Algebra & Number Theory

Volume 19
2025
No. 11

Sym-Noetherianity for powers of GL-varieties

Christopher H. Chiu, Alessandro Danelon,
Jan Draisma, Rob H. Eggermont and Azhar Farooq

msp

ALGEBRA AND NUMBER THEORY 19:11 (2025)
https://doi.org/10.2140/ant.2025.19.2091

Sym-Noetherianity for powers of GL-varieties

Christopher H. Chiu, Alessandro Danelon,
Jan Draisma, Rob H. Eggermont and Azhar Farooq

Much recent literature concerns finiteness properties of infinite-dimensional algebraic varieties equipped with an action of the infinite symmetric group, or of the infinite general linear group. In this paper, we study a common generalisation in which the product of both groups acts on infinite-dimensional spaces, and we show that these spaces are topologically Noetherian with respect to this action.

1. Introduction

1.1. Sym-*Noetherianity and* GL-*Noetherianity.* It has been well-established since the 1980s that if Z is finite-dimensional variety, then the topological space $Z^{\mathbb{N}}$, equipped with the inverse-limit topology of the Zariski topologies, has the property that if

$$X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$$

is a descending chain of closed subvarieties, each stable under the infinite symmetric group $\mathrm{Sym} = \bigcup_n \mathrm{Sym}([n])$ permuting the copies of Z, then $X_n = X_{n+1}$ for all $n \gg 0$. We say that $Z^{\mathbb{N}}$ is Sym-*Noetherian*; see [Cohen 1967; 1987; Aschenbrenner and Hillar 2007; Hillar and Sullivant 2012] for the relevant literature.

On the other hand, the third author proved that if Z is a GL-*variety*: a (typically infinite-dimensional) affine variety equipped with a suitable action of the infinite general linear group $\mathrm{GL} = \bigcup_n \mathrm{GL}_n$ — see below for precise definitions — then Z is topologically GL-Noetherian. See [Draisma 2019] for Noetherianity, and see [Bik et al. 2023a; 2023b] for the structure theory of GL-varieties.

1.2. *Our result:* Sym × GL-*Noetherianity.* Given a GL-variety Z, the group $\mathrm{Sym} \times \mathrm{GL}$ acts naturally on $Z^{\mathbb{N}}$, and our main goal in this paper is to prove the following theorem.

Chui, Danelon, Draisma and Farooq were funded by Vici grant 639.033.514 from the Netherlands Organisation for Scientific Research (NWO). Draisma was also funded by project grant 200021_191981 from the Swiss National Science Foundation. Eggermont was supported by NWO Veni grant 016.Veni.192.113. Draisma thanks the Institute for Advanced Study, where some of the work on this paper took place.
MSC2020: 13A50, 14R20.
Keywords: spaces of tensors, complexity of tensors, stabilisation in varieties of tensors, infinite-dimensional polynomial representations, equivariant algebraic geometry, GL-varieties, Sym-varieties, topological Noetherianity, Vec-varieties, FI-algebras, asymptotic algebraic geometry, representation stability.

Theorem 1.1 (main theorem). *Let Z be a* GL*-variety over a field of characteristic zero. Then $Z^{\mathbb{N}}$ is topologically* Sym × GL*-Noetherian. In other words, every descending chain*

$$X_1 \supseteq X_2 \supseteq \cdots$$

of Sym × GL*-stable closed subvarieties of $Z^{\mathbb{N}}$ stabilises eventually. Equivalently, any* Sym × GL*-stable closed subvariety of $Z^{\mathbb{N}}$ is defined by finitely many* Sym × GL*-orbits of polynomial equations.*

Below we give two examples of Sym × GL-varieties; these illustrate that even when Z is a rather simple GL-variety, $Z^{\mathbb{N}}$ will have many Sym × GL-stable closed subvarieties.

Example 1.2. Consider the space of $\mathbb{N} \times \mathbb{N}$-matrices where Sym permutes the rows, and GL acts simultaneously on all rows. We can think about this space as $Z^{\mathbb{N}}$, where Z is the space $\mathbb{A}^{\mathbb{N}}$ with the obvious GL-action. We write $x_{i,j}$ $(i, j \in \mathbb{N})$ for the coordinates on this space. Let X be a Sym × GL-stable proper closed subvariety of this space. Let f be a nonzero polynomial vanishing identically on X involving only the x_{ij} with $1 \leq i, j \leq n$, chosen such that n is minimal among all defining equations of X. We claim that X is contained in the variety of matrices with rank at most $n-1$. Indeed, suppose that a matrix A in X has rank at least n. Then by basic linear algebra the Sym × GL-orbit of A projects dominantly in the affine space $\mathbb{A}^{n \times n}$ corresponding to the upper left $n \times n$-block. This implies that f is the zero polynomial; a contradiction.

Also, by the minimality of n, there must exist a matrix in X whose rank is $n-1$. However, it is not easy to completely classify the Sym × GL-stable closed subvarieties of $Z^{\mathbb{N}}$ containing a matrix of rank $n-1$ and no matrices of rank n. For instance, fix any matroid M of rank $n-1$ on the ground set $[m] := \{1, \ldots, m\}$ and let $R \subseteq \mathbb{A}^{m \times (n-1)}$ be the variety defined by the determinants of the $(n-1) \times (n-1)$-submatrices whose rows correspond to nonbases of M. Regard R as a subvariety of $\mathbb{A}^{\mathbb{N} \times \mathbb{N}}$ by extending with zeros and set $X_M := \overline{(\mathrm{Sym} \times \mathrm{GL})R}$. This Sym × GL-stable subvariety of $\mathbb{A}^{\mathbb{N} \times \mathbb{N}}$ is the common zero set of two classes of polynomials: all monomials containing variables from at least $m+1$ distinct rows, and the Sym-orbits of all products of the form

$$\prod_{\pi \in \mathrm{Sym}([m])} \det(x[\pi(I_\pi), J_\pi]),$$

where each $I_\pi \subseteq [m]$ is an $(n-1)$-element set that is not a basis of M, each J_π is an arbitrary $(n-1)$-element subset of $\mathbb{N}$, and $x[\pi(I_\pi), J_\pi]$ stands for the matrix of variables x_{ij} with $i \in \pi(I_\pi)$ and $j \in J_\pi$.

Now suppose that M, M' are loopless matroids on ground sets $[m], [m']$, both realisable over the algebraic closure of the ground field. We then claim that $X_M = X_M$ holds (if and) only if M, M' are isomorphic. Indeed, if $X_M = X_{M'}$, then let $p \in \mathbb{A}^{m \times (n-1)} \subseteq \mathbb{A}^{\mathbb{N} \times \mathbb{N}}$ realise M, so that $p \in X_M = X_{M'}$. This means that p satisfies all equations for $X_{M'}$. Since M is loopless, all rows of p are nonzero, and the monomial equations for $X_{M'}$ imply that $m' \geq m$. The converse follows by taking a realisation of M'. That the determinantal equations for $X_{M'}$ vanish on p imply that, after a permutation, all nonbases of M' are also nonbases of M. Again, the converse holds by taking a realisation of M'. Hence M and M' are isomorphic.

We conclude that the considerable combinatorial complexity of the class of realisable matroids is contained in the classification problem for Sym × GL-subvarieties of $Z^{\mathbb{N}}$.

Remark 1.3. Already the classification of Sym-stable closed subvarieties of $(A^1)^{\mathbb{N}}$ is nontrivial [Nagpal and Snowden 2020], so it is not so surprising that also the classification of $\mathrm{Sym} \times \mathrm{GL}$-stable closed subvarieties of $Z^{\mathbb{N}}$ in Example 1.2 is difficult.

Example 1.4. Let Z be the space of symmetric $\mathbb{N} \times \mathbb{N}$ matrices, acted upon by GL via $(g, A) \mapsto gAg^T$. It is not hard to classify the GL-stable closed subvarieties of Z: they are the empty set and the varieties of matrices whose rank is bounded by some $k \in \{0, \dots, \infty\}$.

Now let X be a $\mathrm{Sym} \times \mathrm{GL}$-stable proper closed subvariety of $Z^{\mathbb{N}}$, and let n be minimal such that there exists a nonzero polynomial that vanishes on X and involves only coordinates on the first n copies of Z. Then it follows from [Eggermont 2015, Proposition 3.3] that X is contained in the variety $X_{n,r}$ of $\mathbb{N}$-tuples in which every n-tuple has a nontrivial linear combination whose rank is at most some integer r. However, completely classifying all $\mathrm{Sym} \times \mathrm{GL}$-stable closed subvarieties of $X_{n,r}$ seems completely out of reach.

1.3. *A generalisation:* $\mathrm{Sym}^k \times \mathrm{GL}$*-Noetherianity.* We prove the main theorem by establishing first the following more general result.

Theorem 1.5. *Let $Z_1, \dots, Z_k$ be* GL*-varieties over a field of characteristic zero. Then the variety $Z_1^{\mathbb{N}} \times \cdots \times Z_k^{\mathbb{N}}$ is $\mathrm{Sym}^k \times \mathrm{GL}$-Noetherian.*

Here there is one copy of GL that acts diagonally, and there are k copies of Sym that act on separate copies of $\mathbb{N}$. We believe it is impossible to prove the main theorem without considering multiple copies of Sym. Indeed, covering a proper closed $\mathrm{Sym} \times \mathrm{GL}$-stable subvariety of $Z^{\mathbb{N}}$ requires partitioning $\mathbb{N}$ into finitely many parts such that the points in Z labelled by the indices in one same part behave in a similar fashion. The following example illustrates this point.

Example 1.6. Let Z be the space of $\mathbb{N} \times \mathbb{N}$-matrices over a field of characteristic zero, equipped with the GL-action given by $(g, A) \mapsto gAg^T$. Let X be the closed $\mathrm{Sym} \times \mathrm{GL}$-stable subvariety of $Z^{\mathbb{N}}$ consisting of all infinite matrix tuples $(A_1, A_2, \dots)$ such that each A_i is either symmetric or skew-symmetric. It is easy to see that X is defined by the $\mathrm{Sym} \times \mathrm{GL}$-orbit of the equation $(x_{112} + x_{121})(x_{112} - x_{121})$, where x_{ijk} is the (j, k)-entry of the i-th matrix. We will see that the $\mathrm{Sym} \times \mathrm{GL}$-Noetherianity of X follows from the $\mathrm{Sym}^2 \times \mathrm{GL}$-Noetherianity of the *"smaller"* variety $Z_1^{\mathbb{N}} \times Z_2^{\mathbb{N}}$, where $Z_1 \subseteq Z$ is the GL-subvariety of symmetric matrices, and $Z_2 \subseteq Z$ is the GL-subvariety of skew-symmetric matrices. Here the term "smaller" refers to the fact that both Z_1 and Z_2 are quotients of Z. The exact meaning of smaller varieties is given in Section 2.7.2.

1.4. *Relation to existing literature.* The main theorem generalises the results mentioned in Section 1.1: taking for Z a finite-dimensional affine variety with trivial GL-action, one recovers the Sym-Noetherianity of $Z^{\mathbb{N}}$; and on the other hand, if Z is a GL-variety, then considering chains $X_1 \supseteq X_2 \supseteq \cdots$ in which each X_i is of the form $Z_i^{\mathbb{N}}$ with $Z_i \subseteq Z$ a GL-subvariety, one recovers the GL-Noetherianity of Z.

The proof of the main theorem will reflect these two special cases. We will use the proof method from [Draisma 2019] for the GL-Noetherianity of Z, and similarly, we will use methods for Sym-varieties from [Draisma et al. 2022]. In fact, we do not explicitly use Higman's lemma in our proofs as is classically done [Aschenbrenner and Hillar 2007; Hillar and Sullivant 2012; Draisma 2014], and in passing we give a new proof of the Sym-Noetherianity of $Z^{\mathbb{N}}$ for a finite-dimensional variety Z. However, our proof only yields a *set-theoretic* Noetherianity result, while in the pure Sym-setting (much) stronger results are known: increasing chains of Sym-stable ideals in the coordinate ring of $Z^{\mathbb{N}}$ with Z a finite-dimensional variety stabilise [Cohen 1967; 1987; Aschenbrenner and Hillar 2007; Hillar and Sullivant 2012], and even finitely generated modules over such rings with a compatible Sym-action are Noetherian [Nagel and Römer 2019]. In the pure GL-setting, however, such stronger Noetherianity results are known only for very few classes of GL-varieties: over a field of characteristic zero ring-theoretic Noetherianity holds for a direct sum of copies of the first symmetric power S^1 [Sam and Snowden 2016; 2019], for the second symmetric power S^2, for $\bigwedge^2$ [Nagpal et al. 2016], for $S^1 \oplus S^2$ and for $S^1 \oplus \bigwedge^2$ [Sam and Snowden 2022].

Partitions of $\mathbb{N}$ into finitely many subsets also feature in the classification of symmetric subvarieties of infinite affine space $(\mathbb{A}^1)^{\mathbb{N}}$ [Nagpal and Snowden 2020], and while our proofs do not logically depend on this classification, that paper did serve as an inspiration.

1.5. ***Organisation of this paper.*** This paper is organised as follows. In Sections 2.1 and 2.2 we introduce polynomial functors and affine varieties over the categories **Vec**, $\mathbf{FI^{op}} \times$ **Vec** and $(\mathbf{FI^{op}})^k \times$ **Vec**. This language happens to be more convenient than a purely infinite-dimensional approach, as shown in Remark 2.9. In Section 2.3 we introduce the category **PM** with morphisms between $(\mathbf{FI^{op}})^k \times$ **Vec**-varieties, in which, for the reasons explained in Example 1.6 and above it, k varies. In Section 2.4 we describe $(\mathbf{FI^{op}})^k \times$ **Vec**-varieties of *product type*. The simplest ones among these are of the form

$$Z : (V; S_1, \ldots, S_k) \mapsto \prod_{i=1}^{k} Z_i(V)^{S_i},$$

which are the ones of interest in our Theorems 1.1 and 1.5. Reformulations of our main theorem and its generalisation Theorem 1.5 in this language are in Remark 2.19.

