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On the boundedness of canonical models

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It is conjectured that the canonical models of varieties (not of general type) are bounded when the Iitaka volume is fixed. We confirm this conjecture when a general fiber of the corresponding Iitaka fibration is in a fixed bounded family of polarized log Calabi–Yau pairs.

1. Introduction

Throughout this paper, we work over the complex number field $\mathbb{C}$.

By analogy with the definition of volumes of divisors, the Iitaka volume of a $\mathbb{Q}$-divisor is defined as follows: Let X be a normal projective variety and D be a $\mathbb{Q}$-Cartier divisor. When the Iitaka dimension $\kappa(D)$ of D is nonnegative, the Iitaka volume of D is defined to be

$$\mathrm{Ivol}(D) := \limsup_{m\to\infty} \frac{\kappa(D)!\, h^0(X, \mathcal{O}_X(\lfloor mD \rfloor))}{m^{\kappa(D)}}.$$

For the definition of the Iitaka dimension, see [Lazarsfeld 2004, Definition 2.1.3].

For a pair (X, Δ), if the Iitaka dimension of the log canonical divisor $K_X + \Delta$ is nonnegative, it is conjectured that a general fiber of the Iitaka fibration of $K_X + \Delta$ is birationally equivalent to a log Calabi–Yau pair, according to the abundance conjecture. The main theorem states that, when a general fiber of $K_X + \Delta$ belongs to a fixed bounded family with bounded polarization, the Iitaka volume of the log canonical divisor lies in a set satisfying the descending chain condition (DCC). Furthermore, if the Iitaka volume is fixed, then the canonical model is in a bounded family.

Theorem 1.1. *Fix $\mathscr{C}$ a log bounded class of polarized log Calabi–Yau pairs, $\mathcal{I} \subset [0, 1] \cap \mathbb{Q}$ a DCC set of rational numbers, n a positive integer and v a positive rational number. Suppose (X, Δ) is a projective klt pair of dimension n, L is a divisor on X, and $f : X \to Z$ is a contraction which is birationally equivalent to the Iitaka fibration of $K_X + \Delta$.*

If a general fiber (X_g, Δ_g, L_g) of f is in $\mathscr{C}$ and $\mathrm{coeff}(\Delta) \subset \mathcal{I}$, *then*

(1) $\mathrm{Ivol}(K_X + \Delta)$ *is in a DCC set, and*

(2) *if* $\mathrm{Ivol}(K_X + \Delta) = v$ *is a constant, then*

$$\mathrm{Proj} \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(mK_X + \lfloor m\Delta \rfloor))$$

is in a bounded family.

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Theorem 1.1 is a special case of the following conjecture.

Conjecture 1.2. Let n be a positive integer, v a nonnegative rational number, and $\mathcal{I} \subset [0, 1] \cap \mathbb{Q}$ a DCC set of rational numbers. Let $\mathcal{D}(n, v, \mathcal{I})$ be the set of varieties Z such that

- (X, Δ) is a projective klt pair of dimension n,
- $\operatorname{coeff}(\Delta) \subset \mathcal{I}$,
- $\operatorname{Ivol}(K_X + \Delta) = v$ is a constant, and
- $f : X \dashrightarrow Z$ is the Iitaka fibration associated with $K_X + \Delta$, where

$$Z = \operatorname{Proj} \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(m(K_X + \Delta))).$$

Then $\mathcal{D}(n, v, \mathcal{I})$ is in a bounded family.

An interesting application of Theorem 1.1 is when $X \to Z$ is a Fano-type fibration whose general fibers are ϵ-lc. In this case, a general fiber of f is bounded according to the Birkar-BAB theorem, see [Birkar 2021b], and $-K_X$ will induce a natural polarization on a general fiber. We have the following corollary.

Corollary 1.3. *Let n be a positive integer, v a positive rational number and $\mathcal{I} \subset [0, 1] \cap \mathbb{Q}$ a DCC set of rational numbers. Suppose (X, Δ) is a projective klt pair of dimension n and $f : X \to Z$ is a contraction such that*

- $\operatorname{coeff}(\Delta) \subset \mathcal{I}$,
- $K_X + \Delta \sim_{\mathbb{Q},Z} 0$, *and*
- Δ *is big over* Z.

Then

- $\operatorname{Ivol}(K_X + \Delta)$ *is in a DCC set, and*
- *if* $\operatorname{Ivol}(K_X + \Delta) = v$ *is a constant, then*

$$\operatorname{Proj} \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(mK_X + \lfloor m\Delta \rfloor))$$

  is in a bounded family.

According to [Birkar et al. 2010], the canonical ring $R(X, K_X+\Delta) := \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(m(K_X+\Delta)))$ is finitely generated, which implies that $Z = \operatorname{Proj} \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$ is well-defined and $v = \operatorname{Ivol}(K_X + \Delta)$ is a positive rational number. The validity of Conjecture 1.2 has been established in different scenarios: when $K_X + \Delta$ is big, it was proved in [Hacon et al. 2014]; for the case where a general fiber of f is ϵ-lc Fano-type, it was demonstrated in [Li 2024]; and when f is an elliptic curve, [Filipazzi 2024] shows that X is actually bounded in codimension one. Notably, around the same time this paper was completed, [Birkar 2021a] provided a proof of Conjecture 1.2 for the situation where a general fiber of f belongs to a bounded family.

It is shown in [Hacon et al. 2013] that the boundedness of varieties of general type is connected with the DCC of volumes of the log canonical divisors. We think that the following conjecture is closely related to Conjecture 1.2.

Conjecture 1.4. Let $n \in \mathbb{N}$, and consider a DCC set $\mathcal{I} \subset [0, 1] \cap \mathbb{Q}$. Then the set of Iitaka volumes

$$\{\mathrm{Ivol}(K_X + \Delta) \mid X \text{ is projective, } (X, \Delta) \text{ is klt, } \dim X = n \text{ and } \mathrm{coeff}(\Delta) \subset \mathcal{I}\}$$

is a DCC set.

The main idea is to prove the DCC of Iitaka volumes and the boundedness of the canonical models when the locus of singular fibers of the Iitaka fibrations is "bounded". We show that, in this case, we can choose a uniform base such that the moduli part (see Theorem 2.11) descends.

To be precise, we are interested in the following set of pairs and the corresponding Iitaka fibrations.

Definition 1.5. Fix a DCC set $\mathcal{I} \subset [0, 1] \cap \mathbb{Q}$ and positive integers n, r, l. Let $\mathcal{D}(n, \mathcal{I}, l, r)$ be the set of pairs (X, Δ) satisfying the following conditions:

- (X, Δ) is a projective klt pair of dimension n.
- $\mathrm{coeff}(\Delta) \subset \mathcal{I}$.
- $f : X \to Z$ is the canonical model of (X, Δ).
- A general fiber (X_g, Δ_g) of f has a good minimal model.
- Let $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$ be the generalized pair defined in Theorem 2.12; then $l\boldsymbol{M}_{Z'}$ is nef and Cartier.
- There is a $\mathbb{Q}$-Cartier integral divisor D and a $\mathbb{Q}$-divisor $F \in |K_X + \Delta|_{\mathbb{Q}/Z}$ such that $(X, \mathrm{Supp}(\Delta - F))$ is log smooth over $Z \setminus D$ and
$$\mathrm{Ivol}(K_X + \Delta + f^*D) \le r\,\mathrm{Ivol}(K_X + \Delta).$$

Theorem 1.6. *Fix a DCC set $\mathcal{I} \subset [0, 1] \cap \mathbb{Q}$ and positive integers n, r, l. Then the set*

$$\{\mathrm{Ivol}(K_X + \Delta) \mid (X, \Delta) \in \mathcal{D}(n, \mathcal{I}, l, r)\}$$

satisfies the DCC.

As an application, we prove the following boundedness result.

Theorem 1.7. *Fix a DCC set $\mathcal{I} \subset [0, 1] \cap \mathbb{Q}$, positive integers n, r, l and a positive rational number v. Then the set*

$$\{\mathrm{Proj}\, R(X, K_X + \Delta) \mid (X, \Delta) \in \mathcal{D}(n, \mathcal{I}, l, r),\ \mathrm{Ivol}(K_X + \Delta) = v\}$$

is bounded.

The idea is to prove that we can choose an snc model (see Definition 2.10) of $(X, \Delta - F) \to Z$ to be in a bounded family: this is why we need the last condition in Definition 1.5. We believe that the existence of D and the integer r naturally comes from a suitable moduli space of a general fiber of f. Theorem 1.1 is an application of Theorems 1.6 and 1.7 based on this idea.

2. Preliminaries

Notation and conventions. Let $\mathcal{I} \subset \mathbb{Q}$ be a subset. We say $\mathcal{I}$ satisfies the DCC if there is no strictly decreasing subsequence in $\mathcal{I}$. For a birational morphism $f : Y \to X$ and a divisor B on X, $f_*^{-1}(B)$ denotes the strict transform of B on Y, and $\mathrm{Exc}(f)$ denotes the sum of the reduced exceptional divisors of f. For a $\mathbb{Q}$-divisor D, a map defined by the linear system $|D|$ means a map defined by $|\lfloor D \rfloor|$. Given two $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors A and B, $A \sim_{\mathbb{Q}} B$ means that there is an integer $m > 0$ such that $m(A - B) \sim 0$. For a $\mathbb{Q}$-divisor D, we write $D = D_{\geq 0} - D_{\leq 0}$ as the difference of its positive and negative parts.

A subpair (X, Δ) consists of a normal variety X and a $\mathbb{Q}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We call (X, Δ) a pair if, in addition, Δ is effective. If $g : Y \to X$ is a birational morphism and E is a divisor on Y, the discrepancy $a(E, X, \Delta)$ is $-\mathrm{coeff}_E(\Delta_Y)$, where $K_Y + \Delta_Y := g^*(K_X + \Delta)$. A subpair (X, Δ) is called sub-klt (resp. sub-lc) if, for every birational morphism $Y \to X$ as above, $a(E, X, \Delta) > -1$ (resp. ≥ -1) for every divisor E on Y. A pair (X, Δ) is called klt (resp. lc) if (X, Δ) is sub-klt (resp. sub-lc) and (X, Δ) is a pair.

Let (X, Δ) and (Y, Δ_Y) be two subpairs, and let $h : Y \to X$ be a birational morphism. We say that $(Y, \Delta_Y) \to (X, \Delta)$ is a crepant birational morphism if $K_Y + \Delta_Y \sim_{\mathbb{Q}} h^*(K_X + \Delta)$ and $h_*\Delta_Y = \Delta$. Two pairs (X_i, Δ_i), $i = 1, 2$, are crepant birationally equivalent if there is a subpair (Y, Δ_Y) and two crepant birational morphisms $(Y, \Delta_Y) \to (X_i, \Delta_i)$, $i = 1, 2$.

A generalized pair $(X, \Delta + \boldsymbol{M}_X)$ consists of a normal variety X equipped with a projective morphism $X \to U$, a birational morphism $f : X' \to X$ where X is normal, a $\mathbb{Q}$-boundary Δ, and a $\mathbb{Q}$-Cartier divisor $\boldsymbol{M}_{X'}$ on X' such that $K_X + \Delta + \boldsymbol{M}_X$ is $\mathbb{Q}$-Cartier, $\boldsymbol{M}_{X'}$ is nef over U, and $\boldsymbol{M}_X = f_*\boldsymbol{M}_{X'}$. Let Δ' be the $\mathbb{Q}$-divisor such that

$$K_{X'} + \Delta' + \boldsymbol{M}_{X'} = f^*(K_X + \Delta + \boldsymbol{M}_X).$$

We call $(X, \Delta + \boldsymbol{M}_X)$ a generalized klt (resp. lc) pair if (X', Δ') is sub-klt (resp. sub-lc). When U is a point we drop it by saying X is projective.

A contraction is a projective morphism $f : X \to Z$ with $f_*\mathcal{O}_X = \mathcal{O}_Z$; hence it is surjective with connected fibers. A fibration means a contraction $X \to Z$ such that $\dim X > \dim Z$. Let $X \to Z$ be a fibration and R a $\mathbb{Q}$-divisor on X. We write $R = R_v + R_h$, where R_v is the vertical part and R_h is the horizontal part.

For a scheme X, a stratification of X is a disjoint union $\coprod_i X_i$ of finitely many locally closed subschemes $X_i \hookrightarrow X$ such that the morphism $\coprod_i X_i \to X$ is both a monomorphism and surjective.

The language of the $\boldsymbol{b}$-divisor was introduced by Shokurov.

Definition 2.1. Let X be a projective scheme. We say a formal sum $\boldsymbol{B} = \sum a_\nu \nu$, $a_\nu \in \mathbb{Q}$, where the sum ranges over all divisorial valuations of X, is a $\boldsymbol{b}$-divisor if the set

$$F_X = \{\nu \mid a_\nu \neq 0 \text{ and the center } \nu \text{ on } X \text{ is a divisor}\}$$

is finite. The trace $\boldsymbol{B}_Y$ of $\boldsymbol{B}$ is the sum $\sum a_\nu B_\nu$, where the sum now ranges over the elements of F_Y.

Notice that, by definition, a generalized pair $(X, \Delta + \boldsymbol{M}_X)$ defines a $\boldsymbol{b}$-divisor $\boldsymbol{M}$.

Definition 2.2. For a klt pair (X, Δ) with a projective morphism $\mu : X \to U$, by [Birkar et al. 2010], the canonical ring

$$R(X/U, K_X + \Delta) := \bigoplus_{m \geq 0} \mu_* \mathcal{O}_X(m(K_X + \Delta))$$

is a finitely generated $\mathcal{O}_U$-algebra. We define the canonical model of (X, Δ) over U to be

$$\operatorname{Proj} R(X/U, K_X + \Delta).$$

When U is a point we drop it by saying X is projective.

Next, we state some results that we will use in what follows.

Theorem 2.3 [Hacon et al. 2013, Theorem 2.12]. *Let $f : X \to U$ be a surjective projective morphism and (X, Δ) a dlt pair such that*

- *for a very general point $u \in U$, the fiber (X_u, Δ_u) has a good minimal model, and*
- *the ring $R(X/U, K_X + \Delta)$ is finitely generated.*

Then (X, Δ) has a good minimal model over U.

Theorem 2.4 [Birkar and Zhang 2016, Theorem 1.3]. *Let d and l be two positive integers and $\mathcal{I} \subset [0, 1]$ a DCC set of real numbers. Then there is a positive number m_0 depending only on d, l and $\mathcal{I}$ satisfying the following. Assume that*

- *(Z, B) is a projective lc pair of dimension d,*
- coeff$(B) \in \mathcal{I}$,
- *lM is a nef Cartier divisor, and*
- *$K_Z + B + M$ is big,*

then the linear system $|m(K_Z + B + M)|$ defines a birational map for every positive integer m such that $m_0 \mid m$.

Theorem 2.5 [Birkar and Zhang 2016, Theorem 8.1]. *Let $\mathcal{I}$ be a DCC set of nonnegative real numbers and d a natural number. Then there is a real number $e \in (0, 1)$ depending only on $\mathcal{I}$ and d such that, if*

- *(Z, B) is projective lc of dimension d,*
- *$M = \sum \mu_j M_j$, where M_j are nef Cartier divisors,*
- *the coefficients of B and the μ_j are in $\mathcal{I}$, and*
- *$K_Z + B + M$ is a big divisor,*

then $K_Z + eB + eM$ is a big divisor.

Theorem 2.6 [Filipazzi 2018, Theorem 1.10]. *Let $\mathcal{I} \subset [0, 1] \cap \mathbb{Q}$ be a DCC set, (W, D) a log smooth pair with D reduced, and M a fixed $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on W. Suppose $\mathcal{D}$ is the set of all projective simple normal crossing pairs (Z, B) such that* $\operatorname{coeff}(B) \subset \mathcal{I}$, *there exists a birational morphism $f : Z \to W$ and $f_*B \leq D$. Then, the set*

$$\{\operatorname{vol}(K_Z + B + f^*M) \mid (Z, B) \in \mathcal{D}\}$$

satisfies the DCC.

Theorem 2.7 [Filipazzi 2018, Theorem 1.12]. *Let $(\mathcal{Z}, \operatorname{Supp}(\mathcal{B})) \to T$ be a projective log smooth morphism and $\{x_i\}_{i \geq 1} \subset T$ a set of closed points. Denote by (Z_i, B_i) the pair given by the fiber product $(\mathcal{Z}, \mathcal{B}) \times_T x_i$. Assume that*

- $0 \leq \mathcal{B} \leq \operatorname{red}(\mathcal{B})$, *and*
- *there is a $\mathbb{Q}$-divisor $\mathcal{M}$ on $\mathcal{Z}$ such that $M_i = \mathcal{M}|_{Z_i}$ is nef for every i.*

Then, we have $\operatorname{vol}(K_{Z_i} + B_i + M_i) = \operatorname{vol}(K_{Z_j} + B_j + M_j)$ *for every $i, j \in \mathbb{N}$.*

Definition 2.8. Let X and Z be normal quasiprojective varieties and $f : X \to Z$ a contraction. Let R be a $\mathbb{Q}$-divisor on X such that $K_X + R$ is $\mathbb{Q}$-Cartier. We call $(X, R) \to Z$ an lc-trivial fibration if

- (X, R) is sub-klt over the generic point of Z,
- $K_X + R \sim_{\mathbb{Q},Z} 0$, and
- $h^0(X_\eta, \mathcal{O}_{X_\eta}(\lceil R_{\leq 0} \rceil)) = 1$, where X_η is the generic fiber of f.

Definition 2.9 [Kollár 2007, Definition 8.3.6]. Let $f : X \to Z$ be a projective morphism between normal projective varieties, R be a $\mathbb{Q}$-divisor on X and B be a divisor on Z. We say that $f : X \to Z$ and the divisors R and B satisfy the standard normal crossing assumption if the following hold:

- X and Z are smooth.
- $\operatorname{Supp}(R) + \operatorname{Supp}(f^*B)$ and B are snc divisors.
- $(X, \operatorname{Supp}(R))$ is log smooth over $Z \setminus B$.

In practice, the assumptions on X and the divisors R and B are completely harmless. By contrast, it takes some work to reduce the problems on Z to problems on the following "nice" birational model of Z.

Definition 2.10. An snc model of $f : (X, R) \to Z$ is a birational model $Z' \to Z$ such that there is a reduced divisor D' on Z', a $\mathbb{Q}$-divisor B on Z, and a crepant birational morphism $\phi : (X', R') \to (X, R + f^*B)$, such that the morphism $X' \to Z'$ and R', D' satisfy the standard normal crossing assumption.

The following is a general version of the canonical bundle formula given in [Kollár 2007].

Theorem 2.11 (the canonical bundle formula). *Let X, Z be normal projective varieties and $f : (X, R) \to Z$ an lc-trivial fibration with generic fiber X_η. Suppose B is a reduced divisor on Z such that f has slc fibers in codimension* 1 *over $Z \setminus B$; that is, if D is a prime divisor not contained in B, then*

- *no component of R_v dominates D, and*
- *$(X, R + f^*D)$ is sub-lc over the generic point of D.*

Then one can write

$$K_X + R \sim_{\mathbb{Q}} f^*(K_Z + B_Z + \boldsymbol{M}_Z),$$

where the following hold:

(a) $\boldsymbol{M}_Z = M(X/Z, R)$ *is the moduli part. It is a* $\boldsymbol{b}$*-divisor depending only on the crepant birational equivalence class of* $(X_\eta, R|_{X_\eta})$ *and* Z *such that the following hold*:
   - *There is a birational morphism* $Z' \to Z$ *such that* $\boldsymbol{M}_Z$ *is the pushforward of* $\boldsymbol{M}_{Z'} := M(X'/Z', R')$ *and* $\boldsymbol{M}_{Z''} = M(X''/Z'', R'') = \pi^*\boldsymbol{M}_{Z'}$ *for any birational morphism* $Z'' \to Z'$, *where* X' *is the normalization of the main component of* $X \times_Z Z'$ *and* $(X', R') \to (X, R)$ *is a crepant birational morphism. In this case, we say* $\boldsymbol{M}$ *descends on* Z'.
   - *If* $X \to Z$ *and* R, B *satisfy the standard normal crossing assumption, see Definition 2.9, then* $\boldsymbol{M}$ *descends on* Z.

(b) B_Z *is the unique* $\mathbb{Q}$*-divisor supported on* B *for which there is a codimension* ≥ 2 *closed subset* $W \subset Z$ *such that the following hold*:
   - $(X \setminus f^{-1}(W), R + f^*(B - B_Z))$ *is lc.*
   - *Every irreducible component of* B *is dominated by an lc center of* $(X, R + f^*(B - B_Z))$.

(c) *If the morphism* $X \to Z$ *and the divisors* R *and* B *satisfy the standard normal crossing assumption, see Definition 2.9, then* B_Z *is also the unique smallest* $\mathbb{Q}$*-divisor such that* $R_v + f^*(B - B_Z) \leq \mathrm{red}(f^*B)$.

Proof. Items (a) and (b) follow from [Kollár 2007, Theorem 8.5.1].

For (c): when $R_h \geq 0$, item (c) is [Kollár 2007, Theorem 8.3.7]. We use the idea in this result to tackle the general case. If the morphism $X \to Z$ and the divisors R and B satisfy the standard normal crossing assumption, then $(Z, \mathrm{Supp}(B))$ is log smooth. We replace R with $R + f^*(B - B_Z)$ and B_Z with $B_Z + (B - B_Z) = B$; then

- there is a codimension ≥ 2 closed subset $W \subset Z$ such that $(X \setminus f^{-1}(W), R)$ is sub-lc, and
- every irreducible component of B is dominated by an lc center of (X, R).

It is easy to see that, to prove (c), we only need to prove that W can be chosen to be the empty set, which is equal to saying that (X, R) is lc.

Suppose (X, R) is not lc, and consider the diagram

$$\begin{array}{ccc} (X', R') & \xrightarrow{\pi_X} & (X, R) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow[\pi]{} & Z \end{array}$$

where π is birational, π_X is crepant birational, $f' : X' \to Z'$ is equidimensional and π_X extracts a non-lc place of (X, R), which is denoted by E. Thus we have that $\mathrm{coeff}_E(R') > 1$. Applying (a) and (b) for the lc-trivial fibration $f' : (X', R') \to Z'$, we have

$$K_{X'} + R' \sim_{\mathbb{Q}} f'^*(K_{Z'} + B' + \boldsymbol{M}_{Z'}).$$

By assumption, $X \to Z$ and the divisors R and B satisfy the standard normal crossing assumption. Then $\boldsymbol{M}$ descends on Z, $\pi^*\boldsymbol{M}_Z = \boldsymbol{M}_{Z'}$ and $K_{Z'} + B' \sim_{\mathbb{Q}} \pi^*(K_Z + B)$. Because (Z, B) is lc, (Z', B') is sub-lc.

Let $\widetilde{B}$ be a reduced divisor on Z' such that f' has slc fibers in codimension 1 over $Z' \setminus \widetilde{B}$. By (b), B' is the unique $\mathbb{Q}$-divisor for which there is a codimension ≥ 2 closed subset $W' \subset Z'$ such that

- $(X' \setminus f'^{-1}(W'), R' + f'^*(\widetilde{B} - B'))$ is sub-lc, and
- every irreducible component of B' is dominated by an lc center of $(X', R' + f'^*(\widetilde{B} - B'))$.

Because f' is equidimensional, $\mathrm{coeff}_E(R' + f'^*(\widetilde{B} - B')) \leq 1$ and $\mathrm{coeff}_E(R') > 1$, we then have that $\mathrm{coeff}_E(f'^*(\widetilde{B} - B')) < 0$. Since $\widetilde{B}$ is reduced and E is vertical, we have $\mathrm{coeff}_{f'(E)}(B') > 1$, which contradicts the fact that (Z', B') is sub-lc. □

The next theorem says that the canonical bundle formula works on the Iitaka fibration of a klt pair.

Theorem 2.12. *Let (X, Δ) be an n-dimensional projective klt pair, let $f : X \to Z$ be a contraction such that $\kappa(X_\eta, K_{X_\eta} + \Delta|_{X_\eta}) = 0$, where X_η is the generic fiber of f, and let $g : W \to Z$ be a birational morphism. Then there is a commutative diagram*

$$\begin{array}{ccc} X & \xleftarrow{\ h_X\ } & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Z & \xleftarrow[\ h\ ]{} & Z' \end{array}$$

such that the following hold:

(1) *h and h_X are birational, h factors through g, and f' is equidimensional.*

(2) *Z' is smooth and X' has only quotient singularities.*

(3) *There are a klt pair $K_{X'} + \Delta'$, an effective $\mathbb{Q}$-divisor F' on X', and $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$ a generalized klt pair such that*
   - *$\boldsymbol{M}$ descends on Z',*
   - *$K_{X'} + \Delta' \sim_{\mathbb{Q}} f'^*(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'}) + F'$,*
   - *$f'_*\mathcal{O}_{X'}(mF') \cong \mathcal{O}_{Z'}$ for any $m \geq 0$,*
   - *$h_{X*}\mathcal{O}_{X'}(m(K_{X'} + \Delta')) \cong \mathcal{O}_X(m(K_X + \Delta))$ for all $m \geq 0$,*
   - *(X, Δ), (X', Δ') and $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$ have the same canonical models, and*
   - $\mathrm{Ivol}(K_X + \Delta) = \mathrm{Ivol}(K'_X + \Delta') = \mathrm{vol}(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'})$.

(4) *If* $\mathrm{coeff}(\Delta)$ *is in a DCC set and a general fiber (X_g, Δ_g) of f has a good minimal model, then* $\mathrm{coeff}(B_{Z'})$ *and* $\mathrm{coeff}(B_Z)$ *are in a DCC set, where $B_Z := h_*B_{Z'}$.*

Proof. Fix R with $R_h \leq 0$ such that f is an lc-trivial fibration for the subpair $(X, \Delta + R)$. Notice that such an R exists by the assumption $\kappa(X_\eta, K_{X_\eta} + \Delta|_{X_\eta}) = 0$, and the choice of R_h is unique. Let $\boldsymbol{M}$ be the moduli $\boldsymbol{b}$-divisor of this lc-trivial fibration. By the weak semi-stable reduction theorem by Abramovich and Karu [2000], we can construct X' and Z' satisfying (1) and (2) such that $\boldsymbol{M}$ descends on Z'.

