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Perfectoid towers and their tilts: with an application to the étale cohomology groups of local log-regular rings

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To initiate a systematic study on the applications of perfectoid methods to Noetherian rings, we introduce the notions of perfectoid towers and their tilts. We mainly show that the tilting operation preserves several homological invariants and finiteness properties. Using this, we also provide a comparison result on étale cohomology groups under the tilting. As an application, we prove finiteness of the prime-to-p-torsion subgroup of the divisor class group of a local log-regular ring that appears in logarithmic geometry in the mixed characteristic case.

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1. Introduction

In recent years, the perfectoid technique has become one of the most effective tools in commutative ring theory and singularity theory in mixed characteristic. The *tilting operation* $S \rightsquigarrow S^\flat$ for a perfectoid ring S is a central notion in this method, which makes a bridge between objects in mixed characteristic and objects in positive characteristic. However, perfectoid rings themselves are too big to fit into Noetherian ring theory. Hence, for applications, one often requires distinguished Noetherian ring extensions that approximate perfectoids. Indeed, in many earlier works (such as [7], [8] and [17]), one constructs a highly ramified tower of regular local rings or local log-regular rings:

$$R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$$

that converges to a (pre)perfectoid ring. Our purposes in this paper are to axiomatize the above towers and establish a kind of Noetherization of perfectoid theory. As an application, we show a finiteness result on the divisor class groups of local log-regular rings.

Fix a prime p. The highly ramified towers in the positive characteristic case are of the form

$$R \subseteq R^{1/p} \subseteq R^{1/p^2} \subseteq \cdots .$$

This type of tower also appears when one considers the perfect closure of a reduced $\mathbb{F}_p$-algebra. Thus we formulate this class as a tower-theoretic analogue of perfect $\mathbb{F}_p$-algebras, and call them *perfect towers* (Definition 3.2). Next, we introduce *perfectoid towers* as a generalization of perfect towers, which includes the towers applied so far (cf. Proposition 3.58 and Example 3.62). A perfectoid tower is given by a direct system of rings $R_0 \xrightarrow{t_0} R_1 \xrightarrow{t_1} \cdots$ satisfying seven axioms in Definition 3.4 and Definition 3.21. If we assume that each R_i is Noetherian, then these axioms are essential to cope with two main difficulties which we explain below.

The first difficulty is that the residue ring R_i/pR_i on each layer is not necessarily semiperfect. We overcome it by axioms (b), (c), and (d); these ensure the existence of a surjective ring map $F_i : R_{i+1}/pR_{i+1} \to R_i/pR_i$ which gives a decomposition of the Frobenius endomorphism. We call F_i the *i-th Frobenius projection*, and define a ring $R_j^{s.\flat}$ ($j \geq 0$) as the inverse limit of Frobenius projections starting at R_j/pR_j. Then the resulting tower

$$R_0^{s.\flat} \xrightarrow{t_0^{s.\flat}} R_1^{s.\flat} \xrightarrow{t_1^{s.\flat}} \cdots$$

is perfect, and thus we obtain the tilting operation $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0}) \rightsquigarrow (\{R_i^{s.\flat}\}_{i\geq 0}, \{t_i^{s.\flat}\}_{i\geq 0})$. We remark that this strategy is an axiomatization of the principal arguments in [37].

The second one is that each $R_i^{s.\flat}$ could be imperfect. Because of this, the Witt ring $W(R_i^{s.\flat})$ is often uncontrollable. On the other hand, the definition of Bhatt–Morrow–Scholze's perfectoid rings [5] contains an axiom involving Fontaine's theta map $\theta_S : W(S^\flat) \to S$ (see Definition 3.49(3)), where perfectness of $S^\flat$ is quite effective. Our axioms (f) and (g) are the substitutes for it; these require the Frobenius projections to behave well, especially on the p-torsion parts. This idea is closely related to Gabber and Ramero's characterization of perfectoid rings ([17, Corollary 16.3.75]; see also Theorem 3.50). Indeed, we apply it to deduce that the completed direct limit of a perfectoid tower is a perfectoid ring (Corollary 3.52).

We then verify fundamental properties of the tilting operation for towers. For example, the tilt $(\{R_i^{s.\flat}\}_{i\geq 0}, \{t_i^{s.\flat}\}_{i\geq 0})$ is a perfectoid tower with respect to an ideal $I_0^{s.\flat} \subseteq R_0^{s.\flat}$ which is the kernel of the 0-th projection $R_0^{s.\flat} \to R_0/pR_0$ (Proposition 3.41). It induces an isomorphism between two perfectoid objects of different characteristics modulo the defining ideals (Lemma 3.39). Moreover, this operation preserves several finiteness properties such as Noetherianness on each layer (Proposition 3.42). A key to deducing these statements is the following result (see Remark 3.40 for homological interpretation).

Main Theorem 1 (see Theorem 3.35). *$I_0^{s.\flat}$ is a principal ideal. Moreover, we have isomorphisms of (possibly) nonunital rings $(R_i^{s.\flat})_{I_0^{s.\flat}\text{-tor}} \cong (R_i)_{p\text{-tor}}$ $(i \geq 0)$ that are compatible with $\{t_i^{s.\flat}\}_{i\geq 0}$ and $\{t_i\}_{i\geq 0}$.*

Under certain normality assumptions, we obtain a comparison result on the finiteness of étale cohomology groups under tilting for towers (Proposition 4.7). This proposition is considered to rework the crucial part of the proof of [8, Theorem 3.1.3] in a systematic way. Actually, our proposition applies beyond the regular case.

As a typical example, we investigate certain towers of *local log-regular rings*; this class of rings is defined by Kazuya Kato, and is central to logarithmic geometry (readers interested in logarithmic geometry can refer to [17], [26] and [34]). By Kato's structure theorem, a complete local log-regular ring $(R, \mathcal{Q}, \alpha)$ of mixed characteristic is of the form $C(k)[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]/(p - f)$ where $C(k)$ is a Cohen ring of the residue field k of R (see Theorem 2.22). Gabber and Ramero gave a systematic way to build a perfectoid tower (in our sense) over it, which consists of local log-regular rings (Construction 3.56). In this paper, we reveal that its tilt also consists of local log-regular rings, and arises from $C(k)[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]/(p)$ (Theorem 3.61). It says that these two rings on the starting layers fit into a Noetherian variant of the tilting correspondence in perfectoid theory (e.g. $\mathbb{Z}_p$ corresponds to $\mathbb{F}_p[\![x]\!]$).

We regard Theorem 3.61 to be of fundamental importance in the search on the singularities of Noetherian rings via perfectoid methods. For instance, we can investigate the *divisor class groups* of local log-regular rings.[1] The divisor class group of a Noetherian normal domain is an important invariant, but it is often hard to compute.[2] On the other hand, Polstra recently proved a remarkable result stating that the torsion subgroup of the divisor class group of an F-finite strongly F-regular domain is finite [35]. Based on this result, we obtain the following finiteness theorem.

Main Theorem 2 (Theorem 4.13). *Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring of mixed characteristic with perfect residue field k of characteristic $p > 0$, and denote by $\mathrm{Cl}(R)$ the divisor class group with its torsion subgroup $\mathrm{Cl}(R)_{\mathrm{tor}}$. Assume that $\widehat{R^{\mathrm{sh}}}\left[\frac{1}{p}\right]$ is locally factorial, where $\widehat{R^{\mathrm{sh}}}$ is the completion of the strict Henselization R^{sh}. Then $\mathrm{Cl}(R)_{\mathrm{tor}} \otimes \mathbb{Z}\left[\frac{1}{p}\right]$ is a finite group. In other words, the ℓ-primary subgroup of $\mathrm{Cl}(R)_{\mathrm{tor}}$ is finite for all primes $\ell \neq p$ and vanishes for almost all primes $\ell \neq p$.*

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[1] K. Kato proved that a local log-regular ring is a normal domain [26].

[2] Every abelian group is realized as a divisor class group of some Dedekind domain (due to Claborn's result [9]).

Our approach to the above theorem is a combination of Theorem 3.61 and Proposition 4.7.

Although we formulated the above theorem only in mixed characteristic, it has an analogue in characteristic $p > 0$, which is relatively easy as follows from the fact that F-finite log-regular rings are strongly F-regular (Lemma 2.25) combined with Polstra's theorem.

For a local log-regular ring $(R, \mathcal{Q}, \alpha)$, Gabber and Ramero constructed the isomorphism $\mathrm{Cl}(\mathcal{Q}) \cong \mathrm{Cl}(R)$ where $\mathrm{Cl}(\mathcal{Q})$ is the divisor class group of the associated monoid [17, Corollary 12.6.43]. It induces the finite generation of $\mathrm{Cl}(R)$.[3]

Recently, H. Cai, S. Lee, L. Ma, K. Schwede, and K. Tucker proved that the torsion part of the divisor class group of a BCM-regular ring is finite (see [6, Theorem 7.0.10.]). Since they also proved that local log-regular rings are BCM-regular, their result recovers Main Theorem 2. Although their approach relies on the evaluation of a certain inequality with the perfectoid signature which is defined in [6] as an analogue of F-signature, it does not use a reduction to positive characteristic and is therefore essentially different from our approach.

Outline. In Section 2, we discuss several properties of monoids and local log-regular rings needed in later sections. We also record a shorter proof of the result that *local log-regular rings are splinter* based on the direct summand theorem in Section 2C.

In Section 3, we introduce the notions of perfect towers, perfectoid towers, and their tilts. The most part of this section is devoted to studying fundamental properties of them; in particular, Section 3D deals with Main Theorem 1. In the last subsection Section 3F, we provide explicit examples of perfectoid towers consisting of local log-regular rings, and compute their tilts.

In Section 4, we give a proof for Main Theorem 2 using the tilting operation, which is an application of Sections 2 and 3.

In the Appendix, we review the notion of *maximal sequences* associated to certain differential modules due to Gabber and Ramero [17]. This plays an important role in the construction of perfectoid towers of local log-regular rings (Construction 3.56).

Conventions. • We consistently fix a prime $p > 0$. If we need to refer to another prime, we denote it by ℓ.

• All rings are assumed to be commutative and unital (unless otherwise stated; cf. Theorem 3.35(2)). We mean by a *ring map* a unital ring homomorphism.

• A local ring is a (not necessarily Noetherian) ring with a unique maximal ideal. When a ring R is local, then we use $\mathfrak{m}_R$ (or simply $\mathfrak{m}$ if no confusion is likely) to denote its unique maximal ideal. We say that a ring map $f : R \to S$ is *local* if R and S are local rings and $f^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$.

• Unless otherwise stated, a pair (A, I) consisting of a ring A and an ideal $I \subseteq A$ will be simply called a *pair*.

• The Frobenius endomorphism on an $\mathbb{F}_p$-algebra R is denoted by F_R. If there is no confusion, we denote it by Frob.

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[3] The first-named author recently provided an elementary proof of [17, Corollary 12.6.43]. See [25].

2. Log-regularity

In this section, we discuss several properties of monoids and local log-regular rings. In Section 2A, we review basic terms on monoids, and examine the behavior of p-times maps which are effectively used in Gabber and Ramero's treatment of perfectoid towers (see Construction 3.56). In Section 2B, we review the definition of local log-regular rings and crucial results by K. Kato, and study the relationship with strong F-regularity. In Section 2C, we recall Gabber and Ramero's result which claims that *any local log-regular ring is a splinter* (Theorem 2.29), and give an alternative proof for it using the direct summand Theorem by Y. André [2] (its derived variant is proved by B. Bhatt [4]).

2A. *Preliminaries on monoids.*

2A1. *Basic terms.* Here we review the definition of several notions on monoids.

Definition 2.1. A *monoid* is a semigroup with a unit. A *homomorphism of monoids* is a semigroup homomorphism between monoids that sends a unit to a unit.

Throughout this paper, all monoids are assumed to be commutative. We denote by **Mnd** the category whose objects are (commutative) monoids and whose morphisms are homomorphisms of monoids.

We denote a unit by 0. Let Q be a monoid and Q^* denote the set of all $p \in Q$ such that there exists $q \in Q$ such that $p+q=0$. Let Q^{gp} denote the set of elements $a-b$ for all $a, b \in Q$, where $a-b=a'-b'$ if and only if there exists $c \in Q$ such that $a+b'+c=a'+b+c$. By definition, Q^{gp} is an abelian group. The following conditions yield good classes of monoids.

Definition 2.2. Let Q be a monoid.

(1) Q is called *integral* if for x, x' and $y \in Q$, $x+y=x'+y$ implies $x=x'$.

(2) Q is called *fine* if it is finitely generated and integral.

(3) Q is called *sharp* if $Q^*=0$.

(4) Q is called *saturated* if the following conditions hold.
  (a) Q is integral.
  (b) For any $x \in Q^{gp}$, if $nx \in Q$ for some $n \geq 1$, then $x \in Q$.

For an integral monoid Q, the map $\iota_Q : Q \to Q^{gp}$; $q \mapsto q-0$ is injective (see [34, Chapter I, Proposition 1.3.3]). In Definition 2.2(4), we identify Q with its image in Q^{gp}.

Next we recall the definition of a module over a monoid.[4]

Definition 2.3 (Q-module). Let Q be a monoid.

(1) A *Q-module* is a set M equipped with a binary operation

$$Q \times M \to M\ ;\ (q, x) \mapsto q+x$$

having the following properties:

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[4] This is called a *Q-set in* [34]. We call it as above to follow the convention of the terminology in commutative ring theory.

 (a) $0+x=x$ for any $x \in M$;
 (b) $(p+q)+x=p+(q+x)$ for any $p, q \in \mathcal{Q}$ and $x \in M$.

(2) A *homomorphism of $\mathcal{Q}$-modules* is a (set-theoretic) map $f : M \to N$ between $\mathcal{Q}$-modules such that $f(q+x)=q+f(x)$ for any $q \in \mathcal{Q}$ and $x \in M$. We denote by $\mathcal{Q}$-**Mod** the category of $\mathcal{Q}$-modules and homomorphisms of $\mathcal{Q}$-modules.

For a monoid $\mathcal{Q}$ and a family of $\mathcal{Q}$-modules $\{M_i\}_{i\in I}$, we denote by $\coprod_{i\in I} M_i$ the disjoint union with the binary operation induced by that of each M_i. Then it is the coproduct in $\mathcal{Q}$-**Mod**.

Definition 2.4 (Monoid algebras). Let R be a ring and let $\mathcal{Q}$ be a monoid. Then the *monoid algebra* $R[\mathcal{Q}]$ is the R-algebra which is the free R-module $R^{\oplus \mathcal{Q}}$, endowed with the unique ring structure induced by the homomorphism of monoids

$$\mathcal{Q} \to R[\mathcal{Q}] \ ; \ q \mapsto e^q.$$

For a monoid $\mathcal{Q}$, one obtains the functor

$$\mathcal{Q}\text{-}\mathbf{Mod} \to R[\mathcal{Q}]\text{-}\mathbf{Mod} \ ; \ M \mapsto R[M], \tag{2-1}$$

which is a left adjoint of the forgetful functor $R[\mathcal{Q}]$-**Mod** $\to$ $\mathcal{Q}$-**Mod**. Notice that (2-1) preserves coproducts (we use this property to prove Proposition 2.8).

Like ideals (resp. prime ideals, the Krull dimension) of a ring, an ideal (resp. prime ideals, the dimension) of a monoid is defined as follows.

Definition 2.5. Let $\mathcal{Q}$ be a monoid.

(1) A $\mathcal{Q}$-submodule of $\mathcal{Q}$ is called an *ideal of $\mathcal{Q}$*.

(2) An ideal I is called *prime* if $I \neq \mathcal{Q}$ and $p+q \in I$ implies $p \in I$ or $q \in I$. Remark that the empty set $\varnothing$ is a prime ideal of $\mathcal{Q}$.

(3) The *dimension* of a monoid $\mathcal{Q}$ is the maximal length d of the ascending chain[5] of prime ideals

$$\varnothing = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_d = \mathcal{Q}^+,$$

where $\mathcal{Q}^+$ is the set of non-unit elements of $\mathcal{Q}$ (i.e. $\mathcal{Q}^+ = \mathcal{Q} \setminus \mathcal{Q}^*$). We also denote it by $\dim \mathcal{Q}$.

Next we review a good class of homomorphisms of monoids, called *exact homomorphisms*.

Definition 2.6 (Exact homomorphisms). Let $\mathcal{P}$ and $\mathcal{Q}$ be monoids.

(1) A homomorphism of monoids $\varphi : \mathcal{P} \to \mathcal{Q}$ is said to be *exact* if the diagram of monoids

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "\varphi"] \arrow[d] & \mathcal{Q} \arrow[d] \\
\mathcal{P}^{gp} \arrow[r, "\varphi^{gp}"] & \mathcal{Q}^{gp}
\end{tikzcd}$$

is cartesian.

---

[5] In this paper, the symbol $\subset$ is used to indicate *proper* inclusion for making an analogy to the inequality symbols as in [34].

(2) An *exact submonoid* of $\mathcal{Q}$ is a submonoid $\mathcal{Q}'$ of $\mathcal{Q}$ such that the inclusion map $\mathcal{Q}' \hookrightarrow \mathcal{Q}$ is exact (in other words, $(\mathcal{Q}')^{gp} \cap \mathcal{Q} = \mathcal{Q}'$).

There is a quite useful characterization of exact submonoids (Proposition 2.8). To see this, we recall a graded decomposition of a $\mathcal{Q}$-module attached to a submonoid. For a monoid $\mathcal{Q}$ and a submonoid $\mathcal{Q}' \subseteq \mathcal{Q}$, we denote by $\mathcal{Q} \to \mathcal{Q}/\mathcal{Q}'$ the cokernel of the inclusion map $\mathcal{Q}' \hookrightarrow \mathcal{Q}$.

Definition 2.7. Let $\mathcal{Q}$ be an integral monoid, and let $\mathcal{Q}' \subseteq \mathcal{Q}$ be a submonoid. Then for any $g \in \mathcal{Q}/\mathcal{Q}'$, we denote by $\mathcal{Q}_g$ a $\mathcal{Q}'$-module defined as follows.

- As a set, $\mathcal{Q}_g$ is the inverse image of $g \in \mathcal{Q}/\mathcal{Q}'$ under the cokernel $\mathcal{Q} \to \mathcal{Q}/\mathcal{Q}'$ of $\mathcal{Q}' \hookrightarrow \mathcal{Q}$.
- The operation $\mathcal{Q}' \times \mathcal{Q}_g \to \mathcal{Q}_g$ is defined by the rule $(q, x) \mapsto q + x$ (where $q + x$ denotes the sum of q and x in $\mathcal{Q}$).

By definition, $\mathcal{Q} = \coprod_{g \in \mathcal{Q}/\mathcal{Q}'} \mathcal{Q}_g$ in $\mathcal{Q}'$-**Mod**. Using this, one can refine a characterization of exact embeddings described in [34, Chapter I, Proposition 4.2.7].

Proposition 2.8 (cf. [34, Chapter I, Proposition 4.2.7]). *Let $\mathcal{Q}$ be an integral monoid, and let $\mathcal{Q}' \subseteq \mathcal{Q}$ be a submonoid. Let $\theta : \mathcal{Q}' \hookrightarrow \mathcal{Q}$ be the inclusion map, and let $\mathbb{Z}[\theta] : \mathbb{Z}[\mathcal{Q}'] \to \mathbb{Z}[\mathcal{Q}]$ be the induced ring map. Set $G := \mathcal{Q}/\mathcal{Q}'$.*

(1) *The $\mathbb{Z}[\mathcal{Q}']$-module $\mathbb{Z}[\mathcal{Q}]$ admits a G-graded decomposition $\mathbb{Z}[\mathcal{Q}] = \bigoplus_{g \in G} \mathbb{Z}[\mathcal{Q}_g]$.*

(2) *The following conditions are equivalent.*

  (a) *The inclusion map $\theta : \mathcal{Q}' \hookrightarrow \mathcal{Q}$ is exact. In other words, $(\mathcal{Q}')^{gp} \cap \mathcal{Q} = \mathcal{Q}'$.*
  (b) $\mathcal{Q}_0 = \mathcal{Q}'$.
  (c) $\mathbb{Z}[\theta]$ *splits as a $\mathbb{Z}[\mathcal{Q}']$-linear map.*
  (d) $\mathbb{Z}[\theta]$ *is equal to the canonical embedding $\mathbb{Z}[\mathcal{Q}_0] \hookrightarrow \bigoplus_{g \in G} \mathbb{Z}[\mathcal{Q}_g]$.*
  (e) $\mathbb{Z}[\theta]$ *is universally injective.*

Proof. (1) By applying the functor (2-1) (which admits a right adjoint) to the decomposition $\mathcal{Q} = \coprod_{g \in G} \mathcal{Q}_g$, we find that the assertion follows.

(2) Since $\mathcal{Q}_0 = (\mathcal{Q}')^{gp} \cap \mathcal{Q}$ as sets by definition, the equivalence (a)$\Leftrightarrow$(b) follows. The assertion (a)$\Leftrightarrow$(c)$\Leftrightarrow$(e) is none other than [34, Chapter I, Proposition 4.2.7]. Moreover, (d) implies (c) obviously. Thus it suffices to show the implication (b)$\Rightarrow$(d). Assume that (b) is satisfied. Then one can decompose $\mathcal{Q}$ into the direct sum of $\mathcal{Q}'$-modules $\coprod_{g \in G} \mathcal{Q}_g$ with $\mathcal{Q}_0 = \mathcal{Q}'$. Hence the inclusion map $\mathcal{Q}' \hookrightarrow \mathcal{Q}$ is equal to the canonical embedding $\mathcal{Q}_0 \hookrightarrow \coprod_{g \in G} \mathcal{Q}_g$. Thus the induced homomorphism $\mathbb{Z}[\theta] : \mathbb{Z}[\mathcal{Q}_0] \hookrightarrow \mathbb{Z}\big[\coprod_{g \in G} \mathcal{Q}_g\big]$ satisfies (d), as desired. □

Remark 2.9. In the situation of Proposition 2.8, assume that condition (d) is satisfied. Then the split surjection $\pi : \mathbb{Z}[\mathcal{Q}] \to \mathbb{Z}[\mathcal{Q}']$ has the property that $\pi(e^{\mathcal{Q}}) = e^{\mathcal{Q}'}$ by the construction of the G-graded decomposition $\mathbb{Z}[\mathcal{Q}] = \bigoplus_{g \in G} \mathbb{Z}[\mathcal{Q}_g]$. Moreover, $\pi(e^{\mathcal{Q}^+}) \subseteq e^{(\mathcal{Q}')^+}$ because $\mathcal{Q}^+ \cap \mathcal{Q}' \subseteq (\mathcal{Q}')^+$. We use this fact in our proof for Theorem 2.29.

Proposition 2.8 implies the following useful lemma.

Lemma 2.10. *Let $\mathcal{Q}$ be a fine, sharp, and saturated monoid. Let A be a ring. Then there is an embedding of monoids $\mathcal{Q} \hookrightarrow \mathbb{N}^d$ such that the induced map of monoid algebras*

$$A[\mathcal{Q}] \to A[\mathbb{N}^d] \tag{2-2}$$

splits as an $A[\mathcal{Q}]$-linear map.

Proof. Since $\mathcal{Q}$ is saturated, there exists an embedding $\mathcal{Q}$ into some $\mathbb{N}^d$ as an exact submonoid in view of [34, Chapter I, Corollary 2.2.7]. Then by Proposition 2.8, the associated map of monoid algebras

$$\mathbb{Z}[\mathcal{Q}] \to \mathbb{Z}[\mathbb{N}^d] \tag{2-3}$$

splits as a $\mathbb{Z}[\mathcal{Q}]$-linear map. By tensoring (2-3) with A, we get the desired split map. □

2A2. *c-times maps on integral monoids.* For an integral monoid $\mathcal{Q}$, we denote by $\mathcal{Q}_{\mathbb{Q}}$ the submonoid of $\mathcal{Q}^{gp} \otimes_{\mathbb{Z}} \mathbb{Q}$ defined as

$$\mathcal{Q}_{\mathbb{Q}} := \{x \otimes r \in \mathcal{Q}^{gp} \otimes_{\mathbb{Z}} \mathbb{Q} \mid x \in \mathcal{Q},\ r \in \mathbb{Q}_{\geq 0}\}.$$

Using this, one can define the following monoid which plays a central role in Gabber and Ramero's construction of perfectoid towers consisting of local log-regular rings.

Definition 2.11. Let $\mathcal{Q}$ be an integral monoid. Let c and i be non-negative integers with $c > 0$.

(1) We denote by $\mathcal{Q}_c^{(i)}$ the submonoid of $\mathcal{Q}_{\mathbb{Q}}$ defined as

$$\mathcal{Q}_c^{(i)} := \{\gamma \in \mathcal{Q}_{\mathbb{Q}} \mid c^i\gamma \in \mathcal{Q}\}.$$

(2) We denote by $\iota_c^{(i)} : \mathcal{Q}_c^{(i)} \hookrightarrow \mathcal{Q}_c^{(i+1)}$ the inclusion map, and by $\mathbb{Z}[\iota_c^{(i)}] : \mathbb{Z}[\mathcal{Q}_c^{(i)}] \to \mathbb{Z}[\mathcal{Q}_c^{(i+1)}]$ the induced ring map.

In the rest of this subsection, we fix a positive integer $c > 0$. To prove several properties of $\mathcal{Q}_c^{(i)}$, the following one is important as a starting point.

Lemma 2.12. *Let $\mathcal{Q}$ be an integral monoid. Then for every $i \geq 0$, the following assertions hold.*

(1) *$\mathcal{Q}_c^{(i)}$ is integral.*

(2) $\mathcal{Q}_c^{(i+1)} = (\mathcal{Q}_c^{(i)})_c^{(1)}$.

(3) *The c-times map on $\mathcal{Q}_{\mathbb{Q}}$ restricts to an isomorphism of monoids*:

$$f_c : \mathcal{Q}_c^{(i+1)} \xrightarrow{\cong} \mathcal{Q}_c^{(i)}\ ;\ \gamma \mapsto c\gamma.$$

Proof. (1) Since $\mathcal{Q}^{gp} \otimes_{\mathbb{Z}} \mathbb{Q}$ is an integral monoid, so is $\mathcal{Q}_c^{(i)}$.