Our proofs rely on induction on the "complexity" of product-type $(\mathbf{FI^{op}})^k \times$ **Vec**-varieties. The several well-founded orders used in this induction are the topic of Section 2.7, which builds on **FI**-techniques developed in Sections 2.5 and 2.6. We introduce orders on

(1) polynomial functors (Section 2.7.1),

(2) **Vec**-varieties with a specified closed embedding in $B \times Q$ where B is a finite-dimensional algebraic variety and Q is a suitable polynomial functor (Section 2.7.2),

(3) $(\mathbf{FI^{op}})^k \times$ **Vec**-varieties of product type in the category **PM** (Section 2.7.3),

(4) closed subvarieties of $(\mathbf{FI^{op}})^k \times$ **Vec**-varieties of product type (Section 2.7.4).

Then in Section 3 we formulate and prove the parameterisation theorem, Theorem 3.1, the core technical result of this paper. The statement roughly says that if X is a proper closed $(\mathbf{FI^{op}})^k \times$ **Vec**-subvariety of a variety Z of product type, then X is covered by finitely many morphisms in **PM** from $(\mathbf{FI^{op}})^l \times$ **Vec**-varieties of product form that are smaller than Z in the sense of Section 2.7.3. We prove this theorem by induction on closed subvarieties mentioned in (4). The description of these smaller $(\mathbf{FI}^{\mathbf{op}})^l \times \mathbf{Vec}$-varieties of product type relies on Proposition 3.3. This proposition allows to partition points according to their common behaviour with respect to a specific defining equation, similarly to what happens in [Draisma 2019]. Indeed, our Lemma 3.8 is proven as an iteration of the argument for the embedding theorem in [Bik et al. 2023b], which in turn uses a technique developed in [Draisma 2019]. Essential for applying Proposition 3.3 is the operation of shifting over a tuple of finite sets, described in Section 2.6. In the final Section 4 we use all the above to prove that all $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-varieties of product type are Noetherian via an induction on their order of Section 2.7.3. Theorem 1.5 and the main theorem follow as corollaries.

1.6. ***Notation and conventions.***

- For a nonnegative integer k, we set $[k] := \{1, \dots, k\}$; so in particular $[0] = \varnothing$.
- Let S be a finite set. We denote by $|S|$ the cardinality of S.
- Throughout this paper, we work over a field K of characteristic zero.
- Sym denotes the infinite symmetric group. It is defined as the direct limit over $\mathrm{Sym}(n)$, the symmetric group on the set $[n]$, with the obvious inclusion maps.
- GL denotes the infinite general linear group. It is defined as the direct limit of GL_n, the general linear group on K^n, with inclusion maps $\mathrm{GL}_n \to \mathrm{GL}_{n+1}$ given by

$$g \mapsto \left(\begin{array}{c|c} g & 0 \\ \hline 0 & 1 \end{array}\right).$$

- The category of schemes over K is denoted by $\mathbf{Sch}_K$. A product $X \times Y$ of two schemes will always mean a product in this category.
- A *variety* X here is a reduced affine scheme of finite type over K. By $K[X]$ we denote its coordinate ring, so $X = \mathrm{Spec}\, K[X]$. If Y is a subvariety of X, then we write $\mathcal{I}(Y) \subseteq K[X]$ for the (radical) ideal of functions on X vanishing on Y.
- If $f \in K[X]$ then we write $X[1/f] := \mathrm{Spec}(K[X]_f)$.
- Let $\varphi : X \to Y$ be a morphism of varieties. We denote by $\varphi^{\#} : K[Y] \to K[X]$ the induced morphism on coordinate rings.
- By a point x of a variety X we always mean a closed point of X, i.e., an element of $X(\overline{K})$.

2. The categories of $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-varieties

2.1. **Vec-*varieties.*** Let K be a field of characteristic zero, and let **Vec** be the category of finite-dimensional vector spaces over K with K-linear morphisms. We will be working with **Vec**-varieties, a functorial finite-dimensional counterpart of GL-varieties. Below, we quickly recap the theory of polynomial functors: definitions, relevant properties; and we define the notion of **Vec**-variety. See Remark 2.6 for the connection with GL-varieties.

Definition 2.1. A *polynomial functor* is a functor $P : \mathbf{Vec} \to \mathbf{Vec}$ such that for each $U, V \in \mathbf{Vec}$ the map $P : \mathrm{Hom}_{\mathbf{Vec}}(U, V) \to \mathrm{Hom}_{\mathbf{Vec}}(P(U), P(V))$ is polynomial, and such that the degree of this polynomial map is upper-bounded independently of U, V. The minimal such bound is called the *degree* of P.

We will also regard a polynomial functor P as a functor $\mathbf{Vec} \to \mathbf{Sch}_K$ by composing with the embedding $\mathbf{Vec} \to \mathbf{Sch}_K$ given by $V \mapsto \mathrm{Spec}(\mathrm{Sym}_K(V^*))$, the spectrum of the symmetric algebra on the dual space V^* of V. Every polynomial functor P equals $P_0 \oplus \cdots \oplus P_d$, where d is the degree of P and P_i is defined as

$$P_i(V) := \{v \in P(V) \mid \forall t \in K : P(t\, \mathrm{id}_V)v = t^i v\}.$$

Considering P as a functor $\mathbf{Vec} \to \mathbf{Sch}_K$ we have $P(V) = P_0(V) \times \cdots \times P_d(V)$. We note that P_0 is a constant polynomial functor, which assigns a fixed vector space $P(0) \in \mathbf{Vec}$ to all $V \in \mathbf{Vec}$ and the identity map to each linear map. We call P *pure* if $P_0 = \{0\}$.

Let $X, Y : \mathbf{Vec} \to \mathbf{Sch}_K$ be functors. A *closed immersion* $\iota : X \to Y$ is a natural transformation such that $\iota(V) : X(V) \to Y(V)$ is a closed immersion for all $V \in \mathbf{Vec}$. In particular, X is then a subfunctor of Y.

Definition 2.2. An *affine* **Vec**-*scheme* is a functor $X : \mathbf{Vec} \to \mathbf{Sch}_K$ that admits a closed immersion $X \to P$ with $P : \mathbf{Vec} \to \mathbf{Sch}_K$ a polynomial functor. A **Vec**-*variety* is an affine **Vec**-scheme X such that $X(V)$ is reduced for all $V \in \mathbf{Vec}$. The *category of affine* **Vec**-*schemes* is the full subcategory of the functor category $\mathbf{Sch}_K^{\mathbf{Vec}}$ whose objects are affine **Vec**-schemes.

Spelled out explicitly, a **Vec**-variety X can be described by the data of a polynomial functor P and a subvariety $X(V) \subseteq P(V)$ for each $V \in \mathbf{Vec}$ such that, for each $\varphi \in \mathrm{Hom}_{\mathbf{Vec}}(U, V)$, the linear map $P(\varphi)$ maps $X(U)$ into $X(V)$. A morphism of **Vec**-varieties $\tau : X \to Y$ consists of a morphism of varieties $\tau(V) : X(V) \to Y(V)$ for each $V \in \mathbf{Vec}$ such that, for each $\varphi \in \mathrm{Hom}_{\mathbf{Vec}}(U, V)$, we have $\tau(V) \circ X(\varphi) = Y(\varphi) \circ \tau(U)$.

Remark 2.3. The subcategory of **Vec**-varieties is closed under taking closed immersions and finite products. To see the latter, note that the product of $X, Y : \mathbf{Vec} \to \mathbf{Sch}_K$ in $\mathbf{Sch}_K^{\mathbf{Vec}}$ is given by $V \mapsto X(V) \times Y(V)$; and furthermore, given closed immersions $X \hookrightarrow P$ and $Y \hookrightarrow Q$, the assignment

$$X(V) \times Y(V) \to P(V) \times Q(V)$$

defines a closed immersion of the product $X \times Y$ into the polynomial functor $P \oplus Q$.

Lemma 2.4. *The category of affine* **Vec**-*schemes admits fibre products.*

Proof. First note that for morphisms of affine **Vec**-schemes $X \to Y$ and $Z \to Y$ the fibre product $X \times_Y Z$ of X and Z over Y exists in the functor category $\mathbf{Sch}_K^{\mathbf{Vec}}$ and is given by

$$(X \times_Y Z)(V) := X(V) \times_{Y(V)} Z(V).$$

Moreover, since $Y(V)$ is affine (or more generally since $Y(V)$ is separated; see [Stacks, Tag 01KR]) the natural morphism $X(V) \times_{Y(V)} Z(V) \to X(V) \times Z(V)$ is a closed immersion. The statement then follows by Remark 2.3. □

The main result of [Draisma 2019] says that **Vec**-varieties are topologically Noetherian.

Theorem 2.5 [Draisma 2019, Theorem 1]. *Let X be a* **Vec**-*variety. Then every descending chain of* **Vec**-*subvarieties*

$$X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$$

stabilises, that is, there exists $N \geq 0$ such that for each $n \geq N$ we have $X_n = X_{n+1}$.

Remark 2.6. If X is a **Vec**-variety, then $X_\infty := \lim_{\leftarrow n} X(K^n)$ is a GL-variety in the sense of [Bik et al. 2023b]. This yields an equivalence of categories between **Vec**-varieties and GL-varieties. Most of our reasoning will be in the former terminology, but could be rephrased in the latter.

2.2. $(\mathbf{FI}^{\mathbf{op}})^k \times$ Vec-*varieties.*

Let **FI** be the category of finite sets with injections.

Definition 2.7. Let $k \in \mathbb{Z}_{\geq 0}$. An $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-*variety* is a covariant functor X from $(\mathbf{FI}^{\mathbf{op}})^k$ to the category of **Vec**-varieties.

Explicitly, an $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety is given by the following data: for any k-tuple $(S_1, \dots, S_k)$ we have a **Vec**-variety $X(S_1, \dots, S_k)$, and for any k-tuple of injective maps $\iota = (\iota_1 : S_1 \to T_1, \dots, \iota_k : S_k \to T_k)$, we have a corresponding morphism $X(\iota) : X(T_1, \dots, T_k) \to X(S_1, \dots, S_k)$ of **Vec**-varieties and the usual requirements that $X(\tau \circ \iota) = X(\iota) \circ X(\tau)$ and $X(\mathrm{id}_{S_1}, \dots, \mathrm{id}_{S_k}) = \mathrm{id}_{X(S_1,\dots,S_k)}$.

Again, there are natural notions of morphism and closed immersion of $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-varieties, and we call an $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety Noetherian if every descending chain of closed $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-subvarieties stabilises.

Remark 2.8. In particular, any contravariant functor from **FI** to finite-dimensional affine varieties, i.e., an $\mathbf{FI}^{\mathbf{op}}$-variety, is trivially an $\mathbf{FI}^{\mathbf{op}} \times \mathbf{Vec}$-variety. In this generality, $\mathbf{FI}^{\mathbf{op}}$-varieties are certainly not Noetherian; see [Hillar and Sullivant 2012, Example 3.8].

However, we will be largely concerned with $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-varieties defined as follows. Let $Z_1, \dots, Z_k$ be **Vec**-varieties, define

$$X(S_1, \dots, S_k) := Z_1^{S_1} \times \cdots \times Z_k^{S_k} \tag{1}$$

and for $\iota = (\iota_1, \dots, \iota_k) : (S_1, \dots, S_k) \to (T_1, \dots, T_k)$ define $X(\iota)$ as the product of the natural projections $Z^{T_i} \to Z^{S_i}$ associated to ι_i. We will prove that $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-varieties of this form are, indeed, Noetherian.

Note that we may also regard a $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety as a functor $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec} \to \mathbf{Sch}_K$. For fixed k, the $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-varieties thus form a category by considering it as the full subcategory in the corresponding functor category.

Remark 2.9. If X is an $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety, then the group $\mathrm{Sym}^k \times \mathrm{GL}$ acts on the inverse limit

$$\varprojlim_{n_1,\dots,n_k,n} X([n_1], \dots, [n_k])(K^n).$$

This gives a functor from $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-varieties to (infinite-dimensional) schemes equipped with a $\mathrm{Sym}^k \times \mathrm{GL}$-action. Unlike in Remark 2.6, this is not quite an equivalence of categories (even under reasonable restrictions on the $\mathrm{Sym}^k \times \mathrm{GL}$-action). For example, $X([n_1], \dots, [n_k])$ could be empty for large n_i and a fixed nontrivial GL-variety for smaller n_i. We will consider an explicit example of this type later in Example 2.35. In that case, the inverse limit is empty but the $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety is not trivial. Our theorems will be formulated in the richer category of $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-varieties.

2.3. ***Partition morphisms and the category* PM.** Suppose we are given a point p in some $X(S_1, \dots, S_k)(V)$, where X is as in (1). Then the components of p labelled by one of the finite sets S_i may exhibit different behaviours, which prompts us to further partition S_i into subsets labelling components where the behaviour is similar. See Example 1.6 for an instance of this phenomenon. In that case, p will be in the image of some partition morphism; for Example 1.6 this is further explained in Example 2.15. Partition morphisms are defined below, after another motivating example.

Example 2.10. We revisit a step in the classification of Sym-invariant subvarieties of infinite affine space from [Nagpal and Snowden 2020]. We do so in the **FI**-framework, where this corresponds to closed $\mathbf{FI}^{\mathbf{op}}$-subvarieties of the $\mathbf{FI}^{\mathbf{op}}$-variety $X(S) := (\mathbb{A}^1)^S = \mathbb{A}^S$, where, for an injection $\pi : S \to T$, the map $X(\pi)$ is the corresponding projection $\mathbb{A}^T \to \mathbb{A}^S$. Let Z be a proper closed $\mathbf{FI}^{\mathbf{op}}$-subvariety of X. By (the **FI**-analogue of) [Nagpal and Snowden 2020, Proposition 2.6], the number of distinct coordinates of points in $Z(S)$ is bounded by some natural number l, independently of S. This means that for every $S \in \mathbf{FI}$, $Z(S)$ is contained in the union over all partitions of S into subsets $T_1, \dots, T_l$ of the morphism $\varphi(T_1, \dots, T_l) : \mathbb{A}^l \to X(S)$ that maps $(p_1, \dots, p_l)$ to the tuple $(q_i)_{i \in S}$ with $q_i = p_j$ for the unique $j \in [l]$ with $i \in S_j$. The morphisms $\varphi(T_1, \dots, T_l)$ for varying $(T_1, \dots, T_l) \in \mathbf{FI}^l$ form a partition morphism into X from the constant $(\mathbf{FI}^{\mathbf{op}})^l$-variety $Y : (T_1, \dots, T_l) \mapsto \mathbb{A}^l$, an object that is arguably simpler than X. In the definition that follows, we generalise this notion to the setting where X is an arbitrary $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety.

