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Combing a hedgehog over a field

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We investigate the question of the existence of a nonvanishing section of the tangent bundle on a smooth affine quadric hypersurface Q^o over a given perfect field k. If Q^o admits a k-rational point, we give a number of necessary and sufficient conditions for such existence. We apply these conditions in a number of examples, including the case of the *algebraic n-sphere* over k, $S^n_k \subset \mathbb{A}^{n+1}_k$, defined by the equation $\sum_{i=1}^{n+1} x_i^2 = 1$.

1. Introduction

It is an elementary but nonetheless beautiful result found in nearly all introductory courses in differential topology that, for all $n \geq 1$, the tangent bundle $T_{S^{2n}}$ does not admit a nonvanishing section. One proof uses the Gauss–Bonnet theorem to show that Euler class of $T_{S^{2n}}$ is nonzero by computing its degree as the Euler characteristic of S^{2n}, namely 2, while the existence of a nonvanishing section would force the Euler class to vanish. For the odd-dimensional case, the Euler characteristic vanishes, and hence the Euler class vanishes as well; one can also easily write down explicitly a nonvanishing section of $T_{S^{2n+1}}$.

Writing the n-sphere S^n as the hypersurface in $\mathbb{R}^{n+1}$ defined by the equation $\sum_{i=1}^{n+1} x_i^2 = 1$, one can ask the corresponding question in the algebro-geometric setting: let k be a field of characteristic $\neq 2$ and let $S^n_k \subseteq \mathbb{A}^{n+1}_k$ be the hypersurface defined by the equation $\sum_{i=1}^{n+1} x_i^2 = 1$. Does the tangent bundle $T_{S^n_k}$ admit a nonvanishing section? (To avoid any possible misunderstanding, for $E \to X$ a vector bundle on a k-variety X, a section $s : X \to E$ is said to be nonvanishing if the scheme-theoretic intersection of $s(X)$ with the zero-section of E is the empty scheme. Equivalently, letting $\bar{k}$ denote the algebraic closure of k, the set of $\bar{k}$ points x of X with $s(x) = 0$ is empty.)

This question for $S^2_{\mathbb{Q}_p}$ was originally raised by Umberto Zannier (see Remark 1.7 below for his original formulation). He showed that $S^2_{\mathbb{Q}_p}$ admits a nonvanishing vector field for odd p and he asked if there is a nonvanishing vector field on the 2-sphere over $\mathbb{Q}_2$, motivating our interest in the question of the existence of nonvanishing vector fields on S^n_k for arbitrary n and k.

We give an essentially complete answer to this question; if k is perfect, this is in fact a special case of the more general Theorem 1.4 about smooth affine quadric hypersurfaces.

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Theorem 1.1 (see Theorem 4.1 (1), (3) and Remark 1.10). *Let k be a field of characteristic $\neq 2$.*

(1) *If n is odd, then $T_{S^n_k}$ admits a nonvanishing section.*

(2) *If $n > 0$ is even, then $T_{S^n_k}$ admits a nonvanishing section if and only if -1 is in the subgroup of $k^\times$ generated by the nonzero values of the function $\sum_{i=1}^{n+1} x_i^2$ on k^{n+1}.*

As the condition in (2) is not very explicit, we reformulate this as follows.

Corollary 1.2 (Corollary 4.3). *Let k be a field of characteristic $\neq 2$. For $n > 0$ even, $T_{S^n_k}$ admits a nonvanishing section if and only if the equation*

$$\sum_{i=1}^{2n+1} x_i^2 = -1$$

has a solution with the $x_i \in k$.

Examples 1.3. Let S^n_k be as above.

(1) Suppose that $\operatorname{char} k = p > 2$. Then $T_{S^n_k}$ has a nonvanishing section for all $n > 0$.

(2) Suppose k contains a p-adic field $\mathbb{Q}_p$. Then $T_{S^n_k}$ has a nonvanishing section for all $n > 0$.

(3) Suppose that k is a number field, and take $n > 0$ to be even. Then $T_{S^n_k}$ has a nonvanishing section if and only if k has no real embeddings.

To see this we apply Corollary 1.2. For (1), since $\mathbb{F}_p \subseteq k$, it suffices to take $k = \mathbb{F}_p$. Since every element $x \in \mathbb{F}_p^\times$ is a sum of two squares [Lam 2005, Proposition II.3.4], the condition of Corollary 1.2 is satisfied for all $n \geq 1$. See also Remark 1.10 for an explicit nonvanishing section.

For (2), we reduce as above to the case $k = \mathbb{Q}_p$. If p is odd, then by Hensel's lemma, each solution to $\sum_{i=1}^{2n+1} x_i^2 = -1$ in $\mathbb{F}_p$ lifts to a solution in $\mathbb{Z}_p$, so the criterion is satisfied. For $p = 2$, the class of a unit u in $\mathbb{Z}_2$ modulo squares is given by the image of u in $(\mathbb{Z}/8)^\times$, so it suffices to write 7 as the sum of ≤ 5 squares in $\mathbb{Z}_2$ and it turns out that four squares are enough: $7 = 1+1+1+4$. For those more intrinsically minded, one has the general result that *every* nondegenerate quadratic form ϕ in at least five variables over a local field has a nontrivial zero [Lam 2005, Theorem VI.2.12], which we apply to $\phi = \sum_{i=1}^5 x_i^2$.

For (3), it is clear that the equation $\sum_{i=1}^{2n+1} x_i^2 = -1$ has no solution in k^{2n+1} if k admits a real embedding. Conversely, we may use the Hasse–Minkowski principle for quadratic forms (see, e.g., [Lam 2005, Hasse–Minkowski principle VI.3.1]) to see that $\sum_{i=1}^4 x_i^2 = -1$ has a solution in k if k is a purely imaginary number field. Indeed, it suffices to show that $\sum_{i=1}^4 x_i^2 = -1$ has a solution in k_v for every place v of k. This is clear if v is an infinite place, as $k_v = \mathbb{C}$ by assumption. If v is a finite place, then $k_v \supset \mathbb{Q}_p$ for some prime p, and we have just seen that $\sum_{i=1}^4 x_i^2 = -1$ has a solution in $\mathbb{Q}_p$ for every prime p.

One can also ask about a general smooth affine quadric $Q^o \subseteq \mathbb{A}_k^{n+1}$, with k a field of characteristic $\neq 2$. Since every quadratic form over k can be diagonalized, we may assume that Q^o is defined by an equation of the form $q = 1$, where $q = \sum_{i=1}^{n+1} a_i x_i^2 \in k[x_1, \dots, x_{n+1}]$, with $\prod_i a_i \neq 0$. Here one has a result of essentially the same form as for S^n_k, with the extra condition that, for even n, $Q^o(k)$ should be nonempty, that is, $q = 1$ has a solution in k.

Let $D(q)$ be the set of nonzero values of q on k^{n+1}, let $D(q)^2 = \{a \cdot b \mid a, b \in D(q)\} \subseteq k^\times$, and let $[D(q)]$, $[D(q)^2]$ be the subgroups of $k^\times$ generated by $D(q)$, $D(q)^2$, respectively.

Theorem 1.4 (Theorem 4.1 (1), (3)). *Let k be a perfect field of characteristic $\neq 2$, let $q = \sum_{i=1}^{n+1} a_i x_i^2$ with $a_1, \dots, a_{n+1} \in k^\times$ and let $Q^o \subseteq \mathbb{A}_k^{n+1}$ be the affine quadric hypersurface $q = 1$.*

(1) *If n is odd, then T_{Q^o} has a nonvanishing section.*

(2) *Suppose $Q^o(k) \neq \varnothing$. If $n > 0$ is even, then T_{Q^o} has a nonvanishing section if and only if $-1 \in [D(q)]$.*

If $Q^o(k) = \varnothing$ and n is even, we only have a necessary condition for the existence of a nonvanishing section of T_{Q^o}.

Theorem 1.5 (Theorem 4.1 (2)). *Let k, q and Q^o be as above. If n is even and T_{Q^o} has a nonvanishing section, then $-\prod_{i=1}^{n+1} a_i \in [D(q)^2]$.*

Since $a_i \in D(q)$ for each i, the above condition is the same as asking for $-a_i$ to be in $[D(q)^2]$ for *some* i. Note that $Q^o(k) \neq \varnothing$ if and only if $1 \in D(q)$, so if $Q^o(k) \neq \varnothing$, we have $[D(q)^2] = [D(q)]$, and $-1 \in [D(q)]$ if and only if $-\prod_{i=1}^{n+1} a_i \in [D(q)]$.

Here is a version of Examples 1.3 for general q.

Corollary 1.6 (Corollaries 4.5, 4.7, and 4.9). *Let k, q and Q^o be as in Theorem 1.4.*

(1) *Let $k = \mathbb{F}_{p^m}$ with $p > 2$. Then T_{Q^o} has a nonvanishing section for all $n > 0$.*

(2) *Suppose k is a non-Archimedean local field of characteristic zero, the perfection of a local field of characteristic $p > 2$, or the perfection of a function field of a curve over a finite field of characteristic $p > 2$. Then for n odd, or $n \geq 4$ even, T_{Q^o} has a nonvanishing section. If $n = 2$, then T_{Q^o} has a nonvanishing section if $Q^o(k) \neq \varnothing$.*

(3) *Suppose k is a number field, $Q^o(k) \neq \varnothing$ and $n > 0$ is even. Then T_{Q^o} has a nonvanishing section if and only if the equation $q = 0$ has a nontrivial solution in k_v for every real place v of k. Equivalently, for each real embedding $\sigma : k \hookrightarrow \mathbb{R}$, $\sigma(a_i) < 0$ for some i. T_{Q^o} also has a nonvanishing section if n is odd.*

(4) *Let k be a perfect field of cohomological dimension ≤ 2. Suppose that n is odd, or that $n > 0$ is even and $Q^o(k) \neq \varnothing$. Then T_{Q^o} has a nonvanishing section.*

Remark 1.7 (unimodular rows and unimodular matrices). Let $q = \sum_{i=1}^{n+1} a_i x_i^2 \in k[x_1, \dots, x_{n+1}]$ be a quadratic form, defining $Q^o \subseteq \mathbb{A}_k^{n+1}$ as $V(q-1)$, and let R be the coordinate ring

$$R := k[x_1, \dots, x_{n+1}]/(q-1).$$

We are assuming that q is nondegenerate, that is, $\prod_i a_i \neq 0$, and that $n \geq 1$.

Let $\nabla(q)$ denote the gradient

$$\nabla(q) := (\partial q/\partial x_1, \dots, \partial q/\partial x_{n+1})$$

and let

$$\tilde{\nabla}(q) := (a_1 x_1, \dots, a_{n+1} x_{n+1}),$$

so $2\tilde{\nabla}(q) = \nabla(q)$.