(2) Since any $g \in (\mathcal{Q}_c^{(i)})^{gp}$ satisfies $c^i g \in \mathcal{Q}^{gp}$, the inclusion map $\mathcal{Q}^{gp} \hookrightarrow (\mathcal{Q}_c^{(i)})^{gp}$ becomes an isomorphism $\varphi : \mathcal{Q}^{gp} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} (\mathcal{Q}_c^{(i)})^{gp} \otimes_{\mathbb{Z}} \mathbb{Q}$ by extension of scalars along the flat ring map $\mathbb{Z} \to \mathbb{Q}$. The restriction $\widetilde{\varphi} : \mathcal{Q}_{\mathbb{Q}} \hookrightarrow (\mathcal{Q}_c^{(i)})_{\mathbb{Q}}$ of φ is also an isomorphism, and one can easily check that $\widetilde{\varphi}$ restricts to the desired canonical isomorphism $\mathcal{Q}_c^{(i+1)} \xrightarrow{\cong} (\mathcal{Q}_c^{(i)})_c^{(1)}$.

(3) It is easy to see that the c-times map on $\mathcal{Q}_{\mathbb{Q}}$ restricts to the homomorphism of monoids f_c. Since the abelian group $\mathcal{Q}_{\mathbb{Q}} = \mathcal{Q}^{gp} \otimes_{\mathbb{Z}} \mathbb{Q}$ is torsion-free, f_c is injective. Moreover, any element γ in $\mathcal{Q}_c^{(i)}$ is of the form $x \otimes r$ for some $x \in \mathcal{Q}^{gp}$ and $r \in \mathbb{Q}$, which satisfy $c\big(x \otimes \frac{r}{c}\big) = \gamma$ and $c^{i+1}\big(x \otimes \frac{r}{c}\big) \in \mathcal{Q}$. Hence f_c is also surjective, as desired. □

Let us inspect monoid-theoretic aspects of the inclusion $\iota_c^{(i)} : \mathcal{Q}_c^{(i)} \hookrightarrow \mathcal{Q}_c^{(i+1)}$.

Lemma 2.13. *Let $\mathcal{Q}$ be an integral monoid, and let $\boldsymbol{P} \in \{$fine, sharp, saturated$\}$. If $\mathcal{Q}$ satisfies $\boldsymbol{P}$, then $\mathcal{Q}_c^{(i)}$ also satisfies $\boldsymbol{P}$ for every $i \geq 0$.*

Proof. Assume that $\mathcal{Q}$ is sharp. Pick $x, y \in \mathcal{Q}_c^{(i)}$ such that $x + y = 0$. Then $c^i x = 0$ because $\mathcal{Q}$ is sharp. Since $\mathcal{Q}_c^{(i)}$ is a submonoid of the torsion-free group $\mathcal{Q}^{gp} \otimes_{\mathbb{Z}} \mathbb{Q}$, we have $x = 0$. Next, if $\mathcal{Q}$ is fine or saturated, then it suffices to show the case $i = 1$ by Lemma 2.12(2). If $\mathcal{Q}$ is fine, then there exists a finite system of generators $\{x_1, \dots, x_r\}$ of $\mathcal{Q}$. Hence $\mathcal{Q}_c^{(1)}$ also has a finite system of generators $\big\{x_j \otimes \frac{1}{c}\big\}_{j=1,\dots,r}$. Finally, assume that $\mathcal{Q}_c^{(1)}$ is saturated. Pick an element x of $(\mathcal{Q}_c^{(1)})^{gp}$ such that $nx \in \mathcal{Q}_c^{(1)}$. Then the element cx of $\mathcal{Q}^{gp}$ satisfies $n(cx) = c(nx) \in \mathcal{Q}$. Hence $cx \in \mathcal{Q}$ because $\mathcal{Q}$ is saturated. □

The assumption of fineness on $\mathcal{Q}$ induces several finiteness properties.

Lemma 2.14. *Let $\mathcal{Q}$ be a fine monoid. Then for every $i \geq 0$, the following assertions hold.*

(1) *The ring map $\mathbb{Z}[\iota_c^{(i)}] : \mathbb{Z}[\mathcal{Q}_c^{(i)}] \to \mathbb{Z}[\mathcal{Q}_c^{(i+1)}]$ is module-finite.*

(2) *$\mathcal{Q}_c^{(i+1)}/\mathcal{Q}_c^{(i)} \cong (\mathcal{Q}_c^{(i+1)})^{gp}/(\mathcal{Q}_c^{(i)})^{gp}$ as monoids. Moreover, $\mathcal{Q}_c^{(i+1)}/\mathcal{Q}_c^{(i)}$ forms a finite abelian group.*

(3) *For a prime $p > 0$, we have $\big|\mathcal{Q}_p^{(i+1)}/\mathcal{Q}_p^{(i)}\big| = p^s$ for some $s \geq 0$.*

Proof. In view of Lemma 2.12(2), it suffices to deal with the case when $i = 0$ only. Here notice that $\mathcal{Q}_c^{(0)} = \mathcal{Q}$.

(1) Let $\big\{\frac{1}{c}x_1, \dots, \frac{1}{c}x_r\big\}$ be the system of generators of $\mathcal{Q}_c^{(1)}$ obtained in the proof of Lemma 2.13 where $\frac{1}{c}x_j := x_j \otimes \frac{1}{c}$. Then the $\mathbb{Z}[\mathcal{Q}]$-algebra $\mathbb{Z}[\mathcal{Q}_c^{(1)}]$ is generated by $\{e^{\frac{1}{c}x_1}, \dots, e^{\frac{1}{c}x_r}\}$, and each $e^{\frac{1}{c}x_j} \in \mathbb{Z}[\mathcal{Q}_c^{(1)}]$ is integral over $\mathbb{Z}[\mathcal{Q}]$. Hence $\mathbb{Z}[\iota_c^{(0)}]$ is module-finite, as desired.

(2) By [34, Chapter I, Proposition 1.3.3], $\mathcal{Q}_c^{(1)}/\mathcal{Q}$ is identified with the image of the composition

$$\mathcal{Q}_c^{(1)} \hookrightarrow (\mathcal{Q}_c^{(1)})^{gp} \twoheadrightarrow (\mathcal{Q}_c^{(1)})^{gp}/\mathcal{Q}^{gp}. \tag{2-4}$$

Since $\mathcal{Q}_c^{(1)}$ is generated by $\frac{1}{c}x_1, \dots, \frac{1}{c}x_r$, we see $(\mathcal{Q}_c^{(1)})^{gp}$ is generated by $\frac{1}{c}x_1, \dots, \frac{1}{c}x_r, -\frac{1}{c}x_1, \dots, -\frac{1}{c}x_r$ as a monoid. On the other hand, $-\frac{1}{c}x_j \equiv (c-1)\frac{1}{c}x_j \bmod \mathcal{Q}^{gp}$ for $j = 1, \dots, r$. Hence $(\mathcal{Q}_c^{(1)})^{gp}/\mathcal{Q}^{gp}$ is generated by $\big\{\frac{1}{c}x_j \bmod \mathcal{Q}^{gp}\big\}_{j=1,\dots,r}$ as a monoid. Therefore, the composite map (2-4) is surjective, and $(\mathcal{Q}_c^{(1)})^{gp}/\mathcal{Q}^{gp}$ is a finitely generated torsion abelian group. Thus, $\mathcal{Q}_c^{(1)}/\mathcal{Q}$ coincides with $(\mathcal{Q}_c^{(1)})^{gp}/\mathcal{Q}^{gp}$, which is a finite abelian group, as desired.

(3) Since there exists a surjective group homomorphism

$$f : \underbrace{\mathbb{Z}/p\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z}}_{r} \twoheadrightarrow (\mathcal{Q}_p^{(1)})^{gp}/\mathcal{Q}^{gp} \ ; \ (\overline{n}_1, \dots, \overline{n}_r) \mapsto n_1\Big(\frac{1}{p}x_1\Big) + \cdots + n_r\Big(\frac{1}{p}x_r\Big) \mod \mathcal{Q}^{gp},$$

we have $p^r = \big|(\mathcal{Q}_p^{(1)})^{gp}/\mathcal{Q}^{gp}\big|\,|\operatorname{Ker}(f)|$. Hence $\big|(\mathcal{Q}_p^{(1)})^{gp}/\mathcal{Q}^{gp}\big| = p^s$ for some $s \geq 0$. Thus the assertion follows from (2). ☐

By assuming saturatedness, one finds the exactness of $\iota_c^{(i)} : \mathcal{Q}_c^{(i)} \hookrightarrow \mathcal{Q}_c^{(i+1)}$.

Lemma 2.15. *Let $\mathcal{Q}$ be a saturated monoid. Then for every $i \geq 0$, $\iota_c^{(i)} : \mathcal{Q}_c^{(i)} \hookrightarrow \mathcal{Q}_c^{(i+1)}$ is exact* (*i.e.*, $\mathcal{Q}_c^{(i+1)} \cap (\mathcal{Q}_c^{(i)})^{gp} = \mathcal{Q}_c^{(i)}$).

Proof. It suffices to show that $\mathcal{Q}_c^{(i+1)} \cap (\mathcal{Q}_c^{(i)})^{gp} \subseteq \mathcal{Q}_c^{(i)}$. Pick an element $a \in \mathcal{Q}_c^{(i+1)} \cap (\mathcal{Q}_c^{(i)})^{gp}$. Then $ca \in \mathcal{Q}_c^{(i)}$. Since $\mathcal{Q}_c^{(i)}$ is saturated by Lemma 2.13, it implies that $a \in \mathcal{Q}_c^{(i)}$, as desired. ☐

If further $\mathcal{Q}$ is fine, one can learn more about $\mathbb{Z}[\iota_c^{(i)}] : \mathbb{Z}[\mathcal{Q}_c^{(i)}] \to \mathbb{Z}[\mathcal{Q}_c^{(i+1)}]$ using the exactness of $\iota_c^{(i)}$ assured by Lemma 2.15.

Lemma 2.16. *Let $\mathcal{Q}$ be a fine and saturated monoid. For every $i \geq 0$, set $G_i := \mathcal{Q}_c^{(i+1)}/\mathcal{Q}_c^{(i)}$* (*which is a finite abelian group by Lemma 2.14*(2)) *and $K_i := \operatorname{Frac}(\mathbb{Z}[\mathcal{Q}_c^{(i)}])$.*

(1) *For any $g \in G_i$, we have an isomorphism of $\mathbb{Z}[\mathcal{Q}_c^{(i)}]$-modules $\mathbb{Z}[(\mathcal{Q}_c^{(i+1)})_g] \otimes_{\mathbb{Z}[\mathcal{Q}_c^{(i)}]} K_i \cong K_i$.*

(2) *The base extension $K_i \to \mathbb{Z}[\mathcal{Q}_c^{(i+1)}] \otimes_{\mathbb{Z}[\mathcal{Q}_c^{(i)}]} K_i$ of $\mathbb{Z}[\iota_c^{(i)}]$ is isomorphic to the split injection*

$$K_i \hookrightarrow (K_i)^{\oplus |G_i|}\ ;\ a \mapsto (a, 0, \dots, 0)$$

as a K_i-linear map. In particular, $\dim_{K_i}\big(\mathbb{Z}[\mathcal{Q}_c^{(i+1)}] \otimes_{\mathbb{Z}[\mathcal{Q}_c^{(i)}]} K_i\big) = |\mathcal{Q}_c^{(i+1)}/\mathcal{Q}_c^{(i)}|$.

Proof. In view of Lemma 2.12(2) and Lemma 2.13, it suffices to show the assertions only for the case when $i = 0$.

(1) Let $y_g \in \mathcal{Q}_c^{(1)}$ be an element whose image in $\mathcal{Q}_c^{(1)}/\mathcal{Q}$ is equal to g. Then we obtain an injective homomorphism of $\mathcal{Q}$-modules

$$\iota_g : \mathcal{Q} \hookrightarrow (\mathcal{Q}_c^{(1)})_g\ ;\ x \mapsto x + y_g, \tag{2-5}$$

which induces an injective $\mathbb{Z}[\mathcal{Q}]$-linear map $\mathbb{Z}[\iota_g] : \mathbb{Z}[\mathcal{Q}] \hookrightarrow \mathbb{Z}[(\mathcal{Q}_c^{(1)})_g]$. Thus it suffices to show that $\operatorname{Coker}(\mathbb{Z}[\iota_g]) \otimes_{\mathbb{Z}[\mathcal{Q}]} K_0 = (0)$, i.e., $\operatorname{Coker}(\mathbb{Z}[\iota_g])$ is a torsion $\mathbb{Z}[\mathcal{Q}]$-module. On the other hand, we also have a homomorphism of $\mathcal{Q}$-modules

$$(\mathcal{Q}_c^{(1)})_g \to \mathcal{Q}^{gp}\ ;\ y \mapsto y - y_g,$$

which induces an embedding of $\mathbb{Z}[\mathcal{Q}]$-modules $\operatorname{Coker}(\mathbb{Z}[\iota_g]) \hookrightarrow \mathbb{Z}[\mathcal{Q}^{gp}]/\mathbb{Z}[\mathcal{Q}]$. Since $\mathbb{Z}[\mathcal{Q}^{gp}]/\mathbb{Z}[\mathcal{Q}]$ is $\mathbb{Z}[\mathcal{Q}]$-torsion, the assertion follows.

(2) This follows from the combination of part (1) with Lemma 2.15 and Proposition 2.8(2). ☐

2B. *Local log-regular rings.*

2B1. *Definition of local log-regular rings.* We review the definition and fundamental properties of local log-regular rings. Unless otherwise stated, we always assume that the monoid structure of a commutative ring is specified by the multiplicative structure.

Definition 2.17 [34, Chapter III, Definition 1.2.3]. Let R be a ring and let $\mathcal{Q}$ be a monoid with a homomorphism $\alpha : \mathcal{Q} \to R$ of monoids. Then we say that the triple $(R, \mathcal{Q}, \alpha)$ is a *log ring*. Moreover, we say that $(R, \mathcal{Q}, \alpha)$ is a *local log ring* if $(R, \mathcal{Q}, \alpha)$ is a log ring, where R is a local ring and $\alpha^{-1}(R^\times) = \mathcal{Q}^*$.

In order to preserve the locality of a log structure, we need the locality of a ring map.

Lemma 2.18. *Let $(R, \mathcal{Q}, \alpha)$ be a local log ring and let $(S, \mathfrak{m}_S)$ be a local ring with a local ring map $\phi : R \to S$. Then $(S, \mathcal{Q}, \phi \circ \alpha)$ is also a local log ring.*

Proof. By the locality of ϕ, we obtain the equality $(\phi \circ \alpha)^{-1}(S^\times) = \mathcal{Q}^*$, as desired. □

Now we define *log-regular rings* according to [34].

Definition 2.19. Let $(R, \mathcal{Q}, \alpha)$ be a local log ring, where R is Noetherian and $\overline{\mathcal{Q}} := \mathcal{Q}/\mathcal{Q}^*$ is fine and saturated. Let I_α be the ideal of R generated by the set $\alpha(\mathcal{Q}^+)$. Then $(R, \mathcal{Q}, \alpha)$ is called a *log-regular ring* if the following conditions hold.

(1) R/I_α is a regular local ring.

(2) $\dim R = \dim R/I_\alpha + \dim \mathcal{Q}$.

Remark 2.20. For a monoid $\mathcal{Q}$ such that $\overline{\mathcal{Q}}$ is fine and saturated, the natural projection $\pi : \mathcal{Q} \twoheadrightarrow \overline{\mathcal{Q}}$ splits (see [17, Lemma 6.2.10]). Thus, in the situation of Definition 2.19, α extends to the homomorphism of monoids $\overline{\alpha} : \overline{\mathcal{Q}} \to R$ along π. Namely, we obtain another local log-regular ring $(R, \overline{\mathcal{Q}}, \overline{\alpha})$ with the same underlying ring, where $\overline{\mathcal{Q}}$ is fine, sharp, and saturated.

In his monumental paper [26], Kato considered log structures of schemes on the étale sites, and he then considered them on the Zariski sites [27]. However, we do not need any deep part of logarithmic geometry and the present paper focuses on the local study of schemes with log structures. We should remark that if k is any fixed field and $\mathcal{Q} \subseteq \mathbb{N}^d$ is a fine and saturated monoid, then the monoid algebra $k[\mathcal{Q}]$ is known as an *affine normal semigroup ring* which is actively studied in combinatorial commutative algebra (see the book [30]). The following theorem is a natural extension of the Cohen–Macaulay property for the classical toric singularities over a field proved by Hochster [22].

Theorem 2.21 [27, Theorem 4.1]. *Every local log-regular ring is Cohen–Macaulay and normal.*

Let R be a ring and let $\mathcal{Q}$ be a fine sharp monoid. We denote by $R[\mathcal{Q}^+]$ the ideal of $R[\mathcal{Q}]$ generated by elements $\sum_{q \in \mathcal{Q}^+} a_q e^q$, where a_q is an element of R. Then we denote by $R[\![\mathcal{Q}]\!]$ the adic completion of $R[\mathcal{Q}]$ with respect to the ideal $R[\mathcal{Q}^+]$.

As to the structure of complete local log-regular rings, we have the following result analogous to the classical Cohen's structure theorem, originally proved in [27]. We borrow the presentation from [34, Chapter III, Theorem 1.11.2].

Theorem 2.22 (Kato). *Let $(R, \mathcal{Q}, \alpha)$ be a local log ring such that R is Noetherian and $\mathcal{Q}$ is fine, sharp, and saturated. Let k be the residue field of R and $\mathfrak{m}_R$ its maximal ideal. Let r be the dimension of R/I_α.*

(1) *Suppose that* R *contains a field. Then* $(R, \mathcal{Q}, \alpha)$ *is log-regular if and only if there exists a commutative diagram*

$$\begin{tikzcd} \mathcal{Q} \arrow[r] \arrow[d, "\alpha"] & k[[\mathcal{Q} \oplus \mathbb{N}^r]] \arrow[d, "\psi"] \\ R \arrow[r] & \hat{R} \end{tikzcd}$$

where $\hat{R}$ *is the completion along the maximal ideal and* ψ *is an isomorphism of rings.*

(2) *Assume that* R *is of mixed characteristic* $p > 0$. *Let* $C(k)$ *be a Cohen ring of* k. *Then* $(R, \mathcal{Q}, \alpha)$ *is log-regular if and only if there exists a commutative diagram*

$$\begin{tikzcd} \mathcal{Q} \arrow[r] \arrow[d, "\alpha"] & C(k)[[\mathcal{Q} \oplus \mathbb{N}^r]] \arrow[d, "\psi"] \\ R \arrow[r] & \hat{R} \end{tikzcd}$$

where $\hat{R}$ *is the completion along the maximal ideal and* ψ *is a surjective ring map with* $\operatorname{Ker}(\psi) = (\theta)$ *for some element* $\theta \in \mathfrak{m}_{\hat{R}}$ *whose constant term is* p. *Moreover,* $\operatorname{Ker}(\psi) = (\theta')$ *for any element* $\theta' \in \operatorname{Ker}(\psi)$ *whose constant term is* p.

Proof. Assertion (1) and the first part of (2) are [34, Chapter III, Theorem 1.11.2]. Pick an element $\theta' \in \operatorname{Ker}(\psi)$ whose constant term is p. Note that θ' is a regular element that is not invertible. By [34, Chapter III, Proposition 1.10.13], $C(k)[[\mathcal{Q} \oplus \mathbb{N}^r]]/(\theta')$ is a domain of $\dim \mathcal{Q} + r = \dim R = \dim \hat{R}$. Thus $\operatorname{Ker}(\psi) = (\theta')$ holds.[6] ☐

The completion of a normal affine semigroup ring with respect to the ideal generated by elements of the semigroup is a typical example of local log-regular rings:

Lemma 2.23. *Let* $\mathcal{Q}$ *be a fine, sharp and saturated monoid and let* k *be a field. Then* $(k[[\mathcal{Q}]], \mathcal{Q}, \iota)$ *is a local log-regular ring, where* $\iota : \mathcal{Q} \hookrightarrow k[[\mathcal{Q}]]$ *is the natural injection.*

Proof. By [34, Chapter I, Proposition 3.6.1], $(k[[\mathcal{Q}]], \mathcal{Q}, \iota)$ is a local log ring. Now applying Theorem 2.22, it is a local log-regular ring. ☐

2B2. *Log-regularity and strong* F*-regularity.* Strongly F-regular rings are one of the important classes appearing in the study of F-singularities. Let us recall the definition.

Definition 2.24 (strong F-regularity). Let R be a Noetherian reduced $\mathbb{F}_p$-algebra that is F-finite. Let $F^e_* R$ be the same as R as its underlying abelian groups with its R-module structure via restriction of scalars via the e-th iterated Frobenius endomorphism F^e_R on R. Then we say that R is *strongly* F*-regular*, if for any element $c \in R$ that is not in any minimal prime of R, there exist an $e > 0$ and a map $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ such that $\phi(F^e_* c) = 1$.

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[6] This argument is due to Ogus. See the proof of [34, Chapter III, Theorem 1.11.2(2)].

It is known that strongly F-regular rings are Cohen–Macaulay and normal (for example, see [28, Proposition 4.4 and Theorem 4.6]). Let us show that log-regularity implies strong F-regularity (in positive characteristic cases).

Lemma 2.25. *Let $(R, \mathcal{Q}, \alpha)$ be a local log-regular ring of characteristic $p > 0$ such that R is F-finite. Then R is strongly F-regular.*

Proof. The completion of R with respect to its maximal ideal is isomorphic to the completion of $k[\mathcal{Q} \oplus \mathbb{N}^r]$, and $\mathcal{Q}$ is a fine, sharp and saturated monoid by Theorem 2.22 and [34, Chapter I, Proposition 3.4.1]. Then it follows from Lemma 2.10 that $\mathcal{Q} \oplus \mathbb{N}^r$ can be embedded into $\mathbb{N}^d$ for $d > 0$, and $k[\mathcal{Q} \oplus \mathbb{N}^r] \to k[\mathbb{N}^d] \cong k[x_1, \dots, x_d]$ splits as a $k[\mathcal{Q} \oplus \mathbb{N}^r]$-linear map. Applying [23, Theorem 3.1], we see that $k[\mathcal{Q} \oplus \mathbb{N}^r]$ is strongly F-regular. After completion, the complete local ring $k[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]$ is strongly F-regular in view of [1, Theorem 3.6]. Then by faithful flatness of $R \to k[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]$, [23, Theorem 3.1] applies to yield strong F-regularity of R. ☐

Under the hypothesis in the following proposition, one can easily establish the finiteness of the torsion part of the divisor class group, which is the first assertion of Theorem 4.13.

Proposition 2.26. *Assume that $R \cong C(k)[\![\mathcal{Q}]\!]$, where $C(k)$ is a Cohen ring with F-finite residue field k and $\mathcal{Q}$ is a fine, sharp, and saturated monoid. Let $\mathrm{Cl}(R)_{\mathrm{tor}}$ be the torsion subgroup of $\mathrm{Cl}(R)$. Then $\mathrm{Cl}(R)_{\mathrm{tor}} \otimes \mathbb{Z}_{(\ell)}$ is finite for all $\ell \neq p$, and vanishes for almost all $\ell \neq p$.*

Proof. Since $R \cong C(k)[\![\mathcal{Q}]\!]$, we have

$$R/pR \cong k[\![\mathcal{Q}]\!],$$

which is a local F-finite log-regular ring. There is an induced map $\mathrm{Cl}(R) \to \mathrm{Cl}(R/pR)$. By restriction, we have $\mathrm{Cl}(R)_{\mathrm{tor}} \to \mathrm{Cl}(R/pR)_{\mathrm{tor}}$. Then Lemma 2.25 together with Polstra's result [35] says that $\mathrm{Cl}(R/pR)_{\mathrm{tor}}$ is finite. Let C_ℓ be the maximal ℓ-subgroup of $\mathrm{Cl}(R)_{\mathrm{tor}}$. Since $\ell \neq p$, we find that the map $\mathrm{Cl}(R)_{\mathrm{tor}} \to \mathrm{Cl}(R/pR)_{\mathrm{tor}}$ restricted to C_ℓ is injective in view of [18, Theorem 1.2]. In conclusion, C_ℓ is finite for all $\ell \neq p$, and C_ℓ vanishes for almost all $\ell \neq p$, as desired. ☐

2C. *Log-regularity and splinters.* Local log-regular rings have another notable property; they are *splinters*. Let us recall the definition of splinters.

Definition 2.27. A Noetherian ring A is a *splinter* if every finite ring map $f : A \to B$ such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective admits an A-linear map $h : B \to A$ such that $h \circ f = \mathrm{id}_A$.

In general, it is not easy to see which algebraic operations preserve splinters. In fact, it remains unsolved whether polynomial rings over a splinter are splinters (see [10, Question 1']). Regarding these issues, Datta and Tucker proved remarkable results such as [10, Theorem B], [10, Theorem C], and [10, Example 3.2.1]. See also Murayama's work [32] for the study of purity of ring extensions.

In order to prove the splinter property, we need a lemma on splitting a map under completion.

Lemma 2.28. *Let R be a ring and let $f : M \to N$ be an R-linear map. Consider a decreasing filtration by R-submodules $\{M_\lambda\}_{\lambda\in\Lambda}$ of M and a decreasing filtration by R-submodules $\{N_\lambda\}_{\lambda\in\Lambda}$ of N such that $f(M_\lambda) \subseteq N_\lambda$ for each $\lambda \in \Lambda$. Set*

$$\widehat{M} := \varprojlim_{\lambda\in\Lambda} M/M_\lambda \quad \text{and} \quad \hat{N} := \varprojlim_{\lambda\in\Lambda} N/N_\lambda.$$

Finally, assume that f is a split injection that admits an R-linear map $g : N \to M$ such that $g \circ f = \mathrm{id}_M$, $g(N_\lambda) \subseteq M_\lambda$ for each $\lambda \in \Lambda$. Then f extends to a split injection $\widehat{M} \to \hat{N}$.

Proof. By assumption, there is an induced map

$$M/M_\lambda \xrightarrow{\overline{f}} N/N_\lambda \xrightarrow{\bar{g}} M/M_\lambda$$

which is an identity on M/M_λ. Taking inverse limits, we get an identity map $\widehat{M} \to \hat{N} \to \widehat{M}$, which proves the lemma. ☐

The next result is originally due to Gabber and Ramero [17, Theorem 17.3.12].[7] We give an alternative and short proof, using the direct summand theorem by André [2].