Definition 2.11. Let X be an $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety and let Y be an $(\mathbf{FI}^{\mathbf{op}})^l \times \mathbf{Vec}$-variety. A *partition morphism* $Y \to X$ consists of

(1) a map $\pi : [l] \to [k]$; and

(2) for each l-tuple of finite sets $(T_1, \dots, T_l)$ a morphism

$$\varphi(T_1, \dots, T_l) : Y(T_1, \dots, T_l) \to X\Big(\bigsqcup_{j \in \pi^{-1}(1)} T_j, \dots, \bigsqcup_{j \in \pi^{-1}(k)} T_j\Big)$$

of **Vec**-varieties in such a manner that for any l-tuple $\iota = (\iota_j)_j \in \mathrm{Hom}_{\mathbf{FI}}(S_j, T_j)^l$ the following diagram of **Vec**-variety morphisms commutes:

$$\begin{array}{ccc} Y(T_1, \dots, T_l) & \xrightarrow{\varphi(T_1,\dots,T_l)} & X\Big(\bigsqcup_{j \in \pi^{-1}(1)} T_j, \dots, \bigsqcup_{j \in \pi^{-1}(k)} T_j\Big) \\ \Big\downarrow{\scriptstyle Y(\iota_1,\dots,\iota_l)} & & \Big\downarrow{\scriptstyle X\left(\bigsqcup_{j \in \pi^{-1}(1)} \iota_j, \dots, \bigsqcup_{j \in \pi^{-1}(k)} \iota_j\right)} \\ Y(S_1, \dots, S_l) & \xrightarrow[\varphi(S_1,\dots,S_l)]{} & X\Big(\bigsqcup_{j \in \pi^{-1}(1)} S_j, \dots, \bigsqcup_{j \in \pi^{-1}(k)} S_j\Big) \end{array}$$

Remark 2.12. Note that if we take $k = l$ and $\pi = \mathrm{id}_{[k]}$, then a partition morphism is just a morphism of $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties.

There is a natural way to compose partition morphisms: if (π, φ) is a partition morphism $Y \to X$ as above and (ρ, ψ) is a partition morphism $Z \to Y$, where Z is an $(\mathbf{FI^{op}})^m \times \mathbf{Vec}$-variety, then $(\pi, \varphi) \circ (\rho, \psi)$ is the partition morphism given by the data $\pi \circ \rho : [m] \to [k]$ and the morphisms

$$\varphi\Bigg(\bigsqcup_{n\in\rho^{-1}(1)} R_n, \ldots, \bigsqcup_{n\in\rho^{-1}(l)} R_n\Bigg) \circ \psi(R_1, \ldots, R_m) : Z(R_1, \ldots, R_m) \to X\Bigg(\bigsqcup_{n\in(\pi\circ\rho)^{-1}(1)} R_n, \ldots, \bigsqcup_{n\in(\pi\circ\rho)^{-1}(k)} R_n\Bigg).$$

A tedious but straightforward computation shows that partition morphisms turn the class of $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties, with varying k, into a category. We call this category **PM**.

Definition 2.13. Let X be an $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety, let Y be an $(\mathbf{FI^{op}})^l \times \mathbf{Vec}$-variety, and let $(\pi, \varphi) : Y \to X$ be a partition morphism. Let $S_1, \ldots, S_k \in \mathbf{FI}$ and $V \in \mathbf{Vec}$. The (set-theoretic) *image* of (π, φ) in $X(S_1, \ldots, S_k)(V)$ is defined as the set of all points of the form $(X(\iota_1, \ldots, \iota_k)(V) \circ \varphi(T_1, \ldots, T_l)(V))(q)$, where $T_1, \ldots, T_l$ are finite sets, q is a point in $Y(T_1, \ldots, T_l)(V)$, and each ι_i is a bijection from S_i to $\bigsqcup_{j\in\pi^{-1}(i)} T_j$. The partition morphism (π, φ) is called *surjective* if its image in $X(S_1, \ldots, S_k)(V)$ equals $X(S_1, \ldots, S_k)(V)$ for all choices of $S_1, \ldots, S_k$ and V.

Remark 2.14. In the previous definition, each bijection ι_i induces a partition of the set S_i. Furthermore, if a partition morphism (π, φ) is surjective and for every i the **Vec**-variety

$$X(\varnothing, \ldots, \varnothing, \{*\}, \varnothing, \ldots, \varnothing),$$

where $\{*\}$ is a singleton in the i-th position, is nonempty, then the map π is automatically surjective, so that π induces a partition of $[l]$ into k labelled, nonempty parts. This is our reason for calling the morphisms in **PM** partition morphisms.

The following example rephrases Example 1.6 in the current terminology.

Example 2.15. Let Z be the **Vec**-variety that maps V to $V \otimes V$, and let Z_1, Z_2 be the closed **Vec**-subvarieties consisting of symmetric and skew-symmetric tensors, respectively. Consider the $\mathbf{FI^{op}} \times \mathbf{Vec}$-variety defined by $S \mapsto Z^S$, and for every finite set S let $X(S)$ be the closed **Vec**-subvariety given by the points $x = (x_s)_{s\in S} \in Z(V)^S$ such that each component x_s is either symmetric or skew-symmetric. Note that X is a closed $\mathbf{FI^{op}} \times \mathbf{Vec}$-subvariety. Let Y be the $(\mathbf{FI^{op}})^2 \times \mathbf{Vec}$-variety defined by

$$Y(S_1, S_2) = Z_1^{S_1} \times Z_2^{S_2}.$$

We now construct a partition morphism $\varphi : Y \to X$ as follows. The map $\pi : [2] \to [1]$ is the only possible map, and for every $V \in \mathbf{Vec}$ and $(S_1, S_2) \in \mathbf{FI^{op}}^2$, the map

$$\varphi(S_1, S_2)(V) : Y(S_1, S_2)(V) = Z_1(V)^{S_1} \times Z_2(V)^{S_2} \to X(S_1 \sqcup S_2)(V)$$

is defined by

$$((x_{s_1})_{s_1\in S_1}, (x_{s_2})_{s_2\in S_2}) \mapsto (x_s)_{s\in S_1\sqcup S_2}.$$

Note that the partition morphism φ is surjective. In particular, we say that X *is covered* by Y, and, as we have already hinted in Example 1.6, Y is in some sense smaller than the assignment $S \mapsto Z^S$. The fact that we can do this in general is the content of the *parameterisation theorem* (Theorem 3.1).

The following lemma is immediate.

Lemma 2.16. *Let X be an $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety, let X' be a closed $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-subvariety of X, and let (π, φ) be a partition morphism from an $(\mathbf{FI^{op}})^l \times \mathbf{Vec}$-variety Y to X. Then $Y' := (\pi, \varphi)^{-1}(X')$ defined by*

$$Y'(T_1, \dots, T_l) := \varphi(T_1, \dots, T_l)^{-1}\Big(X'\Big(\bigsqcup_{j\in\pi^{-1}(1)} T_j, \dots, \bigsqcup_{j\in\pi^{-1}(k)} T_j\Big)\Big)$$

is a closed $(\mathbf{FI^{op}})^l \times \mathbf{Vec}$-subvariety of Y, and the data of π together with the restrictions of the morphisms $\varphi(T_1, \dots, T_l)$ gives a partition morphism from Y' to X. Moreover, if (π, φ) is surjective, then so is its restriction to $Y' \to X'$.

The following easy proposition is crucial in our approach to the main theorem.

Proposition 2.17. *If (π, φ) is a surjective partition morphism from Y to X, and Y is a Noetherian $(\mathbf{FI^{op}})^l \times \mathbf{Vec}$-variety, then X is a Noetherian $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety.*

Proof. Let $X_1 \supseteq X_2 \supseteq \cdots$ be a descending chain of closed $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-subvarieties. By Lemma 2.16, the preimages $Y_i := (\pi, \varphi)^{-1}(X_i)$ are closed $(\mathbf{FI^{op}})^l \times \mathbf{Vec}$-subvarieties of Y. Hence the chain $Y_1 \supseteq Y_2 \supseteq \cdots$ stabilises by assumption. The surjectivity of (π, φ) implies the surjectivity of its restriction to $Y_i \to X_i$. This implies that X_i is uniquely determined by Y_i, and hence the chain $X_1 \supseteq X_2 \supseteq \cdots$ stabilises at the same point. □

2.4. *Product type.* We now introduce the $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$ varieties of product type. Essentially, these are the varieties from Remark 2.8, but for our proofs we will need a finer control over these products. Therefore, we will work over a general base $\mathbf{Vec}$-variety Y, and keep track of the "constant parts" B_i of the $\mathbf{Vec}$-varieties whose products we consider.

Definition 2.18. Let Y be a $\mathbf{Vec}$-variety and $k, n_1, \dots, n_k \in \mathbb{Z}_{\geq 0}$. For each $i \in [k]$, let B_i be a $\mathbf{Vec}$-subvariety of $Y \times \mathbb{A}^{n_i}$, and Q_i be a pure polynomial functor. By construction each $\mathbf{Vec}$-variety $B_i \times Q_i$ has a morphism to Y induced by the projection $Y \times \mathbb{A}^{n_i} \to Y$. We define the $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety $Z = [Y; B_1 \times Q_1, \dots, B_k \times Q_k]$ via

$$Z(S_1, \dots, S_k) := \underbrace{(B_1 \times Q_1) \times_Y \cdots \times_Y (B_1 \times Q_1)}_{\text{cardinality-of-}S_1\text{ times}} \times_Y (B_2 \times Q_2) \times_Y \cdots \times_Y (B_k \times Q_k),$$

where for every index $i \in [k]$ the fibre product over Y of $B_i \times Q_i$ with itself is taken $|S_i|$ times, and these copies are labelled by the elements of S_i; and where the morphism $Z(T_1, \dots, T_k) \to Z(S_1, \dots, S_k)$ corresponding to $\iota : S \to T$ is the projection as in Remark 2.8. We also write the above product in a more compact notation as

$$(B_1 \times Q_1)_Y^{S_1} \times_Y \cdots \times_Y (B_k \times Q_k)_Y^{S_k}.$$

We say that Z is an $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety of *product type* (over Y).

Note that $Z(S_1, \dots, S_k)$ is naturally a closed **Vec**-subvariety of

$$Y \times \prod_{i=1}^{k} (\mathbb{A}^{n_i} \times Q_i)^{S_i},$$

where the product is over K. Moreover, if $k = 0$, then by definition $Z = Y$.

When we talk of $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties of product type, we will always specify each B_i together with its closed embedding in $Y \times \mathbb{A}^{n_i}$; the reason being that, in the proof of the main theorem, we aim to argue by induction on both Y and n_i.

Remark 2.19. The settings of Theorems 1.1 and 1.5 can be rephrased in our current terminology as follows. Consider **Vec**-varieties $Z_1, \dots, Z_k$. Then for every $i \in [k]$ there exist $n_i \in \mathbb{Z}_{\geq 0}$, a closed subvariety $A_i \subseteq \mathbb{A}^{n_i}$, and a pure polynomial functor Q_i such that $Z_i \subseteq A_i \times Q_i$. Define Y to be a point, and $B_i := Y \times A_i$. Then the variety $Z_1^{\mathbb{N}} \times \cdots \times Z_k^{\mathbb{N}}$ of Theorem 1.5 is a subvariety of the product-type $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety

$$[Y; B_1 \times Q_1, \dots, B_k \times Q_k],$$

with $k = 1$ being the special case addressed in Theorem 1.1.

Remark 2.20. In [Draisma et al. 2022], for $\mathbf{FI^{op}}$-varieties (no dependence on **Vec**), the notion of product type is more restrictive. Essentially, there the last three authors considered a single finite-dimensional affine variety Z with a morphism to a finite-dimensional, irreducible, affine variety Y, with the additional requirement that $K[Z]$ is a free $K[Y]$-module. This then ensures that each irreducible component of Z^S maps dominantly to Y. In [Draisma et al. 2022] this is used to count the orbits of $\mathrm{Sym}(S)$ on these irreducible components.

The following example describes the partition morphisms between product-type varieties. It is particularly relevant as this is the shape of the partition morphisms we will be dealing with in our proof of the parameterisation theorem (Theorem 3.1).

Example 2.21. Let $Z' := [Y'; B'_1 \times Q'_1, \dots, B'_l \times Q'_l]$ and $Z := [Y; B_1 \times Q_1, \dots, B_k \times Q_k]$ be an $(\mathbf{FI^{op}})^l \times \mathbf{Vec}$-variety and an $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety of product type over Y' and Y, respectively. We want to construct a partition morphism $(\pi, \varphi) : Z' \to Z$. Consider the following data:

- Let $\pi : [l] \to [k]$ be any map.
- Let $\alpha : Y' \to Y$ be a morphism of **Vec**-varieties.
- For each $j \in [l]$ let $\beta_j : B'_j \times Q'_j \to B_{\pi(j)} \times Q_{\pi(j)}$ be a morphism of **Vec**-varieties such that the following diagram commutes:

$$\begin{array}{ccc} B'_j \times Q'_j & \xrightarrow{\beta_j} & B_{\pi(j)} \times Q_{\pi(j)} \\ \downarrow & & \downarrow \\ Y' & \xrightarrow[\alpha]{} & Y \end{array} \tag{2}$$

For each $(T_1, \dots, T_l) \in \mathbf{FI}^l$ we define the morphism of **Vec**-varieties

$$\varphi(T_1, \dots, T_l) : Z'(T_1, \dots, T_l) \to Z\Big(\bigsqcup_{j \in \pi^{-1}(1)} T_j, \dots, \bigsqcup_{j \in \pi^{-1}(k)} T_j \Big)$$

as follows. Let $S_i := \bigsqcup_{j \in \pi^{-1}(i)} T_j$. Then for any $V \in \mathbf{Vec}$ the element

$$((b'_{j,t}, q'_{j,t})_{t \in T_j})_{j \in [l]} \in (B'_1 \times Q'_1)^{T_1}_{Y'}(V) \times_{Y'} \cdots \times_{Y'} (B'_l \times Q'_l)^{T_l}_{Y'}(V)$$

is mapped to the element

$$(((\beta_j(V)(b'_{j,t}, q'_{j,t}))_{t \in T_j})_{j \in \pi^{-1}(i)})_{i \in [k]} \in (B_1 \times Q_1)^{S_1}_{Y}(V) \times_Y \cdots \times_Y (B_k \times Q_k)^{S_k}_{Y}(V).$$

By construction, the pair (π, φ) is a partition morphism $Z' \to Z$. Conversely, every partition morphism $Z' \to Z$ is of this form. Indeed, from a general partition morphism $Z' \to Z$, α is recovered by taking all T_j empty and β_j is recovered by taking T_j a singleton and all $T_{j'}$ with $j' \neq j$ empty. That (2) commutes then follows by applying the commuting diagram from the definition of a partition morphism to the morphism $(\varnothing, \dots, \varnothing, \dots, \varnothing) \to (\varnothing, \dots, \{*\}, \dots, \varnothing)$ in $\mathbf{FI}^l$.

