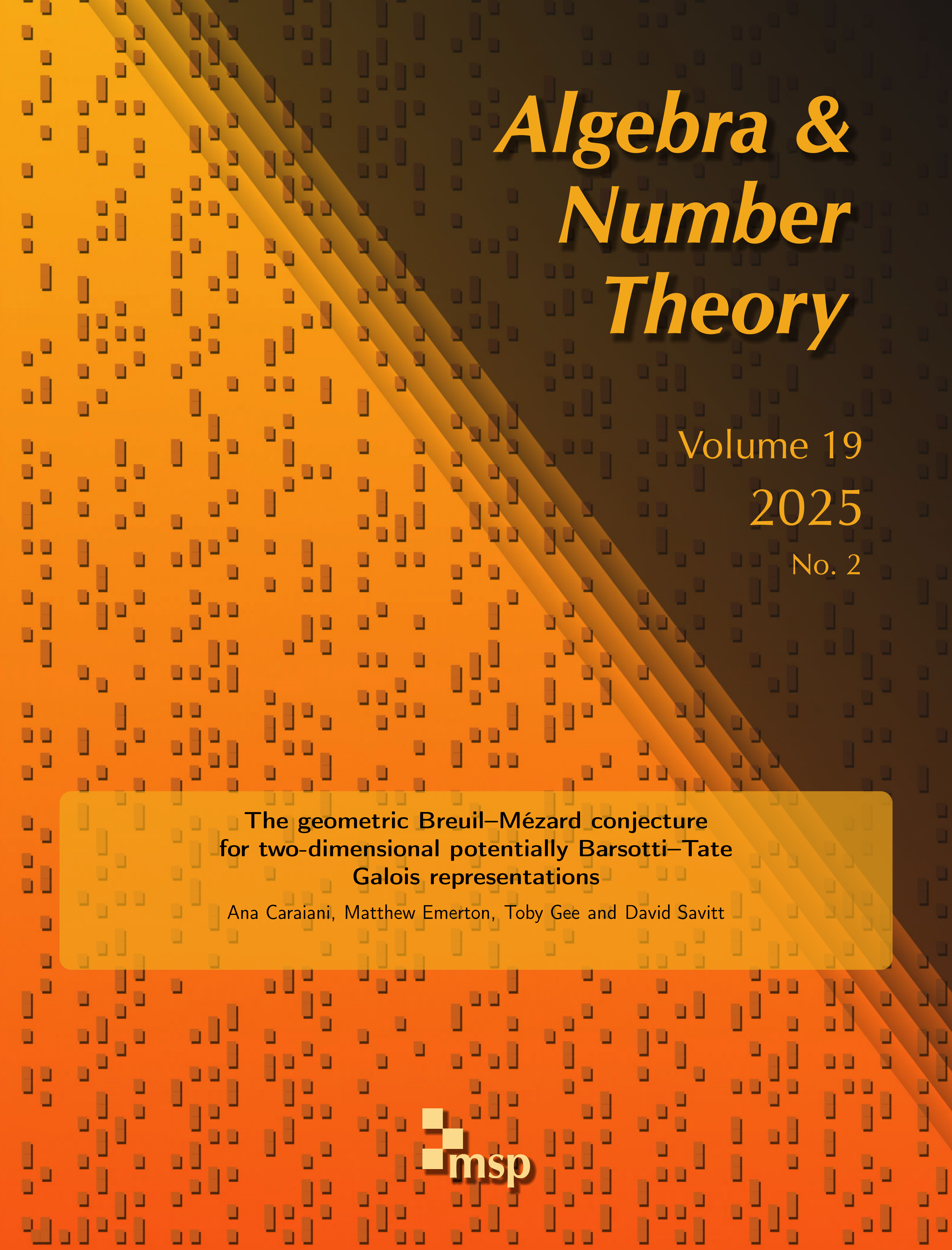

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The geometric Breuil–Mézard conjecture for two-dimensional potentially Barsotti–Tate Galois representations

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We establish a geometrization of the Breuil–Mézard conjecture for potentially Barsotti–Tate representations, as well as of the weight part of Serre's conjecture, for moduli stacks of two-dimensional mod p representations of the absolute Galois group of a p-adic local field. These results are first proved for the stacks of our earlier papers, and then transferred to the stacks of Emerton and Gee by means of a comparison of versal rings.

1. Introduction

Let $K/\mathbb{Q}_p$ be a finite extension with residue field k, let $\overline{K}$ be an algebraic closure of K, and let $d \geq 1$ be a positive integer. Emerton and Gee [2023] have constructed moduli stacks of representations of the absolute Galois group $G_K := \mathrm{Gal}(\overline{K}/K)$, globalizing Mazur's classical deformation theory of Galois representations. These stacks are expected to be the backbone of a categorical p-adic Langlands correspondence, playing the role anticipated by the stacks of [Dat et al. 2020; Zhu 2020] in the $\ell \neq p$ setting.

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To be precise, the book [Emerton and Gee 2023] defines the category $\mathcal{X}_d$ fibered in groupoids over $\operatorname{Spf}\mathbb{Z}_p$ whose A-valued points, for any p-adically complete $\mathbb{Z}_p$-algebra A, are the groupoid of rank d projective étale (φ, Γ)-modules with A-coefficients. Then the finite type points of $\mathcal{X}_d$ correspond to representations $\bar{r} : G_K \to \mathrm{GL}_d(\overline{\mathbb{F}}_p)$, the versal rings of $\mathcal{X}_d$ at finite type points recover classical Galois deformation rings, and one has the following, which is one of the main results of [loc. cit.].

Theorem 1.1 [Emerton and Gee 2023, Theorem 1.2.1]. *Each $\mathcal{X}_d$ is a Noetherian formal algebraic stack. Its underlying reduced substack $\mathcal{X}_{d,\mathrm{red}}$ is an algebraic stack of finite type over $\mathbb{F}_p$, and is equidimensional of dimension $[K : \mathbb{Q}_p]\binom{d}{2}$. The irreducible components of $\mathcal{X}_{d,\mathrm{red}}$ have a natural bijective labeling by Serre weights.*

Recall that a *Serre weight* in this context is an irreducible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_d(k)$ (or rather an isomorphism class thereof). The description in [Emerton and Gee 2023] of the labeling of components of $\mathcal{X}_{d,\mathrm{red}}$ by Serre weights is to some extent combinatorial. Namely, it is shown that each irreducible component of $\mathcal{X}_{d,\mathrm{red}}$ has a dense set of $\overline{\mathbb{F}}_p$-points which are successive extensions of characters, with extensions as nonsplit as possible. The restrictions of these characters to the inertia group yield discrete data (their tame inertia weights) which, together with some further information about peu and très ramifiée extensions, amounts precisely to the data of (a highest weight of) a Serre weight.

It is expected, however, that there is another description of the irreducible components of $\mathcal{X}_{d,\mathrm{red}}$ that is more precise and more informative from the perspective of the p-adic Langlands program. The weight part of Serre's conjecture, as described for instance in [Gee et al. 2018, Section 3], associates to each $\bar{r} : G_K \to \mathrm{GL}_d(\overline{\mathbb{F}}_p)$ a set of Serre weights $W(\bar{r})$. One expects for each Serre weight σ that there is a set of components of $\mathcal{X}_{d,\mathrm{red}}$, including the irreducible component labeled by σ in [Emerton and Gee 2023], the union of whose $\overline{\mathbb{F}}_p$-points are *precisely* the representations $\bar{r}$ with $\sigma \in W(\bar{r})$. Equivalently, after adding additional labels to some of the components (so that components will be labeled by a set of weights, rather than a single weight), the set $W(\bar{r})$ is precisely the collection of labels of the various components of $\mathcal{X}_{d,\mathrm{red}}$ on which $\bar{r}$ lies.

One of the aims of this paper is to establish this expectation in the case $d = 2$, taking as input the weight part of Serre's conjecture for GL_2 [Gee et al. 2015] and the Breuil–Mézard conjecture for two-dimensional potentially Barsotti–Tate representations [Gee and Kisin 2014], and thus obtain a description of *all* the finite type points of each irreducible component of $\mathcal{X}_{2,\mathrm{red}}$, as opposed to just a dense set of points. Indeed we have the following theorem, which can be regarded as a geometrization of the weight part of Serre's conjecture for GL_2. If σ is a Serre weight, let $\mathcal{X}^{\sigma}_{d,\mathrm{red}}$ denote the irreducible component of $\mathcal{X}_{d,\mathrm{red}}$ labeled by σ in [Emerton and Gee 2023]. (We refer the reader to Section 2 for any unfamiliar notation or terminology in what follows.)

Theorem 1.2. *Suppose $p > 2$. For each Serre weight σ we define a cycle Z^{σ} as follows*:

- $Z^{\sigma} = \mathcal{X}^{\sigma}_{2,\mathrm{red}}$ *if the weight σ is not Steinberg.*
- $Z^{\chi\otimes\mathrm{St}} = \mathcal{X}^{\chi}_{2,\mathrm{red}} + \mathcal{X}^{\chi\otimes\mathrm{St}}_{2,\mathrm{red}}$ *if the weight $\sigma \cong \chi \otimes \mathrm{St}$ is Steinberg.*

Then $\sigma \in W(\bar{r})$ if and only if $\bar{r}$ lies in the support of Z^{σ}.

Indeed a stronger statement is true: the cycles $Z^\sigma = \mathcal{X}^\sigma_{2,\mathrm{red}}$ (*for* σ *non-Steinberg*) *and* $Z^{\chi\otimes\mathrm{St}} = \mathcal{X}^{\chi}_{2,\mathrm{red}} + \mathcal{X}^{\chi\otimes\mathrm{St}}_{2,\mathrm{red}}$ *constitute the cycles in a geometric version of the Breuil–Mézard conjecture* (*to be explained below*).

We emphasize that the existence of such a geometric interpretation of the sets $W(\bar{r})$ is far from obvious, and indeed we know of no direct proof using any of the explicit descriptions of $W(\bar{r})$ in the literature; it seems hard to understand in any explicit way which Galois representations arise as the limits of a family of extensions of given characters, and the description of the sets $W(\bar{r})$ is very complicated (for example, the description in [Buzzard et al. 2010] relies on certain Ext groups of crystalline characters). Our proof is indirect, and ultimately makes use of a description of $W(\bar{r})$ given in [Gee and Kisin 2014], which is in terms of potentially Barsotti–Tate deformation rings of $\bar{r}$ and is motivated by the Taylor–Wiles method. We interpret this description in the geometric language of [Emerton and Gee 2014], which we in turn interpret as the formal completion of a "geometric Breuil–Mézard conjecture" for our stacks.

The proof of Theorem 1.2 entwines the main results of the book [Emerton and Gee 2023] with the results of our papers [Caraiani et al. 2022; 2024]. Indeed Theorem 1.2 is (more or less) stated at [Emerton and Gee 2023, Theorem 8.6.2], but the argument given there makes reference to (an earlier version of) this paper.[1] We should therefore explain more precisely what are the contributions of this paper.

For each Hodge type λ and inertial type τ, the book [Emerton and Gee 2023] constructs a closed substack $\mathcal{X}^{\lambda,\tau}_d \subset \mathcal{X}_d$ parametrizing d-dimensional potentially crystalline representations of G_K of Hodge type λ and inertial type τ. When $d = 2$, λ is trivial, and τ is tame, these are stacks of potentially Barsotti–Tate representations of type τ, and we write $\mathcal{X}^{\tau,\mathrm{BT}}_2$ instead.

The papers [Caraiani et al. 2022; 2024] construct and study another stack $\mathcal{Z}^{\mathrm{dd}}$ which can be regarded as a stack of *tamely* potentially Barsotti–Tate representations; as well as a closed substack $\mathcal{Z}^\tau \subset \mathcal{Z}^{\mathrm{dd}}$, for each tame type τ, of potentially Barsotti–Tate representations of type τ. Our stacks $\mathcal{Z}^\tau$ are presumably isomorphic to the stacks $\mathcal{X}^{\tau,\mathrm{BT}}_2$,[2] but literally they are different stacks, constructed differently: the stacks $\mathcal{Z}^\tau$ are stacks of étale φ-modules with tame descent data, constructed by taking the scheme-theoretic image of a stack $\mathcal{C}^{\tau,\mathrm{BT}}$ of Breuil–Kisin modules with tame descent data; whereas the $\mathcal{X}^{\tau,\mathrm{BT}}_2$ are stacks of étale (φ,Γ)-modules, constructed by taking the scheme-theoretic image of a stack of Breuil–Kisin–Fargues modules satisfying a descent condition. In practice it seems to be easier to compute with the stacks $\mathcal{Z}^\tau$ than the stacks $\mathcal{X}^{\tau,\mathrm{BT}}_2$.

The properties of $\mathcal{Z}^{\mathrm{dd}}$ and $\mathcal{Z}^\tau$ that we will use in this paper are recalled in detail in Section 3, but we mention two crucial properties now:

- It is proved in [Caraiani et al. 2024] by a local model argument that the special fiber of $\mathcal{C}^{\tau,\mathrm{BT}}$ is reduced. As a consequence so is its scheme-theoretic image $\mathcal{Z}^{\tau,1}$ in $\mathcal{Z}^\tau$. The stack $\mathcal{Z}^{\mathrm{dd},1}$, the scheme-theoretic image in $\mathcal{Z}^{\mathrm{dd}}$ of the special fiber of $\mathcal{C}^{\mathrm{dd},\mathrm{BT}}$, is similarly reduced.

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[1] The reference [CEGS19, Theorem 5.2.2] in [Emerton and Gee 2023] is Theorem 6.2 of this paper, while the reference [CEGS19, Lemma B.5] in [Emerton and Gee 2023] is [Caraiani et al. 2024, Lemma A.5].

[2] Added in revision: In fact this has now been proved; see [Bellovin et al. 2024, Theorem 4.5].

- It is shown in [Caraiani et al. 2022] that the irreducible components of $\mathcal{Z}^{\tau,1}$ are in bijection with the Jordan–Hölder factors of $\bar{\sigma}(\tau)$; the component corresponding to σ has a dense set of $\overline{\mathbb{F}}_p$-points $\bar{r}$ such that $W(\bar{r}) = \{\sigma\}$.

Here $\sigma(\tau)$ is the representation of $\mathrm{GL}_2(\mathcal{O}_K)$ corresponding to τ under the inertial local Langlands correspondence, and $\bar{\sigma}(\tau)$ is its reduction modulo p. Note the similarity between the second of these two properties, and the labeling by Serre weights in Theorem 1.1.

