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An asymptotic orthogonality relation for $\mathrm{GL}(n, \mathbb{R})$

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Orthogonality is a fundamental theme in representation theory and Fourier analysis. An orthogonality relation for characters of finite abelian groups (now recognized as an orthogonality relation on GL(1)) was used by Dirichlet to prove infinitely many primes in arithmetic progressions. Asymptotic orthogonality relations for $\mathrm{GL}(n)$, with $n \le 3$, and applications to number theory, have been considered by various researchers over the last 45 years. Recently, the authors of the present work have derived an explicit asymptotic orthogonality relation, with a power savings error term, for $\mathrm{GL}(4, \mathbb{R})$. Here we extend those results to $\mathrm{GL}(n, \mathbb{R})$, $n \ge 2$.

For $n \le 5$, our results are contingent on the Ramanujan conjecture at the infinite place, but otherwise are unconditional. In particular, the case $n = 5$ represents a new result. The key new ingredient for the proof of the case $n = 5$ is the theorem of Kim and Shahidi that functorial products of cusp forms on $\mathrm{GL}(2) \times \mathrm{GL}(3)$ are automorphic on $\mathrm{GL}(6)$. For $n > 5$ (assuming again the Ramanujan conjecture holds at the infinite place), our results are conditional on two conjectures, both of which have been verified in various special cases. The first of these conjectures regards lower bounds for Rankin–Selberg L-functions, and the second concerns recurrence relations for Mellin transforms of $\mathrm{GL}(n, \mathbb{R})$ Whittaker functions.

Central to our proof is an application of the Kuznetsov trace formula, and a detailed analysis, utilizing a number of novel techniques, of the various entities — Hecke–Maass cusp forms, Langlands Eisenstein series, spherical principal series Whittaker functions and their Mellin transforms, and so on — that arise in this application.

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1. Introduction

1.1. ***Brief description of the main result of this paper.*** Let $n \geq 1$ be a rational integer, $s \in \mathbb{C}$, and $\mathbb{A}_\mathbb{Q} = \mathbb{R} \times \mathbb{A}_f$ denote the ring of adeles over $\mathbb{Q}$, where $\mathbb{A}_f$ denotes the finite adeles. The family of unitary cuspidal automorphic representations π of $\mathrm{GL}(n, \mathbb{A}_\mathbb{Q})$ and their standard L-functions

$$L(s,\pi) = L_\infty(s,\pi) \cdot \prod_p L_p(s,\pi)$$

were first introduced by Godement and Jacquet [1972] and have played a major role in modern number theory. In the special case of $n = 1$ the Euler products $\prod_p L_p(s,\pi)$ are just Dirichlet L-functions.

In this paper we focus on the unitary cuspidal automorphic representations of $\mathrm{GL}(n, \mathbb{A}_\mathbb{Q})$ with trivial central character which are globally unramified. For $n \geq 2$, these can be studied classically in terms of Hecke–Maass cusp forms on

$$\mathrm{SL}(n,\mathbb{Z}) \backslash \mathrm{GL}(n,\mathbb{R}) / (\mathrm{O}(n,\mathbb{R}) \cdot \mathbb{R}^\times),$$

where

$$\mathfrak{h}^n := \mathrm{GL}(n,\mathbb{R}) / (\mathrm{O}(n,\mathbb{R}) \cdot \mathbb{R}^\times)$$

is a generalization of the classical upper half-plane. In fact $\mathfrak{h}^2 := \left\{ \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix} \right) \,\middle|\, y > 0,\ x \in \mathbb{R} \right\}$ is isomorphic to the classical upper half-plane.

For $n \geq 2$, Hecke–Maass cusp forms are smooth functions $\phi : \mathfrak{h}^n \to \mathbb{C}$ which are automorphic for $\mathrm{SL}(n,\mathbb{Z})$ with moderate growth and which are joint eigenfunctions of the full ring of invariant differential operators on $\mathrm{GL}(n,\mathbb{R})$ and are also joint eigenfunctions of the Hecke operators. Such globally unramified Hecke–Maass forms can be classified in terms of Langlands parameters which (assuming the cusp form is tempered) are n pure imaginary numbers $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in (i \cdot \mathbb{R})^n$ that sum to zero. Further, the Hecke–Maass cusp forms ϕ with Langlands parameters $(\alpha_1, \ldots, \alpha_n)$ can be ordered in terms of their Laplace eigenvalues $\lambda_\Delta(\phi)$ given by

$$\lambda_\Delta(\phi) = \tfrac{1}{24}(n^3 - n) - \tfrac{1}{2}(\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2),$$

as proved by Stephen Miller [2002].

Let ϕ be a Hecke–Maass cusp form for $\mathrm{SL}(n,\mathbb{Z})$ for $n \geq 2$ and set

$$\langle \phi, \phi \rangle := \int\limits_{\mathrm{SL}(n,\mathbb{Z}) \backslash \mathfrak{h}^n} \phi(g) \overline{\phi(g)}\, dg$$

to denote the Petersson norm of ϕ. The Hecke–Maass cusp forms form a Hilbert space over $\mathbb{C}$ with respect to the Petersson inner product.

Definition 1.1.1 (L-function of a Hecke–Maass cusp form). Let ϕ be a Hecke–Maass cusp form for $\mathrm{SL}(n,\mathbb{Z})$. Then for $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ sufficiently large we define the L-function $L(s,\phi) := \sum_{k=1}^{\infty} \lambda(k) k^{-s}$, where $\lambda(k)$ is the k-th Hecke eigenvalue of ϕ.

Definition 1.1.2 (asymptotic orthogonality relation for GL$(n, \mathbb{R})$). Let $\{\phi_j\}_{j=1,2,\ldots}$ (with associated Langlands parameters $\alpha^{(j)} = (\alpha_1^{(j)}, \alpha_2^{(j)}, \ldots, \alpha_n^{(j)})$) denote an orthogonal basis of Hecke–Maass cusp forms for $SL(n, \mathbb{Z})$ with L-function given by $L(s, \phi_j) := \sum_{k=1}^{\infty} \lambda_j(k) k^{-s}$. Fix positive integers ℓ, m. Then, for $T \to \infty$, we have

$$\lim_{T\to\infty} \frac{\sum_{j=1}^{\infty} \lambda_j(\ell)\, \overline{\lambda_j(m)}\, h_T(\alpha^{(j)})/\mathcal{L}_j}{\sum_{j=1}^{\infty} h_T(\alpha^{(j)})/\mathcal{L}_j} = \begin{cases} 1 + o(1) & \text{if } \ell = m, \\ o(1) & \text{if } \ell \neq m, \end{cases}$$

where $\mathcal{L}_j = L(1, \operatorname{Ad}\phi_j)$ and $h_T(\alpha^{(j)})$ is a smooth function of the variables $\alpha^{(j)}$, T (for $T > 0$), with support on the Laplace eigenvalues $\lambda_\Delta(\phi_j)$, where $0 < \lambda_\Delta(\phi_j) \ll T$.

Remark 1.1.3 (power savings error term). The asymptotic orthogonality relation has a power savings error term if $o(1)$ can be replaced with $\mathcal{O}(T^{-\theta})$ for some fixed $\theta > 0$. The error terms $o(1)$, $\mathcal{O}(T^{-\theta})$ will generally depend on L, M. This type of asymptotic orthogonality relation was first conjectured by Fan Zhou [2014].

Remark 1.1.4 (normalization of Hecke–Maass cusp forms). The approach we take in proving asymptotic orthogonality relations for GL$(n, \mathbb{R})$ is the Kuznetsov trace formula presented in Section 4, where $\lambda_j(\ell)\overline{\lambda_j(m)}/\langle \phi_j, \phi_j \rangle$ (which are independent of the way the ϕ_j are normalized) appears naturally on the spectral side of the trace formula leading to an asymptotic orthogonality relation of the form

$$\lim_{T\to\infty} \frac{\sum_{j=1}^{\infty} \lambda_j(\ell)\overline{\lambda_j(m)} h_T(\alpha^{(j)})/\langle \phi_j, \phi_j \rangle}{\sum_{j=1}^{\infty} h_T(\alpha^{(j)})/\langle \phi_j, \phi_j \rangle} = \begin{cases} 1 + o(1) & \text{if } \ell = m, \\ o(1) & \text{if } \ell \neq m. \end{cases} \tag{1.1.5}$$

If we normalize ϕ_j so that its first Fourier coefficient is equal to 1 then it is shown in Proposition 4.1.4 that

$$\langle \phi_j, \phi_j \rangle = c_n L(1, \operatorname{Ad}\phi_j) \prod_{1 \le i \neq k \le n} \Gamma\left(\tfrac{1}{2}(1 + \alpha_i^{(j)} - \alpha_k^{(j)})\right) \quad (c_n \neq 0).$$

This allows us (with a modification of the test function h_T) to replace the inner product $\langle \phi_j, \phi_j \rangle$ appearing in (1.1.5) with the adjoint L-function $\mathcal{L}_j$ as in Definition 1.1.2. The main reason for doing this is that there are much better techniques developed for bounding special values of L-functions, as opposed to bounding inner products of cusp forms. So having $\mathcal{L}_j^{-1}$ in the asymptotic orthogonality relation instead of $\langle \phi_j, \phi_j \rangle^{-1}$ will allow us to obtain better error terms in applications.

Orthogonality relations as in Definition 1.1.2 have a long history going back to Dirichlet (for the case of GL(1)) who introduced the orthogonality relation for Dirichlet characters to prove infinitely many primes in arithmetic progressions. Bruggeman [1978] was the first to obtain an asymptotic orthogonality relation for GL(2), which he presented in the form

$$\lim_{T\to\infty} \sum_{j=1}^{\infty} \frac{\lambda_j(\ell)\overline{\lambda_j(m)} \cdot 4\pi^2 e^{-\lambda_\Delta(\phi_j)/T}}{T \cosh\left(\pi\sqrt{\lambda_\Delta(\phi_j) - \frac{1}{4}}\right)} = \begin{cases} 1 & \text{if } \ell = m, \\ 0 & \text{if } \ell \neq m, \end{cases}$$

where $\{\phi_j\}_{j=1,2,\ldots}$ goes over an orthogonal basis of Hecke–Maass cusp forms for $SL(2, \mathbb{Z})$. This is not quite in the form of Definition 1.1.2 but it can be put into that form with some work. Other versions of

GL(2)-type orthogonality relations with important applications were obtained by Sarnak [1987], and, for holomorphic Hecke modular forms, by Conrey, Duke and Farmer [Conrey et al. 1997] and J. P. Serre [1997].

The first asymptotic orthogonality relations for GL(3) with power savings error term were proved independently by Blomer [2013] and Goldfeld and Kontorovich [2013]. Goldfeld, Stade and Woodbury [Goldfeld et al. 2021b] were the first to obtain a power savings asymptotic orthogonality relation, as in Definition 1.1.2 for GL(4).

A major breakthrough was obtained by Matz and Templier [2021] who unconditionally proved an asymptotic orthogonality relation for $\mathrm{SL}(n, \mathbb{Z})$, as in (1.1.5), for a wide class of test functions for all $n \geq 2$ (with power savings) but without the harmonic weights given by the inverse of the adjoint L-function at 1. Their results were further strengthened in [Finis and Matz 2021]. The principal tool used to prove the asymptotic orthogonality relation in [Matz and Templier 2021] was the Arthur–Selberg trace formula, whereas our approach is the natural generalization of the earlier results [Blomer 2013; Goldfeld and Kontorovich 2013; Goldfeld et al. 2021b], which were based on the Kuznetsov trace formula.

Blomer [2021] presented a very nice exposition comparing the Arthur–Selberg and Kuznetsov trace formulae, which we now briefly summarize for the application to asymptotic orthogonality relations.

- The first key difference between these trace formulae is that the spectral side of the Kuznetsov trace formula has harmonic weights $\mathcal{L}_j^{-1}$, while the Arthur–Selberg trace formula does not have these harmonic weights. For $\mathrm{GL}(n)$ with $n > 3$ it is not currently known how to remove these weights (see [Buttcane and Zhou 2020] for how to remove the weights on GL(3)). Blomer [2021] remarked that "*for applications to L-functions involving period formulae it is often desirable to have an additional factor* $1/L(1, \mathrm{Ad}\,\phi)$ *in the cuspidal spectrum*, *but in other situations one may prefer a summation formula without an extra L-value.*"

- The second major difference between these trace formulae is that the spectral side of the Kuznetsov trace formula does not contain residual spectrum, while the Arthur–Selberg trace formula does. As pointed out by a referee, the bulk of the work in [Matz and Templier 2021] consists in bounding the unipotent contribution on the geometric side of the Arthur trace formula so that it stays in line with the error term coming from the residual Eisenstein contribution on the spectral side given by Lapid and Müller [2009]. These residual Eisenstein series do not appear in the Kuznetsov trace formula, which leads to a very strong conjectural error term in Theorem 1.5.1. In fact, the largest error term on the spectral side of the Kuznetsov trace formula arises from the tempered Eisenstein series coming from the maximal parabolic having $(n-1, 1)$ Levi block decomposition. For explicit comparisons between our main theorem and the results of [Matz and Templier 2021], see Remark 1.5.4.

- There are certain applications of our results using the Kuznetsov trace formula approach that go beyond the results in [Matz and Templier 2021; Finis and Matz 2021]. Recall that $\lambda_j(p)$ denotes the p-th Hecke eigenvalue of the Maass form ϕ_j. Fan's thesis concerns the so-called vertical Sato–Tate problem, which is a conjecture about the distribution of $\lambda_j(p)$, where p is fixed and j varies. This problem was studied by Bruggeman [1978] and Sarnak [1987] (for Maass forms), and Serre [1997] and Conrey, Duke and Farmer [Conrey et al. 1997] (for holomorphic forms), who showed by fixing p and varying j that $\lambda_j(p)$

is an equidistributed sequence with respect to the Plancherel measure which depends on p. Strikingly, as observed by Fan Zhou [2014], if we give each Hecke eigenvalue $\lambda_j(p)$ the weight $\mathcal{L}_j^{-1}$, then the distribution involves the Sato–Tate measure which is independent of p. Jana [2021] generalized the results of Zhou, but he only obtained an asymptotic formula without a power savings error term. A problem for the future would be to combine Jana's approach with the methods of this paper. Jana also obtains bounds toward Sarnak's density hypothesis using this strategy that are stronger than anything known using the Arthur–Selberg trace formula.

The main aim of this paper is to explicitly work out an asymptotic orthogonality relation for $\mathrm{SL}(n,\mathbb{Z})$ via the Kuznetsov trace formula for a special choice of test function $h^{(n)}_{T,R}$ whose form is that of a Gaussian times a fixed polynomial. We do not address applications in this paper and leave that to future research. See [Blomer 2021] for various applications of the Arthur–Selberg and Kuznetsov trace formulae and how they compare. We also point out that the Kuznetsov trace formula was generalized by Jacquet and Lai [1985] who developed the relative trace formula which has had a wide following with new types of applications.

See Theorem 1.5.1 for the statement of our main theorem. The proof we give assumes the Ramanujan conjecture at ∞ but it is possible to prove a weaker result by dropping this assumption. Otherwise the proof is unconditional for $n \le 5$. In particular, the case $n = 5$ represents a complete, new result. For $n > 5$, our result is conditional on two conjectures.

1.2. ***Ishii–Stade conjecture.*** The Ishii–Stade conjecture (see Section 8.2) concerns the normalized Mellin transform $\widetilde{W}_{n,\alpha}(s)$ of the $\mathrm{GL}(n,\mathbb{R})$ Whittaker function $W_{n,\alpha}(y)$ defined in Definition 2.3.3. Here, $s = (s_1, s_2, \ldots, s_{n-1}) \in \mathbb{C}^{n-1}$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) = \mathbb{C}^{n-1}$ satisfies $\sum_{i=1}^{n} \alpha_i = 0$.

Suppose integers m and δ, with $1 \le m \le n-1$ and $\delta \ge 0$, are given. The Ishii–Stade conjecture expresses $\widetilde{W}_{n,\alpha}(s)$ as a finite linear combination, with coefficients that are rational functions of the s_j's and α_k's, of shifted Mellin transforms

$$\widetilde{W}_{n,\alpha}(s+\Sigma),$$

where $\Sigma \in (\mathbb{Z}_{\ge 0})^{n-1}$ and the m-th coordinate of Σ is $\ge \delta$. In other words, for such δ and m, the conjecture expresses the Mellin transform $\widetilde{W}_{n,\alpha}(s)$ in terms of shifts of this Mellin transform by at least δ units to the right in the variable s_m.

Much as recurrence relations of the form

$$\Gamma(s) = [(s+\delta-1)(s+\delta-2)\cdots(s+1)s]^{-1}\Gamma(s+\delta)$$

for Euler's Gamma function imply concrete results concerning analytic continuation, poles, and residues of that function, so will the Ishii–Stade conjecture allow us to obtain explicit information about the behavior of $\widetilde{W}_{n,a}(s)$ beyond its original, a priori domain of definition. This explicit information will be crucial to the analysis of our test function h_T, and consequently, to our derivation of an asymptotic orthogonality relation as in Definition 1.1.2.

We have been able to prove the Ishii–Stade conjecture for $\mathrm{GL}(n,\mathbb{R})$ with $2 \le n \le 5$. See Section 8.2 below.

1.3. ***Lower bound conjecture for Rankin–Selberg L-functions.*** Fix $n \ge 2$. Let $n = n_1 + \cdots + n_r$ be a partition of n with $n_i \in \mathbb{Z}_{>0}$ $(i = 1, \ldots, r)$. The second conjecture we require for the proof of the asymptotic orthogonality relation for $\mathrm{GL}(n, \mathbb{R})$ is a conjecture on the lower bound for Rankin–Selberg L-functions $L(s, \phi_k \times \phi_{k'})$ on the line $\mathrm{Re}(s) = 1$, where $\phi_k, \phi_{k'}$ (for $1 \le k < k' \le r$) are Hecke–Maass cusp forms for $\mathrm{SL}(n_k, \mathbb{Z})$, $\mathrm{SL}(n_{k'}, \mathbb{Z})$, respectively. For a Hecke–Maass cusp form ϕ with Langlands parameters $(\alpha_1, \ldots, \alpha_n)$, let

$$c(\phi) = (1 + |\alpha_1|)(1 + |\alpha_2|) \cdots (1 + |\alpha_n|) \tag{1.3.1}$$

denote the analytic conductor of ϕ as defined by Iwaniec and Sarnak [2000].

Conjecture 1.3.2 (lower bounds for Rankin–Selberg L-functions). *Let $\varepsilon > 0$ be fixed. Then we have the lower bound*

$$|L(1 + it, \phi_k \times \phi_{k'})| \gg_\varepsilon (c(\phi_k) \cdot c(\phi_{k'}))^{-\varepsilon} (|t| + 2)^{-\varepsilon}.$$

Remark 1.3.3. Conjecture 1.3.2 follows from Langlands' conjecture that $\phi_k \times \phi_{k'}$ is automorphic for $\mathrm{SL}(n_k \cdot n_{k'}, \mathbb{Z})$. This can be proved via the method of de la Valée Poussin as in [Sarnak 2004]. Interestingly, Sarnak's approach can be extended to prove Conjecture 1.3.2 if $\phi_{k'}$ is the dual of ϕ_k (see [Goldfeld and Li 2018; Humphries and Brumley 2019]). Stronger bounds can also be obtained if one assumes the Lindelöf or Riemann hypothesis for Rankin–Selberg L-functions.

If $n_k = n'_k = 2$, it was proved by Ramakrishnan [2000] that $\phi_k \times \phi_{k'}$ is automorphic for $\mathrm{SL}(4, \mathbb{Z})$, thus proving the lower bound conjecture for $n \le 4$. Further, for $n_k = 2$ and $n'_k = 3$, it was proved by Kim and Shahidi [2002] that $\phi_k \times \phi_{k'}$ is automorphic for $\mathrm{SL}(6, \mathbb{Z})$, thus proving the lower bound conjecture for $n \le 5$.

1.4. ***Constructing the test functions.*** Fix an integer $n \ge 2$. We now construct two complex-valued test functions on the space of Langlands parameters

$$\{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n \mid \alpha_1 + \cdots + \alpha_n = 0\}$$

that will be used in our proof of the orthogonality relation for $\mathrm{GL}(n, \mathbb{R})$.

We begin by introducing an auxiliary polynomial that is used in constructing the test functions.

Definition 1.4.1 (the polynomial $\mathcal{F}_R^{(n)}(\alpha)$). Let $R \in \mathbb{Z}_{>0}$ and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a Langlands parameter. Then we define

$$\mathcal{F}_R^{(n)}(\alpha) := \prod_{j=1}^{n-2} \prod_{\substack{K, L \subseteq (1,2,\ldots,n) \\ \#K = \#L = j}} \Big(1 + \sum_{k \in K} \alpha_k - \sum_{\ell \in L} \alpha_\ell\Big)^{\frac{R}{2}}.$$

Note that if $\alpha \in (i\mathbb{R})^n$, then $\mathcal{F}_1^{(n)}(\alpha)$ is the square root of a polynomial in α of degree $2D(n)$, where

$$D(n) = \sum_{j=1}^{n-2} \frac{1}{2} \binom{n}{j} \left(\binom{n}{j} - 1 \right) = \frac{1}{2} \binom{2n}{n} - \frac{n(n-1)}{2} - 2^{n-1}. \tag{1.4.2}$$

By abuse of notation, we refer to $\mathcal{F}_R^{(n)}$ as a *polynomial*, although this is not strictly the case unless R is even. For α with bounded real and imaginary parts, say $|\mathrm{Re}(\alpha_j)| < R$ and $|\mathrm{Im}(\alpha_j)| < T^{1+\varepsilon}$, we have

$$|\mathcal{F}_R^{(n)}(\alpha)| \ll T^{\varepsilon+R\cdot D(n)} \quad (T \to +\infty), \tag{1.4.3}$$

with an implicit constant depending on n, ε, R.

Definition 1.4.4 (the test functions $p_{T,R}^{n,\#}(\alpha)$ and $h_{T,R}^{(n)}(\alpha)$). Let $R \in \mathbb{Z}_{>0}$ and $T \to +\infty$. Then for a Langlands parameter $\alpha = (\alpha_1, \dots, \alpha_n)$, we define

$$p_{T,R}^{n,\#}(\alpha) := e^{(\alpha_1^2+\alpha_2^2+\cdots+\alpha_n^2)/(2T^2)} \cdot \mathcal{F}_R^{(n)}\Big(\frac{\alpha}{2}\Big) \prod_{1\le j\ne k\le n} \Gamma\Big(\frac{1+2R+\alpha_j-\alpha_k}{4}\Big),$$

$$h_{T,R}^{(n)}(\alpha) := \frac{|p_{T,R}^{n,\#}(\alpha)|^2}{\prod_{1\le j\ne k\le n} \Gamma((1+\alpha_j-\alpha_k)/2)}.$$

We observe that, by Stirling's formula for the Gamma function and by (1.4.2) and (1.4.3), we have

$$|h_{T,R}^{(n)}(\alpha)| \ll T^{R\cdot(\binom{2n}{n}-2^n)-\frac{n(n-1)}{2}} \tag{1.4.5}$$

whenever $|\mathrm{Re}(\alpha_j)|$ is bounded and $|\mathrm{Im}(\alpha_j)| < T^{1+\varepsilon}$ for $1 \le j \le n$. The implied constant in (1.4.5) depends on n, ε, and R.

Remark 1.4.6 (positivity of $h_{T,R}^{(n)}$). Writing $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $\alpha_j = it_j$ and $t_j \in \mathbb{R}$ for each $j = 1, 2, \dots, n$, the function $h_{T,R}^{(n)}(\alpha)$ is positive. This is the case because $\Gamma\big(\frac{1+iu}{2}\big)\Gamma\big(\frac{1-iu}{2}\big) = \big|\Gamma\big(\frac{1+iu}{2}\big)\big|^2$ for $u \in \mathbb{R}$.

Remark 1.4.7 (Whittaker transform of the test function). The symbol # in the test function $p_{T,R}^{n,\#}$ means this function is the Whittaker transform of $p_{T,R}^{(n)}$. See Section 8.

1.5. *The main theorem.*

Theorem 1.5.1. *Fix $n \ge 2$. Let $\{\phi_j\}_{j=1,2,\dots}$ denote an orthogonal basis of Hecke–Maass cusp forms for* $\mathrm{SL}(n, \mathbb{Z})$ (*assumed to be tempered at* ∞) *with associated Langlands parameter*

$$\alpha^{(j)} = (\alpha_1^{(j)}, \alpha_2^{(j)}, \dots, \alpha_n^{(j)}) \in (i \cdot \mathbb{R})^n$$

and L-function $L(s, \phi_j) := \sum_{k=1}^{\infty} \lambda_j(k)k^{-s}$.

Fix positive integers ℓ, m. Then assuming the Ishii–Stade conjecture (*Conjecture 8.2.3*) *and the lower bound conjecture for Rankin–Selberg L-functions* (*Conjecture 1.3.2*), *we prove that for $T \to \infty$*

$$\sum_{j=1}^{\infty} \lambda_j(\ell)\,\overline{\lambda_j(m)}\,\frac{h_{T,R}^{(n)}(\alpha^{(j)})}{\mathcal{L}_j} = \delta_{\ell,m} \cdot \sum_{i=1}^{n-1} \mathfrak{c}_i \cdot T^{R\cdot(\binom{2n}{n}-2^n)+n-i} + \mathcal{O}_{\varepsilon,R,n}\Big((\ell m)^{\frac{n^2+13}{4}} \cdot T^{R\cdot(\binom{2n}{n}-2^n)+\varepsilon}\Big),$$

where $\delta_{\ell,m}$ is the Kronecker symbol, $\mathcal{L}_j = L(1, \mathrm{Ad}\,\phi_j)$, and $\mathfrak{c}_1, \dots, \mathfrak{c}_{n-1} > 0$ are absolute constants which depend at most on R and n.

Because Conjectures 1.3.2 and 8.2.3 are known to be true for $2 \le n \le 5$ (*see Remark 1.3.3 and Section 8.2*), *the above result is unconditional for such n.*

Remark 1.5.2. Qiao Zhang [2023] recently proved the lower bound

$$|L(1+it, \phi_k \times \phi_{k'})| \gg (c(\phi_k)\cdot c(\phi_{k'}))^{-\theta_{k,k'}}(|t|+2)^{-\frac{1}{2}n_k n_{k'}(1-1/(n_k+n_{k'}))-\varepsilon}, \tag{1.5.3}$$

with $\theta_{k,k'} = n_k + n_{k'} + \varepsilon$. This improves on the bound of Brumley [2006; 2013, Appendix], who obtained nearly the same result but with the term $n_k n_{k'}/2$ replaced by $n_k n_{k'}$. Assuming (1.5.3) we can replace the error term in Theorem 1.5.1 with

$$\mathcal{O}_{\varepsilon,R,n,\ell,m}\Big(T^{R\cdot(\binom{2n}{n}-2^n)+n-1+\frac{n(n-2)}{6}(\theta_{k,k'}-\frac{8}{n^2})}\Big).$$

So if one could prove (1.5.3) with $\theta_{k,k'} < 8/n^2$ this would give a power savings error term in our main theorem and would remove the assumption of the lower bound conjecture (Conjecture 1.3.2). In fact, the proof establishes a black box by which improvements to bounds on Rankin–Selberg L-functions result in better power savings error terms for the continuous spectrum contribution to the asymptotic orthogonality relation.

Remark 1.5.4. A variant of Theorem 1.5.1 is obtained unconditionally in [Matz and Templier 2021; Finis and Matz 2021], without the arithmetic weights $\mathcal{L}_j^{-1}$ and with different test functions, which are indicator functions of $\alpha^{(j)} \in T\Omega$, where Ω is a Weyl group invariant bounded open subset of $i\cdot\mathfrak{a}^*$, where $\mathfrak{a}$ is the Lie algebra of the subgroup of diagonal matrices with positive entries. Additionally, the results of [Matz and Templier 2021; Finis and Matz 2021] do not give the polynomial weights of size $T^{R\cdot(\binom{2n}{n}-2^n)-n(n-1)/2}$ coming from $h_{T,R}^{(n)}(\alpha)$ (see (1.4.5)).

The error term obtained in [Finis and Matz 2021], in the present setting of $\mathrm{SL}(n,\mathbb{Z})$, is $\ll T^{(n-1)(n+2)/2-1}$ as $T\to\infty$. Here, $(n-1)(n+2)/2$ is the dimension of the generalized upper half-plane $\mathfrak{h}^n$, and the error term obtained by Finis and Matz has exponent equal to that dimension minus 1. By comparison, if one removes the polynomial weights $T^{R\cdot(\binom{2n}{n}-2^n)-n(n-1)/2}$ from the error term in Theorem 1.5.1 above, then one obtains an error term that is $\ll T^{n(n-1)/2+\varepsilon}$. Also note that our main term is of a stronger form than that of [Matz and Templier 2021; Finis and Matz 2021], in that ours gives a sum of $n-1$ different high-order asymptotics.

More recently, Jana [2021] obtained a proof of the asymptotic orthogonality relation defined in Definition 1.1.2, using the Kuznetsov trace formula and not the Selberg trace formula, with applications to the equidistribution of Satake parameters with respect to the Sato–Tate measure, second-moment estimates of central values of L-functions as strong as Lindelöf on average, and distribution of low-lying zeros of automorphic L-functions in the analytic conductor aspect. The paper of Jana does not contain a power savings error term.

Remark 1.5.5. It is possible to remove the assumption of Ramanujan at the infinite place with more work, which results in a weaker power savings error term in Theorem 1.5.1. For a Maass form ϕ with Langlands parameter α, note that the test function $h_{T,R}(\alpha)$ is positive. This is true because, even if α is a Langlands parameter of an element in the complementary spectrum, $-\alpha$ is a permutation of $\bar{\alpha}$. A weaker version of Theorem 1.5.1 can be proved if one assumes that almost all (except for a set of zero density) are tempered. Such results have been obtained in [Matz and Templier 2021; Finis and Matz 2021].

Proof of Theorem 1.5.1. Computing the inner product of certain Poincaré series in two ways (see the outline in Section 1.6 below), we obtain a Kuznetsov trace formula relating the so-called geometric and spectral sides. The geometric side consists of a main term $\mathcal{M}$ and a Kloosterman contribution $\mathcal{K}$. The spectral side also consists of two components: a cuspidal (i.e., discrete) contribution $\mathcal{C}$ and an Eisenstein (i.e., continuous) contribution $\mathcal{E}$.

The left-hand side of the theorem is precisely $\mathcal{C}$. The first set of terms on the right-hand side comes from the asymptotic formula for $\mathcal{M}$ given in Proposition 5.0.1. The power of T in the error term comes from the bound for $\mathcal{E}$ given in Theorem 7.1.1 (which also gives a factor of $(\ell m)^{1/2-1/(n^2+1)}$). A bound for $\mathcal{K}$, which is a (finite) sum of terms $\mathcal{I}_w$, with the same power of T but with the given power of ℓm follows as a consequence of Proposition 6.0.1. □

1.6. ***Outline of the key ideas in the proofs.*** Fix $n \geq 2$. The $\mathrm{GL}(n, \mathbb{R})$ orthogonality relation appears directly in the spectral side of the Kuznetsov trace formula for $\mathrm{GL}(n, \mathbb{R})$, which we now discuss. The Kuznetsov trace formula is obtained by computing the inner product of two Poincaré series on $\mathrm{SL}(n, \mathbb{Z})\backslash\mathfrak{h}^n$ in two different ways. The Poincaré series are constructed in a similar manner to Borel Eisenstein series by taking all $U_n(\mathbb{Z})\backslash \mathrm{SL}(n, \mathbb{Z})$ translates of a certain test function which we choose to be the $p^{(n)}_{T,R}$ test function in Definition 1.4.4 multiplied by a character and a power function (see Definition 2.3.7).

The first way of computing the inner product of two Poincaré series is to replace one of the Poincaré series with its spectral expansion into cusp forms and Eisenstein series and then unravel the other Poincaré series with the Rankin–Selberg method. This gives the spectral contribution which has two parts: the cuspidal contribution and the Eisenstein contribution. The second way of computing the inner product is to replace one of the Poincaré series with its Fourier Whittaker expansion and then unravel the other Poincaré series with the Rankin–Selberg method. This is called the geometric contribution to the trace formula, which also consists of two parts: a main term, and the so-called *Kloosterman contribution*. The precise results of these computations are given in Theorems 4.1.1 and 4.2.1, respectively.

Bounding the Eisenstein contribution. The key component of the Eisenstein contribution to the Kuznetsov trace formula is the inner product of an Eisenstein series and the Poincaré series P^M given in Definition 2.3.7. By unraveling the Poincaré series in the inner product (see Proposition 4.1.2) we essentially obtain the M-th Fourier coefficient of the Eisenstein series multiplied by the Whittaker transform of $p^{(n)}_{T,R}$. The explicit formula for the M-th Fourier coefficient of the most general Langlands Eisenstein series given in Proposition 4.1.5 allows us to effectively bound all the terms in the integrals appearing in the Eisenstein contribution except for the product of adjoint L-functions

$$\prod_{\substack{k=1\\ n_k\neq 1}}^{r} L^*(1, \mathrm{Ad}\,\phi_k)^{-\frac{1}{2}} \tag{1.6.1}$$

appearing in that proposition. When considering the Eisenstein contribution to the Kuznetsov trace formula for $\mathrm{GL}(n, \mathbb{R})$, all the adjoint L-functions in the above product are for cusp forms ϕ_k of lower rank $n_k < n$. Now in the special case that $\ell = m = 1$, our main theorem, Theorem 1.5.1, for $\mathrm{GL}(n, \mathbb{R})$

gives a sharp bound for the sum of reciprocals of all adjoint L-functions of lower rank. This allows us to inductively prove a power savings bound for the product (1.6.1).

Asymptotic formula for the geometric contribution. We prove that the geometric contribution is a sum of expressions $\mathcal{I}_w$ over elements w in the Weyl group of $\mathrm{SL}(n,\mathbb{Z})$. The $\mathcal{I}_w$ are complicated multiple sums of multiple integrals weighted by Kloosterman sums (see (4.2.2)). If w_1 is the trivial element of the Weyl group then we obtain an asymptotic formula for $\mathcal{I}_{w_1}$ (see Proposition 5.0.1), while for all other Weyl group elements $\mathcal{I}_{w_i}$, with $i>1$, we obtain error terms with strong bounds for $|\mathcal{I}_{w_i}|$ (see Proposition 6.0.1) which are bounded by the final error term on the right side of our main theorem.

The key terms in (4.2.2), the formula for $\mathcal{I}_w$, are the Kloosterman sums and two appearances of the test function $p^{(n)}_{T,R}$: one that is twisted by the Weyl group element w and one that is not. For the Kloosterman sums, we rely on bounds given by [Dąbrowski and Reeder 1998]. The task of giving strong bounds for $p^{(n)}_{T,R}(y)$ occupies Sections 8, 9 and 10. We deal with the combinatorics of the twisted $p^{(n)}_{T,R}$-function, and we combine the bounds for it, the other $p^{(n)}_{T,R}$-function and the Kloosterman sums in Section 6.

The function $p^{(n)}_{T,R}$ is the inverse Whittaker transform of the test function $p^{n,\#}_{T,R}$ given in Definition 1.4.4 above. Thanks to a formula of [Goldfeld and Kontorovich 2012], we can realize this as an integral of the product of $p^{n,\#}_{T,R}$, the Whittaker function W_α (see Definition 2.3.3), and certain additional gamma factors. We then write the Whittaker function as the inverse Mellin transform of its Mellin transform: $\widetilde{W}_{n,\alpha}(s)$. This leads to the formula (valid for any $\varepsilon>0$)

$$p^{(n)}_{T,R}(y)=\frac{1}{2^{n-1}}\int\limits_{\mathrm{Re}(\alpha_1)=0}\cdots\int\limits_{\mathrm{Re}(\alpha_{n-1})=0} e^{\frac{\alpha_1^2+\alpha_2^2+\cdots+\alpha_n^2}{T^2/2}}\mathcal{F}^{(n)}_R(\alpha)\prod_{1\le j\ne k\le n}\frac{\Gamma\left(\frac{1+2R+\alpha_j-\alpha_k}{4}\right)}{\Gamma\left(\frac{\alpha_j-\alpha_k}{2}\right)}$$
$$\cdot\int\limits_{\mathrm{Re}(s_1)=\varepsilon}\cdots\int\limits_{\mathrm{Re}(s_{n-1})=\varepsilon}\left(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}}(\pi y_j)^{-2s_j}\right)\widetilde{W}_{n,\alpha}(s)\,ds\,d\alpha.$$

To estimate the growth of $p^{(n)}_{T,R}(y)$ uniformly in y and T as $T\to+\infty$, we shift the line of integration in the s-integrals to $\mathrm{Re}(s)=-a$, with $a=(a_1,\dots,a_{n-1})$, where $a_i>0$ for $i=1,\dots,n-1$. We remark that this is precisely where the Ishii–Stade conjecture is required. It is well known that

$$\widetilde{W}_{2,\alpha}(s)=\Gamma(s+\alpha)\Gamma(s-\alpha),$$

and hence understanding the values of $\widetilde{W}_{2,\alpha}(s)$ for $\mathrm{Re}(s)<0$ is straightforward by applying the functional equation for the Gamma function or, equivalently, using an integral representation of the Gamma function valid for $\mathrm{Re}(s)<0$. A similar strategy can be used when $n=3$. However, for $n\ge 4$, the analogous method seems intractable because the Mellin transform is not just a ratio of Gamma functions, but an integral of such. To overcome this difficulty, we apply the Ishii–Stade conjecture to describe the values of $\widetilde{W}_{n,\alpha}(s)$ in terms of sums of the Mellin transform of shifts of the s-variables. See also Remark 8.2.11 below.

The Cauchy residue formula allows us to express $p^{(n)}_{T,R}$ as a sum of the shifted s-integral (termed the *shifted* $p^{(n)}_{T,R}$ *term* and denoted by $p^{(n)}_{T,R}(y;-a)$) and many residue terms. The description of the shifted

$p^{(n)}_{T,R}$ and residue terms is given in Section 8.3. In order to bound $p^{(n)}_{T,R}(y; -a)$ it is convenient to introduce the function $\mathcal{I}_{T,R}(-a) := p^{(n)}_{T,R}(1; -a)$.

The next step is to use a result of Ishii and Stade (see Theorem 8.1.5) which allows us to write the Mellin transform $\widetilde{W}_{n,\alpha}(s)$ as an integral transformation of $\widetilde{W}_{n-1,\beta}(z)$ against certain additional gamma factors. It is important to note that $\beta = (\beta_1, \dots, \beta_{n-1}) \in \mathbb{C}^{n-1}$ can be expressed in terms of $\alpha = (\alpha_1, \dots, \alpha_n)$. By carefully teasing apart the portion of α which determines β and that which doesn't, we are able to separate out the gamma factors that don't depend on β and bound $\mathcal{I}^{(n)}_{T,R}(-a)$ by the product of a power of T and $\mathcal{I}^{(n-1)}_{T,R}(-b)$ for a certain $b = (b_1, \dots, b_{n-1)} \in \mathbb{R}^{n-2}$. This gives an inductive procedure, therefore, for bounding the shifted $p^{(n)}_{T,R}$ term.

In Section 10.2 we set notation for describing the $(r-1)$-fold shifted residue terms. This requires generalizing a result of Stade (see Theorem 10.1.1) on the first set of residues of $\widetilde{W}_{n,\alpha}(s)$ (i.e., those that occur at $\mathrm{Re}(s_i) = 0$) to, first, higher-order residues (i.e., taking the residue with respect to multiple values s_i), and second, to residues which occur along the lines $\mathrm{Re}(s_i) = -k$ for $k \in \mathbb{Z}_{\geq 0}$. This result, together with a teasing out of the variables similar to that described above, allows us to bound an $(r-1)$-fold residue term as the product of certain powers of T and the variables $y_1, \dots, y_{n-1}$ times

$$\prod_{j=1}^{r} \mathcal{I}^{(n_j)}_{T,R}(-a^{(j)}), \quad \text{where } n = n_1 + \cdots + n_r.$$

Applying the bounds on $\mathcal{I}^{(n_j)}_{T,R}$ that we inductively established for bounding the shifted $p^{(n)}_{T,R}$ term, and keeping careful track of all of the exponents and terms $a^{(j)}$, we eventually show that the bound for the shifted main term is in fact valid for every residue term as well.

