

# *Algebra & Number Theory*

Volume 19  
2025  
No. 4

**Odd moments in the distribution of primes**

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Montgomery and Soundararajan showed that the distribution of  $\psi(x+H) - \psi(x)$ , for  $0 \leq x \leq N$ , is approximately normal with mean  $\sim H$  and variance  $\sim H \log(N/H)$ , when  $N^\delta \leq H \leq N^{1-\delta}$ . Their work depends on showing that sums  $R_k(h)$  of  $k$ -term singular series are  $\mu_k(-h \log h + Ah)^{k/2} + O_k(h^{k/2-1/(7k)+\varepsilon})$ , where  $A$  is a constant and  $\mu_k$  are the Gaussian moment constants. We study lower-order terms in the size of these moments. We conjecture that when  $k$  is odd,  $R_k(h) \asymp h^{(k-1)/2}(\log h)^{(k+1)/2}$ . We prove an upper bound with the correct power of  $h$  when  $k = 3$ , and prove analogous upper bounds in the function field setting when  $k = 3$  and  $k = 5$ . We provide further evidence for this conjecture in the form of numerical computations.

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## 1. Introduction

What is the distribution of primes in short intervals? Cramér [1936] modeled the indicator function of the sequence of primes by independent random variables  $X_n$ , for  $n \geq 3$ , which are 1 (“ $n$  is prime”) with probability  $1/\log n$ , and 0 (“ $n$  is composite”) with probability  $1 - 1/\log n$ . Cramér’s model predicts that the distribution of  $\psi(n+h) - \psi(n)$ , a weighted count of the number of primes in an interval of size  $h$  starting at  $n$ , follows a Poisson distribution when  $n$  varies in  $[1, N]$  and when  $h \asymp \log N$ . Gallagher [1976] proved that this follows from a quantitative version of the Hardy–Littlewood prime  $k$ -tuple conjecture: namely, that if  $\mathcal{D} = \{d_1, d_2, \dots, d_k\}$  is a set of  $k$  distinct integers, then

$$\sum_{n \leq N} \prod_{i=1}^k \Lambda(n + d_i) = (\mathfrak{S}(\mathcal{D}) + o(1))N,$$

The author is supported by NSF GRFP grant DGE-1656518 and would like to thank Kannan Soundararajan for many helpful comments and discussions, as well as Régis de la Bretèche, Alexandra Florea, Andrew Granville, Zeev Rudnick, Yuval Wigderson, and the anonymous referee for helpful feedback.

MSC2020: 11N05, 11N13.

Keywords: sums of singular series, distribution of primes.

where  $\mathfrak{S}(\mathcal{D})$  is the singular series, a constant dependent on  $\mathcal{D}$  given by

$$\mathfrak{S}(\mathcal{D}) = \prod_p \left(1 - \frac{1}{p}\right)^{-k} \left(1 - \frac{\nu_p(\mathcal{D})}{p}\right),$$

where  $\nu_p(\mathcal{D})$  denotes the number of distinct residue classes modulo  $p$  among the elements of  $\mathcal{D}$ . The singular series is also given by the formula

$$\mathfrak{S}(\mathcal{D}) = \sum_{\substack{q_1, \dots, q_k \\ 1 \leq q_i < \infty}} \left( \prod_{i=1}^k \frac{\mu(q_i)}{\phi(q_i)} \right) \sum_{\substack{a_1, \dots, a_k \\ 1 \leq a_i \leq q_i \\ (a_i, q_i) = 1 \\ \sum a_i / q_i \in \mathbb{Z}}} e\left(\sum_{i=1}^k \frac{a_i d_i}{q_i}\right). \tag{1}$$

The Hardy–Littlewood prime  $k$ -tuple conjectures give us a better lens through which to understand the distribution of primes: by understanding sums of singular series. For example, Gallagher used the estimate that

$$\sum_{\mathcal{D} \subset [1, h]} \mathfrak{S}(\mathcal{D}) \sim \sum_{\mathcal{D} \subset [1, h]} 1$$

to prove that the Hardy–Littlewood conjectures imply Poisson behavior in intervals of logarithmic length. Our concern is the distribution of primes in somewhat longer intervals, namely, those of size  $H$ , where  $H = o(N)$  and  $H/\log N \rightarrow \infty$  as  $N \rightarrow \infty$ . In this setting, the Cramér model would predict that the distribution of  $\psi(n + H) - \psi(n)$  for  $n \leq N$  is approximately normal, with mean  $\sim H$  and variance  $\sim H \log N$ . However, the Hardy–Littlewood prime  $k$ -tuple conjecture gives a different answer in this case. Montgomery and Soundararajan [2004] provided evidence based on the Hardy–Littlewood prime  $k$ -tuple conjectures that the distribution ought to be approximately normal with variance  $\sim H \log(N/H)$ . They consider the  $K$ -th moment  $M_K(N; H)$  of the distribution of primes in an interval of size  $H$ , given by

$$M_K(N; H) = \sum_{n=1}^N (\psi(n + H) - \psi(n) - H)^K.$$

They conjecture that these moments should be given by the Gaussian moments

$$M_K(N; H) = (\mu_K + o(1))N \left(H \log \frac{N}{H}\right)^{K/2},$$

where  $\mu_K = 1 \cdot 3 \cdots (K - 1)$  if  $K$  is even and 0 if  $K$  is odd, uniformly for  $(\log N)^{1+\delta} \leq H \leq N^{1-\delta}$ . Their technique relies on more refined estimates of sums of the singular series constants  $\mathfrak{S}(\mathcal{D})$ . Instead of the von Mangoldt function  $\Lambda(n)$ , they consider sums of  $\Lambda_0(n) = \Lambda(n) - 1$ , where the main term has been subtracted from the beginning. The corresponding form of the Hardy–Littlewood conjecture states that

$$\sum_{n \leq N} \prod_{i=1}^k \Lambda_0(n + d_i) = (\mathfrak{S}_0(\mathcal{D}) + o(1))N$$

as  $N \rightarrow \infty$ , where  $\mathfrak{S}_0(\mathcal{D})$  is given by

$$\mathfrak{S}_0(\mathcal{D}) = \sum_{\mathcal{J} \subseteq \mathcal{D}} (-1)^{|\mathcal{D} \setminus \mathcal{J}|} \mathfrak{S}(\mathcal{J}),$$

and in turn

$$\mathfrak{S}(\mathcal{D}) = \sum_{\mathcal{J} \subseteq \mathcal{D}} \mathfrak{S}_0(\mathcal{J}).$$

We can combine this with (1) to see that

$$\mathfrak{S}_0(\mathcal{D}) = \sum_{\substack{q_1, \dots, q_k \\ 1 < q_i < \infty}} \left( \prod_{i=1}^k \frac{\mu(q_i)}{\phi(q_i)} \right) \sum_{\substack{a_1, \dots, a_k \\ 1 \leq a_i \leq q_i \\ (a_i, q_i) = 1 \\ \sum a_i / q_i \in \mathbb{Z}}} e\left( \sum_{i=1}^k \frac{a_i d_i}{q_i} \right). \tag{2}$$

Montgomery and Soundararajan considered the sum

$$R_k(h) := \sum_{\substack{d_1, \dots, d_k \\ 1 \leq d_i \leq h \\ d_i \text{ distinct}}} \mathfrak{S}_0(\mathcal{D}), \tag{3}$$

showing that, for any nonnegative integer  $k$ , for any  $h > 1$ , and for any  $\varepsilon > 0$ ,

$$R_k(h) = \mu_k(-h \log h + Ah)^{k/2} + O_k(h^{k/2-1/(7k)+\varepsilon}), \tag{4}$$

where  $A = 2 - \gamma - \log 2\pi$ . Their estimate on  $R_k(h)$  implies their bound on the moments. For more on the distribution of primes in short intervals, see for example [Chan 2006; Granville and Lumley 2023; Montgomery and Soundararajan 2004].

For all  $k$ , the optimal error term in (4) is expected to be smaller. In the case of the variance, this was studied in [Montgomery and Soundararajan 2002]. In this paper, we restrict our attention to the cases when  $k$  is odd. We conjecture the following, which was mentioned in [Lemke Oliver and Soundararajan 2016].

**Conjecture 1.1.** *Let  $k \geq 3$  be an odd integer, and let  $h > 1$ . With  $R_k(h)$  defined as above,*

$$R_k(h) \asymp h^{(k-1)/2} (\log h)^{(k+1)/2}.$$

The conjectured power of  $\log h$  here comes from numerical evidence, which we present in Section 5. For  $k$  odd, we do not know, even heuristically, which terms contribute to the main term in  $R_k(h)$ ; for this reason, we do not know what the constant should be in front of the asymptotic in Conjecture 1.1. Nevertheless, our goal in this paper is to provide evidence for Conjecture 1.1. When  $k = 3$ , we can show an upper bound with the correct power of  $h$ .

**Theorem 1.2.** *For  $h \geq 4$  and  $R_3$  defined in (3),*

$$R_3(h) \ll h(\log h)^5.$$

Another source of evidence for Conjecture 1.1 is the analog of this problem in the function field setting, which is also studied in [Keating and Rudnick 2014]. As we discuss in Section 3, we can consider analogous questions over  $\mathbb{F}[T]$ , where  $\mathbb{F}$  is a finite field, instead of over  $\mathbb{Z}$ . To state the analog, we first revisit the

techniques of Montgomery and Soundararajan in the integer case. Upon expanding (3) using (2), we get

$$R_k(h) = \sum_{\substack{d_1, \dots, d_k \\ 1 \leq d_i \leq h \\ d_i \text{ distinct}}} \sum_{\substack{q_1, \dots, q_k \\ 1 < q_i < \infty}} \left( \prod_{i=1}^k \frac{\mu(q_i)}{\phi(q_i)} \right) \sum_{\substack{a_1, \dots, a_k \\ 1 \leq a_i \leq q_i \\ (a_i, q_i) = 1 \\ \sum a_i / q_i \in \mathbb{Z}}} e\left(\sum_{i=1}^k \frac{a_i d_i}{q_i}\right) = \sum_{\substack{q_1, \dots, q_k \\ 1 < q_i < \infty}} \left( \prod_{i=1}^k \frac{\mu(q_i)}{\phi(q_i)} \right) \sum_{\substack{a_1, \dots, a_k \\ 1 \leq a_i \leq q_i \\ (a_i, q_i) = 1 \\ \sum a_i / q_i \in \mathbb{Z}}} \prod_{i=1}^k E\left(\frac{a_i}{q_i}\right),$$

where  $E(\alpha) = \sum_{m=1}^h e(m\alpha)$ . The sums  $E(\alpha)$  approximately detect when  $\|\alpha\| \leq 1/h$ .

This expression for  $R_k(h)$  is closely related to a quantity studied by in [Montgomery and Vaughan 1986]. They considered the related problem of the  $k$ -th moment of reduced residues modulo a fixed  $q$ , given by

$$m_k(q; h) = \sum_{n=1}^q \left( \sum_{\substack{1 \leq m \leq h \\ (m+n, q) = 1}} 1 - h \frac{\phi(q)}{q} \right)^k.$$

The moment  $m_k$  satisfies  $m_k(q; h) = q(\phi(q)/q)^k V_k(q; h)$ , where  $V_k(q; h)$  is the “singular series sum”,

$$V_k(q; h) = \sum_{\substack{d_1, \dots, d_k \\ 1 \leq d_i \leq h}} \sum_{\substack{q_1, \dots, q_k \\ 1 < q_i | q}} \left( \prod_{i=1}^k \frac{\mu(q_i)}{\phi(q_i)} \right) \sum_{\substack{a_1, \dots, a_k \\ 1 \leq a_i \leq q_i \\ (a_i, q_i) = 1 \\ \sum a_i / q_i \in \mathbb{Z}}} e\left(\sum_{i=1}^k \frac{a_i d_i}{q_i}\right),$$

which differs from  $R_k(h)$  only in that the  $q_i$  are now constrained to divide a fixed  $q$ . In this paper, as well as in the work of Montgomery and Soundararajan, estimating  $V_k(q; h)$  when  $q$  is a product of primes  $p \leq h^{k+1}$  is a key step towards estimating  $R_k(h)$ . Similarly, understanding  $m_k(q; h)$  is closely related to understanding  $R_k(h)$ . For example, Conjecture 1.1 predicts that  $R_k(h) \asymp h^{(k-1)/2}(\log h)^{(k+1)/2}$  when  $k$  is odd; this conjecture is closely related to the prediction that when  $q$  is a product of primes  $p \leq h^A$  for a fixed power  $A$ , and when  $k$  is odd, then we should have  $m_k(q; h) \asymp q(h/(\log h))^{(k-1)/2}$ . Montgomery and Vaughan [1986] predicted that  $m_k(q; h) \ll q(h/(\log h))^{(k-1)/2}$  in this setting. In the function field setting, we study an analog of the moments  $m_k(q; h)$ .

Let  $\mathbb{F}_q$  be a finite field with  $q$  elements, and let  $Q$  be a fixed monic polynomial in  $\mathbb{F}_q[t]$ . Note that  $Q$  in the function field case serves the same role as  $q$  in the integer case, since  $q$  now represents the size of the field. The moment  $m_k(Q; h)$ , an analog of the  $k$ -th moment of reduced residues in short intervals which is defined precisely in (15), is the  $k$ -th moment of the distribution of polynomials that are relatively prime to  $Q$  lying in intervals of size  $q^h$  in the function field  $\mathbb{F}_q[t]$ . In this case an “interval” of size  $q^h$  centered at a polynomial  $G(t)$  consists of all polynomials  $F(t)$  such that  $F(t) \equiv G(t) \pmod{t^h}$ . We can adapt the methods of Montgomery–Vaughan to prove a bound on  $m_k(Q; h)$  that has the same shape as the bounds of Montgomery–Vaughan and Montgomery–Soundararajan.

**Theorem 1.3.** *For any fixed  $k \geq 3$  and for  $Q \in \mathbb{F}_q[t]$  square-free, for  $h \geq 2$ ,*

$$m_k(Q; h) \ll \begin{cases} |Q|(q^h)^{k/2} \left(\frac{\phi(Q)}{|Q|}\right)^{k/2} \left(1 + (q^h)^{-1/(k-2)} \left(\frac{\phi(Q)}{|Q|}\right)^{-2^k + k/2}\right) & \text{if } k \text{ is even,} \\ |Q|((q^h)^{k/2-1/2} + (q^h)^{k/2-1/(k-2)}) \left(\frac{\phi(Q)}{|Q|}\right)^{-2^k + k/2} & \text{if } k \text{ is odd.} \end{cases}$$

The function field exponential sums are cleaner than their integer analogs, making this proof more streamlined than the proof of Montgomery–Vaughan. As a result, the bound is tighter; in fact, for  $k = 3$ , [Theorem 1.3](#) already yields a bound where the exponent of  $q^h$  is 1. This is of the same shape as [Theorem 1.2](#), where the exponent of  $h$  is 1.

Using a more involved argument we can achieve a bound on the fifth moment of reduced residues in short intervals.

**Theorem 1.4.** *Let  $h \geq 2$  and let*

$$Q = \prod_{\substack{P \text{ irred.} \\ |P| \leq q^{6h}}} P.$$

For all  $\varepsilon > 0$ ,

$$m_5(Q; h) \ll |Q|q^{2h+\varepsilon}.$$

As discussed above, [Conjecture 1.1](#) would predict in the integer case that for  $k$  odd and  $q = \prod_{p \leq h^\Lambda} p$ , we have  $m_k(q; h) \asymp q(h/(\log h))^{(k-1)/2}$ . In the function field case, we have a polynomial  $Q(t)$  in place of the modulus  $q$ , and an interval of size  $q^h$  instead of one of size  $h$ , so the analog of [Conjecture 1.1](#) would predict that  $m_5(Q; h) \asymp |Q|q^{2h}(\log q^h)^{-2}$ . In particular, [Theorem 1.4](#) matches the exponent of  $q^h$  in this prediction. Our techniques do not quite succeed in proving such a bound for any higher odd moments, as we note in [Section 4](#). However, we do get as a corollary the following bound on sums of singular series in function fields. The sum  $R_k(q^h)$  of singular series in function fields is defined very analogously to the sum  $R_k(h)$  in the integer setting; a precise definition is given in [\(19\)](#).

**Corollary 1.5.** *Let  $h \geq 2$  and let*

$$Q = \prod_{\substack{P \text{ irred.} \\ |P| \leq q^{6h}}} P.$$

Then

$$R_3(q^h) \ll V_3(Q; h) + q^h \left( \frac{|Q|}{\phi(Q)} \right)^2 \ll q^h \left( \frac{|Q|}{\phi(Q)} \right)^8,$$

and, for all  $\varepsilon > 0$ ,

$$R_5(q^h) \ll V_5(Q; h) + \left( \frac{|Q|}{\phi(Q)} \right)^{21/2} q^{2h} \ll q^{(2+\varepsilon)h}.$$

This paper is organized as follows. In [Section 2](#) we prove [Theorem 1.2](#). In [Section 3](#), we discuss the analogous problem in  $\mathbb{F}_q[T]$ , and adapt the framework of Montgomery and Vaughan to the function field setting to prove [Theorem 1.3](#). In [Section 4](#) we prove [Theorem 1.4](#). Finally, in [Section 5](#) we provide numerical evidence for [Conjecture 1.1](#), and in [Section 6](#) we discuss toy problems, further directions of inquiry, and possible applications of these questions.

## 2. Three-term integer sums: proof of [Theorem 1.2](#)

Our goal is to bound

$$R_3(h) = \sum_{\substack{d_1, d_2, d_3 \\ 1 \leq d_i \leq h \\ d_i \text{ distinct}}} \mathfrak{S}_0(\mathcal{D}).$$

Expanding  $\mathfrak{S}_0(\mathcal{D})$  as an exponential sum yields

$$R_3(h) = \sum_{\substack{d_1, d_2, d_3 \\ 1 \leq d_i \leq h \\ d_i \text{ distinct}}} \sum_{\substack{q_1, q_2, q_3 \\ 1 < q_i < \infty}} \left( \prod_{i=1}^3 \frac{\mu(q_i)}{\phi(q_i)} \right) \sum_{\substack{a_1, a_2, a_3 \\ 1 \leq a_i \leq q_i \\ (a_i, q_i) = 1 \\ \sum a_i / q_i \in \mathbb{Z}}} e \left( \sum_{i=1}^3 \frac{a_i d_i}{q_i} \right).$$

Our argument will follow the same thread as that of [Montgomery and Soundararajan 2004], which in turn relies on the analysis of [Montgomery and Vaughan 1986] of the distribution of reduced residues. To that end, we consider  $V_3(q; h)$ , which is approximately the third centered moment of the number of reduced residues mod  $q$  in an interval of length  $h$ . Precisely,  $V_3(q; h)$  is given by

$$V_3(q; h) = \sum_{\substack{d_1, d_2, d_3 \\ 1 \leq d_i \leq h}} \sum_{\substack{q_1, q_2, q_3 \\ 1 < q_i \mid q}} \left( \prod_{i=1}^3 \frac{\mu(q_i)}{\phi(q_i)} \right) \sum_{\substack{a_1, a_2, a_3 \\ 1 \leq a_i \leq q_i \\ (a_i, q_i) = 1 \\ \sum a_i / q_i \in \mathbb{Z}}} e \left( \sum_{i=1}^3 \frac{a_i d_i}{q_i} \right). \tag{5}$$

This is very similar to the above expression for  $R_3(h)$ ; the two differences are that the outer sum in  $R_3(h)$  is taken over *distinct*  $d_i$ 's, whereas the outer sum for  $V_3(q; h)$  is not, and that the summands  $q_i$  range over all integers for  $R_3(h)$ , but are restricted to factors of  $q$  for  $V_3(q; h)$ .