These properties are combined in Section 4 to prove that the special fiber of $\mathcal{Z}^{\tau}$ is generically reduced. (Note that in general the special fiber of $\mathcal{Z}^{\tau}$ need not be the same as $\mathcal{Z}^{\tau,1}$; and similarly for the special fiber of $\mathcal{Z}^{\mathrm{dd}}$ vis-à-vis $\mathcal{Z}^{\mathrm{dd},1}$.) From this we deduce the following theorem about the special fibers of potentially Barsotti–Tate deformation rings, which seems hard to prove purely in the setting of formal deformations. Let $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbb{Q}_p$, with residue field $\mathbb{F}$.

Theorem 1.3. *Let $\bar{r} : G_K \to \mathrm{GL}_2(\mathbb{F})$ be a continuous representation, and τ a tame type. Let $R_{\bar{r}}^{\tau,\mathrm{BT}}$ be the universal framed deformation $\mathcal{O}$-algebra parametrizing potentially Barsotti–Tate lifts of $\bar{r}$ of type τ. Then $R_{\bar{r}}^{\tau,\mathrm{BT}} \otimes_{\mathcal{O}} \mathbb{F}$ is generically reduced.*

We anticipate that this result will be of independent interest. For example, Caraiani in joint work with James Newton [2023], has used this result in the proof of a modularity lifting theorem in the Barsotti–Tate case for GL_2 over a CM field; this modularity lifting theorem is used, in turn, to deduce the modularity of elliptic curves over $\mathbb{Q}(\sqrt{-d})$ for $d \in \{1, 2, 3, 5\}$.

We remark that très ramifiée representations do not have tamely potentially Barsotti–Tate lifts, hence do not correspond to finite type points on $\mathcal{Z}^{\mathrm{dd},1}$. Equivalently (see [Caraiani et al. 2024, Lemma A.5]), the Jordan–Hölder factors of $\bar{\sigma}(\tau)$ for tame types τ are never Steinberg, and therefore the stacks $\mathcal{Z}^{\tau,1}$ and $\mathcal{Z}^{\mathrm{dd},1}$ do not have irreducible components corresponding to Steinberg weights. So, although $\mathcal{Z}^{\mathrm{dd},1}$ and $\mathcal{X}_{2,\mathrm{red}}$ cannot be isomorphic, we anticipate (but do not prove) that there is an isomorphism between $\mathcal{Z}^{\mathrm{dd},1}$ and the union of the non-Steinberg components of $\mathcal{X}_{2,\mathrm{red}}$, along the same lines as [Bellovin et al. 2024, Theorem 4.5].

If σ is a non-Steinberg weight, then σ can be written as a virtual linear combination of representations $\bar{\sigma}(\tau)$ in the Grothendieck group of $\mathrm{GL}_2(k)$. In Section 5 this observation is translated into a special case of the classical geometric Breuil–Mézard conjecture [Emerton and Gee 2014]; we globalize this in Section 6 to prove the following theorem, which is the main result of this paper.

Theorem 1.4. *The irreducible components of $\mathcal{Z}^{\mathrm{dd},1}$ are in bijection with non-Steinberg Serre weights; write $\overline{\mathcal{Z}}(\sigma)$ for the component corresponding to σ. Then:*

(1) *The finite type points of $\overline{\mathcal{Z}}(\sigma)$ are precisely the representations $\bar{r} : G_K \to \mathrm{GL}_2(\mathbb{F}')$ having σ as a Serre weight.*

(2) *The stack $\mathcal{Z}^{\tau,1}$ is equal to $\bigcup_{\sigma \in \mathrm{JH}(\bar{\sigma}(\tau))} \overline{\mathcal{Z}}(\sigma)$.*

Part (1) of the theorem is the analogue of Theorem 1.2 for the stacks $\mathcal{Z}^{\mathrm{dd},1}$, while part (2) is a geometrization of the Breuil–Mézard conjecture for our tamely potentially Barsotti–Tate stacks. Theorem 1.3

is used crucially in the proof of Theorem 1.4, to confirm that each component $\overline{\mathcal{Z}}(\sigma)$ contributes with multiplicity at most one to the cycle of $R^{\tau,\mathrm{BT}} \otimes_{\mathcal{O}} \mathbb{F}$. We emphasize that, to this point, our results are independent from those of [Emerton and Gee 2023].

As explained above, our construction excludes the très ramifiée representations, which are twists of certain extensions of the trivial character by the mod p cyclotomic character. From the point of view of the weight part of Serre's conjecture, they are precisely the representations which admit a twist of the Steinberg representation as their only Serre weight. In accordance with the picture described above, this means that the full moduli stack of two-dimensional representations of G_K can be obtained from our stack by adding in the irreducible components consisting of the très ramifiée representations. This is carried out by extending our results to the stacks of [Emerton and Gee 2023], using the full strength of [loc. cit.].

In particular, it is proved in [loc. cit.] that the classical (numerical) Breuil–Mézard conjecture is equivalent to a geometrized Breuil–Mézard conjecture for the stacks $\mathcal{X}_d^{\lambda,\tau}$ of [loc. cit.]. Taking the Breuil–Mézard conjecture for potentially Barsotti–Tate representations [Gee and Kisin 2014] as input, they obtain the following theorem.

Theorem 1.5 [Emerton and Gee 2023]. *There exist effective cycles Z^{σ} (elements of the free group on the irreducible components of $\mathcal{X}_{2,\mathrm{red}}$, with nonnegative coefficients) such that for all inertial types τ, the cycle of the special fiber of $\mathcal{X}_2^{\tau,\mathrm{BT}}$ is equal to $\sum_{\sigma} m_{\sigma}(\tau) \cdot Z^{\sigma}$, where $\bar{\sigma}(\tau) = \sum_{\sigma} m_{\sigma}(\tau) \cdot \sigma$ in the Grothendieck group of* $\mathrm{GL}_2(k)$.

We stress that this theorem of [Emerton and Gee 2023] is for all inertial types, in contrast to the Breuil–Mézard result of Theorem 1.4(2) which is only for tame types; in particular the cycles for Steinberg weights σ do occur. In fact the theorem can be (and is) extended to cover potentially semistable representations of Hodge type 0 as well.

It remains to prove that the cycles Z^{σ} are as in Theorem 1.2, i.e., to check that $Z^{\sigma} = \mathcal{X}^{\sigma}_{2,\mathrm{red}}$ when σ is non-Steinberg, whereas $Z^{\chi\otimes\mathrm{St}} = \mathcal{X}^{\chi}_{2,\mathrm{red}} + \mathcal{X}^{\chi\otimes\mathrm{St}}_{2,\mathrm{red}}$. This is where the results of the present paper enter. We argue by transferring results from $\mathcal{Z}^{\tau}$ to $\mathcal{X}_2^{\tau,\mathrm{BT}}$ via a consideration of versal rings, without comparing the two stacks directly.[3] In particular the ring $R_{\bar{r}}^{\tau,\mathrm{BT}}$ is a versal ring to $\mathcal{X}^{\tau,\mathrm{BT}}$ at the point corresponding to $\bar{r}$; and so the formula $Z^{\sigma} = \mathcal{X}^{\sigma}_{2,\mathrm{red}}$ in the non-Steinberg case will follow by an application of Theorem 1.3 for a suitably chosen $\bar{r}$. The Steinberg case is handled directly using a semistable deformation ring. This completes the proof.

2. Notation and conventions

Galois theory. Let $p > 2$ be a prime number, and fix a finite extension $K/\mathbb{Q}_p$, with residue field k of cardinality p^f. In this paper we will study various stacks that are closely related to the representation theory of G_K, the absolute Galois group of K.

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[3] Added in revision: It is now also possible to transfer these results using [Bellovin et al. 2024, Theorem 4.5].

Our representations of G_K will have coefficients in $\overline{\mathbb{Q}}_p$, a fixed algebraic closure of $\mathbb{Q}_p$ whose residue field we denote by $\overline{\mathbb{F}}_p$. Let E be a finite extension of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$. Write $\mathcal{O}$ for the ring of integers in E, with uniformizer ϖ and residue field $\mathbb{F} \subset \overline{\mathbb{F}}_p$.

As is often the case, we assume that our coefficients are "sufficiently large". Specifically, if L is the quadratic unramified extension of K, we assume that E admits an embedding of $K' = L(\pi^{1/(p^{2f}-1)})$ for some uniformizer π of K. Write l for the residue field of L.

Fix an embedding $\sigma_0 : k \hookrightarrow \mathbb{F}$, and recursively define $\sigma_i : k \hookrightarrow \mathbb{F}$ for all $i \in \mathbb{Z}$ so that $\sigma_{i+1}^p = \sigma_i$. For each i we define the fundamental character ω_{σ_i} to be the composite

$$I_K \longrightarrow \mathcal{O}_K^\times \longrightarrow k^\times \xrightarrow{\sigma_i} \overline{\mathbb{F}}_p^\times,$$

where the map $I_K \to \mathcal{O}_K^\times$ is induced by the restriction of the inverse of the Artin map, which we normalize so that uniformizers correspond to geometric Frobenius elements.

Inertial local Langlands. A two-dimensional *tame inertial type* is (the isomorphism class of) a tamely ramified representation $\tau : I_K \to \mathrm{GL}_2(\overline{\mathbb{Z}}_p)$ that extends to a representation of G_K and whose kernel is open. Such a representation is of the form $\tau \simeq \eta \oplus \eta'$, and we say that τ is a *tame principal series type* if η and η' both extend to characters of G_K. Otherwise, $\eta' = \eta^q$, and η extends to a character of G_L. In this case we say that τ is a *tame cuspidal type*. In either case $\tau|_{I_{K'}}$ is trivial, since τ is tame, and therefore a potentially crystalline representation of G_K with inertial type τ will become crystalline over K'.

Henniart [2002] associates a finite-dimensional irreducible E-representation $\sigma(\tau)$ of $\mathrm{GL}_2(\mathcal{O}_K)$ to each inertial type τ; we refer to this association as the *inertial local Langlands correspondence*. Since we are only working with tame inertial types, this correspondence can be made very explicit, as in [Caraiani et al. 2022, Section 1.2]. (Since we will not directly use the explicit description in this paper, we will not repeat it here.)

Serre weights and tame types. By definition, a *Serre weight* is an irreducible $\mathbb{F}$-representation of $\mathrm{GL}_2(k)$. Then, concretely, a Serre weight is of the form

$$\bar{\sigma}_{\vec{t},\vec{s}} := \bigotimes_{j=0}^{f-1} (\det^{t_j} \mathrm{Sym}^{s_j} k^2) \otimes_{k,\sigma_j} \mathbb{F},$$

where $0 \le s_j, t_j \le p-1$ and not all t_j are equal to $p-1$. We say that a Serre weight is *Steinberg* if $s_j = p-1$ for all j, and *non-Steinberg* otherwise.

Let τ be a tame inertial type. Write $\bar{\sigma}(\tau)$ for the semisimplification of the reduction modulo p of a $\mathrm{GL}_2(\mathcal{O}_K)$-stable $\mathcal{O}$-lattice in $\sigma(\tau)$. The action of $\mathrm{GL}_2(\mathcal{O}_K)$ on $\bar{\sigma}(\tau)$ factors through $\mathrm{GL}_2(k)$, so the Jordan–Hölder factors $\mathrm{JH}(\bar{\sigma}(\tau))$ of $\bar{\sigma}(\tau)$ are Serre weights. By the results of [Diamond 2007], these Jordan–Hölder factors of $\bar{\sigma}(\tau)$ are pairwise nonisomorphic, and are parametrized by a certain set $\mathcal{P}_\tau$ that we now recall.

Suppose first that $\tau = \eta \oplus \eta'$ is a tame principal series type. Set $f' = f$ in this case. We define $0 \le \gamma_i \le p-1$ (for $i \in \mathbb{Z}/f\mathbb{Z}$) to be the unique integers not all equal to $p-1$ such that $\eta(\eta')^{-1} = \prod_{i=0}^{f-1} \omega_{\sigma_i}^{\gamma_i}$.

If instead $\tau = \eta \oplus \eta'$ is a cuspidal type, set $f' = 2f$. We define $0 \leq \gamma_i \leq p-1$ (for $i \in \mathbb{Z}/f'\mathbb{Z}$) to be the unique integers such that $\eta(\eta')^{-1} = \prod_{i=0}^{f'-1} \omega_{\sigma'_i}^{\gamma_i}$. Here $\sigma'_0 : l \to \overline{\mathbb{F}}_p^\times$ is a fixed choice of embedding extending σ_0, $(\sigma'_{i+1})^p = \sigma'_i$ for all i, and the fundamental characters $\omega_{\sigma'_i} : I_L \to \overline{\mathbb{F}}_p^\times$ for each $\sigma'_i : l \to \overline{\mathbb{F}}_p^\times$ are defined in the same way as the ω_{σ_i}.

If τ is scalar then we set $\mathcal{P}_\tau = \{\varnothing\}$. Otherwise we have $\eta \neq \eta'$, and we let $\mathcal{P}_\tau$ be the collection of subsets $J \subset \mathbb{Z}/f'\mathbb{Z}$ satisfying the conditions

- if $i-1 \in J$ and $i \notin J$ then $\gamma_i \neq p-1$, and
- if $i-1 \notin J$ and $i \in J$ then $\gamma_i \neq 0$

and, in the cuspidal case, satisfying the further condition that $i \in J$ if and only if $i+f \notin J$.

The Jordan–Hölder factors of $\bar{\sigma}(\tau)$ are by definition Serre weights, and are parametrized by $\mathcal{P}_\tau$ as follows; see [Emerton et al. 2015, Sections 3.2 and 3.3]. For any $J \subseteq \mathbb{Z}/f'\mathbb{Z}$, we let δ_J denote the characteristic function of J, and if $J \in \mathcal{P}_\tau$ we define $s_{J,i}$ by

$$s_{J,i} = \begin{cases} p-1-\gamma_i-\delta_{J^c}(i) & \text{if } i-1 \in J, \\ \gamma_i - \delta_J(i) & \text{if } i-1 \notin J, \end{cases}$$

and we set $t_{J,i} = \gamma_i + \delta_{J^c}(i)$ if $i-1 \in J$ and 0 otherwise. Write $\vec{s}$ for the tuple $(s_{J,i})_{0 \leq i < f}$, suppressing the J from the notation for readability, and similarly for $\vec{t}$.

In the principal series case we let $\bar{\sigma}(\tau)_J := \bar{\sigma}_{\vec{t},\vec{s}} \otimes \eta' \circ \det$ for each $J \in \mathcal{P}_\tau$; the $\bar{\sigma}(\tau)_J$ are precisely the Jordan–Hölder factors of $\bar{\sigma}(\tau)$.