Remark 1.6.2. In comparison to the results of [Goldfeld and Kontorovich 2013; Goldfeld et al. 2021b], we are using a slightly different normalization of the Gamma functions and the auxiliary polynomial $\mathcal{F}^{(n)}_{R}$ in the definition of the test functions $p^{n,\#}_{T,R}$ and $h^{(n)}_{T,R}$ (see Definition 1.4.4). Adjusting for this difference the results obtained here when applied to $n = 3$ and $n = 4$ recover the previously proven asymptotic formulae.

2. Preliminaries

2.1. *Notational conventions.*

Definition 2.1.1 (hat notation for summation). Suppose that $m \in \mathbb{Z}_+$ and $x = (x_1, \dots, x_m) \in \mathbb{C}^m$. For any $0 \leq k \leq m$, define

$$\hat{x}_k := x_1 + \cdots + x_k.$$

Note that empty sums are assumed to be zero.

Definition 2.1.2 (integration notation). Let $n \geq 2$. We will often be working with n- and $(n-1)$-tuples of real or complex numbers. We will denote such tuples without a subscript and use subscripts to refer to the components. For example, we set $y = (y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1}_{>0}$, $s = (s_1, \dots, s_{n-1}) \in \mathbb{C}^{n-1}$ and

$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ such that

$$\alpha_1 + \cdots + \alpha_n = 0.$$

In such cases, we denote integration over all such variables $x = (x_1, \ldots, x_k)$ subject to condition(s) $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_k)$ via

$$\int\limits_{\mathcal{C}} F(x)\, dx := \int\limits_{\mathcal{C}_1} \cdots \int\limits_{\mathcal{C}_k} F(x_1, \ldots, x_k)\, dx_1\, dx_2 \cdots dx_k.$$

For example, given $\beta = (\beta_1, \ldots, \beta_{n-1}) \in \mathbb{C}^{n-1}$ with $\hat{\beta}_{n-1} = 0$, we denote integration over all such β with $\mathrm{Re}(\beta_j) = b_j$ for each $j = 1, \ldots, n-2$ via

$$\int\limits_{\substack{\hat{\beta}_{n-1}=0\\ \mathrm{Re}(\beta)=b}} F(\beta)\, d\beta := \int\limits_{\mathrm{Re}(\beta_1)=b_1} \cdots \int\limits_{\mathrm{Re}(\beta_{n-2})=b_{n-2}} F(\beta_1, \ldots, \beta_{n-2})\, d\beta_1\, d\beta_2 \cdots d\beta_{n-2}.$$

We extend this notation liberally to integrals over s, z and α and apply it also to integrals over the imaginary parts in the sequel.

Definition 2.1.3 (polynomial notation). Our analysis will often require us to bound certain polynomials in a trivial way. Namely, for complex variables x_j, with $j = 1, 2, \ldots, k$, if $|x_j| \ll T^{1+\varepsilon}$ for each j and $P(x_1, x_2, \ldots, x_k)$ is a polynomial, then $|P(x_1, x_2, \ldots, x_k)| \ll T^{\varepsilon+\deg P}$. So, the relevant information about P is its degree. This being the case, we will use the notation $\mathcal{P}_d(x)$ (with $x = (x_1, \ldots, x_k)$) to represent an unspecified polynomial of degree less than or equal to d in the variable(s) x. Note that this notation agrees with the commonly employed practice (also used throughout these notes) of using ε to represent an unspecified positive real number whose precise value is not specified and may differ from one usage to another.

Definition 2.1.4 (vector or matrix notation depending on context). Given a vector $a = (a_1, \ldots, a_{n-1}) \in \mathbb{R}^{n-1}$, we shall define the diagonal matrix

$$t(a) := \mathrm{diag}(a_1 a_2 \cdots a_{n-1}, a_1 a_2 \cdots a_{n-2}, \ldots, a_1, 1).$$

2.2. ***Structure of*** **GL(n).** Suppose n is a positive integer. Let $U_n(\mathbb{R}) \subseteq \mathrm{GL}(n, \mathbb{R})$ denote the set of upper triangular unipotent matrices.

Definition 2.2.1 (character of $U_n(\mathbb{R})$). Let $M = (m_1, \ldots, m_{n-1}) \in \mathbb{Z}^{n-1}$. For an element $x \in U_n(\mathbb{R})$ of the form

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & & x_{1,n} \\ & 1 & x_{2,3} & \cdots & & x_{2,n} \\ & & \ddots & & & \vdots \\ & & & & 1 & x_{n-1,n} \\ & & & & & 1 \end{pmatrix}, \tag{2.2.2}$$

we define the character

$$\psi_M(x) := m_1 x_{1,2} + m_2 x_{2,3} + \cdots + m_{n-1} x_{n-1,n}. \tag{2.2.3}$$

Definition 2.2.4 (generalized upper half-plane). We denote the set of (real) orthogonal matrices $\mathrm{O}(n,\mathbb{R})\subseteq\mathrm{GL}(n,\mathbb{R})$, and we set

$$\mathfrak{h}^n := \mathrm{GL}(n,\mathbb{R})/(\mathrm{O}(n,\mathbb{R})\cdot\mathbb{R}^\times).$$

Every element (via the Iwasawa decomposition of $\mathrm{GL}(n)$ [Goldfeld 2015]) of $\mathfrak{h}^n$ has a coset representative of the form $g = xy$, with x as above and

$$y=\begin{pmatrix} y_1y_2\cdots y_{n-1} & & & & \\ & y_1y_2\cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1\end{pmatrix}, \tag{2.2.5}$$

where $y_i > 0$ for each $1\le i\le n-1$. The group $\mathrm{GL}(n,\mathbb{R})$ acts as a group of transformations on $\mathfrak{h}^n$ by left multiplication.

Definition 2.2.6 (Weyl group and relevant elements). Let $W_n\cong S_n$ denote the Weyl group of $\mathrm{GL}(n,\mathbb{R})$. We consider it as the subgroup of $\mathrm{GL}(n,\mathbb{R})$ consisting of permutation matrices, i.e., matrices that have exactly one 1 in each row/column and all zeros otherwise. An element $w\in W_m$ is called *relevant* if

$$w = w_{(n_1,n_2,\dots,n_r)} := \begin{pmatrix} & & I_{n_r}\\ & \iddots & \\ I_{n_1} & & \end{pmatrix},$$

where I_{n_i} is the identity matrix of size $n_i\times n_i$ and $n = n_1+\cdots+n_r$ is a composition (a way of writing n as a sum of positive integers; see Section 8.3). The *long element of* W_n is $w_{\mathrm{long}} := w_{(1,1,\dots,1)}$.

Definition 2.2.7 (other subgroups of $\mathrm{GL}(n,\mathbb{R})$). We define

$$\begin{aligned}\overline{U}_w &:= \bigl(w^{-1}\cdot {}^tU_n(\mathbb{R})\cdot w\bigr)\cap U_n(\mathbb{R}),\\ \Gamma_w &:= \bigl(w^{-1}\cdot {}^tU_n(\mathbb{Z})\cdot w\bigr)\cap U_n(\mathbb{Z}) = \mathrm{SL}(n,\mathbb{Z})\cap\overline{U}_w,\end{aligned}$$

where ${}^tU_n$ denotes the transpose of U_n, i.e., the set of lower triangular unipotent matrices.

2.3. *Basic functions on the generalized upper half-plane* $\mathfrak{h}^n$.

Definition 2.3.1 (power function). Let $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{C}^n$, with $\hat{\alpha}_n = 0$. Let $\rho=(\rho_1,\dots,\rho_n)$, where $\rho_i = \frac{n+1}{2}-i$ for $i = 1,2,\dots,n$. We define a power function on $xy\in\mathfrak{h}^n$ by

$$I(xy,\alpha)=\prod_{i=1}^{n} d_i^{\alpha_i+\rho_i} = \prod_{i=1}^{n-1} y_i^{\hat{\alpha}_{n-i}+\hat{\rho}_{n-i}}, \tag{2.3.2}$$

where $d_i=\prod_{j\le n-i} y_j$ is the j-th diagonal entry of the matrix $g = xy$ as above.

Definition 2.3.3 (Jacquet's Whittaker function). Let $g\in\mathrm{GL}(n,\mathbb{R})$ with $n\ge 2$. Let $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)\in\mathbb{C}^n$, with $\hat{\alpha}_n = 0$. We define the completed Whittaker function $W_\alpha^\pm : \mathrm{GL}(n,\mathbb{R})/(\mathrm{O}(n,\mathbb{R})\cdot\mathbb{R}^\times)\to\mathbb{C}$ by the integral

$$W_\alpha^{\pm}(g) := \prod_{1\le j<k\le n} \frac{\Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)}{\pi^{(1+\alpha_j-\alpha_k)/2}} \cdot \int\limits_{U_4(\mathbb{R})} I(w_{\text{long}}ug,\alpha)\,\overline{\psi_{1,\dots,1,\pm 1}(u)}\,du,$$

which converges absolutely if $\mathrm{Re}(\alpha_i-\alpha_{i+1})>0$ for $1\le i\le n-1$ (see [Goldfeld et al. 2021a]), and has meromorphic continuation to all $\alpha\in\mathbb{C}^n$ satisfying $\hat{\alpha}_n=0$.

Remark 2.3.4. With the additional gamma factors included in this definition (which can be considered as a "*completed*" Whittaker function) there are $n!$ functional equations, which is equivalent to the fact that the Whittaker function is invariant under all permutations of $\alpha_1,\alpha_2,\dots,\alpha_n$. Moreover, even though the integral (without the normalizing factor) often vanishes identically as a function of α, this normalization never does.

If g is a diagonal matrix in $\mathrm{GL}(n,\mathbb{R})$ then the value of $W_{n,\alpha}^{\pm}(g)$ is independent of sign, so we drop the $\pm$. We also drop the $\pm$ if the sign is $+1$.

Definition 2.3.5 (Whittaker transform and its inverse). Assume $n\ge 2$. Let $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)\in\mathbb{C}^n$ with $\hat{\alpha}_n=0$. Set $y:=(y_1,y_2,\dots y_{n-1})$ and $t(y)$ as in Definition 2.1.4. Let $f:\mathbb{R}_+^{n-1}\to\mathbb{C}$ be an integrable function. Then we define the Whittaker transform $f^{\#}:H^n\to\mathbb{C}$ (where $H^n:=\{\alpha\in\mathbb{C}^n\mid\hat{\alpha}_n=0\}$) by

$$f^{\#}(\alpha) := \int\limits_{y_1=0}^{\infty}\cdots\int\limits_{y_{n-1}=0}^{\infty} f(y)\,W_\alpha(t(y))\prod_{k=1}^{n-1}\frac{dy_k}{y_k^{k(n-k)+1}}, \tag{2.3.6}$$

provided the above integral converges absolutely and uniformly on compact subsets of $\mathbb{R}_+^{n-1}$. The inverse Whittaker transform [Goldfeld and Kontorovich 2012, Theorem 1.6] is

$$f(y) = \frac{1}{\pi^{n-1}} \int\limits_{\substack{\hat{\alpha}_n=0\\ \mathrm{Re}(\alpha)=0}} \frac{f^{\#}(\alpha)W_{-\alpha}(t(y))}{\prod_{1\le k\ne\ell\le n}\Gamma\left(\frac{\alpha_k-\alpha_\ell}{2}\right)}\,d\alpha,$$

provided the above integral converges absolutely and uniformly on compact subsets of $(i\mathbb{R})^n$.

Definition 2.3.7 (normalized Poincaré series). Let $M=(m_1,m_2,\dots,m_{n-1})\in\mathbb{Z}^{n-1}$ with $m_i\ne 0$ for each $i=1,\dots,n-1$. As with y, we may think of M as a matrix. Let $g\in\mathfrak{h}^n$. Then we define

$$P^M(g,\alpha) := \frac{1}{\sqrt{\mathfrak{c}_n}}\cdot\prod_{k=1}^{n-1} m_k^{-\frac{k(n-k)}{2}} \sum_{\gamma\in U_n(\mathbb{Z})\backslash \mathrm{SL}(n,\mathbb{Z})} \psi_M(\gamma g)\cdot p_{T,R}^{(n)}(M\gamma g)\cdot I(\gamma g,\alpha), \tag{2.3.8}$$

where $\mathfrak{c}_n$ is the (nonzero) constant given in Proposition 4.1.4. We extend the definition of ψ_M and $p_{T,R}^{(n)}$ to all of $\mathfrak{h}^n$ by setting $\psi_M(xy):=\psi_M(x)$ and $p_{T,R}^{(n)}(xy):=p_{T,R}^{(n)}(y)$.

Remark 2.3.9. This definition, up to the normalizing factor $\sqrt{\mathfrak{c}_n}\prod_{k=1}^{n-1}m_k^{k(n-k)/2}$, of the Poincaré series agrees with that used in [Goldfeld et al. 2021b] with the minor caveat that $p_{T,R}$ takes on a slightly different normalization in terms of the polynomial $\mathcal{F}_R^{(n)}$ and in the gamma factors appearing in Definition 1.4.4. The normalizing factor is inserted so that in the Kuznetsov trace formula the cuspidal term is precisely the orthogonality relation in Theorem 1.5.1.

2.4. ***Fourier expansion of the Poincaré series.***

Definition 2.4.1 (twisted character). Let

$$V_n := \left\{ v = \begin{pmatrix} v_1 & & & \\ & v_2 & & \\ & & \ddots & \\ & & & v_n \end{pmatrix} \middle| v_1, \dots, v_n \in \{\pm 1\},\ v_1 \cdots v_n = 1 \right\}.$$

Let $M = (m_1, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$, and consider ψ_M the additive character (see (2.2.3)) of $U_n(\mathbb{R})$. Then for $v \in V_n$, we define the twisted character $\psi_M^v : U_n(\mathbb{R}) \to \mathbb{C}$ by $\psi_M^v(g) := \psi_M(v^{-1} g v)$.

Definition 2.4.2 (Kloosterman sum). Fix $L = (\ell_1, \dots, \ell_{n-1}),\ M = (m_1, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$. Let ψ_L, ψ_M be characters of $U_n(\mathbb{R})$. Let $w \in W_n$, where W_n is the Weyl group of GL(n). Let

$$c = \begin{pmatrix} 1/c_{n-1} & & & & \\ & c_{n-1}/c_{n-2} & & & \\ & & \ddots & & \\ & & & c_2/c_1 & \\ & & & & c_1 \end{pmatrix},$$

with $c_i \in \mathbb{Z}_{>0}$. Then the Kloosterman sum is defined as

$$S_w(\psi_L, \psi_M, c) := \sum_{\substack{\gamma = U_n(\mathbb{Z}) \backslash \Gamma \cap G_w / \Gamma_w \\ \gamma = \beta_1 c w \beta_2}} \psi_L(\beta_1)\, \psi_M(\beta_2),$$

with notation as in Definition 11.2.2 of [Goldfeld 2015]. The Kloosterman sum $S_w(\psi, \psi', c)$ is well-defined (i.e., independent of the choice of Bruhat decomposition for γ) if and only if it satisfies the compatibility condition $\psi(cwuw^{-1}) = \psi'(u)$. It is defined to be zero otherwise. (See [Friedberg 1987].)

Proposition 2.4.3 (M-th Fourier coefficient of the Poincaré series P^L). *Let* $L = (\ell_1, \dots, \ell_{n-1})$ *and* $M = (m_1, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$ *satisfy* $\prod_{i=1}^{n-1} \ell_i \neq 0$ *and* $\prod_{i=1}^{n-1} m_i \neq 0$. *If* $\mathrm{Re}(\alpha_k - \alpha_{k+1})$ *is sufficiently large for each* $k = 1, \dots, n-1$, *then*

$$\int_{U_n(\mathbb{Z}) \backslash U_n(\mathbb{R})} P^L(ug, \alpha) \cdot \overline{\psi_M(m)}\, d^*u = \sum_{w \in W_n} \sum_{v \in V_n} \sum_{c_1 = 1}^{\infty} \cdots \sum_{c_{n-1}=1}^{\infty} \frac{S_w(\psi_L, \psi_M^v, c) J_w(g; \alpha, \psi_L, \psi_M^v, c)}{\sqrt{\mathfrak{c}_n} \prod_{k=1}^{n-1} (\ell_k^{\frac{k(n-k)}{2}} c_k^{\alpha_k - \alpha_{k+1} + 1})},$$

where

$$J_w(g; \alpha, \psi_L, \psi_M^v, c) = \int_{U_w(\mathbb{Z}) \backslash U_w(\mathbb{R})} \int_{\overline{U}_w(\mathbb{R})} \psi_L(wug)\, p_{T,R}^{(n)}(Lcwug)\, I(wug, \alpha) \overline{\psi_M^v(u)}\, d^*u,$$

$$U_w(\mathbb{R}) = \big(w^{-1} \cdot U_n(\mathbb{R}) \cdot w\big) \cap U_n(\mathbb{R}), \quad \overline{U}_w(\mathbb{R}) = \big(w^{-1} \cdot {}^t U_n(\mathbb{R}) \cdot w\big) \cap U_n(\mathbb{R}),$$

and ${}^t m$ *denotes the transpose of a matrix* m.

Proof. See Theorem 11.5.4 of [Goldfeld 2015]. ☐

3. Spectral decomposition of $\mathcal{L}^2(\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n)$

3.1. *Hecke–Maass cusp forms for* SL$(n,\mathbb{Z})$.

Definition 3.1.1 (Langlands parameters). Let $n \geq 2$. A vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ is termed a Langlands parameter if $\hat{\alpha}_n = 0$.

Definition 3.1.2 (Hecke–Maass cusp forms). Fix $n \geq 2$. A Hecke–Maass cusp form with Langlands parameter $\alpha \in \mathbb{C}^n$ for $\mathrm{SL}(n,\mathbb{Z})$ is a smooth function $\phi : \mathfrak{h}^n \to \mathbb{C}$ which satisfies $\phi(\gamma g) = \phi(g)$ for all $\gamma \in \mathrm{SL}(n,\mathbb{Z})$, $g \in \mathfrak{h}^n$. In addition ϕ is square integrable, is an eigenfunction of the algebra of Hecke operators on $\mathfrak{h}^n$, and is an eigenfunction of the algebra of $\mathrm{GL}(n,\mathbb{R})$ invariant differential operators on $\mathfrak{h}^n$, with the same eigenvalues under this action as the power function $I(*,\alpha)$. The Laplace eigenvalue of ϕ is given by

$$\frac{n^3-n}{24} - \frac{\alpha_1^2+\alpha_2^2+\cdots+\alpha_n^2}{2}.$$

See Section 6 in [Miller 2002]. The Hecke–Maass cusp form ϕ is said to be tempered at ∞ if the Langlands parameters $\alpha_1, \dots, \alpha_n$ are all pure imaginary.

Proposition 3.1.3 (Fourier expansion of Hecke–Maass cusp forms). *Assume $n \geq 2$. Let $\phi : \mathfrak{h}^n \to \mathbb{C}$ be a Hecke–Maass cusp form for* $\mathrm{SL}(n,\mathbb{Z})$ *with Langlands parameters $\alpha \in \mathbb{C}^n$. Then for $g \in \mathfrak{h}^n$, we have the Fourier–Whittaker expansion*

$$\phi(g) = \sum_{\gamma \in U_{n-1}(\mathbb{Z})\backslash \mathrm{SL}_{n-1}(\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1}\neq 0} \frac{A_\phi(M)}{\prod_{k=1}^{n-1} |m_k|^{\frac{k(n-k)}{2}}} W_\alpha^{\mathrm{sgn}(m_{n-1})}\left(t(M)\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right),$$

where $M = (m_1, m_2, \dots, m_{n-1})$, $t(M)$ is the toric matrix in Definition 2.1.4 and $A_\phi(M)$ is the M-th Fourier coefficient of ϕ.

Proof. See Section 9.1 of [Goldfeld 2015]. □

Definition 3.1.4 (L-function associated to a Hecke–Maass form ϕ). Let $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ sufficiently large. Then the L-function associated to a Hecke–Maass cusp form ϕ is defined as

$$L(s,\phi) := \sum_{m=1}^{\infty} \frac{A_\phi(m,1,\dots,1)}{m^s}$$

and has holomorphic continuation to all $s \in \mathbb{C}$ and satisfies a functional equation $s \to 1-s$. If ϕ is a simultaneous eigenfunction of all the Hecke operators then $L(s,\phi)$ has the Euler product

$$L(s,\phi) = \prod_p \left(1 - \frac{A(p,1,\dots,1)}{p^s} + \frac{A(1,p,1,\dots,1)}{p^{2s}} - \frac{A(1,1,p,\dots,1)}{p^{3s}} + \cdots + (-1)^{n-1}\frac{A(1,\dots,1,p)}{p^{(n-1)s}} + \frac{(-1)^n}{p^{ns}}\right)^{-1}.$$

3.2. *Langlands Eisenstein series for* SL$(n,\mathbb{Z})$.

Definition 3.2.1 (parabolic subgroup). For $n \geq 2$ and $1 \leq r \leq n$, consider a partition of n given by $n = n_1 + \cdots + n_r$ with positive integers $n_1, \dots, n_r$. We define the standard parabolic subgroup

$$\mathcal{P} := \mathcal{P}_{n_1,n_2,\ldots,n_r} := \left\{ \begin{pmatrix} \mathrm{GL}(n_1) & * & \cdots & * \\ 0 & \mathrm{GL}(n_2) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{GL}(n_r) \end{pmatrix} \right\}.$$

Letting I_r denote the $r \times r$ identity matrix, the subgroup

$$N^{\mathcal{P}} := \left\{ \begin{pmatrix} I_{n_1} & * & \cdots & * \\ 0 & I_{n_2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n_r} \end{pmatrix} \right\}$$

is the unipotent radical of $\mathcal{P}$. The subgroup

$$M^{\mathcal{P}} := \left\{ \begin{pmatrix} \mathrm{GL}(n_1) & 0 & \cdots & 0 \\ 0 & \mathrm{GL}(n_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{GL}(n_r) \end{pmatrix} \right\}$$

is the standard choice of Levi subgroup of $\mathcal{P}$.

Definition 3.2.2 (Hecke–Maass form Φ associated to a parabolic $\mathcal{P}$). Let $n \geq 2$. Consider a partition $n = n_1 + \cdots + n_r$, with $1 < r < n$. Let $\mathcal{P} := \mathcal{P}_{n_1,n_2,\ldots,n_r} \subset \mathrm{GL}(n,\mathbb{R})$. For $i = 1, 2, \ldots, r$, let $\phi_i : \mathrm{GL}(n_i,\mathbb{R}) \to \mathbb{C}$ be either the constant function 1 (if $n_i = 1$) or a Hecke–Maass cusp form for $\mathrm{SL}(n_i,\mathbb{Z})$ (if $n_i > 1$). The form $\Phi := \phi_1 \otimes \cdots \otimes \phi_r$ is defined on $\mathrm{GL}(n,\mathbb{R}) = \mathcal{P}(\mathbb{R})$ (where $K = \mathrm{O}(n,\mathbb{R})$) by the formula

$$\Phi(nmk) := \prod_{i=1}^{r} \phi_i(m_i) \quad (n \in N^{\mathcal{P}}, m \in M^{\mathcal{P}}, k \in K)$$

where $m \in M^{\mathcal{P}}$ has the form

$$m = \begin{pmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_r \end{pmatrix},$$

with $m_i \in \mathrm{GL}(n_i,\mathbb{R})$. In fact, this construction works equally well if some or all of the ϕ_i are Eisenstein series.

Definition 3.2.3 (character of a parabolic subgroup). Let $n \geq 2$. Fix a partition $n = n_1 + n_2 + \cdots + n_r$ with associated parabolic subgroup $\mathcal{P} := \mathcal{P}_{n_1,n_2,\ldots,n_r}$. Define

$$\rho_{\mathcal{P}}(j) = \begin{cases} \frac{1}{2}(n - n_1), & j = 1 \\ \frac{1}{2}(n - n_j) - n_1 - \cdots - n_{j-1}, & j \geq 2. \end{cases} \tag{3.2.4}$$

Let $s = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r$ satisfy $\sum_{i=1}^{r} n_i s_i = 0$. Consider the function (see Definition 2.3.1)

$$|\cdot|_{\mathcal{P}}^{s} := I(\,\cdot\,,\alpha)$$

on $\mathrm{GL}(n,\mathbb{R})$, where

$$\alpha = \Bigg(\overbrace{s_1-\rho_{\mathcal{P}}(1)+\frac{1-n_1}{2},\, s_1-\rho_{\mathcal{P}}(1)+\frac{3-n_1}{2},\,\dots,\, s_1-\rho_{\mathcal{P}}(1)+\frac{n_1-1}{2}}^{n_1 \text{ terms}},\ \overbrace{s_2-\rho_{\mathcal{P}}(2)+\frac{1-n_2}{2},\, s_2-\rho_{\mathcal{P}}(2)+\frac{3-n_2}{2},\,\dots,\, s_2-\rho_{\mathcal{P}}(2)+\frac{n_2-1}{2}}^{n_2 \text{ terms}},\dots,\ \overbrace{s_r-\rho_{\mathcal{P}}(r)+\frac{1-n_r}{2},\, s_r-\rho_{\mathcal{P}}(r)+\frac{3-n_r}{2},\,\dots,\, s_r-\rho_{\mathcal{P}}(r)+\frac{n_r-1}{2}}^{n_r \text{ terms}}\Bigg).$$

The conditions $\sum_{i=1}^r n_i s_i = 0$ and $\sum_{i=1}^r n_i \rho_{\mathcal{P}}(i) = 0$ guarantee that α's entries sum to zero. When $g \in \mathcal{P}$, with diagonal block entries $m_i \in \mathrm{GL}(n_i,\mathbb{R})$, one has

$$|g|_{\mathcal{P}}^{s} = \prod_{i=1}^{r} |\det(m_i)|^{s_i},$$

so that $|\cdot|_{\mathcal{P}}^{s}$ restricts to a character of $\mathcal{P}$ which is trivial on $N^{\mathcal{P}}$.

Definition 3.2.5 (Langlands Eisenstein series twisted by Hecke–Maass forms of lower rank). Let $\Gamma = \mathrm{SL}(n,\mathbb{Z})$, with $n \geq 2$. Consider a parabolic subgroup $\mathcal{P} = \mathcal{P}_{n_1,\dots,n_r}$ of $\mathrm{GL}(n,\mathbb{R})$ and functions Φ and $|\cdot|_{\mathcal{P}}^{s}$ as given in Definitions 3.2.2 and 3.2.3, respectively. Let

$$s = (s_1, s_2, \dots, s_r) \in \mathbb{C}^r, \quad \text{where } \sum_{i=1}^{r} n_i s_i = 0.$$

The Langlands Eisenstein series determined by this data is defined by

$$E_{\mathcal{P},\Phi}(g,s) := \sum_{\gamma \in (\mathcal{P}\cap\Gamma)\backslash\Gamma} \Phi(\gamma g)\cdot|\gamma g|_{\mathcal{P}}^{s+\rho_{\mathcal{P}}} \tag{3.2.6}$$

as an absolutely convergent sum for $\mathrm{Re}(s_i)$ sufficiently large, and extends to all $s \in \mathbb{C}^r$ by meromorphic continuation.

For $k = 1, 2, \dots, r$, let $\alpha^{(k)} := (\alpha_{k,1}, \dots, \alpha_{k,n_k})$ denote the Langlands parameters of ϕ_k. We adopt the convention that if $n_k = 1$ then $\alpha_{k,1} = 0$. Then the Langlands parameters of $E_{\mathcal{P},\Phi}(g,s)$ (denoted by $\alpha_{\mathcal{P},\Phi}(s)$) are

$$\Big(\overbrace{\alpha_{1,1}+s_1,\dots,\alpha_{1,n_1}+s_1}^{n_1 \text{ terms}},\quad \overbrace{\alpha_{2,1}+s_2,\dots,\alpha_{2,n_2}+s_2}^{n_2 \text{ terms}},\quad \dots\quad, \overbrace{\alpha_{r,1}+s_r,\dots,\alpha_{r,n_r}+s_r}^{n_r \text{ terms}}\Big). \tag{3.2.7}$$

Definition 3.2.8 (the M-th Fourier coefficient of $E_{\mathcal{P},\Phi}$). Let $s = (s_1, s_2, \dots, s_r) \in \mathbb{C}^r$, where $\sum_{i=1}^r n_i s_i = 0$. Consider $E_{\mathcal{P},\Phi}(*,s)$ with associated Langlands parameters $\alpha_{\mathcal{P},\Phi}(s)$ as defined in (3.2.7). Let $M = (m_1, m_2, \dots, m_{n-1}) \in \mathbb{Z}_{>0}^{n-1}$. Then the M-th term in the Fourier–Whittaker expansion of $E_{\mathcal{P},\Phi}$ is

$$\int_{U_n(\mathbb{Z})\backslash U_n(\mathbb{R})} E_{\mathcal{P},\Phi}(ug,s)\,\overline{\psi_M(m)}\,du = \frac{A_{E_{\mathcal{P},\Phi}}(M,s)}{\prod_{k=1}^{n-1} m_k^{k(n-k)/2}}\, W_{\alpha_{\mathcal{P},\Phi}(s)}(Mg),$$

3.3. *Langlands spectral decomposition for* SL(n, $\mathbb{Z}$).

Definition 3.3.1 (Petersson inner product). Let $n \geq 2$. For $F, G \in \mathcal{L}^2(\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n)$ we define the Petersson inner product to be

$$\langle F, G\rangle := \int_{\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n} F(g)\overline{G(g)}\,dg.$$

For $g = xy \in \mathfrak{h}^n$, with

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & & x_{1,n} \\ & 1 & x_{2,3} & \cdots & & x_{2,n} \\ & & \ddots & & & \vdots \\ & & & & 1 & x_{n-1,n} \\ & & & & & 1 \end{pmatrix}, \quad y = \begin{pmatrix} y_1y_2\cdots y_{n-1} & & & & \\ & y_1y_2\cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix},$$

the measure dg is given by $dx\,dy$, with

$$dx = \prod_{1\leq i<j\leq n} dx_{i,j}, \qquad dy = \prod_{k=1}^{n-1} \frac{dy_k}{y^{k(n-k)+1}}.$$

The Langlands spectral decomposition for SL(n, $\mathbb{Z}$) states that

$$\mathcal{L}^2(\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n) = (\text{cuspidal spectrum}) \oplus (\text{residual spectrum}) \oplus (\text{continuous spectrum}).$$

We shall be applying the Langlands spectral decomposition to Poincaré series which are orthogonal to the residual spectrum.

Theorem 3.3.2 (Langlands spectral decomposition for SL(n, $\mathbb{Z}$)). *Let $\phi_1, \phi_2, \phi_3, \ldots$ denote an orthogonal basis of Hecke–Maass forms for* SL(n, $\mathbb{Z}$). *Assume that $F, G \in \mathcal{L}^2(\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n)$ are orthogonal to the residual spectrum. Then for $g \in \mathrm{GL}(n,\mathbb{R})$ we have*

$$F(g) = \sum_{j=1}^{\infty}\langle F, \phi_j\rangle \frac{\phi_j(g)}{\langle\phi_j,\phi_j\rangle} + \sum_{\mathcal{P}}\sum_{\Phi} c_{\mathcal{P}} \int_{\substack{n_1s_1+\cdots+n_rs_r=0\\ \mathrm{Re}(s)=0}} \langle F, E_{\mathcal{P},\Phi}(*\,,s)\rangle E_{\mathcal{P},\Phi}(g\,,s)\,ds,$$

$$\langle F, G\rangle = \sum_{j=1}^{\infty} \frac{\langle F,\phi_j\rangle\,\langle\phi_j, G\rangle}{\langle\phi_j,\phi_j\rangle} + \sum_{\mathcal{P}}\sum_{\Phi} c_{\mathcal{P}} \int_{\substack{n_1s_1+\cdots+n_rs_r=0\\ \mathrm{Re}(s)=0}} \langle F,\ E_{\mathcal{P},\Phi}(*\,,s)\rangle\langle E_{\mathcal{P},\Phi}(*\,,s),\ G\rangle\,ds,$$

where the sum over $\mathcal{P}$ ranges over parabolics associated to partitions $n = \sum_{k=1}^{r} n_k$, while the sum over Φ (see Definition 3.2.2) ranges over an orthonormal basis of Hecke–Maass forms associated to $\mathcal{P}$. Furthermore, $c_{\mathcal{P}}$ is a fixed nonzero constant.

Proof. For proofs see [Arthur 1979; Langlands 1976; Mœglin and Waldspurger 1995]. □

4. Kuznetsov trace formula

The Kuznetsov trace formula is derived by computing the inner product of two Poincaré series in two different ways. More precisely, let $L = (\ell_1, \ldots, \ell_{m-1})$, $M = (m_1, \ldots, m_{n-1}) \in \mathbb{Z}^{n-1}$, with $\prod_{i=1}^{n-1} m_i \neq 0$ and $\prod_{i=1}^{n-1} \ell_i \neq 0$, and consider the Petersson inner product $\langle P^L, P^M\rangle$.

In particular since $P^L, P^M \in \mathcal{L}^2(\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n)$ (see [Friedberg 1987]), the inner product can be computed with the spectral expansion of the Poincaré series. The geometric approach utilizes the Fourier Whittaker expansion of the Poincaré series which involve Kloosterman sums.

The trace formula takes the form

$$\underbrace{\mathcal{C}+\mathcal{E}}_{\text{spectral side}} = \underbrace{\mathcal{M}+\mathcal{K}}_{\text{geometric side}}. \tag{4.0.1}$$

Here $\mathcal{C}$ is the cuspidal contribution and $\mathcal{E}$ is the Eisenstein contribution. See Theorem 4.1.1 for their precise definitions. The geometric side consists of terms corresponding to elements of the Weyl group. The identity element gives the main term $\mathcal{M}$, and the Kloosterman contribution $\mathcal{K}$ is the sum of the remaining terms. See Theorem 4.2.1 for their precise definitions. The Kloosterman term $\mathcal{K}$ and the Eisenstein contribution $\mathcal{E}$ will be small with the special choice of the test function $p_{T,R}$, and they constitute the error term in the main theorem.

4.1. ***Spectral side of the Kuznetsov trace formula.*** The first way to compute the inner product of the Poincaré series uses the spectral decomposition of the Poincaré series.

Recall also the definition of the adjoint L-function: $L(s, \mathrm{Ad}\,\phi) := L(s, \phi\times\bar{\phi})/\zeta(s)$, where $L(s,\phi\times\bar{\phi})$ is the Rankin–Selberg convolution L-function as in Section 12.1 of [Goldfeld 2015].

Theorem 4.1.1 (spectral decomposition for the inner product of Poincaré series). *Fix $n \ge 2$ and $L = (\ell_1, \dots, \ell_{n-1})$, $M = (m_1, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$. Then for $\alpha_0 := \left(-\frac{n-1}{2} + j - 1\right)_{j=1,\dots,n}$ we have*

$$\langle P^L(*, \alpha_0), P^M(*, \alpha_0)\rangle = \mathcal{C}+\mathcal{E}.$$

With the notation of the spectral decomposition theorem (Theorem 3.3.2) the cuspidal contribution to the Kuznetsov trace formula is

$$\mathcal{C} := \sum_{i=1}^{\infty} \frac{\lambda_{\phi_i}(M)\overline{\lambda_{\phi_i}(L)}\cdot |p^{n,\#}_{T,R}(\alpha^{(i)})|^2}{L(1,\mathrm{Ad}\,\phi_i)\cdot \prod_{1\le j\ne k\le n}\Gamma((1+\alpha^{(i)}_j-\alpha^{(i)}_k)/2)}$$

and the Eisenstein contribution to the Kuznetsov trace formula is

$$\mathcal{E} := \sum_{\mathcal{P}}\sum_{\Phi} c_{\mathcal{P}} \int\limits_{\substack{n_1 s_1+\cdots+n_r s_r=0\\ \mathrm{Re}(s_j)=0}} A_{E_{\mathcal{P},\Phi}}(L,s)\,\overline{A_{E_{\mathcal{P},\Phi}}(M,s)}\cdot |p^{n,\#}_{T,R}(\alpha_{(\mathcal{P},\Phi)}(s))|^2\,ds$$

for constants $c_{\mathcal{P}} > 0$.

Proof. The proof follows from the Langlands spectral decomposition theorem (Theorem 3.3.2) with the choices $F = P^L$ and $G = P^M$. We have

$$\langle P^L, P^M\rangle = \sum_{j=1}^{\infty}\frac{\langle P^L,\phi_j\rangle\,\langle\phi_j, P^M\rangle}{\langle\phi_j,\phi_j\rangle} + \sum_{\mathcal{P}}\sum_{\Phi} c_{\mathcal{P}} \int\limits_{\substack{n_1 s_1+\cdots+n_r s_r=0\\ \mathrm{Re}(s_j)=0}} \langle F,\ E_{\mathcal{P},\Phi}(*,s)\rangle\langle E_{\mathcal{P},\Phi}(*,s),\ G\rangle\,ds.$$

We then insert the inner products given in Proposition 4.1.2 below. Doing so, we see that the cuspidal spectrum is

$$\sum_{i=1}^{\infty}\frac{\langle P^L,\phi_i\rangle\langle\phi_i,P^M\rangle}{\langle\phi_i,\phi_i\rangle}=\sum_{i=1}^{\infty}\frac{A_{\phi_i}(M)\overline{A_{\phi_i}(L)}}{\mathfrak{c}_n\cdot\langle\phi_i,\phi_i\rangle}|p_{T,R}^{n,\#}(\alpha^{(i)})|^2.$$

From Proposition 4.1.4, we see that

$$A_\phi(M)\overline{A_\phi(L)}=|A_\phi(1)|^2\lambda_\phi(M)\overline{\lambda_\phi(L)}=\frac{\mathfrak{c}_n\cdot\langle\phi,\phi\rangle\cdot\lambda_\phi(M)\overline{\lambda_\phi(L)}}{L(1,\operatorname{Ad}\phi)\prod_{1\le j\ne k\le n}\Gamma\bigl(\frac{1+\alpha_j-\alpha_k}{2}\bigr)}.$$

The cuspidal part is now immediate. The contributions from the Eisenstein series are computed in like manner using Proposition 4.1.5. □

Proposition 4.1.2 (the inner product of P^M with an Eisenstein series or Hecke–Maass form). *Let $M=(m_1,m_2,\dots,m_{n-1})$. Consider the Eisenstein series $E_{\mathcal{P},\Phi}(*,s)$, with associated Langlands parameters $\alpha_{\mathcal{P},\Phi}(s)$. Let ϕ denote a Hecke–Maass cusp form for $\mathrm{SL}(n,\mathbb{Z})$ with Langlands parameter α and M-th Fourier coefficient $A_\phi(M)$. Then for $\alpha_0:=\bigl(-\frac{n-1}{2}+j-1\bigr)_{j=1,\dots,n}$,*

$$\langle\phi,P^M(*,\alpha_0)\rangle=\frac{1}{\sqrt{\mathfrak{c}_n}}A_\phi(M)\cdot p_{T,R}^{n,\#}(\alpha),$$
$$\langle E_{\mathcal{P},\Phi}(*,s),P^M(*,\alpha_0)\rangle=\frac{1}{\sqrt{\mathfrak{c}_n}}A_{E_{\mathcal{P},\Phi}}(M,s)\cdot p_{T,R}^{n,\#}(\alpha_{\mathcal{P},\Phi}(s)),$$

where the inner products on the left are defined by analytic continuation and $\mathfrak{c}_n$ is the nonzero constant (depending only on n) from Proposition 4.1.4.