**Theorem 2.1.** *Let  $h \geq 4$  and let  $q$  be the product of primes  $p \leq h^4$ . Then*

$$V_3(q; h) \ll h(\log h)^5.$$

We use Theorem 2.1 to establish Theorem 1.2. In order to derive Theorem 1.2, it suffices to show that terms arising from transforming  $V_3(q; h)$  into  $R_3(h)$  do not contribute more than  $O(h(\log h)^5)$ ; in fact they contribute on the order of  $h(\log h)^2$ , which is the conjectured asymptotic size of  $R_3(h)$ . We begin with this derivation of Theorem 1.2 from Theorem 2.1.

To account for terms where  $d_1, d_2, d_3$  are not necessarily distinct, we make the following definition.

**Definition 2.2.** Let  $k \geq 2$ , and let  $\mathcal{D} = \{d_1, \dots, d_k\}$  be a  $k$ -tuple of not necessarily distinct integers, and fix  $q$  a square-free integer. Then the *singular series at  $\mathcal{D}$  with respect to  $q$*  is given by

$$\mathfrak{S}(\mathcal{D}; q) := \sum_{q_1, \dots, q_k \mid q} \left( \prod_{i=1}^k \frac{\mu(q_i)}{\phi(q_i)} \right) \sum_{\substack{a_1, \dots, a_k \\ 1 \leq a_i \leq q_i \\ (a_i, q_i) = 1 \\ \sum a_i / q_i \in \mathbb{Z}}} e \left( \sum_{i=1}^k \frac{a_i d_i}{q_i} \right).$$

Just as for  $\mathfrak{S}(\mathcal{D})$ , one can subtract off the main term of  $\mathfrak{S}(\mathcal{D}; q)$  to define

$$\mathfrak{S}_0(\mathcal{D}; q) := \sum_{\mathcal{J} \subset \mathcal{D}} (-1)^{|\mathcal{D} \setminus \mathcal{J}|} \mathfrak{S}(\mathcal{J}; q).$$

Combining this with the definition for  $\mathfrak{S}(\mathcal{D}; q)$  yields the formula

$$\mathfrak{S}_0(\mathcal{D}; q) = \sum_{1 < q_1, \dots, q_k \mid q} \left( \prod_{i=1}^k \frac{\mu(q_i)}{\phi(q_i)} \right) \sum_{\substack{a_1, \dots, a_k \\ 1 \leq a_i \leq q_i \\ (a_i, q_i) = 1 \\ \sum a_i / q_i \in \mathbb{Z}}} e \left( \sum_{i=1}^k \frac{a_i d_i}{q_i} \right). \tag{6}$$

If the  $d_i$  are not all distinct, this expression converges for any fixed  $q$  but not in the  $q \rightarrow \infty$  limit. The singular series at  $\mathcal{D}$  with respect to  $q$  is equal to a finite Euler product.

**Lemma 2.3.** *Let  $k \geq 2$ , and let  $\mathcal{D} = \{d_1, \dots, d_k\}$  be a  $k$ -tuple of not necessarily distinct integers, and fix  $q$  a square-free integer. Then*

$$\mathfrak{S}(\mathcal{D}; q) = \prod_{p|q} \left(1 - \frac{1}{p}\right)^{-k} \left(1 - \frac{v_p(\mathcal{D})}{p}\right),$$

where  $v_p(\mathcal{D})$  is the number of distinct residue classes mod  $p$  occupied by elements of  $\mathcal{D}$ .

This lemma is proven in [Montgomery and Soundararajan 2004, Lemma 3]; it is stated there for sets with distinct elements, but their proof holds in this setting as well. They note first that  $\mathfrak{S}(\mathcal{D}; q)$  is multiplicative in  $q$ , so that it suffices to check the lemma for primes  $p$ . For a given prime  $p$ , they express the condition that  $\sum_{i=1}^k a_i/q_i \in \mathbb{Z}$  in terms of additive characters mod  $p$ , and then rearrange to get the result.

Consider the following expression for  $\mathfrak{S}_0$ , which is [Montgomery and Soundararajan 2004, equation (45)]. For all  $y \geq h$ ,

$$\mathfrak{S}_0(\mathcal{D}) = \sum_{\substack{q_1, q_2, q_3 \\ q_i > 1 \\ p|q_i \Rightarrow p \leq y}} \prod_{i=1}^3 \frac{\mu(q_i)}{\phi(q_i)} A(q_1, q_2, q_3; \mathcal{D}) + O\left(\frac{(\log y)}{y}\right),$$

where

$$A(q_1, q_2, q_3; \mathcal{D}) = \sum_{\substack{a_1, a_2, a_3 \\ 1 \leq a_i \leq q_i \\ (a_i, q_i) = 1 \\ \sum a_i/q_i \in \mathbb{Z}}} e\left(\sum_{i=1}^3 \frac{d_i a_i}{q_i}\right).$$

Apply this to  $R_3(h)$  with  $y = h^4$  and  $q = \prod_{p \leq y} p$  to get

$$R_3(h) = \sum_{\substack{q_1, q_2, q_3 \\ q_i > 1 \\ q_i | q}} \prod_{i=1}^3 \frac{\mu(q_i)}{\phi(q_i)} S(q_1, q_2, q_3; h) + O(1),$$

where

$$S(q_1, q_2, q_3; h) := \sum_{\substack{d_1, d_2, d_3 \\ 1 \leq d_i \leq h \\ d_i \text{ distinct}}} A(q_1, q_2, q_3; \{d_1, d_2, d_3\}) = \sum_{\substack{d_1, d_2, d_3 \\ 1 \leq d_i \leq h \\ d_i \text{ distinct}}} \sum_{\substack{a_1, a_2, a_3 \\ 1 \leq a_i \leq q_i \\ (a_i, q_i) = 1 \\ \sum a_i/q_i \in \mathbb{Z}}} e\left(\sum_{i=1}^3 \frac{d_i a_i}{q_i}\right).$$

If the condition that the  $d_i$  should be distinct were omitted, then the main term in  $R_3(h)$  would be exactly  $V_3(q; h)$ . So, it suffices to remove this condition.

Put  $\delta_{i,j} = 1$  if  $d_i = d_j$  and 0 otherwise, so that

$$\prod_{1 \leq i < j \leq 3} (1 - \delta_{i,j}) = \begin{cases} 1 & \text{if the } d_i \text{ are all pairwise distinct,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$S(q_1, q_2, q_3; h) = \sum_{\substack{d_1, d_2, d_3 \\ 1 \leq d_i \leq h}} \left( \prod_{1 \leq i < j \leq 3} (1 - \delta_{i,j}) \right) \sum_{\substack{a_1, a_2, a_3 \\ 1 \leq a_i \leq q_i \\ (a_i, q_i) = 1 \\ \sum a_i / q_i \in \mathbb{Z}}} e \left( \sum_{i=1}^3 \frac{d_i a_i}{q_i} \right).$$

Expanding the product over the  $\delta_{i,j}$  yields

$$1 - \delta_{1,2} - \delta_{1,3} - \delta_{2,3} + \delta_{1,2}\delta_{2,3} + \delta_{1,3}\delta_{1,2} + \delta_{2,3}\delta_{1,3} - \delta_{1,2}\delta_{2,3}\delta_{1,3}.$$

Note that the last four terms each require precisely that  $d_1 = d_2 = d_3$  in order to be nonzero; each of these can be written as  $\delta_{1,2,3}$ , so that their sum is  $2\delta_{1,2,3}$ . The following lemma addresses the contribution of these last four terms.

**Lemma 2.4.** *Let  $h \geq 4$  be an integer. Then*

$$2 \sum_{d \leq h} \sum_{\substack{q_1, q_2, q_3 \\ q_i > 1 \\ q_i \mid q}} \prod_{i=1}^3 \frac{\mu(q_i)}{\phi(q_i)} \sum_{\substack{a_1, a_2, a_3 \\ 1 \leq a_i \leq q_i \\ (a_i, q_i) = 1 \\ \sum a_i / q_i \in \mathbb{Z}}} e \left( \sum_{i=1}^3 \frac{d a_i}{q_i} \right) = 2h \left( \frac{q}{\phi(q)} \right)^2 - 6h \frac{q}{\phi(q)} + 4h.$$

*Proof.* Note that the left-hand expression is precisely  $2 \sum_{d \leq h} \mathfrak{S}_0(\{d, d, d\}; q)$ . Expanding  $\mathfrak{S}_0$  and applying Lemma 2.3 yields

$$\begin{aligned} 2 \sum_{d \leq h} \mathfrak{S}_0(\{d, d, d\}; q) &= 2 \sum_{d \leq h} (\mathfrak{S}(\{d, d, d\}; q) - 3\mathfrak{S}(\{d, d\}; q) + 3\mathfrak{S}(\{d\}; q) - 1) \\ &= 2 \sum_{d \leq h} \left( \prod_{p \mid q} \left( 1 - \frac{1}{p} \right)^{-2} - 3 \prod_{p \mid q} \left( 1 - \frac{1}{p} \right)^{-1} + 2 \right) \\ &= 2h \frac{q^2}{\phi(q)^2} - 6h \frac{q}{\phi(q)} + 4h, \end{aligned}$$

as desired. □

Now consider the contribution to  $R_3(h)$  from the terms  $-\delta_{1,2}$ ,  $-\delta_{1,3}$ , and  $-\delta_{2,3}$ . Via relabeling, it suffices to only consider the term with  $-\delta_{1,2}$ , which is nonzero when  $d_1 = d_2$  and otherwise 0.

**Lemma 2.5.** *Let  $h \geq 4$  be an integer. Then*

$$\sum_{d, d_3 \leq h} \sum_{\substack{q_1, q_2, q_3 \\ q_i > 1 \\ q_i \mid q}} \prod_{i=1}^3 \frac{\mu(q_i)}{\phi(q_i)} \sum_{\substack{a_1, a_2, a_3 \\ 1 \leq a_i \leq q_i \\ (a_i, q_i) = 1 \\ \sum a_i / q_i \in \mathbb{Z}}} e \left( d \left( \frac{a_1}{q_1} + \frac{a_2}{q_2} \right) \right) e \left( \frac{d_3 a_3}{q_3} \right) = \left( \frac{q}{\phi(q)} - 2 \right) \left( h \frac{q}{\phi(q)} - h \log h + Bh + O(h^{1/2+\varepsilon}) \right).$$

*Proof.* As in the previous lemma, we note that the left-hand side is  $\sum_{d, d_3 \leq h} \mathfrak{S}_0(\{d, d, d_3\}; q)$ . We again expand and apply Lemma 2.3, to get

$$\begin{aligned} \sum_{d, d_3 \leq h} \mathfrak{S}_0(\{d, d, d_3\}; q) &= \sum_{d, d_3 \leq h} (\mathfrak{S}(\{d, d, d_3\}; q) - 2\mathfrak{S}(\{d, d_3\}; q) - \mathfrak{S}(\{d, d\}; q) + 2) \\ &= \left( \frac{q}{\phi(q)} - 2 \right) \left( \sum_{d, d_3 \leq h} \mathfrak{S}(\{d, d_3\}; q) - h^2 \right). \end{aligned}$$

By [Montgomery and Soundararajan 2004, Lemma 4],

$$\sum_{d, d_3 \leq h} \mathfrak{S}(\{d, d_3\}; q) = \sum_{q_1 | q} \frac{\mu(q_1)^2}{\phi(q_1)^2} \sum_{\substack{1 \leq a \leq q_1 \\ (a, q_1) = 1}} \left| E\left(\frac{a}{q_1}\right) \right|^2 = h \frac{q}{\phi(q)} + h^2 - h \log h + Bh + O(h^{1/2+\varepsilon}),$$

with  $B = 1 - \gamma - \log 2\pi$ . Thus our expression becomes

$$= \left(\frac{q}{\phi(q)} - 2\right) \left(h \frac{q}{\phi(q)} - h \log h + Bh + O(h^{1/2+\varepsilon})\right),$$

as desired. □

Combining these computations yields

$$\begin{aligned} R_3(h) &= V_3(q; h) + 2h \left(\frac{q}{\phi(q)}\right)^2 - 6h \frac{q}{\phi(q)} + 4h - 3 \left(\frac{q}{\phi(q)} - 2\right) \left(h \frac{q}{\phi(q)} - h \log h + Bh + O(h^{1/2+\varepsilon})\right) \\ &= V_3(q; h) - h \left(\frac{q}{\phi(q)}\right)^2 + 3h \log h \frac{q}{\phi(q)} - 3Bh \frac{q}{\phi(q)} - 6h \log h + 6Bh + 4h + O\left(h^{1/2+\varepsilon} \frac{q}{\phi(q)}\right). \end{aligned}$$

By Theorem 2.1,  $V_3(q; h) \ll h(\log h)^5$ , so  $R_3(h) \ll h(\log h)^5$ , which completes the proof of Theorem 1.2.

**2.1. Preparing for the proof of Theorem 2.1.** The rest of this section will be devoted to the proof of Theorem 2.1; here we begin by fixing some notation and proving several preparatory lemmas. Specifically, Lemmas 2.8, 2.9, 2.10, and 2.11 are general results on adding integer reciprocals along hyperplanes. Lemmas 2.12, 2.13, and 2.14 rely on these general results to prove bounds on specific sums that will appear in the proof of Theorem 2.1.

We begin with a reparametrization of variables into a system of common divisors. Let  $(q_1, q_2, q_3)$  be a triple in the sum in (5) defining  $V_3(q; h)$ . The contribution of the  $(q_1, q_2, q_3)$ -term to  $V_3(q; h)$  is zero unless there are nontrivial solutions to

$$\frac{a_1}{q_1} + \frac{a_2}{q_2} + \frac{a_3}{q_3} \in \mathbb{Z},$$

or equivalently

$$a_1 q_2 q_3 + a_2 q_1 q_3 + a_3 q_1 q_2 \equiv 0 \pmod{q_1 q_2 q_3},$$

where  $(a_i, q_i) = 1$  for all  $i$ . This implies that  $q_1 \mid q_2 q_3$  (and likewise  $q_2 \mid q_1 q_3$  and  $q_3 \mid q_1 q_2$ ), since reducing mod  $q_1$  shows that  $a_1 q_2 q_3 \equiv 0 \pmod{q_1}$ , and by assumption  $(a_1, q_1) = 1$ . Since  $q$  is square-free, so are  $q_1, q_2$ , and  $q_3$ , so we can reparametrize as follows. Let  $g = \gcd(q_1, q_2, q_3)$  be the product of all primes dividing all three  $q_i$ 's. Define  $x = \gcd(q_2/g, q_3/g)$ ,  $y = \gcd(q_1/g, q_3/g)$ , and  $z = \gcd(q_1/g, q_2/g)$ . Then  $q_1 = gyz$ ,  $q_2 = gxz$ , and  $q_3 = gxy$ , with  $g, x, y, z$  pairwise coprime and square-free. This reparametrization is the same as writing the system of *relative greatest common divisors* for  $q_1, q_2$ , and  $q_3$ ; see for example [Elsholtz and Planitzer 2020] for more details.

Then

$$V_3(q; h) = \sum_{\substack{g, x, y, z | q \\ gxy, gxz, gyz > 1}} \frac{\mu(g)\mu(gxyz)^2}{\phi(g)\phi(gxyz)^2} \sum_{\substack{a_1, a_2, a_3 \\ 0 \leq a_1 < gyz, \dots \\ (a_1, gyz) = \dots = 1 \\ a_1/(gyz) + a_2/(gxz) + a_3/(gxy) \in \mathbb{Z}}} E\left(\frac{a_1}{gyz}\right) E\left(\frac{a_2}{gxz}\right) E\left(\frac{a_3}{gxy}\right).$$

We start by taking absolute values, using the bound that, for all  $0 \leq \alpha < 1$ ,  $|E(\alpha)| \leq F(\alpha)$ , where

$$F(\alpha) := \min\{h, \|\alpha\|^{-1}\}, \tag{7}$$

so that

$$V_3(q; h) \leq \sum_{\substack{g, x, y, z | q \\ gxy, gxz, gyz > 1}} \frac{\mu(gxyz)^2}{\phi(g)\phi(gxyz)^2} \sum_{\substack{a_1, a_2, a_3 \\ 0 \leq a_1 < gyz, \dots \\ (a_1, gyz) = \dots = 1 \\ \sum a_i / gyz \in \mathbb{Z}}} F\left(\frac{a_1}{gyz}\right) F\left(\frac{a_2}{gxz}\right) F\left(\frac{a_3}{gxy}\right). \tag{8}$$

We now split the sum  $V_3(q; h)$  into three different sums, addressed separately. Let  $T_1$  consist of all terms  $g, x, y, z$  in (8) with  $gx \geq h$ . Let  $T_2$  consist of all terms  $g, x, y, z$  in (8) with  $gx < h$ ,  $gy < h$ , and  $gz < h$ , and  $\|a_2/q_2\|, \|a_3/q_3\| > 1/h$ . Finally, let  $T_3$  consist of all terms  $g, x, y, z$  in (8) with  $gx < h$ ,  $gy < h$ , and  $gz < h$ , as well as the constraints that  $\|a_1/(gyz)\| \leq 2/h$ ,  $\|a_2/(gxz)\| \leq 2/h$ , and  $\|a_3/(gxy)\| \leq 2/h$ .