For (3), because (X, Δ) is klt, we can choose a sufficiently large integer k such that if we define $\Delta' := (h_X)_*^{-1}\Delta + (1-1/k)E$, where E is the reduced exceptional divisor, then $K_{X'} + \Delta' \geq h_X^*(K_X+\Delta)$. Also by the semistable reduction, X' has a toroidal structure $(X' \setminus \mathrm{Supp}(\Delta')) \subset X'$, and we have that (X', Δ') is klt. It is easy to see that

$$(h_X)_*\mathcal{O}_{X'}(m(K_{X'}+\Delta')) \cong \mathcal{O}_X(m(K_X+\Delta))$$

for all $m \geq 0$; hence $\kappa(X, K_X+\Delta) = \kappa(X', K_{X'}+\Delta')$.

If $\kappa(X, K_X+\Delta) < 0$, choose $a \gg 0$ such that $a(K_{X'}+\Delta')$ is Cartier. Because $\kappa(X_\eta, K_{X_\eta}+\Delta|_{X_\eta}) = 0$, we may also assume that $h^0(X'_\eta, \mathcal{O}_{X'_\eta}(a(K_{X'_\eta}+\Delta'|_{X'_\eta}))) = 1$. Since Z' is smooth and f' is equidimensional, by [Hartshorne 1980, Corollary 1.7], $f'_*\mathcal{O}_{X'}(a(K_{X'}+\Delta'))$ is a reflexive sheaf of rank 1. Moreover, since Z' is smooth, $f'_*\mathcal{O}_{X'}(a(K_{X'}+\Delta'))$ is a line bundle on Z'; denote it by $\mathcal{O}_{Z'}(D)$. Choose a general sufficiently ample divisor A' on Z' such that $\mathcal{O}_{Z'}(A'+D)$ is big. Let $A := (1/a)A'$; then $f'_*\mathcal{O}_{X'}(a(K_{X'}+\Delta'+f'^*A))$ is a big line bundle and $\kappa(X', K_{X'}+\Delta'+f'^*A) = \dim Z' \geq 0$. Because A' is general, $(X', \Delta'+f'^*A)$ is klt. It is easy to see that, to prove (3), we may replace (X', Δ') with $(X', \Delta'+f'^*A)$ and assume $\kappa(X, K_X+\Delta) \geq 0$.

Suppose $\kappa(X, K_X+\Delta) \geq 0$ and choose $a \gg 0$ such that $H^0(X', \mathcal{O}_{X'}(a(K_{X'}+\Delta'))) > 0$; then we can choose $L \in |a(K_{X'}+\Delta')|$. Define

$$G := \max\{N \mid N \text{ is an effective } \mathbb{Q}\text{-divisor such that } L \geq f'^*N\}$$

and

$$D := \frac{1}{a}G \quad \text{and} \quad F' := \frac{1}{a}(L - f'^*G).$$

Then we have $K_{X'}+\Delta' \sim_{\mathbb{Q}} f'^*D + F'$. It is easy to see that $h^0(X'_\eta, \mathcal{O}_{X'_\eta}(mF'|_{X'_\eta})) = 1$ for all $m \geq 0$. Because f' is equidimensional, $f'_*\mathcal{O}_{X'}(mF')$ is a reflexive sheaf of rank 1 for every $m \geq 0$. Moreover, since Z' is smooth, $f'_*\mathcal{O}_{X'}(mF')$ is an invertible sheaf for every $m \geq 0$. Since $\mathrm{Supp}(F')$ does not contain the whole fiber over any codimension 1 point on Z', it is easy to see that $f'_*\mathcal{O}_{X'}(mF') \cong \mathcal{O}_{Z'}$ for all $m \geq 0$.

Let X'_η be the generic fiber of f'; then $(K_{X'}+\Delta')|_{X'_\eta} \sim_{\mathbb{Q}} F'|_{X'_\eta}$. Because $f'_*\mathcal{O}_{X'}(mF') \cong \mathcal{O}_{Z'}$, we have $H^0(X'_\eta, \mathcal{O}_{X'_\eta}(\lceil(\Delta'-F')_{\leq 0}\rceil)) = 1$ and $f' : (X', \Delta'-F') \to Z'$ is an lc-trivial fibration. By the canonical bundle formula, there is a generalized pair $(Z', B_{Z'}+\boldsymbol{M}_{Z'})$ such that

$$K_{X'}+\Delta'-F' \sim_{\mathbb{Q}} f'^*(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'}).$$

Also because $f'_*\mathcal{O}_{X'}(mF') \cong \mathcal{O}_{Z'}$, there is an integer $l > 0$ such that

$$H^0(X', \mathcal{O}_{X'}(ml(K_{X'}+\Delta'))) \cong H^0(Z', \mathcal{O}_{Z'}(ml(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'})))$$

for all $m \geq 0$. Then (X, Δ), (X', Δ') and $(Z', B_{Z'}+\boldsymbol{M}_{Z'})$ all have the same canonical models, and $\mathrm{Ivol}(K_X+\Delta) = \mathrm{Ivol}(K'_X+\Delta') = \mathrm{vol}(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'})$.

For (4), if $\mathrm{coeff}(\Delta)$ is in a DCC set, by the construction of Δ', $\mathrm{coeff}(\Delta')$ is also in a DCC set. Because (X', Δ') is a klt pair, by the main theorem of [Birkar et al. 2010], $R(X'/Z, K_{X'}+\Delta')$ is finitely generated. Because a general fiber (X_g, Δ_g) has a good minimal model and $K_{X'}+\Delta'-h^*(K_X+\Delta)$ is effective and exceptional over X, we have that a general fiber (X'_g, Δ'_g) of $(X', \Delta') \to Z'$ has a good minimal model. By Theorem 2.3, (X', Δ') has a good minimal model over Z'; we denote it by $h_Y : (X', \Delta') \dashrightarrow (Y, \Delta_Y)$. By (2), $K_{X'} + \Delta' \sim_{\mathbb{Q},Z'} F'$ and $f'_*\mathcal{O}_{X'}(mF') \cong \mathcal{O}_{Z'}$ for all $m \geq 0$; therefore Z' is the canonical model of (X', Δ') over Z'. By the definition of good minimal models, we have $K_Y + \Delta_Y \sim_{\mathbb{Q},Z'} 0$ and $(h_Y)_*F' = 0$.

Since coeff(Δ') is in a DCC set and Δ_Y is the pushforward of Δ', coeff(Δ_Y) is also in a DCC set. Let B' be the unique smallest reduced divisor on Z' such that f' has slc fibers in codimension 1 over $Z' \setminus B'$. By Theorem 2.11, there is a codimension ≥ 2 closed subset $W \subset Z'$ such that $B_{Z'}$ is the smallest $\mathbb{Q}$-divisor supported on B' such that $(X' \setminus f'^{-1}(W), \Delta' - F' + f'^*(B' - B_{Z'}))$ is sub-lc.

Because $K_{X'} + \Delta' - F' \sim_{\mathbb{Q},Z'} 0$, $K_Y + \Delta_Y \sim_{\mathbb{Q},Z'} 0$, and $(h_Y)_*(\Delta' - F') = 0$, we have that $B_{Z'}$ is also the smallest $\mathbb{Q}$-divisor supported on B' such that $(Y \setminus f_Y^{-1}(W), \Delta_Y + h_{Y*}f'^*(B' - B_{Z'}))$ is lc. Because coeff(Δ_Y) is in a DCC set, by [Hacon et al. 2014, Theorem 1.1], coeff$(B_{Z'})$ is in a DCC set. □

Remark 2.13. Suppose (X, Δ) is a projective klt pair and $f : X \dashrightarrow Z$ is the canonical model of (X, Δ). Let $g : Y \to X$ be a resolution of the indeterminacy of f. Choose a sufficiently large integer k such that if we define $\Delta_Y := g_*^{-1}\Delta + (1 - 1/k)E$, where E is the reduced exceptional divisor, then $K_Y + \Delta_Y \geq g^*(K_X + \Delta)$.

Because $K_Y + \Delta_Y - g^*(K_X + \Delta)$ is effective and exceptional over X, Z is also the canonical model of (Y, Δ_Y) and $\kappa(Y_\eta, K_{Y_\eta} + \Delta_Y|_{Y_\eta}) = 0$. By Theorem 2.12, the contraction $Y \to Z$ defines a moduli $\boldsymbol{b}$-divisor $\boldsymbol{M}$ and a generalized pair $(Z', B' + \boldsymbol{M}_{Z'})$ with a birational morphism $Z' \to Z$, Z is also the canonical model of $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$, and $\mathrm{Ivol}(K_X + \Delta) = \mathrm{Ivol}(K_Y + \Delta_Y) = \mathrm{vol}(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'})$.

Furthermore, if coeff(Δ) is in a DCC set, then coeff(Δ_Y) is also in a DCC set. If a general fiber (X_g, Δ_g) of f has a good minimal model, then a general fiber (Y_g, Δ_{Y_g}) of f_Y has a good minimal model, and therefore coeff(B') is in a DCC set by Theorem 2.12.

3. DCC of Iitaka volumes

Lemma 3.1. *Fix a positive integer C and a finite set $\mathcal{I} \subset [0, 1] \cap \mathbb{Q}$. Suppose $\mathcal{Z} \to T$ is a family of projective smooth varieties, where T is of finite type. Let $\mathcal{A}$ be a relative very ample divisor on $\mathcal{Z}$ over T. Let $\mathcal{S}$ be a set of generalized pairs such that, for every $(Z, B_Z + \boldsymbol{M}_Z) \in \mathcal{S}$, there is a closed point $t \in T$ such that*

- *there is a birational morphism $\phi : Z \to \mathcal{Z}_t$, and*
- *$\phi_*\boldsymbol{M}_Z \sim_{\mathbb{Q}} D_1 - D_2$ for two effective $\mathbb{Q}$-divisors D_k with* coeff$(D_k) \subset \mathcal{I}$ *and* $\deg_{\mathcal{A}_t}(D_k) \leq C$, $k = 1, 2$.

Then there is a smooth projective morphism $\mathcal{Z}' \to T'$, where T' is of finite type, and finitely many $\mathbb{Q}$-divisors $\mathcal{M}_k$ on $\mathcal{Z}'$ over T' such that, for any $(Z, B_Z + \boldsymbol{M}_Z) \in \mathcal{S}$, there is a closed point $t' \in T'$ and an isomorphism $\psi : \mathcal{Z}_t \to \mathcal{Z}'_{t'}$ such that $\psi_*\phi_*\boldsymbol{M}_Z \sim_{\mathbb{Q}} \mathcal{M}_k|_{\mathcal{Z}'_{t'}}$.

Proof. Since the coefficients of D_k, $k = 1, 2$, are in a finite set $\mathcal{I}$, there is a positive number δ such that coeff$(D_k) > \delta$, which implies $(1/\delta)D_k \geq \lfloor D_k \rfloor$, $k = 1, 2$. Because $\deg_{\mathcal{A}_t}(D_k) \leq C$, we have that $\deg_{\mathcal{A}_t}(\mathrm{Supp}(D_k)) \leq C/\delta$. By boundedness of the Chow variety, see [Kollár 1996, §1.3], there is a morphism $\mathcal{Z}' \to T'$ and a divisor $\mathcal{D}$ on $\mathcal{Z}'$ such that, for every closed point $t \in T$, there is a closed point $t' \in T'$ and an isomorphism $\psi : \mathcal{Z}_{t'} \to \mathcal{Z}'_t$ such that $\mathrm{Supp}(\psi_* D_k) \subset \mathcal{D}|_{\mathcal{Z}'_{t'}}$, $k = 1, 2$.

Let R be a component of $\mathcal{D}$, let $S \to T'$ be the normalization of the Stein factorization of $R \to T'$ such that $S \to T'$ is finite and S is normal, and consider the diagram

$$\begin{array}{ccc} \mathcal{Z}'' & \longrightarrow & \mathcal{Z}' \\ \downarrow & & \downarrow \\ S & \longrightarrow & T' \end{array}$$

Because $S \to T'$ is finite, S is irreducible and $\mathcal{Z}'' \to S$ is flat, we have that $\mathcal{Z}''$ is a quasiprojective variety. $\mathcal{Z}''$ is normal by [EGA IV$_3$ 1966, 5.12.7]. Replacing $\mathcal{Z}' \to T'$ by $\mathcal{Z}'' \to S$ finitely many times, we may assume that the fibers of $R \to T'$ are irreducible for every component R of $\mathcal{D}$.

Since, for every component R of $\mathcal{D}$, the coefficients of R in D_1 and D_2 are in a finite set, there are only finitely many possibilities for D_1, D_2 and $D_1 - D_2$. Then there are only finitely many $\mathbb{Q}$-divisors $\mathcal{M}_k$ on $\mathcal{Z}'$ over T' such that $\psi_* \phi_* \boldsymbol{M}_Z \sim_{\mathbb{Q}} \mathcal{M}_k|_{\mathcal{Z}_t}$ for some k. ☐

The next theorem says that if we bound the Iitaka volume of $(X, \Delta) \in \mathcal{D}(n, \mathcal{I}, l, r)$, then we can choose the snc model of $X \to \mathrm{Proj}\, R(X, K_X + \Delta)$ to be in a bounded family depending only on $n, \mathcal{I}, l$ and r.

Theorem 3.2. *Fix a DCC set $\mathcal{I} \subset [0, 1] \cap \mathbb{Q}$, positive integers $n, l, r, v > 0$, and a positive number $\delta > 0$. Define $\mathcal{D}'(n, \mathcal{I}, l, r, v, \delta)$ to be the set of n-dimensional projective pairs (X, Δ) such that*

- $(X, \Delta) \in \mathcal{D}(n, \mathcal{I}, l, r)$,
- $\mathrm{Ivol}(K_X + \Delta) \le v$, *and*
- *if Z is the canonical model of (X, Δ), then there is an effective ample $\mathbb{Q}$-divisor H on Z with* $\mathrm{coeff}(H) > \delta$ *such that*

$$\mathrm{Ivol}(K_X + \Delta + f^* H) \le r\, \mathrm{Ivol}(K_X + \Delta).$$

Then there is a family of projective log smooth pairs $(\mathcal{Z}, \mathcal{P}) \to T$, where T is a scheme of finite type, and finitely many $\mathbb{Q}$-divisors $\mathcal{M}_k$, $k \in \Lambda$, on $\mathcal{Z}$, where Λ is a finite index set, such that, for every $(X, \Delta) \in \mathcal{D}'(n, \mathcal{I}, l, r, v, \delta)$, there is a closed point $t \in T$ such that the following hold:

- *$\mathcal{Z}_t$ is birationally equivalent to the canonical model of (X, Δ),*
- *If $\boldsymbol{M}$ is the moduli $\boldsymbol{b}$-divisor corresponding to $(X, \Delta) \dashrightarrow Z$ defined in Remark 2.13, then $\boldsymbol{M}_{\mathcal{Z}_t} \sim_{\mathbb{Q}} \mathcal{M}_k|_{\mathcal{Z}_t}$ for some $k \in \Lambda$,*
- *There is a birational morphism $X' \to X$ and a $\mathbb{Q}$-divisor F' on X' such that the morphism $X' \to \mathcal{Z}_t$ and $\Delta' - F'$, $\mathcal{P}_t$ satisfy the standard normal crossing assumption, where Δ' is the strict transform of Δ plus the exceptional divisor and $F' \in |K_{X'} + \Delta'|_{\mathbb{Q}/\mathcal{Z}_t}$. In particular, $\boldsymbol{M}$ descends on $\mathcal{Z}_t$.*
- *If $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$ is the generalized pair defined in Remark 2.13 such that there is a birational morphism $\phi_t : Z' \to \mathcal{Z}_t$, then $B := \phi_{t*} B_{Z'} \le \mathcal{P}_t$.*

Proof. We replace $\mathcal{I}$ by $\mathcal{I}\cup\{1-1/k,\, k\in\mathbb{N}\}$; note that $\mathcal{I}$ is still a DCC set.

Step 1: We prove that Z is birationally bounded.

Suppose $\kappa(X, K_X+\Delta)=d\le n$; then $\dim Z=d$. Let $Z'\to Z$ be a projective birational morphism and $(Z', B_{Z'}+\boldsymbol{M}_{Z'})$ be the generalized klt pair defined in Remark 2.13, and denote the morphism $Z'\to Z$ by h. By Theorem 2.12 (4), $\mathrm{coeff}(B_{Z'})$ is in a DCC set $\mathcal{I}'$ depending only on $\mathcal{I}, d, n$. By assumption, $l\boldsymbol{M}_{Z'}$ is nef and Cartier. We may assume that $\{1-1/k,\, k\in\mathbb{N}\}\subset\mathcal{I}'$.

By Theorem 2.4, there is an integer r' such that $|r'(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'})|$ defines a birational map $\phi: Z' \dashrightarrow W$. Let $h': Z''\to Z'$ be a birational morphism such that ϕ extends to a morphism $\phi': Z''\to W$. Because $(Z', B_{Z'}+\boldsymbol{M}_{Z'})$ is generalized klt, we can choose an integer $k\gg 0$ such that if we define

$$B_{Z''}:=h'^{-1}_{*}B_{Z'}+\Big(1-\frac{1}{k}\Big)E,$$

where E is the reduced h'-exceptional divisor, then $K_{Z''}+B_{Z''}+\boldsymbol{M}_{Z''}-h'^{*}(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'})$ is effective and exceptional over Z'. Then we replace $(Z', B_{Z'}+\boldsymbol{M}_{Z'})$ by $(Z'', B_{Z''}+\boldsymbol{M}_{Z''})$; we may assume that $|r'(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'})|$ defines a birational morphism $\phi: Z'\to W$. Note that we keep $\mathrm{coeff}(B_{Z'})\subset\mathcal{I}'$, Z is still the canonical model of $(Z', B_{Z'}+\boldsymbol{M}_{Z'})$ and $\mathrm{vol}(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'})=\mathrm{Ivol}(K_X+\Delta)\le v$.

Because $|r'(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'})|$ defines a birational morphism $\phi: Z'\to W$, there is a very ample divisor A on W such that $r'(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'})\sim\phi^{*}A+F_1$, where F_1 is a ϕ-exceptional $\mathbb{Q}$-divisor. Because

$$A^d\le\mathrm{vol}(r'(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'}))=r'^{d}\,\mathrm{Ivol}(K_X+\Delta)<r'^{d}C,$$

by boundedness of the Chow variety, see [Kollár 1996, §1.3], W is in a bounded family. Then there exists a projective morphism $\mathcal{W}'\to T$ over a scheme T of finite type and a relative very ample divisor $\mathcal{A}'$ depending only on $n, \mathcal{I}, l, v$, such that there is a closed point $t\in T$ and an isomorphism $\chi: W\to\mathcal{W}'_t$ such that $\chi^{*}\mathcal{A}'_t=A$. Because r' is fixed and the coefficients of $B_{Z'}$ are in a DCC set $\mathcal{I}'$, it is easy to see that the coefficients of F_1 are also in a DCC set $\widetilde{\mathcal{I}}$.

Passing to a stratification of T and a log resolution of the generic fiber of $\mathcal{W}'\to T$, we may assume that there is a birational morphism $\xi:\mathcal{W}\to\mathcal{W}'$, and $\mathcal{W}\to T$ is a smooth morphism. Let $\mathcal{A}$ be a very ample divisor on $\mathcal{W}$ over T. Then there is an integer r'' such that $r''\xi^{*}\mathcal{A}'-\mathcal{A}$ is big over T. After increasing r', replacing Z' by a birational model and (W, A) by $(\mathcal{W}_t, \mathcal{A}_t)$, we may assume W is smooth and there is a very ample divisor A on W such that

$$A^d\le\mathrm{vol}(r'(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'})).$$

Step 2: We construct a birational map $Z'\dashrightarrow Z^{!}$, two morphisms $h^{!}: Z^{!}\to Z$, $\phi^{!}: Z^{!}\to W$ and an ample $\mathbb{Q}$-divisor $L^{!}$ on $Z^{!}$.

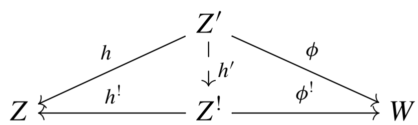
Let m be the Cartier index of H, and define

$$\begin{aligned}L' &:= \frac{1}{r'}(\phi^{*}A+F_1)+(2d+1)\phi^{*}A+(2d+1)mh^{*}H\\ &\sim_{\mathbb{Q}} K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'}+(2d+1)\phi^{*}A+(2d+1)mh^{*}H. \qquad (3\text{-}1)\end{aligned}$$

Because H is an effective ample $\mathbb{Q}$-divisor on Z, by [Birkar et al. 2010], the canonical model of $K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'}+(2d+1)\phi^*A+(2d+1)mh^*H$ exists; denote it by $h':Z'\dashrightarrow Z^!$. Then

$$h'_*(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'}+(2d+1)\phi^*A+(2d+1)mh^*H)\sim_{\mathbb{Q}} h'_*L'$$

is ample, and we write $L^!:=h'_*L'$. Because ϕ^*A and mh^*H are nef Cartier divisors, by [Birkar and Zhang 2016, Lemma 4.4], both ϕ^*A and h^*H are h'-trivial, so there are two birational morphisms $\phi^!:Z^!\to W$ and $h^!:Z^!\to Z$ as in the following diagram:

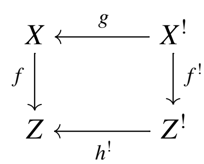


Because $L^!$ is ample and effective and W is smooth, by the negativity lemma, $L^!=\phi^{!*}\phi^!{}_*L^!-F_W$, where F_W is effective and has the same support as $\mathrm{Exc}(\phi^!)$. Then we have

$$\mathrm{Supp}(\phi^{!*}\phi^!{}_*L^!)\supset \mathrm{Exc}(\phi^!) \quad \text{and} \quad Z^!\setminus \mathrm{Supp}(L^!)\supseteq W\setminus \mathrm{Supp}(\phi^!{}_*L^!).$$

<u>Step 3</u>: We use the two birational morphisms $Z^!\to Z$, $Z^!\to W$ and ampleness of $L^!$ to show that if there is a $\mathbb{Q}$-Cartier integral divisor D and a $\mathbb{Q}$-divisor $F\in |K_X+\Delta|_{\mathbb{Q}/Z}$ on X such that $(X,\mathrm{Supp}(\Delta-F))$ is log smooth over $Z\setminus D$, then $(X,\mathrm{Supp}(\Delta-F))$ is log smooth over $W\setminus \mathrm{Supp}(\phi^!_*L^!+\phi^!_*h^{!*}D)$.

Consider the diagram



where $X^!$ is the normalization of the main component of $X\times_Z Z^!$. Because $\mathrm{Supp}(h^{!*}D)=h^{!-1}(\mathrm{Supp}(D))$ and $X\to Z$ is smooth over $Z\setminus\mathrm{Supp}(D)$, we have that $X^!\to Z^!$ is smooth over $Z^!\setminus \mathrm{Supp}(h^{!*}D)$. Because $Z^!$ is normal, $f^{!-1}(Z^!\setminus\mathrm{Supp}(h^{!*}D))$ is normal and

$$f^{!-1}(Z^!\setminus \mathrm{Supp}(h^{!*}D))\cong f^{-1}(Z\setminus\mathrm{Supp}(D))\times_{Z\setminus\mathrm{Supp}(D)} Z^!\setminus \mathrm{Supp}(h^{!*}D).$$

Define $K_{X^!}+\Delta^{!'}-F^{!'}:=g^*(K_X+\Delta-F)$, where $\Delta^{!'}$ and $F^{!'}$ are two effective $\mathbb{Q}$-divisors with no common component. Suppose $\Delta^{!'}=\Delta^{!''}+\Delta_v$ and $F^{!'}=F^{!''}+F_v$, where Δ_v and F_v are $f^!$-vertical and not supported on $f^{!-1}(\mathrm{Supp}(h^{!*}D))$, and the prime components of $\Delta^{!''}$ and $F^{!''}$ are either $f^!$-horizontal or supported on $f^{!-1}(\mathrm{Supp}(h^{!*}D))$. Because $f^!$ is smooth over $Z^!\setminus\mathrm{Supp}(h^{!*}D)$, we have that $\Delta_v|_{f^{!-1}(Z^!\setminus\mathrm{Supp}(h^{!*}D))}$ and $F_v|_{f^{!-1}(Z^!\setminus\mathrm{Supp}(h^{!*}D))}$ are the pullback of two divisors on $Z^!\setminus\mathrm{Supp}(h^{!*}D)$. It is easy to see that there is a $\mathbb{Q}$-divisor $R^!$ on $X^!$ such that $\mathrm{Supp}(R^!)\subset f^{!-1}(\mathrm{Supp}(h^{!*}D))$ and $\Delta_v-F_v-R^!\sim_{\mathbb{Q},f^!}0$. Let $\Delta^!$ and $F^!$ be two effective $\mathbb{Q}$-divisors without common component such that $\Delta^!-F^!=\Delta^{!''}-F^{!''}+R^!$; then $K_{X^!}+\Delta^!-F^!\sim_{\mathbb{Q},f^!}0$.

If P is a component of $\mathrm{Supp}(\Delta^!-F^!)$, then it is either supported on $f^{!-1}(h^{!*}D)$ or is $f^!$-horizontal. Then $\mathrm{Supp}(\Delta^!-F^!)$ does not contain any irreducible component of the fiber over any prime divisor on $Z^!$

that is not contained in $\mathrm{Supp}(h^{!*}D)$, and we have

$$\mathrm{Supp}(\Delta^! - F^!)|_{(Z^! \setminus \mathrm{Supp}(h^{!*}D))} = \mathrm{Supp}(\Delta - F) \times_Z (Z^! \setminus \mathrm{Supp}(h^{!*}D)).$$

Because $(X, \Delta) \subset \mathcal{D}(n, \mathcal{I}, l, r)$, by definition of $\mathcal{D}(n, \mathcal{I}, l, r)$, we know that $(X, \mathrm{Supp}(\Delta - F))$ is log smooth over $Z \setminus D$, and hence $f^! : (X^!, \mathrm{Supp}(\Delta^! - F^!)) \to Z^!$ is log smooth over $Z^! \setminus \mathrm{Supp}(h^{!*}D)$.