Proof. We outline the case of the Hecke–Maass forms. The series definition of the Poincaré series converges absolutely for sufficiently large $\mathrm{Re}(\alpha'_i-\alpha'_{i+1})$ $(1\le i\le n-1)$. It follows that for such α' we may unravel the Poincaré series $P^M(*,\alpha')$ in the inner product $\langle\phi,P^M\rangle$ with the Rankin–Selberg method. The inner product picks out the M-th Fourier coefficient of ϕ multiplied by a certain Whittaker transform of $p_{T,R}^{(n)}(My)\cdot I(y,\alpha')$. This Whittaker transform has analytic continuation in α' to a region including α_0. For sufficiently large $\mathrm{Re}(\alpha'_i-\alpha'_{i+1})$, we have from (2.3.8) that

$$\langle\phi,P^M(*,\alpha')\rangle=\frac{A_\phi(M)}{\sqrt{\mathfrak{c}_n}\prod_{k=1}^{n-1}m_k^{k(n-k)}}\int_{y_1=0}^{\infty}\cdots\int_{y_{n-1}=0}^{\infty}\overline{p_{T,R}^{(n)}(My)\cdot I(y,\alpha')}\cdot W_\alpha(My)\prod_{k=1}^{n-1}\frac{dy_k}{y_k^{k(n-k)+1}}. \quad (4.1.3)$$

Note that $I(y,\alpha_0)=1$. The integral in (4.1.3) converges (as a function of α') to a region which includes α_0. It follows that the analytic continuation in α' to α_0 of the inner product satisfies

$$\langle\phi,P^M(*,\alpha_0)\rangle=\frac{1}{\sqrt{\mathfrak{c}_n}}\cdot A_\phi(M)\cdot p_{T,R}^{n,\#}(\alpha).$$

The proof for $E_{\mathcal{P},\Phi}$ is the same. □

For $n\ge 2$, consider a Hecke–Maass cusp form ϕ for $\mathrm{SL}(n,\mathbb{Z})$ with Fourier Whittaker expansion given by Proposition 3.1.3. Assume ϕ is a Hecke eigenform. Let $A_\phi(1):=A_\phi(1,1,\dots,1)$ denote the first

Fourier–Whittaker coefficient of ϕ. Then we have

$$A_\phi(M) = A_\phi(1) \cdot \lambda_\phi(M),$$

where $\lambda_\phi(M)$ is the Hecke eigenvalue (see Section 9.3 in [Goldfeld 2015]), and $\lambda_\phi(1) = 1$.

Proposition 4.1.4 (first Fourier–Whittaker coefficient of a Hecke–Maass cusp form). *Assume $n \geq 2$. Let ϕ be a Hecke–Maass cusp form for $\mathrm{SL}(n, \mathbb{Z})$ with Langlands parameters $\alpha = (\alpha_1, \ldots, \alpha_n)$. Then the first coefficient $A_\phi(1)$ is given by*

$$|A_\phi(1)|^2 = \frac{\mathfrak{c}_n \cdot \langle \phi, \phi \rangle}{L(1, \mathrm{Ad}\,\phi) \prod_{1\leq j\neq k\leq n} \Gamma\big(\frac{1+\alpha_j-\alpha_k}{2}\big)},$$

where $\mathfrak{c}_n \neq 0$ is a constant depending on n only.

Proof. See [Goldfeld et al. 2021a]. □

Proposition 4.1.5 (the M-th Fourier coefficient of $E_{\mathcal{P},\Phi}$). *Let $s = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r$, where $\sum_{i=1}^r n_i s_i = 0$. Consider $E_{\mathcal{P},\Phi}(*, s)$ with associated Langlands parameters $\alpha_{\mathcal{P},\Phi}(s)$ as defined in* (3.2.7). *Assume that each Hecke–Maass form ϕ_k (with $1 \leq k \leq r$) occurring in Φ has Langlands parameters $\alpha^{(k)} := (\alpha_{k,1}, \ldots, \alpha_{k,n_k})$ with the convention that if $n_k = 1$ then $\alpha_{k,1} = 0$. We also assume that each ϕ_k is normalized to have Petersson norm $\langle \phi_k, \phi_k \rangle = 1$.*

Let $L^*(1+s_j-s_\ell, \phi_j \times \phi_\ell)$ denote the completed Rankin–Selberg L-function if $n_j \neq 1 \neq n_\ell$; otherwise define

$$L^*(1+s_j-s_\ell, \phi_j \times \phi_\ell) = \begin{cases} L^*(1+s_j-s_\ell, \phi_j) & \text{if } n_\ell = 1 \text{ and } n_j \neq 1, \\ L^*(1+s_j-s_\ell, \phi_\ell) & \text{if } n_j = 1 \text{ and } n_\ell \neq 1, \\ \zeta^*(1+s_j-s_\ell) & \text{if } n_j = n_\ell = 1, \end{cases}$$

where $\zeta^*(w) = \pi^{-w/2}\Gamma(w/2)\zeta(w)$ is the completed Riemann ζ-function. Also define

$$L^*(1, \mathrm{Ad}\,\phi_k) = L(1, \mathrm{Ad}\,\phi_k) \prod_{1\leq i\neq j\leq n_k} \Gamma\bigg(\frac{1+\alpha_{k,i}-\alpha_{k,j}}{2}\bigg),$$

with the convention that $L^*(1, \mathrm{Ad}\,1) = 1$.

Let $M = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{Z}_{>0}^{n-1}$. Per our convention (Definition 2.1.4), we may think of M as a vector or a diagonal matrix. Then the M-th term in the Fourier–Whittaker expansion of $E_{\mathcal{P},\Phi}$ is

$$\int_{U_n(\mathbb{Z})\backslash U_n(\mathbb{R})} E_{\mathcal{P},\Phi}(ug, s)\, \overline{\psi_M(m)}\, du = \frac{A_{E_{\mathcal{P},\Phi}}(M, s)}{\prod_{k=1}^{n-1} m_k^{k(n-k)/2}}\, W_{\alpha_{\mathcal{P},\Phi}(s)}(Mg),$$

where $A_{E_{\mathcal{P},\Phi}}(M, s) = A_{E_{\mathcal{P},\Phi}}((1, \ldots, 1), s) \cdot \lambda_{E_{\mathcal{P},\Phi}}(M, s)$,

$$\lambda_{E_{P,\Phi}}((m, 1, \ldots, 1), s) = \sum_{\substack{c_1,\ldots,c_n\in\mathbb{Z}_{>0} \\ c_1c_2\cdots c_n=m}} \lambda_{\phi_1}(c_1) \cdots \lambda_{\phi_r}(c_r) \cdot c_1^{s_1} \cdots c_r^{s_r} \tag{4.1.6}$$

is the $(m, 1, \dots, 1)$*-th* (*or more informally the* m*-th*) *Hecke eigenvalue of* $E_{\mathcal{P},\Phi}$, *and*

$$A_{E_{\mathcal{P},\Phi}}((1,\dots,1),s) = d_0 \prod_{\substack{k=1\\ n_k\neq 1}}^{r} L^*(1,\operatorname{Ad}\phi_k)^{-\frac{1}{2}} \prod_{1\le j<\ell\le r} L^*(1+s_j-s_\ell,\phi_j\times\phi_\ell)^{-1}$$

for some constant $d_0 \neq 0$ *depending only on* n.

Proof. See [Goldfeld et al. 2024]. □

4.2. ***Geometric side of the Kuznetsov trace formula.*** In this section, we obtain explicit descriptions of the terms $\mathcal{M}$ and $\mathcal{K}$ appearing on the geometric side of the Kuznetsov trace formula. In order to do this, we introduce Kloosterman sums for SL$(n,\mathbb{Z})$, which appear in the Fourier expansion of the Poincaré series. In the inner product $\langle P^L, P^M\rangle$ we replace P^L with its Fourier expansion and unravel P^M following the Rankin–Selberg method.

Theorem 4.2.1 (geometric side of the trace formula). *Fix* $L=(\ell_1,\dots,\ell_{n-1})$ *and* $M=(m_1,\dots,m_{n-1})\in\mathbb{Z}^{n-1}$ ($\mathfrak{c}_n$ *is a nonzero constant*; *see Proposition 4.1.4*). *It follows that for* $\alpha_0 := \left(-\frac{n-1}{2}+j-1\right)_{j=1,\dots,n}$

$$\langle P^L(*,\alpha_0), P^M(*,\alpha_0)\rangle = \mathcal{M}+\mathcal{K}.$$

For w_1 *the trivial element in the Weyl group* W_n, *we define*

$$\mathcal{M} := \mathcal{I}_{w_1} \quad \textit{and} \quad \mathcal{K} := \sum_{\substack{w\in W_n\\ w\neq w_1}} \mathcal{I}_w,$$

where

$$\mathcal{I}_w := \sum_{v\in V_n}\sum_{c_1=1}^{\infty}\cdots\sum_{c_{n-1}=1}^{\infty} \frac{S_w(\psi_L,\psi_M^v,c)}{\mathfrak{c}_n\cdot\prod_{k=1}^{n-1}(m_k\ell_k)^{\frac{k(n-k)}{2}}} \int_{y_1=0}^{\infty}\cdots\int_{y_{n-1}=0}^{\infty}\ \int_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})}\ \int_{\bar{U}_w(\mathbb{R})}$$

$$\cdot\,\psi_L(wuy)\,\overline{\psi_M^v(u)}\,p_{T,R}^{(n)}(Lcwuy)\,\overline{p_{T,R}^{(n)}(My)}\,d^*u\,\frac{dy_1\cdots dy_{n-1}}{\prod_{k=1}^{n-1}y_k^{k(n-k)+1}}. \tag{4.2.2}$$

Proof. We compute the inner product

$$\begin{aligned}
&\lim_{\alpha\to\alpha_0}\langle P^L(*,\alpha), P^M(*,\alpha)\rangle\\
&\qquad= \lim_{\alpha\to\alpha_0}\int_{\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n} P^L(g,\alpha)\cdot\overline{P^M(g,\alpha)}\,dg\\
&\qquad=\frac{1}{\sqrt{\mathfrak{c}_n}\prod_{k=1}^{n-1}m_k^{\frac{k(n-k)}{2}}}\lim_{\alpha\to\alpha_0}\int_{U_n(\mathbb{Z})\backslash\mathfrak{h}^n} P^L(g,\alpha)\cdot\overline{\psi_M(g)\,p_{T,R}^{(n)}(Mg)\,I(g,\alpha)}\,dg\\
&\qquad=\frac{1}{\sqrt{\mathfrak{c}_n}}\left(\prod_{k=1}^{n-1}m_k^{-\frac{k(n-k)}{2}}\right)\lim_{\alpha\to\alpha_0}\int_{\substack{y\in\mathbb{R}^{n-1}\\ y>0}}\left(\int_{U_n(\mathbb{Z})\backslash U_n(\mathbb{R})} P^L(uy,\alpha)\cdot\overline{\psi_M(m)}\,du\right)\overline{p_{T,R}^{(n)}(My)\,I(y,\alpha)}\,dy.
\end{aligned}$$

Note that, as $\alpha\to\alpha_0$, the function $I(g,\alpha)\to 1$ (for any $g\in\mathfrak{h}^n$) and $\prod_{k=1}^{n-1}c_k^{\alpha_k-\alpha_{k+1}+1}\to 1$. It follows from this and Proposition 2.4.3 above that

$$
\begin{aligned}
&\mathfrak{c}_n \cdot \prod_{k=1}^{n-1} (m_k\ell_k)^{k(n-k)/2} \cdot \lim_{\alpha\to\alpha_0} \langle P^L(*,\alpha),\, P^M(*,\alpha)\rangle \\
&= \lim_{\alpha\to\alpha_0} \sum_{w\in W_n}\sum_{v\in V_n}\sum_{c_1=1}^{\infty}\cdots\sum_{c_{n-1}=1}^{\infty} \frac{S_w(\psi_L,\psi_M^v,c)}{\mathfrak{c}_n\cdot\prod_{k=1}^{n-1}(m_i\ell_i)^{\frac{i(n-i)}{2}}\prod_{k=1}^{n-1} c_k^{\alpha_k-\alpha_{k+1}+1}} \\
&\qquad \cdot \int_{y_1=0}^{\infty}\cdots\int_{y_{n-1}=0}^{\infty}\int_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})}\int_{\bar{U}_w(\mathbb{R})} \psi_L(wuy)\,\overline{\psi_M^v(u)}\,p_{T,R}^{(n)}(Lcwuy)\,\overline{p_{T,R}^{(n)}(My)} \\
&\qquad\qquad \cdot I(wuy,\alpha)\overline{I(y,\alpha)}\,d^*u\,\frac{dy_1\cdots dy_{n-1}}{\prod_{k=1}^{n-1} y_k^{k(n-k)+1}} \\
&= \sum_{w\in W_n}\sum_{v\in V_n}\sum_{c_1=1}^{\infty}\cdots\sum_{c_{n-1}=1}^{\infty} \frac{S_w(\psi_L,\psi_M^v,c)}{\mathfrak{c}_n\cdot\prod_{k=1}^{n-1}(m_k\ell_k)^{\frac{k(n-k)}{2}}} \\
&\qquad \cdot \int_{y_1=0}^{\infty}\cdots\int_{y_{n-1}=0}^{\infty}\int_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})}\int_{\bar{U}_w(\mathbb{R})} \psi_L(wuy)\,\overline{\psi_M^v(u)}\,p_{T,R}^{(n)}(Lcwuy)\,\overline{p_{T,R}^{(n)}(My)}\,du\,\frac{dy_1\cdots dy_{n-1}}{\prod_{k=1}^{n-1} y_k^{k(n-k)+1}} \\
&= \sum_{w\in W_n} \mathcal{I}_w,
\end{aligned}
$$

as claimed. □

5. Asymptotic formula for the main term

Proposition 5.0.1 (main term in the trace formula). *Let $L=(\ell_1,\dots,\ell_{n-1})$, $M=(m_1,\dots,m_{n-1})\in\mathbb{Z}^{n-1}$ satisfy $\prod_{i=1}^{n-1}\ell_i\neq 0$ and $\prod_{i=1}^{n-1} m_i\neq 0$. There exist fixed constants $\mathfrak{c}_1,\dots,\mathfrak{c}_{n-1}>0$ (depending only on R and n) such that the main term $\mathcal{M}$ in the Kuznetsov trace formula* (4.0.1) *is given by*

$$
\mathcal{M} = \delta_{L,M}\cdot\left(\left(\sum_{i=1}^{n-1}\mathfrak{c}_i\cdot T^{R(2\cdot D(n)+n(n-1))+n-i}\right)+\mathcal{O}(T^{R(2\cdot D(n)+n(n-1))})\right),
$$

where

$$
D(n) = \frac{1}{2}\binom{2n}{n} - \frac{n(n-1)}{2} - 2^{n-1}
$$

and $\delta_{L,M}$ is the Kronecker symbol (i.e., $\delta_{L,M}=0$ if $L\neq M$and $\delta_{L,L}=1$).

Proof. It follows from the definition $\mathcal{M}=\mathcal{I}_{w_1}$, making the change of variables $y\mapsto M^{-1}y$ and noting that $U_{w_1}(\mathbb{Z})=U_n(\mathbb{Z})$ and $U_{w_1}(\mathbb{R})=U_n(\mathbb{R})$, that

$$
\begin{aligned}
\mathcal{M} &= \frac{1}{\mathfrak{c}_n}\cdot\int_{y_1=0}^{\infty}\cdots\int_{y_{n-1}=0}^{\infty}\left(\int_{U_n(\mathbb{Z})\backslash U_n(\mathbb{R})}\psi_L(u)\overline{\psi_M(m)}\,d^*u\right)p_{T,R}(LM^{-1}y)\,\overline{p_{T,R}(y)}\,\frac{dy_1\cdots dy_{n-1}}{\prod_{i=1}^{n-1} y_i^{i(n-i)+1}} \\
&= \delta_{L,M}\cdot\mathfrak{d}_n\int_{y_1=0}^{\infty}\cdots\int_{y_{n-1}=0}^{\infty}|p_{T,R}^{(n)}(y)|^2\,\frac{dy_1\cdots dy_{n-1}}{\prod_{i=1}^{n-1} y_i^{i(n-i)+1}} = \delta_{L,M}\cdot\mathfrak{d}_n\langle p_{T,R},\,p_{T,R}\rangle \\
&= \delta_{L,M}\cdot\mathfrak{d}_n\int_{\substack{\hat{\alpha}_n=0\\ \mathrm{Re}(\alpha)=0}} \frac{|p_{T,R}^{n,\#}(\alpha)|^2}{\prod_{1\le j\neq k\le n}\Gamma\left(\frac{\alpha_j-\alpha_k}{2}\right)}\,d\alpha \\
&= \delta_{L,M}\,\mathfrak{d}_n\cdot\langle p_{T,R}^{n,\#},\,p_{T,R}^{n,\#}\rangle,
\end{aligned}
$$

where the representation in terms of the norm of $p^{n,\#}_{T,R}$ follows from the Plancherel formula in Corollary 1.9 of [Goldfeld and Kontorovich 2012] and $\mathfrak{d}_n$ is a nonzero constant depending only on n. Hence the main term for GL(n) is thus

$$\mathcal{M} = \delta_{L,M}\mathfrak{d}_n \cdot \int\limits_{\substack{\hat{\alpha}_n=0\\ \mathrm{Re}(\alpha_j)=0}} \frac{\left| e^{(\alpha_1^2+\cdots+\alpha_n^2)/(2T^2)} \mathcal{F}^{(n)}_R\left(\frac{\alpha}{2}\right) \prod_{1\le j\ne k\le n} \Gamma\left(\frac{2R+1+\alpha_j-\alpha_k}{4}\right)\right|^2}{\prod_{1\le j\ne k\le n} \Gamma\left(\frac{\alpha_j-\alpha_k}{2}\right)}\, d\alpha.$$

Let $\alpha_j = i\tau_j$ with $\tau_j \in \mathbb{R}$. It then follows from Stirling's asymptotic formula that

$$\mathcal{M} \sim \delta_{L,M}\mathfrak{d}_n \cdot \int\limits_{\hat{\tau}_n=0} e^{(-\tau_1^2-\cdots-\tau_n^2)/T^2} \left(\mathcal{F}^{(n)}_R\left(\frac{i\tau}{2}\right)\right)^2 \prod_{1\le j<k\le n} (1+|\tau_j-\tau_k|)^{2R}\, d\tau.$$

If we now make the change of variables $\tau_j \to \tau_j T$ for each $j = 1, \ldots, n$, and we use the fact that the degree of $\mathcal{F}^{(n)}_1$ is $D(n)$ (see Definition 1.4.1) it follows that, if $L = M$, as $T \to \infty$ we have $\mathcal{M} \sim \mathfrak{c}T^{R\cdot(2D(n)+n(n-1))+n-1}$, where

$$\mathfrak{c} = \mathfrak{d}_n \cdot \int\limits_{\hat{\tau}_n=0} e^{-\tau_1^2-\cdots-\tau_n^2} \left(\mathcal{F}^{(n)}_R\left(\frac{i\tau}{2}\right)\right)^2 \prod_{1\le j<k\le n} (1+|\tau_j-\tau_k|)^{2R}\, d\tau,$$

and otherwise, the main term is zero. This gives the $i = 1$ term in the statement of the proposition. The method of proof can be extended by using additional terms in Stirling's asymptotic expansion for the Gamma function to obtain the additional terms. □

Remark 5.0.2. Note that this doesn't agree with [Goldfeld et al. 2021b] in the case of $n = 4$ because we have used a different normalization. Namely, the linear factors of $\mathcal{F}^{(n)}_R$ agree with those defined previously, but we take a different power of each. Also, the gamma factors which appear in $p^{n,\#}_{T,R}$ have a different R: namely, what was $2 + R$ in each gamma factor previously has been replaced by $2R + 1$ here.

6. Bounding the geometric side

The goal of this section is to use the bound given in Theorem 10.0.1 to prove the following, i.e., to bound $\mathcal{K}$, the geometric side of the Kuznetsov trace formula.

Proposition 6.0.1. *Let $\mathcal{I}_w$ be as above. Let $M = (m_1, \ldots, m_{n-1})$, $L = (\ell_1, \ldots, \ell_{n-1}) \in (\mathbb{Z}_{>0})^{n-1}$. Let $\rho \in \frac{1}{2} + \mathbb{Z}$. Let $D(n) = \frac{1}{2}\binom{2n}{n} - \frac{n(n-1)}{2} - 2^{n-1}$ as in* (1.4.2). *Then for R sufficiently large and any $\varepsilon > 0$, we have*

$$|\mathcal{I}_w| \ll_{\varepsilon,R} T^{\varepsilon+R(2D(n)+n(n-1))+\frac{(n-1)(n+4)}{2}-\left\lfloor\frac{n-1}{2}\right\rfloor-\rho n-\Phi(w)} \cdot \prod_{i=1}^{n-1} (\ell_i m_i)^{2\rho+\frac{n^2+1}{4}},$$

where if $w = w_{(n_1,\ldots,n_r)}$ with $r \ge 2$,

$$\Phi(w) := \Phi(n_1, \ldots, n_r) := \frac{1}{2}\sum_{k=1}^{r-1} (n_k + n_{k+1})(n - \hat{n}_k)\hat{n}_k.$$

Remark 6.0.2. Assuming the lower bound conjecture for Rankin–Selberg L-functions, the resulting bound for the Eisenstein series contribution to the Kuznetsov trace formula (see Theorem 7.1.1) is of the magnitude T to the power $R(2D(n)+n(n-1))+\varepsilon$. Therefore, given Proposition 6.0.1 and Lemma A.13 $\bigl($which says that $\Phi(w) \geq \Phi(1, n-1) = \frac{n(n-1)}{2}\bigr)$, in order for the bound from the geometric side of the trace formula to be less than the Eisenstein series contribution, it suffices that

$$\frac{(n-1)(n+4)}{2} - \left\lfloor \frac{n-1}{2} \right\rfloor - \rho n - \frac{n(n-1)}{2} \leq 0,$$

which simplifies to give

$$\rho \geq \begin{cases} \frac{3}{2} - \frac{3}{2n} & \text{if } n \text{ is odd,} \\ \frac{3}{2} - \frac{1}{n} & \text{if } n \text{ is even.} \end{cases}$$

Since we require that $\rho \in \frac{1}{2} + \mathbb{Z}$, we find that it suffices to take $\rho = \frac{3}{2}$ universally, meaning that the exponent of each term $\ell_i m_i$ can be taken to be $\frac{n^2+13}{4}$. In particular, for the case of $n = 4$, we see that this exponent is $\frac{29}{4}$, which is an improvement on the bound of $\frac{15}{2}$ obtained in [Goldfeld et al. 2021b].

As remarked above, the main result that we will need is Theorem 10.0.1 or, more specifically, Remark 10.0.5, which states that for any $0 < \varepsilon < \frac{1}{2}$, and for $a = (a_1, a_2, \ldots, a_{n-1})$ satisfying $\lfloor a_j \rfloor + \varepsilon < a_j < \lceil a_j \rceil - \varepsilon$ for each $j = 1, \ldots, n-1$, we have

$$|p^{(n)}_{T,R}(y)| \ll \delta^{-\frac{1}{2}}(y) \cdot \|y\|^{2a} \cdot T^{\varepsilon + \frac{(n+4)(n-1)}{4} + R \cdot \left(D(n) + \frac{n(n-1)}{2}\right) - \sum_{j=1}^{n-1} B(a_j)}. \tag{6.0.3}$$

(The terms $\delta^{-1/2}(y)$, $\|y\|^{2a}$ are defined in Section 6.1 below. The function B is defined in Theorem 9.0.2.)

This bound for $p^{(n)}_{T,R}(y)$ is obtained via an integral representation denoted by $p^{(n)}_{T,R}(y; b)$ (see (8.1.4)) over variables $s = (s_1, \ldots, s_n)$ valid for any $b = (b_1, \ldots, b_n)$ with $b_j > 0$ for each $j = 1, \ldots, n-1$. The integral is taken over the lines $\mathrm{Re}(s_j) = b_j$. Essentially, the bound is then obtained by moving the lines of integration to $\mathrm{Re}(s_j) = -a_j$ for some $a = (a_1, \ldots, a_{n_1}) \in (\mathbb{R}_{>0})^{n-1}$.

The strategy for proving Proposition 6.0.1 will be to, first, introduce notation to rewrite $\mathcal{I}_w$ in a simplified form. We do this in Section 6.1. Then, in Section 6.2 we give bounds for $\mathcal{I}_w$ obtained by applying (6.0.3) to $|p_{T,R}(Lcwuy)|$ (with a parameter $a = (a_1, \ldots, a_{n-1})$) and to $|p_{T,R}(My)|$ (with a parameter $b = (b_1, \ldots, b_{n-1})$) for general $a, b \in (\mathbb{R}_{>0})^{n-1}$. In particular, we establish (6.2.2), bounding $|\mathcal{I}_w|$ in terms of the product of three independent quantities $K(c, w; a)$, $X(u, w; a)$ and $Y(y, w; a, b)$. In Section 6.3, we will show that $K(c, w; a)$ will converge provided that a satisfies certain conditions (independent of w), and that for this choice of a, $X(u, w; a)$ also converges. We then determine b (dependent on w and a) for which $Y(y, w; a, b)$ is also convergent. Finally, in Section 6.4, we complete the proof of Proposition 6.0.1 by simplifying the expression for the given choices of a and b.

6.1. ***Rewriting $\mathcal{I}_w$.*** Let $T_n(\mathbb{R})$ and $U_n(\mathbb{R})$ be the subgroups of $\mathrm{GL}_n(\mathbb{R})$ consisting of diagonal matrices (with positive terms) and upper triangular unipotent matrices, respectively. Recall that if $t = \mathrm{diag}(t_1, \ldots, t_n) \in T_n(\mathbb{R})$ and $u \in U_n(\mathbb{R})$, the modular character $\delta : T_n(\mathbb{R}) \to \mathbb{R}$ is defined to satisfy

$d(t^{-1}ut) = \delta(t)\, du$. Explicitly, it is given by

$$\delta(t) = \prod_{i=1}^{n} t_i^{2i-n-1}.$$

More generally, if $a = (a_1, \dots, a_{n-1}) \in \mathbb{R}^{n-1}$, for

$$y = (y_1, \dots, y_{n-1}) := \operatorname{diag}(y_1 \cdots y_{n-2} y_{n-1}, \dots, y_1 y_2, y_1, 1),$$

with $y_1, \dots, y_{n-1} > 0$, we define

$$\|y\|^a := \prod_{k=1}^{n-1} y_k^{a_k}.$$

One checks that in the special case of $a_j = \frac{j(n-j)}{2}$ for $j = 1, \dots, n-1$,

$$\delta^{-\frac{1}{2}}(y) = \|y\|^a. \tag{6.1.1}$$

Similarly, if ${}^tU_n(\mathbb{R})$ is the subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting of lower triangular unipotent matrices and

$$\overline{U}_w := (w^{-1}\, {}^tU_n(\mathbb{R}) w) \cap U_n(\mathbb{R}),$$

then we can consider the character δ_w on $T_n(\mathbb{R})$ which satisfies $d(tut^{-1}) = \delta_w(t)\, du$ upon restricting the measure on $U_n(\mathbb{R})$ to $\overline{U}_w$. It can be checked that

$$\delta_w(y) = \delta^{\frac{1}{2}}(y) \cdot \delta^{-\frac{1}{2}}(wyw^{-1}). \tag{6.1.2}$$

Recall from Theorem 4.2.1 that for $L = (\ell_1, \dots, \ell_{n-1})$, $M = (m_1, \dots, m_{n-1}) \in (\mathbb{Z}_{>0})^{n-1}$ and

$$c = \begin{pmatrix} 1/c_{n-1} & & & & \\ & c_{n-1}/c_{n-2} & & & \\ & & \ddots & & \\ & & & c_2/c_1 & \\ & & & & c_1 \end{pmatrix},$$

where $c_i \in \mathbb{Z}_{>0}$ for $i = 1, \dots, n-1$, the Kloosterman contribution to the Kuznetsov trace formula is given by

$$\mathcal{K} = \sum_{\substack{w \in W_n \\ w \neq w_1}} \mathcal{I}_w,$$

where, using the notation defined above and letting $dy^\times$ denote the measure $\prod_{k=1}^{n-1} dy_k/y_k$,

$$\mathcal{I}_w := \mathfrak{c}_n^{-1} \sum_{v \in V_n} \sum_{c_1=1}^{\infty} \cdots \sum_{c_{n-1}=1}^{\infty} S_w(\psi_L, \psi_M^v, c)$$
$$\cdot \int\limits_{\substack{y=(y_1,\dots,y_{n-1}) \\ y_1,\dots,y_{n-1}>0}} \int\limits_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})} \int\limits_{\overline{U}_w(\mathbb{R})} \delta^{\frac{1}{2}}(LM) \cdot \delta(y) \cdot \psi_L(wuy)\, \overline{\psi_M^v(u)}\, p_{T,R}^{(n)}(Lcwuy)\, \overline{p_{T,R}^{(n)}(My)}\, d^*u\, dy^\times. \tag{6.1.3}$$

We recall that by [Friedberg 1987], $\mathcal{I}_w$ is identically zero unless w is *relevant* (see Definition 2.2.6).

6.2. ***Bounds for $\mathcal{I}_w$ in terms of a and b.*** Since $p^{(n)}_{T,R}(g)$ is determined by the Iwasawa decomposition of g, we first make the change of variables $u \mapsto y^{-1}uy$. Then (6.1.3) implies that

$$|\mathcal{I}_w| \ll \sum_{v\in V_n} \sum_{c_1=1}^{\infty} \cdots \sum_{c_{n-1}=1}^{\infty} |S_w(\psi_L, \psi_M^v, c)| \cdot \int\limits_{\substack{y=(y_1,\dots,y_{n-1})\\ y_1,\dots,y_{n-1}>0}} \int\limits_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})} \int\limits_{\bar{U}_w(\mathbb{R})} \cdot\, \delta^{\frac12}(M)\cdot\delta^{\frac12}(L)\cdot\delta_w(y)\cdot\delta(y)\cdot|p^{(n)}_{T,R}(Lcwyu)|\,|p^{(n)}_{T,R}(My)|\,d^*u\,dy^\times. \quad (6.2.1)$$

For the purposes of our analysis, we break up the integral in the y-variables. To this end, let

$$I_0 := (0,1], \quad I_1 = (1,\infty).$$

For $\tau = (\tau_1,\dots,\tau_{n-1}) \in \{0,1\}^{n-1}$, define

$$I_\tau := I_{\tau_1} \times \cdots \times I_{\tau_{n-1}}.$$

Hence,

$$\int\limits_{\substack{y=(y_1,\dots,y_{n-1})\\ y_1,\dots,y_{n-1}>0}} = \sum_{\tau\in\{0,1\}^{n-1}} \int\limits_{I_\tau},$$

and (6.2.1) becomes

$$|\mathcal{I}_w| \ll \sum_{\tau} |\mathcal{I}_w(\tau)|,$$

where

$$|\mathcal{I}_w(\tau)| := \sum_{v\in V_n} \sum_{c_1=1}^{\infty} \cdots \sum_{c_{n-1}=1}^{\infty} |S_w(\psi_L, \psi_M^v, c)| \cdots \int\limits_{y_1\in I_{\tau_1}} \int\limits_{y_2\in I_{\tau_2}} \cdots \int\limits_{y_{n-1}\in I_{\tau_{n-1}}} \int\limits_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})} \int\limits_{\bar{U}_w(\mathbb{R})} \delta^{\frac12}(M)\cdot\delta^{\frac12}(L)\cdot\delta_w(y)\cdot\delta(y) \cdot|p^{(n)}_{T,R}(Lcwyu)|\,|p^{(n)}_{T,R}(My)|\,d^*u\,dy^\times. \quad (6.2.2)$$

Our strategy is now to, for each choice of τ, replace the terms with $p^{(n)}_{T,R}$ with the bound from (6.0.3) (in the first instance using a choice of $a = (a_1,\dots,a_{n-1}) \in \mathbb{R}^{n-1}$, and in the second instance using $b = (b_1,\dots,b_{n-1}) \in \mathbb{R}^{n-1}$). Then we need to find choices of a and b for which the corresponding integrals converge and give good bounds.

Recall that if $g = utk$ is the Iwasawa decomposition of an element $g \in \mathrm{GL}_n(\mathbb{R})$, then $p^{(n)}_{T,R}(g) = p^{(n)}_{T,R}(t)$. With this in mind, consider the Iwasawa decomposition $wu = u_0tk$, where $u_0 \in U_n(\mathbb{R})$, $t \in T_n(\mathbb{R})$ and $k \in O(n,\mathbb{R})$. Then

$$Lcwyu = Lc(wyw^{-1})u_0tk = u_1Lc(wyw^{-1})tk \quad (u_1 = (Lcwyw^{-1})^{-1}u_0(Lcwyw^{-1}))$$

is the Iwasawa form of $Lcwyu$; hence

$$|p^{(n)}_{T,R}(Lcwyu)| = |p^{(n)}_{T,R}(Lcwyw^{-1}t)|.$$

Recall that the Iwasawa form of wu is assumed to be u_0tk, meaning $wu = u_0tk$, where $u_0 \in U_n(\mathbb{R})$, $t \in T_n(\mathbb{R})$ and $k \in O(n,\mathbb{R})$. It can be shown [Jacquet 1967] that

$$t^2 = \begin{pmatrix} 1/\xi_{n-1} & & & & \\ & \xi_{n-1}/\xi_{n-2} & & & \\ & & \ddots & & \\ & & & \xi_2/\xi_1 & \\ & & & & \xi_1 \end{pmatrix}, \tag{6.2.3}$$

where $\xi_i = \xi_i(wu) \geq 1$ for any $u \in \overline{U}_w$. For example, in the case $n = 4$ and $w = w_{\text{long}} = w_{(1,1,1,1)}$, we find that $\overline{U}_{w_{\text{long}}} = U_4(\mathbb{R})$ and, for

$$u = \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

that

$$\begin{aligned} \xi_1(w_{\text{long}}u) &= 1 + x_{12}^2 + x_{13}^2 + x_{14}^2, \\ \xi_2(w_{\text{long}}u) &= 1 + x_{23}^2 + x_{24}^2 + (x_{12}x_{24} - x_{14})^2 + (x_{12}x_{23} - x_{13})^2 + (x_{13}x_{24} - x_{14}x_{23})^2, \\ \xi_3(w_{\text{long}}u) &= 1 + x_{34}^2 + (x_{23}x_{34} - x_{24})^2 + (x_{12}x_{23}x_{34} - x_{13}x_{34} - x_{12}x_{24} + x_{14})^2. \end{aligned}$$

In general, the values ξ_i are always of the form 1 plus a sum of squares of functions consisting of the entries of u.

From (6.0.3), replacing a with b, we see that $|p_{T,R}^{(n)}(My)|$ is bounded by

$$\ll \delta(M)^{-\frac{1}{2}} \cdot \|M\|^{2b} \cdot \delta(y)^{-\frac{1}{2}} \cdot \|y\|^{2b} \cdot T^{\varepsilon + \frac{(n+4)(n-1)}{4} + R \cdot \left(D(n) + \frac{n(n-1)}{2}\right) - \sum_{j=1}^{n-1} B(b_j)}.$$

To similarly bound $|p_{T,R}^{(n)}(Lcwyw^{-1}t)|$, we first remark that since

$$c = c_1 \begin{pmatrix} d_1 d_2 \cdots d_{n-1} & & & & \\ & d_1 d_2 \cdots d_{n-2} & & & \\ & & \ddots & & \\ & & & d_1 & \\ & & & & 1 \end{pmatrix} =: c_1 d, \quad \text{where } d_i = \frac{c_{i-1}c_{i+1}}{c_i^2},$$

setting $c_0 = c_n := 1$ (and $a_0 = a_n := 0$ as usual), we see that

$$\delta^{-\frac{1}{2}}(c) \cdot \|c\|^{2a} = \delta(c)^{-\frac{1}{2}} \prod_{i=1}^{n-1} \left(\frac{c_{i-1}c_{i+1}}{c_i^2} \right)^{2a_i} = \prod_{k=1}^{n-1} c_k^{-1+2a_{i-1}-4a_i+2a_{i+1}}.$$

Therefore, it follows that

$$\begin{aligned} |p_{T,R}^{(n)}(Lcwyw^{-1}t)| \ll \frac{\delta(L)^{-\frac{1}{2}} \cdot \|L\|^{2a} \cdot \delta(t)^{-\frac{1}{2}} \cdot \|t\|^{2a}}{\prod_{k=1}^{n-1} c_k^{1-2a_{i-1}+4a_i-2a_{i+1}}} \cdot \delta(wyw^{-1})^{-\frac{1}{2}} \\ \cdot \|wyw^{-1}\|^{2a} \cdot T^{\varepsilon + \frac{(n+4)(n-1)}{4} + R \cdot \left(D(n) + \frac{n(n-1)}{2}\right) - \sum_{j=1}^{n-1} B(a_j)}. \end{aligned}$$

Recall that if $t = t(u)$ is as in (6.2.3), if we define, for $a = (a_1, \dots, a_{n-1}), b = (b_1, \dots, b_{n-1}) \in \mathbb{R}^{n-1}$,

$$K(w; a) := \sum_{v \in V_n} \sum_{c_1=1}^{\infty} \cdots \sum_{c_{n-1}=1}^{\infty} \frac{|S_w(\psi_L, \psi_M^v, c)|}{\prod_{i=1}^{n-1} c_i^{1-2a_{i-1}+4a_i-2a_{i+1}}},$$

$$X(w; a) := \int\limits_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})} \int\limits_{\bar{U}_w(\mathbb{R})} \delta(t)^{-\frac{1}{2}} \cdot \|t\|^{2a} \, d^*u,$$

and for a given choice of $\tau = (\tau_1, \dots, \tau_{n-1}) \in \{0, 1\}^{n-1}$

$$Y(\tau, w; a, b) := \int\limits_{y_1 \in I_{\tau_1}} \int\limits_{y_2 \in I_{\tau_2}} \cdots \int\limits_{y_{n-1} \in I_{\tau_{n-1}}} \|y\|^{2b} \cdot \|wyw^{-1}\|^{2a} \, dy^{\times},$$

then the bound on $|\mathcal{I}_w(\tau)|$ given in (6.2.2) can be replaced by

$$|\mathcal{I}_w(\tau)| \ll T^{\varepsilon + \frac{(n+4)(n-1)}{2} + R\cdot(2D(n)+n(n-1)) - \sum_{j=1}^{n-1}(B(a_j)+B(b_j))} \cdot K(w; a) \cdot X(w; a) \cdot Y(\tau, w; a, b) \cdot \|L\|^{2a} \cdot \|M\|^{2b}. \quad (6.2.4)$$

We remark that in simplifying/finding $Y(\tau, w; a, b)$, we have used (6.1.2). The basic strategy to prove Proposition 6.0.1 is now clear: we first find a such that both $K(w; a)$ and $X(w; a)$ converge; then given this choice of a, we determine a particular value of b for which $Y(\tau, w; a, b)$ converges as well; finally, we work out the corresponding bounds on $\|L\|^{2a}$, $\|M\|^{2b}$ and $\sum_{j=1}^{n-1}(B(a_j) + B(b_j))$.

6.3. ***Restrictions on the parameters a and b.*** The trivial bound (see [Dąbrowski and Reeder 1998]) for the Kloosterman sum is given by

$$S(1, 1, c) \ll \delta^{\frac{1}{2}}(c) = c_1 c_2 \cdots c_{n-1}.$$

Hence $K(w; a)$ is convergent whenever a is chosen such that

$$\|c\|^{2a} = \prod_{k=1}^{n-1} c_k^{2a_{k-1}-4a_k+2a_{k+1}} \ll \delta^{-\frac{1}{2}-\varepsilon}(c).$$

From (6.1.1), if we set $a_j = \frac{j(n-j)}{2}(1+\varepsilon)$, then $\|c\|^{2a} = \delta^{-1-\varepsilon}(c) \ll \delta^{-1/2-\varepsilon}(c)$. More generally, $K(w; a)$ converges in the case

$$a_j := \rho + \frac{j(n-j)}{2}(1+\varepsilon), \quad \rho > 0, \ j = 1, \dots, n-1. \quad (6.3.1)$$

That this choice of a makes $K(w; a)$ converge is a consequence of the easily verifiable fact that

$$\|c\|^{2a} = (c_1 c_{n-1})^{-2\rho} \cdot \delta^{-1-\varepsilon}(c).$$

We assume henceforth that a satisfies (6.3.1).

