




Because $\mathrm{coeff}(B_Z)$ is in a DCC set $\mathcal{I}'$ and $r(K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}) \sim g^*A+F$, with

$$\{r(K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}})\} = \{rB_Z\} = \{F\},$$

$\mathrm{coeff}(F)$ is in a DCC set $\mathcal{I}''=\mathcal{I}''(\mathcal{I}', d, r)$. In particular, there is a positive number $\delta$ such that $\mathrm{coeff}(F) > \delta$.

The proof of the following claim is deferred until after the main proof.

Claim: $(W, \mathrm{Supp}(g'_*(A'+B_{Z_c})))$, which is equal to $(W, \mathrm{Supp}(A+g'_*(\phi'^*\Lambda+F_c+B_{Z_c})))$, is log bounded.

Because $A'$ is ample and effective and $W$ is smooth, we have that $g'(\mathrm{Supp}(A'))$ is pure of codimension 1 and $g'(\mathrm{Supp}(A'))=\mathrm{Supp}(g'_*A')$. By the negativity lemma, $A'=g'^*g'_*A'-E'$, where $E'$ is an effective exceptional $\mathbb{Q}$-divisor such that $\mathrm{Supp}(E')=\mathrm{Exc}(g')$. Because $A' \geq 0$, we have that $\mathrm{Exc}(g') \subset \mathrm{Supp}(g'^*g'_*A')$ and

$$W \setminus \mathrm{Supp}(g'_*A') \cong Z_c \setminus \mathrm{Supp}(g'^*g'_*A').$$

By Theorem 6.7, $\phi(Z \setminus \mathrm{Supp}(\boldsymbol{M}_Z^{\mathrm{fix}})) \subset \mathcal{H}^*$. Since $\mathrm{Supp}(\boldsymbol{M}_Z^{\mathrm{fix}}) \subset \mathrm{Supp}(F)$ and $\phi(Z \setminus \mathrm{Supp}(\boldsymbol{M}_Z^{\mathrm{fix}})) \subset \mathcal{H}^*$, we have $\mathrm{Supp}(\boldsymbol{M}_{Z_c}^{\mathrm{fix}}) \subset \mathrm{Supp}(F_c) \subset \mathrm{Supp}(A')$ and $\phi'(Z_c \setminus \mathrm{Supp}(\boldsymbol{M}_{Z_c}^{\mathrm{fix}})) \subset \mathcal{H}^*$. Let

$$U_c := Z_c \setminus \mathrm{Supp}(g'^*g'_*A') = W \setminus \mathrm{Supp}(g'_*A').$$

It is easy to see that $U_c \subset Z_c \setminus \mathrm{Supp}(\boldsymbol{M}_{Z_c}^{\mathrm{fix}})$ and $\phi'(U_c) \subset \mathcal{H}^*$.

Because $(W, \mathrm{Supp}(g'_*(A'+B_{Z_c})))$ is log bounded, there is a family of varieties $\mathcal{U} \to T$ over a scheme of finite type $T$ and a closed point $t \in T$ such that

$$\mathcal{U}_t \cong W \setminus g'_*(A'+B_{Z_c}) \subset U_c.$$

Because $\mathcal{H}^*$ is weakly bounded, by applying Theorem 5.5 with $\mathcal{M}^0 := \mathcal{H}^* \times T$, there exists a finite type scheme $\mathscr{W}$ and a morphism $\mathscr{W} \times \mathcal{U} \to \mathcal{H}^* \times T$ over $T$ such that, if we let $\Theta : \mathscr{W} \times \mathcal{U} \to \mathcal{H}^*$ be the composition of $\mathscr{W} \times \mathcal{U} \to \mathcal{H}^* \times T$ with the projection $\mathcal{H}^* \times T \to \mathcal{H}^*$, then $\phi'|_{\mathcal{U}_t} = \Theta|_{\{p\}\times\mathcal{U}_t}$ for a closed point $p \in \mathscr{W}$. We replace $\mathcal{U} \to T$ by $\mathscr{W} \times \mathcal{U} \to \mathscr{W} \times T$.

Let $V := U \times_{\mathcal{H}} \overline{\mathcal{H}}$; then $V \to U$ is a finite cover. By Theorem 6.6 and the fact that $(X_U, \Delta|_{X_U})$ is crepant birationally equivalent to the pullback of $(\mathcal{Y}_{\mathcal{S}}, \mathcal{R}_{\mathcal{S}}) \to \mathcal{S}$ via $U \to \mathcal{H} \hookrightarrow \mathcal{S}$, for every irreducible component $V_i$ of $V$, the generic fiber of $(X, \Delta) \times_Z V_i \to V_i$ is crepant birationally equivalent to the generic fiber of $(\mathcal{Y}^!_{\mathcal{H}^!}, \mathcal{R}^!_{\mathcal{H}^!}) \times_{\mathcal{H}^!} V_i \to V_i$. Then, up to passing to a stratification of $T$, by Theorem 6.8, there is a compactification $\mathcal{U} \hookrightarrow \mathcal{Z}/T$ and a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\mathcal{M}$ on $\mathcal{Z}$ such that the moduli $\boldsymbol{b}$-divisor $\boldsymbol{M}$ of $(X, \Delta)$ descends on $\mathcal{Z}_t$ and $\boldsymbol{M}_{\mathcal{Z}_t} = \mathcal{M}|_{\mathcal{Z}_t}$.

Let $\mathcal{P} := \mathcal{Z} \setminus \mathcal{U}$; then $\mathcal{P}_t = \mathrm{Supp}(g'_*(A'+B_{Z_c}))$. After passing to a log resolution of the generic fiber and passing to a stratification of $T$, we may assume that $(\mathcal{Z}, \mathcal{P}) \to T$ is a projective log smooth morphism. We also replace $\mathcal{M}$ by its pullback. Note: we still have that $\boldsymbol{M}$ descends on $\mathcal{Z}_t$ and $\boldsymbol{M}_{\mathcal{Z}_t} = \mathcal{M}|_{\mathcal{Z}_t}$.

Let $h : Z' \to Z$ be a log resolution of $(Z, B_Z)$ such that the isomorphism $U_c \cong \mathcal{U}_t$ extends to a morphism $Z' \to \mathcal{Z}_t$. We replace $Z$ with $Z'$ and $B_Z$ with its strict transform plus $(1-1/k)E$, where $E$ denotes the reduced exceptional divisor and $k$ is a sufficiently large integer. Note that we keep $\mathrm{vol}(K_Z+B_Z+\boldsymbol{M}_Z)$ and the canonical model of $(Z, B_Z+\boldsymbol{M}_Z)$, and we still have $\mathrm{coeff}(B_{Z'}) \subset \mathcal{I}'$.

Since $\mathrm{Supp}(g_*B_Z) = \mathrm{Supp}(g'_*B_{Z_c})$ and $\mathrm{Supp}(g'_*(A'+B_{Z_c})) \subset \mathcal{P}_t$, the pushforward of $B_{Z'}$ on $\mathcal{Z}_t$ is contained in $\mathcal{P}_t$. Also because $\boldsymbol{M}$ descends on $\mathcal{Z}_t$, we have that $\boldsymbol{M}$ descends on $Z'$; hence $(Z', B_{Z'}+\boldsymbol{M}_{Z'})$ is a generalized klt pair and $\mathrm{coeff}(B_{Z'}) \subset \mathcal{I}'$ is a DCC set. Then, by Theorems 3.3 and 4.2, conclusions (i) and (ii) follow. □

*Proof of claim.* We use the same notation as in the proof of Theorem 1.1.

Because $W$ is bounded by the construction, $A$ and $\Lambda$ are integral divisors, $\mathrm{coeff}(B_{Z_c})$ is in a DCC set, $\mathrm{coeff}(F_c)$ is bounded from below, and $A$ is very ample on $W$, by boundedness of the Chow variety, we only need to prove that the intersection numbers

$$A^{d-1}.g'_*\phi'^*\Lambda, \quad A^{d-1}.g'_*B_{Z_c} \quad \text{and} \quad A^{d-1}.g'_*F_c$$

are bounded from above.

First we show that there is a constant $C_1$ such that

$$\mathrm{vol}(K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}) \le C_1.$$

By Theorem 2.5, there is a rational number $e \in (0, 1)$ such that $K_Z+B_Z+e\boldsymbol{M}_Z$ is big. By the log-concavity of the volume function, we have that

$$\mathrm{vol}(K_Z+B_Z+\boldsymbol{M}_Z^{\mathrm{fix}}) \ge \lambda^d \,\mathrm{vol}(K_Z+B_Z+e\boldsymbol{M}_Z^{\mathrm{fix}}) + (1-\lambda)^d \,\mathrm{vol}(K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}), \quad (7\text{-}1)$$

where

$$\lambda = \frac{(2d+1)l-1}{(2d+1)l-e} < 1.$$

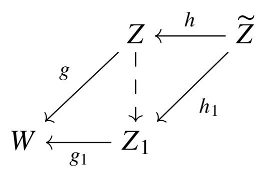
By assumption, $\operatorname{vol}(K_Z + B_Z + \boldsymbol{M}_Z^{\mathrm{fix}}) \le C$, and hence $\operatorname{vol}(K_Z + B_Z + (2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}) \le C/(1-\lambda)^d$.

Second we prove that $A^{d-1}.g'_*\phi'^*\Lambda$ is bounded from above, which is equivalent to proving that $A^{d-1}.g_*\phi^*\Lambda$ is bounded from above. The idea is to show that $A^{d-1}.g_*\phi^*\Lambda$ is equal to the degree of a divisor on a $(g, m)$-curve, with $g + m$ bounded, then apply weak boundedness.

Let $A_1, \ldots, A_{d-1} \in |g^*A|$ be $d-1$ general members of the linear system. Because $g^*A$ is base point free, the elements of $\{\operatorname{Supp}(A_i), i = 1, \ldots, d-1\}$ are smooth divisors and intersect along a smooth curve $C$. By the adjunction formula,

$$(g^*A)^{d-1}.(K_Z + B_Z + (2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}} + (d-1)g^*A) = \deg(K_C + B_Z|_C + (2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}|_C).$$

Consider the diagram




where $Z_1$ is the canonical model of $K_Z + B_Z + (2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}} + (2d+1)g^*A$ and $\widetilde{Z}$ is a resolution of indeterminacies of $Z \dashrightarrow Z_1$. By [Birkar and Zhang 2016, Lemma 4.4], $Z \dashrightarrow Z_1$ is $g^*A$-trivial, so there is a birational morphism $g_1 : Z_1 \to W$. By the projection formula,

$$\begin{aligned}(g^*A)^{d-1}.(K_Z + B_Z + (2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}} + (2d+1)g^*A)&\\ &= (h^*g^*A)^{d-1}.(h^*(K_Z + B_Z + (2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}} + (2d+1)g^*A))\\ &= (g_1^*A)^{d-1}.(h_{1*}h^*(K_Z + B_Z + (2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}} + (2d+1)g^*A))\\ &= (g_1^*A)^{d-1}.(K_{Z_1} + B_{Z_1} + (2d+1)l\boldsymbol{M}_{Z_1}^{\mathrm{fix}} + (2d+1)g_1^*A),\end{aligned}$$

where $B_{Z_1}$ is the pushforward of $B_Z$. Since $Z_1$ is the canonical model of

$$K_Z + B_Z + (2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}} + (2d+1)g^*A,$$

$K_{Z_1} + B_{Z_1} + (2d+1)l\boldsymbol{M}_{Z_1}^{\mathrm{fix}} + (2d+1)g_1^*A$ is ample. By the binomial theorem, we have

$$\begin{aligned}(K_{Z_1} + B_{Z_1} + (2d+1)l\boldsymbol{M}_{Z_1}^{\mathrm{fix}} + (2d+1)g_1^*A + g_1^*A)^d&\\ &= \sum_{0\le i\le d}\binom{d}{i}(g_1^*A)^{d-i}.(K_{Z_1} + B_{Z_1} + (2d+1)l\boldsymbol{M}_{Z_1}^{\mathrm{fix}} + (2d+1)g_1^*A)^i.\end{aligned}$$

Because $g_1^*A$ and $K_{Z_1} + B_{Z_1} + (2d+1)l\boldsymbol{M}_{Z_1}^{\mathrm{fix}} + (2d+1)g_1^*A$ are both nef, we have

$$(g_1^*A)^{d-i}.(K_{Z_1} + B_{Z_1} + (2d+1)l\boldsymbol{M}_{Z_1}^{\mathrm{fix}} + (2d+1)g_1^*A)^i \ge 0$$

for every $0 \le i \le d$. Then

$$
\begin{aligned}
(g_1^*A)^{d-1}.(K_{Z_1}+B_{Z_1}+(2d+1)l\boldsymbol{M}_{Z_1}^{\mathrm{fix}}+(2d+1)g_1^*A) \\
&\le \binom{d}{1}(g_1^*A)^{d-1}.(K_{Z_1}+B_{Z_1}+(2d+1)l\boldsymbol{M}_{Z_1}^{\mathrm{fix}}+(2d+1)g_1^*A)\\
&\le (K_{Z_1}+B_{Z_1}+(2d+1)l\boldsymbol{M}_{Z_1}^{\mathrm{fix}}+(2d+1)g_1^*A+g_1^*A)^d\\
&= \mathrm{vol}(K_{Z_1}+B_{Z_1}+(2d+1)l\boldsymbol{M}_{Z_1}^{\mathrm{fix}}+(2d+2)g_1^*A).
\end{aligned}
$$

Since $Z \dashrightarrow Z_1$ is $g^*A$-trivial and $Z_1$ is also the canonical model of $K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}+(2d+2)g^*A$, we have

$$
\begin{aligned}
\mathrm{vol}(K_{Z_1}+B_{Z_1}+(2d+1)l\boldsymbol{M}_{Z_1}^{\mathrm{fix}}+(2d+2)g_1^*A) &= \mathrm{vol}(K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}+(2d+2)g^*A)\\
&\le \mathrm{vol}(K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}+(2d+2)(g^*A+F))\\
&= \mathrm{vol}((1+(2d+2)r)(K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}))\\
&\le \left(\frac{1+(2d+2)r}{r}\right)^d C_1.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\deg(K_C+B_Z|_C+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}|_C) &= (g^*A)^{d-1}.(K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}+(d-1)g^*A)\\
&\le (g^*A)^{d-1}.(K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}+(2d+1)g^*A)\\
&\le \left(\frac{1+(2d+2)r}{r}\right)^d C_1. \qquad (7\text{-}2)
\end{aligned}
$$

By the construction of $\boldsymbol{M}_Z^{\mathrm{fix}}$, we have that $Z \setminus \mathrm{Supp}(\boldsymbol{M}_Z^{\mathrm{fix}})$ maps into $\mathcal{H}^*$, so $C \setminus \mathrm{Supp}(\boldsymbol{M}_Z^{\mathrm{fix}}|_C)$ maps into $\mathcal{H}^*$. Suppose $C^\circ := C \setminus \mathrm{Supp}(\boldsymbol{M}_Z^{\mathrm{fix}}|_C)$ is a $(g,m)$-curve. Then $m \le \deg_C(l\boldsymbol{M}_Z^{\mathrm{fix}}|_C)$ and

$$
2g-2+(2d+1)m \le \deg(K_C+B_Z|_C+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}|_C)
$$

is bounded. Because $\mathcal{H}^*$ is weakly bounded with respect to $\Lambda$ and $C^\circ$ is a $(g,m)$-curve with $2g+(2d+1)m$ bounded, we have that $(g^*A)^{d-1}.\phi^*\Lambda = C.\phi^*\Lambda = \deg_C(\phi^*\Lambda|_C)$ is bounded and, by the projection formula, $A^{d-1}.g_*\phi^*\Lambda$ is bounded.

Third we show that $A^{d-1}.g'_*B_{Z_c}$ is bounded from above, which is equivalent to proving that $A^{d-1}.g_*B_Z$ is bounded from above. Because $\mathrm{coeff}(B_Z) \subset \mathcal{I}'$ is in a DCC set, $l\boldsymbol{M}_Z^{\mathrm{fix}}$ is nef and Cartier and $K_Z+B_Z+\boldsymbol{M}_Z$ is big, by [Birkar and Zhang 2016, Theorem 8.1], there exists $e$ depending only on $d$ and $\mathcal{I}'$ such that $K_Z+eB_Z+\boldsymbol{M}_Z$ is big. Thus we have

$$
A^{d-1}.g_*B_Z \le \frac{1}{1-e}(g^*A)^{d-1}.((1-e)B_Z+K_Z+eB_Z+\boldsymbol{M}_Z) = \frac{1}{1-e}(g^*A)^{d-1}.(K_Z+B_Z+\boldsymbol{M}_Z).
$$

Since $\boldsymbol{M}_Z^{\mathrm{fix}}$ and $g^*$ are effective, we have

$$
(g^*A)^{d-1}.(K_Z+B_Z+\boldsymbol{M}_Z) \le (g^*A)^{d-1}.(K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}+(2d+1)g^*A).
$$

We then apply the last inequality of (7-2).

Finally we prove that $A^{d-1}.g'_*F_c$ is bounded from above, which is equivalent to proving that $A^{d-1}.g_*F$ is bounded from above. Because $K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}\sim_{\mathbb{Q}}(g^*A+F)/r$, we have

$$A^{d-1}.g_*F=(g^*A)^{d-1}.F\le (g^*A)^{d-1}.r(K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}),$$

which is also bounded by (7-2). □

*Proof of Corollary 1.3.* After replacing $X$ with a $\mathbb{Q}$-factorization and $\Delta$ with its strict transform, we may assume $X$ is $\mathbb{Q}$-factorial. Let $\delta$ be a sufficiently small positive rational number such that $(X,(1+\delta)\Delta)$ is klt.

Since $K_X+(1+\delta)\Delta\sim_{\mathbb{Q},Z}\delta\Delta$ is big over $Z$, by [Birkar et al. 2010], there exists the relative canonical model $X\dashrightarrow X'$ of $K_X+(1+\delta)\Delta$ over $Z$, and hence $K_{X'}+(1+\delta)\Delta'$ is ample over $Z$, where $\Delta'$ is the pushforward of $\Delta$. For a general fiber $(X'_g,\Delta'_g)$ of $f':X'\to Z$, we have that $K_{X'_g}+(1+\delta)\Delta'_g$ is ample.

Because $X\dashrightarrow X'$ is a birational contraction and $K_X+\Delta\sim_{\mathbb{Q},Z}0$, we have $K_{X'}+\Delta'\sim_{\mathbb{Q},Z}0$, which implies $K_{X'_g}+\Delta'_g\sim_{\mathbb{Q}}0$. Thus

$$-K_{X'_g}\sim_{\mathbb{Q}}\Delta'_g\sim_{\mathbb{Q}}\frac{1}{\delta}(K_{X'_g}+(1+\delta)\Delta'_g)$$

is ample. Note $K_X+\Delta$ is crepant birationally equivalent to $K_{X'}+\Delta'$. Then $\mathrm{Ivol}(K_X+\Delta)=\mathrm{Ivol}(K_{X'}+\Delta')$ and $(X,\Delta)$ and $(X',\Delta')$ have the same canonical model. We replace $(X,\Delta)$ with $(X',\Delta')$.

Because $\mathrm{coeff}(\Delta)$ is in a DCC set $\mathcal{I}$, by [Hacon et al. 2014, Theorem 1.5], there exists a finite subset $\mathcal{I}'\subset\mathcal{I}$ such that $\mathrm{coeff}(\Delta_g)\subset\mathcal{I}'$. Furthermore, there is a positive rational number $\epsilon\in(0,1)$ depending only on $\mathcal{I}'$ such that $(X_g,\Delta_g)$ is $\epsilon$-lc. By the Birkar-BAB theorem [Birkar 2021b, Theorem 1.1], $X_g$ is in a bounded family only depending on $\epsilon$ and $\dim X_g$. Because $\dim X_g\le\dim X=n$, by boundedness, there exist positive integers $l$ and $C$ depending only on $\epsilon$ and $n$ such that $-lK_{X_g}$ is very ample without higher cohomology and $\mathrm{vol}(-lK_{X_g})=(-lK_{X_g})^{\dim X_g}\le C$.

Since $\mathrm{coeff}(\Delta_g)$ is in a finite set $\mathcal{I}'$, there exists $\delta'>0$ such that $\mathrm{coeff}(\Delta_g)\ge\delta'$. Because $\Delta_g\sim_{\mathbb{Q}}-K_{X_g}$, we have

$$\mathrm{red}(\Delta_g).(-lK_{X_g})^{\dim X_g-1}\le\frac{1}{\delta'}(-K_{X_g})(-lK_{X_g})^{\dim X_g-1}\le\frac{1}{l\delta'}(-lK_{X_g})^{\dim X_g}\le\frac{C}{l\delta'}.$$

Because $-lK_{X_g}$ is very ample without higher cohomology,

$$(-lK_{X_g})^{\dim X_g}\le C\quad\text{and}\quad \mathrm{red}(\Delta_g).(-lK_{X_g})^{\dim X_g-1}\le\frac{C}{l\delta'},$$

we have that $(X_g,\Delta_g,-lK_{X_g})$ is in a log bounded class of polarized log Calabi–Yau pairs. We define $L:=-lK_X$, then apply Theorem 1.1. □

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# Geometry of PCF parameters in spaces of quadratic polynomials

Laura DeMarco and Niki Myrto Mavraki

We study algebraic relations among postcritically finite (PCF) parameters in the family $f_c(z) = z^2 + c$. It is known that an algebraic curve in $\mathbb{C}^2$ contains infinitely many PCF pairs $(c_1, c_2)$ if and only if the curve is special (i.e., the curve is a vertical or horizontal line through a PCF parameter, or the curve is the diagonal). Here we extend this result to subvarieties of arbitrary dimension in $\mathbb{C}^n$ for any $n \geq 2$. Consequently, we obtain uniform bounds on the number of PCF pairs on nonspecial curves in $\mathbb{C}^2$ and the number of PCF parameters in real algebraic curves in $\mathbb{C}$, depending only on the degree of the curve. We also compute the optimal bound for the general curve of degree $d$. For $d = 1$, we prove that there are only finitely many nonspecial lines in $\mathbb{C}^2$ containing more than two PCF pairs, and similarly, that there are only finitely many (real) lines in $\mathbb{C} = \mathbb{R}^2$ containing more than two PCF parameters.

## 1. Introduction

For each $c \in \mathbb{C}$, let $f_c(z) = z^2 + c$. Recall that the polynomial $f_c$ is postcritically finite (PCF) if the critical point at $z_0 = 0$ has a finite forward orbit. In this article, we study algebraic relations among the PCF parameters $c \in \mathbb{C}$.

Our starting point is the following theorem of Ghioca, Krieger, Nguyen, and Ye, which continued a study of dynamical orbit relations initiated in [Baker and DeMarco 2011]. Generalizations to algebraic curves in other polynomial families were obtained in [Favre and Gauthier 2022].

**Theorem 1.1** [Ghioca et al. 2017]. *Let $C$ be an irreducible complex algebraic curve in $\mathbb{C}^2$. Then $C$ contains infinitely many PCF pairs $(c_1, c_2)$ if and only if $C$ is either*

(1) *a vertical line $\{x = c_1\}$ for a PCF $f_{c_1}$; or*

(2) *a horizontal line $\{y = c_2\}$ for a PCF $f_{c_2}$; or*

(3) *the diagonal $\{x = y\}$*

*in coordinates $(x, y)$ on $\mathbb{C}^2$.*

Note that a real algebraic curve in $\mathbb{R}^2$ passing through a PCF parameter $c_0$ in the Mandelbrot set (identifying $\mathbb{R}^2$ with $\mathbb{C}$) gives rise to a complex algebraic curve in $\mathbb{C}^2$ passing through the PCF pair $(c_0, \bar{c}_0)$. So the above result also controls PCF points on real curves in $\mathbb{C}$. See Figure 1 and Section 5.

*MSC2020:* 11G50, 37F46.
*Keywords:* Mandelbrot set, postcritically finite maps, quadratic polynomials, special points, unlikely intersections, uniformity, bifurcation measure, equidistribution.

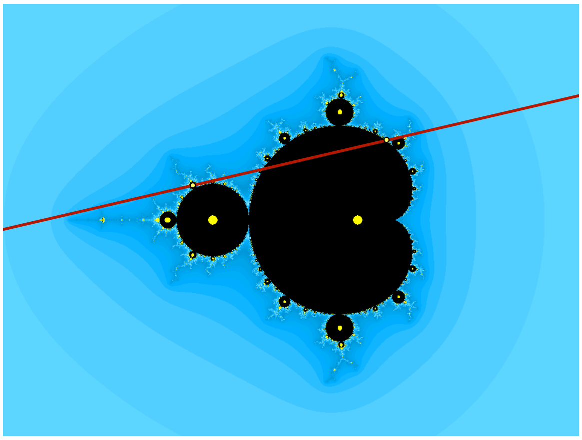

**Figure 1.** The Mandelbrot set with PCF parameters marked in yellow. There are only finitely many real lines in $\mathbb{C}$ containing more than two PCF parameters; see Theorem 1.8.

Theorem 1.1 was motivated by analogies between the PCF maps in the space of quadratic polynomials and the elliptic curves with complex multiplication (CM) in the space of $j$-invariants; see, for example, [Silverman 2012, Chapter 6; Jones 2013, Conjecture 3.11]. It was known that the only algebraic curves in $\mathbb{C}^2$ with infinitely many CM pairs are the modular curves (together with the infinite collection of vertical or horizontal lines through a CM point) [André 1998; Edixhoven 1998].

Our first result is an extension of Theorem 1.1 to arbitrary dimensions, exactly analogously to the classification of special subvarieties in the CM case [Pila 2011; Edixhoven 2005]:

**Theorem 1.2.** *Let $n \geq 2$. Let $X$ be an irreducible complex algebraic subvariety in $\mathbb{C}^n$. There is a Zariski dense set of special points in $X$ if and only if $X$ is special.*

By definition, a parameter $c \in \mathbb{C}$ is *special* if $f_c$ is PCF. For any positive integer $n$, a point $(c_1, \ldots, c_n) \in \mathbb{C}^n$ is *special* if $c_i$ is special for all $i = 1, \ldots, n$. We say that an irreducible curve $C \subset \mathbb{C}^2$ is *special* if it is one of the three types listed in Theorem 1.1. The *special subvarieties* of $\mathbb{C}^n$ are the preimages of special curves from projections to $\mathbb{C}^2$, and their intersections. More precisely, an irreducible subvariety $Z$ of $\mathbb{C}^n$ is special if and only if there exist a partition $S_0 \cup \cdots \cup S_r$ of $\{1, \ldots, n\}$, where $r \geq 0$ and $S_k \neq \varnothing$ for each $k > 0$, and a collection of PCF parameters $c_i \in \mathbb{C}$ for $i \in S_0$ such that

$$Z = \Bigl(\bigcap_{i \in S_0} \{x_i = c_i\}\Bigr) \cap \Bigl(\bigcap_{k=1}^{r} \bigcap_{j \in S_k} \{x_j = x_{i_k}\}\Bigr),$$

where $(x_1, \ldots, x_n)$ are the coordinates of $\mathbb{C}^n$ and $i_k := \min S_k$ for each $k > 0$. Note that the dimension of $Z$ is equal to $r$.

Although Theorem 1.2 is worded the same as statements about modular curves, the proof methods are (necessarily) very different. As in the proof of Theorem 1.1, it is important that the PCF parameters are a set of algebraic numbers with bounded Weil height, which is not the case for singular moduli, and in fact of height 0 for a dynamically defined height on $\mathbb{P}^1(\overline{\mathbb{Q}})$. This allows the use of certain arithmetic equidistribution theorems for points of small height; we rely on the recent equidistribution result of [Yuan and Zhang 2023] (though we could have used the older result of [Yuan 2008] as we explain in Remark 2.5). Focusing then on an archimedean place, and via the slicing of positive currents, we reduce the proof of Theorem 1.2 to Luo's theorem on the inhomogeneity of the Mandelbrot set [Luo 2021].

As an application of Theorem 1.2, we obtain uniform versions of Theorem 1.1, in the spirit of Scanlon's automatic uniformity [Scanlon 2004] (though we give a direct proof, not relying on [Scanlon 2004, Theorem 2.4]).

**Theorem 1.3.** *Fix $d \in \mathbb{N}$. There is a constant $M(d) < \infty$ such that*

$$\#\{\text{special points in } C\} \leq M(d),$$

*for all complex algebraic curves $C \subset \mathbb{C}^2$ of degree $d$ without special components.*

It is natural to ask how many special points can lie on a nonspecial curve in $\mathbb{C}^2$. We obtain an explicit bound for the general curve of degree $d$:

**Theorem 1.4.** *Fix $d \in \mathbb{N}$, and let $X_d$ denote the Chow variety of all plane curves with degree $\leq d$. There exists a Zariski-closed strict subvariety $V_d \subset X_d$ such that*

$$\#\{\text{special points in } C\} \leq \tfrac{1}{2}d(d+3)$$

*for all curves $C \in X_d \setminus V_d$.*

The upper bound in Theorem 1.4 is optimal:

**Theorem 1.5.** *There is a Zariski-dense subset $S_d \subset X_d$ such that*

$$\#\{\text{special points in } C\} = \tfrac{1}{2}d(d+3)$$

*for all curves $C \in S_d$.*

Note that $\frac{1}{2}d(d+3)$ is the dimension of the space $X_d$, and this is no accident. It is well known that there exists a curve of degree $d$ through any collection of $N_d = \frac{1}{2}d(d+3)$ points in $\mathbb{C}^2$. Choosing those points to be special, we can build a Zariski-dense collection of curves in $X_d$ containing at least $N_d$ special points. The upper bound of Theorem 1.4 is obtained by showing there are no unexpected symmetries among general special-point configurations, as a consequence of Theorem 1.2 and the explicit description of the special subvarieties.

Our proof does not give a complete description of the exceptional variety $V_d$ in Theorem 1.4, though the methods can be used to classify its positive-dimensional components. For example, in the case of $d = 1$, we show:

**Theorem 1.6.** *All but finitely many nonspecial lines in $\mathbb{C}^2$ contain at most 2 special points.*

In other words, the subvariety $V_1$ of Theorem 1.4 can be taken to be the union of the 1-parameter families in $X_1$ of horizontal and vertical lines, together with a finite set of points in $X_1$. A more detailed result about lines in $\mathbb{C}^2$ is stated as Proposition 4.4. The analogue of Theorem 1.6 in the setting of CM points in $\mathbb{C}^2$ was proved in [Bilu et al. 2017].

Note that the finite set of nonspecial lines in $\mathbb{C}^2$ containing at least 3 special points is not empty. For example, the line $\{y = -x\}$ passes through $(0, 0)$, $(i, -i)$, and $(-i, i)$; the line $\{y = ix\}$ passes through $(0, 0)$, $(-1, -i)$, and $(i, -1)$; and $\{y = -ix\}$ passes through $(0, 0)$, $(-1, i)$, and $(-i, -1)$.

**Question 1.7.** *How many nonspecial lines in $\mathbb{C}^2$ pass through at least 3 distinct special points, and what is the optimal value of $M(1)$ in Theorem 1.3?*

As mentioned after Theorem 1.1, a real algebraic curve in $\mathbb{R}^2 = \mathbb{C}$ passing through a given parameter $c$ in the Mandelbrot set gives rise to a complex algebraic curve in $\mathbb{C}^2$ passing through the point $(c, \bar{c})$ (see Section 5). Moreover, the subset of such curves is Zariski dense in $X_d$ for each degree $d \geq 1$. Theorems 1.3 and 1.4 therefore apply to bound PCF parameters on real algebraic curves of a given degree in $\mathbb{R}^2 = \mathbb{C}$. For example, we have:

**Theorem 1.8.** *There is a uniform bound on the number of PCF parameters on any real algebraic curve in $\mathbb{R}^2 = \mathbb{C}$ depending only on the degree of the curve (as long as the curve does not contain the real axis). Moreover, there are only finitely many real lines in $\mathbb{C}$ that contain more than two PCF parameters.*

Note that the finite set of real lines in $\mathbb{C}$ containing more than two PCF parameters is not empty: the real axis contains infinitely many and the imaginary axis contains at least 3 (at $c = 0$ and $c = \pm i$).

**Remark 1.9.** Finiteness results analogous to Theorem 1.6, upon replacing "lines" with "curves of degree $d$" and the bound of 2 with $\frac{1}{2}d(d+3)$, do not hold for $d > 1$. For example, for algebraic curves of degree $d = 2$, we know there is a conic through any 5 given points in $\mathbb{C}^2$, and 5 is the optimal bound on special points in general conics (by Theorems 1.4 and 1.5), but there is a Zariski-dense set of curves in the 3-dimensional space of conics

$$x^2 + y^2 + Axy + B(x + y) + C = 0$$

in $\mathbb{C}^2$ containing at least 6 special points. Indeed, 3 given special points in $\mathbb{C}^2$ (generally) determine the coefficients $A$, $B$, $C$, and the $(x, y) \mapsto (y, x)$ symmetry of the curve (generally) guarantees an additional 3 special points. For real conics in $\mathbb{R}^2 = \mathbb{C}$, one can do the same with symmetry under complex conjugation; see Figure 2.

***Outline.*** In Section 2, we prove Theorem 1.2. Section 3 provides a brief review of the Chow variety $X_d$ and basic results on families of curves passing through points. Section 4 contains the proofs of Theorems 1.3, 1.4, 1.5, and 1.6. Finally, in Section 5, we look at real algebraic curves passing through PCF parameters in the Mandelbrot set and prove Theorem 1.8.

## 2. Proof of Theorem 1.2

In this section we prove Theorem 1.2. In fact we prove a stronger result, showing that our classification theorem remains true if we treat small points in addition to the special points. Our notion of size is given by a height function

$$h_{\text{crit}}(c_1, \ldots, c_n) := \sum_{i=1}^{n} \hat{h}_{f_{c_i}}(0) \geq 0 \tag{2-1}$$

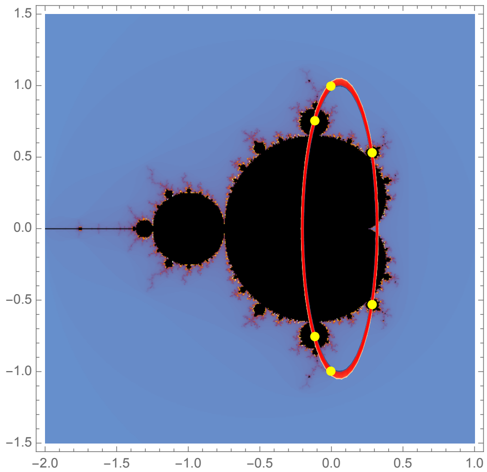


**Figure 2.** A general real conic in $\mathbb{R}^2$ contains no more than 5 PCF parameters, but there are infinitely many symmetric conics with at least 6 special points.

for $(c_1, \dots, c_n) \in \overline{\mathbb{Q}}^n$. Here $\hat{h}_{f_{c_i}}$ is the canonical height associated to the quadratic polynomial $z^2 + c_i$, introduced in [Call and Silverman 1993]. We say that a sequence $\{x_k\} \subset \overline{\mathbb{Q}}^n$ is *small* if $h_{\text{crit}}(x_k) \to 0$ as $k \to \infty$. Notice that our special points of $\mathbb{C}^n$ are precisely the zeros of $h_{\text{crit}}$.

Let $Y \subset \mathbb{C}^n$ be a variety. A sequence $\{x_k\} \subset Y$ is called *generic* if no subsequence lies in a proper subvariety of $Y$.

**Theorem 2.1.** *Let $n \geq 2$. Let $X$ be an irreducible algebraic subvariety in $\mathbb{C}^n$ defined over $\overline{\mathbb{Q}}$. Then $X$ contains a generic sequence of small points if and only if $X$ is special.*

The idea of considering points that are small with respect to some height function originates in [Bogomolov 1980]; in a dynamical context, see, for example, [Ghioca et al. 2015, Conjecture 2.3] or [Zhang 2006].

**Remark 2.2.** In Theorem 2.1 we have assumed that $X$ is defined over $\overline{\mathbb{Q}}$, which is not the case in Theorem 1.2. However, our special points are defined over $\overline{\mathbb{Q}}$ so that a subvariety that contains a Zariski dense set of special points is automatically defined over $\overline{\mathbb{Q}}$. Therefore, Theorem 1.2 follows from Theorem 2.1. Note here that the structure of the special subvarieties ensures that they contain a Zariski dense set of special points.

**Remark 2.3.** Assuming $X$ is a curve in $\mathbb{C}^2$, the conclusion of Theorem 2.1 is not contained in [Ghioca et al. 2017] but follows immediately from the proof of Theorem B in [Favre and Gauthier 2022].

**2.1.** ***Arithmetic equidistribution.*** The first key ingredient in our proof is the following equidistribution theorem. Let $\mathcal{M}$ denote the Mandelbrot set in $\mathbb{C}$. Let $\mu_{\mathcal{M}}$ denote the bifurcation measure on $\mathcal{M}$. As computed in [DeMarco 2001, §6], $\mu_{\mathcal{M}}$ is proportional to the harmonic measure supported on $\partial\mathcal{M}$ for the domain $\hat{\mathbb{C}} \setminus \mathcal{M}$, relative to the point at $\infty$. The support of $\mu_{\mathcal{M}}$ is equal to the boundary of $\mathcal{M}$; it has continuous potentials and total mass equal to $\frac{1}{2}$.

**Theorem 2.4.** *Let $n \geq 2$ and $H \subset \mathbb{C}^n$ be an irreducible hypersurface defined over a number field $K$. Assume that the projection $p_j : H \to \mathbb{C}^{n-1}$ omitting the $j$-th coordinate is dominant. Then for any generic sequence $\{x_k\} \subset H(\overline{K})$ of small points, their* $\mathrm{Gal}(\overline{K}/K)$*-orbits equidistribute to the probability measure*

$$\mu_j := c\,(\pi_1|_H)^*(\mu_{\mathcal{M}}) \wedge \cdots \wedge (\pi_{j-1}|_H)^*(\mu_{\mathcal{M}}) \wedge (\pi_{j+1}|_H)^*(\mu_{\mathcal{M}}) \wedge \cdots \wedge (\pi_n|_H)^*(\mu_{\mathcal{M}})$$

*on $H(\mathbb{C})$, where $\pi_i : \mathbb{C}^n \to \mathbb{C}$ is the projection to the $i$-th coordinate, $(\pi_i|_H)^*\mu_{\mathcal{M}}$ is the pullback as a* $(1,1)$*-current, and $c > 0$ is a positive constant. That is, for any continuous function $\varphi$ on $H$ with compact support in the smooth part of $H$, we have*

$$\frac{1}{\#\,\mathrm{Gal}(\overline{K}/K)\cdot x_k} \sum_{y \in \mathrm{Gal}(\overline{K}/K)\cdot x_k} \varphi(y) \to \int \varphi \, d\mu_j \quad \text{as } k \to \infty.$$

To prove Theorem 2.4, we rely on the recent theory on adelic line bundles developed in [Yuan and Zhang 2023]. We let $f : \mathbb{A}^1 \times \mathbb{P}^1 \to \mathbb{A}^1 \times \mathbb{P}^1$ be the algebraic family of unicritical quadratic polynomials

$$f(t,z) = (t, z^2 + t),$$

defined over $\mathbb{Q}$. Let $L$ be the line bundle on $\mathbb{A}^1 \times \mathbb{P}^1$, isomorphic to $\mathcal{O}(1)$ on fibers $\mathbb{P}^1$ and such that $f^*L = 2L$. We denote by $\bar{L}_f$ the $f$-invariant extension of $L$ as defined in [Yuan and Zhang 2023, Theorem 6.1.1]. Let $i : \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{P}^1$ be defined by $i(t) = (t, 0)$ and define

$$\bar{L}_{\mathrm{crit}} := i^* \bar{L}_f.$$

This is an adelic line bundle on $\mathbb{A}^1$ as in [Yuan and Zhang 2023, §6.2.1]. Furthermore, by [Yuan and Zhang 2023, Lemma 6.2.1], the height associated to $\bar{L}_{\mathrm{crit}}$ is given by

$$h_{\bar{L}_{\mathrm{crit}}}(t) = \hat{h}_{f_t}(0) = h_{\mathrm{crit}}(t) \tag{2-2}$$

for each $t \in \mathbb{A}^1(\overline{\mathbb{Q}})$, where $h_{\mathrm{crit}}$ is the height defined in (2-1) with $n = 1$. By construction, we have

$$c_1(\bar{L}_{\mathrm{crit}}) = \mu_{\mathcal{M}} \tag{2-3}$$

at the archimedean place of $\mathbb{Q}$.

**Remark 2.5.** To prove Theorem 2.4, we use the recent equidistribution theory in quasiprojective varieties developed in [Yuan and Zhang 2023]. However, it is known that $\bar{L}_{\mathrm{crit}}$ extends to define an adelic metrized line bundle on the projective line $\mathbb{P}^1$; see, for example, [Favre and Rivera-Letelier 2006, §6.5]. Arguments similar to the ones in [Mavraki et al. 2023] would allow us to use the equidistribution result established in [Yuan 2008] instead, applied to a projective compactification of the $H$ in Theorem 2.4 and a modification of the metrized line bundles $\overline{M}_j$ we define in (2-4) below, but the results from [Yuan and Zhang 2023] considerably simplify the exposition here.

*Proof of Theorem 2.4.* Fix $j$ as in the statement of the theorem. As $\mu_{\mathcal{M}}$ has continuous potentials, we deduce that $\mu_j$ does not put mass on the singular locus $H^{\mathrm{sing}}$ of $H$, so we may replace $H$ with

$H \setminus H^{\text{sing}}$ and assume that $H$ is smooth. We define a metrized line bundle on the smooth, quasiprojective hypersurface $H$ by

$$\overline{M}_j = \bigotimes_{i \neq j} (\pi_i|_H)^*(\bar{L}_{\text{crit}}). \tag{2-4}$$

This defines an adelic line bundle on $H$, so that $\overline{M}_j \in \widehat{\text{Pic}}(H)_{\mathbb{Q}}$ in the notation of [Yuan and Zhang 2023]. By [Yuan and Zhang 2023, Theorem 6.1.1] we know that $\bar{L}_f$ is nef in the sense of [Yuan and Zhang 2023] and by the functoriality of nefness we also have that $\overline{M}_j$ is nef; see [Yuan and Zhang 2023, page 8]. In what follows we work with the standard absolute on $\mathbb{C}$. As in [Yuan and Zhang 2023, Lemma 6.3.7], we know from (2-3) that the curvature form associated to $\bar{L}_{\text{crit}}$ (at the archimedean place of $\mathbb{Q}$) is equal to $\mu_{\mathcal{M}}$. Since $c_1(\bar{L}_{\text{crit}})^{\wedge 2} \equiv 0$, we thus see that

$$c_1(\overline{M}_j)^{\wedge(n-1)} = (n-1)!\,(\pi_1|_H)^*(\mu_{\mathcal{M}}) \wedge \cdots \wedge \widehat{(\pi_j|_H)^*(\mu_{\mathcal{M}})} \wedge \cdots \wedge (\pi_n|_H)^*(\mu_{\mathcal{M}}),$$

where the $\widehat{\cdot}$ means the $j$-th term is omitted and the pullbacks are defined in the sense of currents. Our assumption that the projection $p_j$ is dominant ensures that this measure is nontrivial. By [Yuan and Zhang 2023, Lemma 5.4.4], we infer that $\overline{M}_j$ is nondegenerate as defined in [Yuan and Zhang 2023, §6.2.2]. In other words, the adelic line bundle $\overline{M}_j$ satisfies all the assumptions of [Yuan and Zhang 2023, Theorem 5.4.3]. Thus, if $\{y_k\} \subset H(\overline{\mathbb{Q}})$ is a generic sequence with $h_{\overline{M}_j}(y_k) \to h_{\overline{M}_j}(H)$, then its Galois conjugates equidistribute with respect to the probability measure associated to $c_1(\overline{M}_j)^{\wedge(n-1)}$.

Now let $\{x_k\} \subset H(\overline{\mathbb{Q}})$ be a generic sequence of small points, as in the statement of the theorem. Note that

$$h_{\overline{M}_j}(x) = \sum_{i \neq j} h_{\text{crit}}(\pi_i(x))$$

for $x \in H(\overline{\mathbb{Q}})$ so that by (2-2) we have

$$\lim_{k \to \infty} h_{\overline{M}_j}(x_k) = 0. \tag{2-5}$$

Therefore, by the number field case of the fundamental inequality [Yuan and Zhang 2023, Theorem 5.3.3], we have that

$$h_{\overline{M}_j}(H) \leq 0. \tag{2-6}$$

Note that here we have used the fact that $\overline{M}_j$ is nef and nondegenerate. By the nefness of $\overline{M}_j$ we also have that $h_{\overline{M}_j}(H) \geq 0$ by [Yuan and Zhang 2023, Proposition 4.1.1]. Thus,

$$h_{\overline{M}_j}(H) = 0.$$

Therefore, the result follows by the equidistribution theorem [Yuan and Zhang 2023, Theorem 5.4.3]. □

**2.2. *Inhomogeneity of $\mathcal{M}$.*** To deduce Theorem 2.1, we will combine Theorem 2.4 with the following result of Luo that the Mandelbrot set has no local symmetries.

**Theorem 2.6** [Luo 2021]. *Let $U$ be an open set in $\mathbb{C}$ with $U \cap \partial\mathcal{M} \neq \varnothing$. Suppose $\varphi : U \to V$ is a conformal isomorphism such that $\varphi(U \cap \partial\mathcal{M}) = V \cap \partial\mathcal{M}$. Then $\varphi$ is the identity.*

We begin by proving Theorem 2.1 for a certain class of hypersurfaces.

**Proposition 2.7.** *Let $n \geq 2$. Assume that $H \subset \mathbb{C}^n$ is an irreducible hypersurface, defined over $\overline{\mathbb{Q}}$, which projects dominantly on each collection of $n-1$ coordinates and which contains a generic sequence of small points. Then $n = 2$ and $H \subset \mathbb{C}^2$ is the diagonal line.*

*Proof.* Let $H$ be a hypersurface as in the statement defined over a number field $K$. In particular $H$ contains a generic small sequence. Since $H$ projects dominantly on each collection of $n-1$ coordinates, we may apply Theorem 2.4 to deduce that the $\mathrm{Gal}(\overline{K}/K)$-orbits of our sequence equidistribute with respect to $\mu_j$ for all $j$ (for the $\mu_j$ in the statement of Theorem 2.4). In particular

$$T \wedge (\pi_{n-1}|_H)^*(\mu_{\mathcal{M}}) = \alpha \cdot T \wedge (\pi_n|_H)^*(\mu_{\mathcal{M}}) \tag{2-7}$$

for some constant $\alpha > 0$, where $T = (\pi_1|_H)^*(\mu_{\mathcal{M}}) \wedge \cdots \wedge (\pi_{n-2}|_H)^*(\mu_{\mathcal{M}})$ is an $(n-2, n-2)$-current on $H$.

For $n = 2$, equation (2-7) means that $(\pi_1|_H)^*(\mu_{\mathcal{M}}) = \alpha \cdot (\pi_2|_H)^*(\mu_{\mathcal{M}})$ on the curve $H$ in $\mathbb{C}^2$. But the projections are locally invertible away from finitely many points, so the measure equality induces a local isomorphism between a neighborhood of a point in $\partial\mathcal{M} \subset \mathbb{C}$ and its image. Theorem 2.6 then implies that this local isomorphism is the identity. That is, the curve $H$ must be the diagonal line in $\mathbb{C}^2$ as claimed.

Assume now that $n \geq 3$. Let $\pi : H \to \mathbb{C}^{n-2}$ be the projection to the first $n-2$ coordinates. Observe that $T = \pi^*(\nu)$ for the measure $\nu = p_1^*(\mu_{\mathcal{M}}) \wedge \cdots \wedge p_{n-2}^*(\mu_{\mathcal{M}})$ on $\mathbb{C}^{n-2}$, where $p_i : \mathbb{C}^{n-2} \to \mathbb{C}$ is the projection to the $i$-th coordinate.

By our assumption, $\pi$ is dominant and the fiber-dimension theorem yields that the fibers

$$H_z := H \cap \{x_1 = z_1, \ldots, x_{n-2} = z_{n-2}\} \subset H$$

are curves for $z = (z_1, \ldots, z_n)$ in a Zariski open and dense subset of $\mathbb{C}^{n-2}$. Note that each $(1, 1)$-current $p_j^*\mu_{\mathcal{M}}$ has continuous potentials on $\mathbb{C}^{n-2}$, so the measure $\nu$ does not charge pluripolar sets. Thus the fiber $H_z$ is a curve for $\nu$-almost every $z$, and, by the characterization of slicing of currents as in [Bassanelli and Berteloot 2007, Proposition 4.3], equation (2-7) implies that

$$\int_{\mathbb{C}^{n-2}} \left( \int_{H_z} \varphi \, d\pi_{n-1}|_{H_z}^*(\mu_{\mathcal{M}})(x) \right) d\nu(z) = \alpha \int_{\mathbb{C}^{n-2}} \left( \int_{H_z} \varphi \, d\pi_n|_{H_z}^*(\mu_{\mathcal{M}})(x) \right) d\nu(z), \tag{2-8}$$

for every continuous and compactly supported function $\varphi$ on $H$. It follows that we have equality of measures

$$\pi_{n-1}|_{H_z}^*(\mu_{\mathcal{M}}) = \alpha \cdot \pi_n|_{H_z}^*(\mu_{\mathcal{M}}) \tag{2-9}$$

for $\nu$-almost every $z := (z_1, \ldots, z_{n-2})$ in $\mathbb{C}^{n-2}$. In detail, suppose there exists a point $z_0$ in the support of $\nu$, where $H_{z_0}$ is a curve such that

$$\pi_{n-1}|_{H_{z_0}}^*(\mu_{\mathcal{M}}) \neq \alpha \cdot \pi_n|_{H_{z_0}}^*(\mu_{\mathcal{M}}).$$

Then we can find a continuous function $\psi_{z_0}$ on $H_{z_0}$ such that

$$\int_{H_{z_0}} \psi_{z_0} \pi_{n-1}|^*_{H_{z_0}}(\mu_{\mathcal{M}}) \neq \alpha \int_{H_{z_0}} \psi_{z_0} \pi_n|^*_{H_{z_0}}(\mu_{\mathcal{M}}).$$

Note that the measures $\pi_i|^*_{H_z}(\mu_{\mathcal{M}})$ vary continuously as functions on $z$ on a neighborhood of $z_0$, by the continuity of the potentials. We thus infer that

$$\int_{H_z} \psi_{z_0} \pi_{n-1}|^*_{H_z}(\mu_{\mathcal{M}}) \neq \alpha \int_{H_z} \psi_{z_0} \pi_n|^*_{H_z}(\mu_{\mathcal{M}})$$

for all $z$ in a small open neighborhood $U$ of $z_0$. We can therefore find $\varphi = h \cdot \psi_{z_0}$, where $h$ is a continuous function supported on $U$ and for which the equality (2-8) fails.

Again by Theorem 2.6, equation (2-9) yields that $H_z$ is special for $\nu$-almost all $z$. Since $T$ does not charge pluripolar sets and since $H$ projects dominantly on each $n-1$ coordinates, we infer that $H \subset \pi^{-1}_{(n-1,n)}(\Delta)$, where $\pi_{(i,j)} : \mathbb{C}^n \to \mathbb{C}^2$ is the projection to the $i$-th and $j$-th coordinates. Repeating the argument using the equalities of all measures $\mu_j$, we get

$$H \subset \bigcap_{i \neq j \in \{1,\dots,n\}} \pi^{-1}_{(i,j)}(\Delta).$$

But since $H$ has dimension $n-1$ and $n \geq 3$ this is impossible. This completes our proof. □

**2.3.** ***Proof of Theorem 2.1.*** We can now complete the proof of Theorem 2.1 (and so also of Theorem 1.2) by reducing it to Proposition 2.7. This argument is inspired by [Ghioca et al. 2018].

First we show that if Theorem 2.1 holds for hypersurfaces $X$, then it holds in general. So assume that the theorem is true when $X$ is a hypersurface and let $X$ be an irreducible subvariety of $\mathbb{C}^n$ with dimension $d < n-1$ which contains a generic sequence of small points. Permuting the coordinates if necessary, we may assume that $X$ projects dominantly to the first $d$ coordinates. Now let $\pi_{(j)} : \mathbb{C}^n \to \mathbb{C}^{d+1}$ denote the projection to the first $d$ and the $j$-th coordinates. Let $X_j$ denote the Zariski closure of $\pi^{-1}_{(j)}(\pi_{(j)}(X))$ in $\mathbb{C}^n$. Each $X_j$ is a hypersurface in $\mathbb{C}^n$ and contains a generic sequence of small points. Therefore by our assumption $X_j$ must be special. If $X \subset X_j$ is special, then our claim follows. Otherwise $X_j = \{(x_1, \dots, x_n) \in \mathbb{C}^n : x_j = c_j\}$ for a special point $c_j$ or $X_j = \{(x_1, \dots, x_n) \in \mathbb{C}^n : x_j = x_k\}$ for some $k \in \{1, \dots, d\}$. From the precise form of each $X_j$, it is easy to see that $\bigcap_{j=d+1}^n X_j$ has dimension $d = \dim X$. But $X \subset \bigcap_{j=d+1}^n X_j$, so we must have $X = \bigcap_{j=d+1}^n X_j$ and our claim follows.

Therefore, it suffices to prove Theorem 2.1 for hypersurfaces $X$. Arguing by induction on $n$, we may further assume that $X$ projects dominantly on each $n-1$ coordinates. Indeed, if $n = 2$ and the curve $X$ is vertical or horizontal then since it contains a generic sequence of small points, it must be special. Assume now that $n \geq 3$, and that $X$ does not project dominantly on, say, the last $n-1$ coordinates. Then it has the form $X = \mathbb{C} \times X_0$ for a hypersurface $X_0 \subset \mathbb{C}^{n-1}$ (see, e.g., [Mavraki et al. 2023, Lemma 3.1]). By induction, Theorem 2.1 follows by Proposition 2.7. As explained in Remark 2.2, Theorem 1.2 also holds.

## 3. Maximal variation and the lower bound

In this section, we provide some basic background on the Chow variety $X_d$ of curves of degree $\leq d$ in $\mathbb{C}^2$, and we prove lower bounds on the number of special points in families of curves.

**3.1.** ***Chow and maximal variation.*** Fix integer $d \geq 1$. We work with the Chow variety $X_d$ of algebraic curves in the plane $\mathbb{C}^2$, defined over the field $\mathbb{C}$ of complex numbers, of degree $\leq d$. As a variety, $X_d$ is simply the complement of a single point in a projective space $\mathbb{P}_{\mathbb{C}}^{N_d}$, where

$$N_d = \binom{d+2}{2} - 1 = \frac{d(d+3)}{2}.$$

Indeed, each curve is the vanishing locus of a nonzero homogeneous polynomial $F(x, y, z)$ of degree $d$, uniquely determined up to scale, and evaluated at points of the form $(x, y, 1)$ for $(x, y) \in \mathbb{C}^2$. We exclude the polynomial $F(x, y, z) = z^d$.

Let $\mathcal{C} \to V$ be a family of plane algebraic curves, parameterized by an algebraic variety $V$ defined over $\mathbb{C}$. There is an induced map from $V$ to $X_d$ for some degree $d$. We say that the family $\mathcal{C} \to V$ is *maximally varying* if the induced map $V \to X_d$ has finite fibers. For each integer $m \geq 1$, we let

$$\mathcal{C}_V^m := \mathcal{C} \times_V \cdots \times_V \mathcal{C} \tag{3-1}$$

denote the $m$-th fiber power of $\mathcal{C}$ over $V$. There is a natural map

$$\rho_m : \mathcal{C}_V^m \to \mathbb{C}^{2m}$$

defined by sending a tuple of $m$ points $x_1, \ldots, x_m$ on a curve $C \in V$ to the $m$-tuple $(x_1, \ldots, x_m)$ in $(\mathbb{C}^2)^m$.

**Proposition 3.1.** *Suppose that $V$ is an irreducible quasiprojective complex algebraic variety of dimension $\ell \geq 1$. If $\mathcal{C} \to V$ is a maximally varying family of curves in $\mathbb{C}^2$ of degree $\leq d$, then the natural map $\rho_m$ is dominant for all $m \leq \ell$ and generically finite for $m = \ell$.*

*Proof.* The result is clear for $m = 1$, because the image of $\mathcal{C}$ in $\mathbb{C}^2$ cannot be contained in a single algebraic curve if the image of $V$ in $X_d$ is not a point. For $m > 1$, it suffices to show that the image of $\rho_m$ contains the union of subvarieties of the form $\{(z_1, \ldots, z_{m-1})\} \times U_{(z_1,\ldots,z_{m-1})}$, where $U_{(z_1,\ldots,z_{m-1})}$ is Zariski open in $\mathbb{C}^2$, over a Zariski dense and open subset of points $(z_1, \ldots, z_{m-1}) \in (\mathbb{C}^2)^{m-1}$. Indeed, the dominance follows because the maps are algebraic, and the generic finiteness for $m = \ell$ follows because $\dim \mathcal{C}_V^m = \ell + m$.

We proceed by induction. We have already seen that the result holds for $m = 1$ and any $\ell \geq 1$. Now assume $\ell > 1$, and fix $1 < m \leq \ell$. Assume the result holds for $\rho_{m-1}$. Then, as the smooth part $V^{\mathrm{sm}}$ of $V$ is Zariski open and dense, the image of $\rho_{m-1}$ restricted to $\mathcal{C}_V^{m-1}$ over $V^{\mathrm{sm}}$ contains a Zariski open set $U \subset \mathbb{C}^{2(m-1)}$. Choose any point $(z_1, \ldots, z_{m-1})$ in $U$. Suppose that $\lambda_0 \in V^{\mathrm{sm}}$ is a parameter for which $C_{\lambda_0}$ contains the points $z_1, \ldots, z_{m-1}$. There is a subvariety $V_1$ of $V$ containing $\lambda_0$ and with codimension $\leq m - 1$ consisting of curves $C_\lambda$ that persistently contain the points $z_1, \ldots, z_{m-1}$. In particular, the dimension of $V_1$ is at least 1. Maximal variation implies that the image of $\rho_1$ on $\mathcal{C}$ over $V_1$ is dominant to $\mathbb{C}^2$. It follows that $\rho_m$ on $\mathcal{C}^m_{V_1}$ is dominant to $\{(z_1, \dots, z_{m-1})\} \times \mathbb{C}^2$. Letting the point $(z_1, \dots, z_{m-1})$ vary over the image of $\rho_{m-1}$, we use the induction hypothesis to see that $\rho_m$ is dominant from $\mathcal{C}^m_V$ to $\mathbb{C}^{2m}$. □

**3.2.** ***A lower bound on the number of special points.*** For a family of plane algebraic curves $\mathcal{C} \to V$ parameterized by $V$, recall the definition of $\mathcal{C}^m_V$ in (3-1).

**Proposition 3.2.** *Suppose that $\mathcal{C} \to V$ is a maximally varying family of irreducible, complex algebraic curves in $\mathbb{C}^2$, over an irreducible quasiprojective complex algebraic variety $V$ of dimension $\ell \geq 1$. Then the preimage $\rho_\ell^{-1}(S)$ of the set of special points $S \subset \mathbb{C}^{2\ell}$ is Zariski dense in $\mathcal{C}^\ell_V$. In particular, there is a Zariski dense set of curves $\lambda \in V$ for which the fiber $C_\lambda$ contains at least $\ell$ distinct special points of $\mathbb{C}^2$.*

*Proof.* By maximal variation, we know that the fiber product $\mathcal{C}^\ell_V$ maps generically finitely and dominantly by $\rho_\ell$ to $\mathbb{C}^{2\ell}$. The special points are Zariski dense in the image. This implies that the set of points $P = (\lambda, x_1, \dots, x_\ell) \in \rho_\ell^{-1}(S) \subset \mathcal{C}^\ell_V$ is Zariski dense in $\mathcal{C}^\ell_V$, where $\lambda \in V$ and $\{x_1, \dots, x_\ell\}$ is a collection of special points on the fiber $C_\lambda$ of $\mathcal{C}$ over $\lambda$. In particular, the $x_i$ must be generally all distinct. □

## 4. Optimal general upper bounds

In this section we prove Theorems 1.3, 1.4, 1.5, and 1.6. For each integer $d \geq 1$, let $X_d$ denote the Chow variety of all algebraic curves in $\mathbb{C}^2$ of degree $\leq d$ defined over $\mathbb{C}$.

**4.1.** ***The uniform bound of Theorem 1.3.*** Let $\mathcal{C} \to V$ denote a family of algebraic curves in $\mathbb{C}^2$, parameterized by an irreducible, quasiprojective variety $V$ over $\mathbb{C}$ of dimension $\ell \geq 1$, for which the general curve in the family is irreducible. As introduced in Section 3.1, there is a natural map

$$\rho_1 : \mathcal{C} \to \mathbb{C}^2,$$

sending each curve to its image in $\mathbb{C}^2$. Recall the definitions of $\mathcal{C}^m_V$ and

$$\rho_m : \mathcal{C}^m_V \to \mathbb{C}^{2m}$$

given there, for each integer $m \geq 1$. Recall also that the family is maximally varying if the induced map $V \to X_d$ has finite fibers. From Proposition 3.1, we know that maximal variation implies that the maps $\rho_m$ are dominant for all $m \leq \dim V$.

**Proposition 4.1.** *Suppose that $\mathcal{C} \to V$ is a maximally varying family of curves in $\mathbb{C}^2$ with $\ell = \dim V > 0$. Assume the general curve in the family is irreducible. Then the Zariski closure of the image in $\mathbb{C}^{2(\ell+1)}$ of the fiber power $\mathcal{C}^{\ell+1}_V$ by $\rho_{\ell+1}$ is not special, unless $\ell = 1$ and $\mathcal{C}$ is a family of horizontal or vertical lines in $\mathbb{C}^2$.*

As a consequence we have:

**Theorem 4.2.** *For any family $\mathcal{C} \to V$ of curves in $\mathbb{C}^2$ — irreducible or not, maximally varying or not — there is a uniform upper bound $M = M(\mathcal{C})$ on the number of special points on $C_\lambda$, for all $\lambda \in V$ over which the fiber $C_\lambda$ of $\mathcal{C}$ has no special irreducible components.*

*Proof of Theorem 4.2.* Assume Proposition 4.1. If $V$ or the generic curve is reducible, we work with irreducible components. If the family fails to be maximally varying, it is convenient to factor through the image of $V$ in the Chow variety $X_d$ for some degree $d$. So we now assume that $V$ is an irreducible subvariety in $X_d$ of dimension $\geq 1$ and the associated curve family $\mathcal{C} \to V$ consists of generally irreducible curves and not exclusively of horizontal or vertical lines.

Consider the fiber powers $\mathcal{C}_V^m \to V$ for each $m \geq 1$. Suppose there is a generic sequence of points $C_n \in V$ for which the number of special points of the curves $C_n \subset \mathbb{C}^2$ is larger than $n$, for each $n \in \mathbb{N}$. This implies that the special points of $\mathbb{C}^{2m}$ are Zariski dense in the images $\rho_m(\mathcal{C}_V^m)$ for every $m \geq 1$. Indeed, this is clear for $m = 1$ because $\rho_1$ maps $\mathcal{C}$ dominantly to $\mathbb{C}^2$. For each positive integer $m \geq 2$, the set of special points in $\rho_m(\mathcal{C}_V^m)$ includes the $m$-tuples formed from the $n$ distinct special points on $C_n$; note that this set of $m$-tuples in $\mathbb{C}^{2m}$ is symmetric under permutation of the $m$ copies of $\mathbb{C}^2$. Let $Z_m$ be the Zariski closure of these special points within $\rho_m(\mathcal{C}_V^m)$; note that $Z_m$ is also symmetric under permutation of the $m$ copies of $\mathbb{C}^2$. Because $\{C_n\}$ is a generic sequence in $V$, note also that $\rho_m^{-1}(Z_m)$ must project dominantly to $V$. If $Z_m$ is not equal to all of $\rho_m(\mathcal{C}_V^m)$, then $\rho_m^{-1}(Z_m)$ is contained in a subvariety $H \subset \mathcal{C}_V^m$ which is a family of hypersurfaces over $V$ that are symmetric with respect to permutation of the components in each fiber $C \times \cdots \times C$. Now consider the projections from $\mathcal{C}_V^m \to \mathcal{C}_V^{m-1}$ forgetting one factor, restricted to the hypersurface $H$. By the symmetry of $H$, each of these projections is generically finite and of the same degree, say $r(m)$. So over a Zariski open subset of $V$, this bounds the number of points on a given curve $C$; in particular this contradicts the assumption on the sequence of curves $C_n$. So the special points of $\mathbb{C}^{2m}$ must be Zariski dense in $\rho_m(\mathcal{C}_V^m)$ for every $m \geq 1$.

From Theorem 1.2, the density of special points in $\rho_m(\mathcal{C}_V^m)$ implies that (the Zariski closure) of $\rho_m(\mathcal{C}_V^m)$ is special. But taking $m = \dim V + 1$, this contradicts Proposition 4.1.

So there is a uniform bound on the number of special points in the curve $C \subset \mathbb{C}^2$ for all curves $C$ in a Zariski open subset $U$ of $V$. We then repeat the argument on each of the finitely many irreducible components of $V \setminus U$. We continue until we are left with families of vertical or horizontal lines. □

*Proof of Proposition 4.1.* From Proposition 3.1, we know that the map $\rho_m : \mathcal{C}_V^m \to \mathbb{C}^{2m}$ is dominant for all $m \leq \ell$ and generically finite for $m = \ell$. Note that $\dim \mathcal{C}_V^{\ell+1} = 2\ell + 1 < 2\ell + 2$, so the map

$$\rho_{\ell+1} : \mathcal{C}_V^{\ell+1} \to \mathbb{C}^{2(\ell+1)}$$

cannot be dominant. Consider the projections $\pi_{ij}$ from $\mathcal{C}_V^{\ell+1}$ to $\mathbb{C}^2$ defined by composing $\rho_{\ell+1}$ with

$$(x_1, \ldots, x_{2\ell+2}) \mapsto (x_i, x_j)$$

for each pair $1 \leq i < j \leq 2\ell + 2$.

Assume $\ell > 1$. The projections $\pi_{ij}$ are dominant for all pairs $i < j$, because they factor through the dominant maps $\mathcal{C}_V^{\ell+1} \to \mathcal{C}_V^{\ell} \to \mathbb{C}^{2\ell}$, where the first arrow forgets the $k$-th factor of $\mathcal{C}$ over $V$ for some choice of indices $\{2k-1, 2k\}$ not containing $i$ or $j$, and the second arrow is $\rho_\ell$. In view of the structure of special subvarieties from Theorem 1.2, we see immediately that $\rho_{\ell+1}(\mathcal{C}_V^{\ell+1})$ cannot be special in $\mathbb{C}^{2\ell+2}$.

Now suppose that $\ell = 1$, and assume that $\mathcal{C}$ is not a family of vertical or horizontal lines. We aim to show that $\rho_2(\mathcal{C}_V^2)$ is not a special hypersurface in $\mathbb{C}^4$. Note that a general curve $C$ in the family $\mathcal{C}$ projects dominantly to both coordinates in $\mathbb{C}^2$. It follows that $\rho_2(\mathcal{C}_V^2)$ cannot be contained in the hyperplane $\{x_i = c_i\}$ for a special parameter $c_i$ and any $i \in \{1, 2, 3, 4\}$. Because the curves over $V$ are not all equal to the diagonal line in $\mathbb{C}^2$, the space $\rho_2(\mathcal{C}_V^2)$ also cannot be contained in the special hypersurfaces defined by $\{x_k = x_{k+1}\}$ for $k \in \{1, 3\}$. Recalling the definition of special subvarieties, it remains to check that $\rho_2(\mathcal{C}_V^2)$ does not lie in any of the hypersurfaces $\{x_1 = x_3\}$, $\{x_1 = x_4\}$, $\{x_2 = x_3\}$, or $\{x_2 = x_4\}$. But for a general choice of curve $C$ in the family, the product $C \times C \subset \mathbb{C}^4$ maps dominantly to the spaces of pairs with coordinates $(x_1, x_3)$, $(x_2, x_4)$, $(x_2, x_3)$, or $(x_1, x_4)$. This proves that $\rho_2(\mathcal{C}_V^2)$ is not special.

Finally suppose that $\mathcal{C}$ is a family of vertical or horizontal lines. For concreteness, we can take $V = \mathbb{C}$ and $\lambda \in V$ corresponding to the vertical line $\{x = \lambda\}$ for $\lambda \in V$. Then the image of $\mathcal{C}_V^2$ in $\mathbb{C}^4$ is the set of all 4-tuples $(\lambda, x_2, \lambda, x_4)$ for any $(\lambda, x_2, x_4) \in \mathbb{C}^3$. In other words, the image of $\mathcal{C}_V^2$ is the special hypersurface defined by $\{x_1 = x_3\}$. Similarly for families of horizontal lines. □

*Proof of Theorem 1.3.* The theorem is an immediate consequence of Theorem 4.2, taking $V = X_d$. □

**4.2.** ***Optimal general bound over Chow; proof of Theorems 1.4 and 1.5.*** Let $V = X_d$ be the Chow variety of all affine curves of degree $\leq d$ in $\mathbb{C}^2$ and $\mathcal{C} \to V$ the universal family of such curves. Recall from Section 3.1 that

$$N_d := \dim V = \tfrac{1}{2} d(d+3).$$

Consider the fiber power $\mathcal{C}_V^{N_d+1} \to V$ and its image under the natural map

$$\rho := \rho_{N_d+1} : \mathcal{C}_V^{N_d+1} \to \mathbb{C}^{2(N_d+1)}.$$

Suppose that $S$ is a special subvariety of $\mathbb{C}^{2(N_d+1)}$ that is contained in the Zariski closure of the image $\rho(\mathcal{C}_V^{N_d+1})$ for which $\rho^{-1}(S)$ projects dominantly to $V$. We aim to show that $S$ must lie in the union of special diagonals

$$\Delta_{i,j} := \{(x_1, \ldots, x_{2N_d+2}) \in \mathbb{C}^{2N_d+2} : (x_i, x_{i+1}) = (x_j, x_{j+1})\} \tag{4-1}$$

for odd integers $i$ and $j$ satisfying $1 \leq i < j \leq 2N_d + 1$.

This classification will imply the two theorems. Indeed, if there were a generic sequence of elements $C_n \in V$ for which the curve $C_n \subset \mathbb{C}^2$ contains at least $N_d + 1$ distinct special points of $\mathbb{C}^2$, then the $(N_d+1)$-tuples of such points will be special in $\mathbb{C}^{2(N_d+1)}$ and will lie outside of the special diagonals $\Delta_{i,j}$. From Theorem 1.2, each irreducible component $Z$ of the Zariski closure of these special points in $\mathbb{C}^{2(N_d+1)}$ is itself a special subvariety, and by construction is contained in the closure of $\rho(\mathcal{C}_V^{N_d+1})$. As $\{C_n\}$ is a generic sequence of points in $V$, the preimage $\rho^{-1}(Z)$ of each component will project dominantly to $V$. In other words, this $Z$ is a special subvariety of the type described, but not contained in the special diagonals $\Delta_{i,j}$, leading to a contradiction. This will prove Theorem 1.4. The equality of Theorem 1.5 is then a consequence of the lower bound in Proposition 3.2.

For the proof, suppose that $S$ is a special subvariety of $\mathbb{C}^{2(N_d+1)}$ that is contained in the Zariski closure of the image $\rho(\mathcal{C}_V^{N_d+1})$ for which $\rho^{-1}(S)$ projects dominantly to $V$. Recall that our goal is to show that $S$ must lie in the union of the special diagonals $\Delta_{i,j}$. We begin with a few important observations. First, note that $\dim \rho^{-1}(S) \geq \dim S$, so the dominance of the projection to $V$ implies that a general fiber of this projection has dimension $\geq \dim S - N_d$. In other words, the intersection of $S$ with $C \times \cdots \times C$ in $\mathbb{C}^{2(N_d+1)}$ has dimension at least

$$\dim S - N_d = N_d + 2 - \operatorname{codim} S,$$

for a general curve $C$ in $V$. Moreover, as the image $\rho(\mathcal{C}_V^{N_d+1})$ is not itself special in $\mathbb{C}^{2(N_d+1)}$ by Proposition 4.1, we have that $S \subsetneq \overline{\rho(\mathcal{C}_V^{N_d+1})} \subsetneq \mathbb{C}^{2(N_d+1)}$. Therefore, the codimension of $S$ in $\mathbb{C}^{2(N_d+1)}$ is at least 2. We begin by working case by case through some examples of special subvarieties, as classified in Theorem 1.2, to see that they either cannot be contained in $\overline{\rho(\mathcal{C}_V^{N_d+1})}$ or that $\rho^{-1}(S)$ cannot project dominantly to $V$, unless $S$ is contained in one of the special diagonals. We then handle the general case.

- $S = \{x_1 = c_1 \text{ and } x_2 = c_2\}$: A general curve $C \in V$ does not pass through the point $(c_1, c_2) \in \mathbb{C}^2$, so $\rho^{-1}(S)$ cannot project dominantly to $V$.
- $S = \{x_1 = c_1 \text{ and } x_3 = c_3\}$: A general curve in $\mathbb{C}^2$ of degree $d$ projects dominantly to its first coordinate, so $\rho^{-1}(S)$ does project dominantly to $V$ in this case. However, the intersection of $S$ with a general fiber $C \times \cdots \times C \subset \mathbb{C}^{2N_d+2}$ has dimension only $N_d - 1 < \dim S - N_d$, so this $S$ could not have been contained in the closure of $\rho(\mathcal{C}_V^{N_d+1})$.
- $S = \{x_1 = c_1 \text{ and } x_2 = x_3\}$: Again this $S$ has codimension 2, while the intersection with $C \times \cdots \times C$ for a general curve $C \in V$ has dimension only $N_d - 1$.
- $S = \Delta_{1,3} = \{x_1 = x_3 \text{ and } x_2 = x_4\}$: These relations again impose conditions on two of the $N_d + 1$ components of $C \times \cdots \times C$. But note that any collection of $N_d + 1$ points $(x_1, x_2), \ldots, (x_{2N_d+1}, x_{2N_d+2})$ in $\mathbb{C}^2$ satisfying $(x_1, x_2) = (x_3, x_4)$ lie on some curve of degree $d$, because at most $N_d$ of the points are distinct. So all of $\Delta_{1,3}$ is contained in $\rho(\mathcal{C}_V^{N_d+1})$. The intersection of $\Delta_{1,3}$ with a general $C \times \cdots \times C$ has dimension $N_d$, which is the expected dimension.
- $S = \{x_1 = c_1 \text{ and } x_2 = x_3 = x_4\}$: Again we impose relations on only two of the $N_d + 1$ components of $C \times \cdots \times C$, but there are too many relations; a general curve does not intersect both $(c_1, y)$ and $(y, y)$ for any choice of $y \in \mathbb{C}$. Consequently, the preimage $\rho^{-1}(S)$ in $\mathcal{C}_V^{N_d+1}$ does not project dominantly to $V$.
- $S = \{x_1 = c_1 \text{ and } x_2 = x_3 \text{ and } x_4 = c_4\}$: These three relations are imposed upon only two of the $N_d + 1$ components of $C \times \cdots \times C$. As in the previous example, a general curve $C$ does not intersect both $(c_1, y)$ and $(y, c_4)$ for any choice of $y \in \mathbb{C}$. The preimage $\rho^{-1}(S)$ in $\mathcal{C}_V^{N_d+1}$ does not project dominantly to $V$.

In general, recall that since $S$ is special it is defined by imposing "special relations" of the form $x_i = c_i$ for a PCF parameter $c_i$ or $x_k = x_\ell$ for $i, k, \ell \in \{1, \ldots, 2N_d + 2\}$. In general, we see that if we define $S$ by imposing up to $N_d + 1$ special relations on the coordinates of the $N_d + 1$ components of a general product $C \times \cdots \times C$ in $\mathbb{C}^{2(N_d+1)}$, as long as no one point is constant (as in the first example) nor that there are three relations on coordinates of two components (as in the last two examples), nor that two of the coordinates are required to agree (so as to be a subvariety of a special diagonal $\Delta_{i,j}$), then the preimage $\rho^{-1}(S)$ in $\mathcal{C}_V^{N_d+1}$ will project dominantly to $V$, but the general intersection of $S$ with $C \times \cdots \times C$ will not have sufficiently large dimension. That is, this $S$ will not lie in $\overline{\rho(\mathcal{C}_V^{N_d+1})}$ in $\mathbb{C}^{2(N_d+1)}$. If we specify that one of the components is a fixed special point, or if there are at least three relations imposed upon a pair of components of $C \times \cdots \times C$ (as is the case if codim $S > N_d + 1$), then the preimage $\rho^{-1}(S)$ will not project dominantly to $V$. This completes the proofs of Theorems 1.4 and 1.5.