Recall that $Z^! \setminus \mathrm{Supp}(L^!) \supseteq W \setminus \mathrm{Supp}({\phi^!}_* L^!)$. Then

$$Z^! \setminus \mathrm{Supp}(h^{!*}D) \supseteq Z^! \setminus \{\mathrm{Supp}(L^! + h^{!*}D)\} \supseteq W \setminus \{\mathrm{Supp}({\phi^!}_* L^! + {\phi^!}_* h^{!*}D)\}$$

and $X^! \to W$ is isomorphic to $X^! \to Z^!$ over $W \setminus \{\mathrm{Supp}({\phi^!}_* L^! + {\phi^!}_* h^{!*}D)\}$. Then $(X^!, \mathrm{Supp}(\Delta^! - F^!)) \to W$ is log smooth over $W \setminus \{\mathrm{Supp}({\phi^!}_* L^! + {\phi^!}_* h^{!*}D)\}$.

<u>Step 4</u>: We prove that $(W, \mathrm{Supp}({\phi^!}_* L^! + \phi_* B_{Z'} + {\phi^!}_* h^{!*}D))$ is log bounded.

Because $\mathrm{Supp}({\phi^!}_* L^! + B_W + {\phi^!}_* h^{!*}D) = \mathrm{Supp}(\phi_*(\phi^* A + F_1 + B_{Z'} + h^*(D + H)))$, we only need to prove that $(W, \mathrm{Supp}(\phi_*(\phi^* A + F_1 + B_{Z'} + h^*(D + H))))$ is log bounded. Recall that W is bounded by A by construction; we only need to work on the boundary.

Recall that the coefficients of F_1 and $B_{Z'}$ are in a DCC set and $\mathrm{coeff}(H) \geq \delta$ by assumption. Then there is a positive number $\delta' < 1$ such that $(F_1 + B_{Z'})/\delta' \geq \mathrm{red}(F_1 + B_{Z'})$. By assumption, A and D are two effective integral divisors, so we only need to prove that there exists a constant $v' > 0$ such that

$$A^{d-1}.\phi_*(\mathrm{red}(\phi^* A + F_1 + B_{Z'} + h^* D) + h^* H) < v'.$$

By the projection formula, this is equivalent to proving

$$(\phi^* A)^{d-1}.(\mathrm{red}(\phi^* A + F_1 + B_{Z'} + h^* D) + h^* H) < v'.$$

Let $G = 2((2d+1)+1)\phi^* A$. By [Hacon et al. 2013, Lemma 3.2], we have

$$G^{d-1}.(\mathrm{red}(\phi^* A + F_1 + B_{Z'} + h^* D)) \leq 2^d \operatorname{vol}\Big(K_{Z'} + \frac{1}{\delta'} B_{Z'} + \phi^* A + \frac{1}{\delta'} F_1 + h^* D + G\Big). \tag{3-2}$$

Recall that the coefficients of $B_{Z'}$ are in a DCC set $\mathcal{I}'$ and the Cartier index of the $\boldsymbol{b}$-divisor $\boldsymbol{M}$ is l, according to the assumption that $(X, \Delta) \subset \mathcal{D}(n, \mathcal{I}, l, r)$. By Theorem 2.5, there is a positive number $e < 1$ depending only on $\mathcal{I}'$ and l such that $K_{Z'} + eB_{Z'} + \boldsymbol{M}_{Z'}$ is big. Because $\boldsymbol{M}_{Z'}$ is pseudo-effective and

$$K_{Z'} + \frac{1}{\delta'} B_{Z'} + \frac{\frac{1}{\delta'} - 1}{1 - e}(K_{Z'} + eB_{Z'} + \boldsymbol{M}_{Z'}) + \boldsymbol{M}_{Z'} \sim_{\mathbb{Q}} \frac{\frac{1}{\delta'} - e}{1 - e}(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'}),$$

for any divisor E, we have that

$$\operatorname{vol}\Big(E + K_{Z'} + \frac{1}{\delta'} B_{Z'}\Big) \leq \operatorname{vol}\Big(E + \frac{\frac{1}{\delta'} - e}{1 - e}(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'})\Big).$$

Because $\phi^* A$ and F_1 are effective, it is easy to see that

$$\phi^* A + \frac{1}{\delta'} F_1 + G \leq \Big(1 + 2((2d+1)+1) + \frac{1}{\delta'}\Big)(\phi^* A + F_1) \sim_{\mathbb{Q}} \Big(1 + 2((2d+1)+1) + \frac{1}{\delta'}\Big) r'(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'}).$$

Then we have

$$\begin{aligned}\operatorname{vol}\Big(K_{Z'}+\frac{1}{\delta'}B_{Z'}+\phi^*A+\frac{1}{\delta'}F_1+h^*D+G\Big)&\\ &\le \operatorname{vol}\Bigg(\frac{\frac{1}{\delta'}-e}{1-e}(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'})+\phi^*A+\frac{1}{\delta'}F_1+h^*D+G\Bigg)\\ &\le \operatorname{vol}\Bigg(\Bigg(\frac{\frac{1}{\delta'}-e}{1-e}+r'+2r'((2d+1)+1)+\frac{r'}{\delta'}\Bigg)(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'})+h^*D\Bigg)\\ &\le v''^d\operatorname{vol}(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'}+h^*D),\end{aligned}\tag{3-3}$$

where

$$v'':=\Bigg(\frac{\frac{1}{\delta'}-e}{1-e}+r'+2r'((2d+1)+1)+\frac{r'}{\delta'}\Bigg)^d.$$

Recall that, by construction, $H^0(X,m(K_X+\Delta+f^*D))\cong H^0(Z',m(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'}+h^*D))$ for all $m\gg 0$ sufficiently divisible. Then we have

$$\operatorname{vol}(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'}+h^*D)=\operatorname{Ivol}(K_X+\Delta+f^*D)\le r\operatorname{Ivol}(K_X+\Delta)\le rC,$$

where the second inequality is from the definition of $\mathcal{D}(n,\mathcal{I},l,r)$. Then we have

$$G^{d-1}.(\operatorname{red}(\phi^*A+F_1+B_{Z'}+h^*D))\le 2^d rC''v.$$

Because ϕ^*A and h^*H are nef, we have that

$$\begin{aligned}(\phi^*A)^{d-1}.h^*H\le(\phi^*A+h^*H)^d&\le r'^d\operatorname{vol}(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'}+H)\\ &\le r'^d\operatorname{Ivol}(K_X+\Delta+f^*H)\le r'^d rC.\end{aligned}\tag{3-4}$$

Let $v':=2^dC''v/(2(2d+1)+1)^d+r'^d rC$. Then $(\phi^*A)^{d-1}.(\operatorname{red}(\phi^*A+F_1+B_{Z'}+h^*D)+h^*H)<v'$. By boundedness of the Chow varieties, see [Kollár 1996, §1.3], $(W,\operatorname{Supp}(\phi_*(\phi^*A+F_1+B_{Z'}+h^*D+h^*H)))$ is log bounded, and therefore $(W,\operatorname{Supp}(\phi_*(\phi^*A+F_1+h^*D)))$ is log bounded.

Step 5: We take a log resolution of $(W,\phi_*(\phi^*A+F_1+h^*D))$ to get a log bounded family $(\mathcal{Z},\mathcal{P})\to T$, then show the moduli part $\boldsymbol{M}$ descends on $\mathcal{Z}_t$ by using the standard normal crossing assumptions.

By the definition of log boundedness, there is a flat morphism $(\mathcal{Z},\mathcal{P})\to T$ such that, for a closed point $t\in T$, we have $(W,\phi_*(\phi^*A+F_1+h^*D))\cong(\mathcal{Z}_t,\mathcal{P}_t)$. Because $f^!:(X^!,\operatorname{Supp}(\Delta^!-F^!))\to W$ is log smooth over $W\setminus\operatorname{Supp}(\phi^!{}_*L^!+\phi^!{}_*h^{!*}D)$ and $\operatorname{Supp}(\phi^!{}_*L^!+\phi^!{}_*h^{!*}D)=\operatorname{Supp}(\phi_*(\phi^*A+F_1+h^*D))$, there is a rational contraction $f_t:X^!\dashrightarrow\mathcal{Z}_t$ such that $(X^!,\operatorname{Supp}(\Delta^!-F^!))$ is log smooth over $\mathcal{Z}_t\setminus\mathcal{P}_t$.

After passing to a stratification of T and log resolution of the generic fiber of $\mathcal{Z}\to T$, we can assume $(\mathcal{Z}_t,\mathcal{P}_t)$ is log smooth for every closed point $t\in T$. We choose a birational model Z' of Z as in Theorem 2.12 such that $\phi:Z'\to\mathcal{Z}_t$ is still a birational morphism.

We replace X by a higher birational model which resolves the indeterminacy of $X^!\dashrightarrow\mathcal{Z}_t$, replace Δ by its strict transform plus the exceptional divisor, and choose $F\in|K_X+\Delta|_{\mathbb{Q}/\mathcal{Z}_t}$. Because $(X^!,\operatorname{Supp}(\Delta^!-F^!))$ is log smooth over $\mathcal{Z}_t\setminus\mathcal{P}_t$, we may assume that the morphism $X\to\mathcal{Z}_t$ and divisors $\Delta-F$, $\mathcal{P}_t$ satisfy the standard normal crossing assumptions. Hence the corresponding moduli $\boldsymbol{b}$-divisor descends on $\mathcal{Z}_t$.

Because $(X, \Delta) \to Z$ has the same generic fiber (X_η, Δ_η) as $f : (X, \Delta) \to \mathcal{Z}_t$ and $\kappa(X_\eta, K_{X_\eta} + \Delta_\eta) = 0$, the moduli $\boldsymbol{b}$-divisor $\boldsymbol{M}$ of $(X, \Delta) \to Z$ descends on $\mathcal{Z}_t$. Also because $\phi : Z' \to \mathcal{Z}_t$ is a birational morphism, we have $\boldsymbol{M}_{Z'} = \phi^*\boldsymbol{M}_{\mathcal{Z}_t}$.

Step 6: We show that the boundary part is $\mathbb{Q}$-linearly equivalent to the difference of two $\mathbb{Q}$-divisors on $\mathcal{Z}_t$ both with bounded degrees. Therefore, the boundary part is bounded up to $\mathbb{Q}$-linear equivalence.

By Theorem 2.5, there is a rational number $e < 1$ depending only on $\mathcal{I}$, d, and l such that both $K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'}$ and $K_{Z'} + B_{Z'} + e\boldsymbol{M}_{Z'}$ are big divisors. By Theorem 2.4, there is an integer $\tilde{r}$ depending only on $\mathcal{I}$, d, l and e such that both $|m\tilde{r}(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'})|$ and $|m\tilde{r}(K_{Z'} + B_{Z'} + e\boldsymbol{M}_{Z'})|$ define birational maps for all integers $m \geq 1$. By assumption, $l\boldsymbol{M}_{\mathcal{Z}_t}$ is Cartier, so we may choose $\tilde{r} = r'l$ for some integer $r' \gg 0$ such that both $\tilde{r}\boldsymbol{M}_{Z'}$ and $\tilde{r}e\boldsymbol{M}_{Z'}$ are Cartier divisors and both $|\tilde{r}(K_{Z'} + \lfloor \tilde{r}B_{Z'} \rfloor/\tilde{r} + \boldsymbol{M}_{Z'})|$ and $|\tilde{r}(K_{Z'} + \lfloor \tilde{r}B_{Z'} \rfloor/\tilde{r} + e\boldsymbol{M}_{Z'})|$ define birational maps. Let

$$D_1'' \in \left|\tilde{r}\left(K_{Z'} + \frac{\lfloor \tilde{r}B_{Z'} \rfloor}{\tilde{r}} + \boldsymbol{M}_{Z'}\right)\right|, \quad D_2'' \in \left|\tilde{r}\left(K_{Z'} + \frac{\lfloor \tilde{r}B_{Z'} \rfloor}{\tilde{r}} + e\boldsymbol{M}_{Z'}\right)\right|$$

be general members. Define two effective $\mathbb{Q}$-divisors

$$D_1' \sim_{\mathbb{Q}} \frac{K_{Z'} + \lfloor \tilde{r}B_{Z'} \rfloor/\tilde{r} + \boldsymbol{M}_{Z'}}{1-e}, \quad D_2' \sim_{\mathbb{Q}} \frac{K_{Z'} + \lfloor \tilde{r}B_{Z'} \rfloor/\tilde{r} + e\boldsymbol{M}_{Z'}}{1-e}.$$

It is easy to see that the coefficients of D_1' and D_2' are in a discrete set that depends only on $r, \tilde{r}, e, \mathcal{I}$. Let $D_1 = \phi_* D_1'$ and $D_2 = \phi_* D_2'$. It is easy to see the degrees of D_1 and D_2 with respect to A in W are bounded. Because the coefficients of D_1 and D_2 are in a finite set and $\boldsymbol{M}_Z = D_1 - D_2$, by Lemma 3.1, up to replacing the family, there are finitely many divisors $\mathcal{M}_k$, $k \in \Lambda$, on $\mathcal{Z}$ such that $\boldsymbol{M}_{\mathcal{Z}_t} = \mathcal{M}_k|_{\mathcal{Z}_t}$ for some $k \in \Lambda$. □

Theorem 3.3. *Suppose $(\mathcal{Z}, \mathcal{P}) \to T$ is a family of projective log smooth pairs, where T is of finite type, and let $\mathcal{M}$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $\mathcal{Z}$. Fix an integer $l > 0$ and a DCC set $\mathcal{I} \subset [0, 1] \cap \mathbb{Q}$. For a closed point $t \in T$, let $\mathcal{S}(\mathcal{I}, \mathcal{Z}, \mathcal{P}, \mathcal{M}, T, t)$ denote the set of generalized pairs $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$ such that*

- $(Z', \mathrm{Supp}(B_{Z'}))$ *is log smooth*,
- $\mathrm{coeff}(B_{Z'}) \in \mathcal{I}$,
- *there is a birational morphism $\phi : Z' \to \mathcal{Z}_t$*,
- $\phi_* B_{Z'} \leq \mathcal{P}|_{\mathcal{Z}_t}$,
- $\boldsymbol{M}$ *descends on $\mathcal{Z}_t$, and*
- $\boldsymbol{M}_{Z'} = \phi^*(\mathcal{M}|_{\mathcal{Z}_t})$.

Let $\mathcal{S}(\mathcal{I}, \mathcal{Z}, \mathcal{P}, \mathcal{M}, T) := \bigcup_{t \in T} \mathcal{S}(\mathcal{I}, \mathcal{Z}, \mathcal{P}, \mathcal{M}, T, t)$. Then the set

$$\{\mathrm{vol}(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'}) \mid (Z', B_{Z'} + \boldsymbol{M}_{Z'}) \in \mathcal{S}(\mathcal{I}, \mathcal{Z}, \mathcal{P}, \mathcal{M}, T)\}$$

satisfies the DCC.

Proof. Let $T' \subset T$ be the subset such that $\boldsymbol{M}|_{\mathcal{Z}_t}$ is nef for every $t \in T'$. Fix a closed point $0 \in T'$. For any closed point $t \in T'$ and $(Z', B_{Z'} + \boldsymbol{M}_{Z'}) \in \mathcal{S}(\mathcal{I}, \mathcal{Z}, \mathcal{P}, \mathcal{M}, T, t)$, because $(\mathcal{Z}_t, \mathcal{P}_t)$ is log smooth, by the proof of [Filipazzi 2018, Theorem 1.10], we may assume that $\phi : Z' \to \mathcal{Z}_t$ only blows up strata of $\mathcal{P}_t$. On the other hand, by the proof of Lemma 3.1, after replacing T by an étale cover, we may assume every stratum of $(\mathcal{Z}, \mathcal{P})$ has irreducible fibers over T. Therefore, we may find a sequence of blowups $g : \mathcal{Z}' \to \mathcal{Z}$ such that $Z' = \mathcal{Z}'_t$. It is easy to see that there is a unique divisor $\mathcal{B}_{Z'}$ supported on the strict transform of $\mathcal{P}$ and the exceptional locus of g such that $B_{Z'} = \mathcal{B}_{Z'}|_{\mathcal{Z}'_t}$. Let $Y = \mathcal{Z}'_0$ be the fiber over 0 of $\mathcal{Z}' \to T$ and $B_Y := \mathcal{B}_{Z'}|_{\mathcal{Z}'_0}$. By Theorem 2.7, we have that

$$\operatorname{vol}(K_{Z'} + B_{Z'} + \phi^*\mathcal{M}|_{\mathcal{Z}_t}) = \operatorname{vol}(K_Y + B_Y + (g^*\mathcal{M})|_{\mathcal{Z}'_0}).$$

Then the set

$$\{\operatorname{vol}(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'}) \mid (Z', B_{Z'} + \boldsymbol{M}_{Z'}) \in \mathcal{S}(\mathcal{I}, \mathcal{Z}, \mathcal{P}, \mathcal{M}, T, t)\}$$

is independent of $t \in T$. Now apply Theorem 2.6. □

Proof of Theorem 1.6. Fix an arbitrary constant $v > 0$, let

$$\mathcal{D}(n, \mathcal{I}, l, r, v^-) := \{(X, \Delta) \in \mathcal{D}(n, \mathcal{I}, l, r), \operatorname{Ivol}(K_X + \Delta) \le v\}.$$

We only need to prove $\{\operatorname{Ivol}(K_X + \Delta) \mid (X, \Delta) \in \mathcal{D}(n, \mathcal{I}, l, r, v^-)\}$ is a DCC set.

Fix $(X, \Delta) \in \mathcal{D}(n, \mathcal{I}, l, r, v^-)$. Because Z is the canonical model of $K_X + \Delta$, by Theorem 2.12, there is a generalized pair $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$ and a birational morphism $h : Z' \to Z$ such that Z is the canonical model of $K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'}$. Let B_Z be the pushforward of $B_{Z'}$; then $K_Z + B_Z + \boldsymbol{M}_Z$ is ample.

By Theorem 2.4, there is an integer $r' > 0$ which only depends on $\mathcal{I}$ and l such that $|r'(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'})|$ defines a birational map. Choose a general member $H' \in |r'(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'})|$, and let $H := h_*H'$. Then H is ample and the coefficients of H are bounded below by a positive number δ'. By definition of the canonical model, $h^*H \le H'$, by Theorem 2.12 (3),

$$H^0(X, \mathcal{O}_X(ml(K_X + \Delta))) \cong H^0(Z', \mathcal{O}_{Z'}(ml(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'}))),$$

and we have that

$$\operatorname{Ivol}(K_X + \Delta + f^*H) \le (1 + r)^d \operatorname{Ivol}(K_X + \Delta). \tag{3-5}$$

Then (X, Δ) and H satisfy the conditions in Theorem 3.2.

Let $(\mathcal{Z}, \mathcal{P}) \to T$ be the bounded family, let $\mathcal{M}_k$, $k \in \Lambda$, be the $\mathbb{Q}$-divisors defined in Theorem 3.2, and let $\mathcal{D}'$ be the set of generalized klt pairs $(W', B_{W'} + \boldsymbol{M}_{W'})$ such that

- $(W', \operatorname{Supp}(B_{W'}))$ is log smooth,
- there is a morphism $\phi : W' \to \mathcal{Z}_t$ for a closed point $t \in T$,
- $\boldsymbol{M}_{W'} = \phi^*(\mathcal{M}_k|_{\mathcal{Z}_t})$ for some $k \in \Lambda$, and
- $\operatorname{coeff}(B_{W'})$ is in a fixed DCC set and $\phi_*(B_{W'}) \le \mathcal{P}_t$.

Since Z is the canonical model of (X, Δ), we have $\mathrm{Ivol}(K_X+\Delta) = \mathrm{vol}(K_Z+B_Z+\boldsymbol{M}_Z)$. Let $(Z', B_{Z'}+\boldsymbol{M}_{Z'})$ be a generalized pair as in Theorem 2.12 such that there is a birational morphism $\psi_t : Z' \to \mathcal{Z}_t$ for a closed point $t \in T$. By Theorem 2.12 (3), $\mathrm{vol}(K_Z+B_Z+\boldsymbol{M}_Z) = \mathrm{vol}(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'})$, and, by Theorem 3.2, $\psi_{t*}B_{Z'} \le \mathcal{P}_t$ and $\boldsymbol{M}_{Z'} = \psi_t^*\mathcal{M}_k$. Then $(Z', B_{Z'}+\boldsymbol{M}_{Z'}) \in \mathcal{D}'$ and

$$\{\mathrm{Ivol}(K_X+\Delta) \mid (X, \Delta) \in \mathcal{D}(n, \mathcal{I}, l, r, v^-)\} \subset \{\mathrm{vol}(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'}) \mid (Z', B_{Z'}+\boldsymbol{M}_{Z'}) \in \mathcal{D}'\}.$$

Because Λ is a finite set, by Theorem 3.3, the set $\{\mathrm{vol}(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'}) \mid (Z', B_{Z'}+\boldsymbol{M}_{Z'}) \in \mathcal{D}'\}$ satisfies the DCC, and hence $\{\mathrm{Ivol}(K_X+\Delta) \mid (X, \Delta) \in \mathcal{D}(n, \mathcal{I}, l, r, v^-)\}$ satisfies the DCC. □

4. Boundedness of canonical models

In this section, we follow the method of [Hacon et al. 2018, Chapter 7].

Definition 4.1. Let (Z, B) be a pair. Define a $\boldsymbol{b}$-divisor $\boldsymbol{M}_B$ by assigning to any divisorial valuation μ

$$\boldsymbol{M}_B(\mu) = \begin{cases} \mathrm{mult}_\Gamma(B) & \text{if the center of } \mu \text{ is a divisor } \Gamma \text{ on } Z, \\ 1 & \text{otherwise.} \end{cases} \tag{4-1}$$

Theorem 4.2. *Let v be a positive rational number, and let $\mathcal{I} \subset [0, 1]$ be a DCC set of positive rational numbers. Suppose $(\mathcal{Z}, \mathcal{P}) \to T$ is a family of projective log smooth pairs, where T is of finite type, and $\mathcal{M}$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $\mathcal{Z}$. Let $\mathcal{S}(v, \mathcal{I}, \mathcal{Z}, \mathcal{P}, \mathcal{M}, T)$ be the set of generalized pairs $(Z', B_{Z'}+\boldsymbol{M}_{Z'})$ such that*

- *$(Z', B_{Z'}+\boldsymbol{M}_{Z'})$ is generalized klt,*
- $\mathrm{vol}(K_{Z'}+B_{Z'}+\boldsymbol{M}_{Z'}) = v$,
- $\mathrm{coeff}(B_{Z'}) \subset \mathcal{I}$,
- *there is a closed point $t \in T$ and a birational morphism $\phi : Z' \to \mathcal{Z}_t$,*
- *$\phi_* B_{Z'} \le \mathcal{P}_t$,*
- *$\boldsymbol{M}$ descends on $\mathcal{Z}_t$, and*
- *$\boldsymbol{M}_{Z'} = \phi^*(\mathcal{M}|_{\mathcal{Z}_t})$.*

Let $(Z, B_Z+\boldsymbol{M}_Z)$ be the canonical model of $(Z', B_{Z'}+\boldsymbol{M}_{Z'})$. Then Z is in a bounded family depending only on v, $\mathcal{I}$, $(\mathcal{Z}, \mathcal{P}) \to T$ and $\mathcal{M}$.

Proof. It suffices to show that, for any generalized pair $(Z', B_{Z'}+\boldsymbol{M}_{Z'}) \subset \mathcal{S}(v, \mathcal{I}, \mathcal{Z}, \mathcal{P}, \mathcal{M}, T)$, there is an integer $N > 0$ such that if $(Z, B_Z+\boldsymbol{M}_Z)$ is the canonical model of $(Z', B_{Z'}+\boldsymbol{M}_{Z'})$, then $N(K_Z+B_Z+\boldsymbol{M}_Z)$ is Cartier and very ample.

Suppose this is not the case: let $\{(Z'_i, B^i_{Z'_i}+\boldsymbol{M}_{Z'_i}), i \ge 1\} \subset \mathcal{S}(v, \mathcal{I}, \mathcal{Z}, \mathcal{P}, \mathcal{M}, T)$ be a sequence and $(Z_i, B^i_{Z_i}+\boldsymbol{M}^i_{Z_i})$ the corresponding canonical model such that $i!(K_{Z_i}+B^i_{Z_i}+\boldsymbol{M}^i_{Z_i})$ is not very ample for every $i \ge 1$. Let $\{t_i \in T, i \ge 1\}$ be the corresponding sequence of closed points, and let $\phi_i : Z'_i \to \mathcal{Z}_{t_i}$ be the corresponding morphisms. By the construction, we have $\phi_{i*}B^i_{Z'_i} \le \mathcal{P}_{t_i}$ and $\boldsymbol{M}^i_{Z'_i} = \phi_i^*(\mathcal{M}|_{\mathcal{Z}_{t_i}})$. After replacing T by a closed subset, we assume that $\{t_i \in T, i \ge 1\}$ is dense in T.

Step 1: We prove that there exists a birational morphism $g:\mathcal{Z}'\to\mathcal{Z}$ such that

- g is obtained by blowing up the corresponding strata of $\boldsymbol{M}_{\mathcal{P}}$, and
- $\operatorname{vol}(K_{\mathcal{Z}'_{t_i}}+\Phi^i|_{\mathcal{Z}'_{t_i}}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_i}})=v$ for every $i\geq 1$, where Φ^i is the $\mathbb{Q}$-divisor supported on $\boldsymbol{M}_{\mathcal{P},\mathcal{Z}'}$ such that $\Phi^i|_{\mathcal{Z}'_{t_i}}=\boldsymbol{M}_{B^i_{Z'_i},\mathcal{Z}'_{t_i}}$.

