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Algebraic cycles and functorial lifts from G_2 to PGSp_6

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We study instances of Beilinson–Tate conjectures for automorphic representations of PGSp_6 whose spin L-function has a pole at $s = 1$. We construct algebraic cycles of codimension 3 in the Siegel–Shimura variety of dimension 6 and we relate its regulator to the residue at $s = 1$ of the L-function of certain cuspidal forms of PGSp_6. Using the exceptional theta correspondence between the split group of type G_2 and PGSp_6 and assuming the nonvanishing of a certain archimedean integral, this allows us to confirm a conjecture of Gross and Savin on rank-7 motives of type G_2.

1. Introduction

We establish a connection between algebraic cycles in Siegel sixfolds and the residue at $s = 1$ of spin L-functions of automorphic representations of GSp_6, as predicted by conjectures of Beilinson and Tate. Moreover, we exploit an exceptional theta correspondence between the split group of type G_2 and PGSp_6 to answer a question of Gross and Savin.

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1.1. ***Motivation.*** Let $\pi = \pi_\infty \otimes \pi_f$ be a cohomological cuspidal automorphic representation of $\mathrm{PGSp}_6(\mathbb{A})$, let $M(\pi_f)$ denote the spin Chow motive with coefficients in a number field L conjecturally attached to π and let $L(s, M(\pi_f)(3))$ be its Hasse–Weil L-function. Let

$$r_{\mathcal{H}} : H^1_{\mathcal{M}}(M(\pi_f)(4)) \oplus N(M(\pi_f)(3)) \to H^1_{\mathcal{H}}(M(\pi_f)(4))$$

denote Beilinson–Deligne regulator. Here $H^1_{\mathcal{M}}(M(\pi_f)(4))$ denotes the first motivic cohomology group of $M(\pi_f)(4)$, the group $N(M(\pi_f)(3))$ denotes algebraic cycles in $M(\pi_f)(3)$ up to homological equivalence and $H^1_{\mathcal{H}}(M(\pi_f)(4))$ denotes the first absolute Hodge cohomology group of $M(\pi_f)(4)$.

Conjecture 1.1 (Beilinson–Tate). (1) The map $r_{\mathcal{H}}$ induces an isomorphism

$$(H^1_{\mathcal{M}}(M(\pi_f)(4)) \oplus N(M(\pi_f)(3))) \otimes_{\mathbb{Q}} \mathbb{R} \to H^1_{\mathcal{H}}(M(\pi_f)(4)).$$

(2) $\mathrm{ord}_{s=0} L(s, M(\pi_f)(3)) = \dim_L H^1_{\mathcal{M}}(M(\pi_f)(4))$.

(3) $-\mathrm{ord}_{s=1} L(s, M(\pi_f)(3)) = \dim_L N(M(\pi_f)(3))$.

(4) $\det(\mathrm{Im}\, r_{\mathcal{H}}) = L^*(1, M(\pi_f)(3))\mathcal{D}(M(\pi_f)(4))$, where $\mathcal{D}(M(\pi_f)(4))$ denotes the Deligne L-structure of $\det(H^1_{\mathcal{H}}(M(\pi_f)(4))$.

In [Burgos Gil et al. 2024], we studied the contribution of the motivic cohomology to this conjecture. This corresponds to the case where $L(s, M(\pi_f)(3))$ does not have a pole at $s = 1$. In this article, we focus on the contribution of algebraic cycles, which corresponds to the case where $L(s, M(\pi_f)(3))$ has a simple pole at $s = 1$.

The ℓ-adic étale realization $M_\ell(\pi_f)$ of $M(\pi_f)$ is expected to be a $\mathrm{GL}_8(\overline{\mathbb{Q}}_\ell)$-valued Galois representation factoring through the spin representation $\mathrm{Spin} : \mathrm{Spin}_7(\overline{\mathbb{Q}}_\ell) \to \mathrm{GL}_8(\overline{\mathbb{Q}}_\ell)$. If $L(s, M(\pi_f)(3))$ has a pole at $s = 1$, Conjecture 1.1(3) implies the existence of an invariant vector in this eight-dimensional Galois representation. As the stabilizer in $\mathrm{Spin}_7(\overline{\mathbb{Q}}_\ell)$ of a generic vector in the spin representation is the exceptional group $G_2(\overline{\mathbb{Q}}_\ell)$, by Langlands reciprocity principle, π should be a functorial lift from a group G of type G_2. In fact, we have $\mathrm{Spin}_{|G_2} = \mathrm{Std} \oplus \mathbf{1}$, where Std denotes the standard representation of G_2 and $\mathbf{1}$ denotes the trivial representation. Then, if σ is a cuspidal automorphic representation of $G(\mathbb{A})$ lifting to π, Gross and Savin [1998] conjectured that the motive $M(\pi_f)$ decomposes as the direct sum of the rank-7 motive $M(\sigma_f)$ attached to σ and the rank-1 trivial motive generated by the class given in Conjecture 1.1. Moreover, inspired by local calculations, they conjectured that this class should arise from a Hilbert modular threefold.

1.2. ***Main results.*** Let F denote a real étale quadratic $\mathbb{Q}$-algebra, i.e., F is either a quadratic extension of $\mathbb{Q}$ or $\mathbb{Q} \times \mathbb{Q}$. Associated to the totally real étale cubic algebra $E = \mathbb{Q} \times F$ of $\mathbb{Q}$ there is a Hilbert modular threefold $\mathrm{Sh}_{\boldsymbol{H}}/\mathbb{Q}$, with underlying reductive group $\boldsymbol{H} = \{g \in \mathrm{Res}_{E/\mathbb{Q}} \mathrm{GL}_{2,E} \mid \det(g) \in \boldsymbol{G}_m\}$. The group $\boldsymbol{H}$ embeds naturally into $\boldsymbol{G} = \mathrm{GSp}_6$ and one has a closed embedding $\iota : \mathrm{Sh}_{\boldsymbol{H}} \hookrightarrow \mathrm{Sh}_{\boldsymbol{G}}$ of codimension 3 in the Shimura variety attached to $\boldsymbol{G}$, which is the Siegel variety of dimension 6. Let V^λ be the irreducible algebraic representation of $\boldsymbol{G}$ of highest weight $\lambda = (\lambda_1, \lambda_2, \lambda_3, c)$ (see Section 2 for notation on algebraic representations). The representation V^λ contains the trivial $\boldsymbol{H}$-representation if and only if $c=0$ and $\lambda_1=\lambda_2+\lambda_3$. When this holds $\iota^* V^\lambda$ contains $\lambda_2-\lambda_3+1$ copies of the trivial representation of $\boldsymbol{H}$, which we index by the values $\lambda_2 \geq \mu \geq \lambda_3$. Then, for any such μ, the cycle $\mathrm{Sh}_{\boldsymbol{H}}$ of $\mathrm{Sh}_{\boldsymbol{G}}$ induces a class

$$\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{M}} \in H^6_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{G}}, \mathscr{V}^\lambda_{\mathcal{M}}(3)),$$

where $\mathscr{V}^\lambda_{\mathcal{M}}$ is the Chow local system associated to V^λ and $H^6_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{G}}, \mathscr{V}^\lambda_{\mathcal{M}}(3))$ is the motivic cohomology group of $\mathrm{Sh}_{\boldsymbol{G}}$ with coefficients in $\mathscr{V}^\lambda_{\mathcal{M}}(3)$. We denote by $\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{H}} \in H^7_{\mathcal{H}}(\mathrm{Sh}_{\boldsymbol{G}}, \mathscr{V}^\lambda_{\mathcal{H}}(4))$ (resp. $\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},B} \in H^6_B(\mathrm{Sh}_{\boldsymbol{G}}, \mathscr{V}^\lambda_B(3)))$ the image of $\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{M}}$ in absolute Hodge cohomology, (resp. Betti cohomology) (see Definition 3.11 for the precise definition of $\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{H}}$). Let π be a cuspidal automorphic representation of $\mathrm{PGSp}_6(\mathbb{A})$ whose archimedean component belongs to the discrete series L-packet of V^λ and has Hodge type $(3,3)$. For a cusp form $\Psi=\Psi_\infty \otimes \Psi_f$ in the space of π, whose archimedean component Ψ_∞ is a highest weight vector in the minimal K-type of π_∞, we have a vector valued harmonic differential form ω_Ψ whose cohomology class $[\omega_\Psi]$ is an element of $H^6_{\mathrm{dR},c}(\mathrm{Sh}_{\boldsymbol{G}}, \mathscr{V}^\lambda_{\mathrm{dR}})$. Poincaré duality induces maps

$$\langle\,\cdot\,, [\omega_\Psi]\rangle_B : H^6_B(\mathrm{Sh}_{\boldsymbol{G}}, \mathscr{V}^\lambda_B(3)) \to \mathbb{C},$$
$$\langle\,\cdot\,, [\omega_\Psi]\rangle_{\mathcal{H}} : H^7_{\mathcal{H}}(\mathrm{Sh}_{\boldsymbol{G}}, \mathscr{V}^\lambda_{\mathcal{H}}(4)) \to \mathbb{C}.$$

The pairings $\langle \mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},B}, [\omega_\Psi]\rangle_B$ and $\langle \mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{H}}, [\omega_\Psi]\rangle_{\mathcal{H}}$ are computed in terms of the residue of a certain adelic integral of Rankin–Selberg type considered in [Pollack and Shah 2018]. Therein it is shown that, if π supports certain Fourier coefficients associated to F, then the local factors at unramified places v of this integral represent the degree 8 local spin L-function $L(s, \pi_v, \mathrm{Spin})$ of π_v. The following result gives evidence for Conjecture 1.1 for the motive associated to π.

Theorem 1.2 (Theorem 5.11). *Let $\pi=\pi_\infty \otimes \pi_f$ be a cuspidal automorphic representation of $\mathrm{PGSp}_6(\mathbb{A})$ such that π_∞ is a discrete series of Hodge type $(3,3)$ in the discrete series L-packet of V^λ. Then*

$$\langle \mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},B}, [\omega_\Psi]\rangle_B = \langle \mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{H}}, [\omega_\Psi]\rangle_{\mathcal{H}} = C \cdot \mathrm{Res}_{s=1}(\mathcal{I}_S(\Phi, \Psi^{[\lambda,\mu]}, s) L^S(s, \pi, \mathrm{Spin})),$$

where C is an explicit nonzero constant independent of π, S is a sufficiently large set of places containing the ramified and archimedean places, $\Psi^{[\lambda,\mu]} = A^{[\lambda,\mu]} \cdot \Psi$ for some weight lowering operator $A^{[\lambda,\mu]}$ defined in Proposition 4.8, Φ is a Schwartz–Bruhat function and $\mathcal{I}_S(\Phi, \Psi^{[\lambda,\mu]}, s)$ is the integral defined in Theorem 5.8.

Remark 1.3. We point out that, according to [Gan and Gurevich 2009, Proposition 12.1] there exist a Schwartz–Bruhat function Φ and a vector $\Psi \in \pi$ such that $\mathcal{I}_S(\Phi, \Psi, 1)$ is nonzero. However we do not know if this holds for Ψ_∞ in the minimal K-type of π_∞. Moreover, one can show that there exists a cusp form $\widetilde{\Psi} \in \pi$, which coincides with Ψ at the archimedean place and away from S, such that

$$\mathcal{I}_S(\Phi, \widetilde{\Psi}^{[\lambda,\mu]}, s) = \mathcal{I}_\infty(\Phi_\infty, \Psi^{[\lambda,\mu]}_\infty, s).$$

Although we have not been able to calculate it, we expect that for a natural choice of Φ_∞ the archimedean integral $\mathcal{I}_\infty(\Phi_\infty, \Psi_\infty^{[\lambda,\mu]}, s)$ is the Gamma factor of the spin motive attached to π by the rule of Serre, and hence holomorphic and nonzero at $s = 1$.

As a corollary of this theorem, one can deduce, under the additional assumption that π is the Steinberg representation at a finite place, a weak version of Conjecture 1.1(1) (Corollary 5.15) and Conjecture 1.1(3) (Corollary 5.14).

When $\mathrm{Res}_{s=1} L^S(s, \pi, \mathrm{Spin})$ is nonzero then (see [Gan and Savin 2020, Theorem 1.1]) π is a weak functorial lift of a cuspidal automorphic representation σ of an exceptional group of type G_2. Moreover (see Proposition 8.1), we have

$$\mathrm{Res}_{s=1} L^S(s, \pi, \mathrm{Spin}) = L^S(1, \sigma, \mathrm{Std}) \mathrm{Res}_{s=1} \zeta^S(s).$$

Hence, up to controlling the value of the archimedean integral at $s = 1$, Theorem 1.2 above gives a cohomological formula for the critical value $L^S(1, \sigma, \mathrm{Std})$.

Our second main result concerns the program of Gross and Savin on rank-7 motives of Galois type G_2. The first step towards the conjecture of Gross and Savin was made in [Kret and Shin 2023], where the authors constructed GSpin-valued Galois representations associated to cohomological cuspidal automorphic forms of symplectic groups. Moreover, based on the calculations of [Gross and Savin 1998], Kret and Shin [2023, Theorem 11.1] verified that, for suitable automorphic representations of $\mathrm{PGSp}_6(\mathbb{A})$ in the image of the exceptional theta correspondence from the compact form G_2^c of type G_2, the image of their Galois representation lies actually in $G_2(\overline{\mathbb{Q}}_\ell)$. More precisely, let ρ_π be the $\mathrm{Spin}_7(\overline{\mathbb{Q}}_\ell)$-valued Galois representation attached to π. Assuming that π is a nontrivial small theta lift of σ, we have

$$\mathrm{Spin} \circ \rho_\pi = \mathrm{Std} \circ \rho_\sigma \oplus \mathbf{1}, \tag{1}$$

where $\mathrm{Std} \circ \rho_\sigma$ is the standard Galois representation attached to σ and $\mathbf{1}$ denotes the one-dimensional trivial representation.

Remark 1.4. Technically speaking, only the dual pair (G_2^c, PGSp_6) is considered in [Gross and Savin 1998], but their conjecture also applies to the dual pair (G_2, PGSp_6). Using the results of [Kret and Shin 2023] and the study of the exceptional theta correspondence for (G_2, PGSp_6) (see Theorem 1.8 below), we construct (Theorem 8.3), under some assumptions, Galois representations associated to cohomological cuspidal automorphic representations σ of $G_2(\mathbb{A})$, which sit in a decomposition as that of (1).

Theorem 1.5 (Theorem 8.6). *Let σ be an irreducible cuspidal automorphic representation of $G_2^c(\mathbb{A})$ or $G_2(\mathbb{A})$ such that the big theta lift $\Theta(\sigma)$ to $\mathrm{PGSp}_6(\mathbb{A})$ has an irreducible subquotient $\pi = \bigotimes_v' \pi_v$, which is a cuspidal automorphic representation such that π_∞ is cohomological for V as above and π_p is the Steinberg representation for some prime number p. Assume that the integral $\mathcal{I}_S(\Phi, \Psi^{[\lambda,\mu]}, 1)$ is nonzero for some Φ and $\Psi^{[\lambda,\mu]}$ as above. Then, the trivial representation $\mathbf{1}$ in (1) is generated by the étale realization of $\mathcal{Z}_{\boldsymbol{H},\mathcal{M}}^{[\lambda,\mu]}$.*

Remark 1.6. Note that the archimedean part π_∞ of π is not necessarily of Hodge type $(3, 3)$. However, it is one of the main results of [Kret and Shin 2023] that the L-packet of π is stable at infinity. In particular, there exists a cuspidal automorphic representation $\pi^{3,3} = \pi_\infty^{3,3} \otimes \pi_f$ whose archimedean part is cohomological and of Hodge type $(3, 3)$ and whose nonarchimedean part is equivalent to π_f. In the integral appearing in the statement of Theorem 1.2, the archimedean part of the cusp form $\Psi^{[\lambda,\mu]}$ is a suitable vector in the minimal K-type of $\pi_\infty^{3,3}$.

Remark 1.7. In Proposition 8.4 we give a list of cases where σ is known to have a small theta lift $\pi = \bigotimes_v' \pi_v$ of σ to $\mathrm{PGSp}_6(\mathbb{A})$ which is a cuspidal automorphic representation such that π_∞ is cohomological for V as above and π_p is the Steinberg representation for some prime number p, as in the previous theorem.

We conclude this introduction explaining a result which provides cases where Theorem 1.5 can be applied and which has its own interest. Indeed, note that a necessary condition for the integral $\mathcal{I}_S(\Phi, \Psi^{[\lambda,\mu]}, 1)$ to be nonzero, is that π supports a rank-2 Fourier coefficient associated to F. By a result of Gan [2005, Theorem 3.1], every cuspidal automorphic representation σ of $G_2(\mathbb{A})$ supports a Fourier coefficient associated to an étale cubic algebra E.

Theorem 1.8 (Theorem 7.2, Proposition 7.13). *Let σ be a cuspidal automorphic representation of $G_2(\mathbb{A})$. Assume that*

- *σ is not globally generic*;
- *σ_p is generic at some finite place p.*

Then the big theta lift $\Theta(\sigma)$ is cuspidal. Moreover $\Theta(\sigma)$ supports a rank-2 Fourier coefficient associated to F (and is in particular nonzero) if and only if σ supports a Fourier coefficient associated to $\mathbb{Q} \times F$.

1.3. ***Overview of the proofs.*** The main difficulty for calculating the pairing of Theorem 1.2 between the motivic class and the cohomology class $[\omega_\Psi]$ resides on the fact that the first class is constructed from the decomposition into irreducible components of the restriction of V to the subgroup $\boldsymbol{H}$, while the test vector is constructed from its restriction to the maximal compact subgroup $\mathrm{U}(3)$ of $\boldsymbol{G}(\mathbb{R})$. One needs to carefully study the relationship between these two different decompositions (Theorem 4.2). As a consequence we get a formula for the pairing in terms of a period integral (Propositions 4.8 and 4.10). These adelic integrals are in turn related to the residue of the partial spin L-function of π by means of the work of Pollack and Shah (Proposition 5.10), which allows to conclude the proof. Theorem 1.5 follows basically from Theorem 1.2 and 1.8. The proof of Theorem 1.8 goes as follows. We first prove (Theorem 7.2 and Corollary 7.3) that σ lifts to a cuspidal representation using the tower of exceptional correspondences for G_2 studied in [Ginzburg et al. 1997b], which reduces the problem to the vanishing of certain automorphic period integrals. Finally, we establish (Proposition 7.13) a correspondence between Fourier coefficients of σ and its theta lift, which in particular implies the nonvanishing of the latter.

1.4. ***Structure of the manuscript.*** In Section 2 we fix notation, conventions, and basic results that will be useful in the body of the article. In particular, we prove that, under some mild assumptions, the localization at a maximal ideal of the Hecke algebra of the cohomology of the Siegel sixfold is cuspidal and concentrated in the middle degree. We also introduce Absolute Hodge cohomology and compute the dimension of its π_f-isotypical component. In Section 3 we explain the construction of the motivic class $\mathcal{Z}_{\mathcal{M}}^{[\lambda,\mu]}$ and its realizations. In Section 4 we construct the harmonic differential form ω_Ψ associated to a suitable cuspidal form Ψ in the space of π and we prove our first main result concerning the calculation of the pairing between the motivic class and the cohomology class $[\omega_\Psi]$. In Section 5, we use the results of Pollack and Shah to relate the pairing to the residue of the spin L-function. Sections 6 and 7 are devoted to the study of the exceptional theta correspondence between G_2 and PGSp_6 and contain the proof of Theorem 1.8. Finally, in Section 8 we relate the pairing to a critical value of the standard L-function of G_2. We also deduce from the work of Kret and Shin the existence of Galois representations attached to certain cuspidal representations of G_2 and we conclude with a proof of Theorem 1.5.

2. Preliminaries

2.1. ***Algebraic groups and algebraic representations.*** Let ψ denote an antisymmetric nondegenerate bilinear form on a finite-dimensional $\mathbb{Q}$-vector space V. The symplectic group $\mathrm{GSp}(V, \psi)$ is the $\mathbb{Q}$-group scheme defined by

$$\mathrm{GSp}(V, \psi) = \{g \in \mathrm{GL}(V) \mid \forall v, w \in V, \psi(gv, gw) = \nu(g)\psi(v, w),\ \nu(g) \in \boldsymbol{G}_m\}.$$

Then $\nu : \mathrm{GSp}(V, \psi) \to \boldsymbol{G}_m$ is a character. Let I_n denote the identity matrix of size n. When V is the $\mathbb{Q}$-vector space $\mathbb{Q}^{2n}$ endowed with the bilinear form whose matrix is $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$, we let GSp_{2n} denote $\mathrm{GSp}(\mathbb{Q}^{2n}, J)$ and we let Sp_{2n} denote $\ker \nu$. In this paper, we are mainly interested in the case $n = 3$. Hence we will denote by $\boldsymbol{G}$ the group GSp_6 and by $\boldsymbol{G}_0$ the group Sp_6. Let $\boldsymbol{T} \subset \boldsymbol{G}$ denote the maximal diagonal torus and $\boldsymbol{B} \subset \boldsymbol{G}$ denote the standard Borel. We have

$$\boldsymbol{T} = \{\mathrm{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_1^{-1}\nu, \alpha_2^{-1}\nu, \alpha_3^{-1}\nu), \alpha_1, \alpha_2, \alpha_3, \nu \in \boldsymbol{G}_m\}.$$

We associate to any 4-tuple $(\lambda_1, \lambda_2, \lambda_3, c) \in \mathbb{Z}^4$ such that $c \equiv \lambda_1 + \lambda_2 + \lambda_3 \pmod 2$ the algebraic character $\lambda(\lambda_1, \lambda_2, \lambda_3, c)$ of $\boldsymbol{T}$ defined by

$$\lambda(\lambda_1, \lambda_2, \lambda_3, c) : \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_1^{-1}\nu, \alpha_2^{-1}\nu, \alpha_3^{-1}\nu) \mapsto \alpha_1^{\lambda_1}\alpha_2^{\lambda_2}\alpha_3^{\lambda_3}\nu^{\frac{1}{2}(c-\lambda_1-\lambda_2-\lambda_3)}.$$

This defines an isomorphism between the group of 4-tuples

$$(\lambda_1, \lambda_2, \lambda_3, c) \in \mathbb{Z}^4 \quad \text{such that} \quad c \equiv \lambda_1 + \lambda_2 + \lambda_3 \pmod 2$$

and the group of algebraic characters of $\boldsymbol{T}$. Let $\rho_1 = \lambda(1, -1, 0, 0)$ and $\rho_2 = \lambda(0, 1, -1, 0)$ denote the short simple roots and let $\rho_3 = \lambda(0, 0, 2, 0)$ denote the long simple root. The set of roots of $\boldsymbol{T}$ in $\boldsymbol{G}$ is $R = R^+ \cup R^-$, where

$$R^+ = \{\rho_1, \rho_2, \rho_1 + \rho_2, \rho_2 + \rho_3, \rho_1 + \rho_2 + \rho_3, \rho_1 + 2\rho_2 + \rho_3, 2\rho_1 + 2\rho_2 + \rho_3, 2\rho_2 + \rho_3, \rho_3\}$$

is the set of positive roots with respect to $\boldsymbol{B}$ and $R^- = -R^+$. A weight $\lambda = \lambda(\lambda_1, \lambda_2, \lambda_3, c)$ is dominant for $\boldsymbol{B}$ if $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$. For any such λ, there exists a unique (up to isomorphism) irreducible algebraic representation V^λ of $\boldsymbol{G}$ of highest weight λ and every irreducible algebraic representation of $\boldsymbol{G}$ is obtained in this way (up to isomorphism). Similarly, irreducible algebraic representations of GSp_4 are classified by their highest weight which is a character of the shape $\lambda(\lambda_1, \lambda_2, c)$ with $\lambda_1 \geq \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 \equiv c \pmod 2$ (see for example [Lemma 2017, §2.3] for more details). We will also use the classification of irreducible algebraic representations of the groups $\boldsymbol{G}_0 = \mathrm{Sp}_6$ and Sp_4. To this end let us recall that the diagonal maximal torus $\boldsymbol{T}_0 = \boldsymbol{T} \cap \boldsymbol{G}_0$ of $\boldsymbol{G}_0$ is

$$\boldsymbol{T}_0 = \{\mathrm{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_1^{-1}, \alpha_2^{-1}, \alpha_3^{-1}), \alpha_1, \alpha_2, \alpha_3 \in \boldsymbol{G}_m\}$$

and that its group of algebraic characters is isomorphic to $\mathbb{Z}^3$ via $(\lambda_1, \lambda_2, \lambda_3) \mapsto \lambda(\lambda_1, \lambda_2, \lambda_3)$, where

$$\lambda(\lambda_1, \lambda_2, \lambda_3) : \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_1^{-1}, \alpha_2^{-1}, \alpha_3^{-1}) \mapsto \alpha_1^{\lambda_1} \alpha_2^{\lambda_2} \alpha_3^{\lambda_3}. \tag{2}$$

A weight $\lambda = \lambda(\lambda_1, \lambda_2, \lambda_3)$ is dominant with respect to the standard Borel $\boldsymbol{B}_0 = \boldsymbol{B} \cap \boldsymbol{G}_0$ if $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$ and for any such λ there exists a unique (up to isomorphism) irreducible algebraic representation V^λ of $\boldsymbol{G}_0$ of highest weight λ and every irreducible algebraic representation of $\boldsymbol{G}_0$ is obtained in this way (up to isomorphism). Similarly, irreducible algebraic representations of Sp_4 are classified by characters $\lambda(\lambda_1, \lambda_2)$ with $\lambda_1 \geq \lambda_2$, with obvious notation.

2.2. ***Compact Lie groups and representations.*** Let $\mathrm{U}(n) = \{g \in \mathrm{GL}_n(\mathbb{C}) \mid {}^t\bar{g} g = I_n\}$ denote the unitary group and let $K_\infty \subset \boldsymbol{G}_0(\mathbb{R})$ be the subgroup defined as

$$K_\infty = \left\{\left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right) \,\middle|\, AA^t + BB^t = 1,\ AB^t = BA^t\right\}.$$

We have an isomorphism $\kappa : \mathrm{U}(3) \simeq K_\infty$ defined by $A + iB \mapsto \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right)$. In fact K_∞ is a maximal compact subgroup of $\boldsymbol{G}_0(\mathbb{R})$. Let $T_\infty \subset K_\infty$ denote $\{\kappa(\mathrm{diag}(z_1, z_2, z_3)), z_1, z_2, z_3 \in \mathrm{U}(1)\}$. Then T_∞ is Cartan subgroup of K_∞. Its group of algebraic characters is isomorphic to $\mathbb{Z}^3$ via $(\lambda_1, \lambda_2, \lambda_3) \mapsto \lambda'(\lambda_1, \lambda_2, \lambda_3)$, where

$$\lambda'(\lambda_1, \lambda_2, \lambda_3) : \kappa(\mathrm{diag}(z_1, z_2, z_3)) \mapsto z_1^{\lambda_1} z_2^{\lambda_2} z_3^{\lambda_3}.$$

An algebraic character is dominant if $\lambda_1 \geq \lambda_2 \geq \lambda_3$. For any dominant integral weight λ', there exists a unique (up to isomorphism) irreducible representation $\tau_{\lambda'}$ of K_∞ in a finite-dimensional $\mathbb{C}$-vector space and every irreducible representation of K_∞ is obtained in this way (up to isomorphism). In what follows, we will simply denote the irreducible representation of highest weight $\lambda'(\lambda_1, \lambda_2, \lambda_3)$ by $\tau_{(\lambda_1, \lambda_2, \lambda_3)}$. Let us explain the connection between the weights λ of $\boldsymbol{T}_0$ defined by (2) in the previous section and the weights λ' defined above. Let $J \in \boldsymbol{G}_0(\mathbb{C})$ denote the matrix $J = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} I_3 & iI_3 \\ iI_3 & I_3 \end{smallmatrix}\right)$. Then we have

$$J^{-1}\kappa(\mathrm{diag}(z_1, z_2, z_3))J = \mathrm{diag}(z_1, z_2, z_3)$$

and so, for any $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3$, we have

$$\lambda(\lambda_1, \lambda_2, \lambda_3)(J^{-1}\kappa(z_1, z_2, z_3)J) = \lambda'(\lambda_1, \lambda_2, \lambda_3)(\mathrm{diag}(z_1, z_2, z_3)).$$

In brief, the character $\lambda'(\lambda_1, \lambda_2, \lambda_3)$ of T_∞ is conjugated to the restriction of $\lambda(\lambda_1, \lambda_2, \lambda_3)$ to $\mathrm{U}(1)^3 \subset \mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{C}^\times = \boldsymbol{T}_0(\mathbb{C})$.

2.3. ***Lie algebras.*** Let $\mathfrak{g}_0$ (resp. $\mathfrak{k}$) denote the Lie algebra of $\boldsymbol{G}_0(\mathbb{R})$ (resp. K_∞) and let $\mathfrak{g}_{0,\mathbb{C}}$ (resp. $\mathfrak{k}_\mathbb{C}$) denote its complexification. Then

$$\mathfrak{g}_0 = \left\{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in M_6(\mathbb{R}) \mid B = B^t, C = C^t, A = -D^t \right\},$$
$$\mathfrak{k} = \left\{ \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right) \in M_6(\mathbb{R}) \mid A = -A^t, B = B^t \right\}.$$

The Lie algebra $\mathfrak{k}$ is the 1-eigenspace for the Cartan involution $\theta(X) = -X^t$. The (-1)-eigenspace is $\mathfrak{p} = \left\{ \left(\begin{smallmatrix} A & B \\ B & -A \end{smallmatrix}\right) \in M_6(\mathbb{R}) \mid A = A^t, B = B^t \right\}$. Letting

$$\mathfrak{p}_\mathbb{C}^\pm = \left\{ \left(\begin{smallmatrix} A & \pm iA \\ \pm iA & -A \end{smallmatrix}\right) \in M_6(\mathbb{C}) \mid A = A^t \right\},$$

we have $\mathfrak{p}_\mathbb{C} = \mathfrak{p}_\mathbb{C}^+ \oplus \mathfrak{p}_\mathbb{C}^-$ and one has the Cartan decomposition

$$\mathfrak{g}_{0,\mathbb{C}} = \mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}_\mathbb{C}^+ \oplus \mathfrak{p}_\mathbb{C}^-.$$

For $1 \le j \le 3$, let $D_j \in M_3(\mathbb{C})$ be the matrix with entry 1 at position (j, j) and 0 elsewhere. Define $T_j = \left(\begin{smallmatrix} 0 & D_j \\ -D_j & 0 \end{smallmatrix}\right)$. Then the Lie algebra $\mathfrak{h}$ of T_∞ is $\mathfrak{h} = \mathbb{R} \cdot T_1 \oplus \mathbb{R} \cdot T_2 \oplus \mathbb{R} \cdot T_3$. This is a compact Cartan subalgebra of $\mathfrak{g}_0$. Let (e_1, e_2, e_3) denote the basis of $\mathfrak{h}_\mathbb{C}^*$ dual to $(-iT_1, -iT_2, -iT_3)$. A system of positive roots for $(\mathfrak{g}_{0,\mathbb{C}}, \mathfrak{h}_\mathbb{C})$ is then given by

$$\{e_1 \pm e_2, e_1 \pm e_3, e_2 \pm e_3, 2e_1, 2e_2, 2e_3\}.$$

The simple roots are $e_1 - e_2$, $e_2 - e_3$ and $2e_3$. We note that $\mathfrak{p}_\mathbb{C}^+$ is spanned by the root spaces corresponding to the positive roots of type $2e_j$ and $e_j + e_k$. We denote by $\Delta = \{\pm 2e_j, \pm(e_j \pm e_k)\}$ the set of all roots, $\Delta_\mathrm{c} = \{\pm(e_j - e_k)\}$ the set of compact roots and $\Delta_\mathrm{nc} = \Delta - \Delta_\mathrm{c}$ the noncompact roots. Finally, we denote by Δ^+, Δ_c^+ and Δ_nc^+ the set of positive, positive compact and positive noncompact roots, respectively.

2.4. ***Weyl groups.*** Recall that the Weyl group of $\boldsymbol{G}_0$ is given by $\mathfrak{W}_{\boldsymbol{G}_0} = \{\pm 1\}^3 \rtimes \mathfrak{S}_3$. The reflection σ_j in the hyperplane orthogonal to $2e_j$ simply reverses the sign of e_j while leaving the other e_k fixed. The reflection σ_{jk} in the hyperplane orthogonal to $e_j - e_k$ exchanges e_j and e_k and leaves the remaining e_ℓ fixed. The Weyl group $\mathfrak{W}_{K_\infty}$ of $K_\infty \cong U(3)$ is isomorphic to $\mathfrak{S}_3$ and, via the embedding into $\boldsymbol{G}$, identifies with the subgroup of $\mathfrak{W}_{\boldsymbol{G}_0}$ generated by the σ_{jk}. With the identification $\mathfrak{W}_{\boldsymbol{G}_0} = N(\boldsymbol{T}_0)/Z(\boldsymbol{T}_0)$, an explicit description of $\mathfrak{W}_{\boldsymbol{G}_0}$ and $\mathfrak{W}_{K_\infty}$ is given as follows. The matrices corresponding to the reflections σ_{jk} are $\left(\begin{smallmatrix} S_{jk} & 0 \\ 0 & -S_{jk} \end{smallmatrix}\right)$, where S_{jk} is the matrix with entry 1 at places (ℓ, ℓ), $\ell \neq j, k$, (k, j) and (j, k) and zeroes elsewhere. The matrices corresponding to the reflection σ_j in the hyperplane orthogonal to $2e_j$ are of the form $\left(\begin{smallmatrix} 0 & U_j \\ -U_j & 0 \end{smallmatrix}\right)$, where U_j denotes the diagonal matrix with -1 at the place (j, j) and ones at the other entries of the diagonal. This gives an explicit description of the elements of $\mathfrak{W}_{K_\infty}$ and their length:

$$\mathfrak{W}_{K_\infty} = \{1, \sigma_{12}, \sigma_{13}, \sigma_{23}, \sigma_{12}\sigma_{13}, \sigma_{12}\sigma_{23}\} \xrightarrow{\ell(\bullet)} \{0, 1, 1, 1, 2, 2\}.$$

2.5. ***Discrete series.*** We recall standard facts on discrete series for $\boldsymbol{G}_0(\mathbb{R}) = \mathrm{Sp}_6(\mathbb{R})$ and for $\mathrm{PGSp}_6(\mathbb{R})$. For any nonsingular weight Λ define

$$\Delta^+(\Lambda) := \{\alpha \in \Delta \mid \langle \alpha, \Lambda \rangle > 0\}, \quad \Delta_c^+(\Lambda) = \Delta^+(\Lambda) \cap \Delta_c,$$

where $\langle\ ,\ \rangle$ is the standard scalar product on $\mathbb{R}^3$. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a weight of T_∞ such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$ and let $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha = (3, 2, 1)$. As $|\mathfrak{W}_{G_0}/\mathfrak{W}_{K_\infty}| = 8$, by [Knapp 1986, Theorem 9.20], the set of equivalence classes of irreducible discrete series representations of $\boldsymbol{G}_0(\mathbb{R})$ with Harish-Chandra parameter $\lambda+\rho$ contains 8 elements. More precisely, choose representatives $\{w_1, \dots, w_8\}$ of $\mathfrak{W}_{G_0}/\mathfrak{W}_{K_\infty}$ of increasing length and such that for any $1 \leq i \leq 8$. Then the weight $w_i(\lambda+\rho)$ is dominant for K_∞. The representatives, defined by their action on ρ, are $w_1(3, 2, 1) = (3, 2, 1)$, $w_2(3, 2, 1) = (3, 2, -1)$, $w_3(3, 2, 1) = (3, 1, -2)$, $w_4(3, 2, 1) = (2, 1, -3)$, $w_5(3, 2, 1) = (3, -1, -2)$, $w_6(3, 2, 1) = (2, -1, -3)$, $w_7(3, 2, 1) = (1, -2, -3)$, $w_8(3, 2, 1) = (-1, -2, -3)$. Then, for any $1 \leq i \leq 8$, there exists an irreducible discrete series π_∞^Λ, where $\Lambda = w_i(\lambda+\rho)$, of Harish-Chandra parameter Λ and containing with multiplicity 1 the minimal K_∞-type with highest weight $\Lambda + \delta_{G_0} - 2\delta_{K_\infty}$, where δ_{G_0} (resp. δ_{K_∞}) is the half-sum of roots (resp. of compact roots) which are positive with respect to the Weyl chamber in which Λ lies, i.e., $2\delta_{G_0} := \sum_{\alpha\in\Delta^+(\Lambda)}\alpha$ (resp. $2\delta_{K_\infty} := \sum_{\alpha\in\Delta_c^+(\Lambda)}\alpha$). Moreover, for $i \neq j$, $\Lambda = w_i(\lambda+\rho)$, $\Lambda = w_j(\lambda+\rho)$, the representations π_∞^Λ and π_∞^Λ are not equivalent and any discrete series of $\boldsymbol{G}_0$ is obtained in this way. Let V^λ be the irreducible algebraic representation of $\boldsymbol{G}_0$ of highest weight $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ (for $\boldsymbol{T}_0$).

