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On two definitions of wave-front sets for p-adic groups

Cheng-Chiang Tsai

The wave-front set for an irreducible admissible representation of a p-adic reductive group is the set of maximal nilpotent orbits which appear in the local character expansion. By a result of Mœglin and Waldspurger, they are also the maximal nilpotent orbits whose associated degenerate Whittaker models are nonzero. However, in the literature there are two versions commonly used, one defining maximality using analytic closure and the other using Zariski closure. We show that these two definitions are inequivalent for $G = \mathrm{Sp}_4$.

1. Introduction

Let F be a finite extension of $\mathbb{Q}_p$ and G be a connected reductive group over F. Write $\mathfrak{g} := \mathrm{Lie}\, G$. The local character expansion of Howe and of Harish-Chandra [1999, Theorem 16.2] asserts that, for any irreducible admissible $\mathbb{C}$-representation π of $G(F)$, there exist constants $c_{\mathcal{O}}(\pi) \in \mathbb{C}$ indexed by nilpotent $\mathrm{Ad}(G(F))$-orbits $\mathcal{O} \subset \mathfrak{g}(F)$, together with a neighborhood $U = U_\pi$ of $0 \in \mathfrak{g}(F)$, such that the character Θ_π of π satisfies the following identity of distributions on U:

$$(\Theta_\pi \circ \log^*)|_U \equiv \sum_{\mathcal{O}} c_{\mathcal{O}}(\pi) \hat{I}_{\mathcal{O}}|_U. \tag{1}$$

Here $I_{\mathcal{O}}$ is the distribution of integrating a function on $\mathcal{O}$ with any $G(F)$-invariant positive measure, and $\hat{I}_{\mathcal{O}}$ its Fourier transform, namely $\hat{I}_{\mathcal{O}}(f) := I_{\mathcal{O}}(\hat{f})$.

Mœglin and Waldspurger [1987] generalized a result of Rodier [1975] and showed that, for $\mathcal{O} \in \max\{\mathcal{O} : c_{\mathcal{O}}(\pi) \neq 0\}$, the quantity $c_{\mathcal{O}}(\pi)$ with suitable normalization is equal to the dimension of the degenerate Whittaker model for π relative to $\mathcal{O}$. Degenerate Whittaker models are local analogues and necessary conditions for existence of Fourier coefficients for automorphic forms. The set $\max\{\mathcal{O} : c_{\mathcal{O}}(\pi) \neq 0\}$ is therefore of particular interest, and is typically called the *wave-front set*. However, there are two partial orders commonly used in the literature: for two nilpotent $\mathrm{Ad}(G(F))$-orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ the partial order $\mathcal{O}_1 < \mathcal{O}_2$ is defined either (i) if the analytic closure (using the Hausdorff p-adic topology on $\mathfrak{g}(F)$) of $\mathcal{O}_1$ is strictly contained in the analytic closure of $\mathcal{O}_2$, or alternatively (ii) if the Zariski closure of $\mathcal{O}_1$ is strictly contained in the Zariski closure of $\mathcal{O}_2$.

Let us denote by $\mathrm{WF}^{\mathrm{rat}}(\pi) := \max\{\mathcal{O} : c_{\mathcal{O}}(\pi) \neq 0\}$ the set given by the first definition, and by $\mathrm{WF}^{\mathrm{Zar}}(\pi)$ the analogous set given by the second definition. Since the Zariski closure is larger than the analytic closure, we have an obvious inclusion $\mathrm{WF}^{\mathrm{rat}}(\pi) \supseteq \mathrm{WF}^{\mathrm{Zar}}(\pi)$. At the same time, there is the notion

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of geometric wave-front sets: Fix an algebraic closure $\bar{F}$ of F and let $\overline{\mathrm{WF}}^{\mathrm{rat}}(\pi)$ (resp. $\overline{\mathrm{WF}}^{\mathrm{Zar}}(\pi)$) be the set of $\mathrm{Ad}(G(\bar{F}))$-orbits in $\mathfrak{g}(\bar{F})$ that meet those in $\mathrm{WF}^{\mathrm{rat}}(\pi)$ (resp. $\mathrm{WF}^{\mathrm{Zar}}(\pi)$). Again we have $\overline{\mathrm{WF}}^{\mathrm{rat}}(\pi) \supseteq \overline{\mathrm{WF}}^{\mathrm{Zar}}(\pi)$. We note that by [Poonen 2017, Proposition 3.5.75], any $G(F)$-orbit $\mathcal{O} \subset \mathfrak{g}(F)$ is Zariski dense in the $G(\bar{F})$-orbit it sits in. Hence $\overline{\mathrm{WF}}^{\mathrm{Zar}}(\pi)$ is equal to the set of maximal geometric orbits that appear in (1). We thank Emile Okada for clarifying this.