In the cuspidal case, one checks that $s_{J,i} = s_{J,i+f}$ for all i, and also that the character

$$\eta' \cdot \prod_{i=0}^{f'-1} (\sigma'_i)^{t_{J,i}} : l^\times \to \mathbb{F}^\times$$

factors as $\theta \circ N_{l/k}$ where $N_{l/k}$ is the norm map. We let $\bar{\sigma}(\tau)_J := \bar{\sigma}_{0,\vec{s}} \otimes \theta \circ \det$, again for $J \in \mathcal{P}_\tau$; the $\bar{\sigma}(\tau)_J$ are precisely the Jordan–Hölder factors of $\bar{\sigma}(\tau)$.

p-adic Hodge theory. We normalize Hodge–Tate weights so that all Hodge–Tate weights of the cyclotomic character are equal to -1. We say that a potentially crystalline representation $r : G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ has *Hodge type* 0, or is *potentially Barsotti–Tate*, if for each $\varsigma : K \hookrightarrow \overline{\mathbb{Q}}_p$, the Hodge–Tate weights of r with respect to ς are 0 and 1. (Note that this is a more restrictive definition of potentially Barsotti–Tate than is sometimes used; however, we will have no reason to deal with representations with nonregular Hodge–Tate weights, and so we exclude them from consideration. Note also that it is more usual in the literature to say that r is potentially Barsotti–Tate if it is potentially crystalline, and $r^\vee$ has Hodge type 0.)

We say that a potentially crystalline representation

$$r : G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

has *inertial type* τ if the traces of elements of I_K acting on τ and on

$$\mathrm{D}_{\mathrm{pcris}}(r) = \varinjlim_{K'/K} (\mathrm{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V_r)^{G_{K'}}$$

are equal (here V_r is the underlying vector space of V_r). A representation $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ *has a potentially Barsotti–Tate lift of type* τ if and only if $\bar{r}$ admits a lift to a representation $r : G_K \to \mathrm{GL}_2(\overline{\mathbb{Z}}_p)$ of Hodge type 0 and inertial type τ.

Serre weights of mod p Galois representations. Given a continuous representation $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, there is an associated (nonempty) set of Serre weights $W(\bar{r})$, defined to be the set of Serre weights $\bar{\sigma}_{\vec{t},\vec{s}}$ such that $\bar{r}$ has a crystalline lift whose Hodge–Tate weights are as follows: for each embedding $\sigma_j : k \hookrightarrow \mathbb{F}$ there is an embedding $\widetilde{\sigma}_j : K \hookrightarrow \overline{\mathbb{Q}}_p$ lifting σ_j such that the $\widetilde{\sigma}_j$-labeled Hodge–Tate weights of r are $\{-s_j - t_j, 1 - t_j\}$, and the remaining $(e-1)f$ pairs of Hodge–Tate weights of r are all $\{0, 1\}$.

There are in fact several different definitions of $W(\bar{r})$ in the literature; as a result of the papers [Barnet-Lamb et al. 2013; Gee and Kisin 2014; Gee et al. 2015], these definitions are known to be equivalent up to normalization. The normalizations in this paper are the same as those of [Caraiani et al. 2022; 2024]; see [Caraiani et al. 2022, Section 1.2]for a detailed discussion of these normalizations. In particular we have normalized the set of Serre weights so that $\bar{r}$ has a potentially Barsotti–Tate lift of type τ if and only if $W(\bar{r}) \cap \mathrm{JH}(\bar{\sigma}(\tau)) \neq \varnothing$ [Caraiani et al. 2024, Lemma A.5].

Stacks. We follow the terminology of [Stacks 2005–]; in particular, we write "algebraic stack" rather than "Artin stack". More precisely, an algebraic stack is a stack in groupoids in the *fppf* topology, whose diagonal is representable by algebraic spaces, which admits a smooth surjection from a scheme. See [Stacks 2005–, Tag 026N] for a discussion of how this definition relates to others in the literature, and [Stacks 2005–, Tag 04XB] for key properties of morphisms representable by algebraic spaces.

For a commutative ring A, an *fppf stack over* A (or *fppf* A-stack) is a stack fibered in groupoids over the big *fppf* site of $\operatorname{Spec} A$. Following [Emerton 2019, Definitions 5.3 and 7.6], an *fppf* stack in groupoids $\mathcal{X}$ over a scheme S is called a *formal algebraic stack* if there is a morphism $U \to \mathcal{X}$, whose domain U is a formal algebraic space over S (in the sense of [Stacks 2005–, Tag 0AIL]), and which is representable by algebraic spaces, smooth, and surjective.

Let $\operatorname{Spf} \mathcal{O}$ denote the affine formal scheme (or affine formal algebraic space, in the terminology of [Stacks 2005–]) obtained by ϖ-adically completing $\operatorname{Spec} \mathcal{O}$. A formal algebraic stack $\mathcal{X}$ over $\operatorname{Spec} \mathcal{O}$ is called ϖ-adic if the canonical map $\mathcal{X} \to \operatorname{Spec} \mathcal{O}$ factors through $\operatorname{Spf} \mathcal{O}$, and if the induced map $\mathcal{X} \to \operatorname{Spf} \mathcal{O}$ is algebraic, i.e., representable by algebraic stacks (in the sense of [Stacks 2005–, Tag 06CF] and [Emerton 2019, Definition 3.1]).

3. Moduli stacks of Breuil–Kisin modules and étale φ-modules

The main object of study in this paper is the ϖ-adic formal algebraic stack $\mathcal{Z}^{\mathrm{dd}}$ that was introduced and studied in [Caraiani et al. 2022; 2024], and whose $\overline{\mathbb{F}}_p$-points are naturally in bijection with the continuous representations $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ admitting a potentially Barsotti–Tate lift of some tame type. In this section we review the construction and known properties of $\mathcal{Z}^{\mathrm{dd}}$, as well as those of several other closely related stacks.

Stacks of Breuil–Kisin modules. For each tame type τ, there is a ϖ-adic formal algebraic stack $\mathcal{C}^{\tau,\mathrm{BT}}$ whose $\mathrm{Spf}(\mathcal{O}_{E'})$-points, for any finite extension E'/E, are the Breuil–Kisin modules corresponding to two-dimensional potentially Barsotti–Tate representations of type τ. This stack is constructed in several steps, which we review in brief. (We refer the reader to [Caraiani et al. 2024] as well as to the summary in [Caraiani et al. 2022, Section 2.3] for complete definitions, recalling here only what will be used in this paper.)

For each integer $a \geq 1$, we write $\mathcal{C}^{\mathrm{dd},a}$ for the *fppf* stack over $\mathcal{O}/\varpi^a$ which associates to any $\mathcal{O}/\varpi^a$-algebra A the groupoid $\mathcal{C}^{\mathrm{dd},a}(A)$ of rank 2 Breuil–Kisin modules of height at most 1 with A-coefficients and descent data from K' to K. Set $\mathcal{C}^{\mathrm{dd}} = \varinjlim_a \mathcal{C}^{\mathrm{dd},a}$. The closed substack $\mathcal{C}^{\mathrm{dd},\mathrm{BT}}$ of $\mathcal{C}^{\mathrm{dd}}$ is cut out by a Kottwitz-type determinant condition, which can be thought of (on the Galois side) as cutting out the tamely potentially Barsotti–Tate representations from among all tamely potentially crystalline representations with Hodge–Tate weights in $\{0, 1\}$. By [Caraiani et al. 2024, Corollary 4.2.13] the stack $\mathcal{C}^{\mathrm{dd},\mathrm{BT}}$ then decomposes as a disjoint union of closed substacks $\mathcal{C}^{\tau,\mathrm{BT}}$, one for each tame type τ, consisting of Breuil–Kisin modules with descent data of type τ. Finally, for each $a \geq 1$ we write $\mathcal{C}^{\tau,\mathrm{BT},a} = \mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$, and similarly for $\mathcal{C}^{\mathrm{dd},\mathrm{BT},a}$. The following properties are established in [loc. cit., Corollaries 3.1.8 and 4.5.3, Proposition 5.2.21].

Theorem 3.1. *The stacks $\mathcal{C}^{\mathrm{dd},a}$, $\mathcal{C}^{\mathrm{dd},\mathrm{BT},a}$, and $\mathcal{C}^{\tau,\mathrm{BT},a}$ are algebraic stacks of finite type over $\mathcal{O}$, while the stacks $\mathcal{C}^{\mathrm{dd}}$, $\mathcal{C}^{\mathrm{dd},\mathrm{BT}}$, and $\mathcal{C}^{\tau,\mathrm{BT}}$ are ϖ-adic formal algebraic stacks. Moreover*:

(1) *$\mathcal{C}^{\tau,\mathrm{BT}}$ is analytically normal, Cohen–Macaulay, and flat over $\mathcal{O}$.*

(2) *The stacks $\mathcal{C}^{\mathrm{dd},\mathrm{BT},a}$ and $\mathcal{C}^{\tau,\mathrm{BT},a}$ are equidimensional of dimension $[K : \mathbb{Q}_p]$.*

(3) *The special fibers $\mathcal{C}^{\mathrm{dd},\mathrm{BT},1}$ and $\mathcal{C}^{\tau,\mathrm{BT},1}$ are reduced.*

Galois moduli stacks. Let $\mathcal{R}^{\mathrm{dd},a}_{K'}$ be the *fppf* $\mathbb{F}$-stack which associates to any $\mathcal{O}/\varpi^a$-algebra A the groupoid $\mathcal{R}^{\mathrm{dd},a}_{K'}(A)$ of rank 2 étale φ-modules with A-coefficients and descent data from K' to K. We will usually suppress K' from the notation. Inverting u on Breuil–Kisin modules gives a proper morphism $\mathcal{C}^{\mathrm{dd},a} \to \mathcal{R}^{\mathrm{dd},a}$, which then restricts to proper morphisms $\mathcal{C}^{\mathrm{dd},\mathrm{BT},a} \to \mathcal{R}^{\mathrm{dd},a}$ as well as $\mathcal{C}^{\tau,\mathrm{BT},a} \to \mathcal{R}^{\mathrm{dd},a}$ for each τ.

Emerton and Gee [2021] developed a theory of scheme-theoretic images of proper morphisms $\mathcal{X} \to \mathcal{F}$ of stacks over a locally Noetherian base-scheme S, where $\mathcal{X}$ is an algebraic stack which is locally of finite presentation over S, and the diagonal of $\mathcal{F}$ is representable by algebraic spaces and locally of finite presentation. This theory applies in particular to each of the morphisms of the previous paragraph (even though $\mathcal{R}^{\mathrm{dd},a}$ is *not* algebraic). We define $\mathcal{Z}^{\mathrm{dd},a}$ and $\mathcal{Z}^{\tau,a}$ to be the scheme-theoretic images of the morphisms $\mathcal{C}^{\mathrm{dd},\mathrm{BT},a} \to \mathcal{R}^{\mathrm{dd},a}$ and $\mathcal{C}^{\tau,\mathrm{BT},a} \to \mathcal{R}^{\mathrm{dd},a}$, respectively. Set $\mathcal{Z}^{\mathrm{dd}} = \varinjlim_a \mathcal{Z}^{\mathrm{dd},a}$ and $\mathcal{Z}^{\tau} = \varinjlim_a \mathcal{Z}^{\tau,a}$. The following theorem combines [Caraiani et al. 2024, Theorem 5.1.2, Proposition 5.1.4, Lemma 5.1.8, Proposition 5.2.20].

Theorem 3.2. *The stacks $\mathcal{Z}^{\mathrm{dd},a}$ and $\mathcal{Z}^{\tau,a}$ are algebraic stacks of finite type over $\mathcal{O}$, while the stacks $\mathcal{Z}^{\mathrm{dd}}$ and $\mathcal{Z}^{\tau}$ are ϖ-adic formal algebraic stacks. Moreover*:

(1) *The stacks* $\mathcal{Z}^{\mathrm{dd},a}$ *and* $\mathcal{Z}^{\tau,a}$ *are equidimensional of dimension* $[K:\mathbb{Q}_p]$.

(2) *The stacks* $\mathcal{Z}^{\mathrm{dd},1}$ *and* $\mathcal{Z}^{\tau,1}$ *are reduced.*

(3) *The* $\overline{\mathbb{F}}_p$*-points of* $\mathcal{Z}^{\mathrm{dd},1}$ *are naturally in bijection with the continuous representations* $\bar{r}:G_K\to\mathrm{GL}_2(\overline{\mathbb{F}}_p)$ *which are not a twist of a très ramifiée extension of the trivial character by the mod p cyclotomic character. Similarly, the* $\overline{\mathbb{F}}_p$*-points of* $\mathcal{Z}^{\tau,1}$ *are naturally in bijection with the continuous representations* $\bar{r}:G_K\to\mathrm{GL}_2(\overline{\mathbb{F}}_p)$ *which have a potentially Barsotti–Tate lift of type* τ.

In particular the stack $\mathcal{Z}^{\mathrm{dd},1}$ *is the underlying reduced substack of* $\mathcal{Z}^{\mathrm{dd},a}$ *for each* $a\geq 1$, *as well as of* $\mathcal{Z}^{\mathrm{dd}}$; *and similarly for the stacks* $\mathcal{Z}^{\tau,1}$.

Remark 3.3. We stress that the morphism $\mathcal{Z}^{\mathrm{dd},a}\hookrightarrow\mathcal{Z}^{\mathrm{dd}}\times_{\mathcal{O}}\mathcal{O}/\varpi^a$ need not be an isomorphism *a priori*, and we have no reason to expect that it is. However, our results in the next section will prove that it is *generically* an isomorphism for all $a\geq 1$.

Versal rings and deformation rings. Let x be an $\mathbb{F}'$-point of $\mathcal{Z}^{\tau,a}$, corresponding to the representation $\bar{r}:G_K\to\mathrm{GL}_2(\mathbb{F}')$. We will usually write $R_{\bar{r}}^{\tau,\mathrm{BT}}$ for the reduced and p-torsion free quotient of the universal framed deformation ring of $\bar{r}$ whose $\overline{\mathbb{Q}}_p$-points correspond to the potentially Barsotti–Tate lifts of $\bar{r}$ of type τ. (In Section 5 we will denote this ring instead by $R_{\bar{r},0,\tau}$, for ease of comparison with the paper [Gee and Kisin 2014].)