We next consider the convergence of $X(w; a)$. Recall that the Iwasawa form of wu is assumed to be $u_0 t k$, meaning $wu = u_0 t k$, where $u_0 \in U_n(\mathbb{R})$, $t \in T_n(\mathbb{R})$ and $k \in O(n, \mathbb{R})$. Indeed, t is given by (6.2.3).

Then

$$X(w;a)=\int\limits_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})}\;\int\limits_{\overline{U}_w(\mathbb{R})}\delta(t)^{-\frac{1}{2}}\cdot\|t\|^{2a}\,d^*u\ll\int\limits_{U_w(\mathbb{Z})\backslash U_w(\mathbb{R})}\;\int\limits_{\overline{U}_w(\mathbb{R})}\delta^{-\frac{3}{2}-\varepsilon}(t)\,d^*u.$$

The fact that the right-hand side converges is a consequence of [Jacquet 1967].

We now turn to the convergence of $Y(\tau,w;a,b)$. Applying Lemma A.1 (which describes $\|wyw^{-1}\|^{2a}$), we see that

$$Y(\tau,w;a,b)=\int\limits_{I_\tau}\biggl(\prod_{i=1}^{s}\prod_{j=1}^{n_i}y_{n-\hat{n}_i+j}^{2b_{(n-\hat{n}_i+j)}-2(a_{\hat{n}_{i-1}}-a_{(\hat{n}_{i-1}+j)}+a_{\hat{n}_i})}\biggr)\,dy^\times=\prod_{i=1}^{s}\prod_{j=1}^{n_i}Y_{n-\hat{n}_i+j}(\tau,w;a,b),$$

where

$$Y_{n-\hat{n}_i+j}(\tau,w;a,b):=\int\limits_{I_{\tau_{n-\hat{n}_i+j}}}y_{n-\hat{n}_i+j}^{2b_{n-\hat{n}_i+j}-2(a_{\hat{n}_{i-1}}-a_{\hat{n}_{i-1}+j}+a_{\hat{n}_i})}\,\frac{dy_{n-\hat{n}_i+j}}{y_{n-\hat{n}_i+j}}.$$

Hence, in order to bound $Y(\tau,w;a,b)$ (and thereby show that $\mathcal{I}_w(\tau)$ converges), we must choose $b=(b_1,\dots,b_{n-1})$ such that $Y_{n-\hat{n}_i+j}(\tau,w;a,b)$ converges. Clearly

$$b_{n-\hat{n}_i+j}=a_{\hat{n}_{i-1}}-a_{\hat{n}_{i-1}+j}+a_{\hat{n}_i}+(-1)^{\tau_{n-\hat{n}_i+j}}\cdot\frac{\varepsilon}{2}\quad(i=1,\dots,s,\;\;j=1,\dots,n_i)\tag{6.3.2}$$

suffices, since making this choice implies that, for each $k=1,\dots,n-1$,

$$Y_k(\tau,w;a,b)=\begin{cases}\int_0^1 y^{\varepsilon}\,(dy/y) & \text{if }\tau_k=0,\\ \int_1^\infty y^{-\varepsilon}\,(dy/y) & \text{if }\tau_k=1,\end{cases}$$

which converges (and gives the same value $\frac{1}{\varepsilon}$) in either case.

6.4. ***Proof of Proposition 6.0.1.*** We have now shown that if $w=w_{(n_1,\dots,n_r)}$ and we choose a as in (6.3.1) and b via (6.3.2) accordingly, the right-hand side of (6.2.4) converges, and hence gives a bound for $|\mathcal{I}_w|$. Therefore, in order to complete the proof of Proposition 6.0.1, we need to first show that

$$\|L\|^{2a}\cdot\|M\|^{2b}\ll\prod_{i=1}^{n-1}(\ell_i m_i)^{2\rho+\frac{n^2+1}{4}},$$

and second that the given choice of a and b gives the claimed bound for the power of T appearing in (6.2.4).

To complete the first of these tasks we note that, by (6.3.1) and the fact that $j(n-j)$ is maximized (in j) when $j=\frac{n}{2}$, we have

$$a_j=\rho+\frac{j(n-j)}{2}(1+\varepsilon)\le\rho+\frac{n^2}{8}(1+\varepsilon)<\rho+\frac{n^2+1}{8}\tag{6.4.1}$$

for $\varepsilon<1/n^2$ and $1\le j\le n-1$. Similarly, using (6.3.1) and (6.3.2) we compute that, for $1\le i\le s$ and $1\le j\le n_i$,

$$b_{n-\hat{n}_i+j}=\rho+\tfrac{1}{2}(j^2+j(2\hat{n}_{i-1}-n)+\hat{n}_i(n-\hat{n}_i))+\varepsilon\tag{6.4.2}$$

for ε sufficiently small. Note that the right-hand side of (6.4.2) is a concave up parabola in j, and therefore, on the interval $1 \le j \le n_i$, can attain its maximum only at $j = 1$ or $j = n_i$. So, if we can show that $b_{n-\hat{n}_i+1}$ and $b_{n-\hat{n}_i+ni}$ both satisfy a suitable upper bound, then the same bound will hold for all $1 \le j \le n_i$.

We consider first the endpoint $j = n_i$. Using (6.4.2) and the fact that $\hat{n}_i - n_i = \hat{n}_{i-1}$, we find that

$$b_{n-\hat{n}_i+n_i} = \rho + \tfrac{1}{2}\hat{n}_{i-1}(n - \hat{n}_{i-1}) + \varepsilon.$$

Again, $j(n - j)$ is maximized when $j = \frac{n}{2}$, so we conclude that

$$b_{n-\hat{n}_i+n_i} \le \rho + \frac{n^2}{8} + \varepsilon < \rho + \frac{n^2+1}{8} \tag{6.4.3}$$

for ε sufficiently small.

Next we consider the endpoint $j = 1$. From (6.4.2) we find that

$$\begin{aligned} b_{n-\hat{n}_i+1} &= \rho + \tfrac{1}{2}(1 - n + \hat{n}_i(n - \hat{n}_i) + 2\hat{n}_{i-1}) + \varepsilon \\ &\le \rho + \tfrac{1}{2}(-1 - n + \hat{n}_i(n - \hat{n}_i) + 2\hat{n}_i) + \varepsilon, \end{aligned} \tag{6.4.4}$$

where the last step follows because $\hat{n}_{i-1} = \hat{n}_i - n_i \le \hat{n}_i - 1$. We find using calculus that, as a function of $\hat{n}_i$, the right-hand side of (6.4.4) is maximized when $\hat{n}_i = \frac{n+2}{2}$. So

$$\begin{aligned} b_{n-\hat{n}_i+1} &\le \rho + \frac{1}{2}\left(-1 - n + \frac{n+2}{2}\left(n - \frac{n+2}{2}\right) + n + 2\right) + \varepsilon \\ &= \rho + \frac{n^2}{8} + \varepsilon \le \rho + \frac{n^2+1}{8} \end{aligned} \tag{6.4.5}$$

for ε small enough. From (6.4.3) and (6.4.5) it follows, again, that

$$b_{n-\hat{n}_i+j} \le \rho + \frac{n^2+1}{8}$$

for all $1 \le i \le s$ and $1 \le j \le n_i$. This and (6.4.1) yield the desired bound on $\|L\|^{2a} \cdot \|M\|^{2b}$.

The second task is accomplished using Lemma A.9. □

7. Bounding the Eisenstein spectrum $\mathcal{E}$

Recall that if $L = (\ell_1, \dots, \ell_{n-1})$, $M = (m_1, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$, with $\prod_{i=1}^{n-1} \ell_i m_i \ne 0$, then, by Theorem 4.1.1, the Eisenstein contribution to the Kuznetsov trace formula is given by

$$\mathcal{E} = \sum_{\mathcal{P}} \sum_{\Phi} \mathcal{E}_{\mathcal{P},\Phi},$$

where

$$\mathcal{E}_{\mathcal{P},\Phi} := c_{\mathcal{P}} \int\limits_{\substack{n_1 s_1 + \cdots + n_r s_r = 0 \\ \mathrm{Re}(s_j) = 0}} A_{E_{\mathcal{P},\Phi}}(L, s)\, \overline{A_{E_{\mathcal{P},\Phi}}(M, s)} \cdot |p^{n,\#}_{T,R}(\alpha_{(\mathcal{P},\Phi)}(s))|^2 \, ds.$$

In this section we give bounds for $\mathcal{E}$ in the case that $L = (\ell, 1, \dots, 1)$ and $M = (m, 1, \dots, 1)$, with $\ell, m \ne 0$.

7.1. ***The Eisenstein contribution $\mathcal{E}$ to the Kuznetsov trace formula.*** The main result of this section is the following.

Theorem 7.1.1 (bounding the Eisenstein contribution $\mathcal{E}$). *Fix $n \geq 2$ and a sufficiently large integer $R > 0$. Let $L = (\ell, 1, \dots, 1)$, $M = (m, 1, \dots, 1) \in \mathbb{Z}^{n-1}$ with $\ell, m \neq 0$. Then, assuming the lower bound conjecture for Rankin–Selberg L-functions* (see Conjecture 1.3.2), *for $T \to \infty$ we have the bound*

$$\sum_{\mathcal{P}} \sum_{\Phi} |\mathcal{E}_{\mathcal{P},\Phi}| \ll (\ell m)^{\frac{1}{2} - \frac{1}{n^2+1} + \varepsilon} \cdot T^{R \cdot (\binom{2n}{n} - 2^n) + \varepsilon}. \qquad \square$$

7.2. ***Proof of Theorem 7.1.1.***

Proof. We proceed by induction on n, beginning with the case $n = 2$. In this case, the only parabolic subgroup is the minimum parabolic, or Borel, subgroup $\mathcal{B} = \mathcal{P}_{1,1}$, and the only function Φ corresponding to $\mathcal{B}$ (see Definition 3.2.2) is the constant function $\Phi = 1$. The Eisenstein contribution in this case, then, is simply the quantity $\mathcal{E}_{\mathcal{B},1}$.

By Theorem 4.1.1 in the case $n = 2$, we have

$$\mathcal{E}_{\mathcal{B},1} = c_{\mathcal{B}} \int\limits_{\operatorname{Re} s_1 = 0} A_{E_{\mathcal{B},1}}(\ell, s)\, \overline{A_{E_{\mathcal{B},1}}(m, s)} \cdot |p^{2,\#}_{T,R}(\alpha_{(\mathcal{B},1)}(s))|^2 \, ds_1,$$

where $s = (s_1, -s_1)$. Now note that, by (3.2.7), $\alpha_{(\mathcal{B},1)}(s) = s$. Moreover, by Definition 1.4.1, we have $\mathcal{F}^{(2)}_R \equiv 1$, so by Definition 1.4.4, we have

$$p^{2,\#}_{T,R}(\alpha_{(\mathcal{B},1)}(s)) = e^{s_1^2/T^2} \Gamma\Big(\frac{2R+1+2s_1}{4}\Big) \Gamma\Big(\frac{2R+1-2s_1}{4}\Big).$$

Furthermore, we see from Proposition 4.1.5 that

$$|A_{E_{\mathcal{B},1}}(\ell, s)| \ll |\zeta^*(1+2s_1)|^{-1} \sum_{\substack{c_1, c_2 \in \mathbb{Z}_{>0} \\ c_1 c_2 = \ell}} |c_1^{\alpha_1} c_2^{\alpha_2}| \ll \ell^{\varepsilon} \left| \Gamma\Big(\frac{1+2s_1}{2}\Big) \zeta(1+2s_1) \right|^{-1}.$$

Then

$$|\mathcal{E}_{\mathcal{B},1}| \ll (\ell m)^{\varepsilon} \int\limits_{\operatorname{Re}(s_1)=0} e^{s_1^2/T^2} \frac{\big|\Gamma\big(\frac{2R+1+2s_1}{4}\big)\Gamma\big(\frac{2R+1-2s_1}{4}\big)\big|^2}{\big|\Gamma\big(\frac{1+2s_1}{2}\big)\zeta(1+2s_1)\big|^2} \, |ds_1|.$$

We may restrict our integration to the domain $|\operatorname{Im}(s)| \leq T$, since $e^{s_1^2/T^2}$ decays exponentially otherwise. On this domain, we use Stirling's bound (9.2.1) for the Gamma function, as well as the Vinogradov bound

$$|\zeta(1+it)|^{-1} \ll (1+|t|)^{\varepsilon} \quad (t \in \mathbb{R}).$$

We get

$$|\mathcal{E}_{\mathcal{B},1}| \ll (\ell m)^{\varepsilon} \int\limits_{\substack{\operatorname{Re}(s_1)=0 \\ \operatorname{Im}(s_1) \leq T}} |1+s_1|^{2R-1+\varepsilon} \, |ds_1|,$$

from which it follows immediately that $|\mathcal{E}_{\mathcal{B},1}| \ll T^{2R+\varepsilon}$. So our desired result holds in the case $n = 2$.

We now proceed to the general case. For $n > 2$, in order to establish bounds for $\mathcal{E}_{\mathcal{P},\Phi}$, we need to know that our main theorem is true for all $k < n$. The reason this is needed is because we have to bound Rankin–Selberg L-functions $L(s, \phi_k \times \phi_{k'})$, with $2 \le k, k' < n$. This will require knowing the Weyl law with harmonic weights (Theorem 7.2.3) for $2 \le k, k' < n$. We may assume by induction, however, that this is indeed the case, i.e., the Weyl law with harmonic weights holds for all $2 \le k < n$.

Now recall that, for the parabolic $\mathcal{P}$ associated to a partition $n = n_1 + \cdots + n_r$, we have

$$\mathcal{E}_{\mathcal{P},\Phi} = \int\limits_{\substack{n_1 s_1 + \cdots + n_r s_r = 0 \\ \mathrm{Re}(s_j)=0}} A_{E_{\mathcal{P},\Phi}}(L, s)\, \overline{A_{E_{\mathcal{P},\Phi}}(M, s)} \cdot |p^{n,\#}_{T,R}(\alpha_{(\mathcal{P},\Phi)}(s))|^2 \, ds$$

where $\alpha_{\mathcal{P},\Phi}(s)$ is given by (see (3.2.7))

$$\Big(\overbrace{\alpha_{1,1}+s_1, \dots, \alpha_{1,n_1}+s_1}^{n_1 \text{ terms}}, \quad \overbrace{\alpha_{2,1}+s_2, \dots, \alpha_{2,n_2}+s_2}^{n_2 \text{ terms}}. \quad \dots \quad , \overbrace{\alpha_{r,1}+s_r, \dots, \alpha_{r,n_r}+s_r}^{n_r \text{ terms}}\Big).$$

Since $\sum_{i=1}^{n_k} \alpha_{k,i} = 0$ for all $1 \le k \le r$ we see that

$$\sum_{k=1}^{r} \sum_{i=1}^{n_k} (\alpha_{k,i} + s_k)^2 = \sum_{k=1}^{r} \sum_{i=1}^{n_k} (\alpha_{k,i}^2 + s_k^2).$$

Now, for any $\beta = (\beta_1, \dots, \beta_n) \in (i\mathbb{R})^n$, where $\hat{\beta}_n = 0$, we have

$$p^{n,\#}_{T,R}(\beta) := \left(\frac{\beta_1^2 + \beta_2^2 + \cdots + \beta_n^2}{2T^2} \right) \cdot \mathcal{F}^{(n)}_R\left(\frac{\beta}{2}\right) \prod_{1 \le i < j \le n} \left| \Gamma\left(\frac{2R + 1 + \beta_i - \beta_j}{4} \right) \right|^2.$$

It follows that

$$\begin{aligned} p^{n,\#}_{T,R}(\alpha_{(\mathcal{P},\Phi)}(s)) = \exp&\left(\frac{\sum_{k=1}^{r} \sum_{i=1}^{n_k} (\alpha_{k,i}^2 + s_k^2)}{2T^2} \right) \mathcal{F}^{(n)}_R\left(\frac{\alpha_{(\mathcal{P},\Phi)}(s)}{2} \right) \\ &\cdot \prod_{\substack{k=1 \\ n_k \ne 1}}^{r} \prod_{1 \le i < j \le n_k} \left| \Gamma\left(\frac{2R + 1 + \alpha_{k,i} - \alpha_{k,j}}{4} \right) \right|^2 \\ &\cdot \prod_{1 \le k < k' \le r} \prod_{i=1}^{n_k} \prod_{j=1}^{n_{k'}} \left| \Gamma\left(\frac{2R + 1 + s_k - s_{k'} + \alpha_{k,i} - \alpha_{k',j}}{4} \right) \right|^2. \end{aligned}$$

By Proposition 4.1.5, the m-th coefficient of $E_{\mathcal{P},\Phi}$ is given by

$$\begin{aligned} A_{E_{\mathcal{P},\Phi}}((m, 1, \dots, 1), s) = \prod_{\substack{k=1 \\ n_k \ne 1}}^{r} L^*(1, \mathrm{Ad}\,\phi_k)^{-\frac{1}{2}} \prod_{1 \le i < j \le r} & L^*(1 + s_i - s_j, \phi_i \times \phi_j)^{-1} \\ & \cdot \sum_{\substack{1 \le c_1, c_2, \dots, c_r \in \mathbb{Z} \\ c_1 c_2 \cdots c_r = m}} \lambda_{\phi_1}(c_1) \cdots \lambda_{\phi_r}(c_r) \cdot c_1^{s_1} \cdots c_r^{s_r} \end{aligned}$$

up to a nonzero constant factor with absolute value depending only on n. To bound the divisor sum above we will use the bound of Luo, Rudnick and Sarnak [Luo et al. 1999] for the m-th Hecke Fourier coefficient of a $\mathrm{GL}(\kappa)$ (for $\kappa \ge 2$) Hecke–Maass cusp form ϕ given by

$$|\lambda_\phi(m, 1, \dots, 1)| \le m^{\frac{1}{2} - 1/(\kappa^2+1) + \varepsilon}.$$

(A slightly stronger result has been obtained by Kim and Sarnak [Kim 2003]. However, the stated result above is sufficient for our purposes.) We immediately obtain the following bound for the divisor sum:

$$\sum_{\substack{1\le c_1,c_2,\dots,c_r\in\mathbb{Z}\\ c_1c_2\cdots c_r=m}} |\lambda_{\phi_1}(c_1)\cdots\lambda_{\phi_r}(c_r)\cdot c_1^{s_1}\cdots c_r^{s_r}| \ll m^{\frac{1}{2}-1/(n^2+1)+\varepsilon}.$$

It follows that

$$\begin{aligned}
|\mathcal{E}_{\mathcal{P},\Phi}| &\ll (m\ell)^{\frac{1}{2}-\frac{1}{n^2+1}+\varepsilon}\cdot\left(\frac{\sum_{k=1}^r\sum_{i=1}^{n_k}\alpha_{k,i}^2}{T^2}\right)\\
&\quad\cdot\int\limits_{\substack{n_1s_1+\cdots+n_rs_r=0\\ \mathrm{Re}(s_j)=0,\mathrm{Im}(s_j)\ll T}} \left|\mathcal{F}_R^{(n)}\left(\frac{\alpha_{(\mathcal{P},\Phi)}(s)}{2}\right)\right|^2\left(\prod_{\substack{k=1\\ n_k\neq1}}^r\prod_{1\le i<j\le n_k}\left|\Gamma\left(\frac{2R+1+\alpha_{k,i}-\alpha_{k,j}}{4}\right)\right|^4\right)\\
&\qquad\cdot\left(\prod_{1\le k<k'\le r}\prod_{i=1}^{n_k}\prod_{j=1}^{n_{k'}}\left|\Gamma\left(\frac{2R+1+s_k-s_{k'}+\alpha_{k,i}-\alpha_{k',j}}{4}\right)\right|^4\right)\\
&\qquad\cdot\left(\prod_{\substack{k=1\\ n_k\neq1}}^r|L^*(1,\mathrm{Ad}\,\phi_k)|^{-1}\right)\cdot\left(\prod_{1\le k<k'\le r}|L^*(1+s_k-s_{k'},\phi_k\times\phi_{k'})|^{-2}\,|ds|\right)\\
&\ll (m\ell)^{\frac{1}{2}-\frac{1}{n^2+1}+\varepsilon}\prod_{\substack{k=1\\ n_k\neq1}}^r\exp\left(\frac{\alpha_{k,1}^2+\cdots+\alpha_{k,n_k}^2}{T^2}\right)\\
&\quad\cdot\int\limits_{\substack{n_1s_1+\cdots+n_rs_r=0\\ \mathrm{Re}(s_j)=0,\mathrm{Im}(s_j)\ll T}} \left|\mathcal{F}_R^{(n)}\left(\frac{\alpha_{(\mathcal{P},\Phi)}(s)}{2}\right)\right|^2\prod_{\substack{k=1\\ n_k\neq1}}^r\frac{1}{|L(1,\mathrm{Ad}\,\phi_k)|}\prod_{1\le i<j\le n_k}\frac{\left|\Gamma\left(\frac{2R+1+\alpha_{k,i}-\alpha_{k,j}}{4}\right)\right|^4}{\left|\Gamma\left(\frac{1+\alpha_{k,i}-\alpha_{k,j}}{2}\right)\right|^2}\\
&\qquad\cdot\prod_{1\le k<k'\le r}\frac{1}{|L(1+s_k-s_{k'},\phi_k\times\phi_{k'})|^2}\prod_{i=1}^{n_k}\prod_{j=1}^{n_{k'}}\frac{\left|\Gamma\left(\frac{2R+1+s_k-s_{k'}+\alpha_{k,i}-\alpha_{k',j}}{4}\right)\right|^4}{\left|\Gamma\left(\frac{1+s_k-s_{k'}+\alpha_{k,i}-\alpha_{k',j}}{2}\right)\right|^2}\,|ds|.
\end{aligned}$$

Lemma 7.2.1. *Assume $|s_k|\ll T^{1+\varepsilon}$ and $|\alpha_{k,j}|\ll T^{1+\varepsilon}$ for $1\le k\le r$ and $1\le j\le n_k$. Then for $\alpha:=\alpha_{(\mathcal{P},\Phi)}(s)$ and $\alpha^{(k)}$ as in Definition A.16, we have*

$$|\mathcal{F}_R^{(n)}(\alpha_{(\mathcal{P},\Phi)}(s))|^2\ll\left(\prod_{\substack{k=1\\ n_k\neq1}}^r|\mathcal{F}_R^{(n_k)}(\alpha^{(k)})|^2\right)\cdot T^{R\cdot B(n)+\varepsilon},$$

where $B(n)=2D(n)-2\sum_{k=1,n_k\neq1}^r D(n_k)$.

Proof. This follows immediately from Lemma A.27. □

It follows from Lemma 7.2.1 that for $|\alpha^{(k)}|^2=\alpha_{k,1}^2+\cdots+\alpha_{k,n_k}^2$, we have

$$\begin{aligned}
|\mathcal{E}_{\mathcal{P},\Phi}| &\ll (m\ell)^{\frac{1}{2}-\frac{1}{n^2+1}+\varepsilon}T^{R\cdot B(n)+\varepsilon}\prod_{\substack{k=1\\ n_k\neq1}}^r\frac{\exp\left(\frac{|\alpha^{(k)}|^2}{T^2}\right)\left|\mathcal{F}_R^{(n_k)}\left(\frac{\alpha^{(k)}}{2}\right)\right|^2\prod_{1\le i<j\le n_k}\frac{|\Gamma((2R+1+\alpha_{k,i}-\alpha_{k,j})/4)|^4}{|\Gamma((1+\alpha_{k,i}-\alpha_{k,j})/2)|^2}}{|L(1,\mathrm{Ad}\,\phi_k)|}\\
&\quad\cdot\int\limits_{\substack{n_1s_1+\cdots+n_rs_r=0\\ \mathrm{Re}(s_j)=0,|\mathrm{Im}(s_j)|\ll T}}\prod_{1\le k<k'\le r}\frac{1}{|L(1+s_k-s_{k'},\phi_k\times\phi_{k'})|^2}\prod_{i=1}^{n_k}\prod_{j=1}^{n_{k'}}\frac{\left|\Gamma\left(\frac{2R+1+s_k-s_{k'}+\alpha_{k,i}-\alpha_{k',j}}{4}\right)\right|^4}{\left|\Gamma\left(\frac{1+s_k-s_{k'}+\alpha_{k,i}-\alpha_{k',j}}{2}\right)\right|^2}\,|ds|
\end{aligned}$$

$$\ll (m\ell)^{\frac{1}{2}-\frac{1}{n^2+1}+\varepsilon}\cdot T^{R\cdot B(n)+\varepsilon}\prod_{\substack{k=1\\ n_k\neq 1}}^{r}\frac{|h_{T,R}^{(n_k)}(\alpha^{(k)})|}{|L(1,\mathrm{Ad}\,\phi_k)|}$$
$$\cdot\int\limits_{\substack{n_1s_1+\cdots+n_rs_r=0\\ \mathrm{Re}(s_j)=0,|\mathrm{Im}(s_j)|\ll T}}\prod_{1\le k<k'\le r}\frac{1}{|L(1+s_k-s_{k'},\phi_k\times\phi_{k'})|^2}\prod_{i=1}^{n_k}\prod_{j=1}^{n_{k'}}\frac{\left|\Gamma\left(\frac{2R+1+s_k-s_{k'}+\alpha_{k,i}-\alpha_{k',j}}{4}\right)\right|^4}{\left|\Gamma\left(\frac{1+s_k-s_{k'}+\alpha_{k,i}-\alpha_{k',j}}{2}\right)\right|^2}\,|ds|.$$

where

$$h_{T,R}^{(n_k)}(\alpha^{(k)})=\exp\left(\frac{|\alpha^{(k)}|^2}{T^2}\right)\mathcal{F}_R^{(n_k)}\left(\frac{\alpha^{(k)}}{2}\right)^2\prod_{1\le i\neq j\le n_k}\frac{\Gamma\left(\frac{2R+1+\alpha_{k,i}-\alpha_{k,j}}{4}\right)^2}{\Gamma\left(\frac{1+\alpha_{k,i}-\alpha_{k,j}}{2}\right)}.$$

Next

$$\prod_{1\le k<k'\le r}\prod_{i=1}^{n_k}\prod_{j=1}^{n_{k'}}\frac{\left|\Gamma\left(\frac{2R+1+s_k-s_{k'}+\alpha_{k,i}-\alpha_{k',j}}{4}\right)\right|^4}{\left|\Gamma\left(\frac{1+s_k-s_{k'}+\alpha_{k,i}-\alpha_{k',j}}{2}\right)\right|^2}\ll T^{(2R-1)\sum_{1\le k<k'\le r}n_k\cdot n_{k'}}.$$

We obtain the bound

$$|\mathcal{E}_{\mathcal{P},\Phi}|\ll (m\ell)^{\frac{1}{2}-\frac{1}{n^2+1}+\varepsilon}\cdot T^{R\cdot B(n)+\varepsilon+(2R-1)\cdot\sum_{1\le k<k'\le r}n_k\cdot n_{k'}}$$
$$\cdot\prod_{k=1}^{r}\frac{|h_{T,R}^{(n_k)}(\alpha^{(k)})|}{|L(1,\mathrm{Ad}\,\phi_k)|}\int\limits_{\substack{n_1s_1+\cdots+n_rs_r=0\\ \mathrm{Re}(s_j)=0,|\mathrm{Im}(s_j)|\ll T}}\prod_{1\le k<k'\le r}\frac{|ds|}{|L(1+s_k-s_{k'},\phi_k\times\phi_{k'})|^2}.$$

Next, we bound the s-integral above. It follows from Langlands' conjecture (see Conjecture 1.3.2) that for $|\mathrm{Im}(s_k)|$, $|\mathrm{Im}(s_{k'})|\ll T$ we have the bound

$$|L(1+s_k-s_{k'},\phi_k\times\phi_{k'})|^{-2}\ll T^{\varepsilon}.$$

This together with the bound

$$\int\limits_{\substack{n_1s_1+\cdots+n_rs_r=0\\ \mathrm{Re}(s_j)=0,\ |\mathrm{Im}(s_j)|\ll T}}|ds|\ \ll\ T^{r-1},$$

implies that

$$|\mathcal{E}_{\mathcal{P},\Phi}|\ll (m\ell)^{\frac{1}{2}-\frac{1}{n^2+1}+\varepsilon}T^{R\cdot B(n)+(2R-1)\sum_{1\le k<k'\le r}n_k\cdot n_{k'}+(r-1)+\varepsilon}\cdot\left(\prod_{k=1}^{r}\frac{|h_{T,R}^{(n_k)}(\alpha^{(k)})|}{|L(1,\mathrm{Ad}\,\phi_k)|}\right).\tag{7.2.2}$$

Since each $n_k<n$ (for $k=1,2,\dots,r$), we can apply our inductive procedure together with the following theorem to bound $\sum_\Phi|\mathcal{E}_{\mathcal{P},\Phi}|$.

Theorem 7.2.3 (Weyl law with harmonic weights for $\mathrm{GL}(n_k)$ with $n_k<n$). *Suppose $n_k\in\mathbb{Z}$ with $2\le n_k<n$. Let $\{\phi_1,\phi_2,\dots\}$ be an orthogonal basis of Hecke–Maass cusp forms for $\mathrm{GL}(n_k)$ ordered by eigenvalue. If $\alpha^{(j)}$ are the Langlands parameters of ϕ_j, then*

$$\sum_{j=1}^{\infty}\frac{h_{T,R}^{(n_k)}(\overline{\alpha^{(j)}})}{\mathcal{L}_j}\ll_n T^{2R\cdot\left(D(k)+\frac{n_k(n_k-1)}{2}\right)+n_k-1}.\tag{7.2.4}$$

Proof. In [Goldfeld et al. 2021b], all that was needed to prove this statement for $n = 4$ was to have it be true for $n_k = 2$ and $n_k = 3$, which was already known. A similar induction argument works in general. □

It immediately follows from the bounds (7.2.2) and (7.2.4) that

$$\sum_{\Phi} |\mathcal{E}_{\mathcal{P},\Phi}| \ll (m\ell)^{\frac{1}{2}-\frac{1}{n^2+1}+\varepsilon}\, T^{R\cdot B(n)+(2R-1)\sum_{1\le k<k'\le r} n_k\cdot n_{k'}+(r-1)+\varepsilon}\cdot T^{\sum_{k=1}^r \left(2R\cdot\left(D(k)+\frac{n_k(n_k-1)}{2}\right)+n_k-1\right)}.$$

Recall that $B(n) = 2D(n) - 2\sum_{k=1}^r D(n_k)$, which implies that

$$\sum_{\Phi} |\mathcal{E}_{\mathcal{P},\Phi}| \ll (m\ell)^{\frac{1}{2}-\frac{1}{n^2+1}+\varepsilon} T^{2R\cdot D(n)+2R\left(\sum_{1\le k<k'\le r} n_k\cdot n_{k'}+\sum_{k=1}^r \frac{n_k(n_k-1)}{2}\right)+\sum_{k=1}^r n_k-\sum_{1\le k<k'\le r} n_k\cdot n_{k'}-1+\varepsilon}.$$

Next, $\sum_{1\le k<k'\le r} n_k\cdot n_{k'} + \sum_{k=1}^r \frac{n_k(n_k-1)}{2} = \frac{n(n-1)}{2}$ by Lemma A.22 and $\sum_{k=1}^r n_k = n$. It follows that

$$\sum_{\Phi} |\mathcal{E}_{\mathcal{P},\Phi}| \ll (m\ell)^{\frac{1}{2}-\frac{1}{n^2+1}+\varepsilon}\, T^{2R\cdot\left(D(n)+\frac{n(n-1)}{2}\right)+n-1-\sum_{1\le k<k'\le r} n_k\cdot n_{k'}+\varepsilon}.$$

To complete the proof, we need to sum over all parabolics $\mathcal{P}$. It suffices, therefore, to consider the "worst case scenario" among the possible partitions $n = n_1 + \cdots + n_r$ for which the expression

$$\sum_{1\le k<k'\le r} n_k n_{k'}$$

is minimized. It is easy to see that this occurs when $r = 2$ and $\{n_1, n_2\} = \{n-1, 1\}$, giving the bound $n-1$. It follows that

$$\sum_{\mathcal{P}}\sum_{\Phi} |\mathcal{E}_{\mathcal{P},\Phi}| \ll (m\ell)^{\frac{1}{2}-\frac{1}{n^2+1}+\varepsilon}\, T^{2R\cdot\left(D(n)+\frac{n(n-1)}{2}\right)+\varepsilon}.$$

Using (1.4.2), this immediately implies the desired result. □

Remark 7.2.5. Jana and Nelson [2019] proved the bound

$$\sum_{c(\phi_j)\le T^n} \frac{1}{\mathcal{L}_j} \ll T^{n^2-n}, \tag{7.2.6}$$

where $c(\phi)$ is the analytic conductor given in (1.3.1). This is an unsmoothed version of Theorem 7.2.3. Our result is a smoothed version, and it doesn't seem possible to derive a bound as in (7.2.6) with a sharp cutoff without using a different approach.

8. An integral representation of $p_{T,R}^{(n)}(y)$

Recall (see 1.4.4) that

$$p_{T,R}^{n,\#}(\alpha) := e^{(\alpha_1^2+\alpha_2^2+\cdots+\alpha_n^2)/(2T^2)}\cdot \mathcal{F}_R^{(n)}\left(\frac{\alpha}{2}\right) \prod_{1\le j\ne k\le n} \Gamma\left(\frac{1+2R+\alpha_j-\alpha_k}{4}\right).$$

Using the formula for the inverse Lebedev–Whittaker transform given in [Goldfeld and Kontorovich 2012], it follows that

$$p_{T,R}^{(n)}(y) := \frac{1}{\pi^{n-1}} \int\limits_{\substack{\hat{\alpha}_n=0\\ \mathrm{Re}(\alpha)=0}} \frac{p_{T,R}^{n,\#}(\alpha)\, \overline{W_{n,\alpha}(y)}}{\prod_{1\le j\ne k\le n} \Gamma\big(\frac{\alpha_j-\alpha_k}{2}\big)}\, d\alpha$$
$$= \frac{1}{\pi^{n-1}} \int\limits_{\substack{\hat{\alpha}_n=0\\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\alpha_2^2+\cdots+\alpha_n^2}{2T^2}} \cdot \mathcal{F}_R^{(n)}\Big(\frac{\alpha}{2}\Big) \prod_{1\le j\ne k\le n} \Gamma_R\Big(\frac{\alpha_j-\alpha_k}{2}\Big) \overline{W_{n,\alpha}(y)}\, d\alpha,$$

where

$$\Gamma_R(z) := \frac{\Gamma\big(\frac{1}{2}\big(\frac{1}{2}+R+z\big)\big)}{\Gamma(z)}.$$

The strategy in this section for giving a representation of $p_{T,R}^{(n)}(y)$ follows the same general outline as was used to obtain the results for GL(3) and GL(4) given in [Goldfeld and Kontorovich 2013] and [Goldfeld et al. 2021b], respectively. As in the prior works, we express the Whittaker function as the inverse Mellin transform of its Mellin transform. (See Section 8.1.) Plugging this into the above formula gives an integral representation of $p_{T,R}^{(n)}(y)$ in terms of an additional variable $s = (s_1, \ldots, s_{n-1})$.

8.1. ***Normalized Mellin transform of Whittaker function.*** We introduce (as in [Ishii and Stade 2007]) the following Mellin transform and its inverse.

Definition 8.1.1 (normalized Mellin transform of Whittaker function). Let $n \in \mathbb{Z}_+$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ such that $\hat{\alpha}_n = 0$. Let $W_{n,\alpha}(y)$ be the Whittaker function of Definition 2.3.3. The *Mellin transform* is

$$\widetilde{W}_{n,\alpha}(s) := 2^{n-1} \int_0^\infty \cdots \int_0^\infty W_{n,2\alpha}(y) \prod_{j=1}^{n-1} (\pi y_j)^{2s_j} \frac{dy_j}{y_j^{1+\frac{j(n-j)}{2}}}, \tag{8.1.2}$$

and the *inverse Mellin transform* is given by

$$W_{n,\alpha}(y) = \frac{1}{2^{n-1}} \int\limits_{\substack{s=(s_1,\ldots,s_{n-1})\\ \mathrm{Re}(s)=2b}} \Bigg(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}} (\pi y_j)^{-s_j}\Bigg) \widetilde{W}_{n,\frac{\alpha}{2}}\big(\tfrac{s}{2}\big)\, ds. \tag{8.1.3}$$

As a consequence of this definition, we have

$$p_{T,R}^{(n)}(y) = \frac{1}{(2\pi)^{n-1}} \int\limits_{\substack{\hat{\alpha}_n=0\\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_n^2}{T^2/2}} \cdot \mathcal{F}_R^{(n)}(\alpha) \Bigg(\prod_{1\le j\ne k\le n} \Gamma_R(\alpha_j-\alpha_k)\Bigg)$$
$$\cdot \int\limits_{\substack{s=(s_1,\ldots,s_{n-1})\\ \mathrm{Re}(s)=b}} \Bigg(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}} (\pi y_j)^{-2s_j}\Bigg) \widetilde{W}_{n,\alpha}(s)\, ds\, d\alpha, \tag{8.1.4}$$

where $b = (b_1, \ldots, b_{n-1})$ with each $b_j > 0$.

We use the following theorem to make (8.1.4) explicit and to begin setting up an inductive method to bound $p_{T,R}^{(n)}(y)$ for all $n \ge 2$.

Theorem 8.1.5 (Ishii–Stade). *Let $m \geq 2$ and $\varepsilon > 0$. Fix a Langlands parameter $\alpha \in \mathbb{C}^m$. Let $s \in \mathbb{C}^{m-1}$ with* $\mathrm{Re}(s) > \varepsilon$. *Then*

$$\widetilde{W}_{m,\alpha}(s) = \int\limits_{\substack{z=(z_1,\dots,z_{m-2})\\ \mathrm{Re}(z)=\varepsilon}} \left(\prod_{j=1}^{m-1} \Gamma\left(s_j - z_{j-1} + \frac{(m-j)\alpha_m}{m-1}\right)\Gamma\left(s_j - z_j - \frac{j\alpha_m}{m-1}\right)\right) \cdot \frac{\widetilde{W}_{m-1,\beta}(z)}{(2\pi i)^{m-2}}\, dz, \quad (8.1.6)$$

where

$$z_0 := -0 + \frac{0 \cdot \alpha_m}{m-1} = 0, \quad z_{m-1} := \alpha_m - \frac{(m-1)\alpha_m}{m-1} = 0$$

and

$$\beta = (\beta_1, \dots, \beta_{m-1}) := \left(\alpha_1 + \frac{\alpha_m}{m-1}, \dots, \alpha_{m-1} + \frac{\alpha_m}{m-1}\right).$$

8.2. *A shifted $p^{(n)}_{T,R}$ term and the Ishii–Stade conjecture.* Our goal is to insert (8.1.6) into (8.1.4) and then shift the lines of integration in s to $\mathrm{Re}(s) = -a$, to the left of some of the poles of $\widetilde{W}_{n,\alpha}(s)$, which (see Theorem 10.1.1) occur at $\mathrm{Re}(s_i) = -\delta$ for every $1 \leq i \leq n-1$ and $\delta \in \mathbb{Z}_{\geq 0}$. By Cauchy's residue formula, this allows us to describe $p^{(n)}_{T,R}(y)$ in terms of a the sum of a *shifted $p^{(n)}_{T,R}$ term* and finitely many *shifted residue terms*.

Definition 8.2.1 (shifted $p^{(n)}_{T,R}$ term). Let $n \geq 2$ be an integer and $a = (a_1, \dots, a_{n-1}) \in \mathbb{R}^{n-1}$. The *shifted $p^{(n)}_{T,R}$ term* is given by the same formula as (8.1.4) but with b replaced by $-a$:

$$p^{(n)}_{T,R}(y; -a) := \frac{1}{(2\pi)^{n-1}} \int\limits_{\substack{\hat{\alpha}_n=0\\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_n^2}{T^2/2}} \cdot \mathcal{F}^{(n)}_R(\alpha)\left(\prod_{1\leq j\neq k\leq n} \Gamma_R(\alpha_j - \alpha_k)\right) \cdot \int\limits_{\substack{s=(s_1,\dots,s_{n-1})\\ \mathrm{Re}(s)=-a}} \left(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}} (\pi y_j)^{-2s_j}\right) \widetilde{W}_{n,\alpha}(s)\, ds\, d\alpha. \quad (8.2.2)$$

One might be tempted to insert (8.1.6) into (8.2.2), but this is invalid if $n > 3$, because Theorem 8.1.5 requires that $\mathrm{Re}(s_i) > \varepsilon$ for each $i = 1, \dots, n-1$. To overcome this difficulty, we use shift equations as given in the following conjecture. This allows us to evaluate $\widetilde{W}_{n,\alpha}(s)$ for $\mathrm{Re}(s) < 0$.