The set $\mathrm{WF}^{\mathrm{rat}}(\pi)$ was used in [Mœglin and Waldspurger 1987; Mœglin 1996; Gomez et al. 2021] and many others. On the other hand, $\overline{\mathrm{WF}}^{\mathrm{Zar}}(\pi)$ was used in, for example, [Waldspurger 2018]. Both $\mathrm{WF}^{\mathrm{rat}}(\pi)$ and $\overline{\mathrm{WF}}^{\mathrm{Zar}}(\pi)$ were discussed in [Ciubotaru et al. 2025], while their main results determine $\overline{\mathrm{WF}}^{\mathrm{Zar}}(\pi)$ but not $\overline{\mathrm{WF}}^{\mathrm{rat}}(\pi)$. Nevertheless, in [Jiang et al. 2022] the main conjecture, Conjecture 1.3, is stated for $\mathrm{WF}^{\mathrm{Zar}}(\pi)$ but it seems that the spirit might work for $\mathrm{WF}^{\mathrm{rat}}(\pi)$ as well. Given the abundance of results on the topic, it is desirable to know how/whether $\mathrm{WF}^{\mathrm{rat}}(\pi)$ and $\mathrm{WF}^{\mathrm{Zar}}(\pi)$ (resp. $\overline{\mathrm{WF}}^{\mathrm{rat}}(\pi)$ and $\overline{\mathrm{WF}}^{\mathrm{Zar}}(\pi)$) could be different. In fact, the longstanding conjecture about geometric wave-front sets, proposed and proved for GL_n in [Mœglin and Waldspurger 1987], asserted that:

Conjecture 1.1. *For any irreducible admissible representation π of $G(F)$, the set $\overline{\mathrm{WF}}^{\mathrm{rat}}(\pi)$ is a singleton.*

Since $\overline{\mathrm{WF}}^{\mathrm{Zar}}(\pi)$ is obviously nonempty, the validity of Conjecture 1.1 for any π is equivalent to the validities of the following two statements:

Conjecture 1.2 (counterexample in [Tsai 2024, Theorem 1.1]). *$\overline{\mathrm{WF}}^{\mathrm{Zar}}(\pi)$ is a singleton.*

Conjecture 1.3. *We have $\overline{\mathrm{WF}}^{\mathrm{rat}}(\pi) = \overline{\mathrm{WF}}^{\mathrm{Zar}}(\pi)$ or equivalently $\mathrm{WF}^{\mathrm{rat}}(\pi) = \mathrm{WF}^{\mathrm{Zar}}(\pi)$.*

As indicated above, the first counterexample for Conjecture 1.1 is a counterexample to Conjecture 1.2. The purpose of this paper is to show that Conjecture 1.3 also has a counterexample, in fact, in the case of split rank 2, which is the smallest absolute rank where Conjecture 1.3 becomes nontrivial.

Let $p \geq 11$ be any prime number, q a power of p with $q \equiv 1 \pmod 4$, and F any nonarchimedean local field with residue field $\mathbb{F}_q$ and fixed uniformizer $\varpi \in F$. Let $G = \mathrm{Sp}_4/_F$ be the group of linear operators on F^4 that preserve the symplectic form

$$\langle \vec{x}, \vec{y} \rangle = x_1 y_4 + x_2 y_3 - x_3 y_2 - x_4 y_1. \tag{2}$$

Denote by $\mathfrak{m}$ the maximal ideal and $\mathfrak{m}^0 = \mathcal{O}_F$ the ring of integers in F. Consider the Moy–Prasad filtration $(G(F)_r)_{r \in (1/2)\mathbb{Z}_{\geq 0}}$ "associated to the Siegel parahoric." It is given by

$$G(F)_n := \left\{ g \in G(F) : g - \mathrm{Id}_4 \in \begin{bmatrix} \mathfrak{m}^n & \mathfrak{m}^n & \mathfrak{m}^n & \mathfrak{m}^n \\ \mathfrak{m}^n & \mathfrak{m}^n & \mathfrak{m}^n & \mathfrak{m}^n \\ \mathfrak{m}^{n+1} & \mathfrak{m}^{n+1} & \mathfrak{m}^n & \mathfrak{m}^n \\ \mathfrak{m}^{n+1} & \mathfrak{m}^{n+1} & \mathfrak{m}^n & \mathfrak{m}^n \end{bmatrix} \right\},$$

$$G(F)_{n+1/2} := \left\{ g \in G(F) : g - \mathrm{Id}_4 \in \begin{bmatrix} \mathfrak{m}^{n+1} & \mathfrak{m}^{n+1} & \mathfrak{m}^n & \mathfrak{m}^n \\ \mathfrak{m}^{n+1} & \mathfrak{m}^{n+1} & \mathfrak{m}^n & \mathfrak{m}^n \\ \mathfrak{m}^{n+1} & \mathfrak{m}^{n+1} & \mathfrak{m}^{n+1} & \mathfrak{m}^{n+1} \\ \mathfrak{m}^{n+1} & \mathfrak{m}^{n+1} & \mathfrak{m}^{n+1} & \mathfrak{m}^{n+1} \end{bmatrix} \right\} \tag{3}$$

for $n \in \mathbb{Z}_{\geq 0}$. The group $G(F)_1$ is a normal subgroup of $G(F)_{1/2}$, and the quotient may be identified as

$$V := G(F)_{1/2}/G(F)_1 \cong \left\{ \begin{bmatrix} 0 & 0 & b & a \\ 0 & 0 & c & b \\ e & d & 0 & 0 \\ f & e & 0 & 0 \end{bmatrix} : a, b, c \in \mathcal{O}_F/\mathfrak{m},\ d, e, f \in \mathfrak{m}/\mathfrak{m}^2 \right\}.$$

Fix an additive character $\psi : F \to \mathbb{C}^\times$ that is trivial on $\mathfrak{m}$ but nontrivial on $\mathcal{O}_F$. Consider

$$A := \begin{bmatrix} 0 & 0 & 0 & \varpi^{-1} \\ 0 & 0 & \varpi^{-1} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \in \mathfrak{g}(F). \tag{4}$$

