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Divisibility of character values of the symmetric group by prime powers

Sarah Peluse and Kannan Soundararajan

In memory of Chandra Sekhar Raju

Let k be a positive integer. We show that, as n goes to infinity, almost every entry of the character table of S_n is divisible by k. This proves a conjecture of Miller.

1. Introduction

It is a standard fact that the irreducible characters of S_n take only integer values for every natural number n. In 2017, Miller [12] computed the character tables of S_n for all $n \leq 38$ and looked at various statistical properties of these integers as n grew. His computations suggested that

(1) the density of even entries tended to 1 as n tended to infinity,

(2) the density of entries divisible by 3, the density of entries divisible by 5, and the density of entries divisible by 7 increase with n,

(3) about half of the nonzero entries were positive,

(4) and the density of zeros in the character table decreased as n grew, but not very quickly.

Based on this first observation, Miller [12; 14] conjectured that as n goes to infinity, almost every entry of the character table of the symmetric group S_n is even. Following partial progress due to McKay [11], Gluck [6], Ganguly, Prasad and Spallone [5], and Morotti [16], Peluse proved this conjecture in [18]. Based on the second observation, Miller [12; 14] also conjectured, more generally, that for any fixed prime p, almost every entry of the character table of S_n is a multiple of p as n goes to infinity. We proved this conjecture in [19], with a uniform upper bound for the number of entries not divisible by a fixed prime. Recently, Miller [13] conjectured, even more generally, that for any fixed prime power q, almost every entry of the character table of S_n is a multiple of q as n goes to infinity. In this paper, we prove this most general of Miller's conjectures.

Theorem 1.1. *Let n be large and $q \leq 10^{-3} \log n/(\log\log n)^2$ be a prime power. The number of entries in the character table of S_n that are not divisible by q is at most*

$$O(p(n)^2 \exp(-(\log\log n)^2)).$$

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It follows immediately from Theorem 1.1 and the union bound that almost every entry of the character table of S_n is divisible by any fixed positive integer as n goes to infinity.

Corollary 1.2. *Let k be any positive integer. Then, as n goes to infinity, the proportion of entries in the character table of S_n that are not divisible by k tends to* 0.

Our methods do not seem to shed any light on Miller's third and fourth observations. Most interesting to us is the question of what proportion of character table entries are zero, and recent large scale simulations of Miller and Scheinerman [15] suggest that the proportion of zeros tends to 0 as n tends to infinity. Combining the Murnaghan–Nakayama rule and an old result of Erdős and Lehner [2] on the distribution of the largest part of a uniformly random partition of n produces a proportion of about $2/\log n$ zeros in the character table of S_n, and no lower bound of a larger order of magnitude seems to be known. In the related setting of finite simple groups of Lie type, Larsen and Miller [8] have shown that almost every character table entry is zero as the rank goes to infinity.

2. Proof outline

For any partitions λ and μ of n, let χ_μ^λ denote the value of the irreducible character of S_n corresponding to λ on the conjugacy class of elements with cycle type corresponding to μ. In [19], our argument proceeded by combining two key facts: (i) if μ contains a part substantially larger than the typical largest part of a random partition, then $\chi_\mu^\lambda = 0$ for almost every λ, and (ii) if ν is another partition of n that is obtained from μ by combining p parts of the same size m into one part of size pm, then $\chi_\mu^\lambda \equiv \chi_\nu^\lambda \pmod{p}$ for every λ. We showed that, for almost every μ, repeatedly combining p parts of the same size in this manner produces a partition $\tilde{\mu}$ containing a very large part, large enough so that $\chi_{\tilde{\mu}}^\lambda$ must be zero for almost every λ. Our main result on the divisibility of character values by primes then followed from the fact that $\chi_\mu^\lambda \equiv \chi_{\tilde{\mu}}^\lambda \pmod{p}$ for every λ.

The second key fact generalizes to a congruence of character value modulo prime powers in a straightforward manner.

Lemma 2.1. *Let p^r be a power of the prime p. Suppose that μ is a partition of n, and that ν is another partition of n obtained from μ by replacing p^r parts of the same size m by p^{r-1} parts of size pm. Then for all partitions λ of n, we have*

$$\chi_\mu^\lambda \equiv \chi_\nu^\lambda \pmod{p^r}.$$

However, when $r > 1$, it is no longer the case that starting from a typical partition μ of n and repeatedly combining p^r parts of the same size m into p^{r-1} parts of size pm produces a partition $\tilde{\mu}$ containing a part substantially larger than the largest part of a typical partition of n. The argument from [19] that worked for primes thus breaks down for all other prime powers.

The key idea used to overcome this barrier is a new condition for character values of the symmetric group to be divisible by a fixed prime power, which we prove by exploiting certain symmetries that appear after applying the Murnaghan–Nakayama rule multiple times.

Theorem 2.2. *Let $n, m_1, \dots, m_r$ be distinct positive integers. Let μ be a partition of n containing parts of size $m_1, \dots, m_r$, each appearing at least p^{r-1} times. If λ is a $\left(\sum_{i=1}^r k_i m_i\right)$-core partition of n for all r-tuples $(k_1, \dots, k_r)$ of integers $0 \le k_1, \dots, k_r \le p^{r-1}$ for which some $k_i = p^{r-1}$, then*

$$p^r \mid \chi_\mu^\lambda.$$

Starting with a partition μ of n, repeatedly combine p^r parts of the same size m into p^{r-1} parts of size pm, until the process terminates in a partition $\tilde{\mu}$ where no part appears more than $p^r - 1$ times. As a preliminary to applying Theorem 2.2 we show that for a typical partition μ, the resulting partition $\tilde{\mu}$ will have r parts that are suitably large, and with each appearing at least p^{r-1} times.

Proposition 2.3. *Starting with a partition μ of n, repeatedly replace every occurrence of p^r parts of the same size m by p^{r-1} parts of size pm until we arrive at a partition $\tilde{\mu}$ where no part appears more than $p^r - 1$ times. Then, except for*

$$O(p(n)\exp(-n^{1/20p^r}))$$

initial partitions μ, the partition $\tilde{\mu}$ contains at least r distinct parts $m_1, \dots, m_r$, each appearing at least p^{r-1} times and satisfying

$$p^{r-1} m_j > \left(1 + \frac{1}{6p^r}\right) \frac{\sqrt{6}}{2\pi} \sqrt{n} \log n.$$

This holds uniformly for $p^r \le 10^{-3} \log n / (\log\log n)^2$.

The significance of the lower bound on $p^{r-1} m_j$ in Proposition 2.3 is that it lies beyond the threshold of values t such that almost every partition of n is a t-core.

Lemma 2.4. *Let $1 \le L \le \log n / \log\log n$ be a real number. Then, for any given integer t with*

$$t \ge \left(1 + \frac{1}{L}\right) \frac{\sqrt{6}}{2\pi} \sqrt{n} \log n,$$

all but

$$O\left(p(n) \frac{\log n}{n^{1/2L}}\right)$$

partitions of n are t-cores.