2.5. ***The leading monomial ideal.*** The following definition gives a size measure for a closed subvariety $B \subseteq Y \times \mathbb{A}^n$.

Definition 2.22. Let Y be a **Vec**-variety, $n \in \mathbb{Z}_{\geq 0}$ and B a closed **Vec**-subvariety of $Y \times \mathbb{A}^n$. For $V \in \mathbf{Vec}$ consider the ideal $\mathcal{I}(B(V))$ of $K[Y(V)][x_1, \dots, x_n]$ defining $B(V)$. We fix the lexicographic order on monomials in $x_1, \dots, x_n$, and denote by $\mathrm{LM}(B)$ the set of those monomials that appear as leading monomials of *monic* polynomials in $\mathcal{I}(B(V))$, i.e., those with leading coefficient $1 \in K[Y(V)]$.

Indeed, $LM(B)$ is well-defined:

Lemma 2.23. *The set* $\mathrm{LM}(B)$ *does not depend on the choice of* V.

Proof. Let $V \in \mathbf{Vec}$ and consider the linear maps $\iota : 0 \to V$ and $\pi : V \to 0$. If $f \in \mathcal{I}(B(V))$ is monic with leading monomial x^u, then applying $Y(\iota)^{\#}$ to all coefficients of f yields a polynomial in $\mathcal{I}(B(0))$ which is monic with leading monomial x^u. This shows that the leading monomials of monic polynomials in $\mathcal{I}(B(V))$ remain leading monomials of monic elements in $\mathcal{I}(B(0))$. One obtains the converse inclusion by applying $Y(\pi)^{\#}$. □

The following lemma monitors the size of LM of the constant parts after a base change in product-type varieties. It is used in Proposition 2.28.

Lemma 2.24. *Let $Y' \to Y$ be a morphism of* **Vec***-varieties, let B be a closed* **Vec***-subvariety of $Y \times \mathbb{A}^n$, and define $B' := Y' \times_Y B \subseteq Y' \times \mathbb{A}^n$. Then* $\mathrm{LM}(B') \supseteq \mathrm{LM}(B)$.

Proof. Pulling back a monic equation for $B(V)$ along $Y'(V) \times \mathbb{A}^n \to Y(V) \times \mathbb{A}^n$ yields a monic equation for $B'(V)$ with the same leading monomial. □

2.6. ***Shifting over tuples of finite sets.*** Shifting over a finite set is a standard technique in the theory of **FI**-modules [Church et al. 2015], and was also used by the last three authors in [Draisma et al. 2022] to turn certain $\mathbf{FI^{op}}$-varieties into products. The third author used the operation of shifting over a vector space in [Draisma 2019] to prove what became "the embedding theorem" for GL-varieties in [Bik et al. 2023b, Theorems 4.1, 4.2]. Here we describe this operation in the context of $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties.

Definition 2.25. Let X be an $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety and let $S = (S_1, \dots, S_k) \in \mathbf{FI}^k$. Then the *shift* $\mathrm{Sh}_S X$ *of* X *over* S is the $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety defined by

$$(\mathrm{Sh}_S X)(T_1, \dots, T_k) := X(S_1 \sqcup T_1, \dots, S_k \sqcup T_k)$$

and, for injections $\iota_i : T_i \to T_i'$,

$$(\mathrm{Sh}_S X)(\iota_1, \dots, \iota_k) := X(\mathrm{id}_{S_1} \sqcup \iota_1, \dots, \mathrm{id}_{S_k} \sqcup \iota_k).$$

Remark 2.26. Consider a tuple $S = (S_1, \dots, S_k)$ in $(\mathbf{FI^{op}})^k$ and define the covariant functor

$$\mathrm{Sh}_S : (\mathbf{FI^{op}})^k \times \mathbf{Vec} \to (\mathbf{FI^{op}})^k \times \mathbf{Vec}$$

by assigning to each tuple $(T_1, \dots, T_k)$ the tuple $(S_1 \sqcup T_1, \dots, S_k \sqcup T_k)$ and assigning to each morphism $\iota : (\iota_1, \dots, \iota_k) : (T_1, \dots, T_k) \to (T_1', \dots, T_k')$ the morphism $\iota \sqcup \mathrm{id}_S$. In particular $\mathrm{Sh}_S X$ is the composition $X \circ \mathrm{Sh}_S$.

Remark 2.27. Let $V \in \mathbf{Vec}$. While $\mathrm{Sh}_S X(T_1, \dots, T_k)(V)$ and $X(S_1 \sqcup T_1, \dots, S_k \sqcup T_k)(V)$ coincide as sets, the action induced by functoriality of the k copies of the symmetric group on them is different. Indeed, the groups $\mathrm{Sym}(T_1) \times \cdots \times \mathrm{Sym}(T_k)$ and $\mathrm{Sym}(S_1 \sqcup T_1) \times \cdots \times \mathrm{Sym}(S_k \sqcup T_k)$ act, respectively, on the former and on the latter.

The following proposition describes the shift operation on product-type varieties.

Proposition 2.28. *The shift* $\mathrm{Sh}_S Z$ *over* $S = (S_1, \dots, S_k)$ *of an* $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$*-variety*

$$Z := [Y; B_1 \times Q_1, \dots, B_k \times Q_k]$$

of product type is itself isomorphic to a variety of product type:

$$\mathrm{Sh}_S Z \cong [Y'; B_1' \times Q_1, \dots, B_k' \times Q_k]$$

with

$$Y' := (B_1 \times Q_1)_Y^{S_1} \times_Y \cdots \times_Y (B_k \times Q_k)_Y^{S_k} \quad \text{and} \quad B_i' := Y' \times_Y B_i.$$

Furthermore, each B_i' *is naturally a* $\mathbf{Vec}$*-subvariety of* $Y' \times \mathbb{A}^{n_i}$*, and we have* $\mathrm{LM}(B_i') \supseteq \mathrm{LM}(B_i)$.

Proof. Straightforward. The last statement follows from Lemma 2.24. □

2.7. ***Well-founded orders.*** In this paper a *preorder* $\preceq$ on a class is a reflexive and transitive relation. We also write $B \succeq A$ for $A \preceq B$. Furthermore, write $A \prec B$ or $B \succ A$ to mean that $A \preceq B$ but not $B \preceq A$. The preorder is well-founded if it admits no infinite strictly decreasing chains $A_1 \succ A_2 \succ \cdots$.

In this section we first recall a well-founded preorder on polynomial functors. Building on it, we define well-founded preorders

- on varieties appearing in the definition of $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-varieties of product type,
- on product-type varieties, and
- on closed subvarieties of a fixed product-type variety.

2.7.1. *Order on polynomial functors.*

Definition 2.29. For polynomial functors P, Q, we write $P \preceq Q$ if $P \cong Q$ or else, for the largest e with $P_e \not\cong Q_e$, P_e is a quotient of Q_e.

This is a well-founded partial order on polynomial functors; see [Draisma 2019, Lemma 12].

2.7.2. *Order on* **Vec***-varieties of type* $B \times Q$. Consider **Vec**-varieties Y, Y', integers n, n', pure polynomial functors Q, Q', and **Vec**-subvarieties $B \subset Y \times \mathbb{A}^n$, $B' \subset Y' \times \mathbb{A}^{n'}$. We say that $B' \times Q' \preceq B \times Q$ if

(1) $Q' \prec Q$ in the order of Definition 2.29; or

(2) $Q' \cong Q$, $n' = n$ and $\mathrm{LM}(B') \supseteq \mathrm{LM}(B)$.

This is a preorder on **Vec**-varieties of this type.

Remark 2.30. We remark that $\preceq$ is defined on **Vec**-varieties *with a specified product decomposition* $B \times Q$ where B is a **Vec**-variety *with a specified closed embedding into a specified product* $Y \times \mathbb{A}^n$ of a **Vec**-variety Y and some n. It is not a preorder on **Vec**-varieties without further data.

Lemma 2.31. *The preorder on* **Vec***-varieties defined as above is well-founded.*

Proof. Suppose we have an infinite strictly decreasing chain

$$B_1 \times Q_1 \succ B_2 \times Q_2 \succ \cdots$$

with $B_i \subseteq Y_i \times \mathbb{A}^{n_i}$. Then we have $Q_1 \succeq Q_2 \succeq \cdots$. By the well-foundedness of $\succeq$ on polynomial functors, there exists a $j \geq 1$ such that both Q_i and n_i are constant for $i \geq j$. But then

$$\mathrm{LM}(B_i) \subsetneq \mathrm{LM}(B_{i+1}) \subsetneq \cdots,$$

which contradicts Dickson's lemma. □

2.7.3. *Order on product-type varieties.* Consider an $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety $Z := [Y;\, B_1 \times Q_1, \ldots, B_k \times Q_k]$, and an $(\mathbf{FI}^{\mathbf{op}})^l \times \mathbf{Vec}$-variety $Z' := [Y';\, B'_1 \times Q'_1, \ldots, B'_l \times Q'_l]$. We say that $Z' \preceq Z$ if there exists a map $\pi : [l] \to [k]$ with the following properties:

(1) $B'_j \times Q'_j \preceq B_{\pi(j)} \times Q_{\pi(j)}$ holds for all $j \in [l]$.

(2) For all j whose π-fibre $\pi^{-1}(\pi(j))$ has cardinality at least 2 we have $B'_j \times Q'_j \prec B_{\pi(j)} \times Q_{\pi(j)}$.

(3) If π is a bijection, then either at least one of the inequalities in (1) is strict, or else Y' is a closed **Vec**-subvariety of Y.

Lemma 2.32. *Suppose $Z' \preceq Z$ is witnessed by $\pi : [l] \to [k]$ and suppose that at least one of the following holds*:

- $l \neq k$, *or*
- *at least one of the inequalities in* (1) *is strict.*

Then we have $Z' \prec Z$.

Proof. Assume, on the contrary, that $\sigma : [k] \to [l]$ witnesses $Z \preceq Z'$. Construct a directed graph Γ with vertex set $[l] \sqcup [k]$ and an arrow from each $j \in [l]$ to $\pi(j)$ and an arrow from each $i \in [k]$ to $\sigma(i)$. Like any digraph in which each vertex has out-degree 1, Γ is a union of disjoint directed cycles (here of even length) plus a number of trees rooted at vertices in those cycles and directed towards those roots. Moreover, those cycles have the same number of vertices in $[l]$ as in $[k]$.

The assumptions imply that at least one of the vertices of Γ does not lie on a directed cycle. Without loss of generality, there exists an $i \in [k]$ not in any cycle such that $j := \sigma(i)$ lies on a cycle. Let n be half the length of that cycle, so that $(\sigma\pi)^n(j) = j$. Then we have

$$B'_j \times Q'_j \preceq B_{\pi(j)} \times Q_{\pi(j)} \preceq \cdots \preceq B_{\pi(\sigma\pi)^{n-1}(j)} \times Q_{\pi(\sigma\pi)^{n-1}(j)} \prec B'_{(\sigma\pi)^n(j)} \times Q'_{(\sigma\pi)^n(j)} = B'_j \times Q'_j,$$

where the strict inequality holds because $\sigma^{-1}(j)$ has at least two elements: i and $\pi(\sigma\pi)^{n-1}(j)$. By transitivity of the preorder from Section 2.7.2, we find $B'_j \times Q'_j \prec B'_j \times Q'_j$, which however contradicts the reflexivity of that preorder. □

Lemma 2.33. *The relation $\preceq$ is a well-founded preorder on varieties in* **PM** *of product type.*

Proof. For reflexivity we may take π equal to the identity. For transitivity, if $\pi : [l] \to [k]$ witnesses $Z' \preceq Z$ and $\sigma : [k] \to [m]$ witnesses $Z \preceq Z''$, then $\tau := \sigma \circ \pi$ witnesses $Z' \preceq Z''$ — here we note that if $|\tau^{-1}(\tau(j))| > 1$ for some $j \in [l]$, then either $|\pi^{-1}(\pi(j))| > 1$ or else $|\sigma^{-1}(\sigma(\pi(j)))| > 1$; in both cases we find that $B'_j \times Q'_j \prec B''_{\tau(j)} \times Q''_{\tau(j)}$.

For well-foundedness, suppose that we had a sequence $Z_1 \succ Z_2 \succ Z_3 \succ \cdots$, where

$$Z_i = [Y_i; B_{i,1} \times Q_{i,1}, \ldots, B_{i,k_i} \times Q_{i,k_i}],$$

and where $\pi_i : [k_{i+1}] \to [k_i]$ is a witness to $Z_i \succ Z_{i+1}$. We note that $k_i > 0$ for all i. Otherwise $0 = k_i = k_{i+1} = \cdots$ and then $Z_i = Y_i \succ Z_{i+1} = Y_{i+1} \succ \cdots$ implies that $Y_i \supsetneq Y_{i+1} \supsetneq \cdots$, which contradicts the Noetherianity of the **Vec**-variety Y_i; see Theorem 2.5.

From the chain, we construct an infinite rooted forest with vertex set $[k_1] \sqcup [k_2] \sqcup \cdots$ as follows: $[k_1]$ is the set of roots, and we attach each $j \in [k_{i+1}]$ via an edge with $\pi_i(j)$; the latter is called the *parent* of the former. We further label each vertex $j \in [k_i]$ with the product $B_{i,j} \times Q_{i,j}$.