Denote by $\psi_A : V \to \mathbb{C}^\times$ the character $B \mapsto \psi(\mathrm{Tr}(AB))$, and by $\tilde{\psi}_A$ its pullback to $G(F)_{1/2}$. Conjecture 1.3 is disproved by:

Theorem 1.4. *For any irreducible component π of the compact induction*

$$\text{c-ind}_{G(F)_{1/2}}^{G(F)} \tilde{\psi}_A,$$

we have that $\mathrm{WF}^{\mathrm{rat}}(\pi)$ contains two regular nilpotent orbits and also a subregular nilpotent orbit. Consequently $\mathrm{WF}^{\mathrm{Zar}}(\pi)$ contains only the two regular nilpotent orbits.

In fact, the subregular orbit is the unique one not contained in the analytic closure of the previous two regular nilpotent orbits. The representation π is one of the so-called epipelagic representations in [Reeder and Yu 2014]. Prior to this work, similar representations for much higher-rank groups had already been studied in a joint work in progress of Chi-Heng Lo and the author to produce a counterexample to Conjecture 1.2 for split groups (rather than for ramified groups as in [Tsai 2024]). We also remark that in the language of the newer paper [Tsai 2023], we have $\mathrm{WF}^{\mathrm{rat}}(\pi) = \mathrm{WF}^{\mathrm{rat}}(A)$ and the result may well be interpreted as for the wave-front set of $A \in \mathfrak{g}(F)$.

2. Nilpotent orbits

For our $G = \mathrm{Sp}_4$, the subregular nilpotent $\mathrm{Ad}(G(F))$-orbits correspond to partition $[2^2]$ and any such orbit has a representative of the form

$$e_{a,b,c} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & b & 0 & 0 \end{bmatrix}, \quad a, b, c \in F.$$

Denote by v_i $(1 \leq i \leq 4)$ the i-th coordinate vector of our 4-dimensional symplectic space. The operator $e_{a,b,c}$ defines a nondegenerate quadratic form on $\mathrm{span}(v_1, v_2)$ by

$$(X, Y)_{a,b,c} := \langle X, e_{a,b,c} Y \rangle, \tag{5}$$

where $\langle\cdot,\cdot\rangle$ is as in (2). The $\mathrm{Ad}(G(F))$-orbit of $e_{a,b,c}$ is uniquely determined [Nevins 2011, Proposition 5] by the isomorphism class of the quadratic form $(\cdot,\cdot)_{a,b,c}$. Similarly, a regular nilpotent $\mathrm{Ad}(G(F))$-orbit has a representative of the form

$$n_d=\begin{bmatrix}0&0&0&0\\1&0&0&0\\0&d&0&0\\0&0&-1&0\end{bmatrix},\quad d\in F^\times. \tag{6}$$

The orbit is again uniquely determined by the image of d in $F^\times/(F^\times)^2$. We show that:

Lemma 2.1. *The element $e_{a,b,c}$ lies in the analytic closure of $\mathrm{Ad}(G(F))n_d$ if and only if the quadratic form $(\cdot,\cdot)_{(a,b,c)}$ represents d, namely $(v,v)_{a,b,c}=d$ for some $v\in\mathrm{span}(v_1,v_2)$.*

Proof. Suppose $(v,v)_{a,b,c}=d$ for some $v\in\mathrm{span}(v_1,v_2)$. Then with a change of basis we may assume $(v_2,v_2)_{a,b,c}=d$, i.e., $a=d$. We have (with all hidden entries being 0's) for $e,f\in F$, $h\in F^\times$ that

$$\begin{bmatrix}1&&&\\-e&1&&\\f&&1&\\&f&e&1\end{bmatrix}\begin{bmatrix}0&&&\\1&0&&\\&d&0&\\&&1&0\end{bmatrix}\begin{bmatrix}1&&&\\e&1&&\\-f&&1&\\&-f&-e&1\end{bmatrix}=\begin{bmatrix}0&&&\\1&0&&\\de&d&0&\\2f+de^2&de&1&0\end{bmatrix},$$

$$\begin{bmatrix}h^{-1}&&&\\&1&&\\&&1&\\&&&h\end{bmatrix}\begin{bmatrix}0&&&\\1&0&&\\de&d&0&\\2f+de^2&de&1&0\end{bmatrix}\begin{bmatrix}h&&&\\&1&&\\&&1&\\&&&h^{-1}\end{bmatrix}=\begin{bmatrix}0&&&\\h&0&&\\deh&d&0&\\2fh^2+de^2h^2&deh&h&0\end{bmatrix}.$$

For arbitrarily small h we can choose $e,f\in F$ so that $deh=b$, $2fh^2+de^2h^2=c$. Hence the above converges to $e_{a,b,c}$ as desired.