We can swiftly deduce our main result, Theorem 1.1, from the results stated above.

Deducing Theorem 1.1. Let μ be a partition of n, and suppose that $\tilde{\mu}$ is as in Proposition 2.3. Then, for all but at most

$$O(p(n)\exp(-n^{1/20p^r}))$$

choices of μ, the partition $\tilde{\mu}$ contains at least r distinct parts $m_1, \dots, m_r$, each appearing at least p^{r-1} times and satisfying

$$p^{r-1} m_j > \left(1 + \frac{1}{6p^r}\right) \frac{\sqrt{6}}{2\pi} \sqrt{n} \log n. \tag{2-1}$$

Consider any r-tuple $(k_1, \dots, k_r)$ with $0 \le k_1, \dots, k_r \le p^{r-1}$ and $k_i = p^{r-1}$ for some i. Then $k_1 m_1 + \cdots + k_r m_r$ also exceeds the bound in (2-1), so that by Lemma 2.4 all but $O(p(n)(\log n)/n^{1/2L})$ partitions λ of n are $(k_1 m_1 + \cdots + k_r m_r)$-cores. Since there are at most $r(p^{r-1}+1)^{r-1}$ such r-tuples $(k_1, \dots, k_r)$, by the union bound we see that all but at most

$$O\left(p(n)\frac{\log n}{n^{1/12p^r}} r(p^{r-1}+1)^{r-1}\right)$$

partitions λ of n are $(k_1 m_1 + \cdots + k_r m_r)$-cores for all choices of the r-tuple $(k_1, \dots, k_r)$.

Theorem 2.2 now shows that p^r divides $\chi_{\tilde{\mu}}^{\lambda}$, and since $\chi_{\mu}^{\lambda} \equiv \chi_{\tilde{\mu}}^{\lambda} \pmod{p^r}$ by Lemma 2.1, it also follows that p^r divides χ_{μ}^{λ}. Putting everything together, we conclude that the number of partitions λ and μ with $p^r \nmid \chi_{\mu}^{\lambda}$ is at most

$$O\left(p(n)^2\left(\exp(-n^{1/(20p^r)}) + \frac{1}{n^{1/13p^r}} r(p^{r-1}+1)^{r-1}\right)\right) = O(p(n)^2 \exp(-(\log\log n)^2)),$$

in the range $p^r \le 10^{-3}\log n/(\log\log n)^2$. □

The rest of the paper is organized as follows. We will prove Lemmas 2.1 and 2.4 in Section 3, Theorem 2.2 in Sections 4, 5, 6, and 7, and Proposition 2.3 in Sections 8 and 9.

3. Proofs of Lemmas 2.1 and 2.4

We begin by proving the two lemmas stated in the previous section.

Proof of Lemma 2.1. We claim that if $Q \in \mathbb{Z}[x_1, \dots, x_k]$ is a polynomial with integer coefficients, then

$$Q(x_1, \dots, x_k)^{p^r} \equiv Q(x_1^p, \dots, x_k^p)^{p^{r-1}} \pmod{p^r}.$$

As is well known, we may write

$$Q(x_1, \dots, x_k)^p = Q(x_1^p, \dots, x_k^p) + p \cdot R(x_1, \dots, x_k) \tag{3-1}$$

for some $R \in \mathbb{Z}[x_1, \dots, x_k]$, which establishes the claim when $r = 1$. For $r > 1$, raise both sides of (3-1) to the power p^{r-1}, and expand using the binomial theorem:

$$\begin{aligned} Q(x_1, \dots, x_k)^{p^r} &= (Q(x_1^p, \dots, x_k^p) + p \cdot R(x_1, \dots, x_k))^{p^{r-1}} \\ &= Q(x_1^p, \dots, x_k^p)^{p^{r-1}} + \sum_{\ell=1}^{p^{r-1}} \binom{p^{r-1}}{\ell} Q(x_1^p, \dots, x_k^p)^{p^{r-1}-\ell} (pR(x_1, \dots, x_k))^{\ell}. \end{aligned}$$

Note that for $1 \le \ell \le p^{r-1}$

$$p^{\ell}\binom{p^{r-1}}{\ell} = p^{\ell}\frac{p^{r-1}}{\ell}\binom{p^{r-1}-1}{\ell-1} \equiv 0 \pmod{p^r},$$

since the power of p dividing ℓ is certainly at most $\ell - 1$. This establishes our claim.

The lemma now follows by applying this observation to the polynomials appearing in Frobenius's formula for the character values χ_{μ}^{λ} and χ_{ν}^{λ}; see Chapter 4 of [4]. □

Proof of Lemma 2.4. The proof is essentially identical to that of Proposition 1 of [19], but we include the short argument for completeness. Since every partition of n is a t-core for $t > n$, we may naturally assume that $t \le n$. From Lemma 5 of [16], we know that at most $(t+1)p(n-t)$ partitions of n are not t-cores. By the asymptotic formula

$$p(m) \sim \frac{1}{4\sqrt{3}m} \exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{m}\right)$$

for the partition function, we have

$$(t+1)p(n-t) \ll \frac{t+1}{n-t+1} \exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n-t}\right) \le \frac{t+1}{n-t+1} \exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n} - \frac{\pi t}{\sqrt{6n}}\right).$$

In the range $n \ge t \ge (1+1/L)(\sqrt{6}/2\pi)\sqrt{n}\log n$, the right-hand side above is maximized at the lower endpoint $t = (1+1/L)(\sqrt{6}/2\pi)\sqrt{n}\log n$. It follows that the number of partitions of n that are not t-cores is

$$\ll \frac{\log n}{\sqrt{n}} n^{-(1+1/L)/2} \exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right) \ll p(n)\frac{\log n}{n^{1/2L}},$$

where the last step uses again the asymptotic for the partition function. □

4. Partitions and Abaci

The proof of Theorem 2.2 requires the machinery of the *abacus* associated to a partition. A good reference for this theory is Section 2.7 of [7], and we recall some salient facts below.

4.1. ***The notion of an abacus.*** An abacus is a bi-infinite sequence of 0's and 1's beginning with an infinite sequence of 1's and ending with an infinite sequence of 0's.

More formally, let

$$\mathcal{S} := \{s : \mathbb{Z} \to \{0, 1\} : \text{there exists a } k \ge 0 \text{ such that } s(-i) = 1 \text{ and } s(i) = 0 \text{ for all } i \ge k\}$$

denote the set of all sequences of 0's and 1's indexed using the integers, that begin with an infinite sequence of 1's and end with an infinite sequence of 0's. For example,

$$\ldots, 1, \ldots, 1, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 0, \ldots, 0, \ldots$$

is in $\mathcal{S}$. We consider two sequences s and s' in $\mathcal{S}$ to be equivalent if there is some integer j such that $s(i) = s'(i-j)$ for all i, that is, if s' can be produced by shifting the terms in s by j. This is an equivalence relation, and an abacus refers to an equivalence class in $\mathcal{S}$ under this relation. We denote by $\mathcal{A}$ the set of such abaci, so that by an element a of $\mathcal{A}$ we mean the equivalence class consisting of some sequence $s \in \mathcal{S}$ together with all its shifts.