**4.3. *Lines and the proof of Theorem 1.6.*** The classification of special subvarieties (Theorem 1.2) and the proof strategy in Section 4.2 for Theorem 1.4 suggest that the general curve in a "generically chosen" maximally varying family $\mathcal{C} \to V$ of curves in $\mathbb{C}^2$ with dimension $\ell = \dim V$, in any degree, intersects at most $\ell$ distinct special points. Here we show this is indeed the case in degree $d = 1$. Before doing so, we give an example where this expectation fails.

**Example 4.3** (an exceptional family of lines). Consider the pencil of lines in $\mathbb{C}^2$ passing through a given special point $P$, parameterized by $V \simeq \mathbb{P}^1$; for example, take $P = (-1, -2) \in \mathbb{C}^2$. Because we can connect any special point in $\mathbb{C}^2$ to $P$ with a line, there are infinitely many lines in this family containing at least 2 special points, though $\dim V = 1$.

Less obvious is the fact that the general bound on the number of special points in a line, for the family of lines in Example 4.3, is also 2. That is, there are at most finitely many lines in $\mathbb{C}^2$ through the special point $P$ containing more than 2 distinct special points of $\mathbb{C}^2$. On the other hand, if we consider the pencil of lines in $\mathbb{C}^2$ passing through a nonspecial point such as $P = (1, 1)$, then all but finitely many lines in the family have at most 1 special point. These facts are contained in the following proposition:

**Proposition 4.4.** *Let $V \subset X_1$ be an irreducible curve in the Chow variety of lines in $\mathbb{C}^2$, not consisting exclusively of vertical or horizontal lines. Then, outside of finitely many parameters $\lambda \in V$, the lines of the family intersect at most 1 special point in $\mathbb{C}^2$, unless $V$ is*

(1) *the family of all lines through a special point $P = (p_1, p_2) \in \mathbb{C}^2$;*

(2) *the family of lines defined by $L_\lambda = \{(x, y) \in \mathbb{C}^2 : x + y = \lambda\}$, for $\lambda \in \mathbb{C}$;*

(3) *the family of lines $L_\lambda$ containing $(c_1, \lambda)$ and $(\lambda, c_2)$ for special parameters $c_1 \neq c_2$, for $\lambda \in \mathbb{C}$.*

*In each of these 3 cases, outside of a finite set of parameters $\lambda \in V$, there are at most 2 special points on the line $L_\lambda$; moreover, there are infinitely many parameters $\lambda \in V$ for which the line $L_\lambda$ contains exactly 2 special points of $\mathbb{C}^2$.*

*Proof.* Let $V \subset X_1$ be an irreducible algebraic curve defined over $\mathbb{C}$, not consisting of vertical or horizontal lines. Let $\mathcal{C} \to V$ denote this family of lines over $V$. Consider the special points in $\rho_2(\mathcal{C}_V^2)$ in $\mathbb{C}^4$, for the map $\rho_2$ defined in Section 3.1. Note that their Zariski closure must contain the special diagonal surface

$$\Delta_{1,3} = \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 : (x_1, x_2) = (x_3, x_4)\},$$

because $\rho_1$ is dominant to $\mathbb{C}^2$ (in which special points are Zariski dense). The general bound on the number of special points on lines $L \in V$ is 1 unless either

(S) there is a special surface contained in the Zariski closure $\overline{\rho_2(\mathcal{C}_V^2)}$, other than the diagonal $\Delta_{1,3}$, intersecting $L \times L$ in a curve for general $L \in V$; or

(C) there is a special curve contained in the Zariski closure $\overline{\rho_2(\mathcal{C}_V^2)}$, not contained in $\Delta_{1,3}$, intersecting $L \times L$ in a nonempty finite set for general $L \in V$.

Indeed, if an infinite collection of lines $L$ in $V$ had at least 2 special points, then the Zariski closure of those pairs of points would not be contained in $\Delta_{1,3}$ and would form a special subvariety lying in $\overline{\rho_2(\mathcal{C}_V^2)}$. Some irreducible component $Z$ of this Zariski closure is positive dimensional (because it contains an infinite collection of points), and it must have dimension $< \dim \overline{\rho_2(\mathcal{C}_V^2)} = 3$ because the hypersurface $\overline{\rho_2(\mathcal{C}_V^2)}$ in $\mathbb{C}^4$ cannot be special by Proposition 4.1. Thus, $Z$ is either a curve or a surface. Because the collection of lines $L$ containing these special points was infinite in the curve $V$, some component $Z$ must have preimage $\rho_2^{-1}(Z)$ that projects dominantly to $V$. The dimension of the intersection of $Z$ with the general $L \times L$, as described in cases (S) and (C), then follows by dimension count, exactly as in Section 4.2.

We will see that cases (1) and (2) of the proposition correspond to the existence of special surfaces of type (S), and case (3) of the proposition gives rise to special curves of type (C). For each of the families (1), (2), and (3), it is clear that there are infinitely many lines in the family containing at least 2 distinct special points. It will remain to show that there are at most 2 special points on all but finitely many lines in each of these families.

We work case by case, considering each type of special surface or curve in $\mathbb{C}^4$:

(S1) $\{x \in \mathbb{C}^4 : x_i = c_i \text{ and } x_j = c_j\}$ for $i < j$ in $\{1, 2, 3, 4\}$: If $\{i, j\}$ is $\{1, 2\}$ or $\{3, 4\}$, then this special surface is contained in $\overline{\rho_2(\mathcal{C}_V^2)}$ if and only if $V$ is the pencil of lines through the special point $P = (c_i, c_j)$, and we denote these surfaces by

$$S_{P,1} := \{P\} \times \mathbb{C}^2 \quad \text{and} \quad S_{P,2} := \mathbb{C}^2 \times \{P\}.$$

If $\{i, j\}$ is $\{1, 3\}$, $\{2, 4\}$, $\{1, 4\}$, or $\{2, 3\}$, then the intersection with $L \times L$ is finite for general $L \in V$, so this surface cannot be of type (S).

(S2) $\{x \in \mathbb{C}^4 : x_i = c_i \text{ and } x_j = x_k\}$ for three distinct indices $i, j, k$: If $\{j, k\}$ is $\{1, 2\}$ or $\{3, 4\}$, then the surface is not contained in $\overline{\rho_2(\mathcal{C}_V^2)}$ because the intersections of $L$ with $\{x = y\}$ and $\{x = c_i\}$ are finite for general $L \in V$. If $\{i, j\} = \{1, 2\}$, the intersection of the special surface with $L \times L$ is again generally finite; other cases are similar, and none can be of type (S).

(S3) $\{x_i = x_j \text{ and } x_k = x_m\}$ with disjoint pairs of indices $\{i, j\}$ and $\{k, m\}$ in $\{1, 2, 3, 4\}$: If the pairs are $\{1, 2\}$ and $\{3, 4\}$, then the intersection with $L \times L$ is finite for general $L \in V$, so it cannot be of type (S). If the pairs are $\{1, 3\}$ and $\{2, 4\}$, then the special subvariety is the special diagonal surface

$$\Delta_{1,3} = \{(x_1, x_2) = (x_3, x_4)\}.$$

If they are $\{1, 4\}$ and $\{2, 3\}$, then the surface lies in $\overline{\rho_2(\mathcal{C}_V^2)}$ if and only if the points on $L$ come in symmetric pairs (so $(x, y) \in L$ if and only if $(y, x) \in L$) for general $L \in V$. In other words, the family of lines is of the form

$$x + y = b$$

for a nonconstant function $b$ on $V$, and we denote this special surface by

$$D := \{x_1 = x_4 \text{ and } x_2 = x_3\}.$$

For this family of lines, a point $(x, y)$ on the line is special if and only if $(y, x)$ is special, so there are infinitely many such lines with at least two distinct special points.

(S4) $\{x_i = x_j = x_k\}$ for three distinct indices $i, j, k$: There is at least one pair of indices which is either $\{1, 2\}$ or $\{3, 4\}$. But then the intersection with $L \times L$ is finite for general $L \in V$, so this surface cannot be of type (S).

We now consider the existence of special curves of type (C). We work case by case again, considering each type of special curve in $\mathbb{C}^4$.

(C1) $\{x \in \mathbb{C}^4 : x_i = c_i, x_j = c_j, x_k = c_k\}$ for $i < j < k$ in $\{1, 2, 3, 4\}$: There is a pair of indices equal to $\{1, 2\}$ or $\{3, 4\}$. So the product $L \times L$ intersects this curve for a general $L \in V$ if and only if the family of lines persistently contains a special point $P$. In other words, the family must be case (1) of the proposition, and the Zariski closure $\overline{\rho_2(\mathcal{C}_V^2)}$ also contains the surfaces $S_{P,1}$ and $S_{P,2}$ of type (S1).

(C2) $\{x_i = c_i, x_j = c_j, x_k = x_m\}$ for disjoint pairs $\{i, j\}$ and $\{k, m\}$: If $\{i, j\} = \{1, 2\}$, then the general intersection with $L \times L$ is empty unless the lines contain $P = (c_1, c_2)$ for all $L \in V$. In particular, this family of lines must be case (1) of the proposition, and the Zariski closure $\overline{\rho_2(\mathcal{C}_V^2)}$ also contains the surfaces $S_{P,1}$ and $S_{P,2}$ of type (S1). Similarly for $\{i, j\} = \{3, 4\}$. If $\{i, j\} = \{1, 3\}$, then the equality $x_2 = x_4$ in $L \times L$ would imply that $c_1 = c_3$ because $L$ is degree 1 and not horizontal for all $L$, making this special curve lie in the diagonal $\Delta_{1,3}$. Similarly for $\{i, j\} = \{2, 4\}$. So, for each of these cases, the curve cannot be of type (C).

For $\{i, j\} = \{1, 4\}$ with $c_1 = c_4$, the relation $x_2 = x_3$ implies that the general line in the family must intersect points of the form $(c_1, y)$ and $(y, c_1)$ for some $y \in \mathbb{C}$ (where the $y$ value can vary with the line). If $y = c_1$ for a general line, then the family of lines must be case (1) with $P = (c_1, c_1)$, and the Zariski closure $\overline{\rho_2(\mathcal{C}_V^2)}$ also contains the surfaces $S_{P,1}$ and $S_{P,2}$ of type (S1). If $y \neq c_1$ for a general line in the family, then the pair of points $(c_1, y)$ and $(y, c_1)$ determine the line uniquely; the family must be of the form $x + y = \lambda$, and the Zariski closure $\overline{\rho_2(\mathcal{C}_V^2)}$ also contains the surface $D$ of type (S3). For $\{i, j\} = \{1, 4\}$ with $c_1 \neq c_4$, the relation $x_2 = x_3$ implies that the general line in the family must intersect points of the form $(c_1, y)$ and $(y, c_4)$ for some $y \in \mathbb{C}$ (where the $y$ value must vary with the line). The family of lines is therefore case (3) of the proposition, and the Zariski closure $\overline{\rho_2(\mathcal{C}_V^2)}$ contains the curve of type (C) defined by

$$C_{c_1,c_4,1} := \{x_1 = c_1 \text{ and } x_4 = c_4 \text{ and } x_2 = x_3\}.$$

By the symmetry of $\overline{\rho_2(\mathcal{C}_V^2)}$, we see that the curve

$$C_{c_1,c_4,2} = \{x_2 = c_4 \text{ and } x_3 = c_1 \text{ and } x_1 = x_4\}$$

is also contained in $\overline{\rho_2(\mathcal{C}_V^2)}$, for the same family of lines. The case of $\{i, j\} = \{2, 3\}$ leads to the same conclusion.

(C3) $\{x_i = c_i, x_j = x_k = x_m\}$: Assume first that $i = 1$. The relations determine a curve of type (C) if the lines of the family contain both $(c_1, y)$ and $(y, y)$ for some $y \in \mathbb{C}$ (where the value of $y$ can vary with the line). If the general line intersects the point $P = (c_1, c_1)$, then the family is case (1), and the Zariski closure $\overline{\rho_2(\mathcal{C}_V^2)}$ also contains the surfaces $S_{P,1}$ and $S_{P,2}$ of type (S1). If $y \neq c_1$ for a general line, then the lines must be horizontal, and this case has been ruled out by assumption. Similarly for $i = 2, 3, 4$.

(C4) $\{x_1 = x_2 = x_3 = x_4\}$: This curve is contained in the diagonal surface $\Delta_{1,3}$.

The case-by-case analysis shows that the three types of families of lines listed in the proposition are the only families of lines that give rise to special subvarieties of type (S) or type (C). Thus, any other 1-parameter maximally varying family of lines has at most 1 special point on the general line in the family.

To see that the bound is at most 2 on the three types of exceptions (1), (2), and (3), we look again at the cases and the structure of the Zariski closure of the special points in $\rho_2(\mathcal{C}_V^2)$. Suppose we are in case (1), and assume there are at least 3 distinct special points on infinitely many lines $L$ in the family. Then, considering pairs of special points on a line $L$, with neither equal to the given special $P \in \mathbb{C}^2$, we build a component of the Zariski closure of special points in $\rho_2(\mathcal{C}_V^2)$ that is neither in $\Delta_{1,3}$ nor in the surfaces $S_{P,1}$ or $S_{P,2}$ of type (S1) above. Similarly for case (2), choosing pairs of distinct special points that are not symmetric (as $(x, y)$ and $(y, x)$) leads to a component of the Zariski closure of special points in $\overline{\rho_2(\mathcal{C}_V^2)}$ that is neither in $\Delta_{1,3}$ nor in the surface $D$ of type (S3). And finally, for case (3), the existence of pairs of points that are distinct and not equal to the pair $(c_1, \lambda)$ and $(\lambda, c_4)$ as described in case (C2) leads to a special component not contained in $\Delta_{1,3}$ nor in the curves $C_{c_1,c_4,1}$ or $C_{c_1,c_4,2}$.

Thus, it remains to observe that a family of lines $\mathcal{C} \to V$ over a curve $V$ cannot be exhibited as a family of the form (1), (2), or (3) in two distinct ways. For example, as there is a unique line through distinct points $P$ and $Q$ in $\mathbb{C}^2$, there cannot be a family of lines exhibited as case (1) of the proposition for two distinct special points $P \neq Q$. It is also clear that there is no point $P \in \mathbb{C}^2$ in every line of the family of case (2), so cases (1) and (2) cannot coincide. Given a family as in case (3), we can easily compute that there are at most two lines in the family containing any given point $P \in \mathbb{C}^2$, so cases (3) and (1) cannot coincide. The family of lines in case (2) has constant slope, while those of case (3) have slopes varying with $\lambda$ because $c_1 \neq c_2$, so the cases (2) and (3) cannot coincide. Finally, we check that a family of lines cannot be exhibited as case (3) for distinct special pairs $(c_1, c_2)$ and $(c_1', c_2')$ with $c_1 \neq c_2$ and $c_1' \neq c_2'$. If so, there would be a quadratic relation that must be satisfied for all parameters $\lambda \in \mathbb{C}$, namely

$$c_1\lambda^2 - c_2\lambda^2 - \lambda^2 c_1' + \lambda^2 c_2' + 2c_2c_1'\lambda - 2c_1c_2'\lambda + c_1^2c_2' - c_1^2c_2 + c_1c_2^2 - c_2^2c_1' = 0,$$

which implies that $(c_1, c_2) = (c_1', c_2')$.

This proves that the bound is at most 2 for each of these exceptional families. To see that the bound is optimal, we observe that the families are constructed to have infinitely many lines containing at least 2 special points. □

*Proof of Theorem 1.6.* The Chow variety of lines $X_1$ has dimension 2. From Theorem 1.4, we know that there is a finite union $V_1$ of irreducible curves and points in $X_1$ such that there are at most 2 special points on each line $L \notin V_1$. Now fix an irreducible curve $C \subset V_1$, and assume it is not the family of vertical or horizontal lines in $\mathbb{C}^2$. From Proposition 4.4, there is again a bound of 2 on the number of special points for all but finitely many $L \in C$. This completes the proof. □

## 5. Real algebraic curves in $\mathbb{R}^2$

In this section, we observe that Theorems 1.3, 1.4, 1.5, and 1.6 apply to real algebraic curves in $\mathbb{R}^2$ passing through PCF parameters in the Mandelbrot set, in particular providing a proof of Theorem 1.8.

Suppose $P(x, y) \in \mathbb{R}[x, y]$ is a polynomial with degree $d \geq 1$. Writing $x = \frac{1}{2}(c+\bar{c})$ and $y = \frac{1}{2i}(c-\bar{c})$, we obtain a polynomial of $c$ and $\bar{c}$ of degree $d$ with complex coefficients. In this way, any real algebraic curve in $\mathbb{R}^2$ passing through a collection $\{c_1, \dots, c_m\}$ of PCF parameters in $\mathbb{C}$ gives rise to a complex algebraic curve in $\mathbb{C}^2$ passing through special points $\{(c_1, \bar{c}_1), \dots, (c_m, \bar{c}_m)\}$. (Recall that the set of special parameters is symmetric under complex conjugation.)

For example, if we begin with the line in $\mathbb{R}^2$ defined by

$$\{(x, y) \in \mathbb{R}^2 : ax + by = r\}$$

with $a, b, r \in \mathbb{R}$, then this line contains $c \in \mathbb{C}$ if and only if the complex line

$$\left\{(x, y) \in \mathbb{C}^2 : \tfrac{1}{2}(a - ib)x + \tfrac{1}{2}(a + ib)y = r\right\}$$

contains the point $(c, \bar{c})$ in $\mathbb{C}^2$. In particular, taking $b = 1$ and $a = r = 0$ shows that the real axis in $\mathbb{C}$ corresponds to the diagonal line $x = y$ in $\mathbb{C}^2$. Note that the vertical and horizontal lines in $\mathbb{C}^2$ cannot arise by this construction.

**Example 5.1.** The imaginary axis in $\mathbb{C}$ contains the three PCF parameters $\{i, 0, -i\}$ and corresponds to the complex line $y = -x$ in $\mathbb{C}^2$. Other than the real and imaginary axes in $\mathbb{C}$, we do not know any examples of real lines with more than two PCF parameters.

We see immediately that Theorem 1.3 applies to real algebraic curves in each degree $d \geq 1$, implying there is a uniform bound on the number of PCF parameters on any such curve, depending only on the degree. And so does Theorem 1.6, as the real axis in $\mathbb{C}$ is the only line containing infinitely many PCF parameters, implying that there are only finitely many real lines in $\mathbb{C}$ passing through more than two PCF parameters. This completes the proof of Theorem 1.8.

Note that the set of all complex algebraic curves of degree $d$ built from real curves in the above way is Zariski-dense in the Chow variety $X_d$ of all complex curves of degree $\leq d$ in $\mathbb{C}^2$. Therefore, Theorem 1.4 also holds for real algebraic curves. Finally, observing that there always exists a real algebraic curve of degree $d$ through any collection of $\frac{1}{2}d(d+3)$ points in $\mathbb{C}$, we see that Theorem 1.5 also holds by choosing the curves to pass through collections of PCF parameters, so the bound of $\frac{1}{2}d(d+3)$ is optimal for a general real curve in degree $d$.

**Example 5.2.** Holly Krieger pointed out to us that Theorem 1.6 also implies there are only finitely many horizontal real lines in $\mathbb{C}$ that contain more than one PCF parameter. Indeed, if two distinct PCF parameters, say $c_1$ and $c_2$, have the same nonzero imaginary part, then $c_2 - c_1 = \bar{c}_2 - \bar{c}_1 \in \mathbb{R}$, and the four pairs $(c_1, \bar{c}_2)$, $(\bar{c}_2, c_1)$, $(c_2, \bar{c}_1)$, and $(\bar{c}_1, c_2)$ are on the complex line

$$x + y = \alpha := c_1 + \bar{c}_2$$

in $\mathbb{C}^2$. We do not know of any examples of horizontal lines, other than the real axis in $\mathbb{C}$, containing more than one PCF parameter.

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# An asymptotic orthogonality relation for $\mathrm{GL}(n,\mathbb{R})$

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Orthogonality is a fundamental theme in representation theory and Fourier analysis. An orthogonality relation for characters of finite abelian groups (now recognized as an orthogonality relation on GL(1)) was used by Dirichlet to prove infinitely many primes in arithmetic progressions. Asymptotic orthogonality relations for $\mathrm{GL}(n)$, with $n \le 3$, and applications to number theory, have been considered by various researchers over the last 45 years. Recently, the authors of the present work have derived an explicit asymptotic orthogonality relation, with a power savings error term, for $\mathrm{GL}(4,\mathbb{R})$. Here we extend those results to $\mathrm{GL}(n,\mathbb{R})$, $n \ge 2$.

For $n \le 5$, our results are contingent on the Ramanujan conjecture at the infinite place, but otherwise are unconditional. In particular, the case $n = 5$ represents a new result. The key new ingredient for the proof of the case $n = 5$ is the theorem of Kim and Shahidi that functorial products of cusp forms on $\mathrm{GL}(2)\times\mathrm{GL}(3)$ are automorphic on $\mathrm{GL}(6)$. For $n > 5$ (assuming again the Ramanujan conjecture holds at the infinite place), our results are conditional on two conjectures, both of which have been verified in various special cases. The first of these conjectures regards lower bounds for Rankin–Selberg L-functions, and the second concerns recurrence relations for Mellin transforms of $\mathrm{GL}(n,\mathbb{R})$ Whittaker functions.

Central to our proof is an application of the Kuznetsov trace formula, and a detailed analysis, utilizing a number of novel techniques, of the various entities — Hecke–Maass cusp forms, Langlands Eisenstein series, spherical principal series Whittaker functions and their Mellin transforms, and so on — that arise in this application.



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# 1. Introduction

**1.1.** ***Brief description of the main result of this paper.*** Let $n \geq 1$ be a rational integer, $s \in \mathbb{C}$, and $\mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \mathbb{A}_f$ denote the ring of adeles over $\mathbb{Q}$, where $\mathbb{A}_f$ denotes the finite adeles. The family of unitary cuspidal automorphic representations $\pi$ of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$ and their standard L-functions

$$L(s,\pi) = L_\infty(s,\pi) \cdot \prod_p L_p(s,\pi)$$

were first introduced by Godement and Jacquet [1972] and have played a major role in modern number theory. In the special case of $n = 1$ the Euler products $\prod_p L_p(s,\pi)$ are just Dirichlet L-functions.

In this paper we focus on the unitary cuspidal automorphic representations of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$ with trivial central character which are globally unramified. For $n \geq 2$, these can be studied classically in terms of Hecke–Maass cusp forms on

$$\mathrm{SL}(n,\mathbb{Z}) \backslash \mathrm{GL}(n,\mathbb{R}) / (\mathrm{O}(n,\mathbb{R}) \cdot \mathbb{R}^\times),$$

where

$$\mathfrak{h}^n := \mathrm{GL}(n,\mathbb{R}) / (\mathrm{O}(n,\mathbb{R}) \cdot \mathbb{R}^\times)$$

is a generalization of the classical upper half-plane. In fact $\mathfrak{h}^2 := \left\{ \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \;\middle|\; y > 0,\ x \in \mathbb{R} \right\}$ is isomorphic to the classical upper half-plane.

For $n \geq 2$, Hecke–Maass cusp forms are smooth functions $\phi : \mathfrak{h}^n \to \mathbb{C}$ which are automorphic for $\mathrm{SL}(n,\mathbb{Z})$ with moderate growth and which are joint eigenfunctions of the full ring of invariant differential operators on $\mathrm{GL}(n,\mathbb{R})$ and are also joint eigenfunctions of the Hecke operators. Such globally unramified Hecke–Maass forms can be classified in terms of Langlands parameters which (assuming the cusp form is tempered) are $n$ pure imaginary numbers $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in (i \cdot \mathbb{R})^n$ that sum to zero. Further, the Hecke–Maass cusp forms $\phi$ with Langlands parameters $(\alpha_1, \ldots, \alpha_n)$ can be ordered in terms of their Laplace eigenvalues $\lambda_\Delta(\phi)$ given by

$$\lambda_\Delta(\phi) = \tfrac{1}{24}(n^3 - n) - \tfrac{1}{2}(\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2),$$

as proved by Stephen Miller [2002].

Let $\phi$ be a Hecke–Maass cusp form for $\mathrm{SL}(n,\mathbb{Z})$ for $n \geq 2$ and set

$$\langle \phi, \phi \rangle := \int\limits_{\mathrm{SL}(n,\mathbb{Z}) \backslash \mathfrak{h}^n} \phi(g) \overline{\phi(g)}\, dg$$

to denote the Petersson norm of $\phi$. The Hecke–Maass cusp forms form a Hilbert space over $\mathbb{C}$ with respect to the Petersson inner product.

**Definition 1.1.1** (L-function of a Hecke–Maass cusp form). Let $\phi$ be a Hecke–Maass cusp form for $\mathrm{SL}(n,\mathbb{Z})$. Then for $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ sufficiently large we define the L-function $L(s,\phi) := \sum_{k=1}^{\infty} \lambda(k) k^{-s}$, where $\lambda(k)$ is the $k$-th Hecke eigenvalue of $\phi$.

**Definition 1.1.2** (asymptotic orthogonality relation for $\mathrm{GL}(n,\mathbb{R})$). Let $\{\phi_j\}_{j=1,2,\dots}$ (with associated Langlands parameters $\alpha^{(j)}=(\alpha_1^{(j)},\alpha_2^{(j)},\dots,\alpha_n^{(j)})$) denote an orthogonal basis of Hecke–Maass cusp forms for $\mathrm{SL}(n,\mathbb{Z})$ with L-function given by $L(s,\phi_j):=\sum_{k=1}^{\infty}\lambda_j(k)k^{-s}$. Fix positive integers $\ell, m$. Then, for $T\to\infty$, we have

$$\lim_{T\to\infty}\frac{\sum_{j=1}^{\infty}\lambda_j(\ell)\,\overline{\lambda_j(m)}\,h_T(\alpha^{(j)})/\mathcal{L}_j}{\sum_{j=1}^{\infty}h_T(\alpha^{(j)})/\mathcal{L}_j}=\begin{cases}1+o(1) & \text{if } \ell=m,\\ o(1) & \text{if } \ell\neq m,\end{cases}$$

where $\mathcal{L}_j=L(1,\mathrm{Ad}\,\phi_j)$ and $h_T(\alpha^{(j)})$ is a smooth function of the variables $\alpha^{(j)}$, $T$ (for $T>0$), with support on the Laplace eigenvalues $\lambda_\Delta(\phi_j)$, where $0<\lambda_\Delta(\phi_j)\ll T$.

**Remark 1.1.3** (power savings error term). The asymptotic orthogonality relation has a power savings error term if $o(1)$ can be replaced with $\mathcal{O}(T^{-\theta})$ for some fixed $\theta>0$. The error terms $o(1)$, $\mathcal{O}(T^{-\theta})$ will generally depend on $L$, $M$. This type of asymptotic orthogonality relation was first conjectured by Fan Zhou [2014].

**Remark 1.1.4** (normalization of Hecke–Maass cusp forms). The approach we take in proving asymptotic orthogonality relations for $\mathrm{GL}(n,\mathbb{R})$ is the Kuznetsov trace formula presented in Section 4, where $\lambda_j(\ell)\overline{\lambda_j(m)}/\langle\phi_j,\phi_j\rangle$ (which are independent of the way the $\phi_j$ are normalized) appears naturally on the spectral side of the trace formula leading to an asymptotic orthogonality relation of the form

$$\lim_{T\to\infty}\frac{\sum_{j=1}^{\infty}\lambda_j(\ell)\overline{\lambda_j(m)}h_T(\alpha^{(j)})/\langle\phi_j,\phi_j\rangle}{\sum_{j=1}^{\infty}h_T(\alpha^{(j)})/\langle\phi_j,\phi_j\rangle}=\begin{cases}1+o(1) & \text{if } \ell=m,\\ o(1) & \text{if } \ell\neq m.\end{cases}\tag{1.1.5}$$

If we normalize $\phi_j$ so that its first Fourier coefficient is equal to 1 then it is shown in Proposition 4.1.4 that

$$\langle\phi_j,\phi_j\rangle=c_nL(1,\mathrm{Ad}\,\phi_j)\prod_{1\le i\neq k\le n}\Gamma\big(\tfrac{1}{2}(1+\alpha_i^{(j)}-\alpha_k^{(j)})\big)\quad(c_n\neq 0).$$

This allows us (with a modification of the test function $h_T$) to replace the inner product $\langle\phi_j,\phi_j\rangle$ appearing in (1.1.5) with the adjoint L-function $\mathcal{L}_j$ as in Definition 1.1.2. The main reason for doing this is that there are much better techniques developed for bounding special values of L-functions, as opposed to bounding inner products of cusp forms. So having $\mathcal{L}_j^{-1}$ in the asymptotic orthogonality relation instead of $\langle\phi_j,\phi_j\rangle^{-1}$ will allow us to obtain better error terms in applications.

Orthogonality relations as in Definition 1.1.2 have a long history going back to Dirichlet (for the case of GL(1)) who introduced the orthogonality relation for Dirichlet characters to prove infinitely many primes in arithmetic progressions. Bruggeman [1978] was the first to obtain an asymptotic orthogonality relation for GL(2), which he presented in the form

$$\lim_{T\to\infty}\sum_{j=1}^{\infty}\frac{\lambda_j(\ell)\overline{\lambda_j(m)}\cdot 4\pi^2e^{-\lambda_\Delta(\phi_j)/T}}{T\cosh\big(\pi\sqrt{\lambda_\Delta(\phi_j)-\frac{1}{4}}\big)}=\begin{cases}1 & \text{if } \ell=m,\\ 0 & \text{if } \ell\neq m,\end{cases}$$

where $\{\phi_j\}_{j=1,2,\dots}$ goes over an orthogonal basis of Hecke–Maass cusp forms for $\mathrm{SL}(2,\mathbb{Z})$. This is not quite in the form of Definition 1.1.2 but it can be put into that form with some work. Other versions of

GL(2)-type orthogonality relations with important applications were obtained by Sarnak [1987], and, for holomorphic Hecke modular forms, by Conrey, Duke and Farmer [Conrey et al. 1997] and J. P. Serre [1997].

The first asymptotic orthogonality relations for GL(3) with power savings error term were proved independently by Blomer [2013] and Goldfeld and Kontorovich [2013]. Goldfeld, Stade and Woodbury [Goldfeld et al. 2021b] were the first to obtain a power savings asymptotic orthogonality relation, as in Definition 1.1.2 for GL(4).

A major breakthrough was obtained by Matz and Templier [2021] who unconditionally proved an asymptotic orthogonality relation for $\mathrm{SL}(n, \mathbb{Z})$, as in (1.1.5), for a wide class of test functions for all $n \geq 2$ (with power savings) but without the harmonic weights given by the inverse of the adjoint L-function at 1. Their results were further strengthened in [Finis and Matz 2021]. The principal tool used to prove the asymptotic orthogonality relation in [Matz and Templier 2021] was the Arthur–Selberg trace formula, whereas our approach is the natural generalization of the earlier results [Blomer 2013; Goldfeld and Kontorovich 2013; Goldfeld et al. 2021b], which were based on the Kuznetsov trace formula.

Blomer [2021] presented a very nice exposition comparing the Arthur–Selberg and Kuznetsov trace formulae, which we now briefly summarize for the application to asymptotic orthogonality relations.

- The first key difference between these trace formulae is that the spectral side of the Kuznetsov trace formula has harmonic weights $\mathcal{L}_j^{-1}$, while the Arthur–Selberg trace formula does not have these harmonic weights. For $\mathrm{GL}(n)$ with $n > 3$ it is not currently known how to remove these weights (see [Buttcane and Zhou 2020] for how to remove the weights on GL(3)). Blomer [2021] remarked that "*for applications to L-functions involving period formulae it is often desirable to have an additional factor* $1/L(1, \mathrm{Ad}\,\phi)$ *in the cuspidal spectrum*, *but in other situations one may prefer a summation formula without an extra L-value.*"

- The second major difference between these trace formulae is that the spectral side of the Kuznetsov trace formula does not contain residual spectrum, while the Arthur–Selberg trace formula does. As pointed out by a referee, the bulk of the work in [Matz and Templier 2021] consists in bounding the unipotent contribution on the geometric side of the Arthur trace formula so that it stays in line with the error term coming from the residual Eisenstein contribution on the spectral side given by Lapid and Müller [2009]. These residual Eisenstein series do not appear in the Kuznetsov trace formula, which leads to a very strong conjectural error term in Theorem 1.5.1. In fact, the largest error term on the spectral side of the Kuznetsov trace formula arises from the tempered Eisenstein series coming from the maximal parabolic having $(n-1, 1)$ Levi block decomposition. For explicit comparisons between our main theorem and the results of [Matz and Templier 2021], see Remark 1.5.4.

- There are certain applications of our results using the Kuznetsov trace formula approach that go beyond the results in [Matz and Templier 2021; Finis and Matz 2021]. Recall that $\lambda_j(p)$ denotes the $p$-th Hecke eigenvalue of the Maass form $\phi_j$. Fan's thesis concerns the so-called vertical Sato–Tate problem, which is a conjecture about the distribution of $\lambda_j(p)$, where $p$ is fixed and $j$ varies. This problem was studied by Bruggeman [1978] and Sarnak [1987] (for Maass forms), and Serre [1997] and Conrey, Duke and Farmer [Conrey et al. 1997] (for holomorphic forms), who showed by fixing $p$ and varying $j$ that $\lambda_j(p)$

is an equidistributed sequence with respect to the Plancherel measure which depends on $p$. Strikingly, as observed by Fan Zhou [2014], if we give each Hecke eigenvalue $\lambda_j(p)$ the weight $\mathcal{L}_j^{-1}$, then the distribution involves the Sato–Tate measure which is independent of $p$. Jana [2021] generalized the results of Zhou, but he only obtained an asymptotic formula without a power savings error term. A problem for the future would be to combine Jana's approach with the methods of this paper. Jana also obtains bounds toward Sarnak's density hypothesis using this strategy that are stronger than anything known using the Arthur–Selberg trace formula.

The main aim of this paper is to explicitly work out an asymptotic orthogonality relation for $\mathrm{SL}(n,\mathbb{Z})$ via the Kuznetsov trace formula for a special choice of test function $h^{(n)}_{T,R}$ whose form is that of a Gaussian times a fixed polynomial. We do not address applications in this paper and leave that to future research. See [Blomer 2021] for various applications of the Arthur–Selberg and Kuznetsov trace formulae and how they compare. We also point out that the Kuznetsov trace formula was generalized by Jacquet and Lai [1985] who developed the relative trace formula which has had a wide following with new types of applications.

See Theorem 1.5.1 for the statement of our main theorem. The proof we give assumes the Ramanujan conjecture at $\infty$ but it is possible to prove a weaker result by dropping this assumption. Otherwise the proof is unconditional for $n \le 5$. In particular, the case $n = 5$ represents a complete, new result. For $n > 5$, our result is conditional on two conjectures.

**1.2.** ***Ishii–Stade conjecture.*** The Ishii–Stade conjecture (see Section 8.2) concerns the normalized Mellin transform $\widetilde{W}_{n,\alpha}(s)$ of the $\mathrm{GL}(n,\mathbb{R})$ Whittaker function $W_{n,\alpha}(y)$ defined in Definition 2.3.3. Here, $s = (s_1, s_2, \ldots, s_{n-1}) \in \mathbb{C}^{n-1}$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) = \mathbb{C}^{n-1}$ satisfies $\sum_{i=1}^{n} \alpha_i = 0$.

Suppose integers $m$ and $\delta$, with $1 \le m \le n-1$ and $\delta \ge 0$, are given. The Ishii–Stade conjecture expresses $\widetilde{W}_{n,\alpha}(s)$ as a finite linear combination, with coefficients that are rational functions of the $s_j$'s and $\alpha_k$'s, of shifted Mellin transforms

$$\widetilde{W}_{n,\alpha}(s+\Sigma),$$

where $\Sigma \in (\mathbb{Z}_{\ge 0})^{n-1}$ and the $m$-th coordinate of $\Sigma$ is $\ge \delta$. In other words, for such $\delta$ and $m$, the conjecture expresses the Mellin transform $\widetilde{W}_{n,\alpha}(s)$ in terms of shifts of this Mellin transform by at least $\delta$ units to the right in the variable $s_m$.

Much as recurrence relations of the form

$$\Gamma(s) = [(s+\delta-1)(s+\delta-2)\cdots(s+1)s]^{-1}\Gamma(s+\delta)$$

for Euler's Gamma function imply concrete results concerning analytic continuation, poles, and residues of that function, so will the Ishii–Stade conjecture allow us to obtain explicit information about the behavior of $\widetilde{W}_{n,a}(s)$ beyond its original, a priori domain of definition. This explicit information will be crucial to the analysis of our test function $h_T$, and consequently, to our derivation of an asymptotic orthogonality relation as in Definition 1.1.2.

We have been able to prove the Ishii–Stade conjecture for $\mathrm{GL}(n,\mathbb{R})$ with $2 \le n \le 5$. See Section 8.2 below.

**1.3.** ***Lower bound conjecture for Rankin–Selberg L-functions.*** Fix $n \geq 2$. Let $n = n_1 + \cdots + n_r$ be a partition of $n$ with $n_i \in \mathbb{Z}_{>0}$ $(i = 1, \ldots, r)$. The second conjecture we require for the proof of the asymptotic orthogonality relation for $\mathrm{GL}(n, \mathbb{R})$ is a conjecture on the lower bound for Rankin–Selberg L-functions $L(s, \phi_k \times \phi_{k'})$ on the line $\mathrm{Re}(s) = 1$, where $\phi_k, \phi_{k'}$ (for $1 \leq k < k' \leq r$) are Hecke–Maass cusp forms for $\mathrm{SL}(n_k, \mathbb{Z})$, $\mathrm{SL}(n_{k'}, \mathbb{Z})$, respectively. For a Hecke–Maass cusp form $\phi$ with Langlands parameters $(\alpha_1, \ldots, \alpha_n)$, let

$$c(\phi) = (1 + |\alpha_1|)(1 + |\alpha_2|) \cdots (1 + |\alpha_n|) \tag{1.3.1}$$

denote the analytic conductor of $\phi$ as defined by Iwaniec and Sarnak [2000].

**Conjecture 1.3.2** (lower bounds for Rankin–Selberg L-functions). *Let $\varepsilon > 0$ be fixed. Then we have the lower bound*

$$|L(1 + it, \phi_k \times \phi_{k'})| \gg_\varepsilon (c(\phi_k) \cdot c(\phi_{k'}))^{-\varepsilon} (|t| + 2)^{-\varepsilon}.$$

**Remark 1.3.3.** Conjecture 1.3.2 follows from Langlands' conjecture that $\phi_k \times \phi_{k'}$ is automorphic for $\mathrm{SL}(n_k \cdot n_{k'}, \mathbb{Z})$. This can be proved via the method of de la Valée Poussin as in [Sarnak 2004]. Interestingly, Sarnak's approach can be extended to prove Conjecture 1.3.2 if $\phi_{k'}$ is the dual of $\phi_k$ (see [Goldfeld and Li 2018; Humphries and Brumley 2019]). Stronger bounds can also be obtained if one assumes the Lindelöf or Riemann hypothesis for Rankin–Selberg L-functions.

If $n_k = n'_k = 2$, it was proved by Ramakrishnan [2000] that $\phi_k \times \phi_{k'}$ is automorphic for $\mathrm{SL}(4, \mathbb{Z})$, thus proving the lower bound conjecture for $n \leq 4$. Further, for $n_k = 2$ and $n'_k = 3$, it was proved by Kim and Shahidi [2002] that $\phi_k \times \phi_{k'}$ is automorphic for $\mathrm{SL}(6, \mathbb{Z})$, thus proving the lower bound conjecture for $n \leq 5$.

**1.4.** ***Constructing the test functions.*** Fix an integer $n \geq 2$. We now construct two complex-valued test functions on the space of Langlands parameters

$$\{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n \mid \alpha_1 + \cdots + \alpha_n = 0\}$$

that will be used in our proof of the orthogonality relation for $\mathrm{GL}(n, \mathbb{R})$.

We begin by introducing an auxiliary polynomial that is used in constructing the test functions.

**Definition 1.4.1** (the polynomial $\mathcal{F}_R^{(n)}(\alpha)$). Let $R \in \mathbb{Z}_{>0}$ and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a Langlands parameter. Then we define

$$\mathcal{F}_R^{(n)}(\alpha) := \prod_{j=1}^{n-2} \prod_{\substack{K, L \subseteq (1,2,\ldots,n) \\ \#K = \#L = j}} \left(1 + \sum_{k \in K} \alpha_k - \sum_{\ell \in L} \alpha_\ell\right)^{\frac{R}{2}}.$$

Note that if $\alpha \in (i\mathbb{R})^n$, then $\mathcal{F}_1^{(n)}(\alpha)$ is the square root of a polynomial in $\alpha$ of degree $2D(n)$, where

$$D(n) = \sum_{j=1}^{n-2} \frac{1}{2}\binom{n}{j}\left(\binom{n}{j} - 1\right) = \frac{1}{2}\binom{2n}{n} - \frac{n(n-1)}{2} - 2^{n-1}. \tag{1.4.2}$$

By abuse of notation, we refer to $\mathcal{F}_R^{(n)}$ as a *polynomial*, although this is not strictly the case unless $R$ is even. For $\alpha$ with bounded real and imaginary parts, say $|\mathrm{Re}(\alpha_j)| < R$ and $|\mathrm{Im}(\alpha_j)| < T^{1+\varepsilon}$, we have

$$|\mathcal{F}_R^{(n)}(\alpha)| \ll T^{\varepsilon + R\cdot D(n)} \quad (T \to +\infty), \tag{1.4.3}$$

with an implicit constant depending on $n, \varepsilon, R$.

**Definition 1.4.4** (the test functions $p_{T,R}^{n,\#}(\alpha)$ and $h_{T,R}^{(n)}(\alpha)$). Let $R \in \mathbb{Z}_{>0}$ and $T \to +\infty$. Then for a Langlands parameter $\alpha = (\alpha_1, \dots, \alpha_n)$, we define

$$p_{T,R}^{n,\#}(\alpha) := e^{(\alpha_1^2+\alpha_2^2+\cdots+\alpha_n^2)/(2T^2)} \cdot \mathcal{F}_R^{(n)}\Big(\frac{\alpha}{2}\Big) \prod_{1\le j\ne k\le n} \Gamma\Big(\frac{1+2R+\alpha_j-\alpha_k}{4}\Big),$$

$$h_{T,R}^{(n)}(\alpha) := \frac{|p_{T,R}^{n,\#}(\alpha)|^2}{\prod_{1\le j\ne k\le n} \Gamma((1+\alpha_j-\alpha_k)/2)}.$$

We observe that, by Stirling's formula for the Gamma function and by (1.4.2) and (1.4.3), we have

$$|h_{T,R}^{(n)}(\alpha)| \ll T^{R\cdot(\binom{2n}{n}-2^n)-\frac{n(n-1)}{2}} \tag{1.4.5}$$

whenever $|\mathrm{Re}(\alpha_j)|$ is bounded and $|\mathrm{Im}(\alpha_j)| < T^{1+\varepsilon}$ for $1 \le j \le n$. The implied constant in (1.4.5) depends on $n$, $\varepsilon$, and $R$.

**Remark 1.4.6** (positivity of $h_{T,R}^{(n)}$). Writing $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $\alpha_j = it_j$ and $t_j \in \mathbb{R}$ for each $j = 1, 2, \dots, n$, the function $h_{T,R}^{(n)}(\alpha)$ is positive. This is the case because $\Gamma\big(\frac{1+iu}{2}\big)\Gamma\big(\frac{1-iu}{2}\big) = \big|\Gamma\big(\frac{1+iu}{2}\big)\big|^2$ for $u \in \mathbb{R}$.

**Remark 1.4.7** (Whittaker transform of the test function). The symbol # in the test function $p_{T,R}^{n,\#}$ means this function is the Whittaker transform of $p_{T,R}^{(n)}$. See Section 8.

### 1.5. *The main theorem.*

**Theorem 1.5.1.** *Fix $n \ge 2$. Let $\{\phi_j\}_{j=1,2,\dots}$ denote an orthogonal basis of Hecke–Maass cusp forms for* $\mathrm{SL}(n, \mathbb{Z})$ (*assumed to be tempered at* $\infty$) *with associated Langlands parameter*

$$\alpha^{(j)} = (\alpha_1^{(j)}, \alpha_2^{(j)}, \dots, \alpha_n^{(j)}) \in (i \cdot \mathbb{R})^n$$

*and L-function $L(s, \phi_j) := \sum_{k=1}^\infty \lambda_j(k) k^{-s}$.*

*Fix positive integers $\ell, m$. Then assuming the Ishii–Stade conjecture* (*Conjecture 8.2.3*) *and the lower bound conjecture for Rankin–Selberg L-functions* (*Conjecture 1.3.2*), *we prove that for $T \to \infty$*

$$\sum_{j=1}^\infty \lambda_j(\ell)\,\overline{\lambda_j(m)}\,\frac{h_{T,R}^{(n)}(\alpha^{(j)})}{\mathcal{L}_j} = \delta_{\ell,m} \cdot \sum_{i=1}^{n-1} \mathfrak{c}_i \cdot T^{R\cdot(\binom{2n}{n}-2^n)+n-i} + \mathcal{O}_{\varepsilon,R,n}\Big((\ell m)^{\frac{n^2+13}{4}} \cdot T^{R\cdot(\binom{2n}{n}-2^n)+\varepsilon}\Big),$$

*where $\delta_{\ell,m}$ is the Kronecker symbol, $\mathcal{L}_j = L(1, \mathrm{Ad}\ \phi_j)$, and $\mathfrak{c}_1, \dots, \mathfrak{c}_{n-1} > 0$ are absolute constants which depend at most on $R$ and $n$.*

*Because Conjectures 1.3.2 and 8.2.3 are known to be true for $2 \le n \le 5$* (*see Remark 1.3.3 and Section 8.2*), *the above result is unconditional for such $n$.*

**Remark 1.5.2.** Qiao Zhang [2023] recently proved the lower bound

$$|L(1+it, \phi_k \times \phi_{k'})| \gg (c(\phi_k)\cdot c(\phi_{k'}))^{-\theta_{k,k'}} (|t|+2)^{-\frac{1}{2}n_k n_{k'}(1-1/(n_k+n_{k'}))-\varepsilon}, \tag{1.5.3}$$

with $\theta_{k,k'} = n_k + n_{k'} + \varepsilon$. This improves on the bound of Brumley [2006; 2013, Appendix], who obtained nearly the same result but with the term $n_k n_{k'}/2$ replaced by $n_k n_{k'}$. Assuming (1.5.3) we can replace the error term in Theorem 1.5.1 with

$$\mathcal{O}_{\varepsilon,R,n,\ell,m}\big(T^{R\cdot(\binom{2n}{n}-2^n)+n-1+\frac{n(n-2)}{6}(\theta_{k,k'}-\frac{8}{n^2})}\big).$$

So if one could prove (1.5.3) with $\theta_{k,k'} < 8/n^2$ this would give a power savings error term in our main theorem and would remove the assumption of the lower bound conjecture (Conjecture 1.3.2). In fact, the proof establishes a black box by which improvements to bounds on Rankin–Selberg L-functions result in better power savings error terms for the continuous spectrum contribution to the asymptotic orthogonality relation.

**Remark 1.5.4.** A variant of Theorem 1.5.1 is obtained unconditionally in [Matz and Templier 2021; Finis and Matz 2021], without the arithmetic weights $\mathcal{L}_j^{-1}$ and with different test functions, which are indicator functions of $\alpha^{(j)} \in T\Omega$, where $\Omega$ is a Weyl group invariant bounded open subset of $i\cdot\mathfrak{a}^*$, where $\mathfrak{a}$ is the Lie algebra of the subgroup of diagonal matrices with positive entries. Additionally, the results of [Matz and Templier 2021; Finis and Matz 2021] do not give the polynomial weights of size $T^{R\cdot(\binom{2n}{n}-2^n)-n(n-1)/2}$ coming from $h^{(n)}_{T,R}(\alpha)$ (see (1.4.5)).

The error term obtained in [Finis and Matz 2021], in the present setting of $\mathrm{SL}(n,\mathbb{Z})$, is $\ll T^{(n-1)(n+2)/2-1}$ as $T\to\infty$. Here, $(n-1)(n+2)/2$ is the dimension of the generalized upper half-plane $\mathfrak{h}^n$, and the error term obtained by Finis and Matz has exponent equal to that dimension minus 1. By comparison, if one removes the polynomial weights $T^{R\cdot(\binom{2n}{n}-2^n)-n(n-1)/2}$ from the error term in Theorem 1.5.1 above, then one obtains an error term that is $\ll T^{n(n-1)/2+\varepsilon}$. Also note that our main term is of a stronger form than that of [Matz and Templier 2021; Finis and Matz 2021], in that ours gives a sum of $n-1$ different high-order asymptotics.

More recently, Jana [2021] obtained a proof of the asymptotic orthogonality relation defined in Definition 1.1.2, using the Kuznetsov trace formula and not the Selberg trace formula, with applications to the equidistribution of Satake parameters with respect to the Sato–Tate measure, second-moment estimates of central values of L-functions as strong as Lindelöf on average, and distribution of low-lying zeros of automorphic L-functions in the analytic conductor aspect. The paper of Jana does not contain a power savings error term.

**Remark 1.5.5.** It is possible to remove the assumption of Ramanujan at the infinite place with more work, which results in a weaker power savings error term in Theorem 1.5.1. For a Maass form $\phi$ with Langlands parameter $\alpha$, note that the test function $h_{T,R}(\alpha)$ is positive. This is true because, even if $\alpha$ is a Langlands parameter of an element in the complementary spectrum, $-\alpha$ is a permutation of $\bar{\alpha}$. A weaker version of Theorem 1.5.1 can be proved if one assumes that almost all (except for a set of zero density) are tempered. Such results have been obtained in [Matz and Templier 2021; Finis and Matz 2021].

*Proof of Theorem 1.5.1.* Computing the inner product of certain Poincaré series in two ways (see the outline in Section 1.6 below), we obtain a Kuznetsov trace formula relating the so-called geometric and spectral sides. The geometric side consists of a main term $\mathcal{M}$ and a Kloosterman contribution $\mathcal{K}$. The spectral side also consists of two components: a cuspidal (i.e., discrete) contribution $\mathcal{C}$ and an Eisenstein (i.e., continuous) contribution $\mathcal{E}$.

The left-hand side of the theorem is precisely $\mathcal{C}$. The first set of terms on the right-hand side comes from the asymptotic formula for $\mathcal{M}$ given in Proposition 5.0.1. The power of $T$ in the error term comes from the bound for $\mathcal{E}$ given in Theorem 7.1.1 (which also gives a factor of $(\ell m)^{1/2-1/(n^2+1)}$). A bound for $\mathcal{K}$, which is a (finite) sum of terms $\mathcal{I}_w$, with the same power of $T$ but with the given power of $\ell m$ follows as a consequence of Proposition 6.0.1. □

**1.6.** ***Outline of the key ideas in the proofs.*** Fix $n \geq 2$. The GL$(n, \mathbb{R})$ orthogonality relation appears directly in the spectral side of the Kuznetsov trace formula for GL$(n, \mathbb{R})$, which we now discuss. The Kuznetsov trace formula is obtained by computing the inner product of two Poincaré series on $\mathrm{SL}(n, \mathbb{Z})\backslash\mathfrak{h}^n$ in two different ways. The Poincaré series are constructed in a similar manner to Borel Eisenstein series by taking all $U_n(\mathbb{Z})\backslash \mathrm{SL}(n, \mathbb{Z})$ translates of a certain test function which we choose to be the $p^{(n)}_{T,R}$ test function in Definition 1.4.4 multiplied by a character and a power function (see Definition 2.3.7).

The first way of computing the inner product of two Poincaré series is to replace one of the Poincaré series with its spectral expansion into cusp forms and Eisenstein series and then unravel the other Poincaré series with the Rankin–Selberg method. This gives the spectral contribution which has two parts: the cuspidal contribution and the Eisenstein contribution. The second way of computing the inner product is to replace one of the Poincaré series with its Fourier Whittaker expansion and then unravel the other Poincaré series with the Rankin–Selberg method. This is called the geometric contribution to the trace formula, which also consists of two parts: a main term, and the so-called *Kloosterman contribution*. The precise results of these computations are given in Theorems 4.1.1 and 4.2.1, respectively.

***Bounding the Eisenstein contribution.*** The key component of the Eisenstein contribution to the Kuznetsov trace formula is the inner product of an Eisenstein series and the Poincaré series $P^M$ given in Definition 2.3.7. By unraveling the Poincaré series in the inner product (see Proposition 4.1.2) we essentially obtain the $M$-th Fourier coefficient of the Eisenstein series multiplied by the Whittaker transform of $p^{(n)}_{T,R}$. The explicit formula for the $M$-th Fourier coefficient of the most general Langlands Eisenstein series given in Proposition 4.1.5 allows us to effectively bound all the terms in the integrals appearing in the Eisenstein contribution except for the product of adjoint L-functions

$$\prod_{\substack{k=1\\ n_k\neq 1}}^{r} L^*(1, \mathrm{Ad}\,\phi_k)^{-\frac{1}{2}} \tag{1.6.1}$$

appearing in that proposition. When considering the Eisenstein contribution to the Kuznetsov trace formula for GL$(n, \mathbb{R})$, all the adjoint L-functions in the above product are for cusp forms $\phi_k$ of lower rank $n_k < n$. Now in the special case that $\ell = m = 1$, our main theorem, Theorem 1.5.1, for GL$(n, \mathbb{R})$

gives a sharp bound for the sum of reciprocals of all adjoint L-functions of lower rank. This allows us to inductively prove a power savings bound for the product (1.6.1).

***Asymptotic formula for the geometric contribution.*** We prove that the geometric contribution is a sum of expressions $\mathcal{I}_w$ over elements $w$ in the Weyl group of $\mathrm{SL}(n,\mathbb{Z})$. The $\mathcal{I}_w$ are complicated multiple sums of multiple integrals weighted by Kloosterman sums (see (4.2.2)). If $w_1$ is the trivial element of the Weyl group then we obtain an asymptotic formula for $\mathcal{I}_{w_1}$ (see Proposition 5.0.1), while for all other Weyl group elements $\mathcal{I}_{w_i}$, with $i>1$, we obtain error terms with strong bounds for $|\mathcal{I}_{w_i}|$ (see Proposition 6.0.1) which are bounded by the final error term on the right side of our main theorem.

The key terms in (4.2.2), the formula for $\mathcal{I}_w$, are the Kloosterman sums and two appearances of the test function $p^{(n)}_{T,R}$: one that is twisted by the Weyl group element $w$ and one that is not. For the Kloosterman sums, we rely on bounds given by [Dąbrowski and Reeder 1998]. The task of giving strong bounds for $p^{(n)}_{T,R}(y)$ occupies Sections 8, 9 and 10. We deal with the combinatorics of the twisted $p^{(n)}_{T,R}$-function, and we combine the bounds for it, the other $p^{(n)}_{T,R}$-function and the Kloosterman sums in Section 6.

The function $p^{(n)}_{T,R}$ is the inverse Whittaker transform of the test function $p^{n,\#}_{T,R}$ given in Definition 1.4.4 above. Thanks to a formula of [Goldfeld and Kontorovich 2012], we can realize this as an integral of the product of $p^{n,\#}_{T,R}$, the Whittaker function $W_\alpha$ (see Definition 2.3.3), and certain additional gamma factors. We then write the Whittaker function as the inverse Mellin transform of its Mellin transform: $\widetilde{W}_{n,\alpha}(s)$. This leads to the formula (valid for any $\varepsilon>0$)

$$p^{(n)}_{T,R}(y)=\frac{1}{2^{n-1}}\int\limits_{\mathrm{Re}(\alpha_1)=0}\cdots\int\limits_{\mathrm{Re}(\alpha_{n-1})=0} e^{\frac{\alpha_1^2+\alpha_2^2+\cdots+\alpha_n^2}{T^2/2}}\mathcal{F}^{(n)}_R(\alpha)\prod_{1\le j\ne k\le n}\frac{\Gamma\big(\frac{1+2R+\alpha_j-\alpha_k}{4}\big)}{\Gamma\big(\frac{\alpha_j-\alpha_k}{2}\big)}\cdot\int\limits_{\mathrm{Re}(s_1)=\varepsilon}\cdots\int\limits_{\mathrm{Re}(s_{n-1})=\varepsilon}\left(\prod_{j=1}^{n-1}y_j^{\frac{j(n-j)}{2}}(\pi y_j)^{-2s_j}\right)\widetilde{W}_{n,\alpha}(s)\,ds\,d\alpha.$$

To estimate the growth of $p^{(n)}_{T,R}(y)$ uniformly in $y$ and $T$ as $T\to+\infty$, we shift the line of integration in the $s$-integrals to $\mathrm{Re}(s)=-a$, with $a=(a_1,\dots,a_{n-1})$, where $a_i>0$ for $i=1,\dots,n-1$. We remark that this is precisely where the Ishii–Stade conjecture is required. It is well known that

$$\widetilde{W}_{2,\alpha}(s)=\Gamma(s+\alpha)\Gamma(s-\alpha),$$

and hence understanding the values of $\widetilde{W}_{2,\alpha}(s)$ for $\mathrm{Re}(s)<0$ is straightforward by applying the functional equation for the Gamma function or, equivalently, using an integral representation of the Gamma function valid for $\mathrm{Re}(s)<0$. A similar strategy can be used when $n=3$. However, for $n\ge 4$, the analogous method seems intractable because the Mellin transform is not just a ratio of Gamma functions, but an integral of such. To overcome this difficulty, we apply the Ishii–Stade conjecture to describe the values of $\widetilde{W}_{n,\alpha}(s)$ in terms of sums of the Mellin transform of shifts of the $s$-variables. See also Remark 8.2.11 below.

The Cauchy residue formula allows us to express $p^{(n)}_{T,R}$ as a sum of the shifted $s$-integral (termed the *shifted* $p^{(n)}_{T,R}$ *term* and denoted by $p^{(n)}_{T,R}(y;-a)$) and many residue terms. The description of the shifted

$p^{(n)}_{T,R}$ and residue terms is given in Section 8.3. In order to bound $p^{(n)}_{T,R}(y; -a)$ it is convenient to introduce the function $\mathcal{I}_{T,R}(-a) := p^{(n)}_{T,R}(1; -a)$.

The next step is to use a result of Ishii and Stade (see Theorem 8.1.5) which allows us to write the Mellin transform $\widetilde{W}_{n,\alpha}(s)$ as an integral transformation of $\widetilde{W}_{n-1,\beta}(z)$ against certain additional gamma factors. It is important to note that $\beta = (\beta_1, \ldots, \beta_{n-1}) \in \mathbb{C}^{n-1}$ can be expressed in terms of $\alpha = (\alpha_1, \ldots, \alpha_n)$. By carefully teasing apart the portion of $\alpha$ which determines $\beta$ and that which doesn't, we are able to separate out the gamma factors that don't depend on $\beta$ and bound $\mathcal{I}^{(n)}_{T,R}(-a)$ by the product of a power of $T$ and $\mathcal{I}^{(n-1)}_{T,R}(-b)$ for a certain $b = (b_1, \ldots, b_{n-1)} \in \mathbb{R}^{n-2}$. This gives an inductive procedure, therefore, for bounding the shifted $p^{(n)}_{T,R}$ term.

In Section 10.2 we set notation for describing the $(r-1)$-fold shifted residue terms. This requires generalizing a result of Stade (see Theorem 10.1.1) on the first set of residues of $\widetilde{W}_{n,\alpha}(s)$ (i.e., those that occur at $\mathrm{Re}(s_i) = 0$) to, first, higher-order residues (i.e., taking the residue with respect to multiple values $s_i$), and second, to residues which occur along the lines $\mathrm{Re}(s_i) = -k$ for $k \in \mathbb{Z}_{\geq 0}$. This result, together with a teasing out of the variables similar to that described above, allows us to bound an $(r-1)$-fold residue term as the product of certain powers of $T$ and the variables $y_1, \ldots, y_{n-1}$ times

$$\prod_{j=1}^{r} \mathcal{I}^{(n_j)}_{T,R}(-a^{(j)}), \qquad \text{where } n = n_1 + \cdots + n_r.$$

Applying the bounds on $\mathcal{I}^{(n_j)}_{T,R}$ that we inductively established for bounding the shifted $p^{(n)}_{T,R}$ term, and keeping careful track of all of the exponents and terms $a^{(j)}$, we eventually show that the bound for the shifted main term is in fact valid for every residue term as well.

**Remark 1.6.2.** In comparison to the results of [Goldfeld and Kontorovich 2013; Goldfeld et al. 2021b], we are using a slightly different normalization of the Gamma functions and the auxiliary polynomial $\mathcal{F}^{(n)}_R$ in the definition of the test functions $p^{n,\#}_{T,R}$ and $h^{(n)}_{T,R}$ (see Definition 1.4.4). Adjusting for this difference the results obtained here when applied to $n = 3$ and $n = 4$ recover the previously proven asymptotic formulae.

## 2. Preliminaries

### 2.1. *Notational conventions.*

**Definition 2.1.1** (hat notation for summation). Suppose that $m \in \mathbb{Z}_+$ and $x = (x_1, \ldots, x_m) \in \mathbb{C}^m$. For any $0 \leq k \leq m$, define

$$\hat{x}_k := x_1 + \cdots + x_k.$$

Note that empty sums are assumed to be zero.

**Definition 2.1.2** (integration notation). Let $n \geq 2$. We will often be working with $n$- and $(n-1)$-tuples of real or complex numbers. We will denote such tuples without a subscript and use subscripts to refer to the components. For example, we set $y = (y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1}_{>0}$, $s = (s_1, \ldots, s_{n-1}) \in \mathbb{C}^{n-1}$ and

$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ such that

$$\alpha_1 + \cdots + \alpha_n = 0.$$

In such cases, we denote integration over all such variables $x = (x_1, \dots, x_k)$ subject to condition(s) $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_k)$ via

$$\int\limits_{\mathcal{C}} F(x)\, dx := \int\limits_{\mathcal{C}_1} \cdots \int\limits_{\mathcal{C}_k} F(x_1, \dots, x_k)\, dx_1\, dx_2 \cdots dx_k.$$

For example, given $\beta = (\beta_1, \dots, \beta_{n-1}) \in \mathbb{C}^{n-1}$ with $\hat{\beta}_{n-1} = 0$, we denote integration over all such $\beta$ with $\mathrm{Re}(\beta_j) = b_j$ for each $j = 1, \dots, n-2$ via

$$\int\limits_{\substack{\hat{\beta}_{n-1}=0 \\ \mathrm{Re}(\beta)=b}} F(\beta)\, d\beta := \int\limits_{\mathrm{Re}(\beta_1)=b_1} \cdots \int\limits_{\mathrm{Re}(\beta_{n-2})=b_{n-2}} F(\beta_1, \dots, \beta_{n-2})\, d\beta_1\, d\beta_2 \cdots d\beta_{n-2}.$$

We extend this notation liberally to integrals over $s$, $z$ and $\alpha$ and apply it also to integrals over the imaginary parts in the sequel.

**Definition 2.1.3** (polynomial notation). Our analysis will often require us to bound certain polynomials in a trivial way. Namely, for complex variables $x_j$, with $j = 1, 2, \dots, k$, if $|x_j| \ll T^{1+\varepsilon}$ for each $j$ and $P(x_1, x_2, \dots, x_k)$ is a polynomial, then $|P(x_1, x_2, \dots, x_k)| \ll T^{\varepsilon + \deg P}$. So, the relevant information about $P$ is its degree. This being the case, we will use the notation $\mathcal{P}_d(x)$ (with $x = (x_1, \dots, x_k)$) to represent an unspecified polynomial of degree less than or equal to $d$ in the variable(s) $x$. Note that this notation agrees with the commonly employed practice (also used throughout these notes) of using $\varepsilon$ to represent an unspecified positive real number whose precise value is not specified and may differ from one usage to another.

**Definition 2.1.4** (vector or matrix notation depending on context). Given a vector $a = (a_1, \dots, a_{n-1}) \in \mathbb{R}^{n-1}$, we shall define the diagonal matrix

$$t(a) := \mathrm{diag}(a_1 a_2 \cdots a_{n-1}, a_1 a_2 \cdots a_{n-2}, \dots, a_1, 1).$$

**2.2.** ***Structure of* GL(*n*).** Suppose $n$ is a positive integer. Let $U_n(\mathbb{R}) \subseteq \mathrm{GL}(n, \mathbb{R})$ denote the set of upper triangular unipotent matrices.

**Definition 2.2.1** (character of $U_n(\mathbb{R})$). Let $M = (m_1, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$. For an element $x \in U_n(\mathbb{R})$ of the form

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ & 1 & x_{2,3} & \cdots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix}, \tag{2.2.2}$$

we define the character

$$\psi_M(x) := m_1 x_{1,2} + m_2 x_{2,3} + \cdots + m_{n-1} x_{n-1,n}. \tag{2.2.3}$$

**Definition 2.2.4** (generalized upper half-plane). We denote the set of (real) orthogonal matrices $\mathrm{O}(n,\mathbb{R}) \subseteq \mathrm{GL}(n,\mathbb{R})$, and we set

$$\mathfrak{h}^n := \mathrm{GL}(n,\mathbb{R})/(\mathrm{O}(n,\mathbb{R})\cdot\mathbb{R}^\times).$$

Every element (via the Iwasawa decomposition of GL$(n)$ [Goldfeld 2015]) of $\mathfrak{h}^n$ has a coset representative of the form $g = xy$, with $x$ as above and

$$y = \begin{pmatrix} y_1y_2\cdots y_{n-1} & & & & \\ & y_1y_2\cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix}, \tag{2.2.5}$$

where $y_i > 0$ for each $1 \le i \le n-1$. The group GL$(n,\mathbb{R})$ acts as a group of transformations on $\mathfrak{h}^n$ by left multiplication.

**Definition 2.2.6** (Weyl group and relevant elements). Let $W_n \cong S_n$ denote the Weyl group of GL$(n,\mathbb{R})$. We consider it as the subgroup of GL$(n,\mathbb{R})$ consisting of permutation matrices, i.e., matrices that have exactly one 1 in each row/column and all zeros otherwise. An element $w \in W_m$ is called *relevant* if

$$w = w_{(n_1,n_2,\ldots,n_r)} := \begin{pmatrix} & & I_{n_r} \\ & \iddots & \\ I_{n_1} & & \end{pmatrix},$$

where $I_{n_i}$ is the identity matrix of size $n_i \times n_i$ and $n = n_1 + \cdots + n_r$ is a composition (a way of writing $n$ as a sum of positive integers; see Section 8.3). The *long element of* $W_n$ is $w_{\mathrm{long}} := w_{(1,1,\ldots,1)}$.

**Definition 2.2.7** (other subgroups of GL$(n,\mathbb{R})$). We define

$$\overline{U}_w := \big(w^{-1}\cdot {}^tU_n(\mathbb{R})\cdot w\big) \cap U_n(\mathbb{R}),$$
$$\Gamma_w := \big(w^{-1}\cdot {}^tU_n(\mathbb{Z})\cdot w\big) \cap U_n(\mathbb{Z}) = \mathrm{SL}(n,\mathbb{Z}) \cap \overline{U}_w,$$

where ${}^tU_n$ denotes the transpose of $U_n$, i.e., the set of lower triangular unipotent matrices.

### 2.3. *Basic functions on the generalized upper half-plane* $\mathfrak{h}^n$.

**Definition 2.3.1** (power function). Let $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{C}^n$, with $\hat{\alpha}_n = 0$. Let $\rho = (\rho_1,\ldots,\rho_n)$, where $\rho_i = \frac{n+1}{2} - i$ for $i = 1,2,\ldots,n$. We define a power function on $xy \in \mathfrak{h}^n$ by

$$I(xy,\alpha) = \prod_{i=1}^{n} d_i^{\alpha_i+\rho_i} = \prod_{i=1}^{n-1} y_i^{\hat{\alpha}_{n-i}+\hat{\rho}_{n-i}}, \tag{2.3.2}$$

where $d_i = \prod_{j\le n-i} y_j$ is the $j$-th diagonal entry of the matrix $g = xy$ as above.

**Definition 2.3.3** (Jacquet's Whittaker function). Let $g \in \mathrm{GL}(n,\mathbb{R})$ with $n \ge 2$. Let $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n) \in \mathbb{C}^n$, with $\hat{\alpha}_n = 0$. We define the completed Whittaker function $W_\alpha^\pm : \mathrm{GL}(n,\mathbb{R})/(\mathrm{O}(n,\mathbb{R})\cdot\mathbb{R}^\times) \to \mathbb{C}$ by the integral

$$W_\alpha^{\pm}(g) := \prod_{1\le j<k\le n} \frac{\Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)}{\pi^{(1+\alpha_j-\alpha_k)/2}} \cdot \int\limits_{U_4(\mathbb{R})} I(w_{\text{long}} u g, \alpha)\, \overline{\psi_{1,\dots,1,\pm 1}(u)}\, du,$$

which converges absolutely if $\text{Re}(\alpha_i - \alpha_{i+1}) > 0$ for $1 \le i \le n-1$ (see [Goldfeld et al. 2021a]), and has meromorphic continuation to all $\alpha \in \mathbb{C}^n$ satisfying $\hat{\alpha}_n = 0$.

**Remark 2.3.4.** With the additional gamma factors included in this definition (which can be considered as a "*completed*" Whittaker function) there are $n!$ functional equations, which is equivalent to the fact that the Whittaker function is invariant under all permutations of $\alpha_1, \alpha_2, \dots, \alpha_n$. Moreover, even though the integral (without the normalizing factor) often vanishes identically as a function of $\alpha$, this normalization never does.

If $g$ is a diagonal matrix in $\text{GL}(n, \mathbb{R})$ then the value of $W_{n,\alpha}^{\pm}(g)$ is independent of sign, so we drop the $\pm$. We also drop the $\pm$ if the sign is $+1$.

**Definition 2.3.5** (Whittaker transform and its inverse). Assume $n \ge 2$. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{C}^n$ with $\hat{\alpha}_n = 0$. Set $y := (y_1, y_2, \dots y_{n-1})$ and $t(y)$ as in Definition 2.1.4. Let $f : \mathbb{R}_+^{n-1} \to \mathbb{C}$ be an integrable function. Then we define the Whittaker transform $f^\# : H^n \to \mathbb{C}$ (where $H^n := \{\alpha \in \mathbb{C}^n \mid \hat{\alpha}_n = 0\}$) by

$$f^\#(\alpha) := \int\limits_{y_1=0}^{\infty} \cdots \int\limits_{y_{n-1}=0}^{\infty} f(y)\, W_\alpha(t(y)) \prod_{k=1}^{n-1} \frac{dy_k}{y_k^{k(n-k)+1}}, \tag{2.3.6}$$

provided the above integral converges absolutely and uniformly on compact subsets of $\mathbb{R}_+^{n-1}$. The inverse Whittaker transform [Goldfeld and Kontorovich 2012, Theorem 1.6] is

$$f(y) = \frac{1}{\pi^{n-1}} \int\limits_{\substack{\hat{\alpha}_n=0\\ \text{Re}(\alpha)=0}} \frac{f^\#(\alpha) W_{-\alpha}(t(y))}{\prod_{1\le k\ne \ell\le n} \Gamma\left(\frac{\alpha_k-\alpha_\ell}{2}\right)}\, d\alpha,$$

provided the above integral converges absolutely and uniformly on compact subsets of $(i\mathbb{R})^n$.

**Definition 2.3.7** (normalized Poincaré series). Let $M = (m_1, m_2, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$ with $m_i \ne 0$ for each $i = 1, \dots, n-1$. As with $y$, we may think of $M$ as a matrix. Let $g \in \mathfrak{h}^n$. Then we define

$$P^M(g, \alpha) := \frac{1}{\sqrt{\mathfrak{c}_n}} \cdot \prod_{k=1}^{n-1} m_k^{-\frac{k(n-k)}{2}} \sum_{\gamma \in U_n(\mathbb{Z})\backslash \text{SL}(n,\mathbb{Z})} \psi_M(\gamma g) \cdot p_{T,R}^{(n)}(M\gamma g) \cdot I(\gamma g, \alpha), \tag{2.3.8}$$

where $\mathfrak{c}_n$ is the (nonzero) constant given in Proposition 4.1.4. We extend the definition of $\psi_M$ and $p_{T,R}^{(n)}$ to all of $\mathfrak{h}^n$ by setting $\psi_M(xy) := \psi_M(x)$ and $p_{T,R}^{(n)}(xy) := p_{T,R}^{(n)}(y)$.

**Remark 2.3.9.** This definition, up to the normalizing factor $\sqrt{\mathfrak{c}_n} \prod_{k=1}^{n-1} m_k^{k(n-k)/2}$, of the Poincaré series agrees with that used in [Goldfeld et al. 2021b] with the minor caveat that $p_{T,R}$ takes on a slightly different normalization in terms of the polynomial $\mathcal{F}_R^{(n)}$ and in the gamma factors appearing in Definition 1.4.4. The normalizing factor is inserted so that in the Kuznetsov trace formula the cuspidal term is precisely the orthogonality relation in Theorem 1.5.1.

**2.4.** ***Fourier expansion of the Poincaré series.***

**Definition 2.4.1** (twisted character). Let

$$V_n := \left\{ v = \begin{pmatrix} v_1 & & & \\ & v_2 & & \\ & & \ddots & \\ & & & v_n \end{pmatrix} \middle| v_1, \dots, v_n \in \{\pm 1\},\ v_1 \cdots v_n = 1 \right\}.$$

Let $M = (m_1, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$, and consider $\psi_M$ the additive character (see (2.2.3)) of $U_n(\mathbb{R})$. Then for $v \in V_n$, we define the twisted character $\psi_M^v : U_n(\mathbb{R}) \to \mathbb{C}$ by $\psi_M^v(g) := \psi_M(v^{-1} g v)$.

**Definition 2.4.2** (Kloosterman sum). Fix $L = (\ell_1, \dots, \ell_{n-1}),\ M = (m_1, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$. Let $\psi_L, \psi_M$ be characters of $U_n(\mathbb{R})$. Let $w \in W_n$, where $W_n$ is the Weyl group of GL$(n)$. Let

$$c = \begin{pmatrix} 1/c_{n-1} & & & & \\ & c_{n-1}/c_{n-2} & & & \\ & & \ddots & & \\ & & & c_2/c_1 & \\ & & & & c_1 \end{pmatrix},$$

with $c_i \in \mathbb{Z}_{>0}$. Then the Kloosterman sum is defined as

$$S_w(\psi_L, \psi_M, c) := \sum_{\substack{\gamma = U_n(\mathbb{Z}) \backslash \Gamma \cap G_w / \Gamma_w \\ \gamma = \beta_1 c w \beta_2}} \psi_L(\beta_1)\, \psi_M(\beta_2),$$

with notation as in Definition 11.2.2 of [Goldfeld 2015]. The Kloosterman sum $S_w(\psi, \psi', c)$ is well-defined (i.e., independent of the choice of Bruhat decomposition for $\gamma$) if and only if it satisfies the compatibility condition $\psi(cwuw^{-1}) = \psi'(u)$. It is defined to be zero otherwise. (See [Friedberg 1987].)