It is explained in [Caraiani et al. 2024, Section 5.2] that there are versal rings $R_x^{\tau,a}$ to $\mathcal{Z}^{\tau,a}$ at the point x, such that the following holds. (These rings are denoted $R^{\tau,a}$ in [Caraiani et al. 2024]; we include the subscript x here to emphasize the dependence on the point x.)

Proposition 3.4 [Caraiani et al. 2024, Proposition 5.2.19]. *We have* $\varprojlim R_x^{\tau,a}=R_{\bar{r}}^{\tau,\mathrm{BT}}$; *thus* $R_{\bar{r}}^{\tau,\mathrm{BT}}$ *is a versal ring to* $\mathcal{Z}^{\tau}$ *at* x.

Similarly there is a versal ring $R_x^{\mathrm{dd},a}$ to $\mathcal{Z}^{\mathrm{dd},a}$ at x, and each $R_x^{\tau,a}$ is a quotient of $R_x^{\mathrm{dd},a}$.

Irreducible components of $\mathcal{C}^{\tau,\mathrm{BT},1}$ ***and*** $\mathcal{Z}^{\tau,1}$**.** Fix a tame type τ, and recall that we set $f'=f$ if the type τ is principal series, while $f'=2f$ if the type τ is cuspidal. We say that a subset $J\subset\mathbb{Z}/f'\mathbb{Z}$ is a *profile* if

- τ is scalar and $J=\varnothing$,
- τ is a nonscalar principal series type and J is arbitrary, or
- τ is cuspidal and J has the property that $i\in J$ if and only if $i+f\notin J$.

Thus there are exactly 2^f profiles if τ is nonscalar. The set $\mathcal{P}_\tau$ introduced in Section 2 is a subset of the set of profiles.

To each profile J, the discussion in [Caraiani et al. 2022, Section 4.2.7] associates a closed substack $\overline{\mathcal{C}}(J)$ of $\mathcal{C}^{\tau,\mathrm{BT},1}$. The stack $\overline{\mathcal{Z}}(J)$ is then defined to be the scheme-theoretic image of $\overline{\mathcal{C}}(J)$ under the map $\mathcal{C}^{\tau,\mathrm{BT},1}\to\mathcal{Z}^{\tau,1}$.

The following description of the irreducible components of $\mathcal{C}^{\tau,\mathrm{BT},1}$ and $\mathcal{Z}^{\tau,1}$ is proved in [Caraiani et al. 2022, Proposition 5.1.13, Theorem 5.1.17, Corollary 5.3.3, Theorem 5.4.3]; the description of the components of $\mathcal{Z}^{\tau,1}$ in part (3) of the theorem is analogous to the description of the components of $\mathcal{X}_{2,\mathrm{red}}$ of Theorem 1.1.

Theorem 3.5. *The irreducible components of* $\mathcal{C}^{\tau,\mathrm{BT},1}$ *and* $\mathcal{Z}^{\tau,1}$ *are as follows*:

1. *The irreducible components of* $\mathcal{C}^{\tau,1}$ *are precisely the* $\bar{\mathcal{C}}(J)$ *for profiles* J, *and if* $J \neq J'$ *then* $\bar{\mathcal{C}}(J) \neq \bar{\mathcal{C}}(J')$.
2. *The irreducible components of* $\mathcal{Z}^{\tau,1}$ *are precisely the* $\overline{\mathcal{Z}}(J)$ *for profiles* $J \in \mathcal{P}_\tau$, *and if* $J \neq J'$ *then* $\overline{\mathcal{Z}}(J) \neq \overline{\mathcal{Z}}(J')$.
3. *For each* $J \in \mathcal{P}_\tau$ *there is a dense open substack* $\mathcal{U}$ *of* $\bar{\mathcal{C}}(J)$ *such that the map* $\bar{\mathcal{C}}(J) \to \overline{\mathcal{Z}}(J)$ *restricts to an open immersion on* $\mathcal{U}$.
4. *For each* $J \in \mathcal{P}_\tau$, *there is a dense set of finite type points of* $\overline{\mathcal{Z}}(J)$ *with the property that the corresponding Galois representations have* $\bar\sigma(\tau)_J$ *as a Serre weight*, *and which furthermore admit a unique Breuil–Kisin model of type* τ *as defined in* [Caraiani et al. 2022, Definition 5.1.16].

Remark 3.6. If $\bar\sigma(\tau)_J = \bar\sigma_{\vec{t},\vec{s}}$, then the dense set of finite type points of $\overline{\mathcal{Z}}(J)$ produced in the proof of [Caraiani et al. 2022, Theorem 5.1.17], as claimed in Theorem 3.5(4), consists of points corresponding to reducible representations $\bar r$ such that $\bar r|_{I_K}$ is an extension of $\bar\varepsilon^{-1}\prod_{i=0}^{f-1}\omega_{\sigma_i}^{t_i}$ by $\prod_{i=0}^{f-1}\omega_{\sigma_i}^{s_i+t_i}$ (necessarily peu ramifiée in case the ratio of the two characters is cyclotomic).

Remark 3.7. We emphasize in Theorem 3.5 that the components of $\mathcal{Z}^{\tau,1}$ are indexed by profiles $J \in \mathcal{P}_\tau$, *not* by all profiles. If $J \notin \mathcal{P}_\tau$, then by [Caraiani et al. 2022, Theorem 5.1.12] the stack $\overline{\mathcal{Z}}(J)$ has dimension strictly smaller than $[K : \mathbb{Q}_p]$, and so is properly contained in some component of $\mathcal{Z}^{\tau,1}$.

Remark 3.8. Strictly speaking, in the principal series case Theorem 3.5 is proved in [Caraiani et al. 2022] for stacks of Breuil–Kisin modules and of étale φ-modules with descent data from $K(\pi^{1/(p^f-1)})$ to K, rather than from our K' to K. But in fact we can replace $K(\pi^{1/(p^f-1)})$ with any extension of prime-to-p degree that remains Galois over K, such as K', without changing the resulting stacks. This follows from [Caraiani et al. 2024, Proposition 4.3.1(2)], together with the isomorphism $\mathcal{R}^{\mathrm{dd}}_{K(\pi^{1/(p^f-1)})} \xrightarrow{\sim} \mathcal{R}^{\mathrm{dd}}_{K'}$ of stacks of étale φ-modules (which is easier than *loc. cit.*, since multiplication by u on an étale φ-module is bijective, and can be proved by the same argument as in [Emerton and Gee 2023, Corollary 2.3.21]).

4. Generic reducedness of Spec $R_{\bar r}^{\tau,\mathrm{BT}}/\varpi$

Fix a Galois representation $\bar r : G_K \to \mathrm{GL}_2(\mathbb{F}')$, where $\mathbb{F}'/\mathbb{F}$ is a finite extension. Our goal in this section is to prove that the scheme Spec $R_{\bar r}^{\tau,\mathrm{BT}}/\varpi$ is generically reduced; this will be a key ingredient in the proof of our geometric Breuil–Mézard result in Section 6. Recall that a scheme is generically reduced if it contains an open reduced subscheme whose underlying topological space is dense; in the case of a Noetherian affine scheme Spec A, this is equivalent to requiring that the localization of A at each of its minimal primes is reduced.

We may of course suppose that $R_{\bar{r}}^{\tau,\mathrm{BT}} \neq 0$, so that $\bar{r}$ has a potentially Barsotti–Tate lift of type τ, and so corresponds to a finite type point $x : \operatorname{Spec} \mathbb{F}' \to \mathcal{Z}^{\tau,a}$ for any $a \geq 1$. It follows from Proposition 3.4 that $\operatorname{Spec} R_x^{\tau,a}$ is a closed subscheme of $\operatorname{Spec} R_{\bar{r}}^{\tau,\mathrm{BT}}/\varpi^a$, but we have no reason to believe that equality holds. It follows from Theorem 3.2(2), together with Lemma 4.5 below, that $\operatorname{Spec} R_x^{\tau,1}$ is the underlying reduced subscheme of $\operatorname{Spec} R_{\bar{r}}^{\tau,\mathrm{BT}}/\varpi$, so that equality holds in the case $a = 1$ if and only if $\operatorname{Spec} R_{\bar{r}}^{\tau,\mathrm{BT}}/\varpi$ is reduced; however, again, we have no reason to believe that this holds in general.

Nevertheless it *is* the case that $\operatorname{Spec} R_{\bar{r}}^{\tau,\mathrm{BT}}/\varpi$ is generically reduced; see Theorem 4.6 below. We will deduce Theorem 4.6 from the following global statement.

Proposition 4.1. *Let τ be a tame type. There is a dense open substack $\mathcal{U}$ of $\mathcal{Z}^{\tau}$ such that $\mathcal{U}_{/\mathbb{F}}$ is reduced.*

Proof. The proposition will follow from an application of Lemma A.6, and the key to this application will be to find a candidate open substack $\mathcal{U}^1$ of $\mathcal{Z}^{\tau,1}$, which we will do using our study of the irreducible components of $\mathcal{C}^{\tau,\mathrm{BT},1}$ and $\mathcal{Z}^{\tau,1}$.

Recall that, for each profile $J \in \mathcal{P}_{\tau}$, we let $\overline{\mathcal{Z}}(J)$ denote the scheme-theoretic image of $\overline{\mathcal{C}}(J)$ under the proper morphism $\mathcal{C}^{\tau,\mathrm{BT},1} \to \mathcal{Z}^{\tau,1}$. Each $\overline{\mathcal{Z}}(J)$ is a closed substack of $\mathcal{Z}^{\tau,1}$, and so, if we let $\mathcal{V}(J)$ be the complement in $\mathcal{Z}^{\tau,1}$ of the union of the $\overline{\mathcal{Z}}(J')$ for all profiles $J' \neq J$ then $\mathcal{V}(J)$ is a dense open substack of $\overline{\mathcal{Z}}(J)$, by Theorem 3.5(2) and Remark 3.7 (the former in consideration of $J' \in \mathcal{P}_{\tau}$, the latter for $J' \notin \mathcal{P}_{\tau}$). The preimage $\mathcal{W}(J)$ of $\mathcal{V}(J)$ in $\mathcal{C}^{\tau,\mathrm{BT},1}$ is therefore a dense open substack of $\overline{\mathcal{C}}(J)$. Possibly shrinking $\mathcal{W}(J)$ further, we may suppose by Theorem 3.5(3) that the morphism $\mathcal{W}(J) \to \mathcal{Z}^{\tau,1}$ is an open immersion.

Write $|\cdot|$ for the underlying topological space of a stack. The complement $|\overline{\mathcal{C}}(J)| \setminus |\mathcal{W}(J)|$ is a closed subset of $|\overline{\mathcal{C}}(J)|$, and thus of $|\mathcal{C}^{\tau,\mathrm{BT},1}|$, and its image under the proper morphism $\mathcal{C}^{\tau,\mathrm{BT},1} \to \mathcal{Z}^{\tau,1}$ is a closed subset of $|\mathcal{Z}^{\tau,1}|$, which is (e.g., for dimension reasons[4]) a proper closed subset of $|\overline{\mathcal{Z}}(J)|$; so if we let $\mathcal{U}(J)$ be the complement in $\mathcal{V}(J)$ of this image, then $\mathcal{U}(J)$ is open and dense in $\overline{\mathcal{Z}}(J)$, and the morphism $\mathcal{C}^{\tau,\mathrm{BT},1} \times_{\mathcal{Z}^{\tau,1}} \mathcal{U}(J) \to \mathcal{U}(J)$ is a monomorphism. Set $\mathcal{U}^1 = \bigcup_J \mathcal{U}(J)$. Since the $\mathcal{U}(J)$ are pairwise disjoint by construction,

$$\mathcal{C}^{\tau,\mathrm{BT},1} \times_{\mathcal{Z}^{\tau,1}} \mathcal{U}^1 \to \mathcal{U}^1 \tag{4.1.1}$$

is again a monomorphism. By construction (taking into account Theorem 3.5(2)), $\mathcal{U}^1$ is dense in $\mathcal{Z}^{\tau,1}$.

Now let $\mathcal{U}$ denote the open substack of $\mathcal{Z}^{\tau}$ corresponding to $\mathcal{U}^1$. Since $|\mathcal{Z}^{\tau}| = |\mathcal{Z}^{\tau,1}|$, we see that $\mathcal{U}$ is dense in $\mathcal{Z}^{\tau}$. We have seen in the previous paragraph that the statement of Lemma A.6(5) holds (taking

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[4] Choose, as we may, a surjective smooth morphism $U \to \overline{\mathcal{Z}}(J)$ with U a finite type $\mathbb{F}$-scheme. Let $V := \overline{\mathcal{C}}(J) \times_{\overline{\mathcal{Z}}(J)} U$. Then the projection $V \to \overline{\mathcal{C}}(J)$ is again smooth and surjective, while the projection $f : V \to U$ is representable by schemes and proper; in particular, V is also a finite type $\mathbb{F}$-scheme. Let $W := \mathcal{W}(J) \times_{\overline{\mathcal{Z}}(J)} U$. Then W is a dense open subscheme of U, equipped with a section $W \to V$ which realizes it as a dense open subscheme of V as well. Write $Y := V \setminus W$ (a closed subset of V) and $X = f(Y)$ (a closed subset of U). If T is any irreducible component of V, then $\dim T = \dim(T \cap W) = \dim(f(T) \cap W) = \dim f(T)$ (in particular, $f(T)$ is an irreducible component of U; and each irreducible component of U arises in this manner), while $\dim(T \cap Y) < \dim T$. Thus also $\dim f(T \cap Y) < \dim T = \dim f(T)$, so that $f(T \cap Y)$ is a proper closed subset of $f(T)$. Since $Y = \bigcup_T (T \cap Y)$, we see that $X = \bigcup_T f(T \cap Y)$ is a proper closed subset of U, whose complement U' is dense in U. The image of U' is then a dense (equivalently, nonempty) open substack of $\overline{\mathcal{Z}}(J)$; and this image coincides with the complement in $\overline{\mathcal{Z}}(J)$ of the image of $\overline{\mathcal{C}}(J) \setminus \mathcal{W}(J)$ (since U' is the preimage of this complement, by construction).

$a = 1$, $\mathcal{X} = \mathcal{C}^{\tau,\mathrm{BT}}$, and $\mathcal{Y} = \mathcal{Z}^{\tau}$); so Lemma A.6 implies that, for each $a \geq 1$, the closed immersion

$$\mathcal{U} \times_{\mathcal{Z}^{\tau}} \mathcal{Z}^{\tau,a} \hookrightarrow \mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^{a} \tag{4.1.2}$$

is an isomorphism.

