Algebra & Number Theory

Volume 19

2025

No. 11

On the equivalence between the effective adjunction conjectures of Prokhorov–Shokurov and of Li

Jingjun Han, Jihao Liu and Qingyuan Xue

msp

ALGEBRA AND NUMBER THEORY 19:11 (2025)
https://doi.org/10.2140/ant.2025.19.2261

On the equivalence between the effective adjunction conjectures of Prokhorov–Shokurov and of Li

Jingjun Han, Jihao Liu and Qingyuan Xue

Prokhorov and Shokurov introduced the effective adjunction conjecture, also known as the effective basepoint-freeness conjecture, which asserts that the moduli component of an lc-trivial fibration is effectively basepoint-free. Li proposed a variation of this conjecture, known as the Γ-effective adjunction conjecture, and demonstrated that a weaker version of his conjecture follows from the original Prokhorov–Shokurov conjecture.

In this paper, we prove the equivalence between Prokhorov–Shokurov's and Li's effective adjunction conjectures. The key to our proof is establishing uniform rational polytopes for canonical bundle formulas. This relies on recent advancements in the minimal model program theory of algebraically integrable foliations, primarily developed by Ambro–Cascini–Shokurov–Spicer and Chen–Han–Liu–Xie.

1. Introduction

We work over the field of complex numbers $\mathbb{C}$.

Prokhorov and Shokurov famously proposed the effective basepoint-freeness conjecture concerning the moduli part of lc-trivial fibrations.

Conjecture 1.1 [Prokhorov and Shokurov 2009, Conjecture 7.13]. *Let d be a positive integer and $\Gamma_0 \subset [0, 1]$ a finite set of rational numbers. Then there exists a positive integer I depending only on d and Γ_0 satisfying the following conditions. Assume that*

(1) *$f : (X, B) \to Z$ is an lc-trivial fibration such that* $\dim X - \dim Z = d$, *and*

(2) *the coefficients of the horizontal$/Z$ part of B belong to Γ_0.*

Then $I\boldsymbol{M}$ is basepoint-free, where $\boldsymbol{M}$ is the moduli part of $f : (X, B) \to Z$.

Conjecture 1.1 is known for its complexity and has only been proven for the case $d = 1$, as detailed in [Prokhorov and Shokurov 2009, Theorem 8.1]. The noneffective version of this conjecture for $d = 2$ was recently proven in [Ascher et al. 2023, Theorem 1.4]. However, for $d \geq 3$, Conjecture 1.1 remains largely unresolved.

MSC2020: 14E30, 37F75.
Keywords: algebraically integrable foliation, canonical bundle formula, uniform rational polytope.

The importance of Conjecture 1.1 lies in its close relationship with moduli theory. Specifically, since the moduli parts of lc-trivial fibrations characterize the moduli space of the general fiber of the family $X \to Z$, Conjecture 1.1 is crucial for the study of the moduli of varieties, especially log Calabi–Yau varieties; see, for example, [Ascher et al. 2023].

Recent developments in moduli theory suggest that, instead of considering only pairs with standard or rational coefficients, it is more natural to include pairs with arbitrary coefficients in $[0, 1]$ or $\left(\frac{1}{2}, 1\right]$; cf. [Kollár 2023, 6.26–6.28]. In particular, pairs with irrational coefficients need to be considered. Since Conjecture 1.1 only considers lc-trivial fibrations with rational horizontal coefficients, it is natural to inquire whether a generalization of Conjecture 1.1 for lc-trivial fibrations with irrational coefficients is feasible. Fortunately, Z. Li has proposed such a variation in [Li 2024, Conjecture 3.5(1)], adopting the notation of Γ-basepoint-freeness. In this paper, we propose a stronger version of [Li 2024, Conjecture 3.5(1)].

Definition 1.2 [Li 2024, Definition 3.4]. Let $\Gamma \subset (0, 1]$ be a set. A $\boldsymbol{b}$-divisor $\boldsymbol{D}$ on a normal projective variety X is called *Γ-basepoint-free* if there exist $a_1, \dots, a_k \in \Gamma$ and basepoint-free $\boldsymbol{b}$-divisors $\boldsymbol{D}_1, \dots, \boldsymbol{D}_k$ such that $\sum_{i=1}^k a_i = 1$ and $\boldsymbol{D} = \sum_{i=1}^k a_i \boldsymbol{D}_i$.

Conjecture 1.3 [Li 2024, Conjecture 3.5(1)]. *Let d be a positive integer and $\Gamma \subset [0, 1]$ a DCC set of real numbers. Then there exist a positive integer I, a finite set $\Gamma_0 \subset (0, 1]$ depending only on d, and Γ satisfying the following conditions. Assume that*

(1) *$f : (X, B) \to Z$ is an lc-trivial fibration such that $\dim X - \dim Z = d$, and*

(2) *the coefficients of the horizontal$/Z$ part of B belong to Γ.*

Then $I\boldsymbol{M}$ is Γ_0-basepoint-free, where $\boldsymbol{M}$ is the moduli part of $f : (X, B) \to Z$.

It is evident that Conjecture 1.3 implies Conjecture 1.1. This raises the intriguing question of whether the two conjectures are, in fact, equivalent. Supporting this possibility, Z. Li introduced a less stringent form of Conjecture 1.3 in [Li 2024, Conjecture 3.5(2)] and proved that Conjecture 1.1 implies this weaker version. However, it remains unproven whether [Li 2024, Conjecture 3.5(2)] implies Conjecture 1.1. Additionally, [Li 2024, Conjecture 3.5(2)] is notably more complex than Conjecture 1.3.

In our paper, we demonstrate that Prokhorov–Shokurov's Conjecture 1.1 and Li's Conjecture 1.3 are, in fact, equivalent.

Theorem 1.4. *For any positive integer d, Conjecture 1.1 in relative dimension d (i.e., $\dim X - \dim Z = d$) and Conjecture 1.3 in relative dimension d are equivalent.*

As an immediate corollary, we prove Conjecture 1.3 when $d = 1$:

Corollary 1.5. *Conjecture 1.3 holds when $d = 1$.*

Idea of the proof. The idea behind the proof of Theorem 1.4 is to establish uniform rational polytopes for canonical bundle formulas (see Theorem 3.4 below). Roughly speaking, given an lc-trivial fibration $f : (X, B) \to Z$ with moduli part $\boldsymbol{M}$, we aim to establish a uniform decomposition $(X, B) = \sum a_i (X, B_i)$, where $\sum a_i = 1$, the horizontal/Z coefficients of B_i are rational, and each $f : (X, B_i) \to Z$ is an lc-trivial fibration with moduli part $\boldsymbol{M}_i$, so that $\boldsymbol{M} = \sum a_i \boldsymbol{M}_i$. By "uniform", we mean that the a_i and the horizontal/Z coefficients of B_i depend only on $\dim X - \dim Z$ and the horizontal/Z coefficients of B.

Establishing such a uniform decomposition is a natural idea. In fact, it is straightforward to establish such a uniform decomposition without the condition "$\boldsymbol{M} = \sum a_i \boldsymbol{M}_i$" using [Han et al. 2024, Theorem 5.6]. Similar results can be found in [Li 2024] as well. However, proving $\boldsymbol{M} = \sum a_i \boldsymbol{M}_i$ is a difficult task. The existence of such a decomposition with $\boldsymbol{M} = \sum a_i \boldsymbol{M}_i$ is already nontrivial even when uniformity is not required [Jiao et al. 2022, Theorem 2.23]. This is because the coefficient of the discriminant part of the canonical bundle formula $f : (X, B) \to Z$ and $f : (X, B_i) \to Z$ are of the forms $1 - \operatorname{lct}_{\eta_D}(X, B; f^*D)$ and $1 - \operatorname{lct}_{\eta_D}(X, B_i; f^*D)$, respectively, yet we only have

$$\operatorname{lct}_{\eta_D}(X, B; f^*D) \geq \sum a_i \operatorname{lct}_{\eta_D}(X, B_i; f^*D)$$

in general. The uniform decomposition shows that not only the inequality becomes equality but also the equality holds for any X, Z, and any divisor D over Z simultaneously (as $\boldsymbol{M}$ is a $\boldsymbol{b}$-divisor).

The key new ingredient we need for the proof of the existence of the *uniform* decomposition is the minimal model program theory for algebraically integrable foliations, which has been established very recently [Ambro et al. 2021; Chen et al. 2023].

More precisely, let $\mathcal{F}$ be the foliation induced by $f : X \to Z$ and B^h the horizontal/Z part of B. The key observation is that if f is equidimensional and $(X, \mathcal{F}, B^h)$ is lc satisfying Property $(*)$ (see [Ambro et al. 2021, Definitions 2.13 and 3.5]), then $K_{\mathcal{F}} + B^h$ is exactly the moduli part of $f : (X, B) \to Z$ [Ambro et al. 2021, Proposition 3.6]. Therefore, if we can decompose $(X, \mathcal{F}, B^h)$ into foliated triples with Property $(*)$ uniformly, then it automatically induces a decomposition of $\boldsymbol{M}$. Such a decomposition is possible (see Theorem 3.2) if we replace "Property $(*)$" with the condition "weak ACSS" (see [Chen et al. 2023, Definition 7.2.3]), thanks to the existence of uniform rational lc polytopes for foliations [Liu et al. 2024a, Theorem 1.8; 2024b, Theorem 1.3; Das et al. 2023, Theorem 1.5; Chen et al. 2023, Theorem 2.4.7]. The rest of the proof involves a series of changes of models that preserve the moduli part of the canonical bundle formula, relying on the fact that for lc-trivial fibrations which are crepant over the generic point of the base, the moduli parts of the canonical bundle formulas are the same (see Lemma 2.14).

2. Preliminaries

Notations and definitions. We adopt the standard notations and definitions from [Kollár and Mori 1998; Birkar et al. 2010] and use them freely. For foliations and generalized foliated quadruples, we follow [Chen et al. 2023], which generally aligns with the notations and definitions from [Cascini and Spicer 2020; 2021; Ambro et al. 2021], but there may be minor differences. For $\boldsymbol{b}$-divisors and generalized pairs, we follow the notations and definitions from [Birkar and Zhang 2016; Han and Li 2022; Hacon and Liu 2023]. For the canonical bundle formula, we adhere to [Chen et al. 2023], which is generally consistent with the classical definitions. For the reader's convenience, we provide the following notations and definitions that are either not commonly used in the literature or have minor differences from the classical definitions:

Definition 2.1. Let m be a positive integer and $\boldsymbol{v} \in \mathbb{R}^m$. The *rational envelope* of $\boldsymbol{v}$ is the minimal rational affine subspace of $\mathbb{R}^m$ which contains $\boldsymbol{v}$. For example, if $m = 2$ and $\boldsymbol{v} = \left(\frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2}\right)$, then the rational envelope of $\boldsymbol{v}$ is $(x_1 + x_2 = 1) \subset \mathbb{R}^2_{x_1x_2}$.

