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Yichen Qin and Daxin Xu

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Irregular Hodge filtration of hypergeometric differential equations

Yichen Qin and Daxin Xu

Fedorov and Sabbah–Yu calculated the (irregular) Hodge numbers of hypergeometric connections. In this paper, we study the irregular Hodge filtrations on hypergeometric connections defined by rational parameters and provide a new proof of the aforementioned results. Our approach is based on a geometric interpretation of hypergeometric connections, which enables us to show that certain hypergeometric sums are everywhere ordinary on $|\mathbb{G}_{m,\mathbb{F}_p}|$; i.e., "Frobenius Newton polygon equals the irregular Hodge polygon".

1. Introduction

Our primary focus is to investigate the Hodge theoretic properties of confluent hypergeometric differential equations. These differential equations have irregular singularities and are equipped with *irregular Hodge filtrations*, which are defined in [Sabbah 2018]. The irregular Hodge theory, initiated by Deligne [2007a; 2007b], extends the classical Hodge theory and has been developed in a series of works; see [Sabbah 2010; Kontsevich and Soibelman 2011; Yu 2014; Esnault et al. 2017; Sabbah and Yu 2015; Sabbah 2018].

Let $n \geq m$ be two nonnegative integers, λ a real number, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_m)$ two nondecreasing sequences of real numbers in $[0, 1)$. Let S be the scheme $\mathbb{G}_m \backslash \{1\}$ (resp. $\mathbb{G}_m$) if $n = m$ (resp. $n > m$) with coordinate z. The *hypergeometric equation* is the linear differential equation defined by the differential operator

$$\mathrm{Hyp}_\lambda(\alpha; \beta) := \lambda \prod_{i=1}^{n} (z\partial_z - \alpha_i) - z \prod_{j=1}^{m} (z\partial_z - \beta_j). \tag{1.0.0.1}$$

The *hypergeometric connection* $\mathcal{H}yp_\lambda(\alpha; \beta)$ is the associated connection on the complex algebraic variety $S_{\mathbb{C}}$; see (2.1.1.1). We say that the pair (α, β) is *nonresonant* if $\alpha_i \neq \beta_j$ for any i and j. In this case, the hypergeometric connection $\mathcal{H}yp_\lambda(\alpha; \beta)$ is irreducible and rigid, as seen by combining the works [Beukers and Heckman 1989] and [Katz 1990].

When $n = m$, hypergeometric connections have regular singularities at 0, 1, and ∞. Simpson [1990, Corollary 8.1] demonstrated that rigid irreducible connections on curves with regular singularities whose eigenvalues of monodromy actions at singularities have norm 1 underlie complex variations of Hodge structure. In this case, Fedorov [2018] computed the Hodge numbers associated with the Hodge filtrations of irreducible hypergeometric connections, and Martin [2021] gave an alternative proof.

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When $n > m$, hypergeometric connections are called *confluent*, indicating the merging of singularities, and have a regular singularity at 0 and an irregular singularity at ∞. Sabbah [2018, Theorem 0.7] showed that a rigid irreducible connection on $\mathbb{P}^1$ with real formal exponents at each singular point admits a variation of irregular Hodge structure away from singularities. For confluent hypergeometric connections, Sabbah and Yu [2019] computed the corresponding irregular Hodge numbers. In addition, Castaño Domínguez and Sevenheck [2021, Theorem 4.7] and Castaño Domínguez, Reichelt and Sevenheck [Castaño Domínguez et al. 2019, Theorem 5.8] explicitly calculated the irregular Hodge filtration for $m = 0$ or 1, respectively.

This article focuses on cases where λ, α, and β are rational numbers. We explicitly construct the irregular Hodge filtration $F^{\bullet}_{\mathrm{irr}}$ on hypergeometric connections in Theorem 3.3.1 and provide a uniform method for reproving the results of Fedorov and Sabbah–Yu.

Theorem 1.0.1 (3.3.1). *Suppose (α, β) is nonresonant. We define a map $\theta : \{1, \dots, n\} \to \mathbb{R}$ by*

$$\theta(k) = (n-m)\alpha_k + \#\{i \mid \beta_i < \alpha_k\} + (n-k) - \sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{m} \beta_j. \tag{1.0.1.1}$$

Then, up to an $\mathbb{R}$-shift,[1] *the jumps of the irregular Hodge filtration on $\mathcal{H}yp_\lambda(\alpha, \beta)$ occur at $\theta(k)$ and, for any $p \in \mathbb{R}$, we have*

$$\operatorname{rk} \operatorname{gr}^p_{F_{\mathrm{irr}}} \mathcal{H}yp_\lambda(\alpha; \beta) = \#\theta^{-1}(p).$$

1.1. ***Application to Frobenius slopes of hypergeometric sums.*** Our method has an arithmetic application to the Frobenius slopes of hypergeometric sums: the arithmetic incarnation of hypergeometric functions [Katz 1990].

Let K be a p-adic field with residue field $\mathbb{F}_p$ containing an element π satisfying $\pi^{p-1} = -p$. Such an element π corresponds to an additive character $\psi : \mathbb{F}_p \to K^\times$ by Dwork's theory [1974]. Suppose that (α, β) is nonresonant and that

$$\alpha_i = \frac{a_i}{p-1}, \ \beta_j = \frac{b_j}{p-1} \in \frac{\mathbb{Z}}{p-1}.$$

Miyatani [2020] showed that there exists a unique Frobenius structure φ (up to a scalar) on the analytification of the hypergeometric connection $\mathcal{H}yp_{(-1)^{m+np}/\pi^{n-m}}(\alpha; \beta)$ on S_K, which underlies an overconvergent F-isocrystal on the special fiber of S (called the *hypergeometric F-isocrystal*). The Frobenius trace of φ at an $\mathbb{F}_q$-point a of S is given by the *hypergeometric sum* $\mathrm{Hyp}(\alpha; \beta)(a)$, defined by

$$\sum_{\substack{x_i, y_j \in \mathbb{F}_q^\times \\ x_1 \cdots x_n = a y_1 \cdots y_m}} \psi\biggl(\mathrm{Tr}\biggl(\sum_{i=1}^{n} x_i - \sum_{j=1}^{m} y_j\biggr)\biggr) \cdot \prod_{i=1}^{n} \omega^{a_i}(\mathrm{Nm}(x_i)) \prod_{j=1}^{m} \omega^{-b_j}(\mathrm{Nm}(y_j)),$$

where $\omega : \mathbb{F}_p^\times \to K^\times$ denotes the Teichmüller lift, $\mathrm{Tr} = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$, and $\mathrm{Nm} = \mathrm{Nm}_{\mathbb{F}_q/\mathbb{F}_p}$.

[1] Our Hodge numbers $\theta(k)$ are normalized according to the geometric interpretation in Proposition 2.4.1, which is different from those of Fedorov and Sabbah–Yu by a shift.

Frobenius eigenvalues of φ at a are Weil numbers and have complex absolute valuations $q^{(n+m-1)/2}$ via an isomorphism $\overline{K} \simeq \mathbb{C}$. When (α, β) is resonant, the above hypergeometric sum can also be written as a sum of n Weil numbers. It is expected that the p-adic valuations of these Frobenius eigenvalues (called *Frobenius slopes*) are related to the (irregular) Hodge filtration. Our geometric construction of hypergeometric connections allows us to show the following result.

Theorem 1.1.1 (4.0.2). *Suppose $n > m$ and that α_i and β_j lie in $\frac{1}{p-1}\mathbb{Z} \cap [0, 1)$. For every p-power q and $a \in \mathbb{G}_m(\mathbb{F}_q)$, the multiset of Frobenius eigenvalues of* $\mathrm{Hyp}(\alpha; \beta)(a)$ (*normalized by* ord_q) *coincides with the multiset of irregular Hodge numbers $\{\theta(1), \ldots, \theta(n)\}$ defined in (1.0.1.1).*

Following [Mazur 1972], we encode the information of the p-adic valuations of Frobenius eigenvalues and (irregular) Hodge numbers into the Newton polygon and the (irregular) Hodge polygon, respectively, as defined in Definition 4.0.1.

For crystalline cohomology groups of a smooth proper variety over k, Mazur and Ogus showed that the associated (Frobenius) Newton polygon lies above the Hodge polygon defined by Hodge numbers [Mazur 1972; Berthelot and Ogus 1978]. For F-isocrystals associated with exponential sums, "Newton above Hodge" type results were studied by Dwork's school. For example, Dwork [1974], Sperber [1977], and Wan [1993] proved that Kloosterman sums (hypergeometric sums of type $(n, 0)$ with $\alpha = (0, \ldots, 0)$) are everywhere ordinary on $|\mathbb{G}_{m,\mathbb{F}_p}|$; i.e., two polygons coincide for every closed point $a \in |\mathbb{G}_m|$. We use a "Newton above Hodge" result of Adolphson and Sperber [1989; 1993] and identify their (combinatorial) Hodge polygon for the above hypergeometric sums with the irregular Hodge polygon of hypergeometric connections. Finally, we deduce "Newton equals Hodge" by a criterion for ordinariness due to Wan [1993].

Remark 1.1.2. (i) One may also consider the Frobenius Newton polygon of hypergeometric sums defined by multiplicative characters of orders dividing p^s-1 for a positive integer s. In this case, Adolphson and Sperber showed that the associated Frobenius Newton polygon lies above their (combinatorial) Hodge polygon, which can be viewed as an average of irregular Hodge polygons. However, the associated hypergeometric sums may not be ordinary in the case $s > 1$. There is an example of hypergeometric sums (of type $(n, m) = (2, 0)$) for which the Frobenius Newton polygon lies strictly above Adolphson and Sperber's Hodge polygon [1987] for every $a \in |\mathbb{G}_{m,\mathbb{F}_p}|$.

(ii) The ordinariness of hypergeometric sums also fails in the nonconfluent case (i.e., $n = m$). For $p = 31$ and the hypergeometric sum defined by $\alpha = (0,\ 0, 0, 0)$, $\beta = \left(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right)$ at $a = 4$ or 17, its Newton polygon $\left(\text{with slope } \left(\frac{5}{2}, \frac{5}{2}, \frac{9}{2}, \frac{9}{2}\right)\right)$ [Drinfeld and Kedlaya 2017, Appendix A.5][2] strictly lies above the irregular Hodge polygon (with slope $(2, 3, 4, 5)$).

1.2. *Strategy of proof.*

The proof of Theorem 1.0.1 can be reduced to calculating the irregular Hodge filtration on each fiber of $\mathcal{H}yp_\lambda(\alpha, \beta)$. We adopt an approach similar to those used in [Fresán et al. 2022; Sabbah and Yu 2023; Qin 2024], where the authors calculated the Hodge numbers of motives attached to Kloosterman and Airy moments. The key ingredient of this argument is an (exponentially) geometric

[2] In [loc. cit.], the Frobenius slopes are normalized and are different from our convention by a shift of 2.

interpretation of hypergeometric connections in Corollary 2.3.3. More precisely, there exists a smooth quasiprojective variety X with a regular function $g : X \times S \to \mathbb{A}^1$ such that the hypergeometric connections are subquotients of the $\mathscr{D}_S$-module $\mathcal{H}^N \mathrm{pr}_+(\mathcal{O}_{X\times S}, \mathrm{d}+\mathrm{d}g)$, where $N = \dim X$ and pr is the projection $\mathrm{pr} : X \times S \to S$. Our construction is motivated by Katz's hypergeometric sums and the function-sheaf dictionary. A related construction can be found in [Kamgarpour and Yi 2021].

Through this geometric interpretation, each fiber $\mathcal{H}yp_\lambda(\alpha, \beta)_a$ at a closed point a of S is identified with a subquotient of the twisted de Rham cohomology of the pair $(X, g_a := g|_{\mathrm{pr}_z^{-1}(a)})$, i.e., the hypercohomology of the twisted de Rham complex $(\Omega^\bullet_X, \mathrm{d}+\mathrm{d}g_a)$. Then, we reduce to calculate the irregular Hodge filtration on the twisted de Rham cohomology of the pair (X, g_a) (up to a shift).

The irregular Hodge filtration on the twisted de Rham cohomology of the pairs (X, g_a) has been studied in [Yu 2014]. In the context of our case, we can select $X = \mathbb{G}_m^{n+m-1}$ and g_a as a Laurent polynomial with good properties; see Corollary 2.3.3. Under these assumptions, Yu showed that the irregular Hodge filtration on $\mathrm{H}^{n+m-1}_{\mathrm{dR}}(X, g_a)$ can be calculated by the Newton polyhedron filtration on the Newton polytope $\Delta(g_a)$ (3.1.1.1). This identification enables us to prove, via a combinatorial approach, a fiberwise version of Theorem 1.0.1 as follows.

Theorem 1.2.1 (3.3.3). *Up to an $\mathbb{R}$-shift, the jumps of the irregular Hodge filtration $F^\bullet_{\mathrm{irr}}$ on the fiber $\mathcal{H}yp(\alpha; \beta)_a$ occur at $\theta(k)$ from (1.0.1.1) for $1 \le k \le n$. Moreover, we have*

$$\dim \mathrm{gr}^p_{F_{\mathrm{irr}}} \mathcal{H}yp(\alpha; \beta)_a = \#\theta^{-1}(p) \quad \textit{for any } p \in \mathbb{R}.$$

In addition, our geometric construction allows us to answer a question of Katz [1990, 6.3.8] on the comparison between modified hypergeometric $\mathscr{D}$-modules and hypergeometric connections in the resonant case (see Proposition 2.4.7) when the parameters are rational.

1.3. ***Organization of this article.*** We present a geometric interpretation of hypergeometric connections in Section 2. Section 3 is devoted to the proofs of Theorems 1.2.1 and 1.0.1. In Section 4, we study hypergeometric sums defined by multiplicative characters of orders dividing $p-1$ and prove that they are ordinary (Theorem 1.1.1).

2. Hypergeometric connections

In this section, we give an (exponentially) geometric interpretation of the hypergeometric connections in Proposition 2.3.1, Corollary 2.3.3, and Proposition 2.4.1. We work with varieties over $\mathbb{C}$ in Sections 2 and 3.

2.1. ***Review of hypergeometric connections following*** **[Katz 1990].**

2.1.1. *Hypergeometric connections.* Let $n \ge m \ge 0$ be two integers, $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_j)$ two sequences of nondecreasing rational numbers (and we don't require that they lie in $[0, 1)$ as in the introduction), and $\lambda \in \mathbb{Q}$. Let $\mathscr{D}_S$ be the sheaf of differential operators on the scheme S, which is $\mathbb{G}_m \backslash \{1\}$

(resp. $\mathbb{G}_m$) if $n = m$ (resp. $n > m$) with coordinate z. Then, the hypergeometric connection $\mathcal{H}yp_\lambda(\alpha;\beta)$ on S is defined by the differential operator in (1.0.0.1) as

$$\mathscr{D}_S/\operatorname{Hyp}_\lambda(\alpha;\beta). \tag{2.1.1.1}$$

By [Katz 1990, (3.1)], one has for $\gamma \in \mathbb{Q}$ that

$$\mathcal{H}yp_\lambda(\alpha;\beta)\otimes\left(\mathcal{O}, \mathrm{d}+\gamma\frac{\mathrm{d}z}{z}\right) \simeq \mathcal{H}yp_\lambda(\alpha+\gamma;\beta+\gamma), \tag{2.1.1.2}$$

where $\alpha+\gamma$ (resp. $\beta+\gamma$) is the sequence consisting of $\alpha_i+\gamma$ (resp. $\beta_j+\gamma$). Furthermore, one has for $\mu \in \mathbb{Q}^\times$ that

$$[x \mapsto \mu\cdot x]^+\mathcal{H}yp_\lambda(\alpha;\beta) \simeq \mathcal{H}yp_{\lambda/\mu}(\alpha;\beta). \tag{2.1.1.3}$$

Thanks to the above relations, we can often assume that $\lambda = 1$ and $\alpha_1 = 0$. For simplicity, we denote by $\mathcal{H}yp(\alpha;\beta)$ the connection $\mathcal{H}yp_1(\alpha;\beta)$.

When the pair (α,β) is nonresonant, i.e., $\alpha_i-\beta_j \notin \mathbb{Z}$ for any i and j, Katz [1990, Proposition 3.2] showed that $\mathcal{H}yp(\alpha;\beta)$ is irreducible and only depends on $\alpha \bmod \mathbb{Z}$ and $\beta \bmod \mathbb{Z}$. In this case, we may assume that α and β are two nondecreasing sequences of rational numbers in $[0, 1)$.

2.1.2. *Modified hypergeometric $\mathscr{D}$-modules.* Given a morphism g between smooth varieties, for a bounded complex of holonomic algebraic $\mathscr{D}$-modules, following [Fresán et al. 2022, Appendix A.1], we denote by g^+, g_+, and $g_\dagger$ the derived pullback functor, the pushforward functor, and the pushforward with compact support functor, respectively. The k-th cohomology of a complex K is denoted by $\mathcal{H}^k(K)$.