Conjecture 8.2.3 (Ishii–Stade). *Let $m, n \in \mathbb{Z}$ with $1 \leq m \leq n-1$; let $\delta \in \mathbb{Z}_{\geq 0}$. Let*

$$(x)_n := \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\cdots(x+n-1).$$

Then there exists a positive integer r and, for each i with $1 \leq i \leq r$, a polynomial $P_i(s, \alpha)$ and an $(n-1)$-tuple $\Sigma_i \in (\mathbb{Z}_{\geq 0})^{n-1}$, such that

$$\widetilde{W}_{n,\alpha}(s) = \left[\prod_{1\leq j_1<j_2<\cdots<j_m\leq n} (s_m + \alpha_{j_1} + \alpha_{j_2} + \cdots + \alpha_{j_m})_\delta\right]^{-1} \sum_{i=1}^{r} P_i(s, \alpha)\widetilde{W}_{n,\alpha}(s + \Sigma_i), \quad (8.2.4)$$

where the mth coordinate of each Σ_i is at least δ. Moreover, for each i, we have

$$\deg(P_i(s, \alpha)) + 2|\Sigma_i| = \delta\binom{n}{m}.$$

Proof of conjecture for $2 \le n \le 5$. Note that the case $\delta = 0$ of the conjecture is trivial. Moreover, for a given m and n with $1 \le m \le n-1$, it's enough to prove the conjecture for $\delta = 1$. The case $\delta > 1$ then follows by applying the case $\delta = 1$ to itself iteratively.

For $\delta = 1$ and $n = 2$ or $n = 3$, the conjecture follows immediately from the explicit formulae

$$\widetilde{W}_{2,\alpha}(s) = \Gamma(s+\alpha)\Gamma(s-\alpha),$$
$$\widetilde{W}_{3,\alpha}(s) = \frac{\Gamma(s_1+\alpha_1)\Gamma(s_1+\alpha_2)\Gamma(s_1+\alpha_3)\Gamma(s_2-\alpha_1)\Gamma(s_2-\alpha_2)\Gamma(s_2-\alpha_3)}{\Gamma(s_1+s_2)},$$

respectively, together with the functional equation $\Gamma(s+1) = s\Gamma(s)$. The case $\delta = 1$ and $n = 4$ is a consequence of [Stade and Trinh 2021, equations (21), (29), and (31)].

We now consider the case $\delta = 1$ and $n = 5$. Note that it suffices to derive the appropriate recurrence relations for $m = 1$ and $m = 2$ (that is, for the variables s_1 and s_2); the cases $m = 3$ and $m = 4$ then follow from the invariance of $\widetilde{W}_{5,\alpha}(s)$ under the involution

$$(s_1, s_2, s_3, s_4, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \to (s_4, s_3, s_2, s_1, -\alpha_1, -\alpha_2, -\alpha_3, -\alpha_4, -\alpha_5).$$

We follow an approach developed by Taku Ishii (personal correspondence). First, consider the case $m = 1$: we wish to show that

$$\left[\prod_{i=1}^{5}(s_1+\alpha_i)\right]\widetilde{W}_{5,\alpha}(s) \tag{8.2.5}$$

is equal to a finite sum of terms $P_i(s,\alpha)\widetilde{W}_{n,\alpha}(s+\Sigma_i)$, where the first coordinate of each $\Sigma_i \in (\mathbb{Z}_{\ge 0})^4$ is at least 1, and $\deg(P_i(s,\alpha)) + 2|\Sigma_i| = 5$ for each i. To this end, let

$$\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) = (-s_1, s_1-s_2, s_2-s_3, s_3-s_4, s_4); \tag{8.2.6}$$

note that $\sum_i \sigma_i = 0$. Since $s_1 + \sigma_1 = 0$, we have

$$\left[\prod_{i=1}^{5}(s_1+\alpha_i)\right]\widetilde{W}_{5,\alpha}(s) = \left[\prod_{i=1}^{5}(s_1+\alpha_i) - \prod_{i=1}^{5}(s_1+\sigma_i)\right]\widetilde{W}_{5,\alpha}(s). \tag{8.2.7}$$

But for indeterminates $T, x_1, x_2, x_3, x_4, x_5$, we have

$$\prod_{i=1}^{5}(T+x_i) = T^5 + T^4P_1(x) + T^3P_2(x) + T^2P_3(x) + TP_4(x) + P_5(x), \tag{8.2.8}$$

where $P_k(x)$ is the elementary symmetric polynomial of degree k in x_1, x_2, x_3, x_4, x_5. So by (8.2.7) above,

$$\begin{aligned}
&\left[\prod_{i=1}^{5}(s_1+\alpha_i)\right]\widetilde{W}_{5,\alpha}(s) \\
&\quad = \left[s_1^5 + s_1^4P_1(\alpha) + s_1^3P_2(\alpha) + s_1^2P_3(\alpha) + sP_4(\alpha) + P_5(\alpha)\right]\widetilde{W}_{5,\alpha}(s) \\
&\qquad\qquad - \left[s_1^5 + s_1^4P_1(\sigma) + s_1^3P_2(\sigma) + s_1^2P_3(\sigma) + sP_4(\sigma) + P_5(\sigma)\right]\widetilde{W}_{5,\alpha}(s) \\
&\quad = \left[s_1^3\{P_2(\alpha)-P_2(\sigma)\} + s_1^2\{P_3(\alpha)-P_3(\sigma)\} + s_1\{P_4(\alpha)-P_4(\sigma)\} + \{P_5(\alpha)-P_5(\sigma)\}\right]\widetilde{W}_{5,\alpha}(s),
\end{aligned} \tag{8.2.9}$$

since $P_1(\alpha) = P_1(\sigma) = 0$.

Now let e_k, for $1\le k\le 4$, be the four-tuple with a 1 in the k-th place and zeroes elsewhere. By [Ishii and Oda 2014, Proposition 3.6], we have

$$P_k(\alpha)-P_k(\sigma)=Z_k-C_k$$

(as operators acting on functions in the variable $s=(s_1,s_2,s_3,s_4)$), where the "Capelli elements" C_k annihilate $\widetilde{W}_{5,\alpha}(s)$, and

$$\begin{aligned}
Z_2 f(s) &= f(s+e_1)+f(s+e_2)+f(s+e_3)+f(s+e_4),\\
Z_3 f(s) &= P_1(\sigma_3,\sigma_4,\sigma_5)f(s+e_1)+P_1(\sigma_1,\sigma_4,\sigma_5)f(s+e_2)+P_1(\sigma_1,\sigma_2,\sigma_5)f(s+e_3)\\
&\qquad +P_1(\sigma_1,\sigma_2,\sigma_3)f(s+e_4),\\
Z_4 f(s) &= P_2(\sigma_3,\sigma_4,\sigma_5)f(s+e_1)+P_2(\sigma_1,\sigma_4,\sigma_5)f(s+e_2)+P_2(\sigma_1,\sigma_2,\sigma_5)f(s+e_3)\\
&\qquad +P_2(\sigma_1,\sigma_2,\sigma_3)f(s+e_4)+f(s+e_1+e_3)+f(s+e_1+e_4)+f(s+e_2+e_4),\\
Z_5 f(s) &= P_3(\sigma_3,\sigma_4,\sigma_5)f(s+e_1)+P_3(\sigma_1,\sigma_4,\sigma_5)f(s+e_2)+P_3(\sigma_1,\sigma_2,\sigma_5)f(s+e_3)\\
&\qquad +P_3(\sigma_1,\sigma_2,\sigma_3)f(s+e_4)+P_1(\sigma_5)f(s+e_1+e_3)+P_1(\sigma_3)f(s+e_1+e_4)+P_1(\sigma_1)f(s+e_2+e_4).
\end{aligned}$$

So by (8.2.9),

$$\begin{aligned}
\Big[\prod_{i=1}^{5}(s_1+\alpha_i)\Big]\widetilde{W}_{5,\alpha}(s) &= \big[s_1^3Z_2+s_1^2Z_3+s_1Z_4+Z_5\big]\widetilde{W}_{5,\alpha}(s)\\
&= \big[s_1^3+s_1^2P_1(\sigma_3,\sigma_4,\sigma_5)+s_1P_2(\sigma_3,\sigma_4,\sigma_5)+P_3(\sigma_3,\sigma_4,\sigma_5)\big]\widetilde{W}_{5,\alpha}(s+e_1)\\
&\quad +\big[s_1^3+s_1^2P_1(\sigma_1,\sigma_4,\sigma_5)+s_1P_2(\sigma_1,\sigma_4,\sigma_5)+P_3(\sigma_1,\sigma_4,\sigma_5)\big]\widetilde{W}_{5,\alpha}(s+e_2)\\
&\quad +\big[s_1^3+s_1^2P_1(\sigma_1,\sigma_2,\sigma_5)+s_1P_2(\sigma_1,\sigma_2,\sigma_5)+P_3(\sigma_1,\sigma_2,\sigma_5)\big]\widetilde{W}_{5,\alpha}(s+e_3)\\
&\quad +\big[s_1^3+s_1^2P_1(\sigma_1,\sigma_2,\sigma_3)+s_1P_2(\sigma_1,\sigma_2,\sigma_3)+P_3(\sigma_1,\sigma_2,\sigma_3)\big]\widetilde{W}_{5,\alpha}(s+e_4)\\
&\quad +\big[s_1+P_1(\sigma_5)\big]\widetilde{W}_{5,\alpha}(s+e_1+e_3)\\
&\quad +\big[s_1+P_1(\sigma_3)\big]\widetilde{W}_{5,\alpha}(s+e_1+e_4)\\
&\quad +\big[s_1+P_1(\sigma_1)\big]\widetilde{W}_{5,\alpha}(s+e_2+e_4).
\end{aligned}\tag{8.2.10}$$

Recalling that the P_k's are the elementary symmetric polynomials of degree k in their arguments, we see

$$s_1^3+s_1^2P_1(a,b,c)+s_1P_2(a,b,c)+P_3(a,b,c)=(s_1+a)(s_1+b)(s_1+c),$$

for indeterminates a,b,c. So (8.2.10) gives

$$\begin{aligned}
\Big[\prod_{i=1}^{5}(s_1+\alpha_i)\Big]\widetilde{W}_{5,\alpha}(s) &= (s_1+\sigma_3)(s_1+\sigma_4)(s_1+\sigma_5)\widetilde{W}_{5,\alpha}(s+e_1)\\
&\quad +(s_1+\sigma_1)(s_1+\sigma_4)(s_1+\sigma_5)\widetilde{W}_{5,\alpha}(s+e_2)\\
&\quad +(s_1+\sigma_1)(s_1+\sigma_2)(s_1+\sigma_5)\widetilde{W}_{5,\alpha}(s+e_3)\\
&\quad +(s_1+\sigma_1)(s_1+\sigma_2)(s_1+\sigma_3)\widetilde{W}_{5,\alpha}(s+e_4)\\
&\quad +(s_1+\sigma_5)\widetilde{W}_{4,\alpha}(s+e_1+e_3)\\
&\quad +(s_1+\sigma_3)\widetilde{W}_{5,\alpha}(s+e_1+e_4)\\
&\quad +(s_1+\sigma_1)\widetilde{W}_{5,\alpha}(s+e_2+e_4)
\end{aligned}$$

$$\begin{aligned}&= (s_1+s_2-s_3)(s_1+s_3-s_4)(s_1+s_4)\widetilde{W}_{5,\alpha}(s+e_1)\\&\quad+(s_1+s_4)\widetilde{W}_{5,\alpha}(s+e_1+e_3)+(s_1+s_2-s_3)\widetilde{W}_{5,\alpha}(s+e_1+e_4),\end{aligned}$$

where we have the last step by the definition (8.2.6) of the σ_i's. This is our desired shift equation in s_1.

The shift equation in s_2 is derived analogously. A fundamental difference in this derivation is that, in place of (8.2.8), we use the following expression involving *Schur polynomials* s_μ (see [Macdonald 1979, Section I.3], especially Exercise 10 of that section):

$$\prod_{1\le i<j\le 5}(T+x_i+x_j)=\sum_{\mu=(\mu_1,\mu_2,\dots,\mu_5)\in S}\left(\frac{T}{2}\right)^{10-(\mu_1+\mu_2+\cdots+\mu_5)} d_\mu\, s_\mu(x_1,x_2,\dots,x_5).$$

Here,

$$S=\big\{(\mu_1,\mu_2,\dots,\mu_5)\in(\mathbb{Z}_{\ge 0})^5 : \mu_i\le 5-i\ (1\le i\le 5)\text{ and }\mu_1\ge\mu_2\ge\cdots\ge\mu_5\big\},$$

and d_μ is the determinant of the matrix

$$\left(\binom{2(5-i)}{\mu_j+5-j}\right)_{1\le i,j\le 5}.$$

The Schur polynomials are symmetric polynomials in the x_k's and are therefore expressible in terms of the elementary symmetric polynomials in the x_k's. Techniques like those employed above, in the case $m=1$, therefore apply. We omit the details. □

Remark 8.2.11. The above proof, in the case $m=1$ (that is, for the variable s_1—and therefore also for the variable s_{n-1}), generalizes to the case of $\mathrm{GL}(n,\mathbb{R})$ for any $n\ge 2$. For $2\le m\le n-2$, we do not yet have a proof that works for all $n\ge 2$, though we expect that the above ideas and techniques should prove relevant. Indeed, using the above methods, and applying Mathematica to help with the more arduous calculations, we have been able to verify Conjecture 8.2.3 in full generality for $n\le 7$.

We further note that, alternatively, one might continue $\widetilde{W}_{n,\alpha}(s)$ in the s_j's by shifting or deforming the lines of integration in (8.1.6). Unfortunately such an approach has, thus far, failed to yield suitable results. In particular, the residues that one obtains in moving these lines of integration past poles of the integrand are quite complicated, and do not seem to lend themselves to bounds of the type required to estimate $p^{(n)}_{T,R}(y)$.

8.3. ***$p^{(n)}_{T,R}(y)$ is a sum of a shifted term and residues.*** Besides the shifted $p^{(n)}_{T,R}$-term (because we cross poles of $\widetilde{W}_{n,\alpha}(s)$ upon shifting the lines of integration) there are also many residue terms. The residue terms will be parametrized by compositions of n. Recall that a *composition of length* r of a positive integer n is a way of writing $n=n_1+\cdots+n_r$ as a sum of strictly positive integers. Two sums that differ in the order define different compositions. Compare this, on the other hand with *partitions* which are compositions of n for which the order doesn't matter.

Definition 8.3.1 (a-admissible compositions). Let $a=(a_1,\dots,a_{n-1})\in\mathbb{R}^{n-1}$. A composition $n=n_1+\cdots+n_r$ is termed *a-admissible* if

$$a_{\hat{n}_i}>0\quad\text{for all } i=1,\dots,r-1.$$

The set of a-admissible compositions of length greater than 1 is

$$\mathcal{C}_a := \{\text{compositions } n = n_1 + \cdots + n_r \mid 2 \le r \le n,\ a_{\hat{n}_i} > 0 \text{ for all } i = 1, \ldots, r-1\}.$$

Remark 8.3.2. At times we may also notate a composition $n = n_1 + \cdots + n_r$ as an ordered list $C = (n_1, \ldots, n_r)$.

Definition 8.3.3 ($(r-1)$-fold residue term). Suppose that $r \ge 2$ and $C \in \mathcal{C}_a$ is given by $n = n_1 + \cdots + n_r$. Let

$$\delta_C := (\delta_1, \delta_2, \ldots, \delta_{r-1}) \in (\mathbb{Z}_{\ge 0})^{r-1},$$

with $0 \le \delta_i \le \lfloor a_{\hat{n}_i} \rfloor$ for each $i = 1, \ldots, r-1$. If C has length 2, we write $\delta_C = \delta$. We define the $(r-1)$*-fold residue term*

$$p_{T,R}^{(n)}(y; -a, \delta_C) := \int\limits_{\substack{\hat{\alpha}_n = 0\\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_n^2}{T^2/2}} \cdot \mathcal{F}_R^{(n)}(\alpha) \Big(\prod_{1 \le j \ne k \le n} \Gamma_R(\alpha_j - \alpha_k) \Big) \cdot \Big(\prod_{i=1}^{r-1} y_i^{\frac{\hat{n}_i(n-\hat{n}_i)}{2} + 2(\hat{\alpha}_{\hat{n}_i} + \delta_i)} \Big) \cdot \int\limits_{\substack{\mathrm{Re}(s_j) = -a_j\\ j \notin \{\hat{n}_1, \ldots, \hat{n}_{r-1}\}}} \Big(\prod_{j \notin \{\hat{n}_1, \ldots, \hat{n}_{r-1}\}} y_j^{\frac{j(n-j)}{2} - 2s_j} \Big) \cdot \operatorname*{Res}_{s_{\hat{n}_1} = -\hat{\alpha}_{\hat{n}_1} - \delta_1} \Big(\operatorname*{Res}_{s_{\hat{n}_2} = -\hat{\alpha}_{\hat{n}_2} - \delta_2} \big(\cdots \big(\operatorname*{Res}_{s_{\hat{n}_{r-1}} = -\hat{\alpha}_{\hat{n}_{r-1}} - \delta_{r-1}} \widetilde{W}_{n,\alpha}(s) \big) \cdots \big) \Big)\, ds\, d\alpha. \tag{8.3.4}$$

Remark 8.3.5. In the shifted integral (8.3.4), if $-a_i > 0$ for some i, there will be no residues coming from the integral in s_i because we are not shifting past any poles. For this reason, one only obtains residue terms $p_{T,R}^{(n)}(y; -a, \delta_C)$ in the case that C is a-admissible. That said, (8.3.4) makes perfect sense even if C is not a-admissible. In this case, $p_{T,R}^{(n)}(y; -a, \delta_C)$ is identically zero.

Proposition 8.3.6. *Suppose that $a = (a_1, \ldots, a_{n-1}) \in \mathbb{R}^{n-1}$. Then there exists constants $\kappa(C)$ such that*

$$p_{T,R}^{(n)}(y) = p_{T,R}^{(n)}(y; -a) + \sum_{C \in \mathcal{C}_a} \kappa(C) \sum_{\substack{\delta_C = (\delta_1, \ldots, \delta_{r-1})\\ 0 \le \delta_i \le \lfloor a_{\hat{n}_i} \rfloor}} p_{T,R}^{(n)}(y; -a, \delta_C).$$

Before giving the proof, we make some preliminary remarks and observations.

Remark 8.3.7. Notice that an element σ of the symmetric group S_n (i.e., the group of permutations of a set of n elements) acts on $\alpha = (\alpha_1, \ldots, \alpha_n)$ and, by extension, on $\hat{\alpha}_k$ via

$$\sigma \cdot \hat{\alpha}_k := \alpha_{\sigma(1)} + \alpha_{\sigma(2)} + \cdots + \alpha_{\sigma(k)}.$$

We can consider the analog to (8.3.4) obtained by replacing each instance of $\hat{\alpha}_m$ with $\sigma \cdot \hat{\alpha}_m$:

$$\int\limits_{\substack{\hat{\alpha}_n = 0\\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_n^2}{T^2/2}} \cdot \mathcal{F}_R^{(n)}(\alpha) \cdot \Big(\prod_{1 \le j \ne k \le n} \Gamma_R(\alpha_j - \alpha_k) \Big) \Big(\prod_{i \in \{\hat{n}_1, \ldots, \hat{n}_{r-1}\}} y_i^{\frac{i(n-i)}{2} + 2(\sigma \cdot \hat{\alpha}_i + \delta_i)} \Big) \cdot \int\limits_{\substack{\mathrm{Re}(s_j) = -a_j\\ j \in \{\hat{n}_1, \ldots, \hat{n}_{r-1}\}}} \Big(\prod_{j \notin \{\hat{n}_1, \ldots, \hat{n}_{r-1}\}} y_j^{\frac{j(n-j)}{2} - 2s_j} \Big) \operatorname*{Res}_{s_{i_1} = -\sigma \cdot \hat{\alpha}_{i_1} - \delta_{i_1}} \big(\operatorname*{Res}_{s_{i_2} = -\sigma \cdot \hat{\alpha}_{i_2} - \delta_{i_2}} \cdots \operatorname*{Res}_{s_{i_k} = -\sigma \cdot \hat{\alpha}_{i_k} - \delta_{i_k}} \widetilde{W}_{n,\alpha}(s) \big)\, ds\, d\alpha$$

We make two observations:

• As C varies over all compositions of length r and σ varies over all possible permutations and δ_C varies over all $(\mathbb{Z}_{\geq 0})^{r-1}$, one obtains all possible $(r-1)$-fold residues coming from shifting the lines of integration in $p^{(n)}_{T,R}(y)$. This is a consequence of Theorem 10.1.1 below.

• The action of S_n on ordered subsets of $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ given by permuting the indices is trivial on $\widetilde{W}_{n,\alpha}(s)$, i.e., $\widetilde{W}_{n,\sigma(\alpha)}(s) = \widetilde{W}_{n,\alpha}(s)$, and on the function

$$e^{\frac{\alpha_1^2+\cdots+\alpha_n^2}{T^2/2}} \cdot \mathcal{F}^{(n)}_R(\alpha) \cdot \Bigg(\prod_{1\leq j\neq k\leq n} \Gamma_R(\alpha_j - \alpha_k) \Bigg).$$

This implies that relabeling the variables $\alpha_1, \alpha_2, \ldots \alpha_n$ by $\alpha_{\sigma^{-1}(1)}, \alpha_{\sigma^{-1}(2)}, \ldots, \alpha_{\sigma^{-1}(n)}$ everywhere doesn't change the value of the integral, and recovers the original integral given in (8.3.4).

Remark 8.3.8. The constant $\kappa(C)$ is the size of the (generic) orbit of the action of S_n on the set

$$A = \{\hat{\alpha}_{\hat{n}_1}, \ldots, \hat{\alpha}_{\hat{n}_{r-1}}\}.$$

Hence, defining the stabilizer of A to be

$$\mathrm{Stab}(A) := \{\sigma \in S_n \mid \sigma \cdot \hat{\alpha}_m = \hat{\alpha}_m \text{ for each } m = \hat{n}_1, \ldots, \hat{n}_{r-1}\},$$

we see that

$$\kappa(C) = \frac{\# S_n}{\#\,\mathrm{Stab}(A)} = \frac{n!}{\prod_{i=1}^{r-1}(n_i!)}.$$

Since the exact value of $\kappa(C)$ is irrelevant to our application, we omit its proof below and leave it instead to the interested reader.

Proof of Proposition 8.3.6. Beginning with (8.1.4), we see that $p^{(n)}_{T,R}(y) = p^{(n)}_{T,R}(y; b)$ for any $b = (b_1, \ldots, b_{n-1})$ with $b_i > 0$ for each $i = 1, \ldots, n-1$. In order to compare this with $p^{(n)}_{T,R}(y; -a)$, we successively shift the lines of integration in the variables s_k for each k such that $-a_k < 0$ (in descending order). If $-a_k > 0$ then shifting the line of integration from $\mathrm{Re}(s) = b_k$ to $\mathrm{Re}(s_k) = -a_k$ doesn't change the value of the integral in s_k. In other words, there is a residue term if and only if the composition C is admissible.

Beginning with the fact that

$$p^{(n)}_{T,R}(y) = p^{(n)}_{T,R}(y; b) \quad \text{for any } b = (b_1, \ldots, b_{n-1}) \text{ for which } b_j > 0 \text{ for all } j,$$

we may shift the line of integration in s_{n-1} to $\mathrm{Re}(s_{n-1}) = -a_{n-1}$. In doing so, provided that $a_{n-1} > 0$, we pass poles at $s_{n-1} = -\sigma \cdot \hat{\alpha}_1 - \delta_1$ for each $0 \leq \delta \leq \lfloor a_1 \rfloor$. Hence, taking into account Remark 8.3.7, and considering $n = (n-1) + 1$ (denoted by $(n-1, 1)$), it follows that

$$p^{(n)}_{T,R}(y) = p^{(n)}_{T,R}(y; (b_1, b_2, b_3, \ldots, -a_{n-1})) \\ + \kappa((n-1, 1)) \cdot \sum_{\delta_{(n-1,1)}} p^{(n)}_{T,R}(y; (b_1, b_2, \ldots, -a_{n-1}), \delta_{(n-1,1)}), \quad (8.3.9)$$

where $\kappa((n-1, 1))$ is a constant (which can be verified to agree with the description given in Remark 8.3.8.)

We now shift the line of integration in s_{n-2} to $\mathrm{Re}(s_{n-2}) = -a_{n-2}$. As before, provided that $a_{n-2} > 0$, the Cauchy residue theorem and Remark 8.3.7 give

$$\begin{aligned} p^{(n)}_{T,R}(y) = p^{(n)}_{T,R}&(y; (b_1, \dots, b_{n-3}, -a_{n-2}, -a_{n-1})) \\ &+ \kappa((n-2, 2)) \sum_{\delta_{(n-2,2)}} p^{(n)}_{T,R}(y; (b_1, \dots, b_{n-3}, -a_{n-2}, -a_{n-1}), \delta_{(n-2,2)}) \\ &\quad + \kappa((n-1, 1)) \sum_{\delta_{(n-1)}} p^{(n)}_{T,R}(y; (b_1, \dots, b_{n-3}, -a_{n-2}, -a_{n-1}), \delta_{(n-1,1)}) \\ &\qquad + \kappa((n-2, 1, 1)) \sum_{\delta_{(n-2,1,1)}} p^{(n)}_{T,R}(y; (b_1, \dots, b_{n-3}, -a_{n-2}, -a_{n-1}), \delta_{(n-2,1,1)}) \end{aligned} \tag{8.3.10}$$

for constants $\kappa(C)$ for each of $C = (n-1, 1), (n-2, 2), (n-2, 1, 1)$ as claimed.

We next repeat this process shifting the integrals in s_{n-3} for each of the terms on the right of (8.3.10), and then again for s_{n-4} and so forth (skipping those s_m for which $a_m < 0$) until all of the lines of integration have been moved to $\mathrm{Re}(s_m) = -a_m$ for every possible integral. The claimed formula is now evident. □

8.4. *Example:* GL(4). We now consider the special case of $\widetilde{W}_{4,\alpha}(s)$ where

$$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in (i\mathbb{R})^4, \quad \hat{\alpha}_4 = 0.$$

Fix $\varepsilon > 0$. Recall that $p^{(4)}_{T,R}(y) = p^{(4)}_{T,R}(y; (\varepsilon, \varepsilon, \varepsilon))$. If we now shift the lines of integration to $\mathrm{Re}(s) = (-a)$ where $a = (a_1, a_2, a_3) \in \mathbb{R}^3$, then we get additional residue terms corresponding to each composition $4 = n_1 + \cdots + n_r$ and each $\delta_C \in (\mathbb{Z}_{\geq 0})^r$ as follows.

In general the composition $n = n_1 + \cdots + n_r$ (by abuse of notation, we also think of this as a vector $(n_1, \dots, n_r)$ so that $\hat{n}_k = n_1 + \cdots + n_k$) corresponds to taking an $(r-1)$-fold residue in the variables $s_{\hat{n}_1}, s_{\hat{n}_2}, \dots, s_{\hat{n}_{r-1}}$. Here is a table of the residues corresponding to the different compositions:

composition C	residues in s-variables	δ_C
$1+3$	$s_1 = -\alpha_1 - \delta_1$	(δ_1)
$2+2$	$s_2 = -\alpha_1 - \alpha_2 - \delta_2$	(δ_2)
$3+1$	$s_3 = -\alpha_1 - \alpha_2 - \alpha_3 - \delta_3$	(δ_3)
$1+1+2$	$s_1 = -\alpha_1 - \delta_1,\ s_2 = -\alpha_1 - \alpha_2 - \delta_2$	(δ_1, δ_2)
$1+2+1$	$s_1 = -\alpha_1 - \delta_1,\ s_3 = -\alpha_1 - \alpha_2 - \alpha_3 - \delta_3$	(δ_1, δ_3)
$2+1+1$	$s_2 = -\alpha_1 - \alpha_2 - \delta_2,\ s_3 = -\alpha_1 - \alpha_2 - \alpha_3 - \delta_3$	(δ_2, δ_3)

In each case $0 \leq \delta_i \leq \lfloor a_i \rfloor$. Not included in the table are the triple residues in $s_i = -\hat{\alpha}_i - \delta_i$ for each $i = 1, 2, 3$. These correspond to the composition $4 = 1+1+1+1$ and $\delta_C = (\delta_1, \delta_2, \delta_3)$.

8.5. *The integral $\mathcal{I}^{(m)}_{T,R}(-a)$ in terms of an explicit recursive formula for $\widetilde{W}_{m,\alpha}(s)$.* At first glance, the following definition appears to be relevant only for the shifted $p^{(n)}_{T,R}$-term, as it is essentially equal to $p^{(n)}_{T,R}((1, \dots, 1); -a)$, and not for the shifted residue terms. However, it will turn out to be pivotal to bounding the residue terms as well.

Definition 8.5.1 (the integral $\mathcal{I}_{T,R}^{(m)}$). Let $m \geq 2$ be an integer and $a=(a_1,\dots,a_{m-1})\in\mathbb{R}^{m-1}$. Then we define

$$\mathcal{I}_{T,R}^{(m)}(-a) := \int\limits_{\substack{\hat{\alpha}_m=0\\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_m^2}{T^2/2}} \cdot \mathcal{F}_R^{(n)}(\alpha)\bigg(\prod_{1\leq j\neq k\leq m}\Gamma_R(\alpha_j-\alpha_k)\bigg) \int\limits_{\substack{s=(s_1,\dots,s_{m-1})\\ \mathrm{Re}(s)=-a}} |\widetilde{W}_{m,\alpha}(s)|\,ds\,d\alpha. \tag{8.5.2}$$

As alluded to above, inserting the result of Theorem 8.1.5 into (8.2.2), we find that

$$|p_{T,R}^{(n)}(y,-a)| \ll \bigg(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}-2a_j}\bigg)\mathcal{I}_{T,R}^{(n)}(-a).$$

Hence, giving a bound for $p_{T,R}^{(n)}(y)$ requires only that we bound $\mathcal{I}_{T,R}^{(m)}(-a)$ in the case of $m=n$. However, much more is true: we will show that if C is the composition $n=n_1+\cdots+n_r$, then $p_{T,R}^{(n)}(y;-a,\delta_C)$ can be bounded by the same product of y_i's as above times a certain power of T and a product of the form

$$\prod_{\ell=1}^{r-1}\mathcal{I}_{T,R}^{(n_\ell)}(-a^{(\ell)})$$

for certain values $a^{(\ell)}=(a_1^{(\ell)},\dots,a_{n_\ell-1}^{(\ell)})$ which depend on the value of $a=(a_1,\dots,a_{n-1})\in\mathbb{R}^{n-1}$.

The significance of this fact should not be understated. Without it, we would be required to treat nearly every possible composition C (hence each possible residue term) individually. Indeed, returning to the case of $n=4$, as noted in Section 8.4 above, there were seven residue terms. The only symmetries that we were able to exploit in [Goldfeld et al. 2021b] to help were that the (1, 3) and (3, 1) residues were equivalent, and the (1, 1, 2) and (2, 1, 1) residues were equivalent as well. This left five individual distinct cases, each of which required several pages of work to bound. So, although the method of this paper does require dealing with some tricky notation and combinatorics, it eliminates the need to treat each residue on its own terms.

9. Bounding $\mathcal{I}_{T,R}^{(m)}$

Recall that for $\alpha=(\alpha_1,\dots,\alpha_m)\in\mathbb{C}^m$ satisfying $\hat{\alpha}_m=0$ and $a=(a_1,a_2,\dots,a_{n-1})\in\mathbb{R}^{n-1}$,

$$\mathcal{I}_{T,R}^{(m)}(-a) := \int\limits_{\substack{\hat{\alpha}_m=0\\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_m^2}{T^2/2}} \cdot \mathcal{F}_R^{(m)}(\alpha)\prod_{1\leq j\neq k\leq m}|\Gamma_R(\alpha_j-\alpha_k)| \int\limits_{\mathrm{Re}(s)=-a} |\widetilde{W}_{m,\alpha}(s)|\,ds\,d\alpha. \tag{9.0.1}$$

Theorem 9.0.2. *Let $\mathcal{I}_{T,R}^{(m)}(-a)$ be as above and set $D(m)=\deg(\mathcal{F}_1^{(m)}(\alpha))$. Then, for any $0<\varepsilon<\frac{1}{2}$,*

$$\mathcal{I}_{T,R}^{(m)}(-a) \ll T^{\varepsilon+\frac{(m+4)(m-1)}{4}+R\cdot\left(D(m)+\frac{m(m-1)}{2}\right)-\sum_{j=1}^{m-1}B(a_j)},$$

where

$$B(c)=\begin{cases} 0 & \text{if } c<0,\\ \lfloor c\rfloor+2(c-\lfloor c\rfloor) & \text{if } 0<\lfloor c\rfloor+\varepsilon<c\leq\lfloor c\rfloor+\frac{1}{2},\\ \lceil c\rceil & \text{if } \frac{1}{2}<\lceil c\rceil-\frac{1}{2}\leq c<\lceil c\rceil-\varepsilon.\end{cases}$$

The implicit constant depends on ε, R and m.

Theorem 8.1.5 allows us to write $\widetilde{W}_{m,\alpha}(s)$ in terms of an integral of the product of several Gamma functions and the lower-rank Mellin transform $\widetilde{W}_{m-1,\beta}(z)$, where

$$\beta=(\beta_1,\ldots,\beta_{m-1}):=\left(\alpha_1+\frac{\alpha_m}{m-1},\ldots,\alpha_{m-1}+\frac{\alpha_m}{m-1}\right).$$

Using this, we are able siphon off the contribution to the integrand of (9.0.1) that is independent of the variable β. This in turn allows us to relate $\mathcal{I}_{T,R}^{(m)}$ to $\mathcal{I}_{T,R}^{(m-1)}$ and prove the result inductively.

9.1. ***Symmetry of integration in α.*** Since the integrand of (9.0.1) is invariant under the action of $\sigma\in S_m$ acting on $\alpha=(\alpha_1,\ldots,\alpha_m)$, we may restrict the integration to a fundamental domain. A choice of such a fundamental domain is

$$\operatorname{Im}(\alpha_1)\geq\operatorname{Im}(\alpha_2)\geq\cdots\geq\operatorname{Im}(\alpha_m). \tag{9.1.1}$$

Hence, (9.0.1) is equal, up to a constant, to the same integral but restricted to α satisfying (9.1.1). In the sequel we will always assume that (9.1.1) holds.

9.2. ***Extended exponential zero set.*** Recall that Stirling's asymptotic formula (for $\sigma\in\mathbb{R}$ fixed and $t\in\mathbb{R}$ with $|t|\to\infty$) is given by

$$\Gamma(\sigma+it)\sim\sqrt{2\pi}\cdot|t|^{\sigma-\frac{1}{2}}\,e^{-\frac{\pi}{2}|t|}. \tag{9.2.1}$$

Definition 9.2.2 (exponential and polynomial factors of a ratio of Gamma functions). We call $|t|^{\sigma-1/2}$ the *polynomial factor* of $\Gamma(\sigma+it)$, and $e^{-(\pi/2)|t|}$ is called the *exponential factor*. For a ratio of Gamma functions, the *polynomial* (*respectively*, *exponential*) *factor* is composed of the polynomial (respectively, exponential) factors of each individual Gamma function.

Lemma 9.2.3 (extended exponential zero set). *Assume that $\alpha\in\mathbb{C}^m$ is a Langlands parameter satisfying*

$$\operatorname{Im}(\alpha_1)\geq\operatorname{Im}(\alpha_2)\geq\cdots\geq\operatorname{Im}(\alpha_m).$$

Then the integrand of $\mathcal{I}_{T,R}^{(m)}$ (as a function of s) has exponential decay outside of the set $I=I_1\times I_2\times\cdots\times I_{m-1}$, where

$$I_j:=\left\{s_j\;\middle|\;-\sum_{k=1}^{j}\operatorname{Im}(\alpha_k)\leq\operatorname{Im}(s_j)\leq-\sum_{k=1}^{j}\operatorname{Im}(\alpha_{m-k+1})\right\}.$$

Remark 9.2.4. See [Goldfeld et al. 2021b] for the definition of the *exponential zero set* of an integral. The extended exponential zero set given in Lemma 9.2.3 contains the exponential zero set for $\mathcal{I}_{T,R}^{(m)}$.

Proof. We first prove Lemma 9.2.3 in the case that $m=2$. In the formula (9.0.1) for $\mathcal{I}_{T,R}^{(n)}$, replace $\widetilde{W}_{2,\alpha}(s_1)$ with $\Gamma(s_1+\alpha_1)\Gamma(s_1+\alpha_2)$. Then assuming (9.1.1), the exponential factor is $e^{(\pi/2)\mathcal{E}(s,\alpha)}$, where

$$\mathcal{E}(s,\alpha)=|\operatorname{Im}(s_1)+\operatorname{Im}(\alpha_1)|+|\operatorname{Im}(s_1)+\operatorname{Im}(\alpha_2)|-2\operatorname{Im}(\alpha_1).$$

We see, therefore, that the exponential factor $\mathcal{E}(s,\alpha)$ is negative unless

$$\operatorname{Im}(s_1)+\operatorname{Im}(\alpha_1)\geq 0\quad\text{and}\quad\operatorname{Im}(s_1)+\operatorname{Im}(\alpha_2)\leq 0\quad\Longleftrightarrow\quad-\operatorname{Im}(\alpha_1)\leq\operatorname{Im}(s_1)\leq-\operatorname{Im}(\alpha_2),$$

as claimed.

Let us suppose that $m \geq 3$ and $c = (c_1, c_2, \dots, c_{m-1})$, with $c_j > 0$ $(j = 1, 2, \dots, m-1)$. In order to prove Lemma 9.2.3 using induction on m, we make use of the change of variables

$$\beta_j = \alpha_j + \frac{\alpha_m}{m-1}, \quad j = 1, \dots, m-1.$$

Observe that

$$\beta_1 + \cdots + \beta_{m-1} = 0.$$

By Lemma A.19 in the case that $k = m-1$,

$$\alpha_1^2 + \cdots + \alpha_m^2 = \beta_1^2 + \cdots + \beta_{m-1}^2 + \frac{m}{m-1}\alpha_m^2.$$

Then in the integrand for $\mathcal{I}^{(m)}_{T,R}(c)$ we may substitute the formula for $\widetilde{W}_{m,\alpha}(s)$ given in Theorem 8.1.5. We also use the fact (see Lemma A.26) that

$$\prod_{1\leq j\neq k\leq m} \Gamma(\alpha_j - \alpha_k) = \biggl(\prod_{1\leq j\neq k\leq m-1} \Gamma(\beta_j - \beta_k)\biggr)\cdot\biggl(\prod_{i=1}^{m-1}\Gamma(\alpha_m - \alpha_i)\Gamma(\alpha_i - \alpha_m)\biggr),$$

and, via Stirling,

$$\prod_{i=1}^{m-1}\Gamma(\alpha_m - \alpha_i)\Gamma(\alpha_i - \alpha_m) \ll e^{\pi\,\mathrm{Im}(\alpha_m)}.$$

Note that (9.1.1) implies that $\mathrm{Im}(\alpha_m) \leq 0$; hence,

$$\begin{aligned}
\mathcal{I}^{(m)}_{T,R}(c) \ll \int\limits_{\mathrm{Re}(\alpha_m)=0} e^{\frac{m}{m-1}\frac{\alpha_m^2}{T^2/2}} \int\limits_{\substack{\hat{\beta}_{m-1}=0\\ \mathrm{Re}(\beta)=0}} e^{\frac{\beta_1^2+\cdots+\beta_{m-1}^2}{T^2/2}} \cdot |\mathcal{P}_{(D(m)-D(m-1))R}(\alpha_m, \beta)| \\
\cdot \mathcal{F}^{(m-1)}_R(\beta) \prod_{1\leq j\neq k\leq m-1} |\Gamma_R(\beta_j - \beta_k)| \int\limits_{\substack{\mathrm{Re}(z_j)=b_j\\ 1\leq j\leq m-2}} |\widetilde{W}_{m-1,\beta}(z)| \\
\cdot \prod_{j=1}^{m-1} \int\limits_{\mathrm{Re}(s_j)=c_j} \biggl|\Gamma\biggl(s_j - z_{j-1} + \frac{(m-j)\alpha_m}{m-1}\biggr)\Gamma\biggl(s_j - z_j - \frac{j\alpha_m}{m-1}\biggr) \\
\cdot \Gamma_R\biggl(\frac{-m}{m-1}\alpha_m - \beta_j\biggr)\Gamma_R\biggl(\beta_j + \frac{m}{m-1}\alpha_m\biggr)\biggr|\, ds_j\, dz\, d\alpha.
\end{aligned}$$

By the induction hypothesis, the second row of this expression has exponential decay outside of the set

$$\biggl\{ z = (z_1, \dots, z_{m-2}) \biggm| -\sum_{k=1}^{j} \beta_k \leq \mathrm{Im}(z_j) \leq -\sum_{j=1}^{k} \beta_{m-j} \biggr\} \tag{9.2.5}$$

for each $k = 1, 2, \dots, m-2$. (Recall that $z_0 = z_{m-1} = 0$.)