We first assume that 2 is invertible in k, so we can rewrite the tangent-normal sequence for $Q^o \subseteq \mathbb{A}_k^{n+1}$ using $\tilde{\nabla}(q)$, as

$$0 \to T_{Q^o} \to \mathcal{O}_{Q^o}^{n+1} \xrightarrow{\tilde{\nabla}(q)^t} \mathcal{O}_{Q^o} \to 0, \tag{1.8}$$

where $\tilde{\nabla}(q)^t \in M_{n+1\times 1}(R)$ is the transpose of $\tilde{\nabla}(q)$. Since Q^o is affine, we can rephrase everything in terms of R-modules, giving the exact sequence

$$0 \to \mathfrak{T}_{Q^o} \to R^{n+1} \xrightarrow{\tilde{\nabla}(q)^t} R \to 0,$$

with $\mathfrak{T}_{Q^o}$ the R-module of global sections of T_{Q^o}. Since

$$(x_1, \ldots, x_{n+1}) \cdot \tilde{\nabla}(q) = 1,$$

we may split the surjection by $(x_1, \ldots, x_{n+1}) : R \to R^{n+1}$, exhibiting $\mathfrak{T}_{Q^o}$ as a stably free R-module, and showing that $(x_1, \ldots, x_{n+1})$ is a *unimodular row*, i.e., $(x_1, \ldots, x_{n+1})R$ is the unit ideal.

It is straightforward to see that the stably free R-module $\mathfrak{T}_{Q^o}$ is free if and only if there is a matrix $M \in \mathrm{GL}_{n+1}(R)$ with the first row $(x_1, \ldots, x_{n+1})$; by dividing the last row of M by $\det M$, we may in fact take M to have $\det M = 1$, so $\mathfrak{T}_{Q^o}$ is a free R-module if and only if there is a unimodular matrix M over R with the first row $(x_1, \ldots, x_{n+1})$.

More generally, we may take k to be an arbitrary commutative ring (even with 2 not a unit), and let $(a_{ij})_{1\le i,j\le n+1} \in \mathrm{GL}_{n+1}(k)$ be an invertible symmetric matrix. Let

$$q := \sum_{i,j=1}^{n+1} a_{ij} x_i x_j, \quad R := k[x_1, \ldots, x_{n+1}]/(q-1), \quad Q^o := \operatorname{Spec} R,$$
$$\tilde{\nabla}(q) := \Big(\sum_j a_{1j} x_j, \sum_j a_{2j} x_j, \ldots, \sum_j a_{n+1\,j} x_j\Big).$$

Then the map $R^{n+1} \xrightarrow{\tilde{\nabla}(q)^t} R$ is surjective, and we may define a stably free R-module $\mathfrak{T}_{Q^o/k}$ by the exact sequence

$$0 \to \mathfrak{T}_{Q^o/k} \to R^{n+1} \xrightarrow{\tilde{\nabla}(q)^t} R \to 0. \tag{1.9}$$

Since $(x_1, \ldots, x_{n+1}) \cdot \tilde{\nabla}(q) = 1 \in R$, $(x_1, \ldots, x_{n+1})$ is a unimodular row over R, and the R-module $\mathfrak{T}_{Q^o/k}$ is free if and only if there is a matrix $M \in \mathrm{SL}_{n+1}(R)$ with first row $(x_1, \ldots, x_{n+1})$. Furthermore, if $2 \in k^\times$, then Q^o is smooth over k, and $\mathfrak{T}_{Q^o/k}$ is the R-module of global sections of the relative tangent bundle $T_{Q^o/k} \to Q^o$ of Q^o over $\operatorname{Spec} k$.

Note that, for $n = 2$ and $2 \in k^\times$, the existence of a unimodular matrix over R with the first row (x_1, x_2, x_3) is equivalent to the existence of a nonvanishing section of $T_{Q^o/k}$. Indeed, $T_{Q^o/k}$ admits a nonvanishing section if and only if we can write $\mathfrak{T}_{Q^o/k} \cong R \oplus P$, with P a rank one projective R-module, which yields the isomorphism $\bigwedge_R^2 \mathfrak{T}_{Q^o/k} \cong P$. The exact sequence (1.9) gives an isomorphism $\bigwedge_R^2 \mathfrak{T}_{Q^o/k} \cong R$, so $P \cong R$ and thus $\mathfrak{T}_{Q^o/k} \cong R^2$. This gives the existence of $M \in \mathrm{SL}_3(R)$ with first row (x_1, x_2, x_3).

The original form of Zannier's question was in the following terms: taking $q := x^2 + y^2 + z^2 \in k[x, y, z]$, for which fields k is the unimodular row (x, y, z) over R the first row in a unimodular matrix over R? He showed this was the case for $k = \mathbb{Q}_p$, p odd, and asked about $k = \mathbb{Q}_2$. Note that one can just as well ask the question for k an arbitrary commutative ring and general q as above; our results only give a criterion for a positive answer to this question if k is a perfect field with 2 invertible and $n = 2$. Noting our positive answer for $k = \mathbb{Q}_2$, Zannier asked in a recent private communication about the case $k = \mathbb{Z}_2$.

Remark 1.10 (explicit sections). (1) If n is odd or if the quadratic form $q := \sum_{i=1}^{n+1} a_i x_i^2$ is isotropic over k (i.e., $q = 0$ has a nontrivial solution in k), one can write down explicit nonvanishing sections of T_{Q^o}.

For n odd, the tangent-normal sequence for $Q^o \subseteq \mathbb{A}_k^{n+1}$,

$$0 \to T_{Q^o} \to \mathcal{O}_{Q^o}^{n+1} \xrightarrow{(2a_1x_1,\dots,2a_{n+1}x_{n+1})^t} \mathcal{O}_{Q_o} \to 0,$$

says a section s of T_{Q^o} is given by an $(n+1)$-tuple of regular functions $(s_1, s_2, \dots, s_{n+1})$ with

$$\sum_{i=1}^{n+1} a_i x_i s_i = 0.$$

One can take

$$s = (a_2x_2, -a_1x_1, \dots, a_{n+1}x_{n+1}, -a_nx_n),$$

which is clearly nonvanishing.

This is a special case of the following general result. Let A be a commutative ring, let $(a_1, \dots, a_{2m})$ be a unimodular row in A^{2m} (i.e., $a_1, \dots, a_{2m}$ generate the unit ideal in A), and let M be the stably free A-module defined by the exactness of the sequence

$$0 \to M \to A^{2m} \xrightarrow{(a_1,\dots,a_{2m})^t} A \to 0.$$

Then M admits the free rank one summand defined by

$$0 \to A \xrightarrow{(-a_2,a_1,\dots,-a_{2m},a_{2m-1})} M \subset A^{2m}.$$

Now take n even and suppose q is isotropic. Then after a k-linear change of coordinates and change of notation, we may assume that

$$q = 2x_1x_2 + \sum_{i=3}^{n+1} a_i x_i^2$$

(see, e.g., [Lam 2005, Theorem I.3.4]). In this case, the tangent-normal sequence for $Q^o \subseteq \mathbb{A}_k^{n+1}$ is

$$0 \to T_{Q^o} \to \mathcal{O}_{Q^o}^{n+1} \xrightarrow{(2x_2,2x_1,2a_3x_3,\dots,2a_{n+1}x_{n+1})^t} \mathcal{O}_{Q_o} \to 0.$$

Letting

$$s = (0, a_3x_3, -x_1, a_5x_5, -a_4x_4, \dots, a_{n+1}x_{n+1}, -a_nx_n)$$

gives a section of T_{Q^o} with $s = 0$ given by $Q^o \cap (x_1 = x_3 = \cdots = x_{n+1} = 0)$, which is clearly empty.

In particular, let k be a field of characteristic $p > 2$. In a finite field, -1 is a sum of two squares [Lam 2005, Proposition II.3.4], whence the quadratic form $x_1^2 + x_2^2 + \cdots + x_{n+1}^2$ is isotropic over k provided that $n \geq 2$. Hence the tangent bundle $T_{S_k^n}$ to the algebraic n-sphere over k admits a nonvanishing section for every $n \geq 1$.

(2) Our main results for even n and q anisotropic only give criteria for the existence of a nonvanishing section, without giving an explicit formula. In the case that Zannier had asked about originally, $S^2_{\mathbb{Q}_2}$, Peter Müller (private communication, Universität Würzburg, June 27, 2024), noting our existence result and following a suggestion of Zannier, found an explicit trivialization of $T_{S^2_{\mathbb{Q}_2}}$. We quote from his private communication:

> Indeed, some sophisticated computations eventually gave an explicit example over $\mathbb{Q}_2$ (in fact even over $\mathbb{Q}(\sqrt{-7})$), ...

Here is Müller's example. Let $R = \mathbb{Q}_2[x, y, z]/(x^2 + y^2 + z^2 - 1)$, the coordinate ring of $S^2_{\mathbb{Q}_2}$. The polynomial $T^2 - T + 2$ has two roots in $\mathbb{Z}_2$, exactly one of which, which we denote by ω, is a unit in $\mathbb{Z}_2$. In particular, $2-\omega$ is also a unit in $\mathbb{Z}_2$. Müller gives his example in the form of a 3×3 matrix over R with determinant $2-\omega$ and first row (x, y, z). The explicit matrix is

$$\begin{pmatrix} x & y & z \\ -y+z+1 & (1-\omega)x+y+(1+\omega)z+\omega & -x-y+2z+(1-\omega) \\ \omega y+(2-\omega)z & (1-2\omega)x+(1+\omega)y+3z+1 & -2x+(2-\omega)z-\omega \end{pmatrix}.$$

Let λ_i be the dot product of (x, y, z) with the i-th row. Noting that $(x, y, z)\cdot(x, y, z) = 1$ in R, this gives the following two independent nonvanishing sections of $T_{S^2_{\mathbb{Q}_2}}$:

$$s_1(x, y, z) = (-y+z+1,\ (1-\omega)x+y+(1+\omega)z+\omega,\ -x-y+2z+(1-\omega)) - \lambda_2\cdot(x, y, z),$$

$$s_2(x, y, z) = (\omega y+(2-\omega)z,\ (1-2\omega)x+(1+\omega)y+3z+1,\ -2x+(2-\omega)z-\omega) - \lambda_3\cdot(x, y, z).$$

Müller notes that this also works over $\mathbb{Q}(\sqrt{-7})$, where we take ω to be either of the two roots of T^2-T+2 in $\mathbb{Q}(\sqrt{-7})$.

In addition, Müller's example gives a positive answer to Zannier's question over $\mathbb{Z}_2$ instead of $\mathbb{Q}_2$, just divide the last row by the determinant $2-\omega \in \mathbb{Z}_2^\times$.

Remark 1.11 (some nonexamples). Suppose n is even. We have already seen in Remark 1.10 that T_{Q^o} has a nonvanishing section if q is isotropic over k. On the other hand, q being isotropic over k implies that Q^o has a k-rational point [Lam 2005, Theorem I.3.4(3)], so if $Q^o(k) = \varnothing$, then q is anisotropic over k and we do not have any explicit method for finding a (possible) nonvanishing section of T_{Q^o}. Moreover, Theorem 1.5 is our only result that considers the case n even and $Q^o(k) = \varnothing$, and it only gives us a necessary condition for T_{Q^o} to have a nonvanishing section. Here is a series of examples that are not covered by any of our results.

Take $k = \mathbb{Q}_p$ with $p > 2$. Let $u \in \mathbb{Z}_p^\times$ be a nonsquare modulo p, and let $q = ux_1^2 + px_2^2 - upx_3^2$. It follows from [Lam 2005, Theorem VI.2.2] that $q - x_0^2$ is anisotropic over $\mathbb{Q}_p$; hence $Q^o(\mathbb{Q}_p) = \varnothing$ and also q is anisotropic over $\mathbb{Q}_p$. Moreover $-(u\cdot p\cdot(-up)) = 1$ in $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$, so $-(u\cdot p\cdot(-up))$ is in $[D(q)^2]$; hence the necessary condition in Theorem 1.5 is satisfied. We do not know whether T_{Q^o} has a nonvanishing section in any of these cases.