Let $\mathrm{mult}: \mathbb{G}_m\times\mathbb{G}_m \to \mathbb{G}_m$ be the product map. The convolution functors $\star_*$ and $\star_!$ on $\mathbb{G}_m$ are defined, for two objects M and N of $\mathrm{D}^b(\mathscr{D}_{\mathbb{G}_m})$, by

$$M\star_* N := \mathrm{mult}_+(M\boxtimes N) \quad\text{and}\quad M\star_! N := \mathrm{mult}_\dagger(M\boxtimes N),$$

respectively. These convolution functors are associative and commutative. Moreover, the duality functor $\mathbb{D}$ interchanges $\star_!$ and $\star_*$.

Definition 2.1.3. Let α and β be two sequences of rational numbers. For $? \in \{!, *\}$, the convolution

$$\mathcal{H}yp(\alpha_1;\varnothing)\star_? \cdots \star_? \mathcal{H}yp(\alpha_n;\varnothing)\star_? \mathcal{H}yp(\varnothing;\beta_1)\star_? \cdots \star_? \mathcal{H}yp(\varnothing;\beta_m)$$

is a holonomic $\mathscr{D}_{\mathbb{G}_m}$-module [Katz 1990, (6.3.6)]. We denote it by $\mathcal{H}yp(?;\alpha;\beta)$ and call it a *modified hypergeometric $\mathscr{D}$-module*.

The restrictions of the above two modified hypergeometric $\mathscr{D}$-modules to S are generally not isomorphic to the hypergeometric connections. When (α,β) is nonresonant, the natural map

$$\mathcal{H}yp(!;\alpha;\beta) \to \mathcal{H}yp(*;\alpha;\beta) \tag{2.1.3.1}$$

is an isomorphism, as seen by using an argument similar to those in [Katz 1990, Theorem 8.4.2 (5)] and [Miyatani 2020, Proposition 3.3.3]. In this case, both modified hypergeometric $\mathscr{D}_{\mathbb{G}_m}$-modules, restricted to S, are isomorphic to the hypergeometric connection $\mathcal{H}yp(\alpha;\beta)$ by [Katz 1990, (5.3.1)].

2.2. ***The Newton polytope of a Laurent polynomial.*** We study the Newton polytope of a Laurent polynomial appearing in the geometric interpretation of hypergeometric connections in Proposition 2.4.1.

Definition 2.2.1. Let N be a positive integer and

$$g(z_1, \dots, z_N) = \sum_{\tau \in \mathbb{Z}^N} c(\tau) z^\tau$$

be a Laurent polynomial in variables $z_1, \dots, z_N$, with $z^\tau = \prod_{i=1}^N z_i^{\tau_i}$ for $\tau = (\tau_1, \dots, \tau_N)$.

(1) The support of g is the subset $\mathrm{Supp}(g) = \{\tau \mid c(\tau) \neq 0\}$ of $\mathbb{Z}^N$.

(2) The *Newton polytope* $\Delta(g)$ is the convex hull of the set $\mathrm{Supp}(g) \cup \{0\}$ in $\mathbb{R}^N$.

(3) The Laurent polynomial g is called *nondegenerate* with respect to $\Delta(g)$ (or simply nondegenerate) if, for each face $\sigma \subset \Delta(g)$ not passing through 0, the Laurent polynomial $g_\sigma := \sum_{\tau \in \sigma \cap \mathbb{Z}^N} c(\tau) z^\tau$ has no critical point in $(\mathbb{C}^\times)^N$.

Let $n \geq m \geq 0$ and $d \geq 1$ be three integers, $f : \mathbb{G}_m^{n+m} \to \mathbb{A}^1$ the Laurent polynomial

$$f : (x_2, \dots, x_n, y_1, \dots, y_m, z) \mapsto \sum_{i=2}^n x_i^d - \sum_{j=1}^m y_j^d + z \cdot \frac{\prod_{j=1}^m y_j^d}{\prod_{i=2}^n x_i^d}, \tag{2.2.1.1}$$

and $\mathrm{pr}_z : \mathbb{G}_m^{n+m} \to \mathbb{G}_m$ the projection onto the z-coordinate. For $a \in \mathbb{C}^\times$, we set $f_a = f|_{\mathrm{pr}_z^{-1}(a)}$.

We denote by $\{u_i, v_j\}_{2 \leq i \leq n, 1 \leq j \leq m}$ the coordinates in $\mathbb{R}^{n+m-1}$, and identify a monomial $\prod_i x_i^{a_i} \cdot \prod_j y_j^{b_j}$ with a lattice point $(a_i, b_j) \in \mathbb{Z}^{n+m-1} \subset \mathbb{R}^{n+m-1}$.

Lemma 2.2.2. *Assume that $n > m = 0$ and $a \in \mathbb{C}^\times$.*

(1) *The Laurent polynomial f_a is convenient*; *i.e.*, *the origin is in the interior of* $\Delta(f_a)$.

(2) *The Newton polytope $\Delta(f_a)$ is defined by*

$$h_{n+1} := \sum_{i=2}^n u_i \leq d \qquad \text{and} \qquad h_{i_0} := \sum_{i=2}^n u_i - (n-m) u_{i_0} \leq d, \quad 2 \leq i_0 \leq n. \tag{2.2.2.1}$$

(3) *The Laurent polynomial f_a is nondegenerate with respect to* $\Delta(f_a)$.

Proof. (1) Let P_i for $2 \leq i \leq n$ and R be the points in $\mathbb{Z}^{n-1}$ corresponding to x_i^d and $1/\prod_{i=2}^n x_i^d$, respectively. Observe that 0 is an interior point of the Newton polytope, as $0 = \big(\sum_{i=2}^n P_i + R\big)/n$.

(2) A face $\sigma \subset \Delta(f_a)$ of dimension $n-2$ must pass through $n-1$ points among $\{P_i, R\}$. So either $R \notin \sigma$ or there exists a $P_{i_0} \notin \sigma$. In the first case, the face lies on the hyperplane defined by the equation $h_{n+1} = d$. In the latter case, the face lies on the hyperplane defined by the equations $h_{i_0} = d$.

(3) Let σ be a face which does not pass through 0. Since the support of f_a has n points, it must pass through at most $n-1$ points in $\mathrm{Supp}(f_a)$. Let $I \subset \{2, \dots, n\}$ be a subset of the indices. Then $f_{a,\sigma}$ is either

$$f_{a,\sigma} = \sum_{i \in I} x_i^d \qquad \text{or} \qquad f_{a,\sigma} = \sum_{i \in I} x_i^d + \frac{a}{\prod_{i=2}^n x_i^d} \quad \text{for } |I| \leq n-2.$$

We can check that they are smooth on $\mathbb{G}_m^{n-1}$. Therefore f_a is nondegenerate. ☐

Lemma 2.2.3. *Assume that $n > m \neq 0$ and $a \in \mathbb{C}^\times$.*

(1) *The cone $\mathbb{R}_{\geq 0} \cdot \Delta(f_a)$ is defined by*

$$u_i + v_j \geq 0, \quad v_j \geq 0$$

for $i = 2, \dots, n$ and $j = 1, \dots, m$,

(2) *The Newton polytope $\Delta(f_a)$ is defined by*

$$u_i + v_j \geq 0, \quad v_j \geq 0, \quad h_{n+1} := \sum u_i + \sum v_j \leq d$$

and

$$h_{i_0} := \sum_i u_i + \sum_j v_j - (n-m)u_{i_0} \leq d, \quad 2 \leq i_0 \leq n. \tag{2.2.3.1}$$

(3) *The Laurent polynomial f_a is nondegenerate with respect to $\Delta(f_a)$.*

Proof. Let P_i and Q_j be the points in $\mathbb{Z}^{n+m-1}$ corresponding to monomials x_i^d and y_j^d for $2 \leq i \leq n$ and $1 \leq j \leq m$, respectively, and R the lattice point corresponding to

$$\prod_{j=1}^{m} y_j^d \Big/ \prod_{i=2}^{n} x_i^d.$$

In this case, the origin 0 is not an interior point of the Newton polytope. So $\Delta(f_a)$ has $(n+m+1)$-many vertices. To determine a face of dimension $n+m-2$, we need to choose $(n+m-1)$-many points among $\{P_i, Q_j, R\}$.

(1) For the first part, it suffices to determine faces $\sigma \subset \Delta(f_a)$ with dimensions $n+m-2$ containing 0.

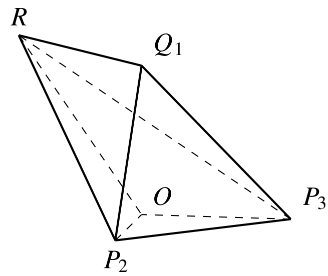
- If σ does not pass through R, it contains $(n+m-2)$ distinct points in $\{P_i, Q_j\}$. In this case, σ misses one point Q_{j_0} and lies on the hyperplane $v_{j_0} = 0$. Otherwise, σ misses one point P_{i_0}. Hence the hyperplane is given by the equation $u_{i_0} = 0$. Therefore, R and P_{i_0} lie on the two sides of the hyperplane, respectively, which is absurd.
- If σ passes through R, it contains $(n+m-3)$ distinct points in $\{P_i, Q_j\}$. In this case, σ has to miss one P_{i_0} and one Q_{j_0} and lies on the hyperplane $u_{i_0} + v_{j_0} = 0$. Otherwise, σ misses two $P_{i_0}, P_{i_0'}$ or $Q_{j_0}, Q_{j_0'}$. So σ lies on the hyperplane $u_{i_0} - u_{i_0'} = 0$ or $v_{j_0} - v_{j_0'} = 0$. However, the points $P_{i_0}, P_{i_0'}$ or $Q_{j_0}, Q_{j_0'}$ lie on different sides of the hyperplane $u_{i_0} - u_{i_0'} = 0$ or $v_{j_0} - v_{j_0'} = 0$, which contradicts the definition of σ.

(2) For the second part, it suffices to determine faces of dimension $n+m-2$ that do not pass through the origin.

- If $R \notin \sigma$, then σ contains all points P_i and Q_j. In this case, σ lies on the hyperplane $\sum u_i + \sum v_j = d$.
- If $R \in \sigma$, then σ contains $n+m-2$ points among $\{P_i, Q_j\}$. In this case, σ misses one P_{i_0} and lies on the hyperplane $h_{i_0} = d$. Otherwise, it misses one Q_{j_0} and lies on the hyperplane

$$\sum_{i=2}^{n} u_i + \sum_{j=1}^{m} v_j + (n-m)v_{j_0} = d.$$

However, the points 0 and Q_{j_0} are on different sides of the hyperplane.



(3) Let σ be a face which does not pass through 0. Since the support of f_a has $n+m$ points, it must pass through at most $n+m-1$ points in $\mathrm{Supp}(f_a)$. Let $I \subset \{2, \dots, n\}$ and $J \subset \{1, \dots, m\}$ be two subsets of the indices. Then $f_{a,\sigma}$ is either

$$f_{a,\sigma} = \sum_{i\in I} x_i^d - \sum_{j\in J} y_j^d \quad \text{or} \quad f_{a,\sigma} = \sum_{i\in I} x_i^d - \sum_{j\in J} y_j^d + a \cdot \frac{\prod_{j=1}^m y_j^d}{\prod_{i=2}^m x_i^d} \quad \text{for } |I|+|J| \le n+m-2.$$

To see that the partial Laurent polynomials $f_{a,\sigma}$ are all smooth on $(\mathbb{G}_m)^{n+m-1}$, it suffices to show that the system of equations

$$\{f_{a,\sigma} = \partial_{x_i} f_{a,\sigma} = \partial_{y_j} f_{a,\sigma} = 0 \mid 2 \le i \le n,\, 1 \le j \le m\}$$

has no solutions in $(\mathbb{G}_m)^{n+m-1}$. In fact, in the first case above, taking any $i_0 \in I$ or $j_0 \in J$, we have the equation $0 = \partial_{x_{i_0}} f_{a,\sigma} = dx_{i_0}^{d-1}$ or $0 = \partial_{y_{j_0}} f_{a,\sigma} = dj_{j_0}^{d-1}$, which is impossible if $d = 1$. If $d \ge 2$, then x_{i_0} or y_{j_0} is forced to be 0. In the second case, for any $i_0 \notin I$ or $j_0 \notin J$, we have

$$0 = \partial_{x_{i_0}} f_{a,\sigma} = -\frac{d}{x_{i_0}} \cdot a \cdot \frac{\prod_{j=1}^m y_j^d}{\prod_{i=2}^n x_i^d} \quad \text{or} \quad 0 = \partial_{y_j} f_{a,\sigma} = \frac{d}{y_{j_0}} \cdot a \cdot \frac{\prod_{j=1}^m y_j^d}{\prod_{i=2}^n x_i^d},$$

which again forces some $y_j = 0$.

Consequently, all the $f_{a,\sigma}$ have no critical points in $\mathbb{G}_m^{n+m-1}$, and therefore f_a is nondegenerate. □

Lemma 2.2.4. *Assume that $n = m$ and $a \in \mathbb{C}^\times$.*

(1) *The cone $\mathbb{R}_{\ge 0} \cdot \Delta(f_a)$ is defined by*

$$u_i + v_j \ge 0, \quad v_j \ge 0$$

for $i = 2, \dots, n$ and $j = 1, \dots, m$.

(2) *The Newton polytope $\Delta(f_a)$ is defined by*

$$u_i + v_j \ge 0, \quad v_j \ge 0, \quad \text{and} \quad h_{n+1} := \sum u_i + \sum v_j \le d. \tag{2.2.4.1}$$

(3) *The Laurent polynomial f_a is nondegenerate with respect to $\Delta(f_a)$ if $a \ne 1$.*

Proof. We use the same notation as in Lemma 2.2.3. The proof of the first assertion is the same as that in Lemma 2.2.3. The second assertion follows from the observation that the points $\{P_i, Q_j, R\}$ all lie on the hyperplane

$$\sum u_i + \sum v_j - d = 0.$$

Let σ be the face passing through $\{P_i, Q_j, R\}$. If a face τ of $\Delta(f_a)$ does not contain 0, it is a face of σ. Similar to the proof of Lemma 2.2.3, one can check that, if τ is a proper face of σ, there is no solution for the system of equations

$$\{f_{a,\tau} = \partial_{x_i} f_{a,\tau} = \partial_{y_j} f_{a,\tau} = 0 \mid 2 \leq i \leq n, 1 \leq j \leq m\}.$$

If $\tau = \sigma$, the system of equations

$$\{f_a = \partial_{x_i} f_a = \partial_{y_j} f_a = 0 \mid 2 \leq i \leq n, 1 \leq j \leq m\}$$

has solutions in $\mathbb{G}_m^{n+m-1}$ if and only if $a = 1$ (in such cases $x_i = y_j = c \in \mathbb{R}$ are solutions). So f_a is nondegenerate with respect to $\Delta(f_a)$ if $a \neq 1$. □

Remark 2.2.5. The volume of $\Delta(f_a)$ is $d^{n+m-1}n/(n+m-1)!$. In fact, the Newton polytope can be decomposed into n-copies $(n+m-1)$-simplexes, and each of them has volume $d^{n+m-1}/(n+m-1)!$.

2.3. ***Geometric interpretations.*** We present some geometric interpretations of hypergeometric connections here. Let d be a common denominator of α_i and β_j, and set $a_i = d \cdot \alpha_i$ and $b_j = d \cdot \beta_j$. To α_i (resp. β_j), we associate the character $\chi_i : \mu_d \to \mathbb{C}^\times$ (resp. ρ_j) which sends ζ_d to $\zeta_d^{a_i}$ (resp. $\zeta_d^{b_j}$). Set

$$\begin{aligned} \chi \times \rho &= \chi_1 \times \cdots \times \chi_n \times \rho_1^{-1} \times \cdots \times \rho_m^{-1}, \\ \tilde{\chi} \times \rho &= \chi_2 \times \cdots \times \chi_n \times \rho_1^{-1} \times \cdots \times \rho_m^{-1} \end{aligned} \tag{2.3.0.1}$$

as products of these characters.

Now we introduce two diagrams as follows:

- Let $\mathbb{G}_m^{n+m}$ be the torus with coordinates x_i and y_j for $1 \leq i \leq n$ and $1 \leq j \leq m$. We consider the diagram

$$\begin{array}{ccc} & \mathbb{G}_m^{n+m} & \\ {}^{\sigma}\swarrow & & \searrow^{\varpi} \\ \mathbb{A}_t^1 & & \mathbb{G}_m \end{array} \tag{2.3.0.2}$$

where

$$\sigma(x_i, y_j) = \sum_{i=1}^n x_i^d - \sum_{j=1}^m y_j^d \quad \text{and} \quad \varpi(x_i, y_j) = \prod_{i=1}^n x_i^d \Big/ \prod_{j=1}^m y_j^d.$$

Let the group μ_d^{n+m} act on $\mathbb{G}_m^{n+m}$ by multiplication and on $\mathbb{A}_t^1$ and $\mathbb{G}_m$ trivially. Then, it can be verified that σ and ϖ are μ_d^{n+m}-equivariant.