Theorem 2.29. *A local log-regular ring $(R, \mathcal{Q}, \alpha)$ is a splinter.*

Proof. First, we prove the theorem when R is complete. By Remark 2.20, we may assume that $\mathcal{Q}$ is fine, sharp, and saturated. By Theorem 2.22, we have

$$R \cong k[[\mathcal{Q} \oplus \mathbb{N}^r]], \text{ or } R \cong C(k)[[\mathcal{Q} \oplus \mathbb{N}^r]]/(p - f),$$

depending on whether R contains a field or not. Let us consider the mixed characteristic case. By Lemma 2.10, there is a split injection $C(k)[\mathcal{Q} \oplus \mathbb{N}^r] \to C(k)[\mathbb{N}^d]$ for some $d > 0$, which comes from an injection $\delta : \mathcal{Q} \oplus \mathbb{N}^r \to \mathbb{N}^d$ that realizes $\delta(\mathcal{Q} \oplus \mathbb{N}^r)$ as an exact submonoid of $\mathbb{N}^d$. After dividing out by the ideal $(p - f)$, we find that the map

$$C(k)[[\mathcal{Q} \oplus \mathbb{N}^r]]/(p - f) \to C(k)[[\mathbb{N}^d]]/(p - f)$$

splits as a $C(k)[[\mathcal{Q} \oplus \mathbb{N}^r]]/(p - f)$-linear map by Remark 2.9 and Lemma 2.28. Hence, R becomes a direct summand of the complete regular local ring $A := C(k)[[x_1, \dots, x_d]]/(p - f)$. By invoking [10, Proposition 2.2.8] and the Direct Summand Theorem [2], we see that R is a splinter. The case where $R = k[[\mathcal{Q} \oplus \mathbb{N}^r]]$ can be treated similarly.

Next let us consider the general case. Then the completion map $R \to \hat{R}$ is faithfully flat and $\hat{R}$ is a complete local log-regular ring (see Theorem 2.22). Hence applying the complete case as above and [10, Proposition 2.2.8] shows that R is a splinter, as desired. ☐

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[7] One notices that the treatment of logarithmic geometry in [17] is topos-theoretic, while [27] considers mostly the Zariski sites.

3. Perfectoid towers and small tilts

In this section, we establish a tower-theoretic framework to deal with perfectoid objects using the notion of *perfectoid towers*. We first introduce the class of *perfect towers* (Definition 3.2) in Section 3A, and then define *inverse perfection of towers* (Definition 3.8) in Section 3B. These notions are tower-theoretic variants of perfect $\mathbb{F}_p$-algebras and inverse perfection of rings, respectively. In Section 3C, we give a set of axioms for perfectoid towers. In Section 3D, we adopt the process of inverse perfection for perfectoid towers as a new tilting operation. Indeed, we verify the invariance of several good properties under the tilting; Main Theorem 1 is discussed here. In Section 3E, we describe the relationship between perfectoid towers and perfectoid rings. This subsection also includes an alternative characterization of perfectoid rings without $\mathbb{A}_{\mathrm{inf}}$. In Section 3F, we calculate the tilts of perfectoid towers consisting of local log-regular rings.

3A. ***Perfect towers.*** First of all, we consider the category of *towers of rings*.

Definition 3.1 (towers of rings).

(1) A *tower of rings* is a direct system of rings of the form

$$R_0 \xrightarrow{t_0} R_1 \xrightarrow{t_1} R_2 \xrightarrow{t_2} \cdots \xrightarrow{t_{i-1}} R_i \xrightarrow{t_i} \cdots,$$

and we denote it by $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ or $\{R_0 \xrightarrow{t_0} R_1 \xrightarrow{t_1} \cdots\}$.

(2) A morphism of towers of rings $f : (\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0}) \to (\{R'_i\}_{i\geq 0}, \{t'_i\}_{i\geq 0})$ is defined as a collection of ring maps $\{f_i : R_i \to R'_i\}_{i\geq 0}$ that is compatible with the transition maps; in other words, f represents the commutative diagram

$$\begin{tikzcd}
R_0 \arrow[r] \arrow[d, "f_0"'] & R_1 \arrow[r] \arrow[d, "f_1"'] & R_2 \arrow[r] \arrow[d, "f_2"'] & \cdots \arrow[r] & R_i \arrow[r] \arrow[d, "f_i"'] & \cdots \\
R'_0 \arrow[r] & R'_1 \arrow[r] & R'_2 \arrow[r] & \cdots \arrow[r] & R'_i \arrow[r] & \cdots .
\end{tikzcd}$$

For a tower of rings $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$, we often denote by R_∞ an inductive limit $\varinjlim_{i\geq 0} R_i$. Clearly, an isomorphism of towers of rings $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0}) \to (\{R'_i\}_{i\geq 0}, \{t'_i\}_{i\geq 0})$ induces the isomorphism of rings $R_\infty \xrightarrow{\cong} R'_\infty$. For every $i \geq 0$, we regard R_{i+1} as an R_i-algebra via the transition map t_i.

Recall that the direct perfection of an $\mathbb{F}_p$-algebra R, which we denote by R^{perf}, is the direct limit of the tower $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ where $R_i = R$ and $t_i = F_R$ for every $i \geq 0$. We denote by $\phi_R : R \to R^{\mathrm{perf}}$ the natural map $R_0 \to \varinjlim_{i\geq 0} R_i$. If R is reduced, this tower can be regarded as ring extensions obtained by adjoining p^i-th roots (cf. Example 3.3). We formulate such towers as follows, and call them *perfect towers*.

Definition 3.2 (perfect towers). A *perfect* $\mathbb{F}_p$*-tower* (or, *perfect tower* as an abbreviated form) is a tower of rings that is isomorphic to a tower of the following form, where R is a reduced $\mathbb{F}_p$-algebra:

$$R \xrightarrow{F_R} R \xrightarrow{F_R} R \xrightarrow{F_R} \cdots . \tag{3-1}$$

Example 3.3. Let R be a reduced $\mathbb{F}_p$-algebra. Let R^{1/p^i} be the ring of p^i-th roots of elements of R for every $i \geq 0$.[8] Then the tower $R \xrightarrow{t_0} R^{1/p} \xrightarrow{t_1} R^{1/p^2} \xrightarrow{t_2} \cdots$ is a perfect tower. Indeed, we have an isomorphism $F_i : R^{1/p^{i+1}} \to R^{1/p^i}$; $x \mapsto x^p$. By putting $F_{0,i+1} := F_0 \circ \cdots \circ F_i$, we obtain the desired commutative ladder:

$$\begin{tikzcd}
R^{1/p^0} \arrow[r, "t_0"] \arrow[d, "F_{0,0}"] & R^{1/p} \arrow[r, "t_1"] \arrow[d, "F_{0,1}"] & \cdots \arrow[r, "t_{i-1}"] & R^{1/p^i} \arrow[r, "t_i"] \arrow[d, "F_{0,i}"] & \cdots \\
R \arrow[r, "F_R"'] & R \arrow[r, "F_R"'] & \cdots \arrow[r, "F_R"'] & R \arrow[r, "F_R"'] & \cdots .
\end{tikzcd}$$

3B. ***Purely inseparable towers and inverse perfection.*** In this subsection, we define *inverse perfection for towers*, which assigns a perfect tower to a tower by arranging a certain type of inverse limits of rings. For this, we introduce the following class of towers that admit distinguished inverse systems of rings.

Definition 3.4 (purely inseparable towers). Let R be a ring, and let $I \subseteq R$ be an ideal.

(1) A tower $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is called a *p-purely inseparable tower arising from* (R, I) if it satisfies the following axioms.

  (a) $R_0 = R$ and $p \in I$.

  (b) For any $i \geq 0$, the ring map $\bar{t}_i : R_i/IR_i \to R_{i+1}/IR_{i+1}$ induced by t_i is injective.

  (c) For any $i \geq 0$, the image of the Frobenius endomorphism on R_{i+1}/IR_{i+1} is contained in the image of $\bar{t}_i : R_i/IR_i \to R_{i+1}/IR_{i+1}$.

(2) Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a p-purely inseparable tower arising from (R, I). For any $i \geq 0$, we denote by $F_i : R_{i+1}/IR_{i+1} \to R_i/IR_i$ the ring map (which uniquely exists by axioms (b) and (c)) such that the following diagram commutes:

$$\begin{tikzcd}
R_{i+1}/IR_{i+1} \arrow[r, "F_{R_{i+1}/IR_{i+1}}"] \arrow[dr, "F_i"'] & R_{i+1}/IR_{i+1} \\
& R_i/IR_i . \arrow[u, "\bar{t}_i"]
\end{tikzcd} \tag{3-2}$$

We call F_i *the i-th Frobenius projection* (of $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ associated to (R, I)).

Hereafter, we leave out "p-" from "p-purely inseparable towers" if no confusion occurs. Similarly, we omit the parenthetic phrase "of … associated to (R, I)" subsequent to "the i-th Frobenius projection" (but we should be careful in some situations; cf. Remark 3.38).

Throughout this paper, when a purely inseparable tower $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is given and its starting layer (R, I) is clear from the context, we denote R_i/IR_i by $\bar{R}_i$ for every $i \geq 0$.

Example 3.5. Any perfect tower is a purely inseparable tower. More precisely, $(\{R\}_{i\geq 0}, \{F_R\}_{i\geq 0})$ appearing in Definition 3.2 is a purely inseparable tower arising from $(R, (0))$. Indeed, axioms (a) and (c)

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[8] For more details of the ring of p-th roots of elements of a reduced ring, we refer to [28].

are obvious, and axiom (b) follows from reducedness of R. The i-th Frobenius projection is given by the identity map on R.

To develop the theory of perfectoid towers, we often use a combination of diagram (3-2) in Definition 3.4 and diagram (3-3) in the following lemma.

Lemma 3.6. *Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a purely inseparable tower arising from some pair (R, I). Then for every $i \geq 0$, the following assertions hold.*

(1) $\operatorname{Ker}(F_i) = \operatorname{Ker}(F_{\overline{R}_{i+1}})$. *In particular, F_i is injective if and only if $\overline{R}_{i+1}$ is reduced.*

(2) *Any element of $\overline{R}_{i+1}$ is a root of a polynomial of the form $X^p - \overline{t}_i(a)$ with $a \in \overline{R}_i$. In particular, the ring map $\overline{t}_i : \overline{R}_i \hookrightarrow \overline{R}_{i+1}$ is integral.*

(3) *The following diagram commutes*:

$$\begin{tikzcd}
\overline{R}_{i+1} \arrow[rd, "F_i"] & \\
\overline{R}_i \arrow[u, "\overline{t}_i"] \arrow[r, "F_{\overline{R}_i}"'] & \overline{R}_i .
\end{tikzcd} \tag{3-3}$$

Proof. Since $\overline{t}_i$ is injective, the commutative diagram (3-2) yields assertion (1). Moreover, (3-2) also yields the equality $x^p - \overline{t}_i(F_i(x)) = 0$ for every $x \in \overline{R}_{i+1}$. Hence assertion (2) follows. To prove (3), let us recall the equalities

$$\overline{t}_i \circ F_{\overline{R}_i} = F_{\overline{R}_{i+1}} \circ \overline{t}_i = \overline{t}_i \circ F_i \circ \overline{t}_i ,$$

where the second one follows from the commutative diagram (3-2). Since $\overline{t}_i$ is injective, we obtain the equality $F_{\overline{R}_i} = F_i \circ \overline{t}_i$, as desired. ☐

Lemma 3.6(3) is essential for defining inverse perfection of towers (cf. Definition 3.8(2)). Moreover, it provides a useful tool for studying direct perfection on each layer. Recall that for an $\mathbb{F}_p$-algebra homomorphism $f : R \to S$, there exists a unique ring map $f^{\mathrm{perf}} : R^{\mathrm{perf}} \to S^{\mathrm{perf}}$ such that the following diagram commutes (the notations are explained just before Definition 3.2):

$$\begin{tikzcd}
R \arrow[r, "f"] \arrow[d, "\phi_R"'] & S \arrow[d, "\phi_S"] \\
R^{\mathrm{perf}} \arrow[r, "f^{\mathrm{perf}}"] & S^{\mathrm{perf}} .
\end{tikzcd}$$

Corollary 3.7. *Keep the notation as in Lemma 3.6. Then $(\overline{t}_i)^{\mathrm{perf}} : (\overline{R}_i)^{\mathrm{perf}} \to (\overline{R}_{i+1})^{\mathrm{perf}}$ is an isomorphism of rings whose inverse map is $(F_i)^{\mathrm{perf}} : (\overline{R}_{i+1})^{\mathrm{perf}} \to (\overline{R}_i)^{\mathrm{perf}}$ up to the Frobenius automorphisms.*

Proof. By Lemma 3.6(3), $F_{(\overline{R}_{i+1})^{\mathrm{perf}}}$ is described as $(F_{\overline{R}_{i+1}})^{\mathrm{perf}} = (\overline{t}_i)^{\mathrm{perf}} \circ F_i^{\mathrm{perf}}$, and it is an automorphism. Similarly, it follows from the commutative diagram (3-2) that $F_i^{\mathrm{perf}} \circ (\overline{t}_i)^{\mathrm{perf}}$ is the Frobenius automorphism of $(\overline{R}_i)^{\mathrm{perf}}$. Hence the assertion follows. ☐

Definition 3.8 (inverse perfection of towers). Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a (p-)purely inseparable tower arising from some pair (R, I).

(1) For any $j \geq 0$, we define the *j-th inverse quasi-perfection of* $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ *associated to* (R, I) as the limit
$$(R_j)_I^{q.\,\mathrm{frep}} := \varprojlim\{\cdots \to \overline{R}_{j+i+1} \xrightarrow{F_{j+i}} \overline{R}_{j+i} \to \cdots \xrightarrow{F_j} \overline{R}_j\}.$$

(2) For any $j \geq 0$, we define an injective ring map $(t_j)_I^{q.\,\mathrm{frep}} : (R_j)_I^{q.\,\mathrm{frep}} \hookrightarrow (R_{j+1})_I^{q.\,\mathrm{frep}}$ by the rule
$$(t_j)_I^{q.\,\mathrm{frep}}((a_i)_{i\geq 0}) := (\overline{t}_{j+i}(a_i))_{i\geq 0}.$$
We call the resulting tower $\big(\{(R_i)_I^{q.\,\mathrm{frep}}\}_{i\geq 0}, \{(t_i)_I^{q.\,\mathrm{frep}}\}_{i\geq 0}\big)$ *the inverse perfection of* $\big(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0}\big)$ *associated to* (R, I).

(3) For any $j \geq 0$, we define a ring map $(F_j)_I^{q.\,\mathrm{frep}} : (R_{j+1})_I^{q.\,\mathrm{frep}} \to (R_j)_I^{q.\,\mathrm{frep}}$ by the rule
$$(F_j)_I^{q.\,\mathrm{frep}}((a_i)_{i\geq 0}) := (F_{j+i}(a_i))_{i\geq 0}. \tag{3-4}$$

(4) For any $j \geq 0$ and for any $m \geq 0$, we denote by $\Phi_m^{(j)}$ the m-th projection map:
$$(R_j)_I^{q.\,\mathrm{frep}} \to \overline{R}_{j+m}\ ;\ (a_i)_{i\geq 0} \mapsto a_m.$$

If no confusion occurs, we also abbreviate $(R_j)_I^{q.\,\mathrm{frep}}$, $(t_j)_I^{q.\,\mathrm{frep}}$, $(F_j)_I^{q.\,\mathrm{frep}}$ to $R_j^{q.\,\mathrm{frep}}$, $t_j^{q.\,\mathrm{frep}}$, $F_j^{q.\,\mathrm{frep}}$.

Example 3.9. Let R be an $\mathbb{F}_p$-algebra. Set $R_i := R$ and $t_i := \mathrm{id}_R$ for every $i \geq 0$. Then the tower $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is a purely inseparable tower arising from $(R, (0))$. Moreover, for every $j \geq 0$, the attached j-th inverse quasi-perfection is a limit
$$R_j^{q.\,\mathrm{frep}} = \varprojlim\{\cdots \xrightarrow{F_R} R \xrightarrow{F_R} R \xrightarrow{F_R} R\},$$
which is none other than the inverse perfection of R.

In the situation of Definition 3.8, we have the commutative diagram:
$$\begin{tikzcd}
(R_{j+1})_I^{q.\,\mathrm{frep}} \arrow[r, "F_{(R_{j+1})_I^{q.\,\mathrm{frep}}}"] \arrow[dr, "(F_j)_I^{q.\,\mathrm{frep}}"'] & (R_{j+1})_I^{q.\,\mathrm{frep}} \\
& (R_j)_I^{q.\,\mathrm{frep}}. \arrow[u, "(t_j)_I^{q.\,\mathrm{frep}}"']
\end{tikzcd} \tag{3-5}$$

Therefore the tower $\big(\{(R_i)_I^{q.\,\mathrm{frep}}\}_{i\geq 0}, \{(t_i)_I^{q.\,\mathrm{frep}}\}_{i\geq 0}\big)$ is also a purely inseparable tower associated to $((R_0)_I^{q.\,\mathrm{frep}}, (0))$.

In the rest of this subsection, we fix a purely inseparable tower $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ arising from some pair (R, I). Keep in mind that the inverse perfection $(\{(R_i)_I^{q.\,\mathrm{frep}}\}_{i\geq 0}, \{(t_i)_I^{q.\,\mathrm{frep}}\}_{i\geq 0})$ is given in Definition 3.8(2), and its Frobenius projections $\{(F_i)_I^{q.\,\mathrm{frep}}\}_{i\geq 0}$ are described in Definition 3.8(3). Some basic properties of inverse quasi-perfection are contained in the following proposition.

Proposition 3.10. *The following assertions hold.*

(1) *For any $j \geq 0$, the following assertions hold.*

  (a) *Let $J \subseteq (R_j)_I^{q.\,\mathrm{frep}}$ be a finitely generated ideal such that $J^k \subseteq \mathrm{Ker}(\Phi_0^{(j)})$ for some $k > 0$ (see Definition 3.8(4) for $\Phi_0^{(j)}$). Then $(R_j)_I^{q.\,\mathrm{frep}}$ is J-adically complete and separated.*

  (b) *Let $x = (x_i)_{i \geq 0}$ be an element of $(R_j)_I^{q.\,\mathrm{frep}}$. Then x is a unit if and only if $x_0 \in R_j/IR_j$ is a unit.*

  (c) *The ring map $(F_j)_I^{q.\,\mathrm{frep}}$ is an isomorphism.*

(2) *$(\{(R_i)_I^{q.\,\mathrm{frep}}\}_{i \geq 0}, \{(t_i)_I^{q.\,\mathrm{frep}}\}_{i \geq 0})$ is a perfect tower. In particular, each $(R_i)_I^{q.\,\mathrm{frep}}$ is reduced.*

Proof. (1) Since $(\{(R_{j+i})^{q.\,\mathrm{frep}}\}_{i \geq 0}, \{(t_{j+i})_I^{q.\,\mathrm{frep}}\}_{i \geq 0})$ is the inverse perfection of $(\{R_{j+i}\}_{i \geq 0}, \{t_{j+i}\}_{i \geq 0})$, we are reduced to showing the assertions in the case when $j = 0$.

(a): By definition, $(R_0)_I^{q.\,\mathrm{frep}}$ is complete and separated with respect to the linear topology induced by the descending filtration

$$\mathrm{Ker}(\Phi_0^{(0)}) \supseteq \mathrm{Ker}(\Phi_1^{(0)}) \supseteq \mathrm{Ker}(\Phi_2^{(0)}) \supseteq \cdots .$$

Moreover, since $J^k \subseteq \mathrm{Ker}(\Phi_0^{(0)})$, we have $(J^k)^{[p^i]} \subseteq \mathrm{Ker}(\Phi_i^{(0)})$ for every $i \geq 0$ by the commutative diagram (3-2).[9] On the other hand, since J^k is finitely generated, $(J^k)^{p^i r} \subseteq (J^k)^{[p^i]}$ for some $r > 0$. Thus the assertion follows from [15, Lemma 2.1.1].

(b): It is obvious that $x_0 \in \overline{R}_0$ is a unit if $x \in (R_0)_I^{q.\,\mathrm{frep}}$ is a unit. Conversely, assume that $x_0 \in \overline{R}_0$ is a unit. Then for every $i \geq 0$, $x_i^{p^i}$ is a unit because it is the image of x_0 in $\overline{R}_i$. Hence x_i is also a unit. Therefore, we have isomorphisms $R_i/IR_i \xrightarrow{\times x_i} R_i/IR_i$ $(i \geq 0)$ that are compatible with the Frobenius projections. Thus we obtain the isomorphism between inverse limits $(R_0)_I^{q.\,\mathrm{frep}} \xrightarrow{\times x} (R_0)_I^{q.\,\mathrm{frep}}$, which yields the assertion.

(c): Consider the shifting map $s_0 : (R_0)_I^{q.\,\mathrm{frep}} \to (R_1)_I^{q.\,\mathrm{frep}}$ defined by the rule $s_0((a_i)_{i \geq 0}) := (a_{i+1})_{i \geq 0}$. Then one can easily check that s_0 is the inverse map of $(F_0)_I^{q.\,\mathrm{frep}}$.

(2) Define $F_{0,i}^{q.\,\mathrm{frep}} : (R_i)_I^{q.\,\mathrm{frep}} \to (R_0)_I^{q.\,\mathrm{frep}}$ as the composite map $(F_0)_I^{q.\,\mathrm{frep}} \circ \cdots \circ (F_{i-1})_I^{q.\,\mathrm{frep}}$ (if $i \geq 1$) or the identity map (if $i = 0$). Then the collection $\{F_{0,i}^{q.\,\mathrm{frep}}\}_{i \geq 0}$ gives a morphism of towers from $(\{(R_i)_I^{q.\,\mathrm{frep}}\}_{i \geq 0}, \{(t_i)_I^{q.\,\mathrm{frep}}\}_{i \geq 0})$ to $\left\{(R_0)_I^{q.\,\mathrm{frep}} \xrightarrow{F_{(R_0)_I^{q.\,\mathrm{frep}}}} (R_0)_I^{q.\,\mathrm{frep}} \xrightarrow{F_{(R_0)_I^{q.\,\mathrm{frep}}}} \cdots\right\}$. Using assertion (1-c) and Lemma 3.6(1), we complete the proof. ☐

The operation of inverse quasi-perfection preserves the locality of rings and ring maps.

Lemma 3.11. *Assume that R_i is a local ring for any $i \geq 0$, and $I \neq R$. Then for any $j \geq 0$, the following assertions hold.*

(1) *The ring maps t_j, $\overline{t}_j$, and F_j are local.*

(2) *$(R_j)_I^{q.\,\mathrm{frep}}$ is a local ring.*

(3) *The ring map $(t_j)_I^{q.\,\mathrm{frep}} : (R_j)_I^{q.\,\mathrm{frep}} \to (R_{j+1})_I^{q.\,\mathrm{frep}}$ is local.*

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[9] The symbol $I^{[p^n]}$ for an ideal I in an $\mathbb{F}_p$-algebra A is the ideal generated by the elements x^{p^n} for $x \in I$.

Proof. As in Proposition 3.10(1), it suffices to show the assertions in the case when $j = 0$.

(1) Since the diagrams (3-2) and (3-3) are commutative, $F_0 \circ \bar{t}_0$ and $\bar{t}_0 \circ F_0$ are local. Hence $\bar{t}_0$ and F_0 are local. In particular, the composition $R_0 \twoheadrightarrow \bar{R}_0 \xrightarrow{\bar{t}_0} \bar{R}_1$ is local. Since this map factors through t_0, t_0 is also local, as desired.

(2) Let $\mathfrak{m}_0$ be the maximal ideal of R_0. Consider the ideal

$$(\mathfrak{m}_0)_I^{q.\,\mathrm{frep}} = \{(x_i)_{i \geq 0} \in (R_0)_I^{q.\,\mathrm{frep}} \mid x_0 \in \mathfrak{m}_0/IR_0\},$$

where $\mathfrak{m}_0/IR_0$ is the maximal ideal of $\bar{R}_0$. Then by Proposition 3.10(1-b), $(\mathfrak{m}_0)_I^{q.\,\mathrm{frep}}$ is a unique maximal ideal of $(R_0)_I^{q.\,\mathrm{frep}}$. The assertion follows.

(3) By assertion (2), $(\{(R_i)_I^{q.\,\mathrm{frep}}\}_{i \geq 0}, \{(t_i)_I^{q.\,\mathrm{frep}}\}_{i \geq 0})$ is a purely inseparable tower of local rings. Hence by (1), $(t_0)_I^{q.\,\mathrm{frep}}$ is local. ☐

A purely inseparable tower also satisfies the following amusing property. This is well-known in positive characteristic, in which case $R_i \to R_{i+1}$ gives a universal homeomorphism (i.e. the induced morphism of schemes $\operatorname{Spec} R_{i+1} \to \operatorname{Spec} R_i$ is a universally homeomorphism). See also Proposition 3.45.

Lemma 3.12. *For every $i \geq 0$, assume that R_i is I-adically Henselian.*[10] *Then the ring map t_i induces an equivalence of categories*:

$$\mathbf{F.\acute{E}t}(R_i) \xrightarrow{\cong} \mathbf{F.\acute{E}t}(R_{i+1}),$$

where $\mathbf{F.\acute{E}t}(A)$ is the category of finite étale A-algebras for a ring A.

Proof. By Corollary 3.7, we obtain the commutative diagram of rings

$$\begin{tikzcd}
R_i \arrow[r, "t_i"] \arrow[d, "\pi_i"'] & R_{i+1} \arrow[d, "\pi_{i+1}"] \\
\bar{R}_i \arrow[r, "\bar{t}_i"] \arrow[d, "\phi_{\bar{R}_i}"'] & \bar{R}_{i+1} \arrow[d, "\phi_{\bar{R}_{i+1}}"] \\
(\bar{R}_i)^{\mathrm{perf}} \arrow[r, "(\bar{t}_i)^{\mathrm{perf}}"] & (\bar{R}_{i+1})^{\mathrm{perf}}
\end{tikzcd} \tag{3-6}$$

where π_j ($j \in \{i, i+1\}$) is the natural projection, and the bottom map is an isomorphism. Since the Frobenius endomorphism on any $\mathbb{F}_p$-algebra gives a universal homeomorphism [38, Tag 0CC6], so does $\phi_{\bar{R}_j}$ by [38, Tag 01YW] and [38, Tag 01YZ]. Hence $\phi_{\bar{R}_j}$ induces an equivalence of categories of finite étale algebras over respective rings in view of [38, Tag 0BQN]. The same assertion holds for π_j by the lifting property of a henselian pair [38, Tag 09ZL]. By going around the diagram (3-6), we finish the proof. ☐

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[10] This condition is realized if R_0 is I-adically Henselian and each $t_i : R_i \to R_{i+1}$ is integral.