In particular, since the closed immersion $\mathcal{U}^{1} = \mathcal{U} \times_{\mathcal{Z}^{\tau}} \mathcal{Z}^{\tau,1} \to \mathcal{U}_{/\mathbb{F}}$ is an isomorphism, we may regard $\mathcal{U}_{/\mathbb{F}}$ as an open substack of $\mathcal{Z}^{\tau,1}$. Since $\mathcal{Z}^{\tau,1}$ is reduced, by Theorem 3.2(2), so is its open substack $\mathcal{U}_{/\mathbb{F}}$. This completes the proof of the proposition. ☐

Corollary 4.2. *Let τ be a tame type. There is a dense open substack $\mathcal{U}$ of $\mathcal{Z}^{\tau}$ such that we have an isomorphism $\mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{Z}^{\tau}} \mathcal{U} \xrightarrow{\sim} \mathcal{U}$, as well as isomorphisms*

$$\mathcal{U} \times_{\mathcal{Z}^{\tau}} \mathcal{C}^{\tau,\mathrm{BT},a} \xrightarrow{\sim} \mathcal{U} \times_{\mathcal{Z}^{\tau}} \mathcal{Z}^{\tau,a} \xrightarrow{\sim} \mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^{a},$$

for each $a \geq 1$.

Proof. Taking $\mathcal{U}$ as constructed in the proof of Proposition 4.1, this follows from Lemma A.6 applied to the monomorphism (4.1.1) and isomorphism (4.1.2). ☐

Remark 4.3. More colloquially, Corollary 4.2 shows that for each tame type τ, there is an open dense substack $\mathcal{U}$ of $\mathcal{Z}^{\tau}$ consisting of Galois representations which have a unique Breuil–Kisin model of type τ.

Lemma 4.4. *If $\mathcal{U}$ is an open substack of $\mathcal{Z}^{\tau}$ satisfying the condition of Proposition 4.1, and if $T \to \mathcal{Z}^{\tau}_{/\mathbb{F}}$ is a smooth morphism whose source is a scheme, then $T \times_{\mathcal{Z}^{\tau}_{/\mathbb{F}}} \mathcal{U}_{/\mathbb{F}}$ is reduced, and is a dense open subscheme of T.*

Proof. Since $\mathcal{Z}^{\tau}_{/\mathbb{F}}$ is a Noetherian algebraic stack (being of finite presentation over $\operatorname{Spec} \mathbb{F}$), the open immersion

$$\mathcal{U}_{/\mathbb{F}} \to \mathcal{Z}^{\tau}_{/\mathbb{F}}$$

is quasicompact [Stacks 2005–, Tag 0CPM], and has dense image by assumption. Again by assumption, it factors through $\mathcal{Z}^{\tau}_{\mathrm{red}}$ $(= (\mathcal{Z}^{\tau}_{/\mathbb{F}})_{\mathrm{red}})$, and the resulting open immersion

$$\mathcal{U}_{/\mathbb{F}} \to \mathcal{Z}^{\tau}_{\mathrm{red}}$$

is again quasicompact with dense image. Since its target is reduced, it is in fact scheme-theoretically dominant.

Now the given smooth morphism $T \to \mathcal{Z}^{\tau}_{/\mathbb{F}}$ base-changes to a smooth morphism

$$T \times_{\mathcal{Z}^{\tau}_{/\mathbb{F}}} \mathcal{Z}^{\tau}_{\mathrm{red}} \to \mathcal{Z}^{\tau}_{\mathrm{red}},$$

whose source is equal to the underlying reduced scheme T_{red} of T. (Indeed the source is reduced, because property of being reduced is local for the smooth topology, [Stacks 2005–, Tag 04YH]; and it is a closed subscheme of T with underlying topological space equal to that of T, by [Stacks 2005–, Tag 04XH].) Since smooth morphisms are in particular flat, the pullback

$$T \times_{\mathcal{Z}^{\tau}_{/\mathbb{F}}} \mathcal{U}_{/\mathbb{F}} = T_{\mathrm{red}} \times_{\mathcal{Z}^{\tau}_{\mathrm{red}}} \mathcal{U}_{/\mathbb{F}} \to T_{\mathrm{red}}$$

is an open immersion with dense image; here we use the fact that for a quasicompact morphism, the property of being scheme-theoretically dominant is preserved by flat base-change. Since the source of this pullback is open in T_{red}, it is itself reduced. $\square$

The following result is standard, but we recall the proof for the sake of completeness.

Lemma 4.5. *Let T be a Noetherian scheme, all of whose local rings at finite type points are G-rings. If T is reduced (resp. generically reduced), then so are all of its complete local rings at finite type points.*

Proof. Let t be a finite type point of T, and write $A := \mathcal{O}_{T,t}$. Then A is a (generically) reduced local G-ring, and we need to show that its completion $\widehat{A}$ is also (generically) reduced. Let $\widehat{\mathfrak{p}}$ be a (minimal) prime of $\widehat{A}$; since $A \to \widehat{A}$ is (faithfully) flat, $\widehat{\mathfrak{p}}$ lies over a (minimal) prime $\mathfrak{p}$ of A by the going-down theorem.

Then $A_{\mathfrak{p}}$ is reduced by assumption, and we need to show that $\widehat{A}_{\widehat{\mathfrak{p}}}$ is reduced. By [Stacks 2005–, Tag 07QK], it is enough to show that the morphism $A \to \widehat{A}_{\widehat{\mathfrak{p}}}$ is regular. Both A and $\widehat{A}$ are G-rings (the latter by [Stacks 2005–, Tag 07PS]), so the composite

$$A \to \widehat{A} \to (\widehat{A}_{\widehat{\mathfrak{p}}})\widehat{\ }$$

is a composite of regular morphisms, and is thus a regular morphism by [Stacks 2005–, Tag 07QI].

This composite factors through the natural morphism $A_{\mathfrak{p}} \to (\widehat{A}_{\widehat{\mathfrak{p}}})\widehat{\ }$, so this morphism is also regular. Factoring it as the composite

$$A_{\mathfrak{p}} \to \widehat{A}_{\widehat{\mathfrak{p}}} \to (\widehat{A}_{\widehat{\mathfrak{p}}})\widehat{\ },$$

it follows from [Stacks 2005–, Tag 07NT] that $A_{\mathfrak{p}} \to \widehat{A}_{\widehat{\mathfrak{p}}}$ is regular, as required. $\square$

Finally, we are ready to prove the main result of this section.

Theorem 4.6. *For any tame type τ, the scheme* Spec $R_{\bar{r}}^{\tau,\mathrm{BT}}/\varpi$ *is generically reduced, with underlying reduced subscheme* Spec $R_{x}^{\tau,1}$.

Proof. By Proposition 3.4, we have a versal morphism

$$\operatorname{Spf} R_{\bar{r}}^{\tau,\mathrm{BT}}/\varpi \to \mathcal{Z}^{\tau}_{/\mathbb{F}}$$

at the point of $\mathcal{Z}^{\tau}_{/\mathbb{F}}$ corresponding to $\bar{r} : G_K \to \mathrm{GL}_2(\mathbb{F}')$. Since $\mathcal{Z}^{\tau}_{/\mathbb{F}}$ is an algebraic stack of finite presentation over $\mathbb{F}$ (as $\mathcal{Z}^{\tau}$ is a ϖ-adic formal algebraic stack of finite presentation over $\operatorname{Spf}\mathcal{O}$), we may apply [Stacks 2005–, Tag 0DR0] to this morphism so as to find a smooth morphism $V \to \mathcal{Z}^{\tau}_{/\mathbb{F}}$ with source a finite type $\mathcal{O}/\varpi$-scheme, and a point $v \in V$ with residue field $\mathbb{F}'$, such that there is an isomorphism $\widehat{\mathcal{O}}_{V,v} \cong R_{\bar{r}}^{\tau,\mathrm{BT}}/\varpi$, compatible with the given morphism to $\mathcal{Z}^{\tau}_{/\mathbb{F}}$. Proposition 4.1 and Lemma 4.4 taken together show that V is generically reduced, and so the result follows from Lemma 4.5. $\square$

5. A case of the classical geometric Breuil–Mézard conjecture

In this section, by combining the methods of [Emerton and Gee 2014] and [Gee and Kisin 2014] we prove a special case of the classical geometric Breuil–Mézard conjecture [Emerton and Gee 2014, Conjecture 4.2.1]. This result is “globalized” in Section 6.

Let $\bar{r} : G_K \to \mathrm{GL}_2(\mathbb{F})$ be a continuous representation, and let $R^{\square}_{\bar{r}}$ be the universal framed deformation $\mathcal{O}$-algebra for $\bar{r}$. In this section we write $R^{\square}_{\bar{r},0,\tau}$ for the quotient of $R^{\square}_{\bar{r}}$ that elsewhere we have denoted $R^{\tau,\mathrm{BT}}_{\bar{r}}$. We use the more cumbersome notation $R^{\square}_{\bar{r},0,\tau}$ here to make it easier for the reader to refer to [Emerton and Gee 2014] and [Gee and Kisin 2014].

By [Emerton and Gee 2014, Proposition 4.1.2], $R^{\square}_{\bar{r},0,\tau}/\varpi$ is zero if $\bar{r}$ has no potentially Barsotti–Tate lifts of type τ, and otherwise it is equidimensional of dimension $4+[K:\mathbb{Q}_p]$. Each Spec $R^{\square}_{\bar{r},0,\tau}/\varpi$ is a closed subscheme of Spec $R^{\square}_{\bar{r}}/\varpi$, and we write $Z(R^{\square}_{\bar{r},0,\tau}/\varpi)$ for the corresponding cycle, as in [loc. cit., Definition 2.2.5]. This is a formal sum of the irreducible components of Spec $R^{\square}_{\bar{r},0,\tau}/\varpi$, weighted by the multiplicities with which they occur.

Lemma 5.1. *If σ is a non-Steinberg Serre weight of* $\mathrm{GL}_2(k)$, *then there are integers $n_\tau(\sigma)$ such that $\sigma = \sum_\tau n_\tau(\sigma)\bar{\sigma}(\tau)$ in the Grothendieck group of mod p representations of* $\mathrm{GL}_2(k)$, *where the τ run over the tame types.*

Proof. This is an immediate consequence of the surjectivity of the natural map from the Grothendieck group of $\overline{\mathbb{Q}}_p$-representations of $\mathrm{GL}_2(k)$ to the Grothendieck group of $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(k)$ [Serre 1977, Section III, Theorem 33], together with the observation that the reduction of the Steinberg representation of $\mathrm{GL}_2(k)$ is precisely $\bar{\sigma}_{\vec{0},\overrightarrow{p-1}}$. ☐

Let σ be a non-Steinberg Serre weight of $\mathrm{GL}_2(k)$, so that by Lemma 5.1 we can write

$$\sigma = \sum_{\tau} n_\tau(\sigma)\bar{\sigma}(\tau) \tag{5.1.1}$$

in the Grothendieck group of mod p representations of $\mathrm{GL}_2(k)$. Note that the integers $n_\tau(\sigma)$ are not uniquely determined; however, all our constructions elsewhere in this paper will be (nonobviously!) independent of the choice of the $n_\tau(\sigma)$. We also write

$$\bar{\sigma}(\tau) = \sum_{\sigma} m_\sigma(\tau)\sigma;$$

since $\bar{\sigma}(\tau)$ is multiplicity-free, each $m_\sigma(\tau)$ is equal to 0 or 1. Then

$$\sigma = \sum_{\sigma'}\Big(\sum_{\tau} n_\tau(\sigma)m_{\sigma'}(\tau)\Big)\sigma',$$

and therefore

$$\sum_{\tau} n_\tau(\sigma)m_{\sigma'}(\tau) = \delta_{\sigma,\sigma'}. \tag{5.1.2}$$

For each non-Steinberg Serre weight σ, we set

$$\mathcal{C}_\sigma := \sum_{\tau} n_\tau(\sigma)Z(R^{\square}_{\bar{r},0,\tau}/\varpi),$$

where the sum ranges over the tame types τ, and the integers $n_\tau(\sigma)$ are as in (5.1.1). By definition this is a formal sum with (possibly negative) multiplicities of irreducible subschemes of Spec $R^{\square}_{\bar{r}}/\varpi$; recall that we say that it is *effective* if all of the multiplicities are nonnegative.

Theorem 5.2. *Let σ be a non-Steinberg Serre weight. Then the cycle $\mathcal{C}_\sigma$ is effective, and is nonzero precisely when $\sigma \in W(\bar{r})$. It is independent of the choice of integers $n_\tau(\sigma)$ satisfying* (5.1.1). *For each tame type τ, we have*

$$Z(R^{\square}_{\bar{r},0,\tau}/\varpi) = \sum_{\sigma \in \mathrm{JH}(\bar{\sigma}(\tau))} \mathcal{C}_\sigma.$$

Proof. We will argue exactly as in the proof of [Emerton and Gee 2014, Theorem 5.5.2] (taking $n = 2$), and we freely use the notation and definitions of [loc. cit.]. Since $p > 2$, we have $p \nmid n$ and thus a suitable globalization $\bar{\rho}$ exists provided that [loc. cit., Conjecture A.3] holds for $\bar{r}$. The latter follows from the proof of Theorem A.2 of [Gee and Kisin 2014] (which shows that $\bar{r}$ has a potentially Barsotti–Tate lift) and Lemma 4.4.1 of [loc. cit.]. (which shows that any potentially Barsotti–Tate representation is potentially diagonalizable). These same results also show that the equivalent conditions of [Emerton and Gee 2014, Lemma 5.5.1] hold in the case that $\lambda_v = 0$ for all v, and in particular in the case that $\lambda_v = 0$ and τ_v is tame for all v, which is all that we will require.