• Let $\mathbb{G}_m^{n+m}$ be the torus with coordinates z, x_i, and y_j for $2 \leq i \leq n$ and $1 \leq j \leq m$, and S be $\mathbb{G}_{m,z}$ (resp. $\mathbb{G}_{m,z} \backslash \{1\}$) if $n \neq m$ (resp. $n = m$). We consider the diagram

$$\begin{array}{ccccc} & \mathbb{G}_m^{n+m} & \hookleftarrow & U := S \times \mathbb{G}_m^{n+m-1} & \\ {\scriptstyle f} \swarrow & & \searrow {\scriptstyle \mathrm{pr}_z} & & \searrow {\scriptstyle \mathrm{pr}_z} \\ \mathbb{A}_t^1 & & \mathbb{G}_{m,z} & \hookleftarrow & S \end{array} \tag{2.3.0.3}$$

where pr_z is the projection to the z-coordinate and f is the Laurent polynomial

$$\sum_{i=2}^{n} x_i^d - \sum_{j=1}^{m} y_j^d + z \cdot \frac{\prod_{j=1}^{m} y_j^d}{\prod_{i=2}^{n} x_i^d}$$

defined in (2.2.1.1). Let the group $G = \mu_d^{n+m-1}$ act on $\mathbb{G}_m^{n+m}$ (resp. $S \times \mathbb{G}_m^{n+m-1}$) by multiplication on the coordinates x_i and y_j and trivially on z, and on $\mathbb{A}_t^1$, $\mathbb{G}_{m,z}$, and S trivially. Then f and pr_z are μ_d^{n+m-1}-equivariant.

Let $\mathcal{E}^t = (\mathcal{O}, \mathrm{d} + \mathrm{d}t)$ be the exponential $\mathscr{D}$-module on $\mathbb{A}_t^1$. For a regular function $h : X \to \mathbb{A}_t^1$, we denote by $\mathcal{E}^h := h^+\mathcal{E}^t$ the connection $(\mathcal{O}_X, \mathrm{d} + \mathrm{d}h)$ on X.

Proposition 2.3.1. *Let α and β be as above. The complexes $\varpi_?\mathcal{E}^\sigma$ are μ_d^{n+m}-equivariant and concentrated in degree 0 for $? \in \{\dagger, +\}$. Moreover, we have isomorphisms of $\mathscr{D}_{\mathbb{G}_m}$-modules*

$$\mathcal{H}yp(*; \alpha; \beta) \simeq (\varpi_+\mathcal{E}^\sigma)^{(\mu_d^{n+m}, \chi \times \rho)},$$
$$\mathcal{H}yp(!; \alpha; \beta) \simeq (\varpi_\dagger\mathcal{E}^\sigma)^{(\mu_d^{n+m}, \chi \times \rho)},$$

where the exponent $(\mu_d^{n+m}, \chi \times \rho)$ means taking the $(\chi \times \rho)$-isotypic component with respect to the action of μ_d^{n+m}.

Proof. The case of $\mathcal{H}yp(!; \alpha; \beta)$ can be deduced from the case of $\mathcal{H}yp(*; \alpha; \beta)$ by applying the duality functor. So, we only prove the latter case. Recall that the action of μ_d^{n+m} on $\mathbb{A}_t^1$ is trivial in diagram (2.3.0.2). So the $\mathscr{D}_{\mathbb{A}_t^1}$-module $\mathcal{E}^t$ is μ_d^{n+m}-equivariant. Since σ and ϖ are both μ_d^{n+m}-equivariant morphisms, φ_+ and σ^+ preserve μ_d^{n+m}-equivariant objects. Hence the complex

$$\varphi_+\mathcal{E}^\sigma = \varphi_+\sigma^+\mathcal{E}^t$$

is μ_d^{n+m}-equivariant.

Assume that $(n, m) = (1, 0)$. Then $\sigma : \mathbb{G}_{m,x_1} \to \mathbb{A}^1$ is the map $x_1 \mapsto x_1^d$ and $\varpi : \mathbb{G}_{m,x_1} \to \mathbb{G}_{m,z}$ is the d-th power map. So by the identity

$$\varpi_+\mathcal{O}_{\mathbb{G}_m} = \bigoplus_{i=0}^{d-1} \Big(\mathcal{O}_{\mathbb{G}_m}, \mathrm{d} + \frac{i}{d}\frac{\mathrm{d}z}{z}\Big)$$

and the projection formula, we have

$$(\varpi_+\mathcal{E}^\sigma) = \mathcal{E}^z \otimes (\varpi_+\mathcal{O}_{\mathbb{G}_m}) = \bigoplus_{i=0}^{d-1} \mathcal{E}^z \otimes \Big(\mathcal{O}_{\mathbb{G}_m}, \mathrm{d} + \frac{i}{d}\frac{\mathrm{d}z}{z}\Big),$$

which is concentrated in degree 0. Taking the isotypic component, we have

$$(\varpi_+\mathcal{E}^\sigma)^{(\mu_d^{n+m},\chi\times\rho)} = (\varpi_+\mathcal{E}^{x_1^d})^{(\mu_d,\chi_1)} = \mathcal{E}^z \otimes (\varpi_+\mathcal{O}_{\mathbb{G}_m})^{(\mu_d,\chi_1)}$$
$$= \left(\mathcal{O}_{\mathbb{G}_m}, \mathrm{d}+\mathrm{d}z+\alpha_1\frac{\mathrm{d}z}{z}\right) = \mathcal{H}yp(*;\alpha_1;\varnothing)$$

in the case where $(n,m)=(1,0)$. The proof of the case where $(n,m)=(0,1)$ is similar. In general, we use induction on $n+m$. The proof follows from the following lemma. □

Lemma 2.3.2. *Let α, α', β and β' be four sequences of rational numbers with common denominator d, whose lengths are n, n', m and m', respectively. We denote by χ_i, χ_i', ρ_j, and ρ_j' characters of μ_d corresponding to α_i, α_i', β_j, and β_j', respectively. Let σ and ϖ (resp. σ' and ϖ') be the maps for (n,m) (resp. (n',m')) in the diagram (2.3.0.2).*

Suppose that $(\varpi_+\mathcal{E}^\sigma)$ and $(\varpi'_+\mathcal{E}^{\sigma'})$ are concentrated in degree 0, and there are isomorphisms of $\mathscr{D}$-modules

$$\mathcal{H}yp(*;\alpha;\beta) \simeq (\varpi_+\mathcal{E}^\sigma)^{(\mu_d^{n+m},\chi\times\rho)},$$
$$\mathcal{H}yp(*;\alpha';\beta') \simeq (\varpi_+\mathcal{E}^{\sigma'})^{(\mu_d^{n+m},\chi'\times\rho')}.$$

Then $((\varpi\cdot\varpi')_+\mathcal{E}^{\sigma\boxplus\sigma'})$ is also concentrated in degree 0, and we have an isomorphism of $\mathscr{D}$-modules

$$\mathcal{H}yp(*;\alpha,\alpha';\beta,\beta') \simeq ((\varpi\cdot\varpi')_+\mathcal{E}^{\sigma\boxplus\sigma})^{(\mu_d^{n+n'+m+m'},\chi\times\chi'\times\rho\times\rho')},$$

where $\varpi\cdot\varpi' = \mathrm{mult}\circ(\varpi\times\varpi')$, pr and pr' are the projections from $\mathbb{G}_m^{n+n'+m+m'}$ to $\mathbb{G}_m^{n+m}$ and $\mathbb{G}_m^{n'+m'}$, respectively, and $\sigma\boxplus\sigma' = \sigma\circ\mathrm{pr}+\sigma'\circ\mathrm{pr}'$ is the Thom–Sebastiani sum.

Proof. The proof of this lemma is essentially that of [Katz 1990, Lemma 5.4.3]. Notice that the exterior product $\mathcal{E}^\sigma\boxtimes\mathcal{E}^{\sigma'}$ is $\mathcal{E}^{\sigma\boxplus\sigma'}$. Then

$$(\varpi_+\mathcal{E}^\sigma)\star_*(\varpi'_+\mathcal{E}^{\sigma'}) = \mathrm{mult}_+((\varpi_+\mathcal{E}^\sigma)\boxtimes(\varpi'_+\mathcal{E}^{\sigma'})) = \mathrm{mult}_+(\varpi\times\varpi')_+(\mathcal{E}^\sigma\boxtimes\mathcal{E}^{\sigma'}) = (\varpi\cdot\varpi')_+\mathcal{E}^{\sigma\boxplus\sigma'}.$$

By the Künneth formula [Hotta et al. 2008, Proposition 1.5.28 (i) and Proposition 1.5.30], we conclude that $(\varpi\cdot\varpi')_+\mathcal{E}^{\sigma\boxplus\sigma'}$ is again concentrated in degree 0.

Viewing μ_d^{n+m}-equivariant and $\mu_d^{n'+m'}$-equivariant objects as $\mu_d^{n+m+n'+m'}$-equivariant via the identifications

$$\mu_d^{n+m} \simeq \mu_d^{n+m}\times 1 \quad\text{and}\quad \mu_d^{n'+m'} \simeq 1\times\mu_d^{n'+m'},$$

we can verify that both $\boxtimes$ and mult are $\mu_d^{n+m+n'+m'}$-equivariant. Hence the convolution product $\star_*$ is also $\mu_d^{n+m+n'+m'}$-equivariant. Therefore, we conclude the lemma by taking the corresponding isotypic components of the above formula. □

Corollary 2.3.3. *Let α and β be as above and $\alpha_1=0$. The complexes of $\mathscr{D}_{\mathbb{G}_m}$-modules $\mathrm{pr}_{z?}\mathcal{E}^f$ are μ_d^{n+m-1}-equivariant and concentrated in degree 0 for $?\in\{\dagger,+\}$. Moreover, we have*

$$\mathcal{H}yp(*;\alpha;\beta) \simeq (\mathcal{H}^0\mathrm{pr}_{z+}\mathcal{E}^f)^{(G,\tilde{\chi}\times\rho)},$$
$$\mathcal{H}yp(!;\alpha;\beta) \simeq (\mathcal{H}^0\mathrm{pr}_{z\dagger}\mathcal{E}^f)^{(G,\tilde{\chi}\times\rho)}.$$

Proof. Similar to Proposition 2.3.1, we only consider the case of $\mathcal{H}yp(*;\alpha;\beta)$. By the construction of the diagram in (2.3.0.3), the morphisms pr_z and f are μ_d^{n+m-1}-equivariant. Hence the complex of $\mathscr{D}_{\mathbb{G}_m}$-modules $\mathrm{pr}_{z+}\mathcal{E}^f = \mathrm{pr}_{z+}f^+\mathcal{E}^t$ is μ_d^{n+m-1}-equivariant.

By assumption, we set $\alpha_1 = 0$, which implies that the character χ_1 is trivial. By Proposition 2.3.1, we have

$$\begin{aligned}(\varpi_+\mathcal{E}^\sigma)^{(\mu_d^{n+m},\chi\times\rho)} &= \left(\left(x_1\cdot\prod_{i=2}^n x_i^d \Big/ \prod_{j=1}^m y_j^d\right)_+ \mathcal{E}^{x_1+\sum_{i=2}^m x_i^d - \sum_j y_j^d}\right)^{(1\times G,1\times\tilde{\chi}\times\rho)} \\ &= (\mathrm{pr}_{z+}\mathcal{E}^f)^{(G,\tilde{\chi}\times\rho)}, \end{aligned} \tag{2.3.3.1}$$

where we performed a change of variable $z = x_1\cdot\prod_{i=2}^n x_i^d/\prod_{j=1}^m y_j^d$ to get rid of the variable x_1 in the last isomorphism. Because $(\varpi_+\mathcal{E}^\sigma)$ is concentrated in degree 0 and isomorphic to $\mathcal{H}yp(*;\alpha;\beta)$, so is $(\mathrm{pr}_{z+}\mathcal{E}^f)^{(G,\tilde{\chi}\times\rho)}$. □

Corollary 2.3.4. *Assume that (α,β) is nonresonant and $\alpha_1 = 0$. Then, the natural map*

$$(\mathcal{H}^0\mathrm{pr}_{z\dagger}\mathcal{E}^f)^{(G,\tilde{\chi}\times\rho)} \to (\mathcal{H}^0\mathrm{pr}_{z+}\mathcal{E}^f)^{(G,\tilde{\chi}\times\rho)}$$

is an isomorphism of $\mathscr{D}_{\mathbb{G}_m}$-modules. In particular, for a closed point[3] a of S, the forget-support map

$$\mathrm{H}^{n+m-1}_{\mathrm{dR},c}(\mathbb{G}_m^{n+m-1}, f_a)^{(G,\tilde{\chi}\times\rho)} \to \mathrm{H}^{n+m-1}_{\mathrm{dR}}(\mathbb{G}_m^{n+m-1}, f_a)^{(G,\tilde{\chi}\times\rho)}$$

is an isomorphism.

Proof. Using induction on the size of α and β, one can verify that the diagram

$$\begin{array}{ccc} \mathcal{H}yp(!;\alpha;\beta) & \xrightarrow{\simeq} & \mathcal{H}yp(*;\alpha;\beta) \\ \downarrow\simeq & & \downarrow\simeq \\ (\mathcal{H}^0\varpi_\dagger\mathcal{E}^\sigma)^{(\mu_d^{n+m},\chi\times\rho)} & \longrightarrow & (\mathcal{H}^0\varpi_+\mathcal{E}^\sigma)^{(\mu_d^{n+m},\chi\times\rho)} \\ \downarrow\simeq & & \downarrow\simeq \\ (\mathcal{H}^0\mathrm{pr}_{z\dagger}\mathcal{E}^f)^{(G,\tilde{\chi}\times\rho)} & \longrightarrow & (\mathcal{H}^0\mathrm{pr}_{z+}\mathcal{E}^f)^{(G,\tilde{\chi}\times\rho)} \end{array}$$

is commutative, where the horizontal morphisms are the canonical forget-support morphisms with the top one being (2.1.3.1), the two upper vertical morphisms are those from Proposition 2.3.1, and the two lower vertical morphisms are (2.3.3.1). So, we deduce the isomorphism

$$(\mathcal{H}^0\mathrm{pr}_{z\dagger}\mathcal{E}^f)^{(G,\tilde{\chi}\times\rho)} \to (\mathcal{H}^0\mathrm{pr}_{z+}\mathcal{E}^f)^{(G,\tilde{\chi}\times\rho)}.$$

At last, we take the noncharacteristic inverse image along $a:\mathrm{Spec}(\mathbb{C})\to\mathbb{G}_m$ and the base change theorem [Hotta et al. 2008, Theorem 1.7.3 & Proposition 1.5.28] to conclude the isomorphism of twisted de Rham cohomology groups. □

[3]The modified hypergeometric $\mathscr{D}_{\mathbb{G}_m}$-modules $\mathcal{H}yp(?,\alpha;\beta)$ are smooth on S, on which the hypergeometric connections $\mathcal{H}yp(\alpha;\beta)$ are defined.

Remark 2.3.5. When (α, β) is nonresonant and $\alpha_1 = 0$, we deduce from Corollary 2.3.3 the isomorphism

$$[z \mapsto (-1)^{n-m} z]^{+} \mathcal{H}yp(\alpha; \beta) \simeq (\mathcal{H}^0 \mathrm{pr}_{z+} \mathcal{E}^{-f})^{(G, \tilde{\chi} \times \rho)}$$

by performing a change of variable by sending x_i and y_j to $-x_i$ and $-y_j$, respectively, in the diagram (2.3.0.3). According to (2.1.1.3), the first term in the above is $\mathcal{H}yp_{(-1)^{n-m}}(\alpha; \beta)$. In particular, the results in Corollary 2.3.4 remain valid if we replace f with $-f$.

2.4. ***Explicit cyclic vectors for hypergeometric connections.*** We present explicit cyclic vectors for $\mathcal{H}yp(\alpha; \beta)$ in terms of sections of some subquotients of some relative de Rham cohomologies equipped with their Gauss–Manin connections. This point of view will be used in the computation of Hodge filtrations in Section 3.

Recall that d is an integer such that $a_i = d\alpha_i$ and $b_j = d\beta_j$ are integers for all i and j, and we take notation from (2.3.0.3). When (α, β) is nonresonant and $\alpha_1 = 0$, there exists an isomorphism between the hypergeometric connection $\mathcal{H}yp(\alpha; \beta)$ and $(\mathcal{H}^0 \mathrm{pr}_{z\dagger} \mathcal{E}^{f})^{(G, \tilde{\chi} \times \rho)}|_S$ by (2.1.3.1) and Corollary 2.3.3. From now on, we will identify the latter with the relative de Rham cohomology $\mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S, f)^{(G, \tilde{\chi} \times \rho)}$ on S equipped with the Gauss–Manin connection, where $U = S \times \mathbb{G}_m^{n+m-1}$.