For any set of nonnegative real numbers Γ, we define

$$\Gamma_+ := \left(\left\{\sum \gamma_i \,\middle|\, \gamma_i \in \Gamma\right\} \cup \{0\}\right) \cap [0, 1], \quad D(\Gamma) := \left\{\frac{m-1+\gamma}{m} \,\middle|\, m \in \mathbb{N}^+, \gamma \in \Gamma_+\right\}.$$

For any real number t and $\mathbb{R}$-divisor $D = \sum d_i D_i$, where D_i are the irreducible components of D, we define $D^{\leq t} := \sum_{d_i \leq t} d_i D_i$ and $D^{>t} := \sum_{d_i > t} d_i D_i$.

Definition 2.2 (lc-trivial fibration [Chen et al. 2023, Definition 11.3.1]). Let (X, B) be a subpair and $f : X \to Z$ a contraction. We say that $f : (X, B) \to Z$ is an *lc-trivial fibration* if

(1) (X, B) is sub-lc over the generic point of Z,

(2) $K_X + B \sim_{\mathbb{R},Z} 0$, and

(3) there exists a birational morphism $h : Y \to X$ with $K_Y + B_Y = h^*(K_X + B)$ such that $-B_Y^{\leq 0}$ is $\mathbb{R}$-Cartier and
$$\kappa_\sigma(Y/Z, -B_Y^{\leq 0}) = 0.$$

We remark that the classical definition of an lc-trivial fibration replaces condition (3) with

(3′) $\operatorname{rank} f_*\mathcal{O}_X(\lceil \boldsymbol{A}^*(X, B)\rceil) = 1$

[Kawamata 1998; Ambro 2004; Kollár 2007; Fujino and Gongyo 2014]. It is worth mentioning that, in this paper, we only consider lc-trivial fibrations $f : (X, B) \to Z$ such that $B \geq 0$ over the generic point of Z. In this case, both (3) and (3′) automatically hold, so there will be no confusion in the notation.

Definition 2.3 (discriminant and moduli parts [Ambro et al. 2021, Definition 2.3]). Let (X, B) be a subpair and $f : X \to Z$ a contraction such that (X, B) is generically sub-lc$/Z$. In the following, we fix a choice of K_X and a choice of K_Z, and suppose that for any birational morphisms $g : \bar{X} \to X$ and $g_Z : \bar{Z} \to Z$, $K_{\bar{X}}$ and $K_{\bar{Z}}$ are chosen as the Weil divisors such that $g_*K_{\bar{X}} = K_X$ and $(g_Z)_*K_{\bar{Z}} = K_Z$.

For any prime divisor D on Z, we define

$$b_D(X, B; f) := 1 - \sup\{t \mid (X, B + tf^*D) \text{ is sub-lc over the generic point of } D\}.$$

Although f^*D may not be well-defined everywhere, it is at least defined near the generic point of D, which suffices for our purposes. We then define the *discriminant part* of $f : (X, B) \to Z$ as

$$B_Z := \sum_{D \text{ is a prime divisor on } Z} b_D(X, B; f)D.$$

Next, we define the trace moduli part of f. By Definition-Theorem 2.6, there exists an equidimensional contraction $f' : X' \to Z'$ associated with birational morphisms $h' : X' \to X$ and $h'_Z : Z' \to Z$ such that Z' is smooth and $f \circ h' = h'_Z \circ f'$. Write $K_{X'} + B' := h'^*(K_X + B)$, and let $B_{Z'}$ be the discriminant part of $f' : (X', B') \to Z'$. Define

$$M_{X'} := K_{X'} + B' - f'^*(K_{Z'} + B_{Z'}).$$

We then define the *trace moduli part* of $f : (X, B) \to Z$ as

$$M_X := h'_* M_{X'}.$$

It is easy to check that M_X does not depend on the choice of f'.

By construction, there exist $\boldsymbol{b}$-divisors $\boldsymbol{B}$ on Z and $\boldsymbol{M}$ on X satisfying the following. For any contraction $f'' : X'' \to Z''$ associated with birational morphisms $h'' : X'' \to X$ and $h''_Z : Z'' \to Z$ such that $f \circ h'' = h''_Z \circ f''$, $\boldsymbol{B}_{Z''}$ is the discriminant part of $f'' : (X'', B'') \to Z''$, and $\boldsymbol{M}_{X''}$ is the trace moduli part of $f'' : (X'', B'') \to Z''$, where $K_{X''} + B'' := h''^*(K_X + B)$. We call $\boldsymbol{B}$ the *discriminant* $\boldsymbol{b}$*-divisor* of $f : (X, B) \to Z$ and $\boldsymbol{M}$ the *moduli part* of $f : (X, B) \to Z$. By construction, $\boldsymbol{B}$ is uniquely determined, and $\boldsymbol{M}$ is uniquely determined for any fixed choices of K_X and K_Z.

Remark 2.4 (base moduli part). The moduli part $\boldsymbol{M}$ defined in Definition 2.3 follows the same notation as in [Ambro et al. 2021], which is defined on X rather than on the base Z.

For any lc-trivial fibration $f : (X, B) \to Z$, the canonical bundle formula indicates that

$$K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + \boldsymbol{M}^Z_Z),$$

where B_Z is the discriminant part and $\boldsymbol{M}^Z$ is a $\boldsymbol{b}$-divisor. Such $\boldsymbol{M}^Z$ is also called the "moduli part" of $f : (X, B) \to Z$ in many references. To avoid any confusion, we call such $\boldsymbol{M}^Z$ the *base moduli part* of $f : (X, B) \to Z$.

It is clear that $\boldsymbol{M} \sim_{\mathbb{R}} f^*\boldsymbol{M}^Z$ as $\boldsymbol{b}$-divisors. Moreover, for lc-trivial fibrations, the effective basepoint-freeness and the effective Γ-basepoint-freeness of the moduli part are equivalent to those of the base moduli part.

Remark 2.5. We can similarly define lc-trivial fibrations, discriminant parts, and base moduli parts for foliations. We refer the reader to [Chen et al. 2023, Definition 11.3.1, Definition-Lemma 11.5.1] for details. We do not need them in the rest of the paper.

Definition-Theorem 2.6 [Ambro et al. 2021, Theorem 2.2; Liu et al. 2023, Definition-Theorem 6.5]. Let X be a normal quasiprojective variety, $X \to U$ a projective morphism, $X \to Z$ a contraction, and B an $\mathbb{R}$-divisor on X. Then there exist a toroidal pair $(X', \Sigma_{X'})/U$, a log smooth pair $(Z', \Sigma_{Z'})$, and a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow{h_Z} & Z \end{array}$$

satisfying the following:

(1) h and h_Z are projective birational morphisms.

(2) $f' : (X', \Sigma_{X'}) \to (Z', \Sigma_{Z'})$ is a toroidal contraction.

(3) $\operatorname{Supp}(h_*^{-1}B) \cup \operatorname{Supp}\operatorname{Exc}(h)$ is contained in $\operatorname{Supp}\Sigma_{X'}$.

(4) X' has at most toric quotient singularities.

(5) f' is equidimensional.

(6) X' is $\mathbb{Q}$-factorial klt.

We call any such $f' : (X', \Sigma_{X'}) \to (Z', \Sigma_{Z'})$ (associated with h and h_Z) which satisfies (1–6) an *equidimensional model* of $f : (X, B) \to Z$.

Definition 2.7 (foliated log smooth [Ambro et al. 2021, §3.2; Das et al. 2023, Definition 2.17]). Let $(X, \mathcal{F}, B)$ be a foliated triple such that $\mathcal{F}$ is algebraically integrable. We say that $(X, \mathcal{F}, B)$ is *foliated log smooth* if there exists a contraction $f : X \to Z$ satisfying the following.

(1) X has at most quotient toric singularities.

(2) $\mathcal{F}$ is induced by f.

(3) (X, Σ_X) is toroidal for some reduced divisor Σ_X such that $\operatorname{Supp} B \subset \Sigma_X$. In particular, $(X, \operatorname{Supp} B)$ is toroidal, and X is $\mathbb{Q}$-factorial klt.

(4) There exists a log smooth pair (Z, Σ_Z) such that
$$f : (X, \Sigma_X) \to (Z, \Sigma_Z)$$
is an equidimensional toroidal contraction.

Definition 2.8 (weak ACSS [Das et al. 2023, Definitions 4.1–4.3]). Let $(X, \mathcal{F}, B)$ be a foliated triple, G a reduced divisor on X, and $f : X \to Z$ a contraction. We say that $(X, \mathcal{F}, B; G)/Z$ is *weak ACSS* if the following conditions hold:

(1) $(X, \mathcal{F}, B)$ is lc.

(2) $(Z, f(G))$ is log smooth and $G = f^{-1}(f(G))$.

(3) f is equidimensional and $\mathcal{F}$ is induced by f.

(4) For any closed point $z \in Z$ and any reduced divisor $\Sigma \geq f(G)$ on Z such that (Z, Σ) is log smooth near z,
$$(X, B + G + f^*(\Sigma - f(G)))$$
is lc over a neighborhood of z.

We say that $(X, \mathcal{F}, B)$ is *weak ACSS* if $(X, \mathcal{F}, B; G)/Z$ is weak ACSS for some reduced divisor G on X and some contraction $f : X \to Z$.

Results on foliations with weak ACSS singularities.

Lemma 2.9 [Chen et al. 2023, Lemma 7.3.3]. *Let $(X, \mathcal{F}, B)$ be a foliated triple such that $\mathcal{F}$ is algebraically integrable. Suppose that $(X, \mathcal{F}, B)$ is foliated log smooth, associated with a contraction* $f : (X, \Sigma_X) \to (Z, \Sigma_Z)$ (as in Definition 2.7(3)). *Let G be the vertical$/Z$ part of Σ_X and B^h the horizontal$/Z$ part of B. Then $(X, \mathcal{F}, \operatorname{Supp} B^h; G)/Z$ is weak ACSS.*

Proposition 2.10 (cf. [Ambro et al. 2021, Proposition 3.6; Chen et al. 2023, Proposition 7.3.6]). *Let $(X, \mathcal{F}, B)$ be a foliated triple, $f : X \to Z$ a contraction, and G a reduced divisor on X such that $(X, \mathcal{F}, B; G)/Z$ is weak ACSS. Let $\boldsymbol{M}$ be the moduli part of $f : (X, B+G) \to Z$. Then*:

(1) $K_{\mathcal{F}} + B \sim \boldsymbol{M}_X$.

(2) $K_{\mathcal{F}} + B \sim_Z K_X + B + G$.