4.2. ***The abacus associated to a partition.*** We now show how abaci are in one-to-one correspondence with partitions of integers. Starting with an integer partition λ, we construct an abacus $a_\lambda \in \mathcal{A}$ as follows. Draw the Young diagram of λ, and trace out the boundary of the diagram, moving from the lower left-hand

Figure 1. The Young diagram of (6, 5, 3, 1, 1, 1).

corner to the upper right-hand corner, writing a 0 for each horizontal move and a 1 for each vertical move. Then prepend an infinite string of 1's and append an infinite string of 0's to find a representative of the corresponding element a_λ of $\mathcal{A}$.

This procedure is easily reversed, and starting with an abacus a in $\mathcal{A}$ we obtain a Young diagram, which corresponds to a partition λ. If $s \in \mathcal{S}$ is a representative of a, then the partition λ is a partition of the integer $n(a)$ which counts the number of pairs of indices (i, j) with $i < j$ such that $s(i) = 0$ and $s(j) = 1$.

To illustrate, consider the partition (6, 5, 3, 1, 1, 1), whose Young diagram is pictured in Figure 1. If we start in the lower left-hand corner of this diagram and move along the boundary to the upper right-hand corner, we move right, up three times, right twice, up, right twice, up, right, and up. The correspondence described above produces the string

$$0, 1, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, \tag{4-1}$$

which we can turn into a bi-infinite sequence by adding an infinite sequence of 1's to the beginning and an infinite sequence of 0's to the end:

$$\ldots, 1, \ldots, 1, 1, 1, 0, 1, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, \ldots, 0, \ldots. \tag{4-2}$$

The equivalence class of this sequence is the abacus associated to (6, 5, 3, 1, 1, 1).

4.3. ***Hooks and border strips.*** Let λ be a partition. The *hook* h associated to a box b in the Young diagram of λ consists of the box b together with all the boxes directly to its right and directly below it. The hook-length of h, denoted by $\ell(h)$, is the number of boxes contained in the hook. The *height* of the hook h, denoted by $\mathrm{ht}(h)$, is one less than the number of rows in the Young diagram of λ that contain a box of h. Associated to each hook is a *border strip* (also known as a skew hook), denoted $\mathrm{bs}(h)$, which is the connected region of boundary boxes of the Young diagram running from the rightmost to the bottommost box of h. Removing such a border strip leaves behind a smaller Young diagram. These notions play a prominent role in the representation theory of the symmetric group, and in particular feature in the Murnaghan–Nakayama rule for computing character values, which we next recall; see Theorem 2.4.7 of [7], and also Chapter 4 of [4].

Theorem 4.1 (the Murnaghan–Nakayama rule). *Let n and t be positive integers, with $t \le n$. Let $\sigma \in S_n$ be of the form $\sigma = \tau \cdot \rho$, where ρ is a t-cycle, and τ is a permutation of S_n with support disjoint from ρ. Let λ be a partition of n. Then*

$$\chi^{\lambda}(\sigma) = \sum_{\substack{h \in \lambda \\ \ell(h)=t}} (-1)^{\mathrm{ht}(h)} \chi^{\lambda \setminus \mathrm{bs}(h)}(\tau). \tag{4-3}$$

Above, $\chi^{\lambda}(\sigma)$ denotes the value of the character of the irreducible representation of S_n corresponding to the partition λ, evaluated on the conjugacy class of σ, $\lambda \setminus \mathrm{bs}(h)$ denotes the partition of $n-t$ obtained by removing the border strip $\mathrm{bs}(h)$ from the Young diagram of λ, and $\chi^{\lambda \setminus \mathrm{bs}(h)}(\tau)$ denotes the character value of the irreducible representation of S_{n-t} corresponding to the partition $\lambda \setminus \mathrm{bs}(h)$ evaluated on the conjugacy class of τ.

The abacus notation helps with thinking about hook lengths and border strips. Let λ be a partition, let a_λ denote the corresponding abacus, and let s be a representative in $\mathcal{S}$ for the abacus a_λ. Each hook h in the Young diagram of λ is in natural one-to-one correspondence with a pair of indices (i, j), $i < j$, with $s(i) = 0$ and $s(j) = 1$. The length of the hook h is $j - i$. In particular, the partition λ contains no hooks of length t (that is, λ is a t-core) if and only if there is no pair of indices $(i, i+t)$ with $s(i) = 0$ and $s(i+t) = 1$. The height of the hook h equals the number of 1's in the sequence s lying strictly between the 0 at index i and the 1 at index j:

$$\mathrm{ht}(h) = \#\{i < k < j : s(k) = 1\}.$$

Further, the abacus notation gives an easy description of the result of removing a border strip from a partition. Define, for any pair of distinct integers (i, j) the operator $T_{ij} : \mathcal{S} \to \mathcal{S}$ that swaps the terms indexed by i and j in a bi-infinite sequence $s \in \mathcal{S}$ and leaves all other entries fixed. Thus for $s \in \mathcal{S}$

$$(T_{ij}s)(k) = \begin{cases} s(k), & k \ne i, j, \\ s(j), & k = i, \\ s(i), & k = j. \end{cases}$$

With this notation in place, suppose λ is a partition, and $s \in a_\lambda$ is a representative of the abacus of λ. Let h be a hook of λ, corresponding to the pair of indices (i, j) (with $i < j$) in s. Then $T_{ij}s$ is a representative of the abacus associated to $\lambda \setminus \mathrm{bs}(h)$.

Returning to our example of the partition $(6, 5, 3, 1, 1, 1)$, Figure 2 contains its Young diagram again, but now with each box filled in with the corresponding hook-length. The unique hook h of length 5 in the diagram corresponds to the pair of indices $(5, 10)$ of the sequence (4-1). If we remove the corresponding border strip, we obtain the diagram pictured in Figure 3, which corresponds to the partition $(6, 5, 3, 1, 1, 1) \setminus \mathrm{bs}(h) = (6, 2, 1, 1, 1, 1)$ and the bi-infinite sequence

$$\ldots, 1, \ldots, 1, 1, 1, 0, 1, 1, 1, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0 \ldots, 0, \ldots$$

of 0's and 1's. Note that if we swap the 0 and 1 corresponding to the hook h in the representative (4-2) of $a_{(6,5,3,1,1,1)}$, then we get an equivalent bi-infinite sequence.

11	7	6	4	3	1
9	5	4	2	1	
6	2	1			
3					
2					
1					

Figure 2. Hook-lengths for (6, 5, 3, 1, 1, 1).

Figure 3. The Young diagram of (6, 2, 1, 1, 1, 1).