Proposition 2.4.1. *Suppose that $\alpha_1 = 0$ and (α, β) is nonresonant. The relative de Rham cohomology $\mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S, f)^{(G, \tilde{\chi} \times \rho)}$ admits a cyclic vector, defined by the cohomology class of the differential form*

$$\omega = \prod_{i=2}^{n} x_i^{a_i} \cdot \prod_{j=1}^{m} y_j^{-b_j} \frac{\mathrm{d}x_2}{x_2} \cdots \frac{\mathrm{d}x_n}{x_n} \frac{\mathrm{d}y_1}{y_1} \cdots \frac{\mathrm{d}y_m}{y_m}. \tag{2.4.1.1}$$

Remark 2.4.2. Under the above assumption, the isomorphism class of $\mathcal{H}yp(\alpha; \beta)$ depends only on the congruence classes of α and β modulo $\mathbb{Z}$. Then, any differential form

$$\prod_{i=2}^{n} x_i^{u_i} \cdot \prod_{j=1}^{m} y_j^{-v_j} \frac{\mathrm{d}x_2}{x_2} \cdots \frac{\mathrm{d}x_n}{x_n} \frac{\mathrm{d}y_1}{y_1} \cdots \frac{\mathrm{d}y_m}{y_m},$$

satisfying $u_i \equiv a_i$, $v_j \equiv b_j$ modulo d, is a cyclic vector of $\mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S, f)^{(G, \tilde{\chi} \times \rho)}$.

Proof. The morphism $\mathrm{pr}_z : U = S \times \mathbb{G}_m^{n+m-1} \to S$ in (2.3.0.3) is smooth. It follows that the relative de Rham cohomologies $\mathcal{H}^i_{\mathrm{dR}}(U/S, f)$ are equipped with the Gauss–Manin connections $D := \nabla_{z\partial_z}$, given by

$$\nabla_{z\partial_z} \omega = z\partial_z \omega + z\partial_z(f)\omega \tag{2.4.2.1}$$

for $0 \leq i \leq n+m-1$. By Lemmas 2.2.2, 2.2.3, and 2.2.4, the Laurent polynomial $f_a := f|_{\mathrm{pr}_z^{-1}(a)}$ is nondegenerate for each closed point a of S. By [Adolphson and Sperber 1997, Theorems 1.4 and 4.1], the cohomology group $\mathcal{H}^i_{\mathrm{dR}}(U/S, f_a)$ vanishes if $i \neq n+m-1$.

Now we consider the $(G, \tilde{\chi} \times \rho)$-isotypic component of the connection $\mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S, f)$. It remains to prove that the cohomology class defined by the differential form ω (2.4.1.1) is a cyclic vector for $\mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S, f)^{(G, \tilde{\chi} \times \rho)}$.

Lemma 2.4.3. *Let $t_2, \dots, t_n$ and $s_1, \dots, s_m$ be integers, and set*

$$\tilde{\omega} := \prod_{i=2}^{n} x_i^{t_i} \cdot \prod_{j=1}^{m} y_j^{s_j} \frac{\mathrm{d}x_2}{x_2} \cdots \frac{\mathrm{d}x_n}{x_n} \frac{\mathrm{d}y_1}{y_1} \cdots \frac{\mathrm{d}y_m}{y_m}$$

as a class in $\mathcal{H}_{\mathrm{dR}}^{n+m-1}(U/S, f)$. For each u and v such that $2 \leq u \leq n$ and $1 \leq v \leq m$, respectively, we have

$$\left(D - \frac{t_u}{d}\right)\tilde{\omega} = x_u^d \cdot \tilde{\omega} \quad \textit{and} \quad \left(D + \frac{s_v}{d}\right)\tilde{\omega} = y_v^d \cdot \tilde{\omega}.$$

Proof. We prove the identity $(D - t_2/d)\tilde{\omega} = x_2^d \cdot \tilde{\omega}$, and the proof for the rest is identical. By (2.4.2.1), we have

$$D\tilde{\omega} = z \cdot \frac{\prod_{j=1}^{m} y_j^d}{\prod_{i=2}^{n} x_i^d} \tilde{\omega}.$$

Since U and S are affine, the image of any $(n+m-2)$-form under the relative differential

$$\nabla_{U/S} : \Omega_{U/S}^{n+m-2} \to \Omega_{U/S}^{n+m-1}$$

in $\mathcal{H}_{\mathrm{dR}}^{n+m-1}(U/S, f)$ is zero. Then, we have

$$\begin{aligned} 0 = \nabla_{U/S}\left(\prod_{i=2}^{n} x_i^{t_i} \cdot \prod_{j=1}^{m} y_j^{s_j} \frac{\mathrm{d}x_3}{x_3} \cdots \frac{\mathrm{d}x_n}{x_n} \frac{\mathrm{d}y_1}{y_1} \cdots \frac{\mathrm{d}y_m}{y_m}\right) = t_2 \cdot \tilde{\omega} + x_2 \cdot \partial_{x_2} f \cdot \tilde{\omega} \\ = t_2 \cdot \tilde{\omega} + x_2 \cdot \left(d x_2^{d-1} - d x_2^{-1} \cdot z \cdot \frac{\prod_{j=1}^{m} y_j^d}{\prod_{i=2}^{n} x_i^d}\right)\tilde{\omega} = d\left(x_2^d - \left(D - \frac{t_2}{d}\right)\right)\tilde{\omega}. \end{aligned}$$

This is exactly what we want to prove. □

We show that ω (2.4.1.1) satisfies the hypergeometric differential equation $\mathrm{Hyp}(\alpha; \beta)$. By Lemma 2.4.3, we have

$$\prod_{i=2}^{n} (D - \alpha_i)\omega = \prod_{i=2}^{n} x_i^d \cdot \omega \quad \text{and} \quad \prod_{j=1}^{m} (D - \beta_j)\omega = \prod_{j=1}^{m} y_j^d \cdot \omega.$$

Then, we deduce from (2.4.2.1) that

$$\prod_{i=1}^{n} (D - \alpha_i)\omega = D\left(\prod_{i=2}^{n} x_i^d \cdot \omega\right) = z \prod_{j=1}^{m} y_j^d \cdot \omega = z \prod_{j=1}^{m} (D - \beta_j)\omega.$$

Lemma 2.4.4. *The cohomology class of ω in $\mathcal{H}_{\mathrm{dR}}^{n+m-1}(U/S, f)^{(G, \tilde{\chi} \times \rho)}$ is nonzero.*

Proof. The lemma is obviously true if $n = 1$ and $m = 0$. In general, assume that $\omega = 0$. Each point $(A, B) \in (a_i, -b_j) + d \cdot \mathbb{Z}^{n+m-1}$ corresponds to a differential form

$$\prod_{i=2}^{n} x_i^{A_i} \cdot \prod_{j=1}^{m} y_j^{B_j} \frac{\mathrm{d}x_2}{x_2} \cdots \frac{\mathrm{d}x_n}{x_n} \frac{\mathrm{d}y_1}{y_1} \cdots \frac{\mathrm{d}y_m}{y_m}. \tag{2.4.4.1}$$

Notice that we can also take the isotypic components on the level of complexes of differential forms, and the relative differential $\nabla_{U/S}$ respects the corresponding isotopic components. Thus, the differential form in (2.4.4.1) defines a cohomology class in $\mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S,\, f)^{(G,\tilde{\chi}\times\rho)}$. On the other hand, any cohomology class has a representative that is a linear combination of such differential forms.

Since $\mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S,\, f)^{(G,\tilde{\chi}\times\rho)}$ is nonzero, we can select a differential form $\omega^{(0)}$ that defines a nonzero cohomology class. Given that $\omega^{(0)}$ is a linear combination of differential forms of the form in (2.4.4.1), at least one of such forms defines a nonzero cohomology class. We may assume, without loss of generality, that $\omega^{(0)}$ itself is of the form in (2.4.4.1). Using Lemma 2.4.3, we obtain a sequence of differential forms $\{\omega^{(i)}\}_{i=0}^{N}$ corresponding to points $(A^{(i)},\, B^{(i)})$ such that $\omega^{(i+1)} = (D-\gamma_i)\omega^{(i)}$ for some rational number γ_i, and

$$(A^{(N)},\, B^{(N)}) \in (a_i,\, -b_j) + d\cdot\mathbb{N}^{n+m-1}.$$

Applying Lemma 2.4.3 again, we observe that $\omega^{(N)}$ can be expressed as a linear combination of $\{D^k\omega\}_{k\in\mathbb{N}}$, and is thus equal to 0. Hence there exists $M < N$ such that $\omega^{(M)}$ has a nonzero cohomology class and $(D-\gamma_M)\omega^{(M)} = 0$. Thus, $\mathcal{O}_S\cdot\omega^{(M)}$ is the hypergeometric connection $\mathcal{H}yp(\gamma_M;\varnothing)$. Since it is a subconnection of the irreducible connection $\mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S,\, f)^{(G,\tilde{\chi}\times\rho)}$, it must be isomorphic to $\mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S,\, f)^{(G,\tilde{\chi}\times\rho)}$, leading to a contradiction. □

In summary, we obtain a nonzero morphism

$$\mathscr{D}_S/\operatorname{Hyp}(\alpha;\beta) \to \bigoplus_{i=0}^{n-1}\mathcal{O}_S\cdot D^i\omega \subset \mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S,\, f)^{(G,\tilde{\chi}\times\rho)} \tag{2.4.4.2}$$

defined by sending 1 to ω. Since the left-hand side of the morphism is irreducible, it must be a subconnection of the irreducible connection $\mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S,\, f)^{(G,\tilde{\chi}\times\rho)}$ on the right-hand side. Since both sides have the same rank, the above morphism is an isomorphism, implying that ω is a cyclic vector of $\operatorname{Hyp}(\alpha;\beta) \simeq \mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S,\, f)^{(G,\tilde{\chi}\times\rho)}$. □

Remark 2.4.5. If we replace $\mathcal{E}^f$ by

$$(\mathbb{G}_m^{n+m-1}, \mathrm{d}-\mathrm{d}f) = (\mathbb{G}_m^{n+m-1}, \mathrm{d}+\mathrm{d}f)^\vee,$$

the direct sum $\bigoplus_{i=0}^{n-1}\mathcal{O}_S\cdot D^i\omega$ is the $(G,\, \tilde{\chi}\times\rho)$-isotypic component of $\mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S, -f)$, isomorphic to the connection $\mathcal{H}yp_{(-1)^{n-m}}(\alpha;\beta)$. To see this, it suffices to notice that the corresponding identities in Lemma 2.4.3 become

$$\left(D-\frac{t_u}{d}\right)\omega_{t,s} = -x_u^d\omega_{t,s} \quad\text{and}\quad \left(D+\frac{s_v}{d}\right)\omega_{t,s} = -y_v^d\omega_{t,s}$$

in this case. The rest of the proof relies on the above calculation and Remark 2.3.5.

2.4.6. *Resonant case.* When (α,β) is resonant, the modified hypergeometric $\mathscr{D}$-module $\mathcal{H}yp(*;\alpha;\beta)$ depends only on the classes of α and β modulo $\mathbb{Z}$. Katz [1990, 6.3.8] asked whether $\mathcal{H}yp(*;\alpha;\beta)|_S$ is isomorphic to the connection $\mathcal{H}yp((\alpha_i+r_i);(\beta_j+s_j))$ (2.1.1.1) for suitable integers $r_i, s_j\in\mathbb{Z}$. We provide a positive answer to this question in the following proposition.

Proposition 2.4.7. *When (α,β) is resonant, there exists a positive integer h depending on $\alpha \bmod \mathbb{Z}$ and $\beta \bmod \mathbb{Z}$, such that, for any integers $r, s > h$, the modified hypergeometric $\mathscr{D}$-module $\mathcal{H}yp(*;\alpha;\beta)|_S$ is isomorphic to the hypergeometric connection $\mathcal{H}yp((\alpha_1,\alpha_2-r,\ldots,\alpha_n-r);\beta+s)$.*

Proof. We may assume that $\alpha_1 = 0$. Let $\tilde{\omega}_1,\ldots,\tilde{\omega}_n$ be a representative of a basis of the connection $\mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S, f)^{(G,\tilde{\chi}\times\rho)}$. More precisely, we can write

$$\tilde{\omega}_k = \sum_{e\in\mathbb{Z}^{n-1},\, f\in\mathbb{Z}^m} \epsilon_{k,e,f} \prod_{i=2}^{n} x_i^{a_i+d\cdot e_i} \prod_{j=1}^{m} y_j^{-b_j+d\cdot f_j} \frac{\mathrm{d}x_2}{x_2}\cdots\frac{\mathrm{d}x_n}{x_n}\frac{\mathrm{d}y_1}{y_1}\cdots\frac{\mathrm{d}y_m}{y_m},$$

where only finitely many $\epsilon_{k,e,f}$ are nonzero. We equip $\mathbb{Z}^{n+m-1}$ with the partial order defined by the relation $a \geq b$ if $a-b\in\mathbb{N}^{n+m-1}$. Let (e_0, f_0) be a maximal element in the set

$$\{(e',f') \mid (e',f') \leq (e,f) \text{ if } \epsilon_{k,e,f}\neq 0\}.$$

Then we take h to be the maximal value among $\{|(e_0)|_i, |(f_0)|_j\}$.

For any $r, s > h$, as in Proposition 2.4.1, we define a morphism of $\mathscr{D}$-modules:

$$\mathscr{D}_S/\operatorname{Hyp}(0,\alpha_2-r,\ldots,\alpha_n-r;\beta+s) \to \bigoplus_{i=0}^{n-1} \mathcal{O}_{\mathbb{G}_m}\cdot D^i\omega \subset \mathcal{H}^{n+m-1}_{\mathrm{dR}}(U/S, f)^{(G,\tilde{\chi}\times\rho)} \tag{2.4.7.1}$$

by sending 1 to

$$\omega = \prod_{i=2}^{n} x_i^{a_i-d\cdot r}\cdot\prod_{j=1}^{m} y_j^{-b_j-d\cdot s}\frac{\mathrm{d}x_2}{x_2}\cdots\frac{\mathrm{d}x_n}{x_n}\frac{\mathrm{d}y_1}{y_1}\cdots\frac{\mathrm{d}y_m}{y_m}.$$

Since, for all (e,f) with $\epsilon_{k,e,f}\neq 0$, we have $a_i+d\cdot e_i \geq a_i - d\cdot r$ and $b_j + d\cdot f_j \geq b_j - d\cdot s$ for any i and j, we deduce that the class defined by

$$\prod_{i=2}^{n} x_i^{a_i+d\cdot e_i} \prod_{j=1}^{m} y_j^{-b_j+d\cdot f_j} \frac{\mathrm{d}x_2}{x_2}\cdots\frac{\mathrm{d}x_n}{x_n}\frac{\mathrm{d}y_1}{y_1}\cdots\frac{\mathrm{d}y_m}{y_m}$$

lies in the image of (2.4.7.1) by Lemma 2.4.3. This morphism is a surjection between two connections of rank n and is, hence, an isomorphism. □

3. Irregular Hodge filtration of hypergeometric connections

This section aims to calculate the (irregular) Hodge filtrations of hypergeometric connections (see Theorems 3.3.1 and 3.3.3). Throughout this section, let $n \geq m \geq 0$ be two integers, and let $\alpha = (\alpha_1,\ldots,\alpha_n)$ and $\beta = (\beta_1,\ldots,\beta_j)$ be two sequences of nondecreasing rational numbers in $[0,1)$.

3.1. ***Exponential mixed Hodge structures.*** To explain certain duality on the irregular Hodge filtration of hypergeometric connections, we use the language of exponential mixed Hodge structures introduced by Kontsevich and Soibelman [2011]. We recall the basic definitions of exponential mixed Hodge structures from [Fresán et al. 2022, Appendix].

Let X be a smooth algebraic variety and K a number field. We denote by $\mathrm{MHM}(X, K)$ the abelian category of *mixed Hodge modules* on X with coefficients in K. In particular, when $X = \mathrm{Spec}(\mathbb{C})$, the category $\mathrm{MHM}(X, K)$ is equivalent to the category of mixed K-Hodge structures. Moreover, the bounded derived categories $\mathrm{D}^b(\mathrm{MHM}(X, K))$ admit the six functor formalism. For more details about mixed Hodge modules, see [Saito 1990].

Let $\pi : \mathbb{A}^1 \to \mathrm{Spec}(\mathbb{C})$ be the structure morphism. The category $\mathrm{EMHS}(K)$ of *exponential mixed Hodge structures* with coefficients in K is defined as the full subcategory of $\mathrm{MHM}(\mathbb{A}^1, K)$, whose objects N^{H} have vanishing cohomology on $\mathbb{A}^1$, i.e., those satisfying $\pi_* N^{\mathrm{H}} = 0$.

There is an exact functor $\Pi : \mathrm{MHM}(\mathbb{A}^1, K) \to \mathrm{MHM}(\mathbb{A}^1, K)$ defined by

$$N^{\mathrm{H}} \mapsto s_*(N^{\mathrm{H}} \boxtimes j_! \mathcal{O}^{\mathrm{H}}_{\mathbb{G}_m}), \tag{3.1.0.1}$$

where $j : \mathbb{G}_{m,\mathbb{C}} \to \mathbb{A}^1$ is the inclusion and $s : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ is the summation map. The functor Π is a projector onto $\mathrm{EMHS}(K)$; i.e., it factors through $\mathrm{EMHS}(K)$ with essential image $\mathrm{EMHS}(K)$. Using this functor, the dual of an object M in $\mathrm{EMHS}(K)$ is defined by $\Pi([t \mapsto -t]^* \mathbb{D}(M))$, where t is the coordinate of $\mathbb{A}^1$.