**Proposition 2.4.3** ($M$-th Fourier coefficient of the Poincaré series $P^L$). *Let $L = (\ell_1, \dots, \ell_{n-1})$ and $M = (m_1, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$ satisfy $\prod_{i=1}^{n-1} \ell_i \neq 0$ and $\prod_{i=1}^{n-1} m_i \neq 0$. If* $\mathrm{Re}(\alpha_k - \alpha_{k+1})$ *is sufficiently large for each $k = 1, \dots, n-1$, then*

$$\int_{U_n(\mathbb{Z}) \backslash U_n(\mathbb{R})} P^L(ug, \alpha) \cdot \overline{\psi_M(m)}\, d^*u = \sum_{w \in W_n} \sum_{v \in V_n} \sum_{c_1=1}^{\infty} \cdots \sum_{c_{n-1}=1}^{\infty} \frac{S_w(\psi_L, \psi_M^v, c)\, J_w(g; \alpha, \psi_L, \psi_M^v, c)}{\sqrt{\mathfrak{c}_n} \prod_{k=1}^{n-1} (\ell_k^{\frac{k(n-k)}{2}} c_k^{\alpha_k - \alpha_{k+1} + 1})},$$

*where*

$$J_w(g; \alpha, \psi_L, \psi_M^v, c) = \int_{U_w(\mathbb{Z}) \backslash U_w(\mathbb{R})} \int_{\overline{U}_w(\mathbb{R})} \psi_L(wug)\, p_{T,R}^{(n)}(Lcwug)\, I(wug, \alpha) \overline{\psi_M^v(u)}\, d^*u,$$

$$U_w(\mathbb{R}) = \big(w^{-1} \cdot U_n(\mathbb{R}) \cdot w\big) \cap U_n(\mathbb{R}), \quad \overline{U}_w(\mathbb{R}) = \big(w^{-1} \cdot {}^t U_n(\mathbb{R}) \cdot w\big) \cap U_n(\mathbb{R}),$$

*and ${}^t m$ denotes the transpose of a matrix $m$.*

*Proof.* See Theorem 11.5.4 of [Goldfeld 2015]. □

## 3. Spectral decomposition of $\mathcal{L}^2(\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n)$

### 3.1. *Hecke–Maass cusp forms for* SL$(n, \mathbb{Z})$.

**Definition 3.1.1** (Langlands parameters). Let $n \geq 2$. A vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ is termed a Langlands parameter if $\hat{\alpha}_n = 0$.

**Definition 3.1.2** (Hecke–Maass cusp forms). Fix $n \geq 2$. A Hecke–Maass cusp form with Langlands parameter $\alpha \in \mathbb{C}^n$ for $\mathrm{SL}(n, \mathbb{Z})$ is a smooth function $\phi : \mathfrak{h}^n \to \mathbb{C}$ which satisfies $\phi(\gamma g) = \phi(g)$ for all $\gamma \in \mathrm{SL}(n, \mathbb{Z})$, $g \in \mathfrak{h}^n$. In addition $\phi$ is square integrable, is an eigenfunction of the algebra of Hecke operators on $\mathfrak{h}^n$, and is an eigenfunction of the algebra of $\mathrm{GL}(n, \mathbb{R})$ invariant differential operators on $\mathfrak{h}^n$, with the same eigenvalues under this action as the power function $I(*, \alpha)$. The Laplace eigenvalue of $\phi$ is given by

$$\frac{n^3 - n}{24} - \frac{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2}{2}.$$

See Section 6 in [Miller 2002]. The Hecke–Maass cusp form $\phi$ is said to be tempered at $\infty$ if the Langlands parameters $\alpha_1, \ldots, \alpha_n$ are all pure imaginary.

**Proposition 3.1.3** (Fourier expansion of Hecke–Maass cusp forms). *Assume $n \geq 2$. Let $\phi : \mathfrak{h}^n \to \mathbb{C}$ be a Hecke–Maass cusp form for* $\mathrm{SL}(n, \mathbb{Z})$ *with Langlands parameters $\alpha \in \mathbb{C}^n$. Then for $g \in \mathfrak{h}^n$, we have the Fourier–Whittaker expansion*

$$\phi(g) = \sum_{\gamma \in U_{n-1}(\mathbb{Z})\backslash \mathrm{SL}_{n-1}(\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1} \neq 0} \frac{A_\phi(M)}{\prod_{k=1}^{n-1} |m_k|^{\frac{k(n-k)}{2}}} \, W_\alpha^{\mathrm{sgn}(m_{n-1})}\left( t(M) \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right),$$

*where $M = (m_1, m_2, \ldots, m_{n-1})$, $t(M)$ is the toric matrix in Definition 2.1.4 and $A_\phi(M)$ is the $M$-th Fourier coefficient of $\phi$.*

*Proof.* See Section 9.1 of [Goldfeld 2015]. □

**Definition 3.1.4** (L-function associated to a Hecke–Maass form $\phi$). Let $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ sufficiently large. Then the L-function associated to a Hecke–Maass cusp form $\phi$ is defined as

$$L(s, \phi) := \sum_{m=1}^{\infty} \frac{A_\phi(m, 1, \ldots, 1)}{m^s}$$

and has holomorphic continuation to all $s \in \mathbb{C}$ and satisfies a functional equation $s \to 1 - s$. If $\phi$ is a simultaneous eigenfunction of all the Hecke operators then $L(s, \phi)$ has the Euler product

$$L(s, \phi) = \prod_p \left( 1 - \frac{A(p, 1, \ldots, 1)}{p^s} + \frac{A(1, p, 1, \ldots, 1)}{p^{2s}} - \frac{A(1, 1, p, \ldots, 1)}{p^{3s}} + \cdots + (-1)^{n-1} \frac{A(1, \ldots, 1, p)}{p^{(n-1)s}} + \frac{(-1)^n}{p^{ns}} \right)^{-1}.$$

### 3.2. *Langlands Eisenstein series for* SL$(n, \mathbb{Z})$.

**Definition 3.2.1** (parabolic subgroup). For $n \geq 2$ and $1 \leq r \leq n$, consider a partition of $n$ given by $n = n_1 + \cdots + n_r$ with positive integers $n_1, \ldots, n_r$. We define the standard parabolic subgroup

$$\mathcal{P} := \mathcal{P}_{n_1,n_2,\ldots,n_r} := \left\{ \begin{pmatrix} \mathrm{GL}(n_1) & * & \cdots & * \\ 0 & \mathrm{GL}(n_2) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{GL}(n_r) \end{pmatrix} \right\}.$$

Letting $I_r$ denote the $r \times r$ identity matrix, the subgroup

$$N^{\mathcal{P}} := \left\{ \begin{pmatrix} I_{n_1} & * & \cdots & * \\ 0 & I_{n_2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n_r} \end{pmatrix} \right\}$$

is the unipotent radical of $\mathcal{P}$. The subgroup

$$M^{\mathcal{P}} := \left\{ \begin{pmatrix} \mathrm{GL}(n_1) & 0 & \cdots & 0 \\ 0 & \mathrm{GL}(n_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{GL}(n_r) \end{pmatrix} \right\}$$

is the standard choice of Levi subgroup of $\mathcal{P}$.

**Definition 3.2.2** (Hecke–Maass form $\Phi$ associated to a parabolic $\mathcal{P}$). Let $n \geq 2$. Consider a partition $n = n_1 + \cdots + n_r$, with $1 < r < n$. Let $\mathcal{P} := \mathcal{P}_{n_1,n_2,\ldots,n_r} \subset \mathrm{GL}(n, \mathbb{R})$. For $i = 1, 2, \ldots, r$, let $\phi_i : \mathrm{GL}(n_i, \mathbb{R}) \to \mathbb{C}$ be either the constant function 1 (if $n_i = 1$) or a Hecke–Maass cusp form for $\mathrm{SL}(n_i, \mathbb{Z})$ (if $n_i > 1$). The form $\Phi := \phi_1 \otimes \cdots \otimes \phi_r$ is defined on $\mathrm{GL}(n, \mathbb{R}) = \mathcal{P}(\mathbb{R})$ (where $K = \mathrm{O}(n, \mathbb{R})$) by the formula

$$\Phi(nmk) := \prod_{i=1}^{r} \phi_i(m_i) \quad (n \in N^{\mathcal{P}}, m \in M^{\mathcal{P}}, k \in K)$$

where $m \in M^{\mathcal{P}}$ has the form

$$m = \begin{pmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_r \end{pmatrix},$$

with $m_i \in \mathrm{GL}(n_i, \mathbb{R})$. In fact, this construction works equally well if some or all of the $\phi_i$ are Eisenstein series.

**Definition 3.2.3** (character of a parabolic subgroup). Let $n \geq 2$. Fix a partition $n = n_1 + n_2 + \cdots + n_r$ with associated parabolic subgroup $\mathcal{P} := \mathcal{P}_{n_1,n_2,\ldots,n_r}$. Define

$$\rho_{\mathcal{P}}(j) = \begin{cases} \frac{1}{2}(n - n_1), & j = 1 \\ \frac{1}{2}(n - n_j) - n_1 - \cdots - n_{j-1}, & j \geq 2. \end{cases} \tag{3.2.4}$$

Let $s = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r$ satisfy $\sum_{i=1}^{r} n_i s_i = 0$. Consider the function (see Definition 2.3.1)

$$|\cdot|_{\mathcal{P}}^{s} := I(\,\cdot\,, \alpha)$$

on $\mathrm{GL}(n, \mathbb{R})$, where

$$\alpha = \Bigg( \overbrace{s_1 - \rho_{\mathcal{P}}(1) + \frac{1-n_1}{2}, s_1 - \rho_{\mathcal{P}}(1) + \frac{3-n_1}{2}, \dots, s_1 - \rho_{\mathcal{P}}(1) + \frac{n_1-1}{2}}^{n_1 \text{ terms}},$$
$$\overbrace{s_2 - \rho_{\mathcal{P}}(2) + \frac{1-n_2}{2}, s_2 - \rho_{\mathcal{P}}(2) + \frac{3-n_2}{2}, \dots, s_2 - \rho_{\mathcal{P}}(2) + \frac{n_2-1}{2}}^{n_2 \text{ terms}}, \dots,$$
$$\overbrace{s_r - \rho_{\mathcal{P}}(r) + \frac{1-n_r}{2}, s_r - \rho_{\mathcal{P}}(r) + \frac{3-n_r}{2}, \dots, s_r - \rho_{\mathcal{P}}(r) + \frac{n_r-1}{2}}^{n_r \text{ terms}} \Bigg).$$

The conditions $\sum_{i=1}^r n_i s_i = 0$ and $\sum_{i=1}^r n_i \rho_{\mathcal{P}}(i) = 0$ guarantee that $\alpha$'s entries sum to zero. When $g \in \mathcal{P}$, with diagonal block entries $m_i \in \mathrm{GL}(n_i, \mathbb{R})$, one has

$$|g|_{\mathcal{P}}^s = \prod_{i=1}^r |\det(m_i)|^{s_i},$$

so that $|\cdot|_{\mathcal{P}}^s$ restricts to a character of $\mathcal{P}$ which is trivial on $N^{\mathcal{P}}$.

**Definition 3.2.5** (Langlands Eisenstein series twisted by Hecke–Maass forms of lower rank). Let $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, with $n \geq 2$. Consider a parabolic subgroup $\mathcal{P} = \mathcal{P}_{n_1,\dots,n_r}$ of $\mathrm{GL}(n, \mathbb{R})$ and functions $\Phi$ and $|\cdot|_{\mathcal{P}}^s$ as given in Definitions 3.2.2 and 3.2.3, respectively. Let

$$s = (s_1, s_2, \dots, s_r) \in \mathbb{C}^r, \quad \text{where } \sum_{i=1}^r n_i s_i = 0.$$

The Langlands Eisenstein series determined by this data is defined by

$$E_{\mathcal{P},\Phi}(g, s) := \sum_{\gamma \in (\mathcal{P} \cap \Gamma) \backslash \Gamma} \Phi(\gamma g) \cdot |\gamma g|_{\mathcal{P}}^{s+\rho_{\mathcal{P}}} \tag{3.2.6}$$

as an absolutely convergent sum for $\mathrm{Re}(s_i)$ sufficiently large, and extends to all $s \in \mathbb{C}^r$ by meromorphic continuation.

For $k = 1, 2, \dots, r$, let $\alpha^{(k)} := (\alpha_{k,1}, \dots, \alpha_{k,n_k})$ denote the Langlands parameters of $\phi_k$. We adopt the convention that if $n_k = 1$ then $\alpha_{k,1} = 0$. Then the Langlands parameters of $E_{\mathcal{P},\Phi}(g, s)$ (denoted by $\alpha_{\mathcal{P},\Phi}(s)$) are

$$\Big( \overbrace{\alpha_{1,1} + s_1, \dots, \alpha_{1,n_1} + s_1}^{n_1 \text{ terms}}, \quad \overbrace{\alpha_{2,1} + s_2, \dots, \alpha_{2,n_2} + s_2}^{n_2 \text{ terms}}, \quad \dots \quad , \overbrace{\alpha_{r,1} + s_r, \dots, \alpha_{r,n_r} + s_r}^{n_r \text{ terms}} \Big). \tag{3.2.7}$$

**Definition 3.2.8** (the $M$-th Fourier coefficient of $E_{\mathcal{P},\Phi}$). Let $s = (s_1, s_2, \dots, s_r) \in \mathbb{C}^r$, where $\sum_{i=1}^r n_i s_i = 0$. Consider $E_{\mathcal{P},\Phi}(*, s)$ with associated Langlands parameters $\alpha_{\mathcal{P},\Phi}(s)$ as defined in (3.2.7). Let $M = (m_1, m_2, \dots, m_{n-1}) \in \mathbb{Z}_{>0}^{n-1}$. Then the $M$-th term in the Fourier–Whittaker expansion of $E_{\mathcal{P},\Phi}$ is

$$\int\limits_{U_n(\mathbb{Z}) \backslash U_n(\mathbb{R})} E_{\mathcal{P},\Phi}(ug, s)\, \overline{\psi_M(m)}\, du = \frac{A_{E_{\mathcal{P},\Phi}}(M, s)}{\prod_{k=1}^{n-1} m_k^{k(n-k)/2}}\, W_{\alpha_{\mathcal{P},\Phi}(s)}(Mg),$$

### 3.3. *Langlands spectral decomposition for* SL$(n, \mathbb{Z})$.

**Definition 3.3.1** (Petersson inner product). Let $n \geq 2$. For $F, G \in \mathcal{L}^2(\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n)$ we define the Petersson inner product to be

$$\langle F, G\rangle := \int_{\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n} F(g)\overline{G(g)}\, dg.$$

For $g = xy \in \mathfrak{h}^n$, with

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ & 1 & x_{2,3} & \cdots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix},$$

the measure $dg$ is given by $dx\, dy$, with

$$dx = \prod_{1 \leq i < j \leq n} dx_{i,j}, \qquad dy = \prod_{k=1}^{n-1} \frac{dy_k}{y^{k(n-k)+1}}.$$

The Langlands spectral decomposition for $\mathrm{SL}(n,\mathbb{Z})$ states that

$$\mathcal{L}^2(\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n) = (\text{cuspidal spectrum}) \oplus (\text{residual spectrum}) \oplus (\text{continuous spectrum}).$$

We shall be applying the Langlands spectral decomposition to Poincaré series which are orthogonal to the residual spectrum.

**Theorem 3.3.2** (Langlands spectral decomposition for $\mathrm{SL}(n,\mathbb{Z})$). *Let $\phi_1, \phi_2, \phi_3, \ldots$ denote an orthogonal basis of Hecke–Maass forms for* $\mathrm{SL}(n,\mathbb{Z})$. *Assume that $F, G \in \mathcal{L}^2(\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n)$ are orthogonal to the residual spectrum. Then for $g \in \mathrm{GL}(n,\mathbb{R})$ we have*

$$F(g) = \sum_{j=1}^{\infty} \langle F, \phi_j\rangle \frac{\phi_j(g)}{\langle \phi_j, \phi_j\rangle} + \sum_{\mathcal{P}} \sum_{\Phi} c_{\mathcal{P}} \int\limits_{\substack{n_1 s_1 + \cdots + n_r s_r = 0 \\ \mathrm{Re}(s)=0}} \langle F, E_{\mathcal{P},\Phi}(*, s)\rangle E_{\mathcal{P},\Phi}(g, s)\, ds,$$

$$\langle F, G\rangle = \sum_{j=1}^{\infty} \frac{\langle F, \phi_j\rangle \langle \phi_j, G\rangle}{\langle \phi_j, \phi_j\rangle} + \sum_{\mathcal{P}} \sum_{\Phi} c_{\mathcal{P}} \int\limits_{\substack{n_1 s_1 + \cdots + n_r s_r = 0 \\ \mathrm{Re}(s)=0}} \langle F,\ E_{\mathcal{P},\Phi}(*, s)\rangle \langle E_{\mathcal{P},\Phi}(*, s),\ G\rangle\, ds,$$

*where the sum over $\mathcal{P}$ ranges over parabolics associated to partitions $n = \sum_{k=1}^{r} n_k$, while the sum over $\Phi$ (see Definition 3.2.2) ranges over an orthonormal basis of Hecke–Maass forms associated to $\mathcal{P}$. Furthermore, $c_{\mathcal{P}}$ is a fixed nonzero constant.*

*Proof.* For proofs see [Arthur 1979; Langlands 1976; Mœglin and Waldspurger 1995]. □

## 4. Kuznetsov trace formula

The Kuznetsov trace formula is derived by computing the inner product of two Poincaré series in two different ways. More precisely, let $L = (\ell_1, \ldots, \ell_{m-1})$, $M = (m_1, \ldots, m_{n-1}) \in \mathbb{Z}^{n-1}$, with $\prod_{i=1}^{n-1} m_i \neq 0$ and $\prod_{i=1}^{n-1} \ell_i \neq 0$, and consider the Petersson inner product $\langle P^L, P^M\rangle$.

In particular since $P^L, P^M \in \mathcal{L}^2(\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n)$ (see [Friedberg 1987]), the inner product can be computed with the spectral expansion of the Poincaré series. The geometric approach utilizes the Fourier Whittaker expansion of the Poincaré series which involve Kloosterman sums.

The trace formula takes the form

$$\underbrace{\mathcal{C}+\mathcal{E}}_{\text{spectral side}} = \underbrace{\mathcal{M}+\mathcal{K}}_{\text{geometric side}}. \tag{4.0.1}$$

Here $\mathcal{C}$ is the cuspidal contribution and $\mathcal{E}$ is the Eisenstein contribution. See Theorem 4.1.1 for their precise definitions. The geometric side consists of terms corresponding to elements of the Weyl group. The identity element gives the main term $\mathcal{M}$, and the Kloosterman contribution $\mathcal{K}$ is the sum of the remaining terms. See Theorem 4.2.1 for their precise definitions. The Kloosterman term $\mathcal{K}$ and the Eisenstein contribution $\mathcal{E}$ will be small with the special choice of the test function $p_{T,R}$, and they constitute the error term in the main theorem.

**4.1.** ***Spectral side of the Kuznetsov trace formula.*** The first way to compute the inner product of the Poincaré series uses the spectral decomposition of the Poincaré series.

Recall also the definition of the adjoint L-function: $L(s,\mathrm{Ad}\,\phi) := L(s,\phi\times\bar{\phi})/\zeta(s)$, where $L(s,\phi\times\bar{\phi})$ is the Rankin–Selberg convolution L-function as in Section 12.1 of [Goldfeld 2015].

**Theorem 4.1.1** (spectral decomposition for the inner product of Poincaré series). *Fix $n\geq 2$ and $L=(\ell_1,\dots,\ell_{n-1})$, $M=(m_1,\dots,m_{n-1})\in\mathbb{Z}^{n-1}$. Then for $\alpha_0 := \left(-\frac{n-1}{2}+j-1\right)_{j=1,\dots,n}$ we have*

$$\langle P^L(*,\alpha_0),\, P^M(*,\alpha_0)\rangle = \mathcal{C}+\mathcal{E}.$$

*With the notation of the spectral decomposition theorem (Theorem 3.3.2) the cuspidal contribution to the Kuznetsov trace formula is*

$$\mathcal{C} := \sum_{i=1}^{\infty} \frac{\lambda_{\phi_i}(M)\overline{\lambda_{\phi_i}(L)}\cdot|p_{T,R}^{n,\#}(\alpha^{(i)})|^2}{L(1,\mathrm{Ad}\,\phi_i)\cdot\prod_{1\leq j\neq k\leq n}\Gamma((1+\alpha_j^{(i)}-\alpha_k^{(i)})/2)}$$

*and the Eisenstein contribution to the Kuznetsov trace formula is*

$$\mathcal{E} := \sum_{\mathcal{P}}\sum_{\Phi} c_{\mathcal{P}} \int\limits_{\substack{n_1s_1+\cdots+n_rs_r=0\\ \mathrm{Re}(s_j)=0}} A_{E_{\mathcal{P},\Phi}}(L,s)\,\overline{A_{E_{\mathcal{P},\Phi}}(M,s)}\cdot|p_{T,R}^{n,\#}(\alpha_{(\mathcal{P},\Phi)}(s))|^2\,ds$$

*for constants $c_{\mathcal{P}}>0$.*

*Proof.* The proof follows from the Langlands spectral decomposition theorem (Theorem 3.3.2) with the choices $F=P^L$ and $G=P^M$. We have

$$\langle P^L, P^M\rangle = \sum_{j=1}^{\infty}\frac{\langle P^L,\phi_j\rangle\,\langle\phi_j,P^M\rangle}{\langle\phi_j,\phi_j\rangle} + \sum_{\mathcal{P}}\sum_{\Phi} c_{\mathcal{P}} \int\limits_{\substack{n_1s_1+\cdots+n_rs_r=0\\ \mathrm{Re}(s_j)=0}} \langle F,\ E_{\mathcal{P},\Phi}(*,s)\rangle\langle E_{\mathcal{P},\Phi}(*,s),\ G\rangle\,ds.$$

We then insert the inner products given in Proposition 4.1.2 below. Doing so, we see that the cuspidal spectrum is

$$\sum_{i=1}^{\infty} \frac{\langle P^L, \phi_i \rangle \langle \phi_i, P^M \rangle}{\langle \phi_i, \phi_i \rangle} = \sum_{i=1}^{\infty} \frac{A_{\phi_i}(M) \overline{A_{\phi_i}(L)}}{\mathfrak{c}_n \cdot \langle \phi_i, \phi_i \rangle} |p^{n,\#}_{T,R}(\alpha^{(i)})|^2.$$

From Proposition 4.1.4, we see that

$$A_\phi(M)\overline{A_\phi(L)} = |A_\phi(1)|^2 \lambda_\phi(M)\overline{\lambda_\phi(L)} = \frac{\mathfrak{c}_n \cdot \langle \phi, \phi \rangle \cdot \lambda_\phi(M)\overline{\lambda_\phi(L)}}{L(1, \mathrm{Ad}\,\phi) \prod_{1 \le j \ne k \le n} \Gamma\big(\frac{1+\alpha_j - \alpha_k}{2}\big)}.$$

The cuspidal part is now immediate. The contributions from the Eisenstein series are computed in like manner using Proposition 4.1.5. □

**Proposition 4.1.2** (the inner product of $P^M$ with an Eisenstein series or Hecke–Maass form). *Let $M = (m_1, m_2, \dots, m_{n-1})$. Consider the Eisenstein series $E_{\mathcal{P},\Phi}(*, s)$, with associated Langlands parameters $\alpha_{\mathcal{P},\Phi}(s)$. Let $\phi$ denote a Hecke–Maass cusp form for* $\mathrm{SL}(n, \mathbb{Z})$ *with Langlands parameter $\alpha$ and $M$-th Fourier coefficient $A_\phi(M)$. Then for $\alpha_0 := \big(-\frac{n-1}{2} + j - 1\big)_{j=1,\dots,n}$,*

$$\langle \phi, P^M(*, \alpha_0) \rangle = \frac{1}{\sqrt{\mathfrak{c}_n}} A_\phi(M) \cdot p^{n,\#}_{T,R}(\alpha),$$
$$\langle E_{\mathcal{P},\Phi}(*, s), P^M(*, \alpha_0) \rangle = \frac{1}{\sqrt{\mathfrak{c}_n}} A_{E_{\mathcal{P},\Phi}}(M, s) \cdot p^{n,\#}_{T,R}(\alpha_{\mathcal{P},\Phi}(s)),$$

*where the inner products on the left are defined by analytic continuation and $\mathfrak{c}_n$ is the nonzero constant (depending only on $n$) from Proposition 4.1.4.*

*Proof.* We outline the case of the Hecke–Maass forms. The series definition of the Poincaré series converges absolutely for sufficiently large $\mathrm{Re}(\alpha'_i - \alpha'_{i+1})$ $(1 \le i \le n-1)$. It follows that for such $\alpha'$ we may unravel the Poincaré series $P^M(*, \alpha')$ in the inner product $\langle \phi, P^M \rangle$ with the Rankin–Selberg method. The inner product picks out the $M$-th Fourier coefficient of $\phi$ multiplied by a certain Whittaker transform of $p^{(n)}_{T,R}(My) \cdot I(y, \alpha')$. This Whittaker transform has analytic continuation in $\alpha'$ to a region including $\alpha_0$. For sufficiently large $\mathrm{Re}(\alpha'_i - \alpha'_{i+1})$, we have from (2.3.8) that

$$\langle \phi, P^M(*, \alpha') \rangle = \frac{A_\phi(M)}{\sqrt{\mathfrak{c}_n} \prod_{k=1}^{n-1} m_k^{k(n-k)}} \int_{y_1=0}^{\infty} \cdots \int_{y_{n-1}=0}^{\infty} \overline{p^{(n)}_{T,R}(My) \cdot I(y, \alpha')} \cdot W_\alpha(My) \prod_{k=1}^{n-1} \frac{dy_k}{y_k^{k(n-k)+1}}. \quad (4.1.3)$$

Note that $I(y, \alpha_0) = 1$. The integral in (4.1.3) converges (as a function of $\alpha'$) to a region which includes $\alpha_0$. It follows that the analytic continuation in $\alpha'$ to $\alpha_0$ of the inner product satisfies

$$\langle \phi, P^M(*, \alpha_0) \rangle = \frac{1}{\sqrt{\mathfrak{c}_n}} \cdot A_\phi(M) \cdot p^{n,\#}_{T,R}(\alpha).$$

The proof for $E_{\mathcal{P},\Phi}$ is the same. □

For $n \ge 2$, consider a Hecke–Maass cusp form $\phi$ for $\mathrm{SL}(n, \mathbb{Z})$ with Fourier Whittaker expansion given by Proposition 3.1.3. Assume $\phi$ is a Hecke eigenform. Let $A_\phi(1) := A_\phi(1, 1, \dots, 1)$ denote the first

Fourier–Whittaker coefficient of $\phi$. Then we have

$$A_\phi(M) = A_\phi(1) \cdot \lambda_\phi(M),$$

where $\lambda_\phi(M)$ is the Hecke eigenvalue (see Section 9.3 in [Goldfeld 2015]), and $\lambda_\phi(1) = 1$.

**Proposition 4.1.4** (first Fourier–Whittaker coefficient of a Hecke–Maass cusp form). *Assume $n \geq 2$. Let $\phi$ be a Hecke–Maass cusp form for $\mathrm{SL}(n, \mathbb{Z})$ with Langlands parameters $\alpha = (\alpha_1, \ldots, \alpha_n)$. Then the first coefficient $A_\phi(1)$ is given by*

$$|A_\phi(1)|^2 = \frac{\mathfrak{c}_n \cdot \langle \phi, \phi \rangle}{L(1, \mathrm{Ad}\,\phi) \prod_{1 \leq j \neq k \leq n} \Gamma\left(\frac{1+\alpha_j - \alpha_k}{2}\right)},$$

*where $\mathfrak{c}_n \neq 0$ is a constant depending on $n$ only.*

*Proof.* See [Goldfeld et al. 2021a]. □

**Proposition 4.1.5** (the $M$-th Fourier coefficient of $E_{\mathcal{P},\Phi}$). *Let $s = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r$, where $\sum_{i=1}^r n_i s_i = 0$. Consider $E_{\mathcal{P},\Phi}(*, s)$ with associated Langlands parameters $\alpha_{\mathcal{P},\Phi}(s)$ as defined in* (3.2.7). *Assume that each Hecke–Maass form $\phi_k$ (with $1 \leq k \leq r$) occurring in $\Phi$ has Langlands parameters $\alpha^{(k)} := (\alpha_{k,1}, \ldots, \alpha_{k,n_k})$ with the convention that if $n_k = 1$ then $\alpha_{k,1} = 0$. We also assume that each $\phi_k$ is normalized to have Petersson norm $\langle \phi_k, \phi_k \rangle = 1$.*

*Let $L^*(1+s_j - s_\ell, \phi_j \times \phi_\ell)$ denote the completed Rankin–Selberg L-function if $n_j \neq 1 \neq n_\ell$; otherwise define*

$$L^*(1+s_j - s_\ell, \phi_j \times \phi_\ell) = \begin{cases} L^*(1+s_j - s_\ell, \phi_j) & \text{if } n_\ell = 1 \text{ and } n_j \neq 1, \\ L^*(1+s_j - s_\ell, \phi_\ell) & \text{if } n_j = 1 \text{ and } n_\ell \neq 1, \\ \zeta^*(1+s_j - s_\ell) & \text{if } n_j = n_\ell = 1, \end{cases}$$

*where $\zeta^*(w) = \pi^{-w/2}\Gamma(w/2)\zeta(w)$ is the completed Riemann $\zeta$-function. Also define*

$$L^*(1, \mathrm{Ad}\,\phi_k) = L(1, \mathrm{Ad}\,\phi_k) \prod_{1 \leq i \neq j \leq n_k} \Gamma\left(\frac{1+\alpha_{k,i} - \alpha_{k,j}}{2}\right),$$

*with the convention that $L^*(1, \mathrm{Ad}\,1) = 1$.*

*Let $M = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{Z}_{>0}^{n-1}$. Per our convention* (*Definition 2.1.4*), *we may think of $M$ as a vector or a diagonal matrix. Then the $M$-th term in the Fourier–Whittaker expansion of $E_{\mathcal{P},\Phi}$ is*

$$\int_{U_n(\mathbb{Z})\backslash U_n(\mathbb{R})} E_{\mathcal{P},\Phi}(ug, s)\, \overline{\psi_M(m)}\, du = \frac{A_{E_{\mathcal{P},\Phi}}(M, s)}{\prod_{k=1}^{n-1} m_k^{k(n-k)/2}} W_{\alpha_{\mathcal{P},\Phi}(s)}(Mg),$$

*where $A_{E_{\mathcal{P},\Phi}}(M, s) = A_{E_{\mathcal{P},\Phi}}((1, \ldots, 1), s) \cdot \lambda_{E_{\mathcal{P},\Phi}}(M, s)$,*

$$\lambda_{E_{P,\Phi}}((m, 1, \ldots, 1), s) = \sum_{\substack{c_1, \ldots, c_n \in \mathbb{Z}_{>0} \\ c_1 c_2 \cdots c_n = m}} \lambda_{\phi_1}(c_1) \cdots \lambda_{\phi_r}(c_r) \cdot c_1^{s_1} \cdots c_r^{s_r} \tag{4.1.6}$$

*is the* $(m, 1, \dots, 1)$*-th* (*or more informally the* $m$*-th*) *Hecke eigenvalue of* $E_{\mathcal{P},\Phi}$, *and*

$$A_{E_{\mathcal{P},\Phi}}((1,\dots,1),s) = d_0 \prod_{\substack{k=1\\ n_k\neq 1}}^{r} L^*(1,\operatorname{Ad}\phi_k)^{-\frac{1}{2}} \prod_{1\le j<\ell\le r} L^*(1+s_j-s_\ell,\phi_j\times\phi_\ell)^{-1}$$

*for some constant* $d_0 \neq 0$ *depending only on* $n$.

*Proof.* See [Goldfeld et al. 2024]. □

**4.2.** ***Geometric side of the Kuznetsov trace formula.*** In this section, we obtain explicit descriptions of the terms $\mathcal{M}$ and $\mathcal{K}$ appearing on the geometric side of the Kuznetsov trace formula. In order to do this, we introduce Kloosterman sums for SL($n$, $\mathbb{Z}$), which appear in the Fourier expansion of the Poincaré series. In the inner product $\langle P^L, P^M\rangle$ we replace $P^L$ with its Fourier expansion and unravel $P^M$ following the Rankin–Selberg method.

**Theorem 4.2.1** (geometric side of the trace formula). *Fix* $L=(\ell_1,\dots,\ell_{n-1})$ *and* $M=(m_1,\dots,m_{n-1})\in\mathbb{Z}^{n-1}$ ($\mathfrak{c}_n$ *is a nonzero constant*; *see Proposition 4.1.4*). *It follows that for* $\alpha_0 := \left(-\frac{n-1}{2}+j-1\right)_{j=1,\dots,n}$

$$\langle P^L(*,\alpha_0), P^M(*,\alpha_0)\rangle = \mathcal{M}+\mathcal{K}.$$

*For* $w_1$ *the trivial element in the Weyl group* $W_n$, *we define*

$$\mathcal{M} := \mathcal{I}_{w_1} \quad \textit{and} \quad \mathcal{K} := \sum_{\substack{w\in W_n\\ w\neq w_1}} \mathcal{I}_w,$$

*where*

$$\mathcal{I}_w := \sum_{v\in V_n}\sum_{c_1=1}^{\infty}\cdots\sum_{c_{n-1}=1}^{\infty} \frac{S_w(\psi_L,\psi_M^v,c)}{\mathfrak{c}_n\cdot\prod_{k=1}^{n-1}(m_k\ell_k)^{\frac{k(n-k)}{2}}} \int_{y_1=0}^{\infty}\cdots\int_{y_{n-1}=0}^{\infty}\ \int_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})}\ \int_{\bar{U}_w(\mathbb{R})}$$
$$\cdot\,\psi_L(wuy)\,\overline{\psi_M^v(u)}\,p_{T,R}^{(n)}(Lcwuy)\,\overline{p_{T,R}^{(n)}(My)}\,d^*u\,\frac{dy_1\cdots dy_{n-1}}{\prod_{k=1}^{n-1}y_k^{k(n-k)+1}}. \tag{4.2.2}$$

*Proof.* We compute the inner product

$$\begin{aligned}
&\lim_{\alpha\to\alpha_0}\langle P^L(*,\alpha), P^M(*,\alpha)\rangle\\
&\qquad= \lim_{\alpha\to\alpha_0}\int_{\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n} P^L(g,\alpha)\cdot\overline{P^M(g,\alpha)}\,dg\\
&\qquad= \frac{1}{\sqrt{\mathfrak{c}_n}\prod_{k=1}^{n-1}m_k^{\frac{k(n-k)}{2}}}\lim_{\alpha\to\alpha_0}\int_{U_n(\mathbb{Z})\backslash\mathfrak{h}^n} P^L(g,\alpha)\cdot\overline{\psi_M(g)\,p_{T,R}^{(n)}(Mg)\,I(g,\alpha)}\,dg\\
&\qquad= \frac{1}{\sqrt{\mathfrak{c}_n}}\left(\prod_{k=1}^{n-1}m_k^{-\frac{k(n-k)}{2}}\right)\lim_{\alpha\to\alpha_0}\int_{\substack{y\in\mathbb{R}^{n-1}\\ y>0}}\left(\int_{U_n(\mathbb{Z})\backslash U_n(\mathbb{R})} P^L(uy,\alpha)\cdot\overline{\psi_M(m)}\,du\right)\overline{p_{T,R}^{(n)}(My)\,I(y,\alpha)}\,dy.
\end{aligned}$$

Note that, as $\alpha\to\alpha_0$, the function $I(g,\alpha)\to 1$ (for any $g\in\mathfrak{h}^n$) and $\prod_{k=1}^{n-1}c_k^{\alpha_k-\alpha_{k+1}+1}\to 1$. It follows from this and Proposition 2.4.3 above that

$$
\begin{aligned}
&\mathfrak{c}_n \cdot \prod_{k=1}^{n-1} (m_k \ell_k)^{k(n-k)/2} \cdot \lim_{\alpha\to\alpha_0} \langle P^L(*,\alpha),\, P^M(*,\alpha)\rangle \\
&= \lim_{\alpha\to\alpha_0} \sum_{w\in W_n} \sum_{v\in V_n} \sum_{c_1=1}^{\infty} \cdots \sum_{c_{n-1}=1}^{\infty} \frac{S_w(\psi_L, \psi_M^v, c)}{\mathfrak{c}_n \cdot \prod_{k=1}^{n-1} (m_i\ell_i)^{\frac{i(n-i)}{2}} \prod_{k=1}^{n-1} c_k^{\alpha_k-\alpha_{k+1}+1}} \\
&\qquad \cdot \int_{y_1=0}^{\infty} \cdots \int_{y_{n-1}=0}^{\infty} \int_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})} \int_{\overline{U}_w(\mathbb{R})} \psi_L(wuy)\, \overline{\psi_M^v(u)}\, p_{T,R}^{(n)}(Lcwuy)\, \overline{p_{T,R}^{(n)}(My)} \\
&\qquad\qquad \cdot I(wuy,\alpha)\overline{I(y,\alpha)}\, d^*u \, \frac{dy_1\cdots dy_{n-1}}{\prod_{k=1}^{n-1} y_k^{k(n-k)+1}} \\
&= \sum_{w\in W_n} \sum_{v\in V_n} \sum_{c_1=1}^{\infty} \cdots \sum_{c_{n-1}=1}^{\infty} \frac{S_w(\psi_L, \psi_M^v, c)}{\mathfrak{c}_n \cdot \prod_{k=1}^{n-1} (m_k\ell_k)^{\frac{k(n-k)}{2}}} \\
&\qquad \cdot \int_{y_1=0}^{\infty} \cdots \int_{y_{n-1}=0}^{\infty} \int_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})} \int_{\overline{U}_w(\mathbb{R})} \psi_L(wuy)\, \overline{\psi_M^v(u)}\, p_{T,R}^{(n)}(Lcwuy)\, \overline{p_{T,R}^{(n)}(My)}\, du \, \frac{dy_1\cdots dy_{n-1}}{\prod_{k=1}^{n-1} y_k^{k(n-k)+1}} \\
&= \sum_{w\in W_n} \mathcal{I}_w,
\end{aligned}
$$

as claimed. □

## 5. Asymptotic formula for the main term

**Proposition 5.0.1** (main term in the trace formula). *Let* $L = (\ell_1, \ldots, \ell_{n-1}),\ M = (m_1, \ldots, m_{n-1}) \in \mathbb{Z}^{n-1}$ *satisfy* $\prod_{i=1}^{n-1} \ell_i \neq 0$ *and* $\prod_{i=1}^{n-1} m_i \neq 0$. *There exist fixed constants* $\mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1} > 0$ (*depending only on* $R$ *and* $n$) *such that the main term* $\mathcal{M}$ *in the Kuznetsov trace formula* (4.0.1) *is given by*

$$
\mathcal{M} = \delta_{L,M} \cdot \left( \left( \sum_{i=1}^{n-1} \mathfrak{c}_i \cdot T^{R(2\cdot D(n)+n(n-1))+n-i} \right) + \mathcal{O}\big(T^{R(2\cdot D(n)+n(n-1))}\big) \right),
$$

*where*

$$
D(n) = \frac{1}{2}\binom{2n}{n} - \frac{n(n-1)}{2} - 2^{n-1}
$$

*and* $\delta_{L,M}$ *is the Kronecker symbol* (*i.e.*, $\delta_{L,M} = 0$ *if* $L \neq M$*and* $\delta_{L,L} = 1$).

*Proof.* It follows from the definition $\mathcal{M} = \mathcal{I}_{w_1}$, making the change of variables $y \mapsto M^{-1}y$ and noting that $U_{w_1}(\mathbb{Z}) = U_n(\mathbb{Z})$ and $U_{w_1}(\mathbb{R}) = U_n(\mathbb{R})$, that

$$
\begin{aligned}
\mathcal{M} &= \frac{1}{\mathfrak{c}_n} \cdot \int_{y_1=0}^{\infty} \cdots \int_{y_{n-1}=0}^{\infty} \left( \int_{U_n(\mathbb{Z})\backslash U_n(\mathbb{R})} \psi_L(u)\overline{\psi_M(m)}\, d^*u \right) p_{T,R}(LM^{-1}y)\, \overline{p_{T,R}(y)}\, \frac{dy_1\cdots dy_{n-1}}{\prod_{i=1}^{n-1} y_i^{i(n-i)+1}} \\
&= \delta_{L,M} \cdot \mathfrak{d}_n \int_{y_1=0}^{\infty} \cdots \int_{y_{n-1}=0}^{\infty} |p_{T,R}^{(n)}(y)|^2 \, \frac{dy_1\cdots dy_{n-1}}{\prod_{i=1}^{n-1} y_i^{i(n-i)+1}} = \delta_{L,M} \cdot \mathfrak{d}_n \langle p_{T,R}, p_{T,R}\rangle \\
&= \delta_{L,M} \cdot \mathfrak{d}_n \int_{\substack{\hat{\alpha}_n=0 \\ \mathrm{Re}(\alpha)=0}} \frac{|p_{T,R}^{n,\#}(\alpha)|^2}{\prod_{1\le j\ne k\le n} \Gamma\big(\frac{\alpha_j-\alpha_k}{2}\big)}\, d\alpha \\
&= \delta_{L,M}\, \mathfrak{d}_n \cdot \langle p_{T,R}^{n,\#}, p_{T,R}^{n,\#}\rangle,
\end{aligned}
$$

where the representation in terms of the norm of $p_{T,R}^{n,\#}$ follows from the Plancherel formula in Corollary 1.9 of [Goldfeld and Kontorovich 2012] and $\mathfrak{d}_n$ is a nonzero constant depending only on $n$. Hence the main term for GL$(n)$ is thus

$$\mathcal{M} = \delta_{L,M}\mathfrak{d}_n \cdot \int\limits_{\substack{\hat{\alpha}_n=0\\ \mathrm{Re}(\alpha_j)=0}} \frac{\left|e^{(\alpha_1^2+\cdots+\alpha_n^2)/(2T^2)}\mathcal{F}_R^{(n)}\left(\frac{\alpha}{2}\right)\prod_{1\le j\neq k\le n}\Gamma\left(\frac{2R+1+\alpha_j-\alpha_k}{4}\right)\right|^2}{\prod_{1\le j\neq k\le n}\Gamma\left(\frac{\alpha_j-\alpha_k}{2}\right)}\,d\alpha.$$

Let $\alpha_j = i\tau_j$ with $\tau_j \in \mathbb{R}$. It then follows from Stirling's asymptotic formula that

$$\mathcal{M} \sim \delta_{L,M}\mathfrak{d}_n \cdot \int\limits_{\hat{\tau}_n=0} e^{(-\tau_1^2-\cdots-\tau_n^2)/T^2}\left(\mathcal{F}_R^{(n)}\left(\frac{i\tau}{2}\right)\right)^2 \prod_{1\le j<k\le n}(1+|\tau_j-\tau_k|)^{2R}\,d\tau.$$

If we now make the change of variables $\tau_j \to \tau_j T$ for each $j = 1,\ldots,n$, and we use the fact that the degree of $\mathcal{F}_1^{(n)}$ is $D(n)$ (see Definition 1.4.1) it follows that, if $L = M$, as $T\to\infty$ we have $\mathcal{M} \sim \mathfrak{c}T^{R\cdot(2D(n)+n(n-1))+n-1}$, where

$$\mathfrak{c} = \mathfrak{d}_n \cdot \int\limits_{\hat{\tau}_n=0} e^{-\tau_1^2-\cdots-\tau_n^2}\left(\mathcal{F}_R^{(n)}\left(\frac{i\tau}{2}\right)\right)^2 \prod_{1\le j<k\le n}(1+|\tau_j-\tau_k|)^{2R}\,d\tau,$$

and otherwise, the main term is zero. This gives the $i = 1$ term in the statement of the proposition. The method of proof can be extended by using additional terms in Stirling's asymptotic expansion for the Gamma function to obtain the additional terms. □

**Remark 5.0.2.** Note that this doesn't agree with [Goldfeld et al. 2021b] in the case of $n = 4$ because we have used a different normalization. Namely, the linear factors of $\mathcal{F}_R^{(n)}$ agree with those defined previously, but we take a different power of each. Also, the gamma factors which appear in $p_{T,R}^{n,\#}$ have a different $R$: namely, what was $2+R$ in each gamma factor previously has been replaced by $2R+1$ here.

## 6. Bounding the geometric side

The goal of this section is to use the bound given in Theorem 10.0.1 to prove the following, i.e., to bound $\mathcal{K}$, the geometric side of the Kuznetsov trace formula.

**Proposition 6.0.1.** *Let $\mathcal{I}_w$ be as above. Let $M = (m_1,\ldots,m_{n-1})$, $L = (\ell_1,\ldots,\ell_{n-1}) \in (\mathbb{Z}_{>0})^{n-1}$. Let $\rho \in \frac{1}{2}+\mathbb{Z}$. Let $D(n) = \frac{1}{2}\binom{2n}{n} - \frac{n(n-1)}{2} - 2^{n-1}$ as in* (1.4.2). *Then for $R$ sufficiently large and any $\varepsilon > 0$, we have*

$$|\mathcal{I}_w| \ll_{\varepsilon,R} T^{\varepsilon+R(2D(n)+n(n-1))+\frac{(n-1)(n+4)}{2}-\left\lfloor\frac{n-1}{2}\right\rfloor-\rho n-\Phi(w)} \cdot \prod_{i=1}^{n-1}(\ell_i m_i)^{2\rho+\frac{n^2+1}{4}},$$

*where if $w = w_{(n_1,\ldots,n_r)}$ with $r \ge 2$,*

$$\Phi(w) := \Phi(n_1,\ldots,n_r) := \frac{1}{2}\sum_{k=1}^{r-1}(n_k+n_{k+1})(n-\hat{n}_k)\hat{n}_k.$$

**Remark 6.0.2.** Assuming the lower bound conjecture for Rankin–Selberg L-functions, the resulting bound for the Eisenstein series contribution to the Kuznetsov trace formula (see Theorem 7.1.1) is of the magnitude $T$ to the power $R(2D(n)+n(n-1))+\varepsilon$. Therefore, given Proposition 6.0.1 and Lemma A.13 $\big($which says that $\Phi(w) \geq \Phi(1, n-1) = \frac{n(n-1)}{2}\big)$, in order for the bound from the geometric side of the trace formula to be less than the Eisenstein series contribution, it suffices that

$$\frac{(n-1)(n+4)}{2} - \left\lfloor \frac{n-1}{2} \right\rfloor - \rho n - \frac{n(n-1)}{2} \leq 0,$$

which simplifies to give

$$\rho \geq \begin{cases} \frac{3}{2} - \frac{3}{2n} & \text{if } n \text{ is odd,} \\ \frac{3}{2} - \frac{1}{n} & \text{if } n \text{ is even.} \end{cases}$$

Since we require that $\rho \in \frac{1}{2} + \mathbb{Z}$, we find that it suffices to take $\rho = \frac{3}{2}$ universally, meaning that the exponent of each term $\ell_i m_i$ can be taken to be $\frac{n^2+13}{4}$. In particular, for the case of $n = 4$, we see that this exponent is $\frac{29}{4}$, which is an improvement on the bound of $\frac{15}{2}$ obtained in [Goldfeld et al. 2021b].

As remarked above, the main result that we will need is Theorem 10.0.1 or, more specifically, Remark 10.0.5, which states that for any $0 < \varepsilon < \frac{1}{2}$, and for $a = (a_1, a_2, \ldots, a_{n-1})$ satisfying $\lfloor a_j \rfloor + \varepsilon < a_j < \lceil a_j \rceil - \varepsilon$ for each $j = 1, \ldots, n-1$, we have

$$|p^{(n)}_{T,R}(y)| \ll \delta^{-\frac{1}{2}}(y) \cdot \|y\|^{2a} \cdot T^{\varepsilon + \frac{(n+4)(n-1)}{4} + R \cdot \left(D(n) + \frac{n(n-1)}{2}\right) - \sum_{j=1}^{n-1} B(a_j)}. \tag{6.0.3}$$

(The terms $\delta^{-1/2}(y)$, $\|y\|^{2a}$ are defined in Section 6.1 below. The function $B$ is defined in Theorem 9.0.2.)

This bound for $p^{(n)}_{T,R}(y)$ is obtained via an integral representation denoted by $p^{(n)}_{T,R}(y; b)$ (see (8.1.4)) over variables $s = (s_1, \ldots, s_n)$ valid for any $b = (b_1, \ldots, b_n)$ with $b_j > 0$ for each $j = 1, \ldots, n-1$. The integral is taken over the lines $\mathrm{Re}(s_j) = b_j$. Essentially, the bound is then obtained by moving the lines of integration to $\mathrm{Re}(s_j) = -a_j$ for some $a = (a_1, \ldots, a_{n_1}) \in (\mathbb{R}_{>0})^{n-1}$.

The strategy for proving Proposition 6.0.1 will be to, first, introduce notation to rewrite $\mathcal{I}_w$ in a simplified form. We do this in Section 6.1. Then, in Section 6.2 we give bounds for $\mathcal{I}_w$ obtained by applying (6.0.3) to $|p_{T,R}(Lcwuy)|$ (with a parameter $a = (a_1, \ldots, a_{n-1})$) and to $|p_{T,R}(My)|$ (with a parameter $b = (b_1, \ldots, b_{n-1})$) for general $a, b \in (\mathbb{R}_{>0})^{n-1}$. In particular, we establish (6.2.2), bounding $|\mathcal{I}_w|$ in terms of the product of three independent quantities $K(c, w; a)$, $X(u, w; a)$ and $Y(y, w; a, b)$. In Section 6.3, we will show that $K(c, w; a)$ will converge provided that $a$ satisfies certain conditions (independent of $w$), and that for this choice of $a$, $X(u, w; a)$ also converges. We then determine $b$ (dependent on $w$ and $a$) for which $Y(y, w; a, b)$ is also convergent. Finally, in Section 6.4, we complete the proof of Proposition 6.0.1 by simplifying the expression for the given choices of $a$ and $b$.

**6.1.** ***Rewriting $\mathcal{I}_w$.*** Let $T_n(\mathbb{R})$ and $U_n(\mathbb{R})$ be the subgroups of $\mathrm{GL}_n(\mathbb{R})$ consisting of diagonal matrices (with positive terms) and upper triangular unipotent matrices, respectively. Recall that if $t = \mathrm{diag}(t_1, \ldots, t_n) \in T_n(\mathbb{R})$ and $u \in U_n(\mathbb{R})$, the modular character $\delta : T_n(\mathbb{R}) \to \mathbb{R}$ is defined to satisfy

$d(t^{-1}ut) = \delta(t)\,du$. Explicitly, it is given by

$$\delta(t) = \prod_{i=1}^{n} t_i^{2i-n-1}.$$

More generally, if $a = (a_1, \dots, a_{n-1}) \in \mathbb{R}^{n-1}$, for

$$y = (y_1, \dots, y_{n-1}) := \operatorname{diag}(y_1 \cdots y_{n-2} y_{n-1}, \dots, y_1 y_2, y_1, 1),$$

with $y_1, \dots, y_{n-1} > 0$, we define

$$\|y\|^a := \prod_{k=1}^{n-1} y_k^{a_k}.$$

One checks that in the special case of $a_j = \frac{j(n-j)}{2}$ for $j = 1, \dots, n-1$,

$$\delta^{-\frac{1}{2}}(y) = \|y\|^a. \tag{6.1.1}$$

Similarly, if ${}^tU_n(\mathbb{R})$ is the subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting of lower triangular unipotent matrices and

$$\overline{U}_w := (w^{-1}\,{}^tU_n(\mathbb{R})w) \cap U_n(\mathbb{R}),$$

then we can consider the character $\delta_w$ on $T_n(\mathbb{R})$ which satisfies $d(tut^{-1}) = \delta_w(t)\,du$ upon restricting the measure on $U_n(\mathbb{R})$ to $\overline{U}_w$. It can be checked that

$$\delta_w(y) = \delta^{\frac{1}{2}}(y) \cdot \delta^{-\frac{1}{2}}(wyw^{-1}). \tag{6.1.2}$$

Recall from Theorem 4.2.1 that for $L = (\ell_1, \dots, \ell_{n-1})$, $M = (m_1, \dots, m_{n-1}) \in (\mathbb{Z}_{>0})^{n-1}$ and

$$c = \begin{pmatrix} 1/c_{n-1} & & & & \\ & c_{n-1}/c_{n-2} & & & \\ & & \ddots & & \\ & & & c_2/c_1 & \\ & & & & c_1 \end{pmatrix},$$

where $c_i \in \mathbb{Z}_{>0}$ for $i = 1, \dots, n-1$, the Kloosterman contribution to the Kuznetsov trace formula is given by

$$\mathcal{K} = \sum_{\substack{w \in W_n \\ w \neq w_1}} \mathcal{I}_w,$$

where, using the notation defined above and letting $dy^\times$ denote the measure $\prod_{k=1}^{n-1} dy_k/y_k$,

$$\mathcal{I}_w := \mathfrak{c}_n^{-1} \sum_{v \in V_n} \sum_{c_1=1}^{\infty} \cdots \sum_{c_{n-1}=1}^{\infty} S_w(\psi_L, \psi_M^v, c)$$
$$\cdot \int\limits_{\substack{y=(y_1,\dots,y_{n-1}) \\ y_1,\dots,y_{n-1}>0}} \int\limits_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})} \int\limits_{\overline{U}_w(\mathbb{R})} \delta^{\frac{1}{2}}(LM) \cdot \delta(y) \cdot \psi_L(wuy)\, \overline{\psi_M^v(u)}\, p_{T,R}^{(n)}(Lcwuy)\, \overline{p_{T,R}^{(n)}(My)}\, d^*u\, dy^\times. \tag{6.1.3}$$

We recall that by [Friedberg 1987], $\mathcal{I}_w$ is identically zero unless $w$ is *relevant* (see Definition 2.2.6).

**6.2.** ***Bounds for $\mathcal{I}_w$ in terms of a and b.*** Since $p_{T,R}^{(n)}(g)$ is determined by the Iwasawa decomposition of $g$, we first make the change of variables $u \mapsto y^{-1}uy$. Then (6.1.3) implies that

$$|\mathcal{I}_w| \ll \sum_{v\in V_n}\sum_{c_1=1}^{\infty}\cdots\sum_{c_{n-1}=1}^{\infty}|S_w(\psi_L,\psi_M^v,c)| \cdot \int\limits_{\substack{y=(y_1,\dots,y_{n-1})\\ y_1,\dots,y_{n-1}>0}}\ \int\limits_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})}\ \int\limits_{\bar{U}_w(\mathbb{R})} \cdot\delta^{\frac12}(M)\cdot\delta^{\frac12}(L)\cdot\delta_w(y)\cdot\delta(y)\cdot|p_{T,R}^{(n)}(Lcwyu)|\,|p_{T,R}^{(n)}(My)|\,d^*u\,dy^{\times}. \quad (6.2.1)$$

For the purposes of our analysis, we break up the integral in the $y$-variables. To this end, let

$$I_0 := (0,1], \quad I_1 = (1,\infty).$$

For $\tau = (\tau_1,\dots,\tau_{n-1}) \in \{0,1\}^{n-1}$, define

$$I_\tau := I_{\tau_1}\times\cdots\times I_{\tau_{n-1}}.$$

Hence,

$$\int\limits_{\substack{y=(y_1,\dots,y_{n-1})\\ y_1,\dots,y_{n-1}>0}} = \sum_{\tau\in\{0,1\}^{n-1}}\int\limits_{I_\tau},$$

and (6.2.1) becomes

$$|\mathcal{I}_w| \ll \sum_{\tau}|\mathcal{I}_w(\tau)|,$$

where

$$|\mathcal{I}_w(\tau)| := \sum_{v\in V_n}\sum_{c_1=1}^{\infty}\cdots\sum_{c_{n-1}=1}^{\infty}|S_w(\psi_L,\psi_M^v,c)| \cdots\int\limits_{y_1\in I_{\tau_1}}\int\limits_{y_2\in I_{\tau_2}}\cdots\int\limits_{y_{n-1}\in I_{\tau_{n-1}}}\ \int\limits_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})}\ \int\limits_{\bar{U}_w(\mathbb{R})} \delta^{\frac12}(M)\cdot\delta^{\frac12}(L)\cdot\delta_w(y)\cdot\delta(y) \cdot|p_{T,R}^{(n)}(Lcwyu)|\,|p_{T,R}^{(n)}(My)|\,d^*u\,dy^{\times}. \quad (6.2.2)$$

Our strategy is now to, for each choice of $\tau$, replace the terms with $p_{T,R}^{(n)}$ with the bound from (6.0.3) (in the first instance using a choice of $a = (a_1,\dots,a_{n-1}) \in \mathbb{R}^{n-1}$, and in the second instance using $b = (b_1,\dots,b_{n-1}) \in \mathbb{R}^{n-1}$). Then we need to find choices of $a$ and $b$ for which the corresponding integrals converge and give good bounds.

Recall that if $g = utk$ is the Iwasawa decomposition of an element $g \in \mathrm{GL}_n(\mathbb{R})$, then $p_{T,R}^{(n)}(g) = p_{T,R}^{(n)}(t)$. With this in mind, consider the Iwasawa decomposition $wu = u_0tk$, where $u_0 \in U_n(\mathbb{R})$, $t \in T_n(\mathbb{R})$ and $k \in O(n,\mathbb{R})$. Then

$$Lcwyu = Lc(wyw^{-1})u_0tk = u_1Lc(wyw^{-1})tk \quad (u_1 = (Lcwyw^{-1})^{-1}u_0(Lcwyw^{-1}))$$

is the Iwasawa form of $Lcwyu$; hence

$$|p_{T,R}^{(n)}(Lcwyu)| = |p_{T,R}^{(n)}(Lcwyw^{-1}t)|.$$

Recall that the Iwasawa form of $wu$ is assumed to be $u_0tk$, meaning $wu = u_0tk$, where $u_0 \in U_n(\mathbb{R})$, $t \in T_n(\mathbb{R})$ and $k \in O(n,\mathbb{R})$. It can be shown [Jacquet 1967] that

$$t^2 = \begin{pmatrix} 1/\xi_{n-1} & & & & \\ & \xi_{n-1}/\xi_{n-2} & & & \\ & & \ddots & & \\ & & & \xi_2/\xi_1 & \\ & & & & \xi_1 \end{pmatrix}, \tag{6.2.3}$$

where $\xi_i = \xi_i(wu) \ge 1$ for any $u \in \overline{U}_w$. For example, in the case $n = 4$ and $w = w_{\text{long}} = w_{(1,1,1,1)}$, we find that $\overline{U}_{w_{\text{long}}} = U_4(\mathbb{R})$ and, for

$$u = \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

that

$$\begin{aligned} \xi_1(w_{\text{long}}u) &= 1 + x_{12}^2 + x_{13}^2 + x_{14}^2, \\ \xi_2(w_{\text{long}}u) &= 1 + x_{23}^2 + x_{24}^2 + (x_{12}x_{24} - x_{14})^2 + (x_{12}x_{23} - x_{13})^2 + (x_{13}x_{24} - x_{14}x_{23})^2, \\ \xi_3(w_{\text{long}}u) &= 1 + x_{34}^2 + (x_{23}x_{34} - x_{24})^2 + (x_{12}x_{23}x_{34} - x_{13}x_{34} - x_{12}x_{24} + x_{14})^2. \end{aligned}$$

In general, the values $\xi_i$ are always of the form 1 plus a sum of squares of functions consisting of the entries of $u$.

From (6.0.3), replacing $a$ with $b$, we see that $|p^{(n)}_{T,R}(My)|$ is bounded by

$$\ll \delta(M)^{-\frac{1}{2}} \cdot \|M\|^{2b} \cdot \delta(y)^{-\frac{1}{2}} \cdot \|y\|^{2b} \cdot T^{\varepsilon + \frac{(n+4)(n-1)}{4} + R\cdot\left(D(n) + \frac{n(n-1)}{2}\right) - \sum_{j=1}^{n-1} B(b_j)}.$$

To similarly bound $|p^{(n)}_{T,R}(Lcwyw^{-1}t)|$, we first remark that since

$$c = c_1 \begin{pmatrix} d_1d_2\cdots d_{n-1} & & & & \\ & d_1d_2\cdots d_{n-2} & & & \\ & & \ddots & & \\ & & & d_1 & \\ & & & & 1 \end{pmatrix} =: c_1 d, \quad \text{where } d_i = \frac{c_{i-1}c_{i+1}}{c_i^2},$$

setting $c_0 = c_n := 1$ (and $a_0 = a_n := 0$ as usual), we see that

$$\delta^{-\frac{1}{2}}(c) \cdot \|c\|^{2a} = \delta(c)^{-\frac{1}{2}} \prod_{i=1}^{n-1} \left(\frac{c_{i-1}c_{i+1}}{c_i^2}\right)^{2a_i} = \prod_{k=1}^{n-1} c_k^{-1+2a_{i-1}-4a_i+2a_{i+1}}.$$

Therefore, it follows that

$$|p^{(n)}_{T,R}(Lcwyw^{-1}t)| \ll \frac{\delta(L)^{-\frac{1}{2}} \cdot \|L\|^{2a} \cdot \delta(t)^{-\frac{1}{2}} \cdot \|t\|^{2a}}{\prod_{k=1}^{n-1} c_k^{1-2a_{i-1}+4a_i-2a_{i+1}}} \cdot \delta(wyw^{-1})^{-\frac{1}{2}} \cdot \|wyw^{-1}\|^{2a} \cdot T^{\varepsilon + \frac{(n+4)(n-1)}{4} + R\cdot\left(D(n) + \frac{n(n-1)}{2}\right) - \sum_{j=1}^{n-1} B(a_j)}.$$

Recall that if $t = t(u)$ is as in (6.2.3), if we define, for $a = (a_1, \dots, a_{n-1}), b = (b_1, \dots, b_{n-1}) \in \mathbb{R}^{n-1}$,

$$K(w; a) := \sum_{v \in V_n} \sum_{c_1=1}^{\infty} \cdots \sum_{c_{n-1}=1}^{\infty} \frac{|S_w(\psi_L, \psi_M^v, c)|}{\prod_{i=1}^{n-1} c_i^{1-2a_{i-1}+4a_i-2a_{i+1}}},$$

$$X(w; a) := \int\limits_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})} \int\limits_{\bar{U}_w(\mathbb{R})} \delta(t)^{-\frac{1}{2}} \cdot \|t\|^{2a} \, d^*u,$$

and for a given choice of $\tau = (\tau_1, \dots, \tau_{n-1}) \in \{0, 1\}^{n-1}$

$$Y(\tau, w; a, b) := \int\limits_{y_1 \in I_{\tau_1}} \int\limits_{y_2 \in I_{\tau_2}} \cdots \int\limits_{y_{n-1} \in I_{\tau_{n-1}}} \|y\|^{2b} \cdot \|wyw^{-1}\|^{2a} \, dy^\times,$$

then the bound on $|\mathcal{I}_w(\tau)|$ given in (6.2.2) can be replaced by

$$|\mathcal{I}_w(\tau)| \ll T^{\varepsilon + \frac{(n+4)(n-1)}{2} + R\cdot(2D(n)+n(n-1)) - \sum_{j=1}^{n-1}(B(a_j)+B(b_j))} \cdot K(w; a) \cdot X(w; a) \cdot Y(\tau, w; a, b) \cdot \|L\|^{2a} \cdot \|M\|^{2b}. \quad (6.2.4)$$

We remark that in simplifying/finding $Y(\tau, w; a, b)$, we have used (6.1.2). The basic strategy to prove Proposition 6.0.1 is now clear: we first find $a$ such that both $K(w; a)$ and $X(w; a)$ converge; then given this choice of $a$, we determine a particular value of $b$ for which $Y(\tau, w; a, b)$ converges as well; finally, we work out the corresponding bounds on $\|L\|^{2a}$, $\|M\|^{2b}$ and $\sum_{j=1}^{n-1}(B(a_j) + B(b_j))$.

**6.3.** ***Restrictions on the parameters a and b.*** The trivial bound (see [Dąbrowski and Reeder 1998]) for the Kloosterman sum is given by

$$S(1, 1, c) \ll \delta^{\frac{1}{2}}(c) = c_1 c_2 \cdots c_{n-1}.$$

Hence $K(w; a)$ is convergent whenever $a$ is chosen such that

$$\|c\|^{2a} = \prod_{k=1}^{n-1} c_k^{2a_{k-1}-4a_k+2a_{k+1}} \ll \delta^{-\frac{1}{2}-\varepsilon}(c).$$

From (6.1.1), if we set $a_j = \frac{j(n-j)}{2}(1+\varepsilon)$, then $\|c\|^{2a} = \delta^{-1-\varepsilon}(c) \ll \delta^{-1/2-\varepsilon}(c)$. More generally, $K(w; a)$ converges in the case

$$a_j := \rho + \frac{j(n-j)}{2}(1+\varepsilon), \quad \rho > 0, \; j = 1, \dots, n-1. \quad (6.3.1)$$

That this choice of $a$ makes $K(w; a)$ converge is a consequence of the easily verifiable fact that

$$\|c\|^{2a} = (c_1 c_{n-1})^{-2\rho} \cdot \delta^{-1-\varepsilon}(c).$$

We assume henceforth that $a$ satisfies (6.3.1).

We next consider the convergence of $X(w; a)$. Recall that the Iwasawa form of $wu$ is assumed to be $u_0 t k$, meaning $wu = u_0 t k$, where $u_0 \in U_n(\mathbb{R})$, $t \in T_n(\mathbb{R})$ and $k \in O(n, \mathbb{R})$. Indeed, $t$ is given by (6.2.3).

Then

$$X(w;a)=\int\limits_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})}\int\limits_{\overline{U}_w(\mathbb{R})}\delta(t)^{-\frac{1}{2}}\cdot\|t\|^{2a}\,d^*u\ll\int\limits_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})}\int\limits_{\overline{U}_w(\mathbb{R})}\delta^{-\frac{3}{2}-\varepsilon}(t)\,d^*u.$$

The fact that the right-hand side converges is a consequence of [Jacquet 1967].

We now turn to the convergence of $Y(\tau,w;a,b)$. Applying Lemma A.1 (which describes $\|wyw^{-1}\|^{2a}$), we see that

$$Y(\tau,w;a,b)=\int\limits_{I_\tau}\Bigg(\prod_{i=1}^{s}\prod_{j=1}^{n_i}y_{n-\hat{n}_i+j}^{2b_{(n-\hat{n}_i+j)}-2(a_{\hat{n}_{i-1}}-a_{(\hat{n}_{i-1}+j)}+a_{\hat{n}_i})}\Bigg)\,dy^\times=\prod_{i=1}^{s}\prod_{j=1}^{n_i}Y_{n-\hat{n}_i+j}(\tau,w;a,b),$$

where

$$Y_{n-\hat{n}_i+j}(\tau,w;a,b):=\int\limits_{I_{\tau_{n-\hat{n}_i+j}}}y_{n-\hat{n}_i+j}^{2b_{n-\hat{n}_i+j}-2(a_{\hat{n}_{i-1}}-a_{\hat{n}_{i-1}+j}+a_{\hat{n}_i})}\,\frac{dy_{n-\hat{n}_i+j}}{y_{n-\hat{n}_i+j}}.$$

Hence, in order to bound $Y(\tau,w;a,b)$ (and thereby show that $\mathcal{I}_w(\tau)$ converges), we must choose $b=(b_1,\dots,b_{n-1})$ such that $Y_{n-\hat{n}_i+j}(\tau,w;a,b)$ converges. Clearly

$$b_{n-\hat{n}_i+j}=a_{\hat{n}_{i-1}}-a_{\hat{n}_{i-1}+j}+a_{\hat{n}_i}+(-1)^{\tau_{n-\hat{n}_i+j}}\cdot\frac{\varepsilon}{2}\quad(i=1,\dots,s,\ \ j=1,\dots,n_i)\tag{6.3.2}$$

suffices, since making this choice implies that, for each $k=1,\dots,n-1$,

$$Y_k(\tau,w;a,b)=\begin{cases}\int_0^1 y^{\varepsilon}\,(dy/y) & \text{if } \tau_k=0,\\ \int_1^\infty y^{-\varepsilon}\,(dy/y) & \text{if } \tau_k=1,\end{cases}$$

which converges $\big($and gives the same value $\frac{1}{\varepsilon}\big)$ in either case.

**6.4. *Proof of Proposition 6.0.1.*** We have now shown that if $w=w_{(n_1,\dots,n_r)}$ and we choose $a$ as in (6.3.1) and $b$ via (6.3.2) accordingly, the right-hand side of (6.2.4) converges, and hence gives a bound for $|\mathcal{I}_w|$. Therefore, in order to complete the proof of Proposition 6.0.1, we need to first show that

$$\|L\|^{2a}\cdot\|M\|^{2b}\ll\prod_{i=1}^{n-1}(\ell_i m_i)^{2\rho+\frac{n^2+1}{4}},$$

and second that the given choice of $a$ and $b$ gives the claimed bound for the power of $T$ appearing in (6.2.4).

To complete the first of these tasks we note that, by (6.3.1) and the fact that $j(n-j)$ is maximized (in $j$) when $j=\frac{n}{2}$, we have

$$a_j=\rho+\frac{j(n-j)}{2}(1+\varepsilon)\le\rho+\frac{n^2}{8}(1+\varepsilon)<\rho+\frac{n^2+1}{8}\tag{6.4.1}$$

for $\varepsilon<1/n^2$ and $1\le j\le n-1$. Similarly, using (6.3.1) and (6.3.2) we compute that, for $1\le i\le s$ and $1\le j\le n_i$,

$$b_{n-\hat{n}_i+j}=\rho+\tfrac{1}{2}(j^2+j(2\hat{n}_{i-1}-n)+\hat{n}_i(n-\hat{n}_i))+\varepsilon\tag{6.4.2}$$

for $\varepsilon$ sufficiently small. Note that the right-hand side of (6.4.2) is a concave up parabola in $j$, and therefore, on the interval $1 \le j \le n_i$, can attain its maximum only at $j = 1$ or $j = n_i$. So, if we can show that $b_{n-\hat{n}_i+1}$ and $b_{n-\hat{n}_i+ni}$ both satisfy a suitable upper bound, then the same bound will hold for all $1 \le j \le n_i$.

We consider first the endpoint $j = n_i$. Using (6.4.2) and the fact that $\hat{n}_i - n_i = \hat{n}_{i-1}$, we find that

$$b_{n-\hat{n}_i+n_i} = \rho + \tfrac{1}{2}\hat{n}_{i-1}(n - \hat{n}_{i-1}) + \varepsilon.$$

Again, $j(n - j)$ is maximized when $j = \frac{n}{2}$, so we conclude that

$$b_{n-\hat{n}_i+n_i} \le \rho + \frac{n^2}{8} + \varepsilon < \rho + \frac{n^2+1}{8} \tag{6.4.3}$$

for $\varepsilon$ sufficiently small.

Next we consider the endpoint $j = 1$. From (6.4.2) we find that

$$\begin{aligned} b_{n-\hat{n}_i+1} &= \rho + \tfrac{1}{2}(1 - n + \hat{n}_i(n - \hat{n}_i) + 2\hat{n}_{i-1}) + \varepsilon \\ &\le \rho + \tfrac{1}{2}(-1 - n + \hat{n}_i(n - \hat{n}_i) + 2\hat{n}_i) + \varepsilon, \end{aligned} \tag{6.4.4}$$

where the last step follows because $\hat{n}_{i-1} = \hat{n}_i - n_i \le \hat{n}_i - 1$. We find using calculus that, as a function of $\hat{n}_i$, the right-hand side of (6.4.4) is maximized when $\hat{n}_i = \frac{n+2}{2}$. So

$$\begin{aligned} b_{n-\hat{n}_i+1} &\le \rho + \frac{1}{2}\left(-1 - n + \frac{n+2}{2}\left(n - \frac{n+2}{2}\right) + n + 2\right) + \varepsilon \\ &= \rho + \frac{n^2}{8} + \varepsilon \le \rho + \frac{n^2+1}{8} \end{aligned} \tag{6.4.5}$$

for $\varepsilon$ small enough. From (6.4.3) and (6.4.5) it follows, again, that

$$b_{n-\hat{n}_i+j} \le \rho + \frac{n^2+1}{8}$$

for all $1 \le i \le s$ and $1 \le j \le n_i$. This and (6.4.1) yield the desired bound on $\|L\|^{2a} \cdot \|M\|^{2b}$.

The second task is accomplished using Lemma A.9. □

## 7. Bounding the Eisenstein spectrum $\mathcal{E}$

Recall that if $L = (\ell_1, \dots, \ell_{n-1})$, $M = (m_1, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$, with $\prod_{i=1}^{n-1} \ell_i m_i \ne 0$, then, by Theorem 4.1.1, the Eisenstein contribution to the Kuznetsov trace formula is given by

$$\mathcal{E} = \sum_{\mathcal{P}} \sum_{\Phi} \mathcal{E}_{\mathcal{P},\Phi},$$

where

$$\mathcal{E}_{\mathcal{P},\Phi} := c_{\mathcal{P}} \int\limits_{\substack{n_1 s_1 + \cdots + n_r s_r = 0 \\ \mathrm{Re}(s_j) = 0}} A_{E_{\mathcal{P},\Phi}}(L, s)\, \overline{A_{E_{\mathcal{P},\Phi}}(M, s)} \cdot |p^{n,\#}_{T,R}(\alpha_{(\mathcal{P},\Phi)}(s))|^2 \, ds.$$

In this section we give bounds for $\mathcal{E}$ in the case that $L = (\ell, 1, \dots, 1)$ and $M = (m, 1, \dots, 1)$, with $\ell, m \ne 0$.

**7.1.** ***The Eisenstein contribution $\mathcal{E}$ to the Kuznetsov trace formula.*** The main result of this section is the following.

**Theorem 7.1.1** (bounding the Eisenstein contribution $\mathcal{E}$). *Fix $n \geq 2$ and a sufficiently large integer $R > 0$. Let $L = (\ell, 1, \dots, 1)$, $M = (m, 1, \dots, 1) \in \mathbb{Z}^{n-1}$ with $\ell, m \neq 0$. Then, assuming the lower bound conjecture for Rankin–Selberg L-functions* (see Conjecture 1.3.2), *for $T \to \infty$ we have the bound*

$$\sum_{\mathcal{P}} \sum_{\Phi} |\mathcal{E}_{\mathcal{P},\Phi}| \ll (\ell m)^{\frac{1}{2}-\frac{1}{n^2+1}+\varepsilon} \cdot T^{R\cdot(\binom{2n}{n}-2^n)+\varepsilon}. \qquad \square$$

**7.2.** ***Proof of Theorem 7.1.1.***

*Proof.* We proceed by induction on $n$, beginning with the case $n = 2$. In this case, the only parabolic subgroup is the minimum parabolic, or Borel, subgroup $\mathcal{B} = \mathcal{P}_{1,1}$, and the only function $\Phi$ corresponding to $\mathcal{B}$ (see Definition 3.2.2) is the constant function $\Phi = 1$. The Eisenstein contribution in this case, then, is simply the quantity $\mathcal{E}_{\mathcal{B},1}$.

By Theorem 4.1.1 in the case $n = 2$, we have

$$\mathcal{E}_{\mathcal{B},1} = c_{\mathcal{B}} \int_{\operatorname{Re} s_1 = 0} A_{E_{\mathcal{B},1}}(\ell, s)\, \overline{A_{E_{\mathcal{B},1}}(m, s)} \cdot |p^{2,\#}_{T,R}(\alpha_{(\mathcal{B},1)}(s))|^2 \, ds_1,$$

where $s = (s_1, -s_1)$. Now note that, by (3.2.7), $\alpha_{(\mathcal{B},1)}(s) = s$. Moreover, by Definition 1.4.1, we have $\mathcal{F}^{(2)}_R \equiv 1$, so by Definition 1.4.4, we have

$$p^{2,\#}_{T,R}(\alpha_{(\mathcal{B},1)}(s)) = e^{s_1^2/T^2}\Gamma\Big(\frac{2R+1+2s_1}{4}\Big)\Gamma\Big(\frac{2R+1-2s_1}{4}\Big).$$

Furthermore, we see from Proposition 4.1.5 that

$$|A_{E_{\mathcal{B},1}}(\ell, s)| \ll |\zeta^*(1+2s_1)|^{-1} \sum_{\substack{c_1,c_2 \in \mathbb{Z}_{>0} \\ c_1 c_2 = \ell}} |c_1^{\alpha_1} c_2^{\alpha_2}| \ll \ell^{\varepsilon} \left| \Gamma\Big(\frac{1+2s_1}{2}\Big)\zeta(1+2s_1)\right|^{-1}.$$

Then

$$|\mathcal{E}_{\mathcal{B},1}| \ll (\ell m)^{\varepsilon} \int_{\operatorname{Re}(s_1)=0} e^{s_1^2/T^2} \frac{\big|\Gamma\big(\frac{2R+1+2s_1}{4}\big)\Gamma\big(\frac{2R+1-2s_1}{4}\big)\big|^2}{\big|\Gamma\big(\frac{1+2s_1}{2}\big)\zeta(1+2s_1)\big|^2} \, |ds_1|.$$

We may restrict our integration to the domain $|\operatorname{Im}(s)| \leq T$, since $e^{s_1^2/T^2}$ decays exponentially otherwise. On this domain, we use Stirling's bound (9.2.1) for the Gamma function, as well as the Vinogradov bound

$$|\zeta(1+it)|^{-1} \ll (1+|t|)^{\varepsilon} \quad (t \in \mathbb{R}).$$

We get

$$|\mathcal{E}_{\mathcal{B},1}| \ll (\ell m)^{\varepsilon} \int_{\substack{\operatorname{Re}(s_1)=0 \\ \operatorname{Im}(s_1) \leq T}} |1+s_1|^{2R-1+\varepsilon} \, |ds_1|,$$

from which it follows immediately that $|\mathcal{E}_{\mathcal{B},1}| \ll T^{2R+\varepsilon}$. So our desired result holds in the case $n = 2$.