4.4. ***Removing several hooks in succession.*** In our work below, we will need to remove several hooks (more precisely, the border strips corresponding to those hooks) in succession from a partition. By removing a sequence of hooks $h_1, \dots, h_R$ from a partition λ, we mean the following: h_1 is a hook of λ, h_2 is a hook of $\lambda \setminus \mathrm{bs}(h_1)$, h_3 is a hook of $\lambda \setminus \mathrm{bs}(h_1) \setminus \mathrm{bs}(h_2)$, and so on, until we arrive at h_R which is a hook of $\lambda \setminus \mathrm{bs}(h_1) \cdots \setminus \mathrm{bs}(h_{R-1})$, and when this is removed we obtain the final partition $\lambda' = \lambda \setminus \mathrm{bs}(h_1) \cdots \setminus \mathrm{bs}(h_R)$.

Let s be a representative of the abacus a_λ associated to λ. Let (i_1, j_1) denote the pair of indices in s corresponding to the hook h_1, (i_2, j_2) the corresponding pair to h_2 (which, recall, is a hook of $\lambda \setminus \mathrm{bs}(h_1)$ corresponding to the bi-infinite sequence $T_{i_1,j_1}s$), and so on. Thus, the sequence of hooks $h_1, \dots, h_R$ may be encoded by the R-tuple of pairs $((i_1, j_1), (i_2, j_2), \dots, (i_R, j_R))$, and the process of removing these hooks results in the sequence

$$s' = T_{i_R,j_R} T_{i_{R-1},j_{R-1}} \cdots T_{i_1,j_1} s.$$

The sequence s' is a representative of the abacus $a_{\lambda'}$ associated to the partition λ'.

Of particular interest for us will be the situation where all the hooks have the same length, m say. Here $j_k = i_k + m$ for all $1 \le k \le R$, and we may encode the sequence of hooks by simply the R-tuple $(i_1, \dots, i_R)$. Note that the indices $i_1, \dots, i_R$ may contain repeats, but there are also constraints, such as $i_2 \neq i_1$ (since $(i_1, i_1 + m)$ is a hook in s and so it cannot be a hook in $T_{i_1,i_1+m}s$).

5. Plan of the proof of Theorem 2.2

We begin by restating Theorem 2.2 in terms of values of irreducible characters at elements of S_n, which will make the notation involved in its proof cleaner.

Theorem 5.1 (an equivalent formulation of Theorem 2.2). *Let $n, m_1, \ldots, m_r$ be distinct positive integers. Let $\sigma \in S_n$ be a permutation of the form*

$$\sigma = \tau \cdot \prod_{i=1}^{r} \prod_{j=1}^{p^{r-1}} \rho_i^{(j)},$$

where each $\rho_i^{(j)}$ is a cycle of length m_i, the supports of all the cycles $\rho_i^{(j)}$ are disjoint, and $\tau \in S_n$ is a permutation with support disjoint from those of the $\rho_i^{(j)}$. Suppose that λ is a $\left(\sum_{i=1}^{r} k_i m_i\right)$-core partition of n for all r-tuples $(k_1, \ldots, k_r)$ of integers $0 \le k_1, \ldots, k_r \le p^{r-1}$ for which some $k_i = p^{r-1}$. Then

$$p^r \mid \chi^\lambda(\sigma).$$

The proof of Theorem 5.1 rests on the following crucial proposition, which is based on applying the Murnaghan–Nakayama rule p^{r-1} times.

Proposition 5.2. *Let r, m and n be positive integers. Let $\sigma \in S_n$ be of the form*

$$\sigma = \tau \cdot \prod_{j=1}^{p^{r-1}} \rho^{(j)},$$

where each $\rho^{(j)}$ is an m-cycle, with all the cycles $\rho^{(j)}$ being disjoint, and with $\tau \in S_n$ being a permutation whose support is disjoint from all the cycles $\rho^{(j)}$. Denote by L the set of partitions of $n - p^{r-1}m$ that can be obtained from λ by removing, in succession, p^{r-1} border strips of length m. If λ is a $p^{r-1}m$-core partition of n, then

$$\chi^\lambda(\sigma) = p \sum_{\lambda' \in L} \epsilon_{\lambda'} \chi^{\lambda'}(\tau),$$

where each $\epsilon_{\lambda'}$ is an integer.

We will quickly deduce Theorem 5.1 (and hence Theorem 2.2) from Proposition 5.2 and the following simple observation.

Lemma 5.3. *Let n, t and m be positive integers. Let λ be a partition of n which is both a t-core and a $(t+m)$-core. Let λ' be a partition of $n-m$ that can be obtained by removing a border strip of length m from λ. Then λ' is a t-core.*

Proof. If λ has no hook (and thus no border strip) of length m then the lemma holds vacuously. Suppose that λ' arises from removing the border strip corresponding to the hook h of length m in λ. Let a_λ be the abacus of λ, and s be a representative bi-infinite sequence in a_λ. Suppose the hook h corresponds to the pair of indices $(i, i+m)$ with $s(i) = 0$ and $s(i+m) = 1$, so that the partition λ' corresponds to the abacus containing $s' = T_{i,i+m}s$.

If λ' is not a t-core, then there must exist a pair of indices $(j, j+t)$ with $s'(j) = 0$ and $s'(j+t) = 1$. Since the entries of s and s' differ only at the indices i and $i+m$, and since λ is a t-core, we must have either $j = i+m$, or $j+t = i$. If $j = i+m$, then $s(i) = 0$ and $s(i+t+m) = s'(j+t) = 1$ which contradicts the assumption that λ is a $(t+m)$-core. If $j = i-t$, then $s(i-t) = s'(j) = 0$ and $s(i+m) = 1$, which again contradicts the assumption that λ is a $(t+m)$-core. □

Deducing Theorem 5.1 from Proposition 5.2. Apply Proposition 5.2 first with $m = m_r$ to obtain

$$\chi^{\lambda}(\sigma) = p \sum_{\lambda' \in L} \epsilon_{\lambda'} \chi^{\lambda'}\Big(\tau \prod_{i=1}^{r-1} \prod_{j=1}^{p^{r-1}} \rho_i^{(j)}\Big).$$

If t is any number of the form $t = \sum_{i=1}^{r-1} k_i m_i$ where the k_i lie in $[0, p^{r-1}]$ with at least one of them being p^{r-1}, then λ is a $(t + k_r m_r)$-core for all $0 \le k_r \le p^{r-1}$. Since any $\lambda' \in L$ arises from λ by removing p^{r-1} border strips of length m_r, it follows by p^{r-1} applications of Lemma 5.3 that λ' is a t-core.

We may now repeat this argument, applying Proposition 5.2 to each $\lambda' \in L$ and now removing p^{r-1} border strips of length m_{r-1}. Applications of Lemma 5.3 show that the new partitions λ'' that arise are $\big(\sum_{i=1}^{r-2} k_i m_i\big)$-cores for all choices of $0 \le k_i \le p^{r-1}$ with some $k_i = p^{r-1}$.

Carrying this argument out r times, we obtain the desired result. □

The proof of Proposition 5.2 depends on the following two lemmas, which we shall prove in the next two sections.