For each object $\Pi(N^{\mathrm{H}})$ of the category $\mathrm{EMHS}(K)$, there exists a weight filtration $W^{\mathrm{EMHS}}_{\bullet}$ on $\Pi(N^{\mathrm{H}})$, defined by the weight filtration on N^{H}: $W^{\mathrm{EMHS}}_n \Pi(N^{\mathrm{H}}) := \Pi(W_n N^{\mathrm{H}})$. We will drop the superscript for simplicity.

The *de Rham fiber* functor from $\mathrm{EMHS}(K)$ to $\mathrm{Vect}_{\mathbb{C}}$ is defined by

$$\Pi(N^{\mathrm{H}}) \mapsto \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1, \Pi(N) \otimes \mathcal{E}^t), \tag{3.1.0.2}$$

where $\Pi(N)$ denotes the underlying $\mathscr{D}$-module of $\Pi(N^{\mathrm{H}})$ and $\mathcal{E}^t$ denotes the exponential $\mathscr{D}$-module $(\mathcal{O}_{\mathbb{A}^1}, \mathrm{d} + \mathrm{d}t)$.

The de Rham fiber functor is faithful, and one can associate an *irregular Hodge filtration* $F^{\bullet}_{\mathrm{irr}}$ on the de Rham fibers of objects in $\mathrm{EMHS}(K)$ by [Fresán et al. 2022, Proposition A.10], constructed using a generalization of Deligne's filtration [Sabbah 2010, §6.b]; see also [Esnault et al. 2017, §1.6].

3.1.1. *Objects of* EMHS *attached to regular functions.* Let X be a smooth affine variety of dimension n and K a number field. We denote by K^{H}_X the trivial Hodge module on X with coefficients in K. For a regular function $g : X \to \mathbb{A}^1$ and an integer r, we consider the exponential mixed Hodge structures

$$\mathrm{H}^r(X, g) := \Pi(\mathcal{H}^{r-n} g_* K^{\mathrm{H}}_X), \quad \mathrm{H}^r_c(X, g) := \Pi(\mathcal{H}^{r-n} g_! K^{\mathrm{H}}_X).$$

The exponential mixed Hodge structures $\mathrm{H}^r(X, g)$ and $\mathrm{H}^r_c(X, g)$ are mixed of weights at least r and mixed of weights at most r, respectively, by [Fresán et al. 2022, A.19].

The de Rham fiber of $\mathrm{H}^r_?(X, g)$ is isomorphic to $\mathrm{H}^r_{\mathrm{dR},?}(X, g)$ for $? \in \{\varnothing, c\}$. In this case, Esnault, Sabbah, and Yu showed [Esnault et al. 2017, Proposition 1.7.4] that the irregular Hodge filtration on the de Rham fiber coincides with the Yu filtration [2014] on the twisted de Rham cohomologies, where the two filtrations are denoted by $F^{\bullet}_{\mathrm{Del}} = F^{\mathrm{Del}}_{-\bullet}$ and $F^{\bullet}_{\mathrm{Yu}} = F^{\mathrm{Yu}}_{-\bullet}$, respectively, in [loc. cit.].

3.1.2. *Irregular Hodge filtration and Newton monomial filtration.* We briefly recall the definition of the irregular Hodge filtration on the twisted de Rham cohomology following [Yu 2014]. Let X and g be as above, $j : X \to \overline{X}$ a smooth compactification of X, and $D := \overline{X}\backslash X$ the boundary divisor. The pair $(\overline{X}, D)$ is called a *good compactification* of the pair (X, g) if D is normal crossing and g extends to a morphism $\bar{g} : \overline{X} \to \mathbb{P}^1$.

Let P be the pole divisor of g. The twisted de Rham complex $(\Omega^\bullet_{\overline{X}}(*D), \nabla = \mathrm{d} + \mathrm{d}g)$ admits a decreasing filtration $F^\lambda(\nabla) := F^0(\lambda)^{\geq\lceil\lambda\rceil}$, indexed by nonnegative real numbers λ, where $F^0(\lambda)$ is the Yu complex

$$\mathcal{O}_{\overline{X}}(\lfloor -\lambda P\rfloor) \xrightarrow{\nabla} \Omega^1_{\overline{X}}(\log D)(\lfloor (1-\lambda)P\rfloor) \to \cdots \to \Omega^p_{\overline{X}}(\log D)(\lfloor (p-\lambda)P\rfloor) \to \cdots .$$

The *irregular Hodge filtration* on the de Rham cohomology $\mathrm{H}^i_{\mathrm{dR}}(X, g)$ is defined by

$$F^\lambda_{\mathrm{irr}}\mathrm{H}^i_{\mathrm{dR}}(X, g) := \mathrm{im}(\mathbb{H}^i(\overline{X}, F^\lambda(\nabla)) \to \mathrm{H}^i_{\mathrm{dR}}(X, g)), \tag{3.1.0.3}$$

which is independent of the choice of the good compactification $(\overline{X}, D)$ [Yu 2014, Theorem 1.7].

When X is isomorphic to a torus $\mathbb{G}^n_m$, the regular function g on X is a Laurent polynomial of the form $\sum_{P=(p_1,\ldots,p_n)} c(P)x^P$. We refine the normal fan of the Newton polytope $\Delta(g)$ to make the associated toric variety X_{tor} smooth proper. Although $(X_{\mathrm{tor}}, D_{\mathrm{tor}} = X_{\mathrm{tor}}\backslash X)$ is not a good compactification for the pair (X, g) in general, we can still define $F^\lambda_{\mathrm{NP}}(\nabla)$ and the *Newton polyhedron filtration* $F^\lambda_{\mathrm{NP}}\mathrm{H}^i_{\mathrm{dR}}(U, \nabla)$ similarly to that in (3.1.0.3) by replacing the good compactification $(\overline{X}, D)$ with $(X_{\mathrm{tor}}, D_{\mathrm{tor}})$,

When g is nondegenerate with respect to $\Delta(g)$, the only nonvanishing twisted de Rham cohomology group of the pair (X, g) is the middle cohomology group $\mathrm{H}^n_{\mathrm{dR}}(X, g)$ by [Adolphson and Sperber 1997, Theorem 1.4]. Moreover, we have the following theorem.

Theorem 3.1.1 [Yu 2014, Theorem 4.6]. *When g is nondegenerate with respect to $\Delta(g)$, the irregular Hodge filtration $F^\bullet_{\mathrm{irr}}$ agrees with the Newton polyhedron filtration $F^\bullet_{\mathrm{NP}}$ on $\mathrm{H}^n_{\mathrm{dR}}(X, g)$.*

In particular, when g is nondegenerate, we have

$$\mathbb{H}^i(X_{\mathrm{tor}}, F^\lambda_{\mathrm{NP}}(\nabla)) = \mathrm{H}^i(\Gamma(X_{\mathrm{tor}}, F^\lambda_{\mathrm{NP}}(\nabla))),$$

which allows us to compute the irregular Hodge filtration using the knowledge of $\Delta(g)$.

Now, we present an explicit way to calculate the Newton polyhedron filtration. For a cohomology class

$$\omega = x^Q \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n}$$

such that the lattice point $Q = (q_1, \ldots, q_n)$ lies in $\mathbb{R}_{\geq 0}\Delta(g)$, we define $w(Q)$ to be the weight of Q in the sense of [Adolphson and Sperber 1997], i.e., the minimal positive real number w such that $Q \in w \cdot \Delta(g)$. The associated cohomology class of ω lies in $F^\lambda_{\mathrm{NP}}\mathrm{H}^n_{\mathrm{dR}}(X, g)$ if

$$\omega \in \Gamma(X_{\mathrm{tor}}, \Omega^n_{X_{\mathrm{tor}}}(\log D_{\mathrm{tor}})(\lfloor (n-\lambda)P\rfloor)).$$

Notice that each ray ρ in the normal fan of $\Delta(g)$ corresponds to an irreducible component P_ρ of P. Let v_ρ be a primitive vector of the ray ρ. Then, the multiplicity of ω along P_ρ is given by $\langle Q, v_\rho\rangle$ [Fulton 1993, p. 61]. Taking the multiplicities of P_ρ in P into account, we have

$$x^Q \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n} \in F_{\mathrm{NP}}^{n-w(Q)} \mathrm{H}^n_{\mathrm{dR}}(X, g), \tag{3.1.1.1}$$

as remarked in [Yu 2014, p. 126 footnote].

3.1.3. *The* EMHS *associated with hypergeometric connections.* In this subsection, we assume $\alpha_1 = 0$ and let $\tilde{\chi} \times \rho$ be the product of characters associated with α_i and β_j in (2.3.0.1).

Definition 3.1.2. Let K be the number field $\mathbb{Q}(\zeta_d^{a_i}, \zeta_d^{b_j})$ and a a closed point of S. For $? \in \{\varnothing, c\}$, we define

$$E_?(a; \alpha; \beta) := \mathrm{H}^{n+m-1}_?(\mathbb{G}_m^{n+m-1}, f_a)^{(G, \tilde{\chi}\times\rho)}$$

as exponential mixed Hodge structures with coefficients in K in the sense of Section 3.1.1.

By Corollary 2.3.3 and the base change theorem, the de Rham fiber of $E(a; \alpha; \beta)$ is isomorphic to the fiber of $\mathcal{H}yp_\lambda(\alpha; \beta)$ at the closed point $a \cdot \lambda$ of S for $\lambda \in \mathbb{Q}^\times$. In other words, the fiber $\mathcal{H}yp_\lambda(\alpha; \beta)_{a\lambda}$ underlies the above exponential mixed Hodge structure and is equipped with an irregular Hodge filtration F_{irr}, which coincides with the Yu filtration on $\mathrm{H}^{n+m-1}_{\mathrm{dR}}(\mathbb{G}_m^{n+m-1}, f_a)^{(G, \tilde{\chi}\times\rho)}$ as explained in Section 3.1.1.

Remark 3.1.3. The geometric interpretations of the hypergeometric connection are not unique, and their associated Yu filtrations on $\mathcal{H}yp(\alpha; \beta)_a$ coincide only up to certain shifts. For example, we can alternatively identify $\mathcal{H}yp(\alpha; \beta)_a$ with

$$\mathrm{H}^{n+m-1}_{\mathrm{dR}}(\mathbb{G}_m^{n+m-1}, f_a)^{(G, \tilde{\chi}\times\rho)} \otimes \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, x),$$

where the Yu filtration on the one-dimensional vector space $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m, x)$ jumps at 1. Consequently, we deduce a new irregular Hodge filtration on $\mathcal{H}yp(\alpha; \beta)_a$ which differs from our current one by a shift of 1. For this reason, we made a choice of a uniform shift for irregular Hodge filtrations on fibers of $\mathcal{H}yp(\alpha; \beta)$ at closed points of S, using the exponential mixed Hodge structures in Definition 3.1.2. Moreover, this specifically chosen shift determines the shift of the function θ in (1.0.1.1).

Let t be the largest natural number such that $\alpha_t = 0$. We let $\bar{\alpha}$ and $\bar{\beta}$ be the sequences of rational numbers defined by

$$\bar{\alpha}_i = \begin{cases} 0, & 1 \le k \le t, \\ 1 - \alpha_{n+t+1-k}, & t+1 \le k \le n, \end{cases} \quad \text{and} \quad \bar{\beta}_k = 1 - \beta_k. \tag{3.1.3.1}$$

Proposition 3.1.4. (1) *The dual of the exponential mixed Hodge structure $E_c(a; \alpha; \beta)$ is isomorphic to $E((-1)^{n-m}a; \bar{\alpha}; \bar{\beta})(n+m-1)$.*

(2) *When (α, β) is nonresonant, the exponential mixed Hodge structures $E_?(a; \alpha; \beta)$ for $? \in \{\varnothing, c\}$ are isomorphic. In particular, they are pure of weight $n+m-1$.*

Proof. (1) The EMHS $\mathrm{H}_c^{n+m-1}(\mathbb{G}_m^{n+m-1}, f_a)$ is dual to

$$\mathrm{H}^{n+m-1}(\mathbb{G}_m^{n+m-1}, -f_a) \otimes \mathrm{H}_c^{2n+2m-2}(\mathbb{G}_m^{n+m-1})^\vee,$$

which is also isomorphic to $\mathrm{H}^{n+m-1}(\mathbb{G}_m^{n+m-1}, f_{(-1)^{n-m}a}) \otimes \mathrm{H}_c^{2n+2m-2}(\mathbb{G}_m^{n+m-1})^\vee$. We deduce the first assertion by taking their corresponding isotypic components.

(2) Since the de Rham fiber functor is faithful, the forget-support morphism

$$E_c(a; \alpha; \beta) \to E(a; \alpha; \beta)$$

is an isomorphism by Corollary 2.3.4. Hence the exponential mixed Hodge structures $E_c(a; \alpha; \beta)$ and $E(a; \alpha; \beta)$ are isomorphic and are pure of weight $n+m-1$. □

3.2. ***A basis in relative twisted de Rham cohomology.*** In this subsection, we assume $\alpha_1 = 0$. We define positive integers $s_1, \dots, s_{m+1}$ by

$$s_r = \begin{cases} 1, & r = 0, \\ \#\{i : \alpha_i < \beta_r\}, & 1 \le r \le m, \\ n+1, & r = m+1, \end{cases} \tag{3.2.0.1}$$

and, for r and ℓ such that $0 \le r \le m$ and $1 \le \ell \le s_{r+1} - s_r$, we set

$$g_{r,\ell} = x_2^{a_2} \cdots x_{s_r+\ell-1}^{a_{s_r+\ell-1}} \cdot x_{s_r+\ell}^{a_{s_r+\ell}-d} \cdots x_n^{a_n-d} \cdot y_1^{d-b_1} \cdots y_r^{d-b_r} \cdot y_{r+1}^{2d-b_{r+1}} \cdots y_m^{2d-b_m}.$$

Let

$$\eta = \frac{\mathrm{d}x_2}{x_2} \cdots \frac{\mathrm{d}x_n}{x_n} \frac{\mathrm{d}y_1}{y_1} \cdots \frac{\mathrm{d}y_m}{y_m}$$

and $\omega_{r,\ell} = g_{r,\ell} \cdot \eta$ be the corresponding differential forms in $\mathcal{H}_{\mathrm{dR}}^{n+m-1}(U/S, \pm f)^{(G, \tilde{\chi} \times \rho)}$, where U and S are defined in (2.3.0.3).

Proposition 3.2.1. *If* (α, β) *is nonresonant, then the cohomology classes defined by*

$$\omega_{r,\ell}, \quad 0 \le r \le m, \quad 1 \le \ell \le s_{r+1} - s_r$$

in $\mathcal{H}_{\mathrm{dR}}^{n+m-1}(U/S, \pm f)^{(G, \tilde{\chi} \times \rho)}$ *form a basis over* $\mathcal{O}_S$.

Proof. It suffices to show that $\mathrm{Span}(\omega_{r,\ell}) = \mathrm{Span}(D^i \omega \mid 0 \le i \le n-1)$ for a cyclic vector ω.

To a Laurent monomial $g = \prod_{i=2}^n x_i^{u_i} \prod_{j=1}^m y_j^{v_j}$ in variables $\{x_i\}_{i=2}^n$ and $\{y_j\}_{j=1}^m$ we associate a lattice point $\mathcal{P}(g) = (u_2, \dots, u_n, v_1, \dots, v_m) \in \mathbb{Z}^{n+m-1} \subset \mathbb{R}^{n+m-1}$. If $\omega = g \cdot \eta$ is the product of a monomial g with the differential form η, we set $\mathcal{P}(\omega) := \mathcal{P}(g)$ for the corresponding point.

Let π_1 and π_2 be the projections from $\mathbb{R}^{n+m-1}$ to $\mathbb{R}_{u_i}^{n-1}$ and $\mathbb{R}_{v_j}^m$, respectively. Then, for the differential forms $\omega_{r,\ell}$, we have

$$\pi_1(\mathcal{P}(\omega_{r,\ell})) = (a_2, \dots, a_{s_r+\ell-1}, a_{s_r+\ell} - d, \dots, a_n - d)$$

and

$$\pi_2(\mathcal{P}(\omega_{r,\ell})) = (d - b_1, \dots, d - b_r, 2d - b_{r+1}, \dots, 2d - b_m).$$

In Lemmas 2.2.2, 2.2.3, and 2.2.4, we have written down the defining inequalities of the cone $\mathbb{R}_{\geq 0} \cdot \Delta(f_a)$ explicitly as $u_i + v_j \geq 0$ and $v_j \geq 0$. For the points $\mathcal{P}(\omega_{r,\ell})$ corresponding to the values of $\omega_{r,\ell}$ at closed points a of S, we can verify that they satisfy $u_i + v_j \geq 0$ and $v_j \geq 0$ using the fact that $a_i, b_j \leq d$ for any i and j, and $a_i \geq b_j$ when $i \geq s_j + 1$. Thus, all the points $\mathcal{P}(\omega_{r,\ell})$ lie within the cone $\mathbb{R}_{\geq 0} \cdot \Delta(f_a)$.