Moreover, we can choose $K_{\mathcal{F}}$ depending only on the choices of K_X and K_Z such that $K_{\mathcal{F}} + B = \boldsymbol{M}_X$.

Proof. This follows from [Ambro et al. 2021, Proposition 3.6] or [Chen et al. 2023, Proposition 7.3.6]. Note that if we choose $K_{\mathcal{F}} = K_{X/Z} - R$ as in the proof of [Chen et al. 2023, Proposition 7.3.6], then we actually obtain the equality $K_{\mathcal{F}} + B = \boldsymbol{M}_X$. □

Proposition 2.11. *Let $(X, \mathcal{F}, B)/Z$ be a foliated triple and G a reduced divisor on X such that Z is $\mathbb{Q}$-factorial, $(X, \mathcal{F}, B; G)/Z$ is weak ACSS, and*

$$\kappa_\sigma(X/Z, K_{\mathcal{F}} + B) = \kappa_\iota(X/Z, K_{\mathcal{F}} + B) = 0.$$

Then we may run a $(K_{\mathcal{F}} + B)$-MMP$/Z$ with scaling of an ample$/Z$ divisor, which terminates with a model $(X', \mathcal{F}', B')/Z$ such that $K_{\mathcal{F}'} + B' \sim_{\mathbb{R},Z} 0$, where $\mathcal{F}'$ and B' are the pushforwards of $\mathcal{F}$ and B on X', respectively. Moreover, $(X', \mathcal{F}', B'; G')/Z$ is weak ACSS, where G' is the pushforward of G on X'.

Proof. The main part of the proposition follows from [Chen et al. 2023, Proposition 11.2.1]. The "moreover" part follows from [Chen et al. 2023, Lemma 9.1.4]. □

Theorem 2.12 [Das et al. 2023, Theorem 1.5; Chen et al. 2023, Theorem 2.4.7]. *Let r be a positive integer, $v_1^0, \dots, v_m^0$ positive real numbers, and $\boldsymbol{v}_0 := (v_1^0, \dots, v_m^0)$. Then there exists an open set $U \ni \boldsymbol{v}_0$ of the rational envelope of $\boldsymbol{v}_0$ depending only on r and $\boldsymbol{v}_0$ satisfying the following statement*:

Let $\left(X, \mathcal{F}, \sum_{j=1}^m v_j^0 B_j\right)$ be an lc foliated triple such that $\mathcal{F}$ is an algebraically integrable foliation of rank r and $B_j \geq 0$ are distinct Weil divisors. Then $\left(X, \mathcal{F}, \sum_{j=1}^m v_j B_j\right)$ is lc for any $(v_1, \dots, v_m) \in U$.

Theorem 2.13 (cf. [Chen et al. 2023, Theorem 11.1.5]). *Let $(X, \mathcal{F}, B)$ be an lc foliated triple, $f : X \to Z$ a contraction, and G a reduced divisor on X such that $(X, \mathcal{F}, B; G)/Z$ is weak ACSS. Let $\boldsymbol{M}$ be the moduli part of $f : (X, B+G) \to Z$. Then $\boldsymbol{M}$ descends to X.*

Proof. This follows from [Chen et al. 2023, Theorem 11.1.5 and Lemma 5.3.2]. □

Lemma 2.14. *Let (X, B) and (X', B') be two subpairs. Let $f : (X, B) \to Z$ and $f' : (X', B') \to Z'$ be two lc-trivial fibrations over U such that f and f' are birationally equivalent* (*i.e., there exist birational *maps $h : X' \dashrightarrow X$ and $h_Z : Z' \dashrightarrow Z$ such that $f \circ h = h_Z \circ f'$), and (X, B) and (X', B') are crepant over the generic point of Z. Let $\boldsymbol{M}$ and $\boldsymbol{M}^Z$ be the moduli part and the base moduli part of $f : (X, B) \to Z$, respectively, and let $\boldsymbol{M}'$ and $\boldsymbol{M}'^{Z'}$ be the moduli part and the base moduli part of $f' : (X', B') \to Z'$, respectively. Then $\boldsymbol{M} = \boldsymbol{M}'$ for any compatible choices of K_X and K_Z, and $\boldsymbol{M}^Z \sim_{\mathbb{R}} \boldsymbol{M}'^{Z'}$.*

Proof. The proof follows along the same lines as the proof of [Chen et al. 2023, Lemma 11.4.3].

Possibly passing to a common base and resolving the indeterminacy of $h : X' \dashrightarrow X$, we may assume that $f = f'$, $X = X'$, and $Z = Z'$. Replacing $f : X \to Z$ with an equidimensional model, we may assume that f is equidimensional and Z is smooth. Now $K_X + B = K_X + B'$ over the generic point of Z, so $B - B'$ is vertical$/Z$. Since $K_X + B \sim_{\mathbb{R},Z} 0$ and $K_X + B' \sim_{\mathbb{R},Z} 0$, we have $B - B' \sim_{\mathbb{R},Z} 0$, so $B - B' = f^*P$ for some $\mathbb{R}$-divisor P on Z (cf. [Chen et al. 2024, Lemma 2.5]).

Let B_Z and B'_Z be the discriminant parts of $f : (X, B) \to Z$ and $f : (X, B') \to Z$, respectively. By the definition of the discriminant part, $B_Z = B'_Z + P$. Therefore,

$$\boldsymbol{M}_X = K_X + B - f^*(K_Z + B_Z) = K_X + B - f^*(K_Z + B'_Z + P) = K_X + B' - f^*(K_Z + B'_Z) = \boldsymbol{M}'_X.$$

Since we may pass to an arbitrarily high model, we have $\boldsymbol{M} = \boldsymbol{M}'$. By the definition of the base moduli part, we have $\boldsymbol{M}^Z \sim_{\mathbb{R}} \boldsymbol{M}'^{Z'}$. □

Lemma 2.15 (cf. [Han et al. 2021, Lemma 3.8; 2022, Lemma 2.18]). *Let $a_1, \dots, a_k \in (0, 1]$ be real numbers such that $\sum_{i=1}^k a_i = 1$. Let $(X, B_1), \dots, (X, B_k)$ be subpairs and $D \geq 0$ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then*

$$\sum_{i=1}^k a_i \operatorname{lct}(X, B_i; D) \leq \operatorname{lct}\Bigl(X, \sum_{i=1}^k a_i B_i; D\Bigr).$$

Proof. We may assume that $\operatorname{lct}(X, B_i; D) > -\infty$ for any i. For any $1 \leq i \leq k$, let $b_i := \operatorname{lct}(X, B_i; D)$ and $s := \sum_{i=1}^k a_i b_i$. Since $(X, B_i + b_i D)$ is lc for any i and

$$\sum_{i=1}^k a_i B_i + sD = \sum_{i=1}^k a_i (B_i + b_i D),$$

it follows that $\bigl(X, \sum_{i=1}^k a_i B_i + sD\bigr)$ is lc. Thus, the lemma follows. □

3. Uniform rational polytopes for canonical bundle formulas

In this section, we establish the existence of uniform rational polytopes for canonical bundle formulas. We present two versions of this uniform decomposition theorem (Theorems 3.3 and 3.4), whose statements and initial proofs are similar but diverge subsequently. The arguments in Theorem 3.3 are more straightforward and clear from the perspective of uniform decomposition theorems. However, we will apply Theorem 3.4 to prove Theorem 1.4.

Lemma 3.1. *Let X and X' be two normal quasiprojective varieties that are birational to each other. Let $D = \sum_{i=1}^m v_i^0 D_i$ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X and $D' = \sum_{i=1}^m v_i^0 D'_i$ an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X' such* *that D and D' are crepant* (*i.e., there are projective birational morphisms $p : W \to X$ and $q : W \to X'$ such that $p^*D = q^*D'$*), *where D_i and D'_i are $\mathbb{Q}$-divisors. Then for any vector $\boldsymbol{v} = (v_1, \dots, v_m)$ in the rational envelope of $\boldsymbol{v}_0 := (v_1^0, \dots, v_m^0)$ in $\mathbb{R}^m$, $D(\boldsymbol{v}) := \sum_{i=1}^m v_i D_i$ and $D'(\boldsymbol{v}) := \sum_{i=1}^m v_i D'_i$ are crepant.*

Proof. We may write $D = \sum_{i=1}^c r_i \bar{D}_i$ and $D' = \sum_{i=1}^c r_i \bar{D}'_i$, where $\bar{D}_i$ and $\bar{D}'_i$ are $\mathbb{Q}$-divisors and $r_1, \dots, r_c$ are linearly independent over $\mathbb{Q}$. By [Han et al. 2024, Lemma 5.3], $\bar{D}_i$ and $\bar{D}'_i$ are $\mathbb{Q}$-Cartier for each i. Let $p : W \to X$ and $q : W \to X'$ be a common resolution. Then

$$\sum_{i=1}^c r_i p^* \bar{D}_i = p^* D = q^* D' = \sum_{i=1}^c r_i q^* \bar{D}'_i,$$

so

$$\sum_{i=1}^c r_i (p^* \bar{D}_i - q^* \bar{D}'_i) = 0.$$

Thus, $p^* \bar{D}_i = q^* \bar{D}'_i$ for each i. In particular, for any $\boldsymbol{u} = (u_1, \dots, u_c) \in \mathbb{R}^c$, $\bar{D}(\boldsymbol{u}) := \sum_{i=1}^c u_i \bar{D}_i$ and $\bar{D}'(\boldsymbol{u}) := \sum_{i=1}^c u_i \bar{D}'_i$ are crepant. Since for any vector $\boldsymbol{v}$ in the rational envelope of $\boldsymbol{v}_0$, there exists a unique vector $\boldsymbol{u} \in \mathbb{R}^c$ such that $\bar{D}(\boldsymbol{u}) = D(\boldsymbol{v})$ and $\bar{D}'(\boldsymbol{u}) = D'(\boldsymbol{v})$, the lemma follows. □

Theorem 3.2 (uniform weak ACSS rational polytope). *Let r be a positive integer and $v_1^0, \dots, v_m^0$ real numbers. Then there exists an open subset $U \ni \boldsymbol{v}_0$ of the rational envelope of $\boldsymbol{v}_0 := (v_1^0, \dots, v_m^0)$ in $\mathbb{R}^m$ depending only on r and $\boldsymbol{v}_0$ satisfying the following.*

Let $(X, \mathcal{F}, B)$ be an lc foliated triple, $f : X \to Z$ a contraction, and G a reduced divisor on X such that $\operatorname{rank} \mathcal{F} = r$ *and $(X, \mathcal{F}, B; G)/Z$ is weak ACSS. Assume that*

- *$B = \sum_{i=1}^m v_i^0 B_i$, where $B_i \geq 0$ are Weil divisors, and*
- *$B(\boldsymbol{v}) := \sum_{i=1}^m v_i B_i$ for any $\boldsymbol{v} = (v_1, \dots, v_m) \in \mathbb{R}^m$.*

Then $(X, \mathcal{F}, B(\boldsymbol{v}); G)/Z$ is weak ACSS for any $\boldsymbol{v} \in U$.