Now suppose $e_{a,b,c}$ is in the analytic closure of $\mathrm{Ad}(G(F))n_d$, i.e., there is a sequence $g_i\in G(F)$ such that $\mathrm{Ad}(g_i)^{-1}n_d$ converges to $e_{a,b,c}$. To show that $(\cdot,\cdot)_{a,b,c}$ represents d we follow the method of [Djoković 1981, Theorem 6] for real groups. The quadratic form

$$(X,Y)_{d,g_i}:=\langle X,\mathrm{Ad}(g_i)^{-1}(n_d)Y\rangle=\langle g_iX,n_dg_iY\rangle$$

has to converge to $(X,Y)_{a,b,c}$ on $\mathrm{span}(v_1,v_2)$. Since being isomorphic to a nondegenerate quadratic form over F is an open condition in the space of (not necessarily nondegenerate) quadratic forms, for $i\gg 0$ we have that $(\cdot,\cdot)_{d,g_i}|_{\mathrm{span}(v_1,v_2)}\cong(\cdot,\cdot)_{a,b,c}$. In particular, there exists a 2-dimensional subspace W in F^4 such that the restriction of the form $(X,Y)_d:=\langle X,n_dY\rangle$ is isomorphic to $(X,Y)_{a,b,c}$.

Observe the form $(X,Y)_d$ restricts to a rank-2 hyperbolic form on $\mathrm{span}(v_1,v_3)$. The orthogonal complement of $\mathrm{span}(v_1,v_3)$ under it is $\mathrm{span}(v_2)\oplus\mathrm{span}(v_4)$, where the form has discriminant d on $\mathrm{span}(v_2)$ and has $\mathrm{span}(v_4)$ in its kernel. The subspace W must not intersect $\mathrm{span}(v_4)$; hence its image to $F^4/\mathrm{span}(v_4)\cong\mathrm{span}(v_1,v_2,v_3)$ is again 2-dimensional. Denote by $W^\perp$ the orthogonal complement of W in $\mathrm{span}(v_1,v_2,v_3)$. Since $(\cdot,\cdot)_d|_W\cong(\cdot,\cdot)_{a,b,c}$, we have that

$$(\cdot,\cdot)_d|_{\mathrm{span}(v_1,v_3)}\oplus(\cdot,\cdot)_d|_{\mathrm{span}(v_2)}\cong(\cdot,\cdot)_d|_{\mathrm{span}(v_1,v_2,v_3)}\cong(\cdot,\cdot)_{a,b,c}\oplus(\cdot,\cdot)_d|_{W^\perp}$$

are isomorphic as quadratic spaces. Since $(\cdot,\cdot)_d|_{\mathrm{span}(v_1,v_3)}$ is hyperbolic, it is isomorphic to the direct sum of $(\cdot,\cdot)_d|_{W^\perp}$ and some other 1-dimensional quadratic space. By the cancellation theorem of quadratic spaces [Serre 1973, p. 34, Theorem 4], we then have $(\cdot,\cdot)_{a,b,c}$ is isomorphic to the direct sum of $(\cdot,\cdot)_d|_{\mathrm{span}(v_2)}$ and this 1-dimensional space, i.e., $(\cdot,\cdot)_{a,b,c}$, represents d, as asserted. □

3. Shalika germs and the proof of Theorem 1.4

We normalize our Fourier transforms as

$$\hat{f}(B) := \int_{\mathfrak{g}(F)} \psi(\mathrm{Tr}(AB)) f(A)\, dA,$$

where elements in $\mathfrak{g}(F) = \mathfrak{sp}_4(F)$ are identified as 4×4 matrices as usual, i.e., as in (2). It is known (see the main result of [Kim and Murnaghan 2003], or [Kaletha 2015, (6.1)] for a more direct exhibition) that, for any π in Theorem 1.4, on some sufficiently small neighborhood U of $0 \in \mathfrak{g}(F)$ we have

$$(\Theta_\pi \circ \log^*)|_U \equiv c \cdot \hat{I}_A|_U \tag{7}$$

for some $c \in \mathbb{Q}_{>0}$.

Since $p \geq 11$, the hypotheses needed for [DeBacker 2002, Theorem 2.1.5] are satisfied and it gives the following analogue of (1), the *Shalika germ expansion*:

$$I_A(f) = \sum c_{\mathcal{O}}(A) I_{\mathcal{O}}(f). \tag{8}$$

Here $\mathcal{O}$ runs over nilpotent $\mathrm{Ad}(G(F))$-orbits in $\mathfrak{g}(F)$ as in (1), and f has to be a function of depth $-\frac{1}{2}$; a condition that will be automatically met if $\hat{f}$ is supported in a small enough neighborhood. Comparing (1), (7) and (8), we see that the coefficients in (1) satisfy $c_{\mathcal{O}}(\pi) = c \cdot c_{\mathcal{O}}(A)$. In particular, $c_{\mathcal{O}}(\pi) \neq 0$ if and only if $c_{\mathcal{O}}(A) \neq 0$, and we have $\mathrm{WF}^{\mathrm{rat}}(\pi) = \max\{\mathcal{O} : c_{\mathcal{O}}(A) \neq 0\}$, where the partial order is given by the (analytic) closure relation. Fix $\epsilon \in \mathcal{O}_F^\times$ any nonsquare and

$$e = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\varpi^{-1}\epsilon & 0 & 0 \\ \epsilon & 0 & 0 & 0 \end{bmatrix}. \tag{9}$$

Theorem 1.4 now follows from:

Proposition 3.1. *The Shalika germ $c_{\mathcal{O}}(A)$ is zero for a regular nilpotent orbit $\mathcal{O}$ if and only if the closure of $\mathcal{O}$ contains e.*

Proposition 3.2. *The Shalika germ $c_{\mathcal{O}}(A)$ is nonzero for the subregular nilpotent orbit $\mathcal{O} = \mathrm{Ad}(G(F))e$.*

Remark 3.3. It might look like there are smart choices behind e and A. In fact, a random choice of A has about $\frac{1}{2}$ probability to work; it secretly needs a certain invariant in $\mathcal{O}_F^\times$ to be a square. Once that is met, Proposition 3.2 will work for any such A and some e it picks out. Our choice merely gives a nicer matrix calculation. The assumption $q \equiv 1 \pmod 4$ is also taken to simplify the exposition and is not essentially needed.