3C. *Axioms for perfectoid towers.*

3C1. *Remarks on torsion.* In the subsequent Section 3C2, we introduce the class of *perfectoid towers* as a generalization of perfect towers. For this purpose, we need to deal with a purely inseparable tower arising from (R, I) in the case when $I = (0)$ at least, and hence plenty of I-torsion elements. Thus we begin by giving several preliminary lemmas on torsion of modules over rings.

Definition 3.13. Let R be a ring, and let M be an R-module.

1. Let $x \in R$ be an element. We say that an element $m \in M$ is *x-torsion* if $x^n m = 0$ for some $n > 0$. We denote by $M_{x\text{-tor}}$ the R-submodule of M consisting of all x-torsion elements in M.
2. Let $I \subseteq R$ be an ideal. We say that an element $m \in M$ is *I-torsion* if m is x-torsion for every $x \in I$. We denote by $M_{I\text{-tor}}$ the R-submodule of M consisting of all I-torsion elements in M. Note that $M_{(x)\text{-tor}} = M_{x\text{-tor}} = M_{x^n\text{-tor}}$ for every $n > 0$.
3. For an element $x \in R$ (resp. an ideal $I \subseteq R$), we say that M *has bounded x-torsion* (resp. *bounded I-torsion*) if there exists some $l > 0$ such that $x^l M_{x\text{-tor}} = (0)$ ($I^l M_{I\text{-tor}} = (0)$).
4. For an ideal $I \subseteq R$, we denote by $\varphi_{I,M} : M_{I\text{-tor}} \to M/IM$ the composition of natural R-linear maps:

$$M_{I\text{-tor}} \hookrightarrow M \twoheadrightarrow M/IM. \tag{3-7}$$

First we record the following fundamental lemma.

Lemma 3.14. *Let R be a ring, and let M be an R-module. Let $x \in R$ be an element. Then for every $n > 0$, we have*

$$M_{x\text{-tor}} \cap x^n M = x^n M_{x\text{-tor}}.$$

Proof. Pick an element $m \in M_{x\text{-tor}} \cap x^n M$. Then $m = x^n m_0$ for some $m_0 \in M$, and $x^l m = 0$ for some $l > 0$. Hence $x^{l+n} m_0 = 0$, which implies that $m_0 \in M_{x\text{-tor}}$ and thus $m \in x^n M_{x\text{-tor}}$. The containment $x^n M_{x\text{-tor}} \subseteq M_{x\text{-tor}} \cap x^n M$ is clear. □

Corollary 3.15. *Keep the notation as in Lemma 3.14, and suppose further that $x M_{x\text{-tor}} = (0)$. Then the map $\varphi_{(x),M} : M_{x\text{-tor}} \to M/xM$* (see Definition 3.13(4)) *is injective.*

Proof. It is clear from Lemma 3.14. □

Lemma 3.14 is also applied to show a half part of the following useful result.

Lemma 3.16. *Keep the notation as in Lemma 3.14, and suppose further that M has bounded x-torsion. Let $\widehat{M}$ be the x-adic completion of M, and let $\psi : M \to \widehat{M}$ be the natural map. Then the restriction $\psi_{\mathrm{tor}} : M_{x\text{-tor}} \to (\widehat{M})_{x\text{-tor}}$ of ψ is an isomorphism of R-modules.*

Proof. By assumption, there exists some $l > 0$ such that $x^l M_{x\text{-tor}} = (0)$. On the other hand, $\operatorname{Ker}(\psi_{\mathrm{tor}}) = M_{x\text{-tor}} \cap \bigcap_{n=0}^{\infty} x^n M$ is contained in $M_{x\text{-tor}} \cap x^l M$, which is equal to $x^l M_{x\text{-tor}}$ by Lemma 3.14. Hence ψ_{tor} is injective.

Let us prove the surjectivity. Let $\hat{N}$ denote the x-adic completion of N for every R-module N. Then we obtain the commutative diagram of R-modules:

$$\begin{tikzcd}
0 \arrow[r] & M_{x\text{-tor}} \arrow[r, "\iota"] \arrow[d, "\psi_{M_{x\text{-tor}}}"'] & M \arrow[r, "\pi"] \arrow[d, "\psi"] & M/M_{x\text{-tor}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widehat{M_{x\text{-tor}}} \arrow[r, "\hat{\iota}"] & \widehat{M} \arrow[r, "\widehat{\pi}"] & \widehat{M/M_{x\text{-tor}}} \arrow[r] & 0
\end{tikzcd} \tag{3-8}$$

where ι is the inclusion map and π is the natural projection. Since $\psi \circ \iota$ factors through ψ_{tor}, it suffices to show that $(\widehat{M})_{x\text{-tor}} \subseteq \operatorname{Im}(\hat{\iota} \circ \psi_{M_{x\text{-tor}}})$. First, $\psi_{M_{x\text{-tor}}}$ is bijective because it is isomorphic to the canonical isomorphism $M_{x\text{-tor}}/(x^l) \xrightarrow{\cong} \widehat{M_{x\text{-tor}}}/(x^l)$. To show that $(\widehat{M})_{x\text{-tor}} \subseteq \operatorname{Im}(\hat{\iota})$, note that the top row of (3-8) forms an exact sequence, and it consists of R-modules that have bounded x-torsion. Then by [38, Tag 0923] and right exactness of derived completion functors, $\operatorname{Ker}(\widehat{\pi}) = \operatorname{Im}(\hat{\iota})$ (in fact, the bottom sequence is also exact because ψ_{tor} is injective). Since $\widehat{M/M_{x\text{-tor}}}$ is x-torsion free by [14, Chapter II, Lemma 1.1.5], $(\widehat{M})_{x\text{-tor}} \subseteq \operatorname{Ker}(\widehat{\pi})$. The assertion follows. ☐

The following lemma is used for proving Main Theorem 1 (cf. Lemma 3.48).

Lemma 3.17. *Let R be a ring, and let M be an R-module. Let $x \in R$ be an element. Then for every $n > 0$, we have*

$$\operatorname{Ann}_{M/x^nM}(x) \subseteq \operatorname{Im}(\varphi_{(x^n),M}) + x^{n-1}(M/x^nM). \tag{3-9}$$

Proof. Pick an element $m \in M$ such that $xm \in x^nM$. Then $x(m - x^{n-1}m') = 0$ for some $m' \in M$. In particular, $m - x^{n-1}m' \in M_{x^n\text{-tor}}$. Hence $m \bmod x^nM$ lies in the right-hand side of (3-9), as desired. ☐

In the case when $M = R$, we can regard $M_{I\text{-tor}}$ as a (possibly) nonunital subring of R. This point of view provides valuable insights. For example, "reducedness" for $R_{I\text{-tor}}$ induces a good property on boundedness of torsions.

Lemma 3.18. *Let (R, I) be a pair such that $R_{I\text{-tor}}$ does not contain any non-zero nilpotent element of R. Then $IR_{I\text{-tor}} = (0)$.*

Proof. It suffices to show that $xR_{I\text{-tor}} = 0$ for every $x \in I$. Pick an element $a \in R_{I\text{-tor}}$. Then for a sufficiently large $n > 0$, $x^na = 0$. Hence $(xa)^n = x^na \cdot a^{n-1} = 0$. Thus we have $xa = 0$ by assumption, as desired. ☐

Corollary 3.19. *Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a purely inseparable tower arising from some pair (R, I). Then for every $i \geq 0$ and every ideal $J \subseteq (R_i)_I^{q.\,\text{frep}}$, we have $J((R_i)_I^{q.\,\text{frep}})_{J\text{-tor}} = (0)$.*

Proof. Since $(R_i)_I^{q.\,\text{frep}}$ is reduced by Proposition 3.10(2), the assertion follows from Lemma 3.18. ☐

Furthermore, we can treat $R_{I\text{-tor}}$ as a positive characteristic object (in the situation of our interest), even if R is not an $\mathbb{F}_p$-algebra.

Lemma-Definition 3.20. *If* (R, I) *is a pair such that* $p \in I$ *and* $IR_{I\text{-tor}} = (0)$, *the multiplicative map*

$$R_{I\text{-tor}} \to R_{I\text{-tor}};\ x \mapsto x^p \tag{3-10}$$

is also additive. We denote by $F_{R_{I\text{-tor}}}$ *the map* (3-10).

Proof. This immediately follows from the binomial theorem. □

3C2. *Perfectoid towers and pillars.*

Definition 3.21 (perfectoid towers). Let R be a ring, and let $I_0 \subseteq R$ be an ideal. A tower $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is called a *(p)-perfectoid tower arising from* (R, I_0) if it is a p-purely inseparable tower arising from (R, I_0) (cf. Definition 3.4(1)) and satisfies the following additional axioms.

(d) For every $i \geq 0$, the i-th Frobenius projection $F_i : R_{i+1}/I_0R_{i+1} \to R_i/I_0R_i$ (cf. Definition 3.4(2)) is surjective.

(e) For every $i \geq 0$, R_i is an I_0-adically Zariskian ring (in other words, I_0R_i is contained in the Jacobson radical of R_i).

(f) I_0 is a principal ideal, and R_1 contains a principal ideal I_1 that satisfies the following axioms.

  (f-1) $I_1^p = I_0R_1$.

  (f-2) For every $i \geq 0$, $\mathrm{Ker}(F_i) = I_1(R_{i+1}/I_0R_{i+1})$.

(g) For every $i \geq 0$, $I_0(R_i)_{I_0\text{-tor}} = (0)$. Moreover, there exists a (unique) bijective map $(F_i)_{\mathrm{tor}} : (R_{i+1})_{I_0\text{-tor}} \to (R_i)_{I_0\text{-tor}}$ such that the diagram

$$\begin{tikzcd}
(R_{i+1})_{I_0\text{-tor}} \arrow[r, "\varphi_{I_0,R_{i+1}}"] \arrow[d, "(F_i)_{\mathrm{tor}}"'] & R_{i+1}/I_0R_{i+1} \arrow[d, "F_i"] \\
(R_i)_{I_0\text{-tor}} \arrow[r, "\varphi_{I_0,R_i}"'] & R_i/I_0R_i
\end{tikzcd} \tag{3-11}$$

commutes (see Definition 3.13 for the notation; see also Corollary 3.15).

Remark 3.22. If I_0 is generated by an element whose image in R_i is a non-zerodivisor for every $i \geq 0$, then axiom (g) is satisfied automatically. If R_1 is reduced and $I_0 = (0)$, then axiom (g) follows from axioms (d) and (f). Consequently, if every t_i is injective and $\varinjlim_{i\geq 0} R_i$ is a domain, one can ignore axiom (g) when checking that $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is a perfectoid tower.

We have some examples of perfectoid towers.

Example 3.23. (1) (cf. [37, Definition 4.4]) Let $(R, \mathfrak{m}, k)$ be a d-dimensional complete unramified regular local ring of mixed characteristic $p > 0$ whose residue field is perfect. Then we have

$$R \cong W(k)[[x_2, \dots, x_d]].$$

For every $i \geq 0$, set $R_i := R[p^{1/p^i}, x_2^{1/p^i}, \dots, x_d^{1/p^i}]$, and let $t_i : R_i \to R_{i+1}$ be the inclusion map. Then the tower $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is a perfectoid tower arising from $(R, (p))$. Indeed, the Frobenius projection $F_i : R_{i+1}/pR_{i+1} \to R_i/pR_i$ is given as the p-th power map.[11]

(2) For some generalization of (1), one can build a perfectoid tower arising from a complete local log-regular ring. For details, see Section 3F.

(3) We note that t_i (resp. F_i) of a perfectoid tower is not necessarily the inclusion map (resp. the p-th power map). For instance, let R be a reduced $\mathbb{F}_p$-algebra. Set $R_i := R$, $t_i := F_R$, and $F_i := \mathrm{id}_R$ for every $i \geq 0$. Then $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is a perfectoid tower arising from $(R, (0))$.

The class of perfectoid towers is a generalization of perfect towers.

Lemma 3.24. *Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a tower of $\mathbb{F}_p$-algebras. The following conditions are equivalent.*

(1) $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ *is a perfect $\mathbb{F}_p$-tower* (cf. Definition 3.2).

(2) $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ *is a p-perfectoid tower arising from* $(R_0, (0))$.

Proof. (1) ⇒ (2) We may assume that $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is of the form $R \xrightarrow{F_R} R \xrightarrow{F_R} R \xrightarrow{F_R} \cdots$ (see Definition 3.2). By Example 3.5, this is a purely inseparable tower arising from $(R, (0))$. Axiom (e) in Definition 3.21 is obvious. Axioms (d), (f), and (g) are also satisfied, since the Frobenius projection F_i (cf. Example 3.5) is an isomorphism for any $i \geq 0$. This yields the assertion.

(2) ⇒ (1) Conversely, assume that $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is a perfectoid tower arising from $(R_0, (0))$. Since F_i is identified with $(F_i)_{\mathrm{tor}}$ in axiom (g), the injectivity of $(F_i)_{\mathrm{tor}}$ implies that F_i is injective. In other words, R_i is reduced by Lemma 3.6(1). Furthermore, F_i is an isomorphism by axiom (d) or the surjectivity of $(F_i)_{\mathrm{tor}}$. Hence we obtained the desired isomorphism of towers:

$$\begin{tikzcd}
R_0 \arrow[r, "t_0"] \arrow[d, "\mathrm{id}_{R_0}"] & R_1 \arrow[r, "t_1"] \arrow[d, "F_0"] & R_2 \arrow[r, "t_2"] \arrow[d, "F_0\circ F_1"] & R_3 \arrow[r, "t_3"] \arrow[d, "F_0\circ F_1\circ F_2"] & \cdots \\
R_0 \arrow[r, "F_{R_0}"'] & R_0 \arrow[r, "F_{R_0}"'] & R_0 \arrow[r, "F_{R_0}"'] & R_0 \arrow[r, "F_{R_0}"'] & \cdots .
\end{tikzcd} \tag{3-12}$$

The assertion follows. □

For a perfectoid tower $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ arising from (R, I_0), an ideal $I_1 \subseteq R_1$ appearing in axiom (f) in Definition 3.21 is unique. Indeed, it contains I_0R_1, and its image via the projection $R_1 \twoheadrightarrow \overline{R}_1$ is a fixed ideal $\mathrm{Ker}(F_0)$.

Definition 3.25. We call I_1 the *first perfectoid pillar of* $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ *arising from* (R, I_0).

The relationship between I_0 and I_1 can be observed also in higher layers (see Proposition 3.26 below). In the rest of this section, we fix a perfectoid tower $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ arising from some pair (R, I_0), and let I_1 denote the first perfectoid pillar.

---

[11] Axiom (f-2) follows from the normality of R_i. The other axioms are clearly satisfied.

Proposition 3.26. (1) *For a sequence of principal ideals* $\{I_i \subseteq R_i\}_{i\geq 2}$, *the following conditions are equivalent.*

- (a) $F_i^{-1}(I_i\overline{R}_i) = I_{i+1}\overline{R}_{i+1}$ *for every* $i \geq 0$.
- (b) $F_i(I_{i+1}\overline{R}_{i+1}) = I_i\overline{R}_i$ *for every* $i \geq 0$.

(2) *Each one of the equivalent conditions in* (1) *implies that* $I_{i+1}^p = I_iR_{i+1}$ *for every* $i \geq 0$.

(3) *There exists a unique sequence of principal ideals* $\{I_i \subseteq R_i\}_{i\geq 0}$ *that satisfies one of the equivalent conditions in* (1). *Moreover, there exists a sequence of elements* $\{\overline{f}_i \in \overline{R}_i\}_{i\geq 0}$ *such that* $I_i\overline{R}_i = (\overline{f}_i)$ *and* $F_i(\overline{f}_{i+1}) = \overline{f}_i$ *for every* $i \geq 0$.

Proof. (1) Since the implication (a) ⇒ (b) follows from axiom (d) in Definition 3.21, it suffices to show the converse. Assume that condition (b) is satisfied. Then for every $i \geq 0$, the compatibility $\overline{t}_i \circ F_i = F_{\overline{R}_{i+1}}$ implies

$$I_{i+1}^p\overline{R}_{i+1} = I_i\overline{R}_{i+1} \tag{3-13}$$

because I_{i+1} is principal. In particular, $\operatorname{Ker}(F_i) = I_1\overline{R}_{i+1} \subseteq I_{i+1}\overline{R}_{i+1}$ (cf. axiom (f-2)). On the other hand, by the surjectivity of F_i and the assumption again, we have $F_i(F_i^{-1}(I_i\overline{R}_i)) = I_i\overline{R}_i = F_i(I_{i+1}\overline{R}_{i+1})$. Hence

$$F_i^{-1}(I_i\overline{R}_i) \subseteq I_{i+1}\overline{R}_{i+1} + \operatorname{Ker}(F_i) \subseteq I_{i+1}\overline{R}_{i+1} \subseteq F_i^{-1}(I_i\overline{R}_i),$$

which yields the assertion.

(2) Let us deduce the assertion from (3-13) by induction. By definition, $I_1^p = I_0R_1$. We then fix some $i \geq 1$. Suppose that for every $1 \leq k \leq i$, $I_k^p = I_{k-1}R_k$. Then $I_0R_i = I_i^{p^i}$. In particular, R_i is I_i-adically Zariskian by axiom (e). Moreover, by (3-13), we have the equalities of R_i-modules:

$$I_iR_{i+1} = I_{i+1}^p + I_0R_{i+1} = I_{i+1}^p + I_i^{p^i-1}(I_iR_{i+1}).$$

Hence by Nakayama's lemma, we obtain $I_{i+1}^p = I_iR_{i+1}$ as desired.

(3) By the axiom of (dependent) choice, the existence follows from axiom (d) in Definition 3.21. Let us show the uniqueness of $\{I_i \subseteq R_i\}_{i\geq 0}$ that satisfies condition (a) in (1). For every $i \geq 0$, $I_iR_{i+1} \subseteq I_{i+1}$ by (2), and hence I_{i+1} is the inverse image of $F_i^{-1}(I_i\overline{R}_i)$ via the projection $R_{i+1} \to \overline{R}_{i+1}$. Since I_0 is fixed, the assertion follows. □

Definition 3.27. In the situation described in Proposition 3.26(3), we call I_i the *i-th perfectoid pillar of* $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ *arising from* (R_0, I_0).

The following property of perfectoid pillars is applied to prove our main result.

Lemma 3.28. *Let* $\{I_i\}_{i\geq 0}$ *denote the system of perfectoid pillars of* $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$, *and let* $\pi_i : R_i/I_0R_i \to R_i/I_iR_i$ $(i \geq 0)$ *be the natural projections. Then for every* $i \geq 0$, *there exists a unique isomorphism of rings*

$$F_i' : R_{i+1}/I_{i+1}R_{i+1} \xrightarrow{\cong} R_i/I_iR_i$$

such that $\pi_i \circ F_i = F_i' \circ \pi_{i+1}$.

Proof. Since F_i and π_i are surjective, the assertion follows from $\operatorname{Ker}(\pi_i \circ F_i) = F_i^{-1}(I_i(R_i/I_0R_i)) = I_{i+1}(R_{i+1}/I_0R_{i+1})$. ☐

3D. *Tilts of perfectoid towers.*

3D1. *Invariance of some properties.* Here we establish tilting operation for perfectoid towers. For this, we first introduce the notion of *small tilt*, which originates in [37].

Definition 3.29 (small tilts). Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a perfectoid tower arising from some pair (R, I_0).

(1) For any $j \geq 0$, we define the *j-th small tilt* of $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ associated to (R, I_0) as the j-th inverse quasi-perfection of $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ associated to (R, I_0) and denote it by $(R_j)^{s.\flat}_{I_0}$.

(2) Let the notation be as in Lemma 3.28. Then we define $I_i^{s.\flat} := \operatorname{Ker}(\pi_i \circ \Phi_0^{(i)})$ for every $i \geq 0$.

Note that the ideal $I_i^{s.\flat} \subseteq R_i^{s.\flat}$ has the following property.

Lemma 3.30. *Keep the notation as in Definition 3.29. Then for every* $i \geq 0$ *and* $j \geq 0$, *we have* $\Phi_i^{(j)}(I_j^{s.\flat}) = I_{j+i}\overline{R}_{j+i}$.

Proof. Since $\Phi_0^{(j)}$ is surjective, we have $\Phi_0^{(j)}(I_j^{s.\flat}) = I_j\overline{R}_j$. On the other hand, since $\Phi_0^{(j)} = F_j \circ \Phi_1^{(j)}$, we have

$$F_j^{-1}(\Phi_0^{(j)}(I_j^{s.\flat})) = \Phi_1^{(j)}(I_j^{s.\flat}) + \operatorname{Ker}(F_j) = \Phi_1^{(j)}(I_j^{s.\flat}).$$

Hence by condition (a) in Proposition 3.26(1), $\Phi_1^{(j)}(I_j^{s.\flat}) = I_{j+1}\overline{R}_{j+1}$. By repeating this procedure recursively, we obtain the assertion. ☐

The next lemma provides some completeness of the small tilts attached to a perfectoid tower of characteristic $p > 0$ (see also Remark 3.33).

Lemma 3.31. *Let* $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ *be a perfectoid tower arising from* $(R, (0))$. *Then, for any element* $f \in R$ *and any* $j \geq 0$, *the inverse limit* $\varprojlim\{\cdots \xrightarrow{\overline{F}_{j+1}} R_{j+1}/fR_{j+1} \xrightarrow{\overline{F}_j} R_j/fR_j\}$ *is isomorphic to the* f*-adic completion of* R_j.

Proof. It suffices to show the assertion when $j = 0$. Let $(\{R_i'\}_{i\geq 0}, \{t_i'\}_{i\geq 0})$ denote the standard perfect tower (3-1) arising from R. By Lemma 3.24, (3-12) gives a canonical isomorphism $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0}) \xrightarrow{\cong} (\{R_i'\}_{i\geq 0}, \{t_i'\}_{i\geq 0})$. If we put $J_0 = fR_0'$, then $R_i'/J_0R_i' = R/f^{p^i}R$ for every $i \geq 0$. Hence we have the desired canonical isomorphisms:

$$\varprojlim\{\cdots \xrightarrow{\overline{F}_1} R_1/fR_1 \xrightarrow{\overline{F}_0} R_0/fR_0\} \xrightarrow{\cong} \varprojlim_{n\geq 0} R/f^{p^n}R \xrightarrow{\cong} \varprojlim_{n\geq 0} R/f^nR. \qquad ☐$$

Example 3.32. Let S be a perfect $\mathbb{F}_p$-algebra. Pick an arbitrary $f \in S$, and let $\hat{S}$ denote the f-adic completion. We obtain a canonical isomorphism $\varprojlim_{\mathrm{Frob}} S/fS \xrightarrow{\cong} \hat{S}$ by applying the above proof to the tower

$$S \xrightarrow{\mathrm{id}_S} S \xrightarrow{\mathrm{id}_S} S \xrightarrow{\mathrm{id}_S} \cdots.$$

Remark 3.33. In the situation of Lemma 3.31, assume further that $\bar{t}_i : R_i/fR_i \to R_{i+1}/fR_{i+1}$ is injective for every $i \geq 0$. Then $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is a purely inseparable tower arising from $(R, (f))$ with Frobenius projections $\{\overline{F}_i : R_{i+1}/fR_{i+1} \to R_i/fR_i\}_{i\geq 0}$. Furthermore, it satisfies axioms (d), (f), and (g) in Definition 3.21. To check this, we may assume that $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is the standard perfect tower (3-1). Then $\overline{F}_i$ is the natural projection $R/f^{p^{i+1}}R \twoheadrightarrow R/f^{p^i}R$. It is clearly surjective, and its kernel is $f^{p^i}(R/f^{p^{i+1}}R)$. Let I_i be the ideal of R_i generated by $f \in R_i\ (= R)$. Then $I_0R_i = f^{p^i}R$ and $I_1R_{i+1} = f^{p^i}R$. Hence $I_1^p = I_0R_1$ and $\operatorname{Ker}(\overline{F}_i) = I_1\overline{R}_{i+1}$. Finally, note that $(R_i)_{I_0\text{-tor}} = R_{f\text{-tor}}$. Then $I_0(R_i)_{I_0\text{-tor}} = f^{p^i}R_{f\text{-tor}} = (0)$ by Lemma 3.18, and we can take $\mathrm{id}_{R_{f\text{-tor}}}$ as the bijection $(\overline{F}_i)_{\mathrm{tor}}$ fitting into the diagram (3-11).

Definition 3.34 (tilts of perfectoid towers). Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a perfectoid tower arising from some pair (R, I). Then we define *the tilt of* $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ associated to (R, I) as the inverse perfection of $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ associated to (R, I), and denote it by $(\{(R_i)_I^{s.\flat}\}_{i\geq 0}, \{(t_i)_I^{s.\flat}\}_{i\geq 0})$. If no confusion occurs, we can abbreviate $(R_i)_I^{s.\flat}$ and $(t_i)_I^{s.\flat})$ to $R_i^{s.\flat}$ and $t_i^{s.\flat}$.

After discussing several basic properties of this tilting operation, we illustrate how to compute the tilts of perfectoid towers in some specific cases; when they consist of *log-regular rings* (see Theorem 3.61 and Example 3.62).

We should remark that all results on the perfection of purely inseparable towers (established in Section 3B) can be applied to the tilts of perfectoid towers.

Let us state Main Theorem 1 in a more refined form. This is an important tool when one wants to see that a certain correspondence holds between Noetherian rings of mixed characteristic and those of positive characteristic.

