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We combine the split torsor method and the hyperbola method for toric varieties to count rational points and Campana points of bounded height on certain subvarieties of toric varieties.

1. Introduction

We combine the split torsor method and the hyperbola method for toric varieties to count rational points and Campana points of bounded height on certain subvarieties of smooth split proper toric varieties. This line of research has been initiated by Blomer and Brüdern [2018] in the setting of diagonal hypersurfaces in products of projective spaces. Other results in this direction include hypersurfaces and complete intersections in products of projective spaces [Schindler 2016], improvements for bihomogeneous hypersurfaces for degree (2, 2) and (1, 2) [Browning and Hu 2019; Hu 2020], as well as generalizations to hypersurfaces in certain toric varieties [Mignot 2015; 2016; 2018].

The versions of the hyperbola method used in all of these articles are rather close to the original [Blomer and Brüdern 2018] for products of projective spaces. In our recent work [Pieropan and Schindler 2024], we established a very general form of the hyperbola method for split toric varieties, in which the height condition can also globally be given by the maximum of several monomials. The goal of this article is to show applications of our new hyperbola method. We develop a refined framework for the split torsor method on split smooth proper toric varieties and show that counting results for subvarieties of projective spaces can be carried over to toric varieties by a direct application of the hyperbola method [Pieropan and Schindler 2024]. With this we can prove new cases of Manin's conjecture [Batyrev and Manin 1990; Franke et al. 1989] on the number of rational points of bounded height on Fano varieties for certain subvarieties in toric varieties.

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The split torsor method provides a parametrization of rational points on Fano varieties via Cox rings [Derenthal and Pieropan 2020; Salberger 1998]. The Cox ring of a smooth proper toric variety X is a polynomial ring endowed with a grading by the Picard group of the toric variety [Cox 1995]. Subvarieties of toric varieties are intersections of hypersurfaces, which are defined by $\mathrm{Pic}(X)$-homogeneous polynomials in the Cox ring of X. The subvarieties considered in this paper are defined by homogeneous elements in the Cox ring of the toric variety such that each polynomial involves only variables of the same degree. With the split torsor method parametrization, the height is given by the maximum of a set of monomials and this is the correct shape to apply our generalized version of the hyperbola method [Pieropan and Schindler 2024]. The hyperbola method reduces the counting problem to counting functions over boxes of different shapes. An advantage of our method is that it is already adapted to the shape of height functions appearing. Also, compared to earlier versions of the hyperbola method, we do not need estimates for lower-dimensional boxes, and with this our proofs are relatively short.

We now illustrate our approach on a number of examples. In a similar fashion, it is possible to apply counting results such as [Birch 1962; Browning and Heath-Brown 2017; Heath-Brown 1996; Rydin Myerson 2018; 2019], and many others, to subvarieties of toric varieties defined by elements of the Cox ring each involving only variables of the same degree.

1.1. *Results.* Let X be a smooth split complete toric $\mathbb{Q}$-variety with open torus T. Let $\boldsymbol{D}_1, \dots, \boldsymbol{D}_s \in \mathrm{Pic}(X)$ be the pairwise distinct classes of the torus-invariant prime divisors on X. For $i \in \{1, \dots, s\}$, let $n_i = \dim_{\mathbb{Q}} H^0(X, \boldsymbol{D}_i)$. Let H_L be the height associated to a semiample torus-invariant divisor L on X as discussed in Section 2.2.

Our first result concerns subvarieties of toric varieties defined by linear forms.

Theorem 1.1. *Let $V \subseteq X$ be a complete intersection of hypersurfaces $H_{i,l}$ with $1 \le i \le s$, $1 \le l \le t_i$. Assume that $[H_{i,l}] = \boldsymbol{D}_i$ in $\mathrm{Pic}(X)$ for $i \in \{1, \dots, s\}$ such that $t_i \neq 0$. Assume that $V \cap T \neq \varnothing$ and $t_i \le n_i - 2$ for all $i \in \{1, \dots, s\}$. Assume that $L = -\big(K_X + \sum_{i=1}^{s} \sum_{l=1}^{t_i} [H_{i,l}]\big)$ is ample. For $B > 0$, let $N_V(B)$ be the number of $\mathbb{Q}$-rational points on $V \cap T$ of H_L-height at most B. Then*

$$N_V(B) = cB(\log B)^{b-1} + O(B(\log B)^{b-2}(\log\log B)^s),$$

where $b = \mathrm{rk}\,\mathrm{Pic}(V)$ and c is a positive constant, which is defined by (3-7) with $k = b-1$, $C_{M,\boldsymbol{d}}$ given by (4-1), and $\varpi_i = n_i - t_i$ for $i \in \{1, \dots, s\}$.

We use this result as a toy example to show how to combine the hyperbola method with the universal torsor method in the context of rather general smooth split toric varieties. We now move on to results which require a deeper understanding of the underlying Diophantine problems via methods from Fourier analysis.

We start with a result that concerns subvarieties of toric varieties defined by bihomogeneous polynomials. It is obtained by combining the framework developed in this paper with the hyperbola method [Pieropan and Schindler 2024] and preliminary counting results in boxes of different side lengths [Schindler 2016].

Theorem 1.2. *Let $V \subseteq X$ be a smooth complete intersection of hypersurfaces $H_1, \dots, H_t$ of the same degree $e_1\boldsymbol{D}_1 + e_2\boldsymbol{D}_2$ in $\operatorname{Pic}(X)$. Assume that $V \cap T \neq \varnothing$, that $n_i - te_i \geq 2$ for $i \in \{1, 2\}$, and that $n_1 + n_2 > \dim V_1^* + \dim V_2^* + 3 \cdot 2^{e_1+e_2} e_1 e_2 t^3$, where $V_1^*, V_2^* \subseteq \mathbb{A}^{n_1+n_2}$ are affine varieties defined in Section 5. Assume that $L = -([K_X] + [H_1 + \cdots + H_t])$ is ample. Then there is an open subset $W \subseteq X$ such that the number $N_{V,W}(B)$ of $\mathbb{Q}$-rational points on $V \cap W \cap T$ of H_L-height at most B satisfies*

$$N_{V,W}(B) = cB(\log B)^{b-1} + O(B(\log B)^{b-2}(\log\log B)^s)$$

for $B > 0$, where $b = \operatorname{rk}\operatorname{Pic}(V)$ and c is defined in (3-7) with $k = b-1$, $C_{M,\boldsymbol{d}}$ given by (5-1), $\varpi_i = n_i - te_i$ for $i \in \{1, 2\}$, and $\varpi_i = n_i$ for $i \in \{3, \dots, s\}$. The constant c is positive if $V(\mathbb{Q}_v) \neq \varnothing$ for all places v of $\mathbb{Q}$.

Theorems 1.1 and 1.2 are compatible with Manin's conjecture [Franke et al. 1989], as $L|_V = -K_V$ by adjunction. The proofs in Sections 4 and 5 yield an asymptotic formula even if we drop the ampleness assumption on L.

Theorem 1.2 as well as work of Mignot [2016; 2018] include the case of certain hypersurfaces in products of projective spaces. However, in comparison to Mignot's work, we do not require the condition that the effective cone of the toric variety is simplicial. An example of a split toric variety with nonsimplicial effective cone — where our theorem applies — is the blow-up at a torus-invariant point of $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times Y$, where n_1 and n_2 are sufficiently large and Y is a split del Pezzo surface of degree 6.

Our last result concerns sets of Campana points in the sense of [Pieropan et al. 2021] for subvarieties defined by diagonal equations. We introduce the following integral models. Let $\mathscr{X}$ be the $\mathbb{Z}$-toric scheme defined by the fan of X. For $i \in \{1, \dots, s\}$, let $\mathscr{D}_{i,1}, \dots, \mathscr{D}_{i,n_i}$ be the torus-invariant prime divisors on $\mathscr{X}$ of class $\boldsymbol{D}_i$.

Theorem 1.3. *Let $V \subseteq X$ be an intersection of hypersurfaces $H_1, \dots, H_t$ such that H_i is defined by a homogeneous diagonal polynomial in the Cox ring of X of degree $e_i\boldsymbol{D}_i$ in $\operatorname{Pic}(X)$ and with none of the coefficients equal to zero. Let $\mathscr{V}$ be the closure of V in $\mathscr{X}$. For $i \in \{1, \dots, s\}$, fix integers $2 \leq m_{i,1} \leq \cdots \leq m_{i,n_i}$. Let $\mathscr{D}_{\boldsymbol{m}} = \sum_{i=1}^{s} \sum_{j=1}^{n_i} (1 - 1/m_{i,j})\mathscr{D}_{i,j}$. Assume that $V \cap T \neq \varnothing$, that $n_1, \dots, n_t \geq 2$, and, for $i \in \{1, \dots, s\}$, that $\sum_{j=1}^{n_i} 1/m_{i,j} > 3$, and that*

$$\sum_{j=1}^{n_i-1} \frac{1}{e_i m_{i,j}(e_i m_{i,j}+1)} \geq 1 \quad \text{if } e_i = 1 \quad \text{and} \quad \sum_{j=1}^{n_i} \frac{1}{2s_0(e_i m_{i,j})} > 1 \quad \text{if } e_i \geq 2,$$

where $s_0(e_i m_{i,j})$ is defined in Lemma 6.1. Let $L = -(K_X + \mathscr{D}_{\boldsymbol{m}}|_X + H_1 + \cdots + H_t)$ be ample. For $B > 0$, let $N_V(B)$ be the number of $\mathbb{Z}$-Campana points on $(\mathscr{V}, \mathscr{D}_{\boldsymbol{m}}|_{\mathscr{V}})$ that lie in T and have H_L-height at most B. Then

$$N_V(B) = cB(\log B)^{b-1} + O(B(\log B)^{b-2}(\log\log B)^s),$$

where $b = \operatorname{rk}\operatorname{Pic}(V)$ and c is defined in (3-7) with $k = b-1$, $C_{M,\boldsymbol{d}}$ given by (6-11), and $\varpi_1, \dots, \varpi_s$ given by (6-10).

The order of growth in Theorem 1.3 is compatible with the Manin-type conjecture for Campana points [Pieropan et al. 2021], as $L|_V$ is the log anticanonical divisor of the pair $(V, \mathscr{D}_{\boldsymbol{m}}|_V)$ by adjunction.

We now give a number of examples where Theorems 1.2 and 1.3 can be applied. Due to the range of application of the circle method, we require the Cox ring of the toric variety to have a large number of variables of the same degree. This holds for toric varieties with several torus-invariant prime divisors of the same degree and for products of such toric varieties. Here are some concrete examples:

- The projective space $\mathbb{P}^n$ has Cox rings with $n+1$ variables of the same degree.
- The blow-up of the projective space $\mathbb{P}^n$ at $l < n+1$ torus-invariant points has Cox rings with $n+1-l$ variables of the same degree.
- The blow-up of a product of toric varieties each with several torus-invariant prime divisors of the same degree. Indeed, if X and Y are smooth split toric varieties such that the Cox ring of X has n_X variables of the same degree d_X, the Cox ring of Y has n_Y variables of the same degree d_Y, and $P \in X \times Y$ is a point where $m_X \leq n_X$ variables of degree d_X vanish and $m_Y \leq n_Y$ variables of degree d_Y vanish, then the Cox ring of the blow-up of $X \times Y$ at P has m_X variables of the same degree $d_X - e$ and m_Y variables of the same degree $d_Y - e$, where e is the class of the exceptional divisor.

The structure of this article is as follows. In Section 2 we reformulate the height function and the multiplicative function μ for Möbius inversion according to the principle of grouping variables of the same degree. In Section 3 we combine the new framework with the hyperbola method developed in [Pieropan and Schindler 2024] to obtain a general counting tool for points of bounded height on subvarieties of toric varieties. Theorems 1.1, 1.2, and 1.3 are proven in Sections 4, 5, and 6, respectively.

2. Toric varieties setting

Here we introduce the geometric setup and notation for the whole paper. We refer the reader to [Salberger 1998, §8] for a concise introduction to toric varieties and their toric models over $\mathbb{Z}$, and to [Cox et al. 2011] for an extensive treatment of toric varieties.

Let Σ be the fan of a complete smooth split toric variety X over a number field $\mathbb{K}$. We denote by $\{\boldsymbol{D}_1, \dots, \boldsymbol{D}_s\} \subseteq \operatorname{Pic}(X)$ the set of degrees of prime torus-invariant divisors of X. For each $i \in \{1, \dots, s\}$, we denote by $D_{i,1}, \dots, D_{i,n_i}$ the torus-invariant divisors of degree $\boldsymbol{D}_i$ and by $\rho_{i,1}, \dots, \rho_{i,n_i}$ the corresponding rays of Σ. Let $\mathcal{I} := \{(i, j) \in \mathbb{N}^2 : 1 \leq i \leq s,\ 1 \leq j \leq n_i\}$. Let $\Sigma_{\max}$ be the set of maximal cones of Σ. For each maximal cone σ of Σ, let $\mathcal{J}_\sigma := \{(i, j) \in \mathcal{I} : \rho_{i,j} \subseteq \sigma\}$, let $\mathcal{I}_\sigma = \mathcal{I} \smallsetminus \mathcal{J}_\sigma$, and let I_σ be the set of indices $i \in \{1, \dots, s\}$ such that $\{(i, 1), \dots, (i, n_i)\} \cap \mathcal{I}_\sigma \neq \varnothing$.

Let $\mathscr{X}$ be the toric scheme defined by Σ over the ring of integers $\mathcal{O}_\mathbb{K}$ of $\mathbb{K}$, and, for each $(i, j) \in \mathcal{I}$, let $\mathscr{D}_{i,j}$ be the closure of $D_{i,j}$ in $\mathscr{X}$.