By [Emerton and Gee 2014, Lemma 5.5.1(5)], we see that for each choice of tame types τ_v, we have

$$Z(\bar{R}_\infty/\varpi) = \sum_{\bigotimes_{v \mid p} \sigma_v} \prod_{v \mid p} m_{\sigma_v}(\tau_v) Z'_{\bigotimes_{v \mid p} \sigma_v}(\bar{\rho}). \tag{5.2.1}$$

Now, by definition we have

$$Z(\bar{R}_\infty/\varpi) = \prod_{v \mid p} Z(R^{\square}_{\bar{r},0,\tau_v}/\varpi) \times Z(\mathbb{F}[\![x_1, \dots, x_{q-[F^+:\mathbb{Q}]n(n-1)/2}, t_1, \dots, t_{n^2}]\!]). \tag{5.2.2}$$

Fix a non-Steinberg Serre weight $\sigma = \bigotimes_v \sigma_v$, and sum over all choices of types τ_v, weighted by $\prod_{v \mid p} n_{\tau_v}(\sigma_v)$. We obtain

$$\begin{aligned}\sum_\tau \prod_{v \mid p} n_{\tau_v}(\sigma_v) \prod_{v \mid p} Z(R^{\square}_{\bar{r},0,\tau_v}/\varpi) \times Z(\mathbb{F}[\![x_1, \dots, x_{q-[F^+:\mathbb{Q}]n(n-1)/2}, t_1, \dots, t_{n^2}]\!])& \\ = \sum_\tau \prod_{v \mid p} n_{\tau_v}(\sigma_v) \sum_{\bigotimes_{v \mid p} \sigma'_v} \prod_{v \mid p} m_{\sigma'_v}(\tau_v) Z'_{\bigotimes_{v \mid p} \sigma'_v}(\bar{\rho}),&\end{aligned}$$

which by (5.1.2) simplifies to

$$\prod_{v \mid p} \mathcal{C}_{\sigma_v} \times Z(\mathbb{F}[\![x_1, \dots, x_{q-[F^+:\mathbb{Q}]n(n-1)/2}, t_1, \dots, t_{n^2}]\!]) = Z'_{\bigotimes_{v \mid p} \sigma_v}(\bar{\rho}). \tag{5.2.3}$$

Since $Z'_{\bigotimes_{v \mid p} \sigma_v}(\bar{\rho})$ is effective by definition (as it is defined as a positive multiple of the support cycle of a patched module), this shows that every $\prod_{v \mid p} \mathcal{C}_{\sigma_v}$ is effective. We conclude that either every $\mathcal{C}_\sigma$ is effective, or that every $-\mathcal{C}_\sigma$ is effective.

Substituting (5.2.3) and (5.2.2) into (5.2.1), we see that

$$\begin{aligned}\prod_{v \mid p} Z(R^{\square}_{\bar{r},0,\tau_v}/\varpi) \times Z(\mathbb{F}[\![x_1, \dots, x_{q-[F^+:\mathbb{Q}]n(n-1)/2}, t_1, \dots, t_{n^2}]\!])& \\ = \prod_{v \mid p} \Biggl(\sum_{\sigma \in \mathrm{JH}(\bar{\sigma}(\tau))} \mathcal{C}_{\sigma_v} \Biggr) \times Z(\mathbb{F}[\![x_1, \dots, x_{q-[F^+:\mathbb{Q}]n(n-1)/2}, t_1, \dots, t_{n^2}]\!]),&\end{aligned} \tag{5.2.4}$$

and we deduce that either

$$Z(R^{\square}_{\bar{r},0,\tau}/\varpi) = \sum_\sigma m_\sigma(\tau) \mathcal{C}_\sigma$$

for all τ, or

$$Z(R^{\square}_{\bar{r},0,\tau}/\varpi) = -\sum_{\sigma} m_{\sigma}(\tau)\mathcal{C}_{\sigma}$$

for all τ.

Since each $Z(R^{\square}_{\bar{r},0,\tau}/\varpi)$ is effective, the second possibility holds if and only if every $-\mathcal{C}_{\sigma}$ is effective (since either all the $-\mathcal{C}_{\sigma}$ are effective, or all the $\mathcal{C}_{\sigma}$ are effective). It remains to show that this possibility leads to a contradiction. Now, if $Z(R^{\square}_{\bar{r},0,\tau}/\varpi) = -\sum_{\sigma} m_{\sigma}(\tau)\mathcal{C}_{\sigma}$ for all τ, then substituting into the definition $\mathcal{C}_{\sigma} = \sum_{\tau} n_{\tau}(\sigma) Z(R^{\square}_{\bar{r},0,\tau}/\varpi)$, we obtain

$$\mathcal{C}_{\sigma} = \sum_{\sigma'} \biggl(\sum_{\tau} n_{\tau}(\sigma) m_{\sigma'}(\tau) \biggr) (-\mathcal{C}_{\sigma'}),$$

and applying (5.1.2), we obtain $\mathcal{C}_{\sigma} = -\mathcal{C}_{\sigma}$, so that $\mathcal{C}_{\sigma} = 0$ for all σ. Thus all the $\mathcal{C}_{\sigma}$ are effective, as claimed.

Since $Z'_{\bigotimes_{v \mid p} \sigma_v}(\bar{\rho})$ by definition depends only on (the global choices in the Taylor–Wiles method, and) $\bigotimes_{v \mid p} \sigma_v$, and *not* on the particular choice of the $n_{\tau}(\sigma)$, it follows from (5.2.3) that $\mathcal{C}_{\sigma}$ is also independent of this choice.

Finally, note that by definition $Z'_{\bigotimes_{v \mid p} \sigma_v}(\bar{\rho})$ is nonzero precisely when σ_v is in the set $W^{\mathrm{BT}}(\bar{r})$ defined in [Gee and Kisin 2014, Section 3]; but by the main result of [Gee et al. 2015], this is precisely the set $W(\bar{r})$. □

Remark 5.3. As we do not use wildly ramified types elsewhere in the paper, we have restricted the statement of Theorem 5.2 to the case of tame types; but the statement admits a natural extension to the case of wildly ramified inertial types (with some components now occurring with multiplicity greater than one), and the proof goes through unchanged in this more general setting.

6. The geometric Breuil–Mézard conjecture for the stacks $\mathcal{Z}^{\mathrm{dd},1}$

We now prove our main results on the irreducible components of $\mathcal{Z}^{\mathrm{dd},1}$. We do this by a slightly indirect method, defining certain formal sums of these irreducible components which we then compute via the geometric Breuil–Mézard conjecture, and in particular the results of Section 5.

By Theorem 3.2, $\mathcal{Z}^{\mathrm{dd},1}$ is reduced and equidimensional, and each $\mathcal{Z}^{\tau,1}$ is a union of some of its irreducible components. Let $K(\mathcal{Z}^{\mathrm{dd},1})$ be the free abelian group generated by the irreducible components of $\mathcal{Z}^{\mathrm{dd},1}$. We say that an element of $K(\mathcal{Z}^{\mathrm{dd},1})$ is *effective* if the multiplicity of each irreducible component is nonnegative. We say that an element of $K(\mathcal{Z}^{\mathrm{dd},1})$ is *reduced and effective* if the multiplicity of each irreducible component is 0 or 1; we will sometimes abuse language by identifying a reduced and effective cycle with the reduced union of the irreducible components appearing in it.

Let x be a finite type point of $\mathcal{Z}^{\mathrm{dd},1}$, corresponding to a representation $\bar{r} : G_K \to \mathrm{GL}_2(\mathbb{F}')$, and recall that $R^{\mathrm{dd},1}_x$ is a versal ring to $\mathcal{Z}^{\mathrm{dd},1}$ having each $R^{\tau,1}_x$ as a quotient. Since $\mathcal{Z}^{\tau,1}$ is a union of irreducible components of $\mathcal{Z}^{\mathrm{dd},1}$, Spec $R^{\tau,1}_x$ is a union of irreducible components of Spec $R^{\mathrm{dd},1}_x$.

Let $K(R_x^{\mathrm{dd},1})$ be the free abelian group generated by the irreducible components of $\operatorname{Spec} R_x^{\mathrm{dd},1}$. By [Stacks 2005–, Tag 0DRB, Tag 0DRD], there is a natural multiplicity-preserving surjection from the set of irreducible components of $\operatorname{Spec} R_x^{\mathrm{dd},1}$ to the set of irreducible components of $\mathcal{Z}^{\mathrm{dd},1}$ which contain x. Using this surjection, we can define a group homomorphism

$$K(\mathcal{Z}^{\mathrm{dd},1}) \to K(R_x^{\mathrm{dd},1})$$

in the following way: we send any irreducible component $\mathcal{Z}$ of $\mathcal{Z}^{\mathrm{dd},1}$ which contains x to the formal sum of the irreducible components of $\operatorname{Spec} R_x^{\mathrm{dd},1}$ in the preimage of $\mathcal{Z}$ under this surjection, and we send every other irreducible component to 0.

Lemma 6.1. *An element $\overline{\mathcal{T}}$ of $K(\mathcal{Z}^{\mathrm{dd},1})$ is effective if and only if for every finite type point x of $\mathcal{Z}^{\mathrm{dd},1}$, the image of $\overline{\mathcal{T}}$ in $K(R_x^{\mathrm{dd},1})$ is effective. We have $\overline{\mathcal{T}} = 0$ if and only if its image is 0 in every $K(R_x^{\mathrm{dd},1})$.*

Proof. The "only if" direction is trivial, so we need only consider the "if" implication. Write $\overline{\mathcal{T}} = \sum_{\overline{\mathcal{Z}}} a_{\overline{\mathcal{Z}}} \overline{\mathcal{Z}}$, where the sum runs over the irreducible components $\overline{\mathcal{Z}}$ of $\mathcal{Z}^{\mathrm{dd},1}$, and the $a_{\overline{\mathcal{Z}}}$ are integers.

Suppose first that the image of $\overline{\mathcal{T}}$ in $K(R_x^{\mathrm{dd},1})$ is effective for every x; we then have to show that each $a_{\overline{\mathcal{Z}}}$ is nonnegative. To see this, fix an irreducible component $\overline{\mathcal{Z}}$, and choose x to be a finite type point of $\mathcal{Z}^{\mathrm{dd},1}$ which is contained in $\overline{\mathcal{Z}}$ and in no other irreducible component of $\mathcal{Z}^{\mathrm{dd},1}$. Then the image of $\overline{\mathcal{T}}$ in $K(R_x^{\mathrm{dd},1})$ is equal to $a_{\overline{\mathcal{Z}}}$ times a nonempty sum of irreducible components of $\operatorname{Spec} R_x^{\mathrm{dd},1}$. By hypothesis, this must be effective, which implies that $a_{\overline{\mathcal{Z}}}$ is nonnegative, as required.

Finally, if the image of $\overline{\mathcal{T}}$ in $K(R_x^{\mathrm{dd},1})$ is 0, then $a_{\overline{\mathcal{Z}}} = 0$; so if this holds for all x, then $\overline{\mathcal{T}} = 0$. □

For each tame type τ, we let $\mathcal{Z}(\tau)$ denote the formal sum of the irreducible components of $\mathcal{Z}^{\tau,1}$, considered as an element of $K(\mathcal{Z}^{\mathrm{dd},1})$. By Lemma 5.1, for each non-Steinberg Serre weight σ of $\mathrm{GL}_2(k)$ there are integers $n_\tau(\sigma)$ such that $\sigma = \sum_\tau n_\tau(\sigma)\bar{\sigma}(\tau)$ in the Grothendieck group of mod p representations of $\mathrm{GL}_2(k)$, where the τ run over the tame types. We set

$$\mathcal{Z}(\sigma) := \sum_\tau n_\tau(\sigma)\mathcal{Z}(\tau) \in K(\mathcal{Z}^{\mathrm{dd},1}).$$

The integers $n_\tau(\sigma)$ are not necessarily unique, but it follows from the following result that $\mathcal{Z}(\sigma)$ is independent of the choice of $n_\tau(\sigma)$, and is reduced and effective.

Theorem 6.2. (1) *Each $\mathcal{Z}(\sigma)$ is an irreducible component of $\mathcal{Z}^{\mathrm{dd},1}$.*

(2) *The finite type points of $\mathcal{Z}(\sigma)$ are precisely the representations $\bar{r} : G_K \to \mathrm{GL}_2(\mathbb{F}')$ having σ as a Serre weight.*

(3) *For each tame type τ, we have $\mathcal{Z}(\tau) = \sum_{\sigma \in \mathrm{JH}(\bar{\sigma}(\tau))} \mathcal{Z}(\sigma)$.*

(4) *Every irreducible component of $\mathcal{Z}^{\mathrm{dd},1}$ is of the form $\mathcal{Z}(\sigma)$ for some unique non-Steinberg Serre weight σ.*

(5) *For each tame type τ, and each $J \in \mathcal{P}_\tau$, we have $\mathcal{Z}(\bar{\sigma}(\tau)_J) = \overline{\mathcal{Z}}(J)$.*

Proof. Let x be a finite type point of $\mathcal{Z}^{\mathrm{dd},1}$ corresponding to $\bar{r} : G_K \to \mathrm{GL}_2(\mathbb{F}')$, and write $\mathcal{Z}(\sigma)_x$, $\mathcal{Z}(\tau)_x$ for the images in $K(R_x^{\mathrm{dd},1})$ of $\mathcal{Z}(\sigma)$ and $\mathcal{Z}(\tau)$ respectively. Each $\operatorname{Spec} R_x^{\tau,1}$ is a closed subscheme of

Spec $R^{\square}$, the universal framed deformation $\mathcal{O}_{E'}$-algebra for $\bar{r}$, so we may regard the $\mathcal{Z}(\tau)_x$ as formal sums (with multiplicities) of irreducible subschemes of Spec $R^{\square}/\pi$.

By definition, $\mathcal{Z}(\tau)_x$ is just the underlying cycle of Spec $R_x^{\tau,1}$. By Theorem 4.6, this is equal to the underlying cycle of Spec $R_{\bar{r}}^{\tau,\mathrm{BT}}/\varpi$. Consequently, $\mathcal{Z}(\sigma)_x$ is the cycle denoted by $\mathcal{C}_\sigma$ in Section 5. It follows from Theorem 5.2 that:

- $\mathcal{Z}(\sigma)_x$ is effective, and is nonzero precisely when σ is a Serre weight for $\bar{r}$.
- For each tame type τ, we have $\mathcal{Z}(\tau)_x = \sum_{\sigma\in\mathrm{JH}(\bar{\sigma}(\tau))} \mathcal{Z}(\sigma)_x$.

Applying Lemma 6.1, we see that each $\mathcal{Z}(\sigma)$ is effective, and that (3) holds. Since $\mathcal{Z}^{\tau,1}$ is reduced, $\mathcal{Z}(\tau)$ is reduced and effective, so it follows from (3) that each $\mathcal{Z}(\sigma)$ is reduced and effective. Since x is a finite type point of $\mathcal{Z}(\sigma)$ if and only if $\mathcal{Z}(\sigma)_x \neq 0$, we have also proved (2).