Proof. We verify all the conditions of Definition 2.8 for $(X, \mathcal{F}, B(\boldsymbol{v}); G)/Z$. Condition (1) follows from Theorem 2.12. Conditions (2) and (3) are obvious. Therefore, we only need to check condition (4). Note that this does not directly follow from [Han et al. 2024, Theorem 5.6] because $\dim X$ is not fixed.

We only need to show that there exists an open subset $U \ni \boldsymbol{v}_0$ of the rational envelope of $\boldsymbol{v}_0$ such that for any closed point $z \in Z$ and any log smooth pair (Z, Σ) such that $\Sigma \geq f(G)$, we have that

$$(X, B(\boldsymbol{v}) + G + f^*(\Sigma - f(G)))$$

is lc over a neighborhood of z for any $\boldsymbol{v} \in U$. Possibly adding components to Σ, we may assume that z is an lc center of (Z, Σ). Let $G_z := G + f^*(\Sigma - f(G))$. Then $G_z = f^{-1}(\Sigma)$, $(X, B(\boldsymbol{v}_0) + G_z)$ is lc over a neighborhood of z, and $(X, B(\boldsymbol{v}_0) + G_z)/Z$ satisfies Property $(*)$ [Ambro et al. 2021, Definition 2.13] over a neighborhood of z.

Let $\Sigma_1, \dots, \Sigma_{\dim Z}$ be the irreducible components of Σ which contain z and let V be an irreducible component of $f^{-1}(z)$. Then V is an irreducible component of $\bigcap_{i=1}^{\dim Z} f^{-1}(\Sigma_i)$. In particular, there exist prime divisors $G_i \subset f^{-1}(\Sigma_i)$ such that V is a component of $\bigcap_{i=1}^{\dim Z} G_i$. Since each G_i is a component of G_z, each G_i is an lc place of $(X, B(\boldsymbol{v}_0)+G_z)$. Thus any component of $\bigcap_{i=1}^n G_i$ is an lc center of $(X, B(\boldsymbol{v}_0)+G_z)$ for any $1\le n\le \dim Z$ [Ambro 2011, Theorem 1.1]. In particular, V is an lc center of $(X, B(\boldsymbol{v}_0)+G_z)$, and there exists a sequence of lc centers

$$V=: V_{\dim Z}\subsetneq V_{\dim Z-1}\subsetneq\cdots\subsetneq V_1 := G_1$$

such that each V_n is an irreducible component of $\bigcap_{i=1}^n G_i$. We let $\nu_n : W_n\to V_n$ be the normalization of V_n and let $\tau_n : V_n\to V_{n-1}$ be the natural inclusions. Then there exist morphisms $\iota_n : W_n\to W_{n-1}$ such that $\tau_n\circ\nu_n=\nu_{n-1}\circ\iota_n$ for any $n\ge 2$. Since f is equidimensional, $\dim V_n=\dim W_n=\dim X-n$ for any n.

For any $\boldsymbol{v}$ that is contained in the rational polytope of $\boldsymbol{v}_0$ and any $0\le n\le \dim Z$, we define an $\mathbb{R}$-divisor $B_n(\boldsymbol{v})$ on W_n in the following way:

- $W_0 := X$ and $B_0(\boldsymbol{v}) := B(\boldsymbol{v})+G_z$.
- Suppose that we have already constructed $B_n(\boldsymbol{v})$ for some $n\le\dim Z-1$ such that the image of W_m in W_n is an lc center of $(W_n, B_n(\boldsymbol{v}_0))$ for any $m>n$, and $\boldsymbol{v}\to B_n(\boldsymbol{v})$ is a $\mathbb{Q}$-affine function from the rational envelope of $\boldsymbol{v}_0$ to $\mathrm{Weil}_{\mathbb{R}}(W_n)$. Since the image of W_{n+1} in W_n is of codimension 1, we may define
$$K_{W_{n+1}}+B_{n+1}(\boldsymbol{v}_0) := (K_{W_n}+B_n(\boldsymbol{v}_0))|_{W_{n+1}}$$
by usual adjunction. Since $\boldsymbol{v}$ is contained in the rational envelope of $\boldsymbol{v}_0$, by [Han et al. 2024, Lemma 5.3], $K_{W_n}+B_n(\boldsymbol{v})$ is $\mathbb{R}$-Cartier for any $\boldsymbol{v}$ that is contained in the rational envelope of $\boldsymbol{v}_0$. Thus there exist uniquely defined $\mathbb{R}$-divisors $B_{n+1}(\boldsymbol{v})$ such that
$$K_{W_{n+1}}+B_{n+1}(\boldsymbol{v})=(K_{W_n}+B_n(\boldsymbol{v}))|_{W_{n+1}}.$$
- By [Kollár 2013, Theorem 4.9(3)], the image of W_m in W_{n+1} is an lc center of $(W_{n+1}, B_{n+1}(\boldsymbol{v}_0))$ for any $m>n+1$. Moreover, since adjunction $D\to D|_{W_{n+1}}$ is a $\mathbb{Q}$-affine function from $\mathrm{Div}_{\mathbb{R}}(W_n)$ to $\mathrm{Div}_{\mathbb{R}}(W_{n+1})$, $\boldsymbol{v}\to B_{n+1}(\boldsymbol{v})$ is a $\mathbb{Q}$-affine function from the rational envelope of $\boldsymbol{v}_0$ to $\mathrm{Weil}_{\mathbb{R}}(W_{n+1})$. Thus we may repeat this process.

We let $W := W_{\dim Z}$ and $B_W(\boldsymbol{v}) := B_{\dim Z}(\boldsymbol{v})$. Let $h : Y\to W$ be a $\mathbb{Q}$-factorial dlt modification of $(W, B_W(\boldsymbol{v}_0))$ and $B_Y(\boldsymbol{v}) := h^{-1}_*B_W(\boldsymbol{v})+E$ for any $\boldsymbol{v}$, where E is the reduced h-exceptional divisor. Then

$$K_Y+B_Y(\boldsymbol{v}_0) := h^*(K_W+B_W(\boldsymbol{v}_0)).$$

By [Han et al. 2024, Lemma 5.4], for any $\boldsymbol{v}$ that is contained in the rational envelope of $\boldsymbol{v}_0$, we have

$$K_Y+B_Y(\boldsymbol{v})=h^*(K_W+B_W(\boldsymbol{v})).$$

Let $\Gamma_0 := \{0, 1, v_1^0, \dots, v_m^0\}$. Then by our construction, the coefficients of $B_Y(\boldsymbol{v}_0)$ belong to $\Gamma := D(\Gamma_0)$. Since Y is $\mathbb{Q}$-factorial, by the ACC for lc thresholds [Hacon et al. 2014, Theorem 1.1], there exists a real number $t<1$ depending only on r and $\boldsymbol{v}_0$ such that $(Y, B'_Y(\boldsymbol{v}_0) := B_Y(\boldsymbol{v}_0)^{\le t}+\mathrm{Supp}\, B_Y^{>t}(\boldsymbol{v}_0))$ is lc.

For any $\boldsymbol{v}$ that is contained in the rational envelope of $\boldsymbol{v}_0$, we define $B'_Y(\boldsymbol{v})$ to be the unique $\mathbb{R}$-divisor such that for any prime divisor D on Y,

$$\operatorname{mult}_D B'_Y(\boldsymbol{v}) := \begin{cases} \operatorname{mult}_D B_Y(\boldsymbol{v}) & \text{if } D \text{ is a component of } B_Y(\boldsymbol{v}_0)^{\leq t}, \\ 1 & \text{if } D \text{ is a component of } B_Y(\boldsymbol{v}_0)^{>t}, \\ 0 & \text{otherwise.} \end{cases}$$

By our construction, $\boldsymbol{v} \to B'_Y(\boldsymbol{v})$ is a $\mathbb{Q}$-affine function from the rational envelope of $\boldsymbol{v}_0$ to $\operatorname{Weil}_{\mathbb{R}}(Y)$. Since $\dim Y = r$ and the coefficients of $B'_Y(\boldsymbol{v}_0)$ belong to a finite set depend only on r and $\boldsymbol{v}_0$, by [Han et al. 2024, Theorem 5.6], there exists an open set $U \ni \boldsymbol{v}_0$ of the rational envelope of $\boldsymbol{v}_0$ depending only on r and $\boldsymbol{v}_0$ such that $(Y, B'_Y(\boldsymbol{v}))$ is lc for any $\boldsymbol{v} \in U$. Possibly shrinking U, we may assume that the coefficients of $B_Y(\boldsymbol{v})$ are ≤ 1 for any $\boldsymbol{v} \in U$.

By our construction, $B'_Y(\boldsymbol{v}) \geq B_Y(\boldsymbol{v}) \geq 0$ for any $\boldsymbol{v} \in U$, so $(Y, B_Y(\boldsymbol{v}))$ is lc for any $\boldsymbol{v} \in U$. Hence $(W, B_W(\boldsymbol{v})) = (W_{\dim Z}, B_{\dim Z}(\boldsymbol{v}))$ is lc for any $\boldsymbol{v} \in U$. Suppose that $(W_n, B_n(\boldsymbol{v}))$ is lc for some $n \geq 1$. Then by inversion of adjunction [Kawakita 2007, Theorem; Hacon 2014, Theorem 1], we have that $(W_{n-1}, B_{n-1}(\boldsymbol{v}))$ is lc near the image of W_n in W_{n-1} for any $\boldsymbol{v} \in U$. Hence, possibly shrinking to a neighborhood of the image of W_n in W_{n-1}, we may assume that $(W_{n-1}, B_{n-1}(\boldsymbol{v}))$ is lc for any $\boldsymbol{v} \in U$. Thus we may repeat this process and deduce that, possibly shrinking X to a neighborhood of V, we have that $(W_0, B_0(\boldsymbol{v})) = (X, B(\boldsymbol{v}) + G_z)$ is lc for any $\boldsymbol{v} \in U$. Since V can be any irreducible component of $f^{-1}(z)$, $(X, B(\boldsymbol{v}) + G_z)$ is lc near $f^{-1}(z)$ for any $\boldsymbol{v} \in U$. Condition (4) follows and we are done. □

It remains interesting to ask whether there exists a uniform ACSS rational polytope, although we do not do this in our paper.