Let R be the polynomial ring over $\mathcal{O}_\mathbb{K}$ with variables $x_{i,j}$ for $(i, j) \in \mathcal{I}$ and endowed with the $\operatorname{Pic}(X)$-grading induced by assigning degree $\boldsymbol{D}_i$ to the variable $x_{i,j}$ for all $(i, j) \in \mathcal{I}$. For every torus-invariant divisor $D = \sum_{i=1}^{s} \sum_{j=1}^{n_i} a_{i,j} D_{i,j}$ on X and every vector $\boldsymbol{x} = (x_{i,j})_{(i,j)\in\mathcal{I}} \in \mathbb{C}^{\mathcal{I}}$, we write

$$\boldsymbol{x}^D := \prod_{i=1}^{s} \prod_{j=1}^{n_i} x_{i,j}^{a_{i,j}}.$$

By [Salberger 1998, §8], $\mathscr{X}$ has a unique universal torsor $\pi : \mathscr{Y} \to \mathscr{X}$, and $\mathscr{Y} \subseteq \mathbb{A}^{\#\mathcal{I}}_{\mathcal{O}_\mathbb{K}}$ is the open subset whose complement is defined by $\boldsymbol{x}^{D_\sigma} = 0$ for all maximal cones σ of Σ, where $D_\sigma := \sum_{(i,j)\in\mathcal{I}_\sigma} D_{i,j}$ for all $\sigma \in \Sigma_{\max}$.

Let r be the rank of $\mathrm{Pic}(X)$. Let $\mathcal{C}$ be a set of ideals of $\mathcal{O}_\mathbb{K}$ that form a system of representatives for the class group of $\mathbb{K}$. As in [Pieropan and Schindler 2024, §6.1], we fix a basis of $\mathrm{Pic}(X)$, and, for every divisor D on X and every tuple $\mathfrak{c} = (\mathfrak{c}_1, \dots, \mathfrak{c}_r) \in \mathcal{C}^r$, we write $\mathfrak{c}^{[D]} := \prod_{i=1}^r \mathfrak{c}_i^{b_i}$, where $[D] = (b_1, \dots, b_r)$ with respect to the fixed basis of $\mathrm{Pic}(X)$. Then, as in [Pieropan 2016, §2],

$$X(\mathbb{K}) = \mathscr{X}(\mathcal{O}_\mathbb{K}) = \bigsqcup_{\mathfrak{c}\in\mathcal{C}^r} \pi^{\mathfrak{c}}(\mathscr{Y}^{\mathfrak{c}}(\mathcal{O}_\mathbb{K})),$$

where $\pi^{\mathfrak{c}} : \mathscr{Y}^{\mathfrak{c}} \to \mathscr{X}$ is the twist of π defined in [Frei and Pieropan 2016, Theorem 2.7]. The fibers of $\pi|_{\mathscr{Y}^{\mathfrak{c}}(\mathcal{O}_\mathbb{K})}$ are all isomorphic to $(\mathcal{O}_\mathbb{K}^\times)^r$, and $\mathscr{Y}^{\mathfrak{c}}(\mathcal{O}_\mathbb{K}) \subseteq \mathcal{O}_\mathbb{K}^{\mathcal{I}}$ is the subset of points $\boldsymbol{x} \in \bigoplus_{(i,j)\in\mathcal{I}} \mathfrak{c}^{[D_{i,j}]}$ that satisfy

$$\sum_{\sigma\in\Sigma_{\max}} \boldsymbol{x}^{D_\sigma} \mathfrak{c}^{-[D_\sigma]} = \mathcal{O}_\mathbb{K}. \tag{2-1}$$

Let N be the lattice of cocharacters of X. Then $\Sigma \subseteq N \otimes_\mathbb{Z} \mathbb{R}$. For every $(i, j) \in \mathcal{I}$, let $\nu_{i,j}$ be the unique generator of $\rho_{i,j} \cap N$. For every torus-invariant $\mathbb{Q}$-divisor $D = \sum_{i=1}^s \sum_{j=1}^{n_i} a_{i,j} D_{i,j}$ of X and for every $\sigma \in \Sigma_{\max}$, let $u_{\sigma,D} \in \mathrm{Hom}_\mathbb{Z}(N, \mathbb{Q})$ be the character determined by $u_{\sigma,D}(\nu_{i,j}) = a_{i,j}$ for all $(i, j) \in \mathcal{J}_\sigma$, and define $D(\sigma) := D - \sum_{i=1}^s \sum_{j=1}^{n_i} u_{\sigma,D}(\nu_{i,j}) D_{i,j}$. Then D and $D(\sigma)$ are linearly equivalent.

2.1. *Torus-invariant divisors.* We collect properties of toric varieties and their torus-invariant divisors.

Lemma 2.1. (i) *Let $\sigma \in \Sigma_{\max}$.*

(a) *For $i \in I_\sigma$, there is a unique index $j_{i,\sigma} \in \{1, \dots, n_i\}$ such that $(i, j_{i,\sigma}) \in \mathcal{I}_\sigma$. So $\#I_\sigma = \#\mathcal{I}_\sigma = r$.*

(b) *For $i \in I_\sigma$, we have $(i, j') \in \mathcal{J}_\sigma$ for all $j' \in \{1, \dots, n_i\} \smallsetminus \{j_{i,\sigma}\}$.*

(c) *For $i \in \{1, \dots, s\} \smallsetminus I_\sigma$, we have $\{(i, 1), \dots, (i, n_i)\} \subseteq \mathcal{J}_\sigma$.*

Let D be a torus-invariant $\mathbb{Q}$-divisor on X. For $\sigma \in \Sigma_{\max}$, write

$$D(\sigma) = \sum_{i=1}^{s} \sum_{j=1}^{n_i} \alpha_{i,j,\sigma} D_{i,j}.$$

For $i \in \{1, \dots, s\}$, let $\alpha_{i,\sigma} = \sum_{j=1}^{n_i} \alpha_{i,j,\sigma}$.

(ii) *Let $\sigma \in \Sigma_{\max}$. Then $D(\sigma) = \sum_{i\in I_\sigma} \alpha_{i,\sigma} D_{i,j_{i,\sigma}}$.*

(iii) *Let $\sigma, \sigma' \in \Sigma_{\max}$. If there are $i \in I_\sigma$ and $j \in \{1, \dots, n_i\}$ such that $\mathcal{J}_\sigma \cap \mathcal{J}_{\sigma'} = \mathcal{J}_\sigma \smallsetminus \{(i, j)\}$, then $I_\sigma = I_{\sigma'}$ and $\alpha_{i',\sigma} = \alpha_{i',\sigma'}$ for all $i' \in \{1, \dots, s\}$.*

(iv) *Let $\sigma \in \Sigma_{\max}$ and, for every $i \in I_\sigma$, let $j_i \in \{1, \dots, n_i\}$. Then there exists a unique $\sigma' \in \Sigma_{\max}$ such that $I_{\sigma'} = I_\sigma$, $(i, j_i) \in \mathcal{I}_{\sigma'}$ for $i \in I_\sigma$, and $\alpha_{i,\sigma'} = \alpha_{i,\sigma}$ for $i \in \{1, \dots, s\}$.*

(v) *The relation $\sigma \sim \sigma'$ if and only if $I_\sigma = I_{\sigma'}$ defines an equivalence relation on $\Sigma_{\max}$, and the equivalence class of σ has cardinality $\prod_{i\in I_\sigma} n_i$.*

(vi) *Let $\mathcal{J} \subseteq \mathcal{I}$ be minimal with respect to inclusion and such that $\mathcal{J} \cap \mathcal{I}_\sigma \neq \varnothing$ for all $\sigma \in \Sigma_{\max}$. Let $i \in \{1, \dots, s\}$ such that $\{(i, 1), \dots, (i, n_i)\} \cap \mathcal{J} \neq \varnothing$. Then $\{(i, 1), \dots, (i, n_i)\} \subseteq \mathcal{J}$.*

Proof. Part (i) follows from the fact that $[D_{i,j}] = \boldsymbol{D}_i$ for all $j \in \{1, \dots, n_i\}$ and that the set $\{\boldsymbol{D}_i : i \in I_\sigma\}$ is a basis of $\operatorname{Pic}(X)$ by [Cox et al. 2011, Theorem 4.2.8] as X is smooth and proper.

Part (ii) follows from part (i) and the fact that, by construction, $\alpha_{i,j,\sigma} = 0$ whenever $(i, j) \in \mathcal{J}_\sigma$.

For part (iii) we observe that if $\sigma \neq \sigma'$, then $\mathcal{J}_\sigma = (\mathcal{J}_\sigma \cap \mathcal{J}_{\sigma'}) \sqcup \{(i, j)\}$ and $\mathcal{J}_{\sigma'} = (\mathcal{J}_\sigma \cap \mathcal{J}_{\sigma'}) \sqcup \{(i, j_{i,\sigma})\}$, where $j_{i,\sigma}$ is the index defined in part (i). Thus $i \in I_\sigma \cap I_{\sigma'}$, and, for every index $i' \in \{1, \dots, s\}$ with $i' \neq i$, we have

$$\mathcal{J}_\sigma \cap \{(i', 1), \dots, (i', n_{i'})\} = \mathcal{J}_{\sigma'} \cap \{(i', 1), \dots, (i', n_{i'})\} \subseteq \mathcal{J}_\sigma \cap \mathcal{J}_{\sigma'}.$$

Recall that

$$[D(\sigma)] = \sum_{i \in I_\sigma} \alpha_{i,\sigma} \boldsymbol{D}_i \quad \text{and} \quad [D(\sigma')] = \sum_{i \in I_{\sigma'}} \alpha_{i,\sigma'} \boldsymbol{D}_i.$$

Now the result follows as $[D(\sigma)] = [D(\sigma')]$ in $\operatorname{Pic}(X)$ and $\{\boldsymbol{D}_i : i \in I_\sigma\}$ is a basis of $\operatorname{Pic}(X)$.

For part (iv), we write $I_\sigma = \{i_1, \dots, i_r\}$. We construct by induction $\sigma_1, \dots, \sigma_r$ such that, for each $l \in \{1, \dots, r\}$,

$$(i_1, j_{i_1}), \dots, (i_l, j_{i_l}) \in \mathcal{I}_{\sigma_l}, \quad I_{\sigma_l} = I_\sigma, \quad \text{and} \quad \alpha_{i,\sigma_l} = \alpha_{i,\sigma} \ \text{ for all } i \in \{1, \dots, s\}.$$

If $(i_1, j_{i_1}) \in \mathcal{I}_\sigma$, let $\sigma_1 = \sigma$. Otherwise, $(i_1, j_{i_1}) \in \mathcal{J}_\sigma$ and by [Salberger 1998, Lemma 8.9] there is $\sigma_1 \in \Sigma_{\max}$ such that $\mathcal{J}_{\sigma_1} \cap \mathcal{J}_\sigma = \mathcal{J}_\sigma \setminus \{(i_1, j_{i_1})\}$. Since $i_1 \in I_\sigma$, by part (iii) we have $I_{\sigma_1} = I_\sigma$ and $\alpha_{i,\sigma_1} = \alpha_{i,\sigma}$ for all $i \in \{1, \dots, s\}$. Assume that we have constructed σ_{l-1} for given $l \le r$. If $(i_l, j_{i_l}) \in \mathcal{I}_{\sigma_{l-1}}$, let $\sigma_l = \sigma_{l-1}$. Otherwise, $(i_l, j_{i_l}) \in \mathcal{J}_{\sigma_{l-1}}$ and by [Salberger 1998, Lemma 8.9] there is $\sigma_l \in \Sigma_{\max}$ such that $\mathcal{J}_{\sigma_l} \cap \mathcal{J}_{\sigma_{l-1}} = \mathcal{J}_{\sigma_{l-1}} \setminus \{(i_l, j_{i_l})\}$. Since $i_l \in I_{\sigma_{l-1}}$, by part (iii) we have $I_{\sigma_l} = I_{\sigma_{l-1}} = I_\sigma$ and $\alpha_{i,\sigma_l} = \alpha_{i,\sigma_{l-1}} = \alpha_{i,\sigma}$ for all $i \in \{1, \dots, s\}$. Since $(i_1, j_{i_1}), \dots, (i_{l-1}, j_{i_{l-1}}) \in \mathcal{I}_{\sigma_{l-1}}$ and $\mathcal{J}_{\sigma_l} = (\mathcal{J}_{\sigma_{l-1}} \cap \mathcal{J}_{\sigma_l}) \cup \{(i_l, j_{i_l,\sigma_{l-1}})\}$, where $j_{i_l,\sigma_{l-1}}$ is the index defined in part (i), we conclude that $(i_1, j_{i_1}), \dots, (i_l, j_{i_l}) \in \mathcal{I}_{\sigma_l}$. Take $\sigma' = \sigma_r$. The uniqueness of σ' follows from part (i), as σ' is completely determined by $\mathcal{I}_{\sigma'}$.

Part (v) is a direct consequence of part (iv).

For part (vi), let $j \in \{1, \dots, n_i\}$ such that $(i, j) \in \mathcal{J}$. By minimality of $\mathcal{J}$, there exists $\sigma \in \Sigma_{\max}$ such that $\mathcal{J} \cap \mathcal{I}_\sigma = \{(i, j)\}$. If $n_i > 1$, let $j' \in \{1, \dots, n_i\} \setminus \{j\}$. By [Salberger 1998, Lemma 8.9] there is $\sigma' \in \Sigma_{\max}$ such that $\mathcal{J}_{\sigma'} \cap \mathcal{J}_\sigma = \mathcal{J}_\sigma \setminus \{(i, j')\}$. Hence $\mathcal{I}_{\sigma'} = (\mathcal{I}_\sigma \setminus \{(i, j)\}) \cup \{(i, j')\}$. Since $\mathcal{J} \cap (\mathcal{I}_\sigma \setminus \{(i, j)\}) = \varnothing$ and $\mathcal{J} \cap \mathcal{I}_{\sigma'} \neq \varnothing$, we conclude that $(i, j') \in \mathcal{J}$. □

2.2. *Heights.* Let L be a semiample torus-invariant $\mathbb{Q}$-divisor on X. Let H_L be the height on X defined by L as in [Pieropan and Schindler 2024, §6.3]. For $\sigma \in \Sigma_{\max}$, write

$$L(\sigma) = \sum_{i=1}^{s} \sum_{j=1}^{n_i} \alpha_{i,j,\sigma} D_{i,j} \quad \text{and} \quad \alpha_{i,\sigma} = \sum_{j=1}^{n_i} \alpha_{i,j,\sigma}$$

for all $i \in \{1, \dots, s\}$. Let $\Omega_{\mathbb{K}}$ be the set of places of $\mathbb{K}$.