Let P_i and Q_j be the points corresponding to monomials x_i^d and y_j^d, respectively, for $2 \leq i \leq n$ and $1 \leq j \leq m$.

Lemma 3.2.2. *For a point $P \in \mathbb{Z}^{n+m-1}$ and two integers $2 \leq i_0 \leq n$ and $1 \leq j_0 \leq m$, let ω_0, ω_1, and ω_2 be the corresponding differential forms of the points P, $P + Q_{j_0}$, and $P + P_{i_0}$ in $\mathbb{Z}^{n+m-1}$. If the i_0-th coordinate of P is different from the negative of the j_0-th coordinate of P, then we have*

$$\operatorname{Span}(\omega_0, \omega_2) = \operatorname{Span}(\omega_1, \omega_2) \quad \text{in } \mathrm{H}_{\mathrm{dR}}^{n+m-1}(U/S, f)^{(G, \tilde{\chi} \times \rho)}.$$

Proof. Let P be the point $(t_i, s_j) \in \mathbb{Z}^{n+m-1}$ and ω_0 be the associated differential form. By assumption, we have $t_{i_0} \neq -s_{j_0}$. Therefore, we can express

$$\omega_0 = \frac{-d}{t_{i_0} + s_{j_0}} \left(\left(D - \frac{t_{i_0}}{d} \right) \omega_0 - \left(D + \frac{s_{j_0}}{d} \right) \omega_0 \right).$$

In particular, we have

$$\operatorname{Span}\left(\omega_0, \left(D - \frac{t_{i_0}}{d} \right) \omega_0 \right) = \operatorname{Span}\left(\left(D - \frac{t_{i_0}}{d} \right) \omega_0, \left(D + \frac{s_{j_0}}{d} \right) \omega_0 \right).$$

At last, notice that we have

$$\omega_1 = \left(D + \frac{s_{j_0}}{d} \right) \omega_0 \quad \text{and} \quad \omega_2 = \left(D - \frac{t_{i_0}}{d} \right) \omega_0$$

by Lemma 2.4.3 and Remark 2.4.5. □

Step 1: If $s_1 - s_0 = 0$, we skip this step and put $\omega_{r,\ell}^{(1)} = \omega_{r,\ell}$ for any r and ℓ. Otherwise, for $r = 0$ and $1 \leq \ell \leq s_1 - s_0$, we replace the differential forms $\omega_{0,\ell}$ by differential forms $\omega_{0,\ell}^{(1)}$ of the form $g \cdot \eta$ for some monomials g such that

$$\mathcal{P}(\omega_{0,\ell}^{(1)}) = \mathcal{P}(\omega_{0,\ell}) - Q_1.$$

More precisely, we keep the first $n-1$ coordinates of $\mathcal{P}(\omega_{0,\ell})$ unchanged and replace the last m coordinates of $\mathcal{P}(\omega_{0,\ell})$ by that of $\mathcal{P}(\omega_{0,\ell}^{(1)})$:

$$(d - b_1, 2d - b_2, \ldots, 2d - b_m).$$

In particular, by Lemma 2.4.3, one has

$$(D + 1 - \beta_1)\omega_{0,\ell}^{(1)} = \omega_{0,\ell}, \quad (D + 1 - \alpha_{\ell+1})\omega_{0,\ell}^{(1)} = \omega_{0,\ell+1}^{(1)},$$

and

$$(D + 1 - \alpha_{s_1 - s_0})\omega_{0,s_1 - s_0}^{(1)} = \omega_{e,1},$$

where e is the least integer such that $s_e > s_0 = 1$.

We also put $\omega^{(1)}_{r,\ell} = \omega_{r,\ell}$ for $r \geq 1$. Then, using Lemma 3.2.2 for $\omega_0 = \omega^{(1)}_{0,s_1-s_0}$, $\omega_1 = \omega_{0,s_1-s_0}$, and $\omega_2 = \omega_{e,1}$, we have

$$\begin{aligned}\mathrm{Span}\{\omega_{r,\ell} \mid r, \ell\} &= \mathrm{Span}\big(\dots, \omega_{0,s_1-s_0}\big(= (D+1-\beta_1)\omega^{(1)}_{0,s_1-s_0}\big), \omega_{e,1}\big(= (D+1-\alpha_{s_1-s_0})\omega^{(1)}_{0,s_1-s_0}\big), \dots\big) \\ &= \mathrm{Span}(\{\omega_{0,1}, \dots, \omega_{0,s_1-s_0-1}, \omega^{(1)}_{0,s_1-s_0}\} \cup \{\omega^{(1)}_{r,\ell} \mid r \geq 1, \ell\}),\end{aligned}$$

where $0 \leq r \leq m$ and $1 \leq \ell \leq s_{r+1} - s_r$. Continuing to use Lemma 3.2.2 for $\omega_0 = \omega^{(1)}_{0,\ell}$, $\omega_1 = \omega_{0,\ell}$, and $\omega_2 = \omega^{(1)}_{0,\ell+1}$ for $\ell = s_1 - s_0 - 1$ and $s_1 - s_0 - 2, \dots, 1$, we have

$$\begin{aligned}\mathrm{Span}\{\omega_{r,\ell} \mid r, \ell\} &= \mathrm{Span}(\{\omega_{0,1}, \dots, \omega_{0,s_1-s_0-1}, \omega^{(1)}_{0,s_1-s_0}\} \cup \{\omega^{(1)}_{r,\ell} \mid r \geq 1, \ell\}) \\ &= \mathrm{Span}(\{\omega_{0,1}, \omega^{(1)}_{0,2}, \dots, \omega^{(1)}_{0,s_1-s_0}\} \cup \{\omega^{(1)}_{r,\ell} \mid r \geq 1, \ell\}) \\ &= \mathrm{Span}(\omega^{(1)}_{r,\ell} \mid r, \ell).\end{aligned}$$

Step $i \geq 2$: Assume that we have already obtained elements $\omega^{(i-1)}_{r,\ell}$ for $i \geq 2$. If $s_i = s_{i-1}$, we skip this step and put $\omega^{(i)}_{r,\ell} = \omega^{(i-1)}_{r,\ell}$ for any r and ℓ. Otherwise, let $\omega^{(i)}_{r,\ell}$ be differential forms of the form $g \cdot \eta$ for some monomials g such that

$$\mathcal{P}(\omega^{(i)}_{r,\ell}) = \begin{cases} \mathcal{P}(\omega^{(i-1)}_{r,\ell}) - Q_i & \text{if } r \leq i-1, \\ \mathcal{P}(\omega^{(i-1)}_{r,\ell}) & \text{if } i \leq r \leq m. \end{cases}$$

More precisely, when $r \leq i-1$, we keep the first $n-1$ coordinates of $\mathcal{P}(\omega^{(i-1)}_{r,\ell})$ unchanged and replace the last m coordinates of $\mathcal{P}(\omega^{(i-1)}_{r,\ell})$ by that of $\mathcal{P}(\omega^{(i)}_{r,\ell})$:

$$(d - b_1, \dots, d - b_i, 2d - b_{i+1}, \dots, 2d - b_m).$$

Similar to Step 1, we use Lemmas 2.4.3 and 3.2.2 $(s_{r+1}-s_r)$-many times to deduce

$$\mathrm{Span}(\omega^{(i)}_{r,\ell} \mid r, \ell) = \mathrm{Span}(\omega^{(i-1)}_{r,\ell} \mid r, \ell) = \mathrm{Span}(\omega_{r,\ell} \mid r, \ell),$$

where $0 \leq r \leq m$ and $1 \leq \ell \leq s_{r+1} - s_r$.

After Step m: After m steps, we get $\omega^{(m)}_{r,\ell}$ such that

$$\mathcal{P}(\omega^{(m)}_{r,\ell}) = (a_2, \dots, a_{s_r+\ell-1}, a_{s_r+\ell} - d, \dots, a_n - d, d - b_1, \dots, d - b_m).$$

Note that there is a bijection between $\{(r, \ell)\}_{0 \leq r \leq m, 1 \leq \ell \leq s_{r+1}-s_r}$ and $\{1, \dots, n\}$ by sending (r, ℓ) to $s_r + \ell - 1$. We set $\tilde{\omega}_{s_r+\ell-1} = \omega^{(m)}_{r,\ell}$ via this map. Then, by Lemma 2.4.3, we have

$$\tilde{\omega}_{i+1} = (D + 1 - \alpha_{i+1})\tilde{\omega}_i \quad \text{for } 1 \leq i \leq n-1.$$

It follows that

$$\begin{aligned}\mathrm{Span}(D^i \tilde{\omega}_1 \mid 0 \leq i \leq n-1) &= \mathrm{Span}(\tilde{\omega}_i \mid 1 \leq i \leq n) \\ &= \mathrm{Span}(\omega^{(m)}_{r,\ell} \mid r, \ell) = \mathrm{Span}(\omega_{r,\ell} \mid r, \ell).\end{aligned}$$

By Proposition 2.4.1 and Remark 2.4.2, $\tilde{\omega}_1$ is a cyclic vector, from which we showed that $\{\omega_{r,\ell}\}_{r,\ell}$ form a basis. This finishes the proof. ∎

3.3. ***Calculation of the irregular Hodge filtration.*** Using the fact that a nonresonant hypergeometric connection is rigid or its geometric interpretation Corollary 2.3.3, it underlies an irregular Hodge module on $\mathbb{P}^1$ of weight $n+m-1$ by [Sabbah 2018, Theorem 0.7 & p. 78] and, therefore, admits a unique irregular Hodge filtration $F_{\mathrm{irr}}^{\bullet}$. When $n=m$, this irregular Hodge module coincides with the variation of Hodge structures on $\mathcal{H}yp(\alpha;\beta)$.

Recall that, for (α,β), we defined in (1.0.1.1) the numbers

$$\theta(k)=(n-m)\alpha_k+\#\{i\mid\beta_i<\alpha_k\}+(n-k)-\sum_{i=1}^{n}\alpha_i+\sum_{j=1}^{m}\beta_j.$$

Theorem 3.3.1. *Assume (α,β) is nonresonant.*

(1) *When $\alpha_1=0$, via the isomorphism $\mathcal{H}yp(\alpha;\beta)\simeq\mathcal{H}_{\mathrm{dR}}^{n+m-1}(U/S,f)^{(G,\tilde\chi\times\rho)}$, the irregular Hodge filtration on $\mathcal{H}yp(\alpha;\beta)$ can be identified with the following filtration of subbundles*:

$$F_{\mathrm{irr}}^{p}\mathcal{H}_{\mathrm{dR}}^{n+m-1}(U/S,f)^{(G,\tilde\chi\times\rho)}=\bigoplus_{n+m-1-w(\omega_{r,s})\ge p}\omega_{r,s}\mathcal{O}_S.$$

(2) *Up to an $\mathbb{R}$-shift, the jumps of the irregular Hodge filtration on $\mathcal{H}yp(\alpha;\beta)$ occur at $\theta(k)$ and, for any $p\in\mathbb{R}$, we have*

$$\operatorname{rk}\operatorname{gr}_{F_{\mathrm{irr}}}^{p}\mathcal{H}yp(\alpha;\beta)=\#\theta^{-1}(p).$$

Remark 3.3.2. (i) By [Sabbah and Yu 2015, Remark 6.3], the irregular Hodge filtration satisfies the Griffiths' transversality; that is, $\nabla(F_{\mathrm{irr}}^{p}\mathcal{H}yp(\alpha;\beta))\subset\Omega_S^1\otimes F_{\mathrm{irr}}^{p-1}\mathcal{H}yp(\alpha;\beta)$ for all $p\in\mathbb{R}$.

(ii) Inspired by the Griffiths' transversality, we expect that there exists an oper structure on the hypergeometric connections which refines the irregular Hodge filtration. An oper structure is essential in the geometric Langlands correspondence [Beilinson and Drinfeld 1997; Zhu 2017; Kamgarpour et al. 2023].

To prove the above theorem, we study the Hodge numbers of the irregular Hodge filtration on fibers as explained in Section 1.2.

Theorem 3.3.3. *Up to an $\mathbb{R}$-shift, the jumps of the irregular Hodge filtration $F_{\mathrm{irr}}^{\bullet}$ on the fiber $\mathcal{H}yp(\alpha;\beta)_a$ occur at $\theta(k)$ for $1\le k\le n$. Moreover, we have* $\dim\operatorname{gr}_{F_{\mathrm{irr}}}^{p}\mathcal{H}yp(\alpha;\beta)_a=\#\theta^{-1}(p)$ *for any $p\in\mathbb{R}$.*

3.3.1. *Proof of Theorem 3.3.3.* We may assume $\alpha_1=0$ by (2.1.1.2). By Corollary 2.3.3 and Definition 3.1.2, we have

$$\begin{aligned}F_{\mathrm{irr}}^{\bullet}\mathcal{H}yp(\alpha;\beta)_a&\simeq F_{\mathrm{irr}}^{\bullet}\mathrm{H}_{\mathrm{dR}}^{n+m-1}(\mathbb{G}_m^{n+m-1},f_a)^{(G,\tilde\chi\times\rho)}\\&\simeq F_{\mathrm{irr}}^{\bullet}\mathrm{H}_{\mathrm{dR}}^{n+m-1}(\mathbb{G}_m^{n+m-1},-f_{(-1)^{n-m}a})^{(G,\tilde\chi\times\rho)},\end{aligned}\tag{3.3.3.1}$$

where $\tilde\chi$ and ρ are products of characters corresponding to α_i and β_j from (2.3.0.1). So it suffices to compute the irregular Hodge filtration on the twisted de Rham cohomologies $\mathrm{H}_{\mathrm{dR}}^{n+m-1}(\mathbb{G}_m^{n+m},\pm f_a)^{(G,\tilde\chi\times\rho)}$. Since f_a is nondegenerate with respect to $\Delta(f_a)$, we can compute the filtration in terms of Newton polyhedron filtration.

Let $\omega_{r,\ell}$ be the basis of $\mathcal{H}yp(\alpha;\beta)_a$ from Proposition 3.2.1. Recall that $w(\omega_{r,\ell})$ is the minimal positive real number w such that $\mathcal{P}(g_{r,\ell})\in w\cdot\Delta(f_a)$. It follows from (3.1.1.1) that

$$\omega_{r,\ell}\in F_{\mathrm{irr}}^{n+m-1-w(\omega_{r,\ell})}\mathrm{H}_{\mathrm{dR}}^{n+m-1}(\mathbb{G}_m^{n+m-1},\pm f_a)^{(G,\tilde{\chi}\times\rho)}.$$

We consider an auxiliary filtration $G^\bullet$ on $\mathrm{H}_{\mathrm{dR}}^{n+m-1}(\mathbb{G}_m^{n+m-1},\pm f_a)^{(G,\tilde{\chi}\times\rho)}$ defined by

$$G^p:=\mathrm{Span}\{\omega_{r,\ell}\mid n+m-1-w(\omega_{r,\ell})\geq p\}. \tag{3.3.3.2}$$

By the following Lemmas 3.3.4, 3.3.5, and 3.3.6, the filtration $F^\bullet$ coincides with $G^\bullet$, which finishes the proof of the theorem. □

Lemma 3.3.4. *We set $\theta(n+1)=\theta(1)$. For $0\leq r\leq m$ and $1\leq\ell\leq s_{r+1}-s_r$, we have*

$$n+m-1-w(\omega_{r,\ell})=\theta(s_r+\ell).$$

Lemma 3.3.5. *For $0\leq p\leq n+m-1$, we have*

$$\dim\mathrm{gr}_G^p\,\mathrm{H}_{\mathrm{dR}}^{n+m-1}(\mathbb{G}_m^{n+m-1},\pm f_a)^{(G,\tilde{\chi}\times\rho)}=\dim\mathrm{gr}_G^{n+m-1-p}\,\mathrm{H}_{\mathrm{dR}}^{n+m-1}(\mathbb{G}_m^{n+m-1},\mp f_a)^{(G,\tilde{\chi}^{-1}\times\rho^{-1})}.$$

Lemma 3.3.6. *The two filtrations $F_{\mathrm{irr}}^\bullet$ and $G^\bullet$ coincide.*

Proof of Lemma 3.3.4. By Lemmas 2.2.2, 2.2.3, and 2.2.4, the weight $w(\omega_{r,\ell})$ equals the number $\max_k\{h_k(g_{r,\ell})/d\}$, where the h_k are defined in (2.2.2.1), (2.2.3.1), and (2.2.4.1). We can check that

$$w(\omega_{r,\ell})=\frac{h_{s_r+\ell}(g_{r,\ell})}{d},$$

where we put $h_1=\cdots=h_n=h_{n+1}$ when $n=m$. Now, it suffices to check that $n+m-1-w(\omega_{r,\ell})$ agrees with one of the jumps of the irregular Hodge numbers of $\mathcal{H}yp(\alpha;\beta)_a$.