Definition 2.1. The discrete series L-packet $P(V^\lambda)$ associated to λ is the set of isomorphism classes of discrete series of $\boldsymbol{G}_0(\mathbb{R})$ whose Harish-Chandra parameter is of the form $\Lambda = w_i(\lambda+\rho)$ as i varies.

By [Borel and Wallach 1980, Theorem II.5.3], for each $\pi_\infty^\Lambda \in P(V^\lambda)$, the space

$$\mathrm{Hom}_{K_\infty}\big(\textstyle\bigwedge^6 \mathfrak{g}_{0,\mathbb{C}}/\mathfrak{k}_\mathbb{C} \otimes V^\lambda, \pi_\infty^\Lambda\big)$$

has dimension 1. This is a consequence of the fact (see the proof of [Borel and Wallach 1980, Theorem II.5.3]) that the minimal K_∞-type of π_∞^Λ appears uniquely in $\bigwedge^6 \mathfrak{g}_{0,\mathbb{C}}/\mathfrak{k}_\mathbb{C} \otimes V^\lambda$. Using the Cartan decomposition, we get

$$\textstyle\bigwedge^6 \mathfrak{g}_{0,\mathbb{C}}/\mathfrak{k}_\mathbb{C} = \bigoplus_{p+q=6} \bigwedge^p \mathfrak{p}_\mathbb{C}^+ \otimes_\mathbb{C} \bigwedge^q \mathfrak{p}_\mathbb{C}^-.$$

Hence, there exists a unique pair (p, q) such that $\mathrm{Hom}_{K_\infty}\big(\bigwedge^p \mathfrak{p}_\mathbb{C}^+ \otimes \bigwedge^q \mathfrak{p}_\mathbb{C}^- \otimes V^\lambda, \pi_\infty^\Lambda\big)$ is nonzero and hence of dimension 1. We call such a pair (p, q) the Hodge type of π_∞^Λ.

Lemma 2.2. *There exist two elements $\pi_{\infty,1}^{3,3}$ and $\bar{\pi}_{\infty,1}^{3,3}$ in $P(V^\lambda)$ of Hodge type $(3, 3)$. They are characterized by having Harish-Chandra parameters $(\lambda_2+2, \lambda_3+1, -\lambda_1-3)$ and $(\lambda_1+3, -\lambda_3-1, -\lambda_2-2)$ and minimal K_∞-types $\tau_{(\lambda_2+2,\lambda_3+2,-\lambda_1-4)}$ and $\tau_{(\lambda_1+4,-\lambda_3-2,-\lambda_2-2)}$ respectively.*

Proof. The discrete series $\pi_{\infty,1}^{3,3}$ and $\bar{\pi}_{\infty,1}^{3,3}$ correspond to the Weyl representatives w_4 and w_5. Since $w_4\lambda = (\lambda_2, \lambda_3, -\lambda_1)$ and $w_5\lambda = (\lambda_1, -\lambda_3, -\lambda_2)$, the Harish-Chandra parameters of $\pi_{\infty,1}^{3,3}$ and $\bar{\pi}_{\infty,1}^{3,3}$ are as desired. When $\Lambda = w_4(\lambda+\rho)$ (resp. $\Lambda = w_5(\lambda+\rho)$), observe that δ_{G_0} is equal to $(2, 1, -3)$ (resp. $(3, -1, -2)$), while $\delta_{K_\infty} = (1, 0, -1)$ in both cases. Hence, using the formula above, the minimal K_∞-types of $\pi_{\infty,1}^{3,3}$ and $\bar{\pi}_{\infty,1}^{3,3}$ are $\tau_{(\lambda_2+2,\lambda_3+2,-\lambda_1-4)}$ and $\tau_{(\lambda_1+4,-\lambda_3-2,-\lambda_2-2)}$ respectively.

Recall that, after [Vogan and Zuckerman 1984, Proposition 6.19], the Hodge type of a discrete series representation of Harish-Chandra parameter Λ is (p, q), where p (resp. q) is the number of positive noncompact roots in $\Delta^+(\Lambda)$ (resp. $\Delta^-(\Lambda)$). Using this, one easily checks that the Hodge type of $\pi^{3,3}_{\infty,1}$ and $\bar{\pi}^{3,3}_{\infty,1}$ is $(3, 3)$. □

The picture for $\mathrm{PGSp}_6(\mathbb{R})$ is similar, but the set of its Harish-Chandra parameters changes slightly. This is due to the fact that, since its maximal compact subgroup has two connected components, the set of parameters has to be considered up to the action of $\mathfrak{W}_{K_\infty}$ and of w_8, as the latter, which is the antidiagonal matrix with all entries -1, now belongs to the connected component away from the identity of the maximal compact subgroup. Concretely, any parameter $\mu = (\mu_1, \mu_2, \mu_3)$ has to be identified with $w_8\mu = (-\mu_3, -\mu_2, -\mu_1)$. If $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$ and $\sum_i \lambda_i \equiv 0 \pmod 2$, then the irreducible algebraic $\boldsymbol{G}$-representation $V^{(\lambda,0)}$ of highest weight $\lambda(\lambda_1, \lambda_2, \lambda_3, 0)$ defines a representation of PGSp_6. The corresponding discrete series L-packet $P(V^{(\lambda,0)})$ for $\mathrm{PGSp}_6(\mathbb{R})$ has thus four elements. Any element $\pi_\infty \in P(V^{(\lambda,0)})$ of Harish-Chandra parameter μ, viewed as a $\boldsymbol{G}(\mathbb{R})$-representation, decomposes when restricted to $\boldsymbol{G}_0(\mathbb{R})$ as the direct sum of two discrete series in $P(V^\lambda)$ of Harish-Chandra parameters μ and $w_8\mu$. As a consequence, for any such π_∞, the space

$$H^6(\mathfrak{g}, K_{\boldsymbol{G}}; \pi_\infty \otimes V^{(\lambda,0)}) = \mathrm{Hom}_{K_G}\big(\textstyle\bigwedge^6 \mathfrak{g}_{\mathbb{C}}/\mathrm{Lie}(K_{\boldsymbol{G}})_{\mathbb{C}}, \pi_\infty \otimes V^{(\lambda,0)}\big),$$

where $\mathfrak{g} = \mathrm{Lie}(\boldsymbol{G})$, $\mathfrak{g}_{\mathbb{C}}$ is its complexification and $K_{\boldsymbol{G}} = \mathbb{R}_+^\times K_\infty$, is two-dimensional. The discussion above implies the following.

Lemma 2.3. *Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a dominant weight for $\boldsymbol{G}_0$ such that $\sum_i \lambda_i \equiv 0 \pmod 2$. Then there exists a unique discrete series $\pi^{3,3}_\infty \in P(V^{(\lambda,0)})$ of $\mathrm{PGSp}_6(\mathbb{R})$, with Harish-Chandra parameter $(\lambda_2+2, \lambda_3+1, -\lambda_1-3)$, such that*

$$\pi^{3,3}_{\infty}|_{\boldsymbol{G}_0(\mathbb{R})} = \pi^{3,3}_{\infty,1} \oplus \bar{\pi}^{3,3}_{\infty,1}.$$

We will refer to $\pi^{3,3}_\infty$ as the discrete series of $\mathrm{PGSp}_6(\mathbb{R})$ in $P(V^{(\lambda,0)})$ of Hodge type $(3, 3)$.

2.6. *Shimura varieties.* Let F denote a real étale quadratic $\mathbb{Q}$-algebra, i.e., F is either a totally real quadratic extension of $\mathbb{Q}$ or $\mathbb{Q} \times \mathbb{Q}$. Denote by $\mathrm{GL}^*_{2,F}/\mathbb{Q}$ the subgroup scheme of $\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GL}_{2,F}$ sitting in the Cartesian diagram

$$\begin{array}{ccc} \mathrm{GL}^*_{2,F} & \hookrightarrow & \mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GL}_{2,F} \\ \downarrow & & \downarrow \scriptstyle\det \\ \boldsymbol{G}_m & \hookrightarrow & \mathrm{Res}_{F/\mathbb{Q}}\,\boldsymbol{G}_{m,F} \end{array}$$

For instance, when $F = \mathbb{Q} \times \mathbb{Q}$, we have

$$\mathrm{GL}^*_{2,F} = \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 \mid \det(g_1) = \det(g_2)\}.$$

Let $\boldsymbol{H}$ denote the group

$$\boldsymbol{H} = \mathrm{GL}_2 \boxtimes \mathrm{GL}^*_{2,F} = \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}^*_{2,F} \mid \det(g_1) = \det(g_2)\}. \tag{3}$$

We embed $\boldsymbol{H}$ into $\boldsymbol{G}$ as follows. Let us consider the $\mathbb{Q}\times F$-module

$$V := \mathbb{Q}e_1 \oplus Fe_2 \oplus \mathbb{Q}f_1 \oplus Ff_2,$$

where $V_1 := \mathbb{Q}e_1 \oplus \mathbb{Q}f_1$ and $V_2 := Fe_2 \oplus Ff_2$ are respectively the standard representations of GL_2 and $\mathrm{GL}^*_{2,F}$. We equip V with the $\mathbb{Q}\times F$-valued alternating form $\psi' : V\times V \to \mathbb{Q}\times F$, such that $\psi'(e_1, f_1) = (1, 0)$, $\psi'(e_2, f_2) = \left(0, \frac{1}{2}\right)$ and V_1 is orthogonal to V_2. The group $\boldsymbol{H}$ acts naturally on V and preserves ψ' up to a scalar. We can regard V as a six-dimensional $\mathbb{Q}$-vector space with $\mathbb{Q}$-valued symplectic form $\psi := \mathrm{tr}_{(\mathbb{Q}\times F)/\mathbb{Q}} \circ \psi'$. Explicitly, we have

$$\psi(ae_1+\alpha e_2, bf_1+\beta f_2) = ab + \tfrac{1}{2}\mathrm{tr}_{F/\mathbb{Q}}(\alpha\beta).$$

This identification defines an embedding $\boldsymbol{H} \hookrightarrow \mathrm{GSp}(V, \psi)$. We now identify $\mathrm{GSp}(V, \psi)$ with $\boldsymbol{G}$ by choosing a suitable $\mathbb{Q}$-basis of V. Recall that the set of real quadratic $\mathbb{Q}$-algebras is parametrized by $D \in \mathbb{Q}^\times_{>0}/(\mathbb{Q}^\times_{>0})^2$, via $D \mapsto F = \mathbb{Q}\oplus\mathbb{Q}\sqrt{D}$. Using the decomposition $F = \mathbb{Q}\oplus\mathbb{Q}\sqrt{D}$, we consider the $\mathbb{Q}$-basis of V given by

$$\{e_1, e_2, e_3, f_1, f_2, f_3\} := \left\{e_1, e_2, \sqrt{D}e_2, f_1, f_2, \tfrac{1}{\sqrt{D}} f_2\right\}.$$

In this basis, ψ is represented by the matrix $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$; thus we obtain an isomorphism $\mathrm{GSp}(V, \psi) \simeq \boldsymbol{G}$ and the embedding

$$\iota : \boldsymbol{H} \hookrightarrow \boldsymbol{G}.$$

Note that the group

$$\boldsymbol{H}' := \mathrm{GL}_2 \boxtimes \mathrm{GSp}_4 := \{(g_1, g_2) \in \mathrm{GL}_2\times\mathrm{GSp}_4 \mid \det(g_1) = \nu(g_2)\}$$

is also naturally embedded in $\boldsymbol{G}$ and ι factors through $\boldsymbol{H}'$.

Recall from [Burgos Gil et al. 2024, §2.2] that there is a three-dimensional Shimura variety $\mathrm{Sh}_{\boldsymbol{H}}$ associated to the $\boldsymbol{H}(\mathbb{R})$-conjugacy class of

$$h : \boldsymbol{S} \to \boldsymbol{H}_{/\mathbb{R}}, \quad x+iy \mapsto \left(\left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right), \left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right), \left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right)\right),$$

where $\boldsymbol{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \boldsymbol{G}_{m/\mathbb{C}}$ is the Deligne torus. The associated Shimura datum has reflex field is $\mathbb{Q}$ and the Shimura variety $\mathrm{Sh}_{\boldsymbol{H}}$ can be described as follows. If $V \subseteq \boldsymbol{H}(\mathbb{A}_f)$ is a fiber product (over the similitude characters) $V_1 \times_{\mathbb{A}_f^\times} V_2$ of sufficiently small subgroups, we have

$$\mathrm{Sh}_{\boldsymbol{H}}(V) = \mathrm{Sh}_{\mathrm{GL}_2}(V_1) \times_{\boldsymbol{G}_m} \mathrm{Sh}_{\mathrm{GL}^*_{2,F}}(V_2),$$

where $\times_{\boldsymbol{G}_m}$ denotes the fiber product over the zero-dimensional Shimura variety of level $W = \det(V_1) = \det(V_2)$. The connected components are given by

$$\pi_0(\mathrm{Sh}_{\boldsymbol{H}}(V)(\mathbb{C})) = \widehat{\mathbb{Z}}^\times/W.$$

Hence, $\mathrm{Sh}_{\boldsymbol{H}}$ can be thought as the fiber product of a modular curve and a Hilbert–Blumenthal modular surface. We also recall that the complex points of $\mathrm{Sh}_{\boldsymbol{H}}(V)$ are given by

$$\mathrm{Sh}_{\boldsymbol{H}}(V)(\mathbb{C}) = \boldsymbol{H}(\mathbb{Q})\backslash \boldsymbol{H}(\mathbb{A})/\mathbb{Z}_{\boldsymbol{H}}(\mathbb{R}) K_{\boldsymbol{H},\infty} V,$$

where $\mathbb{Z}_{\boldsymbol{H}}$ denotes the center of $\boldsymbol{H}$ and $K_{\boldsymbol{H},\infty} \subseteq \boldsymbol{H}(\mathbb{R})$ is the maximal compact defined as the product $\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$.

The embedding $\iota : \boldsymbol{H} \hookrightarrow \boldsymbol{G}$ induces a Shimura datum for $\boldsymbol{G}$ whose reflex field is $\mathbb{Q}$. For any sufficiently small compact open subgroup U of $\boldsymbol{G}(\mathbb{A}_f)$, denote by $\mathrm{Sh}_{\boldsymbol{G}}(U)$ the associated Shimura variety of dimension 6. We also write $\iota : \mathrm{Sh}_{\boldsymbol{H}}(U \cap \boldsymbol{H}) \hookrightarrow \mathrm{Sh}_{\boldsymbol{G}}(U)$ the closed embedding of codimension 3 induced by the group homomorphism $\iota : \boldsymbol{H} \hookrightarrow \boldsymbol{G}$.

Remark 2.4. If $E/\mathbb{Q}$ is a totally real cubic field extension of $\mathbb{Q}$ then one can analogously define $\boldsymbol{H} = \{g \in \mathrm{Res}_{E/\mathbb{Q}} \mathrm{GL}_{2,E} \mid \det(g) \in \boldsymbol{G}_m\}$ and there is a natural embedding $\iota : \boldsymbol{H} \hookrightarrow \boldsymbol{G}$ (see [Piatetski-Shapiro and Rallis 1987, §1] for details) inducing closed embeddings $\iota : \mathrm{Sh}_{\boldsymbol{H}}(U \cap \boldsymbol{H}) \hookrightarrow \mathrm{Sh}_{\boldsymbol{G}}(U)$ for sufficiently small open compact U. All our results up to Section 5 will hold for any real étale cubic algebra E over $\mathbb{Q}$. Our main interest in the case $E = \mathbb{Q} \times F$ for F a real étale quadratic algebra over $\mathbb{Q}$ is motivated by the integral representation of the spin L-function of $\boldsymbol{G}$ of [Pollack and Shah 2018].

2.7. *Cohomology of Siegel sixfolds.* Let π be a cuspidal automorphic representation of $\boldsymbol{G}(\mathbb{A})$ having nonzero fixed vectors by a neat compact open group $U \subseteq \boldsymbol{G}(\mathbb{A}_f)$. We assume that π has trivial central character and hence we regard it as a cuspidal automorphic representation of $\mathrm{PGSp}_6(\mathbb{A})$. Our purpose is to establish that, under mild assumptions, suitable localizations at π of cuspidal, L^2, inner Betti and Betti cohomologies coincide and are concentrated in the middle degree. The assumptions are the following:

(DS) The archimedean component π_∞ is a discrete series representation of $\mathrm{PGSp}_6(\mathbb{R})$.

(St) At a finite place p the component π_p is the Steinberg representation of $\mathrm{PGSp}_6(\mathbb{Q}_p)$.

Let us fix for the rest of this section $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3$ satisfying $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$ and $\sum \lambda_i \equiv 0 \pmod 2$. We will denote by V, without mentioning λ anymore, the irreducible algebraic representation of $\boldsymbol{G}$ of highest weight $(\lambda, 0)$. As V has trivial central character, it will be considered as an irreducible representation of PGSp_6. Then π_∞ belongs to the discrete series L-packet $P(V)$. As a consequence,

$$H^6(\mathfrak{g}, K_{\boldsymbol{G}}; \pi_\infty \otimes V) = \mathrm{Hom}_{K_{\boldsymbol{G}}}\big(\textstyle\bigwedge^6 \mathfrak{g}_{\mathbb{C}}/\mathrm{Lie}(K_{\boldsymbol{G}})_{\mathbb{C}}; \pi_\infty \otimes V\big) \neq 0,$$

where $K_{\boldsymbol{G}} = \mathbb{R}_+^\times K_\infty$.

There are natural inclusions of spaces of $\mathbb{C}$-valued functions

$$\mathcal{C}^\infty_{\mathrm{cusp}}(\boldsymbol{G}(\mathbb{Q})\backslash\boldsymbol{G}(\mathbb{A})) \subseteq \mathcal{C}^\infty_{\mathrm{rd}}(\boldsymbol{G}(\mathbb{Q})\backslash\boldsymbol{G}(\mathbb{A})) \subseteq \mathcal{C}^\infty_{(2)}(\boldsymbol{G}(\mathbb{Q})\backslash\boldsymbol{G}(\mathbb{A})) \subseteq \mathcal{C}^\infty(\boldsymbol{G}(\mathbb{Q})\backslash\boldsymbol{G}(\mathbb{A})),$$

where these spaces denote, respectively, the space of cuspidal square-integrable functions, rapidly decreasing functions, square-integrable functions and smooth functions, and

$$\mathcal{C}^\infty_{\mathrm{c/center}}(\boldsymbol{G}(\mathbb{Q})\backslash\boldsymbol{G}(\mathbb{A})) \subseteq \mathcal{C}^\infty_{\mathrm{rd}}(\boldsymbol{G}(\mathbb{Q})\backslash\boldsymbol{G}(\mathbb{A})),$$

where the first space is the space of compactly supported modulo the center functions (for the precise definition of these spaces, we refer to [Borel 1981]). Tensoring by V the inclusions above and applying the $(\mathfrak{g}, K_{\boldsymbol{G}})$-cohomology functor, we obtain the natural maps

$$\begin{array}{ccccccc} H^\bullet_{\mathrm{cusp}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}}) & \longrightarrow & H^\bullet_{\mathrm{rd}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}}) & \longrightarrow & H^\bullet_{(2)}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}}) & \longrightarrow & H^\bullet(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}}) \\ & & \Big\uparrow & & & & \\ & & H^\bullet_{\mathrm{c}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}}) & & & & \end{array}$$

where $\mathcal{V}_{\mathbb{C}}$ is the $\mathbb{C}$-local system associated to V. Let $H^\bullet_!(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})$ denote the image of $H^\bullet_c(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})$ in $H^\bullet(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})$. Let N denote the positive integer defined as the product of prime numbers ℓ such that π_ℓ is ramified. The fact that π_∞ is cohomological implies that there exists a number field L whose ring of integers $\mathcal{O}_L$ contains the eigenvalues of the spherical Hecke algebra $\mathcal{H}^{\mathrm{sph},N}$ away from N and with coefficients in $\mathbb{Z}$ acting on $\bigotimes'_{\ell\nmid N} \pi_\ell^{\boldsymbol{G}(\mathbb{Z}_\ell)}$. Let $\mathcal{H}_L^{\mathrm{sph},N}$ denote the spherical Hecke algebra away from N with coefficients in L, let $\theta_\pi : \mathcal{H}_L^{\mathrm{sph},N} \to L$ denote the Hecke character of π and let $\mathfrak{m}_\pi := \ker(\theta_\pi)$. Considering the localization at $\mathfrak{m}_\pi$ of the above cohomology groups, we have the following result.

Proposition 2.5. *Let π satisfy the hypotheses* (DS) *and* (St) *above. Then*

$$H^\bullet_{\mathrm{cusp}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})_{\mathfrak{m}_\pi} = H^\bullet_{(2)}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})_{\mathfrak{m}_\pi} = H^\bullet_!(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})_{\mathfrak{m}_\pi} = H^\bullet(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})_{\mathfrak{m}_\pi}.$$

Proof. By [Borel 1981, Theorem 5.3 & Corollary 5.5], the compositions of the horizontal maps

$$H^\bullet_{\mathrm{cusp}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}}) \hookrightarrow H^\bullet_*(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}}),$$

for $* \in \{\mathrm{rd}, (2), \varnothing\}$, are injections. By [Borel 1981, Theorem 5.2], one has an isomorphism

$$H^\bullet_{\mathrm{c}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}}) \cong H^\bullet_{\mathrm{rd}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}}).$$

Hence, if the equality $H^\bullet_{\mathrm{cusp}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})_{\mathfrak{m}_\pi} = H^\bullet_{(2)}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})_{\mathfrak{m}_\pi}$ holds, we have

$$H^\bullet_{\mathrm{cusp}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})_{\mathfrak{m}_\pi} = H^\bullet_{(2)}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})_{\mathfrak{m}_\pi} = H^\bullet_!(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})_{\mathfrak{m}_\pi}.$$

We show the former equality as follows. By [Borel 1980, §4],

$$H^\bullet_{(2)}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}}) = \bigoplus_{\sigma\subset L^2_d} \sigma_f^U \otimes H^\bullet(\mathfrak{g}, K_{\boldsymbol{G}}; \sigma_\infty \otimes V)^{m(\sigma)}, \tag{4}$$

where σ runs over the set of isomorphism classes of automorphic representations appearing in the discrete spectrum L^2_d of $L^2(Z(\mathbb{A})\boldsymbol{G}(\mathbb{Q})\backslash\boldsymbol{G}(\mathbb{A}))$. Similarly,

$$H^\bullet_{\mathrm{cusp}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}}) = \bigoplus_{\sigma\subset L^2_0} \sigma_f^U \otimes H^\bullet(\mathfrak{g}, K_{\boldsymbol{G}}; \sigma_\infty \otimes V)^{m_0(\sigma)},$$

where σ runs over the set of isomorphism classes of automorphic representations in the cuspidal spectrum $L^2_0 \subset L^2_d$. From (4), we can write

$$H^\bullet_{(2)}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})_{\mathfrak{m}_\pi} = \bigoplus_{\sigma=\sigma_\infty\otimes\sigma_f} \sigma_f^U \otimes H^\bullet(\mathfrak{g}, K_{\boldsymbol{G}}; \sigma_\infty \otimes V)^{m(\sigma)},$$

where $\sigma \in L^2_d$ is such that $\sigma_\ell^{\boldsymbol{G}(\mathbb{Z}_\ell)} \simeq \pi_\ell^{\boldsymbol{G}(\mathbb{Z}_\ell)} \neq 0$ at all $\ell \nmid N$. Notice that the latter implies that $\sigma_f^N \simeq \pi_f^N$, where for any automorphic representation τ we have denoted $\tau_f^N = \otimes_{\ell \nmid N} \tau_\ell$. By [Kret and Shin 2023, Lemma 8.1(2)], the Steinberg condition implies that the representation π_ℓ is tempered and unitary at each $\ell \nmid N$ (as π has trivial central character). Thus, if σ contributes nontrivially to the above sum, its local component at a finite place $\ell \nmid N$ is tempered. This implies that σ is necessarily cuspidal and thus appears in $H^\bullet_{\mathrm{cusp}}(\mathrm{Sh}_{\boldsymbol{G}}, \mathcal{V}_{\mathbb{C}})_{\mathfrak{m}_\pi}$ with multiplicity $m_0(\sigma) = m(\sigma)$. This last statement follows from the fact that any noncuspidal automorphic representation appearing in L^2_d is obtained as a residue of an Eisenstein series and in particular it is nontempered at every place (see [Labesse 1999, Proposition 4.5.4]). We are left to show that

$$H^\bullet_{(2)}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})_{\mathfrak{m}_\pi} = H^\bullet(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})_{\mathfrak{m}_\pi}.$$

Recall that Franke's decreasing filtration on the space of automorphic forms for $\boldsymbol{G}(\mathbb{A})$ (see [Waldspurger 1997, §4.7]) yields a spectral sequence $E_1^{p,q} \Rightarrow H^{p+q}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})$, where

$$E_1^{p,q} = \bigoplus_{\substack{(w,P)\in B(p)\\ \ell(w)\le p+q}} \bigoplus_{\sigma=\sigma_\infty\otimes\sigma_f} (\mathrm{Ind}_{P(\mathbb{A}_f)}^{\boldsymbol{G}(\mathbb{A}_f)} \sigma_f)^U \otimes H^{p+q-\ell(w)}(\mathfrak{m}, K_M; \sigma_\infty \otimes W^{w(\lambda+\rho)-\rho}),$$

where, for all $p \in \mathbb{Z}_{\ge 0}$, $B(p)$ denotes a certain subset depending on p of elements (w, P) (see [Waldspurger 1997, §4.8]), with $w \in \mathfrak{W}_{\boldsymbol{G}}$ and $P = M \cdot U_P$ a standard parabolic subgroup of $\boldsymbol{G}$, $W^{w(\lambda+\rho)-\rho}$ denotes the irreducible algebraic representation of M of highest weight $w(\lambda+\rho)-\rho$, and σ runs over the set of isomorphism classes of automorphic representations appearing in the discrete spectrum of $L^2(Z_M(\mathbb{A})M(\mathbb{Q})\backslash M(\mathbb{A}))$. By the proof of [Kret and Shin 2023, Lemma 8.1(1)], we have that $E^{p,q}_{1,\mathfrak{m}_\pi}$ are zero unless when $(w, P) = (1, \boldsymbol{G})$, in which case there exists a unique $p_0 \in \mathbb{Z}_{\ge 0}$, for which

$$E^{p,q}_{1,\mathfrak{m}_\pi} = \begin{cases} H^{p+q}_{(2)}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})_{\mathfrak{m}_\pi} & \text{if } p = p_0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the spectral sequence for the localization degenerates at the first page and gives

$$H^{p_0+\bullet}_{(2)}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})_{\mathfrak{m}_\pi} = E^{p_0,\bullet}_{1,\mathfrak{m}_\pi} = H^{p_0+\bullet}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})_{\mathfrak{m}_\pi}. \qquad \square$$

Proposition 2.6. *Let π satisfy the hypotheses* (DS) *and* (St) *above. Then*

$$H^\bullet(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})_{\mathfrak{m}_\pi} = H^6(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})_{\mathfrak{m}_\pi} \neq 0.$$

Proof. Suppose that τ_f contributes to $H^i(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})_{\mathfrak{m}_\pi}$. As we noted in the proof of Proposition 2.5, this implies that, for every $\ell \nmid N$, $\tau_\ell \simeq \pi_\ell$ is tempered and unitary (see [Kret and Shin 2023, Lemma 8.1(2)]). Let us fix $\ell \nmid N$; the action of the Frobenius correspondence on intersection cohomology Frob_ℓ on $IH^i(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})[\tau_f]$ and thus on $H^i(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})[\tau_f]$ is pure of weight i, i.e., its eigenvalues all have absolute value $\ell^{i/2}$ (see [Morel 2010, Remark 7.2.5]). On the other hand, by the congruence relation conjectured in [Blasius and Rogawski 1994, §6] and verified in [Wedhorn 2000], Frob_ℓ is a root of the Hecke polynomial

$$H_\ell(T) := \det(T - \ell^3 \mathrm{spin}(\mathrm{Fr}_\ell \ltimes \hat{g})),$$

which is a polynomial in T whose coefficients are elements in the coordinate ring of the set of Fr_ℓ-conjugacy classes of semisimple elements of $\widehat{\boldsymbol{G}}(\mathbb{C}) = \mathbf{GSpin}_7(\mathbb{C})$, for Fr_ℓ a Frobenius element in the Weil group of $\mathbb{Q}_\ell$. By the untwisted Satake isomorphism, we can see $H_\ell(T)$ as a polynomial with coefficients in the spherical Hecke algebra $\mathcal{H}(\boldsymbol{G}(\mathbb{Q}_\ell)//\boldsymbol{G}(\mathbb{Z}_\ell), \mathbb{Q})$ (see [Wedhorn 2000, (2.2.1) & Corollary (2.8)]) and thus we can denote by $H_\ell(T; \tau_\ell)$ the specialization of $H_\ell(T)$ to τ_ℓ, i.e.,

$$H_\ell(T; \tau_\ell) = \det(T - \ell^3 \mathrm{spin}(\phi_{\tau_\ell}(\mathrm{Fr}_\ell))),$$

where ϕ_{τ_ℓ} is the unramified Langlands parameter of τ_ℓ. The congruence relation gives that $H_\ell(\mathrm{Frob}_\ell; \tau_\ell)=0$ on $IH^\bullet(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})[\tau_f]$, which implies that the eigenvalues of Frob_ℓ on $IH^\bullet(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})[\tau_f]$ are a subset of the ones of $\ell^3\mathrm{spin}(\phi_{\tau_\ell}(\mathrm{Frob}_\ell))$. As τ_ℓ is tempered, all the eigenvalues of $\mathrm{spin}(\phi_{\tau_\ell}(\mathrm{Fr}_\ell))$ have absolute value equal to 1 (see [Gross 1998, §6]). Hence the eigenvalues of $\ell^3\mathrm{spin}(\phi_{\tau_\ell}(\mathrm{Fr}_\ell))$, and thus of Frob_ℓ, have all absolute value equal to ℓ^3. In particular, $H^i(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})[\tau_f]$ is zero unless $i = 6$. Finally, notice that $H^6(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})_{\mathfrak{m}_\pi} \neq 0$ as the assumption (DS) implies $H^6(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})[\pi_f] \neq 0$. □

Remark 2.7. The proof of Proposition 2.6 is similar to that of [Kret and Shin 2023, Proposition 8.2], where the proof is carried out with a trace formula argument.

2.8. ***Hodge theory.*** We keep the same notation as Section 2.7. In particular, $\pi = \pi_\infty \otimes \pi_f$ is a cuspidal automorphic representation of $\boldsymbol{G}$ with trivial central character which satisfies (DS) and (St), with $\pi_\infty \in P(V)$ for some irreducible algebraic representation V of $\boldsymbol{G}$ as above.

Let $\mathcal{V}$ denote the $\mathbb{Q}$-local system on $\mathrm{Sh}_{\boldsymbol{G}}(U)$ attached to V. We can take the π_f-isotypic component $H^6_{B,*}[\pi_f]$ of $H^6_*(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})$, where $* \in \{\varnothing, !\}$ and where $\mathcal{V}_{\mathbb{C}}$ denotes $\mathcal{V} \otimes_{\mathbb{Q}} \mathbb{C}$. Propositions 2.5 and 2.6 imply

$$H^\bullet_B[\pi_f] = H^\bullet_{B,!}[\pi_f] = H^6_{B,!}[\pi_f] \neq 0. \tag{5}$$

By [Blasius and Rogawski 1994, (2.3.1)] (see also [Shin and Templier 2014, Proposition 2.15]), if L is a sufficiently large number field, $H^6_B[\pi_f]$ appears as a subquotient of $H^6_!(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_L)$, where $\mathcal{V}_L$ denotes $\mathcal{V} \otimes_{\mathbb{Q}} L$. In particular, we have a projection

$$\mathrm{pr}_\pi : H^6(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_L)_{\mathfrak{m}_\pi}(n) \twoheadrightarrow H^6_B[\pi_f](n).$$

Since $H^6_!(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_L)$ is a pure L-Hodge structure of weight 6, we have

$$H^6_B[\pi_f] = \pi_f^U(L) \otimes M_B(\pi_f),$$

with $\pi_f^U(L)$ a realization of π_f^U over L and $M_B(\pi_f)$ a pure L-Hodge structure of weight 6. Thus we have a decomposition

$$M_B(\pi_f) \otimes \mathbb{C} = \bigoplus_{p+q=6} H^{p,q}(\pi_f).$$

Lemma 2.8. *Under the hypotheses* (DS) *and* (St),

$$\dim_{\mathbb{C}} H^{p,q}(\pi_f) = \begin{cases} 1 & \text{if } p \neq 3, \\ 2 & \text{if } p = 3. \end{cases}$$

In particular, $\dim_L M_B(\pi_f) = 8$.

Proof. Thanks to (5), we have that

$$H^6_B[\pi_f]\otimes\mathbb{C} = H^6_{B,!}[\pi_f]\otimes\mathbb{C} = H^6_{B,\mathrm{cusp}}[\pi_f]\otimes\mathbb{C};$$

hence

$$H^6_B[\pi_f]\otimes\mathbb{C} = \pi_f^U \otimes \bigoplus_{\sigma_\infty} H^6(\mathfrak{g}, K_{\boldsymbol{G}}; \sigma_\infty\otimes V)^{m(\sigma)},$$

where σ_∞ runs over the elements of the discrete series L-packet $P(V)$ of $\mathrm{PGSp}_6(\mathbb{R})$ and $m(\sigma)$ denotes the multiplicity of $\sigma=\sigma_\infty\otimes\pi_f$. Notice that $H^6(\mathfrak{g}, K_{\boldsymbol{G}}; \sigma_\infty\otimes V)$ equals

$$\mathrm{Hom}_{K_\infty}\big(\textstyle\bigwedge^6\mathfrak{g}_0/\mathfrak{k}, \sigma^1_\infty\otimes V\big)\oplus\mathrm{Hom}_{K_\infty}\big(\textstyle\bigwedge^6\mathfrak{g}_0/\mathfrak{k}, \bar\sigma^1_\infty\otimes V\big), \tag{6}$$

where we have denoted $\sigma_{\infty|_{G_0(\mathbb{R})}} = \sigma^1_\infty\oplus\bar\sigma^1_\infty$. According to [Borel and Wallach 1980, Theorem II.5.3(b)], each space in the decomposition above is one-dimensional. Moreover there exists a unique pair of integers $(r_{\sigma_\infty}, s_{\sigma_\infty})$ satisfying $r_{\sigma_\infty}+s_{\sigma_\infty}=6$ such that (6) equals

$$\mathrm{Hom}_{K_\infty}\big(\textstyle\bigwedge^{r_{\sigma_\infty}}\mathfrak{p}^+_{\mathbb{C}}\otimes\bigwedge^{s_{\sigma_\infty}}\mathfrak{p}^-_{\mathbb{C}}, \sigma^1_\infty\otimes V\big)\oplus\mathrm{Hom}_{K_\infty}\big(\textstyle\bigwedge^{s_{\sigma_\infty}}\mathfrak{p}^+_{\mathbb{C}}\otimes\bigwedge^{r_{\sigma_\infty}}\mathfrak{p}^-_{\mathbb{C}}, \bar\sigma^1_\infty\otimes V\big).$$

As we remarked in Section 2.5, the set $P(V)$ has four elements and is in bijection with the set of Hodge types up to conjugation. Since the Hodge structure in $H^6_{B,\mathrm{cusp}}[\pi_f]$ is induced by this splitting, we deduce that

$$\dim_{\mathbb{C}} H^{r_{\sigma_\infty}, s_{\sigma_\infty}}(\pi_f) = \begin{cases} m(\sigma) & \text{if } r_{\sigma_\infty}\neq 3,\\ 2m(\sigma) & \text{if } r_{\sigma_\infty}=3.\end{cases}$$

By [Kret and Shin 2023, Theorem 12.1], the multiplicity of σ is either 0 or 1, while thanks to [Kret and Shin 2023, Corollaries 8.4 & 12.4] the dimension of $M_B(\pi_f)$ equals 8. Hence $m(\sigma)=1$ for all $\sigma_\infty\in P(V)$, which concludes the proof. □

2.9. ***Absolute Hodge cohomology.*** Let us first recall some definitions from [Beĭlinson 1986]. A mixed $\mathbb{R}$-Hodge structure consists of a finite-dimensional $\mathbb{R}$-vector space $M_{\mathbb{R}}$ equipped with an increasing finite filtration W_* called the weight filtration and a decreasing finite filtration F^* on $M_{\mathbb{C}} = M_{\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{C}$ called the Hodge filtration, such that each pair $(\mathrm{Gr}^W_n M_{\mathbb{R}}, (\mathrm{Gr}^W_n M_{\mathbb{C}}, F^*))$ is a pure $\mathbb{R}$-Hodge structure of weight n [Deligne 1971, Définition 2.1.10]. The category of mixed $\mathbb{R}$-Hodge structures is an abelian category [Deligne 1971, Théorème (2.3.5)] and we denote it by $\mathrm{MHS}_{\mathbb{R}}$.