Lemma 2.2. *For every $v \in \Omega_{\mathbb{K}}$ and every $\boldsymbol{x} \in \mathscr{Y}(\mathbb{K})$, we have*

$$\sup_{\sigma \in \Sigma_{\max}} |x^{L(\sigma)}|_v = \sup_{\sigma \in \Sigma_{\max}} \prod_{i=1}^{s} \sup_{1 \le j \le n_i} |x_{i,j}|_v^{\alpha_{i,\sigma}}.$$

Proof. Fix $\nu \in \Omega_{\mathbb{K}}$ and $\boldsymbol{x} \in \mathscr{Y}(\mathbb{K})$. By Lemma 2.1 (ii), we have

$$\sup_{\sigma \in \Sigma_{\max}} |x^{L(\sigma)}|_\nu = \sup_{\sigma \in \Sigma_{\max}} \prod_{i \in I_\sigma} |x_{i,j_{i,\sigma}}|_\nu^{\alpha_{i,\sigma}}.$$

For every $i \in \{1, \dots, s\}$, let $j_i \in \{1, \dots, n_i\}$ such that $|x_{i,j_i}|_\nu = \sup_{1 \le j \le n_i} |x_{i,j}|_\nu$. Let $\sigma \in \Sigma_{\max}$. By Lemma 2.1 (iv) there is $\sigma' \in \Sigma_{\max}$ such that $I_{\sigma'} = I_\sigma$, $(i, j_i) \in \mathcal{I}_{\sigma'}$ for all $i \in I_\sigma$, and $\alpha_{i,\sigma'} = \alpha_{i,\sigma}$ for all $i \in \{1, \dots, s\}$. Then

$$|x^{L(\sigma')}|_\nu = \prod_{i \in I_{\sigma'}} |x_{i,j_i}|_\nu^{\alpha_{i,\sigma'}} = \prod_{i \in I_\sigma} \sup_{1 \le j \le n_i} |x_{i,j}|_\nu^{\alpha_{i,\sigma}} = \prod_{i=1}^{s} \sup_{1 \le j \le n_i} |x_{i,j}|_\nu^{\alpha_{i,\sigma}}. \qquad \square$$

Thus

$$H_L(\boldsymbol{x}) = \prod_{\nu \in \Omega_{\mathbb{K}}} \sup_{\sigma \in \Sigma_{\max}} \prod_{i=1}^{s} \sup_{1 \le j \le n_i} |x_{i,j}|_\nu^{\alpha_{i,\sigma}} \quad \text{for all } \boldsymbol{x} \in \mathscr{Y}(\mathbb{K}).$$

2.3. *Coprimality conditions.* We now rewrite the coprimality condition (2-1) in terms of the notation introduced in this paper.

Lemma 2.3. *For all* $\boldsymbol{x} \in \bigoplus_{(i,j) \in \mathcal{I}} \mathfrak{c}^{[D_{i,j}]}$,

$$\sum_{\sigma \in \Sigma_{\max}} \boldsymbol{x}^{D_\sigma} \mathfrak{c}^{-[D_\sigma]} = \sum_{\sigma \in \Sigma_{\max}} \prod_{i \in I_\sigma} (x_{i,1}, \dots, x_{i,n_i}) \mathfrak{c}^{-\boldsymbol{D}_i}.$$

Proof. For $\sigma \in \Sigma_{\max}$, let

$$X_\sigma = \left\{ \prod_{i \in I_\sigma} x_{i,j_i} : j_i \in \{1, \dots, n_i\} \, \forall i \in \{1, \dots, s\} \right\}.$$

The inclusion $\subseteq$ is clear as $\boldsymbol{x}^{D_\sigma} \in X_\sigma$ and $\mathfrak{c}^{-[D_\sigma]} = \prod_{i \in I_\sigma} \mathfrak{c}^{-\boldsymbol{D}_i}$ for all $\sigma \in \Sigma_{\max}$. For the converse inclusion, fix $\sigma \in \Sigma_{\max}$ and $x \in X_\sigma$. For every $i \in I_\sigma$, let $j_i \in \{1, \dots, n_i\}$ such that $x = \prod_{i \in I_\sigma} x_{i,j_i}$. By Lemma 2.1 (iv) there is $\sigma' \in \Sigma_{\max}$ such that $I_{\sigma'} = I_\sigma$ and $(i, j_i) \in \mathcal{I}_{\sigma'}$ for $i \in I_\sigma$. Then $\boldsymbol{x}^{D_{\sigma'}} = x$. $\square$

2.4. *Möbius function.* Let $\mathcal{I}_{\mathbb{K}}$ be the set of nonzero ideals of $\mathcal{O}_{\mathbb{K}}$. Let $\chi : \mathcal{I}_{\mathbb{K}}{}^s \to \{0, 1\}$ be the characteristic function of the subset

$$\left\{ \mathfrak{b} \in \mathcal{I}_{\mathbb{K}}{}^s : \sum_{\sigma \in \Sigma_{\max}} \prod_{i \in I_\sigma} \mathfrak{b}_i = \mathcal{O}_{\mathbb{K}} \right\}. \tag{2-2}$$

For every $\mathfrak{d} \in \mathcal{I}_{\mathbb{K}}{}^s$, let $\chi_{\mathfrak{d}} : \mathcal{I}_{\mathbb{K}}{}^s \to \{0, 1\}$ be the characteristic function of the subset

$$\{\mathfrak{b} \in \mathcal{I}_{\mathbb{K}}{}^s : \mathfrak{b}_i \subseteq \mathfrak{d}_i \, \forall i \in \{1, \dots, s\}\}.$$

As in [Peyre 1995, Lemme 8.5.1], there exists a unique multiplicative function $\mu : \mathcal{I}_{\mathbb{K}}{}^s \to \mathbb{Z}$ such that

$$\chi = \sum_{\mathfrak{d} \in \mathcal{I}_{\mathbb{K}}{}^s} \mu(\mathfrak{d}) \chi_{\mathfrak{d}}.$$

Note that if $X = \mathbb{P}^n_{\mathbb{Q}}$, the function μ defined above coincides with the classical Möbius function.

Remark 2.4. Let $\mathfrak{p} \in \mathcal{I}_{\mathbb{K}}$ be a prime ideal. The function μ is defined recursively by the formula $\mu(\mathfrak{b}) = \chi(\mathfrak{b}) - \sum_{\mathfrak{b} \subsetneq \mathfrak{d}} \mu(\mathfrak{d})$ for every $\mathfrak{b} \in \mathcal{I}_{\mathbb{K}}{}^s$ and satisfies the following properties:

(i) $\mu(\mathbf{1}) = \chi(\mathbf{1}) = 1$.

(ii) If $e_i \geq 2$ for some $i \in \{1, \dots, s\}$, then $\mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) = 0$, as in that case $\chi(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) = \chi(\mathfrak{p}^{e'_1}, \dots, \mathfrak{p}^{e'_s})$ for $e'_i = e_i - 1$ and $e'_l = e_l$ for all $l \neq i$.

(iii) By induction one shows that $\mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) = 0$ whenever $(e_1, \dots, e_s) \neq \mathbf{0}$ and there is $\sigma \in \Sigma_{\max}$ such that $e_i = 0$ for all $i \in I_\sigma$, as $\chi(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) = 1$ if and only if there is $\sigma \in \Sigma_{\max}$ such that $e_i = 0$ for all $i \in I_\sigma$.

(iv) Let

$$\tilde{f} := \min\{\#J : J \subseteq \{1, \dots, s\},\ J \cap I_\sigma \neq \varnothing\ \forall \sigma \in \Sigma_{\max}\}.$$

By property (iii), if $\mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) \neq 0$, then there are at least $\tilde{f}$ indices i with $e_i = 1$. Let $J \subseteq \{1, \dots, s\}$ be smallest with respect to inclusion and such that $J \cap I_\sigma \neq \varnothing$ for all $\sigma \in \Sigma_{\max}$. Let $J' = J \smallsetminus \{j\}$ for some $j \in J$. Let $e_i = 1$ for $i \in J$ and $e_i = 0$ for $i \notin J$. Let $e'_i = e_i$ for $i \neq j$ and $e'_j = 0$. Then $\chi(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) = 0$ and $\chi(\mathfrak{p}^{e'_1}, \dots, \mathfrak{p}^{e'_s}) = 1$ by minimality of J. Thus $\mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) = -1 \neq 0$. Hence

$$\tilde{f} = \min\left\{\sum_{i=1}^{s} e_i : (e_1, \dots, e_s) \neq \mathbf{0},\ \mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) \neq 0\right\}. \tag{2-3}$$

For $\beta = (\beta_1, \dots, \beta_s) \in \mathbb{R}^s_{\geq 0}$, let

$$f_\beta := \min\left\{\sum_{i=1}^{s} \beta_i e_i : (e_1, \dots, e_s) \neq \mathbf{0},\ \mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s}) \neq 0\right\}.$$

Lemma 2.5. (i) *The series*

$$\sum_{\mathfrak{d} \in \mathcal{I}_{\mathbb{K}}^s} \frac{\mu(\mathfrak{d})}{\prod_{i=1}^{s} \mathfrak{N}(\mathfrak{d}_i)^{\beta_i}}$$

converges absolutely if $f_\beta > 1$.

(ii) *If* $f_\beta > 1$ *and* $\beta_1, \dots, \beta_s \in \mathbb{Z}_{>0}$, *then*

$$\sum_{\mathfrak{d} \in \mathcal{I}_{\mathbb{K}}^s} \frac{\mu(\mathfrak{d})}{\prod_{i=1}^{s} \mathfrak{N}(\mathfrak{d}_i)^{\beta_i}} > 0.$$

Proof. For part (i) we follow the proof of [Salberger 1998, Lemma 11.15] and [Pieropan 2016, Proposition 4]. For $\mathfrak{p} \in \mathcal{I}_{\mathbb{K}}$ a prime ideal, let

$$S(\mathfrak{p}) = \sum_{(e_1, \dots, e_s) \in \mathbb{Z}^s_{\geq 0}} \frac{|\mu(\mathfrak{p}^{e_1}, \dots, \mathfrak{p}^{e_s})|}{\prod_{i=1}^{s} \mathfrak{N}(\mathfrak{p})^{\beta_i e_i}}.$$

As in the two results cited,

$$\lim_{b \to \infty} \sum_{\substack{\mathfrak{d} \in \mathcal{I}_{\mathbb{K}}^s \\ \prod_{i=1}^{s} \mathfrak{N}(\mathfrak{d}_i) \leq b}} \frac{|\mu(\mathfrak{d})|}{\prod_{i=1}^{s} \mathfrak{N}(\mathfrak{d}_i)^{\beta_i}} = \prod_{\mathfrak{p}} S(\mathfrak{p}).$$

By Remark 2.4 (ii), the sum $S(\mathfrak{p})$ is finite. By definition of f_β, if $\mu(\mathfrak{p}^{e_1},\dots,\mathfrak{p}^{e_s})\neq 0$ and $(e_1,\dots,e_s)\neq\mathbf{0}$, then $f_\beta\le\sum_{i=1}^s\beta_i e_i$. Thus

$$S(\mathfrak{p})=1+\frac{1}{\mathfrak{N}(\mathfrak{p})^{f_\beta}}Q\left(\frac{1}{\mathfrak{N}(\mathfrak{p})}\right),$$

where $Q:\mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$ is a monotone increasing function. Since $\mu(\mathfrak{p}^{e_1},\dots,\mathfrak{p}^{e_s})$ is independent of the choice of $\mathfrak{p}$, the function Q is independent of the choice of $\mathfrak{p}$. Thus

$$\sum_{\mathfrak{p}}\frac{1}{\mathfrak{N}(\mathfrak{p})^{f_\beta}}Q\left(\frac{1}{\mathfrak{N}(\mathfrak{p})}\right)\le[\mathbb{K}:\mathbb{Q}]Q(1)\sum_{n\in\mathbb{Z}_{>0}}\frac{1}{n^{f_\beta}}.$$

In part (ii) the series is absolutely convergent by part (i); hence it suffices to show that each factor of its Euler product $\prod_{\mathfrak{p}}S_{\mathfrak{p}}$ is positive. For a prime ideal $\mathfrak{p}\in\mathcal{I}_{\mathbb{K}}$, let $\mathcal{O}_{\mathfrak{p}}$ be the ring of integers of the completion $\mathbb{K}_{\mathfrak{p}}$ of $\mathbb{K}$ at the valuation $v_{\mathfrak{p}}$ defined by $\mathfrak{p}$. Endow $\mathbb{K}_{\mathfrak{p}}$ with the Haar measure normalized such that $\mathcal{O}_{\mathfrak{p}}$ has volume 1. Then $\int_{\mathfrak{p}^j\mathcal{O}_{\mathfrak{p}}}\mathrm{d}y=\mathfrak{N}(\mathfrak{p})^{-j}$ for all $j\ge 0$ by [Chambert-Loir et al. 2018, §1.1.13] and [Neukirch 1999, Proposition II.4.3]. We denote by χ the characteristic function of (2-2), where ideals of $\mathcal{O}_{\mathbb{K}}$ are replaced by ideals of $\mathcal{O}_{\mathfrak{p}}$. By Remark 2.4 (ii),

$$\begin{aligned}S_{\mathfrak{p}}&=\sum_{\boldsymbol{e}\in\{0,1\}^s}\mu(\mathfrak{p}^{e_1},\dots,\mathfrak{p}^{e_s})\prod_{i=1}^s\mathfrak{N}(\mathfrak{p})^{-e_i\beta_i}=\sum_{\boldsymbol{e}\in\{0,1\}^s}\mu(\mathfrak{p}^{e_1},\dots,\mathfrak{p}^{e_s})\prod_{i=1}^s\prod_{j=1}^{\beta_i}\int_{\mathfrak{p}^{e_i}}\mathrm{d}y_{i,j}\\&=\int_{\mathcal{O}_{\mathfrak{p}}^{\sum_{i=1}^s\beta_i}}\chi((y_{1,1},\dots,y_{1,\beta_1}),\dots,(y_{s,1},\dots,y_{s,\beta_s}))\prod_{i=1}^s\prod_{j=1}^{\beta_i}\mathrm{d}y_{i,j}\\&\ge\int_{(\mathcal{O}_{\mathfrak{p}}^\times)^{\sum_{i=1}^s\beta_i}}\prod_{i=1}^s\prod_{j=1}^{\beta_i}\mathrm{d}y_{i,j}=\left(1-\frac{1}{\mathfrak{N}(\mathfrak{p})}\right)^{\sum_{i=1}^s\beta_i}>0,\end{aligned}$$

as χ is a nonnegative function with $\chi(\mathcal{O}_{\mathfrak{p}},\dots,\mathcal{O}_{\mathfrak{p}})=1$. □

Definition 2.6. A function $A:\mathbb{Z}_{>0}^s\to\mathbb{R}$ is compatible with Möbius inversion on X if there exist $\beta_1,\dots,\beta_s\in\mathbb{R}_{\ge 0}^s$ such that $A(\boldsymbol{d})\ll\prod_{i=1}^s d_i^{-\beta_i}$ with $f_{(\beta_1,\dots,\beta_s)}>1$.