Lemma 5.4. *Let λ be a partition, and let λ' be obtained from λ by removing a sequence of R border strips of the same length m. Let $h_1, \dots, h_R$ be a sequence of R hooks of length m which may be removed from the initial partition λ to result in the final partition λ'. Then*

$$(-1)^{\mathrm{ht}(h_1)+\cdots+\mathrm{ht}(h_R)} = \epsilon(\lambda, \lambda')$$

where the sign $\epsilon(\lambda, \lambda') = \pm 1$ depends only on the initial and final partitions λ and λ' and is the same for all such possible sequences of hooks.

We are grateful to the referee for pointing out that Lemma 5.4 may be found in the literature as Proposition 2.2 of [17]. In the interest of keeping our exposition self-contained, we include the short proof of Lemma 5.4 in Section 6.

Lemma 5.5. *Let λ be a $p^{r-1}m$-core partition, and let λ' be a partition that can be obtained from λ by removing $R = p^{r-1}$ border strips of length m. The number of tuples $(i_1, \dots, i_R)$ such that*

$$s' = T_{i_R, i_R+m} T_{i_{R-1}, i_{R-1}+m} \cdots T_{i_1, i_1+m} s$$

is a multiple of p. Here s is a representative of the abacus of λ, and the partition λ' corresponds to the abacus containing s'.

Once Lemmas 5.4 and 5.5 are in place, it is a simple matter to deduce Proposition 5.2.

Deducing Proposition 5.2. We apply the Murnaghan–Nakayama rule repeatedly while removing in succession $R = p^{r-1}$ hooks of length m from λ. This will result in an expression for $\chi^{\lambda}(\sigma)$ of the form $\sum_{\lambda' \in L} c_{\lambda'} \chi^{\lambda'}(\tau)$, for suitable integers $c_{\lambda'}$ which we must show are multiples of p. Now

$$c_{\lambda'} = \sum_{(i_1, \dots, i_R)} (-1)^{\mathrm{ht}(h_1)+\cdots+\mathrm{ht}(h_R)}$$

where the sum is over all R-tuples $(i_1, \ldots, i_R)$ corresponding to hooks $h_1, \ldots, h_R$, which when removed from λ in order result in the partition λ'. Lemma 5.4 tells us that the sign $(-1)^{\mathrm{ht}(h_1)+\cdots+\mathrm{ht}(h_R)}$ is the same for all suitable tuples $(i_1, \ldots, i_R)$, and Lemma 5.5 tells us that the number of such R-tuples is a multiple of p. □

6. Parity of heights of hooks: Proof of Lemma 5.4

Let λ be a partition, and s a representative of the abacus a_λ associated to λ. Augment s by coloring a finite number N of 1's in s with distinct colors, taking care to color all the 1's appearing to the right of the first zero in s. The 1's appearing to the left of the first 0 are unimportant, but we allow the flexibility of coloring some of them since this situation may arise at an intermediate step when we remove hooks from λ. Note that the number of 1's appearing to the right of the first zero equals the number of rows in the partition λ. Thus N is at least the number of rows in λ. Color these 1's in the order of their appearance in s using the colors $c_1, \ldots, c_N$. Call the augmented sequence $\hat{s}$.

We begin with a general observation on removing hooks. Suppose (i, j) is a pair of indices corresponding to a hook h in s (at the moment the hook can have any length $j - i$). Removing this hook produces the sequence $T_{i,j}s$. Considering the augmented sequence $\hat{s}$, we have the corresponding augmented sequence $T_{i,j}\hat{s}$ after removing this hook. If we consider the sequence of colors among the 1's in this sequence, we obtain a permutation π_{ij}, say, of the original sequence of colors $(c_1, \ldots, c_N)$ — the 1 appearing in $(T_{i,j}\hat{s})(i)$ has the color of the 1 in $\hat{s}(j)$, and all other 1's in $T_{ij}(\hat{s})$ retain their color in $\hat{s}$. If the height of the hook removed is k, then note that $\hat{s}$ had k colored 1's between $s(i) = 0$ and $s(j) = 1$ and the permutation π_{ij} can be obtained by making k-transpositions, each time swapping the color of the 1 at position j by the color immediately preceding it. Thus $(-1)^k = (-1)^{\mathrm{ht}(h)}$ equals the sign of the permutation π_{ij}.

If we remove hooks $h_1, \ldots, h_\ell$ in succession (again, their lengths could be arbitrary), then the associated permutations of colors multiply, and therefore so do the signs of these permutations. Thus, after removing these hooks in succession we would arrive at a permutation π of the sequence of colors $(c_1, \ldots, c_N)$ and

$$(-1)^{\mathrm{ht}(h_1)+\mathrm{ht}(h_2)+\cdots+\mathrm{ht}(h_\ell)} = \mathrm{sgn}(\pi).$$

We now turn to the situation of the lemma, where a sequence $h_1, \ldots, h_R$ of R hooks is removed all of length m. Our observation above shows that removing these hooks in order leads to the sequence $\hat{s}'$ where the color of the 1's is given by a permutation π of the original sequence of colors $c_1, \ldots, c_N$. Further the sign of this permutation $\mathrm{sgn}(\pi)$ equals $(-1)^{\mathrm{ht}(h_1)+\cdots+\mathrm{ht}(h_R)}$.

To complete the proof, we will show that every way of removing R hooks of length m that leads to the partition λ' results in the same permutation of colors π. Consider the subsequence of $\hat{s}$ obtained by restricting to a progression $\pmod m$: namely, $(\hat{s}(a + \ell m))_{\ell \in \mathbb{Z}}$. There are m such subsequences corresponding to $a = 1, \ldots, m$. Since the hooks removed all have length m, each removal of a hook affects only the terms within one of these subsequences, leaving all the other subsequences unaltered. Further within any particular subsequence $(\hat{s}(a + \ell m))_{\ell \in \mathbb{Z}}$, it is impossible to alter the original sequence

of colors by removing any sequence of hooks of length m. Therefore we can determine uniquely the color of any element in $\hat{s}'$: the 1's appearing in this sequence in the progression $a \pmod{m}$ have colors determined by their order of appearance in the original sequence s.

7. Proof of Lemma 5.5

Let λ be a $p^{r-1}m$-core partition, and let s be a representative of its abacus. Let s' be the sequence obtained by removing a sequence of $R = p^{r-1}$ border strips of length m from λ, and let λ' be the partition associated to s'. Our goal is to show that the number of ways of reaching λ' starting from λ is a multiple of p.

Let us first note that when $r = 1$, it is impossible to remove a border strip of length m from λ, since λ is an m-core partition by assumption. Thus the number of ways here is 0, and the lemma holds (vacuously). Henceforth, assume that $r \geq 2$.