Definition 2.9. A real mixed $\mathbb{R}$-Hodge structure is given by a mixed $\mathbb{R}$-Hodge structure such that $M_{\mathbb{R}}$ is equipped with an involution F^*_∞ stabilizing the weight filtration and whose $\mathbb{C}$-antilinear complexification $\overline{F^*_\infty} = F^*_\infty\otimes c$, where c denotes the complex conjugation, defines an involution on $M_{\mathbb{C}}$ stabilizing the Hodge filtration.

We will refer to F^*_∞ as the real Frobenius and to $\overline{F^*_\infty}$ as the de Rham involution. We denote by $\mathrm{MHS}^+_{\mathbb{R}}$ the abelian category of real mixed Hodge $\mathbb{R}$-structures. For any pair of objects $M, N\in D(\mathrm{MHS}^+_{\mathbb{R}})$, one has $R\,\mathrm{Hom}_{\mathrm{MHS}^+_{\mathbb{R}}}(M, N) = R\,\mathrm{Hom}_{\mathrm{MHS}_{\mathbb{R}}}(M, N)^{\overline{F^*_\infty}}$, since taking invariants by $\overline{F^*_\infty}$ is an exact functor.

Definition 2.10. If $M = (M_{\mathbb{R}}, F_\infty^*) \in C(\mathrm{MHS}_{\mathbb{R}}^+)$ is a complex of real mixed $\mathbb{R}$-Hodge structure, its absolute Hodge cohomology is defined as

$$R\Gamma_{\mathcal{H}}(M) = R\,\mathrm{Hom}_{\mathrm{MHS}_{\mathbb{R}}}(\mathbb{R}(0), M_{\mathbb{R}}).$$

Its real absolute Hodge cohomology is defined as

$$R\Gamma_{\mathcal{H}/\mathbb{R}}(M) := R\,\mathrm{Hom}_{\mathrm{MHS}_{\mathbb{R}}^+}(\mathbb{R}(0), M)) = R\Gamma_{\mathcal{H}}(M_{\mathbb{R}})^{\overline{F_\infty^*}}.$$

The cohomology groups $H_B^i(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{R}})$, where $\mathcal{V}_{\mathbb{R}} = \mathcal{V} \otimes_{\mathbb{Q}} \mathbb{R}$, are equipped with a real Frobenius F_∞^* acting as the complex conjugation on (the complex points) $\mathrm{Sh}_{\boldsymbol{G}}(U)$ and on $\mathcal{V}_{\mathbb{R}}$, define real mixed $\mathbb{R}$-Hodge structures. This can be deduced directly from [Deligne 1971] since the cohomology with coefficients is a direct factor of the cohomology of a fiber product of the universal abelian variety of $\mathrm{Sh}_{\boldsymbol{G}}(U)$, or from the theory of mixed Hodge modules of [Saito 1990]. We let $M \in C(\mathrm{MHS}_{\mathbb{R}}^+)$ be the complex of real mixed $\mathbb{R}$-Hodge structures given by $\big(\bigoplus_{i\in\mathbb{N}} H_B^i(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{R}})[-i], F_\infty^*\big)$ and we define the absolute real Hodge cohomology $H_{\mathcal{H}}^7(\mathrm{Sh}_{\boldsymbol{G}}(U)/\mathbb{R}, \mathcal{V}_{\mathbb{R}}(4))$ of $\mathrm{Sh}_{\boldsymbol{G}}(U)$ and coefficients in $\mathcal{V}_{\mathbb{R}}(4)$ to be $H^1(R\Gamma_{\mathcal{H}/\mathbb{R}}(M(4)))$. Then we have the short exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{\mathrm{MHS}_{\mathbb{R}}^+}(\mathbb{R}(0), H_B^6(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{R}}(4))) \to H_{\mathcal{H}}^7(\mathrm{Sh}_{\boldsymbol{G}}(U)/\mathbb{R}, \mathcal{V}_{\mathbb{R}}(4)) \\ \to \mathrm{Hom}_{\mathrm{MHS}_{\mathbb{R}}^+}(\mathbb{R}(0), H_B^7(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{R}}(4))) \to 0. \end{aligned}$$

If $\pi = \pi_\infty \otimes \pi_f$ is as above, we denote by

$$H_{\mathcal{H}}^1(M(\pi_f)_{\mathbb{R}}(4)) := \big(H_{\mathcal{H}}^7(\mathrm{Sh}_{\boldsymbol{G}}(U)/\mathbb{R}, \mathcal{V}_{\mathbb{R}}(4)) \otimes L\big)[\pi_f]$$

the π_f-isotypical component.

Lemma 2.11. *Under the hypotheses* (DS) *and* (St), *we have a canonical short exact sequence of finite rank-free* $\mathbb{R} \otimes_{\mathbb{Q}} L$*-modules*

$$0 \to F^4 H^6_{\mathrm{dR}}[\pi_f] \to H^6_{\mathrm{B}}[\pi_f]^{F_\infty^\star = -1}(3) \to H^1_{\mathcal{H}}(M(\pi_f)_{\mathbb{R}}(4)) \to 0.$$

Moreover, *we have*

$$\dim_{\mathbb{R}\otimes_{\mathbb{Q}} L} H^1_{\mathcal{H}}(M(\pi_f)_{\mathbb{R}}(4)) = \dim_{\mathbb{C}} \pi_f^U.$$

Proof. It follows from the existence of the short exact sequence above and from Proposition 2.6 that we have a canonical isomorphism

$$H^1_{\mathcal{H}}(M(\pi_f)_{\mathbb{R}}(4)) \simeq \mathrm{Ext}^1_{\mathrm{MHS}_{\mathbb{R}}^+}(\mathbb{R}(0), H_B^6(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{R}}(4))[\pi_f]).$$

Hence, the first statement of the lemma follows as in [Lemma 2017, Lemma 4.11]. In particular, the map $F^4 H^6_{\mathrm{dR}}[\pi_f] \to H^6_{\mathrm{B}}[\pi_f]^{F_\infty^\star = -1}(3)$ is defined by the composition of

$$F^4 H^6_{\mathrm{dR}}[\pi_f] \to H^6_{\mathrm{dR}}[\pi_f] \otimes \mathbb{C} \simeq H_B^6(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})[\pi_f],$$

of the projection to $H^6_B(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})[\pi_f]^{\overline{F^*_\infty}=1}$, where $\overline{F^*_\infty}$ is the complexification $F^*_\infty \otimes c$, with c denoting the complex conjugation, and of the projection

$$H^6_B(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{C}})[\pi_f]^{\overline{F^*_\infty}=1} = H^6_{\mathrm{B}}[\pi_f]^{F^\star_\infty=-1}(3) \oplus H^6_{\mathrm{B}}[\pi_f]^{F^\star_\infty=1}(4) \to H^6_{\mathrm{B}}[\pi_f]^{F^\star_\infty=-1}(3).$$

Finally, by Lemma 2.8, we have that

$$\dim_{\mathbb{R}\otimes_{\mathbb{Q}}L} F^4 H^6_{\mathrm{dR}}[\pi_f] = 3 \dim_L \pi_f^U(L) = 3 \dim_{\mathbb{C}} \pi_f^U.$$

On the other hand,

$$\dim_{\mathbb{R}\otimes_{\mathbb{Q}}L} H^6_{\mathrm{B}}[\pi_f]^{F^\star_\infty=-1}(3) = (3 + h^{3,+}) \dim_{\mathbb{C}} \pi_f^U,$$

where $h^{3,+}$ is the dimension of the $\mathbb{C}$-vector space $\{x \in H^{3,3}(\pi_f) : F^*_\infty(x) = -x\}$ (see [Burgos Gil et al. 2024, §3.4.2]). By the proof of Lemma 2.8, we have $h^{3,+} = 1$, which implies the result. □

3. Construction of the motivic class

3.1. ***Cartan product.*** Before starting, we briefly recall some properties of the Cartan product of irreducible representations that will be needed (see [Sun 2017, §2.5] for more details). Let A denote either a connected compact Lie group or a semisimple algebraic group. Fix a Cartan subgroup of A and an orientation of the roots. Irreducible algebraic representations of A are parametrized by dominant weights. If λ and σ are two dominant weights with corresponding representations V^λ and V^σ, then the representation $V^{\lambda+\sigma}$ appears in $V^\lambda \otimes V^\sigma$ with multiplicity one. We denote it by $V^\lambda \cdot V^\sigma$ and we call it the Cartan component of $V^\lambda \otimes V^\sigma$. Clearly, the tensor product of two highest weight vectors maps to a corresponding highest weight vector. We denote by $v \otimes w \mapsto v \cdot w$ the projection from $V^\lambda \otimes V^\sigma$ to its Cartan component $V^\lambda \cdot V^\sigma$.

Lemma 3.1 [Sun 2017, Lemma 2.12]. *Every nonzero pure tensor in $V^\lambda \otimes V^\sigma$ projects nontrivially to the Cartan component.*

3.2. ***Branching laws.*** In what follows, we fix a totally real field F over which $\boldsymbol{H}$ splits. Since $\boldsymbol{H}$ is split over F, its finite-dimensional irreducible representations are determined by the highest weight theory and we can thus use the branching laws for algebraic representations from $\boldsymbol{G}$ to $\boldsymbol{H}$ established in [Cauchi and Rodrigues Jacinto 2020].

Lemma 3.2. *The $\boldsymbol{G}$-representation V^λ over F of highest weight $\lambda = (\lambda_1, \lambda_2, \lambda_3, c)$ contains the trivial $\boldsymbol{H}$-representation if and only if $c = 0$ and $\lambda_1 = \lambda_2 + \lambda_3$. When this holds the trivial representation of $\boldsymbol{H}$ appears in $(V^\lambda)_{|\boldsymbol{H}}$ with multiplicity $\lambda_2 - \lambda_3 + 1$.*

Proof. From [Cauchi and Rodrigues Jacinto 2020, Lemma 2.10], the sum of all irreducible sub-$\boldsymbol{H}$-representations of V^λ isomorphic (up to a twist) to $\mathrm{Sym}^{(k,0,0)}$ for some $k \geq 0$ is given by

$$\bigoplus_{\substack{k=|\lambda_1-\lambda_2-\lambda_3| \\ k\equiv|\lambda| \ (\mathrm{mod}\ 2)}}^{\lambda_1-\lambda_2+\lambda_3} r \cdot \mathrm{Sym}^{(k,0,0)} \otimes \det{}^{\frac{1}{2}(|\lambda|-k)}$$

for $r = \lambda_2 - \lambda_3 + 1$. From this we deduce that V^λ contains the trivial $\boldsymbol{H}$-representation with multiplicity $r = \lambda_2 - \lambda_3 + 1$ if and only if $\lambda_1 - \lambda_2 - \lambda_3 = 0$. □

It will be useful to construct explicitly generators of the trivial $\boldsymbol{H}$-representations inside V^λ given by the branching laws. We achieve this by constructing some vectors in the representations $V^{(1,1,0,0)}$ and $V^{(2,1,1,0)}$ and then by taking their Cartan product. From now on, all the representations are defined over F. Moreover, since the branching laws are determined by the restriction to the derived subgroups, in the following we work with the groups

$$\boldsymbol{H}_0 := \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 \hookrightarrow \boldsymbol{H}_0' := \mathrm{SL}_2 \times \mathrm{Sp}_4 \hookrightarrow \boldsymbol{G}_0 = \mathrm{Sp}_6.$$

Recall that we associate to any $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2$ such that $\lambda_1 \geq \lambda_2 \geq 0$, the irreducible Sp_4-representation with highest weight λ. Applying the branching laws [Cauchi and Rodrigues Jacinto 2020, Proposition 2.8], we get the following decompositions of representations of $\boldsymbol{H}_0'$:

$$\begin{aligned} V^{(1,1,0)} &= (\mathrm{Sym}^0 \boxtimes V^{(0,0)}) \oplus (\mathrm{Sym}^0 \boxtimes V^{(1,1)}) \oplus (\mathrm{Sym}^1 \boxtimes V^{(1,0)}),\\ V^{(2,1,1)} &= (\mathrm{Sym}^0 \boxtimes V^{(1,1)}) \oplus (\mathrm{Sym}^0 \boxtimes V^{(2,0)}) \oplus (\mathrm{Sym}^1 \boxtimes V^{(1,0)}) \oplus (\mathrm{Sym}^1 \boxtimes V^{(2,1)}) \oplus (\mathrm{Sym}^2 \boxtimes V^{(1,1)}). \end{aligned}$$

By Lemma 3.2, $V^{(1,1,0)}$ contains two copies of the trivial $\boldsymbol{H}_0$-representation, each of which lies respectively in $\mathrm{Sym}^0 \boxtimes V^{(0,0)}$ and $\mathrm{Sym}^0 \boxtimes V^{(1,1)}$, while $V^{(2,1,1)}$ contains a unique trivial $\boldsymbol{H}_0$-representation appearing in $\mathrm{Sym}^0 \boxtimes V^{(1,1)}$. Using these facts, we can explicitly define generators of these three trivial representations of $\boldsymbol{H}_0$.

Let V be the standard representation of $\boldsymbol{G}_0$ with its symplectic basis $\langle e_1, e_2, e_3, f_1, f_2, f_3\rangle$ given in Section 2.6. According to our choice of embedding $\boldsymbol{H}_0' \hookrightarrow \boldsymbol{G}_0$, $\langle e_1, f_1\rangle$ (resp. $\langle e_2, e_3, f_2, f_3\rangle$) defines a basis of the standard representation of SL_2 (resp. Sp_4). We first recall how one can realize the representations $V^{(1,1,0)}$ and $V^{(1,1,1)}$. As explained in [Fulton and Harris 1991, §17.1], $V^{(1,1,0)}$ is realized inside $\bigwedge^2 V$ as the complement of the $\boldsymbol{G}_0$-invariant subspace generated by the vector $e_1 \wedge f_1 + e_2 \wedge f_2 + e_3 \wedge f_3$ corresponding to the symplectic form or, in other words, as the kernel of the map $\bigwedge^2 V \to V$ sending $v_1 \wedge v_2$ to $\psi(v_1, v_2)$. By [Fulton and Harris 1991, Theorem 17.5], the irreducible representation $V^{(1,1,1)}$ is identified with the kernel of the map $\varphi : \bigwedge^3 V \to V,\ v_1 \wedge v_2 \wedge v_3 \mapsto \sum_{i<j, k\neq i,j} \psi(v_i, v_j)(-1)^{i-j+1} v_k$.

Lemma 3.3. *Let $F(0)$ denote the trivial $\boldsymbol{H}_0$-representation. We have*

$$\begin{aligned} v &:= e_2 \wedge f_2 - e_3 \wedge f_3 \in F(0) \subseteq \mathrm{Sym}^0 \boxtimes V^{(1,1)} \subseteq V^{(1,1,0)},\\ w &:= e_2 \wedge f_2 + e_3 \wedge f_3 - 2 e_1 \wedge f_1 \in F(0) \subseteq \mathrm{Sym}^0 \boxtimes V^{(0,0)} \subseteq V^{(1,1,0)},\\ z &:= z_1 - z_2 \in F(0) \subseteq \mathrm{Sym}^0 \boxtimes V^{(1,1)} \subseteq V^{(2,1,1)}, \end{aligned}$$

where

$$\begin{aligned} z_1 &:= e_1 \cdot (f_1 \wedge e_2 \wedge f_2 - f_1 \wedge e_3 \wedge f_3),\\ z_2 &:= f_1 \cdot (e_1 \wedge e_2 \wedge f_2 - e_1 \wedge e_3 \wedge f_3), \end{aligned}$$

and $\cdot$ denotes the Cartan product.

Proof. The vector v is obtained from the highest weight vector $e_1 \wedge e_2$ in $V^{(1,1,0)}$ by applying the composition $X_{(0,1,-1)} \circ X_{(0,-2,0)} \circ X_{(-1,0,1)}$, where $X_{(-1,0,1)}, X_{(0,-2,0)}, X_{(0,1,-1)} \in \mathfrak{sp}_6$ denote the weight vectors for $\lambda(-1,0,1)$, $\lambda(0,-2,0)$, and $\lambda(0,1,-1)$ respectively (see [Fulton and Harris 1991, §16.1] for the precise description). Moreover, the vector $X_{(-1,0,1)}(e_1 \wedge e_2) = -e_2 \wedge e_3$ is of weight $(0,1,1)$, which appears only in the component $\mathrm{Sym}^0 \boxtimes V^{(1,1)}$, and $X_{(0,1,-1)}, X_{(0,-2,0)} \in \mathfrak{sp}_4 \subseteq \mathfrak{sp}_6$ so v still lies inside $\mathrm{Sym}^0 \boxtimes V^{(1,1)}$. The vector w is $\boldsymbol{H}_0'$-invariant and therefore it generates the only trivial $\boldsymbol{H}_0'$-representation in $V^{(1,1,0)}$. We now explain the definition of z. Note that $e_1 \in V^{(1,0,0)}$ and $f_1 \wedge e_2 \wedge f_2 - f_1 \wedge e_3 \wedge f_3 \in V^{(1,1,1)}$. Thus, by the properties of the Cartan product

$$V^{(1,0,0)} \otimes V^{(1,1,1)} = V^{(1,1,0)} \oplus V^{(2,1,1)} \to V^{(2,1,1)}, \quad v_1 \otimes v_2 \mapsto v_1 \cdot v_2,$$

z_1 is a nonzero vector in $V^{(2,1,1)}$ by Lemma 3.1. The vector z_1 is fixed by $\{I_2\} \times \mathrm{SL}_2^2$, but not by $\mathrm{SL}_2 \times \{I_2\} \times \{I_2\}$, however, as it is easy to verify, we have that

$$z = z_1 + h \cdot z_1 = z_1 - z_2 \in F(0) \subset V^{(2,1,1)}, \quad \text{with } h = \left(\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right), I_2, I_2\right),$$

generates the unique trivial $\boldsymbol{H}_0$-representation of $V^{(2,1,1)}$. □

Lemma 3.4. *Let $\lambda = (\lambda_2 + \lambda_3, \lambda_2, \lambda_3, 0)$ with $\lambda_2 \geq \lambda_3 \geq 0$. For each $\lambda_2 \geq \mu \geq \lambda_3$, the vector*

$$v^{[\lambda,\mu]} := v^{\lambda_2 - \mu} \cdot w^{\mu - \lambda_3} \cdot z^{\lambda_3} \in F(0) \subseteq (V^\lambda)_{|\boldsymbol{H}}$$

realizes a distinct copy of the trivial representation $F(0)$ of $\boldsymbol{H}$ inside $(V^\lambda)_{|\boldsymbol{H}}$.

Proof. For $p, q, r \in \mathbb{N}$, we have

$$v^p \cdot w^q \cdot z^r \in F(0) \subseteq \mathrm{Sym}^0 \boxtimes V^{(p+r,p+r)} \subseteq V^{(p+q+2r,p+q+r,r)}.$$

The vectors v, w, z are $\boldsymbol{H}$-highest weight vectors, and thus $v^{[\lambda,\mu]}$ is too. We are left to show that each of the vectors is different. This follows from the fact that each $v^{[\lambda,\mu]}$ lies in $\mathrm{Sym}^0 \boxtimes V^{(\lambda_2+\lambda_3-\mu,\lambda_2+\lambda_3-\mu)} \otimes \nu^{\mu-\lambda_2-\lambda_3}$ and these representations are all different as μ varies. □

3.3. ***The motivic class.*** As in the section above, we fix a totally real field F such that $\boldsymbol{H}$ splits over F. For a smooth quasiprojective scheme S over a field of characteristic zero, let $\mathrm{CHM}_L(S)$ denote the tensor category of relative Chow motives over S with coefficients in a number field L and denote by $M : \mathrm{Var}/S \to \mathrm{CHM}_L(S)$ the contravariant functor from the category of smooth projective schemes over S to the category of relative Chow motives over S (see [Ancona 2015, §2.1]). By [Deninger and Murre 1991, Corollary 3.2], if A/S is an abelian scheme of relative dimension g, there is a decomposition $M(A) = \bigoplus_{i=1}^{2g} h^i(A)$ in $\mathrm{CHM}_L(S)$. Let G temporarily denote one of the groups $\boldsymbol{H}$ or $\boldsymbol{G}$, and denote by $\mathrm{Rep}_F(G)$ the category of finite-dimensional algebraic representations of G defined over F. Ancona [2015] constructed an additive functor

$$\mu_U^G : \mathrm{Rep}_F(G) \to \mathrm{CHM}_F(\mathrm{Sh}_G(U)),$$

where U is a sufficiently small open compact subgroup of $G(\mathbb{A}_f)$. We recall some of its properties.

Proposition 3.5 [Ancona 2015, Théorème 8.6]. *The functor μ_U^G respects duals, tensor products and satisfies the following properties.*

(1) *If V is the standard representation of G, then $\mu_U^G(V) = h^1(\mathscr{A}_G)$, where $\mathscr{A}_G$ is the universal abelian scheme over $\mathrm{Sh}_G(U)$.*

(2) *If $\nu : G \to \boldsymbol{G}_m$ is the multiplier, then $\mu_U^G(\nu)$ is the Lefschetz motive $F(-1)$.*

(3) *For a G-representation V defined over F, the Betti realization of $\mu_U^G(V)$ is the local system $\mathcal{V}_F$ associated to the vector bundle*

$$G(\mathbb{Q})\backslash(X_G \times V \times (G(\mathbb{A}_f)/U)) \to \mathrm{Sh}_G(U)(\mathbb{C}).$$

(4) *For any prime v of F above ℓ and G-representation V, the v-adic étale realization $\mathcal{V}_v$ of $\mu_U^G(V)$ is the étale sheaf associated to $V \otimes_F F_v$, with U acting on the left via $U \hookrightarrow G(\mathbb{A}_f) \to G(\mathbb{Q}_\ell)$.*

Definition 3.6. Let V^λ be the finite-dimensional irreducible algebraic representation over $\mathbb{Q}$ of $\boldsymbol{G}$ of highest weight λ. We denote by $\mathscr{V}_F^\lambda$ the relative Chow motive associated to $V^\lambda \otimes F$.

Let $U \subset \boldsymbol{G}(\mathbb{A}_f)$ be a sufficiently small compact open subgroup and let $U' = U \cap \boldsymbol{H}(\mathbb{A}_f)$. Recall that we have a closed embedding $\iota : \mathrm{Sh}_{\boldsymbol{H}}(U') \hookrightarrow \mathrm{Sh}_{\boldsymbol{G}}(U)$ which is of codimension 3. Let V^λ the algebraic representation of $\boldsymbol{G}$ (over F) of highest weight $\lambda = (\lambda_1, \lambda_2, \lambda_3, c)$ such that $\lambda_1 = \lambda_2 + \lambda_3$ and $c = 0$. Using the branching laws of Lemma 3.2 and [Torzewski 2020, Theorem 1.2], we get the following (see [Cauchi and Rodrigues Jacinto 2020, Proposition 2.17]).

Proposition 3.7. *For any $\lambda_2 \geq \mu \geq \lambda_3$, we have a Gysin morphism*

$$\iota_*^{[\lambda,\mu]} : H^0_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{H}}(U'), F(0)) \to H^6_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathscr{V}_F^\lambda(3)),$$

corresponding to the embedding of $F(0) \subset \iota^* V^\lambda$ given by the $\boldsymbol{H}$-trivial vector $v^{[\lambda,\mu]}$ of Lemma 3.4.

Definition 3.8. We let $\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{M}} \in H^6_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathscr{V}_F^\lambda(3))$ be the image by $\iota_*^{[\lambda,\mu]}$ of

$$\mathbf{1}_{\mathrm{Sh}_{\boldsymbol{H}}(U')} \in \mathrm{CH}^0(\mathrm{Sh}_{\boldsymbol{H}}(U'))_F = H^0_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{H}}(U'), F(0)).$$

3.4. *Realizations.*

3.4.1. *Étale realization.* Let $\mathfrak{l}$ be a prime of F above ℓ. We have an étale cycle class map

$$\mathrm{cl}_{\text{ét}} : \mathrm{H}^6_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{G}}(\mathrm{U}), \mathscr{V}^\lambda_{\mathrm{F}}(3)) \to \mathrm{H}^6_{\text{ét}}(\mathrm{Sh}_{\boldsymbol{G}}(\mathrm{U}), \mathcal{V}^\lambda_{\mathfrak{l}}(3)) \to \mathrm{H}^6_{\text{ét}}(\mathrm{Sh}_{\boldsymbol{G}}(\mathrm{U})_{\overline{\mathbb{Q}}}, \mathcal{V}^\lambda_{\mathfrak{l}}(3))^{\mathrm{G}_{\mathbb{Q}}},$$

where the last arrow is the natural map obtained from the Hochschild–Serre spectral sequence. We define the following.

Definition 3.9. We let $\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\text{ét}} := \mathrm{cl}_{\text{ét}}(\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{M}}) \in \mathrm{H}^6_{\text{ét}}(\mathrm{Sh}_{\boldsymbol{G}}(\mathrm{U})_{\overline{\mathbb{Q}}}, \mathcal{V}^\lambda_{\mathfrak{l}}(3))^{\mathrm{G}_{\mathbb{Q}}}$.

Remark 3.10. • Notice that $\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\text{ét}}$ equals to the image of $\mathbf{1} \in H^0_{\text{ét}}(\mathrm{Sh}_{\boldsymbol{H}}(U')_{\overline{\mathbb{Q}}}, F_{\mathfrak{l}}(0))$ via the étale Gysin map

$$\iota^{[\lambda,\mu]}_{\text{ét},*} : H^0_{\text{ét}}(\mathrm{Sh}_{\boldsymbol{H}}(U')_{\overline{\mathbb{Q}}}, F_{\mathfrak{l}}(0)) \to H^0_{\text{ét}}(\mathrm{Sh}_{\boldsymbol{H}}(U')_{\overline{\mathbb{Q}}}, \iota^*\mathcal{V}^\lambda_{\mathfrak{l}}) \to H^6_{\text{ét}}(\mathrm{Sh}_{\boldsymbol{G}}(U)_{\overline{\mathbb{Q}}}, \mathcal{V}^\lambda_{\mathfrak{l}}(3)).$$

- As the representation V^λ is self dual, we have a Galois equivariant perfect pairing

$$H^6_{\text{ét},c}(\mathrm{Sh}_{\boldsymbol{G}}(U)_{\overline{\mathbb{Q}}}, \mathcal{V}^\lambda_{\mathfrak{l}}(3)) \times H^6_{\text{ét}}(\mathrm{Sh}_{\boldsymbol{G}}(U)_{\overline{\mathbb{Q}}}, \mathcal{V}^\lambda_{\mathfrak{l}}(3)) \to F_{\mathfrak{l}}(0).$$

Hence, by duality, $\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\text{ét}}$ determines a map

$$H^6_{\text{ét},c}(\mathrm{Sh}_{\boldsymbol{G}}(U)_{\overline{\mathbb{Q}}}, \mathcal{V}^\lambda_{\mathfrak{l}}(3)) \to F_{\mathfrak{l}}(0).$$

3.4.2. *Betti realizations.* As in the previous subsection, we define the class

$$\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},B} \in H^6_B(\mathrm{Sh}_{\boldsymbol{G}}(U)(\mathbb{C}), \mathcal{V}^\lambda_F(3))$$

as the image of $\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{M}}$ via the Betti cycle class map

$$\mathrm{cl_B} : \mathrm{H}^6_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{G}}(\mathrm{U}), \mathscr{V}^\lambda_{\mathrm{F}}(3)) \to \mathrm{H}^6_{\mathrm{B}}(\mathrm{Sh}_{\boldsymbol{G}}(\mathrm{U})(\mathbb{C}), \mathcal{V}^\lambda_{\mathrm{F}}(3)).$$

Note that, as F is totally real, the image satisfies

$$\mathrm{Im}(\mathrm{cl_B}) \subset \mathrm{H}^6_{\mathrm{B}}(\mathrm{Sh}_{\boldsymbol{G}}(\mathrm{U})(\mathbb{C}), \mathcal{V}^\lambda_{\mathbb{R}}(3))^{\mathrm{F}^*_\infty=1},$$

where F^*_∞ denotes the composition of the map induced by complex conjugation on the $\mathbb{C}$-points of $\mathrm{Sh}_{\boldsymbol{G}}(U)$ with complex conjugation on the coefficients.

3.4.3. *Absolute Hodge realizations.* Let $H^6_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathscr{V}^\lambda_F(3))^0 = \ker(\mathrm{cl}_B)$ denote the subgroup of homologically trivial classes and let $H^6_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathscr{V}^\lambda_F(3))_{\mathrm{hom}}$ denote the quotient

$$H^6_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathscr{V}^\lambda_F(3))/H^6_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathscr{V}^\lambda_F(3))^0.$$

Note that when $\lambda_2 = \lambda_3 = 0$, i.e., the representation V^λ is the trivial representation, then

$$H^6_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathscr{V}^\lambda_F(3)) = H^6_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{G}}(U), F(3)) = \mathrm{CH}^3(\mathrm{Sh}_{\boldsymbol{G}}(U))_F$$

is the usual Chow group of 3-codimensional cycles modulo rational equivalence and the space

$$H^6_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathscr{V}^\lambda_F(3))_{\mathrm{hom}} = \mathrm{N}^3(\mathrm{Sh}_{\boldsymbol{G}}(U))_F$$

is the space of 3-codimensional cycles modulo homological equivalence, with coefficients in F. In this section, we define a natural injective map

$$H^6_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathscr{V}^\lambda_F(3))_{\mathrm{hom}} \to H^7_{\mathcal{H}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}^\lambda_{\mathbb{R}}(4)). \tag{7}$$

The definition is similar to the one for smooth projective varieties (see [Schneider 1988, §5]) and we recall it for the convenience of the reader. The cycle class map is an injection

$$\mathrm{cl}_B : H^6_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathscr{V}^\lambda_F(3))_{\mathrm{hom}} \to H^6_B(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_F(3))^{F^*_\infty=1} \cap H^6_B(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}^\lambda_{\mathbb{C}}(3))^{0,0},$$

where $H^6_B(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}^\lambda_{\mathbb{C}}(3))^{0,0}$ denotes the subspace of

$$W_0 H^6_B(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}^\lambda_{\mathbb{C}}(3)) = \mathrm{Gr}^W_0 H^6_B(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}^\lambda_{\mathbb{C}}(3))$$

of vectors which have Hodge type $(0,0)$. The composite of the inclusions

$$H^6_B(\mathrm{Sh}_{\boldsymbol{G}}(U),\mathcal{V}_F(3))^{F^*_\infty=1}\cap H^6_B(\mathrm{Sh}_{\boldsymbol{G}}(U),\mathcal{V}^\lambda_{\mathbb{C}}(3))^{0,0} \hookrightarrow W_0H^6_{\mathrm{dR}}(\mathrm{Sh}_{\boldsymbol{G}}(U),\mathcal{V}_{\mathbb{R}}(3))\hookrightarrow W_2H^6_{\mathrm{dR}}(\mathrm{Sh}_{\boldsymbol{G}}(U),\mathcal{V}_{\mathbb{R}}(3))=W_0H^6_{\mathrm{dR}}(\mathrm{Sh}_{\boldsymbol{G}}(U),\mathcal{V}_{\mathbb{R}}(4))$$

and of the projection

$$W_0H^6_{\mathrm{dR}}(\mathrm{Sh}_{\boldsymbol{G}}(U),\mathcal{V}_{\mathbb{R}}(4)) \to W_0H^6_B(\mathrm{Sh}_{\boldsymbol{G}}(U),\mathcal{V}_{\mathbb{R}}(4))^+\backslash W_0H^6_{\mathrm{dR}}(\mathrm{Sh}_{\boldsymbol{G}}(U),\mathcal{V}_{\mathbb{R}}(4))/F^0W_0H^6_{\mathrm{dR}}(\mathrm{Sh}_{\boldsymbol{G}}(U),\mathcal{V}_{\mathbb{R}}(4))$$

is injective. As the last space above is canonically isomorphic to

$$\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0),H^6_B(\mathrm{Sh}_{\boldsymbol{G}}(U),\mathcal{V}_{\mathbb{R}}(4))),$$

we obtain a natural injective map

$$H^6_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{G}}(U),\mathscr{V}^\lambda_F(3))_{\mathrm{hom}}\to \mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0),H^6_B(\mathrm{Sh}_{\boldsymbol{G}}(U),\mathcal{V}_{\mathbb{R}}(4))).$$

Composing this map with the canonical injection

$$\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0),H^6_B(\mathrm{Sh}_{\boldsymbol{G}}(U),\mathcal{V}_{\mathbb{R}}(4)))\to H^7_{\mathcal{H}}(\mathrm{Sh}_{\boldsymbol{G}}(U),\mathcal{V}^\lambda_{\mathbb{R}}(4))$$

we obtain the map (7). We denote by $\overline{\mathrm{cl}}_{\mathcal{H}}:H^6_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{G}}(U),\mathscr{V}^\lambda_F(3))\to H^7_{\mathcal{H}}(\mathrm{Sh}_{\boldsymbol{G}}(U),\mathcal{V}^\lambda_{\mathbb{R}}(4))$ the composition of the map (7) with the projection $H^6_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{G}}(U),\mathscr{V}^\lambda_F(3))\to H^6_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{G}}(U),\mathscr{V}^\lambda_F(3))_{\mathrm{hom}}$.

Definition 3.11. We define

$$\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{H}}:=\overline{\mathrm{cl}}_{\mathcal{H}}(\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{M}})\in H^7_{\mathcal{H}}(\mathrm{Sh}_{\boldsymbol{G}}(U),\mathcal{V}^\lambda_{\mathbb{R}}(4)).$$

Remark 3.12. Let π be a cuspidal automorphic representation of $\mathrm{PGSp}_6(\mathbb{A})$ which satisfies the hypotheses of Lemma 2.11 and let S be a finite set of places containing the ramified places of π_f and ∞. By the conjectures of Beilinson and Tate and the local calculations of Gross and Savin [1998], there should exist a cubic étale algebra $E/\mathbb{Q}$ such that the cycle $\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{H}}$, with $\boldsymbol{H}$ defined by $E/\mathbb{Q}$, and their Hecke translates are expected to generate $H^1_{\mathcal{H}}(M(\pi_f)_{\mathbb{R}}(4))$ when $\mathrm{ord}_{s=1}L^S(s,\pi,\mathrm{Spin})=-1$. Assuming the nonvanishing of the archimedean integral, Corollary 5.15 confirms this expectation.