Remark 2.7. (i) The inequality $f_\beta>1$ holds whenever $\beta_1,\dots,\beta_s>1$.

(ii) If $\beta_1=\dots=\beta_s=1$, then $f_\beta=\tilde{f}$ by (2-3).

(iii) Case $\beta_1=n_1,\dots,\beta_s=n_s$: As in [Salberger 1998, Lemma 11.15 (d)], let f be the smallest positive integer such that there are f rays of the fan Σ that are not contained in a maximal cone. Then $f\ge 2$, as X is proper. Moreover,

$$f=\min\{\#\mathcal{J}:\mathcal{J}\subseteq\mathcal{I},\ \mathcal{J}\cap\mathcal{I}_\sigma\neq\varnothing\ \forall\sigma\in\Sigma_{\max}\},$$

and Remark 2.4 combined with Lemma 2.1 (vi) gives

$$\begin{aligned}f&=\min\left\{\sum_{i\in J}n_i:J\subseteq\{1,\dots,s\},\ J\cap I_\sigma\neq\varnothing\ \forall\sigma\in\Sigma_{\max},\ \#J=\tilde{f}\right\}\\&=\min\left\{\sum_{i=1}^s n_ie_i:(e_1,\dots,e_s)\neq\mathbf{0},\ \mu(\mathfrak{p}^{e_1},\dots,\mathfrak{p}^{e_s})\neq 0\right\}.\end{aligned}$$

3. Subvarieties

Here we want to count rational points or Campana points of bounded height in subvarieties of toric varieties.

From now on $\mathbb{K} = \mathbb{Q}$. Let X be a complete smooth split toric variety as in Section 2. Assume that $\operatorname{rk}\operatorname{Pic}(X) \geq 2$, that is, X is not a projective space. Let L be a semiample toric invariant $\mathbb{Q}$-divisor on X that satisfies [Pieropan and Schindler 2024, Assumption 6.3]. The latter holds, for example, if L is ample. Throughout this section, we will abbreviate [Pieropan and Schindler 2024] as [PS24].

Let $g_1, \dots, g_t \in R$ be $\operatorname{Pic}(X)$-homogeneous elements. Let $V \subseteq X$ be the schematic intersection of the t hypersurfaces defined by $g_1, \dots, g_t$. Let $T \subseteq X$ be the torus. Without loss of generality, we can assume that $V \cap T \neq \varnothing$. Otherwise, V is contained in a complete smooth split toric subvariety X' of X, and we can replace X by X'. Fix $m_{i,j} \in \mathbb{Z}_{\geq 1}$ for each $(i, j) \in \mathcal{I}$. Let $\boldsymbol{m} = (m_{i,j})_{(i,j)\in\mathcal{I}}$ and

$$\mathscr{D}_{\boldsymbol{m}} = \sum_{i=1}^{s}\sum_{j=1}^{n_i}\Big(1 - \frac{1}{m_{i,j}}\Big)\mathscr{D}_{i,j}.$$

Let $\mathscr{V}$ be the Zariski closure of V in $\mathscr{X}$. We define the intersection multiplicity $n_v(\mathscr{D}_i|_{\mathscr{V}}, \boldsymbol{x})$ of a point $\boldsymbol{x} : \operatorname{Spec}\mathcal{O}_{\mathbb{K}} \to \mathscr{V}$ with $\mathscr{D}_i|_{\mathscr{V}}$ at a place v of $\mathbb{K}$ to be the colength of the ideal of the fiber product of $\operatorname{Spec}\mathcal{O}_{\mathbb{K}} \times_{\mathscr{V}} \mathscr{D}_i|_{\mathscr{V}}$ after base change to the completion of $\mathcal{O}_{\mathbb{K}}$ at v. This definition coincides with the one in [Pieropan et al. 2021, §3] whenever $\mathscr{V}$ is regular. Let $(\mathscr{V}, \mathscr{D}_{\boldsymbol{m}}|_{\mathscr{V}})(\mathbb{Z})$ be the set of Campana $\mathbb{Z}$-points on the Campana orbifold $(\mathscr{V}, \mathscr{D}_{\boldsymbol{m}}|_{\mathscr{V}})$ as in [Pieropan et al. 2021, Definition 3.4].

Let $N_V(B)$ be the number of points in $(\mathscr{V}, \mathscr{D}_{\boldsymbol{m}}|_{\mathscr{V}})(\mathbb{Z}) \cap T(\mathbb{Q})$ of height H_L at most B. If $m_{i,j} = 1$ for all $(i, j) \in \mathcal{I}$, then $N_V(B)$ is the set of $\mathbb{Q}$-rational points on $V \cap T$ of height H_L at most B.

For $i \in \{1, \dots, s\}$ and $\boldsymbol{x} \in \mathscr{Y}(\mathbb{Z})$, let $y_i = \sup_{1\leq j\leq n_i} |x_{i,j}|$. For $\sigma \in \Sigma_{\max}$, write

$$L(\sigma) = \sum_{i=1}^{s}\sum_{j=1}^{n_i} \alpha_{i,j,\sigma} D_{i,j} \quad \text{and} \quad \alpha_{i,\sigma} = \sum_{j=1}^{n_i} \alpha_{i,j,\sigma} \qquad \text{for all } i \in \{1, \dots, s\}.$$

Then, by [PS24, Proposition 6.10] and Lemma 2.2,

$$H_L(\boldsymbol{x}) = \sup_{\sigma\in\Sigma_{\max}} \prod_{i=1}^{s} y_i^{\alpha_{i,\sigma}}.$$

By construction,

$$(\mathscr{V}, \mathscr{D}_{\boldsymbol{m}}|_{\mathscr{V}})(\mathbb{Z}) = (\mathscr{X}, \mathscr{D}_{\boldsymbol{m}})(\mathbb{Z}) \cap V(\mathbb{Q}).$$

We use the torsor parametrization of $(\mathscr{X}, \mathscr{D}_{\boldsymbol{m}})(\mathbb{Z})$ from [PS24, §6.4]. For $B > 0$ and $\boldsymbol{d} \in (\mathbb{Z}_{>0})^s$, let $A(B, \boldsymbol{d})$ be the set of points $\boldsymbol{x} = (x_{i,j})_{1\leq i\leq s, 1\leq j\leq n_i} \in (\mathbb{Z}_{\neq 0})^{\mathcal{I}}$ such that

$$H(\boldsymbol{x}) \leq B, \tag{3-1}$$

$$d_i \mid x_{i,j} \quad \text{for all } i \in \{1, \dots, s\} \text{ and for all } j \in \{1, \dots, n_i\}, \tag{3-2}$$

$$x_{i,j} \text{ is } m_{i,j}\text{-full} \quad \text{for all } i \in \{1, \dots, s\} \text{ and for all } j \in \{1, \dots, n_i\}, \tag{3-3}$$

$$g_1 = \cdots = g_t = 0. \tag{3-4}$$

We observe that $A(B, \boldsymbol{d})$ is a finite set by [PS24, Lemma 6.11]. Then

$$N_V(B) = \frac{1}{2^r} \sum_{\boldsymbol{d} \in (\mathbb{Z}_{>0})^s} \mu(d) \# A(B, \boldsymbol{d}) \tag{3-5}$$

by Lemma 2.3 and the definition of μ in Section 2.4.

Write

$$\# A(B, \boldsymbol{d}) = \sum_{\substack{y_1, \dots, y_s \in \mathbb{Z}_{>0} \\ \prod_{i=1}^s y_i^{\alpha_{i,\sigma}} \leq B \ \forall \sigma \in \Sigma_{\max}}} f_{\boldsymbol{d}}(y_1, \dots, y_s),$$

where

$$f_{\boldsymbol{d}}(y_1, \dots, y_s) = \# \Big\{ \boldsymbol{x} \in (\mathbb{Z}_{\neq 0})^{\mathcal{I}} : \text{(3-2), (3-3), (3-4)},\ y_i = \sup_{1 \leq j \leq n_i} |x_{i,j}| \ \forall i \in \{1, \dots, s\} \Big\}.$$

Let

$$F_{\boldsymbol{d}}(B_1, \dots, B_s) = \sum_{1 \leq y_i \leq B_i, 1 \leq i \leq s} f_{\boldsymbol{d}}(y_1, \dots, y_s).$$

Lemma 3.1. *Assume that*

$$F_{\boldsymbol{d}}(B_1, \dots, B_s) = C_{M,\boldsymbol{d}} \prod_{i=1}^s B_i^{\varpi_i} + O\Big(C_{E,\boldsymbol{d}} \big(\min_{1 \leq i \leq s} B_i \big)^{-\epsilon} \prod_{i=1}^s B_i^{\varpi_i} \Big) \tag{3-6}$$

with $C_{M,\boldsymbol{d}}$, $C_{E,\boldsymbol{d}}$, $\varpi_1, \dots, \varpi_s$, $\epsilon > 0$ such that $C_{M,\boldsymbol{d}}$ and $C_{E,\boldsymbol{d}}$ are compatible with Möbius inversion on X as functions of the variables $\boldsymbol{d}$.

Let a be the maximal value of $\sum_{i=1}^s \varpi_i u_i$ on the polytope $\mathcal{P} \subseteq \mathbb{R}^s$ defined by

$$\sum_{i=1}^s \alpha_{i,\sigma} u_i \leq 1 \ \text{ for all } \sigma \in \Sigma_{\max}, \quad u_i \geq 0 \ \text{ for all } i \in \{1, \dots, s\}.$$

Let F be the face of $\mathcal{P}$ where $\sum_{i=1}^s \varpi_i u_i = a$. Let k be the dimension of F.

(i) *If F is not contained in a coordinate hyperplane of $\mathbb{R}^s$, then*

$$N_V(B) = cB^a (\log B)^k + O(B^a (\log B)^{k-1} (\log \log B)^s),$$

where k is the dimension of F and

$$c = (s-1-k)!\, c_{\mathcal{P}} 2^{-r} \sum_{\boldsymbol{d} \in \mathbb{Z}_{>0}^s} \mu(\boldsymbol{d}) C_{M,\boldsymbol{d}}. \tag{3-7}$$

Here, $c_{\mathcal{P}} = \lim_{\delta \to 0} \delta^{k+1-s} \operatorname{meas}_{s-1}(H_\delta \cap \mathcal{P})$, where $H_\delta \subseteq \mathbb{R}^s$ is the hyperplane defined by $\sum_{i=1}^s \varpi_i u_i = a - \delta$ and $\operatorname{meas}_{s-1}$ is the $(s-1)$-dimensional measure on H_δ given by $\prod_{1 \leq i \leq s, i \neq \tilde{i}} (\varpi_i \, \mathrm{d}u_i)$ for any choice of $\tilde{i} \in \{1, \dots, s\}$.

(ii) *If L is ample, then*

$$a = \inf \Big\{ t \in \mathbb{R} : t[L] - \Big[\sum_{i=1}^s \varpi_i \boldsymbol{D}_i \Big] \text{ is effective} \Big\}$$

and $k+1$ is the codimension of the minimal face of the effective cone of X containing $a[L] - \big[\sum_{i=1}^s \varpi_i \boldsymbol{D}_i\big]$.

(iii) *If $[L] = \sum_{i=1}^s \varpi_i \boldsymbol{D}_i$ is ample, then the face F is not contained in a coordinate hyperplane, $a = 1$, and $k = \operatorname{rk} \operatorname{Pic}(X) - 1$.*

Proof. (i) Let $t_i = \varpi_i u_i$ for all $i \in \{1, \dots, s\}$. By the assumptions on L, the polytope $\mathcal{P}$ is bounded and nondegenerate by [PS24, Remark 6.2]. Applying [PS24, Theorem 1.1] to $\#A(B, \boldsymbol{d})$ gives

$$N_V(B) = cB^a(\log B)^k + O\biggl(B^a(\log B)^{k-1}(\log\log B)^s \sum_{\boldsymbol{d}\in(\mathbb{Z}_{>0})^s} \mu(d)C_{E,\boldsymbol{d}}\biggr).$$

The sums $\sum_{\boldsymbol{d}\in(\mathbb{Z}_{>0})^s}$ in the leading constant c and in the error term converge absolutely by Lemma 2.5 as $C_{M,\boldsymbol{d}}$ and $C_{E,\boldsymbol{d}}$ are compatible with Möbius inversion on X.