The assumption $\mathrm{Im}(\alpha_j) \geq \mathrm{Im}(\alpha_m)$ and the definition of β_j above imply that

$$\mathrm{Im}(\alpha_j - \alpha_m) = \mathrm{Im}\biggl(\beta_j + \frac{m}{m-1}\alpha_n\biggr) \geq 0 \quad (j = 1, 2, \dots, m-1).$$

Thus, the exponential factor coming from the final line in the expression above is $e^{(\pi/2)\mathcal{E}(s,z,\beta,\alpha_m)}$, where

$$\begin{aligned}\mathcal{E}(s,z,\beta,\alpha_n) &= \sum_{j=1}^{n-1}\biggl(\Bigl|\mathrm{Im}\Bigl(s_j - z_{j-1} + \frac{n-j}{n-1}\alpha_n\Bigr)\Bigr| + \Bigl|\mathrm{Im}\Bigl(s_j - z_j - \frac{j}{n-1}\alpha_n\Bigr)\Bigr| - \mathrm{Im}\Bigl(\frac{n}{n-1}\alpha_n + \beta_j\Bigr)\biggr)\\ &= \sum_{j=1}^{n-1}\biggl(\Bigl|\mathrm{Im}\Bigl(s_j - z_{j-1} + \frac{n-j}{n-1}\alpha_n\Bigr)\Bigr| + \Bigl|\mathrm{Im}\Bigl(s_j - z_j - \frac{j}{n-1}\alpha_n\Bigr)\Bigr|\biggr) - n\,\mathrm{Im}(\alpha_n).\end{aligned}$$

We know the integral defining $\mathcal{I}_{T,R}^{(m)}(-a)$ is convergent. Therefore, it must be the case that $\mathcal{E}(s,z,\beta,\alpha_m) \le 0$. In order to find where $\mathcal{E}=0$, i.e., where there is *not* exponential decay, we seek values

$$\epsilon_{1,1}, \epsilon_{2,1}, \ldots, \epsilon_{1,m-1}, \epsilon_{2,m-1} \in \{\pm 1\}$$

for which

$$\sum_{j=1}^{m-1}\biggl(\epsilon_{1,j}\,\mathrm{Im}\Bigl(s_j - z_{j-1} + \frac{m-j}{m-1}\alpha_m\Bigr) + \epsilon_{2,j}\,\mathrm{Im}\Bigl(s_j - z_j - \frac{j}{m-1}\alpha_m\Bigr)\biggr) = m\,\mathrm{Im}(\alpha_m). \tag{9.2.6}$$

In order for the s-variables to cancel it is clear that for each $j = 1, 2, \ldots, m-1$ it need be true that $\epsilon_j := \epsilon_{1,j} = -\epsilon_{2,j}$. With this assumption, (9.2.6) simplifies:

$$\sum_{j=1}^{m-1}\biggl(\epsilon_j\,\mathrm{Im}\Bigl(z_j - z_{j-1} + \frac{m}{m-1}\alpha_m\Bigr)\biggr) = m\,\mathrm{Im}(\alpha_m).$$

In order for this to hold true, it is necessary that $\epsilon_j = 1$ for all j, since otherwise, the coefficients of α_m on each side of the inequality wouldn't match. On the other hand, $\epsilon_j = 1$ for all j is sufficient as well since

$$\sum_{j=1}^{m-1}\mathrm{Im}(z_{j-1} - z_j) = \mathrm{Im}(z_0 - z_{m-1}) = 0.$$

This unique solution to (9.2.6) implies, therefore, that there is exponential decay in the integrand of $\mathcal{I}_{T,R}^{(m)}$ above unless $\mathrm{Im}\bigl(z_{j-1} - \frac{m-j}{m-1}\alpha_m\bigr) \le \mathrm{Im}(s_j) \le \mathrm{Im}\bigl(z_j + \frac{j}{m-1}\alpha_m\bigr)$. The inductive assumption (9.2.5) implies

$$\begin{aligned}\mathrm{Im}\Bigl(z_{j-1} - \frac{m-j}{m-1}\alpha_m\Bigr) &\ge -\sum_{k=1}^{j-1}\Bigl(\beta_k - \frac{\alpha_m}{m-1}\Bigr) - \alpha_m = -\sum_{k=1}^{j}\alpha_j,\\ \mathrm{Im}\Bigl(z_j + \frac{j}{m-1}\alpha_m\Bigr) &\le -\sum_{k=1}^{j}\Bigl(\beta_k - \frac{\alpha_m}{m-1}\Bigr) = -\sum_{k=1}^{m}\alpha_k,\end{aligned}$$

thus yielding the desired bounds on $\mathrm{Im}(s_j)$.

To complete the proof, we remark that if $-a < 0$, in order to use the result of Theorem 8.1.5, we need to first apply the shift equations given in Corollary 9.2.8 below. This will allow us to rewrite $\mathcal{I}_{T,R}^{(m)}(-a)$ as a sum over terms all of which have the same basic form as that for $\mathcal{I}_{T,R}^{(m)}(c)$ with $c > 0$. Each of these terms has precisely the same exponential factor since this depends only on the imaginary parts of the arguments of the Gamma functions; hence the same exponential zero set is determined in general. □

For each $j = 1, \ldots, n$, we define

$$\mathcal{B}_j(s_j, \alpha) := \prod_{\substack{K \subseteq \{1,\ldots,n\} \\ \#K = j}} \left(s_j + \sum_{k \in K} \alpha_k \right). \tag{9.2.7}$$

Using this, the following corollary is easily deduced. (See [Goldfeld et al. 2021b] for the case of $n = 4$.)

Corollary 9.2.8. *Let $r = (r_1, \ldots, r_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$. There exists a sequence of shifts $\sigma = (\sigma_1, \ldots, \sigma_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$ and polynomials $Q_{\sigma,r}(s, \alpha)$ such that*

$$|\widetilde{W}_{n,\alpha}(s)| \ll \sum_{\sigma} \frac{|Q_{\sigma,r}(s,\alpha)|}{\prod_{j=1}^{n-1} |\mathcal{B}_j(s_j,\alpha)|^{r_j}} |\widetilde{W}_{n,\alpha}(s + r + \sigma)|,$$

where

$$Q_{\sigma,r}(s,\alpha) = \prod_{j=1}^{n-1} P_{\sigma_j,r_j}(s,\alpha), \quad \deg(P_{\sigma_j,r_j}(s,\alpha)) = r_j\left(\binom{n}{j} - 2\right) - 2\sigma_j.$$

9.3. *Proof of Theorem 9.0.2 in the case $m = 2$.*

Proof. As in the proof of Lemma 9.2.3, we can replace $\widetilde{W}_{2,(\alpha,-\alpha)}(s)$ with $\Gamma(s+\alpha)\Gamma(s-\alpha)$ and estimate using Stirling's bound. We may, moreover, restrict s to the exponential zero set $-\operatorname{Im}(\alpha) \leq \operatorname{Im}(s) \leq \operatorname{Im}(\alpha)$ to see that

$$\begin{aligned}
\mathcal{I}_{T,R}^{(2)}(-a) &= \int\limits_{\substack{\hat{\alpha}_n = 0 \\ \operatorname{Re}(\alpha)=0}} e^{\frac{\alpha^2}{T^2}} \cdot |\Gamma_R(2\alpha)\Gamma_R(-2\alpha)| \int\limits_{\substack{s=(s_1,\ldots,s_{n-1}) \\ \operatorname{Re}(s)=-a}} |\widetilde{W}_{2,(\alpha,-\alpha)}(s)|\, ds\, d\alpha \\
&\ll \int\limits_{\substack{\hat{\alpha}_n = 0 \\ \operatorname{Re}(\alpha)=0}} e^{\frac{\alpha^2}{T^2}} \cdot (1 + |2\operatorname{Im}(\alpha)|)^{R+\frac{1}{2}} \int\limits_{\substack{\operatorname{Re}(s)=-a \\ -\operatorname{Im}(\alpha) \leq \operatorname{Im}(s) \leq \operatorname{Im}(\alpha)}} (1 + |\operatorname{Im}(s) - \operatorname{Im}(\alpha)|)^{-a-\frac{1}{2}} \\
&\qquad\qquad \cdot (1 + |\operatorname{Im}(s) + \operatorname{Im}(\alpha)|)^{-a-\frac{1}{2}}\, ds\, d\alpha.
\end{aligned}$$

Due to the presence of the term e^{α^2/T^2}, we may assume moreover that $\operatorname{Im}(\alpha) \leq T^{1+\varepsilon}$. Thus, we have the bound

$$\begin{aligned}
\mathcal{I}_{T,R}^{(2)}(-a) &\ll \int\limits_{\substack{\operatorname{Re}(\alpha)=0 \\ 0 \leq \operatorname{Im}(\alpha) \leq T^{\varepsilon+1}}} (1 + 2|\alpha|)^{R+\frac{1}{2}} \int\limits_{\substack{\operatorname{Re}(s)=-a \\ -\operatorname{Im}(\alpha) \leq \operatorname{Im}(s) \leq \operatorname{Im}(\alpha)}} (1+\alpha - s)^{-a-\frac{1}{2}} (1 + \alpha - s)^{-a-\frac{1}{2}}\, ds\, d\alpha \\
&\ll \int\limits_{\substack{\operatorname{Re}(\alpha)=0 \\ 0 \leq \operatorname{Im}(\alpha) \leq T^{\varepsilon+1}}} (1+2|\alpha|)^{R+\frac{1}{2}-\min\{a+\frac{1}{2}, 2a\}}\, d\alpha \ll T^{\varepsilon+R+\frac{3}{2}-\min\{a+\frac{1}{2},2a\}}.
\end{aligned}$$

In the statement of Theorem 9.0.2, the claimed bound is $\mathcal{I}_{T,R}^{(2)}(-a) \ll T^{\varepsilon+R+3/2-B(a)}$, where $B(a)$ is as defined in Theorem 9.0.2. We have in fact proved that $\mathcal{I}_{T,R}^{(2)}(-a) \ll T^{\varepsilon+R+3/2-B'(a)}$, where

$$B'(a) = \max\left\{a + \tfrac{1}{2}, 2a\right\} = \begin{cases} 2a & \text{if } \varepsilon < a \leq \frac{1}{2}, \\ a + \frac{1}{2} & \text{if } a \geq \frac{1}{2}. \end{cases}$$

If, $a<0$, then we may shift the integral over $\mathrm{Re}(s)=-a$ to be as close to $\mathrm{Re}(s)=0$ as desired; indeed, we may make the shift to the point that the error can be absorbed into the ε-term in the power of T. Therefore, since $B(a)\le B'(a)$ for all $a>0$, the theorem follows. □

9.4. *Proof of Theorem 9.0.2 for general m.*

Proof. Let $m\ge 3$ and assume that Theorem 9.0.2 has been shown to be true for all integers $2\le k<m$. It follows from Corollary 9.2.8 with $r_j=\lceil a_j\rceil$ that

$$\mathcal{I}_{T,R}^{(m)}(-a)\ll\sum_{\sigma}\int\limits_{\substack{\hat{\alpha}_m=0\\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_m^2}{T^2/2}}\cdot\mathcal{F}_R^{(m)}(\alpha)\prod_{1\le j\ne k\le m}|\Gamma_R(\alpha_j-\alpha_k)|\cdots\int\limits_{\substack{s=(s_1,\dots,s_{m-1})\\ \mathrm{Re}(s)=-a}}\frac{|\mathcal{P}_{d(m)-2|\sigma|}(s,\alpha)|}{\prod_{j=1}^{m-1}|\mathcal{B}_j(s_j,\alpha)|^{\lceil a_j\rceil}}|\widetilde{W}_{m,\alpha}(s+r+\sigma)|\,ds\,d\alpha.$$

By Theorem 8.1.5,

$$\begin{aligned}\mathcal{I}_{T,R}^{(m)}(-a)\ll\sum_{\sigma}\int\limits_{\substack{\hat{\alpha}_m=0\\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_m^2}{T^2/2}}\cdot\mathcal{F}_R^{(m)}(\alpha)\prod_{1\le j\ne k\le m}|\Gamma_R(\alpha_j-\alpha_k)|\int\limits_{\mathrm{Re}(s)=-a}\frac{|\mathcal{P}_{d(m)-2|\sigma|}(s,\alpha)|}{\prod_{j=1}^{m-1}|\mathcal{B}_j(s_j,\alpha)|^{\lceil a_j\rceil}}\\ \cdot\int\limits_{\substack{z=(z_1,\dots,z_{m-2})\\ \mathrm{Re}(z)=b}}\left(\prod_{j=1}^{m-1}\left|\Gamma\left(s_j+\lceil a_j\rceil+\sigma_j-z_{j-1}+\frac{(m-j)\alpha_m}{m-1}\right)\right|\right.\\ \left.\cdot\left|\Gamma\left(s_j+\lceil a_j\rceil+\sigma_j-z_j-\frac{j\alpha_m}{m-1}\right)\right|\right)\cdot|\widetilde{W}_{m-1,\beta}(z)|\,dz\,ds\,d\alpha.\end{aligned}$$

Next, we use the functional equation for the Gamma function to rewrite

$$\begin{aligned}\Gamma\left(s_j+\lceil a_j\rceil+\sigma_j-z_{j-1}+\frac{(m-j)\alpha_m}{m-1}\right)\Gamma\left(s_j+\lceil a_j\rceil+\sigma_j-z_j-\frac{j\alpha_m}{m-1}\right)\\ =\mathcal{P}_{2\sigma_j}(s,z,\alpha)\Gamma\left(s_j+\lceil a_j\rceil-z_{j-1}+\frac{(m-j)\alpha_m}{m-1}\right)\Gamma\left(s_j+\lceil a_j\rceil-z_j-\frac{j\alpha_m}{m-1}\right).\end{aligned}$$

Additionally, we use the fact that the integrand has exponential decay unless $|\alpha_1|,\dots,|\alpha_m|\le T^{1+\varepsilon}$, and by Lemma 9.2.3, each of the variables s_j are bounded in terms of α. This means that we may replace the polynomials $\mathcal{P}_{2\sigma_j}$ with the bound $T^{\varepsilon+2\sigma_j}$. Note that in doing so, the dependence on σ is removed:

$$\begin{aligned}&\mathcal{I}_{T,R}^{(m)}(-a)\\ &\ll T^{\varepsilon+\sum_{j=1}^{m-1}\lceil a_j\rceil(\binom{m}{j}-2)}\int\limits_{\substack{\hat{\alpha}_m=0\\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_m^2}{T^2/2}}\cdot\mathcal{F}_R^{(m)}(\alpha)\prod_{1\le j\ne k\le m}|\Gamma_R(\alpha_j-\alpha_k)|\\ &\int\limits_{\substack{s=(s_1,\dots,s_{m-1})\\ \mathrm{Re}(s)=-a}}\int\limits_{\substack{z=(z_1,\dots,z_{m-2})\\ \mathrm{Re}(z)=b}}\left(\prod_{j=1}^{m-1}\frac{\left|\Gamma\left(s_j+\lceil a_j\rceil-z_{j-1}+\frac{(m-j)\alpha_m}{m-1}\right)\Gamma\left(s_j+\lceil a_j\rceil-z_j-\frac{j\alpha_m}{m-1}\right)\right|}{|\mathcal{B}_j(s_j,\alpha)|^{\lceil a_j\rceil}}\right)\\ &\cdot|\widetilde{W}_{m-1,\beta}(z)|\,dz\,ds\,d\alpha.\end{aligned}$$

Notice that the conclusion of Proposition 9.4.2 follows from the last several steps by simply replacing s by $s+L$ in the integrand (or, equivalently, replacing $\mathrm{Re}(s)=-a$ by $\mathrm{Re}(s)=-a+L$ in the domain of integration), and then at the step where the functional equation of Gamma is used to remove σ from the Gamma functions, we remove L in the exact same fashion.

We deduce that

$$\mathcal{I}_{T,R}^{(m)}(-a)$$
$$\ll T^{\varepsilon+\sum_{j=1}^{m-1}\lceil a_j\rceil(\binom{m}{j}-2)}\cdot \int\limits_{\substack{\hat{\alpha}_m=0\\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_m^2}{T^2/2}}\cdot \mathcal{F}_R^{(m)}(\alpha)\prod_{1\le j\ne k\le m-1}|\Gamma_R(\alpha_j-\alpha_k)|$$
$$\cdot \int\limits_{\substack{z=(z_1,\dots,z_{m-2})\\ \mathrm{Re}(z)=b}}\prod_{j=1}^{m-1}\int\limits_{\mathrm{Re}(s_j)=\lceil a_j\rceil-a_j}\frac{\left|\Gamma\left(s_j-z_{j-1}-\frac{(m-j)\hat{\alpha}}{m-1}\right)\Gamma\left(s_j-z_j+\frac{j\hat{\alpha}}{m-1}\right)\right|}{|\mathcal{B}_j(s_j,\alpha)|^{\lceil a_j\rceil}}\left|\Gamma_R\left(\frac{n}{n-1}\hat{\alpha}-\beta_j\right)\Gamma_R\left(\beta_j-\frac{m}{m-1}\hat{\alpha}\right)\right|$$
$$\cdot|\widetilde{W}_{m-1,\beta}(z)|\,ds_j\,dz\,d\alpha.$$

Note that we have also made the change of variable $s\mapsto s_j-\lceil a_j\rceil$ for each $j=1,2,\dots,m-1$, and we are using the notation $\hat{\alpha}:=-\alpha_m$. (Using the terminology of Lemma A.19 in the case of $k=m-1$, we have $\hat{\alpha}=\hat{\alpha}_{m-1}$.) As in the case of $n=2$, due to the presence of the exponential terms, we see that the integral has exponential decay unless $|\alpha_j|\ll T^{1+\varepsilon}$.

Lemma 9.4.1. *Let $\alpha=(\alpha_1,\dots,\alpha_m)$ and $\beta_j=\alpha_j-\frac{\hat{\alpha}}{m-1}$ be as above. In particular, they are purely imaginary with $|\beta_k|,|\hat{\alpha}_j|<T^{1+\varepsilon}$. Suppose, moreover, that α is in j-general position. Then*

$$\int\limits_{\mathrm{Re}(s_j)=\lceil a_j\rceil-a_j}\frac{\left|\Gamma\left(s_j-z_{j-1}-\frac{(m-j)\hat{\alpha}}{m-1}\right)\Gamma\left(s_j-z_j+\frac{j\hat{\alpha}}{m-1}\right)\right|}{|\mathcal{B}_j(s_j,\alpha)|^{\lceil a_j\rceil}}\left|\Gamma_R\left(\frac{m}{m-1}\hat{\alpha}-\beta_j\right)\Gamma_R\left(\beta_j-\frac{m}{m-1}\hat{\alpha}\right)\right|ds_j$$
$$\ll T^{\varepsilon+R+\frac12+\max\{0,2(\lceil a_j\rceil-a_j)-1\}}\sum_{\substack{L\subseteq\{1,\dots,m\}\\ \#L=j}}\prod_{\substack{K\subseteq\{1,\dots,m\}\\ \#K=j\\ K\ne L}}\left(1+\left|\sum_{\ell\in L}\alpha_\ell-\sum_{k\in K}\alpha_k\right|\right)^{-[a_j]}.$$

Proof. Let $\mathcal{I}_j$ denote the integral we are seeking to bound.

The polynomial part (see Definition 9.2.2) of the Gamma functions in $\mathcal{I}_j$ is

$$|\mathcal{Q}_j(s,z,\alpha)|\ll\left(1+\mathrm{Im}\left(\beta_j-\frac{n}{n-1}\hat{\alpha}\right)\right)^{\varepsilon+R+\frac12}(1+|\mathrm{Im}(s_j-z_j)|)^{\lceil a_j\rceil-a_j-\mathrm{Re}(z_j)-\frac12}$$
$$\cdot(1+|\mathrm{Im}(s_j-z_{j-1})|)^{\lceil a_j\rceil-a_j-\mathrm{Re}(z_{j-1})-\frac12},$$

and the exponential factor (when taking all $\mathcal{I}_j$ in unison) is negative for any s_j outside of the interval I_j defined in Lemma 9.2.3. That lemma together with the presence of the other exponential terms in our integral allow us to take trivial bounds for the polynomial part, namely that

$$\mathcal{Q}_j(s,z,\alpha)\ll T^{\varepsilon+R+\frac12+\max\{0,2(\lceil a_j\rceil-a_j)-1\}}.$$

(Recall that $0 \le \mathrm{Re}(z_j)$.) Thus we see that

$$\mathcal{I}_j \ll T^{\varepsilon+R+\frac{1}{2}+\max\{0,2(\lceil a_j\rceil-a_j)-1\}} \int\limits_{\substack{\mathrm{Re}(s_j)=\lceil a_j\rceil-a_j \\ \mathrm{Im}(s_j)\in I_j}} \prod_{\substack{J\subseteq\{1,\dots,n\} \\ \#J=j}} \Bigl|s_j + \sum_{k\in J}\alpha_k\Bigr|^{-\lceil a_j\rceil} ds_j.$$

The desired result now follows easily from this and the statement of Lemma A.3. □

Combining Lemma 9.4.1 with the bound for $\mathcal{I}_{T,R}^{(n)}(-a)$ given immediately before the statement of the lemma, and applying Lemmas A.19, A.26 and A.27 (in the case that $k = n-1$ and $\gamma_1 = 0$), we now have the bound

$$\mathcal{I}_{T,R}^{(m)}(-a) \ll \sum_{\substack{L\subseteq\{1,\dots,m\} \\ \#L=j}} T^{\varepsilon+\left(R+\frac{1}{2}\right)(m-1)+\sum_{j=1}^{m-1}\left(\max\{0,2(\lceil a_j\rceil-a_j)-1\}+\lceil a_j\rceil\left(\binom{m}{j}-2\right)\right)} \cdot \int\limits_{\mathrm{Re}(\hat\alpha)=0} e^{\frac{m}{m-1}\frac{\hat\alpha^2}{2T^2}}$$
$$\cdot \int\limits_{\substack{\hat\beta_{m-1}=0 \\ \mathrm{Re}(\beta)=0}} e^{\frac{\beta_1^2+\cdots+\beta_{m-1}^2}{T^2/2}} \cdot \mathcal{P}^R_{D(m)-D(m-1)}(\hat\alpha,\beta)\cdot \mathcal{F}_R^{(m-1)}(\beta) \prod_{1\le j\ne k\le m-1} |\Gamma_R(\beta_j-\beta_k)|$$
$$\cdot \prod_{j=1}^{m-1} \prod_{\substack{K\subseteq\{1,\dots,m\} \\ \#K=j \\ K\ne L}} \Bigl(1+\Bigl|\sum_{\ell\in L}\alpha_\ell - \sum_{k\in K}\alpha_k\Bigr|\Bigr)^{-\lceil a_j\rceil} \int\limits_{\substack{z=(z_1,\dots,z_{m-2}) \\ \mathrm{Re}(z)=b}} |\widetilde{W}_{m-1,\beta}(z)|\,dz\,d\beta\,d\hat\alpha.$$

To be more explicit, the polynomial $\mathcal{P}^R_{D(m)-D(m-1)}(\hat\alpha,\beta)$ is the portion of $\mathcal{F}_R^{(m)}(\alpha)$ which involves the terms α_m.

At this point, we combine each of the terms in the final row with the corresponding term in $\mathcal{F}_R^{(m)}(\alpha)$. Strictly speaking, what is actually happening here is that this has the effect of reducing the power of each factor of $\mathcal{F}_R^{(m)}(\alpha)$ by at most

$$\max\{\lceil a_1\rceil,\dots,\lceil a_{m-1}\rceil\}.$$

Since each of the corresponding exponents remains positive, the net result is to reduce the overall power of T by

$$\varepsilon + \sum_{j=1}^{m-1}\lceil a_j\rceil\left(\binom{m}{j}-1\right).$$

Using this, and accounting for the integration in $\hat\alpha$ (which may be assumed to take place only for $|\mathrm{Im}(\hat\alpha)| \le T^{1+\varepsilon}$), we now may write

$$\mathcal{I}_{T,R}^{(m)}(-a) \ll T^{\varepsilon+\left(R+\frac{1}{2}\right)(n-1)+R(D(m)-D(m-1))+1+\sum_{j=1}^{m-1}(\max\{0,2(\lceil a_j\rceil-a_j)-1\}-\lceil a_j\rceil)}$$
$$\int\limits_{\substack{\hat\beta_{m-1}=0 \\ \mathrm{Re}(\beta)=0}} e^{\frac{\beta_1^2+\cdots+\beta_{m-1}^2}{2T^2}} \cdot \mathcal{F}_R^{(m-1)}(\beta) \prod_{1\le j\ne k\le m-1} |\Gamma_R(\beta_j-\beta_k)| \int\limits_{\substack{z=(z_1,\dots,z_{m-2}) \\ \mathrm{Re}(z)=b}} |\widetilde{W}_{m-1,\beta}(z)|\,dz\,d\beta.$$

Obviously, at this point we want to apply the inductive hypothesis. Since at this point we only need to do so in the case that $b_j > 0$ (i.e., $-a_j < 0$) for all $j = 1,\dots,m-2$, the reduction in the powers of the exponents of any one of the factors of $\mathcal{F}_R^{(m)}(\alpha)$, as occurred above, leaves the overall power positive. Therefore, there is no issue, and we can assert (additionally applying Lemma A.5) the bound

$$\mathcal{I}_{T,R}^{(m)}(-a) \ll T^{\varepsilon+\left(R+\frac{1}{2}\right)(m-1)+R(D(m)-D(m-1))+1+A(m-1)} \cdot T^{R\left(D(m-1)+\frac{(m-1)(m-2)}{2}\right)-\sum_{j=1}^{m-1} B(a_j)}$$
$$= T^{\varepsilon+R\left(D(m)+\frac{m(m-1)}{2}\right)+\frac{n+1}{2}+A(m-1)-\sum_{j=1}^{m-1} B(a_j)}.$$

Taking $A(m)=\frac{m+1}{2}+A(m-1)$ gives the claimed bound. Since $A(2)=\frac{3}{2}$, it follows that

$$A(3)=\tfrac{4}{2}+A(2)=\tfrac{1}{2}(4+3),\ \dots,\ A(m)=\tfrac{1}{2}((m+1)+m+\cdots+3)=\tfrac{1}{4}(m+4)(m-1),$$

as claimed. □

In the course of proving Theorem 9.0.2 we also established the following result that we record here since it will be useful in its own right.

Proposition 9.4.2. *Suppose that $L=(\ell_1,\ell_2,\dots,\ell_{m-1})\in(\mathbb{Z}_{\ge 0})^{m-1}$. Then*

$$\int\limits_{\substack{\hat{\alpha}_m=0\\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_m^2}{2T^2}} \cdot \mathcal{F}_R^{(m)}(\alpha) \prod_{1\le j\ne k\le m} |\Gamma_R(\alpha_j-\alpha_k)| \int\limits_{\substack{s=(s_1,\dots,s_{m-1})\\ \mathrm{Re}(s)=-a}} |\widetilde{W}_{m,\alpha}(s+L)|\,ds\,d\alpha$$
$$\ll T^{\varepsilon+2|L|} \cdot \int\limits_{\substack{\hat{\alpha}_m=0\\ \mathrm{Re}(\alpha)=0}} e^{\frac{\alpha_1^2+\cdots+\alpha_m^2}{2T^2}} \cdot \mathcal{F}_R^{(m)}(\alpha) \prod_{1\le j\ne k\le m} |\Gamma_R(\alpha_j-\alpha_k)| \int\limits_{\substack{s=(s_1,\dots,s_{m-1})\\ \mathrm{Re}(s)=-a}} |\widetilde{W}_{m,\alpha}(s)|\,ds\,d\alpha.$$

As a shorthand for this result, we write $\mathcal{I}_{T,R}^{(m)}(-a+L)\ll T^{\varepsilon+2|L|}\cdot\mathcal{I}_{T,R}^{(m)}(-a)$.

10. Bounding $p_{T,R}^{(n)}(y)$

In this section we prove the following.

Theorem 10.0.1. *Let $n\ge 2$ and $\varepsilon\in\left(0,\frac{1}{4}\right)$. Suppose that $a=(a_1,a_2,\dots,a_{n-1})$ satisfies $\lfloor a_j\rfloor+\varepsilon<a_j<\lceil a_j\rceil-\varepsilon$ for each $j=1,\dots,n-1$. Let $\mathcal{C}$ be the set of compositions $n=n_1+\cdots+n_r$ with $r\ge 2$. Then, for*

$$\Delta_a(C):=\{\delta_C=(\delta_1,\dots,\delta_{r-1})\in\mathbb{Z}^{r-1}\mid 0\le\delta_j<a_{\hat{n}_j}\ (j=1,\dots,r-1)\},$$

and $B(c)$ as defined in Theorem 9.0.2, we have

$$|p_{T,R}^{(n)}(y)|\ll|p_{T,R}^{(n)}(y;-a)|+\sum_{C\in\mathcal{C}}\sum_{\delta_C\in\Delta_a(C)}|p_{T,R}^{(n)}(y;-a,\delta_C)|, \tag{10.0.2}$$

where

$$|p_{T,R}^{(n)}(y;-a)|\ll\prod_{j=1}^{n-1} y_j^{\frac{n(n-j)}{2}+2a_j}\cdot T^{\varepsilon+\frac{(n+4)(n-1)}{4}+\frac{R}{2}\cdot\left(\binom{2n}{n}-2^n\right)-\sum_{j=1}^{n-1}B(a_j)} \tag{10.0.3}$$

and

$$|p_{T,R}^{(n)}(y;-a,\delta_C)|\ll\prod_{j=1}^{n-1} y_j^{\frac{n(n-j)}{2}+2a_j}\cdot T^{\varepsilon+\frac{(n+4)(n-1)}{4}+\frac{R}{2}\cdot\left(\binom{2n}{n}-2^n\right)-\sum_{j=1}^{n-1}B(a_j)-\frac{1}{2}\sum_{k=1}^{r-1}(n_k+n_{k+1})(a_{\hat{n}_k}-\delta_k)}. \tag{10.0.4}$$

The implicit constant depends on both ε and n.

Remark 10.0.5. Note that (10.0.4) is bounded by (10.0.3). Therefore, letting $D(n)=\frac{1}{2}\binom{2n}{n}-\frac{n(n-1)}{2}-2^{n-1}$ as in (1.4.2), Theorem 10.0.1 implies that

$$|p_{T,R}^{(n)}(y)| \ll \prod_{j=1}^{n-1} y_j^{\frac{n(n-j)}{2}+2a_j} \cdot T^{\varepsilon+\frac{(n+4)(n-1)}{4}+R\cdot\left(D(n)+\frac{n(n+1)}{2}\right)-\sum_{j=1}^{n-1} B(a_j)}.$$

10.1. ***Explicit single residue formula.*** In order to bound the terms $p_{T,R}^{(n)}(y;-a,\delta_C)$ we need an explicit formula for the residues of the Mellin transform of the GL(n) Whittaker function. The following result establishes this for the case of single residues (i.e., when the composition C has length 2) as a corollary of Conjecture 8.2.3 combined with a theorem of Stade [2001] for the "first" residues, i.e., for those residues corresponding, in the notation of the theorem, to $\delta=0$.

Theorem 10.1.1. *Let $\widetilde{W}_{m,\alpha}(s)$ be the Mellin transform of the Whittaker function on* GL$(n,\mathbb{R})$ *with purely imaginary parameters $\alpha=(\alpha_1,\dots,\alpha_n)$ in general position. Let $\sigma\in S_n$ act on α via*

$$\sigma\cdot\alpha := (\alpha_{\sigma(1)},\alpha_{\sigma(2)},\dots,\alpha_{\sigma(n)}).$$

The poles of $\widetilde{W}_{n,\alpha}(s)$ occur, for each $1\le m\le n-1$, at

$$s_m\in\{-\sigma\cdot\hat{\alpha}_m-\delta \mid \sigma\in S_n,\ \delta\in\mathbb{Z}_{\ge 0}\}.$$

The residue at $s_m=-\hat{\alpha}_m-\delta$ is equal to a sum over shifts $L=(\ell_1,\ell_2,\dots,\ell_{n-1})$ of terms of the form

$$\prod_{\substack{K\subseteq\{1,2,\dots,n\}\\ \#(K\cap\{1,2,\dots,m\})\neq m-1\\ \#K=m}} \left(\left(\sum_{i\in K}\alpha_i\right)-\hat{\alpha}_m-\delta\right)_\delta^{-1} \left(\prod_{i=1}^{m}\prod_{j=m+1}^{n}\Gamma(\alpha_j-\alpha_i-\delta)\right) \cdot \mathcal{P}_{(\binom{n}{m}-2)\delta-2|L|}(s,\alpha)\,\widetilde{W}_{m,\beta}(s'+L')\,\widetilde{W}_{n-m,\gamma}(s''+L''),$$

where

$$s'=\left(s_j+\frac{j}{m}\hat{\alpha}_m\right)\Big|_{1\le j\le m},\qquad s''=\left(s_{m+j}+\frac{n-m-j}{n-m}\hat{\alpha}_m\right)\Big|_{1\le j\le n-m},\tag{10.1.2}$$

with $L'=(\ell_1,\dots,\ell_{m-1})$ and $L''=(\ell_{m+1},\dots,\ell_{n-1})$ being the portion of L corresponding to s' and s'' respectively. It is the case that $\ell_{m-1}=\ell_{m+1}=0$. Note that we take as definition that $\widetilde{W}_1:=1$. The same formula holds for the residue at $s_m=-\sigma\cdot\hat{\alpha}_m-\delta$ by replacing each instance of α_j with $\alpha_{\sigma(j)}$.

Remark 10.1.3. Another way of writing the above expression for the residue would be to take the product over all $K\subseteq\{1,\dots,n\}$ with $\#K=m$ and replace $\Gamma(\alpha_j-\alpha_i-\delta)$ with $\Gamma(\alpha_j-\alpha_i)$. The two versions are equivalent because if $K\setminus\{1,\dots,m\}=\{j\}$, then $\{1,\dots,m\}\setminus K=\{k\}$ and

$$\left(\left(\sum_{i\in K}\alpha_i\right)-\hat{\alpha}_m-\delta\right)_\delta^{-1}\Gamma(\alpha_j-\alpha_k)=\Gamma(\alpha_j-\alpha_k-\delta).$$

Sketch of proof. In the case that $\delta=0$, this result (for $L=(0,\dots,0)\in\mathbb{C}^{n-1}$) agrees with [Stade 2001, Theorem 3.1]. If $\delta>0$, we need to first apply Conjecture 8.2.3 to rewrite the expression for $\widetilde{W}_{n,\alpha}(s)$ around $s_m=-\alpha_m-\delta$ as a sum over shifts $L=(\ell_1,\dots,\ell_{n-1})\in(\mathbb{Z}_{\ge 0})^{n-1}$ (with $\ell_m\ge\delta$ for each L) of terms $\widetilde{W}_{n,\alpha}(s+L)$. Of all of these terms, the only ones for which there is a pole at $s_m = -\hat{\alpha}_m - \delta$ are those for which $\ell_m = \delta$, in which case we can use the above-referenced theorem of Stade to write down the residue. Doing so, we obtain the alternative expression referenced in Remark 10.1.3. □

10.2. ***Explicit higher residue formulae.*** In order to generalize Theorem 10.1.1, we first establish notation related to the $(r-1)$-fold residue of $\widetilde{W}_{n,\alpha}(s)$ at

$$s_{\hat{n}_\ell} = -\hat{\alpha}_{\hat{n}_\ell} - \delta_{\hat{n}_\ell}, \quad \ell = 1, \dots, r-1.$$

To this end, let $s^{(j)} := (s_1^{(j)}, \dots, s_{n_j-1}^{(j)})$, where $s_k^{(j)} = s_{\hat{n}_{j-1}+k}$. By abuse of notation, we write

$$s := \big(\underbrace{s_1^{(1)}, s_2^{(1)}, \dots, s_{n_1-1}^{(1)}}_{=:s^{(1)}}, \underbrace{s_1^{(2)}, s_2^{(2)}, \dots, s_{n_2-1}^{(2)}}_{=:s^{(2)}}, \dots, \underbrace{s_1^{(k)}, s_2^{(k)}, \dots, s_{n_k-1}^{(k)}}_{=:s^{(k)}}\big) \in \mathbb{C}^{n-r},$$

which agrees with the original $s = (s_1, \dots, s_{n-1})$ but removes $s_{\hat{n}_1}, \dots, s_{\hat{n}_{r-1}}$.

Similarly, if $\alpha = (\alpha_1, \dots, \alpha_n)$, we define

$$\alpha^{(\ell)} := (\alpha_1^{(\ell)}, \dots, \alpha_\ell^{(\ell)}) \in \mathbb{C}^{n_\ell}, \qquad \alpha_j^{(\ell)} := \alpha_{\hat{n}_{\ell-1}+j} - \frac{1}{n_\ell}(\hat{\alpha}_{\hat{n}_\ell} - \hat{\alpha}_{\hat{n}_{\ell-1}}),$$

and

$$|\alpha^{(j)}|^2 := (\alpha_1^{(j)})^2 + (\alpha_2^{(j)})^2 + \cdots + (\alpha_{n_j}^{(j)})^2.$$

If $a \in \mathbb{R}^{n-1}$ then by $\mathrm{Re}(s) = -a$ we mean that $\mathrm{Re}(s_j) = -a_j$ for each $j \neq \hat{n}_1, \dots, \hat{n}_{r-1}$.

With this notation in place, we can now state a generalization of Theorem 10.1.1.