One final nonexample. Take $k = \mathbb{R}$, $q = \sum_{i=1}^{n+1} -x_i^2$, with n even, and let R be the coordinate ring of Q^o. Then we have $Q^o(\mathbb{R}) = \varnothing$. By a theorem of Ojanguren, Parimala and Sridharan [Ojanguren et al.

1986, Theorem 3.2], there is an $M \in \mathrm{SL}_{n+1}(R)$ with

$$(x_1, \dots, x_{n+1}) = (1, 0, \dots, 0)M$$

in other words, $(x_1, \dots, x_{n+1})$ is the first row of the unimodular matrix M. Thus the R-module $\mathfrak{T}_{Q^o}$ is free, so T_{Q^o} is a trivial vector bundle over Q^o, and hence admits a nonvanishing section. Our results only yield the necessary condition

$$-(-1)^{n+1} = 1 \in |D(q)^2|,$$

which (fortunately!) is true in this case.

The main idea behind the proofs of our results is quite close to the case of the real spheres. We have already disposed of the case of odd n in Remark 1.10. For $n > 0$ even, we replace the Euler class $e_{\mathrm{top}}(T_{S^n}) \in H^n(S^n, \mathbb{Z})$ with the Euler class $e(T_{Q^o})$ in the *Chow–Witt group* $\widetilde{\mathrm{CH}}^n(Q^o)$. For a smooth k-variety X and a rank-r vector bundle E on X, one has an Euler class $e(E)$ in the (twisted) Chow–Witt group $\widetilde{\mathrm{CH}}^n(X, \det^{-1}(E))$. In our case, the tangent-normal sequence for T_{Q^o} gives a canonical isomorphism $\det T_{Q^o} \cong \mathcal{O}_{Q^o}$, which induces an isomorphism $\widetilde{\mathrm{CH}}^n(Q^o, \det^{-1} T_{Q^o}) \cong \widetilde{\mathrm{CH}}^n(Q^o)$ with the untwisted version of the Chow–Witt group, giving us our Euler class $e(T_{Q^o}) \in \widetilde{\mathrm{CH}}^n(Q^o)$. A fundamental result of Morel [2012, Section 8.2] says that for a smooth affine k-scheme X of dimension n over a perfect field k and a rank n vector bundle E over X, E admits a nonvanishing section if and only if the Euler class $e(E)$ vanishes (in this form, the result also relies on work of Asok and Fasel [2016] and Asok, Hoyois and Wendt [Asok et al. 2017]; see Theorem 2.2 for the discussion).

Since Q^o is not proper over k, we do not have a nice analog of the Gauss–Bonnet theorem for Q^o, so we pass to its projective closure $Q \subseteq \mathbb{P}^{n+1}$, defined by the equation $\sum_{i=1}^{n+1} a_i x_i^2 = x_0^2$, and let $Q^\infty \subseteq Q$ be the hyperplane section defined by $x_0 = 0$.

Let $\mathrm{GW}(k)$ denote the Grothendieck–Witt ring of (virtual) regular quadratic forms over k. For $p : X \to \operatorname{Spec} k$ a smooth projective variety, we have the pushforward map

$$p_* : \widetilde{\mathrm{CH}}_0(X) \to \widetilde{\mathrm{CH}}_0(\operatorname{Spec} k) = \mathrm{GW}(k),$$

which we denote by $\deg_{GW} : \widetilde{\mathrm{CH}}_0(X) \to \mathrm{GW}(k)$; we call this the *quadratic degree map*. X has a *quadratic Euler characteristic* $\chi(X/k) \in \mathrm{GW}(k)$ and we have a quadratic Gauss–Bonnet theorem [Déglise et al. 2021, Theorem 4.6.1; Levine and Raksit 2020, Theorem 5.3]: letting T_X be the tangent bundle of X, we have

$$\deg_{GW}(e(T_X)) = \chi(X/k),$$

so we are all set up to argue as in differential topology.

Getting back to our quadrics, let us first assume that Q^o has a k-rational point. We show in Section 2 that $e(T_{Q^o}) = 0$ if and only if $\chi(Q/k)$ is in the subgroup $\deg_{GW}(\widetilde{\mathrm{CH}}_0(Q^\infty)) \subseteq \mathrm{GW}(k)$, and we have an explicit expression for $\chi(Q/k)$:

$$\chi(Q/k) = \Big\langle 2, 2\prod_i a_i \Big\rangle + \frac{n}{2}\langle 1, -1\rangle,$$

where $\langle a, b\rangle$ is the quadratic form $ax^2 + by^2$, and $m \cdot \langle a, b\rangle$ is the quadratic form $\sum_{i=1}^m ax_i^2 + by_i^2$.

Putting this together, we see that T_{Q^o} admits a nonvanishing section if and only if $\langle 2, 2\prod_i a_i\rangle + \frac{n}{2}\langle 1, -1\rangle$ is in $\deg_{GW}(\widetilde{\mathrm{CH}}_0(Q^\infty)) \subseteq \mathrm{GW}(k)$. We conclude Section 2 by using this criterion to handle the cases discussed in Examples 1.3 (1), (2) above.

The next step is to use the theory of quadratic forms to rephrase the condition

$$\Big\langle 2, 2\prod_i a_i\Big\rangle + \frac{n}{2}\langle 1, -1\rangle \in \deg_{GW}(\widetilde{\mathrm{CH}}_0(Q^\infty)) \subseteq \mathrm{GW}(k)$$

in terms of the subgroups $[D(q)]$ and $[D(q)^2]$. This is done in Section 3, relying on properties of Scharlau's transfer, Knebusch's norm principle and basic facts about Pfister forms and Pfister neighbors. We apply these tools in Section 4 to give our main results, which yield criteria that are much easier to apply than the one derived in Section 2. We conclude by using this to compute the remaining examples described above; the case in which Q^o does not have a k-rational point is trickier to handle, and we are only able to arrive at the necessary condition stated in Theorem 1.5.

Throughout the paper we employ the following notation:

k	a perfect field with $\mathrm{char}\, k \neq 2$
Sm_k	the category of smooth, separated, finite-type k-schemes
T_X	the tangent bundle of $X \in \mathrm{Sm}_k$
$X(F)$	the set of rational points of X_F for $X \in \mathrm{Sm}_k$ and a field extension F/k
$\mathrm{GW}(F)$	the Grothendieck–Witt ring of (virtual) regular quadratic forms over a field F
$\mathrm{W}(F)$	the Witt ring, the quotient of $\mathrm{GW}(F)$ by the hyperbolic forms
$\mathrm{GI}(F)$	the ideal in $\mathrm{GW}(F)$ generated by the even-dimensional forms
$I(F)$	the image in $\mathrm{W}(F)$ of $\mathrm{GI}(F)$
$F^\times$	the multiplicative group of nonzero elements of the field F
$\langle a_1, a_2, \ldots, a_n\rangle$	the quadratic form $a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2$

2. Recollections on Chow–Witt groups and a computational criterion

Definition 2.1. Assume k to be a perfect field. We will use the Chow–Witt groups, also known as Chow groups of oriented cycles, that were introduced in [Barge and Morel 2000]. These groups provide refined cohomological obstructions to the existence of nonvanishing sections of algebraic vector bundles (see, e.g., [Asok and Fasel 2015; 2023]). We refer the reader to the expositions in [Fasel 2020; Déglise 2023; Asok and Fasel 2016, Sections 2, 3] for the properties of these groups that we list below.

We recall from [Morel 2012, Chapter 2] the *Milnor-Witt K-theory sheaves* $\mathcal{K}_n^{\mathrm{MW}}$, $n \in \mathbb{Z}$. These are Nisnevich sheaves of abelian groups on Sm_k, with products $\mathcal{K}_n^{\mathrm{MW}} \times \mathcal{K}_m^{\mathrm{MW}} \to \mathcal{K}_{n+m}^{\mathrm{MW}}$ making the graded object $\mathcal{K}_*^{\mathrm{MW}} := \bigoplus_{n\in\mathbb{Z}} \mathcal{K}_n^{\mathrm{MW}}$ into a sheaf of associative, unital, graded rings on Sm_k. Given $X \in \mathrm{Sm}_k$ and a line bundle L on X, we have the L-twisted version $\mathcal{K}_n^{\mathrm{MW}}(L)$, giving a Nisnevich sheaf on Sm_k/X. Letting $\mathcal{GW}$ denote the Nisnevich sheaf of Grothendieck–Witt rings, there is a canonical isomorphism $\mathcal{K}_0^{\mathrm{MW}} \cong \mathcal{GW}$. For a field F, and L a dimension one F-vector space, we write $K_n^{\mathrm{MW}}(L)(F)$ for $\mathcal{K}_n^{\mathrm{MW}}(L)(\mathrm{Spec}\, F)$.

For a smooth variety X over k, a line bundle L over X and an integer $n \geq 0$, the *Chow–Witt group* $\widetilde{\mathrm{CH}}^n(X, L)$ is defined as

$$\widetilde{\mathrm{CH}}^n(X, L) = H^n_{\mathrm{Zar}}(X; \mathcal{K}^{\mathrm{MW}}_n(L)).$$

We will also use the homological notation with

$$\widetilde{\mathrm{CH}}_n(X, L) = \widetilde{\mathrm{CH}}^{d-n}(X, L \otimes \omega_X),$$

where $d = \dim X$ and ω_X is the canonical bundle of X. We put

$$\widetilde{\mathrm{CH}}^n(X) := \widetilde{\mathrm{CH}}^n(X, \mathcal{O}_X), \quad \widetilde{\mathrm{CH}}_n(X) := \widetilde{\mathrm{CH}}_n(X, \mathcal{O}_X).$$

Chow–Witt groups have the following properties that we will use below.

(1) $\widetilde{\mathrm{CH}}^n(X, L)$ is canonically identified by [Morel 2012, Theorem 5.47] with the n-th cohomology group of the Rost–Schmid complex

$$\bigoplus_{x \in X^{(0)}} K^{\mathrm{MW}}_n(L_x \otimes \omega_{x/X})(k(x)) \to \bigoplus_{x \in X^{(1)}} K^{\mathrm{MW}}_{n-1}(L_x \otimes \omega_{x/X})(k(x)) \to \cdots \to \bigoplus_{x \in X^{(d)}} K^{\mathrm{MW}}_{n-d}(L_x \otimes \omega_{x/X})(k(x)),$$

where the sums are taken over all the points of X of the respective codimension, L_x is the restriction of L to x, $\omega_{x/X}$ is the determinant of the normal bundle for the embedding $x \to X$ and $d = \dim X$.

(2) For a line bundle L' over X there is a canonical isomorphism [Morel 2012, Remark 5.13]

$$\widetilde{\mathrm{CH}}^n(X, L \otimes (L')^{\otimes 2}) \cong \widetilde{\mathrm{CH}}^n(X, L).$$

(3) For a morphism $f : Y \to X$ of smooth varieties over k one has a functorial pullback homomorphism

$$f^* : \widetilde{\mathrm{CH}}^n(X, L) \to \widetilde{\mathrm{CH}}^n(Y, f^*L)$$

given by the pullback in the cohomology of sheaves. Further, if f is proper then one has a functorial pushforward homomorphism [Fasel 2020, Section 2.3]

$$f_* : \widetilde{\mathrm{CH}}_n(Y, f^*L) \to \widetilde{\mathrm{CH}}_n(X, L)$$

induced by the transfers on the Rost–Schmid complexes. For a closed embedding $i : Z \to X$ of smooth varieties, with $j : X \setminus Z \to X$ the open embedding of the complement, the localization sequence

$$\widetilde{\mathrm{CH}}_n(Z, i^*L) \xrightarrow{i_*} \widetilde{\mathrm{CH}}_n(X, L) \xrightarrow{j^*} \widetilde{\mathrm{CH}}_n(X \setminus Z, j^*L)$$

is exact [Fasel 2020, Section 2.2].