(ii) Let

$$\mathbb{R}^r \hookrightarrow \mathbb{R}^s \hookrightarrow \mathbb{R}^{\mathcal{I}}$$

be the sequence of injective linear maps dual to

$$d : \bigoplus_{(i,j)\in\mathcal{I}} D_{i,j}\mathbb{Z} \twoheadrightarrow \bigoplus_{i=1}^{s} \boldsymbol{D}_i\mathbb{Z} \twoheadrightarrow \mathrm{Pic}(X).$$

Here,

$$\mathbb{R}^s \hookrightarrow \mathbb{R}^{\mathcal{I}}, \quad \sum_{i=1}^{s} u_i e_i \mapsto \sum_{i=1}^{s}\sum_{j=1}^{n_i} u_i e_{i,j},$$

where $\{e_1, \dots, e_s\}$ denotes the dual basis to $\{\boldsymbol{D}_1, \dots, \boldsymbol{D}_s\}$ and $\{e_{i,j} : (i, j) \in \mathcal{I}\}$ denotes the dual basis to $\{D_{i,j} : (i, j) \in \mathcal{I}\}$. Let $\widetilde{P}$ be the polytope defined by

$$\sum_{(i,j)\in\mathcal{I}} \alpha_{i,j,\sigma} u_{i,j} \le 1 \ \text{ for all } \sigma \in \Sigma_{\max}, \quad u_{i,j} \ge 0 \ \text{ for all } (i, j) \in \mathcal{I}.$$

Then $\widetilde{P} \cap \mathbb{R}^s = P$ and

$$\sum_{(i,j)\in\mathcal{I}} \frac{\varpi_i}{n_i} u_{i,j}\Big|_P = \sum_{i=1}^{s}\biggl(\sum_{j=1}^{n_i} \varpi_i/n_i\biggr)u_i.$$

By [PS24, Lemma 6.7], the face F of $\widetilde{P}$ where the maximal value of the function

$$\sum_{(i,j)\in\mathcal{I}} \frac{\varpi_i}{n_i} u_{i,j} \tag{3-8}$$

is attained is contained in $\widetilde{P} \cap \mathbb{R}^r$ and hence also in P. Then a is the maximal value of the function (3-8) on P. The dual linear programming problem is given by minimizing $\sum_{\sigma\in\Sigma_{\max}} \lambda_\sigma$ on the polytope given by

$$\sum_{\sigma\in\Sigma_{\max}} \alpha_{i,j,\sigma}\lambda_\sigma \ge \frac{\varpi_i}{n_i} \ \text{ for all } (i, j) \in \mathcal{I}, \quad \lambda_\sigma \ge 0 \ \text{ for all } \sigma \in \Sigma_{\max}.$$

The arguments that can be found in [PS24, §6.5.1] show that a is the smallest real number such that $a[L] - \sum_{i=1}^{s}\sum_{j=1}^{n_i}(\varpi_i/n_i)\boldsymbol{D}_i$ is effective. As in [PS24, Proposition 6.13], the smallest face of $\mathrm{Eff}(X)$ that contains $a[L] - \sum_{i=1}^{s} \varpi_i \boldsymbol{D}_i$ is dual to the cone generated by F in $\mathbb{R}^r$, and the latter is defined by $a\sum_{i=1}^{s} \alpha_{i,\sigma}u_i - \sum_{i=1}^{s} \varpi_i u_i = 0$ for any $\sigma \in \Sigma_{\max}$ such that $F \subseteq \bigl\{\sum_{i=1}^{s} \alpha_{i,j,\sigma}u_{i,j} = 1\bigr\}$. Thus the minimal face of $\mathrm{Eff}(X)$ containing $a[L] - \bigl[\sum_{i=1}^{s} \varpi_i \boldsymbol{D}_i\bigr]$ has codimension $k + 1$.

(iii) We argue as in the proof of [PS24, Lemma 6.7 (ii)]. Let $\widetilde{H} \subseteq \mathbb{R}^s$ be the inclusion dual to the surjection $\bigoplus_{i=1}^s \mathbb{R}\boldsymbol{D}_i \to \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Then $\sum_{i=1}^s \alpha_{i,\sigma} u_i = \sum_{i=1}^s \varpi_i u_i$ for all $\boldsymbol{u} \in \widetilde{H}$ and all $\sigma \in \Sigma_{\max}$. Thus $\mathcal{P} \cap \widetilde{H}$ is the set of elements $\boldsymbol{u}$ of $\widetilde{H}$ such that $u_1, \dots, u_s \geq 0$ and $\sum_{i=1}^s \varpi_i u_i \leq 1$. Since $F \subseteq \widetilde{H}$ by [PS24, Lemma 6.7 (i)], we have $F = \widetilde{H} \cap \{\sum_{i=1}^s \varpi_i u_i = 1\}$. As in the proof of [PS24, Lemma 6.7 (ii)], we conclude that F is not contained in a coordinate hyperplane of $\mathbb{R}^s$. □

4. Rational points on linear complete intersections

Proof of Theorem 1.1. For $1 \leq i \leq s$ and $1 \leq l \leq t_i$, let $g_{i,l} \in R$ be a linear polynomial defining $H_{i,j}$. Then

$$g_{i,l} = \sum_{j=1}^{n_i} c_{i,j,l} x_{i,j}, \quad l \in \{1, \dots, t_i\},$$

with $c_{i,j,l} \in \mathbb{Z}$, and the $g_{i,1}, \dots, g_{i,t_i}$ are linearly independent for all $i \in \{1, \dots, s\}$. Let $m_{i,j} = 1$ for all $(i, j) \in \mathcal{I}$. Then

$$F_{\boldsymbol{d}}(B_1, \dots, B_s) = \prod_{i=1}^s F_{i,d_i}(B_i),$$

where, for $i \in \{1, \dots, s\}$, $d \in \mathbb{Z}_{>0}$, and $B > 0$,

$$F_{i,d}(B) = \#\Big\{(x_{i,1}, \dots, x_{i,n_i}) \in (\mathbb{Z}_{\neq 0})^{n_i} : \sup_{1 \leq j \leq n_i} |x_{i,j}| \leq B,\, d \mid x_{i,j}\ \forall j \in \{1, \dots, n_i\},\, g_{i,1} = \cdots = g_{i,t_i} = 0\Big\}.$$

For $i \in \{1, \dots, s\}$, let $W_i \subseteq \mathbb{R}^{n_i}$ be the linear space defined by $g_{i,1} = \cdots = g_{i,t_i} = 0$, and let $\Lambda_i \subseteq W_i$ be the restriction of the standard lattice $\mathbb{Z}^{n_i} \subseteq \mathbb{R}^{n_i}$ to W_i. Then, by [Bombieri and Gubler 2006, Lemma 11.10.15], for every $T \geq 1$,

$$\begin{aligned}\#(\mathbb{Z}^{n_i} \cap [-T, T]^{n_i} \cap W_i) &= \#(\Lambda_i \cap T([-1, 1]^n \cap W_i)) \\ &= T^{n_i - t_i} \frac{\operatorname{meas}_{n_i - t_i}([-1, 1]^{n_i} \cap W_i)}{\det \Lambda_i} + O(T^{n_i - t_i - 1}),\end{aligned}$$

where $\operatorname{meas}_{n_i - t_i}$ is the $(n_i - t_i)$-dimensional measure induced by the Lebesgue measure on $\mathbb{R}^{n_i}$. Let

$$c_i = \frac{\operatorname{meas}_{n_i - t_i}([-1, 1]^{n_i} \cap W_i)}{\det \Lambda_i}.$$

Then applying this estimate with $T = B/d$ gives

$$F_{i,d}(B) = c_i (B/d)^{n_i - t_i} + O((B/d)^{n_i - t_i - 1})$$

whenever $d \leq B$. If $d > B$, then $F_{i,d}(B) = 0$ and the same estimate holds. Hence, for $\delta > 0$,

$$F_{\boldsymbol{d}}(B_1, \dots, B_s) = C_{M,\boldsymbol{d}} \prod_{i=1}^s B_i^{n_i - t_i} + O\bigg(C_{E,\boldsymbol{d}} \bigg(\prod_{i=1}^s B_i^{n_i - t_i}\bigg) \big(\min_{1 \leq i \leq s} B_i\big)^{-\delta}\bigg),$$

where

$$C_{M,\boldsymbol{d}} = \prod_{i=1}^s \frac{c_i}{d_i^{n_i - t_i}}, \quad C_{E,\boldsymbol{d}} = \prod_{i=1}^s d_i^{-(n_i - t_i) + \delta}. \tag{4-1}$$

We show that, for $\delta > 0$ sufficiently small, the assumptions of Lemma 3.1 are satisfied. Since $n_i - t_i \geq 2$ for all $i \in \{1, \dots, s\}$ such that $t_i \neq 0$, if $f_{(n_1-t_1,\dots,n_s-t_s)} < 2$, by Remark 2.4 (iv), there is $\tilde{i} \in \{1, \dots, s\}$ such that $t_{\tilde{i}} = 0$, $\tilde{i} \in I_\sigma$ for all $\sigma \in \Sigma_{\max}$, and $n_{\tilde{i}} = 1$. Then $\rho_{\tilde{i},1}$ is not contained in any maximal cone of Σ, contradicting the fact that X is proper. Thus $f_{(n_1-t_1,\dots,n_s-t_s)} \geq 2$. By definition and by Remark 2.4 (ii),

$$f_{(n_1-t_1-\delta,\dots,n_s-t_s-\delta)} \geq f_{(n_1-t_1,\dots,n_s-t_s)} - s\delta.$$

Since V is a smooth complete intersection of smooth divisors, by adjunction [Corti 1992, Proposition 16.4], we have $K_V = K_X + \sum_{i=1}^{s} \sum_{l=1}^{t_i} [H_{i,l}]$. Since

$$\sum_{i=1}^{s} (n_i - t_i)\boldsymbol{D}_i = -[K_X] - \sum_{i=1}^{s} t_i \boldsymbol{D}_i = -[K_X] - \sum_{i=1}^{s} \sum_{l=1}^{t_i} [H_{i,l}],$$

Lemma 3.1 gives

$$N_V(B) = cB(\log B)^{b-1} + O(B(\log B)^{b-2}(\log\log B)^s),$$

where $b = \operatorname{rk}\operatorname{Pic}(X)$ and c is defined in (3-7) with $k = b-1$, $C_{M,\boldsymbol{d}}$ given by (4-1), and $\varpi_i = n_i - t_i$ for $i \in \{1, \dots, s\}$. The restriction $\operatorname{Pic}(X) \to \operatorname{Pic}(V)$ is an isomorphism as $t_i \leq n_i - 2$ for all $i \in \{1, \dots, s\}$. The leading constant c is positive by Lemma 2.5 (ii). □

5. Bihomogeneous hypersurfaces

Proof of Theorem 1.2. In the setting of Theorem 1.2, the hypersurfaces $H_1, \dots, H_t$ are defined by bihomogeneous polynomials $g_1, \dots, g_t$ of degree (e_1, e_2) in the two sets of variables $\{x_{1,j} : 1 \leq j \leq n_1\}$ and $\{x_{2,j} : 1 \leq j \leq n_2\}$. Let $m_{i,j} = 1$ for all $(i, j) \in \mathcal{I}$.

We will apply [Schindler 2016, Theorem 4.4] with

$$R = t, \quad F_i = g_i, \quad \mathscr{B}_i = [-1, 1]^{n_i}, \quad P_i = B_i/d_i, \quad d_i = e_i.$$

In order to apply the cited result, we need to restrict the points to an open set. Let $U \subseteq \mathbb{A}^{n_1+n_2}$ be the open set therein. Since the complement of U is the zero set of homogeneous polynomials by [Schindler 2016, Theorems 4.1 and 4.2], the set $W := \pi(\{\boldsymbol{x} \in Y : (x_{1,1}, \dots, x_{1,n_1}, x_{2,1}, \dots, x_{2,n_2}) \in U\})$ is an open subset of X. Then

$$N_{V,W}(B) = \frac{1}{2^r} \sum_{\boldsymbol{d} \in (\mathbb{Z}_{>0})^s} \mu(\boldsymbol{d}) \# A^W(B, \boldsymbol{d}),$$

with

$$A^W(B, \boldsymbol{d}) = \sum_{\substack{y_1,\dots,y_s \in \mathbb{Z}_{>0} \\ \prod_{i=1}^s y_i^{\alpha_{i,\sigma}} \leq B \ \forall \sigma \in \Sigma_{\max}}} f_{\boldsymbol{d}}^W(y_1, \dots, y_s)$$

and

$$f_{\boldsymbol{d}}^W(y_1, \dots, y_s) = \#\Big\{\boldsymbol{x} \in (\mathbb{Z}_{\neq 0})^{\mathcal{I}} : (x_{1,1}, \dots, x_{1,n_1}, x_{2,1}, \dots, x_{2,n_2}) \in U(\mathbb{Q}), \text{(3-2)}, \text{(3-3)}, \text{(3-4)}, \\ y_i = \sup_{1 \leq j \leq n_i} |x_{i,j}| \ \forall i \in \{1, \dots, s\}\Big\}.$$

Let

$$F_{\boldsymbol{d}}^W(B_1, \dots, B_s) = \sum_{1 \le y_i \le B_i, 1 \le i \le s} f_{\boldsymbol{d}}^W(y_1, \dots, y_s).$$

Then

$$F_{\boldsymbol{d}}^W(B_1, \dots, B_s) = \widetilde{F}_{d_1,d_2}^W(B_1, B_2) \prod_{i=3}^{s} F_{i,d_i}(B_i),$$

where

$$\widetilde{F}_{d_1,d_2}^W(B_1, B_2) = \#\Big\{(x_{1,1}, \dots, x_{1,n_1}, x_{2,1}, \dots, x_{2,n_2}) \in (\mathbb{Z}_{\neq 0})^{n_1+n_2} \cap U(\mathbb{Q}) : \sup_{1 \le j \le n_i} |y_{i,j}| \le B_i/d_i \ \forall i \in \{1, 2\},\ g_1 = \dots = g_t = 0\Big\},$$

and, for $d \in \mathbb{Z}_{>0}$ and $B > 0$,

$$F_{i,d}(B) = \#\Big\{(x_{i,1}, \dots, x_{i,n_i}) \in (\mathbb{Z}_{\neq 0})^{n_i} : \sup_{1 \le j \le n_i} |x_{i,j}| \le B,\ d \mid x_{i,j}\ \forall j \in \{1, \dots, n_i\}\Big\}.$$

If $d \le B_i$, then

$$F_{i,d}(B) = 2^{n_i}(B/d)^{n_i} + O((B/d)^{n_i - \delta})$$

with $0 < \delta \le 1$. If $d > B$, then $F_{i,d}(B) = 0$, and the same estimate holds.