Applying [Filipazzi 2018, Proposition 5.1] to $(\mathcal{Z}_{t_1},\mathcal{P}_{t_1}+\mathcal{M}|_{\mathcal{Z}_{t_1}})$, we obtain a model $\mathcal{Z}'_{t_1}\to\mathcal{Z}_{t_1}$ and the morphism $g:(\mathcal{Z}',\mathcal{P}':=\boldsymbol{M}_{\mathcal{P},\mathcal{Z}'})\to\mathcal{Z}$ obtained by blowing up the corresponding strata of $\boldsymbol{M}_{\mathcal{P}}$. We define $\Phi_{t_i}=\boldsymbol{M}_{B^i_{Z'_i},\mathcal{Z}'_{t_i}}$. Passing to a subsequence, we may also assume that, for any irreducible component P of the support of $\mathcal{P}'$, the coefficients of Φ_{t_i} along P_{t_i} are nondecreasing. Let Φ^i be the $\mathbb{Q}$-divisor supported on $\mathcal{P}'$ such that $\Phi^i|_{\mathcal{Z}'_{t_i}}=\Phi_{t_i}$. Then the coefficients of Φ^i are nondecreasing.

We claim that, for any $i\geq 1$, we have

$$\operatorname{vol}(K_{\mathcal{Z}'_{t_i}}+\Phi_{t_i}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_i}})=v.$$

To prove this, we may fix i. Applying the above cited result to $(\mathcal{Z}_{t_i},\mathcal{P}_{t_i}+\mathcal{M}|_{\mathcal{Z}_{t_i}})$, we obtain a model $\mathcal{Z}''_{t_i}\to\mathcal{Z}'_{t_i}$ and the corresponding morphism $g':(\mathcal{Z}'',\mathcal{P}'':=\boldsymbol{M}_{\mathcal{P},\mathcal{Z}''})\to\mathcal{Z}'$ obtained by blowing up the corresponding strata of $\boldsymbol{M}_{\mathcal{P}}$. By the above cited result again, we have

$$\operatorname{vol}(K_{\mathcal{Z}''_{t_i}}+\Psi_{t_i}+g'^*g^*\mathcal{M}|_{\mathcal{Z}''_{t_i}})=v,$$

where $\Psi_{t_i}:=\boldsymbol{M}_{B^i_{Z'_i},\mathcal{Z}''_{t_i}}$. If Ψ is the divisor supported on $\operatorname{Supp}(\boldsymbol{M}_{\mathcal{P}',\mathcal{Z}''})$ such that $\Psi|_{\mathcal{Z}''_{t_i}}=\Psi_{t_i}$, then

$$\begin{aligned}v&=\operatorname{vol}(K_{\mathcal{Z}''_{t_i}}+\Psi_{t_i}+g'^*g^*\mathcal{M}|_{\mathcal{Z}''_{t_i}})\\&=\operatorname{vol}(K_{\mathcal{Z}''_{t_1}}+\Psi|_{\mathcal{Z}''_{t_1}}+g'^*g^*\mathcal{M}|_{\mathcal{Z}''_{t_1}})\\&=\operatorname{vol}(K_{\mathcal{Z}'_{t_1}}+\Phi^i|_{\mathcal{Z}'_{t_1}}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}})\\&=\operatorname{vol}(K_{\mathcal{Z}'_{t_i}}+\Phi_{t_i}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_i}}),\end{aligned}\tag{4-2}$$

where the second and the fourth equalities follow from Theorem 2.7 and the third one follows from [Filipazzi 2018, Proposition 5.1].

Step 2: We show that, after replacing T by an open subset, $\mathcal{Z}'$ by a resolution and $\{t_i, i\geq 1\}$ by a subsequence, there exist effective $\mathbb{Q}$-divisors $\mathcal{A}$ and $\mathcal{E}^i$ on $\mathcal{Z}'$ such that

- $\mathcal{A}$ is ample over T,
- $\mathcal{E}^i:=\mathcal{E}^1+\Phi^i-\Phi^1$,
- $K_{\mathcal{Z}'}+\Phi^i+g^*\mathcal{M}\sim_{\mathbb{Q}}\mathcal{A}+\mathcal{E}^i$, and
- $(\mathcal{Z}',\operatorname{Supp}(\Phi^i+\mathcal{E}^i))$ is log smooth over T

for every $i\geq 1$.

Because $g^*\mathcal{M}|_{\mathcal{Z}'_{t_i}}$ is nef for every $i \in \mathbb{N}$ and $\{t_i \in T, i \geq 1\}$ is dense in T, we have that $g^*\mathcal{M}|_{\mathcal{Z}'_t}$ is nef for a very general point $t \in T$. Note that $K_{\mathcal{Z}'_{t_1}} + \Phi^1|_{\mathcal{Z}'_{t_1}} + g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}}$ is big. Suppose

$$\mathrm{vol}(K_{\mathcal{Z}'_{t_1}} + \Phi^1|_{\mathcal{Z}'_{t_1}} + g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}}) = v > 0.$$

Then by [Filipazzi 2018, Theorem 1.12], we have $\mathrm{vol}(K_{\mathcal{Z}'_t} + \Phi^1|_{\mathcal{Z}'_t} + g^*\mathcal{M}|_{\mathcal{Z}'_t}) = v$ for a very general point $t \in T$. Since sections on the very general fiber agree with sections on the generic fiber by semicontinuity of cohomology groups, $K_{\mathcal{Z}'} + \Phi^1 + g^*\mathcal{M}$ is big over the generic point of T, and we have that $K_{\mathcal{Z}'} + \Phi^1 + g^*\mathcal{M}$ is big over T.

Let $\mathcal{A}$ be a general relatively ample $\mathbb{Q}$-divisor on $\mathcal{Z}'$ and $\mathcal{E}^1$ be an effective $\mathbb{Q}$-divisor on $\mathcal{Z}'$ such that

$$K_{\mathcal{Z}'} + \Phi^1 + g^*\mathcal{M} \sim_{\mathbb{Q}} \mathcal{A} + \mathcal{E}^1.$$

Define $\mathcal{E}^i := \mathcal{E}^1 + \Phi^i - \Phi^1$; then $\mathcal{E}^i$ is effective and $K_{\mathcal{Z}'} + \Phi^i + g^*\mathcal{M} \sim_{\mathbb{Q}} \mathcal{A} + \mathcal{E}^i$.

After taking a log resolution of the generic fiber and replacing T by an open subset, we may assume that there is a fiberwise log resolution $h : \mathcal{Z}^* \to \mathcal{Z}'$ of $\mathcal{P}' + \mathcal{E}$ over T. By the negativity lemma, there exists a $\mathbb{Q}$-divisor $\mathcal{F}$ on $\mathcal{Z}^*$ which is supported on the exceptional divisor over $\mathcal{Z}$ such that $\mathcal{A}^* := h^*\mathcal{A} - \mathcal{F}$ is relatively ample over T. Let $\Phi^{i*} := \boldsymbol{M}_{\Phi^i, \mathcal{Z}^*}$. Because $(\mathcal{Z}', \Phi^1)$ is lc, if we write

$$K_{\mathcal{Z}^*} + \Phi^{1*} + h^*g^*\mathcal{M} \sim_{\mathbb{Q}} \mathcal{A}^* + \mathcal{E}^{1*},$$

then $\mathcal{E}^{1*}$ is effective and supported on the $\mathrm{Supp}(h_*^{-1}\mathcal{E}^1)$ plus the h-exceptional divisors. Therefore, $(\mathcal{Z}^*, \mathrm{Supp}(\mathcal{E}^{1*}))$ is log smooth over T. Notice that $\Phi^i - \Phi^1$ is effective and supported on $\mathcal{P}'$. Define $\mathcal{E}^{i*} := \mathcal{E}^{1*} + h_*^{-1}(\Phi^i - \Phi^1)$; then

$$K_{\mathcal{Z}^*} + \Phi^{i*} + h^*g^*\mathcal{M} \sim_{\mathbb{Q}} \mathcal{A}^* + \mathcal{E}^{i*}$$

and $(\mathcal{Z}^*, \mathrm{Supp}(\mathcal{E}^{i*}))$ is log smooth over T.

Then we replace $\mathcal{Z}'$, Φ^i, g, $\mathcal{A}$ and $\mathcal{E}^i$ by $\mathcal{Z}^*$, Φ^{i*}, $h \circ g$, $\mathcal{A}^*$ and $\mathcal{E}^{i*}$, respectively. Suppose

$$k = \min\{i \mid t_i \in T,\ i \geq 1\}.$$

Then we pass to a subsequence of $\{t_i, i \in \mathbb{N}\}$ and replace t_1, Φ^1 and $\mathcal{E}^1$ by t_k, Φ^k and $\mathcal{E}^k$, respectively.

<u>Step 3</u>: In this step we construct a $\mathbb{Q}$-divisor $\hat{\Phi}$ on $\mathcal{Z}'$ such that

- $\hat{\Phi} \leq \Phi^1$,
- $(\mathcal{Z}'_{t_i}, \hat{\Phi}|_{\mathcal{Z}'_{t_i}})$ is klt for every $i \geq 1$,
- $\mathrm{vol}(K_{\mathcal{Z}'_{t_1}} + \hat{\Phi}|_{\mathcal{Z}'_{t_1}} + g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}}) = v$, and
- $(Z_1, B^1_{Z_1} + \boldsymbol{M}^1_{Z_1})$ is the canonical model of $(\mathcal{Z}'_{t_1}, \hat{\Phi}|_{\mathcal{Z}'_{t_1}} + g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}})$.

Since $(Z_1, B^1_{Z_1} + \boldsymbol{M}^1_{Z_1})$ is generalized klt, we slightly decrease the coefficients of components of Φ^1 corresponding to the exceptional divisor of $\mathcal{Z}'_{t_1} \dashrightarrow Z_1$ to define a $\mathbb{Q}$-divisor $\hat{\Phi}$ such that

- $\hat{\Phi} \leq \Phi^1$,
- $(\mathcal{Z}'_{t_1}, \hat{\Phi}|_{\mathcal{Z}'_{t_1}})$ is klt,

- $\mathrm{vol}(K_{\mathcal{Z}'_{t_1}} + \hat{\Phi}|_{\mathcal{Z}'_{t_1}} + g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}}) = v$, and
- $(Z_1, B^1_{Z_1} + \boldsymbol{M}^1_{Z_1})$ is the canonical model of $(\mathcal{Z}'_{t_1}, \hat{\Phi}|_{\mathcal{Z}'_{t_1}} + g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}})$.

Note we have $\hat{\Phi} \leq \Phi^1 \leq \Phi^2 \leq \cdots$. Because $(\mathcal{Z}', \mathrm{Supp}(\Phi^i + \mathcal{E}^i))$ is log smooth over T and $(\mathcal{Z}'_{t_1}, \hat{\Phi}|_{\mathcal{Z}'_{t_1}})$ is klt, we have that $(\mathcal{Z}'_{t_i}, \hat{\Phi}|_{\mathcal{Z}'_{t_i}})$ is klt for every i.

Step 4: We show that there exist a sufficiently small positive number $\epsilon \in (0, 1)$ and a birational contraction $\psi : \mathcal{Z}' \dashrightarrow \mathcal{W}$ over T such that

- ψ is the relative canonical model of $\big(\mathcal{Z}', \frac{\epsilon}{1+\epsilon}\Phi^1 + \frac{1}{1+\epsilon}\hat{\Phi} + g^*\mathcal{M}\big)$, and
- $\psi_{t_i} : \mathcal{Z}'_{t_i} \dashrightarrow \mathcal{W}_{t_i}$ is the canonical model of $\big(\mathcal{Z}'_{t_i}, \big(\frac{\epsilon}{1+\epsilon}\Phi^1 + \frac{1}{1+\epsilon}\hat{\Phi} + g^*\mathcal{M}\big)\big|_{\mathcal{Z}'_{t_i}}\big)$ for every $i \geq 1$.

Because $\Phi_{t_1} = \boldsymbol{M}_{B^i_{Z'_1}, \mathcal{Z}'_{t_1}}$, for any common resolution Y of Z'_1 and $\mathcal{Z}'_{t_1}$, we have $\boldsymbol{M}_{\Phi_{t_1}, Y} \geq \boldsymbol{M}_{B^1_{Z'_1}, Y}$. Also because $\mathrm{vol}(K_{Z'_1} + B'_{Z'_1} + \boldsymbol{M}^1_{Z'_1}) = \mathrm{vol}(K_{\mathcal{Z}'_{t_1}} + \Phi_{t_1} + g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}})$, by [Filipazzi 2018, Lemma 5.2], $(Z'_1, B'_{Z'_1} + \boldsymbol{M}^1_{Z'_1})$ has the same canonical model as $(\mathcal{Z}'_{t_1}, \Phi_{t_1} + g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}})$, which is $(Z_1, B^1_{Z_1} + \boldsymbol{M}^1_{Z_1})$. In particular, there is a birational contraction $\mathcal{Z}'_{t_1} \dashrightarrow Z_1$.

Since $(\mathcal{Z}'_{t_1}, \hat{\Phi}|_{\mathcal{Z}'_{t_1}})$ is klt and $(\mathcal{Z}', \mathrm{Supp}(\Phi^i + \mathcal{A} + \mathcal{E}^i))$ is log smooth over T, we can choose $\epsilon \ll 1$ such that $(\mathcal{Z}'_{t_i}, \hat{\Phi}|_{\mathcal{Z}'_{t_i}} + \epsilon\mathcal{E}^1|_{\mathcal{Z}'_{t_i}})$ is klt for every $i \geq 1$ and $g^*\mathcal{M} + \epsilon\mathcal{A}$ is ample over T. We then have that $(\mathcal{Z}'_{t_i}, \hat{\Phi}|_{\mathcal{Z}'_{t_i}} + \epsilon\mathcal{E}^1|_{\mathcal{Z}'_{t_i}} + (g^*\mathcal{M} + \epsilon\mathcal{A})|_{\mathcal{Z}'_{t_i}})$ is generalized klt with nef part $(g^*\mathcal{M} + \epsilon\mathcal{A})|_{\mathcal{Z}'_{t_i}}$ for every $i \geq 1$. Because

$$K_{\mathcal{Z}'_{t_1}} + \hat{\Phi}|_{\mathcal{Z}'_{t_1}} + \epsilon\mathcal{E}^1|_{\mathcal{Z}'_{t_1}} + (g^*\mathcal{M} + \epsilon\mathcal{A})|_{\mathcal{Z}'_{t_1}} \sim_{\mathbb{Q}} K_{\mathcal{Z}'_{t_1}} + \hat{\Phi}|_{\mathcal{Z}'_{t_1}} + g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}} + \epsilon(K_{\mathcal{Z}'_{t_1}} + \Phi^1|_{\mathcal{Z}'_{t_1}} + g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}})$$

and Z_1 is both the canonical model of $(\mathcal{Z}'_{t_1}, \hat{\Phi}|_{\mathcal{Z}'_{t_1}} + g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}})$ and $(\mathcal{Z}'_{t_1}, \Phi^1|_{\mathcal{Z}'_{t_1}} + g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}})$, we have that Z_1 is also the canonical model of $(\mathcal{Z}'_{t_1}, \hat{\Phi}|_{\mathcal{Z}'_{t_1}} + \epsilon\mathcal{E}^1|_{\mathcal{Z}'_{t_1}} + (g^*\mathcal{M} + \epsilon\mathcal{A})|_{\mathcal{Z}'_{t_1}})$.

Because $g^*\mathcal{M} + \epsilon\mathcal{A}$ is ample over T, we can choose a general effective $\mathbb{Q}$-divisor $\mathcal{H} \sim_{\mathbb{Q}} g^*\mathcal{M} + \epsilon\mathcal{A}$ and replace T by an open neighborhood of t_1 such that $(\mathcal{Z}'_{t_i}, \hat{\Phi}|_{\mathcal{Z}'_{t_i}} + \epsilon\mathcal{E}^1|_{\mathcal{Z}'_{t_i}} + \mathcal{H}|_{\mathcal{Z}'_{t_i}})$ is klt for every $i \geq 1$ and $(\mathcal{Z}', \mathrm{Supp}(\hat{\Phi} + \epsilon\mathcal{E}^1 + \mathcal{H}))$ is log smooth over T. It is easy to see that Z_1 is also the canonical model of $(\mathcal{Z}'_{t_1}, \hat{\Phi}|_{\mathcal{Z}'_{t_1}} + \epsilon\mathcal{E}^1|_{\mathcal{Z}'_{t_1}} + \mathcal{H}|_{\mathcal{Z}'_{t_1}})$.

Because $\mathcal{H}|_{\mathcal{Z}'_{t_1}}$ is ample and $(\mathcal{Z}'_{t_1}, \hat{\Phi}|_{\mathcal{Z}'_{t_1}} + \epsilon\mathcal{E}^1|_{\mathcal{Z}'_{t_1}} + \mathcal{H}|_{\mathcal{Z}'_{t_1}})$ is klt, $(\mathcal{Z}'_{t_1}, \hat{\Phi}|_{\mathcal{Z}'_{t_1}} + \epsilon\mathcal{E}^1|_{\mathcal{Z}'_{t_1}} + \mathcal{H}|_{\mathcal{Z}'_{t_1}})$ has a good minimal model, according to [Birkar et al. 2010, Theorem 1.2] and [Kollár and Mori 1998, Theorem 3.3]. Because $(\mathcal{Z}', \mathrm{Supp}(\hat{\Phi} + \epsilon\mathcal{E}^1 + \mathcal{H}))$ is log smooth over T, by [Hacon et al. 2018, Corollary 1.4], suppose $\psi : \mathcal{Z}' \dashrightarrow \mathcal{W}$ is the relative canonical model of $(\mathcal{Z}', \hat{\Phi} + \epsilon\mathcal{E}^1 + \mathcal{H})$ over T. Then, fiber by fiber, $\psi_{t_i} : \mathcal{Z}'_{t_i} \dashrightarrow \mathcal{W}_{t_i}$ gives the canonical model for $(\mathcal{Z}'_{t_i}, \hat{\Phi}|_{\mathcal{Z}'_{t_i}} + \epsilon\mathcal{E}^1|_{\mathcal{Z}'_{t_i}} + \mathcal{H}|_{\mathcal{Z}'_{t_i}})$ for all $i \geq 1$. In particular, $\mathcal{W}_{t_1}$ is the canonical model of $(\mathcal{Z}'_{t_1}, \hat{\Phi}|_{\mathcal{Z}'_{t_1}} + \epsilon\mathcal{E}^1|_{\mathcal{Z}'_{t_1}} + \mathcal{H}|_{\mathcal{Z}'_{t_1}})$, and it is isomorphic to Z_1.

By the definition of the canonical model, $K_{\mathcal{W}} + \psi_*(\hat{\Phi} + \epsilon\mathcal{E}^1 + \mathcal{H})$ is ample over T. We recall that $K_{\mathcal{Z}'} + \hat{\Phi} + \epsilon\mathcal{E}^1 + \mathcal{H} \sim_{\mathbb{Q}} K_{\mathcal{Z}'} + \hat{\Phi} + g^*\mathcal{M} + \epsilon(K_{\mathcal{Z}'} + \Phi^1 + g^*\mathcal{M})$. Then

$$K_{\mathcal{W}} + \psi_*(\hat{\Phi} + \epsilon\mathcal{E}^1 + \mathcal{H}) \sim_{\mathbb{Q}} (1+\epsilon)\Big(K_{\mathcal{W}} + \psi_*\Big(\frac{\epsilon}{1+\epsilon}\Phi^1 + \frac{1}{1+\epsilon}\hat{\Phi} + g^*\mathcal{M}\Big)\Big)$$

and $K_{\mathcal{W}} + \psi_*\big(\frac{\epsilon}{1+\epsilon}\Phi^1 + \frac{1}{1+\epsilon}\hat{\Phi} + g^*\mathcal{M}\big)$ is ample over T. Thus $K_{\mathcal{W}_{t_i}} + \psi_*\big(\frac{\epsilon}{1+\epsilon}\Phi^1 + \frac{1}{1+\epsilon}\hat{\Phi} + g^*\mathcal{M}\big)\big|_{\mathcal{W}_{t_i}}$ is ample for every $i \geq 1$.

Because $\mathcal{Z}' \dashrightarrow \mathcal{W}$ is $K_{\mathcal{Z}'}+\hat{\Phi}+\epsilon\mathcal{E}^1+\mathcal{H}$-nonpositive and
$$K_{\mathcal{Z}'}+\hat{\Phi}+\epsilon\mathcal{E}^1+\mathcal{H}\sim_{\mathbb{Q}}(1+\epsilon)\Big(K_{\mathcal{Z}'}+\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}+g^*\mathcal{M}\Big),$$
$\mathcal{Z}' \dashrightarrow \mathcal{W}$ is $K_{\mathcal{Z}'}+\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}+g^*\mathcal{M}$-nonpositive. Also because
$$K_{\mathcal{W}_{t_i}}+\psi_*\Big(\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}+g^*\mathcal{M}\Big)\Big|_{\mathcal{W}_{t_i}}$$
is ample, $\psi_{t_i}:\mathcal{Z}'_{t_i}\dashrightarrow\mathcal{W}_{t_i}$ is the canonical model of
$$\Big(\mathcal{Z}'_{t_i},\Big(\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}+g^*\mathcal{M}\Big)\Big|_{\mathcal{Z}'_{t_i}}\Big).$$

Step 5: We show that ψ_{t_i} is also the canonical model of $(\mathcal{Z}'_{t_i},\Phi^k|_{\mathcal{Z}'_{t_i}}+g^*\mathcal{M}_{\mathcal{Z}'_{t_i}})$ for every $i,k\geq 1$ and finish the proof of the theorem.

Notice that, by Theorem 2.7,
$$v=\operatorname{vol}(K_{\mathcal{Z}'_{t_k}}+\Phi^k|_{\mathcal{Z}'_{t_k}}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_i}})=\operatorname{vol}(K_{\mathcal{Z}'_{t_1}}+\Phi^k|_{\mathcal{Z}'_{t_1}}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}})$$
for all $k>1$. By the construction of $\hat{\Phi}$, we have
$$\hat{\Phi}\leq\Phi^1\leq\Phi^2\leq\Phi^3\leq\cdots$$
and $\operatorname{vol}(K_{\mathcal{Z}'_{t_1}}+\hat{\Phi}|_{\mathcal{Z}'_{t_1}}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}})=v$; hence
$$\operatorname{vol}\Big(K_{\mathcal{Z}'_{t_1}}+\Big(\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}\Big)\Big|_{\mathcal{Z}'_{t_1}}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}}\Big)=v.$$

Because $(\mathcal{Z}',\operatorname{Supp}(\hat{\Phi}+\Phi^1))$ is log smooth over T, by [Filipazzi 2018, Theorem 1.12], we have
$$\operatorname{vol}\Big(K_{\mathcal{Z}'_{t_1}}+\Big(\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}\Big)\Big|_{\mathcal{Z}'_{t_1}}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_1}}\Big)=\operatorname{vol}\Big(K_{\mathcal{Z}'_{t_i}}+\Big(\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}\Big)\Big|_{\mathcal{Z}'_{t_i}}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_i}}\Big)=v$$
for every $i\geq 1$.

It follows from [Filipazzi 2018, Lemma 5.2] that $\psi_{t_i}:\mathcal{Z}'_{t_i}\dashrightarrow\mathcal{W}_{t_i}$ is also the canonical model of $(\mathcal{Z}'_{t_i},\Phi^k|_{\mathcal{Z}'_{t_i}}+g^*\mathcal{M}|_{\mathcal{Z}'_{t_i}})$ for every $k\geq 1$,
$$\psi_{t_i*}\Phi^k|_{\mathcal{Z}'_{t_i}}=\psi_{t_i*}\Big(\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}\Big)\Big|_{\mathcal{Z}'_{t_i}},$$
and there is an isomorphism $\alpha_i:Z_i\to\mathcal{W}_{t_i}$. Let $N>0$ be an integer such that
$$N\Big(K_{\mathcal{W}}+\psi_*\Big(\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}\Big)+\psi_*g^*\mathcal{M}\Big)$$
is very ample over T. Then
$$N\Big(K_{\mathcal{W}_{t_i}}+\psi_*\Big(\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}\Big)\Big|_{\mathcal{Z}'_{t_i}}+\psi_*g^*\mathcal{M}|_{\mathcal{W}_{t_i}}\Big)$$
is very ample for every $i\geq 1$. Since
$$\psi_{t_i*}\Big(\frac{\epsilon}{1+\epsilon}\Phi^1+\frac{1}{1+\epsilon}\hat{\Phi}\Big)\Big|_{\mathcal{Z}'_{t_i}}=\psi_{t_i*}\Phi^i|_{\mathcal{Z}'_{t_i}}=\alpha_{i*}B^i_{Z_i},$$
we have that $N(K_{Z_i}+B^i_{Z_i}+\boldsymbol{M}^i_{Z_i})$ is very ample for every $i\geq 1$, which is the required contradiction. □

Proof of Theorem 1.7. Define $\mathcal{D}(n, \mathcal{I}, l, r, v)$ to be the set

$$\{(X, \Delta) \mid (X, \Delta) \in \mathcal{D}(n, \mathcal{I}, l, r) \text{ and } \operatorname{Ivol}(K_X + \Delta) = v\}.$$

Suppose $(Z, B_Z + \boldsymbol{M}_Z)$ is the canonical model of $(X, \Delta) \in \mathcal{D}(n, \mathcal{I}, l, r, v)$. Let $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$ be the generalized pair defined in Theorem 2.12 (3). Then $(Z, B_Z + \boldsymbol{M}_Z)$ is the canonical model of $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$.

By Theorem 3.2, there is a log bounded log smooth family of projective varieties $(\mathcal{Z}, \mathcal{P}) \to T$ and finitely many $\mathbb{Q}$-divisors $\mathcal{M}_k$, $k \in \Lambda$, on $\mathcal{Z}$ such that there is a closed point $t \in T$ and a birational morphism $\phi : Z' \to \mathcal{Z}_t$ such that $\phi_* B_{Z'} \le \mathcal{P}_t$. Then we have $(Z', B_{Z'} + \boldsymbol{M}_{Z'}) \in \bigcup_{k \in \Lambda} \mathcal{S}(v, \mathcal{I}, \mathcal{Z}, \mathcal{P}, \mathcal{M}_k, T)$. Because Λ is a finite set, Z is in a bounded family according to Theorem 4.2. □

5. Weak boundedness

The definition of weak boundedness is introduced in [Kovács and Lieblich 2010].