4. Construction of the differential form and pairing with the motivic class

The purpose of this section is to study the Betti and Hodge realizations of the cycle constructed in Section 3.3 by relating their pairing with a suitable cuspidal harmonic differential form to an automorphic period.

4.1. ***Test vectors.*** Recall that the discrete series L-packets for $\mathrm{PGSp}_6(\mathbb{R})$ have four elements, each indexed by a Hodge type (and its conjugate). Let π denote a cuspidal automorphic representation of $\mathrm{PGSp}_6(\mathbb{A})$ for which π_∞ is the discrete series of Hodge type (3,3) in the L-packet of V^λ, where $\lambda=(\lambda_1,\lambda_2,\lambda_3,0)$

and $\lambda_1 = \lambda_2 + \lambda_3$. This translates into saying that π is a cuspidal automorphic representation of $\boldsymbol{G}(\mathbb{A})$ with trivial central character for which

$$H^6(\mathfrak{g}, K_{\boldsymbol{G}}; \pi_\infty \otimes V^\lambda) \neq 0,$$

and such that $\pi_\infty|_{\boldsymbol{G}_0(\mathbb{R})} = \pi^{3,3}_{\infty,1} \oplus \bar{\pi}^{3,3}_{\infty,1}$ is the direct sum of the discrete series representations of respective Harish-Chandra parameters $(\lambda_2+2, \lambda_3+1, -\lambda_1-3)$ and $(\lambda_1+3, -\lambda_3-1, -\lambda_2-2)$. Recall that these discrete series contain with multiplicity one their minimal K_∞-types $\tau_{(\lambda_2+2,\lambda_3+2,-\lambda_1-4)}$ and $\tau_{(\lambda_1+4,-\lambda_3-2,-\lambda_2-2)}$ respectively. On the other hand, as K_∞-representations we have

$$\textstyle\bigwedge^6 \mathfrak{p}_{\mathbb{C}} \supseteq \bigwedge^3 \mathfrak{p}^+_{\mathbb{C}} \otimes \bigwedge^3 \mathfrak{p}^-_{\mathbb{C}} = \bigoplus_i \tau_i \supseteq \tau_{(2,2,-4)} \oplus \tau_{(4,-2,-2)},$$

where the equality expresses the decomposition of the tensor product into irreducible K_∞-representations. This fact will be useful to construct an element in

$$H^6(\mathfrak{g}, K_{\boldsymbol{G}}; \pi_\infty \otimes V^\lambda) = \mathrm{Hom}_{K_\infty}\big(\textstyle\bigwedge^6 \mathfrak{p}_{\mathbb{C}}, \pi_\infty \otimes V^\lambda\big) \simeq \mathrm{Hom}_{K_\infty}\big(\textstyle\bigwedge^6 \mathfrak{p}_{\mathbb{C}} \otimes V^\lambda, \pi_\infty\big),$$

where the last equality follows from the fact that V^λ is self-dual. Before stating the next result, let us fix the following data:

- A highest weight vector Ψ_∞ of the minimal K_∞-type $\tau_{(\lambda_2+2,\lambda_3+2,-\lambda_1-4)}$ of $\pi^{3,3}_{\infty,1}$.
- A highest weight vector $\overline{\Psi}_\infty$ of the minimal K_∞-type $\tau_{(\lambda_1+4,-\lambda_3-2,-\lambda_2-2)}$ of $\bar{\pi}^{3,3}_{\infty,1}$.
- A highest weight vector $X_{(2,2,-4)}$ of $\tau_{(2,2,-4)}$.
- A highest weight vector $X_{(4,-2,-2)}$ of $\tau_{(4,-2,-2)}$.
- A highest weight vector $v^{\lambda'}$ of $\tau_{\lambda'} \subseteq V^\lambda$, where $\tau_{\lambda'}$ denotes the irreducible algebraic K_∞-representations of highest weight $\lambda' = (\lambda_2, \lambda_3, -\lambda_1)$.
- A highest weight vector $v^{\bar{\lambda}'}$ of $\tau_{\bar{\lambda}'} \subseteq V^\lambda$, where $\tau_{\bar{\lambda}'}$ denotes the irreducible algebraic K_∞-representations of highest weight $\bar{\lambda}' = (\lambda_1, -\lambda_3, -\lambda_2)$.

Lemma 4.1. *The spaces* $\mathrm{Hom}_{K_\infty}\big(\bigwedge^6 \mathfrak{p}_{\mathbb{C}} \otimes V^\lambda, \pi^{3,3}_{\infty,1}\big)$ *and* $\mathrm{Hom}_{K_\infty}\big(\bigwedge^6 \mathfrak{p}_{\mathbb{C}} \otimes V^\lambda, \bar{\pi}^{3,3}_{\infty,1}\big)$ *are of dimension 1 and the elements*

$$\omega_{\Psi_\infty} \in \mathrm{Hom}_{K_\infty}\big(\textstyle\bigwedge^6 \mathfrak{p}_{\mathbb{C}} \otimes V^\lambda, \pi^{3,3}_{\infty,1}\big), \quad \omega_{\overline{\Psi}_\infty} \in \mathrm{Hom}_{K_\infty}\big(\textstyle\bigwedge^6 \mathfrak{p}_{\mathbb{C}} \otimes V^\lambda, \bar{\pi}^{3,3}_{\infty,1}\big)$$

defined by

$$\omega_{\Psi_\infty}(X_{(2,2,-4)} \otimes v^{\lambda'}) = \Psi_\infty, \quad \omega_{\overline{\Psi}_\infty}(X_{(4,-2,-2)} \otimes v^{\bar{\lambda}'}) = \overline{\Psi}_\infty$$

are generators of these spaces.

Proof. This is a consequence of [Borel and Wallach 1980, Theorem II.5.3 b)] and its proof. □

4.2. ***Restriction to*** $\boldsymbol{H}$**.** Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, 0)$, with $\lambda_1 = \lambda_2 + \lambda_3$ and let V^λ be as above. Let $\mathfrak{h}$ denote the Lie algebra of $\boldsymbol{H}(\mathbb{R})$ and $\mathfrak{k}_{\boldsymbol{H}}$ the maximal compact modulo the center $K_{\boldsymbol{H}}$. Observe that via the embedding $\iota : \boldsymbol{H}(\mathbb{R}) \hookrightarrow \boldsymbol{G}(\mathbb{R})$, the group $K_{\boldsymbol{H}}$ is isomorphic to T_∞. One has a Cartan decomposition $\mathfrak{h}_\mathbb{C} = \mathfrak{k}_{\boldsymbol{H},\mathbb{C}} \oplus \mathfrak{p}_{\boldsymbol{H},\mathbb{C}}$, where $\mathfrak{p}_{\boldsymbol{H},\mathbb{C}}$ is six-dimensional and is spanned by the noncompact root spaces. We fix once and for all a generator X_0 of the one-dimensional $\mathbb{C}$-vector space $\bigwedge^6 \mathfrak{p}_{\boldsymbol{H},\mathbb{C}} \subseteq \bigwedge^6 \mathfrak{p}_\mathbb{C}$ as in [Burgos Gil et al. 2024, §5.2]. The main result of this section is the following.

Theorem 4.2. *Let ω_{Ψ_∞} and $\omega_{\overline{\Psi}_\infty}$ be the elements of $\mathrm{Hom}_{K_\infty}\big(\bigwedge^6 \mathfrak{p}_\mathbb{C} \otimes V^\lambda, \pi_\infty\big)$ defined in Lemma 4.1. Let X_0 be as above and let v be any $\boldsymbol{H}$-invariant vector in V^λ. Then*

$$\omega_{\Psi_\infty}(X_0 \otimes v) \neq 0, \quad \omega_{\overline{\Psi}_\infty}(X_0 \otimes v) \neq 0.$$

The proof of Theorem 4.2 will be constructive and occupies the rest of this section. We start by recalling the following result.

Lemma 4.3 [Burgos Gil et al. 2024, Lemma 5.4]. *Let X_0 be as above. Then the image of X_0 by*

$$\textstyle\bigwedge^6 \mathfrak{p}_{\boldsymbol{H},\mathbb{C}} \to \bigwedge^6 \mathfrak{p}_\mathbb{C} \to \bigwedge^3 \mathfrak{p}^+_\mathbb{C} \otimes \bigwedge^3 \mathfrak{p}^-_\mathbb{C} \to \tau_{(2,2,-4)},$$

where the first map is induced by the embedding $\boldsymbol{H} \to \boldsymbol{G}$ and the second and the third maps are the natural projections, is nonzero.

We next study the interaction between the branching laws of V^λ to the subgroup $\boldsymbol{H}$ of $\boldsymbol{G}$ and to its maximal compact subgroup. More precisely, we show that the $\boldsymbol{H}$-invariant vectors constructed in Lemma 3.4 project nontrivially to $\tau_{\lambda'}$ and $\tau_{\overline{\lambda}'}$ and moreover that their projections form a basis of the corresponding $(0,0,0)$-weight spaces for the action of T_∞.

Lemma 4.4. *Let $\tau_{\lambda'}$ and $\tau_{\overline{\lambda}'}$ be the irreducible algebraic sub-K_∞-representations of V^λ of highest weight $\lambda' = (\lambda_2, \lambda_3, -\lambda_1)$ and $\overline{\lambda}' = (\lambda_1, -\lambda_3, -\lambda_2)$. Then the weight $(0,0,0)$ appears in both $\tau_{\lambda'}$ and $\tau_{\overline{\lambda}'}$ with multiplicity $\lambda_2 - \lambda_3 + 1$.*

Proof. Let $n_0(\lambda')$ denote the multiplicity of the weight $(0,0,0)$ in $\tau_{\lambda'}$. The Kostant multiplicity formula reads as

$$n_0(\lambda') = \sum_{w \in \mathfrak{W}_{K_\infty}} (-1)^{\ell(w)} P(w(\lambda' + \rho_{K_\infty}) - \rho_{K_\infty}),$$

where $\rho_{K_\infty} = \frac{1}{2}\sum_{\alpha \in \Delta_c^+} \alpha = (1,0,-1)$ and the function $\mu \mapsto P(\mu)$ calculates the number of ways for which the weight μ can be expressed as a linear combination

$$\alpha(e_1 - e_2) + \beta(e_1 - e_3) + \gamma(e_2 - e_3),$$

with $\alpha, \beta, \gamma \in \mathbb{Z}_{\geq 0}$ (see [Fulton and Harris 1991]). Using this formula, it is a tedious but straightforward calculation to verify that $n_0(\lambda') = \lambda_2 - \lambda_3 + 1$ and the same for $\overline{\lambda}' = w_8 \lambda'$. □

According to Lemma 4.4, there are $\lambda_2 - \lambda_3 + 1$ linearly independent vectors of weight $(0,0,0)$ in $\tau_{\lambda'}$. We now show that these weight vectors correspond one to one to the $\boldsymbol{H}$-invariant vectors of Lemma 3.2.

Lemma 4.5. *Let v, w be the vectors of $V^{(1,1,0)}$ and let z be the vector of $V^{(2,1,1)}$ defined in Lemma 3.3. The irreducible algebraic representation $\tau_{(1,0,-1)}$ (resp. $\tau_{(1,1,-2)}$ and $\tau_{(2,-1,-1)}$) appear in the restriction* *of* $V^{(1,1,0)}$ *(resp. of* $V^{(1,1,1)}$*) to* K_∞ *with multiplicity* 1. *Moreover, we have* $v, w \in \tau_{(1,0,-1)} \subseteq V^{(1,1,0)}$, *and* $z \in \tau_{(1,1,-2)} \oplus \tau_{(2,-1,-1)} \subseteq V^{(1,1,1)}$, *with* z *projecting nontrivially to each factor of this decomposition.*

Proof. First observe that $v, w \in V^{(1,1,0)}$ and $z \in V^{(2,1,1)}$ are vectors of weight $(0,0,0)$ both for the split and the compact tori of $\boldsymbol{G}_0(\mathbb{R})$. Indeed these vectors are fixed (up to a constant) by the matrix J sending the noncompact torus $\boldsymbol{T}_0$ to the compact torus T_∞ defined in Section 2.2. Using branching laws from $\boldsymbol{G}_0(\mathbb{R})$ to K_∞, we have a decomposition of K_∞-representations

$$V^{(1,1,0)} = \tau_{(1,1,0)} \oplus \tau_{(1,0,-1)} \oplus \tau_{(0,-1,-1)}.$$

The weight $(0,0,0)$ appears only in $\tau_{(1,0,-1)}$ and with multiplicity 2. Since it has also multiplicity 2 in $V^{(1,1,0)}$, we deduce that $\{v, w\}$ forms a basis for the $(0,0,0)$-eigenspace of $\tau_{(1,0,-1)}$. On the other hand, we have

$$V^{(2,1,1)} = \tau_{(-1,-1,-2)} \oplus \tau_{(1,-1,-2)} \oplus \tau_{(1,1,0)} \oplus \tau_{(1,1,-2)} \oplus \tau_{(1,0,-1)} \oplus \tau_{(2,-1,-1)} \oplus \tau_{(2,1,-1)} \oplus \tau_{(2,1,1)} \oplus \tau_{(0,-1,-1)}.$$

The weight $(0,0,0)$ only appears in $\tau_{(1,1,-2)} \oplus \tau_{(1,0,-1)} \oplus \tau_{(2,-1,-1)}$, which implies that

$$z \in \tau_{(1,1,-2)} \oplus \tau_{(1,0,-1)} \oplus \tau_{(2,-1,-1)}.$$

Notice that the decomposition of the standard representation of $\boldsymbol{G}_0$

$$V = \tau_{(1,0,0)} \oplus \tau_{(0,0,-1)}$$

of K_∞-representations can be realized by picking the basis $\{v_1, v_2, v_3, w_1, w_2, w_3\}$, where $v_r := e_r + i f_r$ and $w_r := i e_r + f_r$. The set $\{v_r\}_{1 \le r \le 3}$ (resp. $\{w_r\}_{1 \le r \le 3}$) defines a basis for $\tau_{(1,0,0)}$ (resp. $\tau_{(0,0,-1)}$). We now write z in terms of this basis. By using the relations

$$e_r = \tfrac{1}{2} v_r - \tfrac{i}{2} w_r, \quad f_r = \tfrac{1}{2} w_r - \tfrac{i}{2} v_r,$$

we have that

$$e_1 \otimes f_1 \wedge (e_2 \wedge f_2 - e_3 \wedge f_3) - f_1 \otimes e_1 \wedge (e_2 \wedge f_2 - e_3 \wedge f_3)$$

is equal to

$$\tfrac{1}{4}\big(v_1 \otimes w_1 \wedge (v_2 \wedge w_2 - v_3 \wedge w_3) - w_1 \otimes v_1 \wedge (v_2 \wedge w_2 - v_3 \wedge w_3)\big).$$

Thus,

$$z = z_1 - z_2 = \tfrac{1}{4}\big(v_1 \cdot w_1 \wedge (v_2 \wedge w_2 - v_3 \wedge w_3) - w_1 \cdot v_1 \wedge (v_2 \wedge w_2 - v_3 \wedge w_3)\big).$$

Notice that the vector $w_1 \wedge (v_2 \wedge w_2 - v_3 \wedge w_3) \in V^{(1,1,1)}$ is of weight $(-1,0,0)$ for T_∞, while $v_1 \wedge (v_2 \wedge w_2 - v_3 \wedge w_3) \in V^{(1,1,1)}$ is of weight $(1,0,0)$ for T_∞. As

$$V^{(1,1,1)} = \tau_{(1,1,1)} \oplus \tau_{(1,-1,-1)} \oplus \tau_{(1,1,-1)} \oplus \tau_{(-1,-1,-1)},$$

and as the weight $(-1,0,0)$ appears only in $\tau_{(1,-1,-1)}$ and $(1,0,0)$ only in $\tau_{(1,1,-1)}$, we have that

$$w_1 \wedge (v_2 \wedge w_2 - v_3 \wedge w_3) \in \tau_{(1,-1,-1)}, \quad v_1 \wedge (v_2 \wedge w_2 - v_3 \wedge w_3) \in \tau_{(1,1,-1)}.$$

By the properties of the Cartan product, the vector $s_1 := v_1 \cdot w_1 \wedge (v_2 \wedge w_2 - v_3 \wedge w_3)$ is nonzero in $\tau_{(2,-1,-1)}$, while $s_2 := w_1 \cdot v_1 \wedge (v_2 \wedge w_2 - v_3 \wedge w_3)$ is nonzero in $\tau_{(1,1,-2)}$. This shows that the vector $z \in V^{(2,1,1)}$ lives in $\tau_{(2,-1,-1)} \oplus \tau_{(1,1,-2)}$, thus finishing the proof. □

Proposition 4.6. *The set* $\{\mathrm{pr}_{\tau_{\lambda'}}(v^{[\lambda,\mu]})\}_\mu$ *(resp.* $\{\mathrm{pr}_{\tau_{\bar{\lambda}'}}(v^{[\lambda,\mu]})\}_\mu$*) forms a basis of the weight* $(0,0,0)$*-eigenspace of* $\tau_{\lambda'} \subset V^\lambda$ *(resp.* $\tau_{\bar{\lambda}'} \subset V^\lambda$*).*

Proof. Recall that we have defined

$$v^{[\lambda,\mu]} := v^{\lambda_2-\mu} \cdot w^{\mu-\lambda_3} \cdot z^{\lambda_3} \in F(0) \subseteq (V^\lambda)_{|\boldsymbol{H}}.$$

By Lemma 4.5, we have that $v, w \in \tau_{(1,0,-1)} \subseteq V^{(1,1,0)}$ so that, for any $\lambda_3 \le \mu \le \lambda_2$, we have $v^{\lambda_2-\mu} \otimes w^{\mu-\lambda_3} \in \tau_{(1,0,-1)}^{\otimes\lambda_2-\lambda_3}$ and we deduce that the projection of $v^{\lambda_2-\mu} \cdot w^{\mu-\lambda_3} \in V^{(\lambda_2-\lambda_3,\lambda_2-\lambda_3,0)}$ to $\tau_{(\lambda_2-\lambda_3,0,\lambda_3-\lambda_2)}$ coincides with their Cartan product with respect to K_∞. Moreover, each of these projections is nonzero because of Lemma 3.1. Since the vectors

$$v^{\lambda_2-\mu} \cdot w^{\mu-\lambda_3} \in \tau_{(\lambda_2-\lambda_3,0,\lambda_3-\lambda_2)}$$

are all different as they live in different $\boldsymbol{H}_0'$ subrepresentations (see the proof of Lemma 3.4), we conclude that they span the $\lambda_2 - \lambda_3 + 1$-dimensional weight $(0,0,0)$-eigenspace of $\tau_{(\lambda_2-\lambda_3,0,\lambda_3-\lambda_2)}$. We now show that z^{λ_3} projects nontrivially to both $\tau_{(2\lambda_3,-\lambda_3,-\lambda_3)}$ and $\tau_{(\lambda_3,\lambda_3,-2\lambda_3)}$. Notice that, as the weights $(2\lambda_3,-\lambda_3,-\lambda_3)$ and $(\lambda_3,\lambda_3,-2\lambda_3)$ are extremal in $V^{(2\lambda_3,\lambda_3,\lambda_3)}$ and appear uniquely, we have a commutative diagram

$$\begin{array}{ccc} (V^{(2,1,1)})^{\otimes\lambda_3} & \xrightarrow{\quad\cdot\quad} & V^{(2\lambda_3,\lambda_3,\lambda_3)} \\ {\scriptstyle (\mathrm{pr}_1,\mathrm{pr}_1')}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ (\tau_{(2,-1,-1)})^{\otimes\lambda_3} \oplus (\tau_{(1,1,-2)})^{\otimes\lambda_3} & \xrightarrow{\cdot} & \tau_{(2\lambda_3,-\lambda_3,-\lambda_3)} \oplus \tau_{(\lambda_3,\lambda_3,-2\lambda_3)} \end{array}$$

where the horizontal arrows are the Cartan projections and the vertical arrows are the natural projections given by the decomposition of $V^{(2r,r,r)}$ as K_∞-representations. Thanks to the commutativity of the diagram, we know that the vector $z^{\otimes\lambda_3} \in (V^{(2,1,1)})^{\otimes\lambda_3}$ maps to

$$\mathrm{pr}_2(z^{\lambda_3}) = \mathrm{pr}_1(z)^{\lambda_3} + \mathrm{pr}_1'(z)^{\lambda_3} = s_1^{\lambda_3} + s_2^{\lambda_3},$$

where s_1, s_2 are as in Lemma 4.5. This shows, again by Lemma 3.1, that each $v^{[\lambda,\mu]}$ projects nontrivially to both $\tau_{\lambda'}$ and $\tau_{\bar{\lambda}'}$ and that each of these projections are different by Lemma 3.4. Indeed,

$$\mathrm{pr}_{\tau_{\lambda'}}(v^{[\lambda,\mu]}) = v^{\lambda_2-\mu} \cdot w^{\mu-\lambda_3} \cdot s_1^{\lambda_3}, \quad \mathrm{pr}_{\tau_{\bar{\lambda}'}}(v^{[\lambda,\mu]}) = v^{\lambda_2-\mu} \cdot w^{\mu-\lambda_3} \cdot s_2^{\lambda_3}.$$

By Lemma 4.4, this means that $\{\mathrm{pr}_{\tau_{\lambda'}}(v^{[\lambda,\mu]})\}_\mu$ (resp. $\{\mathrm{pr}_{\tau_{\bar{\lambda}'}}(v^{[\lambda,\mu]})\}_\mu$) defines a basis of the weight $(0,0,0)$-eigenspace of $\tau_{\lambda'}$ (resp. $\tau_{\bar{\lambda}'}$). This finishes the proof. □

We can now conclude the proof of Theorem 4.2

Proof of Theorem 4.2. By construction, the map ω_{Ψ_∞} factors through $\tau_{\lambda'+(2,2,-4)} \subseteq \tau_{(2,2,-4)} \otimes \tau_{\lambda'}$. Lemma 4.3 shows that the projection of X_0 to $\tau_{(2,2,-4)}$ is nonzero, while Proposition 4.6 shows that $\mathrm{pr}_{\tau_{\lambda'}}(v^{[\lambda,\mu]})$ is nonzero. Since $\tau_{\lambda'+(2,2,-4)}$ is the Cartan product of $\tau_{(2,2,-4)}$ and $\tau_{\lambda'}$, we deduce from Lemma 3.1 that the image of the pure tensor $\mathrm{pr}_{\tau_{(2,2,-4)}}(X_0) \otimes \mathrm{pr}_{\tau_{\lambda'}}(v^{[\lambda,\mu]})$ is nonzero. □

4.3. *The pairing.* Let π denote a cuspidal automorphic representation of $\mathrm{PGSp}_6(\mathbb{A})$ for which π_∞ is the discrete series of Hodge type (3,3) in the L-packet of V^λ with $\lambda = (\lambda_2+\lambda_3, \lambda_2, \lambda_3, 0)$. Let $\Psi = \Psi_\infty \otimes \Psi_f$ denote a cusp form in $\pi = \pi_\infty \otimes \pi_f$. We assume that Ψ_∞ is a highest weight vector of the minimal K_∞-type $\tau_{(\lambda_2+2,\lambda_3+2,-\lambda_1-4)}$ of $\pi_\infty|_{\boldsymbol{G}_0(\mathbb{R})}$. We let $[\omega_{\Psi_\infty}] \in H^6(\mathfrak{g}, K_{\boldsymbol{G}}; \pi_\infty \otimes V^\lambda)$ be the cohomology class of the harmonic differential form ω_{Ψ_∞} defined in Lemma 4.1. We also assume that $\Psi_f \in V_{\pi_f}$ is U-invariant. Then we have $[\omega_\Psi] := [\omega_{\Psi_\infty} \otimes \Psi_f] \in H^6(\mathfrak{g}, K_{\boldsymbol{G}}; \pi^U \otimes V^\lambda)$.

Lemma 4.7. *There is a Hecke-equivariant inclusion*

$$H^6(\mathfrak{g}, K_{\boldsymbol{G}}; \pi^U \otimes V^\lambda) \subset H^6_{\mathrm{dR},c}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}^\lambda_{\mathbb{C}}).$$

Moreover, if π_w is the Steinberg representation for some finite place w, such an inclusion is unique.

Proof. Let $\mathcal{C}^\infty_{\mathrm{rd}}(\boldsymbol{G}(\mathbb{Q})\backslash\boldsymbol{G}(\mathbb{A})/U, V^\lambda)$ denote the space of V^λ-valued $\mathcal{C}^\infty$-functions on the double quotient $\boldsymbol{G}(\mathbb{Q})\backslash\boldsymbol{G}(\mathbb{A})/U$ which, together with all their right $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$-derivatives, are rapidly decreasing in the sense of [Harris 1990]. As π is cuspidal and cusp forms are rapidly decreasing, we have $H^6(\mathfrak{g}, K_{\boldsymbol{G}}; \pi_\infty \otimes V^\lambda_{\mathbb{C}})^{m(\pi)} \otimes \pi_f^U \subset H^6(\mathfrak{g}, K_{\boldsymbol{G}}; \mathcal{C}^\infty_{\mathrm{rd}}(\boldsymbol{G}(\mathbb{Q})\backslash\boldsymbol{G}(\mathbb{A})/U, V^\lambda))$. Thus the result follows from the fact that, according to [Borel 1981, Theorem 5.2] (see also [Harris 1990, Theorem 1.4.1]), there exists a canonical Hecke equivariant isomorphism $H^6_{\mathrm{dR},c}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}^\lambda) \simeq H^6(\mathfrak{g}, K_{\boldsymbol{G}}; \mathcal{C}^\infty_{\mathrm{rd}}(\boldsymbol{G}(\mathbb{Q})\backslash\boldsymbol{G}(\mathbb{A})/U, V^\lambda))$. Finally, if π_w is Steinberg at a finite place w, we have, as in Lemma 2.8, that $m(\pi) = 1$. □

4.3.1. *The pairing in Betti cohomology.* Poincaré duality is a perfect pairing

$$\langle\ ,\ \rangle : H^6_B(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}^\lambda_F(3)) \times H^6_{B,c}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}^\lambda_F) \to F(-3),$$

which is a morphism of mixed F-Hodge structures. Fix the choice of a measure dh on $\boldsymbol{H}(\mathbb{A})$ as follows. For each finite place p, we take the Haar measure dh_p on $\boldsymbol{H}(\mathbb{Q}_p)$ that assigns volume 1 to $\boldsymbol{H}(\mathbb{Z}_p)$. For the archimedean place, we let $X_0 \in \bigwedge^6 \mathfrak{p}_{\boldsymbol{H},\mathbb{C}}$ be the generator fixed at the beginning of Section 4.2. The choice of X_0 induces an equivalence between top differential forms on $X_{\boldsymbol{H}} = \boldsymbol{H}(\mathbb{R})/K_{\boldsymbol{H},\infty}$ and invariant measures dh_∞ on $\boldsymbol{H}(\mathbb{R})$ assigning measure one to $K_{\boldsymbol{H},\infty}$ (see [Harris 1997, p. 83] for details). We let dh_∞ denote the measure associated in this way to the pullback of $\iota^{[\lambda,\mu]*}\omega_\Psi$ to $X_{\boldsymbol{H}}$ and we then define $dh = dh_\infty \prod_p dh_p$.

Proposition 4.8. *We have*

$$\langle \mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},B}, [\omega_\Psi]\rangle = \frac{h_{U'}}{(2\pi i)^3 \cdot \mathrm{vol}(U')} \int_{\boldsymbol{H}(\mathbb{Q})\mathbb{Z}_{\boldsymbol{G}}(\mathbb{A})\backslash\boldsymbol{H}(\mathbb{A})} A^{[\lambda,\mu]} \cdot \Psi(h)\, dh,$$

where $h_{U'} = 4^{-1}|\mathbb{Z}_{\boldsymbol{G}}(\mathbb{Q})\backslash\mathbb{Z}_{\boldsymbol{G}}(\mathbb{A}_f)/(\mathbb{Z}_{\boldsymbol{G}}(\mathbb{A}_f) \cap U')|$ and $A^{[\lambda,\mu]} \in U(\mathfrak{k}_{\mathbb{C}})$ is an element for which $A^{[\lambda,\mu]} \cdot \Psi_\infty = \omega_{\Psi_\infty}(X_0 \otimes v^{[\lambda,\mu]})$.

Proof. By [Borel 1981, Corollary 5.5], there exists a $\mathcal{V}^\lambda$-valued rapidly decreasing differential form η of degree five on $\mathrm{Sh}_{\boldsymbol{G}}(U)$ such that $\omega_c := \omega_\Psi + d\eta$ is compactly supported. We have

$$\begin{aligned}\langle \mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},B}, [\omega_\Psi]\rangle &= \langle \mathrm{cl}_B(\iota^{[\lambda,\mu]}_* \mathbf{1}_{\mathrm{Sh}_{\boldsymbol{H}}(U')}), [\omega_c]\rangle \\ &= \langle \iota^{[\lambda,\mu]}_* \mathrm{cl}_B(\mathbf{1}_{\mathrm{Sh}_{\boldsymbol{H}}(U')}), [\omega_c]\rangle \\ &= \langle \mathrm{cl}_B(\mathbf{1}_{\mathrm{Sh}_{\boldsymbol{H}}(U')}), \iota^{[\lambda,\mu]*}[\omega_c]\rangle \\ &= \frac{1}{(2\pi i)^3}\int_{\mathrm{Sh}_{\boldsymbol{H}}(U')} \iota^{[\lambda,\mu]*}\omega_c,\end{aligned}$$

where $\iota^{[\lambda,\mu]*} : \iota^* V^\lambda \to F(0)$ is the $\boldsymbol{H}$-equivariant projection dual to the inclusion $\iota^{[\lambda,\mu]} : F(0) \to \iota^* V^\lambda$ defined by $1 \mapsto v^{[\lambda,\mu]} \in V^\lambda$, where $v^{[\lambda,\mu]}$ is the vector defined in Lemma 3.4. According to [Borel 1981, §5.6], we have

$$\int_{\mathrm{Sh}_{\boldsymbol{H}}(U')} \iota^{[\lambda,\mu]*}\, d\eta = 0.$$

Hence, using Theorem 4.2 we have

$$\begin{aligned}\langle \mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},B}, [\omega_\Psi]\rangle &= \frac{1}{(2\pi i)^3}\int_{\mathrm{Sh}_{\boldsymbol{H}}(U')} \iota^{[\lambda,\mu]*}\omega_\Psi \\ &= \frac{1}{(2\pi i)^3}\int_{\mathrm{Sh}_{\boldsymbol{H}}(U')} \omega_\Psi(X_0 \otimes v^{[\lambda,\mu]})(h)\, dh \\ &= \frac{1}{(2\pi i)^3}\int_{\boldsymbol{H}(\mathbb{Q})\backslash \boldsymbol{H}(\mathbb{A})/\mathbb{Z}_{\boldsymbol{H}}(\mathbb{R})K_{\boldsymbol{H},\infty}U'} A^{[\lambda,\mu]}\cdot \Psi(h)\, dh \\ &= \frac{h_{U'}}{(2\pi i)^3}\int_{\boldsymbol{H}(\mathbb{Q})\mathbb{Z}_{\boldsymbol{G}}(\mathbb{A})\backslash \boldsymbol{H}(\mathbb{A})/U'} A^{[\lambda,\mu]}\cdot \Psi(h)\, dh \\ &= \frac{h_{U'}}{(2\pi i)^3\cdot \mathrm{vol}(U')}\int_{\boldsymbol{H}(\mathbb{Q})\mathbb{Z}_{\boldsymbol{G}}(\mathbb{A})\backslash \boldsymbol{H}(\mathbb{A})} A^{[\lambda,\mu]}\cdot \Psi(h)\, dh,\end{aligned}$$

where the third equality follows from Theorem 4.2 as $\omega_{\Psi_\infty}(X_0 \otimes v^{[\lambda,\mu]})$ is nonzero and thus it is of the form $A^{[\lambda,\mu]}\cdot\Psi_\infty$, for some $A^{[\lambda,\mu]} \in U(\mathfrak{k}_\mathbb{C})$, because Ψ_∞ is the highest weight vector of the minimal K_∞-type $\tau_{(\lambda_2+2,\lambda_3+2,-\lambda_1-4)}$. Moreover, the fourth equality follows from the fact that Ψ is fixed by the center of $\boldsymbol{G}$, whence, using that $|\mathbb{Z}_{\boldsymbol{H}}(\mathbb{R})/(\mathbb{Z}_{\boldsymbol{G}} \cap \boldsymbol{H})(\mathbb{R})| = 4$, the constant $h_{U'}$ is equal to $4^{-1}|\mathbb{Z}_{\boldsymbol{G}}(\mathbb{Q})\backslash\mathbb{Z}_{\boldsymbol{G}}(\mathbb{A}_f)/(\mathbb{Z}_{\boldsymbol{G}}(\mathbb{A}_f)\cap U')|$. □

Remark 4.9. In view of Proposition 4.8, we immediately notice that if π is not $\boldsymbol{H}$-distinguished, namely

$$\int_{\boldsymbol{H}(\mathbb{Q})\mathbb{Z}_{\boldsymbol{G}}(\mathbb{A})\backslash \boldsymbol{H}(\mathbb{A})} \varphi_\pi(h)\, dh = 0$$

for any cusp form φ_π in the space of π, we have that $\mathrm{pr}_{\pi^\vee}\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},B} = 0$. As we discuss later in Section 8, the $\boldsymbol{H}$-distinguishability is related to the property of π being a (functorial) lift from G_2, which is (conjecturally) equivalent to the fact that the spin L-function of π has a pole at $s = 1$.

4.3.2. *The pairing in absolute Hodge cohomology.* Let

$$\langle\ ,\ \rangle_{\mathcal{H}} : H^7_{\mathcal{H}}(\mathrm{Sh}_{\boldsymbol{G}}(U)/\mathbb{R}, \mathcal{V}_{\mathbb{R}}(4)) \times H^6_{\mathcal{H},c}(\mathrm{Sh}_{\boldsymbol{G}}(U)/\mathbb{R}, \mathcal{V}_{\mathbb{R}}(3)) \to \mathbb{R}$$

be the natural pairing between absolute Hodge cohomology and compactly supported cohomology as constructed in [Beĭlinson 1986, §4.2]. In order to ease notation, we will denote by $H^*_{B,c}(i)$ and $H^*_B(i)$ the cohomology groups $H^*_{B,c}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_F(i))$ and $H^*_B(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_F(i))$, respectively. Recall from Section 2.9 that absolute Hodge cohomology and compactly supported cohomology live in exact sequences

$$0 \to \mathrm{Ext}^1_{\mathrm{MHS}_{\mathbb{R}}}(\mathbb{R}(0), H^6_B(4)) \to H^7_{\mathcal{H}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{R}}(4)) \to \mathrm{Hom}_{\mathrm{MHS}_{\mathbb{R}}}(\mathbb{R}(0), H^7_B(4)) \to 0, \tag{8}$$

$$0 \to \mathrm{Ext}^1_{\mathrm{MHS}_{\mathbb{R}}}(\mathbb{R}(0), H^5_{B,c}(3)) \to H^6_{\mathcal{H},c}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{R}}(3)) \to \mathrm{Hom}_{\mathrm{MHS}_{\mathbb{R}}}(\mathbb{R}(0), H^6_{B,c}(3)) \to 0, \tag{9}$$

which are deduced from the description of absolute Hodge cohomology as a cone of a diagram of complexes of Hodge structures. Let $[\omega_\Psi] \in H^6_{B,c}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}^{\lambda}_{\mathbb{R}}(3))$ be the compactly supported cohomology class of the harmonic differential form ω_Ψ. This class is of Hodge type (3, 3) and hence, since $W_0 H^6_{B,c}(3) = H^6_{B,c}(3)$, it naturally lives in the space $\mathrm{Hom}_{\mathrm{MHS}_{\mathbb{R}}}(\mathbb{R}(0), H^6_{B,c}(3)) = W_0 H_{B,c}(3) \cap F^0 H_{B,c}(3)_{\mathbb{C}}$. Denote by $\widetilde{[\omega_\Psi]}$ any lift of $[\omega_\Psi]$ in $H^6_{\mathcal{H},c}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{R}}(3))$ via the surjection of the exact sequence (9).