To compute $\widetilde{F}_{d_1,d_2}^W(B_1, B_2)$, write $x_{i,j} = d_i y_{i,j}$ for all $(i, j) \in \mathcal{I}$. Then

$$\widetilde{F}_{d_1,d_2}^W(B_1, B_2) = \#\Big\{(y_{1,1}, \dots, y_{1,n_1}, y_{2,1}, \dots, y_{2,n_2}) \in (\mathbb{Z}_{\neq 0})^{n_1+n_2} \cap U(\mathbb{Q}) : \sup_{1 \le j \le n_i} |y_{i,j}| \le B_i/d_i \ \forall i \in \{1, 2\},\ g_1 = \dots = g_t = 0\Big\},$$

as the complement of U is the zero set of homogeneous polynomials by [Schindler 2016, Theorems 4.1 and 4.2]. Let $V_i^* \subseteq \mathbb{A}^{n_1+n_2}$ be the locus where the matrix $(\partial g_l/\partial x_{i,j})_{1 \le l \le t, 1 \le j \le n_i}$ does not have full rank. If $n_1 + n_2 > \dim V_1^* + \dim V_2^* + 3 \cdot 2^{e_1+e_2} e_1 e_2 t^3$, then, by [Schindler 2016, Theorem 4.4], there is $\delta > 0$ such that

$$\begin{aligned}\widetilde{F}_{d_1,d_2}^W(B_1, B_2) &= C \prod_{i=1}^{2} (B_i/d_i)^{n_i - te_i} + O\Big(\big(\min_{i=1,2} B_i/d_i\big)^{-\delta} \prod_{i=1}^{2} (B_i/d_i)^{n_i - te_i}\Big) \\ &= C \prod_{i=1}^{2} (B_i/d_i)^{n_i - te_i} + O\Big(\Big(\prod_{i=1}^{2} d_i^{-(n_i - te_i)+\delta}\Big)\big(\min_{i=1,2} B_i\big)^{-\delta} \prod_{i=1}^{2} B_i^{n_i - te_i}\Big)\end{aligned}$$

with $C \in \mathbb{R}_{\ge 0}$ and $C > 0$ whenever V has nonsingular $\mathbb{Q}_v$-points for all places v of $\mathbb{Q}$. Thus

$$F_{\boldsymbol{d}}^W(B_1, \dots, B_s) = C_{M,\boldsymbol{d}} B_1^{n_1 - te_1} B_2^{n_2 - te_2} \prod_{i=3}^{s} B_i^{n_i} + O\Big(C_{E,\boldsymbol{d}} \big(\min_{1 \le i \le s} B_i\big)^{-\delta} B_1^{n_1 - te_1} B_2^{n_2 - te_2} \prod_{i=3}^{s} B_i^{n_i}\Big),$$

where

$$C_{M,\boldsymbol{d}} = C d_1^{-(n_1 - te_1)} d_2^{-(n_2 - te_2)} \prod_{i=3}^{s} d_i^{-n_i}, \quad C_{E,\boldsymbol{d}} = d_1^{-(n_1 - te_1)+\delta} d_2^{-(n_2 - te_2)+\delta} \prod_{i=3}^{s} d_i^{-n_i+\delta}. \tag{5-1}$$

Recall that $n_i - te_i \geq 2$ for $i \in \{1, 2\}$. For $\delta > 0$ sufficiently small, if

$$f_{n_1-te_1-\delta, n_2-te_2-\delta, n_3-\delta, \dots, n_s-\delta} \leq 1,$$

then by Remark 2.4 (iv) there is $\tilde{i} \in \{3, \dots, s\}$ such that $\tilde{i} \in I_\sigma$ for all $\sigma \in \Sigma_{\max}$ and $n_{\tilde{i}} = 1$. Then the ray $\rho_{\tilde{i},1}$ is contained in no maximal cone of Σ, contradicting the fact that X is proper.

Since V is a smooth complete intersection, the adjunction formula [Corti 1992, Proposition 16.4] gives $K_V = K_X + H_1 + \cdots + H_t$. Let $\varpi_i = n_i - te_i$ for $i \in \{1, 2\}$ and $\varpi_i = n_i$ for $i \in \{3, \dots, s\}$. Since

$$\begin{aligned}\sum_{i=1}^{s} \varpi_i \boldsymbol{D}_i &= -[K_X] - t(e_1 \boldsymbol{D}_1 + e_2 \boldsymbol{D}_2) \\ &= -[K_X] - [H_1 + \cdots + H_t],\end{aligned}$$

Lemma 3.1 applied to $F_{\boldsymbol{d}}^W(B_1, \dots, B_s)$ and $N_{V,W}(B)$ gives

$$N_{V,W}(B) = cB(\log B)^{b-1} + O(B^a (\log B)^{b-2} (\log\log B)^s)$$

for $B > 0$, where $b = \operatorname{rk}\operatorname{Pic}(X)$ and c is defined in (3-7) with $k = b-1$, $C_{M,\boldsymbol{d}}$ given by (5-1). Moreover, the restriction $\operatorname{Pic}(X) \to \operatorname{Pic}(V)$ is an isomorphism, as $t \leq \min\{n_1, n_2\} - 2$. By Lemma 2.5 (ii), the leading constant c is positive if $V(\mathbb{Q}_v) \neq \varnothing$ for all places v of $\mathbb{Q}$ as C is positive under the same conditions by [Schindler 2016, Theorems 4.3 and 4.4]. □

6. Campana points on certain diagonal complete intersections

Proof of Theorem 1.3. In the setting of Theorem 1.3, the hypersurfaces $H_1, \dots, H_t$ are defined by homogeneous diagonal polynomials $g_1, \dots, g_t \in R$ with $\deg g_i = e_i \boldsymbol{D}_i$ in $\operatorname{Pic}(X)$ for all $i \in \{1, \dots, t\}$. Then

$$g_i = \sum_{j=1}^{n_i} c_{i,j} x_{i,j}^{e_i}$$

with $c_{i,j} \in \mathbb{Z}_{\neq 0}$, and

$$F_{\boldsymbol{d}}(B_1, \dots, B_s) = \prod_{i=1}^{s} F_{i,d_i}(B_i),$$

where, for $i \leq t$,

$$F_{i,d}(B) = \#\Big\{(x_{i,1}, \dots, x_{i,n_i}) \in (\mathbb{Z}_{\neq 0})^{n_i} : \\ d \mid x_{i,j},\ x_{i,j} \text{ is } m_{i,j}\text{-full } \forall j \in \{1, \dots, n_i\},\ \sup_{1 \leq j \leq n_i} |x_{i,j}| \leq B,\ g_i = 0\Big\}$$

and, for $i > t$,

$$F_{i,d}(B) = \#\Big\{(x_{i,1}, \dots, x_{i,n_i}) \in (\mathbb{Z}_{\neq 0})^{n_i} : \\ \sup_{1 \leq j \leq n_i} |x_{i,j}| \leq B,\ d \mid x_{i,j},\ x_{i,j} \text{ is } m_{i,j}\text{-full } \forall j \in \{1, \dots, n_i\}\Big\}. \tag{6-1}$$

For $i \leq t$, we estimate $F_{i,d}(B)$ via the following lemma.

Lemma 6.1. *Let $n, e, m_1, \dots, m_n \in \mathbb{Z}_{>0}$. Let $c_1, \dots, c_n \in \mathbb{Z}_{\neq 0}$. Let d be a square-free positive integer. Assume that $n \geq 2$ and $2 \leq m_1 \leq \cdots \leq m_n$.*

(1) *If $e = 1$, assume that*

$$\sum_{j=1}^{n} \frac{1}{m_j} > 3 \quad \textit{and} \quad \sum_{j=1}^{n-1} \frac{1}{em_j(em_j+1)} \geq 1.$$

(2) *If $e \geq 2$, assume that*

$$\sum_{j=1}^{n} \frac{1}{em_j} > 3 \quad \textit{and} \quad \sum_{j=1}^{n} \frac{1}{2s_0(em_j)} > 1,$$

where

$$s_0(m) = \min\{2^{m-1}, \tfrac{1}{2}m(m-1) + \lfloor\sqrt{2m+2}\rfloor\}, \quad m \in \mathbb{Z}_{\geq 0}.$$

For $B > 0$, let

$$F_d(B) = \#\Big\{(x_1, \dots, x_n) \in (\mathbb{Z}_{\neq 0})^n : d \mid x_j,\ x_j \textit{ is } m_j\textit{-full } \forall j \in \{1, \dots, n\},\ \sup_{1 \leq j \leq n} |x_j| \leq B,\ \sum_{j=1}^{n} c_j x_j^e = 0\Big\}.$$

Then there is $\eta > 0$ such that

$$F_d(B) = c_{e,d} B^{\Gamma} + O(d^{-1-\eta} B^{\Gamma-\eta}),$$

where $\Gamma = \sum_{j=1}^{n} 1/m_j - e$ and $c_{e,d}$ is defined in (6-5) and satisfies $0 \leq c_{e,d} \ll d^{-1-\eta}$.

Proof. For every $j \in \{1, \dots, n\}$ and $x_j \in \mathbb{Z}_{\neq 0}$ that is m_j-full, there exist unique $u_j, v_{j,1}, \dots, v_{j,m_j-1} \in \mathbb{Z}_{>0}$ such that

$$|x_j| = u_j^{m_j} \prod_{r=1}^{m_j-1} v_{j,r}^{m_j+r}, \quad \mu^2(v_{j,r}) = 1, \quad \gcd(v_{j,r}, v_{j,r'}) = 1 \quad \text{for all } r, r' \in \{1, \dots, m_j - 1\}, \quad r \neq r'.$$

For every choice of u_j and $v_{j,r}$ as above, if $d \mid x_j$ with $d \in \mathbb{Z}_{>0}$ squarefree, then there exist unique $s_j, t_{j,1}, \dots, t_{j,m_j-1} \in \mathbb{Z}_{>0}$ such that

$$d = s_j \prod_{r=1}^{m_j-1} t_{j,r}, \quad \mu^2(s_j) = \mu^2(t_{j,r}) = 1 \quad \text{for all } r \in \{1, \dots, m_j - 1\}$$

$$\gcd(s_j, v_{j,r}) = \gcd(s_j, t_{j,r}) = \gcd(t_{j,r}, t_{j,r'}) = 1 \quad \text{for all } r, r' \in \{1, \dots, m_j - 1\}, \quad r \neq r'$$

$$s_j \mid u_j, \quad t_{j,r} \mid v_{j,r} \quad \text{for all } r \in \{1, \dots, m_j - 1\}.$$

Write $u_j = s_j \tilde{u}_j$ and $v_{j,r} = t_{j,r} \tilde{v}_{j,r}$ for all $r \in \{1, \dots, m_{j-1}\}$. Write

$$\boldsymbol{s} = (s_1, \dots, s_n), \quad \boldsymbol{t} = (t_{j,r})_{1 \leq j \leq n, 1 \leq r \leq m_j - 1}.$$

For $j \in \{1, \dots, n\}$, write

$$\sigma_j = s_j \prod_{r=1}^{m_j-1} t_{j,r}, \quad \tau_j = s_j^{m_j} \prod_{r=1}^{m_j-1} t_{j,r}^{m_j+r}, \quad w_j = \prod_{r=1}^{m_j-1} \tilde{v}_{j,r}^{m_j+r}.$$

Let $\mathcal{T}_d(B)$ be the set of pairs $(\boldsymbol{s}, \boldsymbol{t}) \in \mathbb{Z}_{>0}^n \times \mathbb{Z}_{>0}^{\sum_{j=1}^{n}(m_j-1)}$ that satisfy

$$\mu^2(\sigma_j) = 1, \quad d = \sigma_j, \quad \tau_j \leq B \quad \text{for all } j \in \{1, \dots, s\}.$$

Note that the first two conditions imply

$$\#\mathcal{T}_d(B) \le \prod_{j=1}^{n} m_j^{\omega(d)} \ll d^{\epsilon}, \tag{6-2}$$

where $\omega(d)$ is the number of distinct prime divisors of d. Let $\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)$ be the set of

$$\tilde{\boldsymbol{v}} = (\tilde{v}_{j,r})_{1\le j\le n, 1\le r\le m_j-1} \in \mathbb{Z}_{>0}^{\sum_{j=1}^{n}(m_j-1)}$$

such that

$$\begin{aligned} \mu^2(t_{j,r}\tilde{v}_{j,r}) = 1,\ \ \gcd(s_j, \tilde{v}_{j,r}) = 1 \quad &\text{for all } j \in \{1,\dots,n\},\ \ r \in \{1,\dots,m_j-1\},\\ \gcd(t_{j,r}\tilde{v}_{j,r}, t_{j,r'}\tilde{v}_{j,r'}) = 1 \quad &\text{for all } j \in \{1,\dots,n\},\ \ r, r' \in \{1,\dots,m_j-1\},\ \ r \ne r',\\ \tau_j w_j \le B \quad &\text{for all } j \in \{1,\dots,n\}. \end{aligned}$$

Let $\mathcal{T}_d(\infty) = \bigcup_{B>0} \mathcal{T}_d(B)$ and $\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(\infty) = \bigcup_{B>0} \mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)$.