Definition 5.1. A (g, m)-curve is an irreducible smooth curve C° whose smooth compactification C has genus g and which satisfies the requirement that $C \setminus C^\circ$ consists of m closed points.

Definition 5.2. Let W be a proper scheme with a line bundle $\mathcal{N}$, and let U be an open subset of a proper variety. We say a morphism $\xi : U \to W$ is weakly bounded with respect to $\mathcal{N}$ if there exists a function $b_{\mathcal{N}} : \mathbb{Z}^2_{\ge 0} \to \mathbb{Z}$ such that, for every pair (g, m) of nonnegative integers, for every (g, m)-curve $C^\circ \subseteq C$, and for every morphism $C^\circ \to U$, one has $\deg \xi_C^* \mathcal{N} \le b_{\mathcal{N}}(g, m)$, where $\xi_C : C \to W$ is the induced morphism. The function $b_{\mathcal{N}}$ will be called a weak bound, and we will say that ξ is weakly bounded by $b_{\mathcal{N}}$.

We say a quasiprojective variety U is weakly bounded if there exists a compactification $i : U \hookrightarrow W$ such that $i : U \hookrightarrow W$ is weakly bounded with respect to an ample line bundle $\mathcal{N}$ on W. The following lemma says that if a projective variety U is weakly bounded with respect to an embedding $U \hookrightarrow W$, then it is also weakly bounded with respect to any other compactification $U \hookrightarrow W'$ and any ample line bundle on W' (possibly by a different weak bound).

Lemma 5.3. *Let U be a weakly bounded quasiprojective variety with a compactification $i : U \hookrightarrow W$ such that $i : U \hookrightarrow W$ is weakly bounded with respect to an ample line bundle $\mathcal{N}$ on W. Then, for any compactification $i' : U \hookrightarrow W'$ and any ample line bundle $\mathcal{N}'$ on W', $i' : U \hookrightarrow W'$ is weakly bounded with respect to $\mathcal{N}'$.*

Proof. Let $g : W'' \to W$ and $h : W'' \to W'$ be a common resolution of W and W'. Let A'' be an ample Cartier divisor on W'', A a Cartier divisor on W such that $\mathcal{O}_W(A) = \mathcal{N}$, and A' a Cartier divisor on W' such that $\mathcal{O}_{W'}(A') = \mathcal{N}'$.

Suppose $C^\circ \subset C$ is a (g, m)-curve for a pair of nonnegative integers (g, m) and $C^\circ \to U$ is a morphism that extends to a morphism $\xi : C \to W$. By definition, there exists a function $b_{\mathcal{N}} : \mathbb{Z}^2_{\ge 0} \to \mathbb{Z}$ such that $\deg \xi^* \mathcal{N} \le b_{\mathcal{N}}(g, m)$.

Because A is ample, $g^* A$ is big, and there exist an effective divisor F on W'' and $l, n \in \mathbb{N}$ such that $l g^* A \sim n A'' + F''$. Write $\operatorname{Supp}(F'') = \bigcup_{1 \le i \le k} W''_i$, where the W''_i are reduced divisors.

Suppose $C^\circ \to U$ extends to a morphism $\xi'' : C \to W''$. Then $\xi := g \circ \xi''$. We claim that there exists a positive number c depending only on $U \hookrightarrow W''$ and A'' such that $\deg \xi''^* A'' \le c b_{\mathcal{N}}(g, m)$, which means $i'' : U \hookrightarrow W''$ is weakly bounded.

We argue by induction on the dimension of U. If $\dim(U) = 1$, then W'' is the normalization of W, g^*A and F'' are ample, and

$$\deg \xi''^* A'' = \deg \xi''^* \Big(\frac{l}{n} g^* A - \frac{1}{n} F''\Big) \le \frac{l}{n} \deg \xi^* A \le \frac{l}{n} b_{\mathcal{N}}(g, m).$$

Thus we may assume the claim is true in dimension one less.

Suppose $\dim(U) > 1$. We have the following two cases.

(1) If $\xi''(C) \not\subset \operatorname{Supp}(F'')$, then

$$\deg \xi''^*(nA'') = \deg \xi''^*(lg^*A - F) \le \deg \xi''^*(lg^*A) = \deg \xi^*(lA) \le l b_{\mathcal{N}}(g, m).$$

Let $c_0 := l/n$; then we have

$$\deg \xi''^* A'' \le c_0 b_{\mathcal{N}}(g, m).$$

(2) If $\xi''(C) \subset \operatorname{Supp}(F'')$, let W_i'' be the irreducible component of $\operatorname{Supp}(F)$ that contains $\xi''(C)$. Define $W_i = g(W_i'')$ and $U_i := U \cap W_i$. It is easy to see that $U_i \hookrightarrow W_i$ is naturally weakly bounded with respect to $A|_{W_i}$ by $b_{\mathcal{N}}(g, m)$. Also because $\dim(U_i) < \dim(U)$ and we assume the claim is true in lower dimensions, there exists $c_i > 0$ such that

$$\deg \xi''^* A'' \le c_i b_{\mathcal{N}}(g, m).$$

Because $\operatorname{Supp}(F) = \bigcup_{1 \le i \le k} W_i''$ has only finitely many components, let $c = \max\{c_i, 0 \le i \le k\}$. Then in both cases we have

$$\deg \xi''^* A'' \le c b_{\mathcal{N}}(g, m)$$

and $i'' : U \hookrightarrow W''$ is weakly bounded with respect to $\mathcal{O}_{X''}(A'')$.

Next we use the weak boundedness of $i' : U \hookrightarrow W''$ to show that $i' : U \hookrightarrow W'$ is weakly bounded. Because A'' is ample, there exist $d, r \in \mathbb{N}$ such that $dA'' \sim rg'^*A' + H''$, where H'' is an ample Cartier divisor. Let $\xi' := g' \circ \xi'' : C \to W'$. We have

$$\deg \xi'^*(rA') = \deg \xi''^*(rg'^*A') \le \deg \xi''^*(rg'^*A' + H'') = \deg \xi''^*(dA'') \le dc b_{\mathcal{N}}(g, m).$$

Thus

$$\deg \xi'^* A' \le \frac{dc}{r} b_{\mathcal{N}}(g, m),$$

and $i' : U \hookrightarrow W'$ is weakly bounded with respect to $\mathcal{O}_{X'}(A')$. □

Lemma 5.4. *Let T be a quasiprojective variety. Then we can decompose T into finitely many locally closed subsets, $T = \bigcup T_i$, such that each T_i is weakly bounded.*

Proof. By the definition of weakly bounded, if a variety U is weakly bounded, then any open subset $U^\circ \subset U$ is also weakly bounded. Therefore, we may replace T with a stratification and assume that T is smooth and projective; we only need to show that T has a weakly bounded open subset.

Fix an integer $n \geq 2$, and let A be a general very ample divisor on $\mathbb{P}^n_T$ such that $K_{\mathbb{P}^n_T/T} + A$ is also very ample. Then $\operatorname{Supp}(A)$ is smooth and dominates T and, by the generic smoothness theorem, there is a normal open subset $T_1 \subset T$ such that $\operatorname{Supp}(A_{T_1})$ is smooth over T_1, where $A_{T_1} := A|_{T_1}$.

Since $K_{\mathbb{P}^n_T/T} + A$ is ample and A is smooth, by the adjunction formula, $K_{A/T} = (K_{\mathbb{P}^n_T/T} + A)|_A$ is very ample, and we have that $A_{T_1} \to T_1$ is a family of canonically polarized smooth varieties. We may assume that T_1 is irreducible and every fiber of $A_{T_1} \to T_1$ has Hilbert polynomial $h(m) = \chi(\omega_{A_t}^{\otimes m})$.

Write $\mathcal{M}_h^\circ$ for the (Deligne–Mumford) stack of canonically polarized smooth varieties with Hilbert polynomial h and $\boldsymbol{M}_h^\circ$ for its coarse moduli space. It is easy to see that g maps A_{T_1} to $T_1 \in \mathcal{M}_h^\circ(T_1)$. Let $\psi : T_1 \to \boldsymbol{M}_h^\circ$ be the induced moduli map.

By [Kovács and Patakfalvi 2017, Corollary 6.20], there is a diagram

$$\begin{array}{ccccc} A' & \xleftarrow{g} & A'' & \xrightarrow{h} & A_{T_1} \\ \downarrow & & \downarrow & & \downarrow \\ T' & \longleftarrow & T'' & \longrightarrow & T_1 \end{array}$$

with Cartesian squares such that

- $T'' \to T_1$ is finite surjective, and
- $A' \to T'$ is a family of canonically polarized smooth varieties for which the induced moduli map $\psi' : T' \to \boldsymbol{M}_h^\circ$ is finite.

Since the diagram is Cartesian,

$$K_{A''/T''} = g^* K_{A'/T'} = h^* K_{A_{T_1}/T_1}.$$

Because h is finite and $K_{A_{T_1}/T_1}$ is ample, $K_{A''/T''}$ is ample and $T'' \to T'$ is quasifinite. It is easy to see that both

$$T'' \to T' \xrightarrow{\psi'} \boldsymbol{M}_h^\circ \quad \text{and} \quad T'' \to T_1 \xrightarrow{\psi} \boldsymbol{M}_h^\circ$$

give the moduli map $\psi'' : T'' \to \boldsymbol{M}_h^\circ$ induced by $A'' \to T''$; thus we have that $\psi : T_1 \to \boldsymbol{M}_h^\circ$ is quasifinite.

By [Kovács and Lieblich 2010, Lemma 6.2], the stack $\boldsymbol{M}_h^\circ$ is weakly bounded with respect to $\boldsymbol{M}_h$ and $\lambda \in \operatorname{Pic}(\boldsymbol{M}_h)$ by a function $b(g, d)$, where $\boldsymbol{M}_h$ is a compactification of $\boldsymbol{M}_h^\circ$ and λ is an ample line bundle according to [Kovács and Patakfalvi 2017]. Let $\hat{T}_1$ be a compactification of T_1 such that $\psi : T_1 \to \boldsymbol{M}_h^\circ$ extends to a morphism $\hat{T}_1 \to \boldsymbol{M}_h$. Let T_1^c be the Stein factorization of $\hat{T}_1 \to \boldsymbol{M}_h$ and denote the finite morphism by $\psi^c : T_1^c \to \boldsymbol{M}_h$.

Suppose $C^\circ \subseteq C$ is a (g, d)-curve. Let $C^\circ \to T_1$ be a morphism, and let $\xi : C \to T_1^c$ be its closure. Then $\psi^c \circ \xi : C \to \boldsymbol{M}_h$ is the closure of $C^\circ \to T_1 \xrightarrow{\psi} \boldsymbol{M}_h^\circ$. By the definition of weakly boundedness,

$$\deg(\psi^c \circ \xi)^* \lambda \leq b(g, d),$$

and hence T_1 is weakly bounded with respect to T_1^c and $\psi^* \lambda$. Because ψ is a finite morphism, $\psi^* \lambda$ is ample, and hence T_1 is weakly bounded. $\square$

Theorem 5.5 [Kovács and Lieblich 2010, Proposition 2.14]. *Let T be a quasicompact quasiseparated reduced $\mathbb{C}$-scheme, and let $\mathcal{U} \to T$ be a smooth morphism. Given a projective T-variety and a polarization over T, $(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(1))$, an open subvariety $\mathcal{M}^\circ \hookrightarrow \mathcal{M}$ over T, and a weak bound b, there exists a T-scheme of finite type $\mathcal{W}^b_{\mathcal{M}^\circ}$ and a morphism $\Theta : \mathcal{W}^b_{\mathcal{M}^\circ} \times \mathcal{U} \to \mathcal{M}^\circ$ such that, for every geometric point $t \in T$ and for every morphism $\xi : \mathcal{U}_t \to \mathcal{M}^\circ_t \subset \mathcal{M}_t$ that is weakly bounded by b, there exists a point $p \in \mathcal{W}^b_{\mathcal{M}^\circ_t}$ such that $\xi = \Theta|_{\{p\}\times \mathcal{U}_t}$.*

In particular, if $\mathcal{M}^\circ$ is weakly bounded and $\mathcal{M}$ is the compactification, by definition, every morphism $\xi : \mathcal{U}_t \to \mathcal{M}^\circ_t \subset \mathcal{M}_t$ is weakly bounded by a weak bound b; hence $\xi = \Theta|_{\{p\}\times \mathcal{U}_t}$ for a closed point $p \in \mathcal{W}^b_{\mathcal{M}^\circ_t}$.

6. Hilbert scheme and the moduli part

6.1. ***Parameter space.*** A class of polarized log Calabi–Yau pairs is a set $\mathscr{C}$ consisting of triples (X, Δ, H) such that

- (X, Δ) is a pair,
- H is an effective ample divisor,
- $K_X + \Delta \sim_{\mathbb{Q}} 0$, and
- $(X, \Delta + \epsilon H)$ is lc for a positive number $\epsilon \ll 1$.

A family of polarized log Calabi–Yau pairs over a normal base scheme S consists of a flat, proper morphism $f : X \to S$, a $\mathbb{Q}$-divisor Δ on X and a $\mathbb{Q}$-Cartier divisor H such that $K_{X/S} + \Delta$ is $\mathbb{Q}$-Cartier and all fibers (X_s, Δ_s, H_s) are polarized log Calabi–Yau pairs. We denote it by $(X, \Delta, H) \to S$.

Given a class of polarized log Calabi–Yau pairs $\mathscr{C}$, we define $\mathscr{MC}(S)$ to be the set of families of polarized log Calabi–Yau pairs over S, $(X, \Delta, H) \to S$, such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $(X_s, \Delta_s, H_s) \in \mathscr{C}$ for every closed point $s \in S$.

Suppose $\mathscr{C}$ is a class of n-dimensional polarized log Calabi–Yau pairs. We say $\mathscr{C}$ is bounded if the following two equivalent conditions hold:

- There exists a positive number C and a positive integer d such that, for every $(Y, D, H) \in \mathscr{C}$, dH is very ample without higher cohomology, $(dH)^n \leq C$, and $(dH)^{n-1}.\operatorname{red}(D) \leq C$.
- There is a flat projective morphism $\mathcal{Z} \to S$ over a scheme of finite type, two divisors $\mathcal{P}, \mathcal{L}$ on $\mathcal{Z}$ which are flat over S, and a positive integer d such that, for every $(Y, D, H) \in \mathscr{C}$, there is a closed point $s \in S$ and an isomorphism $\phi : (Y, dH) \to (\mathcal{Z}_s, \mathcal{L}_s)$ such that $\phi_* D \leq \mathcal{P}_s$.

If the first condition holds, then there is a (nonunique) natural choice of the scheme S in the second condition. By boundedness of the Chow variety, see [Kollár 1996, §1.3], we may assume that Y has a fixed Hilbert polynomial $H(t)$ with respect to dH. Let $\mathbb{P}$ be the projective space of dimension $H(1)-1$ with a fixed coordinate system. By the proof of [Kovács and Patakfalvi 2017, Proposition 6.11], because normality is an open condition, we may choose $\mathcal{H}'$ to be the locally closed subset of the Hilbert scheme of $\mathbb{P}$ which parametrizes all irreducible normal subvarieties of $\mathbb{P}$ with Hilbert polynomial $H(t)$, and we let $\mathscr{F} : \mathcal{X}_{\mathcal{H}'} \to \mathcal{H}'$ be the universal family.

Let Λ be a finite set, and let $p_i(t),\ i \in \Lambda$, be $|\Lambda|$ polynomials such that $\deg p_i(t) = \deg H(t) - 1$ for every i. Let

$$\mathcal{H}_i := \mathrm{Hilb}_{p_i(t)}(\mathcal{X}_{\mathcal{H}'}/\mathcal{H}')$$

be the locally closed subset of the relative Hilbert scheme which parametrizes closed pure dimensional subschemes $D_i \subset \mathcal{X}_{\mathcal{H}'}$ such that $D_i \to \mathcal{H}'$ is a flat family of varieties with Hilbert polynomial $p_i(t)$. Let $\mathcal{D}_i \to \mathcal{H}_i$ be its universal family. For simplicity of notation, we define $\mathcal{H} := \mathcal{H}_1 \times_{\mathcal{H}'} \cdots \times_{\mathcal{H}'} \mathcal{H}_{|\Lambda|}$ and

$$(\mathcal{X}_{\mathcal{H}}, \mathcal{D}_{\mathcal{H}}) := \Big(\mathcal{X}_{\mathcal{H}'} \times_{\mathcal{H}'} \mathcal{H}, \sum \mathcal{D}_i \times_{\mathcal{H}_i} \mathcal{H}\Big).$$

Remark 6.1. Let $\mathscr{C}$ be a bounded class of polarized log Calabi–Yau pairs. With the same notation as above, let (X, Δ) be a klt pair and L be a divisor on X, and suppose a general fiber of a contraction $f : (X, \Delta, L) \to Z$ is in $\mathscr{C}$; that is, there is an open subset $U \subset Z$ such that, for every closed point $u \in U$, $(X_u, \Delta|_{X_u}, L|_{X_u}) \in \mathscr{C}$.

Write $\Delta = \sum \Delta_i$ as the sum of irreducible components and define $\Delta_{i,u} := \Delta_i|_{X_u},\ \Delta_i := \Delta_i|_X$, for a closed point $u \in U$. Because the degree of $\mathrm{Supp}(\Delta_{i,u})$ is bounded from above, by boundedness of the Chow varieties, the Hilbert polynomial of $\Delta_{i,u}$ is in a finite set $\{p_i(t), i \in \Lambda\}$; see [Kollár 1996, §1.3]. Let $(\mathcal{X}_{\mathcal{H}}, \mathcal{D}_{\mathcal{H}}) \to \mathcal{H}$ be the family constructed as above. By the construction of $\mathcal{H}$, every closed point $u \in U$ corresponds to a closed point in $\mathcal{H}$, and there is a morphism $U \to \mathcal{H}$.

Notice that $\Delta_{i,u}$ may not be irreducible for every $u \in U$, and two irreducible components of Δ_u may be considered as two divisors or just one divisor, depending on the divisor Δ_i on X. That means, given two contractions $(X^i, \Delta^i) \to Z^i,\ i = 1, 2$, satisfying the given conditions, even if $(X^1_{u_1}, \Delta^1_{u_1}) \cong (X^2_{u_2}, \Delta^2_{u_2})$, u_1 and u_2 may corresponds to different points in $\mathcal{H}$.

Since dL is very ample without higher cohomology and $f : X \to Z$ is flat over U, we have that $f_*\mathcal{O}_X(dL)$ is locally free over U. Replacing U with an open subset, we may assume that $f_*\mathcal{O}_X(dL)$ is in fact free. Fixing a basis in the space of sections then gives a map $U \to \mathcal{H}'$, and $X_U \to U$ is isomorphic to the pullback of the universal family $\mathcal{X}_{\mathcal{H}'} \to \mathcal{H}'$. Similarly, each irreducible component Δ_i of Δ gives a map $U \to \mathcal{H}_i$. Hence there is a morphism $\phi : U \to \mathcal{H}$ such that $f : (X_U, \mathrm{Supp}(\Delta_U)) \to U$ is isomorphic to the pullback of $(\mathcal{X}_{\mathcal{H}}, \mathcal{D}_{\mathcal{H}}) \to \mathcal{H}$ by ϕ.

Suppose $\alpha = (\alpha_1, \dots, \alpha_k)$ is a vector of rational numbers and

$$\Delta_U = \alpha\, \mathrm{Supp}(\Delta_U) := \sum \alpha_i\, \mathrm{Supp}(\Delta_{i,U}).$$

By the construction of $\mathcal{D}_{\mathcal{H}}$, (X_U, Δ_U) is isomorphic to the pullback of $(\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}}) \to \mathcal{H}$ by ϕ. If there is a point $u \in U$ such that (X_u, Δ_u) is a log Calabi–Yau pair, then $(\mathcal{X}_{\phi(u)}, \alpha\mathcal{D}_{\phi(u)})$ is a log Calabi–Yau pair. If $\mathrm{coeff}(\Delta) \subset \mathcal{I}$ is a DCC set, then, by [Hacon et al. 2014, Theorem 1.5], $\alpha\mathcal{D}_{\mathcal{H}}$ is in a finite set and there are only finitely many $\alpha\mathcal{D}_{\mathcal{H}}$.

Moreover, by Lemma 7.4 in the first arXiv version of [Birkar 2023], after replacing $\mathcal{H}$ by a stratification of a locally closed subvariety, we may assume that $\mathcal{H}$ is smooth and $(\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}})$ is klt log Calabi–Yau over $\mathcal{H}$, and hence $(\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}}) \to \mathcal{H}$ is an lc-trivial fibration.

6.2. *Moduli part.* In this section, we deal with algebraic fibrations whose general fibers are log Calabi–Yau pairs. We claim that such a contraction naturally induces an lc-trivial fibration, and then any such fibration has a moduli $\boldsymbol{b}$-divisor by the canonical bundle formula.

Theorem 6.2. *Let (X, Δ) be a projective lc pair, and let $f : (X, \Delta) \to Z$ be a contraction to a projective normal $\mathbb{Q}$-factorial variety. Suppose that a general fiber (X_g, Δ_g) is a log Calabi–Yau pair. Assume there is a subpair (X', Δ'), a crepant birational morphism $g : (X', \Delta') \to (X, \Delta)$ and a divisor D on Z such that the morphism $h := f \circ g : X' \to Z$ is smooth over $Z \setminus D$ and* $\mathrm{Supp}(\Delta')$ *is simple normal crossing over $Z \setminus D$.*

Then there is a $\mathbb{Q}$-divisor Λ' on X' such that

- $(X'_\eta, \Lambda'_{X_\eta}) \cong (X'_\eta, \Delta'_{X_\eta})$, *where η is the generic point of Z,*
- $\mathrm{Supp}(\Lambda')$ *is log smooth over $Z \setminus D$, and*
- $(X', \Lambda') \to Z$ *is an lc-trivial fibration.*

Proof. Since (X_g, Δ_g) is a log Calabi–Yau pair, we have $K_{X'_\eta} + \Delta'_\eta \sim_{\mathbb{Q}} 0$, and hence there exists a vertical $\mathbb{Q}$-divisor B' such that $K_{X'} + \Delta' + B' \sim_{\mathbb{Q}} 0$.

Suppose $B' = R + G$, where $\mathrm{Supp}(R) \not\subset h^{-1}(\mathrm{Supp}(D))$ and $\mathrm{Supp}(G) \subset h^{-1}(\mathrm{Supp}(D))$. Because R is vertical, h is smooth over the generic point of $h(\mathrm{Supp}(R))$ and Z is $\mathbb{Q}$-factorial, $h(R)$ is a well-defined $\mathbb{Q}$-Cartier divisor on Z; denote it by R_Z. Also because h is smooth over $Z \setminus \mathrm{Supp}(D)$, there exists a $\mathbb{Q}$-divisor F_R supported on $h^{-1}(\mathrm{Supp}(D))$ such that $R + F_R = h^* R_Z$. Hence

$$K_{X'} + \Delta' + B' - (R + F_R) \sim_{\mathbb{Q},h} 0.$$

Let $\Lambda' := \Delta' + B' - (R + F_R)$; then $K_{X'} + \Lambda' \sim_{\mathbb{Q},h} 0$ and $\Lambda'_\eta = \Delta'_\eta$. Write $\Delta' = \Delta'_{\geq 0} - \Delta'_{\leq 0}$. Because $\Delta'_{\leq 0}$ is g-exceptional, it is easy to see that $(X', \Lambda') \to Z$ is an lc-trivial fibration. Because $\mathrm{Supp}(\Delta')$ is log smooth over $Z \setminus D$, $\mathrm{Supp}(F_R) \subset h^{-1}(D)$ and $\mathrm{Supp}(B' - R) \subset h^{-1}(D)$, we have that $\mathrm{Supp}(\Lambda')$ is log smooth over $Z \setminus D$. □

Proposition 6.3. *Let $f : (X, \Delta) \to Z$ be an lc-trivial fibration between normal projective varieties, $\rho : Z' \to Z$ a surjective morphism from a projective normal variety Z' and $f' : (X', \Delta') \to Z'$ the lc-trivial fibration induced by the normalization of the main component of the base change.*

$$\begin{array}{ccc} (X, \Delta) & \xleftarrow{\ \rho_X\ } & (X', \Delta') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Z & \xleftarrow{\ \rho\ } & Z' \end{array}$$

Let $\boldsymbol{M}$ and $\boldsymbol{M}'$ be the moduli $\boldsymbol{b}$-divisors of f and f'. Then the following hold:

(1) *If $\boldsymbol{M}$ descends on Z and $\boldsymbol{M}'$ descends on Z', then $\rho^* \boldsymbol{M}_Z = \boldsymbol{M}'_{Z'}$.*

(2) *If ρ is finite, then $\rho^* \boldsymbol{M}_Z = \boldsymbol{M}'_{Z'}$. In particular, $\boldsymbol{M}$ descends on Z if and only if $\boldsymbol{M}'$ descends on Z'.*

Proof. Result (1) is [Ambro 2005, Proposition 3.1].

For (2), let $g' : W' \to Z'$ and $g : W \to Z$ be birational morphisms such that $\boldsymbol{M}'$ descends on W' and $\boldsymbol{M}$ descends on W and $\rho : Z' \to Z$ lifts to a morphism $\rho_W : W' \to W$. Then $\rho_W^* \boldsymbol{M}_W = \boldsymbol{M}'_{W'}$ by (1). Because ρ is finite and any g-exceptional divisor is only dominated by g'-exceptional divisors, the pushforward of $\rho_W^* \boldsymbol{M}_W = \boldsymbol{M}'_{W'}$ to Z' gives $\rho^* \boldsymbol{M}_Z = \boldsymbol{M}'_{Z'}$. □

Theorem 6.4. *Let (X, Δ) be an lc pair, and let $f : (X, \Delta) \to Z$ be an lc-trivial fibration to a smooth projective variety Z. Suppose $X' \to X$ is a log resolution of (X, Δ) and (X', Δ') is a subpair such that $g : (X', \Delta') \to (X, \Delta)$ is a crepant birational morphism. Suppose $D \subset Z$ is a smooth divisor on Z such that $(X', \mathrm{Supp}(\Delta'))$ is log smooth over the generic point η_D of D. Let Y be the normalization of the irreducible component of $f^{-1}(D)$ that dominates D, and let Δ_Y be the $\mathbb{Q}$-divisor on Y such that*

$$K_Y + \Delta_Y = (K_X + \Delta + f^* D)|_Y.$$

Let $\boldsymbol{M}_Z$ denote the moduli part of $(X, \Delta) \to Z$. Suppose there is a smooth divisor B on Z such that $B + D$ is a reduced simple normal crossing divisor and the morphism $h : X' \to Z$ and Δ', B satisfy the standard normal crossing assumptions. Then $(Y, \Delta_Y) \to D$ is an lc-trivial fibration and its moduli ***b****-divisor $\boldsymbol{N}$ is equal to the restriction of $\boldsymbol{M}$ up to $\mathbb{Q}$-linear equivalence.*

Proof. By assumption, h is smooth over $Z \setminus B$, D is smooth and the singular locus of $h^{-1}(D)$ is contained in $h^{-1}(B) \cap h^{-1}(D)$. After blowing up a sequence of smooth subvarieties whose centers are contained in the singular locus of $h^{-1}(D)$, we may assume that $(X', \mathrm{Supp}(\Delta' + h^*(B + D)))$ is log smooth. It is easy to see that the morphism $h : X' \to Z$ and Δ', B also satisfy the standard normal crossing assumption.