Theorem 3.3 (uniform rational polytope for canonical bundle formula, I). *Let d be a positive integer and $v_1^0, \dots, v_m^0$ real numbers. Then there exists an open subset $U \ni \boldsymbol{v}_0$ of the rational envelope of $\boldsymbol{v}_0 := (v_1^0, \dots, v_m^0)$ in $\mathbb{R}^m$, depending only on d and $\boldsymbol{v}_0$, such that the following conditions hold. Assume that*

- *$f : (X, B) \to Z$ is an lc-trivial fibration with* $\dim X = d$,
- *(X, B) is lc,*
- *$B = \sum_{i=1}^m v_i^0 B_i$, where $B_i \geq 0$ are Weil divisors,*
- *$B(\boldsymbol{v}) := \sum_{i=1}^m v_i B_i$ for any $\boldsymbol{v} = (v_1, \dots, v_m) \in \mathbb{R}^m$, and*
- *B_Z and $\boldsymbol{M}$ are the discriminant part and the moduli part of $f : (X, B) \to Z$, respectively.*

Then:

(1) *For any $\boldsymbol{v} \in U$, $(X, B(\boldsymbol{v}))$ is lc and $f : (X, B(\boldsymbol{v})) \to Z$ is an lc-trivial fibration.*

(2) *For any vectors $\boldsymbol{v}^1, \dots, \boldsymbol{v}^k$ in U and positive real numbers $a_1, \dots, a_k$ such that $\sum_{i=1}^k a_i = 1$, we have*
$$B_Z\Big(\sum_{i=1}^k a_i \boldsymbol{v}^i\Big) = \sum_{i=1}^k a_i B_Z(\boldsymbol{v}^i) \quad \text{and} \quad \boldsymbol{M}\Big(\sum_{i=1}^k a_i \boldsymbol{v}^i\Big) = \sum_{i=1}^k a_i \boldsymbol{M}(\boldsymbol{v}^i),$$
where $B_Z(\boldsymbol{v})$ and $\boldsymbol{M}(\boldsymbol{v})$ are the discriminant part and the moduli part of $f : (X, B(\boldsymbol{v})) \to Z$.

Proof. **Step 1**. In this step, we consider an equidimensional model of $f:(X,B)\to Z$. We then run an MMP to achieve an auxiliary model X'' and construct an lc-trivial fibration $f'':(X'',\widetilde{B}''^h(\boldsymbol{v})+G'')\to Z'$.

Let $f':(X',\Sigma_{X'})\to(Z',\Sigma_{Z'})$ be an equidimensional model of $f:(X,B)\to Z$ associated with $h:X'\to X$ and $h_Z:Z'\to Z$. Let $\widetilde{B}':=h^{-1}_*B+\operatorname{Supp}\operatorname{Exc}(h)$ with horizontal$/Z'$ part $\widetilde{B}'^h$, and $\widetilde{B}'(\boldsymbol{v}):=h^{-1}_*B(\boldsymbol{v})+\operatorname{Supp}\operatorname{Exc}(h)$ with horizontal$/Z'$ part $\widetilde{B}'^h(\boldsymbol{v})$ for any $\boldsymbol{v}\in\mathbb{R}^m$. Let G' be the vertical$/Z'$ part of $\Sigma_{X'}$. Denote $K_{X'}+B':=h^*(K_X+B)$ and $K_{X'}+B'(\boldsymbol{v}):=h^*(K_X+B(\boldsymbol{v}))$ for any $\boldsymbol{v}\in\mathbb{R}^m$.

Let $\mathcal{F}'$ be the foliation induced by f'. Then $(X',\mathcal{F}',\widetilde{B}'^h)$ is foliated log smooth. By Lemma 2.9, $(X',\mathcal{F}',\widetilde{B}'^h;G')/Z'$ is weak ACSS. Choose $K_{\mathcal{F}'}$ as in Proposition 2.10. Then $K_{X'}-K_{\mathcal{F}'}$ is vertical$/Z'$ by Proposition 2.10. Therefore,

$$\kappa_\sigma(X'/Z',K_{\mathcal{F}'}+\widetilde{B}'^h)=\kappa_\sigma(X'/Z',K_{\mathcal{F}'}+\widetilde{B}')=\kappa_\sigma(X'/Z',K_{X'}+\widetilde{B}')=\kappa_\sigma(X'/Z',K_{X'}+B')=0$$

and

$$\kappa_\iota(X'/Z',K_{\mathcal{F}'}+\widetilde{B}'^h)=\kappa_\iota(X'/Z',K_{\mathcal{F}'}+\widetilde{B}')=\kappa_\iota(X'/Z',K_{X'}+\widetilde{B}')=\kappa_\iota(X'/Z',K_{X'}+B')=0.$$

By Proposition 2.11, we may run a $(K_{\mathcal{F}'}+\widetilde{B}'^h)$-MMP$/Z'$ with scaling of an ample$/Z'$ divisor, which terminates with a model $(X'',\mathcal{F}'',\widetilde{B}''^h)/Z'$ of $(X',\mathcal{F}',\widetilde{B}'^h)/Z'$ such that $K_{\mathcal{F}''}+\widetilde{B}''^h\sim_{\mathbb{R},Z'}0$. Denote by $f'':X''\to Z'$ the induced morphism. For any $\boldsymbol{v}\in\mathbb{R}^m$, let $\widetilde{B}''^h(\boldsymbol{v})$ and G'' be the strict transforms of $\widetilde{B}'^h(\boldsymbol{v})$ and G' on X'', respectively. By Proposition 2.11, $(X'',\mathcal{F}'',\widetilde{B}''^h;G'')/Z'$ is weak ACSS.

By Theorem 2.12, there exists an open subset $U\ni\boldsymbol{v}_0$ of the rational envelope of $\boldsymbol{v}_0$, depending only on d and $\boldsymbol{v}_0$, such that both $(X'',\mathcal{F}'',\widetilde{B}''^h(\boldsymbol{v}))$ and $(X,B(\boldsymbol{v}))$ are lc for any $\boldsymbol{v}\in U$. By [Han et al. 2024, Lemma 5.3], $K_X+B(\boldsymbol{v})\sim_{\mathbb{R},Z}0$, so $f:(X,B(\boldsymbol{v}))\to Z$ is an lc-trivial fibration for any $\boldsymbol{v}\in U$. Then $f':(X',B'(\boldsymbol{v}))\to Z'$ is an lc-trivial fibration with moduli part $\boldsymbol{M}(\boldsymbol{v})$ for any $\boldsymbol{v}\in U$. Since $(X'',\mathcal{F}'',\widetilde{B}''^h;G'')/Z'$ is weak ACSS by Theorem 3.2, possibly shrinking U, we may assume that $(X'',\mathcal{F}'',\widetilde{B}''^h(\boldsymbol{v});G'')/Z'$ is weak ACSS for any $\boldsymbol{v}\in U$. By Proposition 2.10 and [Han et al. 2024, Lemma 5.3],

$$K_{X''}+\widetilde{B}''^h(\boldsymbol{v})+G''\sim_{\mathbb{R},Z'}K_{\mathcal{F}''}+\widetilde{B}''^h(\boldsymbol{v})\sim_{\mathbb{R},Z'}0$$

for any $\boldsymbol{v}\in U$, so $f'':(X'',\widetilde{B}''^h(\boldsymbol{v})+G'')\to Z'$ is an lc-trivial fibration.

Step 2. In this step, we show that we may let $\boldsymbol{M}(\boldsymbol{v})$ be the moduli part of $f'':(X'',\widetilde{B}''^h(\boldsymbol{v})+G'')\to Z'$ and conclude the proof of the theorem.

Let $p:W\to X'$ and $q:W\to X''$ be a common resolution of $\phi:X'\dashrightarrow X''$. By construction, $p^*(K_{X'}+B')\le p^*(K_{X'}+\widetilde{B}'^h)$ and $q^*(K_{\mathcal{F}''}+\widetilde{B}''^h)\le p^*(K_{\mathcal{F}'}+\widetilde{B}'^h)=p^*(K_{X'}+\widetilde{B}'^h)$ over a nonempty open subset Z'° of Z'. Moreover, we can write $p^*(K_{X'}+B')-q^*(K_{\mathcal{F}''}+\widetilde{B}''^h)=E_\phi-E_h$ such that $E_\phi\ge0$ is supported on $p^{-1}(\operatorname{Exc}(\phi))$ and $E_h\ge0$ is supported on $p^{-1}(\operatorname{Exc}(h))$ over Z'°. Since $p^*(K_{X'}+B')-q^*(K_{\mathcal{F}''}+\widetilde{B}''^h)\sim_{\mathbb{R},Z'}0$, by the negativity lemma, $E_\phi=E_h=0$.

Thus, we deduce that $K_{\mathcal{F}''}+\widetilde{B}''^h$ and $K_{X'}+B'$ are crepant over the generic point of Z'. By Lemma 3.1, $K_{\mathcal{F}''}+\widetilde{B}''^h(\boldsymbol{v})$ and $K_{X'}+B'(\boldsymbol{v})$ are crepant over the generic point of Z' for any $\boldsymbol{v}\in U$. Therefore, $K_{X''}+\widetilde{B}''^h(\boldsymbol{v})+G''$ and $K_{X'}+B'(\boldsymbol{v})$ are crepant over the generic point of Z' for any $\boldsymbol{v}\in U$. We let

$\boldsymbol{M}(\boldsymbol{v})$ be the moduli part of $f'' : (X'', \widetilde{B}''^h(\boldsymbol{v}) + G'') \to Z'$. By Theorem 2.13, $\boldsymbol{M}(\boldsymbol{v})$ descends to X'' for any $\boldsymbol{v} \in U$. By Lemma 2.14, $\boldsymbol{M}(\boldsymbol{v})$ is also the moduli part of $f' : (X', B'(\boldsymbol{v})) \to Z'$ for any $\boldsymbol{v} \in U$. Hence $\boldsymbol{M}(\boldsymbol{v})$ is also the moduli part of $f : (X, B(\boldsymbol{v})) \to Z$ for any $\boldsymbol{v} \in U$.