For each $a = 1, \dots, m$, consider the subsequences of s and s' obtained by restricting to the progression $a \pmod{m}$: thus, set

$$s(a; m) = (s(a + \ell m))_{\ell \in \mathbb{Z}}, \quad s'(a; m) = (s'(a + \ell m))_{\ell \in \mathbb{Z}}.$$

We may think of $s(a; m)$ and $s'(a; m)$ as corresponding to partitions $\lambda(a; m)$ and $\lambda'(a; m)$, and note that a hook of length m in the partition λ corresponds to a hook of length 1 (or simply a border square) in the partition $\lambda(a; m)$ (for some choice of a). Since $\lambda'(a; m)$ arises from $\lambda(a; m)$ by removing some number of hooks of length 1, the Young diagram of the partition $\lambda'(a; m)$ is contained in the Young diagram of the partition $\lambda(a; m)$ (that is, $\lambda_i(a; m) \geq \lambda'_i(a; m)$ for all i). The difference between the Young diagram of $\lambda(a; m)$ and $\lambda'(a; m)$ (in other words, the boxes in $\lambda(a; m)$ that are not in $\lambda'(a; m)$) is a skew diagram, which we denote by $\lambda(a; m)/\lambda'(a; m)$. Let ℓ_a denote the size of this skew diagram $|\lambda(a; m)/\lambda'(a; m)|$, so that ℓ_a hooks of length 1 must be removed from $\lambda(a; m)$ to reach $\lambda'(a; m)$. Since a total of $R = p^{r-1}$ hooks of length m are removed to go from λ to λ', note that

$$R = p^{r-1} = \sum_{a=1}^{m} \ell_a.$$

The number of ways to go from $\lambda(a; m)$ to $\lambda'(a; m)$ by removing successively ℓ_a hooks of length 1 equals the number of standard Young tableaux of skew shape $\lambda(a; m)/\lambda'(a; m)$, which we denote (in the usual notation) by $f_{\lambda(a;m)/\lambda'(a;m)}$. Recall that a standard Young tableau of this skew shape is a numbering of the boxes in the skew diagram using the numbers 1 to ℓ_a such that the entries are increasing from left to right in each row, and increasing down each column. Each such tableau corresponds to a way of removing hooks, by removing boxes in descending order of their entries.

We can now count the number of ways of going from λ to λ' by removing R hooks of length m. Note that removing a hook from one subsequence $s(a; m)$ has no impact on the hooks in any of the other subsequences. Therefore the desired number of ways to proceed from λ to λ' equals

$$\binom{p^{r-1}}{\ell_1, \ell_2, \dots, \ell_m} \prod_{a=1}^{m} f_{\lambda(a;m)/\lambda'(a;m)}.$$

The multinomial coefficient

$$\binom{p^{r-1}}{\ell_1, \ell_2, \ldots, \ell_m}$$

is a multiple of p, except in the situation where $\ell_a = p^{r-1}$ for some a (and all other ℓ_j are 0). Thus we are left with the case when all the hooks of length m in going from λ to λ' are confined to one subsequence $s(a; m)$. So far, we have not made use of the condition that λ is a $p^{r-1}m$-core, and it is only in this case that we need this assumption. The assumption implies that $\lambda(a; m)$ is p^{r-1}-core, and so the skew diagram $\lambda(a; m)/\lambda'(a; m)$ (which has size $\ell_a = p^{r-1}$) cannot be a border strip of $\lambda(a; m)$. In this situation, it turns out that $f_{\lambda(a;m)/\lambda'(a;m)}$ is a multiple of p. This is implied by our next lemma, which is perhaps of independent interest.

Lemma 7.1. *Let π and τ be two partitions, with the Young diagram of π containing the Young diagram of τ (thus $\pi_i \geq \tau_i$ for all i). Suppose the skew diagram π/τ is not a border strip of the partition π (equivalently, either π/τ is disconnected, or it contains a 2×2 square), and that $|\pi/\tau| = p^t$ is a prime power (with $t > 0$). Then the number of standard Young tableaux of skew shape π/τ, denoted $f_{\pi/\tau}$, is a multiple of p.*

Proof. First suppose that π/τ is disconnected, and is composed of $k \geq 2$ maximally connected skew shapes $S_1, \ldots, S_k$, with $|S_j| = s_j \geq 1$. Then

$$f_{\pi/\tau} = \binom{p^t}{s_1, \ldots, s_k} f_{S_1} \cdots f_{S_k},$$

is clearly a multiple of p.

Now suppose that π/τ is a connected skew shape, but contains a 2×2 square so that it is not a border strip of π. Since $f_{\pi/\tau}$ depends only on the shape π/τ, we may assume that π is minimal, having only as many rows and columns as needed for the skew shape π/τ. Then the maximal hook length of π equals the number of border squares of π, which is strictly smaller than $|\pi/\tau| = p^t$ (since π/τ is not a border strip by assumption).

It is a basic fact (see Section I.9 of [9], for example — the identity below follows from (9.1) of [9] by taking the Hall inner product of both sides with the symmetric function $e_1^{p^t}$) that

$$f_{\pi/\tau} = \sum_{\nu \vdash p^t} f_\nu c^\pi_{\tau\nu},$$

where the sum is over partitions ν of $|\pi/\tau| = p^t$, $f_\nu = \chi^\nu_{(1,\ldots,1)}$ is the degree of the irreducible character corresponding to ν and the $c^\pi_{\tau\nu}$ are the Littlewood–Richardson coefficients (which are integers). By Lemma 2.1, $f_\nu \equiv \chi^\nu_{(p^t)} \pmod p$, so that $p \mid f_\nu$ unless ν is a hook of length p^t. Suppose now that ν is a hook of length p^t. Here we use that the Littlewood–Richardson coefficient $c^\pi_{\tau\nu}$ is zero unless the Young diagram of the partition ν is contained in that of π (see Section I.9 of [9] once again). But all the hooks of π have length $< p^t$, and therefore π cannot contain a hook ν of length p^t. Thus either $c^\pi_{\tau\nu} = 0$ or $p \mid f_\nu$, and therefore the lemma follows. □

8. Preliminaries for the proof of Proposition 2.3

As in [19], let $\tilde{p}(k)$ denote the number of partitions of a nonnegative integer k into powers of p, with the convention that $\tilde{p}(0) = 1$. Denote by $F_p(t)$ the associated generating function

$$F_p(t) := \sum_{k=0}^{\infty} \tilde{p}(k) e^{-k/t} = \prod_{j=0}^{\infty} (1 - e^{-p^j/t})^{-1},$$

where $t > 0$ is a real number. We begin by recalling some estimates from our prior work [19].

Lemma 8.1 [19, Lemma 2]. *When $0 < t \leq 1$, we have $F_p(t) = O(1)$, and when $t \geq 1$, we have*

$$\frac{(\log t)^2}{2\log p} + \frac{1}{2}\log t + O(1) \leq \log F_p(t) \leq \frac{(\log t)^2}{2\log p} + \frac{1}{2}\log t + \frac{1}{8}\log p + O(1).$$

More precise results are known for fixed primes p, as partitions into prime powers have been studied extensively since the work of Mahler [10] and de Bruijn [1]. We will only require the estimates of Lemma 8.1, which are cruder but uniform in p.