Let E' be the irreducible component of $h^* D$ that dominates D, and let $\Delta'_{E'}$ be the $\mathbb{Q}$-divisor on E' such that

$$K_{E'} + \Delta'_{E'} = (K_{X'} + \Delta' + h^* D)|_{E'}.$$

It is easy to see that the generic fiber of $(E', \Delta'_{E'}) \to D$ is crepant birationally equivalent to the generic fiber of $(Y, \Delta_Y) \to D$, which means the two lc-trivial fibrations have the same moduli part. Then we only need to prove the result for $(E', \Delta'_{E'}) \to Z$.

By the canonical bundle formula, there is a divisor B_Z supported on B such that

$$K_{X'} + \Delta' + h^* D \sim_{\mathbb{Q}} h^*(K_Z + B_Z + \boldsymbol{M}_Z + D) \tag{6-1}$$

and

$$K_X + \Delta + f^* D \sim_{\mathbb{Q}} f^*(K_Z + B_Z + \boldsymbol{M}_Z + D). \tag{6-2}$$

Because $B + D$ is reduced and $(Z, B + D)$ is log smooth, $(Z, B + D)$ is an lc pair. By the canonical bundle formula,

$$K_{X'} + \Delta' + h^* D + h^*(B - B_Z) \sim_{\mathbb{Q}} h^*(K_Z + B + D + \boldsymbol{M}_Z).$$

Because $h : X' \to Z$ and Δ', B satisfy the standard normal crossing assumptions, the moduli part $\boldsymbol{M}$ descends on Z and $(Z, B + D + \boldsymbol{M}_Z)$ is generalized lc. Thus, by [Ambro 2004, Theorem 3.1],

$$(X', \Delta' + h^* D + h^*(B - B_Z))$$

is sub-lc. Because Z is smooth and $B-B_Z$ is effective and $\mathbb{Q}$-Cartier, after replacing Δ' by $\Delta'+h^*(B-B_Z)$, Δ by $\Delta+f^*(B-B_Z)$ and B_Z by $B_Z+(B-B_Z)=B$, we can assume that $B=B_Z$ and every irreducible component of B is dominated by an irreducible component of Δ' which has coefficient 1, and (X, Δ) is still a pair. Since $K_{X'}+\Delta'+h^*D=g^*(K_X+\Delta+f^*D)$, we have that $(X, \Delta+f^*D)$ is lc.

Let $g(E')=E$, and suppose $h^*D=E'+E'_1$ and $f^*D=E+E_1$. Recall that Y is the normalization of E. Restricting (6-1) to E' and (6-2) to E, by the adjunction formula, there are a $\mathbb{Q}$-divisor $\Delta'_{E'}$ and an effective $\mathbb{Q}$-divisor Δ_Y such that

$$\begin{aligned}(K_{X'}+\Delta'+h^*D)|_{E'} &\sim_{\mathbb{Q}} K_{E'}+\Delta'_{E'}\\ &\sim_{\mathbb{Q}} h^*_{E'}(K_D+B|_D+\boldsymbol{M}_Z|_D),\\ (K_X+\Delta+f^*D)|_Y &\sim_{\mathbb{Q}} K_Y+\Delta_Y\\ &\sim_{\mathbb{Q}} f^*_E(K_D+B|_D+\boldsymbol{M}_Z|_D).\end{aligned}$$

It follows that $\Delta'_{E'}=\Delta'|_{E'}+E'_1|_{E'}$, $(E', \Delta'_{E'})$ is sub-lc, Δ_Y is effective and $K_{E'}+\Delta'_{E'}\sim_{\mathbb{Q}} g^*_{E'}(K_Y+\Delta_Y)$, where $g_{E'}: E'\to Y$ is the birational morphism induced by $g|_{E'}: E'\to E$. It follows that $\Delta'_{E',\le 0}$ is $g_{E'}$-exceptional, and hence $(E', \Delta'_{E'})\to D$ is an lc-trivial fibration.

$$\begin{array}{ccc} (E', \Delta'_{E'}) & \hookrightarrow & (X', \Delta') \\ \downarrow{\scriptstyle g'_{E'}} & & \downarrow{\scriptstyle g} \\ (Y, \Delta_Y) & \hookrightarrow & (X, \Delta) \\ \downarrow{\scriptstyle f_E} & & \downarrow{\scriptstyle f} \\ D & \hookrightarrow & Z \end{array} \qquad (h : (X', \Delta') \to Z)$$

By the canonical bundle formula for $(E', \Delta'_{E'})\to D$, we have

$$K_{E'}+\Delta'_{E'}\sim_{\mathbb{Q}} h^*_{E'}(K_D+B_D+\boldsymbol{N}_D). \tag{6-3}$$

To prove $\boldsymbol{N}_D\sim_{\mathbb{Q}}\boldsymbol{M}_Z|_D$, we only need to prove that $B_D=B|_D$.

Since the morphism $X'\to Z$ and Δ', B satisfy the standard normal crossing assumption, $\boldsymbol{M}$ descends on Z. Similarly, because $B+D$ is snc, $(D, \operatorname{Supp}(B|_D))$ is log smooth and $(E', \operatorname{Supp}(\Delta'_{E'}))$ is log smooth over $D\setminus B\cap D$, we have that $\boldsymbol{N}_D$ descends on D. For the same reason, the morphism $E'\to D$ and $\Delta'_{E'}$, $B|_D$ satisfy the standard normal crossing assumption. By the construction of the boundary divisor, B_D is the unique smallest $\mathbb{Q}$-divisor supported on $B|_D$ such that

$$\Delta'_{E',v}+h^*_{E'}(B|_D-B_D)\le \operatorname{red}(h^*_{E'}(B|_D)),$$

where $\Delta'_{E',v}$ is the vertical part of $\Delta'_{E'}$. Because every irreducible component of B is dominated by an irreducible component of Δ' which has coefficient 1 and every irreducible component of $B|_D$ is dominated by an irreducible component of $\Delta'_{E'}=\Delta'|_{E'}+E'_1|_{E'}$ which has coefficient 1, we have $B|_D=B_D$ and the result follows. $\square$

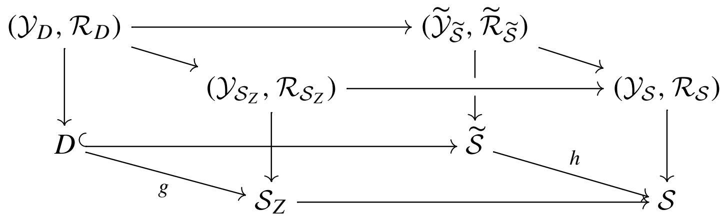
Theorem 6.5. *Let $\mathcal{S}$ be a normal projective variety and $\mathfrak{F} : (\mathcal{Y}_{\mathcal{S}}, \mathcal{R}_{\mathcal{S}}) \to \mathcal{S}$ be an lc-trivial fibration such that the corresponding moduli $\boldsymbol{b}$-divisor $\boldsymbol{\mathcal{M}}$ descends on $\mathcal{S}$. Suppose there exists an open subset $\mathcal{H} \hookrightarrow \mathcal{S}$ such that $(\mathcal{Y}_{\mathcal{S}}, \mathrm{Supp}(\mathcal{R}_{\mathcal{S}}))$ is log smooth over $\mathcal{H}$. Let Z be a projective normal variety and $\phi : Z \to \mathcal{S}$ be a morphism which maps the generic point of Z into $\mathcal{H}$. Assume $(X, \Delta) \to Z$ is an lc-trivial fibration whose generic fiber is crepant birationally equivalent to the generic fiber of $(\mathcal{Y}_{\mathcal{S}}, \mathcal{R}_{\mathcal{S}}) \times_{\mathcal{S}} Z \to Z$. Let $\boldsymbol{M}$ be the moduli $\boldsymbol{b}$-divisor of f. If $\boldsymbol{M}$ descends on Z, then we have*

$$\boldsymbol{M}_Z = \phi^* \boldsymbol{\mathcal{M}}_{\mathcal{S}}.$$

Proof. Let $(\mathcal{Y}_{\phi(Z)}, \mathcal{R}_{\phi(Z)}) \to \phi(Z)$ be the contraction induced by the restriction of $(\mathcal{Y}_{\mathcal{S}}, \mathcal{R}_{\mathcal{S}}) \to \mathcal{S}$ on $\phi(Z)$. Because $(\mathcal{Y}_{\mathcal{S}}, \mathrm{Supp}(\mathcal{R}_{\mathcal{S}}))$ is log smooth over $\mathcal{H}$ and the generic point of $\phi(Z)$ is in $\mathcal{H}$, we have that $(\mathcal{Y}_{\phi(Z)}, \mathcal{R}_{\phi(Z)}) \to \phi(Z)$ is an lc-trivial fibration over an open subset of $\phi(Z)$. We denote the corresponding moduli $\boldsymbol{b}$-divisor by $\boldsymbol{N}$. Let $\mathcal{S}_Z \to \phi(Z)$ be a birational morphism such that $\boldsymbol{N}$ descends on $\mathcal{S}_Z$. We have the following two cases:

Case 1: $\mathcal{S}_Z = \mathcal{S}$. Because ϕ is surjective, $\boldsymbol{M}$ descends on Z and $\boldsymbol{\mathcal{M}}$ descends on $\mathcal{S}$, by Proposition 6.3, we have $\phi^* \boldsymbol{\mathcal{M}}_{\mathcal{S}} \sim_{\mathbb{Q}} \boldsymbol{M}_Z$.

Case 2: $\mathcal{S}_Z$ is a subvariety of $\mathcal{S}$ of codimension ≥ 1. Consider the diagram



where

- $\widetilde{\mathcal{S}} \to \mathcal{S}$ is a log resolution of $(\mathcal{S}, \mathcal{S} \setminus \mathcal{H})$,
- D is a divisor on $\widetilde{\mathcal{S}}$ that dominates $\mathcal{S}_Z$,
- $(\widetilde{\mathcal{S}}, D + h^{-1}(\mathcal{S} \setminus \mathcal{H}))$ is log smooth, and
- $(\mathcal{Y}_{\mathcal{S}_Z}, \mathcal{R}_{\mathcal{S}_Z}) \to \mathcal{S}_Z$, $(\mathcal{Y}_D, \mathcal{R}_D) \to D$ and $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{R}}) \to \widetilde{\mathcal{S}}$ are induced by the pullback of $(\mathcal{Y}, \mathcal{R}) \to \mathcal{S}$.

It is easy to see that $(\widetilde{\mathcal{Y}}, \mathrm{Supp}(\widetilde{\mathcal{R}})) \to \widetilde{\mathcal{S}}$ is log smooth over $\widetilde{\mathcal{S}} \setminus h^{-1}(\mathcal{S} \setminus \mathcal{H})$.

After replacing Z by a higher birational model and $(X, \Delta) \to Z$ by the corresponding pullback, we may assume that $Z \to \mathcal{S}_Z$ is surjective. Because the generic fiber of $(X, \Delta) \to Z$ is crepant birationally equivalent to the generic fiber of the pullback of $(\mathcal{Y}_{\mathcal{S}}, \mathcal{R}_{\mathcal{S}}) \to \mathcal{S}$ via ϕ, it is also crepant birationally equivalent to the generic fiber of the pullback of $(\mathcal{Y}_{\mathcal{S}_Z}, \mathcal{R}_{\mathcal{S}_Z}) \to \mathcal{S}_Z$. Then, by Proposition 6.3, we have $\boldsymbol{M}_Z = \phi^* \boldsymbol{N}_{\mathcal{S}_Z}$.

Because the generic point of $\phi(Z)$ is in $\mathcal{H}$, we have $D \not\subset h^{-1}(\mathcal{S} \setminus \mathcal{H})$. By Theorem 6.4, the induced morphism $(\mathcal{Y}_D, \mathcal{R}_D) \to D$ is an lc-trivial fibration, and the corresponding moduli divisor $\boldsymbol{M}_D$ is equal to $\boldsymbol{\mathcal{M}}_{\widetilde{\mathcal{S}}}|_D = (h^* \boldsymbol{\mathcal{M}}_{\mathcal{S}})|_D$. By Proposition 6.3, $\boldsymbol{M}_D = g^* \boldsymbol{M}_{\mathcal{S}_Z}$, and hence $\boldsymbol{M}_Z = \phi^* \boldsymbol{M}_{\mathcal{S}_Z} = \phi^* \boldsymbol{\mathcal{M}}_{\mathcal{S}}$. □

Theorem 6.6 [Ambro 2005, Theorem 3.3]. *Let $f : (X, \Delta) \to S$ be an lc-trivial fibration over a variety S such that the geometric generic fiber $X_{\bar{\eta}}$ is a projective variety and $\Delta_{\bar{\eta}}$ is effective. Then there exists a diagram*

$$\begin{array}{ccccccc} (X, \Delta) & & & & (X^!, \Delta^!) & & \\ \downarrow f & & & & \downarrow f^! & & \\ S & \xleftarrow{\ \tau\ } & \overline{S} & \xrightarrow{\ \rho\ } & S^! & \xrightarrow{\ \pi\ } & S^* \end{array} \qquad i : S^! \dashrightarrow S, \quad \Phi : S \dashrightarrow S^*$$

such that

(1) *$f^! : (X^!, \Delta^!) \to S^!$ is an lc-trivial fibration,*

(2) *τ and π are generically finite and surjective morphisms and ρ is surjective,*

(3) *there exists a nonempty open subset $U \subset \overline{S}$ and an isomorphism*

$$\begin{array}{ccc} (X, \Delta) \times_S U & \xrightarrow{\ \cong\ } & (X^!, \Delta^!) \times_{S^!} U \\ & \searrow \quad U \quad \swarrow & \end{array}$$

(4) *$\Phi : S \dashrightarrow S^*$ is an extension of the period map defined in* [Ambro 2005, Section 2], *and*

(5) *$i : S^! \dashrightarrow S$ is a rational map such that the generic fiber of $f^!$ is equal to the pullback of f via i.*

Furthermore, if S is proper, then one can choose $\overline{S}$, $S^!$ and S^* to be proper. Let $\boldsymbol{M}$ and $\boldsymbol{M}^!$ be the corresponding moduli $\boldsymbol{b}$-divisors of f and $f^!$. Then we have

(6) *$\boldsymbol{M}^!$ is $\boldsymbol{b}$-nef and big, and*

(7) *if $\boldsymbol{M}$ descends on S and $\boldsymbol{M}^!$ descends on $S^!$, then $\tau^* \boldsymbol{M}_S = \rho^* \boldsymbol{M}^!_{S^!}$.*

Although it is not written in [Ambro 2005], (4) and (5) are implied by its proof.

Theorem 6.7. *Let $(\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}}) \to \mathcal{H}$ be the lc-trivial fibration defined in Remark 6.1. Then, after passing to a stratification of $\mathcal{H}$ and replacing $(\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}}) \to \mathcal{H}$ by the corresponding pullback, we have the diagram*

$$\begin{array}{ccccccc} (\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}}) & & & & (\mathcal{X}^!_{\mathcal{H}^!}, \alpha\mathcal{D}^!_{\mathcal{H}^!}) & & \\ \downarrow \mathcal{F} & & & & \downarrow \mathcal{F}^! & & \\ \mathcal{H} & \xleftarrow{\ \tau\ } & \overline{\mathcal{H}} & \xrightarrow{\ \rho\ } & \mathcal{H}^! & \xrightarrow{\ \pi\ } & \mathcal{H}^* \end{array} \qquad i : \mathcal{H}^! \dashrightarrow \mathcal{H}, \quad \Phi : \mathcal{H} \dashrightarrow \mathcal{H}^*$$

where

- *τ is finite,*
- *π is étale,*
- *Φ is a morphism on $\mathcal{H}$,*
- *$\mathcal{H}^*$ is weakly bounded and smooth,*
- *$(\mathcal{X}_{\mathcal{H}}, \mathcal{D}_{\mathcal{H}}) \times_{\mathcal{H}} \overline{\mathcal{H}} \cong (\mathcal{X}^!_{\mathcal{H}^!}, \mathcal{D}^!_{\mathcal{H}^!}) \times_{\mathcal{H}^!} \overline{\mathcal{H}}$, and*
- *$(\mathcal{X}_{\mathcal{H}}, \mathcal{D}_{\mathcal{H}}) \to \mathcal{H}$ and $(\mathcal{X}^!_{\mathcal{H}^!}, \mathcal{D}^!_{\mathcal{H}^!}) \to \mathcal{H}^!$ have fiberwise log resolutions.*

Furthermore, there exist a smooth compactification $\mathcal{H}^* \hookrightarrow \mathcal{S}^*$ *and a positive integer* l *such that if* $f : (X, \Delta) \to Z$ *is an lc trivial fibration, where*

- Z *is smooth and projective,*
- *there is a rational map* $\phi : Z \dashrightarrow \mathcal{H}$, *and*
- *the generic fiber of* f *is isomorphic to the generic fiber of the pullback of* $(\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}}) \to \mathcal{H}$ *by* ϕ,

then there exists a $\boldsymbol{b}$*-divisor* $\boldsymbol{M}^{\text{fix}}$ *on birational models of* Z *such that*

- $\boldsymbol{M}^{\text{fix}}$ *is effective,*
- $\boldsymbol{M}^{\text{fix}}_{Z'} \sim_{\mathbb{Q}} \boldsymbol{M}_{Z'}$ *for every birational map* $Z' \to Z$,
- $l\boldsymbol{M}^{\text{fix}}$ *is* $\boldsymbol{b}$*-Cartier, and*
- *if* $\Phi \circ \phi$ *extends to a morphism* $Z \to \mathcal{S}^*$, *then* $\operatorname{Supp}(\boldsymbol{M}^{\text{fix}}_Z) \supset Z \setminus U$, *where* $U = (\Phi \circ \phi)^{-1}\mathcal{H}^*$.

Proof. Step 1: We construct the stratification of $\mathcal{H}$, define $\overline{\mathcal{H}}$ and $\mathcal{H}^*$, define the lc-trivial fibration $\mathcal{F}^! : (\mathcal{X}^!_{\mathcal{H}^!}, \alpha\mathcal{D}^!_{\mathcal{H}^!}) \to \mathcal{H}^!$ and construct the diagram satisfying the requirements.

Because $\alpha\mathcal{D}_{\mathcal{H}}$ is effective, by Theorem 6.6, we have the following diagram:

$$\begin{array}{ccccccc} (\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}}) & & & & (\mathcal{X}^!_{\mathcal{H}^!}, \alpha\mathcal{D}^!_{\mathcal{H}^!}) & & \\ \downarrow \mathcal{F} & & \overset{i}{\dashleftarrow\dashrightarrow} & & \downarrow \mathcal{F}^! & & \\ \mathcal{H} & \xleftarrow{\tau} & \overline{\mathcal{H}} & \xrightarrow{\rho} & \mathcal{H}^! & \xrightarrow{\pi} & \mathcal{H}^* \\ & & & \underset{\Phi}{\dashrightarrow} & & & \end{array}$$

We replace $\mathcal{H}$ by an open subset such that

- $\mathcal{F}$ has a fiberwise log resolution.

Then we replace $\mathcal{H}^*$ by an open subset, and $\mathcal{H}^!$ and $\overline{\mathcal{H}}$ by the corresponding preimages such that

- π is étale,
- $\mathcal{H}^*$ is weakly bounded and smooth, and
- $(\mathcal{X}^!_{\mathcal{H}^!}, \mathcal{D}^!_{\mathcal{H}^!}) \to \mathcal{H}^!$ has a fiberwise log resolution.

Next we replace $\mathcal{H}$ by an open subset and $\overline{\mathcal{H}}$ by the corresponding preimage such that

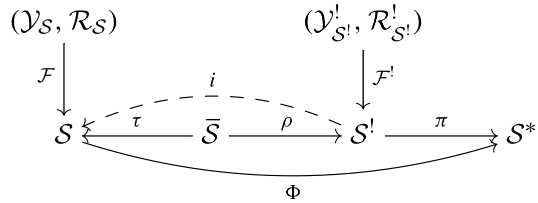
- τ is finite,
- Φ is a morphism on $\mathcal{H}$,
- $(\mathcal{X}_{\mathcal{H}}, \mathcal{D}_{\mathcal{H}}) \times_{\mathcal{H}} \overline{\mathcal{H}} \cong (\mathcal{X}^!_{\mathcal{H}^!}, \mathcal{D}^!_{\mathcal{H}^!}) \times_{\mathcal{H}^!} \overline{\mathcal{H}}$, and
- $(\mathcal{X}_{\mathcal{H}}, \mathcal{D}_{\mathcal{H}}) \to \mathcal{H}$ has a fiberwise log resolution.

Then we repeat this construction with the complement of $\mathcal{H}$. By Noetherian induction, we have a stratification of $\mathcal{H}$ satisfying the properties.

Step 2: We construct smooth compactifications $\mathcal{H} \hookrightarrow \mathcal{S}$, $\mathcal{H}^! \hookrightarrow \mathcal{S}^!$, and $\mathcal{H}^* \hookrightarrow \mathcal{S}^*$, a $\mathbb{Q}$-divisor $\boldsymbol{\mathcal{M}}^{\mathrm{fix}}_{\mathcal{S}}$ on $\mathcal{S}$, and a $\mathbb{Q}$-divisor $\boldsymbol{\mathcal{M}}^{!\mathrm{fix}}_{\mathcal{S}^!}$ on $\mathcal{S}^!$ such that

- $\boldsymbol{\mathcal{M}}^{\mathrm{fix}}_{\mathcal{S}} \sim_{\mathbb{Q}} \boldsymbol{\mathcal{M}}_{\mathcal{S}}$, where $\boldsymbol{\mathcal{M}}$ is the moduli $\boldsymbol{b}$-divisor of $\mathcal{F}$,
- $\boldsymbol{\mathcal{M}}^{!\mathrm{fix}}_{\mathcal{S}^!} \sim_{\mathbb{Q}} \boldsymbol{\mathcal{M}}^!_{\mathcal{S}^!}$, where $\boldsymbol{\mathcal{M}}^!$ is the moduli $\boldsymbol{b}$-divisor of $\mathcal{F}^!$,
- $\mathrm{Supp}(\boldsymbol{\mathcal{M}}^{!\mathrm{fix}}_{\mathcal{S}^!}) \supset \pi^{-1}(\mathcal{S}^* \setminus \mathcal{H}^*)$, and
- $\tau(\mathrm{Supp}(\rho^* \boldsymbol{\mathcal{M}}^{!\mathrm{fix}}_{\mathcal{S}^!})) \subset \mathrm{Supp}(\boldsymbol{\mathcal{M}}^{\mathrm{fix}}_{\mathcal{S}})$.

Let $(\mathcal{Y}_{\mathcal{H}}, \mathcal{R}_{\mathcal{H}}) \to (\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}})$ and $(\mathcal{Y}^!_{\mathcal{H}^!}, \mathcal{R}^!_{\mathcal{H}^!}) \to (\mathcal{X}^!_{\mathcal{H}^!}, \alpha\mathcal{D}^!_{\mathcal{H}^!})$ be crepant birational morphisms which are fiberwise log resolutions of $\mathcal{F}$ and $\mathcal{F}^!$. After taking smooth compactifications of the bases $\mathcal{H}$, $\overline{\mathcal{H}}$, $\mathcal{H}^!$ and $\mathcal{H}^*$, and choosing extensions of the fibrations, we have the following diagram:



Recall that $\mathcal{H}^*$ is weakly bounded. By Lemma 5.3, we may assume that $\mathcal{H}^* \hookrightarrow \mathcal{S}^*$ is weakly bounded with respect to an ample divisor H on $\mathcal{S}^*$.

Furthermore, by choosing the compactification appropriately, we may assume that the moduli part $\boldsymbol{\mathcal{M}}^!$ of $\mathcal{F}^!$ descends on $\mathcal{S}^!$ and the moduli part $\boldsymbol{\mathcal{M}}$ of $\mathcal{F}$ descends on $\mathcal{S}$. By Theorem 6.6, we have $\tau^* \boldsymbol{\mathcal{M}}_{\mathcal{S}} = \rho^* \boldsymbol{\mathcal{M}}^!_{\mathcal{S}^!}$.

Because $\boldsymbol{\mathcal{M}}^!_{\mathcal{S}^!}$ is big, we can fix a section of $\boldsymbol{\mathcal{M}}^{!\mathrm{fix}}_{\mathcal{S}^!} \in |\boldsymbol{\mathcal{M}}^!_{\mathcal{S}^!}|_{\mathbb{Q}}$ such that $\mathrm{Supp}(\boldsymbol{\mathcal{M}}^{!\mathrm{fix}}_{\mathcal{S}^!}) \supset \pi^{-1}(\mathcal{S}^* \setminus \mathcal{H}^*)$. Because $\tau^* \boldsymbol{\mathcal{M}}_{\mathcal{S}} = \rho^* \boldsymbol{\mathcal{M}}^!_{\mathcal{S}^!}$, we can choose $\boldsymbol{\mathcal{M}}^{\mathrm{fix}}_{\mathcal{S}}$ such that $\tau(\mathrm{Supp}(\rho^* \boldsymbol{\mathcal{M}}^{!\mathrm{fix}}_{\mathcal{S}^!})) \subset \mathrm{Supp}(\boldsymbol{\mathcal{M}}^{\mathrm{fix}}_{\mathcal{S}})$.