We claim that, after permuting the names of the variables as necessary, each term  $g, x, y, z, a_1, a_2, a_3$  is contained in sums for  $T_1, T_2$ , or  $T_3$ . Terms where any of  $gx, gy$ , or  $gz$  are  $\geq h$  are included in a copy of  $T_1$ . For remaining terms we have  $gx < h, gy < h$ , and  $gz < h$ . If two of the three fractions  $a_i/q_i$  satisfy  $\|a_i/q_i\| \leq 1/h$  (say  $i = 1, 2$ ), then the third one must satisfy  $\|a_3/q_3\| \leq 2/h$  because  $a_1/q_1 + a_2/q_2 + a_3/q_3 \in \mathbb{Z}$ ; therefore, these terms are included up to permutation of indices in  $T_3$ . The remaining terms must be included, up to permuting the indices, in  $T_2$ . This implies in particular that

$$V_3(q; h) \ll T_1 + T_2 + T_3.$$

We will show in Lemmas 2.15, 2.16, and 2.17 respectively that  $T_1 \ll h(\log h)^5$ ,  $T_2 \ll h(\log h)^4(\log \log h)^2$ , and  $T_3 \ll h(\log h)^4(\log \log h)^2$ , which completes the proof of Theorem 2.1.

In what follows, it will be helpful for us to approximate fractions  $a/q$  by a nearby multiple of  $1/h$ ; to do so, we make the following definition.

**Definition 2.6.** Fix  $h \geq 4$ . Let  $q > 1$  and let  $1 \leq a < q$  with  $(a, q) = 1$ . If  $q > h$ , the  $h$ -approximate numerator  $n(a, q)$  is defined as

$$n(a, q) = \left\lceil h \left\| \frac{a}{q} \right\| \right\rceil = \begin{cases} \left\lceil \frac{ha}{q} \right\rceil & \text{if } \frac{a}{q} \leq \frac{1}{2}, \\ h - \left\lfloor \frac{ha}{q} \right\rfloor & \text{if } \frac{a}{q} > \frac{1}{2}. \end{cases}$$

Meanwhile, if  $q \leq h$ , the  $h$ -approximate numerator  $n(a, q)$  is defined to be  $a$  itself.

For example, if  $q > h$  and  $1/h < a/q \leq 2/h$ , say, then the  $h$ -approximate numerator  $n(a, q)$  is equal to  $2$ , so that  $n(a, q)/(2h) \leq a/q \leq n(a, q)/h$ . The definition is arranged so that  $n(a, q)$  is never zero

when  $(a, q) = 1$ ; if  $0 < a/q \leq 1/h$ , then  $n(a, q) = 1$ . The key consequence of this definition is the following property.

**Claim 2.7.** *Let  $h \geq 4$ . For  $F(\alpha)$  defined in (7), we have*

$$F\left(\frac{a}{q}\right) \leq 2 \left\| \frac{n(a, q)}{\min\{q, h\}} \right\|^{-1}. \tag{9}$$

*Proof.* If  $q \leq h$ , then (9) states that  $\|a/q\|^{-1} \leq 2\|a/q\|^{-1}$ , which is true.

For  $q > h$ , we restrict to considering the case when  $a/q \in (0, \frac{1}{2}]$ , so that  $\|a/q\| = a/q$ ; the case when  $a/q \in (\frac{1}{2}, 1)$  is analogous. Assume first that  $0 < a/q \leq 1/h$ . Then  $F(a/q) = h$  and  $n(a, q) = 1$ , so that (9) states that  $h \leq 2h$ , which is true. Finally assume that  $1/h < a/q$ . By definition,  $n(a, q) = \lceil ha/q \rceil = ha/q + e$ , where  $0 \leq e < 1$ . For any such  $e$ ,

$$\left\| \frac{a}{q} + \frac{e}{h} \right\| \leq \left\| \frac{a}{q} \right\| + \frac{1}{h} \leq 2 \frac{a}{q}.$$

Thus

$$\left(\frac{a}{q}\right)^{-1} \leq 2 \left\| \frac{a}{q} + \frac{e}{h} \right\|^{-1} = 2 \left\| \frac{\lceil ha/q \rceil}{h} \right\|^{-1},$$

which is precisely (9) in this case. □

We write  $\tilde{q} := \min\{q, h\}$ , so that  $F(a/q) \leq 2\|n(a, q)/\tilde{q}\|^{-1}$ . For any fraction  $a/q$ , we then have that  $a/q \approx n(a, q)/\tilde{q}$  in the sense that  $|a/q - n(a, q)/\tilde{q}| < 1/h$ , since if  $q \leq h$  then  $a/q = n(a, q)/\tilde{q}$ , and if  $q > h$  then this follows from the definition of  $n(a, q)$ .

We are now ready to proceed with several lemmas concerning sums of fractions, sums over quantities  $\|a/q\|^{-1}$ , and sums of  $F(\alpha)$ . The following four lemmas are general results on adding integer reciprocals of points lying close to certain hyperplanes. Loosely speaking, these lemmas will appear in our argument in the following way. For each of  $T_1, T_2$ , and  $T_3$ , we will have to evaluate a sum of the form

$$\sum_{\substack{a_1, a_2, a_3 \\ a_1/q_1 + a_2/q_2 + a_3/q_3 \in \mathbb{Z}}} F\left(\frac{a_1}{q_1}\right) F\left(\frac{a_2}{q_2}\right) F\left(\frac{a_3}{q_3}\right),$$

where in practice there will be further constraints on the terms  $a_i$  and  $q_i$ . After applying (9) and the observation that  $a/q \approx n(a, q)/\tilde{q}$ , and dealing with a little casework on the sign of  $n(a_i, q_i)$ , we arrive at a sum that is roughly of the form

$$8 \prod_{i=1}^3 \min\{q_i, h\} \sum_{\substack{a_1, a_2, a_3 \\ \|n(a_1, q_1)/\tilde{q}_1 + n(a_2, q_2)/\tilde{q}_2 + n(a_3, q_3)/\tilde{q}_3\| \approx 0}} \frac{1}{n(a_1, q_1)n(a_2, q_2)n(a_3, q_3)}.$$

In particular, in order to analyze  $T_1, T_2, T_3$ , we will have to understand sums of reciprocals of lattice points. Understanding the precise sums requires some amount of casework, largely coming from the cases  $q_i < h$  versus  $q_i \geq h$  and the cases  $a_i/q_i \leq \frac{1}{2}$  versus  $a_i/q_i > \frac{1}{2}$ . This casework is accomplished by the Lemmas 2.8, 2.10, and 2.11.

**Lemma 2.8.** *Let  $v_2 \geq v_1$  and  $\alpha_1 \geq 1$  be real numbers, and let  $h \in \mathbb{N}$  with  $h \geq 4$ . Then*

$$\sum_{\substack{1 \leq n_1 \leq h/(2\alpha_1) \\ 1 \leq n_2 \leq h/2 \\ 1 \leq n_3 \leq h/2 \\ -\alpha_1 n_1 + n_2 + n_3 \in [v_1, v_2]}} \frac{1}{\alpha_1 n_1 n_2 n_3} \ll \begin{cases} (v_2 - v_1 + 1) \frac{\log h}{\alpha_1} \left( \frac{2 - v_1}{\alpha_1} + 1 \right) & \text{if } v_1 < 0, \\ (v_2 - v_1 + 1) \frac{\log h}{\alpha_1^2} & \text{if } v_1 \geq 0, \end{cases}$$

where  $n_1, n_2,$  and  $n_3$  range over integers.

*Proof.* Since  $n_2 + n_3 \geq v_1 + \alpha_1 n_1$  and  $n_2 + n_3 \geq 2$ ,

$$\frac{1}{\alpha_1 n_1 n_2 n_3} = \frac{1}{\alpha_1 n_1 (n_2 + n_3)} \left( \frac{1}{n_2} + \frac{1}{n_3} \right) \leq \frac{1}{\alpha_1 n_1 \max\{2, v_1 + \alpha_1 n_1\}} \left( \frac{1}{n_2} + \frac{1}{n_3} \right).$$

The sum is then bounded by

$$\begin{aligned} \sum_{\substack{1 \leq n_1 \leq h/(2\alpha_1) \\ 1 \leq n_2 \leq h/2 \\ 1 \leq n_3 \leq h/2 \\ -\alpha_1 n_1 + n_2 + n_3 \in [v_1, v_2]}} \frac{1}{\alpha_1 n_1 n_2 n_3} &\leq \sum_{1 \leq n_1 \leq h/(2\alpha_1)} \frac{1}{\alpha_1 n_1 \max\{v_1 + \alpha_1 n_1, 2\}} \sum_{\substack{1 \leq n_2 \leq h/2 \\ 1 \leq n_3 \leq h/2 \\ -\alpha_1 n_1 + n_2 + n_3 \in [v_1, v_2]}} \frac{1}{n_2} + \frac{1}{n_3} \\ &= \sum_{1 \leq n_1 \leq h/(2\alpha_1)} \frac{1}{\alpha_1 n_1 \max\{v_1 + \alpha_1 n_1, 2\}} \sum_{\substack{1 \leq n_2 \leq h/2 \\ 1 \leq n_3 \leq h/2 \\ -\alpha_1 n_1 + n_2 + n_3 \in [v_1, v_2]}} \frac{2}{n_2}, \end{aligned}$$

where equality follows because the roles of  $n_2$  and  $n_3$  are symmetric. For fixed values of  $n_1$  and  $n_2$ , the integer  $n_3$  must satisfy  $1 \leq n_3 \leq h/2$  and  $n_3 \in [v_1 + \alpha_1 n_1 - n_2, v_2 + \alpha_1 n_1 - n_2]$ ; the number of valid choices of  $n_3$  is  $\ll v_2 - v_1 + O(1)$ . Thus the sum is

$$\begin{aligned} &\ll (v_2 - v_1 + 1) \sum_{1 \leq n_1 \leq h/(2\alpha_1)} \frac{1}{\alpha_1 n_1 \max\{v_1 + \alpha_1 n_1, 2\}} \sum_{1 \leq n_2 \leq h/2} \frac{1}{n_2} \\ &\ll (v_2 - v_1 + 1) \log h \sum_{1 \leq n_1 \leq h/(2\alpha_1)} \frac{1}{\alpha_1 n_1 \max\{v_1 + \alpha_1 n_1, 2\}}. \end{aligned}$$

If  $v_1 \geq 0$ , then  $v_1/\alpha_1 + n_1 \geq 1$  and the sum is

$$\begin{aligned} &\ll (v_2 - v_1 + 1) \log h \sum_{1 \leq n_1 \leq h/(2\alpha_1)} \frac{1}{\alpha_1 n_1 (v_1 + \alpha_1 n_1)} \\ &\ll (v_2 - v_1 + 1) \frac{\log h}{\alpha_1^2} \sum_{1 \leq n_1 \leq h/(2\alpha_1)} \frac{1}{n_1 (v_1/\alpha_1 + n_1)} \ll (v_2 - v_1 + 1) \frac{\log h}{\alpha_1^2}, \end{aligned}$$

since the sum over  $n_1$  is bounded by  $\sum_{n=1}^{\infty} 1/n^2$ , and thus by a constant. This completes the proof for this case.

On the other hand, if  $\nu_1 < 0$ , then the sum is

$$\begin{aligned} &\ll (\nu_2 - \nu_1 + 1) \log h \left( \sum_{\substack{1 \leq n_1 \leq h/(2\alpha_1) \\ n_1 < (2-\nu_1)/\alpha_1 + 1}} \frac{1}{\alpha_1 n_1} + \sum_{\substack{1 \leq n_1 \leq h/(2\alpha_1) \\ \nu_1 + \alpha_1 n_1 \geq 2 + \alpha_1}} \frac{1}{\alpha_1 n_1 (\nu_1 + \alpha_1 n_1)} \right) \\ &\ll (\nu_2 - \nu_1 + 1) \log h \left( \frac{1}{\alpha_1} \left( \frac{2 - \nu_1}{\alpha_1} + 1 \right) + \frac{1}{\alpha_1^2} \sum_{\substack{1 \leq n_1 \leq h/(2\alpha_1) \\ \nu_1/\alpha_1 + n_1 \geq 2/\alpha_1 + 1}} \frac{1}{n_1 (\nu_1/\alpha_1 + n_1)} \right). \end{aligned}$$

The final sum is bounded by  $\sum_{n=1}^{\infty} 1/n^2$ , and thus by a constant. □

**Lemma 2.9.** *Let  $\nu_2 \geq \nu_1 \geq 3$  and  $\alpha_1 \geq 1$  be real numbers, and let  $h \in \mathbb{N}$  with  $h \geq 4$ . Then*

$$\sum_{\substack{1 \leq n_1 \leq h/(2\alpha_1) \\ 1 \leq n_2 \leq h/2 \\ 1 \leq n_3 \leq h/2 \\ \alpha_1 n_1 + n_2 + n_3 \in [v_1, v_2]}} \frac{1}{\alpha_1 n_1 n_2 n_3} \ll \frac{(\nu_2 - \nu_1 + 1)}{\nu_1} \log \min\{\nu_2, h\} \left( \nu_2 - \nu_1 + 1 + \frac{1}{\alpha_1} \log \min\{\nu_1, h\} \right),$$

where  $n_1, n_2$ , and  $n_3$  range over integers.

*Proof.* The first part of this proof follows along identical lines to those of [Lemma 2.8](#), but with  $\alpha_1$  having opposite signs. By following the first part of the argument of [Lemma 2.8](#), we get that the sum we want to bound is

$$\begin{aligned} &\ll (\nu_2 - \nu_1 + 1) \sum_{\substack{1 \leq n_1 \leq h/(2\alpha_1) \\ n_1 \leq (\nu_2 - 2)/\alpha_1}} \frac{1}{\alpha_1 n_1 \max\{\nu_1 - \alpha_1 n_1, 2\}} \sum_{\substack{1 \leq n_2 \leq h/2 \\ n_2 \leq \nu_2 - \alpha_1 n_1}} \frac{1}{n_2} \\ &\ll (\nu_2 - \nu_1 + 1) \log \min\{\nu_2, h\} \sum_{\substack{1 \leq n_1 \leq h/(2\alpha_1) \\ n_1 \leq (\nu_2 - 2)/\alpha_1}} \frac{1}{\alpha_1 n_1 \max\{\nu_1 - \alpha_1 n_1, 2\}}. \end{aligned}$$

If  $\max\{\nu_1 - \alpha_1 n_1, 2\} = 2$ , then  $\nu_1 - 2 < \alpha_1 n_1 \leq \nu_2 - 2$ . The number of such terms is  $\ll \nu_2 - \nu_1$ , and for these terms the summand is  $1/(2\alpha_1 n_1) \ll 1/\nu_1$ , so these terms provide an overall contribution of size  $\ll (\nu_2 - \nu_1 + 1) \log \min\{\nu_2, h\} (\nu_2 - \nu_1)/\nu_1$ . For the remaining terms,  $\alpha_1 n_1 \leq \nu_1 - 2$ .

We rewrite

$$\frac{1}{\alpha_1 n_1 (\nu_1 - \alpha_1 n_1)} = \frac{1}{\nu_1 \alpha_1 n_1} + \frac{1}{\nu_1 (\nu_1 - \alpha_1 n_1)},$$

so that for the remaining terms we have

$$\begin{aligned} \sum_{\substack{1 \leq n_1 \leq h/(2\alpha_1) \\ n_1 \leq (\nu_1 - 2)/\alpha_1}} \frac{1}{\alpha_1 n_1 \max\{\nu_1 - \alpha_1 n_1, 2\}} &= \sum_{\substack{1 \leq n_1 \leq h/(2\alpha_1) \\ n_1 \leq (\nu_1 - 2)/\alpha_1}} \left( \frac{1}{\nu_1 \alpha_1 n_1} + \frac{1}{\nu_1 (\nu_1 - \alpha_1 n_1)} \right) \\ &\ll \frac{1}{\nu_1 \alpha_1} \log \min\{\nu_1, h\} + \frac{1}{\nu_1} \left( 1 + \frac{1}{\alpha_1} \log \min\{\nu_1, h\} \right). \quad \square \end{aligned}$$

**Lemma 2.10.** *Let  $\alpha_1 \geq 1$  and  $v_2 \geq v_1$  be (possibly negative) real numbers, and let  $h \in \mathbb{N}$  with  $h \geq 4$ . Then*

$$\sum_{\substack{1 \leq n_1 \leq h/(2\alpha_1) \\ 1 \leq n_2 \leq h/2 \\ 1 \leq n_3 \leq h/2 \\ \alpha_1 n_1 - n_2 + n_3 \in [v_1, v_2]}} \frac{1}{\alpha_1 n_1 n_2 n_3} \ll (v_2 - v_1 + 1) \left( \frac{\log h}{\alpha_1} + 1 \right) \frac{\log \max\{v_1, \alpha_1 + 1\} + 1}{\max\{v_1, \alpha_1\} + 1},$$

where  $n_1, n_2,$  and  $n_3$  range over integers.

*Proof.* Since  $\alpha_1 n_1 + n_3 \geq v_1 + n_2$  and  $\alpha_1 n_1 + n_3 \geq \alpha_1 + 1$ , we have

$$\frac{1}{\alpha_1 n_1 n_2 n_3} = \frac{1}{n_2(\alpha_1 n_1 + n_3)} \left( \frac{1}{\alpha_1 n_1} + \frac{1}{n_3} \right) \leq \frac{1}{n_2 \max\{v_1 + n_2, \alpha_1 + 1\}} \left( \frac{1}{\alpha_1 n_1} + \frac{1}{n_3} \right).$$

The sum is then bounded by

$$\sum_{\substack{1 \leq n_1 \leq h/(2\alpha_1) \\ 1 \leq n_2 \leq h/2 \\ 1 \leq n_3 \leq h/2 \\ \alpha_1 n_1 - n_2 + n_3 \in [v_1, v_2]}} \frac{1}{\alpha_1 n_1 n_2 n_3} \leq \sum_{1 \leq n_2 \leq h/2} \frac{1}{n_2 \max\{v_1 + n_2, \alpha_1 + 1\}} \sum_{\substack{1 \leq n_1 \leq h/(2\alpha_1) \\ 1 \leq n_3 \leq h/2 \\ \alpha_1 n_1 - n_2 + n_3 \in [v_1, v_2]}} \left( \frac{1}{\alpha_1 n_1} + \frac{1}{n_3} \right).$$

For fixed values of  $n_1$  and  $n_2$ , the integer  $n_3$  must satisfy  $n_3 \in [v_1 - \alpha_1 n_1 + n_2, v_2 - \alpha_1 n_1 + n_2]$  and  $1 \leq n_3 \leq h/2$ ; the number of valid choices of  $n_3$  is  $\ll v_2 - v_1 + O(1)$ . Thus

$$\begin{aligned} \sum_{1 \leq n_2 \leq h/2} \frac{1}{n_2 \max\{v_1 + n_2, \alpha_1 + 1\}} \sum_{\substack{1 \leq n_1 \leq h/(2\alpha_1) \\ 1 \leq n_3 \leq h/2 \\ \alpha_1 n_1 - n_2 + n_3 \in [v_1, v_2]}} \frac{1}{\alpha_1 n_1} &\ll \frac{(v_2 - v_1 + 1)}{\alpha_1} \log h \sum_{1 \leq n_2 \leq h/2} \frac{1}{n_2 \max\{v_1 + n_2, \alpha_1 + 1\}} \\ &\ll (v_2 - v_1 + 1) \frac{\log h \log \max\{v_1, \alpha_1 + 1\} + 1}{\alpha_1 \max\{v_1, \alpha_1\} + 1}. \end{aligned}$$

It remains to evaluate the  $1/n_3$ -term in the sum. Since  $n_3 \geq v_1 - \alpha_1 n_1 + n_2$ , we have

$$\begin{aligned} \sum_{1 \leq n_2 \leq h/2} \frac{1}{n_2 \max\{v_1 + n_2, \alpha_1 + 1\}} \sum_{\substack{1 \leq n_1 \leq h/(2\alpha_1) \\ 1 \leq n_3 \leq h/2 \\ \alpha_1 n_1 - n_2 + n_3 \in [v_1, v_2]}} \frac{1}{n_3} &\ll \sum_{1 \leq n_2 \leq h/2} \frac{1}{n_2 \max\{v_1 + n_2, \alpha_1 + 1\}} \sum_{1 \leq n_1 \leq h/(2\alpha_1)} \frac{v_2 - v_1 + 1}{\lceil v_1 - \alpha_1 n_1 + n_2 \rceil} \\ &\ll \sum_{1 \leq n_2 \leq h/2} \frac{v_2 - v_1 + 1}{n_2 \max\{v_1 + n_2, \alpha_1 + 1\}} \left( \frac{\log h}{\alpha_1} + 1 \right) \\ &\ll (v_2 - v_1 + 1) \left( \frac{\log h}{\alpha_1} + 1 \right) \frac{\log \max\{v_1, \alpha_1 + 1\} + 1}{\max\{v_1, \alpha_1\} + 1}. \end{aligned} \quad \square$$

If  $\alpha_1 = 1$ , we have the following stronger bound.