Corollary 10.2.1. *Let $n = n_1 + \cdots + n_r$ $(r \ge 2)$, and set $\hat{n}_\ell := \sum_{j=1}^{\ell} n_j$ as above. For each $\ell = 1, \dots, r-1$, let $b^{(\ell)} = (b_1^{(\ell)}, b_2^{(\ell)}, \dots, b_{n_\ell-1}^{(\ell)})$ with*

$$b_j^{(\ell)} = \hat{\alpha}_{i_{\ell-1}} + \frac{j}{n_\ell}(\hat{\alpha}_{\hat{n}_\ell} - \hat{\alpha}_{\hat{n}_{\ell-1}}) \quad \textit{for each } 1 \le j \le n_\ell - 1.$$

Let $\delta_j \in \mathbb{Z}_{\ge 0}$ for $j = 1, \dots, r-1$. There exist positive shifts $L = (L^{(1)}, \dots, L^{(r)})$ with $L^{(\ell)} = (L_1^{(\ell)}, \dots, L_{n_\ell-1}^{(\ell)}) \in (\mathbb{Z}_{\ge 0})^r$ such that the iterated residue of $\widetilde{W}_{n,\alpha}(s)$ at

$$s_{\hat{n}_{r-1}} = -\hat{\alpha}_{\hat{n}_{r-1}} - \delta_{r-1}, \dots, s_{\hat{n}_1} = -\hat{\alpha}_{\hat{n}_1} - \delta_1$$

is equal to a sum over all such shifts of

$$\mathcal{P}_d(s,\alpha)\bigg(\prod_{\ell=1}^{r} \widetilde{W}_{n_\ell,\alpha^{(\ell)}}(s^{(\ell)} + b^{(\ell)} + L^{(\ell)})\bigg) \prod_{j=1}^{r-1} \prod_{\substack{K \subseteq \{1,2,\dots,\hat{n}_{j+1}\} \\ \#(K \cap \{1,\dots,\hat{n}_j\}) \neq \hat{n}_j - 1 \\ \#K = \hat{n}_j}} \bigg(\Big(\sum_{i \in K} \alpha_i\Big) - \hat{\alpha}_{\hat{n}_j} - \delta_j\bigg)_{\delta_j}^{-1}$$

$$\cdot \prod_{1 \le k < m \le r} \prod_{i=1}^{n_k} \prod_{j=1}^{n_m} \Gamma\Big(\alpha_j^{(m)} - \alpha_i^{(k)} + \frac{1}{n_m}(\hat{\alpha}_{\hat{n}_m} - \hat{\alpha}_{\hat{n}_{m-1}}) - \frac{1}{n_k}(\hat{\alpha}_{\hat{n}_k} - \hat{\alpha}_{\hat{n}_{k-1}}) - \delta_m\Big),$$

where

$$d = \bigg[\sum_{\ell=1}^{r-1} \delta_\ell\Big(\binom{\hat{n}_{\ell+1}}{\hat{n}_\ell} - 2\Big)\bigg] - 2|L|.$$

Proof. This follows easily by induction with the base case being Theorem 10.1.1. □

Remark 10.2.2. Although it is possible to rewrite each of the terms $\left(\sum_{i\in K}\alpha_i\right)-\hat{\alpha}_{\hat{n}_\ell}-\delta_\ell$ appearing in the statement of Corollary 10.2.1 in terms of the variables $\alpha^{(j)}$ and $\hat{\alpha}_m^{(j)}$ for various j and m, the exact description is unnecessary for our purposes.

10.3. ***Proof of Theorem 10.0.1.*** As a first step, note that Proposition 8.3.6 implies that (10.0.2) follows from (10.0.3) and (10.0.4).

As shown in Section 8.5, the shifted $p_{T,R}^{(n)}$-term satisfies

$$|p_{T,R}^{(n)}(y,-a)| \ll \left(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}-2a_j}\right)\mathcal{I}_{T,R}^{(n)}(-a).$$

Combined with the bound from Theorem 9.0.2, this gives (10.0.3).

To complete the proof, we need to show that (10.0.4) holds. We do this in Section 10.5. Although this proof is valid for any $r\geq 2$, as a warmup, we first prove the special case $r=2$ (i.e., the case of single residues) in Section 10.4. □

10.4. ***Bounds for single residue terms.*** In this section[1] we bound $p_{T,R}^{(n)}(y;-a,\delta_C)$ in the case that $C=(m,n-m)$. Since C is a composition of length 2, we may take (see Definition 8.3.3) $\delta_C=\delta\in\mathbb{Z}_{\geq 0}$.

Proof of (10.0.4) *when* $r=2$. Using Lemmas A.19, A.26 and A.28, we can rewrite

$$e^{\frac{\alpha_1^2+\cdots+\alpha_n^2}{T^2/2}}\mathcal{F}_R^{(n)}(\alpha)\prod_{1\leq j\neq k\leq n}\Gamma_R(\alpha_j-\alpha_k)$$

in terms of β, γ and α_n. Thus, together with Theorem 10.1.1, we see that Definition 8.3.3 in the case of a single residue term (i.e., $r=2$) satisfies the bound

$$\begin{aligned}
&p_{T,R}^{(n)}(y;-a,\delta_C)\\
&\ll \int\limits_{\mathrm{Re}(\hat{\alpha}_m)=0} y_m^{\frac{m(n-m)}{2}+\hat{\alpha}_m+\delta}\cdot e^{\frac{n}{m(n-m)}\frac{\hat{\alpha}_m^2}{T^2/2}} \int\limits_{\substack{\hat{\beta}_m=0\\ \mathrm{Re}(\beta)=0}} e^{\frac{|\beta|^2}{T^2/2}} \int\limits_{\substack{\hat{\gamma}_{n-m}=0\\ \mathrm{Re}(\gamma)=0}} e^{\frac{|\gamma|^2}{T^2/2}}\\
&\quad\cdot\left(\mathcal{F}_R^{(m)}(\beta)\cdot\prod_{1\leq i\neq j\leq m}\Gamma_R(\beta_i-\beta_j)\right)\left(\mathcal{F}_R^{(n-m)}(\gamma)\cdot\prod_{1\leq i\neq j\leq n-m}\Gamma_R(\gamma_i-\gamma_j)\right)\\
&\quad\cdot\prod_{i=1}^{m}\prod_{j=1}^{n-m}\left(\Gamma_R\left(\beta_i-\gamma_j+\frac{n\hat{\alpha}_m}{m(n-m)}\right)\Gamma_R\left(\gamma_j-\beta_i-\frac{n\hat{\alpha}_m}{m(n-m)}\right)\Gamma\left(\gamma_j-\beta_i-\frac{n\hat{\alpha}_m}{m(n-m)}-\delta\right)\right)\\
&\quad\cdot\left(\prod_{j\neq m}\int\limits_{\substack{\mathrm{Re}(s_j)=-a_j\\ j\neq m}} y_j^{\frac{j(n-j)}{2}-s_j}\right)\mathcal{P}_{R(D(n)-D(m)-D(n-m))-\delta\left(\binom{n}{m}-m(n-m)-1\right)}(s,\alpha)\\
&\quad\cdot\mathcal{P}_{\left(\binom{n}{m}-2\right)\delta-2|L|}(s,\alpha)\cdot\widetilde{W}_{m,\beta}(s'+L')\widetilde{W}_{n-m,\gamma}(s''+L'')\,ds\,d\gamma\,d\beta\,d\hat{\alpha}_m.
\end{aligned}$$

[1] Note that this section will be superseded by Section 10.5, which will prove the bound for any admissible C with length$(C)\geq 2$. This section treats the case length$(C)=2$.

In order to have the correct power of y_m, we need to shift the line of integration in $\hat{\alpha}_m$ to $\mathrm{Re}(\hat{\alpha}_m) = a_m - \delta$. Note that by Lemma A.14, no poles are crossed in doing so, and by Lemma A.15, taking $\beta = \beta_i - \gamma_j$ and $z = n\hat{\alpha}_m/(m(n-m))$, we may replace the third-to-last line by

$$\mathcal{P}_{m(n-m)R-n(a_m-\delta)-m(n-m)\delta}(s, \hat{\alpha}_m, \beta, \gamma).$$

Let $|\beta|^2 := \beta_1^2 + \cdots + \beta_m^2$, and define $|\gamma|^2$ similarly. Replacing the integral over $\hat{\alpha}_m$ by $T^{\varepsilon+1}$ and factoring out the powers of y_j, we see that

$$\begin{aligned}|p_{T,R}^{(n)}(y; -a, \delta_C)| \ll \left(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}+a_j}\right) \cdot T^{\varepsilon+(\binom{n}{m}-2)\delta+R(D(n)-D(m)-D(n-m))-\delta(\binom{n}{m}-m(n-m)-1)} \\ \cdot T^{-2|L|+m(n-m)R-n(a_m-\delta)-m(n-m)\delta+1} \cdot \int\limits_{\substack{\hat{\beta}_m=0 \\ \mathrm{Re}(\beta)=0}} e^{\frac{|\beta|^2}{T^2/2}} \int\limits_{\substack{\hat{\gamma}_{n-m}=0 \\ \mathrm{Re}(\gamma)=0}} e^{\frac{|\gamma|^2}{T^2/2}} \\ \cdot \left(\mathcal{F}_R^{(m)}(\beta) \cdot \prod_{1\le i,j\le k} \Gamma_R(\beta_i - \beta_j)\right)\left(\mathcal{F}_R^{(n-m)}(\gamma) \cdot \prod_{1\le i,j\le n-k} \Gamma_R(\gamma_i - \gamma_j)\right) \\ \cdot \int\limits_{\substack{\mathrm{Re}(s_j)=-a_j \\ 1\le j\le n-1 \\ j\ne m}} |\widetilde{W}_{m,\beta}(s'+L')| \cdot |\widetilde{W}_{n-m,\gamma}(s''+L'')| \, ds \, d\gamma \, d\beta.\end{aligned}$$

Note that by Proposition 9.4.2 we may remove the dependence on the shift L. Hence

$$\begin{aligned}|p_{T,R}^{(n)}(y; -a, \delta_C)| \ll \left(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}+a_j}\right) \cdot T^{\varepsilon+R(D(n)-D(m)-D(n-m)+m(n-m))+\delta(n-1)} \\ \cdot T^{-na_m+1} \int\limits_{\substack{\hat{\beta}_m=0 \\ \mathrm{Re}(\beta)=0}} e^{\frac{\beta_1^2+\cdots+\beta_m^2}{T^2/2}} \int\limits_{\substack{\hat{\gamma}_{n-m}=0 \\ \mathrm{Re}(\gamma)=0}} e^{\frac{\gamma_1^2+\cdots+\gamma_n^2}{T^2/2}} \\ \cdot \left(\mathcal{F}_R^{(m)}(\beta) \cdot \prod_{1\le i\ne j\le k} \Gamma_R(\beta_i - \beta_j)\right)\left(\mathcal{F}_R^{(n-m)}(\gamma) \cdot \prod_{1\le i\ne j\le n-k} \Gamma_R(\gamma_i - \gamma_j)\right) \\ \cdot \int\limits_{\substack{\mathrm{Re}(s_j)=-a_j \\ 1\le j\le n-1 \\ j\ne m}} |\widetilde{W}_{m,\beta}(s')| \cdot |\widetilde{W}_{n-m,\gamma}(s'')| \, ds \, d\gamma \, d\beta.\end{aligned}$$

By (10.1.2),

$$s_j' = s_j - \frac{j}{m}(\hat{\alpha}_m - \delta) \quad \text{and} \quad s_j'' = s_{m+j} - \frac{n-m-j}{n-m}(\hat{\alpha}_m - \delta).$$

Thus the integrals in β and γ above are essentially the product of $\mathcal{I}_{T,R}^{(m)}(-a')$ and $\mathcal{I}_{T,R}^{(n-m)}(-a'')$. The only issue is that because, as seen in the fact that the variables s' and s'' are shifted, we have

$$a_j' = a_j - \frac{j}{m}(a_m - \delta) \quad \text{and} \quad a_j'' = a_{m+j} - \frac{n-m-j}{n-m}(a_m - \delta).$$

Therefore, we can rewrite the previous formula as

$$|p^{(n)}_{T,R}(y;-a,\delta_C)| \ll \left(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}+a_j}\right)\cdot T^{\varepsilon+R(D(n)-D(m)-D(n-m)+m(n-m))} \cdot T^{\delta(n-1)-na_m+1}\cdot \mathcal{I}^{(m)}_{T,R}(-a')\cdot \mathcal{I}^{(n-m)}_{T,R}(-a'').$$

By Theorem 9.0.2, we have

$$|p^{(n)}_{T,R}(y;-a,\delta_C)| \ll \left(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}+a_j}\right)\cdot T^{\varepsilon+R(D(n)-D(m)-D(n-m)+m(n-m))} \cdot T^{\delta(n-1)-na_m+1}\cdot T^{\varepsilon+C(m)+R\cdot\left(D(m)+\frac{m(m-1)}{2}\right)-\sum_{j=1}^{m-1}B(a'_j)} \cdot T^{\varepsilon+C(n-m)+R\cdot\left(D(n-m)+\frac{(n-m)(n-m-1)}{2}\right)-\sum_{j=1}^{n-m-1}B(a''_j)},$$

Recall that $C(k)=\frac{(k+4)(k-1)}{4}$. Hence, using the elementary identity

$$C(m)+C(n-m)=C(n)-\frac{m(n-m)}{2}-1$$

together with Lemma A.6,

$$|p^{(n)}_{T,R}(y;-a,\delta_C)| \ll \left(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}+a_j}\right)\cdot T^{\varepsilon+R\left(D(n)+m(n-m)+\frac{m(m-1)}{2}+\frac{(n-m)(n-m-1)}{2}\right)} \cdot T^{\delta(n-1)+C(n)-\frac{m(n-m)}{2}-na_m-\sum_{j=1}^{m-1}B(a_j)+\frac{n-2}{2}(a_m-\delta+1)+B(a_m)}$$

$$\ll \left(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}+a_j}\right)\cdot T^{\varepsilon+C(n)+R\left(D(n)+\frac{n(n-1)}{2}\right)-\sum_{j=1}^{n-1}B(a_j)} \cdot T^{\frac{n-2}{2}(\delta-a_m+1)-\frac{m(n-m)}{2}-n(\delta-a_m)+B(a_m)-\delta}.$$

This gives the desired bound provided that the exponent of the final T is negative. Using the facts that $-\frac{m(n-m)}{2}$ is maximized when $m=1$ or $m=n-1$ and $B(a_m)\le a_m+\frac{1}{2}$, we see that the final exponent is

$$-\frac{n}{2}(a_m-\delta)+\frac{n-1}{2}-\frac{m(n-m)}{2}\le -\frac{n}{2}(a_m-\delta), \tag{10.4.1}$$

as claimed. □

10.5. *Bounds for $(r-1)$-fold residues.* We consider a composition C of n of length $r\ge 2$ given by $n=n_1+\cdots+n_r$. We may also write $C=(n_1,\dots,n_r)$. Let $\hat{n}_\ell=\sum_{j=1}^{\ell} n_j$ as usual.

As a final piece of notation, let $\beta=(\beta_1,\dots,\beta_r)$ be defined via

$$\beta_i := \hat{\alpha}_{\hat{n}_i}-\hat{\alpha}_{\hat{n}_{i-1}}.$$

Note that $\sum_{i=1}^{r}\beta_i=0$ and more generally, defining $\hat{\beta}_m=\sum_{i=1}^{m}\beta_i$, $\hat{\alpha}_{\hat{n}_i}=\hat{\beta}_i$. Since (assuming that $\hat{\alpha}_n=0$) the Jacobians of the changes of variables

$$\alpha\mapsto(\alpha^{(1)},\hat{\alpha}_{\hat{n}_1},\alpha^{(2)},\hat{\alpha}_{\hat{n}_2},\dots,\hat{\alpha}_{\hat{n}_{r-1}},\alpha^{(r)})$$

and

$$(\hat{\alpha}_{\hat{n}_1},\dots,\hat{\alpha}_{\hat{n}_{r-1}})\mapsto(\beta_1,\dots,\beta_{r-1})$$

are trivial, we see that (for $\beta_1+\cdots+\beta_r=0$)

$$d\alpha = d\beta\, d\alpha^{(1)}\, d\alpha^{(2)}\cdots d\alpha^{(r)}. \tag{10.5.1}$$

Proof of (10.0.4) *when* $r \geq 2$. Note that

$$b_j^{(\ell)} = \hat{\alpha}_{\hat{n}_{\ell-1}} + \frac{j}{n_\ell}(\hat{\alpha}_{\hat{n}_\ell} - \hat{\alpha}_{\hat{n}_{\ell-1}}) = \hat{\beta}_{\ell-1} + \frac{j}{n_\ell}\beta_\ell \quad \text{for each } 1 \leq j \leq n_\ell - 1.$$

Recall that by Definition 8.3.3,

$$\begin{aligned} p_{T,R}^{(n)}(y; -a, \delta_C) := \int\limits_{\substack{\hat{\alpha}_n=0\\ \operatorname{Re}(\alpha)=0}} & e^{\frac{\alpha_1^2+\cdots+\alpha_n^2}{T^2/2}} \cdot \mathcal{F}_R^{(n)}(\alpha)\Big(\prod_{1\leq j\neq k\leq n} \Gamma_R(\alpha_j - \alpha_k)\Big) \\ & \cdot \Big(\prod_{i=1}^{r-1} y_{\hat{n}_i}^{\frac{\hat{n}_i(n-\hat{n}_i)}{2}+\hat{\alpha}_{\hat{n}_i}+\delta_i}\Big) \cdot \int\limits_{\substack{\operatorname{Re}(s_j)=-a_j\\ j\notin\{\hat{n}_1,\dots,\hat{n}_{r-1}\}}} \Big(\prod_{j\notin\{\hat{n}_1,\dots,\hat{n}_{r-1}\}} y_j^{\frac{j(n-j)}{2}-s_j}\Big) \\ & \cdot \operatorname*{Res}_{s_{\hat{n}_1}=-\hat{\alpha}_{\hat{n}_1}-\delta_1}\Big(\operatorname*{Res}_{s_{\hat{n}_2}=-\hat{\alpha}_{\hat{n}_2}-\delta_2}\big(\cdots\big(\operatorname*{Res}_{s_{\hat{n}_{r-1}}=-\hat{\alpha}_{\hat{n}_{r-1}}-\delta_{r-1}} \widetilde{W}_{n,\alpha}(s)\big)\cdots\big)\Big)\, ds\, d\alpha. \end{aligned}$$

Using Remark A.29 and Corollary 10.2.1, we can bound $|p_{T,R}^{(n)}(y; -a, \delta_C)|$ by a sum over certain shifts L each of the form

$$\begin{aligned} & \int\limits_{\substack{\hat{\beta}_r=0\\ \operatorname{Re}(\beta)=0}} e^{(\frac{\beta_1^2}{n_1}+\cdots+\frac{\beta_r^2}{n_r})\frac{2}{T^2}} \cdot \Big(\prod_{j=1}^{r-1} y_{\hat{n}_j}^{\frac{\hat{n}_j(n-\hat{n}_j)}{2}+\hat{\beta}_j+\delta_j} \int\limits_{\substack{\hat{\alpha}_{n_j}^{(j)}=0\\ \operatorname{Re}(\alpha^{(j)})=0}} e^{\frac{|\alpha^{(j)}|^2}{T^2/2}}\Big) \\ & \cdot \mathcal{P}_{d_1-2|L|}(\alpha) \cdot \int\limits_{\operatorname{Re}(s)=-a} \Big(\prod_{j\notin\{\hat{n}_1,\dots,\hat{n}_{r-1}\}} y_j^{\frac{j(n-j)}{2}-s_j}\Big) \cdot \mathcal{P}_{d_2}(s,\alpha) \\ & \cdot \prod_{1\leq k<m\leq r} \prod_{i=1}^{n_k}\prod_{j=1}^{n_m} \Gamma\Big(\alpha_j^{(m)} - \alpha_i^{(k)} + \frac{\beta_m}{n_m} - \frac{\beta_k}{n_k} - \delta_k\Big) \prod_{\epsilon\in\{\pm 1\}} \Gamma_R\Big(\epsilon\Big(\alpha_j^{(m)} - \alpha_i^{(k)} + \frac{\beta_m}{n_m} - \frac{\beta_k}{n_k}\Big)\Big) \\ & \cdot \prod_{\ell=1}^{r}\Big(\mathcal{F}_R^{(n_\ell)}(\alpha^{(\ell)})\Big(\prod_{1\leq j\neq k\leq n_\ell} \Gamma_R(\alpha_j^{(\ell)} - \alpha_k^{(\ell)})\Big) \widetilde{W}_{n_\ell,\alpha^{(\ell)}}(s^{(\ell)} + b^{(\ell)} + L^{(\ell)})\Big)\, ds\, d\alpha^{(1)}\, d\alpha^{(2)} \cdots d\alpha^{(r)}\, d\beta, \end{aligned}$$

where

$$d_1 = \sum_{\ell=1}^{r-1} \delta_\ell\Big(\binom{\hat{n}_{\ell+1}}{\hat{n}_\ell} - 2\Big),$$

$$d_2 = R \cdot \Big(D(n) - \sum_{\ell=1}^{r} D(n_\ell)\Big) - \sum_{\ell=1}^{r-1}\Big[\delta_\ell\Big(\binom{\hat{n}_{\ell+1}}{\hat{n}_\ell} - n_{\ell+1}\hat{n}_\ell - 1\Big)\Big]$$

are the degrees coming from Remark A.29 and Corollary 10.2.1 respectively and $b^{(\ell)}$ is as in Corollary 10.2.1. Note that, in addition to using the change of variables (10.5.1), we have used Lemmas A.18 and A.20 to break up $e^{2|\alpha|^2/T^2}$ and rewrite the product of $\Gamma(\alpha_j - \alpha_k)$ in terms of $\alpha^{(1)}, \dots, \alpha^{(r)}$ and β.

The next step is to shift the lines of integration in the variables β_j for $j = 1, \dots, r-1$ (or, equivalently, $\hat{\beta}_j$ for $j = 1, \dots, r-1$) such that the real part of the exponent of each term $y_{\hat{n}_j}$ is $\frac{\hat{n}_j(n-\hat{n}_j)}{2} + a_j$. In particular, this implies that we must shift the line of integration of $\hat{\beta}_j$ to

$$\mathrm{Re}(\hat\beta_j)=a_{\hat n_j}-\delta_j \quad\Longleftrightarrow\quad \mathrm{Re}(\beta_j)=\mathrm{Re}(\hat\beta_j-\hat\beta_{j-1})=(a_{\hat n_j}-\delta_j)-(a_{\hat n_{j-1}}-\delta_{j-1}). \tag{10.5.2}$$

Provided that R is sufficiently large, Lemma A.14 implies that this shift can be made without passing any poles. Moreover, Lemma A.15 implies that

$$\prod_{1\le k<m\le r}\prod_{i=1}^{n_k}\prod_{j=1}^{n_m}\Gamma\Big(\alpha_i^{(k)}-\alpha_j^{(m)}+\frac{\beta_k}{n_k}-\frac{\beta_m}{n_m}-\delta_m\Big)\prod_{\epsilon\in\{\pm1\}}\Gamma_R\Big(\epsilon\Big(\alpha_i^{(k)}-\alpha_j^{(m)}+\frac{\beta_k}{n_k}-\frac{\beta_m}{n_m}\Big)\Big)$$
$$\asymp\prod_{1\le k<m\le r}\prod_{i=1}^{n_k}\prod_{j=1}^{n_m}\Big(1+\Big|\mathrm{Im}\Big(\alpha_i^{(k)}-\alpha_j^{(m)}+\frac{\beta_k}{n_k}-\frac{\beta_m}{n_m}\Big)\Big|\Big)^{R-\mathrm{Re}\,(\frac{\beta_k}{n_k}-\frac{\beta_m}{n_m})-\delta_m}. \tag{10.5.3}$$

Note that the presence of the term $e^{(\beta_1^2/n_1+\cdots+\beta_r^2/n_r)(2/T^2)}$ implies that there is exponential decay for $|\mathrm{Im}(\beta_j)|\gg T^{1+\varepsilon}$. As we will see momentarily, besides the polynomial terms $\mathcal{P}_{d_1}(\alpha)$, $\mathcal{P}_{d_2}(s,\alpha)$ and (10.5.3), we just get a product of $\mathcal{I}_{T,R}^{(n_j)}(-c^{(j)})$ for some (to be determined) values $-c^{(\ell)}$. The upshot is that all of these polynomials can be bounded by T to the degree of the polynomial plus ε. Hence, we can bound the expression above by

$$T^{\varepsilon+r-1+d-2|L|}\cdot\Big(\prod_{j=1}^{n-1}y_j^{\frac{j(n-j)}{2}+a_j}\Big)$$
$$\cdot\prod_{\ell=1}^{r}\Big(\int_{\substack{\hat\alpha_{n_\ell}^{(\ell)}=0\\ \mathrm{Re}(\alpha^{(\ell)})=0}}e^{\frac{|\alpha^{(\ell)}|^2}{T^2/2}}\cdot\mathcal{F}_R^{(n_\ell)}(\alpha^{(\ell)})\cdot\int_{\mathrm{Re}(s^{(\ell)})=-a^{(\ell)}}\Big(\prod_{1\le j\ne k\le n_\ell}\Gamma_R(\alpha_j^{(\ell)}-\alpha_k^{(\ell)})\Big)|\widetilde W_{n_\ell,\alpha^{(\ell)}}(s^{(\ell)}+b^{(\ell)}+L^{(\ell)})|\,ds^{(\ell)}\,d\alpha^{(\ell)}\Big), \tag{10.5.4}$$

where $d=d_1+d_2+d_3$, with d_1 and d_2 as above and

$$d_3=R\cdot\sum_{\ell=1}^{r}n_\ell\hat n_\ell-\sum_{k=1}^{r-1}\big((n_k+n_{k+1})(a_{\hat n_k}-\delta_k)+\delta_kn_{k+1}\hat n_k\big)$$

is the bound coming from the terms described in (10.5.3), simplified using Lemma A.21. Combining everything, we find that d equals

$$R\cdot\Big(D(n)-\sum_{\ell=1}^{r}D(n_\ell)+\sum_{1\le k<m\le r}n_kn_m\Big)-\sum_{k=1}^{r-1}\big(\delta_k+(n_k+n_{k+1})(a_{\hat n_k}-\delta_k)\big).$$

Recall that the bound on $p_{T,R}^{(n)}(y;-a,\delta_C)$ is a *sum* of expressions of the form given in (10.5.4) for various shifts L. However, using Proposition 9.4.2, we can remove the dependence on the shifts. Hence,

$$|p_{T,R}^{(n)}(y;-a,\delta_C)|$$
$$\ll T^{\varepsilon+d+r-1}\cdot\Big(\prod_{j=1}^{n-1}y_j^{\frac{j(n-j)}{2}+a_j}\Big)$$
$$\cdot\prod_{\ell=1}^{r}\Big(\int_{\substack{\hat\alpha_{n_\ell}^{(\ell)}=0\\ \mathrm{Re}(\alpha^{(\ell)})=0}}e^{\frac{|\alpha^{(\ell)}|^2}{T^2/2}}\cdot\mathcal{F}_R^{(n_\ell)}(\alpha^{(\ell)})\cdot\int_{\mathrm{Re}(s^{(\ell)})=-a^{(\ell)}}\Big(\prod_{1\le j\ne k\le n_\ell}\Gamma_R(\alpha_j^{(\ell)}-\alpha_k^{(\ell)})\Big)|\widetilde W_{n_\ell,\alpha^{(\ell)}}(s^{(\ell)}+b^{(\ell)})|\,ds^{(\ell)}\,d\alpha^{(\ell)}\Big). \tag{10.5.5}$$

Thus, setting $c^{(\ell)} = a^{(\ell)} - \mathrm{Re}(b^{(\ell)})$, where

$$b^{(\ell)} = (b_1^{(\ell)}, \ldots, b_{\hat{n}_j}^{(\ell)}), \quad b_j^{(\ell)} = \hat{\beta}_{\ell-1} + \frac{j}{n_\ell}\beta_\ell,$$

we find that

$$|p_{T,R}^{(n)}(y; -a, \delta_C)| \ll \left(\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}+a_j}\right) \cdot T^{\varepsilon+r-1+d} \cdot \prod_{\ell=1}^{r} \mathcal{I}_{T,R}^{(\ell)}(-c^{(\ell)}).$$

Let $C(m) := \frac{(m+4)(m-1)}{4}$. We now apply Theorem 9.0.2 to each $\mathcal{I}_{T,R}^{(n_\ell)}$ to obtain

$$|p_{T,R}^{(n)}(y; -a, \delta_C)| \ll T^{\varepsilon+r-1+d+\sum_{\ell=1}^{r}\left(R\left(D(n_\ell)+\frac{n_\ell(n_\ell-1)}{2}\right)+C(n_\ell)-\sum_{k=1}^{n_\ell-1} B(c_k^{(\ell)})\right)} \cdot \prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}+a_j}.$$

Now we generalize the proof of Lemma A.6, keeping in mind that $a < B(a) < a + \frac{1}{2}$, to simplify the expression

$$\begin{aligned}
\sum_{\ell=1}^{r}\sum_{j=1}^{n_\ell-1} B(c_j^{(\ell)}) &\geq \sum_{\ell=1}^{r}\sum_{j=1}^{n_\ell-1}\left(a_{\hat{n}_\ell+j} - \mathrm{Re}(\hat{\beta}_{\ell-1}) - \frac{j}{n_\ell}\mathrm{Re}(\beta_\ell)\right) \\
&= \left(\sum_{j=1}^{n-1} a_j\right) - \left(\sum_{k=1}^{r-1} a_{\hat{n}_k}\right) - \sum_{\ell=1}^{r}\left[(n_\ell-1)\,\mathrm{Re}(\hat{\beta}_{\ell-1}) + \frac{n_\ell-1}{2}\,\mathrm{Re}(\beta_\ell)\right] \\
&\geq \sum_{j=1}^{n-1}\left(B(a_j) - \tfrac{1}{2}\right) - \sum_{k=1}^{r-1} a_{\hat{n}_k} - \sum_{\ell=1}^{r}\left[(n_\ell-1)\,\mathrm{Re}\left(\hat{\beta}_\ell - \tfrac{1}{2}\beta_\ell\right)\right] \\
&= -\frac{n-1}{2} + \sum_{j=1}^{n-1} B(a_j) - \sum_{k=1}^{r-1} a_{\hat{n}_k} - \sum_{\ell=1}^{r}\left[(n_\ell-1)\left(A_\ell - \tfrac{1}{2}(A_\ell - A_{\ell-1})\right)\right] \\
&= -\frac{n-1}{2} + \sum_{j=1}^{n-1} B(a_j) - \sum_{k=1}^{r-1} a_{\hat{n}_k} - \frac{1}{2}\sum_{\ell=1}^{r}[(n_\ell-1)(A_\ell + A_{\ell-1})].
\end{aligned}$$

Next, we write the sum over ℓ as

$$\begin{aligned}
\sum_{\ell=1}^{r}[(n_\ell-1)(A_\ell + A_{\ell-1})] &= \sum_{\ell=1}^{r}(n_\ell-1)A_\ell + \sum_{\ell=1}^{r}(n_\ell-1)A_{\ell-1} \\
&= \sum_{\ell=1}^{r}(n_\ell-1)A_\ell + \sum_{\ell=0}^{r-1}(n_{\ell+1}-1)A_\ell \\
&= (n_1-1)A_0 + (n_r-1)A_r + \sum_{\ell=1}^{r-1}(n_\ell + n_{\ell+1} - 2)A_\ell \\
&= \sum_{k=1}^{r-1}(n_k + n_{k+1} - 2)(a_{\hat{n}_k} - \delta_k)
\end{aligned}$$

We plug this back in to get

$$-\sum_{\ell=1}^{r}\sum_{j=1}^{n_\ell-1} B(c_j^{(\ell)}) \leq \frac{n-1}{2} - \sum_{j=1}^{n-1} B(a_\ell) + \sum_{k=1}^{r-1} a_{\hat{n}_k} + \frac{1}{2}\sum_{k=1}^{r-1}(n_k + n_{k+1} - 2)(a_{\hat{n}_k} - \delta_k),$$

from which it follows that the exponent of T in the bound for $|p^{(n)}_{T,R}(y;-a,\delta_C)|$ above is

$$\varepsilon+r-1+d+\sum_{\ell=1}^{r}\Bigg(\Bigg(R\Bigg(D(n_\ell)+\frac{n_\ell(n_\ell-1)}{2}\Bigg)+C(n_\ell)\Bigg)-\sum_{k=1}^{n_\ell-1}B(c_k^{(\ell)})\Bigg)$$
$$=\varepsilon+d'+R\Bigg(D(n)+\frac{n(n-1)}{2}\Bigg)+C(n)-\sum_{j=1}^{n-1}B(a_j),$$

where

$$\begin{aligned}d'&=r-1+d''+\frac{n-1}{2}+\frac{1}{2}\sum_{k=1}^{r-1}(n_k+n_{k+1}-2)(a_{\hat{n}_k}-\delta_k)-C(n)+\sum_{\ell=1}^{r}C(n_\ell)+\sum_{k=1}^{r-1}a_{\hat{n}_k}\\&=d''+\frac{n-1}{2}+\frac{1}{2}\sum_{k=1}^{r-1}(n_k+n_{k+1}-2)(a_{\hat{n}_k}-\delta_k)+\sum_{k=1}^{r-1}a_{\hat{n}_k}-\frac{1}{2}\sum_{1\le k<m\le r}n_kn_m\end{aligned}$$

and

$$d''=d-R\cdot\Bigg(D(n)-\sum_{\ell=1}^{r}D(n_\ell)+\sum_{1\le k<m\le r}n_kn_m\Bigg)=-\sum_{k=1}^{r-1}\big(\delta_k+(n_k+n_{k+1})(a_{\hat{n}_k}-\delta_k)\big).$$

Hence,

$$\begin{aligned}d'&=\frac{n-1}{2}-\sum_{k=1}^{r-1}\big(\delta_k+(n_k+n_{k+1})(a_{\hat{n}_k}-\delta_k)\big)\\&\qquad+\frac{1}{2}\sum_{k=1}^{r-1}(n_k+n_{k+1}-2)(a_{\hat{n}_k}-\delta_k)+\sum_{k=1}^{r-1}a_{\hat{n}_k}-\frac{1}{2}\sum_{1\le k<m\le r}n_kn_m\\&=\frac{n-1}{2}-\sum_{k=1}^{r-1}(n_k+n_{k+1})(a_{\hat{n}_k}-\delta_k)+\frac{1}{2}\sum_{k=1}^{r-1}(n_k+n_{k+1})(a_{\hat{n}_k}-\delta_k)-\frac{1}{2}\sum_{1\le k<m\le r}n_kn_m\\&=\frac{1}{2}\Bigg(n-1-\sum_{k=1}^{r-1}(n_k+n_{k+1})(a_{\hat{n}_k}-\delta_k)-\sum_{1\le k<m\le r}n_kn_m\Bigg).\end{aligned}$$

Note that if $r=2$ and $n_1=m$ and $\delta_1=\delta$, then this expression becomes

$$\frac{n-1}{2}-\frac{n}{2}(a_m-\delta)-\frac{m(n-m)}{2},$$

which agrees with (10.4.1).

Therefore, to complete the proof, we need only show that $n-1-\sum_{1\le k<m\le r}n_kn_m\le 0$. Indeed,

$$n-1-\sum_{1\le k<m\le r}n_kn_m=n-1-\sum_{k=1}^{r-1}\sum_{m=k+1}^{r}n_kn_m=n-1-\sum_{k=1}^{r-1}n_k(n-\hat{n}_k)\le n-1-n_1(n-n_1)\le 0,$$

(with the final inequality being equality if and only if $n_1=1$ or $n_1=n-1$), as desired. □

Remark 10.5.6. A critical step in the proof of (10.0.4) (either in the case of single residues, as is proved in Section 10.4 or higher-order residues, as in Section 10.5) is to shift the lines of integration in the variables $\hat{\alpha}_m$ or $\hat{\beta}_j$. A feature of this work that is quite different from the case of GL(4) as proved in [Goldfeld et al.

2021b], is that no poles are crossed when making these shifts. This represents a major simplification. Recall from the discussion of Section 8.4 that in the case of $n = 4$ there are two fundamentally different types of single residues, two different types of double residues and a triple residue. As it turned out, when making the additional shift for each of the single and double residues, one ends up with five *additional* residue terms. Taken all together, it was necessary to complete the analysis of writing down explicitly what the residues are in terms of Gamma functions, finding the exponential zero set, applying Stirling's formula and then obtaining a bound for ten(!) separate residues integrals. All of this was in addition to performing these steps for the shifted $p^{(4)}_{T,R}$-term.

Appendix: Auxiliary results

In an effort to avoid obstructing the flow of the argument in the main body of this paper, we will include here the many technical results that are used throughout. We remind the reader that the notational conventions that are used throughout the paper and this appendix are given in Definition 2.1.1.

Lemma A.1. *Suppose that $w = w_{(n_1,n_2,\ldots,n_r)}$ for some composition $n = n_1 + \cdots + n_2$ with $r \geq 2$. Then, if $y = (y_1, \ldots, y_{n-1})$, it follows that wyw^{-1} is equal to*

$$\Bigg(\underbrace{y_{n-\hat{n}_1+1}, y_{n-\hat{n}_1+2}, \ldots, y_{n-1}}_{n_1-1 \text{ terms}}, \Bigg(\prod_{k=n-\hat{n}_2}^{n-1} y_k\Bigg)^{-1}, \ldots,$$
$$\Bigg(\prod_{k=n-\hat{n}_i}^{n-\hat{n}_{i-2}-1} y_k\Bigg)^{-1}, \underbrace{y_{n-\hat{n}_i+1}, y_{n-\hat{n}_i+2}, \ldots, y_{n-\hat{n}_{i-1}-1}}_{n_i-1 \text{ terms}}, \Bigg(\prod_{k=n-\hat{n}_{i+1}}^{n-\hat{n}_{i-1}-1} y_k\Bigg)^{-1}, \ldots,$$
$$\Bigg(\prod_{k=1}^{n-\hat{n}_{s-2}-1} y_k\Bigg)^{-1}, \underbrace{y_{n-\hat{n}_1+1}, y_{n-\hat{n}_1+2}, \ldots, y_{n-1}}_{n_r-1 \text{ terms}}\Bigg).$$

In particular,

$$\|wyw^{-1}\|^{a_k} = \prod_{i=1}^{r}\prod_{j=1}^{n_i} y_{n-\hat{n}_i+j}^{-a_{\hat{n}_{i-1}}+a_{\hat{n}_{i-1}+j}-a_{\hat{n}_i}}.$$

Proof. Let $w = w_{(n_1,n_2,\ldots,n_r)}$ as above. In order to carefully analyze $y' = wyw^{-1}$, we define $x_i := \prod_{j=1}^{i} y_j$. This notation implies that $y = \mathrm{diag}(x_{n-1}, x_{n-2}, \ldots, x_1, 1)$. Now, let us think of the matrix y as a block diagonal of the form $y = \mathrm{diag}(A_1, A_2, \ldots, A_r)$, where

$$A_i = \mathrm{diag}(x_{n-\hat{n}_{i-1}-1}, x_{n-\hat{n}_{i-1}-2}, \ldots, x_{n-\hat{n}_{i-1}-n_i}) \in \mathrm{GL}(n_i, \mathbb{R}).$$

Thus,

$$y' = wyw^{-1} = \mathrm{diag}(A_r, A_{r-1}, \ldots, A_1) = x_{n-n_1}\,\mathrm{diag}(B_r, B_{r-1}, \ldots, B_1).$$

Let $1 \leq i \leq r$ and $0 \leq j \leq n_i - 1$ and set

$$z_{\hat{n}_{i-1}+j} := \frac{x_{n-\hat{n}_i+j}}{x_{n-n_1}}.$$

Then $(y'_1, y'_2, \ldots, y'_{n-1})$, the Iwasawa y-variables of y' satisfy $y'_i = z_i/z_{i-1}$. For $j \neq 0$, therefore, we see

$$y'_{\hat{n}_{i-1}+j} = \frac{x_{n-\hat{n}_i+j}}{x_{n-\hat{n}_i+j-1}} = \frac{\prod_{k=1}^{n-\hat{n}_i+j} y_k}{\prod_{\ell=1}^{n-\hat{n}_i+j-1} y_\ell} = y_{n-\hat{n}_i+j},$$

and, for $j=0$,

$$y'_{\hat{n}_i} = \frac{x_{n-\hat{n}_{i+1}}}{x_{n-\hat{n}_{i+1}-1}} = \frac{x_{n-\hat{n}_{i+1}}}{x_{n-\hat{n}_i+n_i-1}} = \frac{\prod_{k=1}^{n-\hat{n}_{i+1}} y_k}{\prod_{\ell=1}^{n-\hat{n}_{i-1}-1} y_\ell} = \left(\prod_{k=1}^{n_i+n_{i+1}-1} y_{n-\hat{n}_{i+1}+k}\right)^{-1},$$

from which the statement of the lemma follows directly. □

Definition A.2. We say that $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ is in *j-general position* if the set

$$\left\{ \sum_{k\in J} \alpha_k \;\middle|\; J \subseteq \{1, \ldots, n\},\ \#J = j \right\}$$

consists of $\binom{n}{j}$ distinct elements. We say that α is *in general position* if it is in j-general position for each $j = 1, \ldots, n-1$.