Step 3: We show that, to construct $\boldsymbol{M}^{\mathrm{fix}}$ satisfying the requirements, we are free to replace Z by a higher birational model.

Let $h : Z' \to Z$ be a birational morphism such that $\boldsymbol{M}$ descends on Z'. Suppose there exists a $\boldsymbol{b}$-divisor $\boldsymbol{M}^{\mathrm{fix}}$ satisfying the requirements. Because Z is smooth, $\boldsymbol{M}^{\mathrm{fix}}_Z$ is $\mathbb{Q}$-Cartier. Note $\boldsymbol{M}^{\mathrm{fix}}_{Z'} \sim_{\mathbb{Q}} \boldsymbol{M}_{Z'}$ is nef. By the negativity lemma, $\boldsymbol{M}^{\mathrm{fix}}_{Z'} \leq f^* \boldsymbol{M}^{\mathrm{fix}}_Z$, and we have

$$\mathrm{Supp}(\boldsymbol{M}^{\mathrm{fix}}_{Z'}) \subset f^{-1}\,\mathrm{Supp}(\boldsymbol{M}^{\mathrm{fix}}_Z).$$

Then $\mathrm{Supp}(\boldsymbol{M}^{\mathrm{fix}}_{Z'}) \supset Z' \setminus h^{-1}(U)$ implies that $\mathrm{Supp}(\boldsymbol{M}^{\mathrm{fix}}_Z) \supset Z \setminus U$, so we can replace Z by a higher birational model such that $\boldsymbol{M}$ descends on Z.

Step 4: We construct $\boldsymbol{M}^{\mathrm{fix}}$ and finish the proof.

We have the following two cases:

Case 1: The generic point of $\phi(Z)$ is contained in $\mathrm{Supp}(\boldsymbol{\mathcal{M}}^{\mathrm{fix}}_{\mathcal{S}})$. We stratify $\mathcal{S}$ further to the disjoint union of the irreducible components of $\mathrm{Supp}(\boldsymbol{\mathcal{M}}^{\mathrm{fix}}_{\mathcal{S}})$ and its complement, then replace $(\mathcal{X}, \alpha\mathcal{D}_{\mathcal{H}}) \to \mathcal{S}$ by its restriction and repeat this process. By Noetherian reduction, this will stop.

Case 2: The generic point of $\phi(Z)$ is not contained in $\mathrm{Supp}(\mathcal{M}_{\mathcal{S}}^{\mathrm{fix}})$. Because $(\mathcal{Y}_{\mathcal{S}}, \mathrm{Supp}(\mathcal{R}_{\mathcal{S}}))$ is log smooth over $\mathcal{H}$, the generic fiber of $(X, \Delta) \to Z$ is crepant birationally equivalent to the generic fiber of the pullback of $(\mathcal{Y}_{\mathcal{S}}, \mathcal{R}_{\mathcal{S}}) \to \mathcal{S}$ via ϕ, and the generic point of $\phi(Z)$ is contained in $\mathcal{H}$, by Theorem 6.5, we have $\phi^*\mathcal{M}_{\mathcal{S}} \sim_{\mathbb{Q}} \boldsymbol{M}_Z$. We define the $\boldsymbol{b}$-divisor $\boldsymbol{M}^{\mathrm{fix}}$ by

- $\boldsymbol{M}_Z^{\mathrm{fix}} := \phi^*\mathcal{M}_{\mathcal{S}}^{\mathrm{fix}}$, and
- $\boldsymbol{M}_{Z'}^{\mathrm{fix}} = h^*\boldsymbol{M}_Z^{\mathrm{fix}}$ for any birational morphism $h : Z' \to Z$.

Suppose $l\mathcal{M}_{\mathcal{S}}^{\mathrm{fix}}$ is Cartier and $l\boldsymbol{M}^{\mathrm{fix}}$ is $\boldsymbol{b}$-Cartier.

Because $\mathrm{Supp}(\mathcal{M}_{\mathcal{S}^!}^{!\mathrm{fix}}) \supset \pi^{-1}(\mathcal{S}^* \setminus \mathcal{H}^*)$, $\tau(\mathrm{Supp}(\rho^*\mathcal{M}_{\mathcal{S}^!}^{!\mathrm{fix}})) \subset \mathrm{Supp}(\mathcal{M}_{\mathcal{S}}^{\mathrm{fix}})$, and $\pi \circ \rho = \Phi \circ \tau$, we have $\mathrm{Supp}(\mathcal{M}_{\mathcal{S}}^{\mathrm{fix}}) \supset \tau(\mathrm{Supp}(\rho^*\mathcal{M}_{\mathcal{S}^!}^{!\mathrm{fix}})) \supset \Phi^{-1}(\mathcal{S}^* \setminus \mathcal{H}^*)$, and thus

$$\Phi \circ \phi(Z \setminus \mathrm{Supp}(\phi^*\mathcal{M}_{\mathcal{S}}^{\mathrm{fix}})) \subset \mathcal{H}^*.$$

Also because $\phi^*\mathcal{M}_{\mathcal{S}}^{\mathrm{fix}} \sim_{\mathbb{Q}} \boldsymbol{M}_Z^{\mathrm{fix}}$ and $(\Phi \circ \phi)^{-1}(\mathcal{H}^*) = U$, we have $\mathrm{Supp}(\boldsymbol{M}_Z^{\mathrm{fix}}) \supset Z \setminus U$. □

Suppose there is a family of bases $\mathcal{U} \to T$ of log Calabi–Yau fibrations whose fibers are parametrized by the Hilbert scheme defined in Remark 6.1. That is, every fiber $\mathcal{U}_t$ is the base of a log Calabi–Yau fibration whose fibers belong to the moduli defined in Remark 6.1. Then, for a closed point $t \in T$, we have a moduli map $\phi : \mathcal{U}_t \to \mathcal{H}$. Let $\phi^* : \mathcal{U}_t \to \mathcal{H}^*$ be the composition of $\phi : \mathcal{U}_t \to \mathcal{H}$ with $\Phi : \mathcal{H} \to \mathcal{H}^*$. We define $\overline{\mathcal{U}}_t := \mathcal{U}_t \times_{\mathcal{H}^*} \mathcal{H}^!$ (possibly not connected). Because $\overline{\mathcal{H}} \times_{\mathcal{H}^!} (\mathcal{X}_{\mathcal{H}^!}^!, \alpha\mathcal{D}_{\mathcal{H}^!}^!) \cong \overline{\mathcal{H}} \times_{\mathcal{H}} (\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}})$, there exists a finite cover $V \to \overline{\mathcal{U}}_t$ such that $V \times_{\mathcal{H}^!} (\mathcal{X}_{\mathcal{H}^!}^!, \alpha\mathcal{D}_{\mathcal{H}^!}^!) \cong V \times_{\mathcal{H}} (\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}})$. The next theorem says: if there exists a morphism $\Theta : \mathcal{U} \to \mathcal{H}^*$ such that $\phi^* = \Theta|_{\mathcal{U}_t}$, then we can find a relative compactification of $\mathcal{U} \hookrightarrow \mathcal{Z}$ over T, so that the moduli $\boldsymbol{b}$-divisor of the log Calabi–Yau fibration over $\mathcal{U}_t$ descends on $\mathcal{Z}_t$.

Theorem 6.8. *Consider the diagram*

$$\begin{array}{ccc} (\mathcal{Y}_{\mathcal{S}^!}^!, \mathcal{R}_{\mathcal{S}^!}^!) & & \\ \downarrow & & \\ \mathcal{S}^! & \xrightarrow{\ \pi\ } & \mathcal{S}^* \\ \uparrow & & \uparrow \\ \mathcal{H}^! & \xrightarrow[\pi|_{\mathcal{H}^!}]{} & \mathcal{H}^* \end{array}$$

where

- *$\mathcal{S}^*$ and $\mathcal{S}^!$ are smooth schemes,*
- *$\mathcal{H}^* \hookrightarrow \mathcal{S}^*$ and $\mathcal{H}^! \hookrightarrow \mathcal{S}^!$ are dense open subsets,*
- *$\pi|_{\mathcal{H}^!}$ is étale,*
- *$(\mathcal{Y}_{\mathcal{S}^!}^!, \mathrm{Supp}(\mathcal{R}_{\mathcal{S}^!}^!))$ is log smooth over $\mathcal{H}^!$, and*
- *$(\mathcal{Y}_{\mathcal{H}^!}^!, \mathcal{R}_{\mathcal{H}^!}^!) \to \mathcal{H}^!$ is an lc-trivial fibration whose moduli $\boldsymbol{b}$-divisor $\mathcal{M}^!$ descends on $\mathcal{S}^!$, where $(\mathcal{Y}_{\mathcal{H}^!}^!, \mathcal{R}_{\mathcal{H}^!}^!) := (\mathcal{Y}_{\mathcal{S}^!}^!, \mathcal{R}_{\mathcal{S}^!}^!) \times_{\mathcal{S}^!} \mathcal{H}^!$.*

Suppose there is a family of smooth quasiprojective (possibly not proper) varieties $\mathcal{U} \to T$, where T is of finite type, and a morphism $\Theta : \mathcal{U} \to \mathcal{H}^*$. Let $\overline{\mathcal{U}} := \mathcal{U} \times_{\mathcal{H}^*} \mathcal{H}^!$. Then, after passing to a stratification of T, there is a family of projective varieties $\mathcal{Z} \to T$ and a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\mathcal{M}$ on $\mathcal{Z}$ such that $\mathcal{Z}_s$ is a compactification of $\mathcal{U}_s$ for every closed point $s \in T$ and, for any closed point $t \in T$, if $(X, \Delta) \to Z$ is an lc-trivial fibration such that

- *there is a closed point $t \in T$ together with a birational morphism $Z \to \mathcal{Z}_t$, and*
- *there exist a scheme V and a finite cover $V \to \overline{\mathcal{U}}_t$ such that, for every irreducible component V_i of V, the generic fiber of $(X, \Delta) \times_{\mathcal{Z}_t} V_i \to V_i$ is crepant birationally equivalent to the generic fiber of*

$$(\mathcal{Y}^!_{\mathcal{H}^!}, \mathcal{R}^!_{\mathcal{H}^!}) \times_{\mathcal{H}^!} V_i \to V_i,$$

then the moduli part $\boldsymbol{M}$ of $(X, \Delta) \to Z$ descends on $\mathcal{Z}_t$ and $\boldsymbol{M}_{\mathcal{Z}_t} = \mathcal{M}|_{\mathcal{Z}_t}$

Proof. To prove the result, we may assume $\mathcal{S}^*$ is irreducible.

After passing to a stratification of T, we may assume that T is smooth and $\mathcal{U} \to T$ is a smooth morphism. Because $\mathcal{H}^! \to \mathcal{H}^*$ is étale, $\overline{\mathcal{U}} \to T$ is smooth and $\overline{\mathcal{U}} \to \mathcal{U}$ is étale. Let $K(\widetilde{\mathcal{U}})/K(\mathcal{U})$ be the Galois closure of $K(\overline{\mathcal{U}})/K(\mathcal{U})$ and $\widetilde{\mathcal{U}} \to \mathcal{U}$ be the Galois cover with group G. After replacing $\mathcal{U}$ by an open subset and passing T to a stratification, we assume that $\widetilde{\mathcal{U}}_t \to \mathcal{U}_t$ is an étale morphism for every closed point $t \in T$. Note the fiber of $\widetilde{\mathcal{U}} \to T$ may not be irreducible.

The composition of $\widetilde{\mathcal{U}} \to \overline{\mathcal{U}}$ and base change of $\Theta : \mathcal{U} \to \mathcal{H}^*$ via $\overline{\mathcal{U}} \to \mathcal{U}$ defines a morphism $\widetilde{\Phi}^\circ : \widetilde{\mathcal{U}} \to \mathcal{H}^!$. Suppose $\widetilde{\mathcal{U}} \hookrightarrow \widetilde{\mathcal{Z}}'$ is a compactification over T such that $\widetilde{\Phi}^\circ$ extends to a morphism $\widetilde{\mathcal{Z}}' \to \mathcal{S}^!$. Because $\widetilde{\mathcal{U}}$ is smooth, we may let $\widetilde{\mathcal{Z}} \to \widetilde{\mathcal{Z}}'$ be a G-equivariant log resolution of $(\widetilde{\mathcal{Z}}', \widetilde{\mathcal{Z}}' \setminus \widetilde{\mathcal{U}})$ which is an isomorphism over $\widetilde{\mathcal{U}}$. Note $\widetilde{\Phi}^\circ$ extends to a morphism $\widetilde{\Phi} : \widetilde{\mathcal{Z}} \to \mathcal{S}^!$. After replacing T by a finite cover, we may assume every strata of $(\widetilde{\mathcal{Z}}, \widetilde{\mathcal{Z}} \setminus \widetilde{\mathcal{U}})$ is irreducible over T. By the generic smoothness theorem, after passing to a stratification of T, we may assume that $(\widetilde{\mathcal{Z}}_{s,j}, (\widetilde{\mathcal{Z}}_s \setminus \widetilde{\mathcal{U}}_s)|_{\widetilde{\mathcal{Z}}_{s,j}})$ is log smooth for every closed point $s \in T$ and every connected component $\widetilde{\mathcal{Z}}_{s,j}$ of $\widetilde{\mathcal{Z}}_s$.

Let $\mathcal{Z}$ be the quotient of $\widetilde{\mathcal{Z}}$ by G. Because $\widetilde{\mathcal{Z}}$ is a compactification of $\widetilde{\mathcal{U}}$ over T and the quotient of $\widetilde{\mathcal{U}}$ by G is $\mathcal{U}$, we have that $\mathcal{Z}$ is a compactification of $\mathcal{U}$ over T. Next, we show that $\mathcal{Z}$ satisfies the requirements.

Suppose $(X, \Delta) \to Z$ is an lc-trivial fibration that satisfies the conditions, let $Z \to \mathcal{Z}_t$ be the corresponding birational morphism and $V \to \overline{\mathcal{U}}_t$ the corresponding finite cover, and denote its moduli $\boldsymbol{b}$-divisor by $\boldsymbol{M}$. We replace V by $V \times_{\overline{\mathcal{U}}_t} \widetilde{\mathcal{U}}_t$ and assume $V \to \overline{\mathcal{U}}_t$ factors through $V \to \widetilde{\mathcal{U}}_t$. Because $V \to \widetilde{\mathcal{U}}_t$ and $\widetilde{\mathcal{U}}_t \to \mathcal{U}_t$ are finite covers, we can choose a compactification $V \hookrightarrow W$ such that the induced morphisms $W \to \widetilde{\mathcal{Z}}_t$ and $W \to \mathcal{Z}_t$ are finite covers.

Write

$$(\mathcal{Y}^!_{\widetilde{\mathcal{U}}_t}, \mathcal{R}^!_{\widetilde{\mathcal{U}}_t}) := (\mathcal{Y}^!_{\mathcal{H}^!}, \mathcal{R}^!_{\mathcal{H}^!}) \times_{\mathcal{H}^!} \widetilde{\mathcal{U}}_t,$$

where the morphism $\widetilde{\mathcal{U}}_t \to \mathcal{H}^!$ is $\widetilde{\Phi}^\circ|_{\widetilde{\mathcal{U}}_t}$. Because $(\mathcal{Y}^!_{\mathcal{S}^!}, \mathrm{Supp}(\mathcal{R}^!_{\mathcal{S}^!}))$ is log smooth over $\mathcal{H}^!$, we then have that $(\mathcal{Y}^!_{\widetilde{\mathcal{U}}_t}, \mathrm{Supp}(\mathcal{R}^!_{\widetilde{\mathcal{U}}_t}))$ is log smooth over $\widetilde{\mathcal{U}}_t$. Let $\widetilde{\mathcal{Z}}_{t,i}$ be any irreducible component of $\widetilde{\mathcal{Z}}_t$, and let $\widetilde{\mathcal{U}}_{t,i} := \widetilde{\mathcal{U}}_t \cap \widetilde{\mathcal{Z}}_{t,i}$. Because $(\widetilde{\mathcal{Z}}_{t,i}, (\widetilde{\mathcal{Z}}_t \setminus \widetilde{\mathcal{U}}_t)|_{\widetilde{\mathcal{Z}}_{t,i}})$ is log smooth, the moduli $\boldsymbol{b}$-divisor $\widetilde{\boldsymbol{M}}^i$ of a compactification of $(\mathcal{Y}^!_{\mathcal{H}^!}, \mathcal{R}^!_{\mathcal{H}^!}) \times_{\mathcal{H}^!} \widetilde{\mathcal{U}}_{t,i} \to \widetilde{\mathcal{U}}_{t,i}$ descends on $\widetilde{\mathcal{Z}}_{t,i}$ according to Definition 2.9. We define $\widetilde{\boldsymbol{M}}$ to be the $\boldsymbol{b}$-divisor on $\widetilde{\mathcal{Z}}_t$ whose restriction on $\widetilde{\mathcal{Z}}_{t,i}$ is $\widetilde{\boldsymbol{M}}^i$.

Let V_i be an irreducible component of V which dominates $\widetilde{\mathcal{U}}_{t,i}$. By assumption, the generic fiber of $(X, \Delta) \times_{\mathcal{Z}_t} V_i \to V_i$ is crepant birationally equivalent to the generic fiber of $(\mathcal{Y}^!_{\mathcal{H}^!}, \mathcal{R}^!_{\mathcal{H}^!}) \times_{\mathcal{H}^!} V_i \to V_i$; hence the generic fiber of $(X, \Delta) \times_{\mathcal{Z}_t} W_i \to W_i$ is crepant birationally equivalent to the generic fiber of $(\mathcal{Y}^!_{\widetilde{\mathcal{U}}_t}, \mathcal{R}^!_{\widetilde{\mathcal{U}}_t}) \times_{\widetilde{\mathcal{U}}_t} V_i \to V_i$, where W_i is the irreducible component of W corresponding to V_i. Note that the moduli $\boldsymbol{b}$-divisor only depends on the crepant birational equivalence class of the generic fiber. By Proposition 6.3, because the moduli $\boldsymbol{b}$-divisor $\widetilde{\boldsymbol{M}}^i$ descends on $\widetilde{\mathcal{Z}}_{t,i}$ and $\widetilde{\mathcal{Z}}_{t,i} \to W_i$ is a finite cover, the moduli $\boldsymbol{b}$-divisor of a compactification of $(X, \Delta) \times_{\mathcal{Z}_t} V_i \to V_i$ descends on W_i. Also because $W_i \to \mathcal{Z}_t$ is a finite cover, $\boldsymbol{M}$ descends on $\mathcal{Z}_t$. By considering every irreducible component of $\widetilde{\mathcal{Z}}_t$, we have that $\widetilde{\boldsymbol{M}}_{\widetilde{\mathcal{Z}}_t}$ is equal to the pullback of $\boldsymbol{M}_{\mathcal{Z}_t}$.

Recall that $\widetilde{\Phi}^\circ$ extends to a morphism $\widetilde{\Phi} : \widetilde{\mathcal{Z}} \to \mathcal{S}^!$. Because $(\mathcal{Y}^!_{\mathcal{H}^!}, \operatorname{Supp}(\mathcal{R}^!_{\mathcal{H}^!}))$ is log smooth over $\mathcal{H}^!$, the generic point of $\widetilde{\mathcal{Z}}_{t,i}$ maps into $\mathcal{H}^!$ and $\widetilde{\boldsymbol{M}}^i$ descends on $\widetilde{\mathcal{Z}}_{t,i}$ for every irreducible component $\widetilde{\mathcal{Z}}_{t,i}$ of $\widetilde{\mathcal{Z}}_t$. Then, by Theorem 6.5, we have

$$\widetilde{\boldsymbol{M}}_{\widetilde{\mathcal{Z}}_t} = (\widetilde{\Phi}|_{\widetilde{\mathcal{Z}}_t})^* \boldsymbol{\mathcal{M}}^!_{\mathcal{S}^!}.$$

Let $\widetilde{\mathcal{M}} := \widetilde{\Phi}^* \boldsymbol{\mathcal{M}}^!_{\mathcal{S}^!}$. Because $\widetilde{\mathcal{Z}} \to \mathcal{Z}$ is the quotient by G and $\sum_{g\in G} g^* \widetilde{\mathcal{M}}$ is G-invariant,

$$\frac{1}{|G|} \sum_{g\in G} g^* \widetilde{\mathcal{M}}$$

is equal to the pullback of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\mathcal{M}$ on $\mathcal{Z}$.

Because $\widetilde{\boldsymbol{M}}_{\widetilde{\mathcal{Z}}_t}$ is equal to the pullback of $\boldsymbol{M}_{\mathcal{Z}_t}$ and $\widetilde{\boldsymbol{M}}_{\widetilde{\mathcal{Z}}_t} = \widetilde{\mathcal{M}}|_{\widetilde{\mathcal{Z}}_t}$, we have that $\widetilde{\mathcal{M}}|_{\widetilde{\mathcal{Z}}_t}$ is equal to the pullback of a $\mathbb{Q}$-divisor on $\mathcal{Z}_t$. By the construction of $\mathcal{M}$, we have $\boldsymbol{M}_{\mathcal{Z}_t} = \mathcal{M}|_{\mathcal{Z}_t}$. □

7. Proof of Theorem 1.1

Proof of Theorem 1.1. We use the same notation as in Remark 6.1 and Theorem 6.7.

Let $C > v$ be any fixed number. To prove the DCC, we only need to prove that if $\operatorname{Ivol}(K_X + \Delta) \le C$, then $\operatorname{Ivol}(K_X + \Delta)$ is in a DCC set. By Theorem 2.12, we can construct a generalized pair $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$ and birational morphism $Z' \to Z$ such that

- $\operatorname{coeff}(B_{Z'})$ belongs to a DCC set $\mathcal{I}'$,
- the moduli $\boldsymbol{b}$-divisor $\boldsymbol{M}$ of f descends on Z',
- $\operatorname{Ivol}(K_X + \Delta) = \operatorname{vol}(K_{Z'} + B_{Z'} + \boldsymbol{M}_{Z'})$, and
- (X, Δ) has the same canonical model as $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$.

After replacing Z by Z', and $B_{Z'}$ and $\boldsymbol{M}_{Z'}$ by B_Z and $\boldsymbol{M}_Z$, respectively, we only need to prove that $\operatorname{vol}(K_Z + B_Z + \boldsymbol{M}_Z)$ belongs to a DCC set. To this end, we add $\{1 - 1/k, k \in \mathbb{N}\}$ into $\mathcal{I}'$ and assume that $\{1 - 1/k, k \in \mathbb{N}\} \subset \mathcal{I}'$.

By Remark 6.1, we have an lc-trivial fibration $(\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}}) \to \mathcal{H}$ corresponding to the class $\mathscr{C}$ of polarized log Calabi–Yau pairs. Consider the diagram

$$\begin{array}{ccccccc}
(\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}}) & & & & (\mathcal{X}^!_{\mathcal{H}^!}, \alpha\mathcal{D}^!_{\mathcal{H}^!}) & & \\
\downarrow{\scriptstyle \mathcal{F}} & & \overset{i}{\dashleftarrow\dashrightarrow} & & \downarrow{\scriptstyle \mathcal{F}^!} & & \\
\mathcal{H} & \xleftarrow{\ \tau\ } & \overline{\mathcal{H}} & \xrightarrow{\ \rho\ } & \mathcal{H}^! & \xrightarrow{\ \pi\ } & \mathcal{H}^* \\
 & & & \underset{\Phi}{\longrightarrow} & & &
\end{array}$$

constructed in Theorem 6.7. Because $\mathcal{H}$ has only finitely many irreducible components, to prove the results, we may assume $\mathcal{H}$ is irreducible. Let $\mathcal{S}^*$ be the compactification of $\mathcal{H}^*$, l be the positive integer defined in Theorem 6.7, and $(\mathcal{Y}_{\mathcal{H}}, \mathcal{R}_{\mathcal{H}}) \to (\mathcal{X}_{\mathcal{H}}, \alpha\mathcal{D}_{\mathcal{H}})$ and $(\mathcal{Y}^!_{\mathcal{H}^!}, \mathcal{R}^!_{\mathcal{H}^!}) \to (\mathcal{X}^!_{\mathcal{H}^!}, \alpha\mathcal{D}^!_{\mathcal{H}^!})$ be crepant birational morphisms which are fiberwise log resolutions of $\mathcal{F}$ and $\mathcal{F}^!$.

Since a general fiber (X_g, Δ_g, L_g) is in $\mathscr{C}$, by Remark 6.1, there is an open subset $U \hookrightarrow Z$ such that $(X_U, \Delta|_{X_U})$ is crepant birationally equivalent to the pullback of $(\mathcal{Y}_{\mathcal{H}}, \mathcal{R}_{\mathcal{H}}) \to \mathcal{H}$ by a morphism $U \to \mathcal{H}$. Let $h : Z' \to Z$ be a birational morphism such that $U \to \mathcal{H} \xrightarrow{\Phi} \mathcal{H}^*$ extends to a morphism $\phi : Z' \to \mathcal{S}^*$. Let k be a sufficiently large integer such that $K_{Z'} + h_*^{-1}B_Z + (1-1/k)E + \boldsymbol{M}_{Z'} \geq h^*(K_Z + B_Z + \boldsymbol{M}_Z)$, where E is the exceptional divisor of h. Then we replace Z by Z', B_Z by $h_*^{-1}B_Z + (1-1/k)E$, and $\boldsymbol{M}_Z$ by $\boldsymbol{M}_{Z'}$, and assume that there is a morphism $\phi : Z \to \mathcal{S}^*$. Note that we keep the facts that $\mathrm{coeff}(B_Z)$ is in the DCC set $\mathcal{I}'$, the moduli $\boldsymbol{b}$-divisor $\boldsymbol{M}$ of f descends on Z, $\mathrm{Ivol}(K_X + \Delta) = \mathrm{vol}(K_Z + B_Z + \boldsymbol{M}_Z)$, and (X, Δ) has the same canonical model as $(Z, B_Z + \boldsymbol{M}_Z)$.