If $s_r+\ell=n+1$, the monomial $g_{m,n+1-s_m}$ corresponds to the point

$$(a_2,\ldots,a_n,d-b_1,\ldots,d-b_m).$$

Then we have

$$n+m-1-\frac{h_{n+1}(g_{m,n+1-s_m})}{d}=n-1-\sum_{i=1}^n\alpha_i+\sum_{j=1}^m\beta_j=\theta(1).$$

If $s_r+\ell<n+1$, we have

$$\begin{aligned}n+m-1-\frac{h_{s_r+\ell}(g_{r,\ell})}{d}&\\&=n+m-1-\Big(\sum_{i=1}^n\alpha_i-(n+1-s_r-\ell)-\sum_{j=1}^m\beta_j+(2m-r)-(n-m)(\alpha_{s_r+\ell}-1)\Big)\\&=(n-m)\alpha_{s_r+\ell}+r+(n-s_r-\ell)-\sum_{i=1}^n\alpha_i+\sum_{j=1}^m\beta_j,\end{aligned}$$

which is exactly $\theta(s_r+\ell)$. □

Proof of Lemma 3.3.5. For simplicity, we write

$$\delta_p^{\pm}(\alpha, \beta) := \dim \mathrm{gr}_G^p \mathrm{H}_{\mathrm{dR}}^{n+m-1}(\mathbb{G}_m^{n+m-1}, \pm f_a)^{(G, \tilde{\chi} \times \rho)}. \tag{3.3.6.1}$$

Recall that, in (3.1.3.1), we let t be the biggest natural number such that $\alpha_t = 0$. For $1 \le k \le t$, the numbers α_k and $\bar{\alpha}_{t+1-k}$ are 0. And, for $t+1 \le k \le n$, we have $\bar{\alpha}_{n-k+t+1} = 1 - \alpha_k$. Then

$$\sum_{i=1}^n \alpha_i + \sum_{i=1}^n \bar{\alpha}_i = n - t \quad \text{and} \quad \sum_{j=1}^m \beta_j + \sum_{j=1}^m \bar{\beta}_j = m.$$

Similar to the number $\theta(k)$, we let $\bar{\theta}(k)$ be the numbers

$$(n-m)\bar{\alpha}_k + \#\{i \mid \bar{\beta}_i < \bar{\alpha}_k\} + (n-k) - \sum_{i=1}^n \bar{\alpha}_i + \sum_{j=1}^m \bar{\beta}_j, \quad 1 \le k \le n,$$

for the sequences $\bar{\alpha}$ and $\bar{\beta}$. Then, for $1 \le k \le t$, we have

$$\begin{aligned}\theta(k) + \bar{\theta}(t+1-k) &= \left(n - k - \sum_{i=1}^n \alpha_i + \sum_{j=1}^m \beta_j\right) + \left(n - (t+1-k) - \sum_{i=1}^n \bar{\alpha}_i + \sum_{j=1}^m \bar{\beta}_j\right) \\ &= (2n - t - 1) - (n - t) + m = n + m - 1.\end{aligned}$$

For $t+1 \le k \le n$, we have

$$\begin{aligned}&\theta(k) + \bar{\theta}(n-k+t+1) \\ &\quad = \left((n-m)\alpha_k + \#\{j \mid \beta_j < \alpha_k\} + n - k - \sum_{i=1}^n \alpha_i + \sum_{j=1}^m \beta_j\right) \\ &\qquad + \left((n-m)\bar{\alpha}_{n-k+t+1} + \#\{j \mid \bar{\beta}_j < \bar{\alpha}_{n-k+t+1}\} + n - (n-k+t+1) - \sum_{i=1}^n \bar{\alpha}_i + \sum_{j=1}^m \bar{\beta}_j\right) \\ &\quad = (n-m) + m + (n-t-1) - (n-t) + m = n + m - 1.\end{aligned}$$

So there exists a permutation $\sigma \in S_n$ such that $\theta(k) + \bar{\theta}(\sigma(k)) = n + m - 1$. It follows that

$$\begin{aligned}\delta_p^{\pm}(\alpha, \beta) &= \#\{k \mid \theta(k) = p\} = \#\{k \mid n + m - 1 - p = n + m - 1 - \theta(k)\} \\ &= \#\{k \mid \bar{\theta}(k) = n + m - 1 - p\} = \delta_{n+m-1-p}^{\mp}(\bar{\alpha}, \bar{\beta}).\end{aligned}$$

□

Proof of Lemma 3.3.6. For simplicity, we write

$$h_p^{\pm}(\alpha, \beta) := \dim \mathrm{gr}_{F_{\mathrm{irr}}}^p \mathrm{H}_{\mathrm{dR}}^{n+m-1}(\mathbb{G}_m^{n+m-1}, \pm f_a)^{(G, \tilde{\chi} \times \rho)}. \tag{3.3.6.2}$$

By (3.1.1.1) and the construction of the auxiliary filtration G (3.3.3.2), for every $p \in \mathbb{Q}$, we have

$$G^p \mathrm{H}_{\mathrm{dR}}^{n+m-1}(\mathbb{G}_m^{n+m-1}, \pm f_a)^{(G, \tilde{\chi} \times \rho)} \subset F_{\mathrm{irr}}^p \mathrm{H}_{\mathrm{dR}}^{n+m-1}(\mathbb{G}_m^{n+m-1}, \pm f_a)^{(G, \tilde{\chi} \times \rho)}, \tag{3.3.6.3}$$

which implies that $\sum_{q \le p} \delta_q^{\pm}(\alpha, \beta) \le \sum_{q \le p} h_q^{\pm}(\alpha, \beta)$.

To prove the reverse inclusion, we consider the duality between the two filtered vector spaces

$$(\mathrm{H}^{n+m-1}_{\mathrm{dR}}(\mathbb{G}_m^{n+m-1}, \pm f_a)^{(G,\tilde{\chi}\times\rho)}, F^{\bullet}_{\mathrm{irr}}) \quad \text{and} \quad (\mathrm{H}^{n+m-1}_{\mathrm{dR}}(\mathbb{G}_m^{n+m-1}, \mp f_a)^{(G,\tilde{\chi}^{-1}\times\rho^{-1})}, F^{\bullet}_{\mathrm{irr}}),$$

induced by Proposition 3.1.4. In particular, we have

$$h_p^{\pm}(\alpha,\beta) = h^{\mp}_{n+m-1-p}(\bar{\alpha},\bar{\beta}). \tag{3.3.6.4}$$

Combining Lemma 3.3.5 and equations (3.3.6.3) and (3.3.6.4), we see for any $p \in \mathbb{R}$ that

$$\begin{aligned}\dim G^p\mathrm{H}^{n+m-1}_{\mathrm{dR}}(\mathbb{G}_m^{n+m-1}, \pm f_a)^{(G,\tilde{\chi}\times\rho)} &= \sum_{q\le p}\delta_q^{\pm}(\alpha,\beta) \le \sum_{q\le p} h_q^{\pm}(\alpha,\beta) = \sum_{q\ge n+m-1-p} h_q^{\mp}(\bar{\alpha},\bar{\beta}) \\ &\le \sum_{q\ge n+m-1-p}\delta_q^{\mp}(\bar{\alpha},\bar{\beta}) = \sum_{q\le p}\delta_q^{\pm}(\alpha,\beta) \\ &= \dim G^p\mathrm{H}^{n+m-1}_{\mathrm{dR}}(\mathbb{G}_m^{n+m-1}, \pm f_a)^{(G,\tilde{\chi}\times\rho)}.\end{aligned}$$

Hence both sides in (3.3.6.3) have the same dimension for every p. Then Lemma 3.3.6 follows. □

3.3.2. *Proof of Theorem 3.3.1.* We may assume $\alpha_1 = 0$ by (2.1.1.2). By [Sabbah 2018, Proposition 3.54] and [Mochizuki 2025, Proposition 11.22], the irregular Hodge filtration on $\mathcal{H}yp(\alpha;\beta)$ induces those on fibers $\mathcal{H}yp(\alpha;\beta)_a$ at closed points of S; i.e.,

$$(F^{\bullet}_{\mathrm{irr}}\mathcal{H}yp(\alpha;\beta))_a = F^{\bullet}_{\mathrm{irr}}(\mathcal{H}yp(\alpha;\beta))_a.$$

We have shown in Theorem 3.3.3 that the irregular Hodge filtration on the fibers $\mathcal{H}yp(\alpha;\beta)_a$ are given in terms of the cohomology classes $\omega_{r,s}$ in (3.3.3.2). Hence we deduce that the irregular Hodge filtration on $\mathcal{H}yp(\alpha;\beta)$ is the one in assertion (1).

From (1), we deduce that the irregular Hodge numbers $\mathrm{rk}\,\mathrm{gr}^p_{F_{\mathrm{irr}}}\mathcal{H}yp(\alpha;\beta)$ are given by

$$\#\{(r,s) \mid n+m-1-w(\omega_{r,s}) = p\}.$$

Recall that we have the bijection between the sets

$$\{1,\dots,n\} \quad \text{and} \quad \{(r,\ell) \mid 0\le r\le m,\ 1\le \ell\le s_{r+1}-s_r\},$$

where s_r are numbers defined in (3.2.0.1). Using Lemma 3.3.4, we deduce assertion (2); i.e., the irregular Hodge numbers $\mathrm{rk}\,\mathrm{gr}^p_{F_{\mathrm{irr}}}\mathcal{H}yp(\alpha;\beta)$ coincide with the numbers $\#\theta^{-1}(p)$. □

4. Frobenius structures on hypergeometric connections and *p*-adic estimates

In this section, let p be a prime number and $k = \mathbb{F}_q$ the finite field with $q = p^s$ elements for an integer $s \ge 1$. Let K be a finite extension of $\mathbb{Q}_p$ with residue field k containing an element π satisfying $\pi^{p-1} = -p$. We fix such an element π and denote the associated additive character by $\psi : \mathbb{F}_p \to K^{\times}$ [Berthelot 1984, (1.3)]. The q-th power Frobenius on k admits a lift $\sigma = \mathrm{id}$ on $\mathcal{O}_K$.

Let $n > m$ be two integers,

$$\alpha = \left(\alpha_i = \frac{a_i}{q-1}\right)_{i=1}^{n} \quad \text{and} \quad \beta = \left(\beta_j = \frac{b_j}{q-1}\right)_{j=1}^{m}$$

be two sequences of nondecreasing rational numbers in $[0, 1)$ with denominator $q-1$. Let $\omega : k^\times \to K^\times$ be the Teichmüller character, and set $\chi_i = \omega^{a_i}$ and $\rho_j = \omega^{b_j}$. The hypergeometric sum associated to ψ, $\chi = (\chi_1, \dots, \chi_n)$ and $\rho = (\rho_1, \dots, \rho_m)$ is defined, for $a \in k^\times$, by

$$\mathrm{Hyp}_{(n,m)}(\chi; \rho)(a) = \sum_{\substack{x_i, y_j \in k^\times \\ x_1 \cdots x_n = a y_1 \cdots y_m}} \psi\left(\mathrm{Tr}_{k/\mathbb{F}_p}\left(\sum_{i=1}^{n} x_i - \sum_{j=1}^{m} y_j\right)\right) \cdot \prod_{i=1}^{n} \chi_i(x_i) \prod_{j=1}^{m} \rho_j^{-1}(y_j). \tag{4.0.0.1}$$

When (χ, ρ) is nonresonant, the above sum equals (up to a sign) the Frobenius trace of the hypergeometric overconvergent F-isocrystal $\mathscr{H}yp(\chi, \rho)$ at $a \in \mathbb{G}_m(k)$ [Miyatani 2020] and therefore can be written as a sum of n Frobenius eigenvalues. Its underlying connection is the hypergeometric connection $\mathcal{H}yp_{(-1)^{m+np}/\pi^{n-m}}(\alpha; \beta)$ [Miyatani 2020, Theorem 4.1.3]. When (χ, ρ) is resonant, the above sum can also be written as a sum of n Frobenius eigenvalues (see Section 4.2.1 for a direct proof by induction on n).

We are interested in the p-adic valuation of Frobenius eigenvalues (normalized by ord_q) of the above sum (called *Frobenius slopes*), encoded in the Frobenius Newton polygon [Mazur 1972, §2].

Recall that the irregular Hodge numbers of the hypergeometric connection $\mathcal{H}yp(\alpha; \beta)$ are given by the function $\theta : \{1, \dots, n\} \to \mathbb{Q}$ (1.0.1.1), defined by

$$\theta(k) = (n-m)\alpha_k + \#\{i \mid \beta_i < \alpha_k\} + (n-k) - \sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{m} \beta_j. \tag{4.0.0.2}$$

Definition 4.0.1. Let $\delta_1 < \cdots < \delta_k$ be the Frobenius slopes of $\mathrm{Hyp}_{(n,m)}(\chi; \rho)(a)$, normalized by $\mathrm{ord}_q(q) = 1$, (resp. irregular Hodge numbers of $\mathcal{H}yp(\alpha, \beta)$) with multiplicities $\lambda_1, \dots, \lambda_k$. The Newton polygon (resp. irregular Hodge polygon) is defined as the union of segments in $\mathbb{R}^2$ joining points P_i and P_{i+1} for $0 \le i \le k-1$, where the P_i are given by

$$P_0 = (0, 0), \qquad P_i = \left(\sum_{j=1}^{i} \lambda_j, \sum_{j=1}^{i} \lambda_j \delta_j\right) \quad \text{for } 1 \le i \le k.$$

Theorem 4.0.2. *Suppose $n > m$ and the orders of χ_i, ρ_j divide $p-1$. Then, for each $a \in \mathbb{G}_m(k)$, the Frobenius Newton polygon of* $\mathrm{Hyp}_{(n,m)}(\chi; \rho)(a)$ *coincides with the irregular Hodge polygon defined by* (4.0.0.2).

A "Newton above Hodge" type result for twisted exponential sums was obtained in [Adolphson and Sperber 1993]. In our case, we show that the (combinatorial) Hodge polygon in [loc. cit.] for hypergeometric sums coincides with the irregular Hodge polygon of hypergeometric connections. Then, we apply a result of Wan [1993] to conclude "Newton equals Hodge".

4.1. ***Frobenius Newton polygon above Hodge polygon.*** In this subsection, we revise Adolphson and Sperber's definition [1993] of (combinatorial) Hodge polygons and their result on "Newton above Hodge" for certain twisted exponential sums. Finally, we can identify their Hodge polygon with the irregular Hodge polygon of hypergeometric connections (Proposition 4.1.7).

4.1.1. Let N be a positive integer,

$$\chi = (\chi_1, \dots, \chi_N) : (k^\times)^N \to K^\times$$

a multiplicative character, and $g : \mathbb{G}_m^N \to \mathbb{A}^1$ a morphism defined by a Laurent polynomial

$$g(x_1, \dots, x_N) = \sum_{j=1}^{M} a_j x^{u_j} \in k[x_1^\pm, \dots, x_N^\pm],$$

where $\{u_j\}_{j=1}^M$ is a finite subset of $\mathbb{Z}^N$ and $a_j \in k^\times$. For $m \in \mathbb{N}$, we consider the twisted exponential sum

$$S_m(\chi, g) = \sum_{x \in \mathbb{G}_m^N(\mathbb{F}_{q^m})} \chi^{(m)}(x)\psi^{(m)}(g(x)), \tag{4.1.1.1}$$

where $\chi^{(m)} = \chi \circ \mathrm{Nm}_{\mathbb{F}_{q^m}/k} and\ \psi^{(m)} = \psi \circ \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_p}$. The associated L-function

$$L(\chi, g; T) = \exp\biggl(\sum_{m \geq 1} S_m(\chi, g)\frac{T^m}{m}\biggr) \tag{4.1.1.2}$$

is a rational function in T by the Grothendieck–Lefschetz trace formula (or the Dwork trace formula).

Recall that we denote by $\Delta = \Delta(g)$ the convex closure in $\mathbb{R}^N$ generated by the origin and lattices defined by the exponents $\{u_j\}$ of g in Definition 2.2.1. Let $C(g)$ be the cone over Δ, i.e., the union of all rays in $\mathbb{R}^N$ emanating from the origin and passing through Δ.

We set $M(g) = C(g) \cap \mathbb{Z}^N$. Adolphson and Sperber [1989, (1.7)] considered a subring $R(g)$ of $k[x_1^\pm, \dots, x_N^\pm]$ defined by monomials with exponents in $M(g)$:

$$R(g) = k[x^{M(g)}].$$

We take $d_i \in [0, q-2]$ such that[4] $\chi_i = \omega^{-d_i}$. We set

$$\bar{d}_i = \begin{cases} q - 1 - d_i, & d_i \neq 0, \\ d_i, & d_i = 0, \end{cases}$$

and

$$\boldsymbol{d} = (d_1, \dots, d_N), \qquad \bar{\boldsymbol{d}} = \{\bar{d}_1, \dots, \bar{d}_N\}, \qquad N_{\boldsymbol{d}} = (q-1)^{-1}\boldsymbol{d} + \mathbb{Z}^N.$$

We define an $R(g)$-module $R_{\boldsymbol{d}}(g)$ [Adolphson and Sperber 1989, (1.12)] by

$$R_{\boldsymbol{d}}(g) = \biggl\{\sum_{\text{finite}} b_u x^u \ \bigg|\ u \in N_{\boldsymbol{d}} \cap C(g), b_u \in k\biggr\}.$$

[4]Adolphson–Sperber's convention $\chi_i = \omega^{-d_i}$ is different from our convention in the beginning of Section 4 by a minus sign.