Given a partition μ of k into powers of p, let $\tilde{\mu}$ denote the partition obtained by repeatedly replacing every occurrence of p^r parts of the same size p^j by p^{r-1} parts of size p^{j+1} until no part appears more than $p^r - 1$ times. For every nonnegative integer s, define $\tilde{p}(k; s)$ to be the number of partitions μ of k into powers of p such that $\tilde{\mu}$ does not contain (at least) p^{r-1} parts of the same size p^j for any $j \geq s$. The second lemma of this section gives a useful lower bound for the difference between $\tilde{p}(k)$ and $\tilde{p}(k; s)$.

Lemma 8.2. *For all $s \geq 2$ and $k \geq p^{r+s-1}(1 + 4/s)$, we have*

$$\tilde{p}(k) - \tilde{p}(k; s) \geq \frac{p^{s(s-1)/2}}{(s-1)^{s-1}}.$$

Proof. We will construct at least $p^{s(s-1)/2}/(s-1)^{s-1}$ partitions of k counted in $\tilde{p}(k)$ but not in $\tilde{p}(k; s)$. For each $1 \leq i \leq s-1$, pick an integer a_i in the range

$$0 \leq a_i \leq \frac{p^{s-i}}{s-1}.$$

Each choice of $a_1, \ldots, a_{s-1}$ gives a partition μ counted in $\tilde{p}(k)$ by using a_i copies of p^i for $1 \leq i \leq s-1$ and $k - \sum_{i=1}^{s-1} a_i p^i$ copies of 1. The number of such partitions is

$$\prod_{i=1}^{s-1} \left\lceil \frac{p^{s-i}}{s-1} \right\rceil \geq \prod_{i=1}^{s-1} \frac{p^{s-i}}{s-1} = \frac{p^{s(s-1)/2}}{(s-1)^{s-1}}.$$

Note that if $i > s - \log(s-1)/\log p$, then a_i must be zero, so that all of these partitions have largest part at most $p^s/(s-1)$.

We must check that each such μ is not counted in $\tilde{p}(k; s)$; that is, that the corresponding $\tilde{\mu}$ contains at least p^{r-1} copies of some part p^j with $j \geq s$. Suppose that this is not the case. Notice that, by construction, the number of times any part appears in μ is congruent modulo p^{r-1} to the number of

times it appears in $\tilde{\mu}$. Since no part can appear more than $p^r - 1$ times in $\tilde{\mu}$, it follows that any part that appears fewer than p^{r-1} times or more than $p^r - p^{r-1}$ times in $\tilde{\mu}$ must have appeared in the original partition μ. Since all the parts of μ are below $p^s/(s-1)$, we conclude that $\tilde{\mu}$ can contain (i) at most $p^r - 1$ copies of any part p^j with $p^j \le p^s/(s-1)$, (ii) at most $p^r - p^{r-1}$ copies of any part p^j with $p^s/(s-1) < p^j \le p^{s-1}$, and (iii) no parts of size p^j with $j \ge s$. But these constraints imply that

$$\begin{aligned} k = |\tilde{\mu}| &\le (p^r - 1) \sum_{p^j \le p^s/(s-1)} p^j + (p^r - p^{r-1}) \sum_{p^s/(s-1) < p^j \le p^{s-1}} p^j \\ &< (p^{r-1} - 1)\frac{p^s}{(s-1)}\left(1 - \frac{1}{p}\right)^{-1} + (p^r - p^{r-1})p^{s-1}\left(1 - \frac{1}{p}\right)^{-1} < p^{r+s-1}\left(1 + \frac{4}{s}\right), \end{aligned}$$

which contradicts our assumption on the size of k. □

9. Proof of Proposition 2.3

Let $\mathcal{L}$ be a set of positive integers coprime to p, and define $p(n; \mathcal{L}, s)$ to be the number of partitions μ of n for which $\tilde{\mu}$ contains fewer than p^{r-1} parts of the same size ℓp^j for every $\ell \in \mathcal{L}$ and $j \ge s$. We will prove Proposition 2.3 by obtaining an upper bound for $p(n; \mathcal{L}, s)$ for well-chosen $\mathcal{L}$ and s.

Lemma 9.1. *Suppose that n is large and $p^r \le 10^{-3} \log n / \log\log n$. Put*

$$x = \frac{\sqrt{6n}}{\pi}, \quad s = \left\lfloor \frac{\log\sqrt{n}}{ep^r} \right\rfloor, \tag{9-1}$$

and let $\mathcal{L}$ be the set of integers in the interval $[L, L + x/p^{r+s-1}]$ that are coprime to p, where L is a parameter lying in the range

$$\frac{\sqrt{6n}}{2\pi p^{r+s-1}} \le L \le \left(1 + \frac{1}{5p^r}\right)\frac{\sqrt{6n}}{2\pi p^{r+s-1}} \log n. \tag{9-2}$$

Then

$$p(n; \mathcal{L}, s) \ll p(n) n^{3/4} \exp(-n^{1/(16p^r)}).$$

Before proving the lemma, let us see how Proposition 2.3 would follow. Choose r distinct values L_j (with $1 \le j \le r$) all in the range

$$\left(1 + \frac{1}{6p^r}\right)\frac{\sqrt{6n}}{2\pi p^{r+s-1}} \log n \le L_j \le \left(1 + \frac{1}{5p^r}\right)\frac{\sqrt{6n}}{2\pi p^{r+s-1}} \log n,$$

such that the corresponding sets $\mathcal{L}_j$ are all disjoint. A partition μ for which $\tilde{\mu}$ does not contain r distinct parts $m_1, \dots, m_r$ each appearing at least p^{r-1} times and suitably large as desired in the proposition, must be counted among some $p(n; \mathcal{L}_j, s)$ with $1 \le j \le r$. Thus by Lemma 9.1 the number of such bad partitions μ is

$$\le \sum_{j=1}^{r} p(n; \mathcal{L}_j, s) \ll r p(n) n^{3/4} \exp(-n^{1/(16p^r)}) \ll p(n) n \exp(-n^{1/(16p^r)}) \ll p(n) \exp(-n^{1/(20p^r)}),$$

as claimed.