We claim that π_i is an injection for all $i \gg 0$, i.e., that there are only finitely many vertices with more than one child. Indeed, if not, then by König's lemma the forest would have an infinite path starting at a root in $[k_1]$ and passing through infinitely many vertices with at least two children. By construction, the labels $B \times Q$ decrease weakly along such a path and strictly whenever going from a vertex to one of its more than one children, a contradiction to Lemma 2.31.

For even larger i, the k_i are constant, say equal to k, and hence the π_i are bijections. After reordering, we may assume that the π_i all equal the identity on $[k]$. Moreover, for all such i we still have $B_{i,j} \times Q_{i,j} \succeq B_{i+1,j} \times Q_{i+1,j} \succeq \cdots$ for all $j \in [k]$, and all these chains stabilise. When they do, we have $Y_i \supsetneq Y_{i+1} \supsetneq \cdots$, which is a strictly decreasing chain of **Vec**-varieties — but this again contradicts the Noetherianity of **Vec**-varieties. □

2.7.4. *Order on closed subvarieties of product-type varieties in* **PM**. Consider the $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety $Z = [Y; B_1 \times Q_1, \ldots, B_k \times Q_k]$ and let X be a closed $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-subvariety of Z; X is not required to be of product type. We define

$$\delta_X := \min_{(S_1,\ldots,S_k)\in \mathbf{FI}^k} \left\{ \sum_{i=1}^{k} |S_i| : X(S_1, \ldots, S_k) \neq Z(S_1, \ldots, S_k) \right\}$$

Let X and X' be closed $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-subvarieties of Z. Then we say $X' \preceq X$ if $\delta_{X'} \leq \delta_X$. This is a well-founded preorder on the $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-subvarieties of Z.

Remark 2.34. If f is a nonzero equation for $X(S_1, \ldots, S_k)(V)$ with $\sum_i |S_i| = \delta_X$, then f may still "come from smaller sets". More specifically, there might exist a k-tuple $(S'_1, \ldots, S'_k)$ with $|S'_i| \leq |S_i|$ for all $i \in [k]$ and with strict inequality for at least one i, an $\mathbf{FI}^k$-morphism $\iota := (\iota_1, \ldots, \iota_k) : (S'_1, \ldots, S'_k) \to (S_1, \ldots, S_k)$, and an element $f' \in K[Z(S'_1, \ldots, S'_k)(V)]$ such that $Z(\iota)(V)^{\#}(f') = f$. This is related to Remark 2.9. The following example demonstrates this phenomenon.

Example 2.35. Consider the $\mathbf{FI}^{\mathbf{op}} \times \mathbf{Vec}$-variety $Z := [\mathrm{Spec}(K); \mathbb{A}^1]$. The coordinate ring $K[Z(S)]$ is isomorphic to the polynomial ring over K in $|S|$ variables. Let $n \in \mathbb{Z}_{>0}$ and define the proper closed variety X of Z by

$$X(S) := \begin{cases} Z(S) & \text{for } |S| < n, \\ \varnothing & \text{otherwise.} \end{cases}$$

Then δ_X is equal to n and computed by the element $1 \in K[Z([n])]$, which is the image of $1 \in K[Z(\varnothing)]$ under the natural map $K[Z(\varnothing)] \to K[Z([n])]$.

3. Covering $(\mathbf{FI}^{\mathbf{op}})^k \times$ Vec-varieties by smaller ones

3.1. ***The parameterisation theorem.*** The goal of this section is to prove the following core result, which says that any proper closed subvariety of an $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety of product type is covered by finitely many smaller such varieties.

Theorem 3.1 (parameterisation theorem). *Consider an* $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$*-variety* Z *of product type and let* $X \subsetneq Z$ *be a proper closed* $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$*-subvariety. Then there exist a finite number of quadruples consisting of*

- *an* $l \in \mathbb{Z}_{\geq 0}$;
- *an* $(\mathbf{FI}^{\mathbf{op}})^l \times \mathbf{Vec}$*-variety* Z' *of product type with* $Z' \prec Z$;
- *a* k*-tuple* $S = (S_1, \ldots, S_k) \in \mathbf{FI}^k$; *and*
- *a partition morphism* $(\pi, \varphi) : Z' \to \mathrm{Sh}_S Z$;

such that for any $T_1, \dots, T_k \in \mathbf{FI}^k$, *any* $V \in \mathbf{Vec}$, *and any* $p \in X(T_1, \dots, T_k)(V)$ *there exist*: *one of these finitely many quadruples*; *finite sets* $U_1, \dots, U_k$; *and bijections* $\sigma_i : T_i \to S_i \sqcup U_i$; *such that* p *lies in the image under* $Z(\sigma_1, \dots, \sigma_k)(V)$ *of the image of* (π, φ) *in* $\mathrm{Sh}_S(Z)(U_1, \dots, U_k)(V) = Z(S_1 \sqcup U_1, \dots, S_k \sqcup U_k)(V)$.

Remark 3.2. Recall Definition 2.13 of the image of a partition morphism. Explicitly, the conclusion above means that there exist finite sets $U'_1, \dots, U'_l$ and, for each $i \in [k]$, a bijection $\iota_i : U_i \to \bigsqcup_{j \in \pi^{-1}(i)} U'_j$, and a point $q \in Z'(U'_1, \dots, U'_l)(V)$ such that

$$(Z(\sigma_1, \dots, \sigma_k)(V) \circ (\mathrm{Sh}_S\, Z)(\iota_1, \dots, \iota_l)(V) \circ \varphi(U'_1, \dots, U'_l)(V))(q) = p.$$

Informally, we will say that all points in X are *hit* by finitely many partition morphisms from varieties Z' in **PM** of product type with $Z' \prec Z$.

3.2. ***A key proposition.*** The proof of Theorem 3.1 uses a key proposition that we establish first. The reader may prefer to read only the statement of this proposition and postpone its proof until after reading the proof of Theorem 3.1 in Section 3.5.

Proposition 3.3. *Let* Y *be a* **Vec**-*variety*; $n \in \mathbb{Z}_{\geq 0}$; B *a closed* **Vec**-*subvariety of* $Y \times \mathbb{A}^n$; Q *a pure polynomial functor*; *and* X *a proper closed* **Vec**-*subvariety of* $B \times Q \subseteq Y \times \mathbb{A}^n \times Q$. *Then there exist*

- *a proper closed* **Vec**-*subvariety* Y_0 *of* Y;
- *a* **Vec**-*variety* Y' *together with a morphism* $\alpha : Y' \to Y$;
- $k \in \mathbb{Z}_{\geq 0}$;

and, *for each* $l = 0, \dots, k$,

- *an integer* $n_l \in \mathbb{Z}_{\geq 0}$;
- *a closed* **Vec**-*subvariety* $B_l \subseteq Y' \times \mathbb{A}^{n_l}$;
- *a pure polynomial functor* Q_l;
- *and a morphism* $\beta_l : B_l \times Q_l \to B \times Q$,

such that the following properties hold:

(1) *For each* $l = 0, \dots, k$, *we have that* $B_l \times Q_l \prec B \times Q$ *in the preorder from* Section 2.7.2, *and the following diagram commutes*:

$$\begin{array}{ccc} B_l \times Q_l & \xrightarrow{\beta_l} & B \times Q \\ \downarrow & & \downarrow \\ Y' & \xrightarrow[\alpha]{} & Y \end{array}$$

(2) *Let* $m \in \mathbb{Z}_{\geq 0}$, *let* $V \in \mathbf{Vec}$, *and let* $p_1, \dots, p_m \in X(V) \subseteq Y(V) \times \mathbb{A}^n \times Q(V)$ *be points whose images in* $Y(V)$ *are all equal to the same point* $y \in Y(V) \setminus Y_0(V)$. *Then there exist indices* $l_j \in \{0, \dots, k\}$ *for* $j \in [m]$ *and points* $p'_j \in B_{l_j}(V) \times Q_{l_j}(V)$ *whose images in* $Y'(V)$ *are all equal to the same point* y' *and such that* $\beta_{l_j}(V)(p'_j) = p_j$ *for all* $j \in [m]$.

Remark 3.4. The condition $\beta_{l_j}(V)(p'_j) = p_j$, together with the commuting diagram in (1), implies $\alpha(y') = y$.

Remark 3.5. The labelling by $l \in \{0, \dots, k\}$ rather than by $l \in [k]$ is chosen because in the proof of Proposition 3.3 the data for $l = 0$ are chosen in a slightly different manner than those for $l > 0$. However, in the statement of that proposition, all l play equivalent roles.

To apply Proposition 3.3 in the proof of Theorem 3.1 we will do a shift over an appropriate k-tuple of finite sets. After this shift, we deal with the points of X lying over Y_0 by induction, while we cover those in the complement by a partition morphism constructed with the morphisms α and β_j's, and whose domain is a product-type variety strictly smaller than Z. Before proving Proposition 3.3 in Section 3.4, we demonstrate its statement in two special cases.

Example 3.6. Consider the case where $Y = \operatorname{Spec} K$ and $n = 0$; then $B \subseteq Y \times \mathbb{A}^n$ is also isomorphic to $\operatorname{Spec} K$. Let Q be an arbitrary polynomial functor. In this case, X is a proper closed **Vec**-subvariety of Q and by [Bik et al. 2023b] there exist $k \in \mathbb{Z}_{\geq 0}$, (finite-dimensional) varieties $B_0, \dots, B_k$, pure polynomial functors $Q_0, \dots, Q_k \prec Q$ and morphisms $\beta_l : B_l \times Q_l \to Q$ such that X is the union of the images of the β_l. This is an instance of Proposition 3.3 with $Y_0 = \varnothing$, $Y' = Y$, and $\alpha = \mathrm{id}_Y$. Note that then $B_l \times Q_l \prec Q$ since $Q_l \prec Q$, so the specific choice of embedding $B_l \subseteq \mathbb{A}^{n_l}$ is not relevant.

Example 3.7. Consider the case where Y is constant, that is, just given by a (finite-dimensional) variety, and $Q = 0$. Since X is a proper closed subvariety of $B \subseteq Y \times \mathbb{A}^n$, there exist a $V \in$ **Vec** and a nonzero function $f \in K[B(V)]$ that vanishes identically on $X(V)$.

Then f is represented by a polynomial in $K[Y(V)][x_1, \dots, x_n]$, also denoted by f. We may reduce f modulo $\mathcal{I}(B(V))$ in such a manner that its leading term $c \cdot x^u$ has the property that $c \in K[Y(V)]$ is nonzero and $x^u \notin \mathrm{LM}(B)$. Then we take for Y_0 the closed subvariety of Y defined by the vanishing of c and for Y' the complement $Y \setminus Y_0$, with $\alpha : Y' \to Y$ being the inclusion. Furthermore, we take $k = 0$, and B_0 to be the intersection of B with $Y' \times \mathbb{A}^n$ and with the vanishing locus of f in $Y \times \mathbb{A}^n$. Then $\mathrm{LM}(B_0) \supseteq \mathrm{LM}(B)$ and since c is invertible on Y' and f vanishes on B_0, $x^u \in \mathrm{LM}(B_0) \setminus \mathrm{LM}(B)$. To verify (2) of Proposition 3.3, we observe that the p_j all map to the same point in $Y' = Y \setminus Y_0$, i.e., p_j lies in the set $B_0 \subseteq B$, and we can just take $p'_j := p_j$ for all j.

3.3. *Iterated partial derivatives.* The main technical result for proving Proposition 3.3 is Lemma 3.8 below. This is essentially an iteration of the argument used to establish the embedding theorem in [Bik et al. 2023b], which involves directional derivatives of a function defining a **Vec**-variety along a direction lying in an irreducible subobject of the top-degree part of the ambient polynomial functor.

Lemma 3.8. *Let B be a* **Vec***-variety and Q a pure polynomial functor. Decompose*

$$Q = R_1 \oplus \cdots \oplus R_t,$$

where the R_i are irreducible objects in the abelian category of polynomial functors, arranged in weakly increasing degrees. Denote with $R_{\leq s}$ the functor $\bigoplus_{i=1}^{s} R_i$, so that $R_{\leq 0} = 0$. Let X be a proper closed **Vec***-subvariety of $B \times Q$.*

Then there exist

- *a* $k \in \mathbb{Z}_{\geq 0}$;
- $U_0, \ldots, U_k \in$ **Vec** *with partial sums* $U_{\leq s} := \bigoplus_{i=0}^{s} U_i$ *for* $s \geq 0$;
- *indices* $0 = s_0 < s_1 \leq \cdots \leq s_k \leq t$;
- *for each* $l \in \{0, \ldots, k\}$ *a nonzero function* $h_l \in K[B(U_{\leq l}) \times R_{\leq s_l}(U_{\leq l})]$ (*so that* $h_0 \in K[B(U_0)]$); *and*
- *for each* $l \in \{1, \ldots, k\}$, *a nonzero coordinate* $x_l \in R_{s_l}(U_l)^*$ *and a function* r_l *in* $K[B(U_{\leq l}) \times (R_{\leq s_l}(U_{\leq l})/R_{s_l}(U_l))]$ *such that*
$$h_l = x_l \cdot h_{l-1} + r_l;$$

and such that, *moreover*, *the function* h_k *vanishes on* $X(U_{\leq k})$.

Remark 3.9. It is here that we use the fact that K has characteristic zero, in at least two different ways: the fact that an arbitrary polynomial functor is a direct sum of irreducible ones, and the fact that, by acting with the Lie algebra of GL_n, we can go from an equation to an equation of weight $(1, \ldots, 1)$. We think that our main theorem may be true in positive characteristic as well, but the proof would be more technical and involve techniques from [Bik et al. 2024], where the theory of GL-varieties in positive characteristic is developed.

Proof. Let U be a finite-dimensional vector space for which there exists a nonzero $f \in K[B(U) \times Q(U)]$ that vanishes identically on $X(U)$. Without loss of generality, $U = K^n$ for some n. Since the vanishing ideal of $X(U)$ is a $\mathrm{GL}(U)$-module, we may assume that f is a weight vector with respect to the standard maximal torus in $\mathrm{GL}(U) = \mathrm{GL}_n$. Furthermore, by enlarging U if necessary ($n = \deg(f)$ suffices) we may assume that the weight of f is $(1, \ldots, 1)$ (see [Snowden 2021, Lemma 3.2]; strictly speaking, our $\mathrm{GL}(U)$-action is contragredient to the action there, and writing $(-1, \ldots, -1)$ would be more consistent).