We now proceed to the general case. For $n > 2$, in order to establish bounds for $\mathcal{E}_{\mathcal{P},\Phi}$, we need to know that our main theorem is true for all $k < n$. The reason this is needed is because we have to bound Rankin–Selberg L-functions $L(s, \phi_k \times \phi_{k'})$, with $2 \le k, k' < n$. This will require knowing the Weyl law with harmonic weights (Theorem 7.2.3) for $2 \le k, k' < n$. We may assume by induction, however, that this is indeed the case, i.e., the Weyl law with harmonic weights holds for all $2 \le k < n$.

Now recall that, for the parabolic $\mathcal{P}$ associated to a partition $n = n_1 + \cdots + n_r$, we have

$$\mathcal{E}_{\mathcal{P},\Phi} = \int\limits_{\substack{n_1 s_1 + \cdots + n_r s_r = 0 \\ \mathrm{Re}(s_j)=0}} A_{E_{\mathcal{P},\Phi}}(L, s)\, \overline{A_{E_{\mathcal{P},\Phi}}(M, s)} \cdot |p^{n,\#}_{T,R}(\alpha_{(\mathcal{P},\Phi)}(s))|^2 \, ds$$

where $\alpha_{\mathcal{P},\Phi}(s)$ is given by (see (3.2.7))

$$\Big(\overbrace{\alpha_{1,1} + s_1, \ldots, \alpha_{1,n_1} + s_1}^{n_1 \text{ terms}}, \quad \overbrace{\alpha_{2,1} + s_2, \ldots, \alpha_{2,n_2} + s_2}^{n_2 \text{ terms}}. \quad \ldots \quad , \overbrace{\alpha_{r,1} + s_r, \ldots, \alpha_{r,n_r} + s_r}^{n_r \text{ terms}}\Big).$$

Since $\sum_{i=1}^{n_k} \alpha_{k,i} = 0$ for all $1 \le k \le r$ we see that

$$\sum_{k=1}^{r} \sum_{i=1}^{n_k} (\alpha_{k,i} + s_k)^2 = \sum_{k=1}^{r} \sum_{i=1}^{n_k} (\alpha_{k,i}^2 + s_k^2).$$

Now, for any $\beta = (\beta_1, \ldots, \beta_n) \in (i\mathbb{R})^n$, where $\hat{\beta}_n = 0$, we have

$$p^{n,\#}_{T,R}(\beta) := \left(\frac{\beta_1^2 + \beta_2^2 + \cdots + \beta_n^2}{2T^2}\right) \cdot \mathcal{F}^{(n)}_R\left(\frac{\beta}{2}\right) \prod_{1 \le i < j \le n} \left|\Gamma\left(\frac{2R + 1 + \beta_i - \beta_j}{4}\right)\right|^2.$$

It follows that

$$\begin{aligned} p^{n,\#}_{T,R}(\alpha_{(\mathcal{P},\Phi)}(s)) = \exp&\left(\frac{\sum_{k=1}^{r} \sum_{i=1}^{n_k} (\alpha_{k,i}^2 + s_k^2)}{2T^2}\right) \mathcal{F}^{(n)}_R\left(\frac{\alpha_{(\mathcal{P},\Phi)}(s)}{2}\right) \\ &\cdot \prod_{\substack{k=1 \\ n_k \ne 1}}^{r} \prod_{1 \le i < j \le n_k} \left|\Gamma\left(\frac{2R + 1 + \alpha_{k,i} - \alpha_{k,j}}{4}\right)\right|^2 \\ &\cdot \prod_{1 \le k < k' \le r} \prod_{i=1}^{n_k} \prod_{j=1}^{n_{k'}} \left|\Gamma\left(\frac{2R + 1 + s_k - s_{k'} + \alpha_{k,i} - \alpha_{k',j}}{4}\right)\right|^2. \end{aligned}$$

By Proposition 4.1.5, the $m$-th coefficient of $E_{\mathcal{P},\Phi}$ is given by

$$\begin{aligned} A_{E_{\mathcal{P},\Phi}}((m, 1, \ldots, 1), s) = \prod_{\substack{k=1 \\ n_k \ne 1}}^{r} L^*(1, \mathrm{Ad}\,\phi_k)^{-\frac{1}{2}} \prod_{1 \le i < j \le r} L^*(1 + s_i - s_j, \phi_i \times \phi_j)^{-1} \\ \cdot \sum_{\substack{1 \le c_1, c_2, \ldots, c_r \in \mathbb{Z} \\ c_1 c_2 \cdots c_r = m}} \lambda_{\phi_1}(c_1) \cdots \lambda_{\phi_r}(c_r) \cdot c_1^{s_1} \cdots c_r^{s_r} \end{aligned}$$

up to a nonzero constant factor with absolute value depending only on $n$. To bound the divisor sum above we will use the bound of Luo, Rudnick and Sarnak [Luo et al. 1999] for the $m$-th Hecke Fourier coefficient of a $\mathrm{GL}(\kappa)$ (for $\kappa \ge 2$) Hecke–Maass cusp form $\phi$ given by

$$|\lambda_\phi(m, 1, \ldots, 1)| \le m^{\frac{1}{2} - 1/(\kappa^2 + 1) + \varepsilon}.$$

(A slightly stronger result has been obtained by Kim and Sarnak [Kim 2003]. However, the stated result above is sufficient for our purposes.) We immediately obtain the following bound for the divisor sum:

$$\sum_{\substack{1\le c_1,c_2,\ldots,c_r\in\mathbb{Z}\\ c_1c_2\cdots c_r=m}} |\lambda_{\phi_1}(c_1)\cdots\lambda_{\phi_r}(c_r)\cdot c_1^{s_1}\cdots c_r^{s_r}| \ll m^{\frac12-1/(n^2+1)+\varepsilon}.$$

It follows that

$$\begin{aligned}
|\mathcal{E}_{\mathcal{P},\Phi}| &\ll (m\ell)^{\frac12-\frac{1}{n^2+1}+\varepsilon}\cdot\biggl(\frac{\sum_{k=1}^r\sum_{i=1}^{n_k}\alpha_{k,i}^2}{T^2}\biggr)\\
&\quad\cdot\int\limits_{\substack{n_1s_1+\cdots+n_rs_r=0\\ \mathrm{Re}(s_j)=0,\mathrm{Im}(s_j)\ll T}} \biggl|\mathcal{F}_R^{(n)}\Bigl(\frac{\alpha_{(\mathcal{P},\Phi)}(s)}{2}\Bigr)\biggr|^2\biggl(\prod_{\substack{k=1\\ n_k\ne1}}^r\prod_{1\le i<j\le n_k}\biggl|\Gamma\Bigl(\frac{2R+1+\alpha_{k,i}-\alpha_{k,j}}{4}\Bigr)\biggr|^4\biggr)\\
&\qquad\cdot\biggl(\prod_{1\le k<k'\le r}\prod_{i=1}^{n_k}\prod_{j=1}^{n_{k'}}\biggl|\Gamma\Bigl(\frac{2R+1+s_k-s_{k'}+\alpha_{k,i}-\alpha_{k',j}}{4}\Bigr)\biggr|^4\biggr)\\
&\qquad\cdot\biggl(\prod_{\substack{k=1\\ n_k\ne1}}^r|L^*(1,\mathrm{Ad}\,\phi_k)|^{-1}\biggr)\cdot\biggl(\prod_{1\le k<k'\le r}|L^*(1+s_k-s_{k'},\phi_k\times\phi_{k'})|^{-2}\,|ds|\biggr)\\
&\ll (m\ell)^{\frac12-\frac{1}{n^2+1}+\varepsilon}\prod_{\substack{k=1\\ n_k\ne1}}^r\exp\Bigl(\frac{\alpha_{k,1}^2+\cdots+\alpha_{k,n_k}^2}{T^2}\Bigr)\\
&\quad\cdot\int\limits_{\substack{n_1s_1+\cdots+n_rs_r=0\\ \mathrm{Re}(s_j)=0,\mathrm{Im}(s_j)\ll T}} \biggl|\mathcal{F}_R^{(n)}\Bigl(\frac{\alpha_{(\mathcal{P},\Phi)}(s)}{2}\Bigr)\biggr|^2\prod_{\substack{k=1\\ n_k\ne1}}^r\frac{1}{|L(1,\mathrm{Ad}\,\phi_k)|}\prod_{1\le i<j\le n_k}\frac{\bigl|\Gamma\bigl(\frac{2R+1+\alpha_{k,i}-\alpha_{k,j}}{4}\bigr)\bigr|^4}{\bigl|\Gamma\bigl(\frac{1+\alpha_{k,i}-\alpha_{k,j}}{2}\bigr)\bigr|^2}\\
&\qquad\cdot\prod_{1\le k<k'\le r}\frac{1}{|L(1+s_k-s_{k'},\phi_k\times\phi_{k'})|^2}\prod_{i=1}^{n_k}\prod_{j=1}^{n_{k'}}\frac{\bigl|\Gamma\bigl(\frac{2R+1+s_k-s_{k'}+\alpha_{k,i}-\alpha_{k',j}}{4}\bigr)\bigr|^4}{\bigl|\Gamma\bigl(\frac{1+s_k-s_{k'}+\alpha_{k,i}-\alpha_{k',j}}{2}\bigr)\bigr|^2}\,|ds|.
\end{aligned}$$

**Lemma 7.2.1.** *Assume $|s_k|\ll T^{1+\varepsilon}$ and $|\alpha_{k,j}|\ll T^{1+\varepsilon}$ for $1\le k\le r$ and $1\le j\le n_k$. Then for $\alpha:=\alpha_{(\mathcal{P},\Phi)}(s)$ and $\alpha^{(k)}$ as in Definition A.16, we have*

$$|\mathcal{F}_R^{(n)}(\alpha_{(\mathcal{P},\Phi)}(s))|^2\ll\biggl(\prod_{\substack{k=1\\ n_k\ne1}}^r|\mathcal{F}_R^{(n_k)}(\alpha^{(k)})|^2\biggr)\cdot T^{R\cdot B(n)+\varepsilon},$$

*where $B(n)=2D(n)-2\sum_{k=1,n_k\ne1}^r D(n_k)$.*

*Proof.* This follows immediately from Lemma A.27. □

It follows from Lemma 7.2.1 that for $|\alpha^{(k)}|^2=\alpha_{k,1}^2+\cdots+\alpha_{k,n_k}^2$, we have

$$\begin{aligned}
|\mathcal{E}_{\mathcal{P},\Phi}| &\ll (m\ell)^{\frac12-\frac{1}{n^2+1}+\varepsilon}T^{R\cdot B(n)+\varepsilon}\prod_{\substack{k=1\\ n_k\ne1}}^r\frac{\exp\bigl(\frac{|\alpha^{(k)}|^2}{T^2}\bigr)\bigl|\mathcal{F}_R^{(n_k)}\bigl(\frac{\alpha^{(k)}}{2}\bigr)\bigr|^2\prod_{1\le i<j\le n_k}\frac{|\Gamma((2R+1+\alpha_{k,i}-\alpha_{k,j})/4)|^4}{|\Gamma((1+\alpha_{k,i}-\alpha_{k,j})/2)|^2}}{|L(1,\mathrm{Ad}\,\phi_k)|}\\
&\quad\cdot\int\limits_{\substack{n_1s_1+\cdots+n_rs_r=0\\ \mathrm{Re}(s_j)=0,|\mathrm{Im}(s_j)|\ll T}}\prod_{1\le k<k'\le r}\frac{1}{|L(1+s_k-s_{k'},\phi_k\times\phi_{k'})|^2}\prod_{i=1}^{n_k}\prod_{j=1}^{n_{k'}}\frac{\bigl|\Gamma\bigl(\frac{2R+1+s_k-s_{k'}+\alpha_{k,i}-\alpha_{k',j}}{4}\bigr)\bigr|^4}{\bigl|\Gamma\bigl(\frac{1+s_k-s_{k'}+\alpha_{k,i}-\alpha_{k',j}}{2}\bigr)\bigr|^2}\,|ds|
\end{aligned}$$

$$\ll (m\ell)^{\frac{1}{2}-\frac{1}{n^2+1}+\varepsilon}\cdot T^{R\cdot B(n)+\varepsilon} \prod_{\substack{k=1\\ n_k\neq 1}}^{r} \frac{|h_{T,R}^{(n_k)}(\alpha^{(k)})|}{|L(1,\mathrm{Ad}\,\phi_k)|}$$
$$\cdot \int\limits_{\substack{n_1s_1+\cdots+n_rs_r=0\\ \mathrm{Re}(s_j)=0,|\mathrm{Im}(s_j)|\ll T}} \prod_{1\le k<k'\le r} \frac{1}{|L(1+s_k-s_{k'},\phi_k\times\phi_{k'})|^2} \prod_{i=1}^{n_k}\prod_{j=1}^{n_{k'}} \frac{\left|\Gamma\left(\frac{2R+1+s_k-s_{k'}+\alpha_{k,i}-\alpha_{k',j}}{4}\right)\right|^4}{\left|\Gamma\left(\frac{1+s_k-s_{k'}+\alpha_{k,i}-\alpha_{k',j}}{2}\right)\right|^2}\,|ds|.$$

where

$$h_{T,R}^{(n_k)}(\alpha^{(k)}) = \exp\left(\frac{|\alpha^{(k)}|^2}{T^2}\right)\mathcal{F}_R^{(n_k)}\left(\frac{\alpha^{(k)}}{2}\right)^2 \prod_{1\le i\neq j\le n_k} \frac{\Gamma\left(\frac{2R+1+\alpha_{k,i}-\alpha_{k,j}}{4}\right)^2}{\Gamma\left(\frac{1+\alpha_{k,i}-\alpha_{k,j}}{2}\right)}.$$

Next

$$\prod_{1\le k<k'\le r}\prod_{i=1}^{n_k}\prod_{j=1}^{n_{k'}} \frac{\left|\Gamma\left(\frac{2R+1+s_k-s_{k'}+\alpha_{k,i}-\alpha_{k',j}}{4}\right)\right|^4}{\left|\Gamma\left(\frac{1+s_k-s_{k'}+\alpha_{k,i}-\alpha_{k',j}}{2}\right)\right|^2} \ll T^{(2R-1)\sum_{1\le k<k'\le r} n_k\cdot n_{k'}}.$$

We obtain the bound

$$|\mathcal{E}_{\mathcal{P},\Phi}| \ll (m\ell)^{\frac{1}{2}-\frac{1}{n^2+1}+\varepsilon}\cdot T^{R\cdot B(n)+\varepsilon+(2R-1)\cdot\sum_{1\le k<k'\le r} n_k\cdot n_{k'}}$$
$$\cdot\prod_{k=1}^{r} \frac{|h_{T,R}^{(n_k)}(\alpha^{(k)})|}{|L(1,\mathrm{Ad}\,\phi_k)|} \int\limits_{\substack{n_1s_1+\cdots+n_rs_r=0\\ \mathrm{Re}(s_j)=0,|\mathrm{Im}(s_j)|\ll T}} \prod_{1\le k<k'\le r} \frac{|ds|}{|L(1+s_k-s_{k'},\phi_k\times\phi_{k'})|^2}.$$

Next, we bound the $s$-integral above. It follows from Langlands' conjecture (see Conjecture 1.3.2) that for $|\mathrm{Im}(s_k)|, |\mathrm{Im}(s_{k'})| \ll T$ we have the bound

$$|L(1+s_k-s_{k'},\phi_k\times\phi_{k'})|^{-2} \ll T^{\varepsilon}.$$

This together with the bound

$$\int\limits_{\substack{n_1s_1+\cdots+n_rs_r=0\\ \mathrm{Re}(s_j)=0,\ |\mathrm{Im}(s_j)|\ll T}} |ds| \ \ll\ T^{r-1},$$

implies that

$$|\mathcal{E}_{\mathcal{P},\Phi}| \ll (m\ell)^{\frac{1}{2}-\frac{1}{n^2+1}+\varepsilon} T^{R\cdot B(n)+(2R-1)\sum_{1\le k<k'\le r} n_k\cdot n_{k'}+(r-1)+\varepsilon}\cdot\left(\prod_{k=1}^{r} \frac{|h_{T,R}^{(n_k)}(\alpha^{(k)})|}{|L(1,\mathrm{Ad}\,\phi_k)|}\right). \tag{7.2.2}$$

Since each $n_k < n$ (for $k = 1, 2, \ldots, r$), we can apply our inductive procedure together with the following theorem to bound $\sum_\Phi |\mathcal{E}_{\mathcal{P},\Phi}|$.

**Theorem 7.2.3** (Weyl law with harmonic weights for $\mathrm{GL}(n_k)$ with $n_k < n$). *Suppose $n_k\in\mathbb{Z}$ with $2\le n_k<n$. Let $\{\phi_1,\phi_2,\ldots\}$ be an orthogonal basis of Hecke–Maass cusp forms for $\mathrm{GL}(n_k)$ ordered by eigenvalue. If $\alpha^{(j)}$ are the Langlands parameters of $\phi_j$, then*

$$\sum_{j=1}^{\infty} \frac{h_{T,R}^{(n_k)}(\overline{\alpha^{(j)}})}{\mathcal{L}_j} \ll_n T^{2R\cdot\left(D(k)+\frac{n_k(n_k-1)}{2}\right)+n_k-1}. \tag{7.2.4}$$

*Proof.* In [Goldfeld et al. 2021b], all that was needed to prove this statement for $n=4$ was to have it be true for $n_k=2$ and $n_k=3$, which was already known. A similar induction argument works in general. □

It immediately follows from the bounds (7.2.2) and (7.2.4) that

$$\sum_{\Phi}|\mathcal{E}_{\mathcal{P},\Phi}| \ll (m\ell)^{\frac{1}{2}-\frac{1}{n^2+1}+\varepsilon}\, T^{R\cdot B(n)+(2R-1)\sum_{1\le k<k'\le r} n_k\cdot n_{k'}+(r-1)+\varepsilon}\cdot T^{\sum_{k=1}^r\left(2R\cdot\left(D(k)+\frac{n_k(n_k-1)}{2}\right)+n_k-1\right)}.$$

Recall that $B(n)=2D(n)-2\sum_{k=1}^r D(n_k)$, which implies that

$$\sum_{\Phi}|\mathcal{E}_{\mathcal{P},\Phi}| \ll (m\ell)^{\frac{1}{2}-\frac{1}{n^2+1}+\varepsilon}T^{2R\cdot D(n)+2R\left(\sum_{1\le k<k'\le r} n_k\cdot n_{k'}+\sum_{k=1}^r\frac{n_k(n_k-1)}{2}\right)+\sum_{k=1}^r n_k-\sum_{1\le k<k'\le r} n_k\cdot n_{k'}-1+\varepsilon}.$$

Next, $\sum_{1\le k<k'\le r} n_k\cdot n_{k'}+\sum_{k=1}^r\frac{n_k(n_k-1)}{2}=\frac{n(n-1)}{2}$ by Lemma A.22 and $\sum_{k=1}^r n_k=n$. It follows that

$$\sum_{\Phi}|\mathcal{E}_{\mathcal{P},\Phi}| \ll (m\ell)^{\frac{1}{2}-\frac{1}{n^2+1}+\varepsilon}\, T^{2R\cdot\left(D(n)+\frac{n(n-1)}{2}\right)+n-1-\sum_{1\le k<k'\le r} n_k\cdot n_{k'}+\varepsilon}.$$

To complete the proof, we need to sum over all parabolics $\mathcal{P}$. It suffices, therefore, to consider the "worst case scenario" among the possible partitions $n=n_1+\cdots+n_r$ for which the expression

$$\sum_{1\le k<k'\le r} n_k n_{k'}$$

is minimized. It is easy to see that this occurs when $r=2$ and $\{n_1,n_2\}=\{n-1,1\}$, giving the bound $n-1$. It follows that

$$\sum_{\mathcal{P}}\sum_{\Phi}|\mathcal{E}_{\mathcal{P},\Phi}| \ll (m\ell)^{\frac{1}{2}-\frac{1}{n^2+1}+\varepsilon}\, T^{2R\cdot\left(D(n)+\frac{n(n-1)}{2}\right)+\varepsilon}.$$

Using (1.4.2), this immediately implies the desired result. □

**Remark 7.2.5.** Jana and Nelson [2019] proved the bound

$$\sum_{c(\phi_j)\le T^n}\frac{1}{\mathcal{L}_j} \ll T^{n^2-n}, \tag{7.2.6}$$

where $c(\phi)$ is the analytic conductor given in (1.3.1). This is an unsmoothed version of Theorem 7.2.3. Our result is a smoothed version, and it doesn't seem possible to derive a bound as in (7.2.6) with a sharp cutoff without using a different approach.

## 8. An integral representation of $p_{T,R}^{(n)}(y)$

Recall (see 1.4.4) that

$$p_{T,R}^{n,\#}(\alpha) := e^{(\alpha_1^2+\alpha_2^2+\cdots+\alpha_n^2)/(2T^2)}\cdot\mathcal{F}_R^{(n)}\left(\frac{\alpha}{2}\right)\prod_{1\le j\ne k\le n}\Gamma\left(\frac{1+2R+\alpha_j-\alpha_k}{4}\right).$$

Using the formula for the inverse Lebedev–Whittaker transform given in [Goldfeld and Kontorovich 2012], it follows that

$$p_{T,R}^{(n)}(y) := \frac{1}{\pi^{n-1}} \int\limits_{\substack{\hat{\alpha}_n=0 \\ \mathrm{Re}(\alpha)=0}} \frac{p_{T,R}^{n,\#}(\alpha)\, \overline{W_{n,\alpha}(y)}}{\prod_{1\le j\ne k\le n} \Gamma\left(\frac{\alpha_j-\alpha_k}{2}\right)}\, d\alpha$$

$$= \frac{1}{\pi^{n-1}} \int\limits_{\substack{\hat{\alpha}_n=0 \\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\alpha_2^2+\cdots+\alpha_n^2}{2T^2}} \cdot \mathcal{F}_R^{(n)}\left(\frac{\alpha}{2}\right) \prod_{1\le j\ne k\le n} \Gamma_R\left(\frac{\alpha_j-\alpha_k}{2}\right) \overline{W_{n,\alpha}(y)}\, d\alpha,$$

where

$$\Gamma_R(z) := \frac{\Gamma\left(\frac{1}{2}\left(\frac{1}{2}+R+z\right)\right)}{\Gamma(z)}.$$

The strategy in this section for giving a representation of $p_{T,R}^{(n)}(y)$ follows the same general outline as was used to obtain the results for GL(3) and GL(4) given in [Goldfeld and Kontorovich 2013] and [Goldfeld et al. 2021b], respectively. As in the prior works, we express the Whittaker function as the inverse Mellin transform of its Mellin transform. (See Section 8.1.) Plugging this into the above formula gives an integral representation of $p_{T,R}^{(n)}(y)$ in terms of an additional variable $s = (s_1, \ldots, s_{n-1})$.

**8.1.** ***Normalized Mellin transform of Whittaker function.*** We introduce (as in [Ishii and Stade 2007]) the following Mellin transform and its inverse.

**Definition 8.1.1** (normalized Mellin transform of Whittaker function). Let $n \in \mathbb{Z}_+$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ such that $\hat{\alpha}_n = 0$. Let $W_{n,\alpha}(y)$ be the Whittaker function of Definition 2.3.3. The *Mellin transform* is

$$\widetilde{W}_{n,\alpha}(s) := 2^{n-1} \int_0^\infty \cdots \int_0^\infty W_{n,2\alpha}(y) \prod_{j=1}^{n-1} (\pi y_j)^{2s_j} \frac{dy_j}{y_j^{1+\frac{j(n-j)}{2}}}, \tag{8.1.2}$$

and the *inverse Mellin transform* is given by

$$W_{n,\alpha}(y) = \frac{1}{2^{n-1}} \int\limits_{\substack{s=(s_1,\ldots,s_{n-1}) \\ \mathrm{Re}(s)=2b}} \left( \prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}} (\pi y_j)^{-s_j} \right) \widetilde{W}_{n,\frac{\alpha}{2}}\left(\tfrac{s}{2}\right) ds. \tag{8.1.3}$$

As a consequence of this definition, we have

$$p_{T,R}^{(n)}(y) = \frac{1}{(2\pi)^{n-1}} \int\limits_{\substack{\hat{\alpha}_n=0 \\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_n^2}{T^2/2}} \cdot \mathcal{F}_R^{(n)}(\alpha) \left( \prod_{1\le j\ne k\le n} \Gamma_R(\alpha_j-\alpha_k) \right) \cdot \int\limits_{\substack{s=(s_1,\ldots,s_{n-1}) \\ \mathrm{Re}(s)=b}} \left( \prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}} (\pi y_j)^{-2s_j} \right) \widetilde{W}_{n,\alpha}(s)\, ds\, d\alpha, \tag{8.1.4}$$

where $b = (b_1, \ldots, b_{n-1})$ with each $b_j > 0$.

We use the following theorem to make (8.1.4) explicit and to begin setting up an inductive method to bound $p_{T,R}^{(n)}(y)$ for all $n \ge 2$.

**Theorem 8.1.5** (Ishii–Stade). *Let $m \geq 2$ and $\varepsilon > 0$. Fix a Langlands parameter $\alpha \in \mathbb{C}^m$. Let $s \in \mathbb{C}^{m-1}$ with* $\mathrm{Re}(s) > \varepsilon$. *Then*

$$\widetilde{W}_{m,\alpha}(s) = \int\limits_{\substack{z=(z_1,\dots,z_{m-2})\\ \mathrm{Re}(z)=\varepsilon}} \left(\prod_{j=1}^{m-1} \Gamma\left(s_j - z_{j-1} + \frac{(m-j)\alpha_m}{m-1}\right)\Gamma\left(s_j - z_j - \frac{j\alpha_m}{m-1}\right)\right) \cdot \frac{\widetilde{W}_{m-1,\beta}(z)}{(2\pi i)^{m-2}}\, dz, \quad (8.1.6)$$

*where*

$$z_0 := -0 + \frac{0 \cdot \alpha_m}{m-1} = 0, \qquad z_{m-1} := \alpha_m - \frac{(m-1)\alpha_m}{m-1} = 0$$

*and*

$$\beta = (\beta_1, \dots, \beta_{m-1}) := \left(\alpha_1 + \frac{\alpha_m}{m-1}, \dots, \alpha_{m-1} + \frac{\alpha_m}{m-1}\right).$$

**8.2. *A shifted $p^{(n)}_{T,R}$ term and the Ishii–Stade conjecture.*** Our goal is to insert (8.1.6) into (8.1.4) and then shift the lines of integration in $s$ to $\mathrm{Re}(s) = -a$, to the left of some of the poles of $\widetilde{W}_{n,\alpha}(s)$, which (see Theorem 10.1.1) occur at $\mathrm{Re}(s_i) = -\delta$ for every $1 \leq i \leq n-1$ and $\delta \in \mathbb{Z}_{\geq 0}$. By Cauchy's residue formula, this allows us to describe $p^{(n)}_{T,R}(y)$ in terms of a the sum of a *shifted $p^{(n)}_{T,R}$ term* and finitely many *shifted residue terms*.

**Definition 8.2.1** (shifted $p^{(n)}_{T,R}$ term). Let $n \geq 2$ be an integer and $a = (a_1, \dots, a_{n-1}) \in \mathbb{R}^{n-1}$. The *shifted $p^{(n)}_{T,R}$ term* is given by the same formula as (8.1.4) but with $b$ replaced by $-a$:

$$p^{(n)}_{T,R}(y; -a) := \frac{1}{(2\pi)^{n-1}} \int\limits_{\substack{\hat{\alpha}_n=0\\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_n^2}{T^2/2}} \cdot \mathcal{F}^{(n)}_R(\alpha)\left(\prod_{1\leq j\neq k\leq n} \Gamma_R(\alpha_j - \alpha_k)\right) \cdot \int\limits_{\substack{s=(s_1,\dots,s_{n-1})\\ \mathrm{Re}(s)=-a}} \left(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}} (\pi y_j)^{-2s_j}\right) \widetilde{W}_{n,\alpha}(s)\, ds\, d\alpha. \quad (8.2.2)$$

One might be tempted to insert (8.1.6) into (8.2.2), but this is invalid if $n > 3$, because Theorem 8.1.5 requires that $\mathrm{Re}(s_i) > \varepsilon$ for each $i = 1, \dots, n-1$. To overcome this difficulty, we use shift equations as given in the following conjecture. This allows us to evaluate $\widetilde{W}_{n,\alpha}(s)$ for $\mathrm{Re}(s) < 0$.

**Conjecture 8.2.3** (Ishii–Stade). *Let $m, n \in \mathbb{Z}$ with $1 \leq m \leq n-1$; let $\delta \in \mathbb{Z}_{\geq 0}$. Let*

$$(x)_n := \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\cdots(x+n-1).$$

*Then there exists a positive integer $r$ and, for each $i$ with $1 \leq i \leq r$, a polynomial $P_i(s,\alpha)$ and an $(n-1)$-tuple $\Sigma_i \in (\mathbb{Z}_{\geq 0})^{n-1}$, such that*

$$\widetilde{W}_{n,\alpha}(s) = \left[\prod_{1\leq j_1<j_2<\cdots<j_m\leq n} (s_m + \alpha_{j_1} + \alpha_{j_2} + \cdots + \alpha_{j_m})_\delta\right]^{-1} \sum_{i=1}^{r} P_i(s,\alpha)\widetilde{W}_{n,\alpha}(s+\Sigma_i), \quad (8.2.4)$$

*where the $m$th coordinate of each $\Sigma_i$ is at least $\delta$. Moreover, for each $i$, we have*

$$\deg(P_i(s,\alpha)) + 2|\Sigma_i| = \delta\binom{n}{m}.$$

*Proof of conjecture for* $2 \le n \le 5$. Note that the case $\delta = 0$ of the conjecture is trivial. Moreover, for a given $m$ and $n$ with $1 \le m \le n-1$, it's enough to prove the conjecture for $\delta = 1$. The case $\delta > 1$ then follows by applying the case $\delta = 1$ to itself iteratively.

For $\delta = 1$ and $n = 2$ or $n = 3$, the conjecture follows immediately from the explicit formulae

$$\widetilde{W}_{2,\alpha}(s) = \Gamma(s+\alpha)\Gamma(s-\alpha),$$
$$\widetilde{W}_{3,\alpha}(s) = \frac{\Gamma(s_1+\alpha_1)\Gamma(s_1+\alpha_2)\Gamma(s_1+\alpha_3)\Gamma(s_2-\alpha_1)\Gamma(s_2-\alpha_2)\Gamma(s_2-\alpha_3)}{\Gamma(s_1+s_2)},$$

respectively, together with the functional equation $\Gamma(s+1) = s\Gamma(s)$. The case $\delta = 1$ and $n = 4$ is a consequence of [Stade and Trinh 2021, equations (21), (29), and (31)].

We now consider the case $\delta = 1$ and $n = 5$. Note that it suffices to derive the appropriate recurrence relations for $m = 1$ and $m = 2$ (that is, for the variables $s_1$ and $s_2$); the cases $m = 3$ and $m = 4$ then follow from the invariance of $\widetilde{W}_{5,\alpha}(s)$ under the involution

$$(s_1, s_2, s_3, s_4, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \to (s_4, s_3, s_2, s_1, -\alpha_1, -\alpha_2, -\alpha_3, -\alpha_4, -\alpha_5).$$

We follow an approach developed by Taku Ishii (personal correspondence). First, consider the case $m = 1$: we wish to show that

$$\left[\prod_{i=1}^{5}(s_1+\alpha_i)\right]\widetilde{W}_{5,\alpha}(s) \tag{8.2.5}$$

is equal to a finite sum of terms $P_i(s,\alpha)\widetilde{W}_{n,\alpha}(s+\Sigma_i)$, where the first coordinate of each $\Sigma_i \in (\mathbb{Z}_{\ge 0})^4$ is at least 1, and $\deg(P_i(s,\alpha)) + 2|\Sigma_i| = 5$ for each $i$. To this end, let

$$\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) = (-s_1, s_1 - s_2, s_2 - s_3, s_3 - s_4, s_4); \tag{8.2.6}$$

note that $\sum_i \sigma_i = 0$. Since $s_1 + \sigma_1 = 0$, we have

$$\left[\prod_{i=1}^{5}(s_1+\alpha_i)\right]\widetilde{W}_{5,\alpha}(s) = \left[\prod_{i=1}^{5}(s_1+\alpha_i) - \prod_{i=1}^{5}(s_1+\sigma_i)\right]\widetilde{W}_{5,\alpha}(s). \tag{8.2.7}$$

But for indeterminates $T, x_1, x_2, x_3, x_4, x_5$, we have

$$\prod_{i=1}^{5}(T+x_i) = T^5 + T^4 P_1(x) + T^3 P_2(x) + T^2 P_3(x) + T\, P_4(x) + P_5(x), \tag{8.2.8}$$

where $P_k(x)$ is the elementary symmetric polynomial of degree $k$ in $x_1, x_2, x_3, x_4, x_5$. So by (8.2.7) above,

$$\begin{aligned}
&\left[\prod_{i=1}^{5}(s_1+\alpha_i)\right]\widetilde{W}_{5,\alpha}(s) \\
&\quad = \left[s_1^5 + s_1^4 P_1(\alpha) + s_1^3 P_2(\alpha) + s_1^2 P_3(\alpha) + s\, P_4(\alpha) + P_5(\alpha)\right]\widetilde{W}_{5,\alpha}(s) \\
&\qquad\qquad - \left[s_1^5 + s_1^4 P_1(\sigma) + s_1^3 P_2(\sigma) + s_1^2 P_3(\sigma) + s\, P_4(\sigma) + P_5(\sigma)\right]\widetilde{W}_{5,\alpha}(s) \\
&\quad = \left[s_1^3\{P_2(\alpha)-P_2(\sigma)\} + s_1^2\{P_3(\alpha)-P_3(\sigma)\} + s_1\{P_4(\alpha)-P_4(\sigma)\} + \{P_5(\alpha)-P_5(\sigma)\}\right]\widetilde{W}_{5,\alpha}(s),
\end{aligned} \tag{8.2.9}$$

since $P_1(\alpha) = P_1(\sigma) = 0$.

Now let $e_k$, for $1\le k\le 4$, be the four-tuple with a 1 in the $k$-th place and zeroes elsewhere. By [Ishii and Oda 2014, Proposition 3.6], we have

$$P_k(\alpha)-P_k(\sigma)=Z_k-C_k$$

(as operators acting on functions in the variable $s=(s_1,s_2,s_3,s_4)$), where the "Capelli elements" $C_k$ annihilate $\widetilde{W}_{5,\alpha}(s)$, and

$$\begin{aligned}
Z_2f(s)&=f(s+e_1)+f(s+e_2)+f(s+e_3)+f(s+e_4),\\
Z_3f(s)&=P_1(\sigma_3,\sigma_4,\sigma_5)f(s+e_1)+P_1(\sigma_1,\sigma_4,\sigma_5)f(s+e_2)+P_1(\sigma_1,\sigma_2,\sigma_5)f(s+e_3)\\
&\qquad+P_1(\sigma_1,\sigma_2,\sigma_3)f(s+e_4),\\
Z_4f(s)&=P_2(\sigma_3,\sigma_4,\sigma_5)f(s+e_1)+P_2(\sigma_1,\sigma_4,\sigma_5)f(s+e_2)+P_2(\sigma_1,\sigma_2,\sigma_5)f(s+e_3)\\
&\qquad+P_2(\sigma_1,\sigma_2,\sigma_3)f(s+e_4)+f(s+e_1+e_3)+f(s+e_1+e_4)+f(s+e_2+e_4),\\
Z_5f(s)&=P_3(\sigma_3,\sigma_4,\sigma_5)f(s+e_1)+P_3(\sigma_1,\sigma_4,\sigma_5)f(s+e_2)+P_3(\sigma_1,\sigma_2,\sigma_5)f(s+e_3)\\
&\qquad+P_3(\sigma_1,\sigma_2,\sigma_3)f(s+e_4)+P_1(\sigma_5)f(s+e_1+e_3)+P_1(\sigma_3)f(s+e_1+e_4)+P_1(\sigma_1)f(s+e_2+e_4).
\end{aligned}$$

So by (8.2.9),

$$\begin{aligned}
\Big[\prod_{i=1}^{5}(s_1+\alpha_i)\Big]\widetilde{W}_{5,\alpha}(s)&=\big[s_1^3Z_2+s_1^2Z_3+s_1Z_4+Z_5\big]\widetilde{W}_{5,\alpha}(s)\\
&=\big[s_1^3+s_1^2P_1(\sigma_3,\sigma_4,\sigma_5)+s_1P_2(\sigma_3,\sigma_4,\sigma_5)+P_3(\sigma_3,\sigma_4,\sigma_5)\big]\widetilde{W}_{5,\alpha}(s+e_1)\\
&\quad+\big[s_1^3+s_1^2P_1(\sigma_1,\sigma_4,\sigma_5)+s_1P_2(\sigma_1,\sigma_4,\sigma_5)+P_3(\sigma_1,\sigma_4,\sigma_5)\big]\widetilde{W}_{5,\alpha}(s+e_2)\\
&\quad+\big[s_1^3+s_1^2P_1(\sigma_1,\sigma_2,\sigma_5)+s_1P_2(\sigma_1,\sigma_2,\sigma_5)+P_3(\sigma_1,\sigma_2,\sigma_5)\big]\widetilde{W}_{5,\alpha}(s+e_3)\\
&\quad+\big[s_1^3+s_1^2P_1(\sigma_1,\sigma_2,\sigma_3)+s_1P_2(\sigma_1,\sigma_2,\sigma_3)+P_3(\sigma_1,\sigma_2,\sigma_3)\big]\widetilde{W}_{5,\alpha}(s+e_4)\\
&\quad+\big[s_1+P_1(\sigma_5)\big]\widetilde{W}_{5,\alpha}(s+e_1+e_3)\\
&\quad+\big[s_1+P_1(\sigma_3)\big]\widetilde{W}_{5,\alpha}(s+e_1+e_4)\\
&\quad+\big[s_1+P_1(\sigma_1)\big]\widetilde{W}_{5,\alpha}(s+e_2+e_4).
\end{aligned}\tag{8.2.10}$$

Recalling that the $P_k$'s are the elementary symmetric polynomials of degree $k$ in their arguments, we see

$$s_1^3+s_1^2P_1(a,b,c)+s_1P_2(a,b,c)+P_3(a,b,c)=(s_1+a)(s_1+b)(s_1+c),$$

for indeterminates $a,b,c$. So (8.2.10) gives

$$\begin{aligned}
\Big[\prod_{i=1}^{5}(s_1+\alpha_i)\Big]\widetilde{W}_{5,\alpha}(s)&=(s_1+\sigma_3)(s_1+\sigma_4)(s_1+\sigma_5)\widetilde{W}_{5,\alpha}(s+e_1)\\
&\quad+(s_1+\sigma_1)(s_1+\sigma_4)(s_1+\sigma_5)\widetilde{W}_{5,\alpha}(s+e_2)\\
&\quad+(s_1+\sigma_1)(s_1+\sigma_2)(s_1+\sigma_5)\widetilde{W}_{5,\alpha}(s+e_3)\\
&\quad+(s_1+\sigma_1)(s_1+\sigma_2)(s_1+\sigma_3)\widetilde{W}_{5,\alpha}(s+e_4)\\
&\quad+(s_1+\sigma_5)\widetilde{W}_{4,\alpha}(s+e_1+e_3)\\
&\quad+(s_1+\sigma_3)\widetilde{W}_{5,\alpha}(s+e_1+e_4)\\
&\quad+(s_1+\sigma_1)\widetilde{W}_{5,\alpha}(s+e_2+e_4)
\end{aligned}$$

$$\begin{aligned} &= (s_1+s_2-s_3)(s_1+s_3-s_4)(s_1+s_4)\widetilde{W}_{5,\alpha}(s+e_1) \\ &\quad + (s_1+s_4)\widetilde{W}_{5,\alpha}(s+e_1+e_3) + (s_1+s_2-s_3)\widetilde{W}_{5,\alpha}(s+e_1+e_4), \end{aligned}$$

where we have the last step by the definition (8.2.6) of the $\sigma_i$'s. This is our desired shift equation in $s_1$.

The shift equation in $s_2$ is derived analogously. A fundamental difference in this derivation is that, in place of (8.2.8), we use the following expression involving *Schur polynomials* $s_\mu$ (see [Macdonald 1979, Section I.3], especially Exercise 10 of that section):

$$\prod_{1\le i<j\le 5}(T+x_i+x_j) = \sum_{\mu=(\mu_1,\mu_2,\dots,\mu_5)\in S} \left(\frac{T}{2}\right)^{10-(\mu_1+\mu_2+\cdots+\mu_5)} d_\mu\, s_\mu(x_1,x_2,\dots,x_5).$$

Here,

$$S = \bigl\{(\mu_1,\mu_2,\dots,\mu_5)\in(\mathbb{Z}_{\ge 0})^5 : \mu_i \le 5-i \ (1\le i\le 5) \text{ and } \mu_1\ge\mu_2\ge\cdots\ge\mu_5\bigr\},$$

and $d_\mu$ is the determinant of the matrix

$$\left(\binom{2(5-i)}{\mu_j+5-j}\right)_{1\le i,j\le 5}.$$

The Schur polynomials are symmetric polynomials in the $x_k$'s and are therefore expressible in terms of the elementary symmetric polynomials in the $x_k$'s. Techniques like those employed above, in the case $m=1$, therefore apply. We omit the details. □

**Remark 8.2.11.** The above proof, in the case $m=1$ (that is, for the variable $s_1$—and therefore also for the variable $s_{n-1}$), generalizes to the case of $\mathrm{GL}(n,\mathbb{R})$ for any $n\ge 2$. For $2\le m\le n-2$, we do not yet have a proof that works for all $n\ge 2$, though we expect that the above ideas and techniques should prove relevant. Indeed, using the above methods, and applying Mathematica to help with the more arduous calculations, we have been able to verify Conjecture 8.2.3 in full generality for $n\le 7$.

We further note that, alternatively, one might continue $\widetilde{W}_{n,\alpha}(s)$ in the $s_j$'s by shifting or deforming the lines of integration in (8.1.6). Unfortunately such an approach has, thus far, failed to yield suitable results. In particular, the residues that one obtains in moving these lines of integration past poles of the integrand are quite complicated, and do not seem to lend themselves to bounds of the type required to estimate $p^{(n)}_{T,R}(y)$.

**8.3.** ***$p^{(n)}_{T,R}(y)$ is a sum of a shifted term and residues.*** Besides the shifted $p^{(n)}_{T,R}$-term (because we cross poles of $\widetilde{W}_{n,\alpha}(s)$ upon shifting the lines of integration) there are also many residue terms. The residue terms will be parametrized by compositions of $n$. Recall that a *composition of length* $r$ of a positive integer $n$ is a way of writing $n=n_1+\cdots+n_r$ as a sum of strictly positive integers. Two sums that differ in the order define different compositions. Compare this, on the other hand with *partitions* which are compositions of $n$ for which the order doesn't matter.

**Definition 8.3.1** ($a$-admissible compositions). Let $a=(a_1,\dots,a_{n-1})\in\mathbb{R}^{n-1}$. A composition $n=n_1+\cdots+n_r$ is termed *$a$-admissible* if

$$a_{\hat{n}_i} > 0 \quad \text{for all } i=1,\dots,r-1.$$

The set of $a$-admissible compositions of length greater than 1 is

$$\mathcal{C}_a := \{\text{compositions } n = n_1 + \cdots + n_r \mid 2 \le r \le n,\ a_{\hat{n}_i} > 0 \text{ for all } i = 1, \ldots, r-1\}.$$

**Remark 8.3.2.** At times we may also notate a composition $n = n_1 + \cdots + n_r$ as an ordered list $C = (n_1, \ldots, n_r)$.

**Definition 8.3.3** ($(r-1)$-fold residue term). Suppose that $r \ge 2$ and $C \in \mathcal{C}_a$ is given by $n = n_1 + \cdots + n_r$. Let

$$\delta_C := (\delta_1, \delta_2, \ldots, \delta_{r-1}) \in (\mathbb{Z}_{\ge 0})^{r-1},$$

with $0 \le \delta_i \le \lfloor a_{\hat{n}_i} \rfloor$ for each $i = 1, \ldots, r-1$. If $C$ has length 2, we write $\delta_C = \delta$. We define the $(r-1)$*-fold residue term*

$$p_{T,R}^{(n)}(y; -a, \delta_C) := \int\limits_{\substack{\hat{\alpha}_n=0 \\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_n^2}{T^2/2}} \cdot \mathcal{F}_R^{(n)}(\alpha) \Big( \prod_{1 \le j \ne k \le n} \Gamma_R(\alpha_j - \alpha_k) \Big)$$
$$\cdot \Big( \prod_{i=1}^{r-1} y_i^{\frac{\hat{n}_i(n-\hat{n}_i)}{2} + 2(\hat{\alpha}_{\hat{n}_i} + \delta_i)} \Big) \cdot \int\limits_{\substack{\mathrm{Re}(s_j)=-a_j \\ j \notin \{\hat{n}_1, \ldots, \hat{n}_{r-1}\}}} \Big( \prod_{j \notin \{\hat{n}_1, \ldots, \hat{n}_{r-1}\}} y_j^{\frac{j(n-j)}{2} - 2s_j} \Big)$$
$$\cdot \operatorname*{Res}_{s_{\hat{n}_1} = -\hat{\alpha}_{\hat{n}_1} - \delta_1} \Big( \operatorname*{Res}_{s_{\hat{n}_2} = -\hat{\alpha}_{\hat{n}_2} - \delta_2} \big( \cdots \big( \operatorname*{Res}_{s_{\hat{n}_{r-1}} = -\hat{\alpha}_{\hat{n}_{r-1}} - \delta_{r-1}} \widetilde{W}_{n,\alpha}(s) \big) \cdots \big) \Big)\, ds\, d\alpha. \tag{8.3.4}$$

**Remark 8.3.5.** In the shifted integral (8.3.4), if $-a_i > 0$ for some $i$, there will be no residues coming from the integral in $s_i$ because we are not shifting past any poles. For this reason, one only obtains residue terms $p_{T,R}^{(n)}(y; -a, \delta_C)$ in the case that $C$ is $a$-admissible. That said, (8.3.4) makes perfect sense even if $C$ is not $a$-admissible. In this case, $p_{T,R}^{(n)}(y; -a, \delta_C)$ is identically zero.

**Proposition 8.3.6.** *Suppose that $a = (a_1, \ldots, a_{n-1}) \in \mathbb{R}^{n-1}$. Then there exists constants $\kappa(C)$ such that*

$$p_{T,R}^{(n)}(y) = p_{T,R}^{(n)}(y; -a) + \sum_{C \in \mathcal{C}_a} \kappa(C) \sum_{\substack{\delta_C = (\delta_1, \ldots, \delta_{r-1}) \\ 0 \le \delta_i \le \lfloor a_{\hat{n}_i} \rfloor}} p_{T,R}^{(n)}(y; -a, \delta_C).$$

Before giving the proof, we make some preliminary remarks and observations.

**Remark 8.3.7.** Notice that an element $\sigma$ of the symmetric group $S_n$ (i.e., the group of permutations of a set of $n$ elements) acts on $\alpha = (\alpha_1, \ldots, \alpha_n)$ and, by extension, on $\hat{\alpha}_k$ via

$$\sigma \cdot \hat{\alpha}_k := \alpha_{\sigma(1)} + \alpha_{\sigma(2)} + \cdots + \alpha_{\sigma(k)}.$$

We can consider the analog to (8.3.4) obtained by replacing each instance of $\hat{\alpha}_m$ with $\sigma \cdot \hat{\alpha}_m$:

$$\int\limits_{\substack{\hat{\alpha}_n=0 \\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_n^2}{T^2/2}} \cdot \mathcal{F}_R^{(n)}(\alpha) \cdot \Big( \prod_{1 \le j \ne k \le n} \Gamma_R(\alpha_j - \alpha_k) \Big) \Big( \prod_{i \in \{\hat{n}_1, \ldots, \hat{n}_{r-1}\}} y_i^{\frac{i(n-i)}{2} + 2(\sigma \cdot \hat{\alpha}_i + \delta_i)} \Big)$$
$$\cdot \int\limits_{\substack{\mathrm{Re}(s_j)=-a_j \\ j \in \{\hat{n}_1, \ldots, \hat{n}_{r-1}\}}} \Big( \prod_{j \notin \{\hat{n}_1, \ldots, \hat{n}_{r-1}\}} y_j^{\frac{j(n-j)}{2} - 2s_j} \Big) \operatorname*{Res}_{s_{i_1} = -\sigma \cdot \hat{\alpha}_{i_1} - \delta_{i_1}} \Big( \operatorname*{Res}_{s_{i_2} = -\sigma \cdot \hat{\alpha}_{i_2} - \delta_{i_2}} \cdots \operatorname*{Res}_{s_{i_k} = -\sigma \cdot \hat{\alpha}_{i_k} - \delta_{i_k}} \widetilde{W}_{n,\alpha}(s) \Big)\, ds\, d\alpha$$

We make two observations:

• As $C$ varies over all compositions of length $r$ and $\sigma$ varies over all possible permutations and $\delta_C$ varies over all $(\mathbb{Z}_{\geq 0})^{r-1}$, one obtains all possible $(r-1)$-fold residues coming from shifting the lines of integration in $p^{(n)}_{T,R}(y)$. This is a consequence of Theorem 10.1.1 below.

• The action of $S_n$ on ordered subsets of $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ given by permuting the indices is trivial on $\widetilde{W}_{n,\alpha}(s)$, i.e., $\widetilde{W}_{n,\sigma(\alpha)}(s) = \widetilde{W}_{n,\alpha}(s)$, and on the function

$$e^{\frac{\alpha_1^2+\cdots+\alpha_n^2}{T^2/2}} \cdot \mathcal{F}^{(n)}_R(\alpha) \cdot \Bigg( \prod_{1\leq j\neq k\leq n} \Gamma_R(\alpha_j - \alpha_k) \Bigg).$$

This implies that relabeling the variables $\alpha_1, \alpha_2, \dots \alpha_n$ by $\alpha_{\sigma^{-1}(1)}, \alpha_{\sigma^{-1}(2)}, \dots, \alpha_{\sigma^{-1}(n)}$ everywhere doesn't change the value of the integral, and recovers the original integral given in (8.3.4).

**Remark 8.3.8.** The constant $\kappa(C)$ is the size of the (generic) orbit of the action of $S_n$ on the set

$$A = \{\hat{\alpha}_{\hat{n}_1}, \dots, \hat{\alpha}_{\hat{n}_{r-1}}\}.$$

Hence, defining the stabilizer of $A$ to be

$$\operatorname{Stab}(A) := \{\sigma \in S_n \mid \sigma \cdot \hat{\alpha}_m = \hat{\alpha}_m \text{ for each } m = \hat{n}_1, \dots, \hat{n}_{r-1}\},$$

we see that

$$\kappa(C) = \frac{\# S_n}{\# \operatorname{Stab}(A)} = \frac{n!}{\prod_{i=1}^{r-1}(n_i!)}.$$

Since the exact value of $\kappa(C)$ is irrelevant to our application, we omit its proof below and leave it instead to the interested reader.

*Proof of Proposition 8.3.6.* Beginning with (8.1.4), we see that $p^{(n)}_{T,R}(y) = p^{(n)}_{T,R}(y; b)$ for any $b = (b_1, \dots, b_{n-1})$ with $b_i > 0$ for each $i = 1, \dots, n-1$. In order to compare this with $p^{(n)}_{T,R}(y; -a)$, we successively shift the lines of integration in the variables $s_k$ for each $k$ such that $-a_k < 0$ (in descending order). If $-a_k > 0$ then shifting the line of integration from $\operatorname{Re}(s) = b_k$ to $\operatorname{Re}(s_k) = -a_k$ doesn't change the value of the integral in $s_k$. In other words, there is a residue term if and only if the composition $C$ is admissible.

Beginning with the fact that

$$p^{(n)}_{T,R}(y) = p^{(n)}_{T,R}(y; b) \quad \text{for any } b = (b_1, \dots, b_{n-1}) \text{ for which } b_j > 0 \text{ for all } j,$$

we may shift the line of integration in $s_{n-1}$ to $\operatorname{Re}(s_{n-1}) = -a_{n-1}$. In doing so, provided that $a_{n-1} > 0$, we pass poles at $s_{n-1} = -\sigma \cdot \hat{\alpha}_1 - \delta_1$ for each $0 \leq \delta \leq \lfloor a_1 \rfloor$. Hence, taking into account Remark 8.3.7, and considering $n = (n-1) + 1$ (denoted by $(n-1, 1)$), it follows that

$$p^{(n)}_{T,R}(y) = p^{(n)}_{T,R}(y; (b_1, b_2, b_3, \dots, -a_{n-1})) \\ + \kappa((n-1, 1)) \cdot \sum_{\delta_{(n-1,1)}} p^{(n)}_{T,R}(y; (b_1, b_2, \dots, -a_{n-1}), \delta_{(n-1,1)}), \quad (8.3.9)$$

where $\kappa((n-1, 1))$ is a constant (which can be verified to agree with the description given in Remark 8.3.8.)

We now shift the line of integration in $s_{n-2}$ to $\mathrm{Re}(s_{n-2}) = -a_{n-2}$. As before, provided that $a_{n-2} > 0$, the Cauchy residue theorem and Remark 8.3.7 give

$$\begin{aligned} p^{(n)}_{T,R}(y) = p^{(n)}_{T,R}&(y;\, (b_1, \dots, b_{n-3}, -a_{n-2}, -a_{n-1})) \\ &+ \kappa((n-2, 2)) \sum_{\delta_{(n-2,2)}} p^{(n)}_{T,R}(y;\, (b_1, \dots, b_{n-3}, -a_{n-2}, -a_{n-1}), \delta_{(n-2,2)}) \\ &+ \kappa((n-1, 1)) \sum_{\delta_{(n-1)}} p^{(n)}_{T,R}(y;\, (b_1, \dots, b_{n-3}, -a_{n-2}, -a_{n-1}), \delta_{(n-1,1)}) \\ &+ \kappa((n-2, 1, 1)) \sum_{\delta_{(n-2,1,1)}} p^{(n)}_{T,R}(y;\, (b_1, \dots, b_{n-3}, -a_{n-2}, -a_{n-1}), \delta_{(n-2,1,1)}) \end{aligned} \tag{8.3.10}$$

for constants $\kappa(C)$ for each of $C = (n-1, 1), (n-2, 2), (n-2, 1, 1)$ as claimed.

We next repeat this process shifting the integrals in $s_{n-3}$ for each of the terms on the right of (8.3.10), and then again for $s_{n-4}$ and so forth (skipping those $s_m$ for which $a_m < 0$) until all of the lines of integration have been moved to $\mathrm{Re}(s_m) = -a_m$ for every possible integral. The claimed formula is now evident. □

**8.4.** ***Example:*** **GL(4).** We now consider the special case of $\widetilde{W}_{4,\alpha}(s)$ where

$$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in (i\mathbb{R})^4, \quad \hat{\alpha}_4 = 0.$$

Fix $\varepsilon > 0$. Recall that $p^{(4)}_{T,R}(y) = p^{(4)}_{T,R}(y;\, (\varepsilon, \varepsilon, \varepsilon))$. If we now shift the lines of integration to $\mathrm{Re}(s) = (-a)$ where $a = (a_1, a_2, a_3) \in \mathbb{R}^3$, then we get additional residue terms corresponding to each composition $4 = n_1 + \cdots + n_r$ and each $\delta_C \in (\mathbb{Z}_{\geq 0})^r$ as follows.

In general the composition $n = n_1 + \cdots + n_r$ (by abuse of notation, we also think of this as a vector $(n_1, \dots, n_r)$ so that $\hat{n}_k = n_1 + \cdots + n_k$) corresponds to taking an $(r-1)$-fold residue in the variables $s_{\hat{n}_1}, s_{\hat{n}_2}, \dots, s_{\hat{n}_{r-1}}$. Here is a table of the residues corresponding to the different compositions:

| composition $C$ | residues in $s$-variables | $\delta_C$ |
|---|---|---|
| $1+3$ | $s_1 = -\alpha_1 - \delta_1$ | $(\delta_1)$ |
| $2+2$ | $s_2 = -\alpha_1 - \alpha_2 - \delta_2$ | $(\delta_2)$ |
| $3+1$ | $s_3 = -\alpha_1 - \alpha_2 - \alpha_3 - \delta_3$ | $(\delta_3)$ |
| $1+1+2$ | $s_1 = -\alpha_1 - \delta_1,\ s_2 = -\alpha_1 - \alpha_2 - \delta_2$ | $(\delta_1, \delta_2)$ |
| $1+2+1$ | $s_1 = -\alpha_1 - \delta_1,\ s_3 = -\alpha_1 - \alpha_2 - \alpha_3 - \delta_3$ | $(\delta_1, \delta_3)$ |
| $2+1+1$ | $s_2 = -\alpha_1 - \alpha_2 - \delta_2,\ s_3 = -\alpha_1 - \alpha_2 - \alpha_3 - \delta_3$ | $(\delta_2, \delta_3)$ |

In each case $0 \leq \delta_i \leq \lfloor a_i \rfloor$. Not included in the table are the triple residues in $s_i = -\hat{\alpha}_i - \delta_i$ for each $i = 1, 2, 3$. These correspond to the composition $4 = 1+1+1+1$ and $\delta_C = (\delta_1, \delta_2, \delta_3)$.

**8.5.** ***The integral $\mathcal{I}^{(m)}_{T,R}(-a)$ in terms of an explicit recursive formula for $\widetilde{W}_{m,\alpha}(s)$.*** At first glance, the following definition appears to be relevant only for the shifted $p^{(n)}_{T,R}$-term, as it is essentially equal to $p^{(n)}_{T,R}((1, \dots, 1);\, -a)$, and not for the shifted residue terms. However, it will turn out to be pivotal to bounding the residue terms as well.

**Definition 8.5.1** (the integral $\mathcal{I}_{T,R}^{(m)}$). Let $m \ge 2$ be an integer and $a=(a_1,\dots,a_{m-1})\in\mathbb{R}^{m-1}$. Then we define

$$\mathcal{I}_{T,R}^{(m)}(-a) := \int\limits_{\substack{\hat{\alpha}_m=0\\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_m^2}{T^2/2}} \cdot \mathcal{F}_R^{(n)}(\alpha)\biggl(\prod_{1\le j\ne k\le m}\Gamma_R(\alpha_j-\alpha_k)\biggr) \int\limits_{\substack{s=(s_1,\dots,s_{m-1})\\ \mathrm{Re}(s)=-a}} |\widetilde{W}_{m,\alpha}(s)|\,ds\,d\alpha. \tag{8.5.2}$$

As alluded to above, inserting the result of Theorem 8.1.5 into (8.2.2), we find that

$$|p_{T,R}^{(n)}(y,-a)| \ll \biggl(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}-2a_j}\biggr)\mathcal{I}_{T,R}^{(n)}(-a).$$

Hence, giving a bound for $p_{T,R}^{(n)}(y)$ requires only that we bound $\mathcal{I}_{T,R}^{(m)}(-a)$ in the case of $m=n$. However, much more is true: we will show that if $C$ is the composition $n=n_1+\cdots+n_r$, then $p_{T,R}^{(n)}(y;-a,\delta_C)$ can be bounded by the same product of $y_i$'s as above times a certain power of $T$ and a product of the form

$$\prod_{\ell=1}^{r-1}\mathcal{I}_{T,R}^{(n_\ell)}(-a^{(\ell)})$$

for certain values $a^{(\ell)}=(a_1^{(\ell)},\dots,a_{n_\ell-1}^{(\ell)})$ which depend on the value of $a=(a_1,\dots,a_{n-1})\in\mathbb{R}^{n-1}$.

The significance of this fact should not be understated. Without it, we would be required to treat nearly every possible composition $C$ (hence each possible residue term) individually. Indeed, returning to the case of $n=4$, as noted in Section 8.4 above, there were seven residue terms. The only symmetries that we were able to exploit in [Goldfeld et al. 2021b] to help were that the (1, 3) and (3, 1) residues were equivalent, and the (1, 1, 2) and (2, 1, 1) residues were equivalent as well. This left five individual distinct cases, each of which required several pages of work to bound. So, although the method of this paper does require dealing with some tricky notation and combinatorics, it eliminates the need to treat each residue on its own terms.

## 9. Bounding $\mathcal{I}_{T,R}^{(m)}$

Recall that for $\alpha=(\alpha_1,\dots,\alpha_m)\in\mathbb{C}^m$ satisfying $\hat{\alpha}_m=0$ and $a=(a_1,a_2,\dots,a_{n-1})\in\mathbb{R}^{n-1}$,

$$\mathcal{I}_{T,R}^{(m)}(-a) := \int\limits_{\substack{\hat{\alpha}_m=0\\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_m^2}{T^2/2}} \cdot \mathcal{F}_R^{(m)}(\alpha)\prod_{1\le j\ne k\le m}|\Gamma_R(\alpha_j-\alpha_k)| \int\limits_{\mathrm{Re}(s)=-a} |\widetilde{W}_{m,\alpha}(s)|\,ds\,d\alpha. \tag{9.0.1}$$

**Theorem 9.0.2.** *Let $\mathcal{I}_{T,R}^{(m)}(-a)$ be as above and set $D(m)=\deg(\mathcal{F}_1^{(m)}(\alpha))$. Then, for any $0<\varepsilon<\frac{1}{2}$,*

$$\mathcal{I}_{T,R}^{(m)}(-a) \ll T^{\varepsilon+\frac{(m+4)(m-1)}{4}+R\cdot\left(D(m)+\frac{m(m-1)}{2}\right)-\sum_{j=1}^{m-1}B(a_j)},$$

*where*

$$B(c)=\begin{cases}0 & \text{if } c<0,\\ \lfloor c\rfloor+2(c-\lfloor c\rfloor) & \text{if } 0<\lfloor c\rfloor+\varepsilon<c\le\lfloor c\rfloor+\frac{1}{2},\\ \lceil c\rceil & \text{if } \frac{1}{2}<\lceil c\rceil-\frac{1}{2}\le c<\lceil c\rceil-\varepsilon.\end{cases}$$

*The implicit constant depends on $\varepsilon$, $R$ and $m$.*

Theorem 8.1.5 allows us to write $\widetilde{W}_{m,\alpha}(s)$ in terms of an integral of the product of several Gamma functions and the lower-rank Mellin transform $\widetilde{W}_{m-1,\beta}(z)$, where

$$\beta = (\beta_1, \dots, \beta_{m-1}) := \left(\alpha_1 + \frac{\alpha_m}{m-1}, \dots, \alpha_{m-1} + \frac{\alpha_m}{m-1}\right).$$

Using this, we are able siphon off the contribution to the integrand of (9.0.1) that is independent of the variable $\beta$. This in turn allows us to relate $\mathcal{I}_{T,R}^{(m)}$ to $\mathcal{I}_{T,R}^{(m-1)}$ and prove the result inductively.

**9.1.** ***Symmetry of integration in α.*** Since the integrand of (9.0.1) is invariant under the action of $\sigma \in S_m$ acting on $\alpha = (\alpha_1, \dots, \alpha_m)$, we may restrict the integration to a fundamental domain. A choice of such a fundamental domain is

$$\operatorname{Im}(\alpha_1) \geq \operatorname{Im}(\alpha_2) \geq \cdots \geq \operatorname{Im}(\alpha_m). \tag{9.1.1}$$

Hence, (9.0.1) is equal, up to a constant, to the same integral but restricted to $\alpha$ satisfying (9.1.1). In the sequel we will always assume that (9.1.1) holds.

**9.2.** ***Extended exponential zero set.*** Recall that Stirling's asymptotic formula (for $\sigma \in \mathbb{R}$ fixed and $t \in \mathbb{R}$ with $|t| \to \infty$) is given by

$$\Gamma(\sigma + it) \sim \sqrt{2\pi} \cdot |t|^{\sigma - \frac{1}{2}} e^{-\frac{\pi}{2}|t|}. \tag{9.2.1}$$

**Definition 9.2.2** (exponential and polynomial factors of a ratio of Gamma functions). We call $|t|^{\sigma-1/2}$ the *polynomial factor* of $\Gamma(\sigma + it)$, and $e^{-(\pi/2)|t|}$ is called the *exponential factor*. For a ratio of Gamma functions, the *polynomial* (*respectively*, *exponential*) *factor* is composed of the polynomial (respectively, exponential) factors of each individual Gamma function.

**Lemma 9.2.3** (extended exponential zero set). *Assume that $\alpha \in \mathbb{C}^m$ is a Langlands parameter satisfying*

$$\operatorname{Im}(\alpha_1) \geq \operatorname{Im}(\alpha_2) \geq \cdots \geq \operatorname{Im}(\alpha_m).$$

*Then the integrand of $\mathcal{I}_{T,R}^{(m)}$ (as a function of $s$) has exponential decay outside of the set $I = I_1 \times I_2 \times \cdots \times I_{m-1}$, where*

$$I_j := \left\{ s_j \;\middle|\; -\sum_{k=1}^{j} \operatorname{Im}(\alpha_k) \leq \operatorname{Im}(s_j) \leq -\sum_{k=1}^{j} \operatorname{Im}(\alpha_{m-k+1}) \right\}.$$

**Remark 9.2.4.** See [Goldfeld et al. 2021b] for the definition of the *exponential zero set* of an integral. The extended exponential zero set given in Lemma 9.2.3 contains the exponential zero set for $\mathcal{I}_{T,R}^{(m)}$.

*Proof.* We first prove Lemma 9.2.3 in the case that $m = 2$. In the formula (9.0.1) for $\mathcal{I}_{T,R}^{(n)}$, replace $\widetilde{W}_{2,\alpha}(s_1)$ with $\Gamma(s_1+\alpha_1)\Gamma(s_1+\alpha_2)$. Then assuming (9.1.1), the exponential factor is $e^{(\pi/2)\mathcal{E}(s,\alpha)}$, where

$$\mathcal{E}(s,\alpha) = |\operatorname{Im}(s_1) + \operatorname{Im}(\alpha_1)| + |\operatorname{Im}(s_1) + \operatorname{Im}(\alpha_2)| - 2\operatorname{Im}(\alpha_1).$$

We see, therefore, that the exponential factor $\mathcal{E}(s,\alpha)$ is negative unless

$$\operatorname{Im}(s_1) + \operatorname{Im}(\alpha_1) \geq 0 \quad \text{and} \quad \operatorname{Im}(s_1) + \operatorname{Im}(\alpha_2) \leq 0 \quad \Longleftrightarrow \quad -\operatorname{Im}(\alpha_1) \leq \operatorname{Im}(s_1) \leq -\operatorname{Im}(\alpha_2),$$

as claimed.

Let us suppose that $m \geq 3$ and $c = (c_1, c_2, \ldots, c_{m-1})$, with $c_j > 0$ ($j = 1, 2, \ldots, m-1$). In order to prove Lemma 9.2.3 using induction on $m$, we make use of the change of variables

$$\beta_j = \alpha_j + \frac{\alpha_m}{m-1}, \quad j = 1, \ldots, m-1.$$

Observe that

$$\beta_1 + \cdots + \beta_{m-1} = 0.$$

By Lemma A.19 in the case that $k = m-1$,

$$\alpha_1^2 + \cdots + \alpha_m^2 = \beta_1^2 + \cdots + \beta_{m-1}^2 + \frac{m}{m-1}\alpha_m^2.$$

Then in the integrand for $\mathcal{I}_{T,R}^{(m)}(c)$ we may substitute the formula for $\widetilde{W}_{m,\alpha}(s)$ given in Theorem 8.1.5. We also use the fact (see Lemma A.26) that

$$\prod_{1\leq j\neq k\leq m} \Gamma(\alpha_j - \alpha_k) = \Bigg(\prod_{1\leq j\neq k\leq m-1} \Gamma(\beta_j - \beta_k)\Bigg) \cdot \Bigg(\prod_{i=1}^{m-1} \Gamma(\alpha_m - \alpha_i)\Gamma(\alpha_i - \alpha_m)\Bigg),$$

and, via Stirling,

$$\prod_{i=1}^{m-1} \Gamma(\alpha_m - \alpha_i)\Gamma(\alpha_i - \alpha_m) \ll e^{\pi \operatorname{Im}(\alpha_m)}.$$

Note that (9.1.1) implies that $\operatorname{Im}(\alpha_m) \leq 0$; hence,

$$\begin{aligned}
\mathcal{I}_{T,R}^{(m)}(c) \ll \int\limits_{\operatorname{Re}(\alpha_m)=0} e^{\frac{m}{m-1}\frac{\alpha_m^2}{T^2/2}} \int\limits_{\substack{\hat{\beta}_{m-1}=0\\ \operatorname{Re}(\beta)=0}} e^{\frac{\beta_1^2+\cdots+\beta_{m-1}^2}{T^2/2}} \cdot |\mathcal{P}_{(D(m)-D(m-1))R}(\alpha_m, \beta)| \\
\cdot \mathcal{F}_R^{(m-1)}(\beta) \prod_{1\leq j\neq k\leq m-1} |\Gamma_R(\beta_j - \beta_k)| \int\limits_{\substack{\operatorname{Re}(z_j)=b_j\\ 1\leq j\leq m-2}} |\widetilde{W}_{m-1,\beta}(z)| \\
\cdot \prod_{j=1}^{m-1} \int\limits_{\operatorname{Re}(s_j)=c_j} \Bigg| \Gamma\Bigg(s_j - z_{j-1} + \frac{(m-j)\alpha_m}{m-1}\Bigg)\Gamma\Bigg(s_j - z_j - \frac{j\alpha_m}{m-1}\Bigg) \\
\cdot \Gamma_R\Bigg(\frac{-m}{m-1}\alpha_m - \beta_j\Bigg)\Gamma_R\Bigg(\beta_j + \frac{m}{m-1}\alpha_m\Bigg)\Bigg| \, ds_j \, dz \, d\alpha.
\end{aligned}$$

By the induction hypothesis, the second row of this expression has exponential decay outside of the set

$$\Bigg\{ z = (z_1, \ldots, z_{m-2}) \Bigg| -\sum_{k=1}^{j} \beta_k \leq \operatorname{Im}(z_j) \leq -\sum_{j=1}^{k} \beta_{m-j} \Bigg\} \tag{9.2.5}$$

for each $k = 1, 2, \ldots, m-2$. (Recall that $z_0 = z_{m-1} = 0$.)

The assumption $\operatorname{Im}(\alpha_j) \geq \operatorname{Im}(\alpha_m)$ and the definition of $\beta_j$ above imply that

$$\operatorname{Im}(\alpha_j - \alpha_m) = \operatorname{Im}\Bigg(\beta_j + \frac{m}{m-1}\alpha_n\Bigg) \geq 0 \quad (j = 1, 2, \ldots, m-1).$$

Thus, the exponential factor coming from the final line in the expression above is $e^{(\pi/2)\mathcal{E}(s,z,\beta,\alpha_m)}$, where

$$\begin{aligned}\mathcal{E}(s, z, \beta, \alpha_n) &= \sum_{j=1}^{n-1}\left(\left|\mathrm{Im}\left(s_j - z_{j-1} + \frac{n-j}{n-1}\alpha_n\right)\right| + \left|\mathrm{Im}\left(s_j - z_j - \frac{j}{n-1}\alpha_n\right)\right| - \mathrm{Im}\left(\frac{n}{n-1}\alpha_n + \beta_j\right)\right)\\ &= \sum_{j=1}^{n-1}\left(\left|\mathrm{Im}\left(s_j - z_{j-1} + \frac{n-j}{n-1}\alpha_n\right)\right| + \left|\mathrm{Im}\left(s_j - z_j - \frac{j}{n-1}\alpha_n\right)\right|\right) - n\,\mathrm{Im}(\alpha_n).\end{aligned}$$

We know the integral defining $\mathcal{I}_{T,R}^{(m)}(-a)$ is convergent. Therefore, it must be the case that $\mathcal{E}(s, z, \beta, \alpha_m) \le 0$. In order to find where $\mathcal{E} = 0$, i.e., where there is *not* exponential decay, we seek values

$$\epsilon_{1,1}, \epsilon_{2,1}, \ldots, \epsilon_{1,m-1}, \epsilon_{2,m-1} \in \{\pm 1\}$$

for which

$$\sum_{j=1}^{m-1}\left(\epsilon_{1,j}\,\mathrm{Im}\left(s_j - z_{j-1} + \frac{m-j}{m-1}\alpha_m\right) + \epsilon_{2,j}\,\mathrm{Im}\left(s_j - z_j - \frac{j}{m-1}\alpha_m\right)\right) = m\,\mathrm{Im}(\alpha_m). \tag{9.2.6}$$

In order for the $s$-variables to cancel it is clear that for each $j = 1, 2, \ldots, m-1$ it need be true that $\epsilon_j := \epsilon_{1,j} = -\epsilon_{2,j}$. With this assumption, (9.2.6) simplifies:

$$\sum_{j=1}^{m-1}\left(\epsilon_j\,\mathrm{Im}\left(z_j - z_{j-1} + \frac{m}{m-1}\alpha_m\right)\right) = m\,\mathrm{Im}(\alpha_m).$$

In order for this to hold true, it is necessary that $\epsilon_j = 1$ for all $j$, since otherwise, the coefficients of $\alpha_m$ on each side of the inequality wouldn't match. On the other hand, $\epsilon_j = 1$ for all $j$ is sufficient as well since

$$\sum_{j=1}^{m-1}\mathrm{Im}(z_{j-1} - z_j) = \mathrm{Im}(z_0 - z_{m-1}) = 0.$$

This unique solution to (9.2.6) implies, therefore, that there is exponential decay in the integrand of $\mathcal{I}_{T,R}^{(m)}$ above unless $\mathrm{Im}\big(z_{j-1} - \frac{m-j}{m-1}\alpha_m\big) \le \mathrm{Im}(s_j) \le \mathrm{Im}\big(z_j + \frac{j}{m-1}\alpha_m\big)$. The inductive assumption (9.2.5) implies

$$\begin{aligned}\mathrm{Im}\left(z_{j-1} - \frac{m-j}{m-1}\alpha_m\right) &\ge -\sum_{k=1}^{j-1}\left(\beta_k - \frac{\alpha_m}{m-1}\right) - \alpha_m = -\sum_{k=1}^{j}\alpha_j,\\ \mathrm{Im}\left(z_j + \frac{j}{m-1}\alpha_m\right) &\le -\sum_{k=1}^{j}\left(\beta_k - \frac{\alpha_m}{m-1}\right) = -\sum_{k=1}^{m}\alpha_k,\end{aligned}$$

thus yielding the desired bounds on $\mathrm{Im}(s_j)$.