Proof of Lemma 9.1. Consider the process of going from a partition μ to $\tilde{\mu}$ by combining p^r parts of the same size m into p^{r-1} parts of size pm. Suppose that ℓ is coprime to p, and that the sum of all parts of the form ℓp^j appearing in μ equals ℓk. Restricting our attention to these parts, we may think of μ as giving rise to a partition of k into powers of p, and then $\tilde{\mu}$ correspondingly gives a partition of k into powers of p obtained by repeatedly combining p^r parts of size p^j into p^{r-1} parts of size p^{j+1}. It follows that $p(n; \mathcal{L}, s)$ is the coefficient of z^n in the generating function

$$\prod_{\substack{\ell \notin \mathcal{L} \\ (\ell,p)=1}} \prod_{j=0}^{\infty} (1 - z^{\ell p^j})^{-1} \prod_{\ell \in \mathcal{L}} \Big(\sum_{k=0}^{\infty} \tilde{p}(k; s) z^{\ell k} \Big),$$

which equals

$$\prod_{i=1}^{\infty} (1 - z^i)^{-1} \prod_{\ell \in \mathcal{L}} \Big(\frac{\sum_{k=0}^{\infty} \tilde{p}(k; s) z^{\ell k}}{\sum_{k=0}^{\infty} \tilde{p}(k) z^{\ell k}} \Big).$$

Since all of the coefficients in the generating function for $p(n; \mathcal{L}, s)$ are nonnegative, we must have, for any $0 < z < 1$,

$$p(n; \mathcal{L}, s) \le \frac{1}{z^n} \prod_{i=1}^{\infty} (1 - z^i)^{-1} \prod_{\ell \in \mathcal{L}} \Big(\frac{\sum_{k=0}^{\infty} \tilde{p}(k; s) z^{\ell k}}{\sum_{k=0}^{\infty} \tilde{p}(k) z^{\ell k}} \Big). \tag{9-3}$$

Recall that $x = \sqrt{6n}/\pi$, and take $z = e^{-1/x}$ in the bound (9-3). Then, by the asymptotic formula for the partition function and basic estimates for the generating function of the number of partitions (see Section VIII.6 of [3]), we obtain

$$p(n; \mathcal{L}, s) \ll n^{3/4} p(n) \prod_{\ell \in \mathcal{L}} \Big(\frac{\sum_{k=0}^{\infty} \tilde{p}(k; s) z^{\ell k}}{\sum_{k=0}^{\infty} \tilde{p}(k) z^{\ell k}} \Big) \ll n^{3/4} p(n) \exp(-\Delta), \tag{9-4}$$

where

$$\Delta := \sum_{\ell \in \mathcal{L}} \frac{1}{F_p(x/\ell)} \sum_{k=0}^{\infty} (\tilde{p}(k) - \tilde{p}(k; s)) e^{-\ell k/x}.$$

Our work so far applies to any set $\mathcal{L}$ of integers that are coprime to p, and we now proceed to the situation at hand. The lower bound on L and our choice of x give, for all $\ell \in \mathcal{L}$, the bound

$$F_p\Big(\frac{x}{\ell}\Big) \le F_p\Big(\frac{x}{L}\Big) \le F_p\Big(\frac{2p^{r+s-1}}{\log n}\Big).$$

From this estimate, our choice of $\mathcal{L}$, and Lemma 8.2 it follows that

$$\Delta \ge \frac{1}{F_p(2p^{r+s-1}/\log n)} \sum_{\substack{L \le \ell \le L + x/p^{r+s-1} \\ (\ell,p)=1}} \sum_{k \ge p^{r+s-1}(1+4/s)} \frac{p^{s(s-1)/2}}{(s-1)^{s-1}} e^{-\ell k/x}. \tag{9-5}$$

For ℓ in the range $L \le \ell \le L + x/p^{r+s-1}$, we have

$$\sum_{k \ge p^{r+s-1}(1+4/s)} e^{-\ell k/x} \ge \exp\Big(-\frac{\ell p^{r+s-1}}{x} \Big(1 + \frac{4}{s}\Big)\Big) \frac{e^{-\ell/x}}{1 - e^{-\ell/x}} \ge \frac{x}{2L} \exp\Big(-\Big(\frac{L p^{r+s-1}}{x} + 1 \Big)\Big(1 + \frac{4}{s}\Big)\Big).$$

Inserting this into the right-hand side of (9-5) and noting that (since p^{r+s-1} is small in comparison to x)

$$|\mathcal{L}| \geq \left(1-\frac{1}{p}\right)\frac{x}{p^{r+s-1}} - 2 \geq \frac{x}{3p^{r+s-1}},$$

we obtain (using our choice of x and the range for L)

$$\begin{aligned}\Delta &\geq \frac{1}{F_p(2p^{r+s-1}/\log n)} \cdot \frac{p^{s(s-1)/2}}{(s-1)^{s-1}} \cdot \frac{x}{3p^{r+s-1}} \cdot \frac{x}{2L} \exp\left(-\left(\frac{Lp^{r+s-1}}{x}+1\right)\left(1+\frac{4}{s}\right)\right)\\ &\geq \frac{1}{6}\frac{1}{F_p(2p^{r+s-1}/\log n)} \cdot \frac{p^{s(s-1)/2}}{(s-1)^{s-1}} \cdot \frac{x}{\log n} \cdot \exp\left(-\left(\frac{Lp^{r+s-1}}{x}+1\right)\left(1+\frac{4}{s}\right)\right).\end{aligned}$$

Using Lemma 8.1 and the bound $p^r \leq \log\sqrt{n}$, it follows that

$$\begin{aligned}\log F_p\left(\frac{2p^{r+s-1}}{\log n}\right) &\leq \frac{1}{2\log p}\left(\log\frac{p^{r+s-1}}{\log\sqrt{n}}\right)^2 + \frac{1}{2}\log\frac{p^{r+s-1}}{\log\sqrt{n}} + \frac{1}{8}\log p + O(1)\\ &\leq \frac{1}{2\log p}\left(\log\frac{p^{r+s-1}}{\log\sqrt{n}}\right)^2 + \frac{s}{2}\log p + O(1).\end{aligned}$$

Therefore

$$\begin{aligned}\log\frac{p^{s(s-1)/2}}{F_p(2p^{r+s-1}/\log n)(s-1)^{s-1}} &\geq \frac{s^2}{2}\log p - \frac{1}{2\log p}\left(\log\frac{p^{r+s-1}}{\log\sqrt{n}}\right)^2 - s\log ps + O(1)\\ &\geq s\log\frac{\log\sqrt{n}}{p^r s} - \frac{(\log\log\sqrt{n})^2}{2\log p} + O(1).\end{aligned}$$

Recalling our choice of s, we conclude that

$$\begin{aligned}\log\Delta &\geq s\log\frac{\log\sqrt{n}}{p^r s} + \log\sqrt{n} - \frac{(\log\log\sqrt{n})^2}{2\log p} - \log\log n - \frac{Lp^{r+s-1}}{x}\left(1+\frac{4}{s}\right) + O(1)\\ &\geq \left(1+\frac{1}{ep^r}\right)\log\sqrt{n} - \frac{Lp^{r+s-1}}{x} - \log\log n - \frac{(\log\log\sqrt{n})^2}{2\log p} + O(1)\\ &\geq \left(\frac{1}{ep^r} - \frac{1}{5p^r}\right)\log\sqrt{n} - (\log\log n)^2 + O(1) \geq \frac{\log n}{15p^r} - (\log\log n)^2 + O(1),\end{aligned}$$

upon using the upper bound on L in (9-2). In the range $p^r \leq 10^{-3}\log n/(\log\log n)^2$ we find

$$\log\Delta \geq \frac{\log n}{16p^r} + O(1),$$

which when used in (9-4) yields the lemma. □

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