Choose s_k as the maximal index in $[t]$ such that f involves coordinates in $R_{s_k}(U)^*$; if no such index exists, then k is set to zero, and we may take $U_0 = U$ and $h_0 = f \in K[B(U_0)]$ and we are done.

After acting with the symmetric group $\mathrm{Sym}([n])$ if necessary, we may assume that f contains at least one coordinate in $R_{s_k}(U)^*$ of weight $(0, \ldots, 0, 1, \ldots, 1) =: (0^{n'}, 1^{n_k})$, where there are n' zeroes and n_k ones, with $n' + n_k = n$. We set $U' := K^{n'}$ and $U_k := K^{n_k}$, so that $U = U' \oplus U_k$. Since f has weight $(1, \ldots, 1)$, we can decompose
$$f = \Big(\sum_{i=1}^{N} f_i \cdot y_i\Big) + r$$
where $N \geq 1$, the f_i have weight $(1^{n'}, 0^{n_k})$; the y_i are elements in $R_{s_k}(U)^*$ of weight $(0^{n'}, 1^{n_k})$ and hence lie in $R_{s_k}(U_k)^*$; and r does not contain elements in $R_{s_k}(U_k)^*$. This implies that the f_i are elements of $K[B(U') \times R_{\leq s_k}(U')]$ and r is an element of $K[B(U' \oplus U_k) \times (R_{\leq s_k}(U' \oplus U_k)/R_{s_k}(U_k))]$. Furthermore, we may assume that the f_i are linearly independent over K.

Now act on f with upper triangular elements of $\mathfrak{gl}(U_k)$. With respect to this action, the f_i are constants, the y_i are replaced by higher-weight vectors in $R_{s_k}(U_k)^*$, and r remains an element of

$K[B(U'\oplus U_k)\times(R_{\le s_k}(U'\oplus U_k)/R_{s_k}(U_k))]$. We can choose a sequence of such upper triangular elements that takes y_1 to a nonzero highest weight vector v in $R_{s_k}(U_k)^*$, and the same sequence will take each y_i to a scalar multiple of v. Since the f_i are linearly independent, the term $f_1\cdot v$ in the result is not cancelled. Hence after this action, f has been transformed to the desired shape

$$f = h\cdot x_k + r$$

with $h\in K[B(U')\times R_{\le s_k}(U')]$, x_k a nonzero highest weight vector in $R_{s_k}(U_k)^*$ and r lies in the ring

$$K[B(U'\oplus U_k)\times(R_{\le s_k}(U'\oplus U_k)/R_{s_k}(U_k))].$$

Now we treat the pair (U', h) in exactly the same manner as we treated the pair (U, f), dragging r along in the process: pick s_{k-1} maximal such that h contains elements from $R_{s_{k-1}}(U')^*$. By acting with the symmetric group $\mathrm{Sym}([n'])$ on f we may assume that h contains an element from $R_{s_{k-1}}(U')^*$ of weight $(0^{n''}, 1^{n_{k-1}})$, with $n''+n_{k-1}=n'$. Then set $U''=K^{n''}$ and $U_{k-1}=K^{n_{k-1}}$, so that $U'=U''\oplus U_{k-1}$. By acting on f with upper triangular elements of $\mathfrak{gl}(U_{k-1})$ we transform it into the shape

$$f = (\tilde{h}\cdot x_{k-1}+\tilde{r})\cdot x_k + r,$$

where x_k has not changed, r has changed within the space $K[B(U'\oplus U_k)\times(R_{\le s_k}(U'\oplus U_k)/R_{s_k}(U_k))]$, x_{k-1} is a highest weight vector in $R_{s_{k-1}}(U_{k-1})^*$, $\tilde{h}$ lies in $K[B(U'')\times R_{\le s_{k-1}}(U'')]$, and $\tilde{r}$ lies in the ring

$$K[B(U''\oplus U_{k-1})\times(R_{\le s_{k-1}}(U''\oplus U_{k-1})/R_{s_{k-1}}(U_{k-1}))].$$

Continuing in this fashion, we eventually put f in the form

$$f = x_k(x_{k-1}(\cdots(x_2(x_1h_0+r_1)+r_2)\cdots)+r_{k-1})+r_k$$

where $h_0\in K[B(U_0)]$ and U_0 is the space left over from U after splitting off all the U_i with $i>0$. Now set

$$h_l := x_l(x_{l-1}(\cdots(x_2(x_1h_0+r_1)+r_2)\cdots)+r_{l-1})+r_l$$

and we are done. □

3.4. *Proof of Proposition 3.3.* This section contains the proof of the Proposition 3.3, and, for clarity's sake, we spell it out in a concrete example at the end.

Remark 3.10. We recall that, for any **Vec**-variety Z and any $U\in\mathbf{Vec}$, the shift $\mathrm{Sh}_U\,Z$ of Z over U is the **Vec**-variety defined by $(\mathrm{Sh}_U\,Z)(V)=Z(U\oplus V)$. There is a *natural morphism* $\mathrm{Sh}_U\,Z\to Z$ of **Vec**-varieties: for $V\in\mathbf{Vec}$, this morphism $(\mathrm{Sh}_U\,Z)(V)=Z(U\oplus V)\to Z(V)$ is just $Z(\pi_V)$, where π_V is the projection $U\oplus V\to V$.

Lemma 3.11. *Let Y be a* **Vec**-*variety, $n\in\mathbb{Z}_{\ge 0}$, and B a closed* **Vec**-*subvariety of $Y\times\mathbb{A}^n$. Then for any $U\in$* **Vec**, *$\mathrm{Sh}_U\,B$ is a closed* **Vec**-*subvariety of $(\mathrm{Sh}_U\,Y)\times\mathbb{A}^n$, and $\mathrm{LM}(B)=\mathrm{LM}(\mathrm{Sh}_U(B))$.*

Proof. This follows from Lemma 2.24. □

Remark 3.12. Let X be a **Vec**-variety, $U \in$ **Vec** and $f \in K[X(U)]$. We define $(\mathrm{Sh}_U X)[1/f]$ to be the **Vec**-variety given by $V \mapsto X(U \oplus V)[1/f]$, where we identify f with its image under the natural map $K[X(U)] \to K[X(U \oplus V)]$. Note that the action of the group GL on the coordinate ring of $\mathrm{Sh}_U X$ is the identity on the element f. In particular, for every $V \in$ **Vec**, $(\mathrm{Sh}_U X[1/f])(V) \subseteq \mathrm{Sh}_U X(V)$ is the distinguished open set of points not vanishing on the single f.

Proof of Proposition 3.3. Since X is a proper closed subvariety of $B \times Q$, we apply the machinery of Lemma 3.8. Decompose Q as $R_1 \oplus \cdots \oplus R_t$, where the R_s are irreducible polynomial functors and $\deg(R_s) \le \deg(R_{s+1})$ for all $s = 1, \dots, t-1$. Write $R_{\le s} := R_1 \oplus \cdots \oplus R_s$ and $R_{>s} := R_{s+1} \oplus \cdots \oplus R_t$, so that $R_{\le 0} = \{0\}$ and $R_{>t} = \{0\}$.

By Lemma 3.8, we can construct a sequence of vector spaces $U_0, U_1, \dots, U_k$ with partial sums $U_{\le l} := \bigoplus_{i=0}^{l} U_i$, indices $0 = s_0 < s_1 \le \cdots \le s_k \le t$, nonzero coordinates $x_l \in R_{s_l}(U_l)^*$ for $l \in [k]$, nonzero functions $h_l \in K[B(U_{\le l}) \times R_{\le s_l}(U_{\le l})]$ for $l = 0, \dots, k$ and functions $r_l \in K[B(U_{\le l}) \times (R_{\le s_l}(U_{\le l})/R_{s_l}(U_l))]$ for $l \in [k]$ such that

$$h_l = x_l \cdot h_{l-1} + r_l \tag{A}$$

for each $l = 1, \dots, k$ and such that h_k that vanishes on $X(U_{\le k})$.

Now $h_0 \in K[B(U_0)]$ is represented by a polynomial in $K[Y(U_0)][x_1, \dots, x_n]$, and after reducing modulo $\mathcal{I}(B(U_0))$, we may assume that its leading term equals $c \cdot x^u$ where $c \in K[Y(U_0)]$ is nonzero and $x^u \notin \mathrm{LM}(B)$.

Now set $U := U_{\le k} = U_0 \oplus \cdots \oplus U_k$. Then we construct the relevant data as follows.

(1) Define Y_0 as the closed **Vec**-subvariety of Y defined by the vanishing of c, so that

$$Y_0(V) := \{y \in Y(V) \mid \forall \varphi \in \mathrm{Hom}_{\mathbf{Vec}}(V, U_0) : c(Y(\varphi)y) = 0\}.$$

(2) Set $Y' := (\mathrm{Sh}_U Y)[1/c]$ with $\alpha : Y' \to Y$ the restriction to Y' of the natural morphism $\mathrm{Sh}_U Y \to Y$.

(3) Let B_0 be the closed **Vec**-subvariety of $(\mathrm{Sh}_U B)[1/c]$ defined by the vanishing of the single equation h_0. Note that B_0 is a closed **Vec**-subvariety of $Y' \times \mathbb{A}^{n_0}$ with $n_0 := n$. Define $Q_0 := Q$ and $\beta_0 : B_0 \times Q_0 \to B \times Q$ as the identity on Q and equal to the restriction to B_0 of the natural morphism $\mathrm{Sh}_U B \to B$ on B_0. Note that $\mathrm{LM}(B_0) \supseteq \mathrm{LM}(B)$ by virtue of Lemma 3.11, and since $h_0 \in \mathcal{I}(B_0(U_0))$ has leading term $c \cdot x^u$ and c is invertible on Y', we have $x^u \in \mathrm{LM}(B_0) \setminus \mathrm{LM}(B)$. Thus $B_0 \times Q_0 \prec B \times Q$.

(4) For $l \in [k]$, set

$$Q_l := ((\mathrm{Sh}_U R_{\le s_l})/(R_{\le s_l}(U) \oplus R_{s_l})) \oplus R_{>s_l}.$$

Here we recall that, for any pure polynomial functor R, the top-degree part of $\mathrm{Sh}_U R$ is naturally isomorphic to that of R, and its constant part is isomorphic to $R(U)$ (see [Draisma 2019, Lemma 14] for the first statement; the second is proved in a similar fashion). So, since we ordered the irreducible factors R_s by ascending degrees, R_{s_l} is naturally a subobject of the top-degree part of $\mathrm{Sh}_U R_{\le s_l}$; and the constant polynomial functor $R_{\le s_l}(U)$ is the constant part of $\mathrm{Sh}_U R_{\le s_l}$. Both are modded out, and we have $Q_l \prec Q$.

(5) For $l \in [k]$, we define B_l as

$$B_l := (\mathrm{Sh}_U\, B)[1/c] \times R_{\leq s_l}(U) \times \mathbb{A}^1 \subseteq Y' \times \mathbb{A}^n \times R_{\leq s_l}(U) \times \mathbb{A}^1 \cong Y' \times \mathbb{A}^{n_l}.$$

where $n_l := n + \dim(R_{\leq s_l}(U)) + 1$. Note that the factor $R_{\leq s_l}(U)$ is precisely the constant term modded out in the definition of Q_l; the role of the factor $\mathbb{A}^1$ will become clear below.

(6) To construct $\beta_l : B_l \times Q_l \to B \times Q$ we proceed as follows. Let X_l be the closed **Vec**-subvariety of $B \times R_{\leq s_l}$ defined by the vanishing of h_l. Then (A) shows that, on the distinguished open subvariety $(\mathrm{Sh}_{U_{\leq l-1}}\, X_l)[1/h_{l-1}]$, the coordinate x_l can be expressed as a function on $\mathrm{Sh}_{U_{\leq l-1}}\, B \times ((\mathrm{Sh}_{U_{\leq l-1}}\, R_{\leq s_l})/R_{s_l})$ evaluated at U_l. Since R_{s_l} is irreducible, *each* coordinate on it can be thus expressed; this is a crucial point in the proof of [Draisma 2019, Lemma 25]. This implies that the projection

$$\mathrm{Sh}_{U_{\leq l-1}}\, B \times \mathrm{Sh}_{U_{\leq l-1}}\, R_{\leq s_l} \to (\mathrm{Sh}_{U_{\leq l-1}}\, B) \times (\mathrm{Sh}_{U_{\leq l-1}}\, R_{\leq s_l})/R_{s_l}$$

restricts to a closed immersion of $(\mathrm{Sh}_{U_{\leq l-1}}\, X_l)[1/h_{l-1}]$ into the open subvariety of the right-hand side where h_{l-1} is nonzero. This statement remains true when we replace $U_{\leq l-1}$ everywhere by the larger space U. After also inverting c, we find a closed immersion

$$(\mathrm{Sh}_U\, X_l)[1/h_{l-1}][1/c] \to (\mathrm{Sh}_U\, B)[1/c] \times (\mathrm{Sh}_U\, R_{\leq s_l})/R_{s_l} \times \mathbb{A}^1,$$

where the map to the last factor is given by $1/h_{l-1}$. By [Bik 2020, Proposition 1.3.22] the inverse morphism from the image of this closed immersion lifts to a morphism of ambient **Vec**-varieties

$$\iota : B_l \times (\mathrm{Sh}_U\, R_{\leq s_l})/(R_{\leq s_l}(U) \oplus R_{s_l}) \cong (\mathrm{Sh}_U\, B)[1/c] \times (\mathrm{Sh}_U\, R_{\leq s_l})/R_{s_l} \times \mathbb{A}^1 \to \mathrm{Sh}_U(B \times R_{\leq s_l})$$

that hits all the points in $(\mathrm{Sh}_U\, X_l)[1/h_{l-1}][1/c]$. Finally, we define $\beta_l := \beta_l' \times \mathrm{id}_{R_{>s_l}}$ where β_l' is the composition of ι and the natural morphism $\mathrm{Sh}_U(B \times R_{\leq s_l}) \to B \times R_{\leq s_l}$.