The rest of the section is devoted to the proofs of Propositions 3.1 and 3.2.

Proof of Proposition 3.1. A result of Shelstad [1989], combined with another by Kottwitz [1999, Theorem 5.1] (we thank Alexander Bertoloni Meli for clarifying this), showed that, for a regular nilpotent orbit $\mathcal{O}$, $c_{\mathcal{O}}(A)=0$ if and only if $\mathrm{Ad}(G(F))A$ does not meet the Kostant section associated to any element in $\mathcal{O}$. The theory of the Kostant section also gives that, for any fixed regular $\mathcal{O}$, among the stable orbit of A there is exactly one rational orbit that meets the Kostant section. We have

$$\mathrm{Ad}\left(\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&1\end{bmatrix}\right)A=\begin{bmatrix}0&0&0&\varpi^{-1}\\1&0&0&0\\0&-\varpi^{-1}&0&0\\0&0&-1&0\end{bmatrix}$$

is in the Kostant section for $n_{-\varpi^{-1}}$. Since $q\equiv 1 \pmod 4$, we may fix $i:=\sqrt{-1}$ a square root of -1 in $\mathcal{O}_F$. We have

$$\mathrm{Ad}\left(\begin{bmatrix}1&i&0&0\\1&-i&0&0\\0&0&\frac{1}{2}i&\frac{1}{2}\\0&0&-\frac{1}{2}i&\frac{1}{2}\end{bmatrix}\right)A=\begin{bmatrix}0&0&2\varpi^{-1}&0\\0&0&0&2\varpi^{-1}\\0&\frac{1}{2}i&0&0\\-\frac{1}{2}i&0&0&0\end{bmatrix}.$$

Hence

$$\mathrm{Ad}\left(\begin{bmatrix}2\varpi^{-1}&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&\frac{1}{2}\varpi\end{bmatrix}\begin{bmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\-1&0&0&0\end{bmatrix}\cdot\begin{bmatrix}1&i&0&0\\1&-i&0&0\\0&0&\frac{1}{2}i&\frac{1}{2}\\0&0&-\frac{1}{2}i&\frac{1}{2}\end{bmatrix}\right)A=\begin{bmatrix}0&0&0&2\varpi^{-2}i\\1&0&0&0\\0&\frac{1}{2}i&0&0\\0&0&1&0\end{bmatrix}$$

is in the Kostant section for $n_{i/2}$. We note that both -1 and $\frac{1}{2}i$ are squares in $\mathcal{O}_F^{\times}$, and thus Lemma 2.1 shows that $n_{-\varpi^{-1}}$ and $n_{i/2}$ are exactly the two regular nilpotent orbits whose closure does not contain e. This shows that if $c_{\mathcal{O}}(A)=0$ for a regular nilpotent $\mathcal{O}$, then the closure of $\mathcal{O}$ must contain e. It remains to show that for any regular nilpotent orbit $\mathcal{O}$ different from that of $n_{-\varpi^{-1}}$ and $n_{i/2}$, we have $c_{\mathcal{O}}(A)=0$.

Let $\sqrt{\epsilon}$ be a square root of ϵ in an unramified quadratic extension of F. The element

$$d:=\begin{bmatrix}\sqrt{\epsilon}^{-1}&0&0&0\\0&\sqrt{\epsilon}^{-1}&0&0\\0&0&\sqrt{\epsilon}&0\\0&0&0&\sqrt{\epsilon}\end{bmatrix}$$

has image in $G_{ad}(F)=\mathrm{PSp}_4(F)$. Since the orbit of A meets the Kostant section for $n_{-\varpi^{-1}}$ and $n_{i/2}$, the orbit of $\mathrm{Ad}(d)A$ meets the Kostant section of $\mathrm{Ad}(d)n_{-\varpi^{-1}}$ and $\mathrm{Ad}(d)n_{i/2}$. As $\mathrm{Ad}(d)n_{-\varpi^{-1}}=n_{-\epsilon\varpi^{-1}}$ and $\mathrm{Ad}(d)n_{i/2}=n_{\epsilon i/2}$ are the other two regular nilpotent orbits, using results of Shelstad and Kottwitz and the classical result that a Kostant section meets an $\mathrm{Ad}(G(\bar{F}))$-orbit at one point, it remains to prove that $\mathrm{Ad}(d)A$ and A live in different $\mathrm{Ad}(G(F))$-orbits. The element d defines a class $\alpha_d\in Z^1(F,Z(G))$ and the assertion that $\mathrm{Ad}(d)A$ and A live in different $\mathrm{Ad}(G(F))$-orbits is equivalent to the fact that the image of α_d in $H^1(F,Z_G(A))$ is nontrivial. Observe that α_d is trivial on inertia and sends Frob to

$-1 \in \mu_2 = Z(G)$. Since $Z_G(A)$ is anisotropic over the maximal unramified extension of F, the image of α_d is nontrivial in $H^1(F, Z_G(A)) = H^1(\mathrm{Frob}, X_*(Z_G(A))_{I_F}) = H^1(\mathrm{Frob}, \mu_2^2)$, as claimed. □