Then

$$F_d(B) = \begin{cases} \sum_{\boldsymbol{\varepsilon}\in\{\pm 1\}^n} \sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(B)} \sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)} M_{\boldsymbol{\varepsilon c},\boldsymbol{\gamma}}(B^e) & \text{if } e \text{ is odd},\\ 2^n \sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(B)} \sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)} M_{\boldsymbol{c},\boldsymbol{\gamma}}(B^e) & \text{if } e \text{ is even}, \end{cases} \tag{6-3}$$

where $\boldsymbol{c} = (c_1,\dots,c_n)$, $\boldsymbol{\varepsilon} = (\varepsilon_1,\dots,\varepsilon_n)$, $\boldsymbol{\varepsilon c} = (\varepsilon_1 c_1,\dots,\varepsilon_n c_n)$, $\boldsymbol{\gamma} = (\gamma_1,\dots,\gamma_n)$ with

$$\gamma_j = s_j^{em_j} \prod_{r=1}^{m_j-1} t_{j,r}^{e(m_j+r)} \tilde{v}_{j,r}^{e(m_j+r)} \quad \text{for all } j \in \{1,\dots,n\},$$

and

$$M_{\boldsymbol{\varepsilon c},\boldsymbol{\gamma}}(B^e) = \#\left\{ (\tilde{u}_1,\dots,\tilde{u}_n) \in \mathbb{Z}_{>0}^n : \max_{1\le j\le n} \gamma_j \tilde{u}_j^{em_j} \le B^e,\ \sum_{j=1}^{n} \varepsilon_j c_j \gamma_j \tilde{u}_j^{em_j} = 0 \right\}.$$

An estimate for $M_{\boldsymbol{\varepsilon c},\boldsymbol{\gamma}}(B^e)$ is proven in [Browning and Yamagishi 2021, Theorem 2.7] in the case where

$$\sum_{j=1}^{n-1} \frac{1}{em_j(em_j+1)} \ge 1.$$

The subsequent paper [Balestrieri et al. 2024, Theorem 5.3] extends the range of applicability of [Browning and Yamagishi 2021, Theorem 2.7] to the case where

$$\sum_{j=1}^{n} \frac{1}{em_j} > 3, \qquad \sum_{j=1}^{n} \frac{1}{2s_0(em_j)} > 1.$$

Let

$$\Theta_e = \begin{cases} \frac{1}{m_n(m_n+1)} & \text{if } e = 1,\\ \sum_{j=1}^{n} \frac{1}{2s_0(em_j)} - 1 & \text{if } e \ge 2. \end{cases}$$

For

$$0 < \delta < \frac{1}{(2(n-1)+5)em_n(em_n+1)} \quad \text{and} \quad \epsilon > 0,$$

the two results cited above give

$$\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(B)}\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)} M_{\boldsymbol{\varepsilon c},\boldsymbol{\gamma}}(B^e)=\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(B)}\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)}\frac{\mathfrak{S}_{\boldsymbol{\varepsilon c},\boldsymbol{\gamma}}\mathfrak{I}_{\boldsymbol{\varepsilon c}}}{\prod_{j=1}^n \gamma_j^{1/(em_j)}}B^\Gamma+O(B^\Gamma(F_1+F_2+F_3)), \tag{6-4}$$

where

$$\mathfrak{S}_{\boldsymbol{\varepsilon c},\boldsymbol{\gamma}}=\sum_{q=1}^{\infty}\frac{1}{q^n}\sum_{\substack{a \,(\mathrm{mod}\, q)\\ \gcd(a,q)=1}}\prod_{j=1}^{n}\sum_{r=1}^{q}\exp(2\pi i a\varepsilon_j c_j\gamma_j r^{em_j}/q),$$

$$\mathfrak{I}_{\boldsymbol{\varepsilon c}}=\int_{-\infty}^{\infty}\prod_{j=1}^{n}\biggl(\int_0^1\exp(2\pi i\lambda\varepsilon_j c_j\xi^{em_j})\,\mathrm{d}\xi\biggr)\,\mathrm{d}\lambda,$$

$$F_1=B^{e((2(n-1)+5)\delta-1)-\Gamma}\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(B)}\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)}\biggl(\prod_{j=1}^{n}\frac{B^{1/m_j}}{\gamma_j^{1/(em_j)}}\biggr)\sum_{l=1}^{n}\frac{\gamma_l^{1/(em_l)}}{B^{1/m_l}},$$

$$F_2=B^{-e\delta}\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(B)}\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)}\sum_{q=1}^{\infty}q^{1-\Gamma/e+\epsilon}\prod_{j=1}^{n}\gcd(\gamma_j,q)^{\frac{1}{em_j}}\gamma_j^{-\frac{1}{em_j}},$$

$$F_3=\begin{cases}B^{-e\delta\Theta_e+\epsilon}\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(B)}\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)}\prod_{j=1}^{n}\gamma_j^{-\frac{1}{m_j+1}} & \text{if } e=1,\\ B^{-e\delta\Theta_e+\epsilon}\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(B)}\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)}\prod_{j=1}^{n}\gamma_j^{-\frac{1}{em_j}+\frac{1}{2s_0(em_j)}} & \text{if } e\geq 2.\end{cases}$$

Let

$$c_{e,d}=\begin{cases}\sum_{\boldsymbol{\varepsilon}\in\{\pm1\}^n}\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(\infty)}\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(\infty)}\frac{\mathfrak{S}_{\boldsymbol{\varepsilon c},\boldsymbol{\gamma}}\mathfrak{I}_{\boldsymbol{\varepsilon c}}}{\prod_{j=1}^n\gamma_j^{1/(em_j)}} & \text{if } e \text{ is odd},\\ 2^n\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(\infty)}\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(\infty)}\frac{\mathfrak{S}_{\boldsymbol{c},\boldsymbol{\gamma}}\mathfrak{I}_{\boldsymbol{c}}}{\prod_{j=1}^n\gamma_j^{1/(em_j)}} & \text{if } e \text{ is even}.\end{cases} \tag{6-5}$$

For $T>0$, let

$$f_1(q)=\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(\infty)}\prod_{j=1}^{n}\biggl(\frac{\gcd(\tau_j^e,q)}{\tau_j^e}\biggr)^{\frac{1}{em_j}},\qquad f_2(q)=\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{1},\boldsymbol{1}}(\infty)}\prod_{j=1}^{n}\biggl(\frac{\gcd(w_j^e,q)}{w_j^e}\biggr)^{\frac{1}{em_j}},$$

and

$$f_2(q,T,\boldsymbol{s},\boldsymbol{t})=\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(\infty)\setminus\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(T)}\prod_{j=1}^{n}\biggl(\frac{\gcd(w_j^e,q)}{w_j^e}\biggr)^{\frac{1}{em_j}}.$$

Note that, for $\sum_{j=1}^n 1/(em_j)>1$, we have

$$|\mathfrak{I}_{\boldsymbol{\varepsilon c}}|\ll 1. \tag{6-6}$$

Similarly as in [Browning and Yamagishi 2021, (2.8), (2.9), (2.12)], the difference between $c_{e,d}B^\Gamma$ and the main term obtained by combining (6-3) and (6-4) is bounded by

$$\begin{aligned}B^\Gamma\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(\infty)}\sum_{\tilde{\boldsymbol{v}}\in\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(\infty)\setminus\mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)}&\sum_{q=1}^{\infty}q^{1-\sum_{j=1}^n\frac{1}{em_j}}\prod_{j=1}^{n}\gamma_j^{-\frac{1}{em_j}}\gcd(\gamma_j,q)^{\frac{1}{em_j}}\\ &\ll B^\Gamma\sum_{q=1}^{\infty}q^{-\Gamma/e+\epsilon}\sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(\infty)}\prod_{j=1}^{n}\biggl(\frac{\gcd(\tau_j^e,q)}{\tau_j^e}\biggr)^{\frac{1}{em_j}}f_2(q,B,\boldsymbol{s},\boldsymbol{t}),\end{aligned} \tag{6-7}$$

and $c_{e,d}\ll\sum_{q=1}^{\infty}q^{-\Gamma/e+\epsilon}f_1(q)f_2(q)$.

By [Browning and Yamagishi 2021, (3.10)] and the arguments used to prove [Browning and Yamagishi 2021, (3.9)], we have

$$f_2(q) \ll q^{\epsilon} \tag{6-8}$$

and

$$\begin{aligned} f_2(q,T,\boldsymbol{s},\boldsymbol{t}) &\ll \sum_{i_0=1}^{n} \sum_{\substack{\tilde{v}_{i,r},1\le i\le n,1\le r\le m_i-1\\ \prod_{r=1}^{m_i-1}\tilde{v}_{i,r}^{m_i+r}>T/\tau_i \text{ if } i=i_0}} \prod_{i=1}^{n}\prod_{r=1}^{m_i-1} \frac{\mu^2(\tilde{v}_{i,r})\gcd(\tilde{v}_{i,r}^{e(m_i+r)},q)^{1/(em_i)}}{\tilde{v}_{i,r}^{(m_i+r)/m_i}} \\ &\ll q^{\epsilon}\sum_{i=1}^{n} \sum_{\substack{\tilde{v}_{i,r},1\le r\le m_i-1\\ \prod_{r=1}^{m_i-1}\tilde{v}_{i,r}^{m_i+r}>T/\tau_i}} \prod_{r=1}^{m_i-1} \frac{\mu^2(\tilde{v}_{i,r})\gcd(\tilde{v}_{i,r}^{e(m_i+r)},q)^{1/(em_i)}}{\tilde{v}_{i,r}^{(m_i+r)/m_i}}. \end{aligned}$$

Our next goal is to provide an upper bound for sums of the type occurring in this estimate for $f_2(q,T,\boldsymbol{s},\boldsymbol{t})$.

Lemma 6.2. *Let $m\in\mathbb{N}_{\ge 2}$, $e\in\mathbb{N}$, and let $A>0$ be a real parameter. Then, for every $0<\epsilon<1/(m(m+1))$, we have*

$$\sum_{\substack{v_r\in\mathbb{N},1\le r\le m-1\\ \prod_{r=1}^{m-1}v_r^{m+r}>A}} \prod_{r=1}^{m-1} \frac{\mu^2(v_r)\gcd(v_r^{e(m+r)},q)^{1/(em)}}{v_r^{(m+r)/m}} \ll_{m,\epsilon} A^{-\frac{1}{m(m+1)}+\epsilon} q^{\frac{m-1}{em(m+1)}+\epsilon}.$$

Proof. We first consider the sum

$$S_1 := \sum_{\substack{v_r\in\mathbb{N},1\le r\le m-1\\ \prod_{r=1}^{m-1}v_r^{m+r}>A}} \prod_{r=1}^{m-1} \frac{1}{v_r^{(m+r)/m}}$$

for $A>1$. A dyadic decomposition for each of the variables v_r, $1\le r\le m-1$, leads to the upper bound

$$S_1 \ll \sum_{\substack{l_1,\dots,l_{m-1}\in\mathbb{N}\\ 2^{(m+1)l_1+\cdots+(2m-1)l_{m-1}}>A}} 2^{-\frac{1}{m}l_1-\cdots-\frac{(m-1)}{m}l_{m-1}}.$$

Note that, for each $k\in(1/m)\mathbb{N}$, we have

$$\#\left\{l_1,\dots,l_{m-1}\in\mathbb{N} : \frac{1}{m}l_1+\cdots+\frac{m-1}{m}l_{m-1}=k\right\} \ll_m k^{m-2}.$$

We deduce that

$$S_1 \ll_m \sum_{\substack{k\in(1/m)\mathbb{N}\\ r(k)>0}} k^{m-2}2^{-k},$$

where $r(k)$ is the number of $(l_1,\dots,l_{m-1})\in\mathbb{N}^{m-1}$ such that both

$$\frac{1}{m}l_1+\cdots+\frac{m-1}{m}l_{m-1}=k \quad\text{and}\quad 2^{(m+1)l_1+\cdots+(2m-1)l_{m-1}}>A.$$

Observe that if $r(k)>0$, then there exists $(l_1,\dots,l_{m-1})\in\mathbb{N}^{m-1}$ with

$$\frac{1}{m}l_1+\cdots+\frac{m-1}{m}l_{m-1}=k$$

and

$$m(m+1)k = (m+1)(l_1 + \cdots + (m-1)l_{m-1}) \\ \geq (m+1)l_1 + \frac{m+2}{2}2l_2 + \cdots + \frac{2m-1}{m-1}(m-1)l_{m-1} > \frac{\log A}{\log 2},$$

i.e.,

$$S_1 \ll_m \sum_{\substack{k \in (1/m)\mathbb{N} \\ m(m+1)k > \log A/\log 2}} k^{m-2}2^{-k} \ll_{m,\epsilon} A^{-\frac{1}{m(m+1)}+\epsilon}.$$

Note that the upper bound for S_1 also holds for $A \leq 1$ and $\epsilon < 1/(m(m+1))$.