Property (1) in the proposition holds by construction. We now verify property (2). Thus let $V \in$ **Vec**, $m \in \mathbb{Z}_{\geq 0}$, and let $p_1, \dots, p_m \in X(V) \subseteq Y(V) \times \mathbb{A}^n \times Q(V)$. Assume that the images of $p_1, \dots, p_m$ in $Y(V)$ are all equal to y, and that $y \notin Y_0(V)$. By definition of Y_0, this means that there exists a $\varphi \in \mathrm{Hom}_{\mathbf{Vec}}(V, U)$ such that $c(Y(\varphi)(y)) \neq 0$.

On the other hand, we have $h_k(X(\psi)(p_j)) = 0$ for all j and all $\psi : V \to U$, because h_k vanishes identically on X. For $j \in [k]$ define

$$l_j := \min\{l \mid \forall \psi \in \mathrm{Hom}_{\mathbf{Vec}}(V, U) : h_l(X(\psi)(p_j)) = 0\}.$$

Put differently, l_j is the smallest index l such that the projection of p_j in $B \times R_{\leq s_l}$ lies in $X_l \subseteq B \times R_{\leq s_l}$. Note that, if $l_j > 0$, then there exists a linear map $\psi : V \to U$ such that $h_{l_j - 1}(X(\psi)(p_j)) \neq 0$.

Since $\mathrm{Hom}_{\mathbf{Vec}}(V, U)$ is irreducible, there exists a linear map $\varphi : V \to U$ such that first, $c(Y(\varphi)(y)) \neq 0$; and second, $h_{l_j - 1}(X(\varphi)(p_j)) \neq 0$ for all j with $l_j > 0$.

We now define the p_j' as follows. First, we decompose $p_j = (p_{j,1}, p_{j,2})$, where $p_{j,1} \in B(V) \times R_{\leq s_{l_j}}(V)$ and $p_{j,2} \in R_{>s_{l_j}}(V)$. Similarly, we decompose the point $p_j' = (p_{j,1}', p_{j,2}')$ to be constructed.

(1) Set $p'_{j,2} := p_{j,2}$ for all j. Recall that we had defined $s_0 := 0$, so that this implies that if $l_j = 0$, then the component $p'_{j,2}$ of p'_j in Q equals the component $p_{j,2}$ of p_j in Q.

(2) If $l_j = 0$, then $p_{j,1} \in B(V)$, and $p'_{j,1} \in B_0(V) \subseteq (\mathrm{Sh}_U\, B)[1/c](V)$ is defined as $B(\varphi \oplus \mathrm{id}_V)(p_{j,1})$. Note that $p'_{j,1}$ does indeed lie in $B_0(V)$; this follows from the fact $l_j = 0$, so that $h_0(B(\psi)(p_{j,1})) = 0$ for all $\psi : V \to U_0$, and hence also for all ψ that decompose as $\psi' \circ (\varphi \oplus \mathrm{id}_V)$.

Furthermore, note that $\beta_0(V)(p'_j) = p_j$; this follows from the equality $\pi_V \circ (\varphi \oplus \mathrm{id}_V) = \mathrm{id}_V$. Also, the image of p'_j in $Y'(V)$ equals $Y(\varphi \oplus \mathrm{id}_V)(y) =: y'$.

(3) If $l := l_j > 0$, then $p_{j,1} \in B(V) \times R_{\leq s_l}(V)$ with $s_l \geq 1$, and $p'_{j,1}$ is constructed as follows. First apply $(B \times R_{\leq s_l})(\varphi \oplus \mathrm{id}_V)$ to $p_{j,1}$ and then forget the component in $R_{s_l}(V)$. The morphism β'_l was constructed in such a manner that $\beta'_l(V)(p'_{j,1}) = p_{j,1}$ and therefore $\beta_l(V)(p'_j) = p_j$. Note that also the image of p'_j in $Y'(V)$ equals y'. □

Example 3.13. Write Y for the polynomial functor $V \to V \oplus V$ and write $K[x_i, y_i \mid i \in [n]]$ for the coordinate ring of $Y(K^n)$. Consider the **Vec**-subvariety B of $Y \times \mathbb{A}^1$ defined by $y_1 - t \cdot x_1$, where t is the coordinate of $\mathbb{A}^1$. Then $\mathrm{LM}(B) = \varnothing$ and $B(V)$ is the set of triples $(v, \lambda v, \lambda)$ with $v \in V$ and $\lambda \in K$. Set $Q(V) := S^2 V$, and choose coordinates z_{ij}, $i \leq j$ on $Q(K^n)$ by writing an arbitrary element of $Q(K^n)$ as

$$\sum_{i=1}^{n} z_{ii} e_i^2 + \sum_{1 \leq i < j \leq n} 2 z_{ij} e_i e_j.$$

Note that Q is an irreducible polynomial functor, so, in the notation of Proposition 3.3, we have $R = R_1 = Q$. Define the **Vec**-subvariety

$$X \subset B \times Q \subset Y \times \mathbb{A}^1 \times Q$$

by

$$X(V) := \{(v, w, \lambda, q) \mid (v, w, \lambda) \in B(V) \text{ and } w^2, q \text{ are linearly dependent}\}.$$

An equation for $X(K^2)$ is the determinant

$$f := z_{12} y_1^2 - z_{11} y_1 y_2 = t^2 (z_{12} x_1^2 - z_{11} x_1 x_2) \in K[B(U_0) \times Q(U_0)]$$

with $U_0 := K^2$. Define $U_1 := \langle e_3, e_4 \rangle \cong K^2$, so that $U_0 \oplus U_1 = K^4$. Acting on f equation with the (upper triangular) elements $E_{1,3}$ and $E_{2,4}$ of the Lie algebra $\mathfrak{gl}(U_0 \oplus U_1)$ gives the equation

$$h_1 := z_{34}(x_1^2 t^2) + (2 z_{14} x_1 x_3 - 2 z_{13} x_1 x_4 - z_{11} x_3 x_4) t^2$$

that, by construction, vanishes on $X(U_0 \oplus U_1)$. Note that $z_{34} \in Q(U_1)^*$, $h_0 := x_1^2 t^2 \in K[B(U_0)]$ (and we let c be the leading coefficient: $c := x_1^2$), and the rest belongs to $K[B(U_0 \oplus U_1) \times Q(U_0 \oplus U_1)/Q(U_1)]$.

By acting with permutations $(3, i)$ and $(4, j)$ with $i < j$ on h_1 we find that, where h_0 is nonzero, on X we have

$$z_{ij} = -\frac{1}{h_0} \cdot (2 z_{1j} x_1 x_i - 2 z_{1i} x_1 x_j - z_{11} x_i x_j) t^2. \tag{3}$$

A similar expression can be found for z_{ii}, with the same denominator h_0.

In this case, Y_0 from the proposition is the **Vec**-subvariety of Y defined by $c = x_1^2$. This consists of all pairs $(0, w) \in V \oplus V$. The preimage in X consists of all quadruples $(0, 0, \lambda, q)$ with q arbitrary.

Set $U := U_0 \oplus U_1$, $Y' := \mathrm{Sh}_U\, Y[1/c]$, and let B_0 be the vanishing locus of h_0 in $\mathrm{Sh}_U\, B[1/c] \subset Y' \times \mathbb{A}^1$. Note that we have $t^2 \in \mathrm{LM}(B_0)$ — indeed, t even vanishes identically on B_0. With $Q_0 := Q$ we find $B_0 \times Q_0 \prec B \times Q$, and we define the map

$$\beta_0 : B_0 \times Q_0 \to B \times Q$$

as $B(\pi_V)|_{B_0} \times \mathrm{id}_{Q(V)}$ for every $V \in \mathbf{Vec}$. This covers all the points in $X(V)$ of the form $(v, 0, 0, q)$ with v, q arbitrary.

Finally, consider the map

$$\begin{aligned}\mathrm{Sh}_U(B \times Q)[1/h_0][1/c] \to \mathrm{Sh}_U(B \times Q)/Q \times \mathbb{A}^1 &\cong (\mathrm{Sh}_U\, B \times Q(U) \times \mathbb{A}^1) \times (\mathrm{Sh}_U\, Q/(Q(U) \oplus Q)) \\ &=: B_1 \times Q_1\end{aligned}$$

where the coordinate on $\mathbb{A}^1$ is given by $1/h_0$. This is a closed immersion, because where h_0 is nonzero, coordinates on $Q(V)$ with can be recovered from the coordinates on the right-hand side via (3). We use this to construct the map

$$\beta_1 : B_1 \times Q_1 = \mathrm{Sh}_U(B \times Q)/Q \times \mathbb{A}^1 \to \mathrm{Sh}_U(B \times Q) \to B \times Q.$$

The first arrow is given by the identity on the coordinates not in $Q(V)$, while the coordinates on $Q(V)$ are computed via (3). The second arrow projects into $B(V) \times Q(V)$. This map hits points in $X(V)$ of the form $(v, \lambda v, \lambda, \mu(\lambda v)^2)$ with v, λ nonzero.

3.5. *Proof of Theorem 3.1*. The $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety Z is of product type; hence by Definition 2.18 it can be written as

$$Z = [Y; B_1 \times Q_1, \dots, B_k \times Q_k]$$

for some **Vec**-subvarieties B_i of $Y \times \mathbb{A}^{n_i}$ and pure polynomial functors Q_i. Furthermore, X is a proper closed $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-subvariety of Z.

We prove, by induction on the quantity δ_X, that all points in X can be hit by partition morphisms from finitely many $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties Z' of product type with $Z' \prec Z$. So in the proof we may assume that this is true for all proper closed $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-subvarieties $X' \subsetneq Z$ with $\delta_{X'} < \delta_X$.

Let $(S_1, \dots, S_k) \in \mathbf{FI}^k$ be such that $\sum_i |S_i| = \delta_X$ and $X(S_1, \dots, S_k) \neq Z(S_1, \dots, S_k)$. If all S_i are empty, then set $Y' := X(\varnothing, \dots, \varnothing)$, a proper closed **Vec**-subvariety of Y, $B_i' := Y' \times_Y B_i$, and $Z := [Y'; B_1' \times Q_1, \dots, B_k' \times Q_k]$. The partition morphism $(\mathrm{id}_{[k]}, \varphi)$ with $\varphi(T_1, \dots, T_k)$ the inclusion $\prod_i (B_i' \times Q_i)^{T_i} \to \prod_i (B_i \times Q_i)^{T_i}$ has X in its image, and we have $Z' \prec Z$ because the Q_i remain the same, $\mathrm{LM}(B_i') \supseteq \mathrm{LM}(B_i)$ by Lemma 2.24, and Y' is a proper closed **Vec**-subvariety of Y. In this case, no shift of Z is necessary.

Next assume that not all S_i are empty. First we argue that the points of $X(T_1, \dots, T_k)$, where, for some i, $|T_i|$ is strictly smaller than $|S_i|$, are hit by partition morphisms from finitely many $Z' \prec Z$. We give the argument for $i = k$. Define the k-tuple S to be shifted over as $S := (\varnothing, \dots, \varnothing, T_k) \in \mathbf{FI}^k$, and define the $(\mathbf{FI}^{\mathbf{op}})^{k-1} \times \mathbf{Vec}$- variety Z' of product type

$$Z' := [(B_k \times Q_k)^{T_k}; B_1' \times Q_1, \dots, B_{k-1}' \times Q_{k-1}]$$

with $B_i' = (B_k \times Q_k)^{T_k} \times_Y B_i$. Consider the partition morphism $(\pi, \varphi) : Z' \to \operatorname{Sh}_S Z$, where $\pi : [k-1] \to [k]$ is the inclusion and $\varphi(T_1, \dots, T_{k-1})$ is the natural isomorphism of $\mathbf{Vec}$-varieties

$$Z'(T_1, \dots, T_{k-1}) \to (\operatorname{Sh}_S Z)(T_1, \dots, T_{k-1}, \varnothing) = Z(T_1, \dots, T_{k-1}, T_k).$$

Note that π witnesses $Z' \preceq Z$ since the Q_i with $i \le k-1$ remain the same and $\operatorname{LM}(B_i') \supseteq \operatorname{LM}(B_i)$ by Lemma 2.24. Furthermore, since $k-1 < k$, we have $Z' \prec Z$ by Lemma 2.32. All points in X where the last index set has cardinality $|T_k|$ are hit by this partition morphism. Since there are only finitely many values of $|T_k|$ that are strictly smaller than $|S_k|$, we are done.

So it remains to hit points in $X(T_1, \dots, T_k)$ where $|T_i| \ge |S_i|$ for all i. In this phase we will apply Proposition 3.3.

As by assumption not all S_i are empty, after a permutation of $[k]$ we may assume that $S_k \ne \varnothing$. Let $*$ be an element of S_k and define $\widetilde{S}_k := S_k \setminus \{*\}$. Consider the $\mathbf{Vec}$-varieties

$$Z(S_1, \dots, S_k) = (B_1 \times Q_1)_Y^{S_1} \times_Y \cdots \times_Y (B_k \times Q_k)_Y^{\widetilde{S}_k} \times_Y (B_k \times Q_k)^{\{*\}},$$
$$\widetilde{Y} := Z(S_1, \dots, S_{k-1}, \widetilde{S}_k) = (B_1 \times Q_1)_Y^{S_1} \times_Y \cdots \times_Y (B_k \times Q_k)_Y^{\widetilde{S}_k}.$$

Set $\widetilde{B_k} := \widetilde{Y} \times_Y B_k \subseteq \widetilde{Y} \times \mathbb{A}^{n_k}$, and note that $X(S_1, \dots, S_k)$ is a proper closed $\mathbf{Vec}$-subvariety of $\widetilde{B_k} \times Q_k$. We may therefore apply Proposition 3.3 to $\widetilde{Y}, n_k, \widetilde{B_k}, Q_k$ and $X(S_1, \dots, S_k)$.

First consider the proper closed $\mathbf{Vec}$-subvariety Y_0 of $\widetilde{Y}$ promised by Proposition 3.3, and let X' be the largest closed $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-subvariety of Z that intersects $Z(S_1, \dots, S_{k-1}, \widetilde{S}_k)$ in Y_0. Then $X'(S_1, \dots, \widetilde{S}_k) \ne Z(S_1, \dots, \widetilde{S}_k)$, and thus $\delta_{X'} \le \delta_X - 1 < \delta_X$. Hence, by the induction hypothesis, all points in $X'(T_1, \dots, T_k)$ can be hit by finitely many partition morphisms from varieties $Z' \prec Z$ of product type.