Proof of Proposition 3.2. Consider the characteristic function of the set

$$\left\{ X \in \mathfrak{g}(F) : X \text{ is of the form } \begin{bmatrix} \mathfrak{m}^0 & \mathfrak{m}^0 & \mathfrak{m}^{-1} & \mathfrak{m}^{-1} \\ \mathfrak{m}^0 & \mathfrak{m}^0 & \varpi^{-1}\epsilon + \mathfrak{m}^0 & \mathfrak{m}^{-1} \\ \mathfrak{m}^0 & \mathfrak{m}^0 & \mathfrak{m}^0 & \mathfrak{m}^0 \\ \epsilon + \mathfrak{m} & \mathfrak{m}^0 & \mathfrak{m}^0 & \mathfrak{m}^0 \end{bmatrix} \right\}.$$

Call this function f. It has the property that $f(X+Y) = f(X)$ whenever Y is of the form

$$\begin{bmatrix} \mathfrak{m}^0 & \mathfrak{m}^0 & \mathfrak{m}^{-1} & \mathfrak{m}^{-1} \\ \mathfrak{m}^0 & \mathfrak{m}^0 & \mathfrak{m}^0 & \mathfrak{m}^{-1} \\ \mathfrak{m}^0 & \mathfrak{m}^0 & \mathfrak{m}^0 & \mathfrak{m}^0 \\ \mathfrak{m} & \mathfrak{m}^0 & \mathfrak{m}^0 & \mathfrak{m}^0 \end{bmatrix}.$$

The set of elements of the above form is a Moy–Prasad lattice of depth $-\frac{1}{2}$. Since A is of depth $-\frac{1}{2}$, [DeBacker 2002, Theorem 2.1.5] (or its application to Conjecture 2 of that work) shows that (8) holds for f. Let e be as in (9). We claim that:

Lemma 3.4. *Suppose $I_{\mathcal{O}}(f) \neq 0$ for a nilpotent $\mathrm{Ad}(G(F))$-orbit $\mathcal{O}$. Then e lies in the closure of $\mathcal{O}$.*

Lemma 3.5. $I_A(f) \neq 0$.

With both lemmas, (8) gives $\sum_{\mathcal{O}} c_{\mathcal{O}}(A) I_{\mathcal{O}}(f) = I_A(f) \neq 0$. By Proposition 3.1 and Lemma 3.4, the only nilpotent orbit $\mathcal{O}$ that can contribute to the sum is $\mathcal{O} = \mathrm{Ad}(G(F))e$, which proves Proposition 3.2. □

Proof of Lemma 3.4. We have,

$$\text{for } w = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{Ad}(w)\,\mathrm{supp}(f) = \begin{bmatrix} \mathfrak{m}^0 & \mathfrak{m}^0 & \mathfrak{m}^{-1} & \mathfrak{m}^{-1} \\ \mathfrak{m}^0 & \mathfrak{m}^0 & \mathfrak{m}^0 & \mathfrak{m}^{-1} \\ \mathfrak{m}^0 & -\varpi^{-1}\epsilon + \mathfrak{m}^0 & \mathfrak{m}^0 & \mathfrak{m}^0 \\ \epsilon + \mathfrak{m} & \mathfrak{m}^0 & \mathfrak{m}^0 & \mathfrak{m}^0 \end{bmatrix}.$$

For any $X \in \mathcal{O} \cap \mathrm{Ad}(w)\,\mathrm{supp}(f)$, we observe that

$$\varpi^{2n}\,\mathrm{Ad}\left(\begin{bmatrix} \varpi^n & 0 & 0 & 0 \\ 0 & \varpi^n & 0 & 0 \\ 0 & 0 & \varpi^{-n} & 0 \\ 0 & 0 & 0 & \varpi^{-n} \end{bmatrix}\right) X \in \mathcal{O} \cap \begin{bmatrix} \mathfrak{m}^{2n} & \mathfrak{m}^{2n} & \mathfrak{m}^{4n-1} & \mathfrak{m}^{4n-1} \\ \mathfrak{m}^{2n} & \mathfrak{m}^{2n} & \mathfrak{m}^{4n} & \mathfrak{m}^{4n-1} \\ \mathfrak{m}^0 & -\varpi^{-1}\epsilon + \mathfrak{m}^0 & \mathfrak{m}^{2n} & \mathfrak{m}^{2n} \\ \epsilon + \mathfrak{m} & \mathfrak{m}^0 & \mathfrak{m}^{2n} & \mathfrak{m}^{2n} \end{bmatrix}.$$

As n goes to $+\infty$, such elements converge (or a subsequence does) to an element

$$e' \in \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathfrak{m}^0 & -\varpi^{-1}\epsilon + \mathfrak{m}^0 & 0 & 0 \\ \epsilon + \mathfrak{m} & \mathfrak{m}^0 & 0 & 0 \end{bmatrix}.$$

This element e' lives in the same $\mathrm{Ad}(G(F))$-orbit as e because the bottom-left 2×2 matrix defines an isomorphic quadratic form. □