To complete the proof, we remark that if $-a < 0$, in order to use the result of Theorem 8.1.5, we need to first apply the shift equations given in Corollary 9.2.8 below. This will allow us to rewrite $\mathcal{I}_{T,R}^{(m)}(-a)$ as a sum over terms all of which have the same basic form as that for $\mathcal{I}_{T,R}^{(m)}(c)$ with $c > 0$. Each of these terms has precisely the same exponential factor since this depends only on the imaginary parts of the arguments of the Gamma functions; hence the same exponential zero set is determined in general. □

For each $j = 1, \dots, n$, we define

$$\mathcal{B}_j(s_j, \alpha) := \prod_{\substack{K\subseteq\{1,\dots,n\}\\ \#K=j}} \Bigl(s_j + \sum_{k\in K} \alpha_k\Bigr). \tag{9.2.7}$$

Using this, the following corollary is easily deduced. (See [Goldfeld et al. 2021b] for the case of $n = 4$.)

**Corollary 9.2.8.** *Let $r = (r_1, \dots, r_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$. There exists a sequence of shifts $\sigma = (\sigma_1, \dots, \sigma_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$ and polynomials $Q_{\sigma,r}(s, \alpha)$ such that*

$$|\widetilde{W}_{n,\alpha}(s)| \ll \sum_{\sigma} \frac{|Q_{\sigma,r}(s,\alpha)|}{\prod_{j=1}^{n-1} |\mathcal{B}_j(s_j,\alpha)|^{r_j}} |\widetilde{W}_{n,\alpha}(s + r + \sigma)|,$$

*where*

$$Q_{\sigma,r}(s,\alpha) = \prod_{j=1}^{n-1} P_{\sigma_j,r_j}(s,\alpha), \quad \deg(P_{\sigma_j,r_j}(s,\alpha)) = r_j\Bigl(\binom{n}{j} - 2\Bigr) - 2\sigma_j.$$

### 9.3. *Proof of Theorem 9.0.2 in the case m = 2.*

*Proof.* As in the proof of Lemma 9.2.3, we can replace $\widetilde{W}_{2,(\alpha,-\alpha)}(s)$ with $\Gamma(s+\alpha)\Gamma(s-\alpha)$ and estimate using Stirling's bound. We may, moreover, restrict $s$ to the exponential zero set $-\operatorname{Im}(\alpha) \leq \operatorname{Im}(s) \leq \operatorname{Im}(\alpha)$ to see that

$$\begin{aligned}\mathcal{I}_{T,R}^{(2)}(-a) &= \int\limits_{\substack{\hat{\alpha}_n=0\\ \operatorname{Re}(\alpha)=0}} e^{\frac{\alpha^2}{T^2}} \cdot |\Gamma_R(2\alpha)\Gamma_R(-2\alpha)| \int\limits_{\substack{s=(s_1,\dots,s_{n-1})\\ \operatorname{Re}(s)=-a}} |\widetilde{W}_{2,(\alpha,-\alpha)}(s)|\, ds\, d\alpha \\ &\ll \int\limits_{\substack{\hat{\alpha}_n=0\\ \operatorname{Re}(\alpha)=0}} e^{\frac{\alpha^2}{T^2}} \cdot (1+|2\operatorname{Im}(\alpha)|)^{R+\frac{1}{2}} \int\limits_{\substack{\operatorname{Re}(s)=-a\\ -\operatorname{Im}(\alpha)\leq\operatorname{Im}(s)\leq\operatorname{Im}(\alpha)}} (1+|\operatorname{Im}(s)-\operatorname{Im}(\alpha)|)^{-a-\frac{1}{2}} \\ &\qquad\qquad \cdot (1+|\operatorname{Im}(s)+\operatorname{Im}(\alpha)|)^{-a-\frac{1}{2}}\, ds\, d\alpha.\end{aligned}$$

Due to the presence of the term $e^{\alpha^2/T^2}$, we may assume moreover that $\operatorname{Im}(\alpha) \leq T^{1+\varepsilon}$. Thus, we have the bound

$$\begin{aligned}\mathcal{I}_{T,R}^{(2)}(-a) &\ll \int\limits_{\substack{\operatorname{Re}(\alpha)=0\\ 0\leq\operatorname{Im}(\alpha)\leq T^{\varepsilon+1}}} (1+2|\alpha|)^{R+\frac{1}{2}} \int\limits_{\substack{\operatorname{Re}(s)=-a\\ -\operatorname{Im}(\alpha)\leq\operatorname{Im}(s)\leq\operatorname{Im}(\alpha)}} (1+\alpha-s)^{-a-\frac{1}{2}}(1+\alpha-s)^{-a-\frac{1}{2}}\, ds\, d\alpha \\ &\ll \int\limits_{\substack{\operatorname{Re}(\alpha)=0\\ 0\leq\operatorname{Im}(\alpha)\leq T^{\varepsilon+1}}} (1+2|\alpha|)^{R+\frac{1}{2}-\min\{a+\frac{1}{2},2a\}}\, d\alpha \ll T^{\varepsilon+R+\frac{3}{2}-\min\{a+\frac{1}{2},2a\}}.\end{aligned}$$

In the statement of Theorem 9.0.2, the claimed bound is $\mathcal{I}_{T,R}^{(2)}(-a) \ll T^{\varepsilon+R+3/2-B(a)}$, where $B(a)$ is as defined in Theorem 9.0.2. We have in fact proved that $\mathcal{I}_{T,R}^{(2)}(-a) \ll T^{\varepsilon+R+3/2-B'(a)}$, where

$$B'(a) = \max\bigl\{a+\tfrac{1}{2}, 2a\bigr\} = \begin{cases} 2a & \text{if } \varepsilon < a \leq \frac{1}{2}, \\ a+\frac{1}{2} & \text{if } a \geq \frac{1}{2}. \end{cases}$$

If, $a < 0$, then we may shift the integral over $\mathrm{Re}(s) = -a$ to be as close to $\mathrm{Re}(s) = 0$ as desired; indeed, we may make the shift to the point that the error can be absorbed into the $\varepsilon$-term in the power of $T$. Therefore, since $B(a) \le B'(a)$ for all $a > 0$, the theorem follows. □

**9.4.** ***Proof of Theorem 9.0.2 for general m.***

*Proof.* Let $m \ge 3$ and assume that Theorem 9.0.2 has been shown to be true for all integers $2 \le k < m$. It follows from Corollary 9.2.8 with $r_j = \lceil a_j \rceil$ that

$$\mathcal{I}_{T,R}^{(m)}(-a) \ll \sum_{\sigma} \int_{\substack{\hat{\alpha}_m = 0 \\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_m^2}{T^2/2}} \cdot \mathcal{F}_R^{(m)}(\alpha) \prod_{1 \le j \ne k \le m} |\Gamma_R(\alpha_j - \alpha_k)| \cdots \int_{\substack{s=(s_1,\dots,s_{m-1}) \\ \mathrm{Re}(s)=-a}} \frac{|\mathcal{P}_{d(m)-2|\sigma|}(s,\alpha)|}{\prod_{j=1}^{m-1} |\mathcal{B}_j(s_j,\alpha)|^{\lceil a_j \rceil}} |\widetilde{W}_{m,\alpha}(s+r+\sigma)| \, ds \, d\alpha.$$

By Theorem 8.1.5,

$$\begin{aligned}\mathcal{I}_{T,R}^{(m)}(-a) \ll \sum_{\sigma} \int_{\substack{\hat{\alpha}_m = 0 \\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_m^2}{T^2/2}} \cdot \mathcal{F}_R^{(m)}(\alpha) \prod_{1 \le j \ne k \le m} |\Gamma_R(\alpha_j - \alpha_k)| \int_{\mathrm{Re}(s)=-a} \frac{|\mathcal{P}_{d(m)-2|\sigma|}(s,\alpha)|}{\prod_{j=1}^{m-1} |\mathcal{B}_j(s_j,\alpha)|^{\lceil a_j \rceil}} \\ \cdot \int_{\substack{z=(z_1,\dots,z_{m-2}) \\ \mathrm{Re}(z)=b}} \left( \prod_{j=1}^{m-1} \left| \Gamma\left( s_j + \lceil a_j \rceil + \sigma_j - z_{j-1} + \frac{(m-j)\alpha_m}{m-1} \right) \right| \right. \\ \left. \cdot \left| \Gamma\left( s_j + \lceil a_j \rceil + \sigma_j - z_j - \frac{j\alpha_m}{m-1} \right) \right| \right) \cdot |\widetilde{W}_{m-1,\beta}(z)| \, dz \, ds \, d\alpha.\end{aligned}$$

Next, we use the functional equation for the Gamma function to rewrite

$$\begin{aligned}&\Gamma\left( s_j + \lceil a_j \rceil + \sigma_j - z_{j-1} + \frac{(m-j)\alpha_m}{m-1} \right) \Gamma\left( s_j + \lceil a_j \rceil + \sigma_j - z_j - \frac{j\alpha_m}{m-1} \right) \\ &\qquad = \mathcal{P}_{2\sigma_j}(s,z,\alpha) \Gamma\left( s_j + \lceil a_j \rceil - z_{j-1} + \frac{(m-j)\alpha_m}{m-1} \right) \Gamma\left( s_j + \lceil a_j \rceil - z_j - \frac{j\alpha_m}{m-1} \right).\end{aligned}$$

Additionally, we use the fact that the integrand has exponential decay unless $|\alpha_1|, \dots, |\alpha_m| \le T^{1+\varepsilon}$, and by Lemma 9.2.3, each of the variables $s_j$ are bounded in terms of $\alpha$. This means that we may replace the polynomials $\mathcal{P}_{2\sigma_j}$ with the bound $T^{\varepsilon+2\sigma_j}$. Note that in doing so, the dependence on $\sigma$ is removed:

$$\begin{aligned}&\mathcal{I}_{T,R}^{(m)}(-a) \\ &\ll T^{\varepsilon + \sum_{j=1}^{m-1} \lceil a_j \rceil (\binom{m}{j} - 2)} \int_{\substack{\hat{\alpha}_m = 0 \\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_m^2}{T^2/2}} \cdot \mathcal{F}_R^{(m)}(\alpha) \prod_{1 \le j \ne k \le m} |\Gamma_R(\alpha_j - \alpha_k)| \\ &\qquad \int_{\substack{s=(s_1,\dots,s_{m-1}) \\ \mathrm{Re}(s)=-a}} \int_{\substack{z=(z_1,\dots,z_{m-2}) \\ \mathrm{Re}(z)=b}} \left( \prod_{j=1}^{m-1} \frac{\left| \Gamma\left(s_j + \lceil a_j \rceil - z_{j-1} + \frac{(m-j)\alpha_m}{m-1}\right) \Gamma\left(s_j + \lceil a_j \rceil - z_j - \frac{j\alpha_m}{m-1}\right) \right|}{|\mathcal{B}_j(s_j,\alpha)|^{\lceil a_j \rceil}} \right) \\ &\qquad \cdot |\widetilde{W}_{m-1,\beta}(z)| \, dz \, ds \, d\alpha.\end{aligned}$$

Notice that the conclusion of Proposition 9.4.2 follows from the last several steps by simply replacing $s$ by $s+L$ in the integrand (or, equivalently, replacing $\mathrm{Re}(s)=-a$ by $\mathrm{Re}(s)=-a+L$ in the domain of integration), and then at the step where the functional equation of Gamma is used to remove $\sigma$ from the Gamma functions, we remove $L$ in the exact same fashion.

We deduce that

$$\begin{aligned}
&\mathcal{I}_{T,R}^{(m)}(-a)\\
&\ll T^{\varepsilon+\sum_{j=1}^{m-1}\lceil a_j\rceil(\binom{m}{j}-2)}\cdot \int\limits_{\substack{\hat\alpha_m=0\\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_m^2}{T^2/2}}\cdot \mathcal{F}_R^{(m)}(\alpha)\prod_{1\le j\ne k\le m-1}|\Gamma_R(\alpha_j-\alpha_k)|\\
&\cdot \int\limits_{\substack{z=(z_1,\dots,z_{m-2})\\ \mathrm{Re}(z)=b}}\prod_{j=1}^{m-1}\int\limits_{\mathrm{Re}(s_j)=\lceil a_j\rceil-a_j}\frac{\left|\Gamma\left(s_j-z_{j-1}-\frac{(m-j)\hat\alpha}{m-1}\right)\Gamma\left(s_j-z_j+\frac{j\hat\alpha}{m-1}\right)\right|}{|\mathcal{B}_j(s_j,\alpha)|^{\lceil a_j\rceil}}\left|\Gamma_R\left(\frac{n}{n-1}\hat\alpha-\beta_j\right)\Gamma_R\left(\beta_j-\frac{m}{m-1}\hat\alpha\right)\right|\\
&\qquad\qquad\cdot|\widetilde{W}_{m-1,\beta}(z)|\,ds_j\,dz\,d\alpha.
\end{aligned}$$

Note that we have also made the change of variable $s\mapsto s_j-\lceil a_j\rceil$ for each $j=1,2,\dots,m-1$, and we are using the notation $\hat\alpha:=-\alpha_m$. (Using the terminology of Lemma A.19 in the case of $k=m-1$, we have $\hat\alpha=\hat\alpha_{m-1}$.) As in the case of $n=2$, due to the presence of the exponential terms, we see that the integral has exponential decay unless $|\alpha_j|\ll T^{1+\varepsilon}$.

**Lemma 9.4.1.** *Let $\alpha=(\alpha_1,\dots,\alpha_m)$ and $\beta_j=\alpha_j-\frac{\hat\alpha}{m-1}$ be as above. In particular, they are purely imaginary with $|\beta_k|$, $|\hat\alpha_j|<T^{1+\varepsilon}$. Suppose, moreover, that $\alpha$ is in $j$-general position. Then*

$$\begin{aligned}
&\int\limits_{\mathrm{Re}(s_j)=\lceil a_j\rceil-a_j}\frac{\left|\Gamma\left(s_j-z_{j-1}-\frac{(m-j)\hat\alpha}{m-1}\right)\Gamma\left(s_j-z_j+\frac{j\hat\alpha}{m-1}\right)\right|}{|\mathcal{B}_j(s_j,\alpha)|^{\lceil a_j\rceil}}\left|\Gamma_R\left(\frac{m}{m-1}\hat\alpha-\beta_j\right)\Gamma_R\left(\beta_j-\frac{m}{m-1}\hat\alpha\right)\right|ds_j\\
&\qquad\ll T^{\varepsilon+R+\frac12+\max\{0,2(\lceil a_j\rceil-a_j)-1\}}\sum_{\substack{L\subseteq\{1,\dots,m\}\\ \#L=j}}\prod_{\substack{K\subseteq\{1,\dots,m\}\\ \#K=j\\ K\ne L}}\left(1+\left|\sum_{\ell\in L}\alpha_\ell-\sum_{k\in K}\alpha_k\right|\right)^{-[a_j]}.
\end{aligned}$$

*Proof.* Let $\mathcal{I}_j$ denote the integral we are seeking to bound.

The polynomial part (see Definition 9.2.2) of the Gamma functions in $\mathcal{I}_j$ is

$$\begin{aligned}
|\mathcal{Q}_j(s,z,\alpha)|\ll\left(1+\mathrm{Im}\left(\beta_j-\frac{n}{n-1}\hat\alpha\right)\right)^{\varepsilon+R+\frac12}&(1+|\mathrm{Im}(s_j-z_j)|)^{\lceil a_j\rceil-a_j-\mathrm{Re}(z_j)-\frac12}\\
&\cdot(1+|\mathrm{Im}(s_j-z_{j-1})|)^{\lceil a_j\rceil-a_j-\mathrm{Re}(z_{j-1})-\frac12},
\end{aligned}$$

and the exponential factor (when taking all $\mathcal{I}_j$ in unison) is negative for any $s_j$ outside of the interval $I_j$ defined in Lemma 9.2.3. That lemma together with the presence of the other exponential terms in our integral allow us to take trivial bounds for the polynomial part, namely that

$$\mathcal{Q}_j(s,z,\alpha)\ll T^{\varepsilon+R+\frac12+\max\{0,2(\lceil a_j\rceil-a_j)-1\}}.$$

(Recall that $0 \le \mathrm{Re}(z_j)$.) Thus we see that

$$\mathcal{I}_j \ll T^{\varepsilon+R+\frac{1}{2}+\max\{0,2(\lceil a_j\rceil-a_j)-1\}} \int\limits_{\substack{\mathrm{Re}(s_j)=\lceil a_j\rceil-a_j\\ \mathrm{Im}(s_j)\in I_j}} \prod_{\substack{J\subseteq\{1,\dots,n\}\\ \#J=j}} \left|s_j+\sum_{k\in J}\alpha_k\right|^{-\lceil a_j\rceil} ds_j.$$

The desired result now follows easily from this and the statement of Lemma A.3. □

Combining Lemma 9.4.1 with the bound for $\mathcal{I}^{(n)}_{T,R}(-a)$ given immediately before the statement of the lemma, and applying Lemmas A.19, A.26 and A.27 (in the case that $k=n-1$ and $\gamma_1=0$), we now have the bound

$$\begin{aligned}\mathcal{I}^{(m)}_{T,R}(-a) \ll \sum_{\substack{L\subseteq\{1,\dots,m\}\\ \#L=j}} & T^{\varepsilon+\left(R+\frac{1}{2}\right)(m-1)+\sum_{j=1}^{m-1}\left(\max\{0,2(\lceil a_j\rceil-a_j)-1\}+\lceil a_j\rceil\left(\binom{m}{j}-2\right)\right)}\cdot \int\limits_{\mathrm{Re}(\hat\alpha)=0} e^{\frac{m}{m-1}\frac{\hat\alpha^2}{2T^2}}\\ &\cdot \int\limits_{\substack{\hat\beta_{m-1}=0\\ \mathrm{Re}(\beta)=0}} e^{\frac{\beta_1^2+\cdots+\beta_{m-1}^2}{T^2/2}}\cdot \mathcal{P}^R_{D(m)-D(m-1)}(\hat\alpha,\beta)\cdot \mathcal{F}^{(m-1)}_R(\beta) \prod_{1\le j\ne k\le m-1} |\Gamma_R(\beta_j-\beta_k)|\\ &\cdot \prod_{j=1}^{m-1} \prod_{\substack{K\subseteq\{1,\dots,m\}\\ \#K=j\\ K\ne L}} \left(1+\left|\sum_{\ell\in L}\alpha_\ell-\sum_{k\in K}\alpha_k\right|\right)^{-\lceil a_j\rceil} \int\limits_{\substack{z=(z_1,\dots,z_{m-2})\\ \mathrm{Re}(z)=b}} |\widetilde{W}_{m-1,\beta}(z)|\,dz\,d\beta\,d\hat\alpha.\end{aligned}$$

To be more explicit, the polynomial $\mathcal{P}^R_{D(m)-D(m-1)}(\hat\alpha,\beta)$ is the portion of $\mathcal{F}^{(m)}_R(\alpha)$ which involves the terms $\alpha_m$.

At this point, we combine each of the terms in the final row with the corresponding term in $\mathcal{F}^{(m)}_R(\alpha)$. Strictly speaking, what is actually happening here is that this has the effect of reducing the power of each factor of $\mathcal{F}^{(m)}_R(\alpha)$ by at most

$$\max\{\lceil a_1\rceil,\dots,\lceil a_{m-1}\rceil\}.$$

Since each of the corresponding exponents remains positive, the net result is to reduce the overall power of $T$ by

$$\varepsilon+\sum_{j=1}^{m-1}\lceil a_j\rceil\left(\binom{m}{j}-1\right).$$

Using this, and accounting for the integration in $\hat\alpha$ (which may be assumed to take place only for $|\mathrm{Im}(\hat\alpha)|\le T^{1+\varepsilon}$), we now may write

$$\begin{aligned}\mathcal{I}^{(m)}_{T,R}(-a) \ll\ & T^{\varepsilon+\left(R+\frac{1}{2}\right)(n-1)+R(D(m)-D(m-1))+1+\sum_{j=1}^{m-1}(\max\{0,2(\lceil a_j\rceil-a_j)-1\}-\lceil a_j\rceil)}\\ &\int\limits_{\substack{\hat\beta_{m-1}=0\\ \mathrm{Re}(\beta)=0}} e^{\frac{\beta_1^2+\cdots+\beta_{m-1}^2}{2T^2}}\cdot \mathcal{F}^{(m-1)}_R(\beta) \prod_{1\le j\ne k\le m-1} |\Gamma_R(\beta_j-\beta_k)| \int\limits_{\substack{z=(z_1,\dots,z_{m-2})\\ \mathrm{Re}(z)=b}} |\widetilde{W}_{m-1,\beta}(z)|\,dz\,d\beta.\end{aligned}$$

Obviously, at this point we want to apply the inductive hypothesis. Since at this point we only need to do so in the case that $b_j>0$ (i.e., $-a_j<0$) for all $j=1,\dots,m-2$, the reduction in the powers of the exponents of any one of the factors of $\mathcal{F}_R^{(m)}(\alpha)$, as occurred above, leaves the overall power positive. Therefore, there is no issue, and we can assert (additionally applying Lemma A.5) the bound

$$\begin{aligned}\mathcal{I}_{T,R}^{(m)}(-a) &\ll T^{\varepsilon+\left(R+\frac{1}{2}\right)(m-1)+R(D(m)-D(m-1))+1+A(m-1)}\cdot T^{R\left(D(m-1)+\frac{(m-1)(m-2)}{2}\right)-\sum_{j=1}^{m-1}B(a_j)}\\ &= T^{\varepsilon+R\left(D(m)+\frac{m(m-1)}{2}\right)+\frac{n+1}{2}+A(m-1)-\sum_{j=1}^{m-1}B(a_j)}.\end{aligned}$$

Taking $A(m)=\frac{m+1}{2}+A(m-1)$ gives the claimed bound. Since $A(2)=\frac{3}{2}$, it follows that

$$A(3)=\tfrac{4}{2}+A(2)=\tfrac{1}{2}(4+3),\ \ldots,\ A(m)=\tfrac{1}{2}((m+1)+m+\cdots+3)=\tfrac{1}{4}(m+4)(m-1),$$

as claimed. □

In the course of proving Theorem 9.0.2 we also established the following result that we record here since it will be useful in its own right.

**Proposition 9.4.2.** *Suppose that $L=(\ell_1,\ell_2,\ldots,\ell_{m-1})\in(\mathbb{Z}_{\geq 0})^{m-1}$. Then*

$$\begin{aligned}&\int\limits_{\substack{\hat{\alpha}_m=0\\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_m^2}{2T^2}}\cdot\mathcal{F}_R^{(m)}(\alpha)\prod_{1\leq j\neq k\leq m}|\Gamma_R(\alpha_j-\alpha_k)|\int\limits_{\substack{s=(s_1,\ldots,s_{m-1})\\ \mathrm{Re}(s)=-a}}|\widetilde{W}_{m,\alpha}(s+L)|\,ds\,d\alpha\\ &\qquad\ll T^{\varepsilon+2|L|}\cdot\int\limits_{\substack{\hat{\alpha}_m=0\\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_m^2}{2T^2}}\cdot\mathcal{F}_R^{(m)}(\alpha)\prod_{1\leq j\neq k\leq m}|\Gamma_R(\alpha_j-\alpha_k)|\int\limits_{\substack{s=(s_1,\ldots,s_{m-1})\\ \mathrm{Re}(s)=-a}}|\widetilde{W}_{m,\alpha}(s)|\,ds\,d\alpha.\end{aligned}$$

*As a shorthand for this result, we write $\mathcal{I}_{T,R}^{(m)}(-a+L)\ll T^{\varepsilon+2|L|}\cdot\mathcal{I}_{T,R}^{(m)}(-a)$.*

## 10. Bounding $p_{T,R}^{(n)}(y)$

In this section we prove the following.

**Theorem 10.0.1.** *Let $n\geq 2$ and $\varepsilon\in\left(0,\frac{1}{4}\right)$. Suppose that $a=(a_1,a_2,\ldots,a_{n-1})$ satisfies $\lfloor a_j\rfloor+\varepsilon<a_j<\lceil a_j\rceil-\varepsilon$ for each $j=1,\ldots,n-1$. Let $\mathcal{C}$ be the set of compositions $n=n_1+\cdots+n_r$ with $r\geq 2$. Then, for*

$$\Delta_a(C):=\{\delta_C=(\delta_1,\ldots,\delta_{r-1})\in\mathbb{Z}^{r-1}\mid 0\leq\delta_j<a_{\hat{n}_j}\ (j=1,\ldots,r-1)\},$$

*and $B(c)$ as defined in Theorem 9.0.2, we have*

$$|p_{T,R}^{(n)}(y)|\ll|p_{T,R}^{(n)}(y;-a)|+\sum_{C\in\mathcal{C}}\sum_{\delta_C\in\Delta_a(C)}|p_{T,R}^{(n)}(y;-a,\delta_C)|,\tag{10.0.2}$$

*where*

$$|p_{T,R}^{(n)}(y;-a)|\ll\prod_{j=1}^{n-1}y_j^{\frac{n(n-j)}{2}+2a_j}\cdot T^{\varepsilon+\frac{(n+4)(n-1)}{4}+\frac{R}{2}\cdot\left(\binom{2n}{n}-2^n\right)-\sum_{j=1}^{n-1}B(a_j)}\tag{10.0.3}$$

*and*

$$|p_{T,R}^{(n)}(y;-a,\delta_C)|\ll\prod_{j=1}^{n-1}y_j^{\frac{n(n-j)}{2}+2a_j}\cdot T^{\varepsilon+\frac{(n+4)(n-1)}{4}+\frac{R}{2}\cdot\left(\binom{2n}{n}-2^n\right)-\sum_{j=1}^{n-1}B(a_j)-\frac{1}{2}\sum_{k=1}^{r-1}(n_k+n_{k+1})(a_{\hat{n}_k}-\delta_k)}.\tag{10.0.4}$$

*The implicit constant depends on both $\varepsilon$ and $n$.*

**Remark 10.0.5.** Note that (10.0.4) is bounded by (10.0.3). Therefore, letting $D(n)=\frac{1}{2}\binom{2n}{n}-\frac{n(n-1)}{2}-2^{n-1}$ as in (1.4.2), Theorem 10.0.1 implies that

$$|p_{T,R}^{(n)}(y)| \ll \prod_{j=1}^{n-1} y_j^{\frac{n(n-j)}{2}+2a_j} \cdot T^{\varepsilon+\frac{(n+4)(n-1)}{4}+R\cdot(D(n)+\frac{n(n+1)}{2})-\sum_{j=1}^{n-1}B(a_j)}.$$

**10.1.** ***Explicit single residue formula.*** In order to bound the terms $p_{T,R}^{(n)}(y;-a,\delta_C)$ we need an explicit formula for the residues of the Mellin transform of the GL$(n)$ Whittaker function. The following result establishes this for the case of single residues (i.e., when the composition $C$ has length 2) as a corollary of Conjecture 8.2.3 combined with a theorem of Stade [2001] for the "first" residues, i.e., for those residues corresponding, in the notation of the theorem, to $\delta=0$.

**Theorem 10.1.1.** *Let $\widetilde{W}_{m,\alpha}(s)$ be the Mellin transform of the Whittaker function on* GL$(n,\mathbb{R})$ *with purely imaginary parameters $\alpha=(\alpha_1,\dots,\alpha_n)$ in general position. Let $\sigma\in S_n$ act on $\alpha$ via*

$$\sigma\cdot\alpha := (\alpha_{\sigma(1)},\alpha_{\sigma(2)},\dots,\alpha_{\sigma(n)}).$$

*The poles of $\widetilde{W}_{n,\alpha}(s)$ occur, for each $1\le m\le n-1$, at*

$$s_m\in\{-\sigma\cdot\hat{\alpha}_m-\delta \mid \sigma\in S_n,\ \delta\in\mathbb{Z}_{\ge 0}\}.$$

*The residue at $s_m=-\hat{\alpha}_m-\delta$ is equal to a sum over shifts $L=(\ell_1,\ell_2,\dots,\ell_{n-1})$ of terms of the form*

$$\prod_{\substack{K\subseteq\{1,2,\dots,n\}\\ \#(K\cap\{1,2,\dots,m\})\ne m-1\\ \#K=m}} \left(\left(\sum_{i\in K}\alpha_i\right)-\hat{\alpha}_m-\delta\right)_\delta^{-1} \left(\prod_{i=1}^{m}\prod_{j=m+1}^{n}\Gamma(\alpha_j-\alpha_i-\delta)\right) \cdot \mathcal{P}_{(\binom{n}{m}-2)\delta-2|L|}(s,\alpha)\widetilde{W}_{m,\beta}(s'+L')\widetilde{W}_{n-m,\gamma}(s''+L''),$$

*where*

$$s'=\left(s_j+\frac{j}{m}\hat{\alpha}_m\right)\Big|_{1\le j\le m},\qquad s''=\left(s_{m+j}+\frac{n-m-j}{n-m}\hat{\alpha}_m\right)\Big|_{1\le j\le n-m}, \tag{10.1.2}$$

*with $L'=(\ell_1,\dots,\ell_{m-1})$ and $L''=(\ell_{m+1},\dots,\ell_{n-1})$ being the portion of $L$ corresponding to $s'$ and $s''$ respectively. It is the case that $\ell_{m-1}=\ell_{m+1}=0$. Note that we take as definition that $\widetilde{W}_1:=1$. The same formula holds for the residue at $s_m=-\sigma\cdot\hat{\alpha}_m-\delta$ by replacing each instance of $\alpha_j$ with $\alpha_{\sigma(j)}$.*

**Remark 10.1.3.** Another way of writing the above expression for the residue would be to take the product over all $K\subseteq\{1,\dots,n\}$ with $\#K=m$ and replace $\Gamma(\alpha_j-\alpha_i-\delta)$ with $\Gamma(\alpha_j-\alpha_i)$. The two versions are equivalent because if $K\setminus\{1,\dots,m\}=\{j\}$, then $\{1,\dots,m\}\setminus K=\{k\}$ and

$$\left(\left(\sum_{i\in K}\alpha_i\right)-\hat{\alpha}_m-\delta\right)_\delta^{-1}\Gamma(\alpha_j-\alpha_k)=\Gamma(\alpha_j-\alpha_k-\delta).$$

*Sketch of proof.* In the case that $\delta=0$, this result (for $L=(0,\dots,0)\in\mathbb{C}^{n-1}$) agrees with [Stade 2001, Theorem 3.1]. If $\delta>0$, we need to first apply Conjecture 8.2.3 to rewrite the expression for $\widetilde{W}_{n,\alpha}(s)$ around $s_m=-\alpha_m-\delta$ as a sum over shifts $L=(\ell_1,\dots,\ell_{n-1})\in(\mathbb{Z}_{\ge 0})^{n-1}$ (with $\ell_m\ge\delta$ for each $L$) of terms $\widetilde{W}_{n,\alpha}(s+L)$. Of all of these terms, the only ones for which there is a pole at $s_m = -\hat{\alpha}_m - \delta$ are those for which $\ell_m = \delta$, in which case we can use the above-referenced theorem of Stade to write down the residue. Doing so, we obtain the alternative expression referenced in Remark 10.1.3. □

**10.2.** ***Explicit higher residue formulae.*** In order to generalize Theorem 10.1.1, we first establish notation related to the $(r-1)$-fold residue of $\widetilde{W}_{n,\alpha}(s)$ at

$$s_{\hat{n}_\ell} = -\hat{\alpha}_{\hat{n}_\ell} - \delta_{\hat{n}_\ell}, \quad \ell = 1, \dots, r-1.$$

To this end, let $s^{(j)} := (s_1^{(j)}, \dots, s_{n_j-1}^{(j)})$, where $s_k^{(j)} = s_{\hat{n}_{j-1}+k}$. By abuse of notation, we write

$$s := \big(\underbrace{s_1^{(1)}, s_2^{(1)}, \dots, s_{n_1-1}^{(1)}}_{=:s^{(1)}}, \underbrace{s_1^{(2)}, s_2^{(2)}, \dots, s_{n_2-1}^{(2)}}_{=:s^{(2)}}, \dots, \underbrace{s_1^{(k)}, s_2^{(k)}, \dots, s_{n_k-1}^{(k)}}_{=:s^{(k)}}\big) \in \mathbb{C}^{n-r},$$

which agrees with the original $s = (s_1, \dots, s_{n-1})$ but removes $s_{\hat{n}_1}, \dots, s_{\hat{n}_{r-1}}$.

Similarly, if $\alpha = (\alpha_1, \dots, \alpha_n)$, we define

$$\alpha^{(\ell)} := (\alpha_1^{(\ell)}, \dots, \alpha_\ell^{(\ell)}) \in \mathbb{C}^{n_\ell}, \qquad \alpha_j^{(\ell)} := \alpha_{\hat{n}_{\ell-1}+j} - \frac{1}{n_\ell}(\hat{\alpha}_{\hat{n}_\ell} - \hat{\alpha}_{\hat{n}_{\ell-1}}),$$

and

$$|\alpha^{(j)}|^2 := (\alpha_1^{(j)})^2 + (\alpha_2^{(j)})^2 + \cdots + (\alpha_{n_j}^{(j)})^2.$$

If $a \in \mathbb{R}^{n-1}$ then by $\mathrm{Re}(s) = -a$ we mean that $\mathrm{Re}(s_j) = -a_j$ for each $j \neq \hat{n}_1, \dots, \hat{n}_{r-1}$.

With this notation in place, we can now state a generalization of Theorem 10.1.1.

**Corollary 10.2.1.** *Let $n = n_1 + \cdots + n_r$ $(r \geq 2)$, and set $\hat{n}_\ell := \sum_{j=1}^{\ell} n_j$ as above. For each $\ell = 1, \dots, r-1$, let $b^{(\ell)} = (b_1^{(\ell)}, b_2^{(\ell)}, \dots, b_{n_\ell-1}^{(\ell)})$ with*

$$b_j^{(\ell)} = \hat{\alpha}_{i_{\ell-1}} + \frac{j}{n_\ell}(\hat{\alpha}_{\hat{n}_\ell} - \hat{\alpha}_{\hat{n}_{\ell-1}}) \quad \text{for each } 1 \leq j \leq n_\ell - 1.$$

*Let $\delta_j \in \mathbb{Z}_{\geq 0}$ for $j = 1, \dots, r-1$. There exist positive shifts $L = (L^{(1)}, \dots, L^{(r)})$ with $L^{(\ell)} = (L_1^{(\ell)}, \dots, L_{n_\ell-1}^{(\ell)}) \in (\mathbb{Z}_{\geq 0})^r$ such that the iterated residue of $\widetilde{W}_{n,\alpha}(s)$ at*

$$s_{\hat{n}_{r-1}} = -\hat{\alpha}_{\hat{n}_{r-1}} - \delta_{r-1}, \dots, s_{\hat{n}_1} = -\hat{\alpha}_{\hat{n}_1} - \delta_1$$

*is equal to a sum over all such shifts of*

$$\mathcal{P}_d(s,\alpha)\bigg(\prod_{\ell=1}^{r} \widetilde{W}_{n_\ell,\alpha^{(\ell)}}(s^{(\ell)} + b^{(\ell)} + L^{(\ell)})\bigg) \prod_{j=1}^{r-1} \prod_{\substack{K\subseteq\{1,2,\dots,\hat{n}_{j+1}\} \\ \#(K\cap\{1,\dots,\hat{n}_j\})\neq\hat{n}_j-1 \\ \#K=\hat{n}_j}} \bigg(\Big(\sum_{i\in K}\alpha_i\Big) - \hat{\alpha}_{\hat{n}_j} - \delta_j\bigg)_{\delta_j}^{-1}$$
$$\cdot \prod_{1\leq k<m\leq r} \prod_{i=1}^{n_k} \prod_{j=1}^{n_m} \Gamma\Big(\alpha_j^{(m)} - \alpha_i^{(k)} + \frac{1}{n_m}(\hat{\alpha}_{\hat{n}_m} - \hat{\alpha}_{\hat{n}_{m-1}}) - \frac{1}{n_k}(\hat{\alpha}_{\hat{n}_k} - \hat{\alpha}_{\hat{n}_{k-1}}) - \delta_m\Big),$$

*where*

$$d = \bigg[\sum_{\ell=1}^{r-1} \delta_\ell\bigg(\binom{\hat{n}_{\ell+1}}{\hat{n}_\ell} - 2\bigg)\bigg] - 2|L|.$$

*Proof.* This follows easily by induction with the base case being Theorem 10.1.1. □

**Remark 10.2.2.** Although it is possible to rewrite each of the terms $\left(\sum_{i\in K}\alpha_i\right) - \hat{\alpha}_{\hat{n}_\ell} - \delta_\ell$ appearing in the statement of Corollary 10.2.1 in terms of the variables $\alpha^{(j)}$ and $\hat{\alpha}_m^{(j)}$ for various $j$ and $m$, the exact description is unnecessary for our purposes.

**10.3.** ***Proof of Theorem 10.0.1.*** As a first step, note that Proposition 8.3.6 implies that (10.0.2) follows from (10.0.3) and (10.0.4).

As shown in Section 8.5, the shifted $p_{T,R}^{(n)}$-term satisfies

$$|p_{T,R}^{(n)}(y,-a)| \ll \left(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}-2a_j}\right)\mathcal{I}_{T,R}^{(n)}(-a).$$

Combined with the bound from Theorem 9.0.2, this gives (10.0.3).

To complete the proof, we need to show that (10.0.4) holds. We do this in Section 10.5. Although this proof is valid for any $r \geq 2$, as a warmup, we first prove the special case $r = 2$ (i.e., the case of single residues) in Section 10.4. □

**10.4.** ***Bounds for single residue terms.*** In this section[1] we bound $p_{T,R}^{(n)}(y; -a, \delta_C)$ in the case that $C = (m, n-m)$. Since $C$ is a composition of length 2, we may take (see Definition 8.3.3) $\delta_C = \delta \in \mathbb{Z}_{\geq 0}$.

*Proof of* (10.0.4) *when* $r = 2$. Using Lemmas A.19, A.26 and A.28, we can rewrite

$$e^{\frac{\alpha_1^2+\cdots+\alpha_n^2}{T^2/2}}\mathcal{F}_R^{(n)}(\alpha)\prod_{1\leq j\neq k\leq n}\Gamma_R(\alpha_j-\alpha_k)$$

in terms of $\beta$, $\gamma$ and $\alpha_n$. Thus, together with Theorem 10.1.1, we see that Definition 8.3.3 in the case of a single residue term (i.e., $r = 2$) satisfies the bound

$$\begin{aligned}
&p_{T,R}^{(n)}(y; -a, \delta_C)\\
&\ll \int\limits_{\mathrm{Re}(\hat{\alpha}_m)=0} y_m^{\frac{m(n-m)}{2}+\hat{\alpha}_m+\delta}\cdot e^{\frac{n}{m(n-m)}\frac{\hat{\alpha}_m^2}{T^2/2}} \int\limits_{\substack{\hat{\beta}_m=0\\ \mathrm{Re}(\beta)=0}} e^{\frac{|\beta|^2}{T^2/2}} \int\limits_{\substack{\hat{\gamma}_{n-m}=0\\ \mathrm{Re}(\gamma)=0}} e^{\frac{|\gamma|^2}{T^2/2}}\\
&\quad\cdot\left(\mathcal{F}_R^{(m)}(\beta)\cdot\prod_{1\leq i\neq j\leq m}\Gamma_R(\beta_i-\beta_j)\right)\left(\mathcal{F}_R^{(n-m)}(\gamma)\cdot\prod_{1\leq i\neq j\leq n-m}\Gamma_R(\gamma_i-\gamma_j)\right)\\
&\quad\cdot\prod_{i=1}^{m}\prod_{j=1}^{n-m}\left(\Gamma_R\left(\beta_i-\gamma_j+\frac{n\hat{\alpha}_m}{m(n-m)}\right)\Gamma_R\left(\gamma_j-\beta_i-\frac{n\hat{\alpha}_m}{m(n-m)}\right)\Gamma\left(\gamma_j-\beta_i-\frac{n\hat{\alpha}_m}{m(n-m)}-\delta\right)\right)\\
&\quad\cdot\left(\prod_{j\neq m}\int\limits_{\substack{\mathrm{Re}(s_j)=-a_j\\ j\neq m}} y_j^{\frac{j(n-j)}{2}-s_j}\right)\mathcal{P}_{R(D(n)-D(m)-D(n-m))-\delta\left(\binom{n}{m}-m(n-m)-1\right)}(s,\alpha)\\
&\quad\cdot\mathcal{P}_{(\binom{n}{m}-2)\delta-2|L|}(s,\alpha)\cdot\widetilde{W}_{m,\beta}(s'+L')\widetilde{W}_{n-m,\gamma}(s''+L'')\,ds\,d\gamma\,d\beta\,d\hat{\alpha}_m.
\end{aligned}$$

[1] Note that this section will be superseded by Section 10.5, which will prove the bound for any admissible $C$ with $\mathrm{length}(C) \geq 2$. This section treats the case $\mathrm{length}(C) = 2$.

In order to have the correct power of $y_m$, we need to shift the line of integration in $\hat{\alpha}_m$ to $\mathrm{Re}(\hat{\alpha}_m) = a_m - \delta$. Note that by Lemma A.14, no poles are crossed in doing so, and by Lemma A.15, taking $\beta = \beta_i - \gamma_j$ and $z = n\hat{\alpha}_m/(m(n-m))$, we may replace the third-to-last line by

$$\mathcal{P}_{m(n-m)R-n(a_m-\delta)-m(n-m)\delta}(s, \hat{\alpha}_m, \beta, \gamma).$$

Let $|\beta|^2 := \beta_1^2 + \cdots + \beta_m^2$, and define $|\gamma|^2$ similarly. Replacing the integral over $\hat{\alpha}_m$ by $T^{\varepsilon+1}$ and factoring out the powers of $y_j$, we see that

$$\begin{aligned} |p_{T,R}^{(n)}(y; -a, \delta_C)| \ll \left(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}+a_j}\right) \cdot T^{\varepsilon+(\binom{n}{m}-2)\delta+R(D(n)-D(m)-D(n-m))-\delta(\binom{n}{m}-m(n-m)-1)} \\ \cdot T^{-2|L|+m(n-m)R-n(a_m-\delta)-m(n-m)\delta+1} \cdot \int\limits_{\substack{\hat{\beta}_m=0 \\ \mathrm{Re}(\beta)=0}} e^{\frac{|\beta|^2}{T^2/2}} \int\limits_{\substack{\hat{\gamma}_{n-m}=0 \\ \mathrm{Re}(\gamma)=0}} e^{\frac{|\gamma|^2}{T^2/2}} \\ \cdot \left(\mathcal{F}_R^{(m)}(\beta) \cdot \prod_{1 \le i,j \le k} \Gamma_R(\beta_i - \beta_j)\right)\left(\mathcal{F}_R^{(n-m)}(\gamma) \cdot \prod_{1 \le i,j \le n-k} \Gamma_R(\gamma_i - \gamma_j)\right) \\ \cdot \int\limits_{\substack{\mathrm{Re}(s_j)=-a_j \\ 1 \le j \le n-1 \\ j \ne m}} |\widetilde{W}_{m,\beta}(s'+L')| \cdot |\widetilde{W}_{n-m,\gamma}(s''+L'')| \, ds \, d\gamma \, d\beta. \end{aligned}$$

Note that by Proposition 9.4.2 we may remove the dependence on the shift $L$. Hence

$$\begin{aligned} |p_{T,R}^{(n)}(y; -a, \delta_C)| \ll \left(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}+a_j}\right) \cdot T^{\varepsilon+R(D(n)-D(m)-D(n-m)+m(n-m))+\delta(n-1)} \\ \cdot T^{-na_m+1} \int\limits_{\substack{\hat{\beta}_m=0 \\ \mathrm{Re}(\beta)=0}} e^{\frac{\beta_1^2+\cdots+\beta_m^2}{T^2/2}} \int\limits_{\substack{\hat{\gamma}_{n-m}=0 \\ \mathrm{Re}(\gamma)=0}} e^{\frac{\gamma_1^2+\cdots+\gamma_n^2}{T^2/2}} \\ \cdot \left(\mathcal{F}_R^{(m)}(\beta) \cdot \prod_{1 \le i \ne j \le k} \Gamma_R(\beta_i - \beta_j)\right)\left(\mathcal{F}_R^{(n-m)}(\gamma) \cdot \prod_{1 \le i \ne j \le n-k} \Gamma_R(\gamma_i - \gamma_j)\right) \\ \cdot \int\limits_{\substack{\mathrm{Re}(s_j)=-a_j \\ 1 \le j \le n-1 \\ j \ne m}} |\widetilde{W}_{m,\beta}(s')| \cdot |\widetilde{W}_{n-m,\gamma}(s'')| \, ds \, d\gamma \, d\beta. \end{aligned}$$

By (10.1.2),

$$s'_j = s_j - \frac{j}{m}(\hat{\alpha}_m - \delta) \quad \text{and} \quad s''_j = s_{m+j} - \frac{n-m-j}{n-m}(\hat{\alpha}_m - \delta).$$

Thus the integrals in $\beta$ and $\gamma$ above are essentially the product of $\mathcal{I}_{T,R}^{(m)}(-a')$ and $\mathcal{I}_{T,R}^{(n-m)}(-a'')$. The only issue is that because, as seen in the fact that the variables $s'$ and $s''$ are shifted, we have

$$a'_j = a_j - \frac{j}{m}(a_m - \delta) \quad \text{and} \quad a''_j = a_{m+j} - \frac{n-m-j}{n-m}(a_m - \delta).$$

Therefore, we can rewrite the previous formula as

$$|p_{T,R}^{(n)}(y;-a,\delta_C)| \ll \left(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}+a_j}\right) \cdot T^{\varepsilon+R(D(n)-D(m)-D(n-m)+m(n-m))} \cdot T^{\delta(n-1)-na_m+1} \cdot \mathcal{I}_{T,R}^{(m)}(-a') \cdot \mathcal{I}_{T,R}^{(n-m)}(-a'').$$

By Theorem 9.0.2, we have

$$|p_{T,R}^{(n)}(y;-a,\delta_C)| \ll \left(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}+a_j}\right) \cdot T^{\varepsilon+R(D(n)-D(m)-D(n-m)+m(n-m))} \cdot T^{\delta(n-1)-na_m+1} \cdot T^{\varepsilon+C(m)+R\cdot\left(D(m)+\frac{m(m-1)}{2}\right)-\sum_{j=1}^{m-1} B(a_j')} \cdot T^{\varepsilon+C(n-m)+R\cdot\left(D(n-m)+\frac{(n-m)(n-m-1)}{2}\right)-\sum_{j=1}^{n-m-1} B(a_j'')},$$

Recall that $C(k) = \frac{(k+4)(k-1)}{4}$. Hence, using the elementary identity

$$C(m) + C(n-m) = C(n) - \frac{m(n-m)}{2} - 1$$

together with Lemma A.6,

$$\begin{aligned}|p_{T,R}^{(n)}(y;-a,\delta_C)| &\ll \left(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}+a_j}\right) \cdot T^{\varepsilon+R\left(D(n)+m(n-m)+\frac{m(m-1)}{2}+\frac{(n-m)(n-m-1)}{2}\right)} \cdot T^{\delta(n-1)+C(n)-\frac{m(n-m)}{2}-na_m-\sum_{j=1}^{m-1} B(a_j)+\frac{n-2}{2}(a_m-\delta+1)+B(a_m)} \\ &\ll \left(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}+a_j}\right) \cdot T^{\varepsilon+C(n)+R\left(D(n)+\frac{n(n-1)}{2}\right)-\sum_{j=1}^{n-1} B(a_j)} \cdot T^{\frac{n-2}{2}(\delta-a_m+1)-\frac{m(n-m)}{2}-n(\delta-a_m)+B(a_m)-\delta}.\end{aligned}$$

This gives the desired bound provided that the exponent of the final $T$ is negative. Using the facts that $-\frac{m(n-m)}{2}$ is maximized when $m=1$ or $m=n-1$ and $B(a_m) \le a_m + \frac{1}{2}$, we see that the final exponent is

$$-\frac{n}{2}(a_m-\delta) + \frac{n-1}{2} - \frac{m(n-m)}{2} \le -\frac{n}{2}(a_m-\delta), \tag{10.4.1}$$

as claimed. □

**10.5.** ***Bounds for $(r-1)$-fold residues.*** We consider a composition $C$ of $n$ of length $r \ge 2$ given by $n = n_1 + \cdots + n_r$. We may also write $C = (n_1, \ldots, n_r)$. Let $\hat{n}_\ell = \sum_{j=1}^{\ell} n_j$ as usual.

As a final piece of notation, let $\beta = (\beta_1, \ldots, \beta_r)$ be defined via

$$\beta_i := \hat{\alpha}_{\hat{n}_i} - \hat{\alpha}_{\hat{n}_{i-1}}.$$

Note that $\sum_{i=1}^{r} \beta_i = 0$ and more generally, defining $\hat{\beta}_m = \sum_{i=1}^{m} \beta_i$, $\hat{\alpha}_{\hat{n}_i} = \hat{\beta}_i$. Since (assuming that $\hat{\alpha}_n = 0$) the Jacobians of the changes of variables

$$\alpha \mapsto (\alpha^{(1)}, \hat{\alpha}_{\hat{n}_1}, \alpha^{(2)}, \hat{\alpha}_{\hat{n}_2}, \ldots, \hat{\alpha}_{\hat{n}_{r-1}}, \alpha^{(r)})$$

and

$$(\hat{\alpha}_{\hat{n}_1}, \ldots, \hat{\alpha}_{\hat{n}_{r-1}}) \mapsto (\beta_1, \ldots, \beta_{r-1})$$

are trivial, we see that (for $\beta_1 + \cdots + \beta_r = 0$)

$$d\alpha = d\beta\, d\alpha^{(1)}\, d\alpha^{(2)} \cdots d\alpha^{(r)}. \tag{10.5.1}$$

*Proof of* (10.0.4) *when* $r \geq 2$. Note that

$$b_j^{(\ell)} = \hat{\alpha}_{\hat{n}_{\ell-1}} + \frac{j}{n_\ell}(\hat{\alpha}_{\hat{n}_\ell} - \hat{\alpha}_{\hat{n}_{\ell-1}}) = \hat{\beta}_{\ell-1} + \frac{j}{n_\ell}\beta_\ell \quad \text{for each } 1 \leq j \leq n_\ell - 1.$$

Recall that by Definition 8.3.3,

$$p_{T,R}^{(n)}(y; -a, \delta_C) := \int\limits_{\substack{\hat{\alpha}_n=0\\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_n^2}{T^2/2}} \cdot \mathcal{F}_R^{(n)}(\alpha)\Big(\prod_{1\leq j\neq k\leq n}\Gamma_R(\alpha_j-\alpha_k)\Big)$$
$$\cdot\Big(\prod_{i=1}^{r-1} y_{\hat{n}_i}^{\frac{\hat{n}_i(n-\hat{n}_i)}{2}+\hat{\alpha}_{\hat{n}_i}+\delta_i}\Big)\cdot \int\limits_{\substack{\mathrm{Re}(s_j)=-a_j\\ j\notin\{\hat{n}_1,\ldots,\hat{n}_{r-1}\}}} \Big(\prod_{j\notin\{\hat{n}_1,\ldots,\hat{n}_{r-1}\}} y_j^{\frac{j(n-j)}{2}-s_j}\Big)$$
$$\cdot \operatorname*{Res}_{s_{\hat{n}_1}=-\hat{\alpha}_{\hat{n}_1}-\delta_1}\Big(\operatorname*{Res}_{s_{\hat{n}_2}=-\hat{\alpha}_{\hat{n}_2}-\delta_2}\big(\cdots\big(\operatorname*{Res}_{s_{\hat{n}_{r-1}}=-\hat{\alpha}_{\hat{n}_{r-1}}-\delta_{r-1}} \widetilde{W}_{n,\alpha}(s)\big)\cdots\big)\Big)\,ds\,d\alpha.$$

Using Remark A.29 and Corollary 10.2.1, we can bound $|p_{T,R}^{(n)}(y; -a, \delta_C)|$ by a sum over certain shifts $L$ each of the form

$$\int\limits_{\substack{\hat{\beta}_r=0\\ \mathrm{Re}(\beta)=0}} e^{(\frac{\beta_1^2}{n_1}+\cdots+\frac{\beta_r^2}{n_r})\frac{2}{T^2}}\cdot\Big(\prod_{j=1}^{r-1} y_{\hat{n}_j}^{\frac{\hat{n}_j(n-\hat{n}_j)}{2}+\hat{\beta}_j+\delta_j} \int\limits_{\substack{\hat{\alpha}_{n_j}^{(j)}=0\\ \mathrm{Re}(\alpha^{(j)})=0}} e^{\frac{|\alpha^{(j)}|^2}{T^2/2}}\Big)$$
$$\cdot\mathcal{P}_{d_1-2|L|}(\alpha)\cdot\int\limits_{\mathrm{Re}(s)=-a}\Big(\prod_{j\notin\{\hat{n}_1,\ldots,\hat{n}_{r-1}\}} y_j^{\frac{j(n-j)}{2}-s_j}\Big)\cdot\mathcal{P}_{d_2}(s,\alpha)$$
$$\cdot\prod_{1\leq k<m\leq r}\prod_{i=1}^{n_k}\prod_{j=1}^{n_m}\Gamma\Big(\alpha_j^{(m)}-\alpha_i^{(k)}+\frac{\beta_m}{n_m}-\frac{\beta_k}{n_k}-\delta_k\Big)\prod_{\epsilon\in\{\pm1\}}\Gamma_R\Big(\epsilon\Big(\alpha_j^{(m)}-\alpha_i^{(k)}+\frac{\beta_m}{n_m}-\frac{\beta_k}{n_k}\Big)\Big)$$
$$\cdot\prod_{\ell=1}^{r}\Big(\mathcal{F}_R^{(n_\ell)}(\alpha^{(\ell)})\Big(\prod_{1\leq j\neq k\leq n_\ell}\Gamma_R(\alpha_j^{(\ell)}-\alpha_k^{(\ell)})\Big)\widetilde{W}_{n_\ell,\alpha^{(\ell)}}(s^{(\ell)}+b^{(\ell)}+L^{(\ell)})\Big)\,ds\,d\alpha^{(1)}\,d\alpha^{(2)}\cdots d\alpha^{(r)}\,d\beta,$$

where

$$d_1 = \sum_{\ell=1}^{r-1}\delta_\ell\Big(\binom{\hat{n}_{\ell+1}}{\hat{n}_\ell}-2\Big),$$
$$d_2 = R\cdot\Big(D(n)-\sum_{\ell=1}^{r}D(n_\ell)\Big)-\sum_{\ell=1}^{r-1}\Big[\delta_\ell\Big(\binom{\hat{n}_{\ell+1}}{\hat{n}_\ell}-n_{\ell+1}\hat{n}_\ell-1\Big)\Big]$$

are the degrees coming from Remark A.29 and Corollary 10.2.1 respectively and $b^{(\ell)}$ is as in Corollary 10.2.1. Note that, in addition to using the change of variables (10.5.1), we have used Lemmas A.18 and A.20 to break up $e^{2|\alpha|^2/T^2}$ and rewrite the product of $\Gamma(\alpha_j-\alpha_k)$ in terms of $\alpha^{(1)},\ldots,\alpha^{(r)}$ and $\beta$.

The next step is to shift the lines of integration in the variables $\beta_j$ for $j=1,\ldots,r-1$ (or, equivalently, $\hat{\beta}_j$ for $j=1,\ldots,r-1$) such that the real part of the exponent of each term $y_{\hat{n}_j}$ is $\frac{\hat{n}_j(n-\hat{n}_j)}{2}+a_j$. In particular, this implies that we must shift the line of integration of $\hat{\beta}_j$ to

$$\mathrm{Re}(\hat{\beta}_j) = a_{\hat{n}_j} - \delta_j \quad \iff \quad \mathrm{Re}(\beta_j) = \mathrm{Re}(\hat{\beta}_j - \hat{\beta}_{j-1}) = (a_{\hat{n}_j} - \delta_j) - (a_{\hat{n}_{j-1}} - \delta_{j-1}). \tag{10.5.2}$$

Provided that $R$ is sufficiently large, Lemma A.14 implies that this shift can be made without passing any poles. Moreover, Lemma A.15 implies that

$$\prod_{1\le k<m\le r} \prod_{i=1}^{n_k} \prod_{j=1}^{n_m} \Gamma\Big(\alpha_i^{(k)} - \alpha_j^{(m)} + \frac{\beta_k}{n_k} - \frac{\beta_m}{n_m} - \delta_m\Big) \prod_{\epsilon\in\{\pm 1\}} \Gamma_R\Big(\epsilon\Big(\alpha_i^{(k)} - \alpha_j^{(m)} + \frac{\beta_k}{n_k} - \frac{\beta_m}{n_m}\Big)\Big)$$
$$\asymp \prod_{1\le k<m\le r} \prod_{i=1}^{n_k} \prod_{j=1}^{n_m} \Big(1 + \Big|\mathrm{Im}\Big(\alpha_i^{(k)} - \alpha_j^{(m)} + \frac{\beta_k}{n_k} - \frac{\beta_m}{n_m}\Big)\Big|\Big)^{R-\mathrm{Re}\,(\frac{\beta_k}{n_k} - \frac{\beta_m}{n_m})-\delta_m}. \tag{10.5.3}$$

Note that the presence of the term $e^{(\beta_1^2/n_1+\cdots+\beta_r^2/n_r)(2/T^2)}$ implies that there is exponential decay for $|\mathrm{Im}(\beta_j)| \gg T^{1+\varepsilon}$. As we will see momentarily, besides the polynomial terms $\mathcal{P}_{d_1}(\alpha)$, $\mathcal{P}_{d_2}(s,\alpha)$ and (10.5.3), we just get a product of $\mathcal{I}_{T,R}^{(n_j)}(-c^{(j)})$ for some (to be determined) values $-c^{(\ell)}$. The upshot is that all of these polynomials can be bounded by $T$ to the degree of the polynomial plus $\varepsilon$. Hence, we can bound the expression above by

$$T^{\varepsilon+r-1+d-2|L|} \cdot \Big(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}+a_j}\Big)$$
$$\cdot \prod_{\ell=1}^{r} \Big( \int\limits_{\substack{\hat{\alpha}_{n_\ell}^{(\ell)}=0 \\ \mathrm{Re}(\alpha^{(\ell)})=0}} e^{\frac{|\alpha^{(\ell)}|^2}{T^2/2}} \cdot \mathcal{F}_R^{(n_\ell)}(\alpha^{(\ell)}) \cdot \int\limits_{\mathrm{Re}(s^{(\ell)})=-a^{(\ell)}} \Big(\prod_{1\le j\ne k\le n_\ell} \Gamma_R(\alpha_j^{(\ell)} - \alpha_k^{(\ell)})\Big) |\widetilde{W}_{n_\ell,\alpha^{(\ell)}}(s^{(\ell)} + b^{(\ell)} + L^{(\ell)})| \, ds^{(\ell)} \, d\alpha^{(\ell)}\Big), \tag{10.5.4}$$

where $d = d_1 + d_2 + d_3$, with $d_1$ and $d_2$ as above and

$$d_3 = R \cdot \sum_{\ell=1}^{r} n_\ell \hat{n}_\ell - \sum_{k=1}^{r-1} \big((n_k + n_{k+1})(a_{\hat{n}_k} - \delta_k) + \delta_k n_{k+1} \hat{n}_k\big)$$

is the bound coming from the terms described in (10.5.3), simplified using Lemma A.21. Combining everything, we find that $d$ equals

$$R \cdot \Big(D(n) - \sum_{\ell=1}^{r} D(n_\ell) + \sum_{1\le k<m\le r} n_k n_m\Big) - \sum_{k=1}^{r-1} \big(\delta_k + (n_k + n_{k+1})(a_{\hat{n}_k} - \delta_k)\big).$$

Recall that the bound on $p_{T,R}^{(n)}(y; -a, \delta_C)$ is a *sum* of expressions of the form given in (10.5.4) for various shifts $L$. However, using Proposition 9.4.2, we can remove the dependence on the shifts. Hence,

$$|p_{T,R}^{(n)}(y; -a, \delta_C)|$$
$$\ll T^{\varepsilon+d+r-1} \cdot \Big(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}+a_j}\Big)$$
$$\cdot \prod_{\ell=1}^{r} \Big( \int\limits_{\substack{\hat{\alpha}_{n_\ell}^{(\ell)}=0 \\ \mathrm{Re}(\alpha^{(\ell)})=0}} e^{\frac{|\alpha^{(\ell)}|^2}{T^2/2}} \cdot \mathcal{F}_R^{(n_\ell)}(\alpha^{(\ell)}) \cdot \int\limits_{\mathrm{Re}(s^{(\ell)})=-a^{(\ell)}} \Big(\prod_{1\le j\ne k\le n_\ell} \Gamma_R(\alpha_j^{(\ell)} - \alpha_k^{(\ell)})\Big) |\widetilde{W}_{n_\ell,\alpha^{(\ell)}}(s^{(\ell)} + b^{(\ell)})| \, ds^{(\ell)} \, d\alpha^{(\ell)}\Big). \tag{10.5.5}$$

Thus, setting $c^{(\ell)} = a^{(\ell)} - \mathrm{Re}(b^{(\ell)})$, where

$$b^{(\ell)} = (b_1^{(\ell)}, \dots, b_{\hat{n}_j}^{(\ell)}), \quad b_j^{(\ell)} = \hat{\beta}_{\ell-1} + \frac{j}{n_\ell}\beta_\ell,$$

we find that

$$|p_{T,R}^{(n)}(y; -a, \delta_C)| \ll \left(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}+a_j}\right) \cdot T^{\varepsilon+r-1+d} \cdot \prod_{\ell=1}^{r} \mathcal{I}_{T,R}^{(\ell)}(-c^{(\ell)}).$$

Let $C(m) := \frac{(m+4)(m-1)}{4}$. We now apply Theorem 9.0.2 to each $\mathcal{I}_{T,R}^{(n_\ell)}$ to obtain

$$|p_{T,R}^{(n)}(y; -a, \delta_C)| \ll T^{\varepsilon+r-1+d+\sum_{\ell=1}^{r}\left(R\left(D(n_\ell)+\frac{n_\ell(n_\ell-1)}{2}\right)+C(n_\ell)-\sum_{k=1}^{n_\ell-1} B(c_k^{(\ell)})\right)} \cdot \prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}+a_j}.$$

Now we generalize the proof of Lemma A.6, keeping in mind that $a < B(a) < a + \frac{1}{2}$, to simplify the expression

$$\begin{aligned}
\sum_{\ell=1}^{r}\sum_{j=1}^{n_\ell-1} B(c_j^{(\ell)}) &\geq \sum_{\ell=1}^{r}\sum_{j=1}^{n_\ell-1}\left(a_{\hat{n}_\ell+j} - \mathrm{Re}(\hat{\beta}_{\ell-1}) - \frac{j}{n_\ell}\,\mathrm{Re}(\beta_\ell)\right) \\
&= \left(\sum_{j=1}^{n-1} a_j\right) - \left(\sum_{k=1}^{r-1} a_{\hat{n}_k}\right) - \sum_{\ell=1}^{r}\left[(n_\ell-1)\,\mathrm{Re}(\hat{\beta}_{\ell-1}) + \frac{n_\ell-1}{2}\,\mathrm{Re}(\beta_\ell)\right] \\
&\geq \sum_{j=1}^{n-1}\left(B(a_j) - \tfrac{1}{2}\right) - \sum_{k=1}^{r-1} a_{\hat{n}_k} - \sum_{\ell=1}^{r}\left[(n_\ell-1)\,\mathrm{Re}\left(\hat{\beta}_\ell - \tfrac{1}{2}\beta_\ell\right)\right] \\
&= -\frac{n-1}{2} + \sum_{j=1}^{n-1} B(a_j) - \sum_{k=1}^{r-1} a_{\hat{n}_k} - \sum_{\ell=1}^{r}\left[(n_\ell-1)\left(A_\ell - \tfrac{1}{2}(A_\ell - A_{\ell-1})\right)\right] \\
&= -\frac{n-1}{2} + \sum_{j=1}^{n-1} B(a_j) - \sum_{k=1}^{r-1} a_{\hat{n}_k} - \frac{1}{2}\sum_{\ell=1}^{r}[(n_\ell-1)(A_\ell + A_{\ell-1})].
\end{aligned}$$

Next, we write the sum over $\ell$ as

$$\begin{aligned}
\sum_{\ell=1}^{r}[(n_\ell-1)(A_\ell + A_{\ell-1})] &= \sum_{\ell=1}^{r}(n_\ell-1)A_\ell + \sum_{\ell=1}^{r}(n_\ell-1)A_{\ell-1} \\
&= \sum_{\ell=1}^{r}(n_\ell-1)A_\ell + \sum_{\ell=0}^{r-1}(n_{\ell+1}-1)A_\ell \\
&= (n_1-1)A_0 + (n_r-1)A_r + \sum_{\ell=1}^{r-1}(n_\ell + n_{\ell+1} - 2)A_\ell \\
&= \sum_{k=1}^{r-1}(n_k + n_{k+1} - 2)(a_{\hat{n}_k} - \delta_k)
\end{aligned}$$

We plug this back in to get

$$-\sum_{\ell=1}^{r}\sum_{j=1}^{n_\ell-1} B(c_j^{(\ell)}) \leq \frac{n-1}{2} - \sum_{j=1}^{n-1} B(a_\ell) + \sum_{k=1}^{r-1} a_{\hat{n}_k} + \frac{1}{2}\sum_{k=1}^{r-1}(n_k + n_{k+1} - 2)(a_{\hat{n}_k} - \delta_k),$$

from which it follows that the exponent of $T$ in the bound for $|p^{(n)}_{T,R}(y; -a, \delta_C)|$ above is

$$\varepsilon + r - 1 + d + \sum_{\ell=1}^{r}\left(\left(R\left(D(n_\ell) + \frac{n_\ell(n_\ell - 1)}{2}\right) + C(n_\ell)\right) - \sum_{k=1}^{n_\ell - 1} B(c_k^{(\ell)})\right)$$
$$= \varepsilon + d' + R\left(D(n) + \frac{n(n-1)}{2}\right) + C(n) - \sum_{j=1}^{n-1} B(a_j),$$

where

$$\begin{aligned} d' &= r - 1 + d'' + \frac{n-1}{2} + \frac{1}{2}\sum_{k=1}^{r-1}(n_k + n_{k+1} - 2)(a_{\hat{n}_k} - \delta_k) - C(n) + \sum_{\ell=1}^{r} C(n_\ell) + \sum_{k=1}^{r-1} a_{\hat{n}_k} \\ &= d'' + \frac{n-1}{2} + \frac{1}{2}\sum_{k=1}^{r-1}(n_k + n_{k+1} - 2)(a_{\hat{n}_k} - \delta_k) + \sum_{k=1}^{r-1} a_{\hat{n}_k} - \frac{1}{2}\sum_{1 \le k < m \le r} n_k n_m \end{aligned}$$

and

$$d'' = d - R \cdot \left(D(n) - \sum_{\ell=1}^{r} D(n_\ell) + \sum_{1 \le k < m \le r} n_k n_m\right) = -\sum_{k=1}^{r-1}\big(\delta_k + (n_k + n_{k+1})(a_{\hat{n}_k} - \delta_k)\big).$$

Hence,

$$\begin{aligned} d' &= \frac{n-1}{2} - \sum_{k=1}^{r-1}\big(\delta_k + (n_k + n_{k+1})(a_{\hat{n}_k} - \delta_k)\big) \\ &\qquad + \frac{1}{2}\sum_{k=1}^{r-1}(n_k + n_{k+1} - 2)(a_{\hat{n}_k} - \delta_k) + \sum_{k=1}^{r-1} a_{\hat{n}_k} - \frac{1}{2}\sum_{1 \le k < m \le r} n_k n_m \\ &= \frac{n-1}{2} - \sum_{k=1}^{r-1}(n_k + n_{k+1})(a_{\hat{n}_k} - \delta_k) + \frac{1}{2}\sum_{k=1}^{r-1}(n_k + n_{k+1})(a_{\hat{n}_k} - \delta_k) - \frac{1}{2}\sum_{1 \le k < m \le r} n_k n_m \\ &= \frac{1}{2}\left(n - 1 - \sum_{k=1}^{r-1}(n_k + n_{k+1})(a_{\hat{n}_k} - \delta_k) - \sum_{1 \le k < m \le r} n_k n_m\right). \end{aligned}$$

Note that if $r = 2$ and $n_1 = m$ and $\delta_1 = \delta$, then this expression becomes

$$\frac{n-1}{2} - \frac{n}{2}(a_m - \delta) - \frac{m(n-m)}{2},$$

which agrees with (10.4.1).

Therefore, to complete the proof, we need only show that $n - 1 - \sum_{1 \le k < m \le r} n_k n_m \le 0$. Indeed,

$$n - 1 - \sum_{1 \le k < m \le r} n_k n_m = n - 1 - \sum_{k=1}^{r-1}\sum_{m=k+1}^{r} n_k n_m = n - 1 - \sum_{k=1}^{r-1} n_k(n - \hat{n}_k) \le n - 1 - n_1(n - n_1) \le 0,$$

(with the final inequality being equality if and only if $n_1 = 1$ or $n_1 = n - 1$), as desired. □

**Remark 10.5.6.** A critical step in the proof of (10.0.4) (either in the case of single residues, as is proved in Section 10.4 or higher-order residues, as in Section 10.5) is to shift the lines of integration in the variables $\hat{\alpha}_m$ or $\hat{\beta}_j$. A feature of this work that is quite different from the case of GL(4) as proved in [Goldfeld et al.

2021b], is that no poles are crossed when making these shifts. This represents a major simplification. Recall from the discussion of Section 8.4 that in the case of $n = 4$ there are two fundamentally different types of single residues, two different types of double residues and a triple residue. As it turned out, when making the additional shift for each of the single and double residues, one ends up with five *additional* residue terms. Taken all together, it was necessary to complete the analysis of writing down explicitly what the residues are in terms of Gamma functions, finding the exponential zero set, applying Stirling's formula and then obtaining a bound for ten(!) separate residues integrals. All of this was in addition to performing these steps for the shifted $p^{(4)}_{T,R}$-term.

## Appendix: Auxiliary results

In an effort to avoid obstructing the flow of the argument in the main body of this paper, we will include here the many technical results that are used throughout. We remind the reader that the notational conventions that are used throughout the paper and this appendix are given in Definition 2.1.1.

**Lemma A.1.** *Suppose that $w = w_{(n_1,n_2,\dots,n_r)}$ for some composition $n = n_1 + \cdots + n_2$ with $r \geq 2$. Then, if $y = (y_1, \dots, y_{n-1})$, it follows that $wyw^{-1}$ is equal to*

$$\Bigg(\underbrace{y_{n-\hat{n}_1+1}, y_{n-\hat{n}_1+2}, \dots, y_{n-1}}_{n_1-1 \text{ terms}}, \Bigg(\prod_{k=n-\hat{n}_2}^{n-1} y_k\Bigg)^{-1}, \dots,$$

$$\Bigg(\prod_{k=n-\hat{n}_i}^{n-\hat{n}_{i-2}-1} y_k\Bigg)^{-1}, \underbrace{y_{n-\hat{n}_i+1}, y_{n-\hat{n}_i+2}, \dots, y_{n-\hat{n}_{i-1}-1}}_{n_i-1 \text{ terms}}, \Bigg(\prod_{k=n-\hat{n}_{i+1}}^{n-\hat{n}_{i-1}-1} y_k\Bigg)^{-1}, \dots,$$

$$\Bigg(\prod_{k=1}^{n-\hat{n}_{s-2}-1} y_k\Bigg)^{-1}, \underbrace{y_{n-\hat{n}_1+1}, y_{n-\hat{n}_1+2}, \dots, y_{n-1}}_{n_r-1 \text{ terms}}\Bigg).$$

*In particular,*

$$\|wyw^{-1}\|^{a_k} = \prod_{i=1}^{r}\prod_{j=1}^{n_i} y_{n-\hat{n}_i+j}^{-a_{\hat{n}_{i-1}}+a_{\hat{n}_{i-1}+j}-a_{\hat{n}_i}}.$$

*Proof.* Let $w = w_{(n_1,n_2,\dots,n_r)}$ as above. In order to carefully analyze $y' = wyw^{-1}$, we define $x_i := \prod_{j=1}^{i} y_j$. This notation implies that $y = \mathrm{diag}(x_{n-1}, x_{n-2}, \dots, x_1, 1)$. Now, let us think of the matrix $y$ as a block diagonal of the form $y = \mathrm{diag}(A_1, A_2, \dots, A_r)$, where

$$A_i = \mathrm{diag}(x_{n-\hat{n}_{i-1}-1}, x_{n-\hat{n}_{i-1}-2}, \dots, x_{n-\hat{n}_{i-1}-n_i}) \in \mathrm{GL}(n_i, \mathbb{R}).$$

Thus,

$$y' = wyw^{-1} = \mathrm{diag}(A_r, A_{r-1}, \dots, A_1) = x_{n-n_1}\,\mathrm{diag}(B_r, B_{r-1}, \dots, B_1).$$

Let $1 \leq i \leq r$ and $0 \leq j \leq n_i - 1$ and set

$$z_{\hat{n}_{i-1}+j} := \frac{x_{n-\hat{n}_i+j}}{x_{n-n_1}}.$$

Then $(y'_1, y'_2, \dots, y'_{n-1})$, the Iwasawa $y$-variables of $y'$ satisfy $y'_i = z_i/z_{i-1}$. For $j \neq 0$, therefore, we see

$$y'_{\hat{n}_{i-1}+j} = \frac{x_{n-\hat{n}_i+j}}{x_{n-\hat{n}_i+j-1}} = \frac{\prod_{k=1}^{n-\hat{n}_i+j} y_k}{\prod_{\ell=1}^{n-\hat{n}_i+j-1} y_\ell} = y_{n-\hat{n}_i+j},$$

and, for $j=0$,

$$y'_{\hat{n}_i} = \frac{x_{n-\hat{n}_{i+1}}}{x_{n-\hat{n}_{i+1}-1}} = \frac{x_{n-\hat{n}_{i+1}}}{x_{n-\hat{n}_i+n_i-1}} = \frac{\prod_{k=1}^{n-\hat{n}_{i+1}} y_k}{\prod_{\ell=1}^{n-\hat{n}_{i-1}-1} y_\ell} = \left( \prod_{k=1}^{n_i+n_{i+1}-1} y_{n-\hat{n}_{i+1}+k} \right)^{-1},$$

from which the statement of the lemma follows directly. □

**Definition A.2.** We say that $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ is in *$j$-general position* if the set

$$\left\{ \sum_{k \in J} \alpha_k \;\middle|\; J \subseteq \{1, \dots, n\},\ \#J = j \right\}$$

consists of $\binom{n}{j}$ distinct elements. We say that $\alpha$ is *in general position* if it is in $j$-general position for each $j = 1, \dots, n-1$.

**Lemma A.3.** *Suppose that there exists $\varepsilon > 0$ such that for each $j = 1, \dots, n-1$, the real part of $s_j$ is bounded by at least $\varepsilon$ from any integer. Assume that $\alpha$ is in $j$-general position*, $\mathrm{Re}(\alpha_i) = 0$ *for each $i = 1, \dots, n-1$ , and $r_j \in \mathbb{Z}_{\geq 0}$. Assume that*

$$\mathrm{Im}(\alpha_1) \geq \mathrm{Im}(\alpha_2) \geq \cdots \geq \mathrm{Im}(\alpha_n),$$

*and let $I_j = [-\mathrm{Im}(\alpha_1 + \cdots + \alpha_j), -\mathrm{Im}(\alpha_n + \cdots + \alpha_{n-j+1})]$. If $r_j \geq 2$, then*

$$\int_{\substack{\mathrm{Re}(s_j)=\sigma_j \\ \mathrm{Im}(s_j) \in I_j}} \prod_{\substack{J \subseteq \{1,\dots,n\} \\ \#J=j}} \left| s_j + \sum_{k \in J} \alpha_k \right|^{-r_j} ds_j \ll \sum_{\substack{L \subseteq \{1,\dots,n\} \\ \#L=j}} \prod_{\substack{K \subseteq \{1,\dots,n\} \\ \#K=j \\ K \neq L}} \left( 1 + \left| \sum_{\ell \in L} \alpha_\ell - \sum_{k \in K} \alpha_k \right| \right)^{-r_j}.$$

*If $r_j = 1$ there is an extra power of $\varepsilon$ in the exponent* (*in which case the implicit constant will depend on $\varepsilon$*), *and if $r_j = 0$, the integral is bounded by*

$$\left( 1 + \sum_{k=1}^{j} \alpha_k - \sum_{\ell=1}^{j} \alpha_{n+1-\ell} \right).$$

**Remark A.4.** The implicit $\ll$–constant depends on $\sigma_j$, but in applications this will always be bounded.

*Proof.* The bound in the case of $r_j = 0$ is obvious, so we may assume henceforth that $r_j \geq 1$. Consider the set

$$\mathcal{A}_j := \left\{ \sum_{k \in J} \alpha_k \;\middle|\; J \subseteq \{1, \dots, n\},\ \#J = j \right\}.$$

For a fixed choice $\alpha$ in $j$-general position, let $A_1$ be the element of $\mathcal{A}_j$ that has the greatest imaginary part, $A_2$ the next greatest imaginary part and so on. Hence $-\mathrm{Im}(A_1) < -\mathrm{Im}(A_2) < \cdots < -\mathrm{Im}(A_{\binom{n}{j}})$.