Next we consider the remaining pieces of data from Proposition 3.3. First, we have the $\mathbf{Vec}$-variety Y' with a morphism $\alpha : Y' \to \widetilde{Y}$. Further, we have an integer $s \in \mathbb{Z}_{\ge 0}$ and for each $i = 0, \dots, s$ we have integers n_{k+i}'; $\mathbf{Vec}$-varieties $B_{k+i}' \subseteq Y' \times \mathbb{A}^{n_{k+i}'}$; pure polynomial functors Q_{k+i}'; and morphisms $\beta_{k+i} : B_{k+i}' \times Q_{k+i}' \to \widetilde{B_k} \times Q_k$ satisfying the conditions (1) and (2).

Define $B_i' := Y' \times_Y B_i$ for $i = 1, \dots, k-1$ and the $(\mathbf{FI}^{\mathbf{op}})^{k+s} \times \mathbf{Vec}$-variety

$$Z' := [Y'; B_1' \times Q_1, \dots, B_{k-1}' \times Q_{k-1}, B_k' \times Q_k', \dots, B_{k+s}' \times Q_{k+s}'].$$

Now the map $\pi : [k+s] \to [k]$ that is the identity on $[k-1]$ and maps $[k+s] \setminus [k-1]$ to $\{k\}$ witnesses that $Z' \preceq Z$; here we use that $B_{k+j}' \times Q_{k+j}' \prec B_k \times Q_k$ for $j \in \{0, \dots, s\}$ by the conclusion of Proposition 3.3, and also Lemma 2.24 to show that $B_i' \times Q_i \preceq B_i \times Q_i$ for $i \in [k-1]$. In fact, we have $Z' \prec Z$ by Lemma 2.32.

Now the base variety Y' of Z' comes with a morphism α to the base variety $\widetilde{Y}$ of $\operatorname{Sh}_S Z$; we have morphisms $\beta_i : B_i' \times Q_i \to \widetilde{B}_i \times Q_i$ for $i = 1, \dots, k-1$ (the natural map $B_i' \to \widetilde{B}_i$ times the identity on Q_i) and the morphisms $\beta_{k+j} : B_{k+j}' \times Q_{k+j}' \to \widetilde{B_k} \times Q_k$ defined earlier. By Example 2.21, these data yield a partition morphism $(\pi, \varphi) : Z' \to \mathrm{Sh}_S Z$. We have to show that this partition morphism hits all points in X that are not in X'.

First we show, for a $V \in \mathbf{Vec}$, that a point $p \in \mathrm{Sh}_S X(\widetilde{T}_1, \dots, \widetilde{T}_k)(V)$ whose projection to $\widetilde{Y}(V)$ is not in $Y_0(V)$ lies in the image of $\varphi(\widetilde{T}_1, \dots, \widetilde{T}_k)(V)$. To this end, we write

$$p = ((p_{i,t})_{t \in \widetilde{T}_i})_{i \in [k]}$$

with

$$p_{i,t} \in \mathrm{Sh}_S X(\varnothing, \dots, \varnothing, \{t\}, \varnothing, \dots, \varnothing)(V) = \widetilde{Y}(V) \times_{Y(V)} B_i(V) \times Q_i(V) \subset \widetilde{Y}(V) \times \mathbb{A}^{n_i} \times Q_i(V),$$

where the singleton $\{t\}$ is in the i-th position. We write $p_{i,t} = (\widetilde{y}, a_{i,t}, b_{i,t})$ with $\widetilde{y} \in \widetilde{Y}(V)$, $a_{i,t} \in \mathbb{A}^{n_i}$, and $b_{i,t} \in Q_i(V)$.

By definition of a fibre product, the $p_{i,t}$ all have the same projection $\widetilde{y}$ in $\widetilde{Y}(V) \setminus Y_0(V)$, and hence we can apply (2) of Proposition 3.3 to the points $p_{k,t}$ with $t \in \widetilde{T}_k$. This yields integers $l_t \in \{0, \dots, s\}$ and points $p'_{k,t} \in B'_{k+l_t}(V) \times Q'_{k+l_t}(V)$ for $t \in \widetilde{T}_k$ whose images in $Y'(V)$ are all equal, say to $y' \in Y'(V)$, and which satisfy $\beta_{k+l_t}(V)(p'_{k,t}) = p_{k,t}$ for all t. This implies that $\alpha(y') = \widetilde{y}$.

Define

$$T'_{k+j} := \{t \in \widetilde{T}_k \mid l_t = j\},$$

$j = 0, \dots, s$, and set $T'_i := \widetilde{T}_i$ for $i = 1, \dots, k-1$. In $Z'(T'_1, \dots, T'_{k+s})$ we define the point $q = ((q_{i,t})_{t \in T'_i})_{i \in [k+s]}$ as follows. We set $q_{i,t}$ to be $(y', a_{i,t}, b_{i,t})$ for $i = 1, \dots, k-1$ and $t \in T'_i$, and $q_{i,t} = p'_{k,t}$ for $i = k, \dots, k+s$ and $t \in T'_i$. Then

$$\varphi(T'_1, \dots, T'_{k+s})(q) = p,$$

as desired.

Now, more generally, consider a point p in $X(T_1, \dots, T_k)(V) \setminus X'(T_1, \dots, T_k)(V)$, where the cardinalities satisfy $|T_i| \geq |S_i|$. Then there exists an $\mathbf{FI}^k$-morphism $\iota = (\iota_1, \dots, \iota_k) : S \to (T_1, \dots, T_k)$ such that $X(\iota)(p) \notin Y_0(V)$. Define $\widetilde{T}_i := T_i \setminus \mathrm{Im}(\iota_i)$ and extend ι to an isomorphism $\iota^e : S \sqcup (\widetilde{T}_1, \dots, \widetilde{T}_k) \to (T_1, \dots, T_k)$ by defining ι_i on $\widetilde{T}_i$ to be the inclusion. Consider $X(\iota^e)(p) \in X(S \sqcup (\widetilde{T}_1, \dots, \widetilde{T}_k))(V)$. This is also a point in $\mathrm{Sh}_S X(\widetilde{T}_1, \dots, \widetilde{T}_k)(V)$ whose projection to $\widetilde{Y}(V)$ does not lie in $Y_0(V)$. We can therefore find a point q as described above showing that $X(\iota^e)(p)$ is in the image of $(\pi, \varphi) : Z' \to \mathrm{Sh}_S Z$; by Definition 2.13, then so is p. □

4. Proof of the main theorem

The most general version of our Noetherianity result is the following.

Theorem 4.1. *Any* $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$*-variety of product type is Noetherian.*

Proof. We proceed by induction along the well-founded order on objects of product type in **PM** from Section 2.7.3.

Let Z be an $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety of product type and let $X_1 \supseteq X_2 \supseteq \cdots$ be a descending chain of closed $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-subvarieties. Then either all X_i are equal to Z, or there exists an i_0 such that $X := X_{i_0}$ is a proper closed $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-subvariety of Z. In the latter case, by Theorem 3.1, there exist a finite number of objects $Z_1, \dots, Z_N$ in **PM** of product type, along with k-tuples $S_1, \dots, S_N \in \mathbf{FI}^k$ and partition morphisms $(\pi_j, \varphi_j) : Z_j \to \mathrm{Sh}_{S_j} Z$ such that every point of X is hit by one of these. By the induction hypothesis, all Z_js are Noetherian. For each j, by Lemma 2.16, the preimage in Z_j of the chain $(\mathrm{Sh}_{S_j} X_i)_{i \geq i_0}$ is a chain of closed subvarieties, which therefore stabilises. As soon as these N chains have all stabilised, then so has the chain $(X_i)_i$ — here we have used a version of Proposition 2.17. □

To deduce from this Theorems 1.1 and 1.5, we consider GL-varieties $Z_1, \dots, Z_k$ as well as the product $Z := Z_1^{\mathbb{N}} \times \cdots \times Z_k^{\mathbb{N}}$. Recall Remark 2.9.

Proof of Theorem 1.5. We need to prove that any descending chain $Z \supseteq X_1 \supseteq \dots$ of $\mathrm{Sym}^k \times \mathrm{GL}$-stable closed subvarieties of Z stabilises.

To each Z_i is associated a **Vec**-variety, which by abuse of notation we also denote Z_i; see Remark 2.6. Furthermore, Z_i is a closed subvariety of $B_i \times Q_i$ for some finite-dimensional variety B_i and some pure polynomial functor Q_i, and hence Z is a closed subvariety of

$$(B_1 \times Q_1)^{\mathbb{N}} \times \cdots \times (B_k \times Q_k)^{\mathbb{N}}.$$

Now each X_i defines a closed $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-subvariety $\widetilde{X}_i$ of

$$\widetilde{Z} := [Y; B_1 \times Q_1, \dots, B_k \times Q_k],$$

where Y is a point. By Theorem 4.1, the $\widetilde{X}_i$ stabilise. As soon as they do, so do the X_i. □

Proof of the main theorem. Apply Theorem 1.5 with $k = 1$. □

References

[Aschenbrenner and Hillar 2007] M. Aschenbrenner and C. J. Hillar, "Finite generation of symmetric ideals", *Trans. Amer. Math. Soc.* **359**:11 (2007), 5171–5192. MR

[Bik 2020] M. A. Bik, *Strength and Noetherianity for infinite tensors*, Ph.D. thesis, Universität Bern, 2020, available at https://boristheses.unibe.ch/2031/.

[Bik et al. 2023a] A. Bik, J. Draisma, R. Eggermont, and A. Snowden, "Uniformity for limits of tensors", preprint, 2023. arXiv 2305.19866

[Bik et al. 2023b] A. Bik, J. Draisma, R. H. Eggermont, and A. Snowden, "The geometry of polynomial representations", *Int. Math. Res. Not.* **2023**:16 (2023), 14131–14195. MR

[Bik et al. 2024] A. Bik, J. Draisma, and A. Snowden, "The geometry of polynomial representations in positive characteristic", preprint, 2024. arXiv 2406.07415

[Church et al. 2015] T. Church, J. S. Ellenberg, and B. Farb, "FI-modules and stability for representations of symmetric groups", *Duke Math. J.* **164**:9 (2015), 1833–1910. MR

[Cohen 1967] D. E. Cohen, "On the laws of a metabelian variety", *J. Algebra* **5** (1967), 267–273. MR

[Cohen 1987] D. E. Cohen, "Closure relations: Buchberger's algorithm, and polynomials in infinitely many variables", pp. 78–87 in *Computation theory and logic*, edited by E. Börger, Lecture Notes in Comput. Sci. **270**, Springer, 1987. MR

[Draisma 2014] J. Draisma, "Noetherianity up to symmetry", pp. 33–61 in *Combinatorial algebraic geometry* (Levico Terme, Italy, 2013), edited by S. Di Rocco and B. Sturmfels, Lecture Notes in Math. **2108**, Springer, 2014. MR

[Draisma 2019] J. Draisma, "Topological Noetherianity of polynomial functors", *J. Amer. Math. Soc.* **32**:3 (2019), 691–707. MR

[Draisma et al. 2022] J. Draisma, R. Eggermont, and A. Farooq, "Components of symmetric wide-matrix varieties", *J. Reine Angew. Math.* **793** (2022), 143–184. MR

[Eggermont 2015] R. H. Eggermont, "Finiteness properties of congruence classes of infinite-by-infinite matrices", *Linear Algebra Appl.* **484** (2015), 290–303. MR

[Hillar and Sullivant 2012] C. J. Hillar and S. Sullivant, "Finite Gröbner bases in infinite dimensional polynomial rings and applications", *Adv. Math.* **229**:1 (2012), 1–25. MR

[Nagel and Römer 2019] U. Nagel and T. Römer, "FI- and OI-modules with varying coefficients", *J. Algebra* **535** (2019), 286–322. MR

[Nagpal and Snowden 2020] R. Nagpal and A. Snowden, "Symmetric subvarieties of infinite affine space", preprint, 2020. arXiv 2011.09009

[Nagpal et al. 2016] R. Nagpal, S. V Sam, and A. Snowden, "Noetherianity of some degree two twisted commutative algebras", *Selecta Math.* (*N.S.*) **22**:2 (2016), 913–937. MR

[Sam and Snowden 2016] S. V Sam and A. Snowden, "GL-equivariant modules over polynomial rings in infinitely many variables", *Trans. Amer. Math. Soc.* **368**:2 (2016), 1097–1158. MR

[Sam and Snowden 2019] S. V Sam and A. Snowden, "GL-equivariant modules over polynomial rings in infinitely many variables, II", *Forum Math. Sigma* **7** (2019), art. id. e5. MR

[Sam and Snowden 2022] S. V Sam and A. Snowden, "Sp-equivariant modules over polynomial rings in infinitely many variables", *Trans. Amer. Math. Soc.* **375**:3 (2022), 1671–1701. MR

[Snowden 2021] A. Snowden, "Stable representation theory: beyond the classical groups", preprint, 2021. arXiv 2109.11702

[Stacks] "The Stacks project", electronic reference, available at http://stacks.math.columbia.edu.

Communicated by János Kollár
Received 2022-12-12 Revised 2024-10-29 Accepted 2024-12-05

christopher.chiu@unibe.ch — *Department of Mathematics and Computer Science, Eindhoven University of Technology, Eindhoven, Netherlands*

adanelon@umich.edu — *Department of Mathematics and Computer Science, Eindhoven University of Technology, Eindhoven, Netherlands*

jan.draisma@unibe.ch — *Mathematical Institute, University of Bern, Bern, Switzerland*
Department of Mathematics and Computer Science, Eindhoven University of Technology, Eindhoven, Netherlands

r.h.eggermont@tue.nl — *Department of Mathematics and Computer Science, Eindhoven University of Technology, Eindhoven, Netherlands*

azharbzu11@gmail.com — *Department of Mathematics and Computer Science, Eindhoven University of Technology, Eindhoven, Netherlands*

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2025 is US $565/year for the electronic version, and $820/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

ANT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Algebra & Number Theory

Volume 19 No. 11 2025