By Proposition 2.10, $\boldsymbol{M}_{X''} = K_{\mathcal{F}''} + \widetilde{B}''^h$ and $\boldsymbol{M}(\boldsymbol{v})_{X''} = K_{\mathcal{F}''} + \widetilde{B}''^h(\boldsymbol{v})$ for any $\boldsymbol{v} \in U$. Therefore, for any vectors $\boldsymbol{v}^1, \dots, \boldsymbol{v}^k$ in U and positive real numbers $a_1, \dots, a_k$ such that $\sum_{i=1}^k a_i = 1$, we have

$$\boldsymbol{M}\Big(\sum_{i=1}^k a_i \boldsymbol{v}^i\Big)_{X''} = K_{\mathcal{F}''} + \widetilde{B}''^h\Big(\sum_{i=1}^k a_i \boldsymbol{v}^i\Big) = \sum_{i=1}^k a_i (K_{\mathcal{F}''} + \widetilde{B}''^h(\boldsymbol{v}^i)) = \sum_{i=1}^k a_i \boldsymbol{M}(\boldsymbol{v}^i).$$

Since $(X'', \mathcal{F}'', \widetilde{B}''^h(\boldsymbol{v}); G'')/Z''$ is weak ACSS for any $\boldsymbol{v} \in U$, by Theorem 2.13, $\boldsymbol{M}(\boldsymbol{v})$ descends to X'' for any $\boldsymbol{v} \in U$. Thus

$$\boldsymbol{M}\Big(\sum_{i=1}^k a_i \boldsymbol{v}^i\Big) = \sum_{i=1}^k a_i \boldsymbol{M}(\boldsymbol{v}^i).$$

Let $\boldsymbol{M}^Z(\boldsymbol{v})$ be the base moduli part of $f : (X, B(\boldsymbol{v})) \to Z$ for any $\boldsymbol{v} \in U$. Then for any $\boldsymbol{v} \in U$, $\boldsymbol{M}^Z(\boldsymbol{v})$ descends to Z' and

$$f''^* \boldsymbol{M}^Z(\boldsymbol{v})_{Z'} \sim_{\mathbb{R}} \boldsymbol{M}(\boldsymbol{v})_{X''}.$$

Therefore,

$$\boldsymbol{M}^Z\Big(\sum_{i=1}^k a_i \boldsymbol{v}^i\Big) \sim_{\mathbb{R}} \sum_{i=1}^k a_i \boldsymbol{M}^Z(\boldsymbol{v}^i)$$

and

$$\sum_{i=1}^k a_i (K_Z + B_Z(\boldsymbol{v}^i) + \boldsymbol{M}^Z(\boldsymbol{v}^i)_Z) \sim_{\mathbb{R}} K_Z + B_Z\Big(\sum_{i=1}^k a_i \boldsymbol{v}^i\Big) + \boldsymbol{M}^Z\Big(\sum_{i=1}^k a_i \boldsymbol{v}^i\Big)_Z.$$

Thus,

$$\sum_{i=1}^k a_i B_Z(\boldsymbol{v}^i) \sim_{\mathbb{R}} B_Z\Big(\sum_{i=1}^k a_i \boldsymbol{v}^i\Big).$$

By the definition of the discriminant part and Lemma 2.15,

$$\sum_{i=1}^k a_i B_Z(\boldsymbol{v}^i) \geq B_Z\Big(\sum_{i=1}^k a_i \boldsymbol{v}^i\Big),$$

so

$$\sum_{i=1}^k a_i B_Z(\boldsymbol{v}^i) = B_Z\Big(\sum_{i=1}^k a_i \boldsymbol{v}^i\Big),$$

and we are done. □

Theorem 3.4 (uniform rational polytope for canonical bundle formula, II). *Let d be a positive integer and $v_1^0, \dots, v_m^0$ real numbers. Then there exists an open subset $U \ni \boldsymbol{v}_0$ of the rational envelope of $\boldsymbol{v}_0 := (v_1^0, \dots, v_m^0)$ in $\mathbb{R}^m$, depending only on d and $\boldsymbol{v}_0$, such that the following conditions hold. Assume that*

- *$f : (X, B) \to Z$ is an lc-trivial fibration with* $\dim X - \dim Z = d$,
- *$B = B^h + B^v$, where B^h and B^v are the horizontal$/Z$ part and the vertical$/Z$ part of B, respectively,*

- $B^h=\sum_{i=1}^m v_i^0 B_i^h$, *where* $B_i^h\geq 0$ *are Weil divisors,*
- $B^h(\boldsymbol{v}):=\sum_{i=1}^m v_i B_i^h$ *for any* $\boldsymbol{v}=(v_1,\dots,v_m)\in\mathbb{R}^m$,
- $B^v=\sum_{i=1}^n u_i^0 B_i^v$ *for some real numbers* u_i^0 *and Weil divisors* B_i^v, *and*
- B_Z *and* $\boldsymbol{M}$ *are the discriminant part and the moduli part of* $f:(X,B)\to Z$, *respectively.*

Then there exist $\mathbb{R}$*-affine functions* $s_1,\dots,s_n:\mathbb{R}^m\to\mathbb{R}$ *satisfying the following.*

Let $B^v(\boldsymbol{v}):=\sum_{i=1}^n s_i(\boldsymbol{v})B_i^v$ *and* $B(\boldsymbol{v}):=B^h(\boldsymbol{v})+B^v(\boldsymbol{v})$ *for any* $\boldsymbol{v}\in\mathbb{R}^m$. *Then*:

(1) *For any* $\boldsymbol{v}\in U$, $f:(X,B(\boldsymbol{v}))\to Z$ *is an lc-trivial fibration.*

(2) *For any vectors* $\boldsymbol{v}^1,\dots,\boldsymbol{v}^k$ *in* U *and positive real numbers* $a_1,\dots,a_k$ *such that* $\sum_{i=1}^k a_i=1$, *we have*

$$B_Z\Bigl(\sum_{i=1}^k a_i\boldsymbol{v}^i\Bigr)=\sum_{i=1}^k a_iB_Z(\boldsymbol{v}^i)\quad\text{and}\quad \boldsymbol{M}\Bigl(\sum_{i=1}^k a_i\boldsymbol{v}^i\Bigr)=\sum_{i=1}^k a_i\boldsymbol{M}(\boldsymbol{v}^i),$$

where $B_Z(\boldsymbol{v})$ *and* $\boldsymbol{M}(\boldsymbol{v})$ *are the discriminant part and the moduli part of* $f:(X,B(\boldsymbol{v}))\to Z$.

Proof. **Step 1**. In this step we construct $s_1,\dots,s_n$.

Let V be the rational envelope of $(v_1^0,\dots,v_m^0,u_1^0,\dots,u_n^0)\subset\mathbb{R}^{m+n}$ and let V_m be the image of V in $\mathbb{R}^m$ under the projection $\pi_m:\mathbb{R}^{m+n}\to\mathbb{R}^m:(x_1,\dots,x_{m+n})\to(x_1,\dots,x_m)$. Then V_m is the rational envelope of $\boldsymbol{v}_0$, and there exists an affine function $\tau:V_m\to V$ such that $\pi_m\circ\tau$ is the identity morphism. Now τ naturally extends to an affine function $\beta:\mathbb{R}^m\to V\subset\mathbb{R}^{m+n}$. Let $\pi_n:\mathbb{R}^{m+n}\to\mathbb{R}^n:(x_1,\dots,x_{m+n})\to(x_{m+1},\dots,x_n)$ be the projection and let $s:=\pi_n\circ\beta$. Then we may write

$$s(\boldsymbol{v}):=(s_1(\boldsymbol{v}),\dots,s_n(\boldsymbol{v})),$$

where $s_1,\dots,s_n:\mathbb{R}^m\to\mathbb{R}$ are $\mathbb{R}$-affine functions.

Step 2. This step is almost identical to **Step 1** of the proof of Theorem 3.3, with minor differences. In this step, we consider an equidimensional model of $f:(X,B)\to Z$, run an MMP to achieve an auxiliary model X'', and construct an lc-trivial fibration $f'':(X'',\widetilde{B}''^h(\boldsymbol{v})+G'')\to Z'$.

Let $f':(X',\Sigma_{X'})\to(Z',\Sigma_{Z'})$ be an equidimensional model of $f:(X,B)\to Z$ associated with $h:X'\to X$ and $h_Z:Z'\to Z$. Let $\widetilde{B}':=h_*^{-1}B+\operatorname{Supp}\operatorname{Exc}(h)$ with horizontal$/Z'$ part $\widetilde{B}'^h$, and $\widetilde{B}'(\boldsymbol{v}):=h_*^{-1}B(\boldsymbol{v})+\operatorname{Supp}\operatorname{Exc}(h)$ with horizontal$/Z'$ part $\widetilde{B}'^h(\boldsymbol{v})$ for any $\boldsymbol{v}\in\mathbb{R}^m$. Let G' be the vertical$/Z'$ part of $\Sigma_{X'}$. Denote $K_{X'}+B':=h^*(K_X+B)$ and $K_{X'}+B'(\boldsymbol{v}):=h^*(K_X+B(\boldsymbol{v}))$ for any $\boldsymbol{v}\in\mathbb{R}^m$.

Let $\mathcal{F}'$ be the foliation induced by f'. Then $(X',\mathcal{F}',\widetilde{B}'^h)$ is foliated log smooth. By Lemma 2.9, $(X',\mathcal{F}',\widetilde{B}'^h;G')/Z'$ is weak ACSS. Choose $K_{\mathcal{F}'}$ as in Proposition 2.10. Then $K_{X'}-K_{\mathcal{F}'}$ is vertical$/Z'$ by Proposition 2.10. Therefore,

$$\kappa_\sigma(X'/Z',K_{\mathcal{F}'}+\widetilde{B}'^h)=\kappa_\sigma(X'/Z',K_{\mathcal{F}'}+\widetilde{B}')=\kappa_\sigma(X'/Z',K_{X'}+\widetilde{B}')=\kappa_\sigma(X'/Z',K_{X'}+B')=0,$$

and

$$\kappa_\iota(X'/Z',K_{\mathcal{F}'}+\widetilde{B}'^h)=\kappa_\iota(X'/Z',K_{\mathcal{F}'}+\widetilde{B}')=\kappa_\iota(X'/Z',K_{X'}+\widetilde{B}')=\kappa_\iota(X'/Z',K_{X'}+B')=0.$$

By Proposition 2.11, we may run a $(K_{\mathcal{F}'}+\widetilde{B}'^h)$-MMP$/Z'$ with scaling of an ample$/Z'$ divisor, which terminates with a model $(X'',\mathcal{F}'',\widetilde{B}''^h)/Z'$ of $(X',\mathcal{F}',\widetilde{B}'^h)/Z'$ such that $K_{\mathcal{F}''}+\widetilde{B}''^h\sim_{\mathbb{R},Z'}0$. Denote by $f'':X''\to Z'$ the induced morphism. For any $\boldsymbol{v}\in\mathbb{R}^m$, let $\widetilde{B}''^h(\boldsymbol{v})$ and G'' be the strict transforms of $\widetilde{B}'^h(\boldsymbol{v})$ and G' on X'', respectively. By Proposition 2.11, $(X'',\mathcal{F}'',\widetilde{B}''^h;G'')/Z'$ is weak ACSS.