Write $s_j=\sigma_j+it_j$. Note that $I_j=[-\operatorname{Im}(A_1),-\operatorname{Im}(A_{\binom{n}{j}})]$. Upon applying Lemma A.3 from [Goldfeld et al. 2021b], one obtains the bound

$$\int_{I_j}\prod_{\substack{J\subseteq\{1,\dots,n\}\\ \#J=j}}\Big|s_j+\sum_{k\in J}\alpha_k\Big|^{-r_j}\,ds_j \ll (1+\operatorname{Im}(A_1)-\operatorname{Im}(A_{\binom{n}{j}}))^{\varepsilon}\prod_{k=1}^{\binom{n}{j}-1}(1+\operatorname{Im}(A_k-A_{k+1}))^{-r_j}.$$

This is one of the possible summands on the right-hand side of the statement of the lemma. Hence, regardless of the specific ordering which may arise for the given choice of $\alpha$, the claim follows. □

**Lemma A.5.** *Let $a\in\mathbb{R}$. Then*

$$\max\{0,2(\lceil a\rceil-a)-1\}-\lceil a\rceil\le\begin{cases}-\lceil a\rceil & \text{if } a\in\big(\lceil a\rceil-\frac12,\lceil a\rceil\big],\\ -\lfloor a\rfloor-2(a-\lfloor a\rfloor) & \text{if } a\in\big(\lfloor a\rfloor,\lfloor a\rfloor+\frac12\big].\end{cases}$$

*Proof.* First, let us assume that $a\in\big(\lceil a\rceil-\frac12,\lceil a\rceil\big]$. Then $\lceil a\rceil-a<\frac12$; hence

$$\max\{0,2(\lceil a\rceil-a)-1\}-\lceil a\rceil=-\lceil a\rceil.$$

On the other hand, assuming that $a\in\big(\lfloor a\rfloor,\lfloor a\rfloor+\frac12\big]$, we see that

$$\max\{0,2(\lceil a\rceil-a)-1\}-\lceil a\rceil=\lceil a\rceil-2a-1=\lfloor a\rfloor-2a=-\lfloor a\rfloor-2(a-\lfloor a\rfloor),$$

as claimed. □

**Lemma A.6.** *Suppose that $a_1,\dots,a_n\in\mathbb{R}_{>0}$. Let*

$$B(a):=\begin{cases}0 & \text{if } a<0,\\ \lfloor a\rfloor+2(a-\lfloor a\rfloor) & \text{if } 0<\lfloor a\rfloor+\varepsilon<a\le\lfloor a\rfloor+\frac12,\\ \lceil a\rceil & \text{if } \frac12<\lceil a\rceil-\frac12\le a<\lceil a\rceil-\varepsilon.\end{cases}$$

*Then, for any $\delta_m\in\mathbb{Z}_{\ge0}$ with $0<a_m-\delta_m$,*

$$\sum_{j=1}^{m-1}B\Big(a_j-\frac{j}{m}(a_m-\delta_m)\Big)+\sum_{j=1}^{n-m-1}B\Big(a_{m+j}-\frac{n-m-j}{n-m}(a_m-\delta_m)\Big)\ge\Big(\sum_{j=1}^{n-1}B(a_j)\Big)-\frac{n-2}{2}(a_m-\delta_m+1)-B(a_m).$$

*Proof.* We consider first the case of $r-\frac12\le a_j<r$ for some $r\in\mathbb{Z}$ and all $j=1,2,\dots,n-1$. For any $a\in\mathbb{R}$, note that

$$a\le B(a)\le a+\tfrac12;\tag{A.7}$$

hence

$$\sum_{j=1}^{m-1}B\Big(a_j-\frac{j}{m}(a_m-\delta_m)\Big)\ge\Big(\sum_{j=1}^{m-1}a_j\Big)-\frac{m-1}{2}(a_m-\delta_m)\ge\Big(\sum_{j=1}^{m-1}B(a_j)-\frac12\Big)-\frac{m-1}{2}(a_m-\delta_m)$$
$$=\Big(\sum_{j=1}^{m-1}B(a_j)\Big)-\frac{m-1}{2}(a_m-\delta_m+1).$$

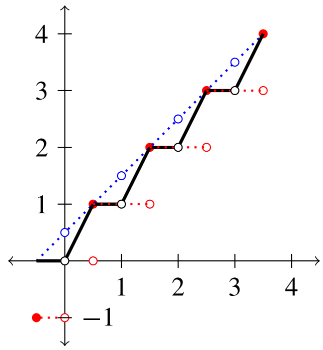


**Figure 1.** Comparing graph of $B(x)$ (thick black) to $B_4(x)$ (dotted red) and $B_3(x)$ (dotted blue) bounds.

Combining this with the other terms (which are easily shown to satisfy the analogous bound), the desired result is immediate. □

**Remark A.8.** The function $B(x)$ appears prominently in Theorem 10.0.1 and is critical in bounding the geometric side of the Kuznetsov trace formula. Its graph is shown in Figure 1 in comparison to two other functions $B_4$ and $B_3$.

In the case of GL(4), the function $B_4$ appears [Goldfeld et al. 2021b] (see Theorem 4.0.1) as a bound for the $p_{T,R}$ function. Indeed, making necessary adjustments due to a different choice of normalization factors (see Remark 1.6.2), the result of [loc. cit.] is that

$$|p^{(4)}_{T,R}(1;-a)| \ll T^{\varepsilon+27R+12-\sum_{i=1}^{3} B_4(a_i)}.$$

Theorem 10.0.1 establishes the same result but with $B_4$ replaced by $B$. Although the improvement is slight, we remark that it is essential in Lemma A.6 and evidently allows the inductive method of the present paper to lead to the same asymptotic orthogonality relation as was established directly in [loc. cit.].

With a bit of work, one can show that the function $B_3$, also graphed in Figure 1, appeared in [Goldfeld and Kontorovich 2013] as a bound for

$$|p^{(3)}_{T,R}(1;-a)| \ll T^{\varepsilon+6R+7-\sum_{i=1}^{2} B_3(a_i)}.$$

Although this looks to be an improvement on our result here, the method of [Goldfeld and Kontorovich 2013] contained an error which the present method (and the method of [Goldfeld et al. 2021b]) corrects.

**Lemma A.9.** *Let $\varepsilon > 0$. Then for any $\rho \in \frac{1}{2}+\mathbb{Z}$ there exists $0 < \varepsilon' < \frac{1}{2}$ sufficiently small such that, setting $\delta = 2\varepsilon'/n^2$, if $a = (a_1, \ldots, a_{n-1})$, where*

$$a_j := \rho + \frac{j(n-j)}{2}(1+\delta),$$

*and, for $w = w_{(n_1,\ldots,n_r)}$, $b(a,w) = b = (b_1, \ldots, b_{n-1})$, where*

$$b_{n-\hat{n}_i+j} := a_{\hat{n}_{i-1}} - a_{\hat{n}_{i-1}+j} + a_{\hat{n}_i} \pm \frac{\delta}{2},$$

(*meaning that $a$ and $b$ satisfy* (6.3.1) *and* (6.3.2), *respectively*), *then*, *letting $B$ be the function defined in Theorem 9.0.2*,

$$\sum_{j=1}^{n-1}(B(a_j)+B(b_j)) \geq \left\lfloor \frac{n-1}{2} \right\rfloor + n\rho + \Phi(n_1,\ldots,n_r) - \varepsilon,$$

*where*

$$\Phi(n_1,\ldots,n_r) := \sum_{k=1}^{r-1}(n_k+n_{k+1})\frac{(n-\hat{n}_k)\hat{n}_k}{2}.$$

*Proof.* We first note that although the bound $B(x) \geq x$ holds for any $x \in \mathbb{R}$, for any $\varepsilon > 0$, $B(x) \geq x + \frac{1}{2} - \varepsilon$ provided that $x$ is sufficiently close to a half integer. Lemma A.11 (as justified in Remark A.12) asserts that if $n$ is odd then $n-1$ elements from the set of all the possible values of $a_k$ and $b_k$ are indeed within $\varepsilon$ of a half integer, and if $n$ is even then $n-2$ of values have this property. Hence,

$$\sum_{k=1}^{n-1}(B(a_k)+B(b_k)) \geq \left\lfloor \frac{n-1}{2} \right\rfloor + \sum_{k=1}^{n-1}(a_k+b_k) - \varepsilon. \tag{A.10}$$

Since $b_{n-\hat{n}_i+j} = a_{\hat{n}_{i-1}} - a_{\hat{n}_{i-1}+j} + a_{\hat{n}_i} \pm \frac{\delta}{2}$, we see that

$$\sum_{j=1}^{n_i}(b_{n-\hat{n}_i+j} + a_{\hat{n}_{i-1}+j}) \sim n_i(a_{\hat{n}_{i-1}} + a_{\hat{n}_i}).$$

Therefore, summing over $i$, we see (making use of the fact that $a_0 = a_n = 0$) that

$$\begin{aligned}\sum_{k=1}^{n-1}(b_k+a_k) &= \sum_{i=1}^{r} n_i(a_{\hat{n}_{i-1}} + a_{\hat{n}_i}) = \sum_{i=1}^{r-1}(n_i+n_{i+1})a_{\hat{n}_i}\\ &= \sum_{k=1}^{r-1}(n_k+n_{k+1})\left(\rho + \frac{(n-\hat{n}_k)\hat{n}_k}{2} + \varepsilon'\right) \sim \rho(2n-n_1-n_r) + \underbrace{\sum_{k=1}^{r-1}(n_k+n_{k+1})\frac{(n-\hat{n}_k)\hat{n}_k}{2}}_{=:\Phi(n_1,\ldots,n_r)}.\end{aligned}$$

Combining this with (A.10), the desired result is now immediate. ☐

**Lemma A.11.** *Let $C = (n_1,\ldots,n_r)$ be a composition of $n$ with $r \geq 2$. Suppose that $\rho \in \frac{1}{2} + \mathbb{Z}$. Set $a_0 := 0$, $a_n := 0$ and for each $1 \leq k \leq n-1$ we have $a_k := \rho + \frac{k(n-k)}{2}$ and for each $1 \leq i \leq r$ and $1 \leq j \leq n_i$ we let $b_{i,j} := a_{\hat{n}_{i-1}} - a_{\hat{n}_{i-1}+j} + a_{\hat{n}_i}$.*

*Then*

$$\#\{k \mid a_k \notin \mathbb{Z}\} + \#\{(i,j) \mid b_{i,j} \notin \mathbb{Z}\} = \begin{cases} 2n - n_1 - n_r - 1 & \text{if } n \text{ is odd},\\ \frac{n}{2} - 1 + \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_r}{2} \right\rfloor + \sum_{i=2}^{r-1}\left\lceil \frac{n_i}{2} \right\rceil & \text{if } n \text{ is even}.\end{cases}$$

**Remark A.12.** Note that the quantity given in Lemma A.11 in the case of $n$ odd is $2n-n_1-n_r-1 \geq n-1$ for any composition $C$ (with equality precisely when $r=2$). If $n$ is even then

$$\frac{n}{2} - 1 + \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_r}{2} \right\rfloor + \sum_{i=2}^{r-1}\left\lceil \frac{n_i}{2} \right\rceil \geq \frac{n}{2} + \frac{n_1}{2} + \frac{n_r}{2} - 2 + \sum_{i=2}^{r-1}\frac{n_i}{2} = n-2.$$

Equality in this case occurs precisely when $n_1$ and $n_r$ are both odd and all other $n_i$ are even.

*Proof.* For notational purposes, set

$$A(n) := \#\{1 \le k \le n-1 \mid a_k \notin \mathbb{Z}\},$$
$$B(C) := \#\{(i,j),\ 1 \le i \le r,\ 1 \le j \le n_i \mid b_{i,j} \notin \mathbb{Z}\}.$$

We first consider the case of $n$ odd, for which $\frac{k(n-k)}{2} \in \mathbb{Z}$ for all integers $k$. Therefore, $A(n) = n-1$. As for $B(C)$, note that $b_{i,j}$ is equal to $\rho$ plus an integer as long as $i \neq 1, r$. Otherwise, $b_{1,j}, b_{r,j} \in \mathbb{Z}$. Hence $B(C) = n - n_1 - n_r$.

In the case of $n$ even, $\frac{k(n-k)}{2} \in \mathbb{Z}$ exactly when $k$ is even. Hence $A(n) = \frac{n}{2} - 1$. To the end of finding $B(C)$, we introduce the notation

$$B_i(C) := \#\{1 \le j \le n_i \mid b_{i,j} \notin \mathbb{Z}\},$$

for which it is clear that $B(C) = \sum_{i=1}^{r} B_i(C)$.

The cardinality of $B_i(C)$ depends, obviously, on the integrality of $b_{i,j}$. To determine this, we first assume that $i = 1$. Then

$$b_{1,j} = -\frac{j(n-j)}{2} + \frac{n_1(n-n_1)}{2}.$$

Therefore (since $n$ is even), $b_{i,j} \in \mathbb{Z}$ if and only if $j \equiv n_1 \pmod 2$. This implies that

$$B_1(C) := \begin{cases} \frac{n_1-1}{2} & \text{if } n_1 \text{ is odd,} \\ \frac{n_1}{2} & \text{if } n_1 \text{ is even,} \end{cases}$$

or more concisely, $\#B_1(C) = \lfloor \frac{n_1}{2} \rfloor$. The determination of $B_r(C)$ is similar: $\#B_r(C) = \lfloor \frac{n_r}{2} \rfloor$.

For $1 < i < r$, we see that

$$\begin{aligned} b_{i,j} &= \rho + \frac{\hat{n}_{i-1}(n-\hat{n}_{i-1})}{2} - \frac{(\hat{n}_{i-1}+j)(n-\hat{n}_{i-1}-j)}{2} + \frac{(\hat{n}_{i-1}+n_i)(n-\hat{n}_{i-1}-n_i)}{2} \\ &= \rho + \hat{n}_{i-1}(n-\hat{n}_{i-1}-n_i) - \frac{(\hat{n}_{i-1}+j)(n-\hat{n}_{i-1}-j)}{2} + \frac{n_i(n-n_i)}{2} \\ &\equiv \frac{1}{2} + \frac{(\hat{n}_{i-1}+j)(n-\hat{n}_{i-1}-j)}{2} + \frac{n_i(n-n_i)}{2} \pmod{\mathbb{Z}}. \end{aligned}$$

We see again that the integrality of $b_{i,j}$ depends on the parity of $n_i$. If $n_i$ is odd,

$$B_i(C) = \#\left\{1 \le j \le n_i \;\middle|\; \frac{(\hat{n}_{i-1}+j)(n-\hat{n}_{i-1}-j)}{2} \notin \mathbb{Z}\right\},$$

and if $n_i$ is even,

$$B_i(C) = \#\left\{1 \le j \le n_i \;\middle|\; \frac{(\hat{n}_{i-1}+j)(n-\hat{n}_{i-1}-j)}{2} \in \mathbb{Z}\right\}.$$

One can check, arguing case by case as above, that in any event, the answer is $B_i(C) = \lceil \frac{n_i}{2} \rceil$. ☐

**Lemma A.13.** *Suppose that* $(n_1, \ldots, n_r) \in \mathbb{C}^r$. *The function*

$$\Phi(n_1, \ldots, n_r) := \sum_{k=1}^{r-1} (n_k + n_{k+1}) \frac{(n_1 + \cdots + n_k)(n_{k+1} + \cdots + n_r)}{2}$$

*is invariant under permutations, i.e., for any* $\sigma \in S_r$, *we have* $\Phi(n_1, \ldots, n_r) = \Phi(n_{\sigma(1)}, \ldots, n_{\sigma(r)})$. *In particular, if* $P = n_1 + \cdots + n_r$ *is a partition of* $n$ *then* $\Phi(P) := \Phi(n_1, \ldots, n_r)$ *is well-defined. Moreover, among all partitions* $P$ *of* $n$ (*with* $r \geq 2$),

$$\Phi(P) \geq \Phi(n-1, 1) = \Phi(1, n-1) = \frac{n(n-1)}{2}.$$

*Proof.* Suppose that $n = n_1 + \cdots + n_r = m_1 + \cdots + m_r$, where

$$m_j = \begin{cases} n_j & \text{if } j \neq k, k+1, \\ n_{k+1} & \text{if } j = k, \\ n_k & \text{if } j = k+1. \end{cases}$$

Then one can show by an elementary (albeit tedious) computation that $\Phi(n_1, \ldots, n_r) = \Phi(m_1, \ldots, m_r)$. In other words, $\Phi$ is invariant under any transposition $\tau \in S_r$, hence invariant under all of $S_r$.

Suppose that $n = n_1 + \cdots + n_r$. If $n_k = n'_k + n''_k$ for some $1 \leq k \leq r$, then one shows via a straightforward computation that

$$\Phi(n_1, \ldots, n_{k-1}, n'_k, n''_k, n_{k+1}, \ldots, n_{r-1}) - \Phi(n_1, \ldots, n_r) = \frac{n_k n'_k n''_k}{2}.$$

If $n = n_1 + \cdots + n_r$ with $r > 2$, it then follows, setting $n_0 := \min\{n_1, n_2, \ldots, n_r\}$, that

$$\Phi(n_1, \ldots, n_r) > \Phi(n_0, n - n_0) = \frac{n n_0 (n - n_0)}{2}.$$

Among all $1 \leq n_0 \leq \frac{n}{2}$, the right-hand side is minimized when $n_0 = 1$. □

Recall that

$$\Gamma_R(z) := \frac{\Gamma\big(\frac{1}{2}\big(\frac{1}{2} + R + z\big)\big)}{\Gamma(z)},$$

as defined at the beginning of Section 8.

**Lemma A.14.** *If* $\delta \in \mathbb{Z}$ *and* $\beta \in i\mathbb{R}$ *are fixed, then the function* $\Gamma_R(\beta + z)\Gamma_R(-\beta - z)\Gamma(-\beta - z - \delta)$ *is holomorphic for all* $z$ *with* $|\mathrm{Re}(z)| < R$.

*Proof.* The fact that $|z| < R$ implies that $\Gamma_R(\pm z)$ is holomorphic is immediate, so the only question is what happens at the (simple) poles of $\Gamma(-\beta - z - \delta)$. But these occur at $z = -\beta + k$ for some integer $k$ which corresponds to zeros of $\Gamma_R(\beta + z)$ or $\Gamma_R(-\beta - z)$. □

**Lemma A.15.** *For* $\delta \in \mathbb{Z}$ *fixed and* $z, \beta \in \mathbb{C}$ *and* $|\mathrm{Re}(z + \beta) + \delta| < R$, *we have the bound*

$$\Gamma_R(\beta + z)\Gamma_R(-\beta - z)\Gamma(-\beta - z - \delta) \asymp (1 + |\mathrm{Im}(\beta + z)|)^{R - \mathrm{Re}(\beta + z) - \delta}.$$

*Proof.* This follows immediately from the Stirling bound $|\Gamma(\sigma + it)| \sim \sqrt{2\pi}|t|^{\sigma - \frac{1}{2}} e^{\pi|t|/2}$. □

**Definition A.16.** Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ be Langlands parameters satisfying $\hat{\alpha}_n = 0$. Let $n = n_1 + \cdots + n_r$ be a partition of $n$ with $n_1, \ldots, n_r \in \mathbb{Z}_+$. Then for each $\ell = 1, \ldots, r$ we define $\alpha^{(\ell)} := (\alpha^{(\ell)}_1, \ldots, \alpha^{(\ell)}_{n_\ell}) \in \mathbb{C}^{n_\ell}$, where

$$\alpha^{(\ell)}_j := \alpha_{\hat{n}_{\ell-1} + j} - \frac{1}{n_\ell}(\hat{\alpha}_{\hat{n}_\ell} - \hat{\alpha}_{\hat{n}_{\ell-1}}), \quad |\alpha^{(\ell)}|^2 := \sum_{j=1}^{n_\ell} (\alpha^{(\ell)}_j)^2.$$

**Remark A.17.** Note that $\sum_{j=1}^{n_\ell}\alpha_j^{(\ell)}=0$ for each $\ell$. In particular $n_\ell=1$ implies $\alpha_1^{(\ell)}=0$.

**Lemma A.18.** *We have* $|\alpha|^2=\sum_{i=1}^{n}\alpha_i^2=\sum_{\ell=1}^{r}\big(|\alpha^{(\ell)}|^2+\frac{1}{n_\ell}(\hat{\alpha}_{\hat{n}_\ell}-\hat{\alpha}_{\hat{n}_{\ell-1}})^2\big)$.

*Proof.* Computing directly, and using the fact that $\sum_{j=1}^{n_\ell}\alpha_j^{(\ell)}=0$, we find that

$$\begin{aligned}\sum_{j=1}^{n}\alpha_j^2&=\sum_{\ell=1}^{r}\sum_{j=1}^{n_\ell}\alpha_{\hat{n}_{\ell-1}+j}^2=\sum_{\ell=1}^{r}\sum_{j=1}^{n_\ell}\left(\alpha_j^{(\ell)}+\frac{1}{n_\ell}(\hat{\alpha}_{\hat{n}_\ell}-\hat{\alpha}_{\hat{n}_{\ell-1}})\right)^2\\&=\sum_{\ell=1}^{r}\sum_{j=1}^{n_\ell}\left((\alpha_j^{(\ell)})^2+\frac{2}{n_\ell}\alpha_j^{(\ell)}(\hat{\alpha}_{\hat{n}_\ell}-\hat{\alpha}_{\hat{n}_{\ell-1}})+\frac{1}{n_\ell^2}(\hat{\alpha}_{\hat{n}_\ell}-\hat{\alpha}_{\hat{n}_{\ell-1}})^2\right)\\&=\sum_{\ell=1}^{r}\left(|\alpha^{(\ell)}|^2+\frac{1}{n_\ell}(\hat{\alpha}_{\hat{n}_\ell}-\hat{\alpha}_{\hat{n}_{\ell-1}})^2\right),\end{aligned}$$

as claimed. □

**Lemma A.19.** *Suppose that* $n\geq 2$ *and* $\alpha_1,\alpha_2,\dots,\alpha_n\in\mathbb{C}$ *satisfies* $\alpha_1+\alpha_2+\cdots+\alpha_n=0$. *Set* $\hat{\alpha}_k=\sum_{j=1}^{k}\alpha_j$ *for fixed* $k\in\{1,2,\dots,n\}$, *and define* $\beta_j:=\alpha_j-\frac{1}{k}\hat{\alpha}_k,\ \gamma_j:=\alpha_{j+k}+\frac{1}{n-k}\hat{\alpha}_k$. *Then*

$$\sum_{i=1}^{n}\alpha_i^2=\sum_{i=1}^{k}\beta_i^2+\sum_{i=1}^{n-k}\gamma_i^2+\frac{n}{k(n-k)}\hat{\alpha}_k^2.$$

*Proof.* This is easily deduced as a special case of Lemma A.18 in the case that $r=2,\ n_1=k,\ n_2=n-k$, $\beta=\alpha^{(1)}$ and $\gamma=\alpha^{(2)}$. □

**Lemma A.20.** *We continue the notation of Lemma A.18. Then*

$$\begin{aligned}\prod_{1\leq i\neq j\leq n}\Gamma_R(\alpha_i-\alpha_j)=&\prod_{\ell=1}^{r}\Bigg(\prod_{1\leq i,j\leq n_\ell}\Gamma_R(\alpha_i^{(\ell)}-\alpha_j^{(\ell)})\Bigg)\\&\cdot\prod_{1\leq k<m\leq r}\prod_{i=1}^{n_k}\prod_{j=1}^{n_m}\prod_{\epsilon\in\{\pm1\}}\Gamma_R\left(\epsilon\left(\alpha_i^{(k)}-\alpha_j^{(m)}+\frac{1}{n_k}(\hat{\alpha}_{\hat{n}_k}-\hat{\alpha}_{\hat{n}_{k-1}})-\frac{1}{n_m}(\hat{\alpha}_{\hat{n}_m}-\hat{\alpha}_{\hat{n}_{m-1}})\right)\right).\end{aligned}$$

*Proof.* Note that if $k\neq m$, then for any $1\leq i\leq n_k$ and $1\leq j\leq n_m$,

$$\alpha_{\hat{n}_{k-1}+i}-\alpha_{\hat{n}_{m-1}+j}=\alpha_i^{(k)}-\alpha_j^{(m)}+\frac{1}{n_k}(\hat{\alpha}_{\hat{n}_k}-\hat{\alpha}_{\hat{n}_{k-1}})-\frac{1}{n_m}(\hat{\alpha}_{\hat{n}_m}-\hat{\alpha}_{\hat{n}_{m-1}}),$$

and for any $1\leq i\neq j\leq n_\ell$ we have $\alpha_{\hat{n}_{\ell-1}+i}-\alpha_{\hat{n}_{\ell-1}+j}=\alpha_i^{(\ell)}-\alpha_j^{(\ell)}$. This immediately implies the desired formula. □

**Lemma A.21.** *Suppose that* $(\beta_1,\dots,\beta_r)$ *satisfies* $\hat{\beta}_r=0$. *Suppose that* $n=n_1+\cdots+n_r$ *and set* $\hat{n}_k=\sum_{j=1}^{k}n_j$. *Then* $\sum_{1\leq k<m\leq r}\sum_{i=1}^{n_k}\sum_{j=1}^{n_m}\big(\beta_k/n_k-\beta_m/n_m\big)=\sum_{j=1}^{r-1}(n_j+n_{j+1})\hat{\beta}_j$.

*Proof.* We calculate

$$\sum_{1\le k<m\le r}\sum_{i=1}^{n_k}\sum_{j=1}^{n_m}\left(\frac{\beta_k}{n_k}-\frac{\beta_m}{n_m}\right)=\sum_{m=2}^{r}\sum_{k=1}^{m-1}\sum_{i=1}^{n_k}\left(n_m\frac{\beta_k}{n_k}-\beta_m\right)=\sum_{m=2}^{r}\sum_{k=1}^{m-1}(n_m\beta_k-n_k\beta_m)$$
$$=\sum_{m=2}^{r}(n_m\hat{\beta}_{m-1}-\hat{n}_{m-1}\beta_m)=\sum_{m=2}^{r}(n_m\hat{\beta}_{m-1}-\hat{n}_{m-1}(\hat{\beta}_m-\hat{\beta}_{m-1}))$$
$$=\sum_{m=2}^{r}((n_m+\hat{n}_{m-1})\hat{\beta}_{m-1}-\hat{n}_{m-1}\hat{\beta}_m)=\sum_{m=2}^{r}(\hat{n}_m\hat{\beta}_{m-1}-\hat{n}_{m-1}\hat{\beta}_m).$$

This final sum telescopes to give $\sum_{j=1}^{r-1}(\hat{n}_{j+1}-\hat{n}_{j-1})\hat{\beta}_j$. Since $\hat{n}_{j+1}-\hat{n}_{j-1}=n_j+n_{j+1}$, this implies the claimed result. □

The following result can be interpreted as a consequence—by counting (half) the number of gamma factors on each side of the equality—of Lemma A.20. Alternatively, proving it independent of Lemma A.20 gives further evidence that the product decomposition is correct.

**Lemma A.22.** *Let $n=n_1+\cdots+n_r$. We have $\sum_{1\le k<k'\le r}n_k\cdot n_{k'}+\sum_{k=1}^{r}\frac{n_k(n_k-1)}{2}=\frac{n(n-1)}{2}$.*

*Proof.* We use induction on $r$. If $r=1$, the formula obviously holds. Let $n=m+n_r$, where $m=n_1+\cdots+n_{r-1}$. Then, by induction,

$$\frac{n(n-1)}{2}=\frac{(m+n_r)(m+n_r-1)}{2}=\frac{m(m-1)}{2}+\frac{mn_r+(m-1)n_r}{2}+\frac{n_r^2}{2}$$
$$=\sum_{k=1}^{r-1}\frac{n_k(n_k-1)}{2}+\sum_{1\le k<k'\le r-1}n_k\cdot n_{k'}+mn_r+\frac{n_r(n_r-1)}{2}.$$

Since $mn_r=n_1n_r+n_2n_r+\cdots+n_{r-1}n_r$, it is evident that the desired formula holds. □

**Lemma A.23.** *Suppose $n=n_1+\cdots+n_r$. Then*

$$n^2+\sum_{\ell=1}^{r}\left(\frac{n_\ell(n_\ell-1)}{2}-n_\ell\hat{n}_\ell\right)=\frac{n(n-1)}{2}.$$

*Proof.* If $r=1$ the result is obviously true. Suppose that the result holds for $r=k$. Write $n=n_1+\cdots+n_k+n_{k+1}=\hat{n}_k+n_{k+1}$. Then

$$n^2+\sum_{\ell=1}^{k+1}\left(\frac{n_\ell(n_\ell-1)}{2}-n_\ell\hat{n}_\ell\right)=n^2+\sum_{\ell=1}^{k}\left(\frac{n_\ell(n_\ell-1)}{2}-n_\ell\hat{n}_\ell\right)+\frac{n_{k+1}(n_{k+1}-1)}{2}-n_{k+1}n$$
$$=n^2-\hat{n}_k^2+\left(\hat{n}_k^2+\sum_{\ell=1}^{k}\left(\frac{n_\ell(n_\ell-1)}{2}-n_\ell\hat{n}_\ell\right)+\frac{n_{k+1}(n_{k+1}-1)}{2}-n_{k+1}n\right)$$
$$=n^2-\hat{n}_k^2+\frac{\hat{n}_k(\hat{n}_k+1)}{2}+\frac{n_{k+1}(n_{k+1}-1)}{2}-n_{k+1}n$$
$$=n^2-\hat{n}_k^2+\frac{\hat{n}_k(\hat{n}_k+1)}{2}+\frac{(n-\hat{n}_k)(n-\hat{n}_k-1)}{2}-(n-\hat{n}_k),$$

which can easily be shown now to simplify to $\frac{n(n-1)}{2}$, as claimed. □

**Remark A.24.** Note that Lemmas A.22 and A.23 are equivalent provided that

$$n^2-\sum_{\ell=1}^{r}n_\ell\hat{n}_\ell=\sum_{1\le k<k'\le r}n_kn_{k'}. \tag{A.25}$$

This can be verified by expanding the left-hand side as follows:

$$n^2-\sum_{\ell=1}^{r}n_\ell\hat{n}_\ell=(n_1+\cdots+n_r)\hat{n}_r-\sum_{\ell=1}^{r}n_\ell\hat{n}_\ell=\sum_{\ell=1}^{r}(n_\ell(n-\hat{n}_\ell)-n_\ell\hat{n}_\ell)=\sum_{\ell=1}^{r}n_\ell(n-n_\ell).$$

That this final expression is equal to right-hand side of (A.25) is clear.

**Lemma A.26.** *Let $\alpha_1,\alpha_2,\dots,\alpha_n\in\mathbb{C}$ satisfy $\alpha_1+\alpha_2+\cdots+\alpha_n=0$. Set $\hat{\alpha}_k:=\sum_{j=1}^{k}\alpha_j$, and let $\beta_i$ $(1\le i\le k)$ and $\gamma_j$ $(1\le j\le n-k)$ be as in the previous lemma. We have*

$$\prod_{1\le i\ne j\le n}\Gamma_R(\alpha_i-\alpha_j)=\Bigg(\prod_{1\le i\ne j\le k}\Gamma_R(\beta_i-\beta_j)\Bigg)\Bigg(\prod_{1\le i\ne j\le n-k}\Gamma_R(\gamma_i-\gamma_j)\Bigg)\cdot\prod_{i=1}^{k}\prod_{j=1}^{n-k}\Gamma_R\Big(\beta_i-\gamma_j+\frac{n}{k(n-k)}\hat{\alpha}_k\Big)\Gamma_R\Big(\gamma_j-\beta_i-\frac{n}{k(n-k)}\hat{\alpha}_k\Big).$$

*Proof.* This is easily deduced as a special case of Lemma A.20 when $r=2,\ n_1=k,\ n_2=n-k,\ \beta=\alpha^{(1)}$ and $\gamma=\alpha^{(2)}$. □

We recall the definition of the polynomial given in Definition 1.4.1:

$$\mathcal{F}_R^{(n)}(\alpha):=\prod_{j=1}^{n-2}\prod_{\substack{K,L\subseteq(1,2,\dots,n)\\ \#K=\#L=j}}\Big(1+\sum_{k\in K}\alpha_k-\sum_{\ell\in L}\alpha_\ell\Big)^{\frac{R}{2}}.$$

Also, we remind the reader of the polynomial notation $\mathcal{P}$ given in Definition 2.1.3.

**Lemma A.27.** *Let $n=n_1+\cdots+n_r$, $\alpha$, and $\alpha^{(\ell)}$ be as in Definition A.16. Set $D(n)=\deg(\mathcal{F}_1^{(n)}(\alpha))$. Then*

$$\mathcal{F}_R^{(n)}(\alpha)=\mathcal{P}_{dR}(\alpha)\prod_{\substack{\ell=1\\ n_\ell\ne1}}^{r}\mathcal{F}_R^{(n_\ell)}(\alpha^{(\ell)}),\quad \text{where } d=d(n_1,\dots,n_r)=D(n)-\sum_{\substack{\ell=1\\ n_\ell\ne1}}^{r}D(n_\ell).$$

*Proof.* This follows from the fact that if $I,J\subseteq\{1,2,\dots,n_\ell\}$ with $\#I=\#J$ then

$$\Big(\sum_{i\in I}\alpha_i^{(\ell)}\Big)-\Big(\sum_{j\in J}\alpha_j^{(\ell)}\Big)=\Big(\sum_{i\in I}\alpha_{\hat{n}_{\ell-1}+i}\Big)-\Big(\sum_{j\in J}\alpha_{\hat{n}_{\ell-1}+j}\Big).$$

Therefore, each $\mathcal{F}_R^{(n_\ell)}(\alpha^{(\ell)})$ constitutes a unique factor of $\mathcal{F}_R^{(n)}(\alpha)$ for each $\ell=1,\dots,r$. □

**Lemma A.28.** *Suppose that $\delta\in\mathbb{Z}_{\ge0}$ and $R>\delta$. Then*

$$\mathcal{F}_R^{(n)}(\alpha)\cdot\prod_{\substack{K\subseteq\{1,2,\dots,n\}\\ \#(K\cap\{1,2,\dots,m\})\ne m-1\\ \#K=m}}\Bigg(\Big(\sum_{i\in K}\alpha_i\Big)-\hat{\alpha}_m-\delta\Bigg)_\delta^{-1}\ll\mathcal{F}_R^{(m)}(\beta)\cdot\mathcal{F}_R^{(n-m)}(\gamma)\cdot\mathcal{P}_d(\alpha),$$

*where $d=R(D(n)-D(m)-D(n-m))-\delta\big(\binom{n}{m}-m(n-m)-1\big)$.*

*Proof.* Let $M := \{1, 2, \ldots, m\}$. Then

$$\#\{K \subseteq \{1, 2, \ldots, n\} \mid \#K = m,\ \#(K \cap M) \neq 0, 1\} = \binom{n}{m} - m(n-m) - 1.$$

From the definition of $\mathcal{F}_R^{(n)}(\alpha)$ given in Definition 1.4.1, we see that for each such $K$ there are factors

$$\Bigl(1 + \sum_{i \in K} \alpha_i - \sum_{j \in M} \alpha_j\Bigr)^{R/2} \Bigl(1 - \sum_{i \in K} \alpha_i + \sum_{j \in M} \alpha_j\Bigr)^{R/2}$$

of $\mathcal{F}_R^{(n)}(\alpha)/[\mathcal{F}_R^{(m)}(\beta)\mathcal{F}_R^{(n-m)}(\gamma)]$ for which

$$\frac{\bigl(1 + \sum_{i \in K} \alpha_i - \sum_{j \in M} \alpha_j\bigr)^{R/2}\bigl(1 - \sum_{i \in K} \alpha_i + \sum_{j \in M} \alpha_j\bigr)^{R/2}}{\bigl(\sum_{i \in K} \alpha_i - \sum_{j \in M} \alpha_j - \delta\bigr)_\delta} \ll \Bigl(1 - \Bigl(\sum_{i \in K} \alpha_i - \sum_{j \in M} \alpha_j\Bigr)^2\Bigr)^{\frac{R-\delta}{2}}.$$

This bound holds because the degree of the Pochhammer symbol in the denominator is $\delta$, and by assumption, the degree of the numerator is $R > \delta$. Combining all such terms with the remaining factors of $\mathcal{F}_R^{(n)}(\alpha)/[\mathcal{F}_R^{(m)}(\beta)\mathcal{F}_R^{(n-m)}(\gamma)]$ gives a polynomial of degree $d$. □

**Remark A.29.** Let $n = n_1 + n_2 + \cdots + n_r$ and $\hat{n}_\ell = \sum_{i=1}^{\ell} n_i$. The result of Lemma A.28 clearly generalizes to the case of taking multiple residues at $s_{\hat{n}_\ell} = -\hat{\alpha}_{\hat{n}_\ell} - \delta_\ell$ for each $\ell = 1, \ldots, r-1$ (in reverse order). In this case, taking the product on the left-hand side over all of the terms we obtain

$$\mathcal{F}_R^{(n)}(\alpha) \cdot \prod_{\ell=1}^{r-1} \prod_{\substack{K \subseteq \{1,2,\ldots,\hat{n}_{\ell+1}\} \\ \#(K \cap \{1,\ldots,\hat{n}_\ell\}) \neq \hat{n}_\ell - 1 \\ \#K = \hat{n}_\ell}} \Bigl(\Bigl(\sum_{i \in K} \alpha_i\Bigr) - \hat{\alpha}_{\hat{n}_\ell} - \delta_\ell\Bigr)_{\delta_\ell}^{-1} \ll \mathcal{P}_d(\alpha) \cdot \prod_{\ell=1}^{r} \mathcal{F}_R^{(n_\ell)}(\alpha^{(\ell)}),$$

where

$$d = R \cdot \Bigl(D(n) - \sum_{\ell=1}^{r} D(n_\ell)\Bigr) - \sum_{\ell=1}^{r-1} \Bigl[\delta_\ell \Bigl(\binom{\hat{n}_{\ell+1}}{\hat{n}_\ell} - n_{\ell+1}\hat{n}_\ell - 1\Bigr)\Bigr].$$

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# On the equivalence between the effective adjunction conjectures of Prokhorov–Shokurov and of Li

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Prokhorov and Shokurov introduced the effective adjunction conjecture, also known as the effective basepoint-freeness conjecture, which asserts that the moduli component of an lc-trivial fibration is effectively basepoint-free. Li proposed a variation of this conjecture, known as the $\Gamma$-effective adjunction conjecture, and demonstrated that a weaker version of his conjecture follows from the original Prokhorov–Shokurov conjecture.

In this paper, we prove the equivalence between Prokhorov–Shokurov's and Li's effective adjunction conjectures. The key to our proof is establishing uniform rational polytopes for canonical bundle formulas. This relies on recent advancements in the minimal model program theory of algebraically integrable foliations, primarily developed by Ambro–Cascini–Shokurov–Spicer and Chen–Han–Liu–Xie.

## 1. Introduction

We work over the field of complex numbers $\mathbb{C}$.

Prokhorov and Shokurov famously proposed the effective basepoint-freeness conjecture concerning the moduli part of lc-trivial fibrations.

**Conjecture 1.1** [Prokhorov and Shokurov 2009, Conjecture 7.13]. *Let $d$ be a positive integer and $\Gamma_0 \subset [0, 1]$ a finite set of rational numbers. Then there exists a positive integer $I$ depending only on $d$ and $\Gamma_0$ satisfying the following conditions. Assume that*

(1) *$f : (X, B) \to Z$ is an lc-trivial fibration such that* $\dim X - \dim Z = d$, *and*

(2) *the coefficients of the horizontal/$Z$ part of $B$ belong to $\Gamma_0$.*

*Then $I\boldsymbol{M}$ is basepoint-free, where $\boldsymbol{M}$ is the moduli part of $f : (X, B) \to Z$.*

Conjecture 1.1 is known for its complexity and has only been proven for the case $d = 1$, as detailed in [Prokhorov and Shokurov 2009, Theorem 8.1]. The noneffective version of this conjecture for $d = 2$ was recently proven in [Ascher et al. 2023, Theorem 1.4]. However, for $d \geq 3$, Conjecture 1.1 remains largely unresolved.

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The importance of Conjecture 1.1 lies in its close relationship with moduli theory. Specifically, since the moduli parts of lc-trivial fibrations characterize the moduli space of the general fiber of the family $X \to Z$, Conjecture 1.1 is crucial for the study of the moduli of varieties, especially log Calabi–Yau varieties; see, for example, [Ascher et al. 2023].

Recent developments in moduli theory suggest that, instead of considering only pairs with standard or rational coefficients, it is more natural to include pairs with arbitrary coefficients in $[0, 1]$ or $\left(\frac{1}{2}, 1\right]$; cf. [Kollár 2023, 6.26–6.28]. In particular, pairs with irrational coefficients need to be considered. Since Conjecture 1.1 only considers lc-trivial fibrations with rational horizontal coefficients, it is natural to inquire whether a generalization of Conjecture 1.1 for lc-trivial fibrations with irrational coefficients is feasible. Fortunately, Z. Li has proposed such a variation in [Li 2024, Conjecture 3.5(1)], adopting the notation of $\Gamma$-basepoint-freeness. In this paper, we propose a stronger version of [Li 2024, Conjecture 3.5(1)].

**Definition 1.2** [Li 2024, Definition 3.4]. Let $\Gamma \subset (0, 1]$ be a set. A $\boldsymbol{b}$-divisor $\boldsymbol{D}$ on a normal projective variety $X$ is called $\Gamma$*-basepoint-free* if there exist $a_1, \dots, a_k \in \Gamma$ and basepoint-free $\boldsymbol{b}$-divisors $\boldsymbol{D}_1, \dots, \boldsymbol{D}_k$ such that $\sum_{i=1}^{k} a_i = 1$ and $\boldsymbol{D} = \sum_{i=1}^{k} a_i \boldsymbol{D}_i$.

**Conjecture 1.3** [Li 2024, Conjecture 3.5(1)]. *Let $d$ be a positive integer and $\Gamma \subset [0, 1]$ a DCC set of real numbers. Then there exist a positive integer $I$, a finite set $\Gamma_0 \subset (0, 1]$ depending only on $d$, and $\Gamma$ satisfying the following conditions. Assume that*

(1) *$f : (X, B) \to Z$ is an lc-trivial fibration such that $\dim X - \dim Z = d$, and*

(2) *the coefficients of the horizontal$/Z$ part of $B$ belong to $\Gamma$.*

*Then $I\boldsymbol{M}$ is $\Gamma_0$-basepoint-free, where $\boldsymbol{M}$ is the moduli part of $f : (X, B) \to Z$.*

It is evident that Conjecture 1.3 implies Conjecture 1.1. This raises the intriguing question of whether the two conjectures are, in fact, equivalent. Supporting this possibility, Z. Li introduced a less stringent form of Conjecture 1.3 in [Li 2024, Conjecture 3.5(2)] and proved that Conjecture 1.1 implies this weaker version. However, it remains unproven whether [Li 2024, Conjecture 3.5(2)] implies Conjecture 1.1. Additionally, [Li 2024, Conjecture 3.5(2)] is notably more complex than Conjecture 1.3.

In our paper, we demonstrate that Prokhorov–Shokurov's Conjecture 1.1 and Li's Conjecture 1.3 are, in fact, equivalent.

**Theorem 1.4.** *For any positive integer $d$, Conjecture 1.1 in relative dimension $d$ (i.e., $\dim X - \dim Z = d$) and Conjecture 1.3 in relative dimension $d$ are equivalent.*

As an immediate corollary, we prove Conjecture 1.3 when $d = 1$:

**Corollary 1.5.** *Conjecture 1.3 holds when $d = 1$.*

*Idea of the proof.* The idea behind the proof of Theorem 1.4 is to establish uniform rational polytopes for canonical bundle formulas (see Theorem 3.4 below). Roughly speaking, given an lc-trivial fibration $f : (X, B) \to Z$ with moduli part $\boldsymbol{M}$, we aim to establish a uniform decomposition $(X, B) = \sum a_i (X, B_i)$, where $\sum a_i = 1$, the horizontal$/Z$ coefficients of $B_i$ are rational, and each $f : (X, B_i) \to Z$ is an lc-trivial fibration with moduli part $\boldsymbol{M}_i$, so that $\boldsymbol{M} = \sum a_i \boldsymbol{M}_i$. By "uniform", we mean that the $a_i$ and the horizontal$/Z$ coefficients of $B_i$ depend only on $\dim X - \dim Z$ and the horizontal$/Z$ coefficients of $B$.

Establishing such a uniform decomposition is a natural idea. In fact, it is straightforward to establish such a uniform decomposition without the condition "$\boldsymbol{M} = \sum a_i \boldsymbol{M}_i$" using [Han et al. 2024, Theorem 5.6]. Similar results can be found in [Li 2024] as well. However, proving $\boldsymbol{M} = \sum a_i \boldsymbol{M}_i$ is a difficult task. The existence of such a decomposition with $\boldsymbol{M} = \sum a_i \boldsymbol{M}_i$ is already nontrivial even when uniformity is not required [Jiao et al. 2022, Theorem 2.23]. This is because the coefficient of the discriminant part of the canonical bundle formula $f : (X, B) \to Z$ and $f : (X, B_i) \to Z$ are of the forms $1 - \mathrm{lct}_{\eta_D}(X, B; f^*D)$ and $1 - \mathrm{lct}_{\eta_D}(X, B_i; f^*D)$, respectively, yet we only have

$$\mathrm{lct}_{\eta_D}(X, B; f^*D) \geq \sum a_i \,\mathrm{lct}_{\eta_D}(X, B_i; f^*D)$$

in general. The uniform decomposition shows that not only the inequality becomes equality but also the equality holds for any $X$, $Z$, and any divisor $D$ over $Z$ simultaneously (as $\boldsymbol{M}$ is a $\boldsymbol{b}$-divisor).

The key new ingredient we need for the proof of the existence of the *uniform* decomposition is the minimal model program theory for algebraically integrable foliations, which has been established very recently [Ambro et al. 2021; Chen et al. 2023].

More precisely, let $\mathcal{F}$ be the foliation induced by $f : X \to Z$ and $B^h$ the horizontal$/Z$ part of $B$. The key observation is that if $f$ is equidimensional and $(X, \mathcal{F}, B^h)$ is lc satisfying Property $(*)$ (see [Ambro et al. 2021, Definitions 2.13 and 3.5]), then $K_{\mathcal{F}} + B^h$ is exactly the moduli part of $f : (X, B) \to Z$ [Ambro et al. 2021, Proposition 3.6]. Therefore, if we can decompose $(X, \mathcal{F}, B^h)$ into foliated triples with Property $(*)$ uniformly, then it automatically induces a decomposition of $\boldsymbol{M}$. Such a decomposition is possible (see Theorem 3.2) if we replace "Property $(*)$" with the condition "weak ACSS" (see [Chen et al. 2023, Definition 7.2.3]), thanks to the existence of uniform rational lc polytopes for foliations [Liu et al. 2024a, Theorem 1.8; 2024b, Theorem 1.3; Das et al. 2023, Theorem 1.5; Chen et al. 2023, Theorem 2.4.7]. The rest of the proof involves a series of changes of models that preserve the moduli part of the canonical bundle formula, relying on the fact that for lc-trivial fibrations which are crepant over the generic point of the base, the moduli parts of the canonical bundle formulas are the same (see Lemma 2.14).

## 2. Preliminaries

***Notations and definitions.*** We adopt the standard notations and definitions from [Kollár and Mori 1998; Birkar et al. 2010] and use them freely. For foliations and generalized foliated quadruples, we follow [Chen et al. 2023], which generally aligns with the notations and definitions from [Cascini and Spicer 2020; 2021; Ambro et al. 2021], but there may be minor differences. For $\boldsymbol{b}$-divisors and generalized pairs, we follow the notations and definitions from [Birkar and Zhang 2016; Han and Li 2022; Hacon and Liu 2023]. For the canonical bundle formula, we adhere to [Chen et al. 2023], which is generally consistent with the classical definitions. For the reader's convenience, we provide the following notations and definitions that are either not commonly used in the literature or have minor differences from the classical definitions:

**Definition 2.1.** Let $m$ be a positive integer and $\boldsymbol{v} \in \mathbb{R}^m$. The *rational envelope* of $\boldsymbol{v}$ is the minimal rational affine subspace of $\mathbb{R}^m$ which contains $\boldsymbol{v}$. For example, if $m = 2$ and $\boldsymbol{v} = \bigl(\frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2}\bigr)$, then the rational envelope of $\boldsymbol{v}$ is $(x_1 + x_2 = 1) \subset \mathbb{R}^2_{x_1 x_2}$.

For any set of nonnegative real numbers $\Gamma$, we define

$$\Gamma_+ := \left(\left\{\sum \gamma_i \;\middle|\; \gamma_i \in \Gamma\right\} \cup \{0\}\right) \cap [0, 1], \quad D(\Gamma) := \left\{\frac{m-1+\gamma}{m} \;\middle|\; m \in \mathbb{N}^+, \gamma \in \Gamma_+\right\}.$$

For any real number $t$ and $\mathbb{R}$-divisor $D = \sum d_i D_i$, where $D_i$ are the irreducible components of $D$, we define $D^{\leq t} := \sum_{d_i \leq t} d_i D_i$ and $D^{>t} := \sum_{d_i > t} d_i D_i$.

**Definition 2.2** (lc-trivial fibration [Chen et al. 2023, Definition 11.3.1]). Let $(X, B)$ be a subpair and $f : X \to Z$ a contraction. We say that $f : (X, B) \to Z$ is an *lc-trivial fibration* if

(1) $(X, B)$ is sub-lc over the generic point of $Z$,

(2) $K_X + B \sim_{\mathbb{R},Z} 0$, and

(3) there exists a birational morphism $h : Y \to X$ with $K_Y + B_Y = h^*(K_X + B)$ such that $-B_Y^{\leq 0}$ is $\mathbb{R}$-Cartier and
$$\kappa_\sigma(Y/Z, -B_Y^{\leq 0}) = 0.$$

We remark that the classical definition of an lc-trivial fibration replaces condition (3) with

(3′) $\operatorname{rank} f_*\mathcal{O}_X(\lceil \boldsymbol{A}^*(X, B)\rceil) = 1$

[Kawamata 1998; Ambro 2004; Kollár 2007; Fujino and Gongyo 2014]. It is worth mentioning that, in this paper, we only consider lc-trivial fibrations $f : (X, B) \to Z$ such that $B \geq 0$ over the generic point of $Z$. In this case, both (3) and (3′) automatically hold, so there will be no confusion in the notation.

**Definition 2.3** (discriminant and moduli parts [Ambro et al. 2021, Definition 2.3]). Let $(X, B)$ be a subpair and $f : X \to Z$ a contraction such that $(X, B)$ is generically sub-lc$/Z$. In the following, we fix a choice of $K_X$ and a choice of $K_Z$, and suppose that for any birational morphisms $g : \overline{X} \to X$ and $g_Z : \overline{Z} \to Z$, $K_{\overline{X}}$ and $K_{\overline{Z}}$ are chosen as the Weil divisors such that $g_* K_{\overline{X}} = K_X$ and $(g_Z)_* K_{\overline{Z}} = K_Z$.

For any prime divisor $D$ on $Z$, we define

$$b_D(X, B; f) := 1 - \sup\{t \mid (X, B + t f^* D) \text{ is sub-lc over the generic point of } D\}.$$

Although $f^* D$ may not be well-defined everywhere, it is at least defined near the generic point of $D$, which suffices for our purposes. We then define the *discriminant part* of $f : (X, B) \to Z$ as

$$B_Z := \sum_{D \text{ is a prime divisor on } Z} b_D(X, B; f) D.$$

Next, we define the trace moduli part of $f$. By Definition-Theorem 2.6, there exists an equidimensional contraction $f' : X' \to Z'$ associated with birational morphisms $h' : X' \to X$ and $h'_Z : Z' \to Z$ such that $Z'$ is smooth and $f \circ h' = h'_Z \circ f'$. Write $K_{X'} + B' := h'^*(K_X + B)$, and let $B_{Z'}$ be the discriminant part of $f' : (X', B') \to Z'$. Define

$$M_{X'} := K_{X'} + B' - f'^*(K_{Z'} + B_{Z'}).$$

We then define the *trace moduli part* of $f : (X, B) \to Z$ as

$$M_X := h'_* M_{X'}.$$

It is easy to check that $M_X$ does not depend on the choice of $f'$.

By construction, there exist $\boldsymbol{b}$-divisors $\boldsymbol{B}$ on $Z$ and $\boldsymbol{M}$ on $X$ satisfying the following. For any contraction $f'' : X'' \to Z''$ associated with birational morphisms $h'' : X'' \to X$ and $h''_Z : Z'' \to Z$ such that $f \circ h'' = h''_Z \circ f''$, $\boldsymbol{B}_{Z''}$ is the discriminant part of $f'' : (X'', B'') \to Z''$, and $\boldsymbol{M}_{X''}$ is the trace moduli part of $f'' : (X'', B'') \to Z''$, where $K_{X''} + B'' := h''^*(K_X + B)$. We call $\boldsymbol{B}$ the *discriminant* $\boldsymbol{b}$*-divisor* of $f : (X, B) \to Z$ and $\boldsymbol{M}$ the *moduli part* of $f : (X, B) \to Z$. By construction, $\boldsymbol{B}$ is uniquely determined, and $\boldsymbol{M}$ is uniquely determined for any fixed choices of $K_X$ and $K_Z$.

**Remark 2.4** (base moduli part). The moduli part $\boldsymbol{M}$ defined in Definition 2.3 follows the same notation as in [Ambro et al. 2021], which is defined on $X$ rather than on the base $Z$.

For any lc-trivial fibration $f : (X, B) \to Z$, the canonical bundle formula indicates that

$$K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + \boldsymbol{M}^Z_Z),$$

where $B_Z$ is the discriminant part and $\boldsymbol{M}^Z$ is a $\boldsymbol{b}$-divisor. Such $\boldsymbol{M}^Z$ is also called the "moduli part" of $f : (X, B) \to Z$ in many references. To avoid any confusion, we call such $\boldsymbol{M}^Z$ the *base moduli part* of $f : (X, B) \to Z$.

It is clear that $\boldsymbol{M} \sim_{\mathbb{R}} f^*\boldsymbol{M}^Z$ as $\boldsymbol{b}$-divisors. Moreover, for lc-trivial fibrations, the effective basepoint-freeness and the effective $\Gamma$-basepoint-freeness of the moduli part are equivalent to those of the base moduli part.

**Remark 2.5.** We can similarly define lc-trivial fibrations, discriminant parts, and base moduli parts for foliations. We refer the reader to [Chen et al. 2023, Definition 11.3.1, Definition-Lemma 11.5.1] for details. We do not need them in the rest of the paper.

**Definition-Theorem 2.6** [Ambro et al. 2021, Theorem 2.2; Liu et al. 2023, Definition-Theorem 6.5]. Let $X$ be a normal quasiprojective variety, $X \to U$ a projective morphism, $X \to Z$ a contraction, and $B$ an $\mathbb{R}$-divisor on $X$. Then there exist a toroidal pair $(X', \Sigma_{X'})/U$, a log smooth pair $(Z', \Sigma_{Z'})$, and a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow{h_Z} & Z \end{array}$$

satisfying the following:

(1) $h$ and $h_Z$ are projective birational morphisms.

(2) $f' : (X', \Sigma_{X'}) \to (Z', \Sigma_{Z'})$ is a toroidal contraction.

(3) $\operatorname{Supp}(h_*^{-1}B) \cup \operatorname{Supp}\operatorname{Exc}(h)$ is contained in $\operatorname{Supp}\Sigma_{X'}$.

(4) $X'$ has at most toric quotient singularities.

(5) $f'$ is equidimensional.

(6) $X'$ is $\mathbb{Q}$-factorial klt.

We call any such $f' : (X', \Sigma_{X'}) \to (Z', \Sigma_{Z'})$ (associated with $h$ and $h_Z$) which satisfies (1–6) an *equidimensional model* of $f : (X, B) \to Z$.

**Definition 2.7** (foliated log smooth [Ambro et al. 2021, §3.2; Das et al. 2023, Definition 2.17]). Let $(X, \mathcal{F}, B)$ be a foliated triple such that $\mathcal{F}$ is algebraically integrable. We say that $(X, \mathcal{F}, B)$ is *foliated log smooth* if there exists a contraction $f : X \to Z$ satisfying the following.

(1) $X$ has at most quotient toric singularities.

(2) $\mathcal{F}$ is induced by $f$.

(3) $(X, \Sigma_X)$ is toroidal for some reduced divisor $\Sigma_X$ such that $\operatorname{Supp} B \subset \Sigma_X$. In particular, $(X, \operatorname{Supp} B)$ is toroidal, and $X$ is $\mathbb{Q}$-factorial klt.

(4) There exists a log smooth pair $(Z, \Sigma_Z)$ such that

$$f : (X, \Sigma_X) \to (Z, \Sigma_Z)$$

is an equidimensional toroidal contraction.

**Definition 2.8** (weak ACSS [Das et al. 2023, Definitions 4.1–4.3]). Let $(X, \mathcal{F}, B)$ be a foliated triple, $G$ a reduced divisor on $X$, and $f : X \to Z$ a contraction. We say that $(X, \mathcal{F}, B; G)/Z$ is *weak ACSS* if the following conditions hold:

(1) $(X, \mathcal{F}, B)$ is lc.

(2) $(Z, f(G))$ is log smooth and $G = f^{-1}(f(G))$.

(3) $f$ is equidimensional and $\mathcal{F}$ is induced by $f$.

(4) For any closed point $z \in Z$ and any reduced divisor $\Sigma \geq f(G)$ on $Z$ such that $(Z, \Sigma)$ is log smooth near $z$,

$$(X, B + G + f^*(\Sigma - f(G)))$$

is lc over a neighborhood of $z$.

We say that $(X, \mathcal{F}, B)$ is *weak ACSS* if $(X, \mathcal{F}, B; G)/Z$ is weak ACSS for some reduced divisor $G$ on $X$ and some contraction $f : X \to Z$.

***Results on foliations with weak ACSS singularities.***

**Lemma 2.9** [Chen et al. 2023, Lemma 7.3.3]. *Let $(X, \mathcal{F}, B)$ be a foliated triple such that $\mathcal{F}$ is algebraically integrable. Suppose that $(X, \mathcal{F}, B)$ is foliated log smooth, associated with a contraction $f : (X, \Sigma_X) \to (Z, \Sigma_Z)$* (as in Definition 2.7(3)). *Let $G$ be the vertical$/Z$ part of $\Sigma_X$ and $B^h$ the horizontal$/Z$ part of $B$. Then $(X, \mathcal{F}, \operatorname{Supp} B^h; G)/Z$ is weak ACSS.*

**Proposition 2.10** (cf. [Ambro et al. 2021, Proposition 3.6; Chen et al. 2023, Proposition 7.3.6]). *Let $(X, \mathcal{F}, B)$ be a foliated triple, $f : X \to Z$ a contraction, and $G$ a reduced divisor on $X$ such that $(X, \mathcal{F}, B; G)/Z$ is weak ACSS. Let $\boldsymbol{M}$ be the moduli part of $f : (X, B+G) \to Z$. Then*:

(1) $K_{\mathcal{F}} + B \sim \boldsymbol{M}_X$.

(2) $K_{\mathcal{F}} + B \sim_Z K_X + B + G$.

*Moreover, we can choose $K_{\mathcal{F}}$ depending only on the choices of $K_X$ and $K_Z$ such that $K_{\mathcal{F}} + B = \boldsymbol{M}_X$.*

*Proof.* This follows from [Ambro et al. 2021, Proposition 3.6] or [Chen et al. 2023, Proposition 7.3.6]. Note that if we choose $K_{\mathcal{F}} = K_{X/Z} - R$ as in the proof of [Chen et al. 2023, Proposition 7.3.6], then we actually obtain the equality $K_{\mathcal{F}} + B = \boldsymbol{M}_X$. □

**Proposition 2.11.** *Let $(X, \mathcal{F}, B)/Z$ be a foliated triple and $G$ a reduced divisor on $X$ such that $Z$ is $\mathbb{Q}$-factorial, $(X, \mathcal{F}, B; G)/Z$ is weak ACSS, and*

$$\kappa_\sigma(X/Z, K_{\mathcal{F}} + B) = \kappa_\iota(X/Z, K_{\mathcal{F}} + B) = 0.$$

*Then we may run a $(K_{\mathcal{F}} + B)$-MMP$/Z$ with scaling of an ample$/Z$ divisor, which terminates with a model $(X', \mathcal{F}', B')/Z$ such that $K_{\mathcal{F}'} + B' \sim_{\mathbb{R},Z} 0$, where $\mathcal{F}'$ and $B'$ are the pushforwards of $\mathcal{F}$ and $B$ on $X'$, respectively. Moreover, $(X', \mathcal{F}', B'; G')/Z$ is weak ACSS, where $G'$ is the pushforward of $G$ on $X'$.*

*Proof.* The main part of the proposition follows from [Chen et al. 2023, Proposition 11.2.1]. The "moreover" part follows from [Chen et al. 2023, Lemma 9.1.4]. □

**Theorem 2.12** [Das et al. 2023, Theorem 1.5; Chen et al. 2023, Theorem 2.4.7]. *Let $r$ be a positive integer, $v_1^0, \dots, v_m^0$ positive real numbers, and $\boldsymbol{v}_0 := (v_1^0, \dots, v_m^0)$. Then there exists an open set $U \ni \boldsymbol{v}_0$ of the rational envelope of $\boldsymbol{v}_0$ depending only on $r$ and $\boldsymbol{v}_0$ satisfying the following statement*:

*Let $\left(X, \mathcal{F}, \sum_{j=1}^m v_j^0 B_j\right)$ be an lc foliated triple such that $\mathcal{F}$ is an algebraically integrable foliation of rank $r$ and $B_j \geq 0$ are distinct Weil divisors. Then $\left(X, \mathcal{F}, \sum_{j=1}^m v_j B_j\right)$ is lc for any $(v_1, \dots, v_m) \in U$.*

**Theorem 2.13** (cf. [Chen et al. 2023, Theorem 11.1.5]). *Let $(X, \mathcal{F}, B)$ be an lc foliated triple, $f : X \to Z$ a contraction, and $G$ a reduced divisor on $X$ such that $(X, \mathcal{F}, B; G)/Z$ is weak ACSS. Let $\boldsymbol{M}$ be the moduli part of $f : (X, B+G) \to Z$. Then $\boldsymbol{M}$ descends to $X$.*

*Proof.* This follows from [Chen et al. 2023, Theorem 11.1.5 and Lemma 5.3.2]. □

**Lemma 2.14.** *Let $(X, B)$ and $(X', B')$ be two subpairs. Let $f : (X, B) \to Z$ and $f' : (X', B') \to Z'$ be two lc-trivial fibrations over $U$ such that $f$ and $f'$ are birationally equivalent (i.e., there exist birational* *maps $h : X' \dashrightarrow X$ and $h_Z : Z' \dashrightarrow Z$ such that $f \circ h = h_Z \circ f'$), and $(X, B)$ and $(X', B')$ are crepant over the generic point of $Z$. Let $\boldsymbol{M}$ and $\boldsymbol{M}^Z$ be the moduli part and the base moduli part of $f : (X, B) \to Z$, respectively, and let $\boldsymbol{M}'$ and $\boldsymbol{M}'^{Z'}$ be the moduli part and the base moduli part of $f' : (X', B') \to Z'$, respectively. Then $\boldsymbol{M} = \boldsymbol{M}'$ for any compatible choices of $K_X$ and $K_Z$, and $\boldsymbol{M}^Z \sim_{\mathbb{R}} \boldsymbol{M}'^{Z'}$.*

*Proof.* The proof follows along the same lines as the proof of [Chen et al. 2023, Lemma 11.4.3].

Possibly passing to a common base and resolving the indeterminacy of $h : X' \dashrightarrow X$, we may assume that $f = f'$, $X = X'$, and $Z = Z'$. Replacing $f : X \to Z$ with an equidimensional model, we may assume that $f$ is equidimensional and $Z$ is smooth. Now $K_X + B = K_X + B'$ over the generic point of $Z$, so $B - B'$ is vertical$/Z$. Since $K_X + B \sim_{\mathbb{R},Z} 0$ and $K_X + B' \sim_{\mathbb{R},Z} 0$, we have $B - B' \sim_{\mathbb{R},Z} 0$, so $B - B' = f^*P$ for some $\mathbb{R}$-divisor $P$ on $Z$ (cf. [Chen et al. 2024, Lemma 2.5]).

Let $B_Z$ and $B'_Z$ be the discriminant parts of $f : (X, B) \to Z$ and $f : (X, B') \to Z$, respectively. By the definition of the discriminant part, $B_Z = B'_Z + P$. Therefore,

$$\boldsymbol{M}_X = K_X + B - f^*(K_Z + B_Z) = K_X + B - f^*(K_Z + B'_Z + P) = K_X + B' - f^*(K_Z + B'_Z) = \boldsymbol{M}'_X.$$

Since we may pass to an arbitrarily high model, we have $\boldsymbol{M} = \boldsymbol{M}'$. By the definition of the base moduli part, we have $\boldsymbol{M}^Z \sim_{\mathbb{R}} \boldsymbol{M}'^{Z'}$. □

**Lemma 2.15** (cf. [Han et al. 2021, Lemma 3.8; 2022, Lemma 2.18]). *Let $a_1, \dots, a_k \in (0, 1]$ be real numbers such that $\sum_{i=1}^k a_i = 1$. Let $(X, B_1), \dots, (X, B_k)$ be subpairs and $D \ge 0$ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on $X$. Then*

$$\sum_{i=1}^k a_i \operatorname{lct}(X, B_i; D) \le \operatorname{lct}\Bigl(X, \sum_{i=1}^k a_i B_i; D\Bigr).$$

*Proof.* We may assume that $\operatorname{lct}(X, B_i; D) > -\infty$ for any $i$. For any $1 \le i \le k$, let $b_i := \operatorname{lct}(X, B_i; D)$ and $s := \sum_{i=1}^k a_i b_i$. Since $(X, B_i + b_i D)$ is lc for any $i$ and

$$\sum_{i=1}^k a_i B_i + sD = \sum_{i=1}^k a_i (B_i + b_i D),$$

it follows that $\bigl(X, \sum_{i=1}^k a_i B_i + sD\bigr)$ is lc. Thus, the lemma follows. □

## 3. Uniform rational polytopes for canonical bundle formulas

In this section, we establish the existence of uniform rational polytopes for canonical bundle formulas. We present two versions of this uniform decomposition theorem (Theorems 3.3 and 3.4), whose statements and initial proofs are similar but diverge subsequently. The arguments in Theorem 3.3 are more straightforward and clear from the perspective of uniform decomposition theorems. However, we will apply Theorem 3.4 to prove Theorem 1.4.

**Lemma 3.1.** *Let $X$ and $X'$ be two normal quasiprojective varieties that are birational to each other. Let $D = \sum_{i=1}^m v_i^0 D_i$ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on $X$ and $D' = \sum_{i=1}^m v_i^0 D'_i$ an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on $X'$ such* *that $D$ and $D'$ are crepant (i.e., there are projective birational morphisms $p : W \to X$ and $q : W \to X'$ such that $p^*D = q^*D'$), where $D_i$ and $D'_i$ are $\mathbb{Q}$-divisors. Then for any vector $\boldsymbol{v} = (v_1, \dots, v_m)$ in the rational envelope of $\boldsymbol{v}_0 := (v_1^0, \dots, v_m^0)$ in $\mathbb{R}^m$, $D(\boldsymbol{v}) := \sum_{i=1}^m v_i D_i$ and $D'(\boldsymbol{v}) := \sum_{i=1}^m v_i D'_i$ are crepant.*

*Proof.* We may write $D = \sum_{i=1}^c r_i \bar{D}_i$ and $D' = \sum_{i=1}^c r_i \bar{D}'_i$, where $\bar{D}_i$ and $\bar{D}'_i$ are $\mathbb{Q}$-divisors and $r_1, \dots, r_c$ are linearly independent over $\mathbb{Q}$. By [Han et al. 2024, Lemma 5.3], $\bar{D}_i$ and $\bar{D}'_i$ are $\mathbb{Q}$-Cartier for each $i$. Let $p : W \to X$ and $q : W \to X'$ be a common resolution. Then

$$\sum_{i=1}^c r_i p^* \bar{D}_i = p^* D = q^* D' = \sum_{i=1}^c r_i q^* \bar{D}'_i,$$

so

$$\sum_{i=1}^c r_i (p^* \bar{D}_i - q^* \bar{D}'_i) = 0.$$

Thus, $p^* \bar{D}_i = q^* \bar{D}'_i$ for each $i$. In particular, for any $\boldsymbol{u} = (u_1, \dots, u_c) \in \mathbb{R}^c$, $\bar{D}(\boldsymbol{u}) := \sum_{i=1}^c u_i \bar{D}_i$ and $\bar{D}'(\boldsymbol{u}) := \sum_{i=1}^c u_i \bar{D}'_i$ are crepant. Since for any vector $\boldsymbol{v}$ in the rational envelope of $\boldsymbol{v}_0$, there exists a unique vector $\boldsymbol{u} \in \mathbb{R}^c$ such that $\bar{D}(\boldsymbol{u}) = D(\boldsymbol{v})$ and $\bar{D}'(\boldsymbol{u}) = D'(\boldsymbol{v})$, the lemma follows. □

**Theorem 3.2** (uniform weak ACSS rational polytope). *Let $r$ be a positive integer and $v_1^0, \dots, v_m^0$ real numbers. Then there exists an open subset $U \ni \boldsymbol{v}_0$ of the rational envelope of $\boldsymbol{v}_0 := (v_1^0, \dots, v_m^0)$ in $\mathbb{R}^m$ depending only on $r$ and $\boldsymbol{v}_0$ satisfying the following.*

*Let $(X, \mathcal{F}, B)$ be an lc foliated triple, $f : X \to Z$ a contraction, and $G$ a reduced divisor on $X$ such that* $\operatorname{rank} \mathcal{F} = r$ *and $(X, \mathcal{F}, B; G)/Z$ is weak ACSS. Assume that*

- *$B = \sum_{i=1}^m v_i^0 B_i$, where $B_i \geq 0$ are Weil divisors, and*
- *$B(\boldsymbol{v}) := \sum_{i=1}^m v_i B_i$ for any $\boldsymbol{v} = (v_1, \dots, v_m) \in \mathbb{R}^m$.*

*Then $(X, \mathcal{F}, B(\boldsymbol{v}); G)/Z$ is weak ACSS for any $\boldsymbol{v} \in U$.*

*Proof.* We verify all the conditions of Definition 2.8 for $(X, \mathcal{F}, B(\boldsymbol{v}); G)/Z$. Condition (1) follows from Theorem 2.12. Conditions (2) and (3) are obvious. Therefore, we only need to check condition (4). Note that this does not directly follow from [Han et al. 2024, Theorem 5.6] because $\dim X$ is not fixed.

We only need to show that there exists an open subset $U \ni \boldsymbol{v}_0$ of the rational envelope of $\boldsymbol{v}_0$ such that for any closed point $z \in Z$ and any log smooth pair $(Z, \Sigma)$ such that $\Sigma \geq f(G)$, we have that

$$(X, B(\boldsymbol{v}) + G + f^*(\Sigma - f(G)))$$

is lc over a neighborhood of $z$ for any $\boldsymbol{v} \in U$. Possibly adding components to $\Sigma$, we may assume that $z$ is an lc center of $(Z, \Sigma)$. Let $G_z := G + f^*(\Sigma - f(G))$. Then $G_z = f^{-1}(\Sigma)$, $(X, B(\boldsymbol{v}_0) + G_z)$ is lc over a neighborhood of $z$, and $(X, B(\boldsymbol{v}_0) + G_z)/Z$ satisfies Property $(*)$ [Ambro et al. 2021, Definition 2.13] over a neighborhood of $z$.

Let $\Sigma_1, \dots, \Sigma_{\dim Z}$ be the irreducible components of $\Sigma$ which contain $z$ and let $V$ be an irreducible component of $f^{-1}(z)$. Then $V$ is an irreducible component of $\bigcap_{i=1}^{\dim Z} f^{-1}(\Sigma_i)$. In particular, there exist prime divisors $G_i \subset f^{-1}(\Sigma_i)$ such that $V$ is a component of $\bigcap_{i=1}^{\dim Z} G_i$. Since each $G_i$ is a component of $G_z$, each $G_i$ is an lc place of $(X, B(\boldsymbol{v}_0)+G_z)$. Thus any component of $\bigcap_{i=1}^n G_i$ is an lc center of $(X, B(\boldsymbol{v}_0)+G_z)$ for any $1\le n\le \dim Z$ [Ambro 2011, Theorem 1.1]. In particular, $V$ is an lc center of $(X, B(\boldsymbol{v}_0)+G_z)$, and there exists a sequence of lc centers

$$V =: V_{\dim Z} \subsetneq V_{\dim Z-1} \subsetneq \cdots \subsetneq V_1 := G_1$$

such that each $V_n$ is an irreducible component of $\bigcap_{i=1}^n G_i$. We let $\nu_n : W_n \to V_n$ be the normalization of $V_n$ and let $\tau_n : V_n \to V_{n-1}$ be the natural inclusions. Then there exist morphisms $\iota_n : W_n \to W_{n-1}$ such that $\tau_n\circ\nu_n = \nu_{n-1}\circ\iota_n$ for any $n\ge 2$. Since $f$ is equidimensional, $\dim V_n=\dim W_n=\dim X-n$ for any $n$.

For any $\boldsymbol{v}$ that is contained in the rational polytope of $\boldsymbol{v}_0$ and any $0\le n\le\dim Z$, we define an $\mathbb{R}$-divisor $B_n(\boldsymbol{v})$ on $W_n$ in the following way:

- $W_0 := X$ and $B_0(\boldsymbol{v}) := B(\boldsymbol{v})+G_z$.
- Suppose that we have already constructed $B_n(\boldsymbol{v})$ for some $n\le\dim Z-1$ such that the image of $W_m$ in $W_n$ is an lc center of $(W_n, B_n(\boldsymbol{v}_0))$ for any $m>n$, and $\boldsymbol{v}\to B_n(\boldsymbol{v})$ is a $\mathbb{Q}$-affine function from the rational envelope of $\boldsymbol{v}_0$ to $\mathrm{Weil}_{\mathbb{R}}(W_n)$. Since the image of $W_{n+1}$ in $W_n$ is of codimension 1, we may define
$$K_{W_{n+1}}+B_{n+1}(\boldsymbol{v}_0) := (K_{W_n}+B_n(\boldsymbol{v}_0))|_{W_{n+1}}$$
by usual adjunction. Since $\boldsymbol{v}$ is contained in the rational envelope of $\boldsymbol{v}_0$, by [Han et al. 2024, Lemma 5.3], $K_{W_n}+B_n(\boldsymbol{v})$ is $\mathbb{R}$-Cartier for any $\boldsymbol{v}$ that is contained in the rational envelope of $\boldsymbol{v}_0$. Thus there exist uniquely defined $\mathbb{R}$-divisors $B_{n+1}(\boldsymbol{v})$ such that
$$K_{W_{n+1}}+B_{n+1}(\boldsymbol{v}) = (K_{W_n}+B_n(\boldsymbol{v}))|_{W_{n+1}}.$$
- By [Kollár 2013, Theorem 4.9(3)], the image of $W_m$ in $W_{n+1}$ is an lc center of $(W_{n+1}, B_{n+1}(\boldsymbol{v}_0))$ for any $m>n+1$. Moreover, since adjunction $D\to D|_{W_{n+1}}$ is a $\mathbb{Q}$-affine function from $\mathrm{Div}_{\mathbb{R}}(W_n)$ to $\mathrm{Div}_{\mathbb{R}}(W_{n+1})$, $\boldsymbol{v}\to B_{n+1}(\boldsymbol{v})$ is a $\mathbb{Q}$-affine function from the rational envelope of $\boldsymbol{v}_0$ to $\mathrm{Weil}_{\mathbb{R}}(W_{n+1})$. Thus we may repeat this process.