Theorem 3.35. *Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a perfectoid tower arising from some pair (R, I_0), and let $\{I_i\}_{i\geq 0}$ be the system of perfectoid pillars. Let $(\{R_i^{s.\flat}\}_{i\geq 0}, \{t_i^{s.\flat}\}_{i\geq 0})$ denote the tilt associated to (R, I_0).*

(1) *For every $j \geq 0$ and every element $f_j^{s.\flat} \in R_j^{s.\flat}$, the following conditions are equivalent.*

  (a) *$f_j^{s.\flat}$ is a generator of $I_j^{s.\flat}$.*

  (b) *For every $i \geq 0$, $\Phi_i^{(j)}(f_j^{s.\flat})$ is a generator of $I_{j+i}\overline{R}_{j+i}$.*

  In particular, $I_j^{s.\flat}$ is a principal ideal, and $(I_{j+1}^{s.\flat})^p = I_j^{s.\flat}R_{j+1}^{s.\flat}$.

(2) *We have isomorphisms of (possibly) nonunital rings $(R_j^{s.\flat})_{I_0^{s.\flat}\text{-tor}} \cong (R_j)_{I_0\text{-tor}}$ that are compatible with $\{t_j\}_{j\geq 0}$ and $\{t_j^{s.\flat}\}_{j\geq 0}$.*

We give its proof in the subsequent Section 3D2. Before that, let us observe that this theorem induces many good properties of tilting. In the rest of this subsection, we keep the notation as in Theorem 3.35.

Lemma 3.36. *For every $i \geq 0$, $R_i^{s.\flat}$ is $I_0^{s.\flat}$-adically complete and separated.*

Proof. By Theorem 3.35, the ideal $I_0^{s.\flat}R_i^{s.\flat} \subseteq R_i^{s.\flat}$ is principal. Hence one can apply Proposition 3.10(1-a) to deduce the assertion. ☐

To discuss perfectoidness for the tilt $(\{R_i^{s.\flat}\}_{i\geq 0}, \{t_i^{s.\flat}\}_{i\geq 0})$, we introduce the following maps.

Definition 3.37. For every $i \geq 0$, we define a ring map $(F_i)^{s.\flat}_{I_0} : (R_{i+1})^{s.\flat}_{I_0}/I_0^{s.\flat}(R_{i+1})^{s.\flat}_{I_0} \to (R_i)^{s.\flat}_{I_0}/I_0^{s.\flat}(R_i)^{s.\flat}_{I_0}$ by the rule

$$(F_i)^{s.\flat}_{I_0}(\alpha_{i+1} \bmod I_0^{s.\flat}(R_{i+1})^{s.\flat}_{I_0}) = (F_i)^{q.\,\mathrm{frep}}_{I_0}(\alpha_{i+1}) \bmod I_0^{s.\flat}(R_i)^{s.\flat}_{I_0},$$

where $\alpha_{i+1} \in (R_{i+1})^{s.\flat}_{I_0}$. If no confusion occurs, we can abbreviate $(F_i)^{s.\flat}_{I_0}$ to $F_i^{s.\flat}$.

Remark 3.38. Although the symbols $(\ \cdot\ )^{s.\flat}$ and $(\ \cdot\ )^{q.\,\mathrm{frep}}$ had been used interchangeably before Definition 3.37, $(F_i)^{s.\flat}_{I_0}$ differs from $(F_i)^{q.\,\mathrm{frep}}_{I_0}$ in general.

The following lemma is an immediate consequence of Theorem 3.35(1), but quite useful.

Lemma 3.39. *For every $j \geq 0$, $\Phi_0^{(j)}$ induces an isomorphism*

$$\overline{\Phi_0^{(j)}} : R_j^{s.\flat}/I_0^{s.\flat}R_j^{s.\flat} \overset{\cong}{\to} R_j/I_0R_j;\ a \bmod I_0^{s.\flat}R_j^{s.\flat} \mapsto \Phi_0^{(j)}(a). \tag{3-14}$$

Moreover, $\{\overline{\Phi_0^{(i)}}\}_{i\geq 0}$ is compatible with $\{t_i\}_{i\geq 0}$ (resp. $\{F_{R_i^{s.\flat}/I_0^{s.\flat}R_i^{s.\flat}}\}_{i\geq 0}$, resp. $\{F_i^{s.\flat}\}_{i\geq 0}$) and $\{t_i^{s.\flat}\}_{i\geq 0}$ (resp. $\{F_{R_i/I_0R_i}\}_{i\geq 0}$, resp. $\{F_i\}_{i\geq 0}$).

Proof. By axiom (d) in Definition 3.21, (3-14) is surjective. We check the injectivity. By Theorem 3.35(1), $I_0^{s.\flat}$ is generated by an element $f_0^{s.\flat} \in R_0^{s.\flat}$ such that $\Phi_i^{(0)}(f_0^{s.\flat})$ is a generator of $I_i\overline{R}_i$ $(i \geq 0)$. Note that $(\{R_{j+i}\}_{i\geq 0}, \{t_{j+i}\}_{i\geq 0})$ is a perfectoid tower arising from (R_j, I_0R_j). Moreover, $\{I_iR_{j+i}\}_{i\geq 0}$ is the system of perfectoid pillars associated to (R_j, I_0R_j) (cf. condition (b) in Proposition 3.26(1)). Put $J_0 := I_0R_j$. Then by Theorem 3.35(1) again, we find that $J_0^{s.\flat} = f_0^{s.\flat}R_j^{s.\flat} = I_0^{s.\flat}R_j^{s.\flat}$. Since $J_0^{s.\flat} = \operatorname{Ker}\Phi_0^{(j)}$, we obtain the first assertion.

One can deduce that $\{\overline{\Phi_0^{(i)}}\}_{i\geq 0}$ is compatible with the Frobenius projections from the commutativity of (3-2), because the other compatibility assertions immediately follow from the construction. ☐

Remark 3.40. Theorem 3.35(2) and Lemma 3.39 can be interpreted as a correspondence of homological invariants between R_i and $R_i^{s.\flat}$ by using Koszul homologies. Indeed, for any generator f_0 (resp. $f_0^{s.\flat}$) of I_0 (resp. $I_0^{s.\flat}$), the Koszul homology $H_q(f_0^{s.\flat}; R_i^{s.\flat})$ is isomorphic to $H_q(f_0; R_i)$ for any $q \geq 0$ as an abelian group.[12]

Now we can show the invariance of several properties of perfectoid towers under tilting. The first one is perfectoidness, which is most important in our framework.

Proposition 3.41. *$(\{R_i^{s.\flat}\}_{i\geq 0}, \{t_i^{s.\flat}\}_{i\geq 0})$ is a perfectoid tower arising from $(R_0^{s.\flat}, I_0^{s.\flat})$.*

Proof. By Lemma 3.39 and Remark 3.33, $(\{R_i^{s.\flat}\}_{i\geq 0}, \{t_i^{s.\flat}\}_{i\geq 0})$ is a purely inseparable tower arising from $(R_0^{s.\flat}, I_0^{s.\flat})$ that also satisfies axioms (d), (f), and (g). Moreover, axiom (e) holds by Lemma 3.36. Hence the assertion follows. ☐

Next we focus on finiteness properties. "Small" in the name of small tilts comes from the following fact.

---

[12] Note that $(R_i)_{I_0\text{-tor}} = \operatorname{Ann}_{R_i}(I_0)$ by axiom (g), and $(R_i^{s.\flat})_{I_0^{s.\flat}\text{-tor}} = \operatorname{Ann}_{R_i^{s.\flat}}(I_0^{s.\flat})$ by Corollary 3.19.

Proposition 3.42. *For every $j \geq 0$, the following assertions hold.*

(1) *If $t_j : R_j \to R_{j+1}$ is module-finite, then so is $t_j^{s.\flat} : R_j^{s.\flat} \to R_{j+1}^{s.\flat}$. Moreover, the converse holds true when R_j is I_0-adically complete and separated.*

(2) *If R_j is a Noetherian ring, then so is $R_j^{s.\flat}$. Moreover, the converse holds true when R_j is I_0-adically complete and separated.*

(3) *Assume that R_j is a Noetherian local ring, and a generator of I_0R_j is regular. Then the dimension of R_j is equal to that of $R_j^{s.\flat}$.*

Proof. (1) By Lemma 3.39, $\overline{t}_j : R_j/I_0R_j \to R_{j+1}/I_0R_{j+1}$ is module-finite if and only if $\overline{t_j^{s.\flat}} : R_j^{s.\flat}/I_0^{s.\flat}R_j^{s.\flat} \to R_{j+1}^{s.\flat}/I_0^{s.\flat}R_{j+1}^{s.\flat}$ is so. Thus by Lemma 3.36 and [29, Theorem 8.4], the assertion follows.

(2) One can prove this assertion by applying Lemma 3.36, Lemma 3.39, and [38, Tag 05GH].

(3) By Theorem 3.35, $I_0^{s.\flat}R_j^{s.\flat}$ is also generated by a regular element. Thus we obtain the equalities $\dim R_j = \dim R_j/I_0R_j + 1$ and $\dim R_j^{s.\flat} = \dim R_j^{s.\flat}/I_0^{s.\flat}R_j^{s.\flat} + 1$. By combining these equalities with Lemma 3.39, we deduce assertion. ☐

Proposition 3.42(2) says that Noetherianness for a perfectoid tower is preserved under tilting.

Definition 3.43. We say that $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is *Noetherian* if R_i is Noetherian for each $i \geq 0$.

Corollary 3.44. *If $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is Noetherian, then so is the tilt $(\{R_i^{s.\flat}\}_{i\geq 0}, \{t_i^{s.\flat}\}_{i\geq 0})$. Moreover, the converse holds true when R_i is I_0-adically complete and separated for each $i \geq 0$.*

Proof. This immediately follows from Proposition 3.42(2). ☐

Finally, let us consider perfectoid towers of henselian rings. Then we obtain the equivalence of categories of finite étale algebras over each layer.

Proposition 3.45. *Assume that R_i is I_0-adically Henselian for any $i \geq 0$. Then we obtain the following equivalences of categories*:

$$\mathbf{F.\acute{E}t}(R_i^{s.\flat}) \xrightarrow{\cong} \mathbf{F.\acute{E}t}(R_i).$$

Proof. This follows from Lemma 3.36, Lemma 3.39 and [38, Tag 09ZL]. ☐

3D2. *Proof of Main Theorem 1.* We keep the notation as above. Furthermore, we set $\overline{I}_i := I_i\overline{R}_i$ for every $i \geq 0$. To prove Theorem 3.35, we investigate some relationship between $(R_i)_{I_0\text{-tor}}$ and $\mathrm{Ann}_{\overline{R}_i}(\overline{I}_i)$. First recall that we can regard $(R_i)_{I_0\text{-tor}}$ as a nonunital subring of $\overline{R}_i$ by Corollary 3.15. Moreover, the map $\overline{t}_i$ naturally restricts to $(R_i)_{I_0\text{-tor}} \hookrightarrow (R_{i+1})_{I_0\text{-tor}}$, as follows.

Lemma 3.46. *For every $i \geq 0$, let $(t_i)_{\mathrm{tor}} : (R_i)_{I_0\text{-tor}} \to (R_{i+1})_{I_0\text{-tor}}$ be the restriction of t_i.*

(1) *$(t_i)_{\mathrm{tor}}$ is the unique map such that $\varphi_{I_0,R_{i+1}} \circ (t_i)_{\mathrm{tor}} = \overline{t}_i \circ \varphi_{I_0,R_i}$.*

(2) $(t_i)_{\mathrm{tor}} \circ (F_i)_{\mathrm{tor}} = (F_{i+1})_{\mathrm{tor}} \circ (t_{i+1})_{\mathrm{tor}} = F_{(R_{i+1})_{I_0\text{-tor}}}$.

Proof. Since φ_{I_0,R_i} is injective by Corollary 3.15, assertion (1) is clear from the construction. Hence we can regard $(t_i)_{\mathrm{tor}}$ and $(F_i)_{\mathrm{tor}}$ as the restrictions of $\overline{t}_i$ and F_i. Thus assertion (2) follows from the compatibility $\overline{t}_i \circ F_i = F_{i+1} \circ \overline{t}_{i+1} = F_{\overline{R}_{i+1}}$ induced by Lemma 3.6(3). ☐

The map $\varphi_{I_0,R_i} : (R_i)_{I_0\text{-tor}} \hookrightarrow R_i/I_0R_i$ restricts to $\mathrm{Ann}_{R_i}(I_i) \hookrightarrow \mathrm{Ann}_{\overline{R}_i}(\overline{I}_i)$. On the other hand, $\mathrm{Ann}_{R_i}(I_i)$ turns out to be equal to $(R_i)_{I_0\text{-tor}}$ by the following lemma.

Lemma 3.47. *For every $i \geq 0$, $I_i(R_i)_{I_0\text{-tor}} = 0$. In particular*, $\mathrm{Im}(\varphi_{I_0,R_i}) \subseteq \mathrm{Ann}_{\overline{R}_i}(\overline{I}_i)$.

Proof. By Lemma 3.46(2) and axiom (g) in Definition 3.21, we find that $F_{(R_i)_{I_0\text{-tor}}}$ is injective. In other words, $(R_i)_{I_0\text{-tor}}$ does not contain any non-zero nilpotent element. Moreover, $(R_i)_{I_0\text{-tor}} = (R_i)_{I_i\text{-tor}}$. Hence the assertion follows from Lemma 3.18. ☐

The following lemma is essential for proving Theorem 3.35.

Lemma 3.48. *For every $i \geq 0$, F_i restricts to a $\mathbb{Z}$-linear map* $\mathrm{Ann}_{\overline{R}_{i+1}}(\overline{I}_{i+1}) \to \mathrm{Ann}_{\overline{R}_i}(\overline{I}_i)$. *Moreover*, *the resulting inverse system* $\{\mathrm{Ann}_{\overline{R}_i}(\overline{I}_i)\}_{i\geq 0}$ *has the following properties.*

1. *For every $j \geq 0$, $\varprojlim^1_{i\geq 0} \mathrm{Ann}_{\overline{R}_{j+i}}(\overline{I}_{j+i}) = (0)$.*
2. *There are isomorphisms of $\mathbb{Z}$-linear maps $\varprojlim_{i\geq 0} \mathrm{Ann}_{\overline{R}_{j+i}}(\overline{I}_{j+i}) \cong (R_j)_{I_0\text{-tor}}$ $(j \geq 0)$ that are multiplicative*, *and compatible with $\{t_j^{s.\flat}\}_{j\geq 0}$ and $\{t_j\}_{j\geq 0}$.*

Proof. Since $F_i(\overline{I}_{i+1}) = \overline{I}_i$, F_i restricts to a $\mathbb{Z}$-linear map $(F_i)_{\mathrm{ann}} : \mathrm{Ann}_{\overline{R}_{i+1}}(\overline{I}_{i+1}) \to \mathrm{Ann}_{\overline{R}_i}(\overline{I}_i)$. Let $\varphi_i : (R_i)_{I_0\text{-tor}} \hookrightarrow \mathrm{Ann}_{\overline{R}_i}(\overline{I}_i)$ be the restriction of φ_{I_0,R_i}. By Lemma 3.17 and Lemma 3.47, we can write $\mathrm{Ann}_{\overline{R}_i}(\overline{I}_i) = \mathrm{Im}(\varphi_i) + \overline{I}_i^{p^i-1}$. Moreover, $\mathrm{Im}(\varphi_i) \cap \overline{I}_i^{p^i-1} = (0)$ by Lemma 3.14 and Lemma 3.47. Hence we have the following ladder with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & (R_{i+1})_{I_0\text{-tor}} \arrow[r, "\varphi_{i+1}"] \arrow[d, "(F_i)_{\mathrm{tor}}"] & \mathrm{Ann}_{\overline{R}_{i+1}}(\overline{I}_{i+1}) \arrow[r] \arrow[d] & \overline{I}_{i+1}^{p^{i+1}-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (R_i)_{I_0\text{-tor}} \arrow[r, "\varphi_i"] & \mathrm{Ann}_{\overline{R}_i}(\overline{I}_i) \arrow[r] & \overline{I}_i^{p^i-1} \arrow[r] & 0
\end{tikzcd} \tag{3-15}$$

where the second and third vertical maps are the restrictions of F_i. Since $F_i(\overline{I}_{i+1}^{p^{i+1}-1}) = 0$, both functors $\varprojlim_{i\geq 0}$ and $\varprojlim^1_{i\geq 0}$ assign (0) to the inverse system $\{\overline{I}_{j+i}^{p^{j+i}-1}\}_{i\geq 0}$. Moreover, since $(F_i)_{\mathrm{tor}}$ is bijective, $\varprojlim_{i\geq 0}(R_{j+i})_{I_0\text{-tor}} \cong (R_j)_{I_0\text{-tor}}$ and $\varprojlim^1_{i\geq 0}(R_{j+i})_{I_0\text{-tor}} = (0)$. Hence we find that $\varprojlim^1_{i\geq 0} \mathrm{Ann}_{\overline{R}_{j+i}}(\overline{I}_{j+i}) = (0)$, which is assertion (1). Furthermore, we obtain the isomorphisms of $\mathbb{Z}$-modules:

$$(R_j)_{I_0\text{-tor}} \xleftarrow{(\Phi_0^{(j)})_{\mathrm{tor}}} \varprojlim_{i\geq 0}(R_{j+i})_{I_0\text{-tor}} \xrightarrow{\varprojlim_{i\geq 0}\varphi_{j+i}} \varprojlim_{i\geq 0} \mathrm{Ann}_{\overline{R}_{j+i}}(\overline{I}_{j+i}) \tag{3-16}$$

(where $(\Phi_0^{(j)})_{\mathrm{tor}}$ denotes the 0-th projection map), which are also multiplicative. Let us deduce (2) from it. Since $t_j^{s.\flat} = \varprojlim_{i\geq 0} \overline{t}_{j+i}$ by definition, the maps $\varprojlim_{i\geq 0} \varphi_{j+i}$ $(j \geq 0)$ are compatible with $\{\varprojlim_{i\geq 0}(t_{j+i})_{\mathrm{tor}}\}_{j\geq 0}$ (induced by Lemma 3.46(2)) and $\{t_j^{s.\flat}\}_{j\geq 0}$ by Lemma 3.46(1). On the other hand, the projections $(\Phi_0^{(j)})_{\mathrm{tor}}$ $(j \geq 0)$ are compatible with $\{\varprojlim_{i\geq 0}(t_{j+i})_{\mathrm{tor}}\}_{j\geq 0}$ and $\{(t_j)_{\mathrm{tor}}\}_{j\geq 0}$. The assertion follows. ☐

Let us complete the proof of Theorem 3.35.

Proof of Theorem 3.35. (1) The implication (a) ⇒ (b) follows from Lemma 3.30. Let us show the converse (b)⇒(a). For every $i \geq 0$, put $\overline{f}_{j+i} := \Phi_i^{(j)}(f_j^{s.\flat})$, and let π_i and F_i' be as in Lemma 3.28. Then, by the assumption, we have the following commutative ladder with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & (\overline{f}_{i+1}) \arrow[r, "\iota_{i+1}"] \arrow[d] & \overline{R}_{i+1} \arrow[r, "\pi_{i+1}"] \arrow[d, "F_i"] & R_{i+1}/I_{i+1} \arrow[r] \arrow[d, "F_i'"] & 0 \\
0 \arrow[r] & (\overline{f}_i) \arrow[r, "\iota_i"] & \overline{R}_i \arrow[r, "\pi_i"] & R_i/I_i \arrow[r] & 0
\end{tikzcd}$$

where ι_i is the inclusion map. Let us consider the exact sequence obtained by taking inverse limits for all columns of the above ladder. Then, since each F_i' is an isomorphism, the map $\varprojlim_{i\geq 0} \pi_{j+i} : R_j^{s.\flat} \to \varprojlim_{i\geq 0} R_{j+i}/I_{j+i}$ is isomorphic to $\pi_j \circ \Phi_0^{(j)}$. Thus we find that $I_j^{s.\flat} = \mathrm{Im}(\varprojlim_{i\geq 0} \iota_{j+i})$. Let us show that the ideal $\mathrm{Im}(\varprojlim_{i\geq 0} \iota_{j+i}) \subseteq R_j^{s.\flat}$ is generated by $f_j^{s.\flat}$. For $i \geq 0$, let $\mu_i : \overline{R}_i \to (\overline{f}_i)$ be the $\overline{R}_i$-linear map induced by multiplication by $\overline{f}_i$. Then we obtain the commutative ladder

$$\begin{tikzcd}
\overline{R}_{i+1} \arrow[r, "\mu_{i+1}"] \arrow[d, "F_i"] & (\overline{f}_{i+1}) \arrow[r, "\iota_{i+1}"] \arrow[d] & \overline{R}_{i+1} \arrow[d, "F_i"] \\
\overline{R}_i \arrow[r, "\mu_i"] & (\overline{f}_i) \arrow[r, "\iota_i"] & \overline{R}_i .
\end{tikzcd}$$

Then, since $\mathrm{Ker}\,\mu_i = \mathrm{Ann}_{\overline{R}_i}(\overline{I}_i)$ for every $i \geq 0$, $\varprojlim_{i\geq 0} \mu_{j+i}$ is surjective by Lemma 3.48(1). Hence we have $\mathrm{Im}(\varprojlim_{i\geq 0} \iota_{j+i}) = \mathrm{Im}(\varprojlim_{i\geq 0}(\iota_{j+i} \circ \mu_{j+i}))$, where the right hand side is the ideal of $R_j^{s.\flat}$ generated by $f_j^{s.\flat}$. Thus we obtain the desired implication. Finally, note that by Proposition 3.26(3), we can take a system of elements $\{f_j^{s.\flat} \in R_j^{s.\flat}\}_{j\geq 0}$ satisfying condition (b) and such that $(f_{j+1}^{s.\flat})^p = f_j^{s.\flat}$ $(j \geq 0)$.

(2) We have $I_j^{s.\flat}(R_j^{s.\flat})_{I_j^{s.\flat}\text{-tor}} = (0)$ by Corollary 3.19. Hence, by assertion (1),

$$(R_j^{s.\flat})_{I_0^{s.\flat}\text{-tor}} = (R_j^{s.\flat})_{I_j^{s.\flat}\text{-tor}} = \mathrm{Ann}_{R_j^{s.\flat}}(I_j^{s.\flat}) = \mathrm{Ker}(\varprojlim_{i\geq 0} \mu_{j+i}) = \varprojlim_{i\geq 0} \mathrm{Ann}_{\overline{R}_{j+i}}(\overline{I}_{j+i}).$$

Thus by Lemma 3.48(2), we obtain an isomorphism $(R_j^{s.\flat})_{I_0^{s.\flat}\text{-tor}} \cong (R_j)_{I_0\text{-tor}}$ with the desired property. □

3E. ***Relation with perfectoid rings.*** In the rest of this paper, for a ring R, we use the following notation. Set the inverse limit

$$R^\flat := \varprojlim\{\cdots \to R/pR \to R/pR \to \cdots \to R/pR\},$$

where each transition map is the Frobenius endomorphism on R/pR. It is called the *tilt* (or *tilting*) of R. Moreover, we denote by $W(R)$ the ring of p-typical Witt vectors over R. If R is p-adically complete and separated, we denote by $\theta_R : W(R^\flat) \to R$ the ring map such that the diagram

$$\begin{tikzcd}
W(R^\flat) \arrow[r, "\theta_R"] \arrow[d] & R \arrow[d] \\
R^\flat \arrow[r] & R/pR
\end{tikzcd} \tag{3-17}$$

(where the vertical maps are induced by reduction modulo p and the bottom map is the first projection) commutes.

Recall the definition of perfectoid rings.

Definition 3.49 [5, Definition 3.5]. A ring S is *perfectoid* if the following conditions hold.

(1) S is ϖ-adically complete and separated for some element $\varpi \in S$ such that ϖ^p divides p.

(2) The Frobenius endomorphism on S/pS is surjective.

(3) The kernel of $\theta_S : W(S^\flat) \to S$ is principal.

We have a connection between perfectoid towers and perfectoid rings. To see this, we use the following characterization of perfectoid rings.

Theorem 3.50 (cf. [17, Corollary 16.3.75]). *Let S be a ring. Then S is a perfectoid ring if and only if S contains an element ϖ with the following properties.*

(1) *ϖ^p divides p, and S is ϖ-adically complete and separated.*

(2) *The ring map $S/\varpi S \to S/\varpi^p S$ induced by the Frobenius endomorphism on $S/\varpi^p S$ is an isomorphism.*

(3) *The multiplicative map*

$$S_{\varpi\text{-tor}} \to S_{\varpi\text{-tor}}\ ;\ s \mapsto s^p \tag{3-18}$$

is bijective.

Proof. The "if" part follows from [17, Corollary 16.3.75].

For the converse, let $\varpi \in S$ be as in Definition 3.49. Such a ϖ clearly has property (1) in the present theorem, and also has property (2) by [5, Lemma 3.10(i)]. To show the remaining part, we set $\widetilde{S} := S/S_{\varpi\text{-tor}}$. By [8, §2.1.3], the diagram of rings:

$$\begin{tikzcd} S \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & (S/\varpi S)_{\mathrm{red}} \arrow[d, "\pi_4"] \\ \widetilde{S} \arrow[r, "\pi_3"'] & (\widetilde{S}/\varpi \widetilde{S})_{\mathrm{red}} \end{tikzcd}$$

(where π_i is the canonical projection map for $i = 1, 2, 3, 4$) is cartesian. Hence $S_{\varpi\text{-tor}}$ $(= \mathrm{Ker}(\pi_1))$ is isomorphic to $\mathrm{Ker}(\pi_4)$ as a (possibly) nonunital ring. Since $(S/\varpi S)_{\mathrm{red}}$ is a perfect $\mathbb{F}_p$-algebra, it admits the Frobenius endomorphism and the inverse Frobenius. Moreover, $\mathrm{Ker}(\pi_4)$ is closed under these operations because $(\widetilde{S}/\varpi \widetilde{S})_{\mathrm{red}}$ is reduced. Consequently, there is a bijection (3-18). Hence ϖ has property (3), as desired. □

Remark 3.51. In view of the above proof, the "only if" part of Theorem 3.50 can be refined as follows. For a perfectoid ring S, an element $\varpi \in S$ such that $p \in \varpi^p S$ and S is ϖ-adically complete and separated satisfies the properties (2) and (3) in Theorem 3.50.