(4) Let F/k be a field extension of finite degree. Since k is perfect, F is separable over k, so the field trace $\mathrm{tr}^F_k : F \to k$ is a nonzero k-linear functional. This gives rise to the Scharlau transfer

$$(\mathrm{tr}^F_k)_* : \mathrm{GW}(F) \to \mathrm{GW}(k),$$

an additive homomorphism, which is given on generators $\langle a \rangle \in \mathrm{GW}(F)$ by defining $(\mathrm{tr}^F_k)_*(\langle a \rangle)$ to be the quadratic form $x \mapsto \mathrm{tr}^F_k(ax^2)$ on the k-vector space F. The pushforward homomorphism in Chow–Witt groups

$$\pi_* : \widetilde{\mathrm{CH}}_0(\mathrm{Spec}\, F) \to \widetilde{\mathrm{CH}}_0(\mathrm{Spec}\, k)$$

for the morphism $\pi : \mathrm{Spec}\, F \to \mathrm{Spec}\, k$ coincides by [Fasel 2020, Example 1.23] with the Scharlau transfer $(\mathrm{tr}_k^F)_*$ under the identifications

$$\widetilde{\mathrm{CH}}_0(\mathrm{Spec}\, F) = K_0^{\mathrm{MW}}(F) \cong \mathrm{GW}(F), \quad \widetilde{\mathrm{CH}}_0(\mathrm{Spec}\, k) = K_0^{\mathrm{MW}}(k) \cong \mathrm{GW}(k).$$

(5) For a rank-n vector bundle E over a smooth variety X over k one has an *Euler class*

$$e(E) = s^* s_*(1_X) \in \widetilde{\mathrm{CH}}^n(X, (\det E)^\vee),$$

where $s : X \to E$ is the zero section. This class is natural with respect to pullbacks [Asok and Fasel 2016, Proposition 3.1.1].

(6) Let E be a rank n vector bundle over a smooth affine variety X of dimension n over k. Suppose that $\det E \cong \mathcal{O}_X$. Then $e(E) = 0$ if and only if E has a nonvanishing section. For $n = 1$ there is nothing to prove, for $n = 2$ this was shown in [Barge and Morel 2000, Theorem 2.2] and for general n this follows from the results of [Morel 2012, Chapter 8; Asok et al. 2017, Theorem 1; Asok and Fasel 2016, Theorem 5.6]; see Theorem 2.2 below for the details.

Theorem 2.2 (Barge–Morel, Morel, Asok–Fasel, Asok–Hoyois–Wendt). *Let k be a perfect field and E be a rank n vector bundle over a smooth affine variety X of dimension n over k. Suppose that* $\det E \cong \mathcal{O}_X$. *Then E admits a nonvanishing section if and only if $e(E) = 0$.*

Proof. If $n = 1$, there is nothing to prove, so assume $n \geq 2$. It follows from [Morel 2012, Theorem 8.14] that E admits a nonvanishing section if and only if a certain obstruction-theoretic Euler class $e_{\mathrm{ob}}(E) \in \widetilde{\mathrm{CH}}^n(X)$ vanishes. Note that in that work it was assumed $n \geq 4$ because of the assumption $r \neq 2$ in [Morel 2012, Theorem 8.1 (3)], which can be removed using [Asok et al. 2017, Theorem 1]. It follows from [Asok and Fasel 2016, Theorem 5.6] that $e_{\mathrm{ob}}(E) = 0$ if and only if $e(E) = 0$, whence the claim. □

Remark 2.3. We expect that Theorem 2.2 generalizes to vector bundles with possibly nontrivial determinant and to general fields, removing the assumptions $\det E \cong \mathcal{O}_X$ and k being perfect.

Remark 2.4. Let X be a smooth hypersurface in $\mathbb{A}_k^{n+1}$. Since $k[x_1, \ldots, x_{n+1}]$ is a UFD, the ideal of X is principal, $I_X = (F)$. We have the tangent-normal sequence describing the tangent bundle T_X as

$$0 \to T_X \to \mathcal{O}_X^{n+1} \xrightarrow{\nabla F} \mathcal{O}_X \to 0,$$

where $\nabla F := (\partial F/\partial x_1, \ldots, \partial F/\partial x_{n+1})$ is the usual gradient of F. Since X is affine, this sequence splits, in particular, T_X is stably trivial and $\det T_X \cong \mathcal{O}_X$, so Theorem 2.2 is applicable to T_X.

Definition 2.5. Let X be a smooth proper variety over a perfect field k with the structure morphism $\pi : X \to \mathrm{Spec}\, k$. Then the *quadratic degree map*

$$\deg_{GW} := \pi_* : \widetilde{\mathrm{CH}}_0(X) \to \widetilde{\mathrm{CH}}_0(\mathrm{Spec}\, k) \cong \mathrm{GW}(k)$$

is the pushforward homomorphism for the structure morphism.

Lemma 2.6. *Let Q be a smooth projective quadric over a perfect field k. Suppose that $Q(k) \neq \varnothing$. Then the quadratic degree map*

$$\deg_{GW} : \widetilde{\mathrm{CH}}_0(Q) \to \mathrm{GW}(k)$$

is an isomorphism.

Proof. Let $n = \dim Q$ and consider the commutative square

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "f"'] & Q \arrow[d, "\pi"] \\ \mathbb{P}^n \arrow[r, "p"] & \operatorname{Spec} k \end{tikzcd}$$

where $Q \xleftarrow{g} Y \xrightarrow{f} \mathbb{P}^n$ is the resolution of the birational morphism $Q \dashrightarrow \mathbb{P}^n$ given by the projection from a rational point on Q. Note that all maps in this square are proper. This gives a commutative diagram of pushforward homomorphisms:

$$\begin{tikzcd} \widetilde{\mathrm{CH}}_0(Y) \arrow[r, "g_*"] \arrow[d, "f_*"'] & \widetilde{\mathrm{CH}}_0(Q) \arrow[d, "\pi_*"] \\ \widetilde{\mathrm{CH}}_0(\mathbb{P}^n) \arrow[r, "p_*"] & \mathrm{GW}(k) \end{tikzcd}$$

It follows from the birational invariance of $\widetilde{\mathrm{CH}}_0$ [Feld 2022, Corollary 2.2.11] that f_* and g_* are isomorphisms. Recall that $\omega_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$, whence

$$\widetilde{\mathrm{CH}}_0(\mathbb{P}^n) = \widetilde{\mathrm{CH}}^n(\mathbb{P}^n, \mathcal{O}(-n-1)) \cong \begin{cases} \widetilde{\mathrm{CH}}^n(\mathbb{P}^n), & n \text{ is odd,} \\ \widetilde{\mathrm{CH}}^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-1)), & n \text{ is even.} \end{cases}$$

The homomorphism p_* is an isomorphism by [Fasel 2013, Corollary 11.8]. Thus $\pi_* = \deg_{GW}$ is an isomorphism as well. □

Definition 2.7. A smooth projective scheme X over k has a *quadratic Euler characteristic* $\chi(X/k) \in \mathrm{GW}(k)$, arising from the categorical Euler characteristic of the dualizable object $\Sigma^\infty_{\mathbb{P}^1} X_+$ in the motivic stable homotopy category $\mathrm{SH}(k)$, together with Morel's theorem [2004, Theorem 6.4.1, Remark 6.4.2] identifying the endomorphisms of the unit in $\mathrm{SH}(k)$ with $\mathrm{GW}(k)$ (see [Hoyois 2014, Section 1; Levine 2020, Section 2.1] for details). The motivic Gauss–Bonnet theorem [Déglise et al. 2021, Theorem 4.6.1; Levine and Raksit 2020, Theorem 5.3] gives the identity

$$\chi(X/k) = \deg_{GW}(e(T_X)) \in \mathrm{GW}(k). \tag{2.8}$$

Theorem 2.9. *Let Q^o be the affine quadric over a perfect field k given by the equation*

$$a_1x_1^2 + a_2x_2^2 + \cdots + a_{n+1}x_{n+1}^2 = 1,$$

with $a_1, \ldots, a_{n+1} \in k^\times$ and let Q^∞ be the projective quadric given by the equation

$$a_1x_1^2 + a_2x_2^2 + \cdots + a_{n+1}x_{n+1}^2 = 0.$$

Then the following hold:

(1) *If n is odd, then the tangent bundle T_{Q^o} has a nonvanishing section.*

(2) *If $n>0$ is even and the tangent bundle T_{Q^o} has a nonvanishing section, then*
$$\frac{n}{2}\cdot\langle 1,-1\rangle+\Big\langle 2, 2\cdot\prod_{i=1}^{n+1} a_i\Big\rangle \in \deg_{GW}(\widetilde{\mathrm{CH}}_0(Q^\infty)) \subseteq \mathrm{GW}(k).$$

(3) *If $n>0$ is even and Q^o has a rational point, then the tangent bundle T_{Q^o} has a nonvanishing section if and only if*
$$\frac{n}{2}\cdot\langle 1,-1\rangle+\Big\langle 2, 2\cdot\prod_{i=1}^{n+1} a_i\Big\rangle \in \deg_{GW}(\widetilde{\mathrm{CH}}_0(Q^\infty)) \subseteq \mathrm{GW}(k).$$

Proof. (1) We have settled the case of odd n in Remark 1.10.

(2), (3) Let Q be the compactification of Q^o given by the equation
$$a_1x_1^2+a_2x_2^2+\cdots+a_{n+1}x_{n+1}^2=x_0^2$$
in the projective space $\mathbb{P}^{n+1}$ and let $j:Q^o\to Q$ be the open embedding. Then $Q^\infty=Q\setminus j(Q^o)$; let $i:Q^\infty\to Q$ be the closed embedding. Consider the localization sequence and the quadratic degree homomorphisms
$$\begin{tikzcd}
\widetilde{\mathrm{CH}}_0(Q^\infty) \arrow[r, "i_*"] \arrow[dr, "\deg_{GW}"'] & \widetilde{\mathrm{CH}}_0(Q) \arrow[r, "j^*"] \arrow[d, "\deg_{GW}"] & \widetilde{\mathrm{CH}}_0(Q^o) \\
& \mathrm{GW}(k) &
\end{tikzcd}$$

We have the identifications
$$\widetilde{\mathrm{CH}}_0(Q)=\widetilde{\mathrm{CH}}^n(Q,\omega_Q)=\widetilde{\mathrm{CH}}^n(Q,(\det T_Q)^\vee),$$
so we consider the Euler class $e(T_Q)\in\widetilde{\mathrm{CH}}^n(Q,(\det T_Q)^\vee)$ as being in $\widetilde{\mathrm{CH}}_0(Q)$.