**Lemma 2.11.** *There exist absolute constants  $C$  and  $D$  such that, for all integers  $v \geq 3$  and  $h \geq 4$ ,*

$$\sum_{\substack{1 \leq n_1 \leq v-2 \\ 1 \leq n_2 \leq v-2 \\ 1 \leq n_3 \leq v-2 \\ n_1+n_2+n_3=v}} \frac{1}{n_1 n_2 n_3} \leq C \quad \text{and} \quad \sum_{\substack{1 \leq n_1 \leq h \\ 1 \leq n_2 \leq v+h \\ 1 \leq n_3 \leq v+h \\ n_2+n_3=v+n_1}} \frac{1}{n_1 n_2 n_3} \leq D,$$

where the sum ranges over integer values of  $n_1, n_2, n_3$ .

*Proof.* For real numbers  $x, x' \geq 1$  with  $|x - x'| \leq 1$ , we have  $|1/x - 1/x'| \leq 2/x$ . Thus

$$\begin{aligned} \sum_{\substack{1 \leq n_1 \leq h/2 \\ 1 \leq n_2 \leq h/2 \\ 1 \leq n_3 \leq h/2 \\ n_1+n_2+n_3=v}} \frac{1}{n_1 n_2 n_3} &\leq 8 \int_1^{v-2} \int_1^{v-x_1-1} \frac{1}{x_1 x_2 (v-x_1-x_2)} dx_2 dx_1 \\ &= 8 \int_1^{v-2} \frac{2 \ln(v-x_1-1)}{x_1 (v-x_1)} dx_1 \\ &\leq 16 \ln v \int_1^{v-2} \frac{1}{x_1 (v-x_1)} dx_1 \\ &= 16 \ln v \frac{2 \ln(v-1)}{v} = 32 \frac{(\ln v)(\ln(v-1))}{v}. \end{aligned}$$

The function  $(\ln v)(\ln(v-1))/v$  has a global maximum  $M$ ; setting  $C = 16M$  completes the proof of the first claim.

For the second claim, we similarly have

$$\begin{aligned} \sum_{\substack{1 \leq n_1 \leq h \\ 1 \leq n_2 \leq v+h \\ 1 \leq n_3 \leq v+h \\ n_2+n_3=v+n_1}} \frac{1}{n_1 n_2 n_3} &\leq 8 \int_1^h \int_1^{v+x_1-1} \frac{1}{x_1 x_2 (v+x_1-x_2)} dx_2 dx_1 \\ &= 16 \int_1^h \frac{\ln(v+x_1-1)}{x_1 (v+x_1)} dx_1 \\ &\leq 16D_1 + 16 \int_{10}^h \frac{\ln(x_1-1)}{x_1^2} dx_1 \end{aligned}$$

for some constant  $D_1$ , since  $\ln(x-1)/x$  is decreasing for  $x \geq 10$ . The integral converges to a constant as  $h \rightarrow \infty$ , so setting  $D = 16D_1 + 16 \int_{10}^{\infty} \ln(x_1-1)/x_1^2 dx_1$  completes the proof.  $\square$

The next two lemmas concern triple sums over  $\|a/q\|^{-1}$ , which arise because of their role in the definition of  $F(\alpha)$  and make use of the previous four lemmas.

**Lemma 2.12.** *Fix an integer  $h \geq 4$ . Then*

$$\sum_{\substack{1 \leq n_i \leq h-1 \\ \|\sum_i n_i/h\| \leq 3/h}} \left\| \frac{n_1}{h} \right\|^{-1} \left\| \frac{n_2}{h} \right\|^{-1} \left\| \frac{n_3}{h} \right\|^{-1} \ll h^3,$$

where  $n_1, n_2$ , and  $n_3$  range over integers.

*Proof.* We will split into cases based on whether  $n_i \leq h/2$  or  $n_i > h/2$ , i.e., based on the value of  $\|n_i/h\|$ .

Assume first that  $1 \leq n_i \leq h/2$  for all  $i = 1, 2, 3$ . Then  $\|n_i/h\| = n_i/h$ , so we have

$$\sum_{\substack{1 \leq n_i \leq h/2 \\ \|\sum_i n_i/h\| \leq 3/h}} \left\| \frac{n_1}{h} \right\|^{-1} \left\| \frac{n_2}{h} \right\|^{-1} \left\| \frac{n_3}{h} \right\|^{-1} = h^3 \sum_{\substack{1 \leq n_i \leq h/2 \\ \|\sum_i n_i/h\| \leq 3/h}} \frac{1}{n_1 n_2 n_3}.$$

To satisfy  $\|\sum_i n_i/h\| \leq 3/h$ , we must have  $n_1 + n_2 + n_3 \in \{3\} \cup [h-3, h+3] \cup [2h-3, 2h+3] \cup \{3h-3\}$ . There are finitely many possible integer values for  $n_1 + n_2 + n_3$ ; for each one, by Lemma 2.11, the sum over  $1/(n_1 n_2 n_3)$  is bounded by an absolute constant. Thus the lemma holds in this case.

Now consider terms where  $h/2 < n_i \leq h-1$  for all  $i$ . For each  $i$ , define  $m_i = h - n_i$ , so that  $1 \leq m_i \leq h/2$ . Then  $\|n_i/h\| = m_i/h$ , and

$$\left\| \sum_i \frac{n_i}{h} \right\| = \left\| 3h - \sum_i \frac{m_i}{h} \right\| = \left\| \sum_i \frac{m_i}{h} \right\|.$$

Then

$$\sum_{\substack{h/2 < n_i \leq h-1 \\ \|\sum_i n_i/h\| \leq 3/h}} \left\| \frac{n_1}{h} \right\|^{-1} \left\| \frac{n_2}{h} \right\|^{-1} \left\| \frac{n_3}{h} \right\|^{-1} \ll \sum_{\substack{1 \leq m_i \leq h/2 \\ \|\sum_i m_i/h\| \leq 3/h}} \left\| \frac{m_1}{h} \right\|^{-1} \left\| \frac{m_2}{h} \right\|^{-1} \left\| \frac{m_3}{h} \right\|^{-1},$$

which is precisely the previous case, since  $1 \leq m_i \leq h/2$  for all  $i$ . Thus this case is also  $\ll h^3$ .

Finally consider terms where, for some  $i$ ,  $n_i \in [1, h/2]$ , whereas for others  $n_i \in (h/2, h-1]$ . As in the previous paragraph, we can always flip all three  $n_i$ 's with  $h - n_i$ . Moreover, the roles of  $n_1, n_2$ , and  $n_3$  are entirely symmetric. Thus it suffices to bound those terms where  $n_2, n_3 \in [1, h/2]$  and  $n_1 \in (h/2, h-1]$ . Set  $m_1 = h - n_1$ . Then

$$\sum_{\substack{h/2 < n_1 \leq h-1 \\ 1 \leq n_2, n_3 \leq h-1 \\ \|\sum_i n_i/h\| \leq 3/h}} \left\| \frac{n_1}{h} \right\|^{-1} \left\| \frac{n_2}{h} \right\|^{-1} \left\| \frac{n_3}{h} \right\|^{-1} = h^3 \sum_{\substack{1 \leq m_1 \leq h/2 \\ 1 \leq n_2, n_3 \leq h/2 \\ \|-m_1/h + n_2/h + n_3/h\| \leq 3/h}} \frac{1}{m_1 n_2 n_3}.$$

Just as before, there are finitely many possible integer values for  $-m_1 + n_2 + n_3$  satisfying the constraint that  $\|\sum_i n_i/h\| \leq 3/h$ . For each value  $v$ , by Lemma 2.11, the sum

$$\sum_{\substack{1 \leq m_1 \leq h/2 \\ 1 \leq n_2, n_3 \leq h/2 \\ -m_1 + n_2 + n_3 = v}} \frac{1}{m_1 n_2 n_3}$$

is bounded by a constant, which completes the proof. □

**Lemma 2.13.** *Let  $h \geq 4$  and  $1 \leq q_1 < h$  be integers. Then*

$$\sum_{\substack{1 \leq n_1 \leq q_1 - 1 \\ 1 \leq n_2, n_3 \leq h - 1 \\ \|n_1/q_1 + n_2/h + n_3/h\| \leq 3/h}} \left\| \frac{n_1}{q_1} \right\|^{-1} \left\| \frac{n_2}{h} \right\|^{-1} \left\| \frac{n_3}{h} \right\|^{-1} \ll h^2 q_1 (\log h),$$

where  $n_1, n_2$ , and  $n_3$  range over integers.

*Proof.* We will split into cases based on whether each of  $n_1/q_1$ ,  $n_2/h$ , and  $n_3/h$  lie in  $(0, \frac{1}{2}]$  or  $(\frac{1}{2}, 1)$ ; for each case, we will show that the bound holds. Assume first that all three of  $n_1/q_1$ ,  $n_2/h$ , and  $n_3/h$  lie in  $(0, \frac{1}{2}]$ . Note that

$$\frac{n_1}{q_1} + \frac{n_2}{h} + \frac{n_3}{h} \geq \frac{1}{q_1} + \frac{2}{h} > \frac{3}{h},$$

so the constraint that  $\|n_1/q_1 + n_2/h + n_3/h\| \leq 3/h$  is equivalent to the constraint that

$$\begin{aligned} \frac{n_1}{q_1} + \frac{n_2}{h} + \frac{n_3}{h} &\in \left[1 - \frac{3}{h}, 1 + \frac{3}{h}\right] \cup \left[2 - \frac{3}{h}, 2 + \frac{3}{h}\right] \cup \left[3 - \frac{3}{h}, 3\right] \\ &\iff \frac{h}{\tilde{q}_1} n_1 + n_2 + n_3 \in [h - 3, h + 3] \cup [2h - 3, 2h + 3] \cup [3h - 3, 3h]. \end{aligned}$$

These are finitely many intervals, each of bounded size. Thus these terms are given by

$$\sum_{\substack{1 \leq n_1 \leq q_1/2 \\ 1 \leq n_2, n_3 \leq h/2 \\ \|n_1/q_1 + n_2/h + n_3/h\| \leq 3/h}} \frac{q_1 h^2}{n_1 n_2 n_3} = \sum_{\substack{[v_1, v_2] \in \{[h-3, h+3], \\ [2h-3, 2h+3], [3h-3, 3h]\}}} \sum_{\substack{1 \leq n_1 \leq q_1/2 \\ 1 \leq n_2, n_3 \leq h/2 \\ (h/q_1)n_1 + n_2 + n_3 \in [v_1, v_2]}} \frac{h^3}{(h/q_1)n_1 n_2 n_3}.$$

We apply [Lemma 2.9](#), with  $\alpha_1 = h/q_1$  and  $[v_1, v_2] = [h - 3, h + 3]$ ,  $[2h - 3, 2h + 3]$ , or  $[3h - 3, 3h]$ , respectively. By [Lemma 2.9](#), each of these three terms is

$$\ll h^3 \frac{1}{h} \log h \left(1 + \frac{\log h}{\alpha_1}\right) \ll h^2 \log h \left(1 + \frac{q_1 \log h}{h}\right),$$

which is  $\ll h^2 q_1 \log h$ , as desired.

Now assume that all three of  $n_1/q_1$ ,  $n_2/h$ , and  $n_3/h$  lie in  $(\frac{1}{2}, 1)$ . Define  $m_1 = q_1 - n_1$ ,  $m_2 = h - n_2$ , and  $m_3 = h - n_3$ , so that

$$\sum_{\substack{q_1/2 < n_1 \leq q_1 - 1 \\ h/2 < n_2, n_3 \leq h - 1 \\ \|n_1/q_1 + n_2/h + n_3/h\| \leq 3/h}} \left\| \frac{n_1}{q_1} \right\|^{-1} \left\| \frac{n_2}{h} \right\|^{-1} \left\| \frac{n_3}{h} \right\|^{-1} = \sum_{\substack{1 \leq m_1 \leq q_1/2 \\ 1 \leq m_2, m_3 \leq h/2 \\ \|m_1/q_1 + m_2/h + m_3/h\| \leq 3/h}} \frac{h^3}{(h/q_1)m_1 m_2 m_3}.$$

This is identical to the previous case, which we have already shown to be  $\ll h^2 q_1 \log h$ .

We now tackle the cases where not all fractions lie in the same half of  $(0, 1)$ . Assume that  $n_1/q_1 \in (\frac{1}{2}, 1)$  but  $n_2/h, n_3/h \in (0, \frac{1}{2}]$ . Define  $m_1 = q_1 - n_1$ , so that

$$\sum_{\substack{q_1/2 < n_1 \leq q_1 - 1 \\ 1 \leq n_2, n_3 \leq h/2 \\ \|n_1/q_1 + n_2/h + n_3/h\| \leq 3/h}} \left\| \frac{n_1}{q_1} \right\|^{-1} \left\| \frac{n_2}{h} \right\|^{-1} \left\| \frac{n_3}{h} \right\|^{-1} = \sum_{\substack{1 \leq m_1 \leq q_1/2 \\ 1 \leq n_2, n_3 \leq h/2 \\ \|-m_1/q_1 + n_2/h + n_3/h\| \leq 3/h}} \frac{h^3}{(h/q_1)m_1 n_2 n_3}.$$

The constraint that

$$\left\| -\frac{m_1}{q_1} + \frac{n_2}{h} + \frac{n_3}{h} \right\| \leq \frac{3}{h}$$

is equivalent to the constraint that  $-h/q_1 m_1 + n_2 + n_3$  lies in one of the intervals  $[-3, 3]$  or  $[h - 3, h + 3]$ . Applying [Lemma 2.8](#) to the sum over  $m_1, n_2, n_3$ , with  $\alpha_1 = h/q_1$  and  $[v_1, v_2]$  equal to each of these intervals respectively, we get that

$$\sum_{\substack{q_1/2 < n_1 \leq q_1 - 1 \\ 1 \leq n_2, n_3 \leq h/2 \\ \|n_1/q_1 + n_2/h + n_3/h\| \leq 3/h}} \left\| \frac{n_1}{q_1} \right\|^{-1} \left\| \frac{n_2}{h} \right\|^{-1} \left\| \frac{n_3}{h} \right\|^{-1} \ll h^3 \frac{\log h}{(h/q_1)} \left( 1 + \frac{1}{(h/q_1)} \right) \ll h^2 q_1 \log h.$$

If  $n_1/q_1 \in (0, \frac{1}{2}]$  but  $n_2/h, n_3/h \in (\frac{1}{2}, 1)$ , then we can once again replace  $n_1$  by  $m_1 = q_1 - n_1$ ,  $n_2$  by  $m_2 = h - n_2$ , and  $n_3$  by  $m_3 = h - n_3$  to revert to the previous case.

Finally assume that  $n_1/q_1 \in (0, \frac{1}{2}]$ ,  $n_2/h \in (\frac{1}{2}, 1)$ , and  $n_3/h \in (0, \frac{1}{2}]$ . The roles of  $n_2$  and  $n_3$  are symmetric, and we can always replace all three  $n_i$ 's by the corresponding  $m_i$ -value, so this is the only remaining case.

Define  $m_2 = h - n_2$ , so that

$$\sum_{\substack{1 \leq n_1 \leq q_1/2 \\ h/2 < n_2 \leq h-1 \\ 1 \leq n_3 \leq h/2 \\ \|n_1/q_1 + n_2/h + n_3/h\| \leq 3/h}} \left\| \frac{n_1}{q_1} \right\|^{-1} \left\| \frac{n_2}{h} \right\|^{-1} \left\| \frac{n_3}{h} \right\|^{-1} = \sum_{\substack{1 \leq n_1 \leq q_1/2 \\ 1 \leq m_2 \leq h/2 \\ 1 \leq n_3 \leq h/2 \\ \|n_1/q_1 - m_2/h + n_3/h\| \leq 3/h}} \frac{h^3}{(h/q_1)n_1 m_2 n_3}.$$

The constraint that  $\|n_1/q_1 - m_2/h + n_3/h\| \leq 3h$  is equivalent to the constraint that  $-(h/q_1)n_1 - m_2 + m_3$  lies in one of the intervals  $[-3, 3]$  or  $[h - 3, h + 3]$ . Applying [Lemma 2.10](#) to the sum over  $n_1, m_2, n_3$  with  $\alpha_1 = h/q_1$  and  $[v_1, v_2]$  equal to each of these intervals respectively, we get that

$$\sum_{\substack{1 \leq n_1 \leq q_1/2 \\ h/2 < n_2 \leq h-1 \\ 1 \leq n_3 \leq h/2 \\ \|n_1/q_1 + n_2/h + n_3/h\| \leq 3/h}} \left\| \frac{n_1}{q_1} \right\|^{-1} \left\| \frac{n_2}{h} \right\|^{-1} \left\| \frac{n_3}{h} \right\|^{-1} \ll h^3 \left( \frac{q_1 \log h}{h} + 1 \right) \frac{q_1 \log(h/q_1 + 1)}{h}.$$

Since  $\log x/x$  is uniformly bounded for  $x \geq 1$ , we have  $q_1/h \log(h/q_1) \ll 1$ , so these terms are also  $\ll h^2 q_1 \log h$ , which completes the proof. □

Finally, the following lemma directly bounds a sum over triple products of  $F(a_i/q_i)$ .