Since every irreducible component of $\mathcal{Z}^{\mathrm{dd},1}$ is an irreducible component of some $\mathcal{Z}^{\tau,1}$, in order to prove (1) and (4) it suffices to show that for each τ, every irreducible component of $\mathcal{Z}^{\tau,1}$ is of the form $\mathcal{Z}(\bar{\sigma}(\tau)_J)$ for some J, and that each $\mathcal{Z}(\bar{\sigma}(\tau)_J)$ is irreducible. Since τ is fixed for the rest of the argument, let us simplify notation by writing $\bar{\sigma}_J$ for $\bar{\sigma}(\tau)_J$. Now, by Theorem 3.5(2), we know that $\mathcal{Z}^{\tau,1}$ has exactly $\#\mathcal{P}_\tau$ irreducible components, namely the $\bar{\mathcal{Z}}(J')$ for $J' \in \mathcal{P}_\tau$. On the other hand, the $\mathcal{Z}(\bar{\sigma}_J)$ are reduced and effective, and since by the results of [Gee et al. 2015] there exist representations admitting $\bar{\sigma}_J$ as their unique non-Steinberg Serre weight,[5] it follows from (2) that for each J, there must be a $J' \in \mathcal{P}_\tau$ such that $\bar{\mathcal{Z}}(J')$ contributes to $\mathcal{Z}(\bar{\sigma}_J)$, but not to any $\mathcal{Z}(\bar{\sigma}_{J''})$ for $J'' \neq J$.

Since $\mathcal{Z}(\tau)$ is reduced and effective, and the sum in (3) is over $\#\mathcal{P}_\tau$ weights σ, it follows that we in fact have $\mathcal{Z}(\bar{\sigma}_J) = \bar{\mathcal{Z}}(J')$. This proves (1) and (4), and to prove (5), it only remains to show that $J' = J$. To see this, note that by (2), $\mathcal{Z}(\bar{\sigma}_J) = \bar{\mathcal{Z}}(J')$ has a dense open substack whose finite type points have $\bar{\sigma}_J$ as their unique non-Steinberg Serre weight (namely the complement of the union of the $\mathcal{Z}(\sigma')$ for all $\sigma' \neq \bar{\sigma}_J$). By Theorem 3.5(4), it also has a dense open substack whose finite type points have $\bar{\sigma}_{J'}$ as a Serre weight. Considering any finite type point in the intersection of these dense open substacks, we see that $\bar{\sigma}_J = \bar{\sigma}_{J'}$, so that $J = J'$, as required. $\square$

7. The geometric Breuil–Mézard conjecture for the stacks $\mathcal{X}_{2,\mathrm{red}}$

We now explain how to transfer our results from the stacks $\mathcal{Z}^{\mathrm{dd},1}$ to the stacks $\mathcal{X}_{2,\mathrm{red}}$ of [Emerton and Gee 2023]. The book [loc. cit.] establishes an equivalence between the classical "numerical" Breuil–Mézard conjecture and the geometric Breuil–Mézard conjecture for the stacks $\mathcal{X}_{2,\mathrm{red}}$. (Indeed the implication from the former to the latter only requires the classical Breuil–Mézard conjecture at a single sufficiently generic point of each component of $\mathcal{X}_{2,\mathrm{red}}$.)

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[5] More precisely, it follows from the results of [Gee et al. 2015] that there exist $[K:\mathbb{Q}]$-dimensional extension spaces V of reducible representations admitting $\bar{\sigma}_J$ as a Serre weight, such that the representations in V admitting at least one other non-Steinberg weight lie in a finite union of proper subspaces; moreover the number of subspaces in this finite union is independent of $\mathbb{F}'$.

The Breuil–Mézard conjecture for two-dimensional potentially Barsotti–Tate representations is established in [Gee and Kisin 2014], and this is extended to two-dimensional potentially semistable representations of Hodge type 0 in [Emerton and Gee 2023, Theorem 8.6.1]. The arguments of [loc. cit., Section 8.3] translate this into the following theorem. Here $\mathcal{X}_2^{\tau,\mathrm{BT},\mathrm{ss}}$ denotes the stack of two-dimensional potentially semistable representations of Hodge type 0 constructed in [loc. cit.], while $\sigma^{\mathrm{ss}}(\tau)$ is as in [loc. cit., Theorem 8.2.1].

Theorem 7.1 [Emerton and Gee 2023]. *There exist effective cycles Z^σ (elements of the free group on the irreducible components of $\mathcal{X}_{2,\mathrm{red}}$, with nonnegative coefficients) such that for all inertial types τ,*

- *the cycle of the special fiber of $\mathcal{X}_2^{\tau,\mathrm{BT}}$ is equal to $\sum_\sigma m_\sigma(\tau)\cdot Z^\sigma$, while*
- *the cycle of the special fiber of $\mathcal{X}_2^{\tau,\mathrm{BT},\mathrm{ss}}$ is equal to $\sum_\sigma m_\sigma^{\mathrm{ss}}(\tau)\cdot Z^\sigma$.*

Here $\bar{\sigma}(\tau)=\sum_\sigma m_\sigma(\tau)\cdot\sigma$ and $\bar{\sigma}^{\mathrm{ss}}(\tau)=\sum_\sigma m_\sigma^{\mathrm{ss}}(\tau)\cdot\sigma$ in the Grothendieck group of $\mathrm{GL}_2(k)$.

Corollary 7.2 [Emerton and Gee 2023]. *Let $\bar{r}: G_K\to\mathrm{GL}_2(\mathbb{F}')$ be a continuous Galois representation, corresponding to a finite type point of $\mathcal{X}_{2,\mathrm{red}}$. For each Serre weight σ we have $\sigma\in W(\bar{r})$ if and only if $\bar{r}$ lies in the support of Z^σ.*

Proof. This follows by the argument in [Emerton and Gee 2023, Section 8.4]: the Breuil–Mézard multiplicity $\mu_\sigma(\bar{r})$ is nonzero if and only if Z^σ is supported at $\bar{r}$. More precisely, invoking Theorem 8.6.1 of [loc. cit.] in place of Conjecture 8.2.2 of [loc. cit.], the next-to-last paragraph of [loc. cit., Section 8.4] shows that $\bar{r}$ lies in the support of Z^σ if and only if σ is an element of the weight set $W_{\mathrm{BT}}(\bar{r})=\{\sigma:\mu_\sigma(\bar{r})>0\}$. But $W_{\mathrm{BT}}(\bar{r})=W(\bar{r})$ by the main results of [Gee et al. 2015]. ☐

Remark 7.3. The cycles Z^σ of Theorem 7.1 are constructed in [Emerton and Gee 2023, Section 8.3] as follows. For each Serre weight σ', the smooth points of $\mathcal{X}_{2,\mathrm{red}}^{\sigma'}$ that furthermore do not lie on any other component of $\mathcal{X}_{2,\mathrm{red}}$ are dense. Choose such a point $\bar{r}_{\sigma'}$ and let $\{\mu_\sigma(\bar{r}_{\sigma'})\}$ be the multiplicities in the Breuil–Mézard conjecture for $\bar{r}_{\sigma'}$. Then $Z^\sigma:=\sum_{\sigma'}\mu_\sigma(\bar{r}_{\sigma'})\cdot\mathcal{X}_{2,\mathrm{red}}^{\sigma'}$.

It remains to compute the cycles Z^σ. We begin with the following observation, which could be proved with modest effort by calculating dimensions of families of extensions and using the results of [Gee et al. 2015], but is also easily deduced from the results of Section 6.

Lemma 7.4. *Let σ,σ' be Serre weights and suppose that σ' is non-Steinberg. Then $\mathcal{X}_{2,\mathrm{red}}^{\sigma}$ contains at least one finite type point corresponding to a representation $\bar{r}$ with $\sigma'\notin W(\bar{r})$.*

Proof. Suppose first that σ is non-Steinberg. By Theorem 6.2 the component $\mathcal{Z}(\sigma)$ of $\mathcal{Z}^{\mathrm{dd},1}$ has a dense open set $U(\sigma)$ whose finite type points $\bar{r}$ have no non-Steinberg Serre weights other than σ: take $\mathcal{Z}(\sigma)\setminus\bigcup_{\sigma'\neq\sigma}\mathcal{Z}(\sigma')$. The finite type points of $\mathcal{Z}(\sigma)$ described in Remark 3.6 are also dense; therefore at least one of them (indeed a dense set of them) lies in $U(\sigma)$. Let $\bar{r}$ be such a representation. The finite type points of $\mathcal{Z}(\sigma)$ described in Remark 3.6 are precisely the family of niveau 1 representations in the description of $\mathcal{X}_{2,\mathrm{red}}^{\sigma}$ of Theorem 1.1 (and whose construction can be found in the proof of [Emerton and Gee 2023, Theorem 5.5.12] and its correction in the errata to [Emerton and Gee 2023]). So $\bar{r}$ lies on $\mathcal{X}^{\sigma}_{2,\mathrm{red}}$ as well, and by construction the only non-Steinberg weight in $W(\bar{r})$ is σ. This completes the non-Steinberg case.

If instead σ is Steinberg, then by construction, and by the second bullet point in [Emerton and Gee 2023, Definition 5.5.1], $\mathcal{X}^{\sigma}_{2,\mathrm{red}}$ contains representations $\bar{r}$ that are très ramifiée. But très ramifiée representations have no non-Steinberg Serre weights by [Caraiani et al. 2024, Lemma A.5]. ☐

Remark 7.5. Once we have proved Theorem 7.6 below, "at least one finite type point" in the statement of Lemma 7.4 can be promoted to "a dense set of finite type points" by taking $\mathcal{X}^{\sigma}_{2,\mathrm{red}} \setminus \bigcup_{\sigma' \neq \sigma} \mathcal{X}^{\sigma'}_{2,\mathrm{red}}$.

We now reach our main theorem.

Theorem 7.6. *Suppose $p > 2$. We have*

- $Z^{\sigma} = \mathcal{X}^{\sigma}_{2,\mathrm{red}}$, *if the weight σ is not Steinberg, while*
- $Z^{\chi\otimes\mathrm{St}} = \mathcal{X}^{\chi}_{2,\mathrm{red}} + \mathcal{X}^{\chi\otimes\mathrm{St}}_{2,\mathrm{red}}$ *if the weight $\sigma \cong \chi \otimes \mathrm{St}$ is Steinberg.*

In particular $\sigma \in W(\bar{r})$ if and only if $\bar{r}$ lies on $\mathcal{X}^{\sigma}_{2,\mathrm{red}}$ if σ is not Steinberg, or on $\mathcal{X}^{\chi}_{2,\mathrm{red}} \cup \mathcal{X}^{\chi\otimes\mathrm{St}}_{2,\mathrm{red}}$ if $\sigma \cong \chi \otimes \mathrm{St}$ is Steinberg.

Essentially the same statement appears at [Emerton and Gee 2023, Theorem 8.6.2], and the argument given there invokes an earlier version of this paper.[6] The proof we give below is independent of the proof of [loc. cit., Theorem 8.6.2], but has the same major beats, and is rearranged to delay until as late as possible making any references to the earlier parts of this paper. Indeed the proof now only invokes the generic reducedness of Theorem 4.6 (but certainly invokes it in a crucial way); we hope that this clarifies the various dependencies involved.

However, we also take the opportunity to repair a small gap in the argument given at [Emerton and Gee 2023, Theorem 8.6.2]. It is claimed there that [CEGS19, Theorem 5.2.2(2)] (i.e., our Theorem 6.2(2)) implies that $\mathcal{X}^{\sigma}_{2,\mathrm{red}}$ has a dense set of finite type points corresponding to representations $\bar{r}$ whose only non-Steinberg Serre weight is σ. Although the conclusion is certainly true, the deduction is incorrect, or at least seems to presume that $\mathcal{X}^{\sigma}_{2,\mathrm{red}}$ can be identified with our $\mathcal{Z}(\sigma)$. We replace this claim with an argument using Lemma 7.4, which does follow from Theorem 6.2 (or from the results of [Gee et al. 2015], as previously noted). The claim about points of $\mathcal{X}^{\sigma}_{2,\mathrm{red}}$ will follow once Theorem 7.6 is proved, as explained in Remark 7.5.

Proof of Theorem 7.6. Consider first the non-Steinberg case. The finite type points in the support of the effective cycle Z^{σ} are precisely the representations having σ as a Serre weight. By Lemma 7.4 each component $\mathcal{X}^{\sigma'}_{2,\mathrm{red}}$ with $\sigma' \neq \sigma$ has a finite type point for which σ is not a Serre weight; therefore $\mathcal{X}^{\sigma'}_{2,\mathrm{red}}$ cannot occur in the cycle Z^{σ}. It follows that $Z^{\sigma} = \mu_{\sigma}(\bar{r}_{\sigma}) \cdot \mathcal{X}^{\sigma}_{2,\mathrm{red}}$ with $\bar{r}_{\sigma}$ as in Remark 7.3, and that $\mu_{\sigma}(\bar{r}_{\sigma'}) = 0$ for all $\sigma' \neq \sigma$. Note that we can already deduce that $\sigma \in W(\bar{r})$ if and only if $\bar{r}$ lies on $\mathcal{X}^{\sigma}_{2,\mathrm{red}}$.

Now consider the Steinberg case; by twisting it will suffice to consider the weight $\sigma = \mathrm{St}$. The type $\sigma^{\mathrm{ss}}(\mathrm{triv})$ is the Steinberg type, and so by Theorem 7.1 the cycle of the special fiber of $\mathcal{X}_2^{\mathrm{triv},\mathrm{BT},\mathrm{ss}}$

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[6] As mentioned in the Introduction, the reference [CEGS19, Theorem 5.2.2] in [Emerton and Gee 2023] is Theorem 6.2 of this paper, while the reference [CEGS19, Lemma B.5] in [Emerton and Gee 2023] is [Caraiani et al. 2024, Lemma A.5].

is equal to Z^{St}. In particular the finite type points of Z^{St} are precisely the representations $\bar{r}$ having a semistable lift of Hodge type 0. Such a lift is either crystalline, in which case the trivial Serre weight is a Serre weight for $\bar{r}$, and by the previous paragraph $\bar{r}$ is a finite type point of $\mathcal{X}_{2,\mathrm{red}}^{\mathrm{triv}}$; or else the lift is semistable noncrystalline, in which case $\bar{r}$ is an unramified twist of an extension of the inverse of the cyclotomic character by the trivial character. Such an extension is either peu ramifiée, in which case again $\mathrm{triv} \in W(\bar{r})$, and $\bar{r}$ lies on $\mathcal{X}_{2,\mathrm{red}}^{\mathrm{triv}}$ by the first paragraph of the proof; or else it is tres̀ ramifiée, and is a finite type point of $\mathcal{X}_{2,\mathrm{red}}^{\mathrm{St}}$ (as a member of the family of niveau 1 representations defining $\mathcal{X}_{2,\mathrm{red}}^{\mathrm{St}}$). Since all the finite type points of the support of Z^{St} are contained in $\mathcal{X}_{2,\mathrm{red}}^{\mathrm{triv}} \cup \mathcal{X}_{2,\mathrm{red}}^{\mathrm{St}}$ it follows that $Z^{\mathrm{St}} = \mu_{\mathrm{St}}(\bar{r}_{\mathrm{triv}})\mathcal{X}_{2,\mathrm{red}}^{\mathrm{triv}} + \mu_{\mathrm{St}}(\bar{r}_{\mathrm{St}})\mathcal{X}_{2,\mathrm{red}}^{\mathrm{St}}$, and that $\mu_{\mathrm{St}}(\bar{r}_{\sigma'}) = 0$ for all $\sigma' \neq \mathrm{triv}, \mathrm{St}$.