Exactness of the localization sequence yields that the Euler class $e(T_{Q^0})=e(j^*T_Q)=j^*e(T_Q)$ vanishes if and only if $e(T_Q)\in i_*\widetilde{\mathrm{CH}}_0(Q^\infty)$. By (2.8) and [Levine 2020, Corollary 12.2] we have
$$\deg_{GW}(e(T_Q))=\chi(Q/k)=\frac{n}{2}\cdot\langle 1,-1\rangle+\Big\langle 2, 2\cdot\prod_{i=1}^{n+1} a_i\Big\rangle. \tag{2.10}$$

Suppose that T_{Q^o} has a nonvanishing section. Then $e(T_{Q^0})=0$ and hence $e(T_Q)$ is in $i_*\widetilde{\mathrm{CH}}_0(Q^\infty)$. Taking quadratic degrees and using formula (2.10) we obtain
$$\deg_{GW}(e(T_Q))=\frac{n}{2}\cdot\langle 1,-1\rangle+\Big\langle 2, 2\cdot\prod_{i=1}^{n+1} a_i\Big\rangle \in \deg_{GW}(i_*\widetilde{\mathrm{CH}}_0(Q^\infty))=\deg_{GW}(\widetilde{\mathrm{CH}}_0(Q^\infty)),$$
proving (2) and one implication in (3).

Now suppose that Q^o has a rational point and $\frac{n}{2}\cdot\langle 1,-1\rangle+\big\langle 2,2\cdot\prod_{i=1}^{n+1}a_i\big\rangle\in\deg_{GW}(\widetilde{\mathrm{CH}}_0(Q^\infty))$. Note that $\deg_{GW}(\widetilde{\mathrm{CH}}_0(Q^\infty))=\deg_{GW}(i_*\widetilde{\mathrm{CH}}_0(Q^\infty))$, whence Lemma 2.6 combined with (2.10) show that

$e(T_Q) \in i_* \widetilde{\mathrm{CH}}_0(Q^\infty)$, yielding $e(T_{Q^o}) = 0$. Remark 2.4 provides an isomorphism $\det T_{Q^o} \cong \mathcal{O}_{Q^o}$, whence Theorem 2.2 implies that T_{Q^o} has a nonvanishing section, completing the proof of (3). $\square$

Example 2.11. Let S_k^2 be the quadric over a field k given by the equation $x^2 + y^2 + z^2 = 1$.

(1) If the equation $x^2 + y^2 = -1$ has a solution over k then the conic $C_k \subseteq \mathbb{P}_k^2$ given by the equation $x^2 + y^2 + z^2 = 0$ has a rational point whence $\deg_{GW}(\widetilde{\mathrm{CH}}_0(C_k)) = \mathrm{GW}(k)$ and Theorem 2.9 yields that $T_{S_k^2}$ has a nonvanishing section. In particular [Lam 2005, Example XI.2.4(2) and (6)] yield that this holds for $k = \mathbb{Q}_p$ the field of p-adic numbers and for $k = \mathbb{F}_{p^n}$ a finite field, with $p \neq 2$ in both cases. By base-change, the same follows if $k \supset \mathbb{Q}_p$ or k has characteristic p, with $p \neq 2$. An explicit nonvanishing section of $T_{S_k^2}$ in these cases may be found as in Remark 1.10.

(2) Let $k = \mathbb{Q}_2$ be the field of dyadic numbers. Then the equation $x^2 + y^2 = -1$ has no solution over k by, e.g., [Lam 2005, Example XI.2.4(7)] and the conic $C_k \subseteq \mathbb{P}_k^2$ given by the equation $x^2 + y^2 + z^2 = 0$ has no rational points. However, it is clear that C_k has a rational point over $\mathbb{Q}_2(\sqrt{-2})$. Moreover, since 2 is equivalent to -14 modulo squares in $\mathbb{Q}_2$ (see, e.g., [Lam 2005, Corollary VI.2.24]), C_k has the point $(6 + \sqrt{-14}, 10, -2 + 3\sqrt{-14})$ over $\mathbb{Q}_2(\sqrt{2})$. A straightforward computation shows that

$$(\mathrm{tr}_{\mathbb{Q}_2}^{\mathbb{Q}_2(\sqrt{\pm 2})})_*(\langle 1\rangle) = \langle 2, \pm 1\rangle,$$

whence

$$\langle 1, -1\rangle + \langle 2, 2\rangle = (\mathrm{tr}_{\mathbb{Q}_2}^{\mathbb{Q}_2(\sqrt{2})})_*(\langle 1\rangle) + (\mathrm{tr}_{\mathbb{Q}_2}^{\mathbb{Q}_2(\sqrt{-2})})_*(\langle 1\rangle) \in \deg_{GW}(\widetilde{\mathrm{CH}}_0(C_k))$$

and Theorem 2.9 (3) yields that $T_{S_k^2}$ has a nonvanishing section. Alternatively, we have

$$(\mathrm{tr}_{\mathbb{Q}_2}^{\mathbb{Q}_2(\sqrt{2})})_*(\langle 1 + \sqrt{2}\rangle) = \langle -2, 1\rangle,$$

whence

$$\langle 1, -1\rangle + \langle 2, 2\rangle = \langle 2, -2\rangle + \langle 1, 1\rangle = (\mathrm{tr}_{\mathbb{Q}_2}^{\mathbb{Q}_2(\sqrt{2})})_*(\langle 1, 1 + \sqrt{2}\rangle) \in \deg_{GW}(\widetilde{\mathrm{CH}}_0(C_k)).$$

Note that C_k does not have a $\mathbb{Q}_2$-point, so we cannot apply Remark 1.10. We were not able to produce an explicit nonvanishing section in this case.

Example 2.12. We can expand on the last example as follows. Let S_k^{2n} be the quadric over a field k given by the equation $\sum_{i=1}^{2n+1} x_i^2 = 1$, $n > 0$, and suppose k contains a p-adic field $\mathbb{Q}_p$. Then $T_{S_k^{2n}}$ has a nonvanishing section. Indeed, letting $C_k^{2n} \subseteq \mathbb{P}_k^{2n}$ be the projective quadric defined by $\sum_{i=1}^{2n+1} x_i^2 = 0$, we have just seen that $\langle 1, -1\rangle + \langle 2, 2\rangle$ is in $\deg_{GW}(\widetilde{\mathrm{CH}}_0(C_k^2)) \subseteq \mathrm{GW}(k)$. But for an arbitrary quadratic extension $k(\sqrt{a})$ of k, we have

$$(\mathrm{tr}_k^{k(\sqrt{a})})_*(\langle \sqrt{a}\rangle) = \langle 1, -1\rangle,$$

so $\langle 1, -1\rangle$ is in $\deg_{GW}(\widetilde{\mathrm{CH}}_0(Q))$ for *every* smooth projective quadric Q over k, and thus we have

$$n \cdot \langle 1, -1\rangle + \langle 2, 2\rangle \in \deg_{GW}(\widetilde{\mathrm{CH}}_0(C_k^2)) \subseteq \deg_{GW}(\widetilde{\mathrm{CH}}_0(C_k^{2n})) \subseteq \mathrm{GW}(k).$$

We then apply Theorem 2.9 (3) to conclude that $T_{S_k^{2n}}$ has a nonvanishing section. Just as in Example 2.11 (1), we can produce an explicit nonvanishing section if C_k^{2n} has a k-rational point. This is the case if $\mathbb{Q}_p \subseteq k$ with p an odd prime or if $n \geq 2$ and $\mathbb{Q}_2 \subseteq k$ (for this last case, see [Lam 2005, Theorem VI.2.12]).

Remark 2.13. Returning to the example of $S^2_{\mathbb{Q}_2}$, Nanjun Yang asked if one could completely compute $\widetilde{\mathrm{CH}}_0(S^2_{\mathbb{Q}_2})$. From the localization exact sequence in the proof of Theorem 2.9, we have the isomorphism

$$\widetilde{\mathrm{CH}}_0(S^2_{\mathbb{Q}_2}) \cong \mathrm{GW}(\mathbb{Q}_2)/\deg_{GW}(\widetilde{\mathrm{CH}}_0(C_{\mathbb{Q}_2})).$$

We may identify $C_{\mathbb{Q}_2}$ with the Severi–Brauer variety associated to the standard quaternions $\mathbb{H}_{\mathbb{Q}_2}$ over $\mathbb{Q}_2$. By [Serre 1962, XIII, Proposition 6], the Brauer group of a local field K is isomorphic to $\mathbb{Q}/\mathbb{Z}$ by the map

$$\mathrm{inv}_K : \mathrm{Br}(K) \to \mathbb{Q}/\mathbb{Z},$$

so, for a degree-2 extension $k \supset \mathbb{Q}_2$, $C_{\mathbb{Q}_2}(k) \neq \varnothing$ if and only if k splits $\mathbb{H}_{\mathbb{Q}_2}$, i.e., if and only if the invariant $\mathrm{inv}_k(\mathbb{H}_k)$ in $\mathbb{Q}/\mathbb{Z}$ is zero. But by [Serre 1962, XIII, Proposition 7],

$$\mathrm{inv}_k(\mathbb{H}_k) = 2 \cdot \mathrm{inv}_{\mathbb{Q}_2}(\mathbb{H}_{\mathbb{Q}_2}) = 2 \cdot \tfrac{1}{2} = 0,$$

so $C_{\mathbb{Q}_2}(\mathbb{Q}_2(\sqrt{a})) \neq \varnothing$ for every nonsquare $a \in \mathbb{Q}_2^\times$.

Thus, $\langle 2, 2a\rangle = \mathrm{tr}_{\mathbb{Q}_2(\sqrt{a})/\mathbb{Q}_2}(\langle 1\rangle)$ is in $\deg_{GW}(\widetilde{\mathrm{CH}}_0(C_{\mathbb{Q}_2}))$ for all nonsquares a, so the ideal I in $\mathrm{GW}(\mathbb{Q}_2)$ generated by the forms

$$\{\langle 2, u\rangle \mid u = 3, 5, 7\} \cup \{\langle 2, 2u\rangle \mid u = 1, 3, 5, 7\}$$

is contained in $\deg_{GW}(\widetilde{\mathrm{CH}}_0(C_{\mathbb{Q}_2}))$. It is easy to see that I is exactly the ideal in $\mathrm{GI}(\mathbb{Q}_2)$ of even-rank forms in $\mathrm{GW}(\mathbb{Q}_2)$; since $\deg_{GW}(\widetilde{\mathrm{CH}}_0(C_{\mathbb{Q}_2}))$ is clearly contained $\mathrm{GI}(\mathbb{Q}_2)$, we have $\deg_{GW}(\widetilde{\mathrm{CH}}_0(C_{\mathbb{Q}_2})) = \mathrm{GI}(\mathbb{Q}_2)$ and

$$\widetilde{\mathrm{CH}}_0(S^2_{\mathbb{Q}_2}) \cong \mathbb{Z}/2$$

via the mod 2 rank map $\mathrm{GW}(\mathbb{Q}_2) \to \mathbb{Z}/2$.

3. Scharlau's transfer for closed points on a quadric

Definition 3.1. Let F be a field, char $F \neq 2$. We denote by $\mathrm{GI}(F) \subseteq \mathrm{GW}(F)$ the ideal consisting of the even-dimensional (virtual) regular quadratic forms in the Grothendieck–Witt ring of F.

Definition 3.2. Let E/F be a field extension of finite degree, char $F \neq 2$, and $s : E \to F$ be a nonzero F-linear functional. Then the *Scharlau transfer* [1969]

$$s_* : \mathrm{GW}(E) \to \mathrm{GW}(F)$$

is the additive homomorphism such that $s_*(\langle a\rangle)$ is the quadratic form $x \mapsto s(ax^2)$ on the F-vector space E. See [Lam 2005, Chapter VII] for some of the properties of the Scharlau transfer.