**Lemma 2.14.** *Let  $h \in \mathbb{N}$  with  $h \geq 4$  and let  $d_1 \geq 1$  and  $d_2 \geq 2$  be positive integers with  $d_1 | d_2$  and  $d_2 < h$ . Then*

$$\sum_{\substack{1 \leq n_1 < d_1 \\ 1 \leq n_2 < d_2}} F\left(\frac{n_1}{d_1}\right) F\left(\frac{n_2}{d_2}\right) F\left(\frac{n_1}{d_1} - \frac{n_2}{d_2}\right) \ll h d_1^2 + d_1^2 d_2 \log d_2,$$

where  $n_1$  and  $n_2$  range over integers.

*Proof.* Write  $f := d_2/d_1$ . Then

$$\frac{n_1}{d_1} - \frac{n_2}{d_2} = \frac{f n_1 - n_2}{d_2}.$$

Since  $d_2 < h$ ,

$$F\left(\frac{fn_1 - n_2}{d_2}\right) = \left\| \frac{fn_1 - n_2}{d_2} \right\|^{-1}$$

unless  $fn_1 - n_2 = 0$ . Moreover, in the range where  $1 \leq n_1 < d_1$  and  $1 \leq n_2 < d_2$ ,

$$F\left(\frac{n_1}{d_1}\right) = \left\| \frac{n_1}{d_1} \right\|^{-1} \quad \text{and} \quad F\left(\frac{n_2}{d_2}\right) = \left\| \frac{n_2}{d_2} \right\|^{-1}.$$

Thus

$$\sum_{\substack{1 \leq n_1 < d_1 \\ 1 \leq n_2 < d_2}} F\left(\frac{n_1}{d_1}\right) F\left(\frac{n_2}{d_2}\right) F\left(\frac{n_1}{d_1} - \frac{n_2}{d_2}\right) = \sum_{\substack{1 \leq n_1 < d_1 \\ 1 \leq n_2 < d_2 \\ fn_1 = n_2}} h \left\| \frac{n_1}{d_1} \right\|^{-1} \left\| \frac{n_2}{d_2} \right\|^{-1} + \sum_{\substack{1 \leq n_1 < d_1 \\ 1 \leq n_2 < d_2 \\ fn_1 \neq n_2}} \left\| \frac{n_1}{d_1} \right\|^{-1} \left\| \frac{n_2}{d_2} \right\|^{-1} \left\| \frac{fn_1 - n_2}{d_2} \right\|^{-1}.$$

The first sum is bounded by

$$\sum_{\substack{1 \leq n_1 < d_1 \\ 1 \leq n_2 < d_2 \\ fn_1 = n_2}} h \left\| \frac{n_1}{d_1} \right\|^{-1} \left\| \frac{n_2}{d_2} \right\|^{-1} = h \sum_{1 \leq n_1 < d_1} \left\| \frac{n_1}{d_1} \right\|^{-2} \leq 2hd_1^2 \sum_{1 \leq n_1 \leq d_1/2} \frac{1}{n_1^2} \ll hd_1^2.$$

It remains to bound the second sum. As in the proofs of Lemmas 2.12 and 2.13, we will split into cases based on whether  $n_1/d_1$  and  $n_2/d_2$  are in  $(0, \frac{1}{2}]$  or  $(\frac{1}{2}, 1)$ .

Assume first that both  $n_1/d_1$  and  $n_2/d_2$  are in  $(0, \frac{1}{2}]$ , or that both  $n_1/d_1$  and  $n_2/d_2$  are in  $(\frac{1}{2}, 1)$ . In the latter case, we can substitute  $m_1 = d_1 - n_1$  and  $m_2 = d_2 - n_2$  to revert precisely to the former case, so it suffices to assume that both  $n_1/d_1$  and  $n_2/d_2$  are in  $(0, \frac{1}{2}]$ . Then

$$\sum_{\substack{1 \leq n_1 \leq d_1/2 \\ 1 \leq n_2 \leq d_2/2 \\ fn_1 \neq n_2}} \frac{d_1 d_2}{n_1 n_2} \left\| \frac{fn_1 - n_2}{d_2} \right\|^{-1} = \sum_{\substack{1 \leq n_1 \leq d_1/2 \\ 1 \leq n_2 \leq d_2/2 \\ fn_1 > n_2}} \frac{d_1 d_2^2}{n_1 n_2 (fn_1 - n_2)} + \sum_{\substack{1 \leq n_1 \leq d_1/2 \\ 1 \leq n_2 \leq d_2/2 \\ fn_1 < n_2}} \frac{d_1 d_2^2}{n_1 n_2 (n_2 - fn_1)}.$$

By applying Lemma 2.8 with  $\alpha_1 = f$  and  $\nu_1 = \nu_2 = 0$ , the first sum is bounded by  $\ll d_2^3 \log d_2 / f^2 = d_1^2 d_2 \log d_2$ . For the second sum, we can achieve a bound that is somewhat stronger than the bound furnished by Lemma 2.10 in this special case. Specifically we have, writing  $n_3 = n_2 - fn_1$ ,

$$\begin{aligned} d_2^3 \sum_{\substack{1 \leq n_1 \leq d_1/2 \\ 1 \leq n_2 \leq d_2/2 \\ 1 \leq n_3 \leq d_2/2 \\ fn_1 - n_2 + n_3 = 0}} \frac{1}{fn_1 n_2 n_3} &= d_2^3 \sum_{1 \leq n_1 \leq d_1/2} \frac{1}{fn_1} \sum_{\substack{fn_1 \leq n_2 \leq d_2/2 \\ 1 \leq n_3 \leq d_2/2 \\ fn_1 + n_3 = n_2}} \frac{1}{n_2 - n_3} \left( \frac{1}{n_3} - \frac{1}{n_2} \right) \\ &= d_2^3 \sum_{1 \leq n_1 \leq d_1/2} \frac{1}{(fn_1)^2} \sum_{1 \leq n_3 \leq d_2/2 - fn_1} \left( \frac{1}{n_3} - \frac{1}{n_3 + fn_1} \right) \\ &\ll d_2^3 \sum_{1 \leq n_1 \leq d_1/2} \frac{1}{(fn_1)^2} \log d_2 \ll d_2^3 \frac{\log d_2}{f^2} = d_1^2 d_2 \log d_2. \end{aligned}$$

Thus in this case, the second sum is  $\ll d_1^2 d_2 \log d_2$ .

Now assume that  $n_1/d_1 \in (\frac{1}{2}, 1)$  but  $n_2/d_2 \in (0, \frac{1}{2}]$ ; by swapping both  $n_i$ 's with  $m_i = d_i - n_i$ , this is the same as the case that  $n_1/d_1 \in (0, \frac{1}{2}]$  but  $n_2/d_2 \in (\frac{1}{2}, 1)$ , so it is our only remaining case.

On substituting  $m_1 = d_1 - n_1$ , the sum in this case becomes

$$\sum_{\substack{1 \leq m_1 \leq d_1/2 \\ 1 \leq n_2 \leq d_2/2 \\ fm_1+n_2 < d_2}} \frac{d_1 d_2}{m_1 n_2} \left\| \frac{fm_1 + n_2}{d_2} \right\|^{-1} = \sum_{\substack{1 \leq m_1 \leq d_1/2 \\ 1 \leq n_2 \leq d_2/2 \\ fm_1+n_2 \leq d_2/2}} \frac{d_1 d_2^2}{m_1 n_2 (fm_1 + n_2)} + \sum_{\substack{1 \leq m_1 \leq d_1/2 \\ 1 \leq n_2 \leq d_2/2 \\ d_2/2 < fm_1+n_2 < d_2}} \frac{d_1 d_2^2}{m_1 n_2 (d_2 - n_2 - fm_1)}.$$

The first sum is

$$\leq d_1 d_2^2 \sum_{\substack{1 \leq m_1 \leq d_1/2 \\ 1 \leq n_2 \leq d_2/2 \\ fm_1+n_2 < d_2}} \frac{1}{fm_1^2 n_2} \ll \frac{d_1 d_2^2}{f} \log d_2 = d_1^2 d_2 \log d_2.$$

As for the second sum, setting  $n_3 = d_2 - n_2 - fm_1$ , we can bound it by applying Lemma 2.9, where  $\alpha_1 = f$  and  $\nu_1 = \nu_2 = d_2$ , to get that

$$d_2^3 \sum_{\substack{1 \leq m_1 \leq d_1/2 \\ 1 \leq n_2 \leq d_2/2 \\ 1 \leq n_3 \leq d_2/2 \\ fm_1+n_2+n_3=d_2}} \frac{1}{fm_1 n_2 n_3} \ll d_2^3 \frac{1}{d_2} \log d_2 \left(1 + \frac{\log d_2}{f}\right) \ll d_2^2 \log d_2 + d_1 d_2 \log d_2,$$

both of which are  $\ll d_1^2 d_2 \log d_2$ . □

**2.2. Bounding  $T_1$ : terms with  $gx \geq h$ .** Define

$$T_1 = \sum_{\substack{g, x, y, z | q \\ gx \geq h}} \frac{\mu(gxyz)^2}{\phi(g)^3 \phi(xyz)^2} \sum_{\substack{a_1, a_2, a_3 \\ (a_1, gyz) = \dots = 1 \\ a_1/gyz + \dots \in \mathbb{Z}}} F\left(\frac{a_1}{gyz}\right) F\left(\frac{a_2}{gxz}\right) F\left(\frac{a_3}{gxy}\right). \tag{10}$$

For these terms, the rough argument that “the probability that each of  $a_2/q_2$  and  $a_3/q_3$  are sufficiently small is about  $1/h$ , making the size of the sum  $h^{1+\varepsilon}$  instead of  $h^{3+\varepsilon}$ ” can be made precise, although some of the counting arguments are rather involved, and rely on the lemmas of the previous section. Nevertheless, we will use this basic idea to prove the following bound.

**Lemma 2.15.** *Let  $h \geq 4$ , let  $q$  be the product of primes  $p \leq h^4$ , and define  $T_1$  by (10). Then*

$$T_1 \ll h(\log h)^5.$$

*Proof.* Recall that  $q_1 = gyz$ ,  $q_2 = gxz$ , and  $q_3 = gxy$ . Since  $gx \geq h$ ,  $gxy$  and  $gxz$  (i.e.,  $q_2$  and  $q_3$ ) must also both be  $\geq h$ . Recall the notation that  $\tilde{q}_i = \min\{q_i, h\}$ , so that  $\tilde{q}_2 = \tilde{q}_3 = h$ .

Since  $a_1/q_1 + a_2/q_2 + a_3/q_3 \in \mathbb{Z}$ , the sum

$$\frac{n(a_1, q_1)}{\tilde{q}_1} + \frac{n(a_2, q_2)}{\tilde{q}_2} + \frac{n(a_3, q_3)}{\tilde{q}_3}$$

satisfies

$$\left\| \frac{n(a_1, q_1)}{\tilde{q}_1} + \frac{n(a_2, q_2)}{\tilde{q}_2} + \frac{n(a_3, q_3)}{\tilde{q}_3} \right\| \leq \left\| \frac{a_1}{q_1} + \frac{a_2}{q_2} + \frac{a_3}{q_3} \right\| + \sum_{i=1}^3 \left\| \frac{n(a_i, q_i)}{\tilde{q}_i} - \frac{a_i}{q_i} \right\| \leq \frac{3}{h},$$

since  $|a/q - n(a, q)/\tilde{q}| < 1/h$  always. We can then bound the sum by replacing the fractions  $a_i/q_i$  by their  $h$ -approximations  $n(a_i, q_i)/q_i$ . Precisely, we have

$$\begin{aligned} T_1 &= \sum_{\substack{g,x,y,z|q \\ gx \geq h}} \frac{\mu(gxyz)^2}{\phi(g)^3 \phi(xyz)^2} \sum_{\substack{a_1, a_2, a_3 \\ (a_i, q_i)=1 \\ \sum_i a_i/q_i \in \mathbb{Z}}} F\left(\frac{a_1}{q_1}\right) F\left(\frac{a_2}{q_2}\right) F\left(\frac{a_3}{q_3}\right) \\ &\ll \sum_{\substack{g,x,y,z|q \\ gx \geq h}} \frac{\mu(gxyz)^2}{\phi(g)^3 \phi(xyz)^2} \sum_{\substack{a_1, a_2, a_3 \\ (a_i, q_i)=1 \\ \sum_i a_i/q_i \in \mathbb{Z}}} \left\| \frac{n(a_1, q_1)}{\tilde{q}_1} \right\|^{-1} \left\| \frac{n(a_2, q_2)}{\tilde{q}_2} \right\|^{-1} \left\| \frac{n(a_3, q_3)}{\tilde{q}_3} \right\|^{-1} \\ &\ll \sum_{\substack{g,x,y,z|q \\ gx \geq h}} \frac{\mu(gxyz)^2}{\phi(g)^3 \phi(xyz)^2} \sum_{\substack{1 \leq n_1, n_2, n_3 \leq \tilde{q}_i - 1 \\ \|\sum_i n_i/\tilde{q}_i\| \leq 3/h}} \left\| \frac{n_1}{\tilde{q}_1} \right\|^{-1} \left\| \frac{n_2}{\tilde{q}_2} \right\|^{-1} \left\| \frac{n_3}{\tilde{q}_3} \right\|^{-1} \sum_{\substack{a_1, a_2, a_3 \\ (a_i, q_i)=1 \\ \sum_i a_i/q_i \in \mathbb{Z} \\ n(a_i, q_i)=n_i}} 1. \end{aligned}$$

The inside sum is the number of triplets  $a_1, a_2, a_3$  with  $n(a_i, q_i) = n_i$  for all  $i$ ,  $(a_i, q_i) = 1$ , and  $\sum_i a_i/q_i \in \mathbb{Z}$ . The constraint that  $n(a_i, q_i) = n_i$  implies that each  $a_i$  lies in an interval of length  $\ll q_i/h + 1$ ; that is, for  $q_i \geq h$ ,  $(q_i/h)n_i \leq a_i \leq (q_i/h)(n_i + 1)$ .

The constraint that  $\sum_i a_i/q_i \in \mathbb{Z}$ , after multiplying out denominators, is equivalent to the constraint

$$a_1x + a_2y + a_3z \equiv 0 \pmod{gxyz}. \tag{11}$$

Once the  $q_i$ 's (or equivalently  $g, x, y$ , and  $z$ ) are fixed, there are  $\ll q_1/h + 1$  choices of  $a_1$  such that  $n(a_1, q_1) = n_1$ . Once  $a_1$  is fixed,  $a_2$  is determined mod  $z$  by (11). Since  $1 \leq a_2 \leq gxz$ , fixing  $a_2$  is equivalent to choosing a congruence class mod  $gx$  for  $a_2$ ; there are  $\ll gx/h + 1$  choices of this congruence class such that  $a_2$  lies within the interval where  $n(a_2, q_2) = n_2$ . Since  $gx \geq h$  by assumption,  $gx/h + 1 \ll gx/h$ . Once  $a_1$  and  $a_2$  have been fixed,  $a_3$  is entirely determined by (11). Thus the total number of triplets  $a_1, a_2, a_3$  satisfying all constraints is  $\ll (q_1/h + 1)(gx/h)$ .

Thus  $T_1$  is bounded by

$$T_1 \ll \sum_{\substack{g,x,y,z|q \\ gx \geq h}} \frac{\mu(gxyz)^2}{\phi(g)^3 \phi(xyz)^2} \left(\frac{q_1}{h} + 1\right) \frac{gx}{h} \sum_{\substack{1 \leq n_i \leq \tilde{q}_i - 1 \\ \|\sum_i n_i/\tilde{q}_i\| \leq 3/h}} \left\| \frac{n_1}{\tilde{q}_1} \right\|^{-1} \left\| \frac{n_2}{h} \right\|^{-1} \left\| \frac{n_3}{h} \right\|^{-1}.$$

Consider first those terms where  $\tilde{q}_1 = h$ . Thus  $q_1/h \gg 1$ , and by Lemma 2.12, the inside sum is  $\ll h^3$ . This implies that the terms with  $\tilde{q}_1 = h$  are bounded by

$$\begin{aligned} &\ll \sum_{\substack{g,x,y,z|q \\ gx \geq h}} \frac{\mu(gxyz)^2}{\phi(g)^3 \phi(xyz)^2} \frac{q_1}{h} \frac{gx}{h} h^3 \\ &\ll h \sum_{\substack{g,x,y,z|q \\ gx \geq h}} \frac{\mu(gxyz)^2}{\phi(g)^3 \phi(xyz)^2} g^2 xyz \quad (\text{since } q_1 = gyz). \end{aligned}$$

Recalling that  $q$  is the product of all primes  $p \leq h^4$ , this sum is

$$\ll h \prod_{p \leq h^4} \left( 1 + \frac{p^2}{(p-1)^3} + \frac{3p}{(p-1)^2} \right) \ll h(\log h)^4.$$

The remaining terms are those where  $\tilde{q}_1 = q_1 < h$ . By applying [Lemma 2.13](#) to the inside sum, the terms with  $\tilde{q}_1 = q_1 < h$  are bounded by

$$\begin{aligned} &\ll \sum_{\substack{g,x,y,z|q \\ gx \geq h}} \frac{\mu(gxyz)^2}{\phi(g)^3 \phi(xyz)^2} \frac{gx}{h} (h^2 q_1 \log h) \\ &\ll h \log h \sum_{\substack{g,x,y,z|q \\ gx \geq h}} \frac{\mu(gxyz)^2 g^2 xyz}{\phi(g)^3 \phi(xyz)^2} \quad (\text{since } q_1 = gyz) \\ &\ll h(\log h) \prod_{p \leq h^4} \left( 1 + \frac{p^2}{(p-1)^3} + \frac{3p}{(p-1)^2} \right) \ll h(\log h)^5. \end{aligned}$$

Thus  $T_1 \ll h(\log h)^4 + h(\log h)^5 \ll h(\log h)^5$ , as desired. □

**2.3. Bounding  $T_2$ : terms with  $gx, gy, gz$  small and  $a_2, a_3$  large.** We now consider  $T_2$ , which is the sum of terms in [\(8\)](#) where  $gx, gy$ , and  $gz$  are all  $< h$  and  $\|a_2/gxz\| \geq 1/h$ , and  $\|a_3/(gxy)\| \geq 1/h$ . That is, define

$$T_2 := \sum_{\substack{g,x,y,z|q \\ x,y,z < h/g}} \frac{\mu(gxyz)^2}{\phi(g)^3 \phi(xyz)^2} \sum_{\substack{a_1, a_2, a_3 \\ (a_1, gyz) = \dots = 1 \\ a_1/gyz + \dots \in \mathbb{Z} \\ \|a_2/gxz\| \geq 1/h \\ \|a_3/gxy\| \geq 1/h}} F\left(\frac{a_1}{gyz}\right) F\left(\frac{a_2}{gxz}\right) F\left(\frac{a_3}{gxy}\right). \tag{12}$$

The strategy for bounding  $T_2$  is very different from that used to bound  $T_1$ . Intuitively, since the fractions  $a_2/(gxz)$  and  $a_3/(gxy)$  are far from an integer, we are now considering terms where the values of  $F(a_2/(gxz))$  and  $F(a_3/(gxy))$  are relatively small, except perhaps at the boundary where  $a_2/(gxz)$  and  $a_3/(gxy)$  are very close to  $1/h$ . Since the denominators are loosely constrained to be small, there cannot be too many points on this boundary. We will prove a precise bound in the following lemma.