Corollary 3.52. *Let* $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ *be a perfectoid tower arising from some pair* (R_0, I_0). *Let* $\widehat{R_\infty}$ *denote the* I_1*-adic completion of* R_∞. *Then* $\widehat{R_\infty}$ *is a perfectoid ring.*

Proof. Since we have $\varinjlim_{i\geq 0} F_{R_i/I_0R_i} = (\varinjlim_{i\geq 0} \overline{t}_i) \circ (\varinjlim_{i\geq 0} F_i)$ and $\varinjlim_{i\geq 0} \overline{t}_i$ is a canonical isomorphism, the Frobenius endomorphism on $\widehat{R_\infty}/I_0\widehat{R_\infty}$ can be identified with $\varinjlim_{i\geq 0} F_i$. Hence one can immediately deduce from the axioms in Definition 3.21 that any generator of $I_1\widehat{R_\infty}$ has the all properties assumed on ϖ in Theorem 3.50. ☐

In view of Theorem 3.50, one can regard perfectoid rings as a special class of perfectoid towers.

Example 3.53. Let S be a perfectoid ring. Let $\varpi \in S$ be such that $p \in \varpi^p S$ and S is ϖ-adically complete and separated. Set $S_i = S$ and $t_i = \mathrm{id}_S$ for every $i \geq 0$, and $I_0 = \varpi^p S$. Then by Remark 3.51, the tower $(\{S_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is a perfectoid tower arising from (S, I_0). In particular, $I_0 S_{I_0\text{-tor}} = (0)$, and $F_{S_{I_0\text{-tor}}}$ is bijective.

Moreover, we can treat more general rings in a tower-theoretic way.

Example 3.54 (Zariskian preperfectoid rings). Let R be a ring that contains an element ϖ such that $p \in \varpi^p R$, R is ϖ-adically Zariskian, and R has bounded ϖ-torsion. Assume that the ϖ-adic completion $\hat{R}$ is a perfectoid ring. Set $R_i = R$ and $t_i = \mathrm{id}_R$ for every $i \geq 0$, and $I_0 = \varpi^p R$. Then the tower $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is a perfectoid tower arising from (R, I_0). Indeed, axioms (a) and (e) are clear from the assumption. Since $\hat{R}$ is perfectoid and $R/\varpi^p R \cong \hat{R}/\varpi^p \hat{R}$, axioms (b), (c), (d) and (f) hold by Example 3.53. Similarly, axiom (g) holds by Lemma 3.16 (the map $\psi_{\mathrm{tor}} : R_{I_0\text{-tor}} \to (\hat{R})_{I_0\text{-tor}}$ is also an isomorphism of nonunital rings).

Recall that we have two types of tilting operation at present; one is defined for perfectoid rings, and the other is for perfectoid towers. The following result asserts that they are compatible.

Lemma 3.55. *Let* $(\{R_i^{s.\flat}\}_{i\geq 0}, \{t_i^{s.\flat}\}_{i\geq 0})$ *be the tilt of* $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ *associated to* (R_0, I_0). *Let* $\widehat{R_\infty^{s.\flat}}$ *be the* $I_0^{s.\flat}$*-adic completion of* $R_\infty^{s.\flat} := \varinjlim_{i\geq 0} R_i^{s.\flat}$. *Let* $(I_0R_\infty)^\flat$ *be the ideal of* $R_\infty^\flat$ *that is the inverse image of* $I_0R_\infty \bmod pR_\infty$ *via the first projection. Then there exist canonical isomorphisms*

$$R_\infty^\flat \xleftarrow{\cong} \varprojlim_{\mathrm{Frob}} R_\infty^{s.\flat}/I_0^{s.\flat}R_\infty^{s.\flat} \xrightarrow{\cong} \widehat{R_\infty^{s.\flat}}$$

under which $(I_0R_\infty)^\flat \subseteq R_\infty^\flat$ *corresponds to* $I_0^{s.\flat}\widehat{R_\infty^{s.\flat}} \subseteq \widehat{R_\infty^{s.\flat}}$.

Proof. Note that $R_\infty^{s.\flat}$ is perfect. By Lemma 3.39 and Example 3.32, we obtain the commutative diagram of rings

$$\begin{tikzcd}
\varprojlim_{\mathrm{Frob}} R_\infty/I_0R_\infty \arrow[d] & \varprojlim_{\mathrm{Frob}} R_\infty^{s.\flat}/I_0^{s.\flat}R_\infty^{s.\flat} \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d] & \widehat{R_\infty^{s.\flat}} \arrow[d] \\
R_\infty/I_0R_\infty & R_\infty^{s.\flat}/I_0^{s.\flat}R_\infty^{s.\flat} \arrow[l, "\cong"'] \arrow[r, equal] & R_\infty^{s.\flat}/I_0^{s.\flat}R_\infty^{s.\flat}
\end{tikzcd} \tag{3-19}$$

where the vertical arrows denote the first projection maps. By [5, Lemma 3.2(i)], we can identify $R^{\flat}_{\infty}$ with $\varprojlim_{\mathrm{Frob}} R_{\infty}/I_0 R_{\infty}$, and the ideal $(I_0 R_{\infty})^{\flat} \subseteq R^{\flat}_{\infty}$ corresponds to the kernel of the leftmost vertical map. Since the kernel of the rightmost vertical map is $I_0^{s.\flat}\widehat{R^{s.\flat}_{\infty}}$, the assertion follows. ☐

3F. *Examples: complete local log-regular rings.*

3F1. *Calculation of the tilts.* As an example of tilts of Noetherian perfectoid towers, we calculate them for certain towers of local log-regular rings. Firstly, we review a perfectoid tower constructed in [17].

Construction 3.56. Let $(R, \mathcal{Q}, \alpha)$ be a *complete* local log-regular ring with perfect residue field of characteristic $p > 0$. Assume that $\mathcal{Q}$ is fine, sharp, and saturated (see Remark 2.20). Let $I_{\alpha} \subseteq R$ be the ideal defined in Definition 2.19. Set $A := R/I_{\alpha}$. Let $(f_1, \dots, f_r)$ be a sequence of elements of R whose image in A is *maximal* (see Definition A.4). Since the residue field of R is perfect, r is the dimension of A (see the Appendix). For every $i \geq 0$, we consider the ring

$$A_i := A[T_1, \dots, T_r]/(T_1^{p^i} - \overline{f}_1, \dots, T_r^{p^i} - \overline{f}_r),$$

where each $\overline{f}_j$ denotes the image of f_j in A $(j = 1, \dots, r)$. Notice that A_i is regular by Theorem A.3. Moreover, we set $\mathcal{Q}^{(i)} := \mathcal{Q}^{(i)}_p$ (see Definition 2.11). Furthermore, we define

$$R'_i := \mathbb{Z}[\mathcal{Q}^{(i)}] \otimes_{\mathbb{Z}[\mathcal{Q}]} R, \;\; R''_i := R[T_1, \dots, T_r]/(T_1^{p^i} - f_1, \dots, T_r^{p^i} - f_r), \tag{3-20}$$

and

$$R_i := R'_i \otimes_R R''_i. \tag{3-21}$$

Let $t_i : R_i \to R_{i+1}$ be the ring map that is naturally induced by the inclusion map $\iota^{(i)} : \mathcal{Q}^{(i)} \hookrightarrow \mathcal{Q}^{(i+1)}$. Since R''_{i+1} is a free R''_i-module, t_i is universally injective by Lemma 2.15 and condition (e) in Proposition 2.8(2).

Proposition 3.57. *Keep the notation as in Construction 3.56. Let $\alpha_i : \mathcal{Q}^{(i)} \to R_i$ be the natural map. Then $(R_i, \mathcal{Q}^{(i)}, \alpha_i)$ is a local log-regular ring.*

Proof. We refer the reader to [17, 17.2.5]. ☐

By construction, we obtain the tower of rings $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ (see Definition 3.1).

Proposition 3.58. *Keep the notation as in Construction 3.56. Then the tower $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ of local log-regular rings defined above is a perfectoid tower arising from $(R, (p))$.*

Proof. We verify (a)–(g) in Definition 3.4 and Definition 3.21. Axiom (a) is trivial. Since t_i is universally injective, axiom (b) follows. Axioms (c) and (d) follow from [17, (17.2.10) and Lemma 17.2.11]. Since R is of residual characteristic p, axiom (e) follows from the locality. Since t_i is injective and R_i is a domain for any $i \geq 0$, axiom (g) holds by Remark 3.22.

Finally, let us check that axiom (f) holds. In the case when $p = 0$, it follows from [17, Theorem 17.2.14(i)]. Otherwise, there exists an element $\varpi \in R_1$ that satisfies $\varpi^p = pu$ for some unit $u \in R_1$ by [17, Theorem 17.2.14(ii)]. Set $I_1 := (\varpi)$. Then axiom (f-1) holds. Axiom (f-2) follows from [17, Theorem 17.2.14(iii)]. Thus the assertion follows. ☐

For calculating the tilt of the perfectoid tower constructed above, the following lemma is quite useful.

Lemma 3.59. *Keep the notation as in Proposition 3.57. Let k be the residue field of R. Then there exists a family of ring maps $\{\phi_i : C(k)\llbracket \mathcal{Q}^{(i)} \oplus (\mathbb{N}^r)^{(i)}\rrbracket \to R_i\}_{i\geq 0}$ which is compatible with the log structures of $\{(R_i, \mathcal{Q}^{(i)}, \alpha_i)\}_{i\geq 0}$ such that the following diagram commutes for every $i \geq 0$:*

$$\begin{tikzcd}
C(k)\llbracket \mathcal{Q}^{(i)} \oplus (\mathbb{N}^r)^{(i)}\rrbracket \arrow[r, hook] \arrow[d, two heads, "\phi_i"] & C(k)\llbracket \mathcal{Q}^{(i+1)} \oplus (\mathbb{N}^r)^{(i+1)}\rrbracket \arrow[d, two heads, "\phi_{i+1}"] \\
R_i \arrow[r, hook, "t_i"] & R_{i+1}
\end{tikzcd} \tag{3-22}$$

(*where the top arrow is the natural inclusion*). *Moreover*, *there exists an element $\theta \in C(k)\llbracket \mathcal{Q} \oplus \mathbb{N}^r\rrbracket$ whose constant term is p such that the kernel of ϕ_i is generated by θ for every $i \geq 0$.*

Proof. First we remark the following. Let k_i be the residue field of R_i. Then by Lemma 3.11(1) and Lemma 3.6(2), the transition maps induce a purely inseparable extension $k \hookrightarrow k_i$. Moreover, this extension is trivial because k is perfect. Therefore, we can identify k_i with k, and the Cohen ring of R_i with $C(k)$.

Next, let us show the existence of a family of ring maps $\{\phi_i\}_{i\geq 0}$ with the desired compatibility. Since $(R_i, \mathcal{Q}^{(i)}, \alpha_i)$ is a complete local log-regular ring, we can take a surjective ring map $\psi_i : C(k)\llbracket \mathcal{Q}^{(i)} \oplus \mathbb{N}^r\rrbracket \to R_i$ as in Theorem 2.22; its kernel is generated by an element θ_i whose constant term is p, and the diagram

$$\begin{tikzcd}
\mathcal{Q}^{(i)} \arrow[r] \arrow[dr, "\alpha_i"'] & C(k)\llbracket \mathcal{Q}^{(i)} \oplus \mathbb{N}^r\rrbracket \arrow[d, two heads, "\psi_i"] \\
 & R_i
\end{tikzcd}$$

commutes. For $j = 1, \dots, r$, let us denote by f_j^{1/p^i} the image of $T_j \in R[T_1, \dots, T_r]$ in R_i (see (3-20) and (3-21)). Note that the sequence $f_1^{1/p^i}, \dots, f_r^{1/p^i}$ in R_i becomes a regular system of parameters of R_i/I_{α_i} by the reduction modulo I_{α_i} (see [17, 17.2.3] and [17, 17.2.5]). Thus, for the set of the canonical basis $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_r\}$ of $\mathbb{N}^r$, we may assume $\psi_i(e^{\boldsymbol{e}_j}) = f_j^{1/p^i}$ by the construction of ψ_i (see the proof of [34, Chapter III, Theorem 1.11.2]). Hence we can choose $\{\psi_i\}_{i\geq 0}$ so that the diagram:

$$\begin{tikzcd}
C(k)\llbracket \mathcal{Q}^{(i)} \oplus \mathbb{N}^r\rrbracket \arrow[r, hook] \arrow[d, two heads, "\psi_i"'] & C(k)\llbracket \mathcal{Q}^{(i+1)} \oplus \mathbb{N}^r\rrbracket \arrow[d, two heads, "\psi_{i+1}"] \\
R_i \arrow[r, hook, "t_i"] & R_{i+1}
\end{tikzcd} \tag{3-23}$$

commutes. Thus it suffices to define $\phi_i : C(k)\llbracket \mathcal{Q}^{(i)} \oplus (\mathbb{N}^r)^{(i)}\rrbracket \to R_i$ as the composite map of the isomorphism $C(k)\llbracket \mathcal{Q}^{(i)} \oplus (\mathbb{N}^r)^{(i)}\rrbracket \xrightarrow{\cong} C(k)\llbracket \mathcal{Q}^{(i)} \oplus \mathbb{N}^r\rrbracket$ obtained by Lemma 2.12(3) and ψ_i.

Finally, note that the image of $\theta_0 \in \operatorname{Ker}(\psi_0)$ in $C(k)\llbracket \mathcal{Q}^{(i)} \oplus \mathbb{N}^r\rrbracket$ is contained in $\operatorname{Ker}(\psi_i)$, and its constant term is still p. Thus, by the latter assertion of Theorem 2.22(2), $\operatorname{Ker}(\psi_i)$ is generated by θ_0. Hence by taking θ_0 as θ, we complete the proof. $\square$

Let us consider the monoids $\mathcal{Q}^{(i)}$ for an integral sharp monoid $\mathcal{Q}$. Since there is the natural inclusion $\iota^{(i)} : \mathcal{Q}^{(i)} \hookrightarrow \mathcal{Q}^{(i+1)}$ for any $i \geq 0$, we obtain a direct system of monoids $(\{\mathcal{Q}^{(i)}\}_{i\geq 0}, \{\iota^{(i)}\}_{i\geq 0})$. Moreover, the p-times map on $\mathcal{Q}^{(i+1)}$ gives a factorization:

$$\begin{tikzcd}
\mathcal{Q}^{(i+1)} \arrow[r, "\times p"] \arrow[dr, two heads, "\times p"'] & \mathcal{Q}^{(i+1)} \\
& \mathcal{Q}^{(i)}. \arrow[u, hook, "\iota^{(i)}"']
\end{tikzcd}$$

From this discussion, we define the small tilt of $\{\mathcal{Q}^{(i)}\}_{i\geq 0}$.

Definition 3.60. Let $\mathcal{Q}$ be an integral monoid, and let $(\{\mathcal{Q}^{(i)}\}_{i\geq 0}, \{\iota^{(i)}\}_{i\geq 0})$ be as above. Then for an integer $j \geq 0$, we define the *j-th small tilt of* $(\{\mathcal{Q}^{(i)}\}_{i\geq 0}, \{\iota^{(i)}\}_{i\geq 0})$ as the inverse limit

$$\mathcal{Q}_j^{s.\flat} := \varprojlim\{\cdots \to \mathcal{Q}^{(j+1)} \to \mathcal{Q}^{(j)}\}, \tag{3-24}$$

where the transition map $\mathcal{Q}^{(i+1)} \to \mathcal{Q}^{(i)}$ is the p-times map of monoids.

Now we can derive important properties of the tilt of the perfectoid tower given in Construction 3.56.

Theorem 3.61. *Keep the notation as in Lemma 3.59.*

1. *The tower* $(\{(R_i)^{s.\flat}_{(p)}\}_{i\geq 0}, \{(t_i)^{s.\flat}_{(p)}\}_{i\geq 0})$ *is isomorphic to* $(\{k[\![\mathcal{Q}^{(i)} \oplus (\mathbb{N}^r)^{(i)}]\!]\}_{i\geq 0}, \{u_i\}_{i\geq 0})$, *where u_i is the ring map induced by the natural inclusion* $\mathcal{Q}^{(i)} \oplus (\mathbb{N}^r)^{(i)} \hookrightarrow \mathcal{Q}^{(i+1)} \oplus (\mathbb{N}^r)^{(i+1)}$.
2. *For every $j \geq 0$, there exists a homomorphism of monoids*
$$\alpha_j^{s.\flat} : \mathcal{Q}_j^{s.\flat} \to (R_j)^{s.\flat}_{(p)}$$
such that $((R_j)^{s.\flat}_{(p)}, \mathcal{Q}_j^{s.\flat}, \alpha_j^{s.\flat})$ *is a local log-regular ring.*
3. *For every $j \geq 0$,* $(t_j)^{s.\flat}_{(p)} : (R_j)^{s.\flat}_{(p)} \to (R_{j+1})^{s.\flat}_{(p)}$ *is module-finite and* $(R_j)^{s.\flat}_{(p)}$ *is F-finite.*

Proof. (1) By Lemma 3.59, each R_i is isomorphic to $C(k)[\![\mathcal{Q}^{(i)} \oplus (\mathbb{N}^r)^{(i)}]\!]/(p - f)C(k)[\![\mathcal{Q}^{(i)} \oplus (\mathbb{N}^r)^{(i)}]\!]$ where f is an element of $C(k)[\![\mathcal{Q} \oplus \mathbb{N}^r]\!]$ which has no constant term. Set $S_i := k[\![\mathcal{Q}^{(i)} \oplus (\mathbb{N}^r)^{(i)}]\!]$ for any $i \geq 0$ and let $u_i : S_i \hookrightarrow S_{i+1}$ be the inclusion map induced by the natural inclusion $\mathcal{Q}^{(i)} \oplus (\mathbb{N}^r)^{(i)} \hookrightarrow \mathcal{Q}^{(i+1)} \oplus (\mathbb{N}^r)^{(i+1)}$. Then the tower $(\{S_i\}_{i\geq 0}, \{u_i\}_{i\geq 0})$ is a perfect tower. Indeed, each S_i is reduced by Theorem 2.21; moreover, by the perfectness of k and Lemma 2.12(3), the Frobenius endomorphism on S_{i+1} factors through a surjection $G_i : S_{i+1} \twoheadrightarrow S_i$. In particular, $(\{S_i\}_{i\geq 0}, \{u_i\}_{i\geq 0})$ is a perfectoid tower arising from $(S_0, (0))$ and G_i is the i-th Frobenius projection (cf. Lemma 3.24).

Put $\overline{f} := f \bmod pC(k)[\![\mathcal{Q} \oplus \mathbb{N}^r]\!] \in S_0$. Then each S_i is $\overline{f}$-adically complete and separated by [15, Lemma 2.1.1]. Moreover, the commutative diagram (3-22) yields the commutative squares ($i \geq 0$):

$$\begin{tikzcd}
S_{i+1}/\overline{f}S_{i+1} \arrow[r, "\cong"] \arrow[d, "\overline{G}_i"] & R_{i+1}/pR_{i+1} \arrow[d, "F_i"] \\
S_i/\overline{f}S_i \arrow[r, "\cong"] & R_i/pR_i
\end{tikzcd}$$

that are compatible with $\{\overline{u}_i : S_i/\overline{f}S_i \to S_{i+1}/\overline{f}S_{i+1}\}_{i\geq 0}$ and $\{\overline{t}_i\}_{i\geq 0}$. Hence by Lemma 3.31, we obtain the isomorphisms

$$(R_j)^{s.\flat}_{(p)} \xleftarrow{\cong} \varprojlim\{\cdots \xrightarrow{\overline{G}_{j+1}} S_{j+1}/\overline{f}S_{j+1} \xrightarrow{\overline{G}_j} S_j/\overline{f}S_j\} \xrightarrow{\cong} S_j \qquad (j \geq 0) \tag{3-25}$$

that are compatible with the transition maps of the towers. Thus the assertion follows.

(2) Considering the inverse limit of the composite maps $\mathcal{Q}^{(j+i)} \xrightarrow{\alpha_{j+i}} R_{j+i} \twoheadrightarrow R_{j+i}/pR_{j+i}$ $(i \geq 0)$, we obtain a homomorphism of monoids $\alpha_j^{s.\flat} : \mathcal{Q}_j^{s.\flat} \to (R_j)^{s.\flat}_{(p)}$. On the other hand, let $\overline{\alpha}_j : \mathcal{Q}^{(j)} \to S_j$ be the natural inclusion. Then, since S_j is canonically isomorphic to $k[\![\mathcal{Q}^{(j)} \oplus \mathbb{N}^r]\!]$, $(S_j, \mathcal{Q}^{(j)}, \overline{\alpha}_j)$ is a local log-regular ring by Theorem 2.22(1). Thus it suffices to show that $((R_j)^{s.\flat}_{(p)}, \mathcal{Q}_j^{s.\flat}, \alpha_j^{s.\flat})$ is isomorphic to $(S_j, \mathcal{Q}^{(j)}, \overline{\alpha}_j)$ as a log ring. Since the transition maps in (3-24) are isomorphisms by Lemma 2.12(3), we obtain the isomorphisms of monoids

$$\mathcal{Q}_j^{s.\flat} \xleftarrow{\mathrm{id}_{\mathcal{Q}_j^{s.\flat}}} \mathcal{Q}_j^{s.\flat} \xrightarrow{\cong} \mathcal{Q}^{(j)} \quad (j \geq 0). \tag{3-26}$$

Then one can connect (3-26) to (3-25) to construct a commutative diagram using $\alpha_j^{s.\flat}$ and $\overline{\alpha}_j$. Hence the assertion follows.

(3) By Lemma 2.14(1), $t_j : R_j \to R_{j+1}$ is module-finite. Hence by Proposition 3.42(1), $(t_j)^{s.\flat}_{(p)} : (R_j)^{s.\flat}_{(p)} \to (R_{j+1})^{s.\flat}_{(p)}$ is also module-finite. Finally let us show that $(R_j)^{s.\flat}_{(p)}$ is F-finite. By assertion (2), $(R_j)^{s.\flat}_{(p)}$ is a complete Noetherian local ring, and the residue field is F-finite because it is perfect. Thus the assertion follows from [29, Theorem 8.4]. □

Example 3.62. (1) A tower of regular local rings which is treated in [7] and [8] is a perfectoid tower in our sense. Let $(R, \mathfrak{m}, k)$ be a d-dimensional regular local ring whose residue field k is perfect and let $x_1, \ldots, x_d$ be a regular sequence of parameters. Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$ be the canonical basis of $\mathbb{N}^d$. Then $(R, \mathbb{N}^d, \alpha)$ is a local log-regular ring where $\alpha : \mathbb{N}^d \to R$ is a homomorphism of monoids which maps $\boldsymbol{e}_i$ to x_i. Furthermore, assume that R is $\mathfrak{m}$-adically complete. Then, by Cohen's structure theorem, R is isomorphic to

$$W(k)[\![x_1, \ldots, x_d]\!]/(p - f)$$

where $f = x_1$ or $f \in (p, x_1, \ldots, x_d)^2$ (the former case is called *unramified*, and the latter *ramified*). Let us construct a perfectoid tower arising from $(R, (p))$ along Construction 3.56. Since k is perfect, $\boldsymbol{\Omega}_k$ is zero by the short exact sequences (A-4) and the definition of itself. This implies that the image of the empty subset of R in k forms a maximal sequence. Hence R_i'' in Construction 3.56 is equal to R. Moreover, $(\mathbb{N}^d)^{(i)}$ is generated by $\frac{1}{p^i}\boldsymbol{e}_1, \ldots, \frac{1}{p^i}\boldsymbol{e}_d$. Applying Construction 3.56, we obtain

$$\begin{aligned} R_i = R_i' = \mathbb{Z}[(\mathbb{N}^d)^{(i)}] \otimes_{\mathbb{Z}[\mathbb{N}^d]} R &\cong R[T_1, \ldots, T_d]/(T_1^{p^i} - x_1, \ldots, T_d^{p^i} - x_d) \\ &\cong W(k)[\![x_1^{1/p^i}, \ldots, x_d^{1/p^i}]\!]/(p - f). \end{aligned}$$

Set the natural injection $t_i : R_i \to R_{i+1}$ for any $i \geq 0$. Then, by Proposition 3.58, $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is a perfectoid tower arising from $(R, (p))$. By Theorem 3.61, its tilt $(\{(R_i)^{s.\flat}_{(p)}\}_{i\geq 0}, \{(t_i)^{s.\flat}_{(p)}\}_{i\geq 0})$ is isomorphic to the tower $k[\![\mathbb{N}^d]\!] \hookrightarrow k[\![(\mathbb{N}^d)^{(1)}]\!] \hookrightarrow k[\![(\mathbb{N}^d)^{(2)}]\!] \hookrightarrow \cdots$, which can be written as

$$k[\![x_1, \dots, x_d]\!] \hookrightarrow k[\![x_1^{1/p}, \dots, x_d^{1/p}]\!] \hookrightarrow k[\![x_1^{1/p^2}, \dots, x_d^{1/p^2}]\!] \hookrightarrow \cdots .$$

(2) Consider the surjection

$$\begin{aligned} S := W(k)[\![x, y, z, w]\!]/(xy - zw) \twoheadrightarrow R &:= W(k)[\![x, y, z, w]\!]/(xy - zw, p - w) \\ &= W(k)[\![x, y, z]\!]/(xy - pz). \end{aligned}$$

where k is a perfect field. Let $\mathcal{Q} \subseteq \mathbb{N}^4$ be a saturated submonoid generated by $(1, 1, 0, 0)$, $(0, 0, 1, 1)$, $(1, 0, 0, 1)$ and $(0, 1, 1, 0)$. Then S admits a homomorphism of monoids $\alpha_S : \mathcal{Q} \to S$ by letting $(1, 1, 0, 0) \mapsto x$, $(0, 0, 1, 1) \mapsto y$, $(1, 0, 0, 1) \mapsto z$ and $(0, 1, 1, 0) \mapsto w$. With this, $(S, \mathcal{Q}, \alpha_S)$ is a local log-regular ring. The composite map $\alpha_R : \mathcal{Q} \to S \to R$ makes R into a local log ring. Indeed, we can write $R \cong W(k)[\![\mathcal{Q}]\!]/(p - e^{(0,1,1,0)})$; hence $(R, \mathcal{Q}, \alpha_R)$ is log-regular by Theorem 2.22.