We let $W := W_{\dim Z}$ and $B_W(\boldsymbol{v}) := B_{\dim Z}(\boldsymbol{v})$. Let $h : Y\to W$ be a $\mathbb{Q}$-factorial dlt modification of $(W, B_W(\boldsymbol{v}_0))$ and $B_Y(\boldsymbol{v}) := h^{-1}_* B_W(\boldsymbol{v})+E$ for any $\boldsymbol{v}$, where $E$ is the reduced $h$-exceptional divisor. Then

$$K_Y+B_Y(\boldsymbol{v}_0) := h^*(K_W+B_W(\boldsymbol{v}_0)).$$

By [Han et al. 2024, Lemma 5.4], for any $\boldsymbol{v}$ that is contained in the rational envelope of $\boldsymbol{v}_0$, we have

$$K_Y+B_Y(\boldsymbol{v}) = h^*(K_W+B_W(\boldsymbol{v})).$$

Let $\Gamma_0 := \{0, 1, v_1^0, \dots, v_m^0\}$. Then by our construction, the coefficients of $B_Y(\boldsymbol{v}_0)$ belong to $\Gamma := D(\Gamma_0)$. Since $Y$ is $\mathbb{Q}$-factorial, by the ACC for lc thresholds [Hacon et al. 2014, Theorem 1.1], there exists a real number $t<1$ depending only on $r$ and $\boldsymbol{v}_0$ such that $(Y, B'_Y(\boldsymbol{v}_0) := B_Y(\boldsymbol{v}_0)^{\le t} + \mathrm{Supp}\, B_Y^{>t}(\boldsymbol{v}_0))$ is lc.

For any $\boldsymbol{v}$ that is contained in the rational envelope of $\boldsymbol{v}_0$, we define $B'_Y(\boldsymbol{v})$ to be the unique $\mathbb{R}$-divisor such that for any prime divisor $D$ on $Y$,

$$\operatorname{mult}_D B'_Y(\boldsymbol{v}) := \begin{cases} \operatorname{mult}_D B_Y(\boldsymbol{v}) & \text{if } D \text{ is a component of } B_Y(\boldsymbol{v}_0)^{\leq t}, \\ 1 & \text{if } D \text{ is a component of } B_Y(\boldsymbol{v}_0)^{>t}, \\ 0 & \text{otherwise.} \end{cases}$$

By our construction, $\boldsymbol{v} \to B'_Y(\boldsymbol{v})$ is a $\mathbb{Q}$-affine function from the rational envelope of $\boldsymbol{v}_0$ to $\operatorname{Weil}_{\mathbb{R}}(Y)$. Since $\dim Y = r$ and the coefficients of $B'_Y(\boldsymbol{v}_0)$ belong to a finite set depend only on $r$ and $\boldsymbol{v}_0$, by [Han et al. 2024, Theorem 5.6], there exists an open set $U \ni \boldsymbol{v}_0$ of the rational envelope of $\boldsymbol{v}_0$ depending only on $r$ and $\boldsymbol{v}_0$ such that $(Y, B'_Y(\boldsymbol{v}))$ is lc for any $\boldsymbol{v} \in U$. Possibly shrinking $U$, we may assume that the coefficients of $B_Y(\boldsymbol{v})$ are $\leq 1$ for any $\boldsymbol{v} \in U$.

By our construction, $B'_Y(\boldsymbol{v}) \geq B_Y(\boldsymbol{v}) \geq 0$ for any $\boldsymbol{v} \in U$, so $(Y, B_Y(\boldsymbol{v}))$ is lc for any $\boldsymbol{v} \in U$. Hence $(W, B_W(\boldsymbol{v})) = (W_{\dim Z}, B_{\dim Z}(\boldsymbol{v}))$ is lc for any $\boldsymbol{v} \in U$. Suppose that $(W_n, B_n(\boldsymbol{v}))$ is lc for some $n \geq 1$. Then by inversion of adjunction [Kawakita 2007, Theorem; Hacon 2014, Theorem 1], we have that $(W_{n-1}, B_{n-1}(\boldsymbol{v}))$ is lc near the image of $W_n$ in $W_{n-1}$ for any $\boldsymbol{v} \in U$. Hence, possibly shrinking to a neighborhood of the image of $W_n$ in $W_{n-1}$, we may assume that $(W_{n-1}, B_{n-1}(\boldsymbol{v}))$ is lc for any $\boldsymbol{v} \in U$. Thus we may repeat this process and deduce that, possibly shrinking $X$ to a neighborhood of $V$, we have that $(W_0, B_0(\boldsymbol{v})) = (X, B(\boldsymbol{v}) + G_z)$ is lc for any $\boldsymbol{v} \in U$. Since $V$ can be any irreducible component of $f^{-1}(z)$, $(X, B(\boldsymbol{v}) + G_z)$ is lc near $f^{-1}(z)$ for any $\boldsymbol{v} \in U$. Condition (4) follows and we are done. □

It remains interesting to ask whether there exists a uniform ACSS rational polytope, although we do not do this in our paper.

**Theorem 3.3** (uniform rational polytope for canonical bundle formula, I). *Let $d$ be a positive integer and $v_1^0, \ldots, v_m^0$ real numbers. Then there exists an open subset $U \ni \boldsymbol{v}_0$ of the rational envelope of $\boldsymbol{v}_0 := (v_1^0, \ldots, v_m^0)$ in $\mathbb{R}^m$, depending only on $d$ and $\boldsymbol{v}_0$, such that the following conditions hold. Assume that*

- *$f : (X, B) \to Z$ is an lc-trivial fibration with* $\dim X = d$,
- *$(X, B)$ is lc,*
- *$B = \sum_{i=1}^m v_i^0 B_i$, where $B_i \geq 0$ are Weil divisors,*
- *$B(\boldsymbol{v}) := \sum_{i=1}^m v_i B_i$ for any $\boldsymbol{v} = (v_1, \ldots, v_m) \in \mathbb{R}^m$, and*
- *$B_Z$ and $\boldsymbol{M}$ are the discriminant part and the moduli part of $f : (X, B) \to Z$, respectively.*

*Then*:

(1) *For any $\boldsymbol{v} \in U$, $(X, B(\boldsymbol{v}))$ is lc and $f : (X, B(\boldsymbol{v})) \to Z$ is an lc-trivial fibration.*

(2) *For any vectors $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k$ in $U$ and positive real numbers $a_1, \ldots, a_k$ such that $\sum_{i=1}^k a_i = 1$, we have*

$$B_Z\Bigl(\sum_{i=1}^k a_i \boldsymbol{v}^i\Bigr) = \sum_{i=1}^k a_i B_Z(\boldsymbol{v}^i) \quad \text{and} \quad \boldsymbol{M}\Bigl(\sum_{i=1}^k a_i \boldsymbol{v}^i\Bigr) = \sum_{i=1}^k a_i \boldsymbol{M}(\boldsymbol{v}^i),$$

*where $B_Z(\boldsymbol{v})$ and $\boldsymbol{M}(\boldsymbol{v})$ are the discriminant part and the moduli part of $f : (X, B(\boldsymbol{v})) \to Z$.*

*Proof.* **Step 1**. In this step, we consider an equidimensional model of $f:(X,B)\to Z$. We then run an MMP to achieve an auxiliary model $X''$ and construct an lc-trivial fibration $f'':(X'',\widetilde{B}''^h(\boldsymbol{v})+G'')\to Z'$.

Let $f':(X',\Sigma_{X'})\to(Z',\Sigma_{Z'})$ be an equidimensional model of $f:(X,B)\to Z$ associated with $h:X'\to X$ and $h_Z:Z'\to Z$. Let $\widetilde{B}':=h^{-1}_*B+\operatorname{Supp}\operatorname{Exc}(h)$ with horizontal$/Z'$ part $\widetilde{B}'^h$, and $\widetilde{B}'(\boldsymbol{v}):=h^{-1}_*B(\boldsymbol{v})+\operatorname{Supp}\operatorname{Exc}(h)$ with horizontal$/Z'$ part $\widetilde{B}'^h(\boldsymbol{v})$ for any $\boldsymbol{v}\in\mathbb{R}^m$. Let $G'$ be the vertical$/Z'$ part of $\Sigma_{X'}$. Denote $K_{X'}+B':=h^*(K_X+B)$ and $K_{X'}+B'(\boldsymbol{v}):=h^*(K_X+B(\boldsymbol{v}))$ for any $\boldsymbol{v}\in\mathbb{R}^m$.

Let $\mathcal{F}'$ be the foliation induced by $f'$. Then $(X',\mathcal{F}',\widetilde{B}'^h)$ is foliated log smooth. By Lemma 2.9, $(X',\mathcal{F}',\widetilde{B}'^h;G')/Z'$ is weak ACSS. Choose $K_{\mathcal{F}'}$ as in Proposition 2.10. Then $K_{X'}-K_{\mathcal{F}'}$ is vertical$/Z'$ by Proposition 2.10. Therefore,

$$\kappa_\sigma(X'/Z',K_{\mathcal{F}'}+\widetilde{B}'^h)=\kappa_\sigma(X'/Z',K_{\mathcal{F}'}+\widetilde{B}')=\kappa_\sigma(X'/Z',K_{X'}+\widetilde{B}')=\kappa_\sigma(X'/Z',K_{X'}+B')=0$$

and

$$\kappa_\iota(X'/Z',K_{\mathcal{F}'}+\widetilde{B}'^h)=\kappa_\iota(X'/Z',K_{\mathcal{F}'}+\widetilde{B}')=\kappa_\iota(X'/Z',K_{X'}+\widetilde{B}')=\kappa_\iota(X'/Z',K_{X'}+B')=0.$$

By Proposition 2.11, we may run a $(K_{\mathcal{F}'}+\widetilde{B}'^h)$-MMP$/Z'$ with scaling of an ample$/Z'$ divisor, which terminates with a model $(X'',\mathcal{F}'',\widetilde{B}''^h)/Z'$ of $(X',\mathcal{F}',\widetilde{B}'^h)/Z'$ such that $K_{\mathcal{F}''}+\widetilde{B}''^h\sim_{\mathbb{R},Z'}0$. Denote by $f'':X''\to Z'$ the induced morphism. For any $\boldsymbol{v}\in\mathbb{R}^m$, let $\widetilde{B}''^h(\boldsymbol{v})$ and $G''$ be the strict transforms of $\widetilde{B}'^h(\boldsymbol{v})$ and $G'$ on $X''$, respectively. By Proposition 2.11, $(X'',\mathcal{F}'',\widetilde{B}''^h;G'')/Z'$ is weak ACSS.

By Theorem 2.12, there exists an open subset $U\ni\boldsymbol{v}_0$ of the rational envelope of $\boldsymbol{v}_0$, depending only on $d$ and $\boldsymbol{v}_0$, such that both $(X'',\mathcal{F}'',\widetilde{B}''^h(\boldsymbol{v}))$ and $(X,B(\boldsymbol{v}))$ are lc for any $\boldsymbol{v}\in U$. By [Han et al. 2024, Lemma 5.3], $K_X+B(\boldsymbol{v})\sim_{\mathbb{R},Z}0$, so $f:(X,B(\boldsymbol{v}))\to Z$ is an lc-trivial fibration for any $\boldsymbol{v}\in U$. Then $f':(X',B'(\boldsymbol{v}))\to Z'$ is an lc-trivial fibration with moduli part $\boldsymbol{M}(\boldsymbol{v})$ for any $\boldsymbol{v}\in U$. Since $(X'',\mathcal{F}'',\widetilde{B}''^h;G'')/Z'$ is weak ACSS by Theorem 3.2, possibly shrinking $U$, we may assume that $(X'',\mathcal{F}'',\widetilde{B}''^h(\boldsymbol{v});G'')/Z'$ is weak ACSS for any $\boldsymbol{v}\in U$. By Proposition 2.10 and [Han et al. 2024, Lemma 5.3],

$$K_{X''}+\widetilde{B}''^h(\boldsymbol{v})+G''\sim_{\mathbb{R},Z'}K_{\mathcal{F}''}+\widetilde{B}''^h(\boldsymbol{v})\sim_{\mathbb{R},Z'}0$$

for any $\boldsymbol{v}\in U$, so $f'':(X'',\widetilde{B}''^h(\boldsymbol{v})+G'')\to Z'$ is an lc-trivial fibration.

**Step 2**. In this step, we show that we may let $\boldsymbol{M}(\boldsymbol{v})$ be the moduli part of $f'':(X'',\widetilde{B}''^h(\boldsymbol{v})+G'')\to Z'$ and conclude the proof of the theorem.

Let $p:W\to X'$ and $q:W\to X''$ be a common resolution of $\phi:X'\dashrightarrow X''$. By construction, $p^*(K_{X'}+B')\le p^*(K_{X'}+\widetilde{B}'^h)$ and $q^*(K_{\mathcal{F}''}+\widetilde{B}''^h)\le p^*(K_{\mathcal{F}'}+\widetilde{B}'^h)=p^*(K_{X'}+\widetilde{B}'^h)$ over a nonempty open subset $Z'^\circ$ of $Z'$. Moreover, we can write $p^*(K_{X'}+B')-q^*(K_{\mathcal{F}''}+\widetilde{B}''^h)=E_\phi-E_h$ such that $E_\phi\ge0$ is supported on $p^{-1}(\operatorname{Exc}(\phi))$ and $E_h\ge0$ is supported on $p^{-1}(\operatorname{Exc}(h))$ over $Z'^\circ$. Since $p^*(K_{X'}+B')-q^*(K_{\mathcal{F}''}+\widetilde{B}''^h)\sim_{\mathbb{R},Z'}0$, by the negativity lemma, $E_\phi=E_h=0$.

Thus, we deduce that $K_{\mathcal{F}''}+\widetilde{B}''^h$ and $K_{X'}+B'$ are crepant over the generic point of $Z'$. By Lemma 3.1, $K_{\mathcal{F}''}+\widetilde{B}''^h(\boldsymbol{v})$ and $K_{X'}+B'(\boldsymbol{v})$ are crepant over the generic point of $Z'$ for any $\boldsymbol{v}\in U$. Therefore, $K_{X''}+\widetilde{B}''^h(\boldsymbol{v})+G''$ and $K_{X'}+B'(\boldsymbol{v})$ are crepant over the generic point of $Z'$ for any $\boldsymbol{v}\in U$. We let

$\boldsymbol{M}(\boldsymbol{v})$ be the moduli part of $f'' : (X'', \widetilde{B}''^h(\boldsymbol{v}) + G'') \to Z'$. By Theorem 2.13, $\boldsymbol{M}(\boldsymbol{v})$ descends to $X''$ for any $\boldsymbol{v} \in U$. By Lemma 2.14, $\boldsymbol{M}(\boldsymbol{v})$ is also the moduli part of $f' : (X', B'(\boldsymbol{v})) \to Z'$ for any $\boldsymbol{v} \in U$. Hence $\boldsymbol{M}(\boldsymbol{v})$ is also the moduli part of $f : (X, B(\boldsymbol{v})) \to Z$ for any $\boldsymbol{v} \in U$.

By Proposition 2.10, $\boldsymbol{M}_{X''} = K_{\mathcal{F}''} + \widetilde{B}''^h$ and $\boldsymbol{M}(\boldsymbol{v})_{X''} = K_{\mathcal{F}''} + \widetilde{B}''^h(\boldsymbol{v})$ for any $\boldsymbol{v} \in U$. Therefore, for any vectors $\boldsymbol{v}^1, \dots, \boldsymbol{v}^k$ in $U$ and positive real numbers $a_1, \dots, a_k$ such that $\sum_{i=1}^k a_i = 1$, we have

$$\boldsymbol{M}\Big(\sum_{i=1}^k a_i \boldsymbol{v}^i\Big)_{X''} = K_{\mathcal{F}''} + \widetilde{B}''^h\Big(\sum_{i=1}^k a_i \boldsymbol{v}^i\Big) = \sum_{i=1}^k a_i (K_{\mathcal{F}''} + \widetilde{B}''^h(\boldsymbol{v}^i)) = \sum_{i=1}^k a_i \boldsymbol{M}(\boldsymbol{v}^i).$$

Since $(X'', \mathcal{F}'', \widetilde{B}''^h(\boldsymbol{v}); G'')/Z''$ is weak ACSS for any $\boldsymbol{v} \in U$, by Theorem 2.13, $\boldsymbol{M}(\boldsymbol{v})$ descends to $X''$ for any $\boldsymbol{v} \in U$. Thus

$$\boldsymbol{M}\Big(\sum_{i=1}^k a_i \boldsymbol{v}^i\Big) = \sum_{i=1}^k a_i \boldsymbol{M}(\boldsymbol{v}^i).$$

Let $\boldsymbol{M}^Z(\boldsymbol{v})$ be the base moduli part of $f : (X, B(\boldsymbol{v})) \to Z$ for any $\boldsymbol{v} \in U$. Then for any $\boldsymbol{v} \in U$, $\boldsymbol{M}^Z(\boldsymbol{v})$ descends to $Z'$ and

$$f''^* \boldsymbol{M}^Z(\boldsymbol{v})_{Z'} \sim_{\mathbb{R}} \boldsymbol{M}(\boldsymbol{v})_{X''}.$$

Therefore,

$$\boldsymbol{M}^Z\Big(\sum_{i=1}^k a_i \boldsymbol{v}^i\Big) \sim_{\mathbb{R}} \sum_{i=1}^k a_i \boldsymbol{M}^Z(\boldsymbol{v}^i)$$

and

$$\sum_{i=1}^k a_i (K_Z + B_Z(\boldsymbol{v}^i) + \boldsymbol{M}^Z(\boldsymbol{v}^i)_Z) \sim_{\mathbb{R}} K_Z + B_Z\Big(\sum_{i=1}^k a_i \boldsymbol{v}^i\Big) + \boldsymbol{M}^Z\Big(\sum_{i=1}^k a_i \boldsymbol{v}^i\Big)_Z.$$

Thus,

$$\sum_{i=1}^k a_i B_Z(\boldsymbol{v}^i) \sim_{\mathbb{R}} B_Z\Big(\sum_{i=1}^k a_i \boldsymbol{v}^i\Big).$$

By the definition of the discriminant part and Lemma 2.15,

$$\sum_{i=1}^k a_i B_Z(\boldsymbol{v}^i) \geq B_Z\Big(\sum_{i=1}^k a_i \boldsymbol{v}^i\Big),$$

so

$$\sum_{i=1}^k a_i B_Z(\boldsymbol{v}^i) = B_Z\Big(\sum_{i=1}^k a_i \boldsymbol{v}^i\Big),$$

and we are done. □

**Theorem 3.4** (uniform rational polytope for canonical bundle formula, II). *Let $d$ be a positive integer and $v_1^0, \dots, v_m^0$ real numbers. Then there exists an open subset $U \ni \boldsymbol{v}_0$ of the rational envelope of $\boldsymbol{v}_0 := (v_1^0, \dots, v_m^0)$ in $\mathbb{R}^m$, depending only on $d$ and $\boldsymbol{v}_0$, such that the following conditions hold. Assume that*

- *$f : (X, B) \to Z$ is an lc-trivial fibration with* $\dim X - \dim Z = d$,
- *$B = B^h + B^v$, where $B^h$ and $B^v$ are the horizontal$/Z$ part and the vertical$/Z$ part of $B$, respectively,*

- $B^h = \sum_{i=1}^m v_i^0 B_i^h$, *where* $B_i^h \geq 0$ *are Weil divisors,*
- $B^h(\boldsymbol{v}) := \sum_{i=1}^m v_i B_i^h$ *for any* $\boldsymbol{v} = (v_1, \dots, v_m) \in \mathbb{R}^m$,
- $B^v = \sum_{i=1}^n u_i^0 B_i^v$ *for some real numbers* $u_i^0$ *and Weil divisors* $B_i^v$, *and*
- $B_Z$ *and* $\boldsymbol{M}$ *are the discriminant part and the moduli part of* $f : (X, B) \to Z$, *respectively.*

*Then there exist* $\mathbb{R}$*-affine functions* $s_1, \dots, s_n : \mathbb{R}^m \to \mathbb{R}$ *satisfying the following.*

*Let* $B^v(\boldsymbol{v}) := \sum_{i=1}^n s_i(\boldsymbol{v}) B_i^v$ *and* $B(\boldsymbol{v}) := B^h(\boldsymbol{v}) + B^v(\boldsymbol{v})$ *for any* $\boldsymbol{v} \in \mathbb{R}^m$. *Then*:

(1) *For any* $\boldsymbol{v} \in U$, $f : (X, B(\boldsymbol{v})) \to Z$ *is an lc-trivial fibration.*

(2) *For any vectors* $\boldsymbol{v}^1, \dots, \boldsymbol{v}^k$ *in* $U$ *and positive real numbers* $a_1, \dots, a_k$ *such that* $\sum_{i=1}^k a_i = 1$, *we have*

$$B_Z\left(\sum_{i=1}^k a_i \boldsymbol{v}^i\right) = \sum_{i=1}^k a_i B_Z(\boldsymbol{v}^i) \quad \textit{and} \quad \boldsymbol{M}\left(\sum_{i=1}^k a_i \boldsymbol{v}^i\right) = \sum_{i=1}^k a_i \boldsymbol{M}(\boldsymbol{v}^i),$$

*where* $B_Z(\boldsymbol{v})$ *and* $\boldsymbol{M}(\boldsymbol{v})$ *are the discriminant part and the moduli part of* $f : (X, B(\boldsymbol{v})) \to Z$.

*Proof.* **Step 1**. In this step we construct $s_1, \dots, s_n$.

Let $V$ be the rational envelope of $(v_1^0, \dots, v_m^0, u_1^0, \dots, u_n^0) \subset \mathbb{R}^{m+n}$ and let $V_m$ be the image of $V$ in $\mathbb{R}^m$ under the projection $\pi_m : \mathbb{R}^{m+n} \to \mathbb{R}^m : (x_1, \dots, x_{m+n}) \to (x_1, \dots, x_m)$. Then $V_m$ is the rational envelope of $\boldsymbol{v}_0$, and there exists an affine function $\tau : V_m \to V$ such that $\pi_m \circ \tau$ is the identity morphism. Now $\tau$ naturally extends to an affine function $\beta : \mathbb{R}^m \to V \subset \mathbb{R}^{m+n}$. Let $\pi_n : \mathbb{R}^{m+n} \to \mathbb{R}^n : (x_1, \dots, x_{m+n}) \to (x_{m+1}, \dots, x_n)$ be the projection and let $s := \pi_n \circ \beta$. Then we may write

$$s(\boldsymbol{v}) := (s_1(\boldsymbol{v}), \dots, s_n(\boldsymbol{v})),$$

where $s_1, \dots, s_n : \mathbb{R}^m \to \mathbb{R}$ are $\mathbb{R}$-affine functions.

**Step 2**. This step is almost identical to **Step 1** of the proof of Theorem 3.3, with minor differences. In this step, we consider an equidimensional model of $f : (X, B) \to Z$, run an MMP to achieve an auxiliary model $X''$, and construct an lc-trivial fibration $f'' : (X'', \widetilde{B}''^h(\boldsymbol{v}) + G'') \to Z'$.

Let $f' : (X', \Sigma_{X'}) \to (Z', \Sigma_{Z'})$ be an equidimensional model of $f : (X, B) \to Z$ associated with $h : X' \to X$ and $h_Z : Z' \to Z$. Let $\widetilde{B}' := h_*^{-1} B + \operatorname{Supp} \operatorname{Exc}(h)$ with horizontal/$Z'$ part $\widetilde{B}'^h$, and $\widetilde{B}'(\boldsymbol{v}) := h_*^{-1} B(\boldsymbol{v}) + \operatorname{Supp} \operatorname{Exc}(h)$ with horizontal/$Z'$ part $\widetilde{B}'^h(\boldsymbol{v})$ for any $\boldsymbol{v} \in \mathbb{R}^m$. Let $G'$ be the vertical/$Z'$ part of $\Sigma_{X'}$. Denote $K_{X'} + B' := h^*(K_X + B)$ and $K_{X'} + B'(\boldsymbol{v}) := h^*(K_X + B(\boldsymbol{v}))$ for any $\boldsymbol{v} \in \mathbb{R}^m$.

Let $\mathcal{F}'$ be the foliation induced by $f'$. Then $(X', \mathcal{F}', \widetilde{B}'^h)$ is foliated log smooth. By Lemma 2.9, $(X', \mathcal{F}', \widetilde{B}'^h; G')/Z'$ is weak ACSS. Choose $K_{\mathcal{F}'}$ as in Proposition 2.10. Then $K_{X'} - K_{\mathcal{F}'}$ is vertical/$Z'$ by Proposition 2.10. Therefore,

$$\kappa_\sigma(X'/Z', K_{\mathcal{F}'} + \widetilde{B}'^h) = \kappa_\sigma(X'/Z', K_{\mathcal{F}'} + \widetilde{B}') = \kappa_\sigma(X'/Z', K_{X'} + \widetilde{B}') = \kappa_\sigma(X'/Z', K_{X'} + B') = 0,$$

and

$$\kappa_\iota(X'/Z', K_{\mathcal{F}'} + \widetilde{B}'^h) = \kappa_\iota(X'/Z', K_{\mathcal{F}'} + \widetilde{B}') = \kappa_\iota(X'/Z', K_{X'} + \widetilde{B}') = \kappa_\iota(X'/Z', K_{X'} + B') = 0.$$

By Proposition 2.11, we may run a $(K_{\mathcal{F}'}+\widetilde{B}'^h)$-MMP$/Z'$ with scaling of an ample$/Z'$ divisor, which terminates with a model $(X'', \mathcal{F}'', \widetilde{B}''^h)/Z'$ of $(X', \mathcal{F}', \widetilde{B}'^h)/Z'$ such that $K_{\mathcal{F}''}+\widetilde{B}''^h \sim_{\mathbb{R},Z'} 0$. Denote by $f'': X'' \to Z'$ the induced morphism. For any $\boldsymbol{v} \in \mathbb{R}^m$, let $\widetilde{B}''^h(\boldsymbol{v})$ and $G''$ be the strict transforms of $\widetilde{B}'^h(\boldsymbol{v})$ and $G'$ on $X''$, respectively. By Proposition 2.11, $(X'', \mathcal{F}'', \widetilde{B}''^h; G'')/Z'$ is weak ACSS.

By Theorem 2.12, there exists an open subset $U \ni \boldsymbol{v}_0$ of the rational envelope of $\boldsymbol{v}_0$ depending only on $d$ and $\boldsymbol{v}_0$ such that $(X'', \mathcal{F}'', \widetilde{B}''^h(\boldsymbol{v}))$ is lc for any $\boldsymbol{v} \in U$. Since $(X'', \mathcal{F}'', \widetilde{B}''^h; G'')/Z'$ is weak ACSS by Theorem 3.2, possibly shrinking $U$, we may assume that $(X'', \mathcal{F}'', \widetilde{B}''^h(\boldsymbol{v}); G'')/Z'$ is weak ACSS for any $\boldsymbol{v} \in U$. By Proposition 2.10 and [Han et al. 2024, Lemma 5.3],

$$K_{X''}+\widetilde{B}''^h(\boldsymbol{v})+G'' \sim_{\mathbb{R},Z'} K_{\mathcal{F}''}+\widetilde{B}''^h(\boldsymbol{v}) \sim_{\mathbb{R},Z'} 0$$

for any $\boldsymbol{v} \in U$, so $f'': (X'', \widetilde{B}''^h(\boldsymbol{v})+G'') \to Z'$ is an lc-trivial fibration.

**Step 3**. This step is almost identical to **Step 2** of the proof of Theorem 3.3 with minor differences. In this step, we show that we may let $\boldsymbol{M}(\boldsymbol{v})$ be the moduli part of $f'': (X'', \widetilde{B}''^h(\boldsymbol{v})+G'') \to Z'$ and conclude the proof of the theorem.

Let $p: W \to X'$ and $q: W \to X''$ be a common resolution of $\phi: X' \dashrightarrow X''$. By construction, $p^*(K_{X'}+B') \le p^*(K_{X'}+\widetilde{B}'^h)$ and $q^*(K_{\mathcal{F}''}+\widetilde{B}''^h) \le p^*(K_{\mathcal{F}'}+\widetilde{B}'^h) = p^*(K_{X'}+\widetilde{B}'^h)$ over an open subset $Z'^\circ$ of $Z'$. Moreover, we can write $p^*(K_{X'}+B') - q^*(K_{\mathcal{F}''}+\widetilde{B}''^h) = E_\phi - E_h$ such that $E_\phi \ge 0$ is supported on $p^{-1}(\mathrm{Exc}(\phi))$ and $E_h \ge 0$ is supported on $p^{-1}(\mathrm{Exc}(h))$ over $Z'^\circ$. Since

$$p^*(K_{X'}+B') - q^*(K_{\mathcal{F}''}+\widetilde{B}''^h) \sim_{\mathbb{R},Z'} 0,$$

by the negativity lemma, $E_\phi = E_h = 0$.

Thus, $K_{\mathcal{F}''}+\widetilde{B}''^h$ and $K_{X'}+B'$ are crepant over the generic point of $Z'$. By Lemma 3.1, $K_{\mathcal{F}''}+\widetilde{B}''^h(\boldsymbol{v})$ and $K_{X'}+B'(\boldsymbol{v})$ are crepant over the generic point of $Z'$ for any $\boldsymbol{v} \in U$. Therefore, $K_{X''}+\widetilde{B}''^h(\boldsymbol{v})+G''$ and $K_{X'}+B'(\boldsymbol{v})$ are crepant over the generic point of $Z'$ for any $\boldsymbol{v} \in U$. We let $\boldsymbol{M}(\boldsymbol{v})$ be the moduli part of $f'': (X'', \widetilde{B}''^h(\boldsymbol{v})+G'') \to Z'$. By Theorem 2.13, $\boldsymbol{M}(\boldsymbol{v})$ descends to $X''$ for any $\boldsymbol{v} \in U$. By Lemma 2.14, $\boldsymbol{M}(\boldsymbol{v})$ is also the moduli part of $f': (X', B'(\boldsymbol{v})) \to Z'$ for any $\boldsymbol{v} \in U$. Hence $\boldsymbol{M}(\boldsymbol{v})$ is also the moduli part of $f: (X, B(\boldsymbol{v})) \to Z$ for any $\boldsymbol{v} \in U$.

By Proposition 2.10, $\boldsymbol{M}_{X''} = K_{\mathcal{F}''}+\widetilde{B}''^h$ and $\boldsymbol{M}(\boldsymbol{v})_{X''} = K_{\mathcal{F}''}+\widetilde{B}''^h(\boldsymbol{v})$ for any $\boldsymbol{v} \in U$. Therefore, for any vectors $\boldsymbol{v}^1, \dots, \boldsymbol{v}^k$ in $U$ and positive real numbers $a_1, \dots, a_k$ such that $\sum_{i=1}^k a_i = 1$, we have

$$\boldsymbol{M}\left(\sum_{i=1}^k a_i \boldsymbol{v}^i\right)_{X''} = K_{\mathcal{F}''}+\widetilde{B}''^h\left(\sum_{i=1}^k a_i \boldsymbol{v}^i\right) = \sum_{i=1}^k a_i (K_{\mathcal{F}''}+\widetilde{B}''^h(\boldsymbol{v}^i)) = \sum_{i=1}^k a_i \boldsymbol{M}(\boldsymbol{v}^i).$$

Since $(X'', \mathcal{F}'', \widetilde{B}''^h(\boldsymbol{v}); G'')/Z''$ is weak ACSS for any $\boldsymbol{v} \in U$, by Theorem 2.13, $\boldsymbol{M}(\boldsymbol{v})$ descends to $X''$ for any $\boldsymbol{v} \in U$. Thus

$$\boldsymbol{M}\left(\sum_{i=1}^k a_i \boldsymbol{v}^i\right) = \sum_{i=1}^k a_i \boldsymbol{M}(\boldsymbol{v}^i).$$

Let $\boldsymbol{M}^Z(\boldsymbol{v})$ be the base moduli part of $f:(X, B(\boldsymbol{v}))\to Z$ for any $\boldsymbol{v}\in U$. Then for any $\boldsymbol{v}\in U$, $\boldsymbol{M}^Z(\boldsymbol{v})$ descends to $Z'$ and

$$f''^{*}\boldsymbol{M}^Z(\boldsymbol{v})_{Z'}\sim_{\mathbb{R}}\boldsymbol{M}(\boldsymbol{v})_{X''}.$$

Therefore,

$$\boldsymbol{M}^Z\Bigl(\sum_{i=1}^{k}a_i\boldsymbol{v}^i\Bigr)\sim_{\mathbb{R}}\sum_{i=1}^{k}a_i\boldsymbol{M}^Z(\boldsymbol{v}^i)$$

and

$$\sum_{i=1}^{k}a_i(K_Z+B_Z(\boldsymbol{v}^i)+\boldsymbol{M}^Z(\boldsymbol{v}^i)_Z)\sim_{\mathbb{R}}K_Z+B_Z\Bigl(\sum_{i=1}^{k}a_i\boldsymbol{v}^i\Bigr)+\boldsymbol{M}^Z\Bigl(\sum_{i=1}^{k}a_i\boldsymbol{v}^i\Bigr)_Z.$$

Thus,

$$\sum_{i=1}^{k}a_iB_Z(\boldsymbol{v}^i)\sim_{\mathbb{R}}B_Z\Bigl(\sum_{i=1}^{k}a_i\boldsymbol{v}^i\Bigr).$$

By the definition of the discriminant part and Lemma 2.15,

$$\sum_{i=1}^{k}a_iB_Z(\boldsymbol{v}^i)\ge B_Z\Bigl(\sum_{i=1}^{k}a_i\boldsymbol{v}^i\Bigr),$$

so

$$\sum_{i=1}^{k}a_iB_Z(\boldsymbol{v}^i)=B_Z\Bigl(\sum_{i=1}^{k}a_i\boldsymbol{v}^i\Bigr),$$

and we are done. □

**Remark 3.5.** The arguments used in Theorems 3.3 and 3.4 also apply to lc-trivial fibrations of generalized pairs, as all results from [Chen et al. 2023] remain applicable. Due to the technical nature of the arguments, we omit detailed statements and proofs here.

**Remark 3.6.** It is also possible to establish uniform rational polytopes for canonical bundle formulas for lc-trivial fibrations with DCC coefficients, similar to [Li 2024, Theorem 4.1; Chen et al. 2024, Theorem 1.9; 2025, Theorem 1.1]. The proof follows almost identically to those of Theorems 3.3 and 3.4. Again, due to the technical nature of the arguments, we omit detailed statements and proofs here.

Since the moduli part is determined by the horizontal part of $B$, the following direct consequence of Theorem 1.4 might be useful. Indeed, Corollary 3.7 could also be applied to prove Theorem 1.4 by either [Li 2024, Proposition 3.3] or [Birkar 2021, Lemma 3.5].

**Corollary 3.7.** *Let $d$ be a positive integer and $v_1^0,\dots,v_m^0$ real numbers. Then there exists an open subset $U\ni\boldsymbol{v}_0$ of the rational envelope of $\boldsymbol{v}_0:=(v_1^0,\dots,v_m^0)$ in $\mathbb{R}^m$, depending only on $d$ and $\boldsymbol{v}_0$, such that the following conditions hold. Assume that*

- *$f:(X,B)\to Z$ is an lc-trivial fibration with* $\dim X-\dim Z=d$,
- *$B=\sum_{i=1}^{m}v_i^0B_i$, where $B_i\ge 0$ are Weil horizontal/$Z$ divisors,*
- *$B(\boldsymbol{v}):=\sum_{i=1}^{m}v_iB_i$ for any $\boldsymbol{v}=(v_1,\dots,v_m)\in\mathbb{R}^m$, and*
- *$B_Z$ and $\boldsymbol{M}$ are the discriminant part and the moduli part of $f:(X,B)\to Z$, respectively.*

*Then*:

(1) *For any $\boldsymbol{v} \in U$, $f : (X, B(\boldsymbol{v})) \to Z$ is an lc-trivial fibration.*

(2) *For any vectors $\boldsymbol{v}^1, \dots, \boldsymbol{v}^k$ in $U$ and positive real numbers $a_1, \dots, a_k$ such that $\sum_{i=1}^k a_i = 1$, we have*

$$B_Z\left(\sum_{i=1}^k a_i \boldsymbol{v}^i\right) = \sum_{i=1}^k a_i B_Z(\boldsymbol{v}^i) \quad \text{and} \quad \boldsymbol{M}\left(\sum_{i=1}^k a_i \boldsymbol{v}^i\right) = \sum_{i=1}^k a_i \boldsymbol{M}(\boldsymbol{v}^i),$$

*where $B_Z(\boldsymbol{v})$ and $\boldsymbol{M}(\boldsymbol{v})$ are the discriminant part and the moduli part of $f : (X, B(\boldsymbol{v})) \to Z$.*

*Proof.* This is a special case of Theorem 3.4 as there is no vertical$/Z$ part of $B$. □

When $B$ is a sum of horizontal$/Z$ divisors, Corollary 3.7 is stronger than Theorem 3.3, as Corollary 3.7 requires that $\dim X - \dim Z = d$ while Theorem 3.3 requires that $\dim X = d$.

## 4. Proof of the main theorem

*Proof of Theorem 1.4.* It is evident that Conjecture 1.3 in relative dimension $d$ implies Conjecture 1.1 in relative dimension $d$. Therefore, we only need to show that Conjecture 1.1 in relative dimension $d$ implies Conjecture 1.3 in relative dimension $d$.

Under the notations and assumptions of Conjecture 1.3, suppose that Conjecture 1.1 holds in relative dimension $d$. By [Hacon et al. 2014, Theorem 1.5], we may assume that $\Gamma$ is a finite set $\{v_1^0, \dots, v_m^0\}$ for some nonnegative integer $m$. Let $B^h$ be the horizontal$/Z$ part of $B$, and write $B^h = \sum_{i=1}^m v_i^0 B_i^h$, where $v_i^0 \in \Gamma$ for each $i$ and $B_i^h \geq 0$ are Weil divisors. Let $B^h(\boldsymbol{v}) := \sum_{i=1}^m v_i B_i^h$ for any $\boldsymbol{v} := (v_1, \dots, v_m) \in \mathbb{R}^m$, and $\boldsymbol{v}_0 := (v_1^0, \dots, v_m^0)$. Let $B_Z$ and $\boldsymbol{M}$ be the discriminant part and the moduli part of $f : (X, B) \to Z$, respectively.

Let $U \ni \boldsymbol{v}_0$ be an open subset of the rational envelope of $\boldsymbol{v}_0$ as in Theorem 3.4 which depends only on $d$ and $\boldsymbol{v}_0$, and we let $s_1, \dots, s_n$, $B^h(\boldsymbol{v})$, $B^v(\boldsymbol{v})$, $B(\boldsymbol{v})$ be as in Theorem 3.4 for any $\boldsymbol{v} \in U$. Let $k := \dim U + 1$ and $\boldsymbol{v}^1, \dots, \boldsymbol{v}^k \in U \cap \mathbb{Q}^m$ be vectors depending only on $d$ and $\boldsymbol{v}_0$ such that $\boldsymbol{v}_0$ is contained in the interior of the convex hull of $\boldsymbol{v}^1, \dots, \boldsymbol{v}^k$. Then there exist unique real numbers $a_1, \dots, a_k \in (0, 1]$ such that $\sum_{i=1}^k a_i = 1$ and $\sum_{i=1}^k a_i \boldsymbol{v}^i = \boldsymbol{v}_0$. By Theorem 3.4,

- $B^h(\boldsymbol{v}^i)$ is the horizontal$/Z$ part of $B(\boldsymbol{v}^i)$ for each $i$,
- $f : (X, B(\boldsymbol{v}^i)) \to Z$ is an lc-trivial fibration for each $i$, and
- $B_Z = \sum_{i=1}^k a_i B_Z(\boldsymbol{v}^i)$ and $\boldsymbol{M} = \sum_{i=1}^k a_i \boldsymbol{M}(\boldsymbol{v}^i)$, where $B_Z(\boldsymbol{v}^i)$ and $\boldsymbol{M}(\boldsymbol{v}^i)$ are the discriminant part and the moduli part of $f : (X, B(\boldsymbol{v}^i)) \to Z$, respectively.

By Conjecture 1.1 in relative dimension $d$, there exists a positive integer $I$ depending only on $d$ and $\Gamma$ such that $I\boldsymbol{M}(\boldsymbol{v}^i)$ is basepoint-free. Therefore, $I\boldsymbol{M}$ is $\{a_1, \dots, a_k\}$-basepoint-free. Let $\Gamma_0 := \{a_1, \dots, a_k\}$, and Conjecture 1.3 in relative dimension $d$ follows. □

*Proof of Corollary 1.5.* This now follows from Theorem 1.4 and [Prokhorov and Shokurov 2009, Theorem 8.1]. □

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# Points of bounded height on certain subvarieties of toric varieties

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We combine the split torsor method and the hyperbola method for toric varieties to count rational points and Campana points of bounded height on certain subvarieties of toric varieties.



## 1. Introduction

We combine the split torsor method and the hyperbola method for toric varieties to count rational points and Campana points of bounded height on certain subvarieties of smooth split proper toric varieties. This line of research has been initiated by Blomer and Brüdern [2018] in the setting of diagonal hypersurfaces in products of projective spaces. Other results in this direction include hypersurfaces and complete intersections in products of projective spaces [Schindler 2016], improvements for bihomogeneous hypersurfaces for degree (2, 2) and (1, 2) [Browning and Hu 2019; Hu 2020], as well as generalizations to hypersurfaces in certain toric varieties [Mignot 2015; 2016; 2018].

The versions of the hyperbola method used in all of these articles are rather close to the original [Blomer and Brüdern 2018] for products of projective spaces. In our recent work [Pieropan and Schindler 2024], we established a very general form of the hyperbola method for split toric varieties, in which the height condition can also globally be given by the maximum of several monomials. The goal of this article is to show applications of our new hyperbola method. We develop a refined framework for the split torsor method on split smooth proper toric varieties and show that counting results for subvarieties of projective spaces can be carried over to toric varieties by a direct application of the hyperbola method [Pieropan and Schindler 2024]. With this we can prove new cases of Manin's conjecture [Batyrev and Manin 1990; Franke et al. 1989] on the number of rational points of bounded height on Fano varieties for certain subvarieties in toric varieties.

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*Keywords:* hyperbola method, m-full numbers, Campana points, toric varieties.

The split torsor method provides a parametrization of rational points on Fano varieties via Cox rings [Derenthal and Pieropan 2020; Salberger 1998]. The Cox ring of a smooth proper toric variety $X$ is a polynomial ring endowed with a grading by the Picard group of the toric variety [Cox 1995]. Subvarieties of toric varieties are intersections of hypersurfaces, which are defined by $\mathrm{Pic}(X)$-homogeneous polynomials in the Cox ring of $X$. The subvarieties considered in this paper are defined by homogeneous elements in the Cox ring of the toric variety such that each polynomial involves only variables of the same degree. With the split torsor method parametrization, the height is given by the maximum of a set of monomials and this is the correct shape to apply our generalized version of the hyperbola method [Pieropan and Schindler 2024]. The hyperbola method reduces the counting problem to counting functions over boxes of different shapes. An advantage of our method is that it is already adapted to the shape of height functions appearing. Also, compared to earlier versions of the hyperbola method, we do not need estimates for lower-dimensional boxes, and with this our proofs are relatively short.

We now illustrate our approach on a number of examples. In a similar fashion, it is possible to apply counting results such as [Birch 1962; Browning and Heath-Brown 2017; Heath-Brown 1996; Rydin Myerson 2018; 2019], and many others, to subvarieties of toric varieties defined by elements of the Cox ring each involving only variables of the same degree.

**1.1.** ***Results.*** Let $X$ be a smooth split complete toric $\mathbb{Q}$-variety with open torus $T$. Let $\boldsymbol{D}_1, \dots, \boldsymbol{D}_s \in \mathrm{Pic}(X)$ be the pairwise distinct classes of the torus-invariant prime divisors on $X$. For $i \in \{1, \dots, s\}$, let $n_i = \dim_{\mathbb{Q}} H^0(X, \boldsymbol{D}_i)$. Let $H_L$ be the height associated to a semiample torus-invariant divisor $L$ on $X$ as discussed in Section 2.2.

Our first result concerns subvarieties of toric varieties defined by linear forms.

**Theorem 1.1.** *Let $V \subseteq X$ be a complete intersection of hypersurfaces $H_{i,l}$ with $1 \le i \le s,\ 1 \le l \le t_i$. Assume that $[H_{i,l}] = \boldsymbol{D}_i$ in $\mathrm{Pic}(X)$ for $i \in \{1, \dots, s\}$ such that $t_i \ne 0$. Assume that $V \cap T \ne \varnothing$ and $t_i \le n_i - 2$ for all $i \in \{1, \dots, s\}$. Assume that $L = -\big(K_X + \sum_{i=1}^{s} \sum_{l=1}^{t_i} [H_{i,l}]\big)$ is ample. For $B > 0$, let $N_V(B)$ be the number of $\mathbb{Q}$-rational points on $V \cap T$ of $H_L$-height at most $B$. Then*

$$N_V(B) = cB(\log B)^{b-1} + O(B(\log B)^{b-2}(\log\log B)^s),$$

*where $b = \mathrm{rk}\,\mathrm{Pic}(V)$ and $c$ is a positive constant, which is defined by* (3-7) *with $k = b-1,\ C_{M,\boldsymbol{d}}$ given by* (4-1), *and $\varpi_i = n_i - t_i$ for $i \in \{1, \dots, s\}$.*

We use this result as a toy example to show how to combine the hyperbola method with the universal torsor method in the context of rather general smooth split toric varieties. We now move on to results which require a deeper understanding of the underlying Diophantine problems via methods from Fourier analysis.

We start with a result that concerns subvarieties of toric varieties defined by bihomogeneous polynomials. It is obtained by combining the framework developed in this paper with the hyperbola method [Pieropan and Schindler 2024] and preliminary counting results in boxes of different side lengths [Schindler 2016].

**Theorem 1.2.** *Let $V \subseteq X$ be a smooth complete intersection of hypersurfaces $H_1, \dots, H_t$ of the same degree $e_1\boldsymbol{D}_1 + e_2\boldsymbol{D}_2$ in $\operatorname{Pic}(X)$. Assume that $V \cap T \neq \varnothing$, that $n_i - te_i \geq 2$ for $i \in \{1, 2\}$, and that $n_1 + n_2 > \dim V_1^* + \dim V_2^* + 3 \cdot 2^{e_1+e_2} e_1 e_2 t^3$, where $V_1^*, V_2^* \subseteq \mathbb{A}^{n_1+n_2}$ are affine varieties defined in Section 5. Assume that $L = -([K_X] + [H_1 + \dots + H_t])$ is ample. Then there is an open subset $W \subseteq X$ such that the number $N_{V,W}(B)$ of $\mathbb{Q}$-rational points on $V \cap W \cap T$ of $H_L$-height at most $B$ satisfies*

$$N_{V,W}(B) = cB(\log B)^{b-1} + O(B(\log B)^{b-2}(\log\log B)^s)$$

*for $B > 0$, where $b = \operatorname{rk}\operatorname{Pic}(V)$ and $c$ is defined in* (3-7) *with $k = b-1$, $C_{M,\boldsymbol{d}}$ given by* (5-1), *$\varpi_i = n_i - te_i$ for $i \in \{1, 2\}$, and $\varpi_i = n_i$ for $i \in \{3, \dots, s\}$. The constant $c$ is positive if $V(\mathbb{Q}_v) \neq \varnothing$ for all places $v$ of $\mathbb{Q}$.*

Theorems 1.1 and 1.2 are compatible with Manin's conjecture [Franke et al. 1989], as $L|_V = -K_V$ by adjunction. The proofs in Sections 4 and 5 yield an asymptotic formula even if we drop the ampleness assumption on $L$.

Theorem 1.2 as well as work of Mignot [2016; 2018] include the case of certain hypersurfaces in products of projective spaces. However, in comparison to Mignot's work, we do not require the condition that the effective cone of the toric variety is simplicial. An example of a split toric variety with nonsimplicial effective cone — where our theorem applies — is the blow-up at a torus-invariant point of $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times Y$, where $n_1$ and $n_2$ are sufficiently large and $Y$ is a split del Pezzo surface of degree 6.

Our last result concerns sets of Campana points in the sense of [Pieropan et al. 2021] for subvarieties defined by diagonal equations. We introduce the following integral models. Let $\mathscr{X}$ be the $\mathbb{Z}$-toric scheme defined by the fan of $X$. For $i \in \{1, \dots, s\}$, let $\mathscr{D}_{i,1}, \dots, \mathscr{D}_{i,n_i}$ be the torus-invariant prime divisors on $\mathscr{X}$ of class $\boldsymbol{D}_i$.

**Theorem 1.3.** *Let $V \subseteq X$ be an intersection of hypersurfaces $H_1, \dots, H_t$ such that $H_i$ is defined by a homogeneous diagonal polynomial in the Cox ring of $X$ of degree $e_i\boldsymbol{D}_i$ in $\operatorname{Pic}(X)$ and with none of the coefficients equal to zero. Let $\mathscr{V}$ be the closure of $V$ in $\mathscr{X}$. For $i \in \{1, \dots, s\}$, fix integers $2 \leq m_{i,1} \leq \dots \leq m_{i,n_i}$. Let $\mathscr{D}_{\boldsymbol{m}} = \sum_{i=1}^{s} \sum_{j=1}^{n_i} (1 - 1/m_{i,j})\mathscr{D}_{i,j}$. Assume that $V \cap T \neq \varnothing$, that $n_1, \dots, n_t \geq 2$, and, for $i \in \{1, \dots, s\}$, that $\sum_{j=1}^{n_i} 1/m_{i,j} > 3$, and that*

$$\sum_{j=1}^{n_i-1} \frac{1}{e_i m_{i,j}(e_i m_{i,j}+1)} \geq 1 \quad \text{if } e_i = 1 \quad \text{and} \quad \sum_{j=1}^{n_i} \frac{1}{2s_0(e_i m_{i,j})} > 1 \quad \text{if } e_i \geq 2,$$

*where $s_0(e_i m_{i,j})$ is defined in Lemma 6.1. Let $L = -(K_X + \mathscr{D}_{\boldsymbol{m}}|_X + H_1 + \dots + H_t)$ be ample. For $B > 0$, let $N_V(B)$ be the number of $\mathbb{Z}$-Campana points on $(\mathscr{V}, \mathscr{D}_{\boldsymbol{m}}|_{\mathscr{V}})$ that lie in $T$ and have $H_L$-height at most $B$. Then*

$$N_V(B) = cB(\log B)^{b-1} + O(B(\log B)^{b-2}(\log\log B)^s),$$

*where $b = \operatorname{rk}\operatorname{Pic}(V)$ and $c$ is defined in* (3-7) *with $k = b - 1$, $C_{M,\boldsymbol{d}}$ given by* (6-11), *and $\varpi_1, \dots, \varpi_s$ given by* (6-10).

The order of growth in Theorem 1.3 is compatible with the Manin-type conjecture for Campana points [Pieropan et al. 2021], as $L|_V$ is the log anticanonical divisor of the pair $(V, \mathscr{D}_{\boldsymbol{m}}|_V)$ by adjunction.

We now give a number of examples where Theorems 1.2 and 1.3 can be applied. Due to the range of application of the circle method, we require the Cox ring of the toric variety to have a large number of variables of the same degree. This holds for toric varieties with several torus-invariant prime divisors of the same degree and for products of such toric varieties. Here are some concrete examples:

- The projective space $\mathbb{P}^n$ has Cox rings with $n+1$ variables of the same degree.

- The blow-up of the projective space $\mathbb{P}^n$ at $l < n+1$ torus-invariant points has Cox rings with $n+1-l$ variables of the same degree.

- The blow-up of a product of toric varieties each with several torus-invariant prime divisors of the same degree. Indeed, if $X$ and $Y$ are smooth split toric varieties such that the Cox ring of $X$ has $n_X$ variables of the same degree $d_X$, the Cox ring of $Y$ has $n_Y$ variables of the same degree $d_Y$, and $P \in X \times Y$ is a point where $m_X \leq n_X$ variables of degree $d_X$ vanish and $m_Y \leq n_Y$ variables of degree $d_Y$ vanish, then the Cox ring of the blow-up of $X \times Y$ at $P$ has $m_X$ variables of the same degree $d_X - e$ and $m_Y$ variables of the same degree $d_Y - e$, where $e$ is the class of the exceptional divisor.

The structure of this article is as follows. In Section 2 we reformulate the height function and the multiplicative function $\mu$ for Möbius inversion according to the principle of grouping variables of the same degree. In Section 3 we combine the new framework with the hyperbola method developed in [Pieropan and Schindler 2024] to obtain a general counting tool for points of bounded height on subvarieties of toric varieties. Theorems 1.1, 1.2, and 1.3 are proven in Sections 4, 5, and 6, respectively.

## 2. Toric varieties setting

Here we introduce the geometric setup and notation for the whole paper. We refer the reader to [Salberger 1998, §8] for a concise introduction to toric varieties and their toric models over $\mathbb{Z}$, and to [Cox et al. 2011] for an extensive treatment of toric varieties.

Let $\Sigma$ be the fan of a complete smooth split toric variety $X$ over a number field $\mathbb{K}$. We denote by $\{\boldsymbol{D}_1, \dots, \boldsymbol{D}_s\} \subseteq \operatorname{Pic}(X)$ the set of degrees of prime torus-invariant divisors of $X$. For each $i \in \{1, \dots, s\}$, we denote by $D_{i,1}, \dots, D_{i,n_i}$ the torus-invariant divisors of degree $\boldsymbol{D}_i$ and by $\rho_{i,1}, \dots, \rho_{i,n_i}$ the corresponding rays of $\Sigma$. Let $\mathcal{I} := \{(i, j) \in \mathbb{N}^2 : 1 \leq i \leq s, 1 \leq j \leq n_i\}$. Let $\Sigma_{\max}$ be the set of maximal cones of $\Sigma$. For each maximal cone $\sigma$ of $\Sigma$, let $\mathcal{J}_\sigma := \{(i, j) \in \mathcal{I} : \rho_{i,j} \subseteq \sigma\}$, let $\mathcal{I}_\sigma = \mathcal{I} \smallsetminus \mathcal{J}_\sigma$, and let $I_\sigma$ be the set of indices $i \in \{1, \dots, s\}$ such that $\{(i, 1), \dots, (i, n_i)\} \cap \mathcal{I}_\sigma \neq \varnothing$.

Let $\mathscr{X}$ be the toric scheme defined by $\Sigma$ over the ring of integers $\mathcal{O}_\mathbb{K}$ of $\mathbb{K}$, and, for each $(i, j) \in \mathcal{I}$, let $\mathscr{D}_{i,j}$ be the closure of $D_{i,j}$ in $\mathscr{X}$.

Let $R$ be the polynomial ring over $\mathcal{O}_\mathbb{K}$ with variables $x_{i,j}$ for $(i, j) \in \mathcal{I}$ and endowed with the $\operatorname{Pic}(X)$-grading induced by assigning degree $\boldsymbol{D}_i$ to the variable $x_{i,j}$ for all $(i, j) \in \mathcal{I}$. For every torus-invariant divisor $D = \sum_{i=1}^{s} \sum_{j=1}^{n_i} a_{i,j} D_{i,j}$ on $X$ and every vector $\boldsymbol{x} = (x_{i,j})_{(i,j)\in\mathcal{I}} \in \mathbb{C}^{\mathcal{I}}$, we write

$$\boldsymbol{x}^D := \prod_{i=1}^{s} \prod_{j=1}^{n_i} x_{i,j}^{a_{i,j}}.$$

By [Salberger 1998, §8], $\mathscr{X}$ has a unique universal torsor $\pi : \mathscr{Y} \to \mathscr{X}$, and $\mathscr{Y} \subseteq \mathbb{A}^{\#\mathcal{I}}_{\mathcal{O}_\mathbb{K}}$ is the open subset whose complement is defined by $\boldsymbol{x}^{D_\sigma} = 0$ for all maximal cones $\sigma$ of $\Sigma$, where $D_\sigma := \sum_{(i,j)\in\mathcal{I}_\sigma} D_{i,j}$ for all $\sigma \in \Sigma_{\max}$.

Let $r$ be the rank of $\operatorname{Pic}(X)$. Let $\mathcal{C}$ be a set of ideals of $\mathcal{O}_\mathbb{K}$ that form a system of representatives for the class group of $\mathbb{K}$. As in [Pieropan and Schindler 2024, §6.1], we fix a basis of $\operatorname{Pic}(X)$, and, for every divisor $D$ on $X$ and every tuple $\mathfrak{c} = (\mathfrak{c}_1, \dots, \mathfrak{c}_r) \in \mathcal{C}^r$, we write $\mathfrak{c}^{[D]} := \prod_{i=1}^r \mathfrak{c}_i^{b_i}$, where $[D] = (b_1, \dots, b_r)$ with respect to the fixed basis of $\operatorname{Pic}(X)$. Then, as in [Pieropan 2016, §2],

$$X(\mathbb{K}) = \mathscr{X}(\mathcal{O}_\mathbb{K}) = \bigsqcup_{\mathfrak{c}\in\mathcal{C}^r} \pi^{\mathfrak{c}}(\mathscr{Y}^{\mathfrak{c}}(\mathcal{O}_\mathbb{K})),$$

where $\pi^{\mathfrak{c}} : \mathscr{Y}^{\mathfrak{c}} \to \mathscr{X}$ is the twist of $\pi$ defined in [Frei and Pieropan 2016, Theorem 2.7]. The fibers of $\pi|_{\mathscr{Y}^{\mathfrak{c}}(\mathcal{O}_\mathbb{K})}$ are all isomorphic to $(\mathcal{O}_\mathbb{K}^\times)^r$, and $\mathscr{Y}^{\mathfrak{c}}(\mathcal{O}_\mathbb{K}) \subseteq \mathcal{O}_\mathbb{K}^{\mathcal{I}}$ is the subset of points $\boldsymbol{x} \in \bigoplus_{(i,j)\in\mathcal{I}} \mathfrak{c}^{[D_{i,j}]}$ that satisfy

$$\sum_{\sigma\in\Sigma_{\max}} \boldsymbol{x}^{D_\sigma} \mathfrak{c}^{-[D_\sigma]} = \mathcal{O}_\mathbb{K}. \tag{2-1}$$

Let $N$ be the lattice of cocharacters of $X$. Then $\Sigma \subseteq N \otimes_\mathbb{Z} \mathbb{R}$. For every $(i, j) \in \mathcal{I}$, let $\nu_{i,j}$ be the unique generator of $\rho_{i,j} \cap N$. For every torus-invariant $\mathbb{Q}$-divisor $D = \sum_{i=1}^s \sum_{j=1}^{n_i} a_{i,j} D_{i,j}$ of $X$ and for every $\sigma \in \Sigma_{\max}$, let $u_{\sigma,D} \in \operatorname{Hom}_\mathbb{Z}(N, \mathbb{Q})$ be the character determined by $u_{\sigma,D}(\nu_{i,j}) = a_{i,j}$ for all $(i, j) \in \mathcal{J}_\sigma$, and define $D(\sigma) := D - \sum_{i=1}^s \sum_{j=1}^{n_i} u_{\sigma,D}(\nu_{i,j}) D_{i,j}$. Then $D$ and $D(\sigma)$ are linearly equivalent.

**2.1. *Torus-invariant divisors.*** We collect properties of toric varieties and their torus-invariant divisors.

**Lemma 2.1.** (i) *Let $\sigma \in \Sigma_{\max}$.*

(a) *For $i \in I_\sigma$, there is a unique index $j_{i,\sigma} \in \{1, \dots, n_i\}$ such that $(i, j_{i,\sigma}) \in \mathcal{I}_\sigma$. So $\#I_\sigma = \#\mathcal{I}_\sigma = r$.*

(b) *For $i \in I_\sigma$, we have $(i, j') \in \mathcal{J}_\sigma$ for all $j' \in \{1, \dots, n_i\} \smallsetminus \{j_{i,\sigma}\}$.*

(c) *For $i \in \{1, \dots, s\} \smallsetminus I_\sigma$, we have $\{(i, 1), \dots, (i, n_i)\} \subseteq \mathcal{J}_\sigma$.*

*Let $D$ be a torus-invariant $\mathbb{Q}$-divisor on $X$. For $\sigma \in \Sigma_{\max}$, write*

$$D(\sigma) = \sum_{i=1}^{s} \sum_{j=1}^{n_i} \alpha_{i,j,\sigma} D_{i,j}.$$

*For $i \in \{1, \dots, s\}$, let $\alpha_{i,\sigma} = \sum_{j=1}^{n_i} \alpha_{i,j,\sigma}$.*

(ii) *Let $\sigma \in \Sigma_{\max}$. Then $D(\sigma) = \sum_{i\in I_\sigma} \alpha_{i,\sigma} D_{i,j_{i,\sigma}}$.*

(iii) *Let $\sigma, \sigma' \in \Sigma_{\max}$. If there are $i \in I_\sigma$ and $j \in \{1, \dots, n_i\}$ such that $\mathcal{J}_\sigma \cap \mathcal{J}_{\sigma'} = \mathcal{J}_\sigma \smallsetminus \{(i, j)\}$, then $I_\sigma = I_{\sigma'}$ and $\alpha_{i',\sigma} = \alpha_{i',\sigma'}$ for all $i' \in \{1, \dots, s\}$.*

(iv) *Let $\sigma \in \Sigma_{\max}$ and, for every $i \in I_\sigma$, let $j_i \in \{1, \dots, n_i\}$. Then there exists a unique $\sigma' \in \Sigma_{\max}$ such that $I_{\sigma'} = I_\sigma$, $(i, j_i) \in \mathcal{I}_{\sigma'}$ for $i \in I_\sigma$, and $\alpha_{i,\sigma'} = \alpha_{i,\sigma}$ for $i \in \{1, \dots, s\}$.*

(v) *The relation $\sigma \sim \sigma'$ if and only if $I_\sigma = I_{\sigma'}$ defines an equivalence relation on $\Sigma_{\max}$, and the equivalence class of $\sigma$ has cardinality $\prod_{i\in I_\sigma} n_i$.*

(vi) *Let $\mathcal{J} \subseteq \mathcal{I}$ be minimal with respect to inclusion and such that $\mathcal{J} \cap \mathcal{I}_\sigma \neq \varnothing$ for all $\sigma \in \Sigma_{\max}$. Let $i \in \{1, \dots, s\}$ such that $\{(i, 1), \dots, (i, n_i)\} \cap \mathcal{J} \neq \varnothing$. Then $\{(i, 1), \dots, (i, n_i)\} \subseteq \mathcal{J}$.*

*Proof.* Part (i) follows from the fact that $[D_{i,j}] = \boldsymbol{D}_i$ for all $j \in \{1, \dots, n_i\}$ and that the set $\{\boldsymbol{D}_i : i \in I_\sigma\}$ is a basis of $\mathrm{Pic}(X)$ by [Cox et al. 2011, Theorem 4.2.8] as $X$ is smooth and proper.

Part (ii) follows from part (i) and the fact that, by construction, $\alpha_{i,j,\sigma} = 0$ whenever $(i, j) \in \mathcal{J}_\sigma$.

For part (iii) we observe that if $\sigma \neq \sigma'$, then $\mathcal{J}_\sigma = (\mathcal{J}_\sigma \cap \mathcal{J}_{\sigma'}) \sqcup \{(i, j)\}$ and $\mathcal{J}_{\sigma'} = (\mathcal{J}_\sigma \cap \mathcal{J}_{\sigma'}) \sqcup \{(i, j_{i,\sigma})\}$, where $j_{i,\sigma}$ is the index defined in part (i). Thus $i \in I_\sigma \cap I_{\sigma'}$, and, for every index $i' \in \{1, \dots, s\}$ with $i' \neq i$, we have

$$\mathcal{J}_\sigma \cap \{(i', 1), \dots, (i', n_{i'})\} = \mathcal{J}_{\sigma'} \cap \{(i', 1), \dots, (i', n_{i'})\} \subseteq \mathcal{J}_\sigma \cap \mathcal{J}_{\sigma'}.$$

Recall that

$$[D(\sigma)] = \sum_{i \in I_\sigma} \alpha_{i,\sigma} \boldsymbol{D}_i \quad \text{and} \quad [D(\sigma')] = \sum_{i \in I_{\sigma'}} \alpha_{i,\sigma'} \boldsymbol{D}_i.$$

Now the result follows as $[D(\sigma)] = [D(\sigma')]$ in $\mathrm{Pic}(X)$ and $\{\boldsymbol{D}_i : i \in I_\sigma\}$ is a basis of $\mathrm{Pic}(X)$.

For part (iv), we write $I_\sigma = \{i_1, \dots, i_r\}$. We construct by induction $\sigma_1, \dots, \sigma_r$ such that, for each $l \in \{1, \dots, r\}$,

$$(i_1, j_{i_1}), \dots, (i_l, j_{i_l}) \in \mathcal{I}_{\sigma_l}, \quad I_{\sigma_l} = I_\sigma, \quad \text{and} \quad \alpha_{i,\sigma_l} = \alpha_{i,\sigma} \ \text{ for all } i \in \{1, \dots, s\}.$$

If $(i_1, j_{i_1}) \in \mathcal{I}_\sigma$, let $\sigma_1 = \sigma$. Otherwise, $(i_1, j_{i_1}) \in \mathcal{J}_\sigma$ and by [Salberger 1998, Lemma 8.9] there is $\sigma_1 \in \Sigma_{\max}$ such that $\mathcal{J}_{\sigma_1} \cap \mathcal{J}_\sigma = \mathcal{J}_\sigma \smallsetminus \{(i_1, j_{i_1})\}$. Since $i_1 \in I_\sigma$, by part (iii) we have $I_{\sigma_1} = I_\sigma$ and $\alpha_{i,\sigma_1} = \alpha_{i,\sigma}$ for all $i \in \{1, \dots, s\}$. Assume that we have constructed $\sigma_{l-1}$ for given $l \leq r$. If $(i_l, j_{i_l}) \in \mathcal{I}_{\sigma_{l-1}}$, let $\sigma_l = \sigma_{l-1}$. Otherwise, $(i_l, j_{i_l}) \in \mathcal{J}_{\sigma_{l-1}}$ and by [Salberger 1998, Lemma 8.9] there is $\sigma_l \in \Sigma_{\max}$ such that $\mathcal{J}_{\sigma_l} \cap \mathcal{J}_{\sigma_{l-1}} = \mathcal{J}_{\sigma_{l-1}} \smallsetminus \{(i_l, j_{i_l})\}$. Since $i_l \in I_{\sigma_{l-1}}$, by part (iii) we have $I_{\sigma_l} = I_{\sigma_{l-1}} = I_\sigma$ and $\alpha_{i,\sigma_l} = \alpha_{i,\sigma_{l-1}} = \alpha_{i,\sigma}$ for all $i \in \{1, \dots, s\}$. Since $(i_1, j_{i_1}), \dots, (i_{l-1}, j_{i_{l-1}}) \in \mathcal{I}_{\sigma_{l-1}}$ and $\mathcal{J}_{\sigma_l} = (\mathcal{J}_{\sigma_{l-1}} \cap \mathcal{J}_{\sigma_l}) \cup \{(i_l, j_{i_l,\sigma_{l-1}})\}$, where $j_{i_l,\sigma_{l-1}}$ is the index defined in part (i), we conclude that $(i_1, j_{i_1}), \dots, (i_l, j_{i_l}) \in \mathcal{I}_{\sigma_l}$. Take $\sigma' = \sigma_r$. The uniqueness of $\sigma'$ follows from part (i), as $\sigma'$ is completely determined by $\mathcal{I}_{\sigma'}$.

Part (v) is a direct consequence of part (iv).

For part (vi), let $j \in \{1, \dots, n_i\}$ such that $(i, j) \in \mathcal{J}$. By minimality of $\mathcal{J}$, there exists $\sigma \in \Sigma_{\max}$ such that $\mathcal{J} \cap \mathcal{I}_\sigma = \{(i, j)\}$. If $n_i > 1$, let $j' \in \{1, \dots, n_i\} \smallsetminus \{j\}$. By [Salberger 1998, Lemma 8.9] there is $\sigma' \in \Sigma_{\max}$ such that $\mathcal{J}_{\sigma'} \cap \mathcal{J}_\sigma = \mathcal{J}_\sigma \smallsetminus \{(i, j')\}$. Hence $\mathcal{I}_{\sigma'} = (\mathcal{I}_\sigma \smallsetminus \{(i, j)\}) \cup \{(i, j')\}$. Since $\mathcal{J} \cap (\mathcal{I}_\sigma \smallsetminus \{(i, j)\}) = \varnothing$ and $\mathcal{J} \cap \mathcal{I}_{\sigma'} \neq \varnothing$, we conclude that $(i, j') \in \mathcal{J}$. □

**2.2.** ***Heights.*** Let $L$ be a semiample torus-invariant $\mathbb{Q}$-divisor on $X$. Let $H_L$ be the height on $X$ defined by $L$ as in [Pieropan and Schindler 2024, §6.3]. For $\sigma \in \Sigma_{\max}$, write

$$L(\sigma) = \sum_{i=1}^{s} \sum_{j=1}^{n_i} \alpha_{i,j,\sigma} D_{i,j} \quad \text{and} \quad \alpha_{i,\sigma} = \sum_{j=1}^{n_i} \alpha_{i,j,\sigma}$$

for all $i \in \{1, \dots, s\}$. Let $\Omega_{\mathbb{K}}$ be the set of places of $\mathbb{K}$.

**Lemma 2.2.** *For every $\nu \in \Omega_{\mathbb{K}}$ and every $\boldsymbol{x} \in \mathscr{Y}(\mathbb{K})$, we have*

$$\sup_{\sigma \in \Sigma_{\max}} |x^{L(\sigma)}|_\nu = \sup_{\sigma \in \Sigma_{\max}} \prod_{i=1}^{s} \sup_{1 \leq j \leq n_i} |x_{i,j}|_\nu^{\alpha_{i,\sigma}}.$$

*Proof.* Fix $\nu \in \Omega_{\mathbb{K}}$ and $\boldsymbol{x} \in \mathscr{Y}(\mathbb{K})$. By Lemma 2.1 (ii), we have

$$\sup_{\sigma\in\Sigma_{\max}} |x^{L(\sigma)}|_\nu = \sup_{\sigma\in\Sigma_{\max}} \prod_{i\in I_\sigma} |x_{i,j_{i,\sigma}}|_\nu^{\alpha_{i,\sigma}}.$$

For every $i \in \{1, \dots, s\}$, let $j_i \in \{1, \dots, n_i\}$ such that $|x_{i,j_i}|_\nu = \sup_{1\le j\le n_i} |x_{i,j}|_\nu$. Let $\sigma \in \Sigma_{\max}$. By Lemma 2.1 (iv) there is $\sigma' \in \Sigma_{\max}$ such that $I_{\sigma'} = I_\sigma$, $(i, j_i) \in \mathcal{I}_{\sigma'}$ for all $i \in I_\sigma$, and $\alpha_{i,\sigma'} = \alpha_{i,\sigma}$ for all $i \in \{1, \dots, s\}$. Then

$$|x^{L(\sigma')}|_\nu = \prod_{i\in I_{\sigma'}} |x_{i,j_i}|_\nu^{\alpha_{i,\sigma'}} = \prod_{i\in I_\sigma} \sup_{1\le j\le n_i} |x_{i,j}|_\nu^{\alpha_{i,\sigma}} = \prod_{i=1}^{s} \sup_{1\le j\le n_i} |x_{i,j}|_\nu^{\alpha_{i,\sigma}}. \qquad \square$$

Thus

$$H_L(\boldsymbol{x}) = \prod_{\nu\in\Omega_{\mathbb{K}}} \sup_{\sigma\in\Sigma_{\max}} \prod_{i=1}^{s} \sup_{1\le j\le n_i} |x_{i,j}|_\nu^{\alpha_{i,\sigma}} \quad \text{for all } \boldsymbol{x} \in \mathscr{Y}(\mathbb{K}).$$

**2.3.** ***Coprimality conditions.*** We now rewrite the coprimality condition (2-1) in terms of the notation introduced in this paper.

**Lemma 2.3.** *For all* $\boldsymbol{x} \in \bigoplus_{(i,j)\in\mathcal{I}} \mathfrak{c}^{[D_{i,j}]}$,

$$\sum_{\sigma\in\Sigma_{\max}} \boldsymbol{x}^{D_\sigma} \mathfrak{c}^{-[D_\sigma]} = \sum_{\sigma\in\Sigma_{\max}} \prod_{i\in I_\sigma} (x_{i,1}, \dots, x_{i,n_i}) \mathfrak{c}^{-\boldsymbol{D}_i}.$$

*Proof.* For $\sigma \in \Sigma_{\max}$, let

$$X_\sigma = \left\{ \prod_{i\in I_\sigma} x_{i,j_i} : j_i \in \{1, \dots, n_i\}\, \forall i \in \{1, \dots, s\} \right\}.$$

The inclusion $\subseteq$ is clear as $\boldsymbol{x}^{D_\sigma} \in X_\sigma$ and $\mathfrak{c}^{-[D_\sigma]} = \prod_{i\in I_\sigma} \mathfrak{c}^{-\boldsymbol{D}_i}$ for all $\sigma \in \Sigma_{\max}$. For the converse inclusion, fix $\sigma \in \Sigma_{\max}$ and $x \in X_\sigma$. For every $i \in I_\sigma$, let $j_i \in \{1, \dots, n_i\}$ such that $x = \prod_{i\in I_\sigma} x_{i,j_i}$. By Lemma 2.1 (iv) there is $\sigma' \in \Sigma_{\max}$ such that $I_{\sigma'} = I_\sigma$ and $(i, j_i) \in \mathcal{I}_{\sigma'}$ for $i \in I_\sigma$. Then $\boldsymbol{x}^{D_{\sigma'}} = x$. $\square$

**2.4.** ***Möbius function.*** Let $\mathcal{I}_{\mathbb{K}}$ be the set of nonzero ideals of $\mathcal{O}_{\mathbb{K}}$. Let $\chi : \mathcal{I}_{\mathbb{K}}{}^s \to \{0, 1\}$ be the characteristic function of the subset

$$\left\{ \mathfrak{b} \in \mathcal{I}_{\mathbb{K}}{}^s : \sum_{\sigma\in\Sigma_{\max}} \prod_{i\in I_\sigma} \mathfrak{b}_i = \mathcal{O}_{\mathbb{K}} \right\}. \tag{2-2}$$

For every $\mathfrak{d} \in \mathcal{I}_{\mathbb{K}}{}^s$, let $\chi_{\mathfrak{d}} : \mathcal{I}_{\mathbb{K}}{}^s \to \{0, 1\}$ be the characteristic function of the subset

$$\{\mathfrak{b} \in \mathcal{I}_{\mathbb{K}}{}^s : \mathfrak{b}_i \subseteq \mathfrak{d}_i\ \forall i \in \{1, \dots, s\}\}.$$

As in [Peyre 1995, Lemme 8.5.1], there exists a unique multiplicative function $\mu : \mathcal{I}_{\mathbb{K}}{}^s \to \mathbb{Z}$ such that

$$\chi = \sum_{\mathfrak{d}\in\mathcal{I}_{\mathbb{K}}{}^s} \mu(\mathfrak{d}) \chi_{\mathfrak{d}}.$$

Note that if $X = \mathbb{P}^n_{\mathbb{Q}}$, the function $\mu$ defined above coincides with the classical Möbius function.

**Remark 2.4.** Let $\mathfrak{p} \in \mathcal{I}_{\mathbb{K}}$ be a prime ideal. The function $\mu$ is defined recursively by the formula $\mu(\mathfrak{b}) = \chi(\mathfrak{b}) - \sum_{\mathfrak{b} \subsetneq \mathfrak{d}} \mu(\mathfrak{d})$ for every $\mathfrak{b} \in \mathcal{I}_{\mathbb{K}}{}^s$ and satisfies the following properties:

(i) $\mu(\mathbf{1}) = \chi(\mathbf{1}) = 1$.