**Lemma 2.16.** *Let  $h \geq 4$ , let  $q$  be the product of primes  $p \leq h^4$ , and let  $T_2$  be defined as in [\(12\)](#). Then*

$$T_2 \ll h(\log h)^4 (\log \log h)^2.$$

*Proof.* We begin by reparametrizing the sum in [\(12\)](#) over  $a_1, a_2, a_3$ . For fixed  $g, x, y, z$  and fixed  $a_1, a_2, a_3$  satisfying the constraints of the sums in [\(12\)](#), we will fix parameters  $a, b, c$  as follows. By the Chinese remainder theorem, and since  $g, x$ , and  $y$  are pairwise relatively prime, there exist unique values  $1 \leq a \leq x$  and  $1 \leq b \leq gy$  such that

$$\frac{a_3}{gxy} \equiv \frac{a}{x} - \frac{b}{gy} \pmod{1}.$$

Similarly, there exist unique values  $1 \leq a' \leq x$  and  $1 \leq c \leq gz$  such that

$$\frac{a_2}{gxz} \equiv \frac{c}{gz} - \frac{a'}{x} \pmod{1}.$$

Since  $a_1/(gyz) + a_2/(gxz) + a_3/(gxy) \in \mathbb{Z}$ , we have

$$gyz \left( \frac{a_2}{gxz} + \frac{a_3}{gxy} \right) \in \mathbb{Z} \implies gyz \left( \frac{a}{x} - \frac{b}{gy} + \frac{c}{gz} - \frac{a'}{x} \right) \in \mathbb{Z} \implies gyz \frac{(a - a')}{x} \in \mathbb{Z}.$$

Since  $(gyz, x) = 1$ , this implies  $x|(a - a')$ ; thus  $a = a'$ . Finally,  $a_1/(gyz) + a_2/(gxz) + a_3/(gxy) \in \mathbb{Z}$  implies that

$$\frac{a_1}{gyz} \equiv -\frac{a_2}{gxz} - \frac{a_3}{gxy} \equiv \frac{b}{gy} - \frac{c}{gz} \pmod{1},$$

so that the triple  $a_1, a_2, a_3$  uniquely determines (and is uniquely determined by) a triple  $a, b, c$  with  $1 \leq a \leq x$ ,  $1 \leq b \leq gy$ , and  $1 \leq c \leq gz$  such that

$$\frac{a_1}{gyz} \equiv \frac{b}{gy} - \frac{c}{gz} \pmod{1}, \quad \frac{a_2}{gxz} \equiv \frac{c}{gz} - \frac{a}{x} \pmod{1}, \quad \text{and} \quad \frac{a_3}{gxy} \equiv \frac{a}{x} - \frac{b}{gy} \pmod{1}.$$

Upon moving the sums over  $y$  and  $z$  in (12) inside, we get

$$T_2 = \sum_{\substack{g,x|q \\ x < h/g}} \frac{\mu(gx)^2}{\phi(g)^3 \phi(x)^2} \sum_{\substack{a \\ (a,x)=1}} S_2(g, x, a),$$

where  $S_2(g, x, a)$  denotes the sum

$$S_2(g, x, a) = \sum_{\substack{y,z|q \\ y,z < h/g}} \frac{\mu(gxyz)^2}{\phi(yz)^2} \sum_{\substack{b,c \\ (b,gy)=(c,gz)=1 \\ \|c/gz - a/x\| \geq 1/h \\ \|a/x - b/gy\| \geq 1/h}} F\left(\frac{a}{x} - \frac{b}{gy}\right) F\left(\frac{b}{gy} - \frac{c}{gz}\right) F\left(\frac{c}{gz} - \frac{a}{x}\right). \tag{13}$$

Since  $gy < h$  and  $gz < h$ , the product  $yz$  is less than  $h^2$ , so that

$$\frac{yz}{\phi(yz)} \ll \log \log(h^2) \ll \log \log h.$$

Thus we can replace the expression  $1/\phi(yz)^2$  in (13) with  $(\log \log h)^2/(y^2 z^2)$ .

Let  $\ell$  and  $m$  be such that  $2^\ell < y \leq 2^{\ell+1}$  and  $2^m < z \leq 2^{m+1}$ , and further define  $n_\ell$  and  $n_m$  to be variables ranging from 1 to  $g2^\ell$  and 1 to  $g2^m$  respectively.

If

$$\frac{n_\ell}{g2^{\ell+1}} \leq \left\| \frac{a}{x} - \frac{b}{gy} \right\| \leq \frac{n_\ell + 1}{g2^{\ell+1}},$$

then

$$F\left(\frac{a}{x} - \frac{b}{gy}\right) \ll F\left(\frac{n_\ell}{g2^{\ell+1}}\right);$$

crucially, this upper bound depends only on  $\ell$  and  $n_\ell$ , and does not depend on  $b$  or  $y$ . Similarly, if

$$\frac{n_m}{g2^{m+1}} \leq \left\| \frac{c}{gz} - \frac{a}{x} \right\| \leq \frac{n_m + 1}{g2^{m+1}},$$

then

$$F\left(\frac{c}{gz} - \frac{a}{x}\right) \ll F\left(\frac{n_m}{g2^{m+1}}\right).$$

Because of the assumption that  $\|a/x - b/(gy)\| \geq 1/h$ , the constraint

$$\frac{n_\ell}{g2^{\ell+1}} \leq \left\| \frac{a}{x} - \frac{b}{gy} \right\| \leq \frac{n_\ell + 1}{g2^{\ell+1}}$$

is satisfied for some  $n_\ell$  with  $1 \leq n_\ell \leq g2^\ell$ ; in particular, the case that  $n_\ell = 0$  is ruled out. Similarly, the case that  $n_m = 0$  is ruled out by our assumptions on  $c/(gz) - a/x$ .

Thus

$$S_2(g, x, a) \ll (\log \log h)^2 \sum_{\ell, m=1}^{\log_2(h/g)} \sum_{n_\ell=1}^{g2^{\ell+1}-1} \sum_{n_m=1}^{g2^{m+1}-1} \frac{1}{2^{2\ell+2m}} \times F\left(\frac{n_\ell}{g2^{\ell+1}}\right) F\left(\frac{n_m}{g2^{m+1}}\right) F\left(\frac{n_\ell 2^m - n_m 2^\ell}{g2^{\ell+m+1}}\right) \sum_{\substack{2^\ell < y \leq 2^{\ell+1} \\ 2^m < z \leq 2^{m+1} \\ n_\ell \leq g2^{\ell+1} \|a/x - b/(gy)\| \leq n_\ell + 1 \\ n_m \leq g2^{m+1} \|c/(gz) - a/x\| \leq n_m + 1}} 1.$$

Define

$$C_{\ell, n_\ell} = \#\left\{ b, y : \frac{b}{y} \in \left( \frac{ga - n_\ell + 1}{x} - \frac{n_\ell}{y}, \frac{ga - n_\ell}{x} - \frac{n_\ell}{y} \right) \cup \left( \frac{ga + n_\ell}{x} + \frac{n_\ell}{y}, \frac{ga + n_\ell + 1}{x} + \frac{n_\ell}{y} \right), 1 \leq b < g2^{\ell+1}, 2^\ell < y \leq 2^{\ell+1} \right\},$$

and define  $C_{m, n_m}$  in the same way, so that the inside sum of  $S_2(g, x, a)$  is  $C_{\ell, n_\ell} C_{m, n_m}$ . The minimum spacing of two distinct points  $b_1/y_1$  and  $b_2/y_2$  with denominators  $y_i \leq 2^{\ell+1}$  is  $O(2^{-2\ell})$ , so

$$C_{\ell, n_\ell} \ll \frac{2^{2\ell}}{2^\ell} \ll 2^\ell,$$

and similarly  $C_{m, n_m} \ll 2^m$ . This implies that

$$S_2(g, x, a) \ll (\log \log h)^2 \sum_{\ell, m=1}^{\log_2(h/g)} \frac{2^{\ell+m}}{2^{2\ell+2m}} \sum_{n_\ell=1}^{g2^{\ell+1}} \sum_{n_m=1}^{g2^{m+1}} F\left(\frac{n_\ell}{g2^{\ell+1}}\right) F\left(\frac{n_m}{g2^{m+1}}\right) F\left(\frac{n_\ell 2^m - n_m 2^\ell}{g2^{\ell+m+1}}\right).$$

By the symmetry of  $\ell$  and  $m$ , we can restrict the sum to the terms where  $\ell \leq m$ . Applying [Lemma 2.14](#) to the sums over  $n_\ell, n_m$  with  $d_1 = g2^\ell$  and  $d_2 = g2^m$  gives

$$S_2(g, x, a) \ll (\log \log h)^2 \sum_{\substack{\ell, m=1 \\ \ell \leq m}}^{\log_2(h/g)} \frac{1}{2^{\ell+m}} (hg^2 2^{2\ell} + g^3 2^{2\ell+m} m) \ll h (\log \log h)^2 g^2 \sum_{\substack{\ell, m=1 \\ \ell \leq m}}^{\log_2(h/g)} \frac{1}{2^{m-\ell}} + (\log \log h)^2 g^3 \sum_{\substack{\ell, m=1 \\ \ell \leq m}}^{\log_2(h/g)} m 2^\ell \ll h (\log \log h)^2 g^2 \left(\log \frac{h}{g}\right)^2,$$

and thus

$$\begin{aligned}
 T_2 &\ll h(\log \log h)^2 \sum_{\substack{g,x|q \\ x < h/g}} \frac{\mu(gx)^2}{\phi(g)^3 \phi(x)^2} \sum_{\substack{a \\ (a,x)=1}} g^2 \left( \log \frac{h}{g} \right)^2 \\
 &\ll h(\log h)^2 (\log \log h)^2 \sum_{\substack{g,x|q \\ x < h/g}} \frac{\mu(gx)^2 g^2}{\phi(g)^3 \phi(x)} \\
 &\ll h(\log h)^2 (\log \log h)^2 \prod_{p \leq h^4} \left( 1 + \frac{p^2}{(p-1)^3} + \frac{1}{p-1} \right) \left( \text{since } q = \prod_{p \leq h^4} p \right) \\
 &\ll h(\log h)^4 (\log \log h)^2. \quad \square
 \end{aligned}$$

**2.4. Bounding  $T_3$ : terms with  $gx, gy, gz$  small and each  $a_i$  small.** All that remains is to analyze the sum  $T_3$ , which consists of the terms in (8) where  $gx, gy,$  and  $gz < h$ , and, for each  $i, \|a_i/q_i\| \leq 2/h$ . Precisely, we define

$$T_3 := \sum_{\substack{g,x,y,z|q \\ x,y,z < h/g}} \frac{\mu(gxyz)^2}{\phi(g)^3 \phi(xyz)^2} \sum_{\substack{a_1, a_2, a_3 \\ (a_1, gyz) = \dots = 1 \\ a_1/gyz + \dots \in \mathbb{Z} \\ \|a_1/gyz\| < 2/h \\ \|a_2/gxz\| < 2/h \\ \|a_3/gxy\| < 2/h}} F\left(\frac{a_1}{gyz}\right) F\left(\frac{a_2}{gxz}\right) F\left(\frac{a_3}{gxy}\right). \quad (14)$$

Intuitively, there are simply not many triples of fractions  $a_i/q_i$  where the denominators are not too big, each fraction is close to an integer, and the sum of all three is in  $\mathbb{Z}$ . We will make this precise in the following lemma bounding  $T_3$ , where the key savings come from bounding the number of satisfactory triples.

**Lemma 2.17.** *Let  $h \geq 4$ , let  $q$  be the product of all primes  $p \leq h^4$  and define  $T_3$  by (14). Then*

$$T_3 \ll h(\log h)^4 (\log \log h)^2.$$

*Proof.* Since  $\|a_3/(gxy)\| < 2/h$ , we must have  $(1/gxy) < 2/h$ , so if  $y < \sqrt{h/(2g)}$ , then  $x > \sqrt{h/(2g)}$ . By the same logic with  $a_1$  and  $a_2$ , at most one of  $x, y, z$  can be  $< \sqrt{h/(2g)}$ . By relabeling if necessary, we get that

$$T_3 \ll \sum_{\substack{g,x,y,z|q \\ x,y,z < h/g \\ y,z \geq \sqrt{h/(2g)}}} \frac{\mu(gxyz)^2}{\phi(g)^3 \phi(xyz)^2} \sum_{\substack{a_1, a_2, a_3 \\ (a_1, gyz) = \dots = 1 \\ a_1/gyz + \dots \in \mathbb{Z} \\ \|a_1/gyz\| < 2/h \\ \|a_2/gxz\| < 2/h \\ \|a_3/gxy\| < 2/h}} F\left(\frac{a_1}{gyz}\right) F\left(\frac{a_2}{gxz}\right) F\left(\frac{a_3}{gxy}\right).$$

As in the proof of Lemma 2.16, there are unique values  $a, b, c$  with

$$\frac{a_1}{gyz} \equiv \frac{b}{gy} - \frac{c}{gz} \pmod{1}, \quad \frac{a_2}{gxz} \equiv \frac{c}{gz} - \frac{a}{gx} \pmod{1}, \quad \text{and} \quad \frac{a_3}{gxy} \equiv \frac{a}{gx} - \frac{b}{gy} \pmod{1},$$

and we can reparametrize  $T_3$  in terms of sums over  $a, b, c$  instead of  $a_1, a_2, a_3$ . Doing so, and moving the sums over  $b, y, c,$  and  $z$  inside, we get that

$$T_3 \ll h^3 \sum_{\substack{g, x | q \\ gx \leq h}} \frac{\mu(gx)^2}{\phi(g)^3 \phi(x)^2} \sum_{\substack{a \leq x \\ (a, x) = 1}} S_3(g, x, a),$$

where

$$S_3(g, x, a) := \sum_{\substack{\sqrt{h/(2g)} \leq y \leq h/(2g) \\ \sqrt{h/(2g)} \leq z \leq h/(2g)}} \frac{\mu(yz)^2}{\phi(yz)^2} \# \left\{ b, c : \frac{b}{gy}, \frac{c}{gz} \in \left( \frac{a}{x} - \frac{2}{h}, \frac{a}{x} + \frac{2}{h} \right) \right\}.$$

Since  $y, z \leq h$ , the product  $yz$  is  $\leq h^2$ , and thus

$$\frac{1}{\phi(yz)^2} \ll \frac{(\log \log h)^2}{y^2 z^2},$$

when this term appears in  $S_3(g, x, a)$ . In order to bound  $S_3(g, x, a)$ , we split the sums over  $y$  and  $z$  dyadically, defining  $\ell$  such that  $2^\ell < y \leq 2^{\ell+1}$  and  $2^m < z \leq 2^{m+1}$ .

Then

$$S_3(g, x, a) \ll (\log \log h)^2 \sum_{\ell, m = (\log_2(h/g))/2}^{\log_2(h/g)} \frac{C_\ell C_m}{2^{2\ell} 2^{2m}},$$

where

$$C_\ell := \# \left\{ b, y : \frac{b}{y} \in \left( \frac{ga}{x} - \frac{2g}{h}, \frac{ga}{x} + \frac{2g}{h} \right), 1 \leq b < y, y \leq 2^{\ell+1} \right\},$$

and  $C_m$  is defined identically, with  $m$  in place of  $\ell$ . The minimum spacing of two distinct points  $b_1/y_1$  and  $b_2/y_2$  with denominators at most  $2^{\ell+1}$  is  $O(1/2^{2\ell})$ , so  $C_\ell \ll 2^{2\ell}(g/h) + 1$ . Since  $\ell \geq \frac{1}{2}(\log_2(h/g))$ ,  $2^{2\ell}(g/h) \geq 1$ , so in particular  $C_\ell \ll 2^{2\ell}(g/h)$ , and similarly  $C_m \ll 2^{2m}(g/h)$ .

Plugging this in gives

$$S_3(g, x, a) \ll (\log \log h)^2 \sum_{\ell, m = (\log_2(h/g))/2}^{\log_2(h/g)} \frac{2^{2\ell} 2^{2m} g^2}{2^{2\ell} 2^{2m} h^2} \ll \frac{g^2}{h^2} \left( \log \frac{h}{g} \right)^2 (\log \log h)^2,$$

so that

$$\begin{aligned} T_3 &\ll h (\log \log h)^2 \sum_{\substack{g, x | q \\ gx \leq h}} \frac{\mu(gx)^2 g^2}{\phi(g)^3 \phi(x)^2} \sum_{\substack{a \leq x \\ (a, x) = 1}} \left( \log \frac{h}{g} \right)^2 \\ &\ll h (\log h)^2 (\log \log h)^2 \sum_{\substack{g, x | q \\ gx \leq h}} \frac{\mu(gx)^2 g^2}{\phi(g)^3 \phi(x)} \\ &\ll h (\log h)^2 (\log \log h)^2 \prod_{p \leq h^4} \left( 1 + \frac{p^2}{(p-1)^3} + \frac{1}{p-1} \right), \end{aligned}$$

recalling that  $q = \prod_{p \leq h^4} p$ . Thus  $T_3 \ll h (\log h)^4 (\log \log h)^2$ . □

Putting Lemmas 2.15, 2.16, and 2.17 together completes the proof of Theorem 2.1.

### 3. Function field analogs: proof of Theorem 1.3

We now turn to considering analogous questions when working in  $\mathbb{F}_q[t]$  rather than in  $\mathbb{Z}$ . To begin with, let us set up the situation in the function field case. Fix a finite field  $\mathbb{F}_q$ . Rather than primes in  $\mathbb{N}$ , consider monic irreducible polynomials in  $\mathbb{F}_q[t]$ .