In the remainder of the argument we can and do assume that $\bar{r}_{\mathrm{triv}}$ is an extension of an unramified twist of the inverse cyclotomic character by a *different* unramified character: these are dense in $\mathcal{X}_{2,\mathrm{red}}^{\mathrm{triv}}$ by the dominance of the eigenvalue morphism of [Emerton and Gee 2023, Theorem 5.5.12(2)].

It remains to determine the multiplicities $\mu_\sigma(\bar{r}_\sigma)$, $\mu_{\mathrm{St}}(\bar{r}_{\mathrm{triv}})$, and $\mu_{\mathrm{St}}(\bar{r}_{\mathrm{St}})$. Each of these is positive because $\bar{r}_\sigma$ does have σ as a Serre weight, while $\bar{r}_{\mathrm{triv}}$ has St as a Serre weight because each extension as in the previous paragraph has a crystalline lift with labeled Hodge–Tate weights $(p, 0)$ at one embedding lifting each embedding $k \hookrightarrow \overline{\mathbb{F}}_p$, and labeled Hodge–Tate weights $(1, 0)$ at all other embeddings.

Suppose that σ is non-Steinberg. Choose any tame type τ such that $\bar{\sigma}(\tau)$ has σ as a Jordan–Hölder factor. The ring $R_{\bar{r}_\sigma}^{\tau,\mathrm{BT}}$ is versal to $\mathcal{X}_2^{\tau,\mathrm{BT}}$ at $\bar{r}_\sigma$, and since $\bar{r}_\sigma$ is a smooth point of $\mathcal{X}_{2,\mathrm{red}}$ the underlying reduced of $\operatorname{Spec} R_{\bar{r}_\sigma}^{\tau,\mathrm{BT}}/\varpi$ is smooth. But $\operatorname{Spec} R_{\bar{r}_\sigma}^{\tau,\mathrm{BT}}/\varpi$ is generically reduced by Theorem 4.6. We deduce that the Hilbert–Samuel multiplicity of $R_{\bar{r}_\sigma}^{\tau,\mathrm{BT}}/\varpi$ is 1, and therefore $\mu_\sigma(\bar{r}_\sigma) \leq 1$. Since $\mu_\sigma(\bar{r}_\sigma)$ is positive it must be equal to 1. Alternately, it follows from Theorem 6.2 and the isomorphism $\mathcal{Z}^\tau \cong \mathcal{X}_2^{\tau,\mathrm{BT}}$ of [Bellovin et al. 2024, Theorem 4.5] that the cycle of the special fiber of $\mathcal{X}_2^{\tau,\mathrm{BT}}$ is reduced and effective; since by Theorem 7.1 this cycle contains $\mathcal{Z}^\sigma$ as a summand, we see again that $\mu_\sigma(\bar{r}_\sigma) \leq 1$.

Our chosen $\bar{r}_{\mathrm{triv}}$ does not have any semistable noncrystalline lifts, and the semistable Hodge type 0 deformation ring of $\bar{r}_{\mathrm{triv}}$ is simply a crystalline deformation ring, indeed one of the flat deformation rings studied by Kisin [2009]. The argument in the previous paragraph showed that the Hilbert–Samuel multiplicity of $R_{\bar{r}_{\mathrm{cris}}}^{\mathrm{triv},\mathrm{BT}}/\varpi$ is 1. It follows that $\mu_{\mathrm{St}}(\bar{r}_{\mathrm{triv}}) \leq 1$, and since it is positive it must be precisely 1. On the other hand the semistable Hodge type 0 deformation ring of the très ramifiée representation $\bar{r}_{\mathrm{St}}$ is an ordinary deformation ring, hence formally smooth, and we obtain $\mu_{\mathrm{St}}(\bar{r}_{\mathrm{St}}) = 1$. ☐

Remark 7.7. The finite type points of $X_{2,\mathrm{red}}^{\mathrm{St}}$ are precisely those $\bar{r}$ having a semistable noncrystalline lift, i.e., the unramified twists of an extension of the inverse of the cyclotomic character by the trivial character; for the details see [Emerton and Gee 2023, Lemma 8.6.4].

Appendix: A lemma on formal algebraic stacks

We suppose given a commutative diagram of morphisms of formal algebraic stacks:

$$\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[dr] & \mathcal{Y} \arrow[d] \\
 & \operatorname{Spf} \mathcal{O}
\end{tikzcd}$$

We suppose that each of $\mathcal{X}$ and $\mathcal{Y}$ is quasicompact and quasiseparated, and that the horizontal arrow is scheme-theoretically dominant, in the sense of [Emerton 2019, Definition 6.13]. We furthermore suppose that the morphism $\mathcal{X} \to \operatorname{Spf} \mathcal{O}$ realizes $\mathcal{X}$ as a finite type ϖ-adic formal algebraic stack.

Concretely, if we write $\mathcal{X}^a := \mathcal{X} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$, then each $\mathcal{X}^a$ is an algebraic stack, locally of finite type over $\operatorname{Spec} \mathcal{O}/\varpi^a$, and there is an isomorphism $\varinjlim_a \mathcal{X}^a \xrightarrow{\sim} \mathcal{X}$. Furthermore, the assumption that the horizontal arrow is scheme-theoretically dominant means that we may find an isomorphism $\mathcal{Y} \cong \varinjlim_a \mathcal{Y}^a$, with each $\mathcal{Y}^a$ being a quasicompact and quasiseparated algebraic stack, and with the transition morphisms being thickenings, such that the morphism $\mathcal{X} \to \mathcal{Y}$ is induced by a compatible family of morphisms $\mathcal{X}^a \to \mathcal{Y}^a$, each of which is scheme-theoretically dominant. (The $\mathcal{Y}^a$ are uniquely determined by the requirement that for all $b \geq a$ large enough so that the morphism $\mathcal{X}^a \to \mathcal{Y}$ factors through $\mathcal{Y} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^b$, $\mathcal{Y}^a$ is the scheme-theoretic image of the morphism $\mathcal{X}^a \to \mathcal{Y} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^b$. In particular, $\mathcal{Y}^a$ is a closed substack of $\mathcal{Y} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$.)

It is often the case, in the preceding situation, that $\mathcal{Y}$ is also a ϖ-adic formal algebraic stack. For example, we have the following result. (Note that the usual graph argument shows that the morphism $\mathcal{X} \to \mathcal{Y}$ is necessarily algebraic, i.e., representable by algebraic stacks, in the sense of [Stacks 2005–, Tag 06CF] and [Emerton 2019, Definition 3.1]. Thus it makes sense to speak of it being proper, following [loc. cit., Definition 3.11].)

Proposition A.1. *Suppose that the morphism $\mathcal{X} \to \mathcal{Y}$ is proper, and that $\mathcal{Y}$ is locally Ind-finite type over* $\operatorname{Spec} \mathcal{O}$ (*in the sense of* [Emerton 2019, Remark 8.30]). *Then $\mathcal{Y}$ is a ϖ-adic formal algebraic stack.*

Proof. This is an application of [loc. cit., Proposition 10.5]. ☐

A key point is that, because the formation of scheme-theoretic images is not generally compatible with nonflat base-change, the closed immersion

$$\mathcal{Y}^a \hookrightarrow \mathcal{Y} \times_{\mathcal{O}} \mathcal{O}/\varpi^a \tag{A.2}$$

is typically *not* an isomorphism, even if $\mathcal{Y}$ is a ϖ-adic formal algebraic stack. Our goal in the remainder of this discussion is to give a criterion (involving the morphism $\mathcal{X} \to \mathcal{Y}$) on an open substack $\mathcal{U} \hookrightarrow \mathcal{Y}$ which guarantees that the closed immersion $\mathcal{U} \times_{\mathcal{Y}} \mathcal{Y}^a \hookrightarrow \mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$ induced by (A.2) *is* an isomorphism.

We begin by establishing a simple lemma. For any $a \geq 1$, we have the 2-commutative diagram:

$$\begin{tikzcd} \mathcal{X}^a \arrow[r] \arrow[d] & \mathcal{Y}^a \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd} \tag{A.3}$$

Similarly, if $b \geq a \geq 1$, then we have the 2-commutative diagram:

$$\begin{tikzcd} \mathcal{X}^a \arrow[r] \arrow[d] & \mathcal{Y}^a \arrow[d] \\ \mathcal{X}^b \arrow[r] & \mathcal{Y}^b \end{tikzcd} \tag{A.4}$$

Lemma A.5. *Each of the diagrams* (A.3) *and* (A.4) *is* 2-*Cartesian.*

Proof. We may embed the diagram (A.3) in the larger 2-commutative diagram:

$$\begin{tikzcd}
\mathcal{X}^a \arrow[r] \arrow[d] & \mathcal{Y}^a \arrow[r, "\text{(A.2)}"] \arrow[d] & \mathcal{Y} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^a \arrow[d] \\
\mathcal{X} \arrow[r] & \mathcal{Y} \arrow[r, equal] & \mathcal{Y}
\end{tikzcd}$$

Since the outer rectangle is manifestly 2-Cartesian, and since (A.2) is a closed immersion (and thus a monomorphism), we conclude that (A.3) is indeed 2-Cartesian.

A similar argument shows that (A.4) is 2-Cartesian. ☐

We next note that, since each of the closed immersions $\mathcal{Y}^a \hookrightarrow \mathcal{Y}$ is a thickening, giving an open substack $\mathcal{U} \hookrightarrow \mathcal{Y}$ is equivalent to giving an open substack $\mathcal{U}^a \hookrightarrow \mathcal{Y}^a$ for some, or equivalently, every, choice of $a \geq 1$; the two pieces of data are related by the formulas $\mathcal{U}^a := \mathcal{U} \times_{\mathcal{Y}} \mathcal{Y}^a$ and $\varinjlim_a \mathcal{U}^a \xrightarrow{\sim} \mathcal{U}$.

Lemma A.6. *Suppose that $\mathcal{X} \to \mathcal{Y}$ is proper. If $\mathcal{U}$ is an open substack of $\mathcal{Y}$, then the following conditions are equivalent*:

(1) *The morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{U} \to \mathcal{U}$ is a monomorphism.*

(2) *The morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{U} \to \mathcal{U}$ is an isomorphism.*

(3) *For every $a \geq 1$, the morphism $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \to \mathcal{U}^a$ is a monomorphism.*

(4) *For every $a \geq 1$, the morphism $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \to \mathcal{U}^a$ is an isomorphism.*

(5) *For some $a \geq 1$, the morphism $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \to \mathcal{U}^a$ is a monomorphism.*

(6) *For some $a \geq 1$, the morphism $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \to \mathcal{U}^a$ is an isomorphism.*

Furthermore, if these equivalent conditions hold, then the closed immersion $\mathcal{U}^a \hookrightarrow \mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$ is an isomorphism, for each $a \geq 1$.

Proof. The key point is that Lemma A.5 implies that the diagram

$$\begin{tikzcd}
\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \arrow[r] \arrow[d] & \mathcal{U}^a \arrow[d] \\
\mathcal{X} \times_{\mathcal{Y}} \mathcal{U} \arrow[r] & \mathcal{U}
\end{tikzcd}$$

is 2-Cartesian, for any $a \geq 1$, and similarly, that if $b \geq a \geq 1$, then the diagram

$$\begin{tikzcd}
\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \arrow[r] \arrow[d] & \mathcal{U}^a \arrow[d] \\
\mathcal{X}^b \times_{\mathcal{Y}^b} \mathcal{U}^b \arrow[r] & \mathcal{U}^b
\end{tikzcd}$$

is 2-Cartesian. Since the vertical arrows of this latter diagram are finite order thickenings, we find (by applying the analogue of [Stacks 2005–, Tag 09ZZ] for algebraic stacks, whose straightforward deduction from that result we leave to the reader) that the top horizontal arrow is a monomorphism if and only if the bottom horizontal arrow is. This shows the equivalence of (3) and (5). Since the morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{U} \to \mathcal{U}$ is obtained as the inductive limit of the various morphisms $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \to \mathcal{U}^a$, we find that (3) implies (1) (by applying e.g., [Emerton 2019, Lemma 4.11(1)], which shows that the inductive limit of monomorphisms is a monomorphism), and also that (4) implies (2) (the inductive limit of isomorphisms being again an isomorphism).

Conversely, if (1) holds, then the base-changed morphism

$$\mathcal{X} \times_{\mathcal{Y}} (\mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^a) \to \mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$$

is a monomorphism. The source of this morphism admits an alternative description as $\mathcal{X}^a \times_{\mathcal{Y}} \mathcal{U}$, which the 2-Cartesian diagram at the beginning of the proof allows us to identify with $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a$. Thus we obtain a monomorphism

$$\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \hookrightarrow \mathcal{U} \times_{\mathcal{O}} \mathcal{O}_{/}\varpi^a.$$

Since this monomorphism factors through the closed immersion $\mathcal{U}^a \hookrightarrow \mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$, we find that each of the morphisms of (3) is a monomorphism; thus (1) implies (3). Similarly, (2) implies (4), and also implies that the closed immersion $\mathcal{U}^a \hookrightarrow \mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$ is an isomorphism, for each $a \geq 1$.

Since clearly (4) implies (6), while (6) implies (5), to complete the proof of the proposition, it suffices to show that (5) implies (6). Suppose then that $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \to \mathcal{U}^a$ is a monomorphism. Since $\mathcal{U}^a \hookrightarrow \mathcal{Y}^a$ is an open immersion, it is in particular flat. Since $\mathcal{X}^a \to \mathcal{Y}^a$ is scheme-theoretically dominant and quasicompact (being proper), any flat base-change of this morphism is again scheme-theoretically dominant, as well as being proper. Thus we see that $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \to \mathcal{U}^a$ is a scheme-theoretically dominant proper monomorphism, i.e., a scheme-theoretically dominant closed immersion, i.e., an isomorphism, as required. ☐

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