Proposition 4.10. *The pairing $\langle \mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{H}}, \widetilde{[\omega_\Psi]}\rangle_{\mathcal{H}}$ depends only on $[\omega_\Psi]$ and not on the choice of lift. We denote this value by $\langle \mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{H}}, [\omega_\Psi]\rangle_{\mathcal{H}}$. Moreover, the pairing is given by the natural Poincaré duality pairing. In particular, we have*

$$\langle \mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{H}}, [\omega_\Psi]\rangle_{\mathcal{H}} = \frac{h_{U'}}{(2\pi i)^3 \cdot \mathrm{vol}(U')} \int_{\boldsymbol{H}(\mathbb{Q})\mathbb{Z}_{\boldsymbol{G}}(\mathbb{A})\backslash \boldsymbol{H}(\mathbb{A})} A^{[\lambda,\mu]} \cdot \Psi(h)\, dh.$$

Proof. We give a sketch of the proof and we refer to [Beĭlinson 1986] or to [Burgos Gil et al. 2007, §5.1] for the facts used here. It follows from the description of the pairing between absolute Hodge cohomology and compactly supported cohomology given in [Beĭlinson 1986, §4.2] that, since our cycle class $\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{H}}$ lives in the subspace $\mathrm{Ext}^1_{\mathrm{MHS}_{\mathbb{R}}}(\mathbb{R}(0), H^6_B(4))$ of $H^7_{\mathcal{H}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{R}}(4))$, the map

$$\langle [\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{H}}], -\rangle : H^6_{\mathcal{H},c}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathcal{V}_{\mathbb{R}}(3)) \to \mathbb{R}$$

factors through $\mathrm{Hom}_{\mathrm{MHS}_{\mathbb{R}}}(\mathbb{R}(0), H^6_{B,c}(3))$ and coincides with the natural Poincaré duality pairing

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{MHS}_{\mathbb{R}}}(\mathbb{R}(0), H^6_B(4)) \otimes \mathrm{Hom}_{\mathrm{MHS}_{\mathbb{R}}}(\mathbb{R}(0), H^6_{B,c}(3)) &\longrightarrow \mathrm{Ext}^1_{\mathrm{MHS}_{\mathbb{R}}}(\mathbb{R}(0), H^6_B(4) \otimes H^6_{B,c}(3)) \\ &\xrightarrow{\cup} \mathrm{Ext}^1_{\mathrm{MHS}_{\mathbb{R}}}(\mathbb{R}(0), H^{12}_{B,c}(7)) \\ &\xrightarrow{\mathrm{Tr}} \mathrm{Ext}^1_{\mathrm{MHS}_{\mathbb{R}}}(\mathbb{R}(0), \mathbb{R}(1)) = \mathbb{R}.\end{aligned}$$

This shows the first two assertions. The last formula follows from Proposition 4.8. □

5. Integral representation and residue of the spin L-function

In this section, using the result of [Pollack and Shah 2018], we explain the precise connection between the period integral appearing in the statement of Proposition 4.8 and the residue of the spin L-function of π in the case where the cubic totally real étale algebra E over $\mathbb{Q}$ defining $\boldsymbol{H}$ is of the form $\mathbb{Q} \times F$, with F a quadratic real étale algebra over $\mathbb{Q}$. We start by recalling well-known analytic properties of some Eisenstein series for GL_2.

5.1. *Eisenstein series for* GL_2. Let $\boldsymbol{T}_2$ denote the maximal diagonal torus of GL_2 and let $\boldsymbol{B}_2 = \boldsymbol{T}_2\boldsymbol{U}_2$ denote the standard Borel. We denote by δ the character of $\boldsymbol{T}_2$ defined by $\mathrm{diag}(t_1, t_2) \mapsto t_1/t_2$ and we regard δ as a character of $\boldsymbol{B}_2$ by extending it trivially to the unipotent radical. Let $\Phi \in \mathcal{S}(\mathbb{A}^2)$ be a Schwartz–Bruhat function. Following Jacquet, for any $s \in \mathbb{C}$, we attach to Φ the function $f_\Phi \in \mathrm{Ind}_{\boldsymbol{B}_2(\mathbb{A})}^{\mathrm{GL}_2(\mathbb{A})} \delta^s$ defined by

$$f_\Phi(h, s) = |\det(h)|^s \int_{\mathbb{A}^\times} \Phi((0, t)h)|t|^{2s}\, d^\times t$$

and the Eisenstein series

$$E_\Phi(h, s) = \sum_{\gamma \in \boldsymbol{B}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q})} f_\Phi(\gamma h, s).$$

In the statement of the following lemma, we denote by $\widehat{\Phi}(0) = \int_{\mathbb{A}^2} \Phi(x, y)\, dx\, dy$ the value at 0 of the Fourier transform of Φ.

Lemma 5.1. (1) *The Eisenstein series $E_\Phi(h, s)$ is absolutely convergent for* $\mathrm{Re}(s)$ *big enough and has a meromorphic continuation to* $\mathbb{C}$.

(2) *We have*

$$E_\Phi(h, s) = \frac{|\det(h)|^{s-1}\widehat{\Phi}(0)}{2(s-1)} + R(h, s),$$

where $R(h, s)$ is an entire function in s for any $h \in \mathrm{GL}_2(\mathbb{A})$.

Proof. Statement (1) is [Jacquet 1972, Proposition 19.2]. According to [Jacquet and Shalika 1981, Lemma (4.2)] and its proof, we have

$$E_\Phi(h, s) = \frac{c|\det(h)|^{s-1}\widehat{\Phi}(0)}{s-1} + R(h, s),$$

where $R(h, s)$ is holomorphic for $\mathrm{Re}(s) > 0$ and $c = (s-1)\int_{|t|\le 1} |t|^{2(s-1)}\, d^\times t$, the integral being over the set $\{t \in \mathbb{A}^\times/\mathbb{Q}^\times : |t| \le 1\}$. By Iwasawa–Tate we have $c = \frac{1}{2}$. □

5.2. *Fourier coefficients.* Here we discuss the definition and basic properties of some Fourier coefficients for cusp forms for $\boldsymbol{G}$, which appear in the integral representation of the spin L-function of [Pollack and Shah 2018].

5.2.1. *The Siegel parabolic.* We let $Q = L_3U_3$ denote the standard Siegel parabolic subgroup of $\boldsymbol{G}$, with Levi $L_3 \simeq \mathrm{GL}_3 \times \mathrm{GL}_1$. Explicitly,

$$L_3 = \left\{m(g, \mu) = \left(\begin{smallmatrix} g & \\ & \mu\, {}^t g^{-1}\end{smallmatrix}\right) \mid g \in \mathrm{GL}_3, \mu \in \mathrm{GL}_1\right\},$$
$$U_3 = \left\{n(u) = \left(\begin{smallmatrix} I_3 & u \\ & I_3\end{smallmatrix}\right),\ u \in M_3 \mid u^t = u\right\}.$$

Denote $\mathrm{Sym}(3) = \{\alpha \in M_3 \mid \alpha^t = \alpha\}$. To each $\alpha \in \mathrm{Sym}(3)(\mathbb{Q})$, we associate the unitary character $\psi_\alpha : U_3(\mathbb{Q})\backslash U_3(\mathbb{A}) \to \mathbb{C}^\times$ by $n(u) \in U_3(\mathbb{A}) \mapsto e(\mathrm{Tr}(\alpha u))$, where $e : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$ is the additive character with $e_\infty(x) := e^{2\pi i x}$ for $x \in \mathbb{R}$, and conductor 1 at the finite places. For each $\alpha \in \mathrm{Sym}(3)(\mathbb{Q})$, we define a Fourier coefficient along U_3 for a cuspidal automorphic representation π of $\boldsymbol{G}(\mathbb{A})$ as follows.

Definition 5.2. Let Ψ be a cusp form in the space of π. Define

$$\Psi_{U_3,\psi_\alpha}(g) := \int_{U_3(\mathbb{Q})\backslash U_3(\mathbb{A})} \psi_\alpha^{-1}(u)\Psi(ug)\,du.$$

We let $L_3(\mathbb{Q})$ acts on $\mathrm{Sym}(3)(\mathbb{Q})$ via the right action $\alpha \cdot m(g,\mu) = \mu^{-1} g^t \alpha g$.

Lemma 5.3. *Let $\alpha, \beta \in \mathrm{Sym}(3)(\mathbb{Q})$. If there exists $m \in L_3(\mathbb{Q})$ such that $\beta = \alpha \cdot m$, then*

$$\Psi_{U_3,\psi_\beta}(g) = \Psi_{U_3,\psi_\alpha}(mg).$$

Proof. Suppose that $\beta = \alpha \cdot m$ with $m = m(g,\mu)$. The result follows from the equality

$$\psi_\beta(n(u)) = e(\mathrm{Tr}(\mu^{-1} g^t \alpha g u)) = e(\mathrm{Tr}(\alpha g u \mu^{-1} g^t)) = \psi_\alpha(m n(u) m^{-1}). \qquad \square$$

In this manuscript, we are interested in Fourier coefficients associated to the set of rank-2 elements of $\mathrm{Sym}(3)(\mathbb{Q})$, which we denote by $\mathrm{Sym}^{\mathrm{rk2}}(3)(\mathbb{Q})$. Let $D \in \mathbb{Q}^\times$ and let F denote the étale quadratic extension $\mathbb{Q}(\sqrt{D})$ of $\mathbb{Q}$. If D is not a square then F is a field, else $F = \mathbb{Q} \times \mathbb{Q}$.

Definition 5.4. We let $\psi_D : U_3(\mathbb{Q})\backslash U_3(\mathbb{A}) \to \mathbb{C}^\times$ be the unitary character

$$\psi_D : n(u) \mapsto e(\mathrm{Tr}(\alpha_D u)) = e(u_{33} - D u_{22})$$

associated to $\alpha_D = \begin{pmatrix} 0 & & \\ & -D & \\ & & 1 \end{pmatrix} \in \mathrm{Sym}^{\mathrm{rk2}}(3)(\mathbb{Q})$.

Lemma 5.5. *A set of representatives of* $\mathrm{Sym}^{\mathrm{rk2}}(3)(\mathbb{Q})/_{\sim M_3(\mathbb{Q})}$ *is given by*

$$\{\alpha_D : D \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2\}.$$

In view of Lemmas 5.3 and 5.5, the set of Fourier coefficients associated to the Siegel parabolic and a rank-2 symmetric matrix is parametrized by the set of étale quadratic algebras of $\mathbb{Q}$.

5.2.2. *Fourier coefficients of type* (4 2). We now turn our attention to Fourier coefficients associated to the unipotent orbit of $\boldsymbol{G}$ associated to the partition (4 2). The corresponding unipotent subgroup is the unipotent radical subgroup of the nonmaximal standard parabolic $P = L_P \cdot U_P$, which arises as the intersection of the Siegel parabolic Q with the Klingen parabolic. Notice that P has Levi $L_P = \mathrm{GL}_2 \times \mathrm{GL}_1^2$, given by

$$\left\{ \begin{pmatrix} a & & & \\ & g & & \\ & & \mu a^{-1} & \\ & & & \mu^t g^{-1} \end{pmatrix} : a, \mu \in \mathrm{GL}_1,\ g \in \mathrm{GL}_2 \right\}.$$

Following [Pollack and Shah 2018, §2.1], we define a unitary character which we still denote

$$\psi_D : U_P(\mathbb{Q})\backslash U_P(\mathbb{A}) \to \mathbb{C}^\times$$

as follows. Every element of $U_P/[U_P, U_P]$ can be expressed as the product of $n(v)\tilde{n}(u)$, where

$$n(v) = \begin{pmatrix} 1 & v_1 & v_2 & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & -v_1 & 1 & \\ & & & -v_2 & & 1 \end{pmatrix} \in \boldsymbol{G}, \quad \tilde{n}(u) = \begin{pmatrix} 1 & & & & & \\ & 1 & & & u_{22} & u_{23} \\ & & 1 & & u_{23} & u_{33} \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \in U_3.$$

We will denote by N_v (resp. N_u) the set of the $n(v)$'s (resp. $\tilde{n}(u)$'s). If $n \equiv n(v)\tilde{n}(u)$ modulo $[U_P, U_P]$, define

$$\psi_D(n) := e(v_1 + u_{33} - Du_{22}) = e(v_1)\psi_D(n(u)).$$

Let π be a cuspidal automorphic representation of $\boldsymbol{G}(\mathbb{A})$. We define the following Fourier coefficients.

Definition 5.6. Let Ψ be a cusp form in the space of π. Define

$$\Psi_{U_P,\psi_D}(g) := \int_{U_P(\mathbb{Q})\backslash U_P(\mathbb{A})} \psi_D^{-1}(u)\Psi(ug)\,du.$$

In the following proposition, we relate these Fourier coefficients to the ones for the Siegel parabolic associated to rank-2 symmetric matrices.

Proposition 5.7. *For a cusp form Ψ in the space of π, the following two conditions are equivalent.*

(1) $\Psi_{U_P,\psi_D}(g) \not\equiv 0$.

(2) *There exists $\alpha \in \mathrm{Sym}^{\mathrm{rk}2}(3)(\mathbb{Q})$ with $\alpha \sim_{L(\mathbb{Q})} \alpha_D$ such that $\Psi_{U_3,\alpha}(g) \not\equiv 0$.*

Proof. Fourier expand $\Psi_{U_3,\psi_D}(g)$ over N_v to get

$$\Psi_{U_3,\psi_D}(g) = \int_{(\mathbb{Q}\backslash\mathbb{A})^2} \Psi_{U_3,\psi_D}(n(v)g)\,dv + \sum_{\gamma \in \mathrm{Stab}_L(\psi_D)(\mathbb{Q})\backslash L(\mathbb{Q})} \Psi_{U_P,\psi_D}(\gamma g).$$

The term

$$\int_{(\mathbb{Q}\backslash\mathbb{A})^2} \Psi_{U_3,\psi_D}(n(v)g)\,dv = \int_{N_u(\mathbb{Q})\backslash N_u(\mathbb{A})} \psi_D^{-1}(\tilde{n}(u)) \int_{U_K(\mathbb{Q})\backslash U_K(\mathbb{A})} \Psi(n_k\tilde{n}(u)g)\,dn_k\,d\tilde{n}(u)$$

and the inner integral vanishes because of cuspidality of Ψ along the unipotent radical U_K of the Klingen parabolic. Thus

$$\Psi_{U_3,\psi_D}(g) = \sum_{\gamma} \Psi_{U_P,\psi_D}(\gamma g).$$

This relation implies the result as follows. If $\Psi_{U_3,\psi_D}(g) \not\equiv 0$, the Fourier coefficient $\Psi_{U_P,\psi_D}(g)$ does not vanish identically. Vice versa, if $\Psi_{U_P,\psi_D}(g) \not\equiv 0$ then there is a character ψ' in the $L(\mathbb{Q})$-orbit of ψ_D such that $\Psi_{U_3,\psi'}(g) \not\equiv 0$. □

5.3. ***The spin L-function and its residue at s = 1.*** Let π denote any cuspidal automorphic representation of $\boldsymbol{G}(\mathbb{A})$ with trivial central character. Let S denote a finite set of places of $\mathbb{Q}$ containing the ones where π

is ramified and the archimedean place. If Spin : $\mathrm{Spin}_7(\mathbb{C}) \to \mathrm{GL}(V_8)$ denotes the eight-dimensional spin representation, the partial spin L-function of π is defined to be

$$L^S(s, \pi, \mathrm{Spin}) := \prod_{\ell \notin S} \frac{1}{\det(1 - \ell^{-s}\,\mathrm{Spin}(s_{\pi_\ell}))},$$

where s_{π_ℓ} denotes the Satake parameter of the unramified local component π_ℓ. Let $\boldsymbol{H}$ be the group (3) associated to the étale cubic algebra $\mathbb{Q} \times F$, where $F = \mathbb{Q}(\sqrt{D})$ with either $D \not\equiv 1 \in \mathbb{Q}_{>0}^\times/(\mathbb{Q}^\times)^2$, in which case F is a real quadratic field, or $D \equiv 1 \bmod (\mathbb{Q}^\times)^2$, in which case $F = \mathbb{Q} \times \mathbb{Q}$. For any cusp form $\Psi \in V_\pi$, Pollack and Shah [2018] gave an integral representation

$$\mathcal{I}(\Phi, \Psi, s) = \int_{\mathbb{Z}(\mathbb{A})\boldsymbol{H}(\mathbb{Q})\backslash \boldsymbol{H}(\mathbb{A})} E_\Phi(h_1, s)\Psi(h)\,dh$$

of $L^S(s, \pi, \mathrm{Spin})$. For any Φ and Ψ, the integral $\mathcal{I}(\Phi, \Psi, s)$ is absolutely convergent for $\mathrm{Re}(s)$ big enough and has a meromorphic continuation to $\mathbb{C}$. According to [Gan and Gurevich 2009, Proposition 7.1], for $\mathrm{Re}(s)$ big enough we have the unfolding

$$\mathcal{I}(\Phi, \Psi, s) = \int_{U_{B_{\boldsymbol{H}}}(\mathbb{A})Z(\mathbb{A})\backslash \boldsymbol{H}(\mathbb{A})} f_\Phi(h_1, s)\Psi_{U_P,\psi_D}(h)\,dh,$$

where $U_{B_{\boldsymbol{H}}}$ is the unipotent radical of the upper triangular Borel subgroup $B_{\boldsymbol{H}}$ of $\boldsymbol{H}$ and Ψ_{U_P,ψ_D} is the Fourier coefficient of Definition 5.6.

Theorem 5.8 [Pollack and Shah 2018]. *For a set Σ of places of $\mathbb{Q}$, denote*

$$\mathcal{I}_\Sigma(\Phi, \Psi, s) = \int_{U_{B_{\boldsymbol{H}}(\mathbb{Q}_\Sigma)} Z_{\boldsymbol{G}}(\mathbb{Q}_\Sigma)\backslash \boldsymbol{H}(\mathbb{Q}_\Sigma)} f(h_1, \Phi_\Sigma, s)\Psi_{U_P,\psi_D}(h)\,dh.$$

Let Ψ be a cusp form in the space of π. Then, for any factorizable Schwartz–Bruhat function Φ on $\mathbb{A}^2$ and up to enlarging S, we have

$$\mathcal{I}(\Phi, \Psi, s) = \mathcal{I}_S(\Phi, \Psi, s)L^S(s, \pi, \mathrm{Spin}).$$

Moreover, there exists a cusp form $\widetilde{\Psi}$ in the space of π and a factorizable Schwartz–Bruhat function Φ on $\mathbb{A}^2$ such that

$$\mathcal{I}(\Phi, \widetilde{\Psi}, s) = \mathcal{I}_\infty(\Phi, \Psi, s)L^S(s, \pi, \mathrm{Spin}).$$

Note that if π does not support a rank-2 Fourier coefficient (for the Siegel parabolic Q) and thus, by Proposition 5.7, a Fourier coefficient for P, the integral $\mathcal{I}(\Phi, \Psi, s)$ is identically zero.

Corollary 5.9 [Pollack and Shah 2018]. *Suppose that π supports a rank-2 Fourier coefficient. Then the partial spin L-function $L^S(s, \pi, \mathrm{Spin})$ has meromorphic continuation in s, is holomorphic outside $s = 1$, and has at worst a simple pole at $s = 1$.*

As we explain in later sections, using results of Gan and Gurevich, Pollack and Shah further proved that when $L^S(s, \pi, \mathrm{Spin})$ has a simple pole at $s = 1$, π lifts to the split G_2 under the exceptional theta correspondence. This observation is based on the following key relation between the residue at $s=1$ of $L^S(s,\pi,\mathrm{Spin})$ and the automorphic period we have introduced in Section 4.3.

Proposition 5.10. *For any factorizable Schwartz–Bruhat function Φ on $\mathbb{A}^2$, we have*

$$\frac{\widehat{\Phi}(0)}{2}\cdot\int_{\mathbb{Z}(\mathbb{A})\boldsymbol{H}(\mathbb{Q})\backslash\boldsymbol{H}(\mathbb{A})}\Psi(h)\,dh=\mathrm{Res}_{s=1}\big(\mathcal{I}_S(\Phi,\Psi,s)L^S(s,\pi,\mathrm{Spin})\big).$$

Proof. Thanks to Lemma 5.1, the residue at $s=1$ of $\mathcal{I}(\Phi,\Psi,s)$ equals

$$\frac{\widehat{\Phi}(0)}{2}\cdot\int_{\mathbb{Z}(\mathbb{A})\boldsymbol{H}(\mathbb{Q})\backslash\boldsymbol{H}(\mathbb{A})}\Psi(h)\,dh.$$

The result then follows from Theorem 5.8. □

We now state our first main result. Let π denote a cuspidal automorphic representation of $\mathrm{PGSp}_6(\mathbb{A})$ for which π_∞ is the discrete series of Hodge type (3,3) in the L-packet of V^λ with $\lambda=(\lambda_2+\lambda_3,\lambda_2,\lambda_3,0)$. Let $\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},B}$, $\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{H}}$, and ω_Ψ be as in Sections 3.4 and 4.3. Let $\Psi^{[\lambda,\mu]}$ denote $A^{[\lambda,\mu]}\cdot\Psi$, where $A^{[\lambda,\mu]}\in U(\mathfrak{k}_\mathbb{C})$ is the operator defined in Proposition 4.8.

Theorem 5.11. *We have*

$$\langle\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{H}},[\omega_\Psi]\rangle_{\mathcal{H}}=\langle\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},B},[\omega_\Psi]\rangle=C\cdot\mathrm{Res}_{s=1}\big(\mathcal{I}_S(\Phi,\Psi^{[\lambda,\mu]},s)L^S(s,\pi,\mathrm{Spin})\big),$$

where

$$C=\frac{\widehat{\Phi}(0)h_{U'}}{2(2\pi i)^3\cdot\mathrm{vol}(U')}.$$

Proof. By Propositions 4.8 and 4.10, we have that

$$\langle\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{H}},[\omega_\Psi]\rangle_{\mathcal{H}}=\langle\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},B},[\omega_\Psi]\rangle=\frac{h_{U'}}{(2\pi i)^3\cdot\mathrm{vol}(U')}\int_{\boldsymbol{H}(\mathbb{Q})\mathbb{Z}_{\boldsymbol{G}}(\mathbb{A})\backslash\boldsymbol{H}(\mathbb{A})}\Psi^{[\lambda,\mu]}(h)\,dh,$$

where $U'=U\cap\boldsymbol{H}(\mathbb{A}_f)$ and $h_{U'}=4^{-1}|\mathbb{Z}_{\boldsymbol{G}}(\mathbb{Q})\backslash\mathbb{Z}_{\boldsymbol{G}}(\mathbb{A}_f)/(\mathbb{Z}_{\boldsymbol{G}}(\mathbb{A}_f)\cap U')|$. By Proposition 5.10, we have

$$\langle\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},B},[\omega_\Psi]\rangle=C\cdot\mathrm{Res}_{s=1}\big(\mathcal{I}_S(\Phi,\Psi^{[\lambda,\mu]},s)L^S(s,\pi,\mathrm{Spin})\big),$$

where

$$C=\frac{\widehat{\Phi}(0)h_{U'}}{2(2\pi i)^3\cdot\mathrm{vol}(U')}.$$

This finishes the proof. □

Let us fix a Schwartz–Bruhat function Φ such that $\widehat{\Phi}(0)\neq 0$.

Corollary 5.12. *Suppose that π satisfies the following hypotheses*:

- $\mathcal{I}_S(\Phi,\Psi^{[\lambda,\mu]},1)\neq 0$ *for some μ.*
- *The partial L-function $L^S(s,\pi,\mathrm{Spin})$ has a pole at $s=1$.*

Then

$$\langle\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},B},[\omega_\Psi]\rangle=\langle\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{H}},[\omega_\Psi]\rangle_{\mathcal{H}}\neq 0.$$

Proof. By [Pollack and Shah 2018, Theorem 1.3] the function $L^S(s, \pi, \mathrm{Spin}) = 1$ has at most a simple pole. As a consequence $\mathrm{Res}_{s=1} L^S(s, \pi, \mathrm{Spin}) \neq 0$. By Theorem 5.11, under the assumption that $\mathcal{I}_S(\Phi, \Psi^{[\lambda,\mu]}, 1) \neq 0$, this implies that $\langle \mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},B}, [\omega_\Psi]\rangle \neq 0$. □

Remark 5.13. If the automorphic representation π supports a rank-2 Fourier coefficient and its partial spin L-function has a (necessarily simple) pole at $s = 1$, by the results of [Pollack and Shah 2018] it is $\boldsymbol{H}$-distinguished, namely the map $\mathcal{P}_{\boldsymbol{H}} \in \mathrm{Hom}_{\boldsymbol{H}(\mathbb{A})}(\pi, \mathbf{1})$ defined by

$$\Psi \mapsto \mathcal{P}_{\boldsymbol{H}}(\Psi) := \int_{Z(\mathbb{A})\boldsymbol{H}(\mathbb{Q})\backslash \boldsymbol{H}(\mathbb{A})} \Psi(h)\, dh$$

is not identically zero. Then, asking $\mathcal{I}_S(\Phi, \Psi^{[\lambda,\mu]}, 1) \neq 0$ for some μ is equivalent to asking that the map obtained as the composition of $\mathcal{P}_{\boldsymbol{H}}$ with an $\boldsymbol{H}(\mathbb{R})$-equivariant embedding $\pi_\infty \to \pi$ restricts nontrivially to the minimal K_∞-type of π_∞.

Denote by $H^6_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathscr{V}^\lambda_F(3))_{\mathrm{hom}}$ the F-vector space defined in Section 3.4.3 and denote by $H^6_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathscr{V}^\lambda_F(3))_{\mathrm{hom}}[\pi_f^\vee]$ its $\pi_f^\vee$-isotypical component. This is a finite-dimensional L-vector space, where L is the number field introduced in Section 2.8. The Tate conjecture for the motive attached to π (see Conjecture 1.1(3)) predicts the equality

$$-\mathrm{ord}_{s=1} L(s, \pi, \mathrm{Spin}) = \dim_L H^6_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathscr{V}^\lambda_F(3))_{\mathrm{hom}}[\pi_f^\vee].$$

Corollary 5.14. *If* $\mathcal{I}_S(\Phi, \Psi^{[\lambda,\mu]}, 1) \neq 0$, *then*

$$-\mathrm{ord}_{s=1} L^S(s, \pi, \mathrm{Spin}) \leq \dim_L H^6_{\mathcal{M}}(\mathrm{Sh}_{\boldsymbol{G}}(U), \mathscr{V}^\lambda_F(3))_{\mathrm{hom}}[\pi_f^\vee].$$

Proof. If $L^S(s, \pi, \mathrm{Spin})$ does not have a pole at $s = 1$, there is nothing to prove. If not, Corollary 5.12 implies that the projection of $\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},B}$ to the $\pi_f^\vee$-isotypical component is nonzero, showing the result. □

The following result verifies a weaker form of Conjecture 1.1(3) for the motive $M(\pi_f^\vee)(3)$ at the cost of supposing that $\mathcal{I}_S(\Phi, \Psi^{[\lambda,\mu]}, 1)$ is nonzero for some μ.

Corollary 5.15. *Suppose that* π *satisfies the hypotheses of Corollary 5.12 and that* (St) *holds. Then* $\mathrm{pr}_{\pi^\vee} \mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{H}}$ *and its Hecke translates generate* $H^1_{\mathcal{H}}(M(\pi_f^\vee)_{\mathbb{R}}(4))$.

Proof. If π_p is the Steinberg representation, it follows from the second statement of Lemma 2.11 and its proof that $H^1_{\mathcal{H}}(M(\pi_f^\vee)_{\mathbb{R}}(4))$ is a rank-1 module over the full Hecke algebra of level U. Hence the result follows by Corollary 5.12. □

6. Exceptional theta lifts from G_2 to PGSp_6

In this section, we discuss the exceptional theta correspondence for the dual reductive pair (G_2, PGSp_6) and describe the set of Fourier coefficients associated to the Heisenberg parabolic for cuspidal automorphic forms of $G_2(\mathbb{A})$. Its sole purpose is to fix notation and to recall some well-known results that will be used later, so the knowledgeable reader might skip it.

	s_1	s_2	s_3	t_1	t_2	t_3	s_4	t_4
s_1	0	$-t_3$	t_2	s_4	0	0	0	s_1
s_2	t_3	0	$-t_1$	0	s_4	0	0	s_2
s_3	$-t_2$	t_1	0	0	0	s_4	0	s_3
t_1	t_4	0	0	0	s_3	$-s_2$	t_1	0
t_2	0	t_4	0	$-s_3$	0	s_1	t_2	0
t_3	0	0	t_4	s_2	$-s_1$	0	t_3	0
s_4	s_1	s_2	s_3	0	0	0	s_4	0
t_4	0	0	0	t_1	t_2	t_3	0	t_4

Table 1. Multiplication table for the basis $\{s_1, s_2, s_3, s_4, t_1, t_2, t_3, t_4\}$.

6.1. ***Split G_2 and E_7.*** In this section we will follow the exposition of the Appendix of [Harris et al. 2023] by Savin.

6.1.1. *The group G_2.* Let $\mathbb{H}$ be the algebra of Hamilton quaternions over $\mathbb{Q}$ with the usual basis $\{1, i, j, k\}$. The conjugate $\bar{a}$ of an element $a = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \in \mathbb{H}$ is $\bar{a} = \alpha_0 - \alpha_1 i - \alpha_2 j - \alpha_3 k$. The split octonion algebra over $\mathbb{Q}$ is $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$ with multiplication

$$(a, b) \cdot (c, d) = (ac + d\bar{b}, \bar{a}d + cb).$$

Then $\mathbb{O}$ is a noncommutative, nonassociative $\mathbb{Q}$-algebra. However it is alternative, which means that for any $x, y \in \mathbb{O}$ we have $x \cdot (x \cdot y) = (x \cdot x) \cdot y$ and $(x \cdot y) \cdot y = x \cdot (y \cdot y)$ (see [Jacobson 1958]). If $x = (a, b)$, let $\bar{x} = (\bar{a}, -b)$. Then $x \mapsto \bar{x}$ is a $\mathbb{Q}$-linear involution on $\mathbb{O}$ satisfying $\overline{x \cdot y} = \bar{y} \cdot \bar{x}$. The norm $\mathrm{N} : \mathbb{O} \to \mathbb{Q}$ is the quadratic form defined by $x \mapsto x \cdot \bar{x} = \bar{x} \cdot x$. The trace $\mathrm{Tr} : \mathbb{O} \to \mathbb{Q}$ is defined by $x \mapsto x + \bar{x}$. For any $x, y, z \in \mathbb{O}$, the properties

$$\mathrm{N}(x \cdot y) = \mathrm{N}(x)\mathrm{N}(y),$$
$$\mathrm{Tr}(x \cdot y) = \mathrm{Tr}(y \cdot x),$$
$$\mathrm{Tr}(x \cdot (y \cdot z)) = \mathrm{Tr}((x \cdot y) \cdot z)$$

are satisfied. For $x, y \in \mathbb{O}$, we write $y \in x^{\perp}$ if y is orthogonal to x with respect to the bilinear form $(x, y) \mapsto \mathrm{Tr}(x \cdot \bar{y})$, which means that $x \cdot \bar{y} + y \cdot \bar{x} = 0$.

Let $l = (0, 1) \in \mathbb{O}$ so that $\{1, i, j, k, l, li, lj, lk\}$ is a basis of $\mathbb{O}$. From this, one constructs another useful basis $\{s_1, s_2, s_3, s_4, t_1, t_2, t_3, t_4\}$, where

$$s_1 = \tfrac{1}{2}(i + li), \quad s_2 = \tfrac{1}{2}(j + lj), \quad s_3 = \tfrac{1}{2}(k + lk), \quad s_4 = \tfrac{1}{2}(1 + l),$$
$$t_1 = \tfrac{1}{2}(i - li), \quad t_2 = \tfrac{1}{2}(j - lj), \quad t_3 = \tfrac{1}{2}(k - lk), \quad t_4 = \tfrac{1}{2}(1 - l).$$

See Table 1 for the multiplication table, as given in Table 1 of the Appendix of [Harris et al. 2023].

We define

$$G_2 := \{g \in \mathrm{GL}(\mathbb{O}) \mid g(x \cdot y) = (gx) \cdot (gy), \forall x, y \in \mathbb{O}\}.$$

to be the group of automorphisms of $\mathbb{O}$. We note that G_2 acts transitively on nonzero elements of trace zero and norm zero. We will denote the set of trace zero octonions by either $\mathbb{O}^0$ or V_7, where the latter notation emphasizes that this set defines the standard irreducible seven-dimensional representation of G_2 and induces an embedding

$$G_2 \hookrightarrow \mathrm{SO}_7.$$

6.1.2. *The dual reductive pair.* We consider the Albert algebra J over $\mathbb{Q}$, which is the set of matrices

$$A = \begin{pmatrix} d & \bar{z} & y \\ z & e & \bar{x} \\ \bar{y} & x & f \end{pmatrix},$$

where $d, e, f \in \mathbb{Q}$ and $x, y, z \in \mathbb{O}$. The algebra J is equipped with a cubic form, called the determinant, which is given by

$$\det(A) = def - dN(x) - eN(y) - fN(z) + \mathrm{Tr}(zyx).$$

The group of isogenies of this form is a group of type E_6 and its orbits on J are classified by the rank. We will need to consider the set Ω of rank-1 elements $A \in J$, i.e., those $A \neq 0$ such that $A^2 = \mathrm{Tr}(A) \cdot A$. This condition means that the entries of A satisfy the equalities

$$\begin{aligned} \mathrm{N}(x) &= ef, & \mathrm{N}(y) &= df, & \mathrm{N}(z) &= de, \\ dx &= \bar{y} \cdot \bar{z}, & ey &= \bar{z} \cdot \bar{x}, & fz &= \bar{x} \cdot \bar{y}. \end{aligned} \tag{10}$$

Let G denote the split adjoint group of type E_7, which is constructed from J by the Koecher–Tits construction (see Section 3 of [Kobayashi and Savin 2015]). The group G has a maximal parabolic $P = MN$ and its opposite $\overline{P} = M\overline{N}$, with $N \simeq J$ and such that the action under conjugation of the Levi M on N gives an isomorphism of M and the group of similitudes of the cubic form on J

$$M \cong \{g \in \mathrm{GL}(J) \mid \det(gA) = \lambda \det(A) \text{ for some } \lambda \in \boldsymbol{G}_m \text{ and all } A \in J\}.$$

The group G_2 can be realized as a subgroup of M via its action on J by the rule

$$g \cdot \begin{pmatrix} d & \bar{z} & y \\ z & e & \bar{x} \\ \bar{y} & x & f \end{pmatrix} = \begin{pmatrix} d & g\bar{z} & gy \\ gz & e & g\bar{x} \\ g\bar{y} & gx & f \end{pmatrix}.$$

This action has fixed points J_3, the Jordan algebra of symmetric 3×3 matrices with entries in $\mathbb{Q}$. Note that the left action of GL_3 on $J_3 \cong N$ given by

$$g \cdot A = \det(g)^{-1} g A g^t \tag{11}$$

extends to an action on J preserving the determinant form up to scalar, thus defining an embedding of GL_3 into M. Then GL_3 is the centralizer of G_2 in M and $Q = \mathrm{GL}_3 U_3$ (which is the Siegel parabolic of PGSp_6) is the centralizer of G_2 in P. Similarly, the opposite $\overline{Q}$ is the centralizer of G_2 in $\overline{P}$. This gives the dual reductive pair (G_2, PGSp_6) in G.