Proof of Lemma 3.5. We look at

$$\operatorname{Ad}\left(\begin{bmatrix}1&0&0&0\\z&1&0&0\\0&0&1&0\\0&0&-z&1\end{bmatrix}\right)A=\begin{bmatrix}0&0&\varpi^{-1}z&\varpi^{-1}\\0&0&\varpi^{-1}(1+z^2)&\varpi^{-1}z\\1&0&0&0\\-2z&1&0&0\end{bmatrix},$$

$$\operatorname{Ad}\left(\begin{bmatrix}x&0&0&0\\0&y^{-1}&0&0\\0&0&y&0\\0&0&0&x^{-1}\end{bmatrix}\cdot\begin{bmatrix}1&0&0&0\\z&1&0&0\\0&0&1&0\\0&0&-z&1\end{bmatrix}\right)A=\begin{bmatrix}0&0&\varpi^{-1}xy^{-1}z&\varpi^{-1}x^2\\0&0&\varpi^{-1}y^{-2}(1+z^2)&\varpi^{-1}xy^{-1}z\\x^{-1}y&0&0&0\\-2x^{-2}z&x^{-1}y&0&0\end{bmatrix}.$$

Suppose $x, y\in\mathcal{O}_F^\times$ and $z\in\mathcal{O}_F$. Denote by $\bar{x},\bar{y},\bar{z},\bar{\epsilon}\in\mathbb{F}_q$ the respective reductions. The right-hand side of the last equation lies in the support of f if and only if

$$\begin{cases}-2\bar{x}^{-2}\bar{z}=\bar{\epsilon},\\ \bar{y}^{-2}(1+\bar{z}^2)=\bar{\epsilon}\end{cases}\iff\begin{cases}\bar{z}=-\frac{1}{2}\bar{\epsilon}\bar{x}^2,\\ \frac{1}{4}\bar{\epsilon}^2\bar{x}^4+1=\bar{\epsilon}\bar{y}^2.\end{cases}$$

That is, as long as the curve $E=\left(\bar{\epsilon}\bar{y}^2=\frac{1}{4}\bar{\epsilon}^2\bar{x}^4+1:(\bar{x},\bar{y})\in(\mathbb{G}_m)^2/_{\mathbb{F}_q}\right)$ has an $\mathbb{F}_q$-point, there exists $g\in G(F)$ such that $\operatorname{Ad}(g)A\in\operatorname{supp}(f)$, i.e., $I_A(f)\neq 0$. Such an $\mathbb{F}_q$-point always exists. Indeed, E differs from its smooth completion E^c by eight $\overline{\mathbb{F}}_q$-points (two for $\bar{x}=0$, four for $\bar{y}=0$ and two at infinity), and none of them is defined over $\mathbb{F}_q$ because $\bar{\epsilon}\in\mathbb{F}_q$ is a nonsquare. Hence $E(\mathbb{F}_q)=E^c(\mathbb{F}_q)$, while E^c is a geometrically connected projective smooth genus-1 curve and always has an $\mathbb{F}_q$-point. □

Remark 3.6. This "none of the boundary points is defined over the residue field" phenomenon seems to be related to the vanishing of $c_{\mathcal{O}}(A)$ for those $\mathcal{O}>\operatorname{Ad}(G(F))e$.

Remark 3.7. Using a special case of [Kim and Murnaghan 2003, Theorem 2.3.1] that $\operatorname{Ad}(g)A\in\mathfrak{g}(F)_{-1/2}\Longrightarrow g\in G(F)_0$, one may reduce the computation of orbital integrals and thus $c_{\mathcal{O}}(A)$ and $c_{\mathcal{O}}(\pi)$ (for $\mathcal{O}=\operatorname{Ad}(G(F))e$) to $\#E(\mathbb{F}_q)$. We predict the dimension of the associated degenerate Whittaker model to be $\frac{1}{4}\#E(\mathbb{F}_q)$, analogous to [Tsai 2017, Theorem 4.10 and Corollary 6.2].

Remark 3.8. We may also work with representations of depth $n+\frac{1}{2}$ by replacing A by $\varpi^{-n}A$ and replacing $G(F)_{1/2}$ by $G(F)_{n+1/2}$ in Theorem 1.4. The same proof works, except that e needs to be replaced by $\varpi^{-n}e$, resulting in every $\mathcal{O}\in\mathrm{WF}^{\mathrm{rat}}(\pi)$ being replaced by $\varpi^{-n}\mathcal{O}$.

4. Langlands parameters

The determination of the Langlands parameter corresponding to an individual π in Theorem 1.4 is part of the difficult problem solved in [Kaletha 2015] with deep insight into the rectifying characters and their relation with transfer factors. The *collection* of all Langlands parameters corresponding to components π in Theorem 1.4 is nevertheless simpler, because it happens in this case that the rectifying characters can be absorbed into the choice of an irreducible component in $\text{c-ind}_{G(F)_{1/2}}^{G(F)}\tilde{\psi}_A$. We describe the collection of such Langlands parameters, in the hope that it may be useful to interested readers.

Consider the ramified quadratic extension $E = F(\sqrt{\varpi})$. Write $\varpi_E = \sqrt{\varpi}$. A Langlands parameter we seek for is a homomorphism $\rho : W_F \to \mathrm{SO}_5(\mathbb{C})$. It has image in $\mathrm{O}_2(\mathbb{C}) \times \mathrm{O}_2(\mathbb{C}) \times \mathrm{SO}_1(\mathbb{C})$, i.e., ρ can be viewed as the sum of two orthogonal self-dual representations and a trivial representation. We have $\rho = \rho_1 \oplus \rho_2 \oplus \mathrm{triv}$, where $\rho_j = \mathrm{Ind}_{W_E}^{W_F} \chi_j$ for $j = 1, 2$. Write $\alpha_1 = 1$ and $\alpha_2 = \sqrt{-1}$ for any choice of square root of -1 in $\mathcal{O}_F^\times$. Then χ_j is a character on $E^\times$ satisfying:

(a) $\chi_j|_{F^\times} \equiv 1$.