The *norm* of a polynomial  $F \in \mathbb{F}_q[t]$  is given by  $|F| = q^{\deg F}$ . We consider intervals in norm, where the interval  $I(F, h)$  of degree  $h$  is defined as

$$I(F, h) := \{G \in \mathbb{F}_q[t] : |F - G| < q^h\}.$$

For a fixed monic polynomial  $Q$ , we define

$$\begin{aligned} \mathcal{C}(Q) &:= \left\{ \frac{A}{Q} \in \mathbb{F}_q[t] : |A| < |Q| \right\}, \\ \mathcal{R}(Q) &:= \left\{ \frac{A}{Q} \in \mathbb{F}_q[t] : |A| < |Q|, (A, Q) = 1 \right\}. \end{aligned}$$

For  $Q = 1$ , we instead for convenience define  $\mathcal{C}(Q) = \{1\} = \mathcal{R}(Q)$ . If  $\deg Q > 0$ , the set of polynomials  $F$  with  $\deg F < \deg Q$  is a canonical set of representatives of  $\mathbb{F}_q[t]/(Q)$ ; in what follows, we will identify  $\{F \in \mathbb{F}_q[t] : \deg F < \deg Q\}$  with  $\mathbb{F}_q[t]/(Q)$ . If  $Q = 1$ , we will take 1 to represent the unique equivalence class of  $\mathbb{F}_q[t]/(Q)$ .

We consider the  $k$ -th moment of the distribution of irreducible polynomials in intervals  $I(F, h)$ . As in the integer case, we begin by considering the related quantity of the distribution of reduced residues modulo a square-free monic polynomial  $Q$ . That is, for  $Q$  a fixed square-free monic polynomial, we consider

$$m_k(Q; h) = \sum_{F \in \mathcal{C}(Q)} \left( \left( \sum_{\substack{G \in I(F, h) \\ (G, Q) = 1}} 1 \right) - \frac{q^h \phi(Q)}{|Q|} \right)^k. \tag{15}$$

Here we are taking the centered moment  $m_k(Q; h)$  by subtracting  $q^h \phi(Q)/|Q|$ , which is the mean value of  $\sum_{G \in I(F, h), (G, Q) = 1} 1$ .

As in the integer case, we can express the moment  $m_k(Q; h)$  in terms of exponential sums. For  $\alpha = F/G \in \mathbb{F}_q(t)$  a rational function, let  $\text{res}(\alpha)$  denote the coefficient of  $1/t$  when  $\alpha$  is written as a Laurent series with finitely many positive terms. Then define

$$e(\alpha) := e_q(\text{res}(\alpha)) = \exp(2\pi i \cdot \text{tr}(\text{res}(\alpha))/p),$$

where  $q$  is a power of the prime  $p$  and  $\text{tr} : \mathbb{F}_q \rightarrow \mathbb{F}_p$  is the trace function. This exponential function, like its integer analog, satisfies the crucial property that, for a monic polynomial  $F \in \mathbb{F}_q[t]$ ,

$$\sum_{\alpha \in \mathcal{C}(F)} e(\alpha) = \begin{cases} 1 & \text{if } F = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We then have the following lemma, analogous to [Montgomery and Vaughan 1986, Lemma 2].

**Lemma 3.1.** *Let  $Q \in \mathbb{F}_q[t]$  be square-free and let  $h \in \mathbb{N}_{\geq 1}$ . Define  $m_k(Q; h)$  by (15). Then*

$$m_k(Q; h) = |Q| \left( \frac{\phi(Q)}{|Q|} \right)^k V_k(Q; h),$$

where

$$V_k(Q; h) := \sum_{\substack{R_1, \dots, R_k \\ |R_i| > 1 \\ R_i \text{ monic}}} \prod_{i=1}^k \frac{\mu(R_i)}{\phi(R_i)} \sum_{\substack{\rho_1, \dots, \rho_k \\ \rho_i \in \mathcal{R}(R_i) \\ \sum_i \rho_i / R_i = 0}} E\left(\frac{\rho_1}{R_1}\right) \cdots E\left(\frac{\rho_k}{R_k}\right),$$

and where, for  $\alpha \in \mathbb{F}_q(t)$  a rational function,

$$E(\alpha) := \sum_{M \in I(0, h)} e(M\alpha).$$

The proof follows that of [Montgomery and Vaughan 1986, Lemma 2] very closely.

*Proof.* Let  $\kappa(R) = 1$  when  $(R, Q) = 1$ , and  $\kappa(R) = 0$  otherwise. Then

$$\kappa(R) = \sum_{S|(R, Q)} \mu(S) = \sum_{S|Q} \frac{\mu(S)}{|S|} \sum_{\sigma \in \mathcal{C}(S)} e(R\sigma) = \sum_{T|Q} \left( \sum_{\substack{A \in \mathcal{C}(T) \\ (A, T) = 1}} e(RA) \right) \left( \sum_{T|S|Q} \frac{\mu(S)}{|S|} \right).$$

Here the second factor is  $(\phi(Q)/|Q|)(\mu(T)/|T|)$ . The function  $\kappa(R)$  has mean value  $\phi(Q)/|Q|$ , so we subtract  $\phi(Q)/|Q|$  from both sides, which removes the term when  $T = 1$ . We then substitute  $R = M + N$ , and sum over  $M$  to see that

$$\sum_{\substack{|M| < q^h \\ (M+N, Q) = 1}} 1 - h \frac{\phi(Q)}{|Q|} = \frac{\phi(Q)}{|Q|} \sum_{\substack{R|Q \\ |R| > 1}} \frac{\mu(R)}{\phi(R)} \sum_{\substack{A \in \mathcal{C}(R) \\ (A, R) = 1}} E\left(\frac{A}{R}\right) e\left(\frac{NA}{R}\right).$$

The argument is completed upon raising both sides to the  $k$ -th power, summing over  $N$ , multiplying out the right-hand side, and appealing to the fact that

$$\sum_{|N| < q^d} e(N(\alpha_1 + \cdots + \alpha_k)) = \begin{cases} q^d & \text{if } \sum \alpha_i \in \mathbb{Z}, \\ 0 & \text{else.} \end{cases} \quad \square$$

One important difference between the integer setting and the function field setting is the behavior of the sums  $E(\alpha)$ , which are particularly well-behaved in  $\mathbb{F}_q[t]$ . These sums have also been studied in [Hayes 1966, Theorem 3.5].

**Lemma 3.2.** *Let  $\alpha \in \mathbb{F}_q(t)$  be a rational function with  $\deg \alpha \leq -1$ . Then*

$$E(\alpha) = \begin{cases} q^h & \text{if } \deg \alpha < -h, \\ 0 & \text{if } \deg \alpha \geq -h. \end{cases}$$

*Proof.* Let  $\mathcal{P}_h \subseteq \mathbb{F}_q[t]$  be the set of polynomials of degree less than  $h$ . Assume first that  $\deg \alpha < -h$ . Then, for all  $M \in \mathcal{P}_h$ ,  $\deg M\alpha = \deg M + \deg \alpha \leq h - 1 - h - 1 = -2$ , so the Laurent series for  $M\alpha$  has

no  $1/t$ -term, and thus  $\text{res}(M\alpha) = 0$ . But then

$$E(\alpha) = \sum_{M \in \mathcal{P}_h} e(M\alpha) = \sum_{M \in \mathcal{P}_h} e_q(\text{res}(M\alpha)) = \sum_{M \in \mathcal{P}_h} 1 = q^h.$$

Now assume that  $\deg \alpha \geq -h$ . Consider the map  $\text{res}_\alpha : \mathcal{P}_h \rightarrow \mathbb{F}_q$  which at a polynomial  $M$  returns the residue of  $M\alpha$ . This map is linear over  $\mathbb{F}_q$ , so its image is either 0 or all of  $\mathbb{F}_q$ . Let  $M = t^{-\deg \alpha - 1}$ . Since  $-h \leq \deg \alpha \leq -1$ , we have  $0 \leq -\deg \alpha - 1 \leq h - 1$ , so  $M$  indeed is a polynomial in  $\mathcal{P}_h$ . On the other hand,  $\text{res}(M\alpha)$  is precisely the leading coefficient of  $\alpha$ , which must be nonzero. Thus the image of  $\text{res}_\alpha$  is nonzero, so it is all of  $\mathbb{F}_q$ . In particular,  $\text{res}_\alpha(M)$  takes each value in  $\mathbb{F}_q$  equally often. Thus

$$E(\alpha) = \sum_{M \in \mathcal{P}_h} e_q(\text{res}(M\alpha))$$

is a balanced exponential sum, which has sum 0. □

This fact and other properties of the sums  $E(\alpha)$  mean that the analysis in [Montgomery and Vaughan 1986] in the function field setting is more streamlined. In fact, their work automatically gives the analog of our desired bound for the third moment in the function field case.

**3.1. The analog of [Montgomery and Vaughan 1986] in the function field setting.** We begin with the following fundamental lemma, with an identical proof to the integer case.

**Lemma 3.3** (Fundamental lemma). *Let  $R_1, \dots, R_k \in \mathbb{F}_q[t]$  be square-free monic polynomials with  $R = [R_1, \dots, R_k]$ . Suppose, for all irreducible  $P \mid R$ ,  $P$  divides at least two  $R_i$ 's. Let  $G_i$  be positive real-valued function defined on  $\mathcal{C}(R_i)$ . Then*

$$\left| \sum_{\substack{A_i \in \mathcal{C}(R_i) \\ \sum_i A_i/R_i = 0}} G_1\left(\frac{A_1}{R_1}\right) \cdots G_k\left(\frac{A_k}{R_k}\right) \right| \leq \frac{1}{|R|} \prod_{i=1}^k \left( |R_i| \sum_{A_i \in \mathcal{C}(R_i)} \left| G_i\left(\frac{A_i}{R_i}\right) \right|^2 \right)^{1/2}.$$

The proof follows [Montgomery and Vaughan 1986] very closely.

*Proof.* We proceed by induction on  $k$ .

Assume first that  $k = 2$ . Then we must have  $R_1 = R_2 = R$ . By Cauchy–Schwarz,

$$\left| \sum_{|A| < |R|} G_1\left(\frac{A}{R}\right) G_2\left(\frac{A}{R}\right) \right| \leq \left( \sum_{|A| < |R|} \left| G_1\left(\frac{A}{R}\right) \right|^2 \right)^{1/2} \left( \sum_{|A| < |R|} \left| G_2\left(\frac{A}{R}\right) \right|^2 \right)^{1/2},$$

which after a bit of rearranging gives the desired result.

Now assume by induction that the result holds for  $j \leq k - 1$ . For arbitrary  $k$ , set  $D = (R_1, R_2)$ , and write  $D = ST$ , with  $S \mid R_3 \cdots R_k$  and  $(T, R_3 \cdots R_k) = 1$ . Furthermore, write  $R_1 = DR'_1$  and  $R_2 = DR'_2$ . Consider any term in the sum. Since  $\sum_i A_i/R_i = 0$ , we have  $T \mid (A_1/R_1 + A_2/R_2)$ . Thus  $A_1/(STR'_1) + A_2/(STR'_2)$  can be expressed as a fraction  $A/(R'_1R'_2S)$ .

By the Chinese remainder theorem,

$$\frac{A_1}{STR'_1} = \frac{\alpha_1}{R'_1} + \frac{\beta_1}{ST} \quad \text{and} \quad \frac{A_2}{STR'_2} = \frac{\alpha_2}{R'_2} + \frac{\beta_2}{ST},$$

where

$$\frac{\beta_2}{ST} = -\frac{\beta_1}{ST} + \frac{\gamma}{S}$$

because  $T|(A_1/R_1 + A_2/R_2)$ . Thus  $A_1/R_1$  and  $A_2/R_2$  can be written as

$$\frac{A_1}{R_1} = \frac{A'_1}{R'_1} + \frac{\delta}{D} \quad \text{and} \quad \frac{A_2}{R_2} = \frac{A'_2}{R'_2} + \frac{\sigma}{S} - \frac{\delta}{D},$$

with each rational function of degree less than 0.

Let  $R^* = R'_1 R'_2 S$ . For each  $A^*$  with  $|A^*| < |R^*|$ ,  $A^*/R^*$  is uniquely of the form

$$\frac{A^*}{R^*} = \frac{A'_1}{R'_1} + \frac{A'_2}{R'_2} + \frac{\sigma}{S}.$$

Define

$$G^*\left(\frac{A^*}{R^*}\right) = \sum_{\delta \in \mathcal{C}(D)} G_1\left(\frac{A'_1}{R'_1} + \frac{\delta}{D}\right) G_2\left(\frac{A'_2}{R'_2} + \frac{\sigma}{S} - \frac{\delta}{D}\right).$$

Then the sum in question is

$$\sum_{\substack{A^* \in \mathcal{C}(R^*) \\ A_i \in \mathcal{C}(R_i) \\ A^*/R^* + \sum_{i=3}^k A_i/R_i = 0}} G^*\left(\frac{A^*}{R^*}\right) G_3\left(\frac{A_3}{R_3}\right) \cdots G_k\left(\frac{A_k}{R_k}\right).$$

Via Cauchy–Schwarz as well as the induction hypothesis, the above is

$$\leq \frac{|T|}{|R|} \left( |R^*| \sum_{A^* \in \mathcal{C}(R^*)} G^*\left(\frac{A^*}{R^*}\right)^2 \right)^{1/2} \prod_{i=3}^k \left( |R_i| \sum_{A_i \in \mathcal{C}(R_i)} G_i\left(\frac{A_i}{R_i}\right)^2 \right)^{1/2}.$$

It remains to bound the sum over  $G^*$  in terms of  $G_1$  and  $G_2$ . By Cauchy–Schwarz,

$$G^*\left(\frac{A^*}{R^*}\right)^2 \leq \left( \sum_{\delta \in \mathcal{C}(D)} G_1\left(\frac{A'_1}{R'_1} + \frac{\delta}{D}\right)^2 \right) \left( \sum_{\delta \in \mathcal{C}(D)} G_2\left(\frac{A'_2}{R'_2} + \frac{\sigma}{S} - \frac{\delta}{D}\right)^2 \right),$$

so summing over  $A^*$  gives

$$\sum_{A^* \in \mathcal{C}(R^*)} G^*\left(\frac{A^*}{R^*}\right)^2 \leq |S| \left( \sum_{A_1 \in \mathcal{C}(R_1)} G_1\left(\frac{A_1}{R_1}\right)^2 \right) \left( \sum_{A_2 \in \mathcal{C}(R_2)} G_2\left(\frac{A_2}{R_2}\right)^2 \right). \quad \square$$

We now present several preliminary lemmas about the sums  $E(\alpha)$ . The following lemma is analogous to [Montgomery and Vaughan 1986, Lemma 4].

**Lemma 3.4.** *For any polynomial  $R \in \mathbb{F}_q[t]$ ,*

$$\sum_{S \in \mathcal{C}(R)} E\left(\frac{S}{R}\right)^2 = \max\{q^{2h}, |R|q^h\}.$$

Moreover, for any polynomial  $R \in \mathbb{F}_q[t]$  and any rational function  $\alpha \in \mathbb{F}_q(t)$ ,

$$\sum_{S \in \mathcal{C}(R)} E\left(\frac{S}{R} + \alpha\right)^2 \begin{cases} = \max\{q^{2h}, |R|q^h\} & \text{if } |\alpha| < q^{-h}, \\ \leq |R|q^{h-1} & \text{if } |\alpha| \geq q^{-h}. \end{cases}$$

*Proof.* If  $\deg R \leq h$ , then, for all  $S$  with  $0 \neq |S| < |R|$ , we have  $h \geq \deg R - \deg S$ , and thus  $E(S/R)^2 = 0$ . Meanwhile,  $E(0)^2 = q^{2h}$ , so in this case  $\sum_{S \in \mathcal{C}(R)} E(S/R)^2 = q^{2h}$ .

Now suppose  $\deg R > h$ . Then  $E(S/R)$  is nonzero if and only if  $\deg S < \deg R - h$ . Thus

$$\sum_{S \in \mathcal{C}(R)} E\left(\frac{S}{R}\right)^2 = \sum_{\substack{S \in \mathcal{C}(R) \\ |S| < |R|/q^h}} E\left(\frac{S}{R}\right)^2 = \sum_{\substack{S \in \mathcal{C}(R) \\ |S| < |R|/q^h}} q^{2h} = |R|q^h,$$

which completes the first portion.

Fix a rational function  $\alpha$ . For all  $S/R$ ,  $E(S/R + \alpha)$  is unchanged by replacing  $\alpha$  with its fractional part, i.e., subtracting off the polynomial portion of  $\alpha$  so that  $|\alpha| < 1$ , including the possibility that  $\alpha = 0$ .

If a term  $E(S/R + \alpha)$  is nonzero, then  $|S/R + \alpha| < q^{-h}$ . We'll split into two cases, when  $|\alpha| < q^{-h}$  and when  $|\alpha| \geq q^h$ . First, if  $|\alpha| < q^{-h}$ , then  $|S/R + \alpha| < q^{-h}$  if and only if  $|S/R| < q^{-h}$ . If  $|R| \geq q^h$ , there are  $|R|/q^h$  values of  $S$  satisfying this; if not, there is one value. Thus if  $|\alpha| < q^{-h}$ , we have

$$\sum_{S \in \mathcal{C}(R)} E\left(\frac{S}{R} + \alpha\right)^2 E\left(\frac{S}{R} + \alpha\right) = \max(q^{2h}, |R|q^h).$$

Now assume  $|\alpha| \geq q^{-h}$ . If  $|S/R + \alpha| < q^{-h}$ , we must have  $|S/R| = |\alpha| \geq q^{-h}$ . Also, the first  $\deg \alpha + h + 1$  terms of  $S/R$  are fixed, because they must cancel with the corresponding terms of  $\alpha$  to yield a rational function of small enough degree. Correspondingly, the first  $\deg \alpha + h + 1$  terms of  $S$  are determined. Since  $|S| = |R\alpha|$ , there are at most  $|R\alpha| \cdot 1/(|\alpha| \cdot |q^{h+1}|) = |R|q^{-h-1}$  nonzero choices of  $S$ . Thus in this case,  $\sum_{S \in \mathcal{C}(R)} E(S/R + \alpha)^2 \leq |R|q^{h-1}$ . □

The following lemma corresponds to Lemma 6 of [Montgomery and Vaughan 1986].