Let X be a variety over k. The *transfer ideal of* X is given by

$$\mathrm{GI}_X^{\mathrm{tr}} = \sum_{x \in X_{(0)}} (s_x)_*(\mathrm{GW}(k(x))) \subseteq \mathrm{GW}(k),$$

with the sum taken over all the closed points of X and $\{s_x : k(x) \to k\}_{x \in X_{(0)}}$ being a chosen set of nonzero k-linear functionals. Note that $\mathrm{GI}_X^{\mathrm{tr}}$ does not depend on the choices of s_x [Lam 2005, Remark VII.1.6(C)]. It is easy to see that the transfer ideal admits an alternative description as

$$\mathrm{GI}_X^{\mathrm{tr}} = \sum_{\substack{F/k \text{ finite} \\ X(F) \neq \varnothing}} (s_F)_*(\mathrm{GW}(F)) \subseteq \mathrm{GW}(k),$$

with the sum taken over all the (isomorphism classes) of field extensions F/k of finite degree such that X_F has a rational point and $\{s_F : F \to k\}_F$ being some chosen set of nonzero k-linear functionals.

Remark 3.3. The transfer ideal $s_*(\mathrm{GW}(F)) \subseteq \mathrm{GW}(k)$ for an extension of fields F/k of finite degree is a classical object of study; see, e.g., [Lam 2005, Chapter VII]. This agrees with the notion introduced above if one considers F as a zero-dimensional variety $\operatorname{Spec} F$ over k.

Lemma 3.4. *Let X be a smooth proper variety over a perfect field k. Then*

$$\mathrm{GI}_X^{\mathrm{tr}} = \deg_{GW}(\widetilde{\mathrm{CH}}_0(X)).$$

Proof. This follows from the description of $\widetilde{\mathrm{CH}}_0(X)$ via the cohomology of the Rost–Schmid complex and the fact that the pushforward in $\widetilde{\mathrm{CH}}_0$ for a separable field extension of finite degree coincides with the Scharlau transfer for the field trace. □

Lemma 3.5. *Let E/F be a field extension of even degree*, char $F \neq 2$, *and $s : E \to F$ be an F-linear nonzero functional. Then $s_*(\mathrm{GW}(E)) \subseteq s_*(\mathrm{GI}(E)) + \langle 1, -1\rangle \cdot \mathrm{GW}(F)$.*

Proof. Note that the claim does not depend on the choice of s (see [Lam 2005, Remark VII.1.6(C)]). Without loss of generality we may assume that $E = F(\alpha)/F$ is a simple extension. Then there is a nonzero functional s such that $s_*(\langle\alpha\rangle) = \frac{1}{2}[E : F] \cdot \langle 1, -1\rangle$ [Lam 2005, Theorem VII.2.3]. The claim follows, since $s_*(\phi) = s_*(\phi + \langle\alpha\rangle) - \frac{1}{2}[E : F] \cdot \langle 1, -1\rangle$. □

Definition 3.6. Let q be a regular quadratic form over k. Then we use the following notation:

- $D(q)$ is the set of nonzero values of q.
- $D(q)^2 = \{a \cdot b \mid a, b \in D(q)\}$ is the set of products of pairs of nonzero values of q.
- $[D(q)]$ and $[D(q)^2]$ are the multiplicative subgroups of $k^\times$ generated by the respective sets.

Note that if $1 \in D(q)$ then $[D(q)] = [D(q)^2]$. Since all the sets introduced above are stable under multiplication by squares $(k^\times)^2$, we will sometimes abuse the notation and denote in the same way the corresponding subsets of $k^\times/(k^\times)^2$.

Definition 3.7. For a regular quadratic form q over k of dimension n the *signed discriminant* $\delta_{\pm}(q)$ is given by the formula

$$\delta_{\pm}(q) := (-1)^{n(n-1)/2} \det A_q,$$

where A_q is a symmetric matrix representing q. This gives a well-defined map

$$\delta_{\pm} : \mathrm{GW}(k) \to k^{\times}/(k^{\times})^2.$$

When restricted to the ideal $\mathrm{GI}(k)$, this map becomes a homomorphism.

Lemma 3.8. *Let Q be a smooth projective quadric over k defined by a quadratic form q and take $\phi \in \mathrm{GI}_Q^{\mathrm{tr}}$. Then $\delta_{\pm}(\phi)$ is in $[D(q)^2]$.*

Proof. We may assume $Q(k) = \varnothing$; otherwise q is isotropic and $D(q) = k^{\times}$, whence there is nothing to prove. Springer's theorem [Lam 2005, Theorem VII.2.7] yields that for every closed point $x \in Q_{(0)}$ the degree $[k(x) : k]$ is even. Hence $\mathrm{GI}_Q^{\mathrm{tr}} \subseteq \mathrm{GI}(k)$, whence $\delta_{\pm}$ restricted to $\mathrm{GI}_Q^{\mathrm{tr}}$ is a homomorphism. Thus it is sufficient to check the claim for $\phi = s_*(\psi)$ with $\psi \in \mathrm{GW}(k(x))$ for a closed point $x \in Q_{(0)}$ and s a chosen k-linear functional $s : k(x) \to k$. Furthermore, by Lemma 3.5, we may assume $\psi \in \mathrm{GI}(k(x))$. By [Scharlau 1985, Chapter II, Theorem 5.12] we have

$$\delta_{\pm}(s_*(\psi)) = N_{k(x)/k}(\delta_{\pm}(\psi)) \in k^{\times}/(k^{\times})^2.$$

The quadric $Q_{k(x)}$ has a rational point, whence $q_{k(x)}$ is isotropic and $D(q_{k(x)}) = k(x)^{\times}$; in particular, $\delta_{\pm}(\psi) \in D(q_{k(x)})$. Then Knebusch's norm principle [Lam 2005, Theorem VII.5.1] implies that $N_{k(x)/k}(\delta_{\pm}(\psi)) \in [D(q)^2]$. □

Remark 3.9. If q is a Pfister form then the last result was obtained in [Bhatwadekar et al. 2014, Lemma 3.6].

Lemma 3.10. *Let Q be a smooth projective quadric over k defined by a quadratic form q. Then $\langle a, b\rangle \in \mathrm{GI}_Q^{\mathrm{tr}}$ if and only if $-ab \in [D(q)^2]$.*

Proof. Since $\delta_{\pm}(\langle a, b\rangle) = -ab$, one implication follows from Lemma 3.8. For the other implication, first note that we may assume $Q(k) = \varnothing$, since otherwise $\mathrm{GI}_Q^{\mathrm{tr}} = \mathrm{GW}(k)$ and there is nothing to prove. Then there exists a closed point $x \in Q_{(0)}$ such that $[k(x) : k] = 2$ and we may choose $\alpha \in k$ such that $k(x) \cong k(\sqrt{\alpha})$. Then for the k-linear functional

$$s : k(\sqrt{\alpha}) \to k, \quad s(1) = 0,\ s(\sqrt{\alpha}) = 1,$$

one has $s_*(\langle 1\rangle) = \langle 1, -1\rangle$, whence

$$\langle 1, -1\rangle \in \mathrm{GI}_Q^{\mathrm{tr}}.$$

Taking $c_1, c_2 \in k^{\times}$, we have

$$\langle 1, -c_1c_2\rangle = \langle c_1\rangle\langle 1, -c_2\rangle + \langle 1, -c_1\rangle - \langle 1, -1\rangle.$$

Recalling that $\mathrm{GI}^{\mathrm{tr}}_Q$ is an ideal in $\mathrm{GW}(k)$, it follows that

$$\langle 1, -c_1\rangle, \langle 1, -c_2\rangle \in \mathrm{GI}^{\mathrm{tr}}_Q \quad \Longrightarrow \quad \langle 1, -c_1c_2\rangle \in \mathrm{GI}^{\mathrm{tr}}_Q\,. \tag{3.11}$$

We claim that

$$c, d \in D(q) \quad \Longrightarrow \quad \langle 1, -cd\rangle \in \mathrm{GI}^{\mathrm{tr}}_Q\,. \tag{3.12}$$

Accepting our claim for the moment, write $-ab \in [D(q)^2]$ as a product, $-ab := \prod_i a_ib_i$, with $a_i, b_i \in D(q)$. By (3.12), we have $\langle 1, -a_ib_i\rangle \in \mathrm{GI}^{\mathrm{tr}}_Q$ for each i. By (3.11), it follows that $\big\langle 1, -\prod_i a_ib_i\big\rangle = \langle 1, ab\rangle$ is in $\mathrm{GI}^{\mathrm{tr}}_Q$. We proceed to prove (3.12).

First suppose that $\dim Q = 0$. Then we may assume $q = x_1^2 - \alpha x_2^2$ and $Q \cong \operatorname{Spec} k(\sqrt{\alpha})$. A straightforward computation with the same functional s as above shows that

$$s_*(\langle w_1 + \sqrt{\alpha}w_2\rangle) = \langle 1, -(u_1^2 - \alpha u_2^2)(v_1^2 - \alpha v_2^2)\rangle$$

for $w_1 = (u_1v_1 + \alpha u_2v_2)/(u_1v_2 + u_2v_1)$ and $w_2 = 1$, which proves (3.12) in this case.

Now suppose that $\dim Q \geq 1$. Let $q(u) = a,\ q(v) = b$ and choose some $w \neq 0$ such that $\psi_q(u, w) = \psi_q(v, w) = 0$, where ψ_q is the symmetric bilinear form associated to q. Put $c = q(w)$ and let $\alpha = -a/c$. Then

$$q(u + \sqrt{\alpha}w) = q(u) + \alpha q(w) + 2\sqrt{\alpha}\psi_q(u, w) = q(u) + \alpha q(w) = 0.$$

Thus there is a closed point $x \in Q_{(0)}$ such that $k(x) \cong k(\sqrt{-a/c})$. Then for the same functional s as above one has

$$s_*(\langle\sqrt{-a/c}\rangle) = \langle 1, -a/c\rangle \in \mathrm{GI}^{\mathrm{tr}}_Q\,.$$

The same argument shows that $\langle 1, -b/c\rangle$ is in $\mathrm{GI}^{\mathrm{tr}}_Q$. Using (3.11), we see that $\langle 1, -ab\rangle$ also belongs to $\mathrm{GI}^{\mathrm{tr}}_Q$ and (3.12) follows. □

Remark 3.13. Let Q be a smooth projective quadric over a field k defined by a quadratic form q. Then the group $[D(q)^2]$ coincides with the group of norms $N_Q(k)$ of Q, i.e., with the multiplicative subgroup of $k^\times$ generated by the norms $N_{F/k}(a)$, with $a \in F^\times$ and F/k being an extension of fields of finite degree such that Q_F has a rational point [Colliot-Thélène and Skorobogatov 1993, Lemma 2.2].

Definition 3.14. For $a_1, a_2, \ldots, a_n \in k^\times$ an *n-fold Pfister form* $\langle\langle a_1, a_2, \ldots, a_n\rangle\rangle$ is the quadratic form $\prod_{i=1}^n \langle 1, -a_i\rangle$ of dimension 2^n. A regular quadratic form q over k is called a *Pfister neighbor* if there exists $a \in k^\times$ such that $\langle a\rangle \cdot q$ is a subform of an n-fold Pfister form with $2^{n-1} < \dim q$. Note that the Pfister form containing $\langle a\rangle \cdot q$ for a Pfister neighbor q is unique [Lam 2005, Proposition X.4.17].