6.2. *Fourier coefficients for G_2.*

6.2.1. *Root system and parabolic subgroups.* Let T be a (rank-2) maximal split torus over $\mathbb{Q}$ in G_2 and let Δ (resp. $\Delta^+ \subset \Delta$) be the set of roots (resp. a subset of positive roots) for G_2. Let a (resp. b) denote the long (resp. short) simple root in Δ^+. Then

$$\Delta^+ = \{a, b, a+b, a+2b, a+3b, 2a+3b\}.$$

We let $B = TU$ denote the Borel subgroup of G_2 associated to Δ^+. Other than B, there are two proper standard parabolic subgroups P_a and P_b of G_2, such that $P_a \cap P_b = B$. They are characterized by the following. For any $\alpha \in \Delta^+$, denote by $x_\alpha : \boldsymbol{G}_a \hookrightarrow U$ the one parameter unipotent subgroup associated to α. Then, for each $r \in \{a, b\}$, the Levi L_r of P_r is isomorphic to GL_2 and contains x_r. We fix an isomorphism $\mathrm{GL}_2 \simeq L_r$ such that $\left(\begin{smallmatrix} 1 & u \\ & 1 \end{smallmatrix}\right) \mapsto x_r(u)$.

Let U_a be the unipotent radical of P_a. It is a 3-step nilpotent group of dimension 5 with filtration

$$U_a \supset U_1 \supset U_2 \supset \{1\},$$

where U_a/U_1 is generated by $\{x_b, x_{a+b}\}$, U_1/U_2 is isomorphic to the one parameter unipotent subgroup x_{a+2b}, and U_2 is generated by $\{x_{a+3b}, x_{2a+3b}\}$. As representations of L_a, U_a/U_1 is the standard representation, while U_1/U_2 is the determinant (see [Gan and Savin 2023, §2.4]).

We denote by $H := P_b$ the so-called Heisenberg parabolic and let $L_H U_H$ denote its Levi decomposition. The unipotent radical U_H is of dimension 5 and admits the filtration

$$U_H \supset [U_H, U_H] \supset \{1\},$$

with $U_H/[U_H, U_H]$ being the four-dimensional abelian unipotent group generated by

$$\{x_a, x_{a+b}, x_{a+2b}, x_{a+3b}\},$$

while $[U_H, U_H]$ is isomorphic to the one parameter unipotent subgroup x_{2a+3b}.

6.2.2. *An embedding of* SL_3 *into* G_2. The group G_2 acts transitively on the set

$$\Gamma_c := \{x \in \mathbb{O}^0 \mid N(x) = -c\}.$$

By [Jacobson 1958, Theorem 4], the stabilizer of an element $y_0 \in \Gamma_1$ is isomorphic to SL_3. Choose y_0 such that the unipotent radical U_{SL_3} of the upper triangular Borel of SL_3 is generated by the one-parameter subgroups

$$\{x_a, x_{a+3b}, x_{2a+3b}\}.$$

In terms of the basis chosen in Section 6.1.1, this is achieved by choosing $y_0 = s_4 - t_4$. In this case, one shows (see [Rallis and Schiffmann 1989, Lemma 2]) that the stabilizer of y_0 leaves invariant the subspace $\langle s_1, s_2, s_3\rangle$ and is identified with $\mathrm{SL}_3 = \mathrm{SL}(\langle s_1, s_2, s_3\rangle)$.

6.2.3. *The Lie algebra of* G_2. The multiplication map on $\mathbb{O}$ induces a map $V_7 \otimes V_7 \to V_7$ given by $x \otimes y \mapsto \frac{1}{2}(xy - yx)$. This map is alternating; hence it induces a G_2-equivariant map $\bigwedge^2 V_7 \to V_7$ which is surjective. Then the Lie algebra $\mathfrak{g}_2$ of G_2 can be identified with the kernel of this map. Under this identification, one has an explicit description of the action of $\mathfrak{g}_2$ on V_7, namely

$$(w \wedge x) \cdot v = \langle x, v\rangle w - \langle w, v\rangle x.$$

We will also need (see [Fulton and Harris 1991, §22.2]) the decomposition

$$\mathfrak{g}_2 = \mathfrak{sl}_3 \oplus \mathrm{Std}_3 \oplus \mathrm{Std}_3^*, \tag{12}$$

where Std_3 is the standard representation of SL_3 with basis $\{v_1, v_2, v_3\}$ and Std_3^* is its dual with basis $\{\delta_1, \delta_2, \delta_3\}$ and where we denote by E_{ij}, $1 \le i < j \le 3$ the standard basis vectors of $\mathfrak{sl}_3$. The identification between the two descriptions (see [Pollack 2021, §2.2]) of $\mathfrak{g}_2$ is given by $E_{ij} = t_j \wedge s_i$, $1 \le i < j \le 3$, $v_i = (s_4 - t_4) \wedge s_i + t_{i+1} \wedge t_{i+2}$ and $\delta_i = (s_4 - t_4) \wedge t_i + s_{i+1} \wedge s_{i+2}$, $1 \le i \le 3$, where indices are taken modulo 3. Moreover, the component $\mathfrak{sl}_3$ is the Lie algebra of the copy of SL_3 embedded into G_2 as above. In particular, E_{12}, E_{13} and E_{23} are root vectors for the roots a, $2a+3b$ and $a+3b$ respectively. Moreover, the vectors v_1, v_2 and δ_3 are root vectors for the roots $a+b$, b and $a+2b$, respectively. Via (12), the Lie algebra $\mathfrak{u}_H$ of U_H is

$$\mathfrak{u}_H = \mathfrak{u}_{\mathrm{SL}_3} \oplus \mathbb{Q}v_1 \oplus \mathbb{Q}\delta_3. \tag{13}$$

Under (12) the Lie algebra $\mathfrak{l}_H$ of the Levi L_H is generated by the Cartan subalgebra and the root vectors v_2, δ_2.

6.2.4. *Fourier coefficients.* We now describe the Fourier coefficients for G_2 associated to the Heisenberg parabolic. We closely follow [Pollack 2021] and refer to it for more details. In order to describe the Fourier coefficients associated to H, we need to study the L_H-representation $V_H := U_H/[U_H, U_H]$. As a GL_2-representation, V_H is isomorphic to $\mathrm{Sym}^3(\mathrm{Std}_2) \otimes \det^{-1}(\mathrm{Std}_2)$, where Std_2 denotes the standard representation of GL_2. Under the identification of (13), (a representative of) an element of $V_H(\mathbb{Q})$ can be written as

$$x_a(\lambda_1)x_{a+b}(\lambda_2/3)x_{a+2b}(\lambda_3/3)x_{a+3b}(\lambda_4), \quad \text{with } \lambda_i \in \mathbb{Q},$$

which corresponds to the binary cubic polynomial

$$p(x, y) = \lambda_1 x^3 + \lambda_2 x^2 y + \lambda_3 x y^2 + \lambda_4 y^3,$$

where x, y form a basis of Std_2. Associated to p, there is the cubic $\mathbb{Q}$-algebra R with basis $\{1, i, j\}$ with multiplicative table

$$\begin{aligned} ij &= -ad \\ i^2 &= -ac + bi - aj \\ j^2 &= -bd + di - cj. \end{aligned}$$

Example 6.1. (1) [Gross and Lucianovic 2009, 3.2] If $p(x, y) = x^2y - xy^2$ then the associated $\mathbb{Q}$-algebra R is isomorphic to $\mathbb{Q}^3$.

(2) [Gross and Lucianovic 2009, 3.3] If $p(x, y) = x^3 - Dxy^2$ $\big($or equivalently $p(x, y) = -Dx^2y + y^3$ using the action of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\big)$ then the associated $\mathbb{Q}$-algebra R is isomorphic to $\mathbb{Q} \oplus \mathbb{Q}(\sqrt{D})$.

There is an action of $\mathrm{GL}_2(\mathbb{Q})$ on the set of bases $\{1, i, j\}$ of a given cubic algebra R, which makes the association $p(X, Y) \mapsto (R, \{1, i, j\})$ $\mathrm{GL}_2(\mathbb{Q})$-equivariant. Since any cubic algebra admits a basis of this shape, we have the following.

Proposition 6.2 [Gross and Lucianovic 2009, Proposition 2.1]. *There is a bijection between the* $\mathrm{GL}_2(\mathbb{Q})$*-orbits on* $V_H(\mathbb{Q})$ *and the set of isomorphism classes of cubic* $\mathbb{Q}$*-algebras. Moreover, each orbit has a well-defined discriminant in* $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$.

Let $e : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$ be the additive character introduced in Section 5.2.1. Let $\langle\ ,\ \rangle$ denote the symplectic pairing on V_H defined as follows. If $v, v' \in V_H$ correspond to $p(x, y)$ and $p'(x, y)$ respectively, then

$$\langle v, v'\rangle = \lambda_1\lambda'_4 - \tfrac{1}{3}\lambda_2\lambda'_3 + \tfrac{1}{3}\lambda_3\lambda'_2 - \lambda_4\lambda'_1.$$

Any character $\psi : U_H(\mathbb{Q})\backslash U_H(\mathbb{A}) \to \mathbb{C}^\times$ factors through $V_H(\mathbb{A})$; hence we consider the projection $\bar{n}$ of $n \in U_H(\mathbb{A})$ to $V_H(\mathbb{A})$, which, by (13), can be written as

$$\bar{n} = x_a(\lambda'_1)x_{a+b}\tfrac{1}{3}\lambda'_2 x_{a+2b}\tfrac{1}{3}\lambda'_3 x_{a+3b}(\lambda'_4).$$

If $v \in V_H(\mathbb{Q})$ corresponds to $p(x, y)$, we then define $\psi_v : U_H(\mathbb{Q})\backslash U_H(\mathbb{A}) \to \mathbb{C}^\times$ by

$$n \mapsto e(\langle v, \bar{n}\rangle) = e\big(\lambda_1\lambda'_4 - \tfrac{1}{3}\lambda_2\lambda'_3 + \tfrac{1}{3}\lambda_3\lambda'_2 - \lambda_4\lambda'_1\big).$$

The character ψ_v is nondegenerate if and only if v corresponds to an étale cubic algebra over $\mathbb{Q}$. In this manuscript, we are interested in étale cubic algebras of the form $\mathbb{Q} \times F$, with F of either the form $\mathbb{Q}(\sqrt{D})$ (with $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2 \ni D \not\equiv 1$) or $\mathbb{Q} \times \mathbb{Q}$ (with $D \equiv 1 \bmod (\mathbb{Q}^\times)^2$).

Definition 6.3. Let $\psi_{H,D} : U_H(\mathbb{Q})\backslash U_H(\mathbb{A}) \to \mathbb{C}^\times$ denote the character associated to $\mathbb{Q} \times F$. Given a cusp form φ for $G_2(\mathbb{A})$, define

$$\varphi_{U_H,\psi_{H,D}}(g) := \int_{U_H(\mathbb{Q})\backslash U_H(\mathbb{A})} \psi^{-1}_{H,D}(n)\varphi(ng)\, dn.$$

6.3. ***The theta lift from G_2 to*** $\mathbf{PGSp_6}$**.** Let $\Pi = \bigotimes'_v \Pi_v$ denote the restricted tensor product of the minimal representations Π_v of $E_7(\mathbb{Q}_v)$ over all places v of $\mathbb{Q}$. A unitary model of the minimal representation is given by $L^2(\Omega)$, where recall that Ω denotes the subset of rank-1 elements in J. There is a unique up to a nonzero scalar embedding

$$\theta : \Pi \to \mathcal{A}(E_7(\mathbb{Q})\backslash E_7(\mathbb{A}))$$

of Π in the space $\mathcal{A}(E_7(\mathbb{Q})\backslash E_7(\mathbb{A}))$ of automorphic forms of E_7 (see [Ginzburg et al. 1997a; Kobayashi and Savin 2015]). For $f \in \Pi$ and $\varphi \in \mathcal{A}(G_2(\mathbb{Q})\backslash G_2(\mathbb{A}))$, we define a function $\Theta(f, \varphi)$ on $\mathrm{PGSp}_6(\mathbb{A})$ by

$$\Theta(f, \varphi)(g) = \int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} \theta(f)(g'g)\varphi(g')\, dg'.$$

Definition 6.4. Let σ be a cuspidal automorphic representation of $G_2(\mathbb{A})$.

(1) Define $\Theta(\sigma)$ to be the span of the functions $\Theta(f, \varphi)$, where $f \in \Pi$ and φ runs through the cusp forms in the contragredient $\sigma^\vee$ of σ.

(2) We say that a cuspidal automorphic representation π of $\mathrm{PGSp}_6(\mathbb{A})$ is a Θ-lift of σ if it appears as an irreducible subquotient of $\Theta(\sigma)$.

If a Θ-lift of σ exists, then its local constituents are compatible with the local theta correspondence between G_2 and PGSp_6.

Proposition 6.5. *Let π be a Θ-lift of σ. Then π_v is an irreducible subquotient of $\Theta(\sigma_v)$.*

Proof. See [Harris et al. 2023, Theorem 1.7(i)]. □

After imposing certain local conditions on σ, in the next section we use one of the main results of [Ginzburg et al. 1997b] to show that $\Theta(\sigma)$ is nonzero and cuspidal, thus proving the existence of a nontrivial Θ-lift of σ. Before doing so, we first recall the properties of the local theta correspondence needed later.

6.3.1. *Discrete series and a conjecture of Gross.* Let T_c denote a compact torus in $G_2(\mathbb{R})$, which is contained in the maximal compact subgroup $K_{G_2} \simeq (\mathrm{SU}_2 \times \mathrm{SU}_2)/\mu_2$ of $G_2(\mathbb{R})$. We abuse notation denoting again by a, b the simple positive roots for T_c (with the short root b which we assume to be compact) and Δ^+ the resulting set of positive roots. Then, $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha = 3a+5b$. The set of positive compact roots is given by

$$\Delta_c^+ = \{b, 2a+3b\},$$

which, in the notation of [Li 1997], is $\{2\varepsilon_2, 2\varepsilon_1\}$. The Weyl group $\mathfrak{W}_{G_2}$ is isomorphic to the dihedral group D_6 of 12 elements and it is generated by w_a and w_b, where w_α denotes the reflections around the line orthogonal to α. The Weyl group $\mathfrak{W}_{K_{G_2}} \simeq (\mathbb{Z}/2\mathbb{Z})^2$ is generated by w_b and $w_{2a+3b} = w_a w_b w_a w_b w_a$.

Let γ be a dominant weight for G_2 with respect to T_c. The set of equivalence classes of irreducible discrete series of $G_2(\mathbb{R})$ associated to γ has cardinality equal to $|\mathfrak{W}_{G_2}/\mathfrak{W}_{K_{G_2}}| = 3$. Choose representatives $\{w_1, w_2, w_3\}$ of $\mathfrak{W}_{G_2}/\mathfrak{W}_{K_{G_2}}$ such that $w_i\rho$ is dominant for K_{G_2}. Then, for any $1 \le i \le 3$, there exists an irreducible discrete series σ_∞^Γ of Harish-Chandra parameter $\Gamma = w_i(\gamma+\rho)$ and minimal K_{G_2}-type $\Gamma+\delta_{G_2}-2\delta_{K_{G_2}}$, where δ_{G_2} (resp. $\delta_{K_{G_2}}$) is the half-sum of roots (resp. compact roots) which are positive with respect to the Weyl chamber in which Γ lies. Precisely, if we let $w_1 = \mathrm{id}$, $w_2 = w_a$, and $w_3 = w_b w_a$, then

$$\begin{aligned} w_1\rho &= \rho = 3\varepsilon_1 + \varepsilon_2, \\ w_2\rho &= 2a+5b = 2\varepsilon_1 + 4\varepsilon_2, \\ w_3\rho &= a+4b = \varepsilon_1 + 5\varepsilon_2. \end{aligned}$$

We let $\mathcal{D}_{3,1}$, $\mathcal{D}_{2,4}$, and $\mathcal{D}_{1,5}$ denote the sets of discrete series of $G_2(\mathbb{R})$ whose Harish-Chandra parameter lies in the Weyl chamber corresponding to $w_1\rho$, $w_2\rho$, and $w_3\rho$ respectively. Elements of $\mathcal{D}_{3,1}$ are the quaternionic discrete series, while elements of $\mathcal{D}_{2,4}$ are the generic discrete series.

Gross has given a precise conjectural description of the entire discrete spectrum of the dual pair (G_2, PGSp_6) (see [Li 1997, Conjecture 1.2]). Recall that there are four families of discrete series for $\mathrm{PGSp}_6(\mathbb{R})$, indexed by the set of Hodge types up to conjugation. In particular, the discrete series of $\mathrm{PGSp}_6(\mathbb{R})$ of Hodge type $(4, 2)$ (resp. $(6, 0)$) are the generic (resp. holomorphic) discrete series.

Conjecture 6.6 (Gross). Let Π_∞ be the minimal representation of $E_7(\mathbb{R})$. The discrete spectrum of the restriction of Π_∞ to the dual pair $G_2(\mathbb{R}) \times \mathrm{PGSp}_6(\mathbb{R})$ is the direct sum of all tensor products $\sigma_\infty \otimes \theta(\sigma_\infty)$, where σ_∞ belongs to the discrete series of G_2. If σ_∞ has infinitesimal character $\gamma+\rho = r\varepsilon_1 + s\varepsilon_2$ and belongs to either $\mathcal{D}_{3,1}$, $\mathcal{D}_{2,4}$, or $\mathcal{D}_{1,5}$, then $\theta(\sigma_\infty)$ is the discrete series of $\mathrm{PGSp}_6(\mathbb{R})$ with infinitesimal character $\big(r, \tfrac{1}{2}(r+s), \tfrac{1}{2}(r-s)\big)$ and Hodge type $(3,3)$, $(4,2)$, or $(5,1)$ respectively.

Partial results towards the conjecture of Gross were shown by Li for discrete series in $\mathcal{D}_{3,1}$ (see Section 6.3.2 below) and for the generic family $\mathcal{D}_{2,4}$ by Harris, Khare and Thorne [Harris et al. 2023, Theorems 1.5 and 1.7(ii)] using the main result of Savin's appendix to [Harris et al. 2023] and the nonvanishing of the global theta lift given by [Ginzburg et al. 1997b, Corollary 4.2]. Li [1997, Theorem 4.3] also gave evidence to the predictions of Gross for a proper subset $\mathcal{D}'_{1,5}$ of $\mathcal{D}_{1,5}$. We also note that the remaining equivalence class of holomorphic discrete series of $\mathrm{PGSp}_6(\mathbb{R})$ (of Hodge type $(6,0)$) is realized in an exceptional theta correspondence studied by Gross and Savin between the compact real form $G_2^c(\mathbb{R})$ of G_2 and $\mathrm{PGSp}_6(\mathbb{R})$ and moreover this is the only Hodge type that appears in that correspondence (see [Gross and Savin 1998, Theorem 3.5]).

6.3.2. *Quaternionic discrete series and their theta lift.* We describe the main result of [Li 1997]. We first notice that a discrete series $\sigma_\infty^{x,y}$ of Harish-Chandra parameter $x\varepsilon_1 + y\varepsilon_2$ lies in the set of quaternionic discrete series $\mathcal{D}_{3,1}$ if x, y are two nonnegative integers such that $x-3 \geq y-1 \geq 0$ and $x-y$ is even. The minimal K_{G_2}-type of $\sigma_\infty^{x,y} \in \mathcal{D}_{3,1}$ is given by

$$\mathrm{Sym}^{x+1}(\mathrm{Std}_{\varepsilon_1}) \boxtimes \mathrm{Sym}^{y-1}(\mathrm{Std}_{\varepsilon_2}),$$

where $\mathrm{Std}_{\varepsilon_1}$ (resp. $\mathrm{Std}_{\varepsilon_2}$) is the standard representation of the SU_2 corresponding to the long root ε_1 (resp. the short root ε_2).

Proposition 6.7. *Let Π_∞ denote the minimal representation of $E_7(\mathbb{R})$. We have*

$$\Pi_\infty|_{G_2(\mathbb{R})\times \mathrm{PGSp}_6(\mathbb{R})} \supseteq \bigoplus_{\sigma_\infty^{x,y}\in\mathcal{D}_{3,1}} \sigma_\infty^{x,y} \otimes \theta(\sigma_\infty^{x,y}),$$

where $\theta(\sigma_\infty^{x,y}) \in P(V^\lambda)$, with $\lambda = \big(x-3, \tfrac{1}{2}(x+y)-2, \tfrac{1}{2}(x-y)-1, 0\big)$, is the discrete series $\pi_\infty^{3,3}$ of Hodge type $(3,3)$ and Harish-Chandra parameter $\big(\tfrac{1}{2}(x+y), \tfrac{1}{2}(x-y), -x\big)$.

Proof. See [Li 1997, Theorem 1.1; Huang et al. 1996, Theorem 5.4]. □

The set $\mathcal{D}_{3,1}$ contains an important family of discrete series, which were studied by Gross and Wallach [1994; 1996].

Definition 6.8. For every $n \geq 2$, the quaternionic discrete series σ_n is the element of $\mathcal{D}_{3,1}$ of Harish-Chandra parameter $(2n-1)\varepsilon_1 + \varepsilon_2$ and minimal K_{G_2}-type

$$\mathrm{Sym}^{2n}(\mathrm{Std}_{\varepsilon_1}) \boxtimes \mathbf{1}.$$

A fundamental property of the members of this family is that they admit (unique) models with respect to the unipotent radical of the Heisenberg parabolic and nondegenerate characters corresponding to totally real étale cubic algebras. Recall, as in Section 6.2.4, that a nondegenerate character $\psi : U_H(\mathbb{R}) \to \mathbb{C}^\times$

corresponds to a cubic algebra, whose discriminant is either positive or negative. The first type corresponds to the $\mathrm{GL}_2(\mathbb{R})$-orbit on $V_H(\mathbb{R})$ given by $\mathbb{R}^3$, while the second to the $\mathrm{GL}_2(\mathbb{R})$-orbit of $\mathbb{R}\times\mathbb{C}$. A representative $\psi : U_H(\mathbb{R}) \to \mathbb{C}^\times$ of the totally real orbit is given by $e^{2\pi i f}$, where $f : U_H(\mathbb{R}) \to \mathbb{R}$ is nonzero on the one parameter unipotent subgroups x_{a+b} and x_{a+2b} and trivial on x_a and x_{a+3b} (see [Gan et al. 2002, §6]). A special case of the main result of [Wallach 2003] gives the following.

Proposition 6.9. *Let ψ be a nondegenerate character of $V_H(\mathbb{R})$. There is (at most) a one-dimensional space of ψ-equivariant linear functionals on σ_n. Moreover,*

$$\dim \mathrm{Hom}_{U_H(\mathbb{R})}(\sigma_n, \psi) = 1,$$

exactly when ψ corresponds to a totally real cubic algebra.

6.3.3. *The nonarchimedean theta correspondence.* We describe the properties of the nonarchimedean theta correspondence which will be later needed to study the global theta correspondence. Let σ be an irreducible admissible representation of $G_2(\mathbb{Q}_p)$. The maximal σ-isotypic quotient of the minimal representation Π_p of $E_7(\mathbb{Q}_p)$ can be expressed as $\sigma \boxtimes \Theta(\sigma)$, with $\Theta(\sigma)$ a smooth representation of $\mathrm{PGSp}_6(\mathbb{Q}_p)$ which is called the big theta lift of σ.

Proposition 6.10. *For an irreducible admissible representation σ of $G_2(\mathbb{Q}_p)$, $\Theta(\sigma)$ has finite length with unique irreducible quotient (if nonzero) $\theta(\sigma)$. Moreover, one has the following.*

(1) *Let σ be an unramified generic representation of $G_2(\mathbb{Q}_p)$ with Satake parameter s. Then $\pi = \theta(\sigma)$ is the unramified representation of $\mathrm{PGSp}_6(\mathbb{Q}_p)$ whose Satake parameter is $\varphi \circ s$, where $\varphi : G_2 \hookrightarrow \mathrm{Spin}_7$ is the map of L-groups.*

(2) *Let St_{G_2} (resp. $\mathrm{St}_{\mathrm{PGSp}_6}$) be the Steinberg representation of $G_2(\mathbb{Q}_p)$ (resp. $\mathrm{PGSp}_6(\mathbb{Q}_p)$). Then $\theta(\mathrm{St}_{G_2}) = \mathrm{St}_{\mathrm{PGSp}_6}$.*

Proof. See [Gan and Savin 2023, Theorems 1.2, 15.3(v); Gross and Savin 1998, Proposition 3.1]. □

7. Cuspidality and Fourier coefficients of the global theta lift

In this section, based on [Ginzburg et al. 1997b; Gross and Savin 1998], and the appendix of Savin in [Harris et al. 2023], we give a criterion on the cuspidality of representations in the image of the exceptional theta lift and on their possession of Fourier coefficients of type (4 2).

7.1. ***Cuspidality of the global lift.*** Let V denote the unipotent subgroup of SL_3 (embedded into G_2 as in Section 6.2.2) generated by the roots $a+3b$ and $2a+3b$. We further consider the subgroup SL_2 embedded into G_2 via the Levi of the "long root" parabolic P_a and denote, for any cusp form φ for $G_2(\mathbb{A})$,

$$\varphi^{\mathrm{SL}_2 V}(g) := \int_{\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})} \int_{V(\mathbb{Q})\backslash V(\mathbb{A})} \varphi(vmg)\, dv\, dm.$$

We will now show that the above period vanishes whenever φ is not globally generic. We are thankful to David Ginzburg for kindly sharing with us a proof of this fact.

Lemma 7.1. *Let σ be a cuspidal automorphic representation of $G_2(\mathbb{A})$, which is not globally generic. For any cusp form $\varphi \in V_\sigma$ and $g \in G_2(\mathbb{A})$, we have $\varphi^{\mathrm{SL}_2 V}(g) = 0$.*

Proof. Let Z denote the unipotent subgroup of G_2 generated by the roots $a+2b$, $a+3b$, and $2a+3b$. Let $\varphi \in V_\sigma$. If we Fourier expand the period $\varphi^{\mathrm{SL}_2 V}(g)$ along the one-dimensional unipotent subgroup $x_{a+2b}(r)$ of G_2, we get

$$\varphi^{\mathrm{SL}_2 V}(g) = \varphi^{\mathrm{SL}_2 Z}(g) + \sum_{\psi} \varphi^{\mathrm{SL}_2 Z,\psi}(g),$$

where the sum runs over nontrivial additive characters $\psi : Z(\mathbb{Q})\backslash Z(\mathbb{A}) \to \mathbb{C}^\times$ supported on the root $a+2b$, $\varphi^{\mathrm{SL}_2 Z}(g)$ is the period of φ over $[\mathrm{SL}_2 Z]$, and

$$\varphi^{\mathrm{SL}_2 Z,\psi}(g) := \int_{\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})} \int_{Z(\mathbb{Q})\backslash Z(\mathbb{A})} \varphi(umg)\psi(u)\, du\, dm.$$

By [Ginzburg et al. 1997b, Lemma 2.1], $\varphi^{\mathrm{SL}_2 Z}(g) = 0$ for all φ in σ and $g \in G_2(\mathbb{A})$. Hence $\varphi^{\mathrm{SL}_2 V}(g) = 0$ if and only if $\varphi^{\mathrm{SL}_2 Z,\psi}(g) = 0$ for all nontrivial ψ. We now argue by contradiction. Suppose that $\varphi^{\mathrm{SL}_2 Z,\psi}(g) \neq 0$ for a certain ψ. We claim that this implies that σ supports Whittaker Fourier coefficients, thus contradicting our hypothesis.

Let U_a be the unipotent radical of P_a introduced in Section 6.2.1. Since V is normal in U_a, we can consider the quotient $V_0 = U_a/V$, which is isomorphic to the Heisenberg group in three variables and it is generated by the roots b, $a+b$, and $a+2b$. The center of V_0 is generated by the root $a+2b$ and is identified with the quotient $Z_0 := Z/V$. As SL_2 is embedded into G_2 via the Levi L_a of P_a, it acts trivially on the quotient Z_0. Therefore, $D := \mathrm{SL}_2 V_0$ is a Jacobi group in the sense of [Ikeda 1994, Definition on p. 619]. Let

$$\widetilde{D(\mathbb{A})} := \widetilde{\mathrm{SL}_2(\mathbb{A})}\, V_0(\mathbb{A}),$$

with $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ denoting the metaplectic cover of $\mathrm{SL}_2(\mathbb{A})$, and denote by $C^\infty_\psi(D(\mathbb{Q})\backslash \widetilde{D(\mathbb{A})})$ the space of functions f on $D(\mathbb{Q})\backslash\widetilde{D(\mathbb{A})}$ such that $f(zvh) = \psi(z) f(vh)$ for any $z \in Z_0(\mathbb{A})$, $v \in V_0(\mathbb{A})$, $h \in \widetilde{\mathrm{SL}_2(\mathbb{A})}$. For any Schwartz function $\Phi \in \mathcal{S}(\mathbb{A})$, we let $\theta^\Phi_{\mathrm{SL}_2} \in C^\infty_\psi(D(\mathbb{Q})\backslash\widetilde{D(\mathbb{A})})$ be the theta function defined in [Ikeda 1994, p. 620]. By [Ikeda 1994, Proposition 1.3], if W is a closed subspace of $C^\infty_{\psi^{-1}}(D(\mathbb{Q})\backslash\widetilde{D(\mathbb{A})})$ which is invariant under right translation of $V_0(\mathbb{A})$, the functions of the form

$$vh \mapsto \overline{\theta^{\Phi_1}_{\mathrm{SL}_2}}(vh) \int_{V_0(\mathbb{Q})\backslash V_0(\mathbb{A})} f(uh)\theta^{\Phi_2}_{\mathrm{SL}_2}(uh)\, du, \tag{14}$$

with $v \in V_0(\mathbb{A})$, $h \in \widetilde{\mathrm{SL}_2(\mathbb{A})}$, $f \in W$, $\Phi_1, \Phi_2 \in \mathcal{S}(\mathbb{A})$, generate a dense subspace of W. We apply this to the space W given by the closure of the subspace generated by the right $V_0(\mathbb{A})$-translations of

$$\varphi^{Z,\psi}(g) := \int_{Z(\mathbb{Q})\backslash Z(\mathbb{A})} \varphi(ug)\psi(u)\, du.$$

Assume that $\varphi^{\mathrm{SL}_2 Z,\psi}$ is not identically zero. By considering right translates of φ we can assume that $\varphi^{\mathrm{SL}_2 Z,\psi}(1)$ is nonzero. This implies that the integral

$$I_1(\varphi, \Phi_1, \Phi_2) := \int_{\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})} \overline{\theta^{\Phi_1}_{\mathrm{SL}_2}}(m) \int_{V_0(\mathbb{Q})\backslash V_0(\mathbb{A})} \varphi^{Z,\psi}(um)\theta^{\Phi_2}_{\mathrm{SL}_2}(um)\, du\, dm$$

is nonzero for some choice of data (Φ_1, Φ_2). Note that the integral $I_1(\varphi, \Phi_1, \Phi_2)$ is well-defined because the functions in (14) are not genuine for our space W. Since $\theta^{\Phi_2}_{\mathrm{SL}_2}(zg) = \psi(z)\theta^{\Phi_2}_{\mathrm{SL}_2}(g)$ for all $z \in Z_0(\mathbb{A})$, we can write $I_1(\varphi, \Phi_1, \Phi_2)$ as

$$\int_{\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})} \int_{(\mathbb{Q}\backslash\mathbb{A})^5} \varphi(x_b(v_1)x_{a+b}(v_2)x_{a+2b}(r_1)x_{a+3b}(r_2)x_{2a+3b}(r_3)m) \cdot \theta^{\Phi_2}_{\mathrm{SL}_2}(x_b(v_1)x_{a+b}(v_2)x_{a+2b}(r_1)m)\overline{\theta^{\Phi_1}_{\mathrm{SL}_2}(m)}\, dv_i\, dr_i\, dm.$$

The integral $I_1(\varphi, \Phi_1, \Phi_2)$ is the residue of the global zeta integral which calculates the standard L-function for φ when φ admits Whittaker coefficients. Namely, by the Siegel–Weil formula $\overline{\theta^{\Phi_1}_{\mathrm{SL}_2}(m)}$ is the residue at $s = \frac{3}{4}$ of an Eisenstein series $\mathrm{Eis}_{\widetilde{\mathrm{SL}}_2}(m, s)$ (depending on Φ_1) on the metaplectic cover of SL_2 normalized as in [Ginzburg 1993, §2]. By [Ginzburg 1993, Theorem 4] $I_1(\varphi, \Phi_1, \Phi_2)$ is the residue of

$$I_2(\varphi, \Phi_1, \Phi_2, s) := \int_{\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})} \int_{(\mathbb{Q}\backslash\mathbb{A})^5} \varphi(x_b(v_1)x_{a+b}(v_2)x_{a+2b}(r_1)x_{a+3b}(r_2)x_{2a+3b}(r_3)m) \cdot \overline{\theta^{\Phi_2}_{\mathrm{SL}_2}}(x_b(v_1)x_{a+b}(v_2)x_{a+2b}(r_1)m)\mathrm{Eis}_{\widetilde{\mathrm{SL}}_2}(m, s)\, dv_i\, dr_i\, dm.$$

We can now prove our claim. Suppose that $\varphi^{\mathrm{SL}_2 Z,\psi}(1) \neq 0$. Then $I_1(\varphi, \Phi_1, \Phi_2)$ is not zero for some choice (Φ_1, Φ_2). This implies that, for $\mathrm{Re}(s)$ large enough, the integral $I_2(\varphi, \Phi_1, \Phi_2, s)$ is not zero. By [Ginzburg 1993, Theorem 1], $I_2(\varphi, \Phi_1, \Phi_2, s)$ unfolds to the Whittaker model and thus contains a Whittaker coefficient of φ as an inner integration. This shows that if $\varphi^{\mathrm{SL}_2 Z,\psi}(1) \neq 0$ for some choice of data, the Whittaker coefficient for φ is nontrivial and thus σ is globally generic. This finishes the proof. □

Theorem 7.2. *Let σ be a cuspidal automorphic representation of $G_2(\mathbb{A})$. Assume that*

(1) *σ is not globally generic*;

(2) *there exists a finite place p such that σ_p is generic.*

Then the big theta lift $\Theta(\sigma)$ of σ to PGSp_6 *is cuspidal.*

Proof. We show the result by using the tower of theta lifts from G_2 and its properties studied in [Ginzburg et al. 1997b]. If σ lifts trivially to PGSp_6 then there is nothing to prove, so suppose that σ has a nonzero theta lift π to PGSp_6. Then, by [Ginzburg et al. 1997b, Theorem A] π is cuspidal if and only if the lifts of σ to PGSp_4 and PGL_3 are both zero. By [Ginzburg et al. 1997b, Theorem 4.1(3)], the lift to PGSp_4 is zero if and only if

$$\varphi^{\mathrm{SL}_3}(g) = \int_{[\mathrm{SL}_3]} \varphi(xg)\, dx = 0 \text{ and } \varphi^{\mathrm{SU}(2,1)}(g) = \int_{[\mathrm{SU}(2,1)]} \varphi(xg)\, dx = 0$$

for any $g \in G_2(\mathbb{A})$, any $\varphi \in V_{\sigma^\vee}$. Here, SL_3 embeds into G_2 as the stabilizer of a norm -1 vector (see Section 6.2.2), while $\mathrm{SU}(2, 1)$ is realized as the stabilizer of a norm $-c$ vector, with c not a square in $\mathbb{Q}$. We argue by contradiction. Suppose that $\sigma^\vee$ has a nontrivial $\mathrm{SU}(2, 1)$-functional. This implies that, at every finite v, σ_v admits one. By Frobenius reciprocity,

$$\mathrm{Hom}_{\mathrm{SU}(2,1)}(\sigma_v^\vee, \mathbb{C}) = \mathrm{Hom}_{G_2}(\text{c-Ind}^{G_2}_{\mathrm{SU}(2,1)}(\mathbb{C}), \sigma_v)$$

and hence, since σ_v is irreducible, one deduces that each local component σ_v of σ is a quotient of $C_c^\infty(G_2(\mathbb{Q}_v)/\mathrm{SU}(2,1)(\mathbb{Q}_v))$. In particular, σ_p is identified with such a quotient. This is a contradiction as, by hypothesis, σ_p is generic but, by [Gross and Savin 1998, Lemma 4.10], $C_c^\infty(G_2(\mathbb{Q}_p)/\mathrm{SU}(2,1)(\mathbb{Q}_p))$ does not admit a Whittaker functional. The same argument also shows the vanishing of φ^{SL_3}. We claim finally that the theta lift of σ to PGL_3 also vanishes. Since σ is not globally generic, Lemma 7.1 shows that, for all $\varphi \in \sigma$, $\varphi^{\mathrm{SL}_2 V}(g) = 0$. We can then apply [Ginzburg et al. 1997b, Theorem 4.1(4)] to deduce that the theta lift of σ to PGL_3 is zero and conclude the proof. □

Corollary 7.3. *Let σ be a cuspidal automorphic representation of $G_2(\mathbb{A})$. Assume that*

(1) *σ_∞ is a discrete series*;

(2) *there exists a finite place p such that σ_p is Steinberg.*

Then $\Theta(\sigma)$ is cuspidal.