There exists a (minimal) positive integer M such that, for all $u \in \left(\frac{1}{q-1}\mathbb{Z}\right)^{\mathbb{N}} \cap C(g)$, the weight function $w(u)$, defined as the minimal positive real number w such that $u \in w\Delta(g)$, is a rational number with denominator dividing M. Then w defines an increasing filtration on $R(g)$ by

$$R(g)_{i/M} = \left\{ \sum_{u \in M(g)} b_u x^u : w(u) \leq \frac{i}{M} \text{ for all } u \text{ with } b_u \neq 0 \right\}.$$

We denote the associated graded module by

$$\bar{R}(g) = \bigoplus_{i \geq 0} \bar{R}(g)_{i/M},$$
$$\bar{R}(g)_{i/M} = R(g)_{i/M} / R(g)_{(i-1)/M}.$$

Similarly, we equip $R_{\boldsymbol{d}}(g)$ with a filtration compatible with that of $R(g)$ and let $\bar{R}_{\boldsymbol{d}}(g)$ be the associated graded $\bar{R}(g)$-module.

4.1.2. In the following, we assume that g is *nondegenerate* and that $\dim \Delta(g) = N$.

For $1 \leq i \leq N$, let $\bar{g}_i$ be the image of $x_i\, \partial/\partial x_i\, g$ in $\bar{R}(g)_1$, and set

$$\bar{I}_{\boldsymbol{d}} = \bar{g}_1 \bar{R}(g)_{\boldsymbol{d}} + \cdots + \bar{g}_N \bar{R}(g)_{\boldsymbol{d}},$$

a graded submodule of $\bar{R}(g)_{\boldsymbol{d}}$. For each $i \geq 0$, we define a finite set

$$S_{\boldsymbol{d}}^{i/M} \subset N_{\boldsymbol{d}} \cap C(g)$$

of exponents as follows [Adolphson and Sperber 1991, §3]: Take a k-linearly independent set of monomials $\{x^\mu \mid \mu \in S_{\boldsymbol{d}}^{i/M}\}$ of weight i/M which spans a k-subspace $\bar{V}_{\boldsymbol{d},i/M}$ complement to $\bar{R}(g)_{\boldsymbol{d},i/M} \cap \bar{I}_{\boldsymbol{d}}$; i.e.,

$$\bar{R}(g)_{\boldsymbol{d},i/M} = \bar{V}_{\boldsymbol{d},i/M} \oplus (\bar{R}(g)_{\boldsymbol{d},i/M} \cap \bar{I}_{\boldsymbol{d},i/M}).$$

Set

$$S_{\boldsymbol{d}} = \bigcup_{i \geq 0} S_{\boldsymbol{d}}^{i/M},$$

which we also denote by $S_{\boldsymbol{d}}(g)$, and let $V(g)$ be the volume of $\Delta(g)$. The quotient $\bar{R}(g)_{\boldsymbol{d}}/\bar{I}_{\boldsymbol{d}}$ admits a basis of monomials in $S_{\boldsymbol{d}}$ and has dimension [loc. cit., Lemma 2.8]

$$\dim \bar{R}(g)_{\boldsymbol{d}}/\bar{I}_{\boldsymbol{d}} = N!V(g).$$

In this case, the L-function $L(\chi, g; T)^{(-1)^{N-1}}$ (4.1.1.2) is a polynomial of degree $N!V(g)$ [loc. cit., Corollary 2.12]. The q-order of roots of this polynomial are called *Frobenius slopes* of the twisted exponential sums $S_m(\chi, g)$.

Adolphson and Sperber studied the *Frobenius Newton polygon* defined by Frobenius slopes of this L-function (Definition 4.0.1) and compared it with a Hodge polygon defined as below.

For an integer $0 \leq d \leq q-2$, let d' be the nonnegative residue of pd modulo $q-1$. Recall that $q = p^s$ for an integer $s \geq 1$. For $\boldsymbol{d} = (d_1, \ldots, d_N)$, we set $\boldsymbol{d}' = (d_1', \ldots, d_N')$ and $\boldsymbol{d}^{(i)}$ the i-th composition of $(-)'$ on $\boldsymbol{d}$ for $i \geq 1$. Note that $\boldsymbol{d}^{(s)} = \boldsymbol{d}$.

We arrange elements of $S_{\boldsymbol{d}} = \{u_{\boldsymbol{d}}(1), \dots, u_{\boldsymbol{d}}(N!V(g))\}$ by $w(u_{\boldsymbol{d}}(1)) \leq \cdots \leq w(u_{\boldsymbol{d}}(N!V(g)))$, and we repeat this ordering for $S_{\boldsymbol{d}'}, \dots, S_{\boldsymbol{d}^{(s-1)}}$. For an integer $\ell \geq 0$, we set [loc. cit., Theorem 3.17]

$$W(\ell) = \operatorname{card}\left\{ j \;\middle|\; \sum_{i=0}^{s-1} w(u_{\boldsymbol{d}^{(i)}}(j)) = \frac{\ell}{M} \right\}.$$

When $\ell > sNM$, we have $W(\ell) = 0$.

Definition 4.1.3 (Adolphson–Sperber). The Hodge polygon $\mathrm{HP}(\Delta(g)_{\boldsymbol{d}})$ is defined by the convex polygon in $\mathbb{R}^2$ with vertices $(0, 0)$ and

$$\left(\sum_{\ell=0}^{m} W(\ell), \frac{1}{sM} \sum_{\ell=0}^{m} \ell W(\ell) \right), \qquad m = 0, 1, \dots, sNM.$$

Theorem 4.1.4 [Adolphson and Sperber 1993, Corollary 3.18]. *If g is nondegenerate and* $\dim(\Delta(g)) = N$, *the Frobenius Newton polygon of $L(\chi, g; T)^{(-1)^{N-1}}$ lies above the Hodge polygon* $\mathrm{HP}(\Delta(g)_{\boldsymbol{d}})$, *and their endpoints coincide.*

Definition 4.1.5. We say that (g, χ) is *ordinary* if these two polygons coincide. When the character χ is trivial, we simply say g is *ordinary*.

4.1.6. In the following, we apply the above theory to the case of hypergeometric sums at the beginning of Section 4. We may assume that χ_1 is trivial (i.e., $\alpha_1 = 0$). Let a be an element of $k^\times$. We take $N = n + m - 1$, $\boldsymbol{d} = (\bar{a}_2, \dots, \bar{a}_n, b_1, \dots, b_m)$, and g to be the nondegenerate function (2.2.1.1)

$$f_a = a \frac{y_1 \cdots y_m}{x_2 \cdots x_n} + x_2 + \cdots + x_n - y_1 - \cdots - y_m.$$

Then, we recover the hypergeometric sum (4.0.0.1) from (4.1.1.1).

Proposition 4.1.7. *If (χ, ρ) is nonresonant and the orders of the characters χ_i and ρ_j divide $p - 1$, then the Hodge polygon* $\mathrm{HP}(\Delta(f_a)_{\boldsymbol{d}})$ *coincides with the irregular Hodge polygon defined by* (4.0.0.2) *associated to*

$$\left(0, \alpha_2 = \frac{a_2}{p-1}, \dots, \alpha_n = \frac{a_n}{p-1}\right), \quad \left(\beta_1 = \frac{b_1}{p-1}, \dots, \beta_m = \frac{b_m}{p-1}\right).$$

Proof. Since α_i and β_j have denominators dividing $p - 1$, the numbers $\boldsymbol{d}^{(i)}$ are equal to $\boldsymbol{d}$ for every $i \geq 1$. In particular, the multiset of slopes of $\mathrm{HP}(\Delta(f_a)_{\boldsymbol{d}})$ coincides with $w(S_{\boldsymbol{d}}) = \{\omega(u) \mid u \in S_{\boldsymbol{d}}\}$.

The cohomology classes $\omega_{r,\ell} = g_{r,\ell} \cdot \eta$ in Proposition 3.2.1 form a basis of the de Rham cohomology group $\mathrm{H}^{n+m-1}_{\mathrm{dR}}(U_a, f_a)^{(G, \tilde{\chi} \times \rho)}$. By the calculation of cohomology groups [loc. cit., §3, Theorem 3.14], the functions $\{g_{r,\ell}\}$ also form a basis of $\overline{V}_{\bar{\boldsymbol{d}}}$, with $\bar{\boldsymbol{d}} = (a_2, \dots, a_n, \bar{b}_1, \dots, \bar{b}_m)$. Hence

$$w(S_{\bar{\boldsymbol{d}}}) = \{w(g_{r,\ell}) \mid 0 \leq r \leq m, 1 \leq \ell \leq s_{r+1} - s_r\}.$$

By (3.1.1.1), Lemma 3.3.4 and the duality (3.3.6.4), the set of weights $w(S_{\boldsymbol{d}})$ coincides with the set of irregular Hodge numbers (4.0.0.2). Then, the proposition follows. □

4.2. ***Frobenius slopes of hypergeometric sums: proof of Theorem 4.0.2.*** We proceed by induction on n. Suppose the theorem holds when the rank of the hypergeometric F-isocrystal is less than n.

4.2.1. *Resonant case.* We first show that we can deduce the assertion in the resonant case from the induction hypothesis. We assume there exists i and j such that $\alpha_i = \beta_j$.

We slightly modify our convention on α and β by replacing those $\alpha_i, \beta_j = 0$ by 1 and then arranging them as $0 < \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le 1$ and $0 < \beta_1 \le \cdots \le \beta_m \le 1$. Note that this modification does not change the multiset $\{\theta(1), \ldots, \theta(n)\}$ of irregular Hodge numbers. After twisting by a multiplicative character, we may assume that $\chi_n = \rho_m = 1$ are the trivial characters (i.e., $\alpha_n = \beta_m = 1$). Then we have the following identities:

$$\begin{aligned}\mathrm{Hyp}_{(n,m)}(\chi;\rho)(a) &= \sum_{x_i, y_j \in k^\times} \psi\Bigl(\sum_{i=1}^{n-1} x_i + a\frac{y_1\cdots y_m}{x_1\cdots x_{n-1}} - \sum_{j=1}^{m} y_j\Bigr)\cdot \prod_{i=1}^{n-1}\chi_i(x_i)\prod_{j=1}^{m-1}\rho_j^{-1}(y_j)\\ &= \sum_{x_i, y_j \in k^\times, y_m\in k} \psi\Bigl(\sum_{i=1}^{n-1} x_i - \sum_{j=1}^{m-1} y_j + y_m\Bigl(a\frac{y_1\cdots y_{m-1}}{x_1\cdots x_{n-1}} - 1\Bigr)\Bigr)\cdot \prod_{i=1}^{n-1}\chi_i(x_i)\prod_{j=1}^{m-1}\rho_j^{-1}(y_j)\\ &\qquad - \sum_{x_i, y_j \in k^\times} \psi\Bigl(\sum_{i=1}^{n-1} x_i - \sum_{j=1}^{m-1} y_j\Bigr)\cdot \prod_{i=1}^{n-1}\chi_i(x_i)\prod_{j=1}^{m-1}\rho_j^{-1}(y_j)\\ &= q\,\mathrm{Hyp}_{(n-1,m-1)}(\chi';\rho')(a) - \psi(-1)^{m-1}\prod_{i=1}^{n-1} G(\psi,\chi_i)\prod_{j=1}^{m-1} G(\psi,\rho_j^{-1}),\end{aligned} \tag{4.2.1.1}$$

where $\chi' = (\chi_1, \ldots, \chi_{n-1})$ and $\rho' = (\rho_1, \ldots, \rho_{m-1})$, and

$$G(\psi,\chi_i) = \sum_{x\in k^\times} \psi(x)\chi_i(x)$$

denotes the Gauss sum. In particular, the above sum can be written as a sum of n Frobenius eigenvalues by induction.

Let θ' be the function (4.0.0.2) defined by rational numbers $\alpha_1, \ldots, \alpha_{n-1}$ and $\beta_1, \ldots, \beta_{m-1}$. Then, we have

$$\theta(k) = \theta'(k) + 1 \quad \text{for all } 1 \le k \le n-1$$

and

$$\theta(n) = \sum_{i=1}^{n}(1-\alpha_i) + \sum_{\beta_j<1}\beta_j = \mathrm{ord}_q\Bigl(\prod_{i=1}^{n-1} G(\psi,\chi_i)\prod_{j=1}^{m-1} G(\psi,\rho_j^{-1})\Bigr),$$

where the second identity follows from Stickelberger's theorem, saying that

$$\mathrm{ord}_q\, G(\psi,\omega^{-k}) = \frac{k}{p-1}.$$

Then, the theorem in the resonant case follows from the induction hypothesis and decomposition (4.2.1.1).

4.2.2. *Nonresonant case.* By the previous argument, we may assume that the assertion for the hypergeometric sum of type (n, m) defined by a resonant pair (α, β) is already proved. It suffices to treat the nonresonant case. We may assume $\chi_1 = 1$ is trivial.

We set $\tilde{f}_a(x_2, \ldots, x_n, y_1, \ldots, y_m) = f_a(x_2^{p-1}, \ldots, x_n^{p-1}, y_1^{p-1}, \ldots, y_m^{p-1})$. We first prove the ordinariness of exponential sums associated to $\tilde{f}_a$ (Definition 4.1.5) using a theorem of Wan [1993].

Let $\delta_1, \dots, \delta_{m+n}$ be all the facets of $\Delta = \Delta(\tilde{f}_a)$ which do not contain the origin. Let $\tilde{f}_a^{\delta_i}$ be the restriction of $\tilde{f}_a$ to δ_i [Wan 2004, §1.1], which is also nondegenerate [Wan 2004, §3.1]. By [Wan 2004, Theorem 3.1], $\tilde{f}_a$ is ordinary if and only if each $\tilde{f}_a^{\delta_i}$ is ordinary.

Each Laurent polynomial $\tilde{f}_a^{\delta_i}$ is diagonal; that is, $\tilde{f}_a^{\delta_i}$ has exactly $n+m-1$ nonconstant terms of monomials and $\Delta(\tilde{f}_a^{\delta_i})$ is $(n+m-1)$-dimensional [Wan 2004, §2]. Indeed, if $V_1, \dots, V_{m+n-1}$ denote the vertex of δ_i written as column vectors, the set $S(\delta_i)$ of solutions of

$$(V_1, \dots, V_{m+n-1}) \begin{pmatrix} r_1 \\ \vdots \\ r_{m+n-1} \end{pmatrix} \equiv 0 \pmod 1, \quad r_i \text{ rational}, \quad 0 \le r_i < 1,$$

forms an abelian group, which is isomorphic to $(\mathbb{Z}/(p-1)\mathbb{Z})^{n+m-1}$. We deduce that, for each δ_i, $\tilde{f}_a^{\delta_i}$ is ordinary by [Wan 2004, Corollary 2.6].

We have a decomposition of exponential sums as follows:

$$\sum_{x_i, y_j \in k^\times} \psi(\tilde{f}_a(x_i, y_j)) = \sum_{\chi_i, \rho_j} \mathrm{Hyp}_{(n,m)}(\chi, \rho)(a), \tag{4.2.2.1}$$

where the sum is taken over all multiplicative characters χ_i and ρ_j with $2 \le i \le n$ and $1 \le j \le m$ of orders dividing $p-1$. We have a similar decomposition for $S_{\boldsymbol{d}}$ (Section 4.1.2) given by

$$S_1(\tilde{f}_a) = \bigsqcup_{\boldsymbol{d}} S_{\boldsymbol{d}}(f_a),$$

where $1 = (0, 0, \dots, 0)$ and $\boldsymbol{d}$ is taken over all $(n+m-1)$-tuples of rational numbers with denominators $p-1$ in $[0, 1)$.

On the left-hand side of (4.2.2.1), we have shown "Newton equals Hodge" (i.e., the ordinariness of $\tilde{f}_a$). Together with the "Newton above Hodge" for each hypergeometric sum (Theorem 4.1.4), we deduce that "Newton equals Hodge" for each component of the right-hand side. Then, the assertion in the nonresonant case follows from Proposition 4.1.7. □

In particular, our proof shows Proposition 4.1.7 in the resonant case.

Corollary 4.2.3. *Proposition 4.1.7 holds without the nonresonant assumption.*

Proof. In the resonant case, the Frobenius Newton polygon equals the irregular Hodge polygon by Section 4.2.1. By the proof in Section 4.2.2, the Frobenius Newton polygon equals the (combinatorial) Hodge polygon defined by Adolphson–Sperber. Then, the assertion follows. □

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yichenqin@fudan.edu.cn School of Mathematical Sciences, Fudan University, Shanghai, China

daxin.xu@amss.ac.cn Morningside Center of Mathematics and State Key Laboratory of Mathematical Sciences, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China

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