Next, note that $R/I_{\alpha_R} \cong k$. Then, for the same reason in (1), R''_i is equal to R. Moreover, $\mathcal{Q}^{(i)}$ is generated by

$$\Big(\frac{1}{p^i}, \frac{1}{p^i}, 0, 0\Big), \ \Big(0, 0, \frac{1}{p^i}, \frac{1}{p^i}\Big), \ \Big(\frac{1}{p^i}, 0, 0\frac{1}{p^i}\Big), \ \Big(0, \frac{1}{p^i}, \frac{1}{p^i}, 0\Big).$$

Thus, applying Construction 3.56, we obtain

$$\begin{aligned} R_i &= R[\![\mathcal{Q}^{(i)}]\!] \\ &\cong W(k)[\![\mathcal{Q}^{(i)}]\!]/(p - e^{(0,1,1,0)}) \\ &\cong W(k)[\![x^{1/p^i}, y^{1/p^i}, z^{1/p^i}, w^{1/p^i}]\!]/(x^{1/p^i} y^{1/p^i} - z^{1/p^i} w^{1/p^i}, p - w). \end{aligned}$$

Set a natural injection $t_i : R_i \to R_{i+1}$. Then, by Proposition 3.58, $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ is a perfectoid tower arising from $(R, (p))$. Hence

$$R_\infty = \varinjlim_{i\geq 0} R_i \cong \bigcup_{i\geq 0} W(k)[\![x^{1/p^i}, y^{1/p^i}, z^{1/p^i}, w^{1/p^i}]\!]/(x^{1/p^i} y^{1/p^i} - z^{1/p^i} w^{1/p^i}, p - w),$$

and its p-adic completion is perfectoid. One can calculate the tilt $\big(\{R_i^{s.\flat}\}_{i\geq 0}, \{t_i^{s.\flat}\}_{i\geq 0}\big)$ to be $k[\![\mathcal{Q}]\!] \hookrightarrow k[\![\mathcal{Q}^{(1)}]\!] \hookrightarrow k[\![\mathcal{Q}^{(2)}]\!] \hookrightarrow \cdots$ by Theorem 3.61, or, more explicitly,

$$k[\![x, y, z, w]\!]/(xy - zw) \hookrightarrow k[\![x^{1/p}, y^{1/p}, z^{1/p}, w^{1/p}]\!]/(x^{1/p} y^{1/p} - z^{1/p} w^{1/p}) \hookrightarrow \cdots .$$

3F2. *Towers of split maps and sousperfectoid rings.* Recall that Hansen and Kedlaya introduced a new class of topological rings that guarantees sheafiness on the associated adic spectra (see [21, Definition 7.1]).

Definition 3.63. Let A be a complete and separated Tate ring such that a prime $p \in A$ is topologically nilpotent. We say that A is *sousperfectoid* if there exists a perfectoid ring B in the sense of Fontaine (see [21, Definition 2.13]) with a continuous A-linear map $f : A \to B$ that splits in the category of topological A-modules. That is, there is a continuous A-linear map $\sigma : B \to A$ such that $\sigma \circ f = \mathrm{id}_A$.

Let us show that a perfectoid tower consisting of split maps induces sousperfectoid rings. In view of Theorem 2.29, one can apply this result to the towers discussed above. See [33] for detailed studies on algebraic aspects of Tate rings.

Proposition 3.64. *Let* $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ *be a perfectoid tower arising from some pair* $(R, (f_0))$*. Assume that* f_0 *is regular, R is* f_0*-adically complete and separated, and* t_i *splits as an* R_i*-linear map for every* $i \geq 0$*. We equip* $R\bigl[\frac{1}{f_0}\bigr]$ *with the linear topology in such a way that* $\{f_0^n R\}_{n\geq 1}$ *defines a fundamental system of open neighborhoods at* $0 \in R\bigl[\frac{1}{f_0}\bigr]$*. Then* $R\bigl[\frac{1}{f_0}\bigr]$ *is a sousperfectoid Tate ring, and hence stably uniform.*

In order to prove this, we need the following lemma.

Lemma 3.65. *Keep the notations and assumptions as in Proposition 3.64. Then the natural map* $R_0 \to \varinjlim_{i\geq 0} R_i$ *splits as an* R_0*-linear map.*

Proof. We use the fact that each $t_i : R_i \to R_{i+1}$ splits as an R_i-linear map by assumption. This implies that the short exact sequence of R-modules

$$0 \to R_0 \to R_i \to R_i/R \to 0$$

splits for any $i \geq 0$. It induces a commutative diagram of R-modules

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{R_0}(R_{i+1}/R_0, R_0) \arrow[r] \arrow[d, "\alpha_i"] & \mathrm{Hom}_{R_0}(R_{i+1}, R_0) \arrow[r] \arrow[d, "\beta_i"] & \mathrm{Hom}_{R_0}(R_0, R_0) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{R_0}(R_i/R_0, R_0) \arrow[r] & \mathrm{Hom}_{R_0}(R_i, R_0) \arrow[r] & \mathrm{Hom}_{R_0}(R_0, R_0) \arrow[r] & 0
\end{tikzcd}$$

where each horizontal sequence is split exact and each vertical map forms an inverse system induced by $t_i : R_i \to R_{i+1}$. Thus β_i is surjective and it follows from the snake lemma that α_i is surjective as well. By taking inverse limits, we obtain the short exact sequence

$$0 \to \varprojlim_{i\geq 0} \mathrm{Hom}_{R_0}(R_i/R_0, R_0) \to \varprojlim_{i\geq 0} \mathrm{Hom}_{R_0}(R_i, R_0) \xrightarrow{h} \mathrm{Hom}_{R_0}(R_0, R_0) \to 0.$$

It follows from [36, Lemma 4.1] that h is the canonical surjection $\mathrm{Hom}_{R_0}(R_\infty, R_0) \twoheadrightarrow \mathrm{Hom}_{R_0}(R_0, R_0)$. Then choosing an inverse image of $\mathrm{id}_{R_0} \in \mathrm{Hom}_{R_0}(R_0, R_0)$ gives a splitting of $R_0 \to R_\infty$. □

Proof of Proposition 3.64. We have constructed an infinite extension $R \to R_\infty$ such that if $\widehat{R}_\infty$ is the f_0-adic completion, then the associated Tate ring $\widehat{R}_\infty\bigl[\frac{1}{f_0}\bigr]$ is a perfectoid ring in the sense of Fontaine by Corollary 3.52 and [5, Lemma 3.21].

By Lemma 2.28 and Lemma 3.65, it follows that the map $R\bigl[\frac{1}{f_0}\bigr] \to \widehat{R}_\infty\bigl[\frac{1}{f_0}\bigr]$ splits in the category of topological $R\bigl[\frac{1}{f_0}\bigr]$-modules (notice that R is f_0-adically complete and separated). Thus, $R\bigl[\frac{1}{f_0}\bigr]$ is a sousperfectoid Tate ring. The combination of [21, Corollary 8.10], [21, Proposition 11.3] and [21, Lemma 11.9] allows us to conclude that $R\bigl[\frac{1}{f_0}\bigr]$ is stably uniform. □

As a corollary, one can obtain the stable uniformity for complete local log-regular rings (see also Construction 3.56 and Theorem 2.29).

Corollary 3.66. *Let* $(R, \mathcal{Q}, \alpha)$ *is a complete local log-regular ring of mixed characteristic with perfect residue field. We equip* $R[\frac{1}{p}]$ *with the structure of a complete and separated Tate ring in such a way that* $\{p^n R\}_{n\geq 1}$ *defines a fundamental system of open neighborhoods at* $0 \in R[\frac{1}{p}]$. *Then* $R[\frac{1}{p}]$ *is stably uniform.*

4. Applications to étale cohomology of Noetherian rings

In this section, we establish several results on étale cohomology of Noetherian rings, as applications of the theory of perfectoid towers developed in Section 3. In Section 4A, for a ring that admits a certain type of perfectoid tower, we prove that finiteness of étale cohomology groups on the positive characteristic side carries over to the mixed characteristic side (Proposition 4.7). In Section 4B, we apply this result to a problem on divisor class groups of log-regular rings.

We prepare some notation. Let X be a scheme and let $X_{\text{ét}}$ denote the category of schemes that are étale over X, and for any étale X-scheme Y, we specify the covering $\{Y_i \to Y\}_{i\in I}$ so that Y_i is étale over Y and the family $\{Y_i\}_{i\in I}$ covers surjectively Y. For an abelian sheaf $\mathcal{F}$ on $X_{\text{ét}}$, we denote by $H^i(X_{\text{ét}}, \mathcal{F})$ the value of the i-th derived functor of $U \in X_{\text{ét}} \mapsto \Gamma(U, \mathcal{F})$. For the most part of applications, we consider *torsion* sheaves, such as $\mathbb{Z}/n\mathbb{Z}$ and μ_n for $n \in \mathbb{N}$. However, for the multiplicative group scheme $\mathbb{G}_m$, we often use the following isomorphism:

$$H^1(X_{\text{ét}}, \mathbb{G}_m) \cong \operatorname{Pic}(X).$$

For the basics on étale cohomology, we often use [12] or [31] as references.

4A. *Tilting étale cohomology groups.*

Let A be a ring with an ideal J, let $\hat{A}$ be the J-adic completion of A, and let $U \subseteq \operatorname{Spec}(A)$ be an open subset. We define the J-adic completion of U to be the open subset $\hat{U} \subseteq \operatorname{Spec}(\hat{A})$, which is the inverse image of U via $\operatorname{Spec}(\hat{A}) \to \operatorname{Spec}(A)$. We will use the following result for deriving results on the behavior of étale cohomology under the tilting operation as well as some interesting results on the divisor class groups of Noetherian normal domains (see Proposition 4.10 and Proposition 4.11).

Theorem 4.1 (Fujiwara and Gabber). *Let* (A, J) *be a Henselian pair with* $X := \operatorname{Spec}(A)$ *and let* $\hat{A}$ *be the* J*-adic completion of* A.

(1) *For any abelian torsion sheaf* $\mathscr{F}$ *on* $X_{\text{ét}}$, *we have* $\boldsymbol{R}\Gamma(\operatorname{Spec}(A)_{\text{ét}}, \mathscr{F}) \simeq \boldsymbol{R}\Gamma(\operatorname{Spec}(A/J)_{\text{ét}}, \mathscr{F}|_{\operatorname{Spec}(A/J)})$.

(2) *Assume that* J *is finitely generated. Then for any abelian torsion sheaf* $\mathscr{F}$ *on* $X_{\text{ét}}$ *and any open subset* $U \subseteq X$ *such that* $X \setminus V(J) \subseteq U$, *we have* $\boldsymbol{R}\Gamma(U_{\text{ét}}, \mathscr{F}) \simeq \boldsymbol{R}\Gamma(\hat{U}_{\text{ét}}, \mathscr{F})$.

Proof. The first statement is known as *Affine analog of proper base change* in [16], while the second one is known as *Formal base change theorem* which is [13, Theorem 7.1.1] in the Noetherian case, and [24, XX, 4.4] in the non-Noetherian case. ☐

We will need the tilting invariance of (local) étale cohomology from [8, Theorem 2.2.7]. To state the theorem and establish a variant of it, we give some notations.

Definition 4.2. Let (A, I) and (B, J) be pairs such that there exists a ring isomorphism $\Phi : A/I \xrightarrow{\cong} B/J$. Then for any open subset $U \subseteq \operatorname{Spec}(B)$ containing $\operatorname{Spec}(B) \setminus V(J)$, we define an open subset $F_{A,\Phi}(U) \subseteq \operatorname{Spec}(A)$ as the complement of the closed subset $\operatorname{Spec}(\Phi)(\operatorname{Spec}(B) \setminus U) \subseteq \operatorname{Spec}(A)$.

One can define small tilts of Zariski-open subsets.

Definition 4.3. Let $(\{R_i\}_{i\geq 0}, \{t_i\}_{i\geq 0})$ be a perfectoid tower arising from some pair (R, I_0), and let $(\{R_i^{s.\flat}\}_{i\geq 0}, \{t_i^{s.\flat}\}_{i\geq 0})$ be the tilt associated to (R, I_0). Recall that we then have an isomorphism of rings $\overline{\Phi_0^{(i)}} : R_i^{s.\flat}/I_0^{s.\flat} R_i^{s.\flat} \xrightarrow{\cong} R_i/I_0R_i$ for every $i \geq 0$. For every $i \geq 0$ and every open subset $U \subseteq \operatorname{Spec}(R_i)$ containing $\operatorname{Spec}(R_i) \setminus V(I_0R_i)$, we define

$$U_{I_0}^{s.\flat} := F_{R_i^{s.\flat}, \overline{\Phi_0^{(i)}}}(U).$$

We also denote $U_{I_0}^{s.\flat}$ by $U^{s.\flat}$ as an abbreviated form.

Note that by the compatibility described in Lemma 3.39, the operation $U \rightsquigarrow U^{s.\flat}$ is compatible with the base extension along the transition maps of a perfectoid tower.

Let us give some examples of $U^{s.\flat}$.

Example 4.4 (punctured spectra of regular local rings). Keep the notation as in Example 3.62(1). In this situation, the isomorphism $\overline{\Phi_0^{(0)}} : R_0^{s.\flat}/I_0^{s.\flat} \xrightarrow{\cong} R_0/I_0$ in Definition 4.3 can be written as

$$k[\![x_1, \dots, x_d]\!]/(p^{s.\flat}) \xrightarrow{\cong} R/pR, \tag{4-1}$$

where $p^{s.\flat} \in k[\![x_1, \dots, x_d]\!]$ is some element. Set $U := \operatorname{Spec}(R) \setminus V(\mathfrak{m})$. Then, since the maximal ideal $\overline{\mathfrak{m}} \subseteq R/pR$ corresponds to the (unique) maximal ideal of $k[\![x_1, \dots, x_d]\!]/(p^{s.\flat})$, we have

$$U^{s.\flat} \cong \operatorname{Spec}(k[\![x_1, \dots, x_d]\!]) \setminus V((x_1, \dots, x_d)).$$

Example 4.5 (tilting for preperfectoid rings). Keep the notation as in Example 3.54. Then by Lemma 3.55, $\overline{\Phi_0^{(0)}} : R_0^{s.\flat}/I_0^{s.\flat} \xrightarrow{\cong} R_0/I_0$ is identified with the isomorphism

$$\overline{\theta}_{\hat{R}} : (\hat{R})^\flat/I_0^\flat(\hat{R})^\flat \xrightarrow{\cong} \hat{R}/I_0\hat{R} \tag{4-2}$$

which is induced by the bottom map in the diagram (3-17). In this case, we denote $F_{R^\flat, \overline{\Phi_0^{(0)}}}(U)$ by $U^\flat$ in distinction from $U^{s.\flat}$.

The comparison theorem we need, due to Česnavičius and Scholze, is stated as follows.

Theorem 4.6 [8, Theorem 2.2.7]. *Let A be a ϖ-adically Henselian ring with bounded ϖ-torsion for an element $\varpi \in A$ such that $p \in \varpi^p A$. Assume that the ϖ-adic completion of A is perfectoid. Let $U \subseteq$* $\operatorname{Spec}(A)$ *be a Zariski-open subset such that* $\operatorname{Spec}(A) \setminus V(\varpi A) \subseteq U$, *and let $U^\flat \subseteq$* $\operatorname{Spec}(A^\flat)$ *be its tilt* (*see Example 4.5*).

(1) *For every torsion abelian group G, we have $\boldsymbol{R}\Gamma(U_{\text{ét}}, G) \cong \boldsymbol{R}\Gamma(U^\flat_{\text{ét}}, G)$ in a functorial manner with respect to A, U, and G.*

(2) *Let Z be the complement of $U \subseteq \operatorname{Spec}(A)$. Then for a torsion abelian group G, we have*

$$\boldsymbol{R}\Gamma_Z(\operatorname{Spec}(A)_{\text{ét}}, G) \cong \boldsymbol{R}\Gamma_Z(\operatorname{Spec}(A^\flat)_{\text{ét}}, G).$$

Now we come to the main result on tilting étale cohomology groups. Recall that we have fixed a prime $p > 0$.

Proposition 4.7. *Let $(\{R_j\}_{j\geq 0}, \{t_j\}_{j\geq 0})$ be a perfectoid tower arising from some pair (R, I_0). Suppose that R_j is I_0-adically Henselian for every $j \geq 0$. Let ℓ be a prime different from p. Suppose further that for every $j \geq 0$, $t_j : R_j \to R_{j+1}$ is a module-finite extension of Noetherian normal domains whose generic extension is of p-power degree.*[13] *Fix a Zariski-open subset $U \subseteq \operatorname{Spec}(R)$ such that $\operatorname{Spec}(R) \setminus V(pR) \subseteq U$ and the corresponding open subset $U^{s.\flat} \subseteq \operatorname{Spec}(R^{s.\flat})$ (cf. Definition 4.3). Then, for any fixed $i, n \geq 0$ such that $|H^i(U^{s.\flat}_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})| < \infty$, one has*

$$|H^i(U_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})| \leq |H^i(U^{s.\flat}_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})|.$$

In particular, if $H^i(U^{s.\flat}_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) = 0$, then $H^i(U_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) = 0$.

Proof. Since each R_j is a p-adically Henselian normal domain, so is $R_\infty = \varinjlim_{j\geq 0} R_j$. Moreover, every prime ℓ different from p is a unit in R_j and R_∞. Attached to the tower $(\{R_j\}_{j\geq 0}, \{t_j\}_{j\geq 0})$, we get a tower of finite (not necessarily flat) maps of normal schemes:

$$U = U_0 \leftarrow \cdots \leftarrow U_j \leftarrow U_{j+1} \leftarrow \cdots . \tag{4-3}$$

More precisely, let $h_j : \operatorname{Spec}(R_{j+1}) \to \operatorname{Spec}(R_j)$ be the associated scheme map. Then the open set U_{j+1} is defined as the inverse image $h_j^{-1}(U_j)$, thus defining the map $U_{j+1} \to U_j$ in the tower (4-3). Since h_j is a finite morphism of normal schemes, Lemma 3.4 of [3] applies to yield a well-defined trace map $\operatorname{Tr} : h_{j*}h_j^*\mathbb{Z}/\ell^n\mathbb{Z} \to \mathbb{Z}/\ell^n\mathbb{Z}$ such that

$$\mathbb{Z}/\ell^n\mathbb{Z} \xrightarrow{h_j^*} h_{j*}h_j^*\mathbb{Z}/\ell^n\mathbb{Z} \xrightarrow{\operatorname{Tr}} \mathbb{Z}/\ell^n\mathbb{Z} \tag{4-4}$$

is multiplication by the generic degree of h_j (=p-power order). Then this is bijective, as the multiplication map by p on $\mathbb{Z}/\ell^n\mathbb{Z}$ is bijective. We have the natural map: $H^i(U_{j,\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \to H^i(U_{j+1,\text{ét}}, h_j^*\mathbb{Z}/\ell^n\mathbb{Z})$. Since h_j is affine, the Leray spectral sequence gives $H^i(U_{j+1,\text{ét}}, h_j^*\mathbb{Z}/\ell^n\mathbb{Z}) \cong H^i(U_{j,\text{ét}}, h_{j*}h_j^*\mathbb{Z}/\ell^n\mathbb{Z})$. Composing these maps, the composite map (4-4) induces

$$H^i(U_{j,\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \to H^i(U_{j+1,\text{ét}}, h_j^*\mathbb{Z}/\ell^n\mathbb{Z}) \xrightarrow{\cong} H^i(U_{j,\text{ét}}, h_{j*}h_j^*\mathbb{Z}/\ell^n\mathbb{Z}) \xrightarrow{\operatorname{Tr}} H^i(U_{j,\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})$$

and the composition is bijective. Since $h_j^*\mathbb{Z}/\ell^n\mathbb{Z} \cong \mathbb{Z}/\ell^n\mathbb{Z}$, we get an injection

$$H^i(U_{j,\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \hookrightarrow H^i(U_{j+1,\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}). \tag{4-5}$$

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[13] The existence of such towers is quite essential for applications to étale cohomology, because the extension degree of each $R_j \to R_{j+1}$ is controlled in such a way that the p-adic completion of its colimit is a perfectoid ring.

Set $U_\infty = \varprojlim_j U_j$. Since each morphism $U_{j+1} \to U_j$ is affine, by using (4-5) and [38, Tag 09YQ], we have

$$H^i(U_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \hookrightarrow \varinjlim_j H^i(U_{j,\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong H^i(U_{\infty,\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}).$$

Thus, it suffices to show that $|H^i(U_{\infty,\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})| \leq |H^i(U^{s.\flat}_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})|$. Hence by tilting étale cohomology using Theorem 4.6, we are reduced to showing

$$|H^i(U^{\flat}_{\infty,\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})| \leq |H^i(U^{s.\flat}_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})|, \tag{4-6}$$

where $U^\flat_\infty$ is the open subset of $\operatorname{Spec}(R^\flat_\infty)$ that corresponds to $U_\infty \subseteq \operatorname{Spec}(R_\infty)$ in view of Example 4.5. On the other hand, considering the tilt of $(\{R_j\}_{j\geq 0}, \{t_j\}_{j\geq 0})$ associated to (R_0, I_0), we have a perfect $\mathbb{F}_p$-tower $(\{R^{s.\flat}_j\}_{j\geq 0}, \{t^{s.\flat}_j\}_{j\geq 0})$. Note that each $R^{s.\flat}_j$ is $I^{s.\flat}_0$-adically Henselian Noetherian ring[14] by Lemma 3.36 and Proposition 3.42(2), and $t^{s.\flat}_j$ is module-finite by Proposition 3.42(1). Considering the small tilts of the Zariski-open subsets appearing in (4-3) (see Definition 4.3), we get a tower of finite maps:

$$U^{s.\flat} = U^{s.\flat}_0 \leftarrow \cdots \leftarrow U^{s.\flat}_j \leftarrow U^{s.\flat}_{j+1} \leftarrow \cdots .$$

So let $U^{s.\flat}_\infty$ be the inverse image of $U^{s.\flat}$ under $\operatorname{Spec}(R^{s.\flat}_\infty) \to \operatorname{Spec}(R^{s.\flat})$. Since $U^{s.\flat}_\infty \to U^{s.\flat}$ is a universal homeomorphism, the preservation of the small étale sites [38, Tag 03SI] gives an isomorphism:

$$H^i(U^{s.\flat}_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong H^i(U^{s.\flat}_{\infty,\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}). \tag{4-7}$$

Now the combination of Lemma 3.55 and Theorem 4.1(2) together with the assumption finishes the proof of the theorem. □

Remark 4.8. One can formulate and prove the version of Proposition 4.7 for the étale cohomology with support in a closed subscheme of $\operatorname{Spec}(R)$, using Theorem 4.6. Then the resulting assertion gives a generalization of Česnavičius-Scholze's argument in [7, Theorem 3.1.3] which is a key part of their proof for the absolute cohomological purity theorem. One of the advantages of Proposition 4.7 is that it can be used to answer some cohomological questions on possibly singular Noetherian schemes (e.g. log-regular schemes) in mixed characteristic.

4B. *Tilting the divisor class groups of local log-regular rings.*

We need a lemma of Grothendieck on the relationship between the divisor class group and the Picard group via direct limit. Its proof is found in [19, Proposition (21.6.12)] or [20, XI Proposition 3.7.1].

Lemma 4.9. *Let X be an integral Noetherian normal scheme, and let $\{U_i\}_{i\in I}$ be a family of open subsets of X. Consider the following conditions.*

(1) *$\{U_i\}_{i\in I}$ forms a filter base. In particular, one can define a partial order on I so that it is a directed set and $\{U_i\}_{i\in I}$ together with the inclusion maps forms an inverse system.*

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[14] It is not obvious whether $R^{s.\flat}_j$ is normal. However, the normality was used only in the trace argument and we do not need it in the following argument.

(2) *Let* $V_i := X \setminus U_i$ *for any* $i \in I$. *Then* $\operatorname{codim}_X(V_i) \geq 2$.

(3) *For any* $x \in \bigcap_{i \in I} U_i$, *the local ring* $\mathcal{O}_{X,x}$ *is factorial.*

If $\{U_i\}_{i \in I}$ *satisfies condition* (2), *then the natural map* $\operatorname{Pic}(U_i) \to \operatorname{Cl}(X)$ *is injective for any* $i \in I$. *If* $\{U_i\}_{i \in I}$ *satisfies conditions* (1), (2) *and* (3), *then* $\varinjlim_{i \in I} \operatorname{Pic}(U_i) \cong \operatorname{Cl}(X)$. *Thus, if* $U \subseteq X$ *is any open subset that is locally factorial with* $\operatorname{codim}_X(X \setminus U) \geq 2$, *then* $\operatorname{Pic}(U) \cong \operatorname{Cl}(X)$.

Next we establish two results on the torsion part of the divisor class group of a (Noetherian) normal domain; they are examples of numerous applications of Theorem 4.1 of independent interest.

Proposition 4.10. *Let* $(R, \mathfrak{m}, k)$ *be a strictly Henselian Noetherian local normal* $\mathbb{F}_p$*-domain of dimension* ≥ 2, *let* $X := \operatorname{Spec}(R)$ *and fix an ideal* $J \subseteq \mathfrak{m}$. *Let* $\{U_i\}_{i \in I}$ *be any family of open subsets of* X *satisfying* (1), (2) *and* (3) *as in the hypothesis of Lemma 4.9 and let* U_i^∞ *be the* $\mathbb{F}_p$*-scheme which is the perfection of* U_i.

(1) *For any prime* $\ell \neq p$,

$$\operatorname{Cl}(X)[\ell^n] \cong \varinjlim_{i \in I} H^1\big((U_i^\infty)_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}\big).$$

(2) *Let* $\widehat{R}^{1/p^\infty}$ *denote the* J*-adic completion of* R^{1/p^∞}. *If each* U_i *has the property that* $X \setminus V(J) \subseteq U_i$, *then for any prime* $\ell \neq p$,

$$\operatorname{Cl}(X)[\ell^n] \cong \varinjlim_{i \in I} H^1\big((\widehat{U}_i^\infty)_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}\big),$$

where $\widehat{U}_i^\infty$ *is inverse image of* U_i^∞ *via the scheme map* $\operatorname{Spec}(\widehat{R}^{1/p^\infty}) \to \operatorname{Spec}(R^{1/p^\infty})$.

Proof. Let us begin with a remark on the direct limit of étale cohomology groups. For the transition morphism $g : U_i^\infty \to U_j^\infty$ which is affine, there is a functorial map $H^1\big((U_j^\infty)_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}\big) \to H^1\big((U_i^\infty)_{\text{ét}}, g^*(\mathbb{Z}/\ell^n\mathbb{Z})\big) \cong H^1\big((U_i^\infty)_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}\big)$, which defines the direct system of cohomology groups.