Proof. We distinguish two cases. We first suppose that σ is globally generic. Then we apply [Harris et al. 2023, Theorem 1.7(ii)] to deduce that its theta lift is cuspidal. If, instead, σ is not globally generic, the result follows from Theorem 7.2 as the Steinberg representation $\sigma_p = \mathrm{St}_{G_2}$ is generic. □

7.2. ***Calculation of orbits.*** This preparatory section presents an elementary but crucial calculation needed in the proof of Proposition 7.7.

Let $e : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$ be the standard nontrivial character introduced in Section 5.2 and let $A \in J(\mathbb{Q})$. We define the character $\psi_A : \mathrm{N}(\mathbb{Q})\backslash\mathrm{N}(\mathbb{A}) \to \mathbb{C}^\times$ by $\psi_A(X) = e(\mathrm{Tr}(A \circ X))$, where $A \circ X = \frac{1}{2}(AX + XA)$ is the Jordan product. Recall from Section 5.2 that, for any $B \in J_3(\mathbb{Q})$, we define a character

$$\psi_B : U_3(\mathbb{Q})\backslash U_3(\mathbb{A}) \to \mathbb{C}^\times$$

by $\psi_B(n(X)) = e(\mathrm{Tr}(BX))$. In particular, we have denoted by ψ_D the character associated to

$$\alpha_D = \begin{pmatrix} 0 & & \\ & -D & \\ & & 1 \end{pmatrix} \in J_3(\mathbb{Q}).$$

Define

$$\omega(\mathbb{Q}) := \{A \in \Omega(\mathbb{Q}) \mid \psi_A|_{U_3(\mathbb{A})} = \psi_D\},$$

i.e., the set of rank-1 matrices in $J(\mathbb{Q})$ inducing the same character as α_D on the unipotent radical of the Siegel parabolic. In the following, we will always see $\omega(\mathbb{Q})$ inside $\overline{N}(\mathbb{Q})$. In particular, if $g \in \mathrm{GL}_3(\mathbb{Q}) \subseteq M(\mathbb{Q})$, its action on A is the dual action to (11), namely $g \cdot A = \det(g)(g^t)^{-1}Ag^{-1}$. Finally, denote by $A(x, y, z)$ the matrix

$$\begin{pmatrix} 0 & \bar{z} & y \\ z & -D & \bar{x} \\ \bar{y} & x & 1 \end{pmatrix} \in J.$$

Lemma 7.4. *We have*

$$\omega(\mathbb{Q}) = \bigl\{A(x, y, z) : \mathrm{Tr}(x) = \mathrm{Tr}(z) = 0, \mathrm{N}(x) = -D, \mathrm{N}(z) = 0, z \in x^\perp, y = -D^{-1}zx\bigr\}.$$

Proof. Let

$$A = \begin{pmatrix} d & \bar{z} & y \\ z & e & \bar{x} \\ \bar{y} & x & f \end{pmatrix} \in J.$$

Similarly to the proof of [Gross and Savin 1998, Lemma 3.4], the condition $\psi_A|_{U_3(\mathbb{A})} = \psi_D$ is equivalent to

$$d = 0, \qquad e = -D, \quad f = 1,$$
$$\bar{x} = -x, \quad \bar{y} = -y, \quad \bar{z} = -z.$$

This together with the condition that A has rank 1 (equation (10)) give

$$\mathrm{N}(x) = -D, \quad \mathrm{N}(y) = \mathrm{N}(z) = 0,$$
$$yz = 0, \qquad zx = -Dy, \qquad xy = z.$$

We claim that these conditions imply that $z \in x^{\perp}$, which means that $z\bar{x} + x\bar{z} = 0$, or equivalently $zx = -xz$. Indeed, multiplying $z = xy$ on the left by x and using alternativity, we obtain

$$xz = x(xy) = (xx)y = Dy = -zx.$$

Finally, as $\mathrm{N}(x) = -D$ and $\mathrm{Tr}(x) = 0$, we have $x^2 = D$ and hence $x^{-1} = D^{-1}x$, which implies that

$$y = x^{-1}z = D^{-1}xz.$$

This shows one inclusion of the statement.

In the other direction let $x, z \in \mathbb{O}$ be as in the right-hand side of the statement. We have to show that $y := -D^{-1}zx$ has norm and trace equal to zero and that $xy = z$. We have

$$\mathrm{N}(y) = (-D)^{-2}\mathrm{N}(z)\mathrm{N}(x) = 0 \quad \text{and} \quad \mathrm{Tr}(y) = -D^{-1}\,\mathrm{Tr}(zx) = D^{-1}\,\mathrm{Tr}(z\bar{x}) = 0$$

as $z \in x^{\perp}$. Hence $\mathrm{Tr}(y) = 0$. Moreover

$$xy = -D^{-1}x(zx) = D^{-1}x(xz) = D^{-1}(xx)z = z.$$

This shows that $A \in \omega(\mathbb{Q})$ and concludes the proof of the lemma. □

As for any $A(x, y, z) \in \omega(\mathbb{Q})$, the octonion $y = -D^{-1}zx$ is determined by x and z and we will often denote $A(x, y, z)$ by $A(x, z)$. Note that there is an action of $G_2(\mathbb{Q})$ on the set $\omega(\mathbb{Q})$ given by the action on the coefficients. The following proposition describing the orbits of this action will be essential.

Proposition 7.5. *The group $G_2(\mathbb{Q})$ acts on $\omega(\mathbb{Q})$ with a finite number of orbits. Moreover, representatives of the orbits and their respective stabilizers are given as follows.*

(1) *If D is a square in $\mathbb{Q}^{\times}$:*
   (a) $A_3 = A(x, 0)$, *where* $x = (s_4 - t_4)\sqrt{D}$ *and* $\mathrm{Stab}_{G_2(\mathbb{Q})}(A(x, 0)) \cong \mathrm{SL}_3$, *where* SL_3 *is embedded into* G_2 *as Section 6.2.2.*
   (b) $A_2 = A(x, t_3)$ *with* $\mathrm{Stab}_{G_2}(A_2) = \mathrm{SL}_2 V \subset \mathrm{SL}_3$, *where* SL_2 *and* V *embed into* SL_3 *as in Section 7.1.*

(c) $A_1 = A(x, s_3)$ *with* $\mathrm{Stab}_{G_2}(A_1) = \mathrm{SL}_2\overline{V} \subset \mathrm{SL}_3$, *where* SL_2 *is as in* (1)(*b*) *and* $\overline{V}$ *is the opposite unipotent subgroup to* V.

(d) $A_0 = A(x, s_1 + t_3)$ *with* $\mathrm{Stab}_{G_2}(A_0) = U_D$, *where* U_D *denotes the unipotent radical of the upper-triangular Borel of* SL_3 (*denoted by* U_{SL_3} *in Section 7.1*).

(2) *If* D *is not square in* $\mathbb{Q}^\times$:

(a) $A_1 = A(x, 0) \in \omega(\mathbb{Q})$, *for any* $x \neq 0$ *for which* $N(x) = -D$, *with*

$$\mathrm{Stab}_{G_2(\mathbb{Q})}(A(x, 0)) \cong \mathrm{SU}_3^D,$$

where $\mathrm{SU}_3^D = \mathrm{SU}(x^\perp)$ *is the unitary group for the restriction of the norm form to the three-dimensional* $\mathbb{Q}(\sqrt{D})$*-subspace of* $\mathbb{O}^0$ *orthogonal to* x.[1]

(b) $A_0 = A(x, z)$, *for any norm zero* z *in* $x^\perp$, *with* $\mathrm{Stab}_{G_2}(A_0) \simeq U_D$, *where* U_D *denotes the unipotent radical of the upper-triangular Borel of* SU_3^D.

Proof. Step 1. By [Rallis and Schiffmann 1989, Theorem 1], the group G_2 acts transitively on the set of trace zero elements of norm $-D$ and hence on the sets $A(x, 0)$. The description of the stabilizer in (1)(a) follows from [Jacobson 1958, Theorem 4] or [Rallis and Schiffmann 1989, Lemma 2]. The description of the stabilizer in (2)(a) follows from [Jacobson 1958, Theorem 3] or [Rallis and Schiffmann 1989, Lemma 3]. More precisely, according to [Rallis and Schiffmann 1989, Lemma 3] the subspace $x^\perp$ of $\mathbb{O}^0$ of elements which are orthogonal to x has the structure of a three-dimensional $\mathbb{Q}(\sqrt{D})$-vector space and the action of $\mathrm{Stab}_{G_2}(x)$ on $x^\perp$ induces an isomorphism $\mathrm{Stab}_{G_2}(x) \simeq \mathrm{SU}_3^D$.

Step 2. We now study the remaining G_2-orbits when D is a square in $\mathbb{Q}$. Again, we can assume that $D = 1$. Recall from Section 6.2.2 that SL_3 embeds into G_2 as the stabilizer of $s_4 - t_4$. This identification is explicitly given as follows (see [Rallis and Schiffmann 1989, Lemma 2]). An element of $g \in \mathrm{SL}_3$ induces an action on $\mathbb{O}^0$ fixing $s_4 - t_4$ and given by the left multiplication by g on $\langle s_1, s_2, s_3\rangle$ and by $(g^t)^{-1}$ on $\langle t_1, t_2, t_3\rangle$. One verifies that this actions respects multiplication and hence defines an element in G_2. Assume $z \neq 0$ is such that $A(x, z) \in \omega(\mathbb{Q})$. Since z is trace zero and orthogonal to $x = s_4 - t_4$ we can write $z = z_1 + z_2$ with $z_1 = \sum_i \alpha_i s_i$ and $z_2 = \sum_i \beta_i t_i$. Since the group SL_3 acts transitively on the nonzero elements of $\langle s_1, s_2, s_3\rangle$ and $\langle t_1, t_2, t_3\rangle$, then the cases where $z_1 = 0$ or $z_2 = 0$ give rise to exactly two orbits. When $z_1 = 0$, taking $z_2 = t_3$ as a generator of this orbit, the corresponding stabilizer is

$$\left\{\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix}\right\} \subset \mathrm{SL}_3,$$

which coincides with $\mathrm{SL}_2 V$ as in (1)(b). Similarly, when $z_2 = 0$, taking $z_1 = s_3$ as the generator of the orbit, then the stabilizer is

$$\left\{\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 1 \end{pmatrix}\right\} \subset \mathrm{SL}_3.$$

[1]The unitary group SU_3^D is a form of SL_3 which splits over $\mathbb{Q}(\sqrt{D})$ and which is isomorphic to $\mathrm{SU}(2, 1)$ (resp. SL_3) if $D < 0$ (resp. $D > 0$) over $\mathbb{R}$.

This is nothing but $\mathrm{SL}_2\overline{V}$, with SL_2 which again embeds in the Levi of the long root parabolic P_a and $\overline{V}$ is the opposite unipotent subgroup to V generated by the negative roots $-a-3b$, $-2a-3b$. Finally we treat the case $z_1, z_2 \neq 0$. Write

$$z = \alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3 + \beta_1 t_1 + \beta_2 t_2 + \beta_3 t_3.$$

The condition $N(z) = 0$ translates then in

$$\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0. \tag{15}$$

We can assume that $z_2 = t_3$. Then $\alpha_3 = 0$ by (15) and, using the action of the stabilizer of t_3, we can assume that $z_1 = s_1$. It is then immediate to check that the stabilizer of $A(s_4 - t_4, s_1 + t_3)$ is as in (1)(d). This concludes the proof of (1).

Step 3. We finally deal with the case where D is not a square in $\mathbb{Q}$. By Witt's theorem, the group SU_3^D acts transitively on the isotropic vectors of the three-dimensional $\mathbb{Q}(\sqrt{D})$ vector space $x^\perp$. We thus have two orbits for $G_2(\mathbb{Q})$ on $\omega(\mathbb{Q})$, generated by $A(x, 0)$ and $A(x, z)$, where z is any nonzero vector in $x^\perp$ with zero norm. We are now left with calculating the stabilizer of the latter orbit. The action of SU_3^D on $x^\perp$ is given by its natural action on $\mathbb{Q}(\sqrt{D})^3$. More precisely, after extending scalars to $\mathbb{Q}(\sqrt{D})$, we can decompose

$$x^\perp \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{D}) = \mathbb{Q}(\sqrt{D})\langle s_1, s_2, s_3\rangle \oplus \mathbb{Q}(\sqrt{D})\langle t_1, t_2, t_3\rangle.$$

The projection to the first component induces an isomorphism of $\mathbb{Q}(\sqrt{D})$-vector spaces

$$x^\perp \simeq \mathbb{Q}(\sqrt{D})\langle s_1, s_2, s_3\rangle$$

(see [Rallis and Schiffmann 1989, Lemma 3]), with SU_3^D acting naturally on the basis $\{s_1, s_2, s_3\}$. Here, we choose the Hermitian form (with respect to the extension $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$) defining SU_3^D given by

$$\begin{pmatrix} & & \sqrt{D}^{-1} \\ & 1 & \\ -\sqrt{D}^{-1} & & \end{pmatrix} \in \mathrm{GL}_3(\mathbb{Q}(\sqrt{D})).$$

We can then suppose that z is sent to s_1 and the corresponding stabilizer is given by

$$\left\{\begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}\right\} \cap \mathrm{SU}_3^D = U_D. \qquad \square$$

7.3. ***Nonvanishing of Fourier coefficients, I.*** Recall that we have denoted by $\Pi = \bigotimes_v' \Pi_v$ the minimal representation of the group E_7. Moreover, in Section 6.3, for $f \in \Pi$ and $\varphi \in \mathcal{A}(G_2(\mathbb{Q})\backslash G_2(\mathbb{A}))$, we have defined the function $\Theta(f, \varphi)$ on $\mathrm{PGSp}_6(\mathbb{A})$ by

$$\Theta(f, \varphi)(g) = \int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} \theta(f)(g'g)\varphi(g')\, dg'. \tag{16}$$

For any $A \in J(\mathbb{Q})$ and $f \in \Pi$, consider the Fourier coefficient

$$\theta(f)_A(g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \theta(f)(ng)\psi_A^{-1}(n)\, dn.$$

We then have the Fourier expansion (see [Harris et al. 2023, §A.3])

$$\theta(f)(g) = \theta(f)_0(g) + \sum_{A \in \Omega(\mathbb{Q})} \theta(f)_A(g), \tag{17}$$

where $\Omega(\mathbb{Q}) \subset J(\mathbb{Q})$ is the subset of rank-1 elements.

The following lemma will be used in the proof of Proposition 7.7. Its proof is similar to that of [Gross and Savin 1998, Lemma 4.6] but we give details for the convenience of the reader. Let A_0 be the representative of the open G_2-orbit on $\omega(\mathbb{Q})$ given in Proposition 7.5. Note that there is no harm in conjugating $A_0 \in J(\mathbb{Q})$ by an element of the Levi $\mathrm{GL}_3(\mathbb{Q})$ of the Siegel parabolic of PGSp_6. Thus, conjugating by $\mathrm{diag}(n, n, n)$, A_0 gets multiplied by n^2 and so we can assume that the entries x, y, z of A_0 are in $\mathbb{O}(\mathbb{Z})$.

Lemma 7.6. *Let S denote a finite number of places containing 2 and ∞, and let $f = \otimes'_v f_v \in \Pi$ be such that, for $v \notin S$, we have $f_v = f_v^0$, where f_v^0 denotes the spherical vector normalized such that $f_v^0(A_0) = 1$. Let $\mathbb{Q}_S = \prod_{v \in S} \mathbb{Q}_v$. If $g \in G_2(\mathbb{A})$, we write $g = g_S g^S$ where $g_S \in G_2(\mathbb{Q}_S)$ and $g^S \in \prod_{v \notin S} G_2(\mathbb{Q}_v)$. Then there exists a nonzero constant c_{A_0} such that for every $g \in G_2(\mathbb{A})$ we have*

$$\theta(f)_{A_0}(g) = c_{A_0} f_S(g_S^{-1} A_0) \prod_{v \notin S} \chi_v(g_v),$$

where $f_S = \bigotimes_{v \in S} f_v$ and χ_v is the characteristic function of $U_D(\mathbb{Z}_v)\backslash G_2(\mathbb{Z}_v)$.

Proof. By uniqueness of local functionals [Harris et al. 2023, Theorem A.4], there exists a nonzero scalar c_{A_0} such that, for any $g \in E_7(\mathbb{A})$, we have $\theta(f)_{A_0}(g) = c_{A_0}(\Pi(g)f)(A_0)$. For $g \in G_2(\mathbb{A})$ we have $(\Pi(g)f)(A_0) = f(g^{-1}A_0)$, where $g^{-1}A_0$ is the result of the natural action of g^{-1} on the off-diagonal entries of A_0. Hence $\theta(f)_{A_0}(g) = c_{A_0} f(g^{-1}A_0) = c_{A_0} \prod_v f_v(g_v^{-1}A_0)$ for $g \in G_2(\mathbb{A})$. Let us prove that for any $p \notin S$, we have $f_p(g_p^{-1}A_0) = \chi_p(g_p)$. So let $g_p \in G_2(\mathbb{Q}_p)$ be such that $f_p^0(g_p^{-1}A_0) \neq 0$ and let x', y', z' denote the off-diagonal entries of $g_p^{-1}A_0$. According to [Harris et al. 2023, Theorem A.5] the spherical vector f_p^0 is supported in $J(\mathbb{Z}_p)$. Hence $x', y', z' \in \mathbb{O}(\mathbb{Z}_p)$. Consider $\mathbb{O}(\mathbb{F}_p)$ the split octonion algebra over $\mathbb{F}_p$. The projections of (x, y, z) and (x', y', z') to $\mathbb{O}(\mathbb{F}_p)$ are $G_2(\mathbb{F}_p)$-conjugated by the proof of Step 1 in Proposition 7.5, which is still valid over the base field $\mathbb{F}_p$ as long as $p \neq 2$. It follows from Hensel's lemma that (x, y, z) and (x', y', z') are $G_2(\mathbb{Z}_p)$-conjugated. Therefore the function $g_p \mapsto f_p^0(g_p^{-1}A_0)$ is supported in $U_D(\mathbb{Z}_p)\backslash G_2(\mathbb{Z}_p) \subset U_D(\mathbb{Q}_p)\backslash G_2(\mathbb{Q}_p)$. Since f_p^0 is $G_2(\mathbb{Z}_p)$-invariant, for $g_p \in G_2(\mathbb{Z}_p)$ we have $f_p^0(g_p^{-1}A_0) = f_p^0(A_0) = 1$. This completes the proof. □

Proposition 7.7. *Let σ be a cuspidal automorphic representation of $G_2(\mathbb{A})$ as in Theorem 7.2 and let $\varphi \in \sigma^\vee$ be a cuspidal form. Then the following conditions are equivalent*:

(1) $\Theta(f, \varphi)_{U_P, \psi_D}(1) \neq 0$ *for some choice of f.*

(2) $\varphi^{U_D}(g) \neq 0$ *for some $g \in G_2(\mathbb{A})$.*

In particular, if any of the conditions holds then $\Theta(\sigma)$ is nonzero.

Proof. Recall first that, according to Proposition 5.7, we have $\Theta(f,\varphi)_{U_P,\psi_D}\neq 0$ if and only if $\Theta(f,\varphi)_{U_3,\alpha}\neq 0$ for some $\alpha\in\mathrm{Sym}^{\mathrm{rk}2}(3)(\mathbb{Q})$ with $\alpha\sim_{L(\mathbb{Q})}\alpha_D$. We write

$$\begin{aligned}\Theta(f,\varphi)_{U_3,\psi_D}(1)&=\int_{U_3(\mathbb{Q})\backslash U_3(\mathbb{A})}\Theta(f,\varphi)(u)\psi_D^{-1}(u)\,du\\&=\int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})}\int_{U_3(\mathbb{Q})\backslash U_3(\mathbb{A})}\sum_{A\in\Omega(\mathbb{Q})}\theta(f)_A(ug)\varphi(g)\psi_D^{-1}(u)\,du\,dg,\end{aligned}$$

where in the second equality we used the definition (16) of $\Theta(f,\varphi)$ and the Fourier expansion (17) of $\theta(f)$. Since $U_3\subseteq N$, we have that $\theta(f)_A(ug)=\psi_A(u)\theta(f)_A(g)$ and

$$\int_{U_3(\mathbb{Q})\backslash U_3(\mathbb{A})}\psi_A(u)\psi_D^{-1}(u)=\begin{cases}\mathrm{vol}(U_3(\mathbb{Q})\backslash U_3(\mathbb{A})) & \text{if } \psi_D=\psi_A|_{U_3(\mathbb{A})},\\ 0 & \text{otherwise.}\end{cases}$$

Hence

$$\Theta(f,\varphi)_{U_3,\psi_D}(1)=\mathrm{vol}(U_3(\mathbb{Q})\backslash U_3(\mathbb{A}))\int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})}\sum_{A\in\omega(\mathbb{Q})}\theta(f)_A(g)\varphi(g)\,dg. \tag{18}$$

Let $(A_i)_i$ be the finite representatives of the orbits of the action of $G_2(\mathbb{Q})$ on $\omega(\mathbb{Q})$ as given by Proposition 7.5, and write Stab_{A_i} for the stabilizers of A_i in G_2. The integral on the right-hand side of (18) becomes

$$\sum_i\int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})}\sum_{g'\in\mathrm{Stab}_{A_i}(\mathbb{Q})\backslash G_2(\mathbb{Q})}\theta(f)_{A_i}(g'g)\varphi(g)\,dg=\sum_i\int_{\mathrm{Stab}_{A_i}(\mathbb{Q})\backslash G_2(\mathbb{A})}\theta(f)_{A_i}(g)\varphi(g)\,dg.$$

Observe now that, by [Harris et al. 2023, Theorem A.4], we have $\theta(f)_{A_i}(g)=c_{A_i}f(g^{-1}A_i)$ for any $g\in G_2$. Hence, since $\mathrm{Stab}_{A_i}(\mathbb{A})$ fixes the matrix A_i, we deduce that the function $g\mapsto\theta(f)_{A_i}(g)$ is left $\mathrm{Stab}_{A_i}(\mathbb{A})$-invariant. Making an inner integration over $\mathrm{Stab}_{A_i}(\mathbb{Q})\backslash\mathrm{Stab}_{A_i}(\mathbb{A})$ in each term of the outer sum, we deduce that the above equals

$$\sum_i\int_{\mathrm{Stab}_{A_i}(\mathbb{A})\backslash G_2(\mathbb{A})}\theta(f)_{A_i}(g)\varphi^{\mathrm{Stab}_{A_i}}(g)\,dg,$$

where $\varphi^{\mathrm{Stab}_{A_i}}(g)$ denotes the period of φ over $\mathrm{Stab}_{A_i}(\mathbb{Q})\backslash\mathrm{Stab}_{A_i}(\mathbb{A})$. We now analyze two different possibilities. If D is not a square in $\mathbb{Q}$, then, by Proposition 7.5(2), $G_2(\mathbb{Q})$ acts on $\omega(\mathbb{Q})$ with two orbits, one closed and one open. Let A_0, A_1 denote representatives of these two orbits with stabilizers $\mathrm{Stab}_{A_0}=U_D$ and $\mathrm{Stab}_{A_1}=\mathrm{SU}_3^D$ in G_2. By the proof of Theorem 7.2, $\varphi^{\mathrm{SU}_3^D}(g)=0$, and hence the only surviving term is the one corresponding to the orbit represented by A_0. If D is a square in $\mathbb{Q}$, then by Proposition 7.5(1), $G_2(\mathbb{Q})$ acts on the set $\omega(\mathbb{Q})$ with four orbits, three closed and one open. Let A_i, $0\le i\le 3$ denote representatives of those orbits, with A_0 representing the open one. The corresponding stabilizers are U_D, SL_3, SL_2V and its conjugate $\mathrm{SL}_2\overline{V}$. By the proof of Theorem 7.2, we have $\varphi^{\mathrm{SL}_3}(g)=0$. By hypothesis σ (and $\sigma^\vee$) is not globally generic; hence Lemma 7.1 implies that $\varphi^{\mathrm{SL}_2V}(g)=\varphi^{\mathrm{SL}_2\overline{V}}(g)=0$. From this, we deduce that, for any D,

$$\Theta(f,\varphi)_{U_3,\psi_D}(1)=\int_{U_D(\mathbb{A})\backslash G_2(\mathbb{A})}\theta(f)_{A_0}(g)\varphi^{U_D}(g)\,dg, \tag{19}$$

where $\varphi^{U_D}(g)$ is the constant term of φ along U_D. This shows that if $\Theta(f,\varphi)_{U_3,\psi_D}(1)\neq 0$ then $\varphi^{U_D}\neq 0$ since the period appears as an inner integral of the Fourier coefficient.

We now show the converse, i.e., that if $\varphi^{U_D}\neq 0$ then, for some choice of $f\in\Pi$, the Fourier coefficient $\Theta(f,\varphi)_{U_3,\psi_D}$ does not vanish. Let S be as in Lemma 7.6. By enlarging S if necessary, we can assume that the cusp form φ is $G_2(\mathbb{Z}_v)$-invariant for all $v\notin S$. By Lemma 7.6, the integral of (19) equals

$$c_{A_0}\cdot\left(\int_{U_D(\mathbb{Q}_S)\backslash G_2(\mathbb{Q}_S)} f_S(g^{-1}A_0)\varphi^{U_D}(g)\,dg\right)\cdot\prod_{v\notin S}\mathrm{vol}(U_D(\mathbb{Z}_v)\backslash G_2(\mathbb{Z}_v),dg_v).$$

It remains to show that, when $\varphi^{U_D}\neq 0$, then for a good choice of f at the places in S, the integral satisfies

$$\int_{U_D(\mathbb{Q}_S)\backslash G_2(\mathbb{Q}_S)} f_S(g^{-1}A_0)\varphi^{U_D}(g)\,dg\neq 0.$$

It follows from [Harris et al. 2023, Theorem A.4] that f_S can be any smooth compactly supported function on $\Omega(\mathbb{Q}_S)$. Let $g_0\in G_2(\mathbb{Q}_S)$ be such that $\varphi^{U_D}(g_0)\neq 0$. We can take a nonnegative f supported in a sufficiently small neighborhood of g_0 to ensure the nonvanishing of the integral. This finishes the proof of the proposition. □

7.4. ***Nonvanishing of Fourier coefficients, II.*** The purpose of this section is to prove the following result.

Theorem 7.8. *Let F denote a quadratic étale algebra and $\sigma=\sigma_\infty\otimes\sigma_f$ be a cuspidal automorphic representation of $G_2(\mathbb{A})$ such that*

- *σ_∞ is a nongeneric discrete series with infinitesimal character $r\varepsilon_1+s\varepsilon_2$;*
- *there exists a finite prime p such that σ_p is Steinberg;*
- *the representation σ supports Fourier coefficient associated to the cubic algebra $\mathbb{Q}\times F$.*

The theta lift $\Theta(\sigma)=\otimes'_v\Theta(\sigma_v)$ is a nonzero cuspidal automorphic representation of $\mathrm{PGSp}_6(\mathbb{A})$. *Moreover, if π denotes any nonzero irreducible subquotient of $\Theta(\sigma)$, then*

- *π_∞ is a discrete series of infinitesimal character $\left(r,\tfrac{1}{2}(r+s),\tfrac{1}{2}(r-s)\right)$;*
- *π_p is Steinberg;*
- *the representation π supports a nontrivial Fourier coefficient of type* $(4\,2)$ *associated to F.*

Remark 7.9. As it will follow from the proof, the condition of σ_∞ being nongeneric can be replaced by σ not being locally generic, i.e., that there exists one local component of σ_v of σ which is not generic.

Let us first fix some notation first. Recall from Section 6.1.2 that the centralizer of G_2 in M is GL_3 and let

$$U_0=\left\{\begin{pmatrix}1&a&b\\0&1&0\\0&0&1\end{pmatrix}\right\}$$

be the unipotent radical of its Borel subgroup of upper triangular matrices. Note that the unipotent subgroup U_0U_3 is the unipotent radical of the parabolic subgroup P of PGSp_6 of Levi $\mathrm{GL}_2\times\mathrm{GL}_1^2$ appearing in Section 5.2.2.

Definition 7.10. Define the character $\psi_0 : U_0(\mathbb{Q})\backslash U_0(\mathbb{A}) \to \mathbb{C}^\times$ by sending

$$\psi_0(u) = e(a).$$

Note that $\psi_0\psi_D$ is the character (simply denoted by ψ_D) on $U_P(\mathbb{Q})\backslash U_P(\mathbb{A})$ introduced in Section 5.2.2.

As explained in Section 7.2, we view the space Ω of rank-1 elements in J inside $\overline{N}$ so that U_0 acts on Ω via the natural right action of $\mathrm{GL}_3 \subseteq M$ on $\overline{N}$. Then we let U_0 act on the left on ω and hence on the triples (x, y, z) of off-diagonal terms by the rule

$$u^{-1} \cdot (x, y, z) = (x + ay + bz, y, z). \tag{20}$$

7.4.1. *The relation between U_P and U_H.* In what follows, we relate the unipotent subgroup U_0 to the unipotent radical U_H of the Heisenberg parabolic. Such a relation will be employed in Proposition 7.13 to establish a relation between Fourier coefficients for the Heisenberg parabolic of G_2-cusp forms and Fourier coefficients of type (4 2) of their theta lifts.

Before stating our result, we make the following comments on the choice of representatives of the open orbits in Proposition 7.5. First, suppose that $D = d^2$, with $d \in \mathbb{Q}^\times$. There is no harm in assuming $d \in \mathbb{Z}$. Recall that the stabilizer in G_2 of the vector $s_4 - t_4$ can be identified with $\mathrm{SL}_3 = \mathrm{SL}(\langle s_1, s_2, s_3\rangle)$. Since the Heisenberg parabolic $H = L_H \cdot U_H$ is the stabilizer of the flag $\langle s_1, t_3\rangle$, its unipotent radical U_H contains $U_D = \mathrm{Stab}_{G_2}(A_0)$, where

$$A_0 = A(d(s_4 - t_4), s_1 - t_3, d(s_1 + t_3)) \in J(\mathbb{Z})$$

is the representative of the open orbit of the action of G_2 on $\omega(\mathbb{Q})$ as in Proposition 7.5. Moreover, U_H/U_D is two-dimensional and supported on the roots $a + b$ and $a + 2b$. Let us now suppose that D is not a square in $\mathbb{Q}^\times$. The vector $x = s_2 + Dt_2$ is a trace zero octonion of norm $-D$ and orthogonal to t_3. We choose the representative of the open orbit to be

$$A_0 = A(s_2 + Dt_2, s_1, t_3) \in J(\mathbb{Z}).$$

Lemma 7.11. *There is a natural surjection $p : U_H \to U_0$ inducing an isomorphism*

$$U_H/U_D \to U_0.$$

Proof. By the description of the action in (20) and the linear independence of the coordinates (x, y, z) of the representative of the open orbit, one sees that U_0 acts freely on it. Hence, the result follows from showing that any element in U_H acts on the triple (x, y, z) as an element of U_0 and vice versa.

Case 1. We start with the case where D is a square in $\mathbb{Q}^\times$. The action of U_0 is given by

$$u^{-1} \cdot (d(s_4 - t_4), s_1 - t_3, d(s_1 + t_3)) = (d(s_4 - t_4) + (a + db)s_1 + (db - a)t_3, s_1 - t_3, d(s_1 + t_3)). \tag{21}$$

Since any element of U_H fixes s_1 and t_3, it suffices to show that U_H acts on $(s_4 - t_4)$ as an element of U_0. We verify this by studying the action of the Lie algebra. By (13), we know that the Lie algebra of U_H is generated by the Lie algebra of the unipotent upper-triangular subgroup U_D in SL_3 and by the vectors v_1 and δ_3. Using the explicit action of the action of the Lie algebra given in Section 6.2.3, one checks that

$$E_{ij}\cdot(s_4-t_4)=0,\quad v_1\cdot(s_4-t_4)=s_1,\quad \delta_3\cdot(s_4-t_4)=t_3.$$

The above equations show that, for $u_1=x_{a+b}(\lambda_1)$ and $u_2=x_{a+2b}(\lambda_2)$ for some scalars λ_1,λ_2, we have

$$u_1\cdot(d(s_4-t_4))=d(s_4-t_4+\lambda_1 s_1),\quad u_2\cdot(d(s_4-t_4))=d(s_4-t_4+\lambda_2 t_3).$$

This gives the desired isomorphism: if $u\in U_H/U_D$ is identified with the product of $x_{a+b}(\lambda_1)x_{a+2b}(\lambda_2)$, then, from (21), we see that it gets sent to the element

$$\begin{pmatrix}1 & \frac{1}{2}d(\lambda_1-\lambda_2) & \frac{1}{2}(\lambda_1+\lambda_2)\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\in U_0.$$

Case 2. We now suppose that D is not a square in $\mathbb{Q}^\times$. Similarly to Case 1, it suffices to calculate $u\cdot(s_2+Dt_2)$ for any $u\in U_H$. As above, one checks that

$$E_{12}\cdot(s_2+Dt_2)=s_1,\quad E_{23}\cdot(s_2+Dt_2)=Dt_3,\quad E_{13}\cdot(s_2+Dt_2)=0,$$
$$v_1\cdot(s_2+Dt_2)=t_3,\quad \delta_3\cdot(s_2+Dt_2)=-Ds_1.$$

This implies that if $u\in V_H=U_H/[U_H,U_H]$ is equal to $x_a(\lambda_1)x_{a+b}(\lambda_2)x_{a+2b}(\lambda_3)x_{a+3b}(\lambda_4)$, then

$$u\cdot(s_2+Dt_2)=s_2+Dt_2+(\lambda_1-\lambda_3 D)s_1+(\lambda_2+D\lambda_4)t_3.$$

In particular, U_D embeds into U_H as the subgroup of matrices with $\lambda_1=\lambda_3 D$ and $\lambda_2=-\lambda_4 D$, and the map $p:U_H/U_D\to U_0$ sends u to the element

$$\begin{pmatrix}1 & \lambda_1-\lambda_3 D & \lambda_2+\lambda_4 D\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\in U_0. \qquad\square$$

Corollary 7.12. *Under the isomorphism $p:U_H/U_D\to U_0$, we have*

$$\psi_{H,D}=\psi_0\circ p,$$

where $\psi_{H,D}:U_H(\mathbb{Q})\backslash U_H(\mathbb{A})\to\mathbb{C}^\times$ is the character corresponding to the étale cubic algebra $\mathbb{Q}\times\mathbb{Q}(\sqrt{D})$.