Lemma 3.15. *Let q be a Pfister neighbor over a field k with the associated Pfister form ϕ and let Q and Φ be the projective quadrics given by $q = 0$ and $\phi = 0$ respectively. Then $\mathrm{GI}^{\mathrm{tr}}_Q = \mathrm{GI}^{\mathrm{tr}}_\Phi$ and $[D(q)^2] = [D(\phi)^2] = D(\phi)$.*

Proof. Suppose ϕ is an n-fold Pfister form and q has dimension $m > 2^{n-1}$. Let $a \in k^\times$ be such that $\langle a\rangle \cdot q$ is a subform of ϕ. Since the quadrics associated to q and $\langle a\rangle \cdot q$ are the same and $[D(q)^2] = [D(\langle a\rangle \cdot q)^2]$, we may assume that q is a subform of ϕ. We claim that for a field extension F/k, $Q(F) \neq \varnothing$ if and only if

$\Phi(F) \neq \varnothing$. Indeed, since q is a subform of ϕ, we have $Q \subseteq \Phi \subset \mathbb{P}_k^{2^n-1}$, and, moreover, $Q = \Phi \cap L$, where L is some codimension-$(2^n - m)$ linear subspace of $\mathbb{P}_k^{2^n-1}$. Thus if $Q(F) \neq \varnothing$ then $\Phi(F) \neq \varnothing$. Now let F be such that $\Phi(F) \neq \varnothing$. Then ϕ_F is isotropic, whence hyperbolic [Lam 2005, Theorem X.1.7], so Φ_F contains a linear subspace $L' \subset \mathbb{P}_F^{2^n-1}$ of dimension $2^{n-1} - 1$. Letting $L_F \subset \mathbb{P}_F^{2^n-1}$ be the base-extension of $L \subset \mathbb{P}_k^{2^n-1}$ to F, we see that Q_F contains the linear subspace $L' \cap L_F \subset \mathbb{P}_F^{2^n-1}$ of dimension at least $2^{n-1} - 1 - (2^n - m) = m - 2^{n-1} - 1 \geq 0$, whence $Q(F) \neq \varnothing$.

The transfer ideals are generated by the Scharlau transfers for the field extensions F/k of finite degree such that $Q(F) \neq \varnothing$ (respectively, $\Phi(F) \neq \varnothing$), whence it follows from the above that $\mathrm{GI}_Q^{\mathrm{tr}} = \mathrm{GI}_\Phi^{\mathrm{tr}}$. Then the equality $[D(q)^2] = [D(\phi)^2]$ follows from Lemma 3.10 because $[D(q)^2]$ and $[D(\phi)^2]$ coincide with the sets of signed discriminants of the binary forms from the respective transfer ideals. Since $1 \in D(\phi)$, we have $[D(\phi)^2] = [D(\phi)]$ and the last equality $[D(\phi)] = D(\phi)$ follows from [Lam 2005, Theorem XI.1.1].

Alternatively, for the equality $[D(q)^2] = [D(\phi)^2]$ one could apply the description of these groups as the groups of norms [Colliot-Thélène and Skorobogatov 1993, Lemma 2.2]. □

4. Nonvanishing vector fields on affine quadrics via groups of values

Using the results of the previous section, we can reformulate Theorem 2.9 in a more manageable form.

Theorem 4.1. *Let $q = a_1x_1^2 + a_2x_2^2 + \cdots + a_{n+1}x_{n+1}^2$ be a quadratic form over a perfect field k with $a_1, \ldots, a_{n+1} \in k^\times$, and let Q^o be the affine quadric given by the equation*

$$a_1x_1^2 + a_2x_2^2 + \cdots + a_{n+1}x_{n+1}^2 = 1.$$

Then the following hold:

(1) *If n is odd then the tangent bundle T_{Q^o} has a nonvanishing section.*

(2) *If $n > 0$ is even and the tangent bundle T_{Q^o} has a nonvanishing section then*
$$-\prod_{i=1}^{n+1} a_i \in [D(q)^2].$$

(3) *If $n > 0$ is even and $Q^o(k) \neq \varnothing$ then the tangent bundle T_{Q^o} has a nonvanishing section if and only if $-1 \in [D(q)]$.*

Proof. The case of odd n follows from Theorem 2.9 (1), so we assume $n > 0$ is even. Let $Q^\infty \subseteq \mathbb{P}^n$ be the quadric given by $q = 0$. By Lemma 3.4 we have

$$\frac{n}{2} \cdot \langle 1, -1\rangle + \Big\langle 2, 2 \cdot \prod_i a_i \Big\rangle \in \deg_{GW}(\widetilde{\mathrm{CH}}_0(Q^\infty)) \quad \Longleftrightarrow \quad \frac{n}{2} \cdot \Big\langle 1, -1\rangle + \langle 2, 2 \cdot \prod_i a_i \Big\rangle \in \mathrm{GI}_{Q^\infty}^{\mathrm{tr}}.$$

Note one trivially has $1 \in [D(q)^2]$. Thus, Lemma 3.10 yields $\langle 1, -1\rangle \in \mathrm{GI}_{Q^\infty}^{\mathrm{tr}}$ and implies in addition that

$$\frac{n}{2} \cdot \langle 1, -1\rangle + \Big\langle 2, 2 \cdot \prod_i a_i \Big\rangle \in \mathrm{GI}_{Q^\infty}^{\mathrm{tr}} \quad \Longleftrightarrow \quad -\prod_{i=1}^{n+1} a_i \in [D(q)^2].$$

Applying Theorem 2.9(2) proves (2).

For (3), note that $Q^o(k) \neq \varnothing$ if and only if $1 \in D(q)$, which implies that $[D(q)] = [D(q)^2]$. Since each a_i is in $D(q)$, we see that $-\prod_{i=1}^{n+1} a_i \in [D(q)^2]$ if and only if $-1 \in [D(q)]$. Thus, $-1 \in [D(q)]$ if and only if $\frac{n}{2}\langle 1, -1\rangle + \langle 2, 2 \cdot \prod_i a_i\rangle$ is in $\deg_{GW}(\widetilde{\mathrm{CH}}_0(Q^\infty))$, and then Theorem 2.9 (3) implies the claim. □

Definition 4.2. Let F be a field. The *level* of F, denoted by $s(F)$, is the minimal integer n such that $-1 \in D(x_1^2 + x_2^2 + \cdots + x_n^2)$. If no such n exists then $s(F) = \infty$. The level of a field is either infinite or a power of 2 [Lam 2005, Pfister's Level Theorem XI.2.2].

Corollary 4.3. *Let S_k^n, $n \geq 1$, be the affine quadric over a field k given by the equation*

$$x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1.$$

Then the tangent bundle $T_{S_k^n}$ has a nonvanishing section if and only if one of the following holds:

(1) *n is odd.*

(2) *$n > 0$ is even and $s(k) \leq 2n + 1$.*

Proof. First assume that $\operatorname{char} k > 2$. Then Remark 1.10 yields that the tangent bundle $T_{S_k^n}$ has a nonvanishing section for every $n \geq 1$. At the same time -1 is a square or a sum of two squares in a finite field, whence in k, thus $s(k)$ is 1 or 2 and, in particular, $s(k) \leq 2n + 1$ for $n \geq 1$. This yields the claim in the positive characteristic.

Assume $\operatorname{char} k = 0$, in particular, k is perfect. By Theorem 4.1 we need to show that for even n one has

$$-1 \in [D(x_1^2 + x_2^2 + \cdots + x_{n+1}^2)]$$

if and only if $s(k) \leq 2n + 1$. Let m be such that $2^{m-1} < n + 1 \leq 2^m$. Then $x_1^2 + x_2^2 + \cdots + x_{n+1}^2$ is a Pfister neighbor with the associated Pfister form $x_1^2 + x_2^2 + \cdots + x_{2^m}^2$. Lemma 3.15 yields

$$[D(x_1^2 + x_2^2 + \cdots + x_{n+1}^2)] = D(x_1^2 + x_2^2 + \cdots + x_{2^m}^2).$$

Thus $-1 \in [D(x_1^2 + x_2^2 + \cdots + x_{n+1}^2)]$ if and only if $s(k) \leq 2^m$. The claim follows since $s(k)$ is a power of 2. □

Example 4.4. Let S_k^2 be the quadric over a field k given by the equation $x^2 + y^2 + z^2 = 1$. If $k = \mathbb{R}$ then $T_{S_k^2}$ has no nonvanishing sections since by a classical result of Poincaré the real vector bundle $T_{S_k^2}(\mathbb{R})$ has no nonvanishing continuous sections in the Euclidean topology [tom Dieck 2008, Theorem 6.5.5]. More generally, Corollary 4.3 yields that $T_{S_k^2}$ has no nonvanishing sections if and only if $s(k) \geq 8$ (including the case of $s(k) = \infty$). In particular, $T_{S_k^2}$ has a nonvanishing section for the following fields (cf. Example 2.11):

(1) k a quadratically closed field,

(2) k a field of characteristic $p > 2$,

(3) k a non-Archimedean local field,

(4) k a purely imaginary number field.

See [Lam 2005, Example XI.2.4] for the relevant computations of $s(k)$.

Corollary 4.5. *Let k be a perfect field of characteristic $\neq 2$ such that every Pfister form of dimension 8 is hyperbolic, i.e., that $I(k)^3 = 0$,[1] and let Q^o be the affine quadric over k given by the equation*

$$a_1x_1^2 + a_2x_2^2 + \cdots + a_{n+1}x_{n+1}^2 = 1,$$

with $a_i \in k^\times$. Suppose that n is odd, or that $n > 0$ is even and $Q^o(k) \neq \varnothing$. Then the tangent bundle T_{Q^o} has a nonvanishing section.

Proof. By Theorem 4.1 it is sufficient to show that for an even $n > 0$ one has $-1 \in [D(\langle a_1, a_2, \ldots, a_{n+1}\rangle)]$. We claim that already

$$-1 \in [D(\langle a_1, a_2, a_3\rangle)^2] \subseteq [D(\langle a_1, a_2, \ldots, a_{n+1}\rangle)^2] = [D(\langle a_1, a_2, \ldots, a_{n+1}\rangle)].$$

Indeed, note that the quadratic form $\langle a_1, a_2, a_3\rangle$ is a Pfister neighbor since

$$\langle a_1\rangle \cdot \langle a_1, a_2, a_3\rangle = \langle 1, a_1a_2, a_1a_3\rangle$$

is a subform of the 2-fold Pfister form $\langle\langle -a_1a_2, -a_1a_3\rangle\rangle = \langle 1, a_1a_2, a_1a_3, a_2a_3\rangle$. Then Lemma 3.15 yields

$$[D(\langle a_1, a_2, a_3\rangle)^2] = D(\langle 1, a_1a_2, a_1a_3, a_2a_3\rangle). \tag{4.6}$$

Now, the form $\langle 1, a_1a_2, a_1a_3, a_2a_3, 1\rangle$ is again a Pfister neighbor with the associated 3-fold Pfister form $\langle\langle -a_1a_2, -a_1a_3, -1\rangle\rangle$. The latter form is hyperbolic by the assumption, whence its subform $\langle 1, a_1a_2, a_1a_3, a_2a_3, 1\rangle$ is isotropic by the same dimension count argument as in the proof of Lemma 3.15. It follows that the equation

$$x_1^2 + a_1a_2x_2^2 + a_1a_3x_3^2 + a_2a_3x_4^2 + x_5^2 = 0$$

has a solution over k. This means that

$$-1 \in D(\langle 1, a_1a_2, a_1a_3, a_2a_3\rangle),$$

yielding by (4.6) that $-1 \in [D(\langle a_1, a_2, a_3\rangle)^2]$ and thereby the claim. □

Corollary 4.7. *Let k be a number field and let Q^o be the affine quadric over k given by the equation*

$$a_1x_1^2 + a_2x_2^2 + \cdots + a_{n+1}x_{n+1}^2 = 1,$$

with $a_i \in k^\times$. Suppose $Q^o(k) \neq \varnothing$.