(1) We prove that *for any* $n \in \mathbb{N}$, *there is an injection of abelian groups*

$$H^1(U_{et}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong \operatorname{Pic}(U)[\ell^n] \subseteq \operatorname{Cl}(X)[\ell^n],$$

where $U \subseteq X$ *is an open subset whose complement is of codimension* ≥ 2. Indeed, consider the Kummer exact sequence

$$0 \to \mathbb{Z}/\ell^n\mathbb{Z} \cong \mu_{\ell^n} \to \mathbb{G}_m \xrightarrow{(\ )^{\ell^n}} \mathbb{G}_m \to 0,$$

where the identification of étale sheaves $\mu_{\ell^n} \cong \mathbb{Z}/\ell^n\mathbb{Z}$ follows from the fact that R is strict Henselian (one simply sends $1 \in \mathbb{Z}/\ell^n\mathbb{Z}$ to the primitive ℓ^n-th root of unity in R). Let $U \subseteq X$ be an open subset with its complement $V = X \setminus U$ having codimension ≥ 2. Then we have an exact sequence (see [31, Chapter III, Proposition 4.9])

$$\Gamma(U_{\text{ét}}, \mathbb{G}_m) \xrightarrow{(\ )^{\ell^n}} \Gamma(U_{\text{ét}}, \mathbb{G}_m) \to H^1(U_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \to \operatorname{Pic}(U) \xrightarrow{(\ )^{\ell^n}} \operatorname{Pic}(U).$$

Since R is strict local and $\ell \neq p$, Hensel's lemma yields that $R^\times = (R^\times)^{\ell^n}$. Since $\operatorname{codim}_X(V) \geq 2$ and X

is normal, we have $\Gamma(U_{\text{ét}}, \mathbb{G}_m) = R^{\times}$. Thus, $H^1(U_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong \operatorname{Pic}(U)[\ell^n]$. Note that $\operatorname{Pic}(U) \hookrightarrow \operatorname{Cl}(U)$ restricts to $\operatorname{Pic}(U)[\ell^n] \hookrightarrow \operatorname{Cl}(U)[\ell^n]$. Moreover, the natural homomorphism $\operatorname{Cl}(X) \to \operatorname{Cl}(U)$ is an isomorphism, thanks to $\operatorname{codim}_X(V) \geq 2$. Hence $H^1(U_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong \operatorname{Pic}(U)[\ell^n] \subseteq \operatorname{Cl}(X)[\ell^n]$, which proves the claim.

Since R is normal, the regular locus has complement with codimension ≥ 2. Using this fact, we can apply Lemma 4.9 to get an isomorphism $\operatorname{Cl}(X)[\ell^n] \cong \varinjlim_{i \in I} H^1\big((U_i)_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}\big)$. By étale invariance of cohomology under taking perfection of $\mathbb{F}_p$-schemes [38, Tag 03SI], we get

$$\operatorname{Cl}(X)[\ell^n] \cong \varinjlim_{i \in I} H^1\big((U_i)_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}\big) \cong \varinjlim_{i \in I} H^1\big((U_i^{\infty})_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}\big),$$

as desired.

(2) Since R is Henselian along $\mathfrak{m}$ and $J \subseteq \mathfrak{m}$, it is Henselian along J by [38, Tag 0DYD]. The perfect closure of R still preserves the Henselian property along J. Theorem 4.1 yields

$$H^1\big((U_i^{\infty})_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}\big) \cong H^1\big((\widehat{U}_i^{\infty})_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}\big)$$

and the conclusion follows from (1). ☐

Proposition 4.11. *Let A be a Noetherian ring with a regular element $t \in A$ such that A is t-adically Henselian and $A \to A/tA$ is the natural surjection between locally factorial domains. Pick an integer $n > 0$ that is invertible on A. Then if* $\operatorname{Cl}(A)$ *has no torsion element of order n, the same holds for* $\operatorname{Cl}(A/tA)$. *If moreover A is a $\mathbb{Q}$-algebra and* $\operatorname{Cl}(A)$ *is torsion-free, then so is* $\operatorname{Cl}(A/tA)$.

Proof. The Kummer exact sequence $0 \to \mu_n \to \mathbb{G}_m \xrightarrow{(\ )^n} \mathbb{G}_m \to 0$ induces the commutative diagram

$$\begin{tikzcd}
H^1(\operatorname{Spec}(A)_{\text{ét}}, \mu_n) \arrow[r, "\delta_1"] \arrow[d, "\alpha"] & \operatorname{Pic}(A) \arrow[r, "(\ )^n"] \arrow[d] & \operatorname{Pic}(A) \arrow[d] \\
H^1(\operatorname{Spec}(A/tA)_{\text{ét}}, \mu_n) \arrow[r, "\delta_2"] & \operatorname{Pic}(A/tA) \arrow[r, "(\ )^n"] & \operatorname{Pic}(A/tA).
\end{tikzcd}$$

By Theorem 4.1, the map α is an isomorphism. Then if $\operatorname{Pic}(A)$ has no torsion element of order n, δ_1 is the zero map. This implies that δ_2 is also the zero map and hence, $\operatorname{Pic}(A/tA)$ has no element of order n. Since both A and A/tA are locally factorial by assumption, we have $\operatorname{Cl}(A) \cong \operatorname{Pic}(A)$ and $\operatorname{Cl}(A/tA) \cong \operatorname{Pic}(A/tA)$. The assertion follows. ☐

It is not necessarily true that δ_1 or δ_2 are injective, because we do not assume A to be strictly Henselian.

Lemma 4.12. *Let $(R, \mathcal{Q}, \alpha)$ be a log-regular ring. Then strict Henselization $(R^{\text{sh}}, \mathcal{Q}, \alpha^{\text{sh}})$ is also a log-regular ring, where $\alpha^{\text{sh}} : \mathcal{Q} \to R \to R^{\text{sh}}$ is the composition of homomorphisms.*

Proof. Since $R \to R^{\text{sh}}$ is a local ring map, $(R^{\text{sh}}, \mathcal{Q}, \alpha^{\text{sh}})$ is a local log ring by Lemma 2.18. Note that we have the equality $I_{\alpha^{\text{sh}}} = I_\alpha R^{\text{sh}}$. Since we have the isomorphism $R^{\text{sh}}/I_{\alpha^{\text{sh}}} \cong (R/I_\alpha)^{\text{sh}}$ by [38, Tag 05WS] and $(R/I_\alpha)^{\text{sh}}$ is a regular local ring by [38, Tag 06LN], $R^{\text{sh}}/I_{\alpha^{\text{sh}}}$ is a regular local ring. Since the dimension of R is equal to the dimension of a strict henselization R^{sh}, we obtain the equalities

$$\dim R^{\mathrm{sh}} - \dim(R^{\mathrm{sh}}/I_{\alpha^{\mathrm{sh}}}) = \dim R^{\mathrm{sh}} - \dim(R/I_\alpha)^{\mathrm{sh}} = \dim R - \dim(R/I_\alpha) = \dim \mathcal{Q}.$$

So the local log ring $(R^{\mathrm{sh}}, \mathcal{Q}, \alpha^{\mathrm{sh}})$ is log-regular. □

Now we can prove the following result on the divisor class groups of local log-regular rings, as an application of the theory of perfectoid towers.

Theorem 4.13. *Let* $(R, \mathcal{Q}, \alpha)$ *be a local log-regular ring of mixed characteristic with perfect residue field* k *of characteristic* $p > 0$, *and denote by* $\mathrm{Cl}(R)$ *the divisor class group with its torsion subgroup* $\mathrm{Cl}(R)_{\mathrm{tor}}$.

(1) *Assume that* $R \cong W(k)[\![\mathcal{Q}]\!]$ *for a fine, sharp, and saturated monoid* $\mathcal{Q}$, *where* $W(k)$ *is the ring of Witt vectors over* k. *Then* $\mathrm{Cl}(R)_{\mathrm{tor}} \otimes \mathbb{Z}\bigl[\frac{1}{p}\bigr]$ *is a finite group. In other words, the* ℓ*-primary subgroup of* $\mathrm{Cl}(R)_{\mathrm{tor}}$ *is finite for all primes* $\ell \neq p$ *and vanishes for almost all primes* $\ell \neq p$.

(2) *Assume that* $\widehat{R^{\mathrm{sh}}}\bigl[\frac{1}{p}\bigr]$ *is locally factorial, where* $\widehat{R^{\mathrm{sh}}}$ *is the completion of the strict Henselization* R^{sh}. *Then* $\mathrm{Cl}(R)_{\mathrm{tor}} \otimes \mathbb{Z}\bigl[\frac{1}{p}\bigr]$ *is a finite group. In other words, the* ℓ*-primary subgroup of* $\mathrm{Cl}(R)_{\mathrm{tor}}$ *is finite for all primes* $\ell \neq p$ *and vanishes for almost all primes* $\ell \neq p$.

Proof. Assertion (1) was already proved in Proposition 2.26. So let us prove assertion (2). We may assume that $\mathcal{Q}$ is fine, sharp, and saturated by Remark 2.20. The proof given below works for the first case under the assumption of local factoriality of $\widehat{R^{\mathrm{sh}}}\bigl[\frac{1}{p}\bigr]$.

Since $R \to \widehat{R^{\mathrm{sh}}}$ is a local flat ring map, the induced map $\mathrm{Cl}(R) \to \mathrm{Cl}(\widehat{R^{\mathrm{sh}}})$ is injective by Mori's theorem (cf. [11, Corollary 6.5.2]). Thus, it suffices to prove the theorem for $\widehat{R^{\mathrm{sh}}}$. Moreover, $\widehat{R^{\mathrm{sh}}}$ is log-regular with respect to the induced log ring structure $\alpha : \mathcal{Q} \to R \to \widehat{R^{\mathrm{sh}}}$ by Lemma 4.12. So without loss of generality, we may assume that the residue field of R is separably closed (hence algebraically closed in our case).

Henceforth, we denote $\widehat{R^{\mathrm{sh}}}$ by R for brevity and fix a prime ℓ that is different from p. By Lemma 4.9 and the local factoriality of $R\bigl[\frac{1}{p}\bigr]$, we claim that there is an open subset $U \subseteq X := \mathrm{Spec}(R)$ such that

$$\mathrm{Pic}(U) \cong \mathrm{Cl}(X), \quad X \setminus V(pR) \subseteq U \quad \text{and} \quad \mathrm{codim}_X(X \setminus U) \geq 2. \tag{4-8}$$

Indeed, X is a normal integral scheme by Kato's theorem (Theorem 2.21). Let U be the union of the regular locus of X and the open $\mathrm{Spec}\bigl(R\bigl[\frac{1}{p}\bigr]\bigr) \subseteq X$. Then by Serre's normality criterion, we see that $\mathrm{codim}_X(X \setminus U) \geq 2$. We fix such an open $U \subseteq X$ once and for all. Taking the cohomology sequence associated to the exact sequence

$$0 \to \mathbb{Z}/\ell^n\mathbb{Z} \to \mathbb{G}_m \xrightarrow{(\,)^{\ell^n}} \mathbb{G}_m \to 0$$

on the strict local scheme X and arguing as in the proof of Proposition 4.10, we have an isomorphism

$$H^1(U_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong \mathrm{Pic}(U)[\ell^n] \cong \mathrm{Cl}(X)[\ell^n]. \tag{4-9}$$

On the other hand, there is a perfectoid tower of module-finite extensions of local log-regular rings arising from $(R, (p))$:

$$(R, \mathcal{Q}, \alpha) = (R_0, \mathcal{Q}^{(0)}, \alpha_0) \to \cdots \to (R_j, \mathcal{Q}^{(j)}, \alpha_j) \to (R_{j+1}, \mathcal{Q}^{(j+1)}, \alpha_{j+1}) \to \cdots . \tag{4-10}$$

Each map is generically of p-power rank in view of Lemma 2.16(2) and Lemma 2.14(3). Moreover, the tilt of (4-10) (associated to $(R, (p))$) is given by

$$(R^{s.\flat}, \mathcal{Q}^{s.\flat}, \alpha^{s.\flat}) = ((R_0)^{s.\flat}_{(p)}, \mathcal{Q}^{s.\flat}_0, \alpha^{s.\flat}_0) \to \cdots \to ((R_j)^{s.\flat}_{(p)}, \mathcal{Q}^{s.\flat}_j, \alpha^{s.\flat}_j) \to ((R_{j+1})^{s.\flat}_{(p)}, \mathcal{Q}^{s.\flat}_{j+1}, \alpha^{s.\flat}_{j+1}) \to \cdots ,$$

where $((R_j)^{s.\flat}_{(p)}, \mathcal{Q}^{s.\flat}_j, \alpha^{s.\flat}_j)$ is a complete local log-regular ring of characteristic $p > 0$ in view of Theorem 3.61. The local ring $R^{s.\flat}$ is strictly Henselian and the complement of $U^{s.\flat} (= U^{s.\flat}_{(p)})$ has codimension ≥ 2 in $\operatorname{Spec}(R^{s.\flat})$. By repeating the proof of Proposition 4.10, we obtain an isomorphism

$$H^1(U^{s.\flat}_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong \operatorname{Pic}(U^{s.\flat})[\ell^n]. \tag{4-11}$$

By Lemma 4.9, the map

$$\operatorname{Pic}(U^{s.\flat})[\ell^n] \to \operatorname{Cl}(R^{s.\flat})[\ell^n] \tag{4-12}$$

is injective. Combining (4-9), (4-11), (4-12) and Proposition 4.7, it is now sufficient to check that there exists an integer $N > 0$ depending only on $R^{s.\flat}$ such that

$$\operatorname{Cl}(R^{s.\flat})[\ell^N] = \bigcup_{n>0} \operatorname{Cl}(R^{s.\flat})[\ell^n], \text{ and } \operatorname{Cl}(R^{s.\flat})[\ell^N] \text{ is finite for all } \ell \text{ and zero for almost all } \ell \neq p.$$

Since we know that $R^{s.\flat}$ is strongly F-regular by Theorem 3.61 and Lemma 2.25, the aforementioned result of Polstra finishes the proof. □

Appendix: Construction of differential modules and maximality

The content of this appendix is taken from Gabber and Ramero's treatise [17], whose purpose is to supply a corrected version of Grothendieck's original presentation in EGA. So we give only a sketch of the constructions of relevant modules and maps. Readers are encouraged to look into [17] for more details as well as proofs. We are motivated by the following specific problem.

Problem A.1. *Let $(A, \mathfrak{m}_A)$ be a Noetherian regular local ring and fix a system of elements $f_1, \dots, f_n \in A$ and a system of integers $e_1, \dots, e_n$ with $e_i > 1$ for every $i = 1, \dots, n$. We set*

$$B := A[T_1, \dots, T_n]/(T_1^{e_1} - f_1, \dots, T_d^{e_n} - f_n).$$

Then find a sufficient condition that ensures that the localization B with respect to a maximal ideal $\mathfrak{n}$ with $\mathfrak{m}_A = A \cap \mathfrak{n}$ is regular.

From the construction, it is obvious that the induced ring map $A \to B$ is a flat finite injective extension. Let now $(A, \mathfrak{m}_A, k)$ be a Noetherian local ring with residue field $k_A := A/\mathfrak{m}_A$ of characteristic $p > 0$.

Following the presentation in [17, (9.6.15)], we define a certain $k_A^{1/p}$-vector space $\boldsymbol{\Omega}_A$ together with a map $\boldsymbol{d}_A : A \to \boldsymbol{\Omega}_A$ as follows.

Case I: $p \notin \mathfrak{m}_A^2$. Let $W_2(k_A)$ denote the p-typical ring of length 2 Witt vectors over k_A. Then there is the ghost component map $\overline{\omega}_0 : W_2(k_A) \to k_A$, and set $V_1(k_A) := \operatorname{Ker}(\overline{\omega}_0)$. More specifically, we have $W_2(k_A) = k_A \times k_A$ as sets with addition and multiplication given respectively by

$$(a, b) + (c, d) = \left(a + c, b + d + \frac{a^p + c^p - (a + c)^p}{p}\right) \quad \text{and} \quad (a, b)(c, d) = (ac, a^p d + c^p b).$$

Using this structure, we see that $V_1(k_A) = 0 \times k_A$ as sets, which is an ideal of $W_2(k_A)$ and $V_1(k_A)^2 = 0$. This makes $V_1(k_A)$ equipped with the structure as a k_A-vector space by letting $x(0, a) := (x, 0)(0, a)$ for $x \in k_A$. One can define the map of k_A-vector spaces

$$k_A^{1/p} \to V_1(k_A)\ ;\ a \mapsto (0, a^p), \tag{A-1}$$

which is a bijection. With this isomorphism, we may view $V_1(k_A)$ as a $k_A^{1/p}$-vector space. Next we form the fiber product ring:

$$A_2 := A \times_{k_A} W_2(k_A).$$

It gives rise to a short exact sequence of A_2-modules

$$0 \to V_1(k_A) \to A_2 \to A \to 0, \tag{A-2}$$

where $A_2 \to A$ is the natural projection, and the A_2-module structure of $V_1(k_A)$ is via the restriction of rings $A_2 \to W_2(k_A)$. From (A-2), we obtain an exact sequence of A-modules:

$$V_1(k_A) \to \overline{\Omega}_A \to \Omega^1_{A/\mathbb{Z}} \to 0,$$

where we put $\overline{\Omega}_A = \Omega^1_{A_2/\mathbb{Z}} \otimes_{A_2} A$. After applying $(\ ) \otimes_A k_A$ to this sequence, we have another sequence of k_A-vector spaces:

$$0 \to V_1(k_A) \xrightarrow{j_A} \overline{\Omega}_A \otimes_A k_A \to \Omega^1_{A/\mathbb{Z}} \otimes_A k_A \to 0. \tag{A-3}$$

Then this is right exact. Moreover, (A-1) yields a unique k_A-linear map $\psi_A : V_1(k_A) \otimes_{k_A} k_A^{1/p} \to V_1(k_A)$. Define $\boldsymbol{\Omega}_A$ as the push-out of the diagram:

$$V_1(k_A) \xleftarrow{\psi_A} V_1(k_A) \otimes_{k_A} k_A^{1/p} \xrightarrow{j_A \otimes k_A^{1/p}} \overline{\Omega}_A \otimes_A k_A^{1/p}.$$

More concretely, we have

$$\boldsymbol{\Omega}_A = \frac{V_1(k_A) \oplus (\overline{\Omega}_A \otimes_A k_A^{1/p})}{T},$$

where $T = \left\{(\psi(x), -(j_A \otimes k_A^{1/p})(x)) \mid \in V_1(k_A) \otimes_{k_A} k_A^{1/p}\right\}$. By the universality of push-outs, we get the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & V_1(k_A) \otimes_{k_A} k_A^{1/p} \arrow[r] \arrow[d, "\psi_A"] & \overline{\Omega}_A \otimes_A k_A^{1/p} \arrow[r] \arrow[d, "\boldsymbol{\psi}_A"] & \Omega^1_{A/\mathbb{Z}} \otimes_A k_A^{1/p} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & V_1(k_A) \arrow[r] & \boldsymbol{\Omega}_A \arrow[r] & \Omega^1_{A/\mathbb{Z}} \otimes_A k_A^{1/p} \arrow[r] & 0.
\end{tikzcd}$$

We define the map

$$\boldsymbol{d}_A : A \to \boldsymbol{\Omega}_A$$

as the composite mapping

$$A \xrightarrow{1\times\tau_{k_A}} A_2 = A \times_{k_A} W_2(k_A) \xrightarrow{d} \Omega^1_{A_2/\mathbb{Z}} \xrightarrow{\mathrm{id}\otimes 1} \overline{\Omega}_A = \Omega^1_{A_2/\mathbb{Z}} \otimes_A k_A^{1/p} \xrightarrow{\boldsymbol{\psi}_A} \boldsymbol{\Omega}_A.$$

Here, $d : A_2 \to \Omega^1_{A_2/\mathbb{Z}}$ is the universal derivation and $\tau_{k_A} : A \to k_A \to W_2(k_A)$, where the first map is the natural projection and the second one is the Teichmüller map.

Case II: $p \in \mathfrak{m}_A^2$. We just set $\boldsymbol{\Omega}_A := \Omega^1_{A/\mathbb{Z}} \otimes_A k_A^{1/p}$, and define $\boldsymbol{d}_A : A \to \boldsymbol{\Omega}_A$ as the map induced by the universal derivation $d_A : A \to \Omega^1_{A/\mathbb{Z}}$.

Combining Cases I and II, we have a map $\boldsymbol{d}_A : A \to \boldsymbol{\Omega}_A$. If $\phi : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ is a local ring map of local rings, this gives rise to the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\boldsymbol{d}_A"] \arrow[d, "\phi"] & \boldsymbol{\Omega}_A \arrow[d, "\boldsymbol{\Omega}_\phi"] \\
B \arrow[r, "\boldsymbol{d}_B"] & \boldsymbol{\Omega}_B.
\end{tikzcd}$$

With this in mind, one can consider the functor $A \mapsto \boldsymbol{\Omega}_A$ from the category of local rings $(A, \mathfrak{m}_A)$ of residual characteristic $p > 0$ to the category of the $k_A^{1/p}$-vector spaces $\boldsymbol{\Omega}_A$. Some distinguished features in this construction are as follows:

Proposition A.2 [17, Proposition 9.6.20]. *Let $\phi : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ be a local ring map of Noetherian local rings such that the residual characteristic of A is $p > 0$. Then*

(1) *Suppose that ϕ is formally smooth for the $\mathfrak{m}_A$-adic topology on A and the $\mathfrak{m}_B$-adic topology on B. Then the maps induced by ϕ and $\boldsymbol{\Omega}_\phi$, namely*

$$(\mathfrak{m}_A/\mathfrak{m}_A^2) \otimes_{k_A} k_B \to \mathfrak{m}_B/\mathfrak{m}_B^2 \quad \textit{and} \quad \boldsymbol{\Omega}_A \otimes_{K_A^{1/p}} k_B^{1/p} \to \boldsymbol{\Omega}_B,$$

are injective.

(2) *Suppose that*

   (a) $\mathfrak{m}_A B = \mathfrak{m}_B$,

   (b) *the residue field extension $k_A \to k_B$ is separable algebraic,*

   (c) *ϕ is flat.*

Then $\mathbf{\Omega}_\phi$ *induces an isomorphism of* $k_A^{1/p}$*-vector spaces*:

$$\mathbf{\Omega}_A \otimes_A B \cong \mathbf{\Omega}_B.$$

(3) *If* $B = A/\mathfrak{m}_A^2$ *and* $\phi : A \to B$ *is the natural map, then* $\mathbf{\Omega}_\phi$ *is an isomorphism.*

(4) *The functor* $\mathbf{\Omega}_\bullet$ *and the natural transformation* $\boldsymbol{d}_\bullet$ *commute with filtered colimits.*

We provide an answer to Problem A.1 as follows.

Theorem A.3 [17, Corollary 9.6.34]. *Let* $f_1, \dots, f_n$ *be a sequence of elements in* A, *and let* $e_1, \dots, e_n$ *be a system of integers with* $e_i > 1$ *for every* $i = 1, \dots, n$. *Set*

$$C := A[T_1, \dots, T_n]/(T_1^{e_1} - f_1, \dots, T_n^{e_n} - f_n).$$

Fix a prime ideal $\mathfrak{n} \subseteq C$ *such that* $\mathfrak{n} \cap A = \mathfrak{m}_A$, *and let* $B := C_\mathfrak{n}$. *Let* $E \subseteq \mathbf{\Omega}_A$ *be the* $k_A^{1/p}$*-vector space spanned by* $\boldsymbol{d}_A f_1, \dots, \boldsymbol{d}_A f_n$. *The following conditions are equivalent.*

(1) A *is a regular local ring, and* $\dim_{k_A^{1/p}} E = n$.

(2) B *is a regular local ring.*

In particular, in the situation of the above theorem, B is a regular local ring if A is a regular local ring and $f_1, \dots, f_n$ is *maximal* in the sense of the following definition.

Definition A.4. Let $(A, \mathfrak{m}_A, k_A)$ be a local ring with residual characteristic $p > 0$. Then we say that a sequence of elements $f_1, \dots, f_n$ in A is *maximal* if $\boldsymbol{d}_A f_1, \dots, \boldsymbol{d}_A f_n$ forms a basis of the $k_A^{1/p}$-vector space $\mathbf{\Omega}_A$.

In general, we have the following fact.

Lemma A.5. *Let* $(A, \mathfrak{m}_A, k_A)$ *be a regular local ring of mixed characteristic and assume that* $f_1, \dots, f_d$ *is a regular system of parameters of* A.

(1) $f_1, \dots, f_d$ *satisfies condition* (1) *of Theorem A.3.*

(2) *If the residue field* k_A *of* A *is perfect, then the sequence* $f_1, \dots, f_d$ *is maximal.*

Proof. (1) In the case that $p \notin \mathfrak{m}_A^2$, [17, Proposition 9.6.17] gives a short exact sequence

$$0 \to \mathfrak{m}_A/\mathfrak{m}_A^2 \otimes_{k_A} k_A^{1/p} \to \mathbf{\Omega}_A \to \Omega^1_{k_A/\mathbb{Z}} \otimes_{k_A} k_A^{1/p} \to 0. \tag{A-4}$$

Then the images $\overline{f}_1, \dots, \overline{f}_d$ form a basis of the $k_A^{1/p}$-vector space $\mathfrak{m}_A/\mathfrak{m}_A^2 \otimes_{k_A} k_A^{1/p}$. The desired claim follows from the left exactness of (A-4).

In the case that $p \in \mathfrak{m}_A^2$, [17, Lemma 9.6.6] gives a short exact sequence

$$0 \to \mathfrak{m}_A/(\mathfrak{m}_A^2 + p\mathfrak{m}_A) \to \Omega_A \to \Omega^1_{k_A/\mathbb{Z}} \to 0. \tag{A-5}$$

and we can argue as in the case $p \notin \mathfrak{m}_A^2$.

(2) If k_A is perfect, then $\Omega^1_{k_A/\mathbb{Z}} = 0$. Therefore, (A-4) and (A-5) (in the latter case, one tensors it with $k_A^{1/p}$ over k_A) gives the desired conclusion. ☐

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