Proof. We start with the case where D is a square in $\mathbb{Q}^\times$. For simplicity, we can (and do) assume that $D=1$. From Lemma 7.11, if $n\in U_H/U_D$ is identified with the product of $x_{a+b}(\lambda_1)x_{a+2b}(\lambda_2)$, it is sent via p to

$$\begin{pmatrix}1 & \frac{1}{2}(\lambda_1-\lambda_2) & \frac{1}{2}(\lambda_1+\lambda_2)\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\in U_0.$$

Hence, the character $\psi_0\circ p:U_H(\mathbb{Q})\backslash U_H(\mathbb{A})\to\mathbb{C}^\times$ sends $n\mapsto e\big(\frac{1}{2}(\lambda_1-\lambda_2)\big)$. We now show that this corresponds to the character $\psi_{H,D}$ associated to $\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}$ as in Section 6.2.4. Recall that each character on $U_H(\mathbb{Q})\backslash U_H(\mathbb{A})$ is of the form $n \mapsto e(\langle w, \bar{n}\rangle)$, where $\bar{n}$ denotes the projection of n to $U_H/[U_H, U_H]$ and $w \in U_H(\mathbb{Q})/[U_H(\mathbb{Q}), U_H(\mathbb{Q})]$ corresponds to a binary cubic form

$$f_w(x, y) = \lambda_1 x^3 + \lambda_2 x^2 y + \lambda_3 x y^2 + \lambda_4 y^3,$$

with $\lambda_i \in \mathbb{Q}$. Furthermore, as $\bar{n} = x_a(\lambda'_1)x_{a+b}(\lambda'_2/3)x_{a+2b}(\lambda'_3/3)x_{a+3b}(\lambda'_4)$ corresponds to $f'(x, y) = \lambda'_1 x^3 + \lambda'_2 x^2 y + \lambda'_3 x y^2 + \lambda'_4 y^3$, the pairing is

$$\langle w, \bar{n}\rangle = \lambda_1\lambda'_4 - \tfrac{1}{3}\lambda_2\lambda'_3 + \tfrac{1}{3}\lambda_3\lambda'_2 - \lambda_4\lambda'_1.$$

Then, the character $\psi_0 \circ p$ corresponds to an element w_D for which $\lambda_1, \lambda_4 = 0$ and $\lambda_2, \lambda_3 = \frac{1}{2}$, namely the binary cubic polynomial $f_D(x, y) = \frac{1}{2}(x^2 y + x y^2)$. The latter is in the $L_H(\mathbb{Q})$-orbit corresponding to the cubic algebra $\mathbb{Q}^3$. Indeed, if we let $g = \left(\begin{smallmatrix} 2 & \\ & -2 \end{smallmatrix}\right) \in L_H(\mathbb{Q})$ act on f_D, we get

$$g \cdot f_D(x, y) = -\tfrac{1}{4} f_D(2x, -2y) = \tfrac{1}{8}(8x^2 y - 8 x y^2) = x^2 y - x y^2,$$

which corresponds to $\mathbb{Q}^3$ by Example 6.1(1).

We now suppose that D is not a square in $\mathbb{Q}^\times$. Then, by Lemma 7.11, if

$$n \equiv x_a(\lambda_1)x_{a+b}(\lambda_2)x_{a+2b}(\lambda_3)x_{a+3b}(\lambda_4) \bmod [U_H, U_H],$$

the character $\psi_0 \circ p : U_H(\mathbb{Q})\backslash U_H(\mathbb{A}) \to \mathbb{C}^\times$ sends $n \mapsto e(\lambda_1 - \lambda_3 D)$. This character is associated to the binary cubic polynomial $f_D(x, y) = D x^2 y - y^3$, which corresponds to $\mathbb{Q} \times \mathbb{Q}(\sqrt{D})$ by Example 6.1(2). □

7.4.2. *Comparison of Fourier coefficients.* The following proposition can be paired with Proposition 7.7 to give three equivalent ways of proving that the theta lift of an automorphic representation of G_2 does not vanish.

Proposition 7.13. *Let σ be a cuspidal automorphic representation of $G_2(\mathbb{A})$ as in Theorem 7.2 and let $\varphi \in \sigma^\vee$ be a cuspidal form. The following conditions are equivalent*:

(1) $\Theta(f, \varphi)_{U_P,\psi_D}(1) \neq 0$ *for some choice of* $f \in \Pi$.

(2) $\varphi_{U_H,\psi_{H,D}}(g) \neq 0$ *for some* $g \in G_2(\mathbb{A})$.

In particular, if any of the conditions holds then $\Theta(\sigma)$ is nonzero.

Proof. Decomposing $U_P = U_0 U_3$, we have

$$\Theta(f, \varphi)_{U_P,\psi_D}(1) = \int_{U_0(\mathbb{Q})\backslash U_0(\mathbb{A})} \int_{U_3(\mathbb{Q})\backslash U_3(\mathbb{A})} \Theta(f, \varphi)(uu')\psi_D^{-1}(u')\psi_{U_0}^{-1}(u)\, du'\, du.$$

As in the proof of Proposition 7.7, this equals

$$\int_{U_0(\mathbb{Q})\backslash U_0(\mathbb{A})} \int_{U_D(\mathbb{A})\backslash G_2(\mathbb{A})} \theta(f)_{A_0}(ug)\varphi^{U_D}(g)\psi_{U_0}^{-1}(u)\, dg\, du.$$

Exchanging integrals and making an inner integration over $U_D(\mathbb{A})\backslash U_H(\mathbb{A})$, we get

$$\int_{U_H(\mathbb{A})\backslash G_2(\mathbb{A})} \int_{U_D(\mathbb{A})\backslash U_H(\mathbb{A})} \left(\int_{U_0(\mathbb{Q})\backslash U_0(\mathbb{A})} \theta(f)_{A_0}(uu'g)\psi_{U_0}^{-1}(u)\, du \right) \varphi^{U_D}(u'g)\, du'\, dg.$$

The isomorphism $p : U_H/U_D \cong U_0$ of Lemma 7.11 induces

$$U_0(\mathbb{Q})\backslash U_0(\mathbb{A}) \cong U_H(\mathbb{Q})U_D(\mathbb{A})\backslash U_H(\mathbb{A})$$

such that $\psi_{H,D} = \psi_0 \circ p$ (see Corollary 7.12). Thus, we can write the integral as

$$\int_{U_H(\mathbb{A})\backslash G_2(\mathbb{A})} \int_{U_D(\mathbb{A})\backslash U_H(\mathbb{A})} \left(\int_{U_H(\mathbb{Q})U_D(\mathbb{A})\backslash U_H(\mathbb{A})} \theta(f)_{A_0}(uu'g)\psi_{H,D}^{-1}(u)\, du \right) \varphi^{U_D}(u'g)\, du'\, dg.$$

Exchanging integrals, we have

$$\begin{aligned}
&\int_{U_H(\mathbb{A})\backslash G_2(\mathbb{A})} \int_{U_H(\mathbb{Q})U_D(\mathbb{A})\backslash U_H(\mathbb{A})} \left(\int_{U_D(\mathbb{A})\backslash U_H(\mathbb{A})} \theta(f)_{A_0}(uu'g)\varphi^{U_D}(u'g)\, du' \right) \psi_{H,D}^{-1}(u)\, du\, dg \\
&= \int_{U_H\backslash G_2(\mathbb{A})} \int_{U_H(\mathbb{Q})U_D(\mathbb{A})\backslash U_H(\mathbb{A})} \left(\int_{U_H(\mathbb{Q})U_D(\mathbb{A})\backslash U_H(\mathbb{A})} \sum_{\gamma\in U_D\backslash U_H(\mathbb{Q})} \theta(f)_{A_0}(u\gamma u'g)\varphi^{U_D}(\gamma u'g)\, du' \right) \\
&\qquad\qquad \cdot \psi_{H,D}^{-1}(u)\, du\, dg \\
&= \int_{U_H\backslash G_2(\mathbb{A})} \int_{U_H(\mathbb{Q})U_D(\mathbb{A})\backslash U_H(\mathbb{A})} \sum_{\gamma} \left(\int_{U_H(\mathbb{Q})U_D(\mathbb{A})\backslash U_H(\mathbb{A})} \theta(f)_{A_0}(\gamma uu'g)\varphi^{U_D}(\gamma u'g)\, du' \right) \psi_{H,D}^{-1}(u)\, du\, dg
\end{aligned}$$

Changing variable $u' \mapsto u'' = \gamma uu' = u\gamma u''$ in the inner integral, the above becomes

$$\int_{U_H(\mathbb{A})\backslash G_2(\mathbb{A})} \int_{U_H(\mathbb{Q})U_D(\mathbb{A})\backslash U_H(\mathbb{A})} \left(\int_{U_D(\mathbb{A})\backslash U_H(\mathbb{A})} \theta(f)_{A_0}(u''g)\varphi^{U_D}(u^{-1}u''g)\, du'' \right) \psi_{H,D}^{-1}(u)\, du\, dg$$

which, after rearranging the integrals, is equal to

$$\int_{U_H(\mathbb{A})\backslash G_2(\mathbb{A})} \int_{U_D(\mathbb{A})\backslash U_H(\mathbb{A})} \theta(f)_{A_0}(u''g)\varphi_{U_H,\psi_{H,D}}(u''g)\, du''\, dg = \int_{U_D(\mathbb{A})\backslash G_2(\mathbb{A})} \theta(f)_{A_0}(g)\varphi_{U_H,\psi_{H,D}}(g)\, dg.$$

This shows that (1) implies (2). The proof of the converse is identical as the one given in Proposition 7.7. □

Proof of Theorem 7.8. Let σ be a cuspidal automorphic representation satisfying the hypotheses of the Theorem. We first apply Corollary 7.3 to deduce that $\Theta(\sigma)$ is cuspidal. Moreover, by Proposition 7.13, the theta lift supports a Fourier coefficient of type (4 2) and, in particular, it is nonzero. Let π be an irreducible subquotient of $\Theta(\sigma)$. Its component at p is the Steinberg representation by the compatibility between the global and local correspondences (Proposition 6.5) and by Proposition 6.10(2). We are now left to prove the statement on its archimedean component. As π is unitary, π_∞ is a unitarizable Harish-Chandra module by [Flath 1979, Theorem 4]. Moreover, as σ_∞ is a discrete series with infinitesimal character $r\varepsilon_1+s\varepsilon_2$, it follows from the discussion in [Li 1997, p. 204] and by Table 1 on [Li 1999, p. 375] that $\Theta(\sigma_\infty)$ has infinitesimal character $\big(r, \frac{1}{2}(r+s), \frac{1}{2}(r-s)\big)$, which is strongly regular in the sense of [Salamanca-Riba 1999, Definition 1.5]. By another application of Proposition 6.5, π_∞ is a subquotient of $\Theta(\sigma_\infty)$, and hence has a strongly regular infinitesimal character. As a consequence, we can apply [Salamanca-Riba 1999, Theorem 1.8] to deduce that π_∞ is cohomological. By [Kret and Shin 2023, Corollary 2.8], since π_p is Steinberg and π_∞ is cohomological, π_∞ is a discrete series with infinitesimal character $\big(r, \frac{1}{2}(r+s), \frac{1}{2}(r-s)\big)$. □

Remark 7.14. Suppose that σ_∞ is a discrete series in $\mathcal{D}_{3,1}$. If $\Theta(\sigma_\infty)$ admits a unique irreducible quotient $\theta(\sigma_\infty)$, then by the results of [Li 1997], $\theta(\sigma_\infty)$ is a discrete series of Hodge type $(3, 3)$. This implies that $\pi_\infty = \theta(\sigma_\infty)$ is a discrete series of Hodge type $(3, 3)$. Although Howe duality conjecture for the pair (G_2, PGSp_6) is known at nonarchimedean places [Gan and Savin 2023], the conjecture is still open at the archimedean place.

8. The cycle class formula and the standard motive for G_2

We conclude this article with the arithmetic applications described in the introduction.

8.1. ***The relation between L-functions of G_2 and* PGSp_6.** The dual group of G_2 is $G_2(\mathbb{C})$, which can be realized as the intersection $\mathrm{SO}_7(\mathbb{C}) \cap \mathrm{Spin}_7(\mathbb{C})$. More precisely, we have the commutative diagram

$$\begin{array}{ccccc} & & \xrightarrow{\quad\mathrm{Std}\quad} & & \\ G_2(\mathbb{C}) & \hookrightarrow & \mathrm{SO}_7(\mathbb{C}) & \hookrightarrow & \mathrm{GL}_7(\mathbb{C}) \\ \zeta \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathrm{Spin}_7(\mathbb{C}) & \hookrightarrow & \mathrm{SO}_8(\mathbb{C}) & \hookrightarrow & \mathrm{GL}_8(\mathbb{C}) \\ & & \xrightarrow[\quad\mathrm{Spin}\quad]{} & & \end{array} \tag{22}$$

where $\mathrm{Std} : G_2(\mathbb{C}) \to \mathrm{GL}(V_7)$ is the standard representation given by trace zero octonions, $\mathrm{Spin} : \mathrm{Spin}_7(\mathbb{C}) \to \mathrm{GL}(V_8)$ is the eight-dimensional spin representation, while the embedding ζ is defined from the fact that the stabilizer in $\mathrm{Spin}_7(\mathbb{C})$ of a generic vector of V_8 is isomorphic to $G_2(\mathbb{C})$. From the commutative diagram, one immediately sees that

$$V_{8|_{G_2}} = V_7 \oplus \mathbf{1}.$$

In particular, if π_ℓ is an unramified smooth representation of $\mathrm{PGSp}_6(\mathbb{Q}_\ell)$ with Satake parameter s_{π_ℓ} belonging to $\zeta(G_2(\mathbb{C}))$, then

$$L(s, \pi_\ell, \mathrm{Spin}) = L(s, \pi_\ell, \mathrm{Std})\zeta_\ell(s),$$

where

$$L(s, \pi_\ell, \mathrm{Std}) := \frac{1}{\det(1 - \ell^{-s}\,\mathrm{Std}(s_{\pi_\ell}))}$$

denotes the Euler factor at ℓ of the seven-dimensional standard L-function for G_2.

Let now π be a cuspidal automorphic representation of PGSp_6, which is unramified outside a finite set of places S containing the archimedean place. As a special case of Langlands functoriality, one expects that if $L^S(s, \pi, \mathrm{Spin})$ has a simple pole at $s = 1$, then π is a functorial lift from either G_2 or G_2^c, recalling that G_2^c denotes the form of G_2 which is compact at ∞ and split at all finite places of $\mathbb{Q}$. We invite the reader to consult [Ginzburg and Jiang 2001; Gan and Gurevich 2009; Pollack and Shah 2018; Gan and Savin 2020] for results in this direction. Moreover, the existence of a pole is usually related to the nonvanishing of a certain period. The following result,[2] summarizing and complementing known results in this direction, gives equivalence conditions for π to be a weak functorial lift from G_2.

Proposition 8.1. *Suppose that π satisfies the hypotheses* (DS) *and* (St) *of Section 2.7. Then π is tempered and the following statements are equivalent*:

(1) *The partial L-function $L^S(s,\pi,\mathrm{Spin})$ has a simple pole at $s=1$.*

(2) *For almost all ℓ, the Satake parameter $s_{\pi_\ell}\in\zeta(G_2(\mathbb{C}))$.*

(3) *There exists a cuspidal automorphic representation σ of either G_2 or G_2^c such that π is a weak functorial lift of σ.*

Moreover, *if π supports a Fourier coefficient of rank-2 associated to the quadratic extension F these conditions are equivalent to*

(4) *π is $\boldsymbol{H}$-distinguished*, *with $\boldsymbol{H}=\mathrm{GL}_2\boxtimes\mathrm{GL}_{2,F}^*$*, *i.e.*, *that there exists a cusp form Ψ in π such that*

$$\int_{\mathbb{Z}(\mathbb{A})\boldsymbol{H}(\mathbb{Q})\backslash\boldsymbol{H}(\mathbb{A})}\Psi(h)\,dh\neq 0.$$

If one of the first three conditions hold, *the residue at $s=1$ of the partial L-function $L^S(s,\pi,\mathrm{Spin})$ is given by*

$$\mathrm{Res}_{s=1}L^S(s,\pi,\mathrm{Spin})=L^S(1,\sigma,\mathrm{Std})\prod_{\ell\in S}(1-\ell^{-1}).$$

Proof. Since π is cohomological and it is Steinberg at a finite place, we can apply [Kret and Shin 2023, Lemma 2.7] to deduce that π is essentially tempered at all places. As π has trivial central character, this is equivalent to being tempered. The equivalence between (2) and (3) and the implication (1) $\Longrightarrow$ (3) follow from [Gan and Savin 2020, Theorem 1.1]. By [Gan and Gurevich 2009, Proposition 5.2], if π is $\boldsymbol{H}$-distinguished then its big theta lift to G_2 is nonzero and is contained in the space of cusp forms on G_2. By the compatibility between the local and global theta correspondence, this implies that every local component π_v appears in the local theta correspondence. When v is a finite unramified place for π, [Gan and Gurevich 2009, Proposition 5.1] implies that $s_{\pi_v}\in\zeta(G_2(\mathbb{C}))$. This shows (4) $\Longrightarrow$ (3).

We next prove that (1) $\Longrightarrow$ (4), for which we'll use the hypothesis on the existence of a Fourier coefficient of rank-2. By [Pollack and Shah 2018, Theorem 2.7] (see Theorem 5.8), given a cusp form Ψ in π, there exists a cusp form $\widetilde{\Psi}$ and a Schwartz–Bruhat function Φ such that

$$\mathcal{I}(\Phi,\widetilde{\Psi},s)=\mathcal{I}_\infty(\Phi,\Psi,s)L^S(s,\pi,\mathrm{Spin}).$$

By Proposition 5.10, taking residues at $s=1$ on both sides we have

$$\frac{\widehat{\Phi}(0)}{2}\cdot\int_{\mathbb{Z}(\mathbb{A})\boldsymbol{H}(\mathbb{Q})\backslash\boldsymbol{H}(\mathbb{A})}\widetilde{\Psi}(h)\,dh=\mathrm{Res}_{s=1}\big(\mathcal{I}_\infty(\Phi,\Psi,s)L^S(s,\pi,\mathrm{Spin})\big),$$

[2] We point out that Proposition 8.1 is not really needed in the following (it is cited in the proof of Theorem 8.6, but only to show that the L-function of the lift from G_2 to PGSp_6 has a pole at $s=1$), but it might be of independent interest.

where $c > 0$ is the constant of Lemma 5.1. We now use [Gan and Gurevich 2009, Proposition 12.1] to deduce that there exists local data Φ_∞ and Ψ_∞ such that $\mathcal{I}_\infty(\Phi, \Psi, 1) \neq 0$. Hence, up to modifying Ψ and Φ at ∞, we obtain

$$\widehat{\Phi}(0) \cdot \int_{\mathbb{Z}(\mathbb{A})\boldsymbol{H}(\mathbb{Q})\backslash \boldsymbol{H}(\mathbb{A})} \widetilde{\Psi}(h)\, dh = C \cdot \mathrm{Res}_{s=1} L^S(s, \pi, \mathrm{Spin}),$$

with C a certain nonzero constant in $\mathbb{C}$. Note finally that we have the freedom to choose Φ such that $\widehat{\Phi}(0) \neq 0$. This follows from the fact that, given the two nonzero linear maps

$$l_1 : \mathcal{S}(\mathbb{A}^2) \to \mathbb{C},\ \Phi \mapsto \mathcal{I}_\infty(\Phi, \Psi, 1) \quad \text{and} \quad l_2 : \mathcal{S}(\mathbb{A}^2) \to \mathbb{C},\ \Phi \mapsto \widehat{\Phi}(0), \quad \ker(l_1) \cup \ker(l_2) \neq \mathcal{S}(\mathbb{A}^2).$$

This shows that if $L^S(s, \pi, \mathrm{Spin})$ has a simple pole and π supports a Fourier coefficient of rank 2, then π is $\boldsymbol{H}$-distinguished.

We finally show the implication (2) $\Longrightarrow$ (1). The commutative diagram (22) implies that

$$L^S(s, \pi, \mathrm{Spin}) = L^S(s, \pi, \mathrm{Std})\zeta^S(s),$$

where $L^S(s, \pi, \mathrm{Std})$ is the partial L-function of π associated to the standard seven-dimensional representation of Spin_7. By [Labesse and Schwermer 2019, Theorem 1.1.1], the restriction to $\mathrm{Sp}_6(\mathbb{A})$ of π contains a cuspidal automorphic representation $\pi^\flat$, such that (up to possibly enlarging S)

$$L^S(s, \pi, \mathrm{Std}) = L^S(s, \pi^\flat, \mathrm{Std}).$$

By [Kret and Shin 2023, Corollary 2.2 & Lemma 2.3], there exists a cuspidal automorphic representation $\pi^\sharp$ of $\mathrm{GL}_7(\mathbb{A})$ such that

$$L^S(s, \pi^\flat, \mathrm{Std}) = L^S(s, \pi^\sharp),$$

where $L^S(s, \pi^\sharp)$ denotes the standard L-function of $\pi^\sharp$. We claim that $L^S(1, \pi^\sharp) \neq 0$. By [Jacquet and Shalika 1976, Theorem (1.3)], $L(s, \pi^\sharp) \neq 0$ for any s with $\mathrm{Re}(s) = 1$. If we write

$$L^S(s, \pi^\sharp) = L(s, \pi^\sharp) \prod_{\ell \in S} L(s, \pi^\sharp_\ell)^{-1},$$

then our claim follows from the fact that each $L(s, \pi^\sharp_\ell)$ has no pole at $s = 1$ (see [Rudnick and Sarnak 1996, p. 317]). This implies that

$$L^S(1, \pi, \mathrm{Std}) \neq 0.$$

Thus, $L^S(s, \pi, \mathrm{Spin})$ has a simple pole at $s = 1$. This proves that (2) implies (1).

If now we assume (3), i.e., that π is a weak functorial lift of σ, then (up to possibly enlarging S)

$$L^S(1, \sigma, \mathrm{Std}) = L^S(1, \pi, \mathrm{Std}) \neq 0,$$

where the first equality is a consequence of the fact that the Satake parameters of σ and π agree almost everywhere. In particular

$$\mathrm{Res}_{s=1} L^S(s, \pi, \mathrm{Spin}) = L^S(1, \sigma, \mathrm{Std})\, \mathrm{Res}_{s=1} \zeta^S(s) = L^S(1, \sigma, \mathrm{Std}) \prod_{\ell \in S} (1 - \ell^{-1}),$$

showing the final claim. □

Remark 8.2. Let π be as in Corollary 5.12. Assuming (St), π is a weak functorial lift of σ as in Proposition 8.1 and Theorem 5.11 reads as

$$\langle \mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{H}}, [\omega_\Psi]\rangle_{\mathcal{H}} = C\cdot \mathcal{I}_S(\Phi, \Psi^{[\lambda,\mu]}, 1)\cdot \prod_{\ell\in S}(1-\ell^{-1})\cdot L^S(1,\sigma,\mathrm{Std}).$$

8.2. ***Galois representations of G_2-type.*** The following result for the compact form of G_2 is shown in [Kret and Shin 2023, Theorem 11.1 and Corollary 11.3]. The same proof works for the split form of G_2 as long as one has some information on its lift to PGSp_6, and we only sketch it for the convenience of the reader.

Theorem 8.3. *Let σ be a cuspidal automorphic representation of $G_2(\mathbb{A})$ or $G_2^c(\mathbb{A})$ which lifts to a nonzero cuspidal automorphic representation π of $\boldsymbol{G}(\mathbb{A})$ such that*

- *π_∞ is cohomological,*
- *π_p is the Steinberg representation at some finite prime p.*

Then, for each prime ℓ and $\iota:\mathbb{C}\cong\overline{\mathbb{Q}}_\ell$, there exists a Galois representation $\rho_\sigma=\rho_{\sigma,\iota}:\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to G_2(\overline{\mathbb{Q}}_\ell)$ such that:

- *For every finite place $v\neq\ell$ where σ is unramified, ρ_σ is unramified at v. Moreover, the semisimple part of $\rho_\sigma(\mathrm{Frob}_v)$ is conjugate to the Satake parameter $\iota(s_{\sigma_v})$ in $G_2(\overline{\mathbb{Q}}_\ell)$.*
- *ρ_{σ_ℓ} is de Rham, and it is crystalline if σ is unramified at ℓ.*
- *$\zeta\circ\rho_\sigma=\rho_\pi$, where π is a theta lift of σ, and $\zeta:G_2(\mathbb{C})\to\mathrm{Spin}_7(\mathbb{C})$ is the embedding appearing in* (22).
- *The Zariski closure of the image of ρ_σ maps onto either the image of a principal SL_2 in G_2 or onto G_2.*

Proof. By [Kret and Shin 2023, Theorem A], there exists a representation $\rho_\pi:\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to\mathrm{Spin}_7(\overline{\mathbb{Q}}_\ell)$ attached to π. By the proof of [Kret and Shin 2023, Theorem 11.1], one has that the image of ρ_π is contained in $G_2(\overline{\mathbb{Q}}_\ell)$, and thus we have ρ_σ such that $\zeta\circ\rho_\sigma=\rho_\pi$ for a suitable choice of embedding $\zeta:G_2(\mathbb{C})\to\mathrm{Spin}_7(\mathbb{C})$ fitting in the diagram (22). Hence, by [Kret and Shin 2023, Theorem A] and Proposition 6.10(1), the representation $\rho_\sigma:\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to G_2(\overline{\mathbb{Q}}_\ell)$ satisfies the desired first three properties. Finally, by [Kret and Shin 2023, Theorem A(v)], the Zariski closure of ρ_π must map onto either a principal SL_2 in $\mathrm{SO}_7\cap\mathrm{G}_2$, or G_2. □

In the following proposition, we describe several cases where Theorem 8.3 applies. Before doing that, we need to introduce some notation. Let ω_1,ω_2 denote the two fundamental weights for G_2, where ω_1 is the highest weight of the standard representation and ω_2 of the fourteen-dimensional adjoint representation. According to our convention on the root system for G_2 in Section 6.2.1, $\omega_1=a+2b$ and $\omega_2=2a+3b$.

Proposition 8.4. *Let σ be a cuspidal automorphic representation of $G_2^\circ(\mathbb{A})$ with $\circ\in\{\varnothing,c\}$ such that σ_∞ is a discrete series of infinitesimal character (r,s), where $r-3\geq s-1\geq 0$ and $r-s$ is even $\big($if $\circ=c$ then σ_∞ is the irreducible algebraic representation of $G_2^c(\mathbb{R})$ of highest weight $(s-1)\omega_1+\frac{1}{2}(r-s-2)\omega_2\big)$ and σ_p is Steinberg at some finite place p. Suppose that one of the following conditions holds.*

(1) *We have* $\circ = c$ *and there exists* $\alpha \in \sigma$ *and a quaternion subalgebra* D *of the nonsplit octonions* $\mathbb{O}^c$ *such that*

$$P^C_\alpha := \int_{C(\mathbb{Q})\backslash C(\mathbb{A})} \alpha(v)\,dv \neq 0,$$

where C *is the centralizer of* D *in* G_2^c.

(2) *We have* $\circ = \varnothing$ *and either* σ *is globally generic or* σ_∞ *is nongeneric and* σ *supports a Fourier coefficient of type* (4 2) *corresponding to* $\mathbb{Q} \times F$, *where* F *is a real quadratic étale* $\mathbb{Q}$*-algebra.*

Then there exists a nontrivial small theta lift π *of* σ *to* $\boldsymbol{G}(\mathbb{A})$ *which is a cuspidal automorphic representation, such that* π_∞ *is a discrete series with infinitesimal character* $\big(r, \frac{1}{2}(r+s), \frac{1}{2}(r-s)\big)$ *and* π_p *is Steinberg.*

Proof. Let σ be as in assumption (1). Since the Steinberg representation is generic, then by [Gross and Savin 1998, Corollary 4.9] the big theta lift of σ to $\boldsymbol{G}(\mathbb{A})$ has a nontrivial cuspidal irreducible subquotient π, which is unramified at almost all places. The infinitesimal character of its archimedean component is given in [Gross and Savin 1998, Theorem 3.5], and by Propositions 6.5 and 6.10 we have that π_p is Steinberg. Under the assumption (2), if σ is globally generic, the result follows from [Harris et al. 2023, Theorem 1.7], and if σ_∞ is not generic and σ supports a Fourier coefficient as in the statement, the result follows from Theorem 7.8. □

By construction, the composition of the Galois representation ρ_π (and thus ρ_σ) with the spin representation appears in $H^6_{\text{ét}}(\mathrm{Sh}_{\boldsymbol{G},\overline{\mathbb{Q}}}, \mathcal{V}^\lambda_\ell(3))$, where the latter denotes the direct limit of the cohomology at level U in coefficients in the ℓ-adic lisse sheaf associated to an irreducible algebraic representation V^λ of $\boldsymbol{G}$, as U varies. This direct limit is a smooth admissible $\overline{\mathbb{Q}}_\ell$-representation of $\boldsymbol{G}(\mathbb{A}_f)$, endowed with an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ commuting with the one of $\boldsymbol{G}(\mathbb{A}_f)$. Let σ and π be as in the statement of Theorem 8.3. Choose an embedding of the rationality field L of π in $\overline{\mathbb{Q}}_\ell$. Then by Lemma 2.8, the $\pi_f^\vee$-isotypic component of $H^6_{\text{ét},!}(\mathrm{Sh}_{\boldsymbol{G},\overline{\mathbb{Q}}}, \mathcal{V}^\lambda_\ell(3))$ is eight-dimensional $\overline{\mathbb{Q}}_\ell$-vector space, and we have

$$H^6_{\text{ét},!}(\mathrm{Sh}_{\boldsymbol{G},\overline{\mathbb{Q}}}, \mathcal{V}^\lambda_\ell(3))[\pi_f^\vee] = V_{\mathrm{Spin}\circ\rho_\pi} \otimes \pi_f^\vee = V_{\mathrm{Spin}\circ\zeta\circ\rho_\sigma} \otimes \pi_f^\vee.$$

If the image of ρ_σ is Zariski dense in $G_2(\overline{\mathbb{Q}}_\ell)$, we have $\mathrm{Spin}\circ\zeta\circ\rho_\sigma = \mathrm{Std}\circ\rho_\sigma \oplus \mathbf{1}$, where $\mathrm{Std}\circ\rho_\sigma$ is the irreducible "standard" Galois representation attached to σ. If not, by Theorem 8.3, the image of ρ_σ is Zariski dense onto a principal $\xi : \mathrm{SL}_2(\overline{\mathbb{Q}}_\ell) \to G_2(\overline{\mathbb{Q}}_\ell)$. Then the branching law of [Gross 2000, (7.1)] gives that $\mathrm{Spin}\circ\zeta\circ\rho_\sigma = \mathrm{Sym}^6\circ\rho_\sigma \oplus \mathbf{1}$, where $\mathrm{Sym}^6\circ\rho_\sigma$ is the irreducible symmetric sixth power Galois representation attached to σ. Denote by $M_\ell(\pi_f)$ the Galois representation $V_{\mathrm{Spin}\circ\rho_\pi}$ and let $M_\ell(\sigma_f)$ be either the Galois representation $V_{\mathrm{Std}\circ\rho_\sigma}$ or $V_{\mathrm{Sym}^6\circ\rho_\sigma}$. Then we have that $M_\ell(\sigma_f)^{G_\mathbb{Q}} = 0$ and $M_\ell(\pi_f)$ decomposes as the direct sum

$$M_\ell(\pi_f) = M_\ell(\sigma_f) \oplus \mathbf{1}, \tag{23}$$

where $\mathbf{1}$ denotes the one-dimensional trivial representation.

Remark 8.5. In the case where ρ_σ is not Zariski dense in $G_2(\overline{\mathbb{Q}}_\ell)$, the Satake parameter $s_{\sigma_p} \in \xi(\mathrm{SL}_2(\mathbb{C}))$ for any unramified prime p. By Langlands reciprocity principle, σ should be the functorial lift of a cuspidal automorphic representation τ of $\mathrm{PGL}_2(\mathbb{A})$, while $V_{\mathrm{Sym}^6\circ\rho_\sigma}$ should be a geometric realization of the motive of the symmetric sixth power of τ.

8.3. ***On a question of Gross and Savin.*** Tate conjecture predicts the existence of a cycle which gives rise to the trivial representation appearing in the decomposition of (23). Gross and Savin [1998], inspired by local computations, conjectured that this cycle should come from a Hilbert modular 3-fold inside $\mathrm{Sh}_{\boldsymbol{G}}$. Theorem 8.6 below supports this expectation for certain cuspidal automorphic representations σ of G_2 and G_2^c.

Let σ be a cuspidal automorphic representation of $G_2^\circ(\mathbb{A})$ with $\circ \in \{\varnothing, c\}$ such that σ_∞ is a discrete series of infinitesimal character (r, s) with $r-3 \geq s-1 \geq 0$ and $r-s$ even and σ_p is Steinberg at some finite place p and let π be the small theta lift of σ given by Proposition 8.4. Let V^λ denote an irreducible algebraic representation of $\boldsymbol{G}$ of highest weight $\lambda = \big(r-3, \frac{1}{2}(r+s)-2, \frac{1}{2}(r-s)-1, 0\big)$. Note that $V^\lambda|_{\boldsymbol{H}}$ contains the trivial representation by Lemma 3.2. Let $U \subset \boldsymbol{G}(\mathbb{A}_f)$ denote a neat compact open subgroup such that $\pi_f^U \neq 0$. For any $\frac{1}{2}(r+s)-2 \geq \mu \geq \frac{1}{2}(r-s)-1$, let

$$\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\text{ét}} := \mathrm{cl}_{\text{ét}}(\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{M}}) \in \mathrm{H}^6_{\text{ét}}(\mathrm{Sh}_{\boldsymbol{G}}(\mathrm{U})_{\overline{\mathbb{Q}}}, \mathcal{V}^\lambda_\ell(3))^{\mathrm{G}_{\mathbb{Q}}}$$

be the étale realization of the motivic class $\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\mathcal{M}}$ (see Definition 3.9), where $\boldsymbol{H} = \mathrm{GL}_2 \boxtimes \mathrm{GL}^*_{2,F}$. Fix a vector $\Psi_f \in \pi_f^U$. By composing the projection to the $\pi_f^\vee$-isotypic component together with the projection given by the vector Ψ_f, we get

$$\mathcal{Z}^\sigma_{\boldsymbol{H},\text{ét}} := \Psi_f(\mathrm{pr}_{\pi^\vee}(\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\text{ét}})) \in (M_\ell(\sigma_f) \oplus \mathbf{1})^{G_{\mathbb{Q}}} = \mathbf{1}.$$

By (the proof of) Lemma 2.8, there exists a cuspidal automorphic representation $\pi^{3,3} = \pi^{3,3}_\infty \otimes \pi_f$ of $\boldsymbol{G}(\mathbb{A})$ whose archimedean component is a discrete series of Hodge type $(3, 3)$ with the same infinitesimal character of π_∞ and whose nonarchimedean part π_f is the same as the one of π. Let $\Psi = \Psi_\infty \otimes \Psi_f$ be the cusp form in the space of $\pi^{3,3}$ such that Ψ_∞ is a highest weight vector of the minimal K_∞-type of $\pi^{3,3}_{\infty,1} \subseteq \pi^{3,3}_\infty|_{\mathrm{Sp}_6(\mathbb{R})}$. For any μ as above, recall that we denote $\Psi^{[\lambda,\mu]} = A^{[\lambda,\mu]} \cdot \Psi_\infty \otimes \Psi_f$, where $A^{[\lambda,\mu]}$ is the operator that appeared in Proposition 4.8.

Theorem 8.6. *Assume that the integral $\mathcal{I}_S(\Phi, \Psi^{[\lambda,\mu]}, 1)$ is nonzero for some Schwartz–Bruhat function Φ. Then the class $\mathcal{Z}^\sigma_{\boldsymbol{H},\acute{e}t}$ generates the trivial subrepresentation $\mathbf{1}$ of $M_\ell(\pi_f)$.*

Proof. By the comparison theorem between étale and Betti cohomology [SGA 4_3 1973, Exposé XI, Theorem 4.4(iii)], Proposition 8.1 and Corollary 5.12, we know that the projection $\mathrm{pr}_{\pi^\vee}\mathcal{Z}^{[\lambda,\mu]}_{\boldsymbol{H},\text{ét}}$ to $M_\ell(\pi_f) \otimes (\pi_f^U)^\vee$ generates a one-dimensional subspace, which is trivial for the action of the Galois group. As we have explained above, the image of ρ_σ is either dense in $G_2(\overline{\mathbb{Q}}_\ell)$ or in $\mathrm{SL}_2(\overline{\mathbb{Q}}_\ell) \to \mathrm{PGL}_2(\overline{\mathbb{Q}}_\ell) \hookrightarrow G_2(\overline{\mathbb{Q}}_\ell)$. In either case, the representation $M_\ell(\sigma_f)$ is irreducible and the trivial factor $\mathbf{1}$ in $M_\ell(\pi_f)$ is hence generated by the image of $\mathcal{Z}^\sigma_{\boldsymbol{H},\text{ét}}$. □

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