Lemma A.3. *Suppose that there exists $\varepsilon > 0$ such that for each $j = 1, \ldots, n-1$, the real part of s_j is bounded by at least ε from any integer. Assume that α is in j-general position,* $\mathrm{Re}(\alpha_i) = 0$ *for each $i = 1, \ldots, n-1$, and $r_j \in \mathbb{Z}_{\geq 0}$. Assume that*

$$\mathrm{Im}(\alpha_1) \geq \mathrm{Im}(\alpha_2) \geq \cdots \geq \mathrm{Im}(\alpha_n),$$

and let $I_j = [-\mathrm{Im}(\alpha_1 + \cdots + \alpha_j), -\mathrm{Im}(\alpha_n + \cdots + \alpha_{n-j+1})]$. If $r_j \geq 2$, then

$$\int\limits_{\substack{\mathrm{Re}(s_j)=\sigma_j \\ \mathrm{Im}(s_j)\in I_j}} \prod_{\substack{J\subseteq\{1,\ldots,n\} \\ \#J=j}} \left| s_j + \sum_{k\in J} \alpha_k \right|^{-r_j} ds_j \ll \sum_{\substack{L\subseteq\{1,\ldots,n\} \\ \#L=j}} \prod_{\substack{K\subseteq\{1,\ldots,n\} \\ \#K=j \\ K\neq L}} \left(1 + \left| \sum_{\ell\in L} \alpha_\ell - \sum_{k\in K} \alpha_k \right| \right)^{-r_j}.$$

If $r_j = 1$ there is an extra power of ε in the exponent (*in which case the implicit constant will depend on ε*), *and if $r_j = 0$, the integral is bounded by*

$$\left(1 + \sum_{k=1}^{j} \alpha_k - \sum_{\ell=1}^{j} \alpha_{n+1-\ell}\right).$$

Remark A.4. The implicit $\ll$–constant depends on σ_j, but in applications this will always be bounded.

Proof. The bound in the case of $r_j = 0$ is obvious, so we may assume henceforth that $r_j \geq 1$. Consider the set

$$\mathcal{A}_j := \left\{ \sum_{k\in J} \alpha_k \;\middle|\; J \subseteq \{1, \ldots, n\},\ \#J = j \right\}.$$

For a fixed choice α in j-general position, let A_1 be the element of $\mathcal{A}_j$ that has the greatest imaginary part, A_2 the next greatest imaginary part and so on. Hence $-\mathrm{Im}(A_1) < -\mathrm{Im}(A_2) < \cdots < -\mathrm{Im}(A_{\binom{n}{j}})$.

Write $s_j = \sigma_j + it_j$. Note that $I_j = [-\operatorname{Im}(A_1), -\operatorname{Im}(A_{\binom{n}{j}})]$. Upon applying Lemma A.3 from [Goldfeld et al. 2021b], one obtains the bound

$$\int_{I_j} \prod_{\substack{J\subseteq\{1,\dots,n\}\\ \#J=j}} \Big| s_j + \sum_{k\in J} \alpha_k \Big|^{-r_j} ds_j \ll (1+\operatorname{Im}(A_1)-\operatorname{Im}(A_{\binom{n}{j}}))^{\varepsilon} \prod_{k=1}^{\binom{n}{j}-1} (1+\operatorname{Im}(A_k - A_{k+1}))^{-r_j}.$$

This is one of the possible summands on the right-hand side of the statement of the lemma. Hence, regardless of the specific ordering which may arise for the given choice of α, the claim follows. □

Lemma A.5. *Let $a \in \mathbb{R}$. Then*

$$\max\{0, 2(\lceil a\rceil - a) - 1\} - \lceil a\rceil \le \begin{cases} -\lceil a\rceil & \text{if } a \in \big(\lceil a\rceil - \frac{1}{2}, \lceil a\rceil\big],\\ -\lfloor a\rfloor - 2(a - \lfloor a\rfloor) & \text{if } a \in \big(\lfloor a\rfloor, \lfloor a\rfloor + \frac{1}{2}\big].\end{cases}$$

Proof. First, let us assume that $a \in \big(\lceil a\rceil - \frac{1}{2}, \lceil a\rceil\big]$. Then $\lceil a\rceil - a < \frac{1}{2}$; hence

$$\max\{0, 2(\lceil a\rceil - a) - 1\} - \lceil a\rceil = -\lceil a\rceil.$$

On the other hand, assuming that $a \in \big(\lfloor a\rfloor, \lfloor a\rfloor + \frac{1}{2}\big]$, we see that

$$\max\{0, 2(\lceil a\rceil - a) - 1\} - \lceil a\rceil = \lceil a\rceil - 2a - 1 = \lfloor a\rfloor - 2a = -\lfloor a\rfloor - 2(a - \lfloor a\rfloor),$$

as claimed. □

Lemma A.6. *Suppose that $a_1, \dots, a_n \in \mathbb{R}_{>0}$. Let*

$$B(a) := \begin{cases} 0 & \text{if } a < 0,\\ \lfloor a\rfloor + 2(a - \lfloor a\rfloor) & \text{if } 0 < \lfloor a\rfloor + \varepsilon < a \le \lfloor a\rfloor + \frac{1}{2},\\ \lceil a\rceil & \text{if } \frac{1}{2} < \lceil a\rceil - \frac{1}{2} \le a < \lceil a\rceil - \varepsilon.\end{cases}$$

Then, for any $\delta_m \in \mathbb{Z}_{\ge 0}$ with $0 < a_m - \delta_m$,

$$\sum_{j=1}^{m-1} B\Big(a_j - \frac{j}{m}(a_m - \delta_m)\Big) + \sum_{j=1}^{n-m-1} B\Big(a_{m+j} - \frac{n-m-j}{n-m}(a_m - \delta_m)\Big) \ge \Big(\sum_{j=1}^{n-1} B(a_j)\Big) - \frac{n-2}{2}(a_m - \delta_m + 1) - B(a_m).$$

Proof. We consider first the case of $r - \frac{1}{2} \le a_j < r$ for some $r \in \mathbb{Z}$ and all $j = 1, 2, \dots, n-1$. For any $a \in \mathbb{R}$, note that

$$a \le B(a) \le a + \tfrac{1}{2}; \tag{A.7}$$

hence

$$\sum_{j=1}^{m-1} B\Big(a_j - \frac{j}{m}(a_m - \delta_m)\Big) \ge \Big(\sum_{j=1}^{m-1} a_j\Big) - \frac{m-1}{2}(a_m - \delta_m) \ge \Big(\sum_{j=1}^{m-1} B(a_j) - \frac{1}{2}\Big) - \frac{m-1}{2}(a_m - \delta_m)$$
$$= \Big(\sum_{j=1}^{m-1} B(a_j)\Big) - \frac{m-1}{2}(a_m - \delta_m + 1).$$

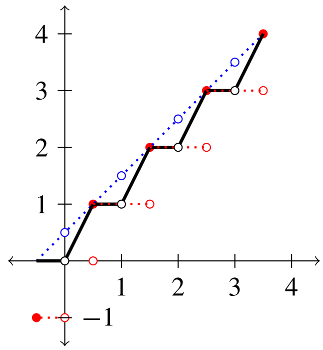

Figure 1. Comparing graph of $B(x)$ (thick black) to $B_4(x)$ (dotted red) and $B_3(x)$ (dotted blue) bounds.

Combining this with the other terms (which are easily shown to satisfy the analogous bound), the desired result is immediate. □

Remark A.8. The function $B(x)$ appears prominently in Theorem 10.0.1 and is critical in bounding the geometric side of the Kuznetsov trace formula. Its graph is shown in Figure 1 in comparison to two other functions B_4 and B_3.

In the case of GL(4), the function B_4 appears [Goldfeld et al. 2021b] (see Theorem 4.0.1) as a bound for the $p_{T,R}$ function. Indeed, making necessary adjustments due to a different choice of normalization factors (see Remark 1.6.2), the result of [loc. cit.] is that

$$|p^{(4)}_{T,R}(1; -a)| \ll T^{\varepsilon+27R+12-\sum_{i=1}^{3} B_4(a_i)}.$$

Theorem 10.0.1 establishes the same result but with B_4 replaced by B. Although the improvement is slight, we remark that it is essential in Lemma A.6 and evidently allows the inductive method of the present paper to lead to the same asymptotic orthogonality relation as was established directly in [loc. cit.].

With a bit of work, one can show that the function B_3, also graphed in Figure 1, appeared in [Goldfeld and Kontorovich 2013] as a bound for

$$|p^{(3)}_{T,R}(1; -a)| \ll T^{\varepsilon+6R+7-\sum_{i=1}^{2} B_3(a_i)}.$$

Although this looks to be an improvement on our result here, the method of [Goldfeld and Kontorovich 2013] contained an error which the present method (and the method of [Goldfeld et al. 2021b]) corrects.

Lemma A.9. *Let $\varepsilon > 0$. Then for any $\rho \in \frac{1}{2} + \mathbb{Z}$ there exists $0 < \varepsilon' < \frac{1}{2}$ sufficiently small such that, setting $\delta = 2\varepsilon'/n^2$, if $a = (a_1, \ldots, a_{n-1})$, where*

$$a_j := \rho + \frac{j(n-j)}{2}(1+\delta),$$

and, for $w = w_{(n_1,\ldots,n_r)}$, $b(a, w) = b = (b_1, \ldots, b_{n-1})$, where

$$b_{n-\hat{n}_i+j} := a_{\hat{n}_{i-1}} - a_{\hat{n}_{i-1}+j} + a_{\hat{n}_i} \pm \frac{\delta}{2},$$

(*meaning that* a *and* b *satisfy* (6.3.1) *and* (6.3.2), *respectively*), *then*, *letting* B *be the function defined in Theorem 9.0.2*,

$$\sum_{j=1}^{n-1}(B(a_j)+B(b_j)) \geq \left\lfloor \frac{n-1}{2} \right\rfloor + n\rho + \Phi(n_1,\dots,n_r) - \varepsilon,$$

where

$$\Phi(n_1,\dots,n_r) := \sum_{k=1}^{r-1}(n_k+n_{k+1})\frac{(n-\hat{n}_k)\hat{n}_k}{2}.$$

Proof. We first note that although the bound $B(x)\geq x$ holds for any $x\in\mathbb{R}$, for any $\varepsilon>0$, $B(x)\geq x+\frac{1}{2}-\varepsilon$ provided that x is sufficiently close to a half integer. Lemma A.11 (as justified in Remark A.12) asserts that if n is odd then $n-1$ elements from the set of all the possible values of a_k and b_k are indeed within ε of a half integer, and if n is even then $n-2$ of values have this property. Hence,

$$\sum_{k=1}^{n-1}(B(a_k)+B(b_k)) \geq \left\lfloor \frac{n-1}{2} \right\rfloor + \sum_{k=1}^{n-1}(a_k+b_k) - \varepsilon. \tag{A.10}$$

Since $b_{n-\hat{n}_i+j} = a_{\hat{n}_{i-1}} - a_{\hat{n}_{i-1}+j} + a_{\hat{n}_i} \pm \frac{\delta}{2}$, we see that

$$\sum_{j=1}^{n_i}(b_{n-\hat{n}_i+j}+a_{\hat{n}_{i-1}+j}) \sim n_i(a_{\hat{n}_{i-1}}+a_{\hat{n}_i}).$$

Therefore, summing over i, we see (making use of the fact that $a_0=a_n=0$) that

$$\begin{aligned}\sum_{k=1}^{n-1}(b_k+a_k) &= \sum_{i=1}^{r} n_i(a_{\hat{n}_{i-1}}+a_{\hat{n}_i}) = \sum_{i=1}^{r-1}(n_i+n_{i+1})a_{\hat{n}_i}\\ &= \sum_{k=1}^{r-1}(n_k+n_{k+1})\left(\rho+\frac{(n-\hat{n}_k)\hat{n}_k}{2}+\varepsilon'\right) \sim \rho(2n-n_1-n_r) + \underbrace{\sum_{k=1}^{r-1}(n_k+n_{k+1})\frac{(n-\hat{n}_k)\hat{n}_k}{2}}_{=:\Phi(n_1,\dots,n_r)}.\end{aligned}$$

Combining this with (A.10), the desired result is now immediate. □

Lemma A.11. *Let* $C=(n_1,\dots,n_r)$ *be a composition of* n *with* $r\geq 2$. *Suppose that* $\rho\in\frac{1}{2}+\mathbb{Z}$. *Set* $a_0:=0$, $a_n:=0$ *and for each* $1\leq k\leq n-1$ *we have* $a_k:=\rho+\frac{k(n-k)}{2}$ *and for each* $1\leq i\leq r$ *and* $1\leq j\leq n_i$ *we let* $b_{i,j}:=a_{\hat{n}_{i-1}}-a_{\hat{n}_{i-1}+j}+a_{\hat{n}_i}$.

Then

$$\#\{k \mid a_k\notin\mathbb{Z}\} + \#\{(i,j)\mid b_{i,j}\notin\mathbb{Z}\} = \begin{cases} 2n-n_1-n_r-1 & \text{if } n \text{ is odd},\\ \frac{n}{2}-1+\left\lfloor\frac{n_1}{2}\right\rfloor+\left\lfloor\frac{n_r}{2}\right\rfloor+\sum_{i=2}^{r-1}\left\lceil\frac{n_i}{2}\right\rceil & \text{if } n \text{ is even}.\end{cases}$$

Remark A.12. Note that the quantity given in Lemma A.11 in the case of n odd is $2n-n_1-n_r-1\geq n-1$ for any composition C (with equality precisely when $r=2$). If n is even then

$$\frac{n}{2}-1+\left\lfloor\frac{n_1}{2}\right\rfloor+\left\lfloor\frac{n_r}{2}\right\rfloor+\sum_{i=2}^{r-1}\left\lceil\frac{n_i}{2}\right\rceil \geq \frac{n}{2}+\frac{n_1}{2}+\frac{n_r}{2}-2+\sum_{i=2}^{r-1}\frac{n_i}{2} = n-2.$$

Equality in this case occurs precisely when n_1 and n_r are both odd and all other n_i are even.

Proof. For notational purposes, set

$$A(n) := \#\{1 \le k \le n-1 \mid a_k \notin \mathbb{Z}\},$$
$$B(C) := \#\{(i,j),\ 1 \le i \le r,\ 1 \le j \le n_i \mid b_{i,j} \notin \mathbb{Z}\}.$$

We first consider the case of n odd, for which $\frac{k(n-k)}{2} \in \mathbb{Z}$ for all integers k. Therefore, $A(n) = n-1$. As for $B(C)$, note that $b_{i,j}$ is equal to ρ plus an integer as long as $i \ne 1, r$. Otherwise, $b_{1,j}, b_{r,j} \in \mathbb{Z}$. Hence $B(C) = n - n_1 - n_r$.

In the case of n even, $\frac{k(n-k)}{2} \in \mathbb{Z}$ exactly when k is even. Hence $A(n) = \frac{n}{2} - 1$. To the end of finding $B(C)$, we introduce the notation

$$B_i(C) := \#\{1 \le j \le n_i \mid b_{i,j} \notin \mathbb{Z}\},$$

for which it is clear that $B(C) = \sum_{i=1}^{r} B_i(C)$.

The cardinality of $B_i(C)$ depends, obviously, on the integrality of $b_{i,j}$. To determine this, we first assume that $i = 1$. Then

$$b_{1,j} = -\frac{j(n-j)}{2} + \frac{n_1(n-n_1)}{2}.$$

Therefore (since n is even), $b_{i,j} \in \mathbb{Z}$ if and only if $j \equiv n_1 \pmod 2$. This implies that

$$B_1(C) := \begin{cases} \frac{n_1-1}{2} & \text{if } n_1 \text{ is odd}, \\ \frac{n_1}{2} & \text{if } n_1 \text{ is even}, \end{cases}$$

or more concisely, $\#B_1(C) = \left\lfloor \frac{n_1}{2} \right\rfloor$. The determination of $B_r(C)$ is similar: $\#B_r(C) = \left\lfloor \frac{n_r}{2} \right\rfloor$.

For $1 < i < r$, we see that

$$\begin{aligned} b_{i,j} &= \rho + \frac{\hat{n}_{i-1}(n-\hat{n}_{i-1})}{2} - \frac{(\hat{n}_{i-1}+j)(n-\hat{n}_{i-1}-j)}{2} + \frac{(\hat{n}_{i-1}+n_i)(n-\hat{n}_{i-1}-n_i)}{2} \\ &= \rho + \hat{n}_{i-1}(n-\hat{n}_{i-1}-n_i) - \frac{(\hat{n}_{i-1}+j)(n-\hat{n}_{i-1}-j)}{2} + \frac{n_i(n-n_i)}{2} \\ &\equiv \frac{1}{2} + \frac{(\hat{n}_{i-1}+j)(n-\hat{n}_{i-1}-j)}{2} + \frac{n_i(n-n_i)}{2} \pmod{\mathbb{Z}}. \end{aligned}$$

We see again that the integrality of $b_{i,j}$ depends on the parity of n_i. If n_i is odd,

$$B_i(C) = \#\left\{1 \le j \le n_i \;\middle|\; \frac{(\hat{n}_{i-1}+j)(n-\hat{n}_{i-1}-j)}{2} \notin \mathbb{Z}\right\},$$

and if n_i is even,

$$B_i(C) = \#\left\{1 \le j \le n_i \;\middle|\; \frac{(\hat{n}_{i-1}+j)(n-\hat{n}_{i-1}-j)}{2} \in \mathbb{Z}\right\}.$$

One can check, arguing case by case as above, that in any event, the answer is $B_i(C) = \left\lceil \frac{n_i}{2} \right\rceil$. □

Lemma A.13. *Suppose that* $(n_1, \dots, n_r) \in \mathbb{C}^r$. *The function*

$$\Phi(n_1, \dots, n_r) := \sum_{k=1}^{r-1} (n_k + n_{k+1}) \frac{(n_1 + \cdots + n_k)(n_{k+1} + \cdots + n_r)}{2}$$

is invariant under permutations, i.e., for any $\sigma \in S_r$, *we have* $\Phi(n_1, \ldots, n_r) = \Phi(n_{\sigma(1)}, \ldots, n_{\sigma(r)})$. *In particular, if* $P = n_1 + \cdots + n_r$ *is a partition of* n *then* $\Phi(P) := \Phi(n_1, \ldots, n_r)$ *is well-defined. Moreover, among all partitions* P *of* n (*with* $r \geq 2$),

$$\Phi(P) \geq \Phi(n-1, 1) = \Phi(1, n-1) = \frac{n(n-1)}{2}.$$

Proof. Suppose that $n = n_1 + \cdots + n_r = m_1 + \cdots + m_r$, where

$$m_j = \begin{cases} n_j & \text{if } j \neq k, k+1, \\ n_{k+1} & \text{if } j = k, \\ n_k & \text{if } j = k+1. \end{cases}$$

Then one can show by an elementary (albeit tedious) computation that $\Phi(n_1, \ldots, n_r) = \Phi(m_1, \ldots, m_r)$. In other words, Φ is invariant under any transposition $\tau \in S_r$, hence invariant under all of S_r.

Suppose that $n = n_1 + \cdots + n_r$. If $n_k = n'_k + n''_k$ for some $1 \leq k \leq r$, then one shows via a straightforward computation that

$$\Phi(n_1, \ldots, n_{k-1}, n'_k, n''_k, n_{k+1}, \ldots, n_{r-1}) - \Phi(n_1, \ldots, n_r) = \frac{n_k n'_k n''_k}{2}.$$

If $n = n_1 + \cdots + n_r$ with $r > 2$, it then follows, setting $n_0 := \min\{n_1, n_2, \ldots, n_r\}$, that

$$\Phi(n_1, \ldots, n_r) > \Phi(n_0, n - n_0) = \frac{n n_0 (n - n_0)}{2}.$$

Among all $1 \leq n_0 \leq \frac{n}{2}$, the right-hand side is minimized when $n_0 = 1$. □

Recall that

$$\Gamma_R(z) := \frac{\Gamma\big(\frac{1}{2}\big(\frac{1}{2} + R + z\big)\big)}{\Gamma(z)},$$

as defined at the beginning of Section 8.

Lemma A.14. *If* $\delta \in \mathbb{Z}$ *and* $\beta \in i\mathbb{R}$ *are fixed, then the function* $\Gamma_R(\beta + z)\Gamma_R(-\beta - z)\Gamma(-\beta - z - \delta)$ *is holomorphic for all* z *with* $|\mathrm{Re}(z)| < R$.

Proof. The fact that $|z| < R$ implies that $\Gamma_R(\pm z)$ is holomorphic is immediate, so the only question is what happens at the (simple) poles of $\Gamma(-\beta - z - \delta)$. But these occur at $z = -\beta + k$ for some integer k which corresponds to zeros of $\Gamma_R(\beta + z)$ or $\Gamma_R(-\beta - z)$. □

Lemma A.15. *For* $\delta \in \mathbb{Z}$ *fixed and* $z, \beta \in \mathbb{C}$ *and* $|\mathrm{Re}(z + \beta) + \delta| < R$, *we have the bound*

$$\Gamma_R(\beta + z)\Gamma_R(-\beta - z)\Gamma(-\beta - z - \delta) \asymp (1 + |\mathrm{Im}(\beta + z)|)^{R - \mathrm{Re}(\beta + z) - \delta}.$$

Proof. This follows immediately from the Stirling bound $|\Gamma(\sigma + it)| \sim \sqrt{2\pi}\,|t|^{\sigma - \frac{1}{2}} e^{\pi|t|/2}$. □

Definition A.16. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ be Langlands parameters satisfying $\hat{\alpha}_n = 0$. Let $n = n_1 + \cdots + n_r$ be a partition of n with $n_1, \ldots, n_r \in \mathbb{Z}_+$. Then for each $\ell = 1, \ldots, r$ we define $\alpha^{(\ell)} := (\alpha_1^{(\ell)}, \ldots, \alpha_{n_\ell}^{(\ell)}) \in \mathbb{C}^{n_\ell}$, where

$$\alpha_j^{(\ell)} := \alpha_{\hat{n}_{\ell-1} + j} - \frac{1}{n_\ell}(\hat{\alpha}_{\hat{n}_\ell} - \hat{\alpha}_{\hat{n}_{\ell-1}}), \quad |\alpha^{(\ell)}|^2 := \sum_{j=1}^{n_\ell} (\alpha_j^{(\ell)})^2.$$

Remark A.17. Note that $\sum_{j=1}^{n_\ell}\alpha_j^{(\ell)}=0$ for each ℓ. In particular $n_\ell=1$ implies $\alpha_1^{(\ell)}=0$.

Lemma A.18. *We have* $|\alpha|^2=\sum_{i=1}^{n}\alpha_i^2=\sum_{\ell=1}^{r}\big(|\alpha^{(\ell)}|^2+\frac{1}{n_\ell}(\hat\alpha_{\hat n_\ell}-\hat\alpha_{\hat n_{\ell-1}})^2\big)$.

Proof. Computing directly, and using the fact that $\sum_{j=1}^{n_\ell}\alpha_j^{(\ell)}=0$, we find that

$$\begin{aligned}\sum_{j=1}^{n}\alpha_j^2=\sum_{\ell=1}^{r}\sum_{j=1}^{n_\ell}\alpha_{\hat n_{\ell-1}+j}^2&=\sum_{\ell=1}^{r}\sum_{j=1}^{n_\ell}\left(\alpha_j^{(\ell)}+\frac{1}{n_\ell}(\hat\alpha_{\hat n_\ell}-\hat\alpha_{\hat n_{\ell-1}})\right)^2\\&=\sum_{\ell=1}^{r}\sum_{j=1}^{n_\ell}\left((\alpha_j^{(\ell)})^2+\frac{2}{n_\ell}\alpha_j^{(\ell)}(\hat\alpha_{\hat n_\ell}-\hat\alpha_{\hat n_{\ell-1}})+\frac{1}{n_\ell^2}(\hat\alpha_{\hat n_\ell}-\hat\alpha_{\hat n_{\ell-1}})^2\right)\\&=\sum_{\ell=1}^{r}\left(|\alpha^{(\ell)}|^2+\frac{1}{n_\ell}(\hat\alpha_{\hat n_\ell}-\hat\alpha_{\hat n_{\ell-1}})^2\right),\end{aligned}$$

as claimed. □

Lemma A.19. *Suppose that* $n\ge 2$ *and* $\alpha_1,\alpha_2,\dots,\alpha_n\in\mathbb{C}$ *satisfies* $\alpha_1+\alpha_2+\cdots+\alpha_n=0$. *Set* $\hat\alpha_k=\sum_{j=1}^{k}\alpha_j$ *for fixed* $k\in\{1,2,\dots,n\}$, *and define* $\beta_j:=\alpha_j-\frac{1}{k}\hat\alpha_k$, $\gamma_j:=\alpha_{j+k}+\frac{1}{n-k}\hat\alpha_k$. *Then*

$$\sum_{i=1}^{n}\alpha_i^2=\sum_{i=1}^{k}\beta_i^2+\sum_{i=1}^{n-k}\gamma_i^2+\frac{n}{k(n-k)}\hat\alpha_k^2.$$

Proof. This is easily deduced as a special case of Lemma A.18 in the case that $r=2$, $n_1=k$, $n_2=n-k$, $\beta=\alpha^{(1)}$ and $\gamma=\alpha^{(2)}$. □

Lemma A.20. *We continue the notation of Lemma A.18. Then*

$$\begin{aligned}\prod_{1\le i\ne j\le n}\Gamma_R(\alpha_i-\alpha_j)=\prod_{\ell=1}^{r}\Bigg(\prod_{1\le i,j\le n_\ell}\Gamma_R(\alpha_i^{(\ell)}-\alpha_j^{(\ell)})\Bigg)\\\cdot\prod_{1\le k<m\le r}\prod_{i=1}^{n_k}\prod_{j=1}^{n_m}\prod_{\epsilon\in\{\pm1\}}\Gamma_R\left(\epsilon\left(\alpha_i^{(k)}-\alpha_j^{(m)}+\frac{1}{n_k}(\hat\alpha_{\hat n_k}-\hat\alpha_{\hat n_{k-1}})-\frac{1}{n_m}(\hat\alpha_{\hat n_m}-\hat\alpha_{\hat n_{m-1}})\right)\right).\end{aligned}$$

Proof. Note that if $k\ne m$, then for any $1\le i\le n_k$ and $1\le j\le n_m$,

$$\alpha_{\hat n_{k-1}+i}-\alpha_{\hat n_{m-1}+j}=\alpha_i^{(k)}-\alpha_j^{(m)}+\frac{1}{n_k}(\hat\alpha_{\hat n_k}-\hat\alpha_{\hat n_{k-1}})-\frac{1}{n_m}(\hat\alpha_{\hat n_m}-\hat\alpha_{\hat n_{m-1}}),$$

and for any $1\le i\ne j\le n_\ell$ we have $\alpha_{\hat n_{\ell-1}+i}-\alpha_{\hat n_{\ell-1}+j}=\alpha_i^{(\ell)}-\alpha_j^{(\ell)}$. This immediately implies the desired formula. □

Lemma A.21. *Suppose that* $(\beta_1,\dots,\beta_r)$ *satisfies* $\hat\beta_r=0$. *Suppose that* $n=n_1+\cdots+n_r$ *and set* $\hat n_k=\sum_{j=1}^{k}n_j$. *Then* $\sum_{1\le k<m\le r}\sum_{i=1}^{n_k}\sum_{j=1}^{n_m}\big(\beta_k/n_k-\beta_m/n_m\big)=\sum_{j=1}^{r-1}(n_j+n_{j+1})\hat\beta_j$.

Proof. We calculate

$$\sum_{1\le k<m\le r}\sum_{i=1}^{n_k}\sum_{j=1}^{n_m}\left(\frac{\beta_k}{n_k}-\frac{\beta_m}{n_m}\right)=\sum_{m=2}^{r}\sum_{k=1}^{m-1}\sum_{i=1}^{n_k}\left(n_m\frac{\beta_k}{n_k}-\beta_m\right)=\sum_{m=2}^{r}\sum_{k=1}^{m-1}(n_m\beta_k-n_k\beta_m)$$
$$=\sum_{m=2}^{r}(n_m\hat\beta_{m-1}-\hat n_{m-1}\beta_m)=\sum_{m=2}^{r}(n_m\hat\beta_{m-1}-\hat n_{m-1}(\hat\beta_m-\hat\beta_{m-1}))$$
$$=\sum_{m=2}^{r}((n_m+\hat n_{m-1})\hat\beta_{m-1}-\hat n_{m-1}\hat\beta_m)=\sum_{m=2}^{r}(\hat n_m\hat\beta_{m-1}-\hat n_{m-1}\hat\beta_m).$$

This final sum telescopes to give $\sum_{j=1}^{r-1}(\hat n_{j+1}-\hat n_{j-1})\hat\beta_j$. Since $\hat n_{j+1}-\hat n_{j-1}=n_j+n_{j+1}$, this implies the claimed result. □

The following result can be interpreted as a consequence—by counting (half) the number of gamma factors on each side of the equality—of Lemma A.20. Alternatively, proving it independent of Lemma A.20 gives further evidence that the product decomposition is correct.

Lemma A.22. *Let $n=n_1+\cdots+n_r$. We have $\sum_{1\le k<k'\le r}n_k\cdot n_{k'}+\sum_{k=1}^{r}\frac{n_k(n_k-1)}{2}=\frac{n(n-1)}{2}$.*

Proof. We use induction on r. If $r=1$, the formula obviously holds. Let $n=m+n_r$, where $m=n_1+\cdots+n_{r-1}$. Then, by induction,

$$\frac{n(n-1)}{2}=\frac{(m+n_r)(m+n_r-1)}{2}=\frac{m(m-1)}{2}+\frac{mn_r+(m-1)n_r}{2}+\frac{n_r^2}{2}$$
$$=\sum_{k=1}^{r-1}\frac{n_k(n_k-1)}{2}+\sum_{1\le k<k'\le r-1}n_k\cdot n_{k'}+mn_r+\frac{n_r(n_r-1)}{2}.$$

Since $mn_r=n_1n_r+n_2n_r+\cdots+n_{r-1}n_r$, it is evident that the desired formula holds. □

Lemma A.23. *Suppose $n=n_1+\cdots+n_r$. Then*

$$n^2+\sum_{\ell=1}^{r}\left(\frac{n_\ell(n_\ell-1)}{2}-n_\ell\hat n_\ell\right)=\frac{n(n-1)}{2}.$$

Proof. If $r=1$ the result is obviously true. Suppose that the result holds for $r=k$. Write $n=n_1+\cdots+n_k+n_{k+1}=\hat n_k+n_{k+1}$. Then

$$n^2+\sum_{\ell=1}^{k+1}\left(\frac{n_\ell(n_\ell-1)}{2}-n_\ell\hat n_\ell\right)=n^2+\sum_{\ell=1}^{k}\left(\frac{n_\ell(n_\ell-1)}{2}-n_\ell\hat n_\ell\right)+\frac{n_{k+1}(n_{k+1}-1)}{2}-n_{k+1}n$$
$$=n^2-\hat n_k^2+\left(\hat n_k^2+\sum_{\ell=1}^{k}\left(\frac{n_\ell(n_\ell-1)}{2}-n_\ell\hat n_\ell\right)+\frac{n_{k+1}(n_{k+1}-1)}{2}-n_{k+1}n\right)$$
$$=n^2-\hat n_k^2+\frac{\hat n_k(\hat n_k+1)}{2}+\frac{n_{k+1}(n_{k+1}-1)}{2}-n_{k+1}n$$
$$=n^2-\hat n_k^2+\frac{\hat n_k(\hat n_k+1)}{2}+\frac{(n-\hat n_k)(n-\hat n_k-1)}{2}-(n-\hat n_k),$$

which can easily be shown now to simplify to $\frac{n(n-1)}{2}$, as claimed. □

Remark A.24. Note that Lemmas A.22 and A.23 are equivalent provided that

$$n^2-\sum_{\ell=1}^{r}n_\ell\hat{n}_\ell=\sum_{1\le k<k'\le r}n_kn_{k'}. \tag{A.25}$$

This can be verified by expanding the left-hand side as follows:

$$n^2-\sum_{\ell=1}^{r}n_\ell\hat{n}_\ell=(n_1+\cdots+n_r)\hat{n}_r-\sum_{\ell=1}^{r}n_\ell\hat{n}_\ell=\sum_{\ell=1}^{r}(n_\ell(n-\hat{n}_\ell)-n_\ell\hat{n}_\ell)=\sum_{\ell=1}^{r}n_\ell(n-n_\ell).$$

That this final expression is equal to right-hand side of (A.25) is clear.

Lemma A.26. *Let $\alpha_1,\alpha_2,\dots,\alpha_n\in\mathbb{C}$ satisfy $\alpha_1+\alpha_2+\cdots+\alpha_n=0$. Set $\hat{\alpha}_k:=\sum_{j=1}^{k}\alpha_j$, and let β_i $(1\le i\le k)$ and γ_j $(1\le j\le n-k)$ be as in the previous lemma. We have*

$$\prod_{1\le i\ne j\le n}\Gamma_R(\alpha_i-\alpha_j)=\Bigg(\prod_{1\le i\ne j\le k}\Gamma_R(\beta_i-\beta_j)\Bigg)\Bigg(\prod_{1\le i\ne j\le n-k}\Gamma_R(\gamma_i-\gamma_j)\Bigg)\cdot\prod_{i=1}^{k}\prod_{j=1}^{n-k}\Gamma_R\Big(\beta_i-\gamma_j+\frac{n}{k(n-k)}\hat{\alpha}_k\Big)\Gamma_R\Big(\gamma_j-\beta_i-\frac{n}{k(n-k)}\hat{\alpha}_k\Big).$$

Proof. This is easily deduced as a special case of Lemma A.20 when $r=2,\ n_1=k,\ n_2=n-k,\ \beta=\alpha^{(1)}$ and $\gamma=\alpha^{(2)}$. □

We recall the definition of the polynomial given in Definition 1.4.1:

$$\mathcal{F}_R^{(n)}(\alpha):=\prod_{j=1}^{n-2}\prod_{\substack{K,L\subseteq(1,2,\dots,n)\\ \#K=\#L=j}}\Big(1+\sum_{k\in K}\alpha_k-\sum_{\ell\in L}\alpha_\ell\Big)^{\frac{R}{2}}.$$

Also, we remind the reader of the polynomial notation $\mathcal{P}$ given in Definition 2.1.3.

Lemma A.27. *Let $n=n_1+\cdots+n_r$, α, and $\alpha^{(\ell)}$ be as in Definition A.16. Set $D(n)=\deg(\mathcal{F}_1^{(n)}(\alpha))$. Then*

$$\mathcal{F}_R^{(n)}(\alpha)=\mathcal{P}_{dR}(\alpha)\prod_{\substack{\ell=1\\ n_\ell\ne1}}^{r}\mathcal{F}_R^{(n_\ell)}(\alpha^{(\ell)}),\quad \text{where } d=d(n_1,\dots,n_r)=D(n)-\sum_{\substack{\ell=1\\ n_\ell\ne1}}^{r}D(n_\ell).$$

Proof. This follows from the fact that if $I,J\subseteq\{1,2,\dots,n_\ell\}$ with $\#I=\#J$ then

$$\Big(\sum_{i\in I}\alpha_i^{(\ell)}\Big)-\Big(\sum_{j\in J}\alpha_j^{(\ell)}\Big)=\Big(\sum_{i\in I}\alpha_{\hat{n}_{\ell-1}+i}\Big)-\Big(\sum_{j\in J}\alpha_{\hat{n}_{\ell-1}+j}\Big).$$

Therefore, each $\mathcal{F}_R^{(n_\ell)}(\alpha^{(\ell)})$ constitutes a unique factor of $\mathcal{F}_R^{(n)}(\alpha)$ for each $\ell=1,\dots,r$. □

Lemma A.28. *Suppose that $\delta\in\mathbb{Z}_{\ge0}$ and $R>\delta$. Then*

$$\mathcal{F}_R^{(n)}(\alpha)\cdot\prod_{\substack{K\subseteq\{1,2,\dots,n\}\\ \#(K\cap\{1,2,\dots,m\})\ne m-1\\ \#K=m}}\Bigg(\Big(\sum_{i\in K}\alpha_i\Big)-\hat{\alpha}_m-\delta\Bigg)_\delta^{-1}\ll\mathcal{F}_R^{(m)}(\beta)\cdot\mathcal{F}_R^{(n-m)}(\gamma)\cdot\mathcal{P}_d(\alpha),$$

where $d=R(D(n)-D(m)-D(n-m))-\delta\big(\binom{n}{m}-m(n-m)-1\big)$.

Proof. Let $M := \{1, 2, \dots, m\}$. Then

$$\#\{K \subseteq \{1, 2, \dots, n\} \mid \#K = m,\ \#(K \cap M) \neq 0, 1\} = \binom{n}{m} - m(n-m) - 1.$$

From the definition of $\mathcal{F}_R^{(n)}(\alpha)$ given in Definition 1.4.1, we see that for each such K there are factors

$$\Big(1 + \sum_{i \in K} \alpha_i - \sum_{j \in M} \alpha_j\Big)^{R/2} \Big(1 - \sum_{i \in K} \alpha_i + \sum_{j \in M} \alpha_j\Big)^{R/2}$$

of $\mathcal{F}_R^{(n)}(\alpha)/[\mathcal{F}_R^{(m)}(\beta)\mathcal{F}_R^{(n-m)}(\gamma)]$ for which

$$\frac{\big(1 + \sum_{i \in K} \alpha_i - \sum_{j \in M} \alpha_j\big)^{R/2}\big(1 - \sum_{i \in K} \alpha_i + \sum_{j \in M} \alpha_j\big)^{R/2}}{\big(\sum_{i \in K} \alpha_i - \sum_{j \in M} \alpha_j - \delta\big)_\delta} \ll \bigg(1 - \Big(\sum_{i \in K} \alpha_i - \sum_{j \in M} \alpha_j\Big)^2\bigg)^{\frac{R-\delta}{2}}.$$

This bound holds because the degree of the Pochhammer symbol in the denominator is δ, and by assumption, the degree of the numerator is $R > \delta$. Combining all such terms with the remaining factors of $\mathcal{F}_R^{(n)}(\alpha)/[\mathcal{F}_R^{(m)}(\beta)\mathcal{F}_R^{(n-m)}(\gamma)]$ gives a polynomial of degree d. □

Remark A.29. Let $n = n_1 + n_2 + \cdots + n_r$ and $\hat{n}_\ell = \sum_{i=1}^{\ell} n_i$. The result of Lemma A.28 clearly generalizes to the case of taking multiple residues at $s_{\hat{n}_\ell} = -\hat{\alpha}_{\hat{n}_\ell} - \delta_\ell$ for each $\ell = 1, \dots, r-1$ (in reverse order). In this case, taking the product on the left-hand side over all of the terms we obtain

$$\mathcal{F}_R^{(n)}(\alpha) \cdot \prod_{\ell=1}^{r-1} \prod_{\substack{K \subseteq \{1,2,\dots,\hat{n}_{\ell+1}\} \\ \#(K \cap \{1,\dots,\hat{n}_\ell\}) \neq \hat{n}_\ell - 1 \\ \#K = \hat{n}_\ell}} \bigg(\Big(\sum_{i \in K} \alpha_i\Big) - \hat{\alpha}_{\hat{n}_\ell} - \delta_\ell\bigg)_{\delta_\ell}^{-1} \ll \mathcal{P}_d(\alpha) \cdot \prod_{\ell=1}^{r} \mathcal{F}_R^{(n_\ell)}(\alpha^{(\ell)}),$$

where

$$d = R \cdot \Big(D(n) - \sum_{\ell=1}^{r} D(n_\ell)\Big) - \sum_{\ell=1}^{r-1} \Big[\delta_\ell \Big(\binom{\hat{n}_{\ell+1}}{\hat{n}_\ell} - n_{\ell+1}\hat{n}_\ell - 1\Big)\Big].$$

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