(ii) If $e_i \geq 2$ for some $i \in \{1, \dots, s\}$, then $\mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) = 0$, as in that case $\chi(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) = \chi(\mathfrak{p}^{e'_1}, \dots, \mathfrak{p}^{e'_s})$ for $e'_i = e_i - 1$ and $e'_l = e_l$ for all $l \neq i$.

(iii) By induction one shows that $\mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) = 0$ whenever $(e_1, \dots, e_s) \neq \mathbf{0}$ and there is $\sigma \in \Sigma_{\max}$ such that $e_i = 0$ for all $i \in I_\sigma$, as $\chi(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) = 1$ if and only if there is $\sigma \in \Sigma_{\max}$ such that $e_i = 0$ for all $i \in I_\sigma$.

(iv) Let

$$\tilde{f} := \min\{\#J : J \subseteq \{1, \dots, s\},\ J \cap I_\sigma \neq \varnothing\ \forall \sigma \in \Sigma_{\max}\}.$$

By property (iii), if $\mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) \neq 0$, then there are at least $\tilde{f}$ indices $i$ with $e_i = 1$. Let $J \subseteq \{1, \dots, s\}$ be smallest with respect to inclusion and such that $J \cap I_\sigma \neq \varnothing$ for all $\sigma \in \Sigma_{\max}$. Let $J' = J \smallsetminus \{j\}$ for some $j \in J$. Let $e_i = 1$ for $i \in J$ and $e_i = 0$ for $i \notin J$. Let $e'_i = e_i$ for $i \neq j$ and $e'_j = 0$. Then $\chi(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) = 0$ and $\chi(\mathfrak{p}^{e'_1}, \dots, \mathfrak{p}^{e'_s}) = 1$ by minimality of $J$. Thus $\mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) = -1 \neq 0$. Hence

$$\tilde{f} = \min\left\{\sum_{i=1}^{s} e_i : (e_1, \dots, e_s) \neq \mathbf{0},\ \mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) \neq 0\right\}. \tag{2-3}$$

For $\beta = (\beta_1, \dots, \beta_s) \in \mathbb{R}^s_{\geq 0}$, let

$$f_\beta := \min\left\{\sum_{i=1}^{s} \beta_i e_i : (e_1, \dots, e_s) \neq \mathbf{0},\ \mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) \neq 0\right\}.$$

**Lemma 2.5.** (i) *The series*

$$\sum_{\mathfrak{d} \in \mathcal{I}_{\mathbb{K}}^s} \frac{\mu(\mathfrak{d})}{\prod_{i=1}^{s} \mathfrak{N}(\mathfrak{d}_i)^{\beta_i}}$$

*converges absolutely if* $f_\beta > 1$.

(ii) *If* $f_\beta > 1$ *and* $\beta_1, \dots, \beta_s \in \mathbb{Z}_{>0}$, *then*

$$\sum_{\mathfrak{d} \in \mathcal{I}_{\mathbb{K}}^s} \frac{\mu(\mathfrak{d})}{\prod_{i=1}^{s} \mathfrak{N}(\mathfrak{d}_i)^{\beta_i}} > 0.$$

*Proof.* For part (i) we follow the proof of [Salberger 1998, Lemma 11.15] and [Pieropan 2016, Proposition 4]. For $\mathfrak{p} \in \mathcal{I}_{\mathbb{K}}$ a prime ideal, let

$$S(\mathfrak{p}) = \sum_{(e_1, \dots, e_s) \in \mathbb{Z}^s_{\geq 0}} \frac{|\mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s})|}{\prod_{i=1}^{s} \mathfrak{N}(\mathfrak{p})^{\beta_i e_i}}.$$

As in the two results cited,

$$\lim_{b \to \infty} \sum_{\substack{\mathfrak{d} \in \mathcal{I}_{\mathbb{K}}^s \\ \prod_{i=1}^{s} \mathfrak{N}(\mathfrak{d}_i) \leq b}} \frac{|\mu(\mathfrak{d})|}{\prod_{i=1}^{s} \mathfrak{N}(\mathfrak{d}_i)^{\beta_i}} = \prod_{\mathfrak{p}} S(\mathfrak{p}).$$

By Remark 2.4 (ii), the sum $S(\mathfrak{p})$ is finite. By definition of $f_\beta$, if $\mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) \neq 0$ and $(e_1, \dots, e_s) \neq \mathbf{0}$, then $f_\beta \le \sum_{i=1}^s \beta_i e_i$. Thus

$$S(\mathfrak{p}) = 1 + \frac{1}{\mathfrak{N}(\mathfrak{p})^{f_\beta}} Q\left(\frac{1}{\mathfrak{N}(\mathfrak{p})}\right),$$

where $Q : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is a monotone increasing function. Since $\mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s})$ is independent of the choice of $\mathfrak{p}$, the function $Q$ is independent of the choice of $\mathfrak{p}$. Thus

$$\sum_{\mathfrak{p}} \frac{1}{\mathfrak{N}(\mathfrak{p})^{f_\beta}} Q\left(\frac{1}{\mathfrak{N}(\mathfrak{p})}\right) \le [\mathbb{K} : \mathbb{Q}] Q(1) \sum_{n \in \mathbb{Z}_{>0}} \frac{1}{n^{f_\beta}}.$$

In part (ii) the series is absolutely convergent by part (i); hence it suffices to show that each factor of its Euler product $\prod_{\mathfrak{p}} S_{\mathfrak{p}}$ is positive. For a prime ideal $\mathfrak{p} \in \mathcal{I}_{\mathbb{K}}$, let $\mathcal{O}_{\mathfrak{p}}$ be the ring of integers of the completion $\mathbb{K}_{\mathfrak{p}}$ of $\mathbb{K}$ at the valuation $v_{\mathfrak{p}}$ defined by $\mathfrak{p}$. Endow $\mathbb{K}_{\mathfrak{p}}$ with the Haar measure normalized such that $\mathcal{O}_{\mathfrak{p}}$ has volume 1. Then $\int_{\mathfrak{p}^j \mathcal{O}_{\mathfrak{p}}} \mathrm{d}y = \mathfrak{N}(\mathfrak{p})^{-j}$ for all $j \ge 0$ by [Chambert-Loir et al. 2018, §1.1.13] and [Neukirch 1999, Proposition II.4.3]. We denote by $\chi$ the characteristic function of (2-2), where ideals of $\mathcal{O}_{\mathbb{K}}$ are replaced by ideals of $\mathcal{O}_{\mathfrak{p}}$. By Remark 2.4 (ii),

$$\begin{aligned} S_{\mathfrak{p}} &= \sum_{\boldsymbol{e} \in \{0,1\}^s} \mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) \prod_{i=1}^s \mathfrak{N}(\mathfrak{p})^{-e_i \beta_i} = \sum_{\boldsymbol{e} \in \{0,1\}^s} \mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) \prod_{i=1}^s \prod_{j=1}^{\beta_i} \int_{\mathfrak{p}^{e_i}} \mathrm{d}y_{i,j} \\ &= \int_{\mathcal{O}_{\mathfrak{p}}^{\sum_{i=1}^s \beta_i}} \chi((y_{1,1}, \dots, y_{1,\beta_1}), \dots, (y_{s,1}, \dots, y_{s,\beta_s})) \prod_{i=1}^s \prod_{j=1}^{\beta_i} \mathrm{d}y_{i,j} \\ &\ge \int_{(\mathcal{O}_{\mathfrak{p}}^\times)^{\sum_{i=1}^s \beta_i}} \prod_{i=1}^s \prod_{j=1}^{\beta_i} \mathrm{d}y_{i,j} = \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\right)^{\sum_{i=1}^s \beta_i} > 0, \end{aligned}$$

as $\chi$ is a nonnegative function with $\chi(\mathcal{O}_{\mathfrak{p}}, \dots, \mathcal{O}_{\mathfrak{p}}) = 1$. □

**Definition 2.6.** A function $A : \mathbb{Z}_{>0}^s \to \mathbb{R}$ is compatible with Möbius inversion on $X$ if there exist $\beta_1, \dots, \beta_s \in \mathbb{R}_{\ge 0}^s$ such that $A(\boldsymbol{d}) \ll \prod_{i=1}^s d_i^{-\beta_i}$ with $f_{(\beta_1, \dots, \beta_s)} > 1$.

**Remark 2.7.** (i) The inequality $f_\beta > 1$ holds whenever $\beta_1, \dots, \beta_s > 1$.

(ii) If $\beta_1 = \dots = \beta_s = 1$, then $f_\beta = \tilde{f}$ by (2-3).

(iii) Case $\beta_1 = n_1, \dots, \beta_s = n_s$: As in [Salberger 1998, Lemma 11.15 (d)], let $f$ be the smallest positive integer such that there are $f$ rays of the fan $\Sigma$ that are not contained in a maximal cone. Then $f \ge 2$, as $X$ is proper. Moreover,

$$f = \min\{\#\mathcal{J} : \mathcal{J} \subseteq \mathcal{I},\ \mathcal{J} \cap \mathcal{I}_\sigma \neq \varnothing\ \forall \sigma \in \Sigma_{\max}\},$$

and Remark 2.4 combined with Lemma 2.1 (vi) gives

$$\begin{aligned} f &= \min\left\{\sum_{i \in J} n_i : J \subseteq \{1, \dots, s\},\ J \cap I_\sigma \neq \varnothing\ \forall \sigma \in \Sigma_{\max},\ \#J = \tilde{f}\right\} \\ &= \min\left\{\sum_{i=1}^s n_i e_i : (e_1, \dots, e_s) \neq \mathbf{0},\ \mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) \neq 0\right\}. \end{aligned}$$

## 3. Subvarieties

Here we want to count rational points or Campana points of bounded height in subvarieties of toric varieties.

From now on $\mathbb{K} = \mathbb{Q}$. Let $X$ be a complete smooth split toric variety as in Section 2. Assume that $\operatorname{rk}\operatorname{Pic}(X) \geq 2$, that is, $X$ is not a projective space. Let $L$ be a semiample toric invariant $\mathbb{Q}$-divisor on $X$ that satisfies [Pieropan and Schindler 2024, Assumption 6.3]. The latter holds, for example, if $L$ is ample. Throughout this section, we will abbreviate [Pieropan and Schindler 2024] as [PS24].

Let $g_1, \dots, g_t \in R$ be $\operatorname{Pic}(X)$-homogeneous elements. Let $V \subseteq X$ be the schematic intersection of the $t$ hypersurfaces defined by $g_1, \dots, g_t$. Let $T \subseteq X$ be the torus. Without loss of generality, we can assume that $V \cap T \neq \varnothing$. Otherwise, $V$ is contained in a complete smooth split toric subvariety $X'$ of $X$, and we can replace $X$ by $X'$. Fix $m_{i,j} \in \mathbb{Z}_{\geq 1}$ for each $(i, j) \in \mathcal{I}$. Let $\boldsymbol{m} = (m_{i,j})_{(i,j)\in\mathcal{I}}$ and

$$\mathscr{D}_{\boldsymbol{m}} = \sum_{i=1}^{s}\sum_{j=1}^{n_i}\Big(1 - \frac{1}{m_{i,j}}\Big)\mathscr{D}_{i,j}.$$

Let $\mathscr{V}$ be the Zariski closure of $V$ in $\mathscr{X}$. We define the intersection multiplicity $n_v(\mathscr{D}_i|_{\mathscr{V}}, \boldsymbol{x})$ of a point $\boldsymbol{x} : \operatorname{Spec}\mathcal{O}_{\mathbb{K}} \to \mathscr{V}$ with $\mathscr{D}_i|_{\mathscr{V}}$ at a place $v$ of $\mathbb{K}$ to be the colength of the ideal of the fiber product of $\operatorname{Spec}\mathcal{O}_{\mathbb{K}} \times_{\mathscr{V}} \mathscr{D}_i|_{\mathscr{V}}$ after base change to the completion of $\mathcal{O}_{\mathbb{K}}$ at $v$. This definition coincides with the one in [Pieropan et al. 2021, §3] whenever $\mathscr{V}$ is regular. Let $(\mathscr{V}, \mathscr{D}_{\boldsymbol{m}}|_{\mathscr{V}})(\mathbb{Z})$ be the set of Campana $\mathbb{Z}$-points on the Campana orbifold $(\mathscr{V}, \mathscr{D}_{\boldsymbol{m}}|_{\mathscr{V}})$ as in [Pieropan et al. 2021, Definition 3.4].

Let $N_V(B)$ be the number of points in $(\mathscr{V}, \mathscr{D}_{\boldsymbol{m}}|_{\mathscr{V}})(\mathbb{Z}) \cap T(\mathbb{Q})$ of height $H_L$ at most $B$. If $m_{i,j} = 1$ for all $(i, j) \in \mathcal{I}$, then $N_V(B)$ is the set of $\mathbb{Q}$-rational points on $V \cap T$ of height $H_L$ at most $B$.

For $i \in \{1, \dots, s\}$ and $\boldsymbol{x} \in \mathscr{Y}(\mathbb{Z})$, let $y_i = \sup_{1\leq j\leq n_i} |x_{i,j}|$. For $\sigma \in \Sigma_{\max}$, write

$$L(\sigma) = \sum_{i=1}^{s}\sum_{j=1}^{n_i}\alpha_{i,j,\sigma} D_{i,j} \quad\text{and}\quad \alpha_{i,\sigma} = \sum_{j=1}^{n_i}\alpha_{i,j,\sigma} \qquad \text{for all } i \in \{1, \dots, s\}.$$

Then, by [PS24, Proposition 6.10] and Lemma 2.2,

$$H_L(\boldsymbol{x}) = \sup_{\sigma\in\Sigma_{\max}} \prod_{i=1}^{s} y_i^{\alpha_{i,\sigma}}.$$

By construction,

$$(\mathscr{V}, \mathscr{D}_{\boldsymbol{m}}|_{\mathscr{V}})(\mathbb{Z}) = (\mathscr{X}, \mathscr{D}_{\boldsymbol{m}})(\mathbb{Z}) \cap V(\mathbb{Q}).$$

We use the torsor parametrization of $(\mathscr{X}, \mathscr{D}_{\boldsymbol{m}})(\mathbb{Z})$ from [PS24, §6.4]. For $B > 0$ and $\boldsymbol{d} \in (\mathbb{Z}_{>0})^s$, let $A(B, \boldsymbol{d})$ be the set of points $\boldsymbol{x} = (x_{i,j})_{1\leq i\leq s, 1\leq j\leq n_i} \in (\mathbb{Z}_{\neq 0})^{\mathcal{I}}$ such that

$$H(\boldsymbol{x}) \leq B, \tag{3-1}$$

$$d_i \mid x_{i,j} \quad \text{for all } i \in \{1, \dots, s\} \text{ and for all } j \in \{1, \dots, n_i\}, \tag{3-2}$$

$$x_{i,j} \text{ is } m_{i,j}\text{-full} \quad \text{for all } i \in \{1, \dots, s\} \text{ and for all } j \in \{1, \dots, n_i\}, \tag{3-3}$$

$$g_1 = \cdots = g_t = 0. \tag{3-4}$$

We observe that $A(B,\boldsymbol{d})$ is a finite set by [PS24, Lemma 6.11]. Then

$$N_V(B)=\frac{1}{2^r}\sum_{\boldsymbol{d}\in(\mathbb{Z}_{>0})^s}\mu(d)\#A(B,\boldsymbol{d}) \tag{3-5}$$

by Lemma 2.3 and the definition of $\mu$ in Section 2.4.

Write

$$\#A(B,\boldsymbol{d})=\sum_{\substack{y_1,\dots,y_s\in\mathbb{Z}_{>0}\\ \prod_{i=1}^s y_i^{\alpha_{i,\sigma}}\le B\ \forall\sigma\in\Sigma_{\max}}} f_{\boldsymbol{d}}(y_1,\dots,y_s),$$

where

$$f_{\boldsymbol{d}}(y_1,\dots,y_s)=\#\Big\{\boldsymbol{x}\in(\mathbb{Z}_{\neq 0})^{\mathcal{I}}:\text{(3-2), (3-3), (3-4)},\ y_i=\sup_{1\le j\le n_i}|x_{i,j}|\ \forall i\in\{1,\dots,s\}\Big\}.$$

Let

$$F_{\boldsymbol{d}}(B_1,\dots,B_s)=\sum_{1\le y_i\le B_i,1\le i\le s} f_{\boldsymbol{d}}(y_1,\dots,y_s).$$

**Lemma 3.1.** *Assume that*

$$F_{\boldsymbol{d}}(B_1,\dots,B_s)=C_{M,\boldsymbol{d}}\prod_{i=1}^s B_i^{\varpi_i}+O\Big(C_{E,\boldsymbol{d}}\big(\min_{1\le i\le s}B_i\big)^{-\epsilon}\prod_{i=1}^s B_i^{\varpi_i}\Big) \tag{3-6}$$

*with $C_{M,\boldsymbol{d}},\ C_{E,\boldsymbol{d}},\ \varpi_1,\dots,\varpi_s,\ \epsilon>0$ such that $C_{M,\boldsymbol{d}}$ and $C_{E,\boldsymbol{d}}$ are compatible with Möbius inversion on $X$ as functions of the variables $\boldsymbol{d}$.*

*Let $a$ be the maximal value of $\sum_{i=1}^s\varpi_i u_i$ on the polytope $\mathcal{P}\subseteq\mathbb{R}^s$ defined by*

$$\sum_{i=1}^s\alpha_{i,\sigma}u_i\le 1\ \ \text{for all }\sigma\in\Sigma_{\max},\qquad u_i\ge 0\ \ \text{for all } i\in\{1,\dots,s\}.$$

*Let $F$ be the face of $\mathcal{P}$ where $\sum_{i=1}^s\varpi_i u_i=a$. Let $k$ be the dimension of $F$.*

(i) *If $F$ is not contained in a coordinate hyperplane of $\mathbb{R}^s$, then*

$$N_V(B)=cB^a(\log B)^k+O(B^a(\log B)^{k-1}(\log\log B)^s),$$

*where $k$ is the dimension of $F$ and*

$$c=(s-1-k)!\,c_{\mathcal{P}}2^{-r}\sum_{\boldsymbol{d}\in\mathbb{Z}_{>0}^s}\mu(\boldsymbol{d})C_{M,\boldsymbol{d}}. \tag{3-7}$$

*Here, $c_{\mathcal{P}}=\lim_{\delta\to 0}\delta^{k+1-s}\operatorname{meas}_{s-1}(H_\delta\cap\mathcal{P})$, where $H_\delta\subseteq\mathbb{R}^s$ is the hyperplane defined by $\sum_{i=1}^s\varpi_i u_i=a-\delta$ and $\operatorname{meas}_{s-1}$ is the $(s-1)$-dimensional measure on $H_\delta$ given by $\prod_{1\le i\le s,i\ne\tilde{i}}(\varpi_i\,\mathrm{d}u_i)$ for any choice of $\tilde{i}\in\{1,\dots,s\}$.*

(ii) *If $L$ is ample, then*

$$a=\inf\Big\{t\in\mathbb{R}:t[L]-\Big[\sum_{i=1}^s\varpi_i\boldsymbol{D}_i\Big]\text{ is effective}\Big\}$$

*and $k+1$ is the codimension of the minimal face of the effective cone of $X$ containing $a[L]-\big[\sum_{i=1}^s\varpi_i\boldsymbol{D}_i\big]$.*

(iii) *If $[L]=\sum_{i=1}^s\varpi_i\boldsymbol{D}_i$ is ample, then the face $F$ is not contained in a coordinate hyperplane, $a=1$, and $k=\operatorname{rk}\operatorname{Pic}(X)-1$.*

*Proof.* (i) Let $t_i = \varpi_i u_i$ for all $i \in \{1, \dots, s\}$. By the assumptions on $L$, the polytope $\mathcal{P}$ is bounded and nondegenerate by [PS24, Remark 6.2]. Applying [PS24, Theorem 1.1] to $\#A(B, \boldsymbol{d})$ gives

$$N_V(B) = cB^a(\log B)^k + O\biggl(B^a(\log B)^{k-1}(\log\log B)^s \sum_{\boldsymbol{d}\in(\mathbb{Z}_{>0})^s} \mu(d)C_{E,\boldsymbol{d}}\biggr).$$

The sums $\sum_{\boldsymbol{d}\in(\mathbb{Z}_{>0})^s}$ in the leading constant $c$ and in the error term converge absolutely by Lemma 2.5 as $C_{M,\boldsymbol{d}}$ and $C_{E,\boldsymbol{d}}$ are compatible with Möbius inversion on $X$.

(ii) Let

$$\mathbb{R}^r \hookrightarrow \mathbb{R}^s \hookrightarrow \mathbb{R}^{\mathcal{I}}$$

be the sequence of injective linear maps dual to

$$d : \bigoplus_{(i,j)\in\mathcal{I}} D_{i,j}\mathbb{Z} \twoheadrightarrow \bigoplus_{i=1}^{s} \boldsymbol{D}_i\mathbb{Z} \twoheadrightarrow \operatorname{Pic}(X).$$

Here,

$$\mathbb{R}^s \hookrightarrow \mathbb{R}^{\mathcal{I}}, \quad \sum_{i=1}^{s} u_i e_i \mapsto \sum_{i=1}^{s}\sum_{j=1}^{n_i} u_i e_{i,j},$$

where $\{e_1, \dots, e_s\}$ denotes the dual basis to $\{\boldsymbol{D}_1, \dots, \boldsymbol{D}_s\}$ and $\{e_{i,j} : (i, j) \in \mathcal{I}\}$ denotes the dual basis to $\{D_{i,j} : (i, j) \in \mathcal{I}\}$. Let $\widetilde{P}$ be the polytope defined by

$$\sum_{(i,j)\in\mathcal{I}} \alpha_{i,j,\sigma} u_{i,j} \le 1 \ \text{ for all } \sigma \in \Sigma_{\max}, \quad u_{i,j} \ge 0 \ \text{ for all } (i, j) \in \mathcal{I}.$$

Then $\widetilde{P} \cap \mathbb{R}^s = P$ and

$$\sum_{(i,j)\in\mathcal{I}} \frac{\varpi_i}{n_i} u_{i,j}\Big|_P = \sum_{i=1}^{s}\biggl(\sum_{j=1}^{n_i} \varpi_i/n_i\biggr)u_i.$$

By [PS24, Lemma 6.7], the face $F$ of $\widetilde{P}$ where the maximal value of the function

$$\sum_{(i,j)\in\mathcal{I}} \frac{\varpi_i}{n_i} u_{i,j} \tag{3-8}$$

is attained is contained in $\widetilde{P} \cap \mathbb{R}^r$ and hence also in $P$. Then $a$ is the maximal value of the function (3-8) on $P$. The dual linear programming problem is given by minimizing $\sum_{\sigma\in\Sigma_{\max}} \lambda_\sigma$ on the polytope given by

$$\sum_{\sigma\in\Sigma_{\max}} \alpha_{i,j,\sigma}\lambda_\sigma \ge \frac{\varpi_i}{n_i} \ \text{ for all } (i, j) \in \mathcal{I}, \quad \lambda_\sigma \ge 0 \ \text{ for all } \sigma \in \Sigma_{\max}.$$

The arguments that can be found in [PS24, §6.5.1] show that $a$ is the smallest real number such that $a[L] - \sum_{i=1}^{s}\sum_{j=1}^{n_i}(\varpi_i/n_i)\boldsymbol{D}_i$ is effective. As in [PS24, Proposition 6.13], the smallest face of $\operatorname{Eff}(X)$ that contains $a[L] - \sum_{i=1}^{s} \varpi_i \boldsymbol{D}_i$ is dual to the cone generated by $F$ in $\mathbb{R}^r$, and the latter is defined by $a\sum_{i=1}^{s} \alpha_{i,\sigma}u_i - \sum_{i=1}^{s} \varpi_i u_i = 0$ for any $\sigma \in \Sigma_{\max}$ such that $F \subseteq \bigl\{\sum_{i=1}^{s} \alpha_{i,j,\sigma}u_{i,j} = 1\bigr\}$. Thus the minimal face of $\operatorname{Eff}(X)$ containing $a[L] - \bigl[\sum_{i=1}^{s} \varpi_i \boldsymbol{D}_i\bigr]$ has codimension $k + 1$.

(iii) We argue as in the proof of [PS24, Lemma 6.7 (ii)]. Let $\widetilde{H} \subseteq \mathbb{R}^s$ be the inclusion dual to the surjection $\bigoplus_{i=1}^s \mathbb{R}\boldsymbol{D}_i \to \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Then $\sum_{i=1}^s \alpha_{i,\sigma} u_i = \sum_{i=1}^s \varpi_i u_i$ for all $\boldsymbol{u} \in \widetilde{H}$ and all $\sigma \in \Sigma_{\max}$. Thus $\mathcal{P} \cap \widetilde{H}$ is the set of elements $\boldsymbol{u}$ of $\widetilde{H}$ such that $u_1, \ldots, u_s \geq 0$ and $\sum_{i=1}^s \varpi_i u_i \leq 1$. Since $F \subseteq \widetilde{H}$ by [PS24, Lemma 6.7 (i)], we have $F = \widetilde{H} \cap \{\sum_{i=1}^s \varpi_i u_i = 1\}$. As in the proof of [PS24, Lemma 6.7 (ii)], we conclude that $F$ is not contained in a coordinate hyperplane of $\mathbb{R}^s$. □

## 4. Rational points on linear complete intersections

*Proof of Theorem 1.1.* For $1 \leq i \leq s$ and $1 \leq l \leq t_i$, let $g_{i,l} \in R$ be a linear polynomial defining $H_{i,j}$. Then

$$g_{i,l} = \sum_{j=1}^{n_i} c_{i,j,l} x_{i,j}, \quad l \in \{1, \ldots, t_i\},$$

with $c_{i,j,l} \in \mathbb{Z}$, and the $g_{i,1}, \ldots, g_{i,t_i}$ are linearly independent for all $i \in \{1, \ldots, s\}$. Let $m_{i,j} = 1$ for all $(i, j) \in \mathcal{I}$. Then

$$F_{\boldsymbol{d}}(B_1, \ldots, B_s) = \prod_{i=1}^s F_{i,d_i}(B_i),$$

where, for $i \in \{1, \ldots, s\}$, $d \in \mathbb{Z}_{>0}$, and $B > 0$,

$$F_{i,d}(B) = \#\Big\{(x_{i,1}, \ldots, x_{i,n_i}) \in (\mathbb{Z}_{\neq 0})^{n_i} : \sup_{1 \leq j \leq n_i} |x_{i,j}| \leq B,\, d \mid x_{i,j}\ \forall j \in \{1, \ldots, n_i\},\, g_{i,1} = \cdots = g_{i,t_i} = 0\Big\}.$$

For $i \in \{1, \ldots, s\}$, let $W_i \subseteq \mathbb{R}^{n_i}$ be the linear space defined by $g_{i,1} = \cdots = g_{i,t_i} = 0$, and let $\Lambda_i \subseteq W_i$ be the restriction of the standard lattice $\mathbb{Z}^{n_i} \subseteq \mathbb{R}^{n_i}$ to $W_i$. Then, by [Bombieri and Gubler 2006, Lemma 11.10.15], for every $T \geq 1$,

$$\begin{aligned}\#(\mathbb{Z}^{n_i} \cap [-T, T]^{n_i} \cap W_i) &= \#(\Lambda_i \cap T([-1, 1]^n \cap W_i)) \\ &= T^{n_i - t_i} \frac{\mathrm{meas}_{n_i - t_i}([-1, 1]^{n_i} \cap W_i)}{\det \Lambda_i} + O(T^{n_i - t_i - 1}),\end{aligned}$$

where $\mathrm{meas}_{n_i - t_i}$ is the $(n_i - t_i)$-dimensional measure induced by the Lebesgue measure on $\mathbb{R}^{n_i}$. Let

$$c_i = \frac{\mathrm{meas}_{n_i - t_i}([-1, 1]^{n_i} \cap W_i)}{\det \Lambda_i}.$$

Then applying this estimate with $T = B/d$ gives

$$F_{i,d}(B) = c_i (B/d)^{n_i - t_i} + O((B/d)^{n_i - t_i - 1})$$

whenever $d \leq B$. If $d > B$, then $F_{i,d}(B) = 0$ and the same estimate holds. Hence, for $\delta > 0$,

$$F_{\boldsymbol{d}}(B_1, \ldots, B_s) = C_{M,\boldsymbol{d}} \prod_{i=1}^s B_i^{n_i - t_i} + O\bigg(C_{E,\boldsymbol{d}} \bigg(\prod_{i=1}^s B_i^{n_i - t_i}\bigg) \big(\min_{1 \leq i \leq s} B_i\big)^{-\delta}\bigg),$$

where

$$C_{M,\boldsymbol{d}} = \prod_{i=1}^s \frac{c_i}{d_i^{n_i - t_i}}, \quad C_{E,\boldsymbol{d}} = \prod_{i=1}^s d_i^{-(n_i - t_i) + \delta}. \tag{4-1}$$

We show that, for $\delta > 0$ sufficiently small, the assumptions of Lemma 3.1 are satisfied. Since $n_i - t_i \geq 2$ for all $i \in \{1, \dots, s\}$ such that $t_i \neq 0$, if $f_{(n_1-t_1,\dots,n_s-t_s)} < 2$, by Remark 2.4 (iv), there is $\tilde{i} \in \{1, \dots, s\}$ such that $t_{\tilde{i}} = 0$, $\tilde{i} \in I_\sigma$ for all $\sigma \in \Sigma_{\max}$, and $n_{\tilde{i}} = 1$. Then $\rho_{\tilde{i},1}$ is not contained in any maximal cone of $\Sigma$, contradicting the fact that $X$ is proper. Thus $f_{(n_1-t_1,\dots,n_s-t_s)} \geq 2$. By definition and by Remark 2.4 (ii),

$$f_{(n_1-t_1-\delta,\dots,n_s-t_s-\delta)} \geq f_{(n_1-t_1,\dots,n_s-t_s)} - s\delta.$$

Since $V$ is a smooth complete intersection of smooth divisors, by adjunction [Corti 1992, Proposition 16.4], we have $K_V = K_X + \sum_{i=1}^{s} \sum_{l=1}^{t_i} [H_{i,l}]$. Since

$$\sum_{i=1}^{s} (n_i - t_i) \boldsymbol{D}_i = -[K_X] - \sum_{i=1}^{s} t_i \boldsymbol{D}_i = -[K_X] - \sum_{i=1}^{s} \sum_{l=1}^{t_i} [H_{i,l}],$$

Lemma 3.1 gives

$$N_V(B) = cB(\log B)^{b-1} + O(B(\log B)^{b-2}(\log\log B)^s),$$

where $b = \operatorname{rk}\operatorname{Pic}(X)$ and $c$ is defined in (3-7) with $k = b-1$, $C_{M,\boldsymbol{d}}$ given by (4-1), and $\varpi_i = n_i - t_i$ for $i \in \{1, \dots, s\}$. The restriction $\operatorname{Pic}(X) \to \operatorname{Pic}(V)$ is an isomorphism as $t_i \leq n_i - 2$ for all $i \in \{1, \dots, s\}$. The leading constant $c$ is positive by Lemma 2.5 (ii). □

## 5. Bihomogeneous hypersurfaces

*Proof of Theorem 1.2.* In the setting of Theorem 1.2, the hypersurfaces $H_1, \dots, H_t$ are defined by bihomogeneous polynomials $g_1, \dots, g_t$ of degree $(e_1, e_2)$ in the two sets of variables $\{x_{1,j} : 1 \leq j \leq n_1\}$ and $\{x_{2,j} : 1 \leq j \leq n_2\}$. Let $m_{i,j} = 1$ for all $(i, j) \in \mathcal{I}$.

We will apply [Schindler 2016, Theorem 4.4] with

$$R = t, \quad F_i = g_i, \quad \mathscr{B}_i = [-1, 1]^{n_i}, \quad P_i = B_i/d_i, \quad d_i = e_i.$$

In order to apply the cited result, we need to restrict the points to an open set. Let $U \subseteq \mathbb{A}^{n_1+n_2}$ be the open set therein. Since the complement of $U$ is the zero set of homogeneous polynomials by [Schindler 2016, Theorems 4.1 and 4.2], the set $W := \pi(\{\boldsymbol{x} \in Y : (x_{1,1}, \dots, x_{1,n_1}, x_{2,1}, \dots, x_{2,n_2}) \in U\})$ is an open subset of $X$. Then

$$N_{V,W}(B) = \frac{1}{2^r} \sum_{\boldsymbol{d} \in (\mathbb{Z}_{>0})^s} \mu(\boldsymbol{d}) \# A^W(B, \boldsymbol{d}),$$

with

$$A^W(B, \boldsymbol{d}) = \sum_{\substack{y_1,\dots,y_s \in \mathbb{Z}_{>0} \\ \prod_{i=1}^s y_i^{\alpha_{i,\sigma}} \leq B \ \forall \sigma \in \Sigma_{\max}}} f_{\boldsymbol{d}}^W(y_1, \dots, y_s)$$

and

$$f_{\boldsymbol{d}}^W(y_1, \dots, y_s) = \#\Big\{\boldsymbol{x} \in (\mathbb{Z}_{\neq 0})^{\mathcal{I}} : (x_{1,1}, \dots, x_{1,n_1}, x_{2,1}, \dots, x_{2,n_2}) \in U(\mathbb{Q}), \text{(3-2), (3-3), (3-4)}, \\ y_i = \sup_{1 \leq j \leq n_i} |x_{i,j}| \ \forall i \in \{1, \dots, s\}\Big\}.$$

Let

$$F_{\boldsymbol{d}}^W(B_1,\dots,B_s)=\sum_{1\le y_i\le B_i,1\le i\le s} f_{\boldsymbol{d}}^W(y_1,\dots,y_s).$$

Then

$$F_{\boldsymbol{d}}^W(B_1,\dots,B_s)=\widetilde{F}_{d_1,d_2}^W(B_1,B_2)\prod_{i=3}^{s}F_{i,d_i}(B_i),$$

where

$$\widetilde{F}_{d_1,d_2}^W(B_1,B_2)=\#\Big\{(x_{1,1},\dots,x_{1,n_1},x_{2,1},\dots,x_{2,n_2})\in(\mathbb{Z}_{\neq 0})^{n_1+n_2}\cap U(\mathbb{Q}):\\ \sup_{1\le j\le n_i}|y_{i,j}|\le B_i/d_i\ \forall i\in\{1,2\},\ g_1=\dots=g_t=0\Big\},$$

and, for $d\in\mathbb{Z}_{>0}$ and $B>0$,

$$F_{i,d}(B)=\#\Big\{(x_{i,1},\dots,x_{i,n_i})\in(\mathbb{Z}_{\neq 0})^{n_i}:\sup_{1\le j\le n_i}|x_{i,j}|\le B,\ d\mid x_{i,j}\ \forall j\in\{1,\dots,n_i\}\Big\}.$$

If $d\le B_i$, then

$$F_{i,d}(B)=2^{n_i}(B/d)^{n_i}+O((B/d)^{n_i-\delta})$$

with $0<\delta\le 1$. If $d>B$, then $F_{i,d}(B)=0$, and the same estimate holds.

To compute $\widetilde{F}_{d_1,d_2}^W(B_1,B_2)$, write $x_{i,j}=d_iy_{i,j}$ for all $(i,j)\in\mathcal{I}$. Then

$$\widetilde{F}_{d_1,d_2}^W(B_1,B_2)=\#\Big\{(y_{1,1},\dots,y_{1,n_1},y_{2,1},\dots,y_{2,n_2})\in(\mathbb{Z}_{\neq 0})^{n_1+n_2}\cap U(\mathbb{Q}):\\ \sup_{1\le j\le n_i}|y_{i,j}|\le B_i/d_i\ \forall i\in\{1,2\},\ g_1=\dots=g_t=0\Big\},$$

as the complement of $U$ is the zero set of homogeneous polynomials by [Schindler 2016, Theorems 4.1 and 4.2]. Let $V_i^*\subseteq\mathbb{A}^{n_1+n_2}$ be the locus where the matrix $(\partial g_l/\partial x_{i,j})_{1\le l\le t,1\le j\le n_i}$ does not have full rank. If $n_1+n_2>\dim V_1^*+\dim V_2^*+3\cdot 2^{e_1+e_2}e_1e_2t^3$, then, by [Schindler 2016, Theorem 4.4], there is $\delta>0$ such that

$$\begin{aligned}\widetilde{F}_{d_1,d_2}^W(B_1,B_2)&=C\prod_{i=1}^{2}(B_i/d_i)^{n_i-te_i}+O\Big(\big(\min_{i=1,2}B_i/d_i\big)^{-\delta}\prod_{i=1}^{2}(B_i/d_i)^{n_i-te_i}\Big)\\&=C\prod_{i=1}^{2}(B_i/d_i)^{n_i-te_i}+O\Big(\Big(\prod_{i=1}^{2}d_i^{-(n_i-te_i)+\delta}\Big)\big(\min_{i=1,2}B_i\big)^{-\delta}\prod_{i=1}^{2}B_i^{n_i-te_i}\Big)\end{aligned}$$

with $C\in\mathbb{R}_{\ge 0}$ and $C>0$ whenever $V$ has nonsingular $\mathbb{Q}_v$-points for all places $v$ of $\mathbb{Q}$. Thus

$$F_{\boldsymbol{d}}^W(B_1,\dots,B_s)=C_{M,\boldsymbol{d}}B_1^{n_1-te_1}B_2^{n_2-te_2}\prod_{i=3}^{s}B_i^{n_i}+O\Big(C_{E,\boldsymbol{d}}\big(\min_{1\le i\le s}B_i\big)^{-\delta}B_1^{n_1-te_1}B_2^{n_2-te_2}\prod_{i=3}^{s}B_i^{n_i}\Big),$$

where

$$C_{M,\boldsymbol{d}}=Cd_1^{-(n_1-te_1)}d_2^{-(n_2-te_2)}\prod_{i=3}^{s}d_i^{-n_i},\quad C_{E,\boldsymbol{d}}=d_1^{-(n_1-te_1)+\delta}d_2^{-(n_2-te_2)+\delta}\prod_{i=3}^{s}d_i^{-n_i+\delta}.\tag{5-1}$$

Recall that $n_i - te_i \geq 2$ for $i \in \{1, 2\}$. For $\delta > 0$ sufficiently small, if

$$f_{n_1-te_1-\delta, n_2-te_2-\delta, n_3-\delta, \dots, n_s-\delta} \leq 1,$$

then by Remark 2.4 (iv) there is $\tilde{i} \in \{3, \dots, s\}$ such that $\tilde{i} \in I_\sigma$ for all $\sigma \in \Sigma_{\max}$ and $n_{\tilde{i}} = 1$. Then the ray $\rho_{\tilde{i},1}$ is contained in no maximal cone of $\Sigma$, contradicting the fact that $X$ is proper.

Since $V$ is a smooth complete intersection, the adjunction formula [Corti 1992, Proposition 16.4] gives $K_V = K_X + H_1 + \cdots + H_t$. Let $\varpi_i = n_i - te_i$ for $i \in \{1, 2\}$ and $\varpi_i = n_i$ for $i \in \{3, \dots, s\}$. Since

$$\begin{aligned}\sum_{i=1}^{s} \varpi_i \boldsymbol{D}_i &= -[K_X] - t(e_1 \boldsymbol{D}_1 + e_2 \boldsymbol{D}_2) \\ &= -[K_X] - [H_1 + \cdots + H_t],\end{aligned}$$

Lemma 3.1 applied to $F_{\boldsymbol{d}}^W(B_1, \dots, B_s)$ and $N_{V,W}(B)$ gives

$$N_{V,W}(B) = cB(\log B)^{b-1} + O(B^a (\log B)^{b-2} (\log\log B)^s)$$

for $B > 0$, where $b = \operatorname{rk} \operatorname{Pic}(X)$ and $c$ is defined in (3-7) with $k = b - 1$, $C_{M,\boldsymbol{d}}$ given by (5-1). Moreover, the restriction $\operatorname{Pic}(X) \to \operatorname{Pic}(V)$ is an isomorphism, as $t \leq \min\{n_1, n_2\} - 2$. By Lemma 2.5 (ii), the leading constant $c$ is positive if $V(\mathbb{Q}_v) \neq \varnothing$ for all places $v$ of $\mathbb{Q}$ as $C$ is positive under the same conditions by [Schindler 2016, Theorems 4.3 and 4.4]. □

## 6. Campana points on certain diagonal complete intersections

*Proof of Theorem 1.3.* In the setting of Theorem 1.3, the hypersurfaces $H_1, \dots, H_t$ are defined by homogeneous diagonal polynomials $g_1, \dots, g_t \in R$ with $\deg g_i = e_i \boldsymbol{D}_i$ in $\operatorname{Pic}(X)$ for all $i \in \{1, \dots, t\}$. Then

$$g_i = \sum_{j=1}^{n_i} c_{i,j} x_{i,j}^{e_i}$$

with $c_{i,j} \in \mathbb{Z}_{\neq 0}$, and

$$F_{\boldsymbol{d}}(B_1, \dots, B_s) = \prod_{i=1}^{s} F_{i,d_i}(B_i),$$

where, for $i \leq t$,

$$F_{i,d}(B) = \#\Big\{(x_{i,1}, \dots, x_{i,n_i}) \in (\mathbb{Z}_{\neq 0})^{n_i} : \\ d \mid x_{i,j},\ x_{i,j} \text{ is } m_{i,j}\text{-full } \forall j \in \{1, \dots, n_i\},\ \sup_{1 \leq j \leq n_i} |x_{i,j}| \leq B,\ g_i = 0\Big\}$$

and, for $i > t$,

$$F_{i,d}(B) = \#\Big\{(x_{i,1}, \dots, x_{i,n_i}) \in (\mathbb{Z}_{\neq 0})^{n_i} : \\ \sup_{1 \leq j \leq n_i} |x_{i,j}| \leq B,\ d \mid x_{i,j},\ x_{i,j} \text{ is } m_{i,j}\text{-full } \forall j \in \{1, \dots, n_i\}\Big\}. \tag{6-1}$$

For $i \leq t$, we estimate $F_{i,d}(B)$ via the following lemma.

**Lemma 6.1.** *Let $n, e, m_1, \dots, m_n \in \mathbb{Z}_{>0}$. Let $c_1, \dots, c_n \in \mathbb{Z}_{\neq 0}$. Let $d$ be a square-free positive integer. Assume that $n \geq 2$ and $2 \leq m_1 \leq \cdots \leq m_n$.*

(1) *If $e = 1$, assume that*

$$\sum_{j=1}^{n} \frac{1}{m_j} > 3 \quad \text{and} \quad \sum_{j=1}^{n-1} \frac{1}{em_j(em_j+1)} \geq 1.$$

(2) *If $e \geq 2$, assume that*

$$\sum_{j=1}^{n} \frac{1}{em_j} > 3 \quad \text{and} \quad \sum_{j=1}^{n} \frac{1}{2s_0(em_j)} > 1,$$

*where*

$$s_0(m) = \min\bigl\{2^{m-1}, \tfrac{1}{2}m(m-1) + \lfloor\sqrt{2m+2}\rfloor\bigr\}, \quad m \in \mathbb{Z}_{\geq 0}.$$

*For $B > 0$, let*

$$F_d(B) = \#\Bigl\{(x_1, \dots, x_n) \in (\mathbb{Z}_{\neq 0})^n : d \mid x_j,\ x_j \text{ is } m_j\text{-full } \forall j \in \{1, \dots, n\},\ \sup_{1 \leq j \leq n} |x_j| \leq B,\ \sum_{j=1}^{n} c_j x_j^e = 0\Bigr\}.$$

*Then there is $\eta > 0$ such that*

$$F_d(B) = c_{e,d} B^{\Gamma} + O(d^{-1-\eta} B^{\Gamma - \eta}),$$

*where $\Gamma = \sum_{j=1}^{n} 1/m_j - e$ and $c_{e,d}$ is defined in (6-5) and satisfies $0 \leq c_{e,d} \ll d^{-1-\eta}$.*

*Proof.* For every $j \in \{1, \dots, n\}$ and $x_j \in \mathbb{Z}_{\neq 0}$ that is $m_j$-full, there exist unique $u_j, v_{j,1}, \dots, v_{j,m_j-1} \in \mathbb{Z}_{>0}$ such that

$$|x_j| = u_j^{m_j} \prod_{r=1}^{m_j-1} v_{j,r}^{m_j+r}, \quad \mu^2(v_{j,r}) = 1, \quad \gcd(v_{j,r}, v_{j,r'}) = 1 \quad \text{for all } r, r' \in \{1, \dots, m_j - 1\}, \quad r \neq r'.$$

For every choice of $u_j$ and $v_{j,r}$ as above, if $d \mid x_j$ with $d \in \mathbb{Z}_{>0}$ squarefree, then there exist unique $s_j, t_{j,1}, \dots, t_{j,m_j-1} \in \mathbb{Z}_{>0}$ such that

$$\begin{aligned} d = s_j \prod_{r=1}^{m_j-1} t_{j,r}, \quad \mu^2(s_j) = \mu^2(t_{j,r}) = 1 \quad &\text{for all } r \in \{1, \dots, m_j - 1\} \\ \gcd(s_j, v_{j,r}) = \gcd(s_j, t_{j,r}) = \gcd(t_{j,r}, t_{j,r'}) = 1 \quad &\text{for all } r, r' \in \{1, \dots, m_j - 1\}, \quad r \neq r' \\ s_j \mid u_j, \quad t_{j,r} \mid v_{j,r} \quad &\text{for all } r \in \{1, \dots, m_j - 1\}. \end{aligned}$$

Write $u_j = s_j \tilde{u}_j$ and $v_{j,r} = t_{j,r} \tilde{v}_{j,r}$ for all $r \in \{1, \dots, m_{j-1}\}$. Write

$$\boldsymbol{s} = (s_1, \dots, s_n), \quad \boldsymbol{t} = (t_{j,r})_{1 \leq j \leq n, 1 \leq r \leq m_j - 1}.$$

For $j \in \{1, \dots, n\}$, write

$$\sigma_j = s_j \prod_{r=1}^{m_j-1} t_{j,r}, \quad \tau_j = s_j^{m_j} \prod_{r=1}^{m_j-1} t_{j,r}^{m_j+r}, \quad w_j = \prod_{r=1}^{m_j-1} \tilde{v}_{j,r}^{m_j+r}.$$

Let $\mathcal{T}_d(B)$ be the set of pairs $(\boldsymbol{s}, \boldsymbol{t}) \in \mathbb{Z}_{>0}^n \times \mathbb{Z}_{>0}^{\sum_{j=1}^n (m_j-1)}$ that satisfy

$$\mu^2(\sigma_j) = 1, \quad d = \sigma_j, \quad \tau_j \leq B \quad \text{for all } j \in \{1, \dots, s\}.$$

Note that the first two conditions imply

$$\#\mathcal{T}_d(B) \le \prod_{j=1}^{n} m_j^{\omega(d)} \ll d^{\epsilon}, \tag{6-2}$$

where $\omega(d)$ is the number of distinct prime divisors of $d$. Let $\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)$ be the set of

$$\tilde{\boldsymbol{v}} = (\tilde{v}_{j,r})_{1\le j\le n, 1\le r\le m_j-1} \in \mathbb{Z}_{>0}^{\sum_{j=1}^{n}(m_j-1)}$$

such that

$$\begin{aligned} \mu^2(t_{j,r}\tilde{v}_{j,r}) = 1,\ \ \gcd(s_j, \tilde{v}_{j,r}) = 1 \quad &\text{for all } j \in \{1,\dots,n\},\ \ r \in \{1,\dots,m_j-1\}, \\ \gcd(t_{j,r}\tilde{v}_{j,r}, t_{j,r'}\tilde{v}_{j,r'}) = 1 \quad &\text{for all } j \in \{1,\dots,n\},\ \ r, r' \in \{1,\dots,m_j-1\},\ \ r \ne r', \\ \tau_j w_j \le B \quad &\text{for all } j \in \{1,\dots,n\}. \end{aligned}$$

Let $\mathcal{T}_d(\infty) = \bigcup_{B>0}\mathcal{T}_d(B)$ and $\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(\infty) = \bigcup_{B>0}\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)$.

Then

$$F_d(B) = \begin{cases} \sum_{\boldsymbol{\varepsilon}\in\{\pm1\}^n}\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(B)}\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)} M_{\boldsymbol{\varepsilon c},\boldsymbol{\gamma}}(B^e) & \text{if } e \text{ is odd}, \\ 2^n \sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(B)}\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)} M_{\boldsymbol{c},\boldsymbol{\gamma}}(B^e) & \text{if } e \text{ is even}, \end{cases} \tag{6-3}$$

where $\boldsymbol{c} = (c_1,\dots,c_n)$, $\boldsymbol{\varepsilon} = (\varepsilon_1,\dots,\varepsilon_n)$, $\boldsymbol{\varepsilon c} = (\varepsilon_1 c_1,\dots,\varepsilon_n c_n)$, $\boldsymbol{\gamma} = (\gamma_1,\dots,\gamma_n)$ with

$$\gamma_j = s_j^{em_j}\prod_{r=1}^{m_j-1} t_{j,r}^{e(m_j+r)}\tilde{v}_{j,r}^{e(m_j+r)} \quad \text{for all } j \in \{1,\dots,n\},$$

and

$$M_{\boldsymbol{\varepsilon c},\boldsymbol{\gamma}}(B^e) = \#\left\{(\tilde{u}_1,\dots,\tilde{u}_n) \in \mathbb{Z}_{>0}^n : \max_{1\le j\le n}\gamma_j\tilde{u}_j^{em_j} \le B^e,\ \sum_{j=1}^{n}\varepsilon_j c_j\gamma_j\tilde{u}_j^{em_j} = 0\right\}.$$

An estimate for $M_{\boldsymbol{\varepsilon c},\boldsymbol{\gamma}}(B^e)$ is proven in [Browning and Yamagishi 2021, Theorem 2.7] in the case where

$$\sum_{j=1}^{n-1}\frac{1}{em_j(em_j+1)} \ge 1.$$

The subsequent paper [Balestrieri et al. 2024, Theorem 5.3] extends the range of applicability of [Browning and Yamagishi 2021, Theorem 2.7] to the case where

$$\sum_{j=1}^{n}\frac{1}{em_j} > 3, \qquad \sum_{j=1}^{n}\frac{1}{2s_0(em_j)} > 1.$$

Let

$$\Theta_e = \begin{cases} \frac{1}{m_n(m_n+1)} & \text{if } e = 1, \\ \sum_{j=1}^{n}\frac{1}{2s_0(em_j)} - 1 & \text{if } e \ge 2. \end{cases}$$

For

$$0 < \delta < \frac{1}{(2(n-1)+5)em_n(em_n+1)} \quad \text{and} \quad \epsilon > 0,$$

the two results cited above give

$$\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(B)}\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)} M_{\boldsymbol{\varepsilon c},\boldsymbol{\gamma}}(B^e)=\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(B)}\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)}\frac{\mathfrak{S}_{\boldsymbol{\varepsilon c},\boldsymbol{\gamma}}\mathfrak{I}_{\boldsymbol{\varepsilon c}}}{\prod_{j=1}^n \gamma_j^{1/(em_j)}}B^\Gamma+O(B^\Gamma(F_1+F_2+F_3)), \tag{6-4}$$

where

$$\mathfrak{S}_{\boldsymbol{\varepsilon c},\boldsymbol{\gamma}}=\sum_{q=1}^\infty\frac{1}{q^n}\sum_{\substack{a\,(\mathrm{mod}\,q)\\ \gcd(a,q)=1}}\prod_{j=1}^n\sum_{r=1}^q \exp(2\pi i a\varepsilon_j c_j\gamma_j r^{em_j}/q),$$

$$\mathfrak{I}_{\boldsymbol{\varepsilon c}}=\int_{-\infty}^{\infty}\prod_{j=1}^n\biggl(\int_0^1\exp(2\pi i\lambda\varepsilon_j c_j\xi^{em_j})\,\mathrm{d}\xi\biggr)\,\mathrm{d}\lambda,$$

$$F_1=B^{e((2(n-1)+5)\delta-1)-\Gamma}\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(B)}\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)}\biggl(\prod_{j=1}^n\frac{B^{1/m_j}}{\gamma_j^{1/(em_j)}}\biggr)\sum_{l=1}^n\frac{\gamma_l^{1/(em_l)}}{B^{1/m_l}},$$

$$F_2=B^{-e\delta}\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(B)}\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)}\sum_{q=1}^\infty q^{1-\Gamma/e+\epsilon}\prod_{j=1}^n\gcd(\gamma_j,q)^{\frac{1}{em_j}}\gamma_j^{-\frac{1}{em_j}},$$

$$F_3=\begin{cases}B^{-e\delta\Theta_e+\epsilon}\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(B)}\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)}\prod_{j=1}^n\gamma_j^{-\frac{1}{m_j+1}} & \text{if } e=1,\\ B^{-e\delta\Theta_e+\epsilon}\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(B)}\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)}\prod_{j=1}^n\gamma_j^{-\frac{1}{em_j}+\frac{1}{2s_0(em_j)}} & \text{if } e\ge 2.\end{cases}$$

Let

$$c_{e,d}=\begin{cases}\sum_{\boldsymbol{\varepsilon}\in\{\pm1\}^n}\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(\infty)}\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(\infty)}\frac{\mathfrak{S}_{\boldsymbol{\varepsilon c},\boldsymbol{\gamma}}\mathfrak{I}_{\boldsymbol{\varepsilon c}}}{\prod_{j=1}^n\gamma_j^{1/(em_j)}} & \text{if } e \text{ is odd},\\ 2^n\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(\infty)}\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(\infty)}\frac{\mathfrak{S}_{\boldsymbol{c},\boldsymbol{\gamma}}\mathfrak{I}_{\boldsymbol{c}}}{\prod_{j=1}^n\gamma_j^{1/(em_j)}} & \text{if } e \text{ is even}.\end{cases} \tag{6-5}$$

For $T>0$, let

$$f_1(q)=\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(\infty)}\prod_{j=1}^n\biggl(\frac{\gcd(\tau_j^e,q)}{\tau_j^e}\biggr)^{\frac{1}{em_j}},\quad f_2(q)=\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{1},\boldsymbol{1}}(\infty)}\prod_{j=1}^n\biggl(\frac{\gcd(w_j^e,q)}{w_j^e}\biggr)^{\frac{1}{em_j}},$$

and

$$f_2(q,T,\boldsymbol{s},\boldsymbol{t})=\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(\infty)\smallsetminus\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(T)}\prod_{j=1}^n\biggl(\frac{\gcd(w_j^e,q)}{w_j^e}\biggr)^{\frac{1}{em_j}}.$$

Note that, for $\sum_{j=1}^n 1/(em_j)>1$, we have

$$|\mathfrak{I}_{\boldsymbol{\varepsilon c}}|\ll 1. \tag{6-6}$$

Similarly as in [Browning and Yamagishi 2021, (2.8), (2.9), (2.12)], the difference between $c_{e,d}B^\Gamma$ and the main term obtained by combining (6-3) and (6-4) is bounded by

$$\begin{aligned}B^\Gamma\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(\infty)}\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(\infty)\smallsetminus\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)}\sum_{q=1}^\infty q^{1-\sum_{j=1}^n\frac{1}{em_j}}\prod_{j=1}^n\gamma_j^{-\frac{1}{em_j}}\gcd(\gamma_j,q)^{\frac{1}{em_j}}&\\ \ll B^\Gamma\sum_{q=1}^\infty q^{-\Gamma/e+\epsilon}\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(\infty)}\prod_{j=1}^n\biggl(\frac{\gcd(\tau_j^e,q)}{\tau_j^e}\biggr)^{\frac{1}{em_j}}f_2(q,B,\boldsymbol{s},\boldsymbol{t}),&\end{aligned} \tag{6-7}$$

and $c_{e,d}\ll\sum_{q=1}^\infty q^{-\Gamma/e+\epsilon}f_1(q)f_2(q)$.

By [Browning and Yamagishi 2021, (3.10)] and the arguments used to prove [Browning and Yamagishi 2021, (3.9)], we have

$$f_2(q) \ll q^{\epsilon} \tag{6-8}$$

and

$$\begin{aligned} f_2(q,T,\boldsymbol{s},\boldsymbol{t}) &\ll \sum_{i_0=1}^{n} \sum_{\substack{\tilde{v}_{i,r},1\le i\le n,1\le r\le m_i-1\\ \prod_{r=1}^{m_i-1}\tilde{v}_{i,r}^{m_i+r}>T/\tau_i \text{ if } i=i_0}} \prod_{i=1}^{n}\prod_{r=1}^{m_i-1} \frac{\mu^2(\tilde{v}_{i,r})\gcd(\tilde{v}_{i,r}^{e(m_i+r)},q)^{1/(em_i)}}{\tilde{v}_{i,r}^{(m_i+r)/m_i}} \\ &\ll q^{\epsilon}\sum_{i=1}^{n} \sum_{\substack{\tilde{v}_{i,r},1\le r\le m_i-1\\ \prod_{r=1}^{m_i-1}\tilde{v}_{i,r}^{m_i+r}>T/\tau_i}} \prod_{r=1}^{m_i-1} \frac{\mu^2(\tilde{v}_{i,r})\gcd(\tilde{v}_{i,r}^{e(m_i+r)},q)^{1/(em_i)}}{\tilde{v}_{i,r}^{(m_i+r)/m_i}}. \end{aligned}$$

Our next goal is to provide an upper bound for sums of the type occurring in this estimate for $f_2(q,T,\boldsymbol{s},\boldsymbol{t})$.

**Lemma 6.2.** *Let $m\in\mathbb{N}_{\ge 2}$, $e\in\mathbb{N}$, and let $A>0$ be a real parameter. Then, for every $0<\epsilon<1/(m(m+1))$, we have*

$$\sum_{\substack{v_r\in\mathbb{N},1\le r\le m-1\\ \prod_{r=1}^{m-1}v_r^{m+r}>A}} \prod_{r=1}^{m-1} \frac{\mu^2(v_r)\gcd(v_r^{e(m+r)},q)^{1/(em)}}{v_r^{(m+r)/m}} \ll_{m,\epsilon} A^{-\frac{1}{m(m+1)}+\epsilon} q^{\frac{m-1}{em(m+1)}+\epsilon}.$$

*Proof.* We first consider the sum

$$S_1 := \sum_{\substack{v_r\in\mathbb{N},1\le r\le m-1\\ \prod_{r=1}^{m-1}v_r^{m+r}>A}} \prod_{r=1}^{m-1}\frac{1}{v_r^{(m+r)/m}}$$

for $A>1$. A dyadic decomposition for each of the variables $v_r$, $1\le r\le m-1$, leads to the upper bound

$$S_1 \ll \sum_{\substack{l_1,\dots,l_{m-1}\in\mathbb{N}\\ 2^{(m+1)l_1+\cdots+(2m-1)l_{m-1}}>A}} 2^{-\frac{1}{m}l_1-\cdots-\frac{(m-1)}{m}l_{m-1}}.$$

Note that, for each $k\in(1/m)\mathbb{N}$, we have

$$\#\Big\{l_1,\dots,l_{m-1}\in\mathbb{N} : \frac{1}{m}l_1+\cdots+\frac{m-1}{m}l_{m-1}=k\Big\} \ll_m k^{m-2}.$$

We deduce that

$$S_1 \ll_m \sum_{\substack{k\in(1/m)\mathbb{N}\\ r(k)>0}} k^{m-2}2^{-k},$$

where $r(k)$ is the number of $(l_1,\dots,l_{m-1})\in\mathbb{N}^{m-1}$ such that both

$$\frac{1}{m}l_1+\cdots+\frac{m-1}{m}l_{m-1}=k \quad \text{and} \quad 2^{(m+1)l_1+\cdots+(2m-1)l_{m-1}}>A.$$

Observe that if $r(k)>0$, then there exists $(l_1,\dots,l_{m-1})\in\mathbb{N}^{m-1}$ with

$$\frac{1}{m}l_1+\cdots+\frac{m-1}{m}l_{m-1}=k$$

and

$$m(m+1)k = (m+1)(l_1+\cdots+(m-1)l_{m-1})$$
$$\geq (m+1)l_1 + \frac{m+2}{2}2l_2+\cdots+\frac{2m-1}{m-1}(m-1)l_{m-1} > \frac{\log A}{\log 2},$$

i.e.,

$$S_1 \ll_m \sum_{\substack{k\in(1/m)\mathbb{N}\\ m(m+1)k>\log A/\log 2}} k^{m-2}2^{-k} \ll_{m,\epsilon} A^{-\frac{1}{m(m+1)}+\epsilon}.$$

Note that the upper bound for $S_1$ also holds for $A \leq 1$ and $\epsilon < 1/(m(m+1))$.

We now turn to the sum in the statement of the lemma. If $v_r$ is a square-free natural number and $d_r = \gcd(v_r^{e(m+r)}, q)$, then we can write

$$d_r = d_{r,1}d_{r,2}^2\cdots d_{r,e(m+r)}^{e(m+r)}, \quad \mu^2(d_{r,j})=1 \text{ for all } 1\leq j\leq e(m+r), \quad \gcd(d_{r,j},d_{r,j'})=1 \text{ for all } j\neq j'.$$

Writing $v_r = v'_r\prod_{j=1}^{e(m+r)} d_{r,j}$ and $d'_r = \prod_{j=1}^{e(m+r)} d_{r,j}$, we find that

$$S_2 := \sum_{\substack{v_r\in\mathbb{N},1\leq r\leq m-1\\ \prod_{r=1}^{m-1} v_r^{m+r}>A}} \prod_{r=1}^{m-1} \frac{\mu^2(v_r)\gcd(v_r^{e(m+r)},q)^{1/(em)}}{v_r^{(m+r)/m}}$$
$$\ll \sum_{\substack{d_{r,1}d_{r,2}^2\cdots d_{r,e(m+r)}^{e(m+r)}\mid q\\ 1\leq r\leq m-1}} \sum_{\substack{v'_r\in\mathbb{N},1\leq r\leq m-1\\ \prod_{r=1}^{m-1}(d'_rv'_r)^{m+r}>A}} \prod_{r=1}^{m-1} \frac{d_r^{1/(em)}}{(d'_rv'_r)^{(m+r)/m}}$$
$$\ll \sum_{\substack{d_{r,1}d_{r,2}^2\cdots d_{r,e(m+r)}^{e(m+r)}\mid q\\ 1\leq r\leq m-1}} \prod_{r=1}^{m-1}\left(\frac{d_r^{1/(em)}}{d_r'^{(m+r)/m}}\right) \sum_{\substack{v'_r\in\mathbb{N},1\leq r\leq m-1\\ \prod_{r=1}^{m-1}(d'_rv'_r)^{m+r}>A}} \prod_{r=1}^{m-1}\frac{1}{(v'_r)^{(m+r)/m}}.$$

By using the upper bound for $S_1$, we find that, for $\epsilon>0$ sufficiently small,

$$S_2 \ll_{\epsilon,m} \sum_{\substack{d_{r,1}d_{r,2}^2\cdots d_{r,e(m+r)}^{e(m+r)}\mid q\\ 1\leq r\leq m-1}} \prod_{r=1}^{m-1}\left(\frac{d_r^{1/(em)}}{d_r'^{(m+r)/m}}\right) A^{-\frac{1}{m(m+1)}+\epsilon}\left(\prod_{r=1}^{m-1}(d'_r)^{m+r}\right)^{\frac{1}{m(m+1)}}$$
$$\ll_{\epsilon,m} A^{-\frac{1}{m(m+1)}+\epsilon} \sum_{\substack{d_{r,1}d_{r,2}^2\cdots d_{r,e(m+r)}^{e(m+r)}\mid q\\ 1\leq r\leq m-1}} \prod_{r=1}^{m-1}\left(d_r^{\frac{1}{em}}(d'_r)^{-\frac{m+r}{m+1}}\right)$$
$$\ll_{\epsilon,m} A^{-\frac{1}{m(m+1)}+\epsilon} \sum_{\substack{d_{r,1}d_{r,2}^2\cdots d_{r,e(m+r)}^{e(m+r)}\mid q\\ 1\leq r\leq m-1}} \prod_{r=1}^{m-1} d_r^{\frac{1}{em}-\frac{m+r}{e(m+r)(m+1)}}$$
$$\ll_{\epsilon,m} A^{-\frac{1}{m(m+1)}+\epsilon} \sum_{\substack{d_r\mid q\\ 1\leq r\leq m-1}} \prod_{r=1}^{m-1} d_r^{\frac{1}{em(m+1)}} \ll_{\epsilon,m} A^{-\frac{1}{m(m+1)}+\epsilon} q^{\frac{m-1}{em(m+1)}+\epsilon}. \qquad \square$$

Lemma 6.2 shows that we can bound $f_2(q, T, \boldsymbol{s}, \boldsymbol{t})$ by

$$f_2(q, T, \boldsymbol{s}, \boldsymbol{t}) \ll \sum_{i=1}^{n} \Big(\frac{T}{\tau_i}\Big)^{-\frac{1}{m_i(m_i+1)}+\epsilon} q^{\frac{m_i-1}{em_i(m_i+1)}+\epsilon}.$$

In the following we write

$$\Delta_i = \frac{m_i - 1}{em_i(m_i+1)}.$$

Then (6-7) is bounded by

$$S_3 := B^{\Gamma} \sum_{i=1}^{n} \sum_{q=1}^{\infty} q^{-\Gamma/e+\Delta_i+\epsilon} \sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(\infty)} \prod_{j=1}^{n} \Big(\frac{\gcd(\tau_j^e, q)}{\tau_j^e}\Big)^{\frac{1}{em_j}} \Big(\frac{B}{\tau_i}\Big)^{-\frac{1}{m_i(m_i+1)}+\epsilon}$$
$$\ll B^{\Gamma} \sum_{i=1}^{n} B^{-\frac{1}{m_i(m_i+1)}+\epsilon} \sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(\infty)} \sum_{q=1}^{\infty} q^{-\Gamma/e+\Delta_i+\epsilon} \tau_i^{\frac{1}{m_i(m_i+1)}} \prod_{j=1}^{n} \Big(\frac{\gcd(\tau_j^e, q)}{\tau_j^e}\Big)^{\frac{1}{em_j}}.$$

As we will encounter similar expressions in our further analysis, we introduce, for $E, D > 0$ and $d$ squarefree, the sum

$$S_d(D, E) := d^E \sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(\infty)} \sum_{q=1}^{\infty} q^{-\Gamma/e+D+\epsilon} \prod_{j=1}^{n} \Big(\frac{\gcd(\tau_j^e, q)}{\tau_j^e}\Big)^{\frac{1}{em_j}}.$$

We write $q = q_1 q_2$ with $\gcd(q_1, d) = 1$ and such that all prime divisors of $q_2$ divide $d$. We then obtain

$$S_d(D, E) \ll d^E \sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(\infty)} \sum_{q_1=1}^{\infty} q_1^{-\Gamma/e+D+\epsilon} \sum_{\substack{q_2=1 \\ p\mid q_2 \Rightarrow p\mid d}}^{\infty} q_2^{-\Gamma/e+D+\epsilon} \prod_{j=1}^{n} \Big(\frac{\gcd(\tau_j^e, q_2)}{\tau_j^e}\Big)^{\frac{1}{em_j}}.$$

If we assume $-\Gamma/e + D < -1$, then the sum over $q_1$ is absolutely convergent. For a given vector $(\boldsymbol{s}, \boldsymbol{t}) \in \mathcal{T}_d(\infty)$ and a prime $p$, we write $\tau_{j,p}$ for the power of $p$ which exactly divides $\tau_j$. We find that

$$S_d(D, E) \ll \sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(\infty)} d^E \prod_{p\mid d} \Big(\sum_{l=0}^{\infty} p^{l(-\Gamma/e+D+\epsilon)} \prod_{j=1}^{n} \Big(\frac{\gcd(\tau_{j,p}^e, p^l)}{\tau_{j,p}^e}\Big)^{\frac{1}{em_j}}\Big).$$

We now split the summation over $l$ into the term $l = 0$, where we use the inequality $\tau_{j,p} \geq p^{m_j}$, and we bound the rest by a geometric sum for $l \geq 1$ using $\gcd(\tau_{j,p}^e, p^l) \leq \tau_{j,p}^e$:

$$S_d(D, E) \ll_D \sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(\infty)} d^{E+\epsilon} \prod_{p\mid d} (p^{-n} + p^{-\Gamma/e+D+\epsilon})$$
$$\ll_D d^{\epsilon} \prod_{p\mid d} (p^{E-n} + p^{-\Gamma/e+D+E+\epsilon}).$$

If $-\Gamma/e + D + E < -1$, then we deduce that

$$S_d(D, E) \ll_D d^{-1-\eta} \tag{6-9}$$

for some $\eta > 0$.

Applying (6-9) to $S_3$ with

$$D = \Delta_i \quad \text{and} \quad E = \frac{2m_i - 1}{m_i(m_i+1)},$$

we obtain $S_3 \ll B^{\Gamma-\eta} d^{-1-\eta}$ for some $\eta > 0$. Hence

$$F_d(B) = c_{e,d} B^{\Gamma} + O(B^{\Gamma}(d^{-1-\eta} B^{-\eta} + F_1 + F_2 + F_3)).$$

We use the bound in (6-8) and apply (6-9) with $D = E = 0$ to get $c_{e,d} \ll d^{-1-\eta}$.

It remains to estimate the error terms $F_1$, $F_2$, and $F_3$. We rewrite $F_1$ as

$$F_1 = B^{e\delta(2(n-1)+5)} \sum_{l=1}^{n} B^{-\frac{1}{m_l}} \sum_{(\boldsymbol{s},\boldsymbol{t}) \in \mathcal{T}_d(B)} \sum_{\tilde{\boldsymbol{v}} \in \mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)} \prod_{\substack{1 \le j \le n \\ j \ne l}} \gamma_j^{-\frac{1}{e m_j}}.$$

As in [Browning and Yamagishi 2021, §3] and [Balestrieri et al. 2024, §6], we have

$$\begin{aligned} F_1 &\ll B^{-\frac{1}{m_n(m_n+1)} + e\delta(2(n-1)+5)} \sum_{(\boldsymbol{s},\boldsymbol{t}) \in \mathcal{T}_d(B)} \prod_{j=1}^{n} \tau_j^{-\frac{1}{m_j+1}} \\ &\ll d^{-\sum_{j=1}^{n} \frac{m_j}{m_j+1} + \varepsilon} B^{-\frac{1}{m_n(m_n+1)} + e\delta(2(n-1)+5)}, \end{aligned}$$

where the last estimate follows from

$$\begin{aligned} \sum_{(\boldsymbol{s},\boldsymbol{t}) \in \mathcal{T}_d(B)} \prod_{j=1}^{n} \tau_j^{-\frac{1}{m_j+1}} &\le \sum_{(\boldsymbol{s},\boldsymbol{t}) \in \mathcal{T}_d(B)} \prod_{j=1}^{n} \sigma_j^{-\frac{m_j}{m_j+1}} \\ &\le d^{-\sum_{j=1}^{n} \frac{m_j}{m_j+1}} \# \mathcal{T}_d(B) \ll d^{-\sum_{j=1}^{n} \frac{m_j}{m_j+1} + \varepsilon} \end{aligned}$$

by (6-2). Combining the arguments for $F_3$ in [Browning and Yamagishi 2021, §3] and in [Balestrieri et al. 2024, §6] and the estimate above, we have

$$F_3 \ll B^{-e\delta\Theta_e + \epsilon} \sum_{(\boldsymbol{s},\boldsymbol{t}) \in \mathcal{T}_d(B)} \prod_{j=1}^{n} \tau_j^{-\frac{1}{m_j+1}} \ll d^{-\sum_{j=1}^{n} \frac{m_j}{m_j+1} + \epsilon} B^{-e\delta\Theta_e + \epsilon}.$$

Since $\sum_{j=1}^{n} (m_j/(m_j+1)) \ge \frac{2}{3} n > 1$ is satisfied for $n \ge 2$, we have $F_1, F_3 \ll d^{-1-\eta} B^{-\eta}$ for a suitable $\eta > 0$. Since

$$F_2 = B^{-e\delta} \sum_{q=1}^{\infty} q^{1-\Gamma/e+\epsilon} f_1(q) f_2(q),$$

the estimate (6-8) combined with (6-9) for $D = 1$ and $E = 0$ yields $F_2 \ll d^{-1-\eta} B^{-e\delta}$, as $\Gamma/e > 2$. □

By Lemma 6.1 and [Pieropan and Schindler 2024, Lemma 5.6],

$$F_{\boldsymbol{d}}(B_1, \dots, B_s) = \prod_{i=1}^{s} (c_{M,i} B_i^{\varpi_i} + O(d_i^{\nu_i+\varepsilon} B_i^{\varpi_i - \delta})),$$

where

$$\varpi_i = \begin{cases} \sum_{j=1}^{n_i} \frac{1}{m_{i,j}} - e_i & \text{if } i \le t, \\ \sum_{j=1}^{n_i} \frac{1}{m_{i,j}} & \text{if } i > t, \end{cases} \tag{6-10}$$

$\nu_i < -1$ for $i \le t$, $\nu_i = -\frac{2}{3} n_i$ if $i > t$, $c_{M,i}$ is the constant $c_{e_i,d_i}$ defined in (6-5) if $i \le t$, and also $c_{M,i} = 2^{n_i}\big(\prod_{j=1}^{n_i} c_{m_{i,j},d_i}\big)$, where $c_{m_{i,j},d_i}$ is the constant defined in [Pieropan and Schindler 2024, (5.11)].

Thus

$$F_{\boldsymbol{d}}(B_1, \dots, B_s) = C_{M,\boldsymbol{d}} \prod_{i=1}^{s} B_i^{\varpi_i} + O\biggl(C_{E,\boldsymbol{d}}\bigl(\min_{1 \le i \le s} B_i\bigr)^{-\delta} \prod_{i=1}^{s} B_i^{\varpi_i}\biggr),$$

where

$$C_{M,\boldsymbol{d}} = \prod_{i=1}^{s} c_{M,i}. \tag{6-11}$$

Lemma 6.1 and [Pieropan and Schindler 2024, (5.14), (5.15)] give

$$C_{M,\boldsymbol{d}}, C_{E,\boldsymbol{d}} \ll \prod_{i=1}^{s} d_i^{-\beta_i}$$

with $\beta_i > 1$ whenever $n_i \ge 2$, and $\beta_i > \frac{2}{3} - \varepsilon$ otherwise. For $\varepsilon > 0$ sufficiently small, $\beta_i + \beta_j > 1$ for every $i, j \in \{1, \dots, s\}$. Thus, by Remark 2.4 (iv), if $f_{\beta_1,\dots,\beta_s} \le 1$, then there exists an index $\tilde{i} \in \{1, \dots, s\}$ such that $\tilde{i} \in I_\sigma$ for all $\sigma \in \Sigma_{\max}$ and $n_{\tilde{i}} = 1$. Then the ray $\rho_{\tilde{i},1}$ is contained in no maximal cone of $\Sigma$, contradicting the fact that $X$ is proper.

Since $c_{i,j} \neq 0$ for all $i \in \{1, \dots, t\}$, $j \in \{1, \dots, n_i\}$, the adjunction formula [Corti 1992, Proposition 16.4] gives $K_V = (K_X + H_1 + \dots + H_t)|_V$. Since

$$\begin{aligned}\sum_{i=1}^{s} \varpi_i \boldsymbol{D}_i &= -K_X - \sum_{i=1}^{s} \sum_{j=1}^{n_i} \Bigl(1 - \frac{1}{m_{i,j}}\Bigr) \boldsymbol{D}_i + \sum_{i=1}^{t} e_i \boldsymbol{D}_i \\ &= -(K_X + [\mathscr{D}_{\boldsymbol{m}}|_X] + [H_1 + \dots + H_t]),\end{aligned}$$

Lemma 3.1 gives

$$N_V(B) = cB(\log B)^{b-1} + O(B(\log B)^{b-2}(\log \log B)^s),$$

where $b = \operatorname{rk} \operatorname{Pic}(X)$ and $c$ is defined in (3-7) with $k = b - 1$, $C_{M,\boldsymbol{d}}$ given by (6-11), and $\varpi_1, \dots, \varpi_s$ given by (6-10). Moreover, the restriction $\operatorname{Pic}(X) \to \operatorname{Pic}(V)$ is an isomorphism as $n_i \ge 3$ for $1 \le i \le t$. □

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