We now turn to the sum in the statement of the lemma. If v_r is a square-free natural number and $d_r = \gcd(v_r^{e(m+r)}, q)$, then we can write

$$d_r = d_{r,1}d_{r,2}^2 \cdots d_{r,e(m+r)}^{e(m+r)}, \quad \mu^2(d_{r,j}) = 1 \text{ for all } 1 \leq j \leq e(m+r), \quad \gcd(d_{r,j}, d_{r,j'}) = 1 \text{ for all } j \neq j'.$$

Writing $v_r = v_r' \prod_{j=1}^{e(m+r)} d_{r,j}$ and $d_r' = \prod_{j=1}^{e(m+r)} d_{r,j}$, we find that

$$S_2 := \sum_{\substack{v_r \in \mathbb{N}, 1 \leq r \leq m-1 \\ \prod_{r=1}^{m-1} v_r^{m+r} > A}} \prod_{r=1}^{m-1} \frac{\mu^2(v_r)\gcd(v_r^{e(m+r)}, q)^{1/(em)}}{v_r^{(m+r)/m}}$$

$$\ll \sum_{\substack{d_{r,1}d_{r,2}^2 \cdots d_{r,e(m+r)}^{e(m+r)} \mid q \\ 1 \leq r \leq m-1}} \sum_{\substack{v_r' \in \mathbb{N}, 1 \leq r \leq m-1 \\ \prod_{r=1}^{m-1}(d_r'v_r')^{m+r} > A}} \prod_{r=1}^{m-1} \frac{d_r^{1/(em)}}{(d_r'v_r')^{(m+r)/m}}$$

$$\ll \sum_{\substack{d_{r,1}d_{r,2}^2 \cdots d_{r,e(m+r)}^{e(m+r)} \mid q \\ 1 \leq r \leq m-1}} \prod_{r=1}^{m-1}\left(\frac{d_r^{1/(em)}}{d_r'^{(m+r)/m}}\right) \sum_{\substack{v_r' \in \mathbb{N}, 1 \leq r \leq m-1 \\ \prod_{r=1}^{m-1}(d_r'v_r')^{m+r} > A}} \prod_{r=1}^{m-1} \frac{1}{(v_r')^{(m+r)/m}}.$$

By using the upper bound for S_1, we find that, for $\epsilon > 0$ sufficiently small,

$$S_2 \ll_{\epsilon,m} \sum_{\substack{d_{r,1}d_{r,2}^2 \cdots d_{r,e(m+r)}^{e(m+r)} \mid q \\ 1 \leq r \leq m-1}} \prod_{r=1}^{m-1}\left(\frac{d_r^{1/(em)}}{d_r'^{(m+r)/m}}\right) A^{-\frac{1}{m(m+1)}+\epsilon}\left(\prod_{r=1}^{m-1}(d_r')^{m+r}\right)^{\frac{1}{m(m+1)}}$$

$$\ll_{\epsilon,m} A^{-\frac{1}{m(m+1)}+\epsilon} \sum_{\substack{d_{r,1}d_{r,2}^2 \cdots d_{r,e(m+r)}^{e(m+r)} \mid q \\ 1 \leq r \leq m-1}} \prod_{r=1}^{m-1}\left(d_r^{\frac{1}{em}}(d_r')^{-\frac{m+r}{m+1}}\right)$$

$$\ll_{\epsilon,m} A^{-\frac{1}{m(m+1)}+\epsilon} \sum_{\substack{d_{r,1}d_{r,2}^2 \cdots d_{r,e(m+r)}^{e(m+r)} \mid q \\ 1 \leq r \leq m-1}} \prod_{r=1}^{m-1} d_r^{\frac{1}{em} - \frac{m+r}{e(m+r)(m+1)}}$$

$$\ll_{\epsilon,m} A^{-\frac{1}{m(m+1)}+\epsilon} \sum_{\substack{d_r \mid q \\ 1 \leq r \leq m-1}} \prod_{r=1}^{m-1} d_r^{\frac{1}{em(m+1)}} \ll_{\epsilon,m} A^{-\frac{1}{m(m+1)}+\epsilon} q^{\frac{m-1}{em(m+1)}+\epsilon}. \qquad \square$$

Lemma 6.2 shows that we can bound $f_2(q, T, \boldsymbol{s}, \boldsymbol{t})$ by

$$f_2(q, T, \boldsymbol{s}, \boldsymbol{t}) \ll \sum_{i=1}^{n} \left(\frac{T}{\tau_i}\right)^{-\frac{1}{m_i(m_i+1)}+\epsilon} q^{\frac{m_i-1}{em_i(m_i+1)}+\epsilon}.$$

In the following we write

$$\Delta_i = \frac{m_i - 1}{em_i(m_i+1)}.$$

Then (6-7) is bounded by

$$\begin{aligned} S_3 &:= B^\Gamma \sum_{i=1}^{n} \sum_{q=1}^{\infty} q^{-\Gamma/e+\Delta_i+\epsilon} \sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(\infty)} \prod_{j=1}^{n} \left(\frac{\gcd(\tau_j^e, q)}{\tau_j^e}\right)^{\frac{1}{em_j}} \left(\frac{B}{\tau_i}\right)^{-\frac{1}{m_i(m_i+1)}+\epsilon} \\ &\ll B^\Gamma \sum_{i=1}^{n} B^{-\frac{1}{m_i(m_i+1)}+\epsilon} \sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(\infty)} \sum_{q=1}^{\infty} q^{-\Gamma/e+\Delta_i+\epsilon} \tau_i^{\frac{1}{m_i(m_i+1)}} \prod_{j=1}^{n} \left(\frac{\gcd(\tau_j^e, q)}{\tau_j^e}\right)^{\frac{1}{em_j}}. \end{aligned}$$

As we will encounter similar expressions in our further analysis, we introduce, for $E, D > 0$ and d squarefree, the sum

$$S_d(D, E) := d^E \sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(\infty)} \sum_{q=1}^{\infty} q^{-\Gamma/e+D+\epsilon} \prod_{j=1}^{n} \left(\frac{\gcd(\tau_j^e, q)}{\tau_j^e}\right)^{\frac{1}{em_j}}.$$

We write $q = q_1 q_2$ with $\gcd(q_1, d) = 1$ and such that all prime divisors of q_2 divide d. We then obtain

$$S_d(D, E) \ll d^E \sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(\infty)} \sum_{q_1=1}^{\infty} q_1^{-\Gamma/e+D+\epsilon} \sum_{\substack{q_2=1 \\ p\mid q_2 \Rightarrow p\mid d}}^{\infty} q_2^{-\Gamma/e+D+\epsilon} \prod_{j=1}^{n} \left(\frac{\gcd(\tau_j^e, q_2)}{\tau_j^e}\right)^{\frac{1}{em_j}}.$$

If we assume $-\Gamma/e + D < -1$, then the sum over q_1 is absolutely convergent. For a given vector $(\boldsymbol{s}, \boldsymbol{t}) \in \mathcal{T}_d(\infty)$ and a prime p, we write $\tau_{j,p}$ for the power of p which exactly divides τ_j. We find that

$$S_d(D, E) \ll \sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(\infty)} d^E \prod_{p\mid d} \left(\sum_{l=0}^{\infty} p^{l(-\Gamma/e+D+\epsilon)} \prod_{j=1}^{n} \left(\frac{\gcd(\tau_{j,p}^e, p^l)}{\tau_{j,p}^e}\right)^{\frac{1}{em_j}} \right).$$

We now split the summation over l into the term $l = 0$, where we use the inequality $\tau_{j,p} \ge p^{m_j}$, and we bound the rest by a geometric sum for $l \ge 1$ using $\gcd(\tau_{j,p}^e, p^l) \le \tau_{j,p}^e$:

$$\begin{aligned} S_d(D, E) &\ll_D \sum_{(\boldsymbol{s},\boldsymbol{t})\in\mathcal{T}_d(\infty)} d^{E+\epsilon} \prod_{p\mid d} (p^{-n} + p^{-\Gamma/e+D+\epsilon}) \\ &\ll_D d^\epsilon \prod_{p\mid d} (p^{E-n} + p^{-\Gamma/e+D+E+\epsilon}). \end{aligned}$$

If $-\Gamma/e + D + E < -1$, then we deduce that

$$S_d(D, E) \ll_D d^{-1-\eta} \tag{6-9}$$

for some $\eta > 0$.

Applying (6-9) to S_3 with

$$D = \Delta_i \quad \text{and} \quad E = \frac{2m_i - 1}{m_i(m_i+1)},$$

we obtain $S_3 \ll B^{\Gamma-\eta} d^{-1-\eta}$ for some $\eta > 0$. Hence

$$F_d(B) = c_{e,d} B^{\Gamma} + O(B^{\Gamma}(d^{-1-\eta} B^{-\eta} + F_1 + F_2 + F_3)).$$

We use the bound in (6-8) and apply (6-9) with $D = E = 0$ to get $c_{e,d} \ll d^{-1-\eta}$.

It remains to estimate the error terms F_1, F_2, and F_3. We rewrite F_1 as

$$F_1 = B^{e\delta(2(n-1)+5)} \sum_{l=1}^{n} B^{-\frac{1}{m_l}} \sum_{(\boldsymbol{s},\boldsymbol{t}) \in \mathcal{T}_d(B)} \sum_{\tilde{\boldsymbol{v}} \in \mathcal{V}_{\boldsymbol{s},\boldsymbol{t}}(B)} \prod_{\substack{1 \le j \le n \\ j \ne l}} \gamma_j^{-\frac{1}{em_j}}.$$

As in [Browning and Yamagishi 2021, §3] and [Balestrieri et al. 2024, §6], we have

$$\begin{aligned} F_1 &\ll B^{-\frac{1}{m_n(m_n+1)} + e\delta(2(n-1)+5)} \sum_{(\boldsymbol{s},\boldsymbol{t}) \in \mathcal{T}_d(B)} \prod_{j=1}^{n} \tau_j^{-\frac{1}{m_j+1}} \\ &\ll d^{-\sum_{j=1}^{n} \frac{m_j}{m_j+1} + \varepsilon} B^{-\frac{1}{m_n(m_n+1)} + e\delta(2(n-1)+5)}, \end{aligned}$$

where the last estimate follows from

$$\begin{aligned} \sum_{(\boldsymbol{s},\boldsymbol{t}) \in \mathcal{T}_d(B)} \prod_{j=1}^{n} \tau_j^{-\frac{1}{m_j+1}} &\le \sum_{(\boldsymbol{s},\boldsymbol{t}) \in \mathcal{T}_d(B)} \prod_{j=1}^{n} \sigma_j^{-\frac{m_j}{m_j+1}} \\ &\le d^{-\sum_{j=1}^{n} \frac{m_j}{m_j+1}} \#\mathcal{T}_d(B) \ll d^{-\sum_{j=1}^{n} \frac{m_j}{m_j+1} + \varepsilon} \end{aligned}$$

by (6-2). Combining the arguments for F_3 in [Browning and Yamagishi 2021, §3] and in [Balestrieri et al. 2024, §6] and the estimate above, we have

$$F_3 \ll B^{-e\delta\Theta_e + \epsilon} \sum_{(\boldsymbol{s},\boldsymbol{t}) \in \mathcal{T}_d(B)} \prod_{j=1}^{n} \tau_j^{-\frac{1}{m_j+1}} \ll d^{-\sum_{j=1}^{n} \frac{m_j}{m_j+1} + \epsilon} B^{-e\delta\Theta_e + \epsilon}.$$

Since $\sum_{j=1}^{n} (m_j/(m_j+1)) \ge \frac{2}{3} n > 1$ is satisfied for $n \ge 2$, we have $F_1, F_3 \ll d^{-1-\eta} B^{-\eta}$ for a suitable $\eta > 0$. Since

$$F_2 = B^{-e\delta} \sum_{q=1}^{\infty} q^{1-\Gamma/e+\epsilon} f_1(q) f_2(q),$$

the estimate (6-8) combined with (6-9) for $D = 1$ and $E = 0$ yields $F_2 \ll d^{-1-\eta} B^{-e\delta}$, as $\Gamma/e > 2$. □

By Lemma 6.1 and [Pieropan and Schindler 2024, Lemma 5.6],

$$F_{\boldsymbol{d}}(B_1, \dots, B_s) = \prod_{i=1}^{s} (c_{M,i} B_i^{\varpi_i} + O(d_i^{\nu_i+\varepsilon} B_i^{\varpi_i - \delta})),$$

where

$$\varpi_i = \begin{cases} \sum_{j=1}^{n_i} \frac{1}{m_{i,j}} - e_i & \text{if } i \le t, \\ \sum_{j=1}^{n_i} \frac{1}{m_{i,j}} & \text{if } i > t, \end{cases} \tag{6-10}$$

$\nu_i < -1$ for $i \le t$, $\nu_i = -\frac{2}{3}n_i$ if $i > t$, $c_{M,i}$ is the constant c_{e_i,d_i} defined in (6-5) if $i \le t$, and also $c_{M,i} = 2^{n_i}\bigl(\prod_{j=1}^{n_i} c_{m_{i,j},d_i}\bigr)$, where $c_{m_{i,j},d_i}$ is the constant defined in [Pieropan and Schindler 2024, (5.11)].

Thus

$$F_{\boldsymbol{d}}(B_1, \dots, B_s) = C_{M,\boldsymbol{d}} \prod_{i=1}^{s} B_i^{\varpi_i} + O\Bigl(C_{E,\boldsymbol{d}}\bigl(\min_{1\le i\le s} B_i\bigr)^{-\delta} \prod_{i=1}^{s} B_i^{\varpi_i}\Bigr),$$

where

$$C_{M,\boldsymbol{d}} = \prod_{i=1}^{s} c_{M,i}. \tag{6-11}$$

Lemma 6.1 and [Pieropan and Schindler 2024, (5.14), (5.15)] give

$$C_{M,\boldsymbol{d}}, C_{E,\boldsymbol{d}} \ll \prod_{i=1}^{s} d_i^{-\beta_i}$$

with $\beta_i > 1$ whenever $n_i \ge 2$, and $\beta_i > \frac{2}{3} - \varepsilon$ otherwise. For $\varepsilon > 0$ sufficiently small, $\beta_i + \beta_j > 1$ for every $i, j \in \{1, \dots, s\}$. Thus, by Remark 2.4 (iv), if $f_{\beta_1,\dots,\beta_s} \le 1$, then there exists an index $\tilde{i} \in \{1, \dots, s\}$ such that $\tilde{i} \in I_\sigma$ for all $\sigma \in \Sigma_{\max}$ and $n_{\tilde{i}} = 1$. Then the ray $\rho_{\tilde{i},1}$ is contained in no maximal cone of Σ, contradicting the fact that X is proper.

Since $c_{i,j} \ne 0$ for all $i \in \{1, \dots, t\}$, $j \in \{1, \dots, n_i\}$, the adjunction formula [Corti 1992, Proposition 16.4] gives $K_V = (K_X + H_1 + \cdots + H_t)|_V$. Since

$$\begin{aligned}\sum_{i=1}^{s} \varpi_i \boldsymbol{D}_i &= -K_X - \sum_{i=1}^{s}\sum_{j=1}^{n_i}\Bigl(1 - \frac{1}{m_{i,j}}\Bigr)\boldsymbol{D}_i + \sum_{i=1}^{t} e_i \boldsymbol{D}_i \\ &= -(K_X + [\mathscr{D}_{\boldsymbol{m}}|_X] + [H_1 + \cdots + H_t]),\end{aligned}$$

Lemma 3.1 gives

$$N_V(B) = cB(\log B)^{b-1} + O(B(\log B)^{b-2}(\log\log B)^s),$$

where $b = \operatorname{rk}\operatorname{Pic}(X)$ and c is defined in (3-7) with $k = b - 1$, $C_{M,\boldsymbol{d}}$ given by (6-11), and $\varpi_1, \dots, \varpi_s$ given by (6-10). Moreover, the restriction $\operatorname{Pic}(X) \to \operatorname{Pic}(V)$ is an isomorphism as $n_i \ge 3$ for $1 \le i \le t$. □

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