By Theorem 2.12, there exists an open subset $U\ni\boldsymbol{v}_0$ of the rational envelope of $\boldsymbol{v}_0$ depending only on d and $\boldsymbol{v}_0$ such that $(X'',\mathcal{F}'',\widetilde{B}''^h(\boldsymbol{v}))$ is lc for any $\boldsymbol{v}\in U$. Since $(X'',\mathcal{F}'',\widetilde{B}''^h;G'')/Z'$ is weak ACSS by Theorem 3.2, possibly shrinking U, we may assume that $(X'',\mathcal{F}'',\widetilde{B}''^h(\boldsymbol{v});G'')/Z'$ is weak ACSS for any $\boldsymbol{v}\in U$. By Proposition 2.10 and [Han et al. 2024, Lemma 5.3],

$$K_{X''}+\widetilde{B}''^h(\boldsymbol{v})+G''\sim_{\mathbb{R},Z'}K_{\mathcal{F}''}+\widetilde{B}''^h(\boldsymbol{v})\sim_{\mathbb{R},Z'}0$$

for any $\boldsymbol{v}\in U$, so $f'':(X'',\widetilde{B}''^h(\boldsymbol{v})+G'')\to Z'$ is an lc-trivial fibration.

Step 3. This step is almost identical to **Step 2** of the proof of Theorem 3.3 with minor differences. In this step, we show that we may let $\boldsymbol{M}(\boldsymbol{v})$ be the moduli part of $f'':(X'',\widetilde{B}''^h(\boldsymbol{v})+G'')\to Z'$ and conclude the proof of the theorem.

Let $p:W\to X'$ and $q:W\to X''$ be a common resolution of $\phi:X'\dashrightarrow X''$. By construction, $p^*(K_{X'}+B')\le p^*(K_{X'}+\widetilde{B}'^h)$ and $q^*(K_{\mathcal{F}''}+\widetilde{B}''^h)\le p^*(K_{\mathcal{F}'}+\widetilde{B}'^h)=p^*(K_{X'}+\widetilde{B}'^h)$ over an open subset Z'° of Z'. Moreover, we can write $p^*(K_{X'}+B')-q^*(K_{\mathcal{F}''}+\widetilde{B}''^h)=E_\phi-E_h$ such that $E_\phi\ge 0$ is supported on $p^{-1}(\mathrm{Exc}(\phi))$ and $E_h\ge 0$ is supported on $p^{-1}(\mathrm{Exc}(h))$ over Z'°. Since

$$p^*(K_{X'}+B')-q^*(K_{\mathcal{F}''}+\widetilde{B}''^h)\sim_{\mathbb{R},Z'}0,$$

by the negativity lemma, $E_\phi=E_h=0$.

Thus, $K_{\mathcal{F}''}+\widetilde{B}''^h$ and $K_{X'}+B'$ are crepant over the generic point of Z'. By Lemma 3.1, $K_{\mathcal{F}''}+\widetilde{B}''^h(\boldsymbol{v})$ and $K_{X'}+B'(\boldsymbol{v})$ are crepant over the generic point of Z' for any $\boldsymbol{v}\in U$. Therefore, $K_{X''}+\widetilde{B}''^h(\boldsymbol{v})+G''$ and $K_{X'}+B'(\boldsymbol{v})$ are crepant over the generic point of Z' for any $\boldsymbol{v}\in U$. We let $\boldsymbol{M}(\boldsymbol{v})$ be the moduli part of $f'':(X'',\widetilde{B}''^h(\boldsymbol{v})+G'')\to Z'$. By Theorem 2.13, $\boldsymbol{M}(\boldsymbol{v})$ descends to X'' for any $\boldsymbol{v}\in U$. By Lemma 2.14, $\boldsymbol{M}(\boldsymbol{v})$ is also the moduli part of $f':(X',B'(\boldsymbol{v}))\to Z'$ for any $\boldsymbol{v}\in U$. Hence $\boldsymbol{M}(\boldsymbol{v})$ is also the moduli part of $f:(X,B(\boldsymbol{v}))\to Z$ for any $\boldsymbol{v}\in U$.

By Proposition 2.10, $\boldsymbol{M}_{X''}=K_{\mathcal{F}''}+\widetilde{B}''^h$ and $\boldsymbol{M}(\boldsymbol{v})_{X''}=K_{\mathcal{F}''}+\widetilde{B}''^h(\boldsymbol{v})$ for any $\boldsymbol{v}\in U$. Therefore, for any vectors $\boldsymbol{v}^1,\dots,\boldsymbol{v}^k$ in U and positive real numbers $a_1,\dots,a_k$ such that $\sum_{i=1}^k a_i=1$, we have

$$\boldsymbol{M}\Big(\sum_{i=1}^k a_i\boldsymbol{v}^i\Big)_{X''}=K_{\mathcal{F}''}+\widetilde{B}''^h\Big(\sum_{i=1}^k a_i\boldsymbol{v}^i\Big)=\sum_{i=1}^k a_i(K_{\mathcal{F}''}+\widetilde{B}''^h(\boldsymbol{v}^i))=\sum_{i=1}^k a_i\boldsymbol{M}(\boldsymbol{v}^i).$$

Since $(X'',\mathcal{F}'',\widetilde{B}''^h(\boldsymbol{v});G'')/Z''$ is weak ACSS for any $\boldsymbol{v}\in U$, by Theorem 2.13, $\boldsymbol{M}(\boldsymbol{v})$ descends to X'' for any $\boldsymbol{v}\in U$. Thus

$$\boldsymbol{M}\Big(\sum_{i=1}^k a_i\boldsymbol{v}^i\Big)=\sum_{i=1}^k a_i\boldsymbol{M}(\boldsymbol{v}^i).$$

Let $\boldsymbol{M}^Z(\boldsymbol{v})$ be the base moduli part of $f:(X, B(\boldsymbol{v}))\to Z$ for any $\boldsymbol{v}\in U$. Then for any $\boldsymbol{v}\in U$, $\boldsymbol{M}^Z(\boldsymbol{v})$ descends to Z' and

$$f''^{*}\boldsymbol{M}^Z(\boldsymbol{v})_{Z'}\sim_{\mathbb{R}}\boldsymbol{M}(\boldsymbol{v})_{X''}.$$

Therefore,

$$\boldsymbol{M}^Z\left(\sum_{i=1}^k a_i\boldsymbol{v}^i\right)\sim_{\mathbb{R}}\sum_{i=1}^k a_i\boldsymbol{M}^Z(\boldsymbol{v}^i)$$

and

$$\sum_{i=1}^k a_i(K_Z+B_Z(\boldsymbol{v}^i)+\boldsymbol{M}^Z(\boldsymbol{v}^i)_Z)\sim_{\mathbb{R}}K_Z+B_Z\left(\sum_{i=1}^k a_i\boldsymbol{v}^i\right)+\boldsymbol{M}^Z\left(\sum_{i=1}^k a_i\boldsymbol{v}^i\right)_Z.$$

Thus,

$$\sum_{i=1}^k a_iB_Z(\boldsymbol{v}^i)\sim_{\mathbb{R}}B_Z\left(\sum_{i=1}^k a_i\boldsymbol{v}^i\right).$$

By the definition of the discriminant part and Lemma 2.15,

$$\sum_{i=1}^k a_iB_Z(\boldsymbol{v}^i)\geq B_Z\left(\sum_{i=1}^k a_i\boldsymbol{v}^i\right),$$

so

$$\sum_{i=1}^k a_iB_Z(\boldsymbol{v}^i)=B_Z\left(\sum_{i=1}^k a_i\boldsymbol{v}^i\right),$$

and we are done. □

Remark 3.5. The arguments used in Theorems 3.3 and 3.4 also apply to lc-trivial fibrations of generalized pairs, as all results from [Chen et al. 2023] remain applicable. Due to the technical nature of the arguments, we omit detailed statements and proofs here.

Remark 3.6. It is also possible to establish uniform rational polytopes for canonical bundle formulas for lc-trivial fibrations with DCC coefficients, similar to [Li 2024, Theorem 4.1; Chen et al. 2024, Theorem 1.9; 2025, Theorem 1.1]. The proof follows almost identically to those of Theorems 3.3 and 3.4. Again, due to the technical nature of the arguments, we omit detailed statements and proofs here.

Since the moduli part is determined by the horizontal part of B, the following direct consequence of Theorem 1.4 might be useful. Indeed, Corollary 3.7 could also be applied to prove Theorem 1.4 by either [Li 2024, Proposition 3.3] or [Birkar 2021, Lemma 3.5].

Corollary 3.7. *Let d be a positive integer and $v_1^0,\dots,v_m^0$ real numbers. Then there exists an open subset $U\ni\boldsymbol{v}_0$ of the rational envelope of $\boldsymbol{v}_0:=(v_1^0,\dots,v_m^0)$ in $\mathbb{R}^m$, depending only on d and $\boldsymbol{v}_0$, such that the following conditions hold. Assume that*

- *$f:(X,B)\to Z$ is an lc-trivial fibration with* $\dim X-\dim Z=d$,
- *$B=\sum_{i=1}^m v_i^0B_i$, where $B_i\geq 0$ are Weil horizontal/Z divisors,*
- *$B(\boldsymbol{v}):=\sum_{i=1}^m v_iB_i$ for any $\boldsymbol{v}=(v_1,\dots,v_m)\in\mathbb{R}^m$, and*
- *B_Z and $\boldsymbol{M}$ are the discriminant part and the moduli part of $f:(X,B)\to Z$, respectively.*

Then:

(1) *For any $\boldsymbol{v} \in U$, $f : (X, B(\boldsymbol{v})) \to Z$ is an lc-trivial fibration.*

(2) *For any vectors $\boldsymbol{v}^1, \dots, \boldsymbol{v}^k$ in U and positive real numbers $a_1, \dots, a_k$ such that $\sum_{i=1}^k a_i = 1$, we have*

$$B_Z\left(\sum_{i=1}^{k} a_i \boldsymbol{v}^i\right) = \sum_{i=1}^{k} a_i B_Z(\boldsymbol{v}^i) \quad \text{and} \quad \boldsymbol{M}\left(\sum_{i=1}^{k} a_i \boldsymbol{v}^i\right) = \sum_{i=1}^{k} a_i \boldsymbol{M}(\boldsymbol{v}^i),$$

where $B_Z(\boldsymbol{v})$ and $\boldsymbol{M}(\boldsymbol{v})$ are the discriminant part and the moduli part of $f : (X, B(\boldsymbol{v})) \to Z$.

Proof. This is a special case of Theorem 3.4 as there is no vertical/Z part of B. □

When B is a sum of horizontal/Z divisors, Corollary 3.7 is stronger than Theorem 3.3, as Corollary 3.7 requires that $\dim X - \dim Z = d$ while Theorem 3.3 requires that $\dim X = d$.

4. Proof of the main theorem

Proof of Theorem 1.4. It is evident that Conjecture 1.3 in relative dimension d implies Conjecture 1.1 in relative dimension d. Therefore, we only need to show that Conjecture 1.1 in relative dimension d implies Conjecture 1.3 in relative dimension d.