(b) $\chi_j(1 + x\varpi) = 1$ for all $x \in \mathcal{O}_E$.

(c) $\chi_j(1 + x\varpi_E) = \psi(2x\alpha_j)$ for all $x \in \mathcal{O}_E$.

Here ψ is as chosen before (4). We note that each χ_j is determined up to a freedom of $\chi_j(\varpi_E) \in \{\pm 1\}$, and consequently there are $2^2 = 4$ candidates for such ρ. Relatedly, there are also 2^2 components of π in Theorem 1.4.

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References

[Ciubotaru et al. 2025] D. Ciubotaru, L. Mason-Brown, and E. Okada, "Wavefront sets of unipotent representations of reductive p-adic groups, II", *J. Reine Angew. Math.* **823** (2025), 191–253. MR Zbl

[DeBacker 2002] S. DeBacker, "Homogeneity results for invariant distributions of a reductive p-adic group", *Ann. Sci. École Norm. Sup.* (4) **35**:3 (2002), 391–422. MR Zbl

[Djoković 1981] D. Ž. Djoković, "Closures of conjugacy classes in classical real linear Lie groups", pp. 63–83 in *Algebra*: *Lie algebras*, *group theory*, *and partially ordered algebraic structures* (Carbondale, IL, 1980), edited by R. K. Amayo, Lecture Notes in Math. **848**, Springer, 1981. MR Zbl

[Gomez et al. 2021] R. Gomez, D. Gourevitch, and S. Sahi, "Whittaker supports for representations of reductive groups", *Ann. Inst. Fourier* (*Grenoble*) **71**:1 (2021), 239–286. MR Zbl

[Harish-Chandra 1999] Harish-Chandra, *Admissible invariant distributions on reductive p-adic groups*, Univ. Lect. Ser. **16**, Amer. Math. Soc., Providence, RI, 1999. MR Zbl

[Jiang et al. 2022] D. Jiang, D. Liu, and L. Zhang, "Arithmetic wavefront sets and generic L-packets", preprint, 2022. arXiv 2207.04700

[Kaletha 2015] T. Kaletha, "Epipelagic L-packets and rectifying characters", *Invent. Math.* **202**:1 (2015), 1–89. MR Zbl

[Kim and Murnaghan 2003] J.-L. Kim and F. Murnaghan, "Character expansions and unrefined minimal K-types", *Amer. J. Math.* **125**:6 (2003), 1199–1234. MR Zbl

[Kottwitz 1999] R. E. Kottwitz, "Transfer factors for Lie algebras", *Represent. Theory* **3** (1999), 127–138. MR Zbl

[Mœglin 1996] C. Mœglin, "Front d'onde des représentations des groupes classiques p-adiques", *Amer. J. Math.* **118**:6 (1996), 1313–1346. MR Zbl

[Mœglin and Waldspurger 1987] C. Mœglin and J.-L. Waldspurger, "Modèles de Whittaker dégénérés pour des groupes p-adiques", *Math. Z.* **196**:3 (1987), 427–452. MR Zbl

[Nevins 2011] M. Nevins, "On nilpotent orbits of SL_n and Sp_{2n} over a local non-Archimedean field", *Algebr. Represent. Theory* **14**:1 (2011), 161–190. MR Zbl

[Poonen 2017] B. Poonen, *Rational points on varieties*, Grad. Stud. in Math. **186**, Amer. Math. Soc., Providence, RI, 2017. MR Zbl

[Reeder and Yu 2014] M. Reeder and J.-K. Yu, "Epipelagic representations and invariant theory", *J. Amer. Math. Soc.* **27**:2 (2014), 437–477. MR Zbl

[Rodier 1975] F. Rodier, "Modèle de Whittaker et caractères de représentations", pp. 151–171 in *Non-commutative harmonic analysis* (Marseille/Luminy, 1974), edited by J. Carmona et al., Lecture Notes in Math. **466**, Springer, 1975. MR Zbl

[Serre 1973] J.-P. Serre, *A course in arithmetic*, Grad. Texts in Math. **7**, Springer, 1973. MR Zbl

[Shelstad 1989] D. Shelstad, "A formula for regular unipotent germs", pp. 275–277 in *Orbites unipotentes et représentations*, *II*, Astérisque **171-172**, Soc. Math. France, Paris, 1989. MR Zbl

[Tsai 2017] C.-C. Tsai, "A formula for certain Shalika germs of ramified unitary groups", *Compos. Math.* **153**:1 (2017), 175–213. MR Zbl

[Tsai 2023] C.-C. Tsai, "Wave-front sets for p-adic Lie algebras", preprint, 2023. arXiv 2311.08078

[Tsai 2024] C.-C. Tsai, "Geometric wave-front set may not be a singleton", *J. Amer. Math. Soc.* **37**:1 (2024), 281–304. MR Zbl

[Waldspurger 2018] J.-L. Waldspurger, "Représentations de réduction unipotente pour $SO(2n+1)$, III: Exemples de fronts d'onde", *Algebra Number Theory* **12**:5 (2018), 1107–1171. MR Zbl

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