(1) *If n is odd, then T_{Q^o} has a nonvanishing section.*

(2) *If $n > 0$ is even, then T_{Q^o} has a nonvanishing section if and only if, for each real embedding $\sigma : k \to \mathbb{R}$, $\sigma(a_i) < 0$ for some i.*

[1] It follows from the Milnor conjecture [Voevodsky 2003, Corollary 7.5; Orlov et al. 2007, Theorem 4.1; Röndigs and Østvær 2016, Theorem 1.1] that $I(k)^3 = 0$ is equivalent to k being of 2-cohomological dimension at most 2.

Proof. The case of odd n follows from Theorem 4.1 (1).

For even $n > 0$, put $q = a_1x_1^2 + a_2x_2^2 + \cdots + a_{n+1}x_{n+1}^2$. Let $\sigma : k \to \mathbb{R}$ be an embedding such that $\sigma(a_i) > 0$ for all i and let $q_{\mathbb{R}}$ be q extended to $\mathbb{R}$ using this embedding. Then $[D(q_{\mathbb{R}})] \subseteq \mathbb{R}_{>0}$, whence $-1 \notin [D(q)]$. Thus Theorem 4.1 (3) yields one direction of the desired implication.

For the other direction it suffices to show that if $n > 0$ is even and for every embedding $\sigma : k \to \mathbb{R}$ one has $\sigma(a_i) < 0$ for some i then $-1 \in [D(q)]$. Note that the assumption that $Q^o(k) \neq \varnothing$ implies that, for each real embedding σ of k, there is a j with $\sigma(a_j) > 0$.

First assume $n \geq 4$. Let v be a place of k and consider the quadratic form $q + x_{n+2}^2 = a_1x_1^2 + a_2x_2^2 + \cdots + a_{n+1}x_{n+1}^2 + x_{n+2}^2$ over k_v. If v is a finite place, then $q + x_{n+2}^2$ is isotropic since every quadratic form of dimension ≥ 5 is isotropic over a local field [Lam 2005, Theorem VI.2.12]. If v is an infinite place then the assumption $\sigma(a_i) < 0$ for some i implies that $q + x_{n+2}^2$ is isotropic. Then [Lam 2005, Hasse–Minkowski Principle VI.3.1] implies that $q + x_{n+2}^2$ is isotropic over k, whence $-1 \in D(q)$.

Now assume $n = 2$. The form $\langle a_1, a_2, a_3\rangle = a_1x_1^2 + a_2x^2 + a_3x^2$ is a Pfister neighbor with the associated Pfister form $\langle 1, a_1a_2, a_1a_3, a_2a_3\rangle$ and Lemma 3.15 yields

$$[D(\langle a_1, a_2, a_3\rangle)^2] = D(\langle 1, a_1a_2, a_1a_3, a_2a_3\rangle). \tag{4.8}$$

Let v be a place of k and consider the form $q' = \langle 1, a_1a_2, a_1a_3, a_2a_3, 1\rangle$ over k_v. As above, if v is a finite place then the form q' is isotropic by [Lam 2005, Theorem VI.2.12]. If v is a complex place then q' is clearly isotropic. If v is a real place with the real embedding $\sigma_v : k \to \mathbb{R}$ then as $Q^o(k) \neq \varnothing$, we have $Q^o(k_v) \neq \varnothing$; hence there is a j such that $\sigma(a_j) > 0$. Combined with our assumption that $\sigma_v(a_i) < 0$ for some i, we see that at least one of $\sigma_v(a_1a_2)$, $\sigma_v(a_1a_3)$ and $\sigma_v(a_2a_3)$ is negative, whence q' is isotropic over k_v. Then [Lam 2005, Hasse–Minkowski Principle VI.3.1] implies that q' is isotropic over k meaning that $-1 \in D(\langle 1, a_1a_2, a_1a_3, a_2a_3\rangle)$. By (4.8), we thus have $-1 \in [D(\langle a_1, a_2, a_3\rangle)^2]$ and the claim follows. $\square$

Corollary 4.9. *Let k be a field of one of the following types*:

(1) *a finite field $\mathbb{F}_{p^n}$, $p > 2$,*

(2) *a non-Archimedean local field of characteristic zero,*

(3) *the perfection of a local field of characteristic $p > 2$,*

(4) *the perfection of the function field of a curve over a finite field.*

Let Q^o be the affine quadric over k given by the equation

$$q := a_1x_1^2 + a_2x_2^2 + \cdots + a_{n+1}x_{n+1}^2 = 1,$$

with $a_i \in k^\times$. Suppose that $n > 0$, and if $n = 2$ and k is of type (2, 3, 4), *suppose in addition that Q^o has a k-rational point. Then T_{Q^o} has a nonvanishing section.*

Proof. In all the above cases, k is a perfect field of cohomological dimension ≤ 2, and the result follows from Corollary 4.5, once we know that $Q^o(k) \neq \varnothing$ if $n \geq 2$ is even. For k of type (1) every regular quadratic form in at least three variables is isotropic [Lam 2005, Proposition I.3.4] and for k of type (2, 3, 4), every regular quadratic form in at least five variables is isotropic [Lam 2005, Theorem VI.2.12, Corollary VI.3.5]; applying this to $q - x_0^2$ shows that $Q^o(k) \neq \varnothing$ in all cases to be considered. □

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References

[Asok and Fasel 2015] A. Asok and J. Fasel, "Splitting vector bundles outside the stable range and $\mathbb{A}^1$-homotopy sheaves of punctured affine spaces", *J. Amer. Math. Soc.* **28**:4 (2015), 1031–1062. MR

[Asok and Fasel 2016] A. Asok and J. Fasel, "Comparing Euler classes", *Q. J. Math.* **67**:4 (2016), 603–635. MR

[Asok and Fasel 2023] A. Asok and J. Fasel, "Vector bundles on algebraic varieties", pp. 2146–2170 in *International Congress of Mathematicians*, *III* (Helsinki, 2022), edited by D. Beliaev and S. Smirnov, Eur. Math. Soc., Berlin, 2023. MR

[Asok et al. 2017] A. Asok, M. Hoyois, and M. Wendt, "Affine representability results in $\mathbb{A}^1$-homotopy theory, I: Vector bundles", *Duke Math. J.* **166**:10 (2017), 1923–1953. MR

[Barge and Morel 2000] J. Barge and F. Morel, "Groupe de Chow des cycles orientés et classe d'Euler des fibrés vectoriels", *C. R. Acad. Sci. Paris Sér. I Math.* **330**:4 (2000), 287–290. MR

[Bhatwadekar et al. 2014] S. M. Bhatwadekar, J. Fasel, and S. Sane, "Euler class groups and 2-torsion elements", *J. Pure Appl. Algebra* **218**:1 (2014), 112–120. MR

[Colliot-Thélène and Skorobogatov 1993] J.-L. Colliot-Thélène and A. N. Skorobogatov, "Groupe de Chow des zéro-cycles sur les fibrés en quadriques", *K-Theory* **7**:5 (1993), 477–500. MR

[Déglise 2023] F. Déglise, "Notes on Milnor–Witt K-theory", preprint, 2023. arXiv 2305.18609

[Déglise et al. 2021] F. Déglise, F. Jin, and A. A. Khan, "Fundamental classes in motivic homotopy theory", *J. Eur. Math. Soc.* **23**:12 (2021), 3935–3993. MR

[tom Dieck 2008] T. tom Dieck, *Algebraic topology*, Eur. Math. Soc., Zürich, 2008. MR

[Fasel 2013] J. Fasel, "The projective bundle theorem for $\mathbf{I}^j$-cohomology", *J. K-Theory* **11**:2 (2013), 413–464. MR

[Fasel 2020] J. Fasel, "Lectures on Chow–Witt groups", pp. 83–121 in *Motivic homotopy theory and refined enumerative geometry* (Essen, Germany, 2018), edited by F. Binda et al., Contemp. Math. **745**, Amer. Math. Soc., Providence, RI, 2020. MR

[Feld 2022] N. Feld, "Birational invariance of the Chow–Witt group of zero-cycles", preprint, 2022. arXiv 2210.03995

[Hoyois 2014] M. Hoyois, "A quadratic refinement of the Grothendieck–Lefschetz–Verdier trace formula", *Algebr. Geom. Topol.* **14**:6 (2014), 3603–3658. MR

[Lam 2005] T. Y. Lam, *Introduction to quadratic forms over fields*, Grad. Stud. in Math. **67**, Amer. Math. Soc., Providence, RI, 2005. MR

[Levine 2020] M. Levine, "Aspects of enumerative geometry with quadratic forms", *Doc. Math.* **25** (2020), 2179–2239. MR

[Levine and Raksit 2020] M. Levine and A. Raksit, "Motivic Gauss–Bonnet formulas", *Algebra Number Theory* **14**:7 (2020), 1801–1851. MR

[Morel 2004] F. Morel, "An introduction to $\mathbb{A}^1$-homotopy theory", pp. 357–441 in *Contemporary developments in algebraic K-theory* (Trieste, Italy, 2002), edited by M. Karoubi et al., ICTP Lect. Notes **15**, Abdus Salam Int. Cent. Theoret. Phys., Trieste, Italy, 2004. MR

[Morel 2012] F. Morel, *$\mathbb{A}^1$-algebraic topology over a field*, Lecture Notes in Math. **2052**, Springer, 2012. MR

[Ojanguren et al. 1986] M. Ojanguren, R. Parimala, and R. Sridharan, "Symplectic bundles over affine surfaces", *Comment. Math. Helv.* **61**:3 (1986), 491–500. MR

[Orlov et al. 2007] D. Orlov, A. Vishik, and V. Voevodsky, "An exact sequence for $K_*^M/2$ with applications to quadratic forms", *Ann. of Math.* (2) **165**:1 (2007), 1–13. MR

[Röndigs and Østvær 2016] O. Röndigs and P. A. Østvær, "Slices of Hermitian K-theory and Milnor's conjecture on quadratic forms", *Geom. Topol.* **20**:2 (2016), 1157–1212. MR

[Scharlau 1969] W. Scharlau, "Zur Pfisterschen Theorie der quadratischen Formen", *Invent. Math.* **6** (1969), 327–328. MR

[Scharlau 1985] W. Scharlau, *Quadratic and Hermitian forms*, Grundl. Math. Wissen. **270**, Springer, 1985. MR

[Serre 1962] J.-P. Serre, *Corps locaux*, Publ. Inst. Math. Univ. Nancago **8**, Hermann, Paris, 1962. MR

[Voevodsky 2003] V. Voevodsky, "Motivic cohomology with $\mathbb{Z}/2$-coefficients", *Publ. Math. Inst. Hautes Études Sci.* **98** (2003), 59–104. MR

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