Under the notations and assumptions of Conjecture 1.3, suppose that Conjecture 1.1 holds in relative dimension d. By [Hacon et al. 2014, Theorem 1.5], we may assume that Γ is a finite set $\{v_1^0, \dots, v_m^0\}$ for some nonnegative integer m. Let B^h be the horizontal/Z part of B, and write $B^h = \sum_{i=1}^m v_i^0 B_i^h$, where $v_i^0 \in \Gamma$ for each i and $B_i^h \geq 0$ are Weil divisors. Let $B^h(\boldsymbol{v}) := \sum_{i=1}^m v_i B_i^h$ for any $\boldsymbol{v} := (v_1, \dots, v_m) \in \mathbb{R}^m$, and $\boldsymbol{v}_0 := (v_1^0, \dots, v_m^0)$. Let B_Z and $\boldsymbol{M}$ be the discriminant part and the moduli part of $f : (X, B) \to Z$, respectively.

Let $U \ni \boldsymbol{v}_0$ be an open subset of the rational envelope of $\boldsymbol{v}_0$ as in Theorem 3.4 which depends only on d and $\boldsymbol{v}_0$, and we let $s_1, \dots, s_n$, $B^h(\boldsymbol{v})$, $B^v(\boldsymbol{v})$, $B(\boldsymbol{v})$ be as in Theorem 3.4 for any $\boldsymbol{v} \in U$. Let $k := \dim U + 1$ and $\boldsymbol{v}^1, \dots, \boldsymbol{v}^k \in U \cap \mathbb{Q}^m$ be vectors depending only on d and $\boldsymbol{v}_0$ such that $\boldsymbol{v}_0$ is contained in the interior of the convex hull of $\boldsymbol{v}^1, \dots, \boldsymbol{v}^k$. Then there exist unique real numbers $a_1, \dots, a_k \in (0, 1]$ such that $\sum_{i=1}^k a_i = 1$ and $\sum_{i=1}^k a_i \boldsymbol{v}^i = \boldsymbol{v}_0$. By Theorem 3.4,

- $B^h(\boldsymbol{v}^i)$ is the horizontal/Z part of $B(\boldsymbol{v}^i)$ for each i,
- $f : (X, B(\boldsymbol{v}^i)) \to Z$ is an lc-trivial fibration for each i, and
- $B_Z = \sum_{i=1}^k a_i B_Z(\boldsymbol{v}^i)$ and $\boldsymbol{M} = \sum_{i=1}^k a_i \boldsymbol{M}(\boldsymbol{v}^i)$, where $B_Z(\boldsymbol{v}^i)$ and $\boldsymbol{M}(\boldsymbol{v}^i)$ are the discriminant part and the moduli part of $f : (X, B(\boldsymbol{v}^i)) \to Z$, respectively.

By Conjecture 1.1 in relative dimension d, there exists a positive integer I depending only on d and Γ such that $I\boldsymbol{M}(\boldsymbol{v}^i)$ is basepoint-free. Therefore, $I\boldsymbol{M}$ is $\{a_1, \dots, a_k\}$-basepoint-free. Let $\Gamma_0 := \{a_1, \dots, a_k\}$, and Conjecture 1.3 in relative dimension d follows. □

Proof of Corollary 1.5. This now follows from Theorem 1.4 and [Prokhorov and Shokurov 2009, Theorem 8.1]. □

Acknowledgements

The authors are grateful to Zhan Li for introducing them to Conjecture 1.3. They would also like to thank Guodu Chen, Junpeng Jiao, Fanjun Meng, and Lingyao Xie for useful discussions. The third author extends his thanks to his advisor Christopher D. Hacon for his constant support and many helpful discussions. This work was partially supported by the National Key Research and Development Program of China (#2024YFA1014400, #2023YFA1010600, #2020YFA0713200). The first author was supported by NSFC for Excellent Young Scientists (#12322102). The third author was partially supported by NSF research grants no. DMS-1801851, DMS-1952522, DMS-2301374. The authors extend their gratitude to the referees for their thorough review of this paper and for their numerous valuable comments.

References

[Ambro 2004] F. Ambro, "Shokurov's boundary property", *J. Differential Geom.* **67**:2 (2004), 229–255. MR

[Ambro 2011] F. Ambro, "Basic properties of log canonical centers", pp. 39–48 in *Classification of algebraic varieties*, Eur. Math. Soc., Zürich, 2011. MR

[Ambro et al. 2021] F. Ambro, P. Cascini, V. Shokurov, and C. Spicer, "Positivity of the moduli part", preprint, 2021. arXiv 2111.00423

[Ascher et al. 2023] K. Ascher, D. Bejleri, H. Blum, K. DeVleming, G. Inchiostro, Y. Liu, and X. Wang, "Moduli of boundary polarized Calabi–Yau pairs", preprint, 2023. arXiv 2307.06522

[Birkar 2021] C. Birkar, "Singularities of linear systems and boundedness of Fano varieties", *Ann. of Math.* (2) **193**:2 (2021), 347–405. MR

[Birkar and Zhang 2016] C. Birkar and D.-Q. Zhang, "Effectivity of Iitaka fibrations and pluricanonical systems of polarized pairs", *Publ. Math. Inst. Hautes Études Sci.* **123** (2016), 283–331. MR

[Birkar et al. 2010] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan, "Existence of minimal models for varieties of log general type", *J. Amer. Math. Soc.* **23**:2 (2010), 405–468. MR

[Cascini and Spicer 2020] P. Cascini and C. Spicer, "On the MMP for rank one foliations on threefolds", preprint, 2020. arXiv 2012.11433

[Cascini and Spicer 2021] P. Cascini and C. Spicer, "MMP for co-rank one foliations on threefolds", *Invent. Math.* **225**:2 (2021), 603–690. MR

[Chen et al. 2023] G. Chen, J. Han, J. Liu, and L. Xie, "Minimal model program for algebraically integrable foliations and generalized pairs", preprint, 2023. arXiv 2309.15823

[Chen et al. 2024] G. Chen, J. Han, and J. Liu, "On effective log Iitaka fibrations and existence of complements", *Int. Math. Res. Not.* **2024**:10 (2024), 8329–8349. MR

[Chen et al. 2025] G. Chen, J. Han, and J. Liu, "Uniform rational polytopes for Iitaka dimensions", pp. 43–68 in *Higher dimensional algebraic geometry* (Baltimore, MD, 2022), Lond. Math. Soc. Lect. Note Ser. **489**, Cambridge Univ. Press, 2025. MR

[Das et al. 2023] O. Das, J. Liu, and R. Mascharak, "ACC for lc thresholds for algebraically integrable foliations", preprint, 2023. arXiv 2307.07157

[Fujino and Gongyo 2014] O. Fujino and Y. Gongyo, "On the moduli b-divisors of lc-trivial fibrations", *Ann. Inst. Fourier* (*Grenoble*) **64**:4 (2014), 1721–1735. MR

[Hacon 2014] C. D. Hacon, "On the log canonical inversion of adjunction", *Proc. Edinb. Math. Soc.* (2) **57**:1 (2014), 139–143. MR

[Hacon and Liu 2023] C. D. Hacon and J. Liu, "Existence of flips for generalized lc pairs", *Camb. J. Math.* **11**:4 (2023), 795–828. MR

[Hacon et al. 2014] C. D. Hacon, J. McKernan, and C. Xu, "ACC for log canonical thresholds", *Ann. of Math.* (2) **180**:2 (2014), 523–571. MR

[Han and Li 2022] J. Han and Z. Li, "Weak Zariski decompositions and log terminal models for generalized pairs", *Math. Z.* **302**:2 (2022), 707–741. MR

[Han et al. 2021] J. Han, Z. Li, and L. Qi, "ACC for log canonical threshold polytopes", *Amer. J. Math.* **143**:3 (2021), 681–714. MR

[Han et al. 2022] J. Han, C. Jiang, and Y. Luo, "Shokurov's conjecture on conic bundles with canonical singularities", *Forum Math. Sigma* **10** (2022), art. id. e38. MR

[Han et al. 2024] J. Han, J. Liu, and V. V. Shokurov, "ACC for minimal log discrepancies of exceptional singularities", *Peking Math. J.* (online publication November 2024).

[Jiao et al. 2022] J. Jiao, J. Liu, and L. Xie, "On generalized lc pairs with b-log abundant nef part", preprint, 2022. arXiv 2202.11256

[Kawakita 2007] M. Kawakita, "Inversion of adjunction on log canonicity", *Invent. Math.* **167**:1 (2007), 129–133. MR

[Kawamata 1998] Y. Kawamata, "Subadjunction of log canonical divisors, II", *Amer. J. Math.* **120**:5 (1998), 893–899. MR

[Kollár 2007] J. Kollár, "Kodaira's canonical bundle formula and adjunction", pp. 134–162 in *Flips for* 3*-folds and* 4*-folds*, Oxford Lect. Ser. Math. Appl. **35**, Oxford Univ. Press, 2007. MR

[Kollár 2013] J. Kollár, *Singularities of the minimal model program*, Cambridge Tracts in Math. **200**, Cambridge Univ. Press, 2013. MR

[Kollár 2023] J. Kollár, *Families of varieties of general type*, Cambridge Tracts in Math. **231**, Cambridge Univ. Press, 2023. MR

[Kollár and Mori 1998] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Math. **134**, Cambridge Univ. Press, 1998. MR

[Li 2024] Z. Li, "A variant of the effective adjunction conjecture with applications", *J. Pure Appl. Algebra* **228**:6 (2024), art. id. 107626. MR

[Liu et al. 2023] J. Liu, Y. Luo, and F. Meng, "On global ACC for foliated threefolds", *Trans. Amer. Math. Soc.* **376**:12 (2023), 8939–8972. MR

[Liu et al. 2024a] J. Liu, F. Meng, and L. Xie, "Complements, index theorem, and minimal log discrepancies of foliated surface singularities", *Eur. J. Math.* **10**:1 (2024), art. id. 6. MR

[Liu et al. 2024b] J. Liu, F. Meng, and L. Xie, "Uniform rational polytopes of foliated threefolds and the global ACC", *J. Lond. Math. Soc.* (2) **109**:6 (2024), art. id. e12950. MR

[Prokhorov and Shokurov 2009] Y. G. Prokhorov and V. V. Shokurov, "Towards the second main theorem on complements", *J. Algebraic Geom.* **18**:1 (2009), 151–199. MR

Communicated by János Kollár
Received 2023-12-23 Revised 2024-07-31 Accepted 2024-09-26

hanjingjun@fudan.edu.cn *Shanghai Center for Mathematical Sciences, Fudan University, Shanghai, China*

liujihao@math.pku.edu.cn *Department of Mathematics, Peking University, Beijing, China*

xueqy1121@qq.com *Shanghai Center for Mathematical Sciences, Fudan University, Shanghai, China*

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2025 is US $565/year for the electronic version, and $820/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Algebra & Number Theory

Volume 19 No. 11 2025