Because $\dim Z \leq \dim X = n$, to prove the results, we may assume $\dim Z = d$ is fixed. Let $\boldsymbol{M}^{\mathrm{fix}}$ be the $\boldsymbol{b}$-divisor defined in Theorem 6.7. Then

- $\boldsymbol{M}^{\mathrm{fix}}_Z$ is effective and nef,
- $\boldsymbol{M}^{\mathrm{fix}}_{Z'} \sim_{\mathbb{Q}} \boldsymbol{M}_{Z'}$ for every birational map $Z' \dashrightarrow Z$, and
- $l\boldsymbol{M}^{\mathrm{fix}}_Z$ is Cartier.

By Step 1 of the proof of Theorem 3.2, there is a positive integer r depending only on d, l, and $\mathcal{I}'$ such that, after replacing Z by a birational model and B_Z by the strict transform plus $(1-1/k)E$, where E denotes the reduced exceptional divisor and k is a sufficiently large integer, there is a birational contraction $g : Z \to W$ and a very ample divisor A on W such that $g^*A + F' \sim r(K_Z + B_Z + \boldsymbol{M}^{\mathrm{fix}}_Z)$ for an effective $\mathbb{Q}$-divisor $F' \geq 0$. Because

$$\mathrm{vol}(A) \leq \mathrm{vol}(r(K_Z + B_Z + \boldsymbol{M}^{\mathrm{fix}}_Z)) = r^d\, \mathrm{Ivol}(K_X + \Delta) \leq r^d C,$$

W is in a bounded family $\mathcal{W} \to S$ and there is a relative very ample divisor $\mathcal{A}$ on $\mathcal{W}$ such that $\mathcal{A}|_{\mathcal{W}_0} \sim A$, where 0 is a closed point of S such that $W \cong \mathcal{W}_0$.

After passing to a stratification of S, we may assume $\mathcal{W} \to S$ has a fiberwise log resolution $\mathcal{W}' \xrightarrow{\mathscr{G}} \mathcal{W} \to S$. Because $\mathcal{A}$ is relatively very ample, we can stratify S further, so that there exists a sufficiently large integer r', a relative very ample divisor $\mathcal{A}'$ on $\mathcal{W}'$ and an effective divisor $\mathcal{E} \sim r'\mathscr{G}^*\mathcal{A} - \mathcal{A}'$ such that $\mathcal{E}|_{\mathcal{W}'_s}$ is effective for every closed point $s \in S$. Then we replace W by $\mathcal{W}'_0$, A by $\mathcal{A}'|_{\mathcal{W}'_0}$, F' by $r'F' + \mathcal{E}|_{\mathcal{W}'_0}$,

Z by a birational model, and B_Z by the strict transform plus $(1-1/k)E$, where E denotes the reduced exceptional divisor and k is a sufficiently large integer. We have

- A is very ample,
- W is smooth, and
- $g^*A + F' \sim r(K_Z + B_Z + \boldsymbol{M}_Z^{\mathrm{fix}})$ for an effective $\mathbb{Q}$-divisor $F' \geq 0$.

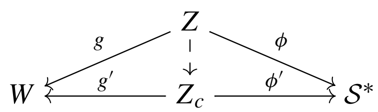
We define $F := F' + r((2d+1)l - 1)\boldsymbol{M}_Z^{\mathrm{fix}}$. Then F is effective, $\mathrm{Supp}(\boldsymbol{M}_Z^{\mathrm{fix}}) \subset \mathrm{Supp}(F)$, and $g^*A + F \sim r(K_Z + B_Z + (2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}})$.

Next, we construct a birational open subset of Z which maps into $\mathcal{H}^*$ via ϕ and belongs to a bounded family of quasiprojective varieties. This is similar to Step 2 of the proof of Theorem 3.2.

Recall that $\mathcal{H}^* \hookrightarrow \mathcal{S}^*$ is weakly bounded with respect to an ample Cartier divisor Λ on $\mathcal{S}^*$. Let $Z \dashrightarrow Z_c$ be the canonical model of $K_Z + B_Z + (2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}} + (2d+1)\phi^*\Lambda + (2d+1)g^*A$. By [Birkar and Zhang 2016, Lemma 4.4], $Z \dashrightarrow Z_c$ is $\boldsymbol{M}_Z^{\mathrm{fix}}$-, g^*A- and $\phi^*\Lambda$-trivial. Then there are two morphisms $g' : Z_c \to W$ and $\phi' : Z_c \to \mathcal{S}^*$. Let B_{Z_c} and F_c be the pushforward of B_Z and F on Z_c. Then $K_{Z_c} + B_{Z_c} + (2d+1)l\boldsymbol{M}_{Z_c}^{\mathrm{fix}} + (2d+1)\phi'^*\Lambda + (2d+1)g'^*A$ is ample: note $l\boldsymbol{M}_{Z_c}^{\mathrm{fix}}$ is nef, effective and Cartier. Because $K_{Z_c} + B_{Z_c} + (2d+1)l\boldsymbol{M}_{Z_c}^{\mathrm{fix}} \sim_{\mathbb{Q}} (g'^*A + F_c)/r$, we have

$$\frac{1}{r}(g'^*A + F_c) + (2d+1)\phi'^*\Lambda + (2d+1)g'^*A$$

is ample. We denote it by A'; clearly A' is effective.



Because $\mathrm{coeff}(B_Z)$ is in a DCC set $\mathcal{I}'$ and $r(K_Z + B_Z + (2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}) \sim g^*A + F$, with

$$\{r(K_Z + B_Z + (2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}})\} = \{rB_Z\} = \{F\},$$

$\mathrm{coeff}(F)$ is in a DCC set $\mathcal{I}'' = \mathcal{I}''(\mathcal{I}', d, r)$. In particular, there is a positive number δ such that $\mathrm{coeff}(F) > \delta$.

The proof of the following claim is deferred until after the main proof.

Claim: $(W, \mathrm{Supp}(g'_*(A' + B_{Z_c})))$, which is equal to $(W, \mathrm{Supp}(A + g'_*(\phi'^*\Lambda + F_c + B_{Z_c})))$, is log bounded.

Because A' is ample and effective and W is smooth, we have that $g'(\mathrm{Supp}(A'))$ is pure of codimension 1 and $g'(\mathrm{Supp}(A')) = \mathrm{Supp}(g'_*A')$. By the negativity lemma, $A' = g'^*g'_*A' - E'$, where E' is an effective exceptional $\mathbb{Q}$-divisor such that $\mathrm{Supp}(E') = \mathrm{Exc}(g')$. Because $A' \geq 0$, we have that $\mathrm{Exc}(g') \subset \mathrm{Supp}(g'^*g'_*A')$ and

$$W \setminus \mathrm{Supp}(g'_*A') \cong Z_c \setminus \mathrm{Supp}(g'^*g'_*A').$$

By Theorem 6.7, $\phi(Z \setminus \mathrm{Supp}(\boldsymbol{M}_Z^{\mathrm{fix}})) \subset \mathcal{H}^*$. Since $\mathrm{Supp}(\boldsymbol{M}_Z^{\mathrm{fix}}) \subset \mathrm{Supp}(F)$ and $\phi(Z \setminus \mathrm{Supp}(\boldsymbol{M}_Z^{\mathrm{fix}})) \subset \mathcal{H}^*$, we have $\mathrm{Supp}(\boldsymbol{M}_{Z_c}^{\mathrm{fix}}) \subset \mathrm{Supp}(F_c) \subset \mathrm{Supp}(A')$ and $\phi'(Z_c \setminus \mathrm{Supp}(\boldsymbol{M}_{Z_c}^{\mathrm{fix}})) \subset \mathcal{H}^*$. Let

$$U_c := Z_c \setminus \mathrm{Supp}(g'^*g'_*A') = W \setminus \mathrm{Supp}(g'_*A').$$

It is easy to see that $U_c \subset Z_c \setminus \mathrm{Supp}(\boldsymbol{M}_{Z_c}^{\mathrm{fix}})$ and $\phi'(U_c) \subset \mathcal{H}^*$.

Because $(W, \operatorname{Supp}(g'_*(A'+B_{Z_c})))$ is log bounded, there is a family of varieties $\mathcal{U} \to T$ over a scheme of finite type T and a closed point $t \in T$ such that

$$\mathcal{U}_t \cong W \setminus g'_*(A'+B_{Z_c}) \subset U_c.$$

Because $\mathcal{H}^*$ is weakly bounded, by applying Theorem 5.5 with $\mathcal{M}^0 := \mathcal{H}^* \times T$, there exists a finite type scheme $\mathscr{W}$ and a morphism $\mathscr{W} \times \mathcal{U} \to \mathcal{H}^* \times T$ over T such that, if we let $\Theta : \mathscr{W} \times \mathcal{U} \to \mathcal{H}^*$ be the composition of $\mathscr{W} \times \mathcal{U} \to \mathcal{H}^* \times T$ with the projection $\mathcal{H}^* \times T \to \mathcal{H}^*$, then $\phi'|_{\mathcal{U}_t} = \Theta|_{\{p\}\times \mathcal{U}_t}$ for a closed point $p \in \mathscr{W}$. We replace $\mathcal{U} \to T$ by $\mathscr{W} \times \mathcal{U} \to \mathscr{W} \times T$.

Let $V := U \times_{\mathcal{H}} \overline{\mathcal{H}}$; then $V \to U$ is a finite cover. By Theorem 6.6 and the fact that $(X_U, \Delta|_{X_U})$ is crepant birationally equivalent to the pullback of $(\mathcal{Y}_{\mathcal{S}}, \mathcal{R}_{\mathcal{S}}) \to \mathcal{S}$ via $U \to \mathcal{H} \hookrightarrow \mathcal{S}$, for every irreducible component V_i of V, the generic fiber of $(X, \Delta) \times_Z V_i \to V_i$ is crepant birationally equivalent to the generic fiber of $(\mathcal{Y}^!_{\mathcal{H}^!}, \mathcal{R}^!_{\mathcal{H}^!}) \times_{\mathcal{H}^!} V_i \to V_i$. Then, up to passing to a stratification of T, by Theorem 6.8, there is a compactification $\mathcal{U} \hookrightarrow \mathcal{Z}/T$ and a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\mathcal{M}$ on $\mathcal{Z}$ such that the moduli $\boldsymbol{b}$-divisor $\boldsymbol{M}$ of (X, Δ) descends on $\mathcal{Z}_t$ and $\boldsymbol{M}_{\mathcal{Z}_t} = \mathcal{M}|_{\mathcal{Z}_t}$.

Let $\mathcal{P} := \mathcal{Z} \setminus \mathcal{U}$; then $\mathcal{P}_t = \operatorname{Supp}(g'_*(A'+B_{Z_c}))$. After passing to a log resolution of the generic fiber and passing to a stratification of T, we may assume that $(\mathcal{Z}, \mathcal{P}) \to T$ is a projective log smooth morphism. We also replace $\mathcal{M}$ by its pullback. Note: we still have that $\boldsymbol{M}$ descends on $\mathcal{Z}_t$ and $\boldsymbol{M}_{\mathcal{Z}_t} = \mathcal{M}|_{\mathcal{Z}_t}$.

Let $h : Z' \to Z$ be a log resolution of (Z, B_Z) such that the isomorphism $U_c \cong \mathcal{U}_t$ extends to a morphism $Z' \to \mathcal{Z}_t$. We replace Z with Z' and B_Z with its strict transform plus $(1-1/k)E$, where E denotes the reduced exceptional divisor and k is a sufficiently large integer. Note that we keep $\operatorname{vol}(K_Z + B_Z + \boldsymbol{M}_Z)$ and the canonical model of $(Z, B_Z + \boldsymbol{M}_Z)$, and we still have $\operatorname{coeff}(B_{Z'}) \subset \mathcal{I}'$.

Since $\operatorname{Supp}(g_*B_Z) = \operatorname{Supp}(g'_*B_{Z_c})$ and $\operatorname{Supp}(g'_*(A'+B_{Z_c})) \subset \mathcal{P}_t$, the pushforward of $B_{Z'}$ on $\mathcal{Z}_t$ is contained in $\mathcal{P}_t$. Also because $\boldsymbol{M}$ descends on $\mathcal{Z}_t$, we have that $\boldsymbol{M}$ descends on Z'; hence $(Z', B_{Z'} + \boldsymbol{M}_{Z'})$ is a generalized klt pair and $\operatorname{coeff}(B_{Z'}) \subset \mathcal{I}'$ is a DCC set. Then, by Theorems 3.3 and 4.2, conclusions (i) and (ii) follow. □

Proof of claim. We use the same notation as in the proof of Theorem 1.1.

Because W is bounded by the construction, A and Λ are integral divisors, $\operatorname{coeff}(B_{Z_c})$ is in a DCC set, $\operatorname{coeff}(F_c)$ is bounded from below, and A is very ample on W, by boundedness of the Chow variety, we only need to prove that the intersection numbers

$$A^{d-1}.g'_*\phi'^*\Lambda, \quad A^{d-1}.g'_*B_{Z_c} \quad \text{and} \quad A^{d-1}.g'_*F_c$$

are bounded from above.

First we show that there is a constant C_1 such that

$$\operatorname{vol}(K_Z + B_Z + (2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}) \le C_1.$$

By Theorem 2.5, there is a rational number $e \in (0, 1)$ such that $K_Z + B_Z + e\boldsymbol{M}_Z$ is big. By the log-concavity of the volume function, we have that

$$\operatorname{vol}(K_Z + B_Z + \boldsymbol{M}_Z^{\mathrm{fix}}) \ge \lambda^d \operatorname{vol}(K_Z + B_Z + e\boldsymbol{M}_Z^{\mathrm{fix}}) + (1-\lambda)^d \operatorname{vol}(K_Z + B_Z + (2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}), \quad (7\text{-}1)$$

where

$$\lambda = \frac{(2d+1)l-1}{(2d+1)l-e} < 1.$$

By assumption, $\operatorname{vol}(K_Z + B_Z + \boldsymbol{M}_Z^{\mathrm{fix}}) \le C$, and hence $\operatorname{vol}(K_Z + B_Z + (2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}) \le C/(1-\lambda)^d$.

Second we prove that $A^{d-1}.g'_*\phi'^*\Lambda$ is bounded from above, which is equivalent to proving that $A^{d-1}.g_*\phi^*\Lambda$ is bounded from above. The idea is to show that $A^{d-1}.g_*\phi^*\Lambda$ is equal to the degree of a divisor on a (g, m)-curve, with $g+m$ bounded, then apply weak boundedness.

Let $A_1, \dots, A_{d-1} \in |g^*A|$ be $d-1$ general members of the linear system. Because g^*A is base point free, the elements of $\{\operatorname{Supp}(A_i), i = 1, \dots, d-1\}$ are smooth divisors and intersect along a smooth curve C. By the adjunction formula,

$$(g^*A)^{d-1}.(K_Z + B_Z + (2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}} + (d-1)g^*A) = \deg(K_C + B_Z|_C + (2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}|_C).$$

Consider the diagram

$$\begin{array}{ccccc} & & Z & \xleftarrow{\;h\;} & \widetilde{Z} \\ & {}^{g}\swarrow & \downarrow & \swarrow_{h_1} & \\ W & \xleftarrow[g_1]{} & Z_1 & & \end{array}$$

where Z_1 is the canonical model of $K_Z + B_Z + (2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}} + (2d+1)g^*A$ and $\widetilde{Z}$ is a resolution of indeterminacies of $Z \dashrightarrow Z_1$. By [Birkar and Zhang 2016, Lemma 4.4], $Z \dashrightarrow Z_1$ is g^*A-trivial, so there is a birational morphism $g_1 : Z_1 \to W$. By the projection formula,

$$\begin{aligned}(g^*A)^{d-1}.(K_Z + B_Z + (2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}} + (2d+1)g^*A)& \\ &= (h^*g^*A)^{d-1}.(h^*(K_Z + B_Z + (2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}} + (2d+1)g^*A)) \\ &= (g_1^*A)^{d-1}.(h_{1*}h^*(K_Z + B_Z + (2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}} + (2d+1)g^*A)) \\ &= (g_1^*A)^{d-1}.(K_{Z_1} + B_{Z_1} + (2d+1)l\boldsymbol{M}_{Z_1}^{\mathrm{fix}} + (2d+1)g_1^*A),\end{aligned}$$

where B_{Z_1} is the pushforward of B_Z. Since Z_1 is the canonical model of

$$K_Z + B_Z + (2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}} + (2d+1)g^*A,$$

$K_{Z_1} + B_{Z_1} + (2d+1)l\boldsymbol{M}_{Z_1}^{\mathrm{fix}} + (2d+1)g_1^*A$ is ample. By the binomial theorem, we have

$$\begin{aligned}(K_{Z_1} + B_{Z_1} + (2d+1)l\boldsymbol{M}_{Z_1}^{\mathrm{fix}} + (2d+1)g_1^*A + g_1^*A)^d& \\ &= \sum_{0\le i\le d} \binom{d}{i} (g_1^*A)^{d-i}.(K_{Z_1} + B_{Z_1} + (2d+1)l\boldsymbol{M}_{Z_1}^{\mathrm{fix}} + (2d+1)g_1^*A)^i.\end{aligned}$$

Because g_1^*A and $K_{Z_1} + B_{Z_1} + (2d+1)l\boldsymbol{M}_{Z_1}^{\mathrm{fix}} + (2d+1)g_1^*A$ are both nef, we have

$$(g_1^*A)^{d-i}.(K_{Z_1} + B_{Z_1} + (2d+1)l\boldsymbol{M}_{Z_1}^{\mathrm{fix}} + (2d+1)g_1^*A)^i \ge 0$$

for every $0 \le i \le d$. Then

$$
\begin{aligned}
(g_1^*A)^{d-1}.(K_{Z_1}+B_{Z_1}+(2d+1)l\boldsymbol{M}_{Z_1}^{\mathrm{fix}}+(2d+1)g_1^*A)&\\
&\le \binom{d}{1}(g_1^*A)^{d-1}.(K_{Z_1}+B_{Z_1}+(2d+1)l\boldsymbol{M}_{Z_1}^{\mathrm{fix}}+(2d+1)g_1^*A)\\
&\le (K_{Z_1}+B_{Z_1}+(2d+1)l\boldsymbol{M}_{Z_1}^{\mathrm{fix}}+(2d+1)g_1^*A+g_1^*A)^d\\
&= \mathrm{vol}(K_{Z_1}+B_{Z_1}+(2d+1)l\boldsymbol{M}_{Z_1}^{\mathrm{fix}}+(2d+2)g_1^*A).
\end{aligned}
$$

Since $Z \dashrightarrow Z_1$ is g^*A-trivial and Z_1 is also the canonical model of $K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}+(2d+2)g^*A$, we have

$$
\begin{aligned}
\mathrm{vol}(K_{Z_1}+B_{Z_1}+(2d+1)l\boldsymbol{M}_{Z_1}^{\mathrm{fix}}+(2d+2)g_1^*A) &= \mathrm{vol}(K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}+(2d+2)g^*A)\\
&\le \mathrm{vol}(K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}+(2d+2)(g^*A+F))\\
&= \mathrm{vol}((1+(2d+2)r)(K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}))\\
&\le \left(\frac{1+(2d+2)r}{r}\right)^d C_1.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\deg(K_C+B_Z|_C+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}|_C) &= (g^*A)^{d-1}.(K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}+(d-1)g^*A)\\
&\le (g^*A)^{d-1}.(K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}+(2d+1)g^*A)\\
&\le \left(\frac{1+(2d+2)r}{r}\right)^d C_1. \qquad (7\text{-}2)
\end{aligned}
$$

By the construction of $\boldsymbol{M}_Z^{\mathrm{fix}}$, we have that $Z \setminus \mathrm{Supp}(\boldsymbol{M}_Z^{\mathrm{fix}})$ maps into $\mathcal{H}^*$, so $C \setminus \mathrm{Supp}(\boldsymbol{M}_Z^{\mathrm{fix}}|_C)$ maps into $\mathcal{H}^*$. Suppose $C^\circ := C \setminus \mathrm{Supp}(\boldsymbol{M}_Z^{\mathrm{fix}}|_C)$ is a (g, m)-curve. Then $m \le \deg_C(l\boldsymbol{M}_Z^{\mathrm{fix}}|_C)$ and

$$
2g-2+(2d+1)m \le \deg(K_C+B_Z|_C+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}|_C)
$$

is bounded. Because $\mathcal{H}^*$ is weakly bounded with respect to Λ and C° is a (g, m)-curve with $2g+(2d+1)m$ bounded, we have that $(g^*A)^{d-1}.\phi^*\Lambda = C.\phi^*\Lambda = \deg_C(\phi^*\Lambda|_C)$ is bounded and, by the projection formula, $A^{d-1}.g_*\phi^*\Lambda$ is bounded.

Third we show that $A^{d-1}.g'_*B_{Z_c}$ is bounded from above, which is equivalent to proving that $A^{d-1}.g_*B_Z$ is bounded from above. Because $\mathrm{coeff}(B_Z) \subset \mathcal{I}'$ is in a DCC set, $l\boldsymbol{M}_Z^{\mathrm{fix}}$ is nef and Cartier and $K_Z+B_Z+\boldsymbol{M}_Z$ is big, by [Birkar and Zhang 2016, Theorem 8.1], there exists e depending only on d and $\mathcal{I}'$ such that $K_Z+eB_Z+\boldsymbol{M}_Z$ is big. Thus we have

$$
A^{d-1}.g_*B_Z \le \frac{1}{1-e}(g^*A)^{d-1}.((1-e)B_Z+K_Z+eB_Z+\boldsymbol{M}_Z) = \frac{1}{1-e}(g^*A)^{d-1}.(K_Z+B_Z+\boldsymbol{M}_Z).
$$

Since $\boldsymbol{M}_Z^{\mathrm{fix}}$ and g^* are effective, we have

$$
(g^*A)^{d-1}.(K_Z+B_Z+\boldsymbol{M}_Z) \le (g^*A)^{d-1}.(K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\mathrm{fix}}+(2d+1)g^*A).
$$

We then apply the last inequality of (7-2).

Finally we prove that $A^{d-1}.g'_*F_c$ is bounded from above, which is equivalent to proving that $A^{d-1}.g_*F$ is bounded from above. Because $K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\text{fix}}\sim_{\mathbb{Q}}(g^*A+F)/r$, we have

$$A^{d-1}.g_*F=(g^*A)^{d-1}.F\le(g^*A)^{d-1}.r(K_Z+B_Z+(2d+1)l\boldsymbol{M}_Z^{\text{fix}}),$$

which is also bounded by (7-2). □

Proof of Corollary 1.3. After replacing X with a $\mathbb{Q}$-factorization and Δ with its strict transform, we may assume X is $\mathbb{Q}$-factorial. Let δ be a sufficiently small positive rational number such that $(X,(1+\delta)\Delta)$ is klt.

Since $K_X+(1+\delta)\Delta\sim_{\mathbb{Q},Z}\delta\Delta$ is big over Z, by [Birkar et al. 2010], there exists the relative canonical model $X\dashrightarrow X'$ of $K_X+(1+\delta)\Delta$ over Z, and hence $K_{X'}+(1+\delta)\Delta'$ is ample over Z, where Δ' is the pushforward of Δ. For a general fiber (X'_g,Δ'_g) of $f':X'\to Z$, we have that $K_{X'_g}+(1+\delta)\Delta'_g$ is ample.

Because $X\dashrightarrow X'$ is a birational contraction and $K_X+\Delta\sim_{\mathbb{Q},Z}0$, we have $K_{X'}+\Delta'\sim_{\mathbb{Q},Z}0$, which implies $K_{X'_g}+\Delta'_g\sim_{\mathbb{Q}}0$. Thus

$$-K_{X'_g}\sim_{\mathbb{Q}}\Delta'_g\sim_{\mathbb{Q}}\frac{1}{\delta}(K_{X'_g}+(1+\delta)\Delta'_g)$$

is ample. Note $K_X+\Delta$ is crepant birationally equivalent to $K_{X'}+\Delta'$. Then $\operatorname{Ivol}(K_X+\Delta)=\operatorname{Ivol}(K_{X'}+\Delta')$ and (X,Δ) and (X',Δ') have the same canonical model. We replace (X,Δ) with (X',Δ').

Because $\operatorname{coeff}(\Delta)$ is in a DCC set $\mathcal{I}$, by [Hacon et al. 2014, Theorem 1.5], there exists a finite subset $\mathcal{I}'\subset\mathcal{I}$ such that $\operatorname{coeff}(\Delta_g)\subset\mathcal{I}'$. Furthermore, there is a positive rational number $\epsilon\in(0,1)$ depending only on $\mathcal{I}'$ such that (X_g,Δ_g) is ϵ-lc. By the Birkar-BAB theorem [Birkar 2021b, Theorem 1.1], X_g is in a bounded family only depending on ϵ and $\dim X_g$. Because $\dim X_g\le\dim X=n$, by boundedness, there exist positive integers l and C depending only on ϵ and n such that $-lK_{X_g}$ is very ample without higher cohomology and $\operatorname{vol}(-lK_{X_g})=(-lK_{X_g})^{\dim X_g}\le C$.

Since $\operatorname{coeff}(\Delta_g)$ is in a finite set $\mathcal{I}'$, there exists $\delta'>0$ such that $\operatorname{coeff}(\Delta_g)\ge\delta'$. Because $\Delta_g\sim_{\mathbb{Q}}-K_{X_g}$, we have

$$\operatorname{red}(\Delta_g).(-lK_{X_g})^{\dim X_g-1}\le\frac{1}{\delta'}(-K_{X_g})(-lK_{X_g})^{\dim X_g-1}\le\frac{1}{l\delta'}(-lK_{X_g})^{\dim X_g}\le\frac{C}{l\delta'}.$$

Because $-lK_{X_g}$ is very ample without higher cohomology,

$$(-lK_{X_g})^{\dim X_g}\le C\quad\text{and}\quad\operatorname{red}(\Delta_g).(-lK_{X_g})^{\dim X_g-1}\le\frac{C}{l\delta'},$$

we have that $(X_g,\Delta_g,-lK_{X_g})$ is in a log bounded class of polarized log Calabi–Yau pairs. We define $L:=-lK_X$, then apply Theorem 1.1. □

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