**Lemma 3.5.** *Let  $R \in \mathbb{F}_q[t]$  be a polynomial, and let  $\alpha, \beta \in \mathbb{F}_q(t)$  be rational functions. Then*

$$\sum_{S \in \mathcal{C}(R)} E\left(\frac{S}{R} + \alpha\right)E\left(\frac{S}{R} + \beta\right) \ll E(\alpha - \beta)q^{-h} \sum_{S \in \mathcal{C}(R)} E\left(\frac{S}{R} + \alpha\right)^2.$$

*Proof.* Again, we split into two cases. Assume first that  $|\alpha - \beta| \geq q^{-h}$ , so  $E(\alpha - \beta) = 0$ . Then either  $|S/R + \beta| \geq q^{-h}$ , or  $|S/R + \alpha| \geq q^{-h}$ . Thus for each  $S/R$ , either  $E(S/R + \alpha) = 0$  or  $E(S/R + \beta) = 0$ , so the product must be 0, and thus the sum is 0.

Now assume that  $|\alpha - \beta| < q^{-h}$ , so  $E(\alpha - \beta) = q^h$ . By Lemma 3.2, if  $|\alpha - \beta| < q^{-h}$ , then  $E(S/R + \alpha) = E(S/R + \beta)$  for all  $S$ . This gives the result. □

We are now ready to prove the following lemma, which is analogous to [Montgomery and Vaughan 1986, Lemma 7].

**Lemma 3.6.** *Let  $k \geq 3$ , and let  $R_1, \dots, R_k \in \mathbb{F}_q[t]$  be square-free polynomials with  $|R_i| > 1$  for all  $i$ . Let  $R = [R_1, \dots, R_k]$ . Let  $D = (R_1, R_2)$  and  $D = ST$ , with  $S|R_3 \cdots R_k$  and  $(T, R_3 \cdots R_k) = 1$ . Write*

$R_1 = DR'_1$ ,  $R_2 = DR'_2$ , and  $R^* = R'_1 R'_2 S$ . Define

$$S(R_1, \dots, R_k) := \sum_{\substack{A_i \in \mathcal{R}(R_i) \\ \sum_i A_i/R_i = 0}} \prod_{i=1}^k E\left(\frac{A_i}{R_i}\right).$$

If, for some  $i$ ,  $|R_i| \leq q^h$ , then  $S(R_1, \dots, R_k) = 0$ . Otherwise,

$$S(R_1, \dots, R_k) \ll |R_1 \cdots R_k| \cdot |R|^{-1} (q^h)^{k/2} (X_1 + X_2 + X_3),$$

where

$$X_1 = q^{-h/2}, \quad X_2 = \begin{cases} |D|^{-1} & \text{if } |R'_1| > q^h, \\ 0 & \text{otherwise,} \end{cases} \quad X_3 = \begin{cases} |S|^{-1/2} & \text{if } R_1 = R_2, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Assume first that, for some  $i$ ,  $|R_i| \leq q^h$ . Then  $E(A_i/R_i) = 0$  whenever  $A_i \neq 0$ , so in particular for all  $A_i$  with  $(A_i, R_i) = 1$ , so the sum is 0. Assume from now on that  $|R_i| > q^h$  for all  $i$ .

We now return to the proof of the fundamental lemma. For  $A^*/R^* = A'_1/R'_1 + A'_2/R'_2 + \sigma/S$ , define

$$G^*\left(\frac{A^*}{R^*}\right) = \sum_{\substack{\delta \in \mathcal{C}(D) \\ (DA'_1 + \delta R'_1, R_1) = 1 \\ (DA'_2 + R'_2 T\sigma - R'_2 \delta, R_2) = 1}} E\left(\frac{A'_1}{R'_1} + \frac{\delta}{D}\right) E\left(\frac{A'_2}{R'_2} + \frac{\sigma}{S} - \frac{\delta}{D}\right).$$

For this sum to be nonempty,  $(A'_1, R'_1) = (A'_2, R'_2) = 1$ . Then

$$S(R_1, \dots, R_k) \leq \frac{|T|}{|R|} \left( |R^*| \sum_{A^* \in \mathcal{C}(R^*)} G^*\left(\frac{A^*}{R^*}\right)^2 \right)^{1/2} \prod_{i=3}^k \left( |R_i| \sum_{\substack{A_i \in \mathcal{R}(R_i) \\ |A_i| < |R_i|/q^h}} 1 \right)^{1/2}.$$

By Lemma 3.4, the product is  $\ll |R_3 \cdots R_k| q^{hk/2-h}$ . Thus it suffices to show that

$$\sum_{A^* \in \mathcal{C}(R^*)} G^*\left(\frac{A^*}{R^*}\right)^2 \ll |R_1| \cdot |R_2| \cdot |S| q^{2h} (X_1^2 + X_2^2 + X_3^2).$$

By Lemma 3.5,

$$G^*\left(\frac{A^*}{R^*}\right) \ll E\left(\frac{A^*}{R^*}\right) q^{-h} \sum_{\delta \in \mathcal{C}(D)} E\left(\frac{\delta}{D} + \frac{A'_1}{R'_1}\right),$$

so by Lemma 3.4,

$$G^*\left(\frac{A^*}{R^*}\right) \ll \begin{cases} E\left(\frac{A^*}{R^*}\right) \max\{q^h, |D|\} & \text{if } \left| \frac{A'_1}{R'_1} \right| < q^{-h}, \\ E\left(\frac{A^*}{R^*}\right) |D| q^{-1} & \text{if } \left| \frac{A'_1}{R'_1} \right| \geq q^{-h}. \end{cases}$$

Summing over  $A^*$  then gives

$$\sum_{A^* \in \mathcal{C}(R^*)} G^*\left(\frac{A^*}{R^*}\right)^2 \ll \sum_{\substack{A^* \in \mathcal{C}(R^*) \\ |A^*/R^*| < q^{-h} \\ |A'_1/R'_1| < q^{-h}}} E\left(\frac{A^*}{R^*}\right)^2 \max\{q^{2h}, |D|^2\} + \sum_{\substack{A^* \in \mathcal{C}(R^*) \\ |A^*/R^*| < q^{-h} \\ |A'_1/R'_1| \geq q^{-h}}} E\left(\frac{A^*}{R^*}\right)^2 |D|^2. \tag{16}$$

Here as in the definition of  $G^*$ , for any nonzero term we must have  $(A'_1, R'_1) = (A'_2, R'_2) = 1$ . In particular,  $A'_1 \equiv 0 \pmod{R'_1}$  only if  $R'_1 = 1$ . We now split into cases based on whether or not  $|R^*| > q^h$  and whether or not  $|R'_1| > q^h$ .

First assume that  $|R^*| > q^h$  and  $|R'_1| > q^h$ . Then

$$\begin{aligned} \sum_{A^* \in \mathcal{C}(R^*)} G^* \left( \frac{A^*}{R^*} \right)^2 &\ll \max\{q^{2h}, |D|^2\} q^{2h} \frac{|R'_1|}{q^h} \frac{|R'_2 S|}{q^h} + |D|^2 \sum_{\substack{A^* \in \mathcal{C}(R^*) \\ |A^*/R^*| < q^{-h} \\ |A'_1/R'_1| \geq q^h}} E \left( \frac{A^*}{R^*} \right)^2 \\ &\ll \max\{q^{2h}, |D|^2\} |R^*| + |D|^2 |R^*| q^h \\ &\ll |R_1| \cdot |R_2| \cdot |S| q^{2h} (X_1^2 + X_2^2). \end{aligned}$$

Now assume that  $|R^*| > q^h$  but  $|R'_1| \leq q^h$ . The first sum in (16) is empty unless  $R'_1 = 1$  (and  $A'_1 = 0$ ). If  $R'_1 = 1$ , then  $R_1 = D$ , so  $|D| > q^h$ . Equation (16) then becomes

$$\sum_{A^* \in \mathcal{C}(R^*)} G^* \left( \frac{A^*}{R^*} \right)^2 \ll q^{2h} |D|^2 + \frac{|R^*|}{q^h} q^{2h} |D|^2 = |R_1 R_2 S| q^{2h} \left( \frac{1}{|R^*|} + q^{-h} \right) \ll |R_1 R_2 S| q^{2h} (X_1^2).$$

If  $R'_1 \neq 1$ , then the first sum is empty, so (16) becomes

$$\sum_{A^* \in \mathcal{C}(R^*)} G^* \left( \frac{A^*}{R^*} \right)^2 \ll \frac{|R^*|}{q^h} q^{2h} |D|^2 = |R_1 R_2 S| q^{2h} (X_1^2).$$

Finally, assume that  $|R^*| \leq q^h$  and thus  $|R'_1| \leq q^h$ . In this case the only nonzero term in (16) in either sum is when  $A^* = 0$ , which forces  $A'_1 = A'_2 = \sigma = 0$ . But then since  $(A'_1, R'_1) = (A'_2, R'_2) = 1$ , we also have  $R'_1 = R'_2 = 1$ , and thus  $R_1 = R_2 = D$ , which has magnitude  $> q^h$ . Thus

$$\sum_{A^* \in \mathcal{C}(R^*)} G^* \left( \frac{A^*}{R^*} \right)^2 \ll q^{2h} |D|^2 = |R_1 R_2 S| q^{2h} \cdot |S|^{-1} = |R_1 R_2 S| q^{2h} X_3^2. \quad \square$$

We now turn to the proof of [Theorem 1.3](#), which corresponds to [\[Montgomery and Vaughan 1986, Lemma 8\]](#). The main strategy here is a careful application of [Lemma 3.6](#), keeping in mind that we can choose which variables play the roles of  $R_1$  and  $R_2$ .

**Lemma 3.7.** *For any fixed  $k \geq 3$ , for  $Q \in \mathbb{F}_q[t]$  square-free, for  $h \geq 1$  and  $m_k(Q; h)$  defined by (15),*

$$m_k(Q; h) \ll |Q| (q^h)^{k/2} \left( \frac{\phi(Q)}{|Q|} \right)^{k/2} \left( 1 + ((q^h)^{-1/2} + (q^h)^{-1/(k-2)}) \left( \frac{\phi(Q)}{|Q|} \right)^{-2k+k/2} \right).$$

*Proof.* We begin with the bound that

$$m_k(Q; h) \ll |Q| \left( \frac{\phi(Q)}{|Q|} \right)^k \sum_{\substack{R|Q \\ R \text{ monic}}} \sum_{\substack{R_i|Q \\ R_i \text{ monic} \\ |R_i| > 1 \\ [R_1, \dots, R_k] = R}} \frac{S(R_1, \dots, R_k)}{\phi(R_1) \cdots \phi(R_k)},$$

where

$$S(R_1, \dots, R_k) = \sum_{\substack{A_i \in \mathcal{R}(R_i) \\ \sum_i A_i/R_i = 0}} \prod_{i=1}^k E \left( \frac{A_i}{R_i} \right).$$

We apply [Lemma 3.6](#), but while using the fact that we have flexibility in how we label  $R_1, \dots, R_k$  in our application of [Lemma 3.6](#). For clarity, we will write  $\tilde{R}_1$  and  $\tilde{R}_2$  to be the  $R_i$ 's that serve as the first two in our application of [Lemma 3.6](#). Choose  $\tilde{R}_1$  and  $\tilde{R}_2$  as follows.

If, for any  $i$ ,  $|R_i| < q^h$ , then  $S(R_1, \dots, R_k)$  must be 0, so assume that  $|R_i| \geq q^h$  for all  $i$ . Let  $R_{ij} = (R_i, R_j)$ . For all  $i$ , since  $R_i \mid \prod_{i \neq j} R_j$ , we have  $R_i \mid \prod_{i \neq j} R_{ij}$  as well. Thus for all  $i$ , there exists  $j \neq i$  such that  $|R_{ij}| \geq |R_i|^{1/(k-1)}$ . If, for some  $i, j$ ,  $|R_{ij}| \geq |R_i|^{1/(k-1)}$  but  $R_i \neq R_j$ , then pick  $\tilde{R}_1$  and  $\tilde{R}_2$  to be  $R_i$  and  $R_j$ , respectively.

If no such  $i$  exists, then, for each  $i$ , there is some  $j \neq i$  with  $R_i = R_j$ . If there exists any triple  $R_i = R_j = R_l$ , then pick  $\tilde{R}_1 = R_i$ ,  $\tilde{R}_2 = R_j$ . If not, then the  $R_i$ 's must be equal in pairs and otherwise disjoint, and  $k$  must be even. Without loss of generality, say that  $R_1 = R_2, R_3 = R_4, \dots, R_{k-1} = R_k$ . Write  $R = UV$ , where  $V$  is the product of all primes dividing at least two  $R_{2i}$ 's, and  $U$  is the product of all primes dividing exactly one  $R_{2i}$ . Then

$$V^2 \mid \prod_{i=1}^{k/2} \left( R_{2i}, \prod_{j \neq i} R_{2j} \right),$$

so there exists some  $i$  with  $\left| (R_{2i}, \prod_{j \neq i} R_{2j}) \right| \geq |V|^{4/k}$ . Take  $\tilde{R}_1$  and  $\tilde{R}_2$  to be  $R_{2i}$  and  $R_{2i-1}$ .

Now we return to our bound on  $m_k(Q; h)$ . We have

$$m_k(Q; h) \ll |Q| \left( \frac{\phi(Q)}{|Q|} \right)^k (q^h)^{k/2} \sum_{\substack{R \mid Q \\ R \text{ monic}}} \frac{1}{|R|} \sum_{\substack{R_i \mid Q \\ R_i \text{ monic} \\ |R_i| > 1 \\ [R_1, \dots, R_k] = R}} \frac{|R_1 \cdots R_k|}{\phi(R_1) \cdots \phi(R_k)} (X_1 + X_2 + X_3),$$

where the  $X_i$  arise by use of [Lemma 3.6](#) as described above.

Consider the contribution from each  $X_i$ . Since  $X_1 = q^{-h/2}$ , the  $X_1$ -terms contribute

$$\begin{aligned} &\ll |Q| \left( \frac{\phi(Q)}{|Q|} \right)^k (q^h)^{k/2-1/2} \sum_{\substack{R \mid Q \\ R \text{ monic}}} \frac{1}{|R|} \sum_{\substack{R_i \mid Q \\ R_i \text{ monic} \\ |R_i| \geq q^h \\ [R_1, \dots, R_k] = R}} \frac{|R_1 \cdots R_k|}{\phi(R_1) \cdots \phi(R_k)} \\ &\ll |Q| \left( \frac{\phi(Q)}{|Q|} \right)^k (q^h)^{k/2-1/2} \prod_{P \mid Q} \left( 1 + \frac{1}{|P|} \left( 2 + \frac{1}{|P|-1} \right)^k \right) \\ &\ll |Q| (q^h)^{k/2-1/2} \left( \frac{\phi(Q)}{|Q|} \right)^{-2^k+k}. \end{aligned}$$

Now consider the  $X_2$ -contribution. If  $X_2 \neq 0$ , then  $|R'_1| > q^h$ , and by our choice of  $R_1, R_2$ , we have  $|D| \geq |R_1|^{1/(k-1)} = |R'_1 \cdot D|^{1/(k-1)}$ . But then  $|D|^{-1} \leq q^{-h/(k-2)}$ , so in turn  $X_2 \leq q^{-h/(k-2)}$ . By the same logic as for the  $X_1$ -terms, the  $X_2$  terms contribute  $\ll |Q| (q^h)^{k/2-1/(k-2)} (\phi(Q)/|Q|)^{-2^k+k}$ .

Finally, consider  $X_3$ . If  $X_3 \neq 0$ , then  $R_1 = R_2$ . By our choice of  $R_1$  and  $R_2$  for the application of [Lemma 3.6](#), in this case each  $R_i$  is equal to some  $R_j$ . If there exists some  $R_i = R_1 = R_2$ , with  $i \geq 3$ , then

$S = R_1 = R_2$ , so  $|S| > q^h$ , and thus for these terms we get a saving of  $q^{-h/2}$  and the bound for  $X_1$  applies. If not, then  $k$  is even and the  $R_i$ 's must be equal in pairs. Let  $R = UV$  as above, where  $U$  is the product of irreducibles  $P$  dividing exactly one pair of  $R_i$ 's, and  $V$  is the product of all other irreducibles  $P$  dividing  $R$ . Write  $R_i = U_i V_i$ , where  $U_i = (R_i, U)$  and  $V_i = (R_i, V)$ . For fixed  $U, V$ , let  $C(U, V)$  be the set of  $k$ -tuples  $(R_1, \dots, R_k)$  yielding  $U$  and  $V$ . There are at most  $\tau_{k/2}(U)$  choices for  $U_2, U_4, \dots, U_k$ , where  $\tau_{k/2}$  is the  $k/2$ -fold divisor function. Since  $V_i | V$ , there are at most  $\tau(V)^{k/2}$  choices for  $V_2, V_4, \dots, V_k$ . Thus  $\#|C(U, V)| \leq \tau_{k/2}(U)d(V)^{k/2}$ . In our application of [Lemma 3.6](#) we have  $|S| \geq |V|^{4/k}$ , so

$$\begin{aligned} \sum_{\substack{UV|Q \\ \text{monic}}} \frac{1}{|UV|} \sum_{(R_1, \dots, R_k) \in C(U, V)} \left( \prod_{i=1}^k \frac{|R_i|}{\phi(R_i)} \right) X_3 &\ll \sum_{\substack{UV|Q \\ \text{monic}}} \frac{\tau_{k/2}(U)(|U|/\phi(U))^2 \tau(V)^{k/2} (|V|/\phi(V))^k}{|U| \cdot |V|^{1+2/k}} \\ &= \prod_{P|Q} \left( 1 + \frac{k|P|}{2(|P|-1)^2} + \frac{2^{k/2}(|P|/(|P|-1))^k}{|P|^{1+2/k}} \right) \\ &\ll \left( \frac{\phi(Q)}{|Q|} \right)^{-k/2}, \end{aligned}$$

so the  $X_3$ -terms contribute  $\ll |Q|(q^h)^{k/2}(\phi(Q)/|Q|)^{k/2}$ , which completes the proof. □

The final contribution of  $X_3$  only arises when  $k$  is even, so when  $k$  is odd we have the estimate

$$m_k(Q; h) \ll |Q|((q^h)^{k/2-1/2} + (q^h)^{k/2-1/(k-2)}) \left( \frac{\phi(Q)}{|Q|} \right)^{k-2k}.$$

For  $k = 3$  this implies that

$$m_3(Q; h) \ll |Q|q^h \left( \frac{\phi(Q)}{|Q|} \right)^{-5}.$$

In the case when  $k = 5$ , we can bound  $m_5(Q; h)$  via a more involved argument.

#### 4. The fifth moment of reduced residues in the function field setting

Our goal in this section is to prove [Theorem 1.4](#), which is a stronger bound on  $m_5(Q; h)$  when  $Q = \prod_{|P| \leq q^{6h}} P$ . We will also prove [Corollary 1.5](#), bounding  $R_3(q^h)$  and  $R_5(q^h)$  in the ring  $\mathbb{F}_q[t]$ .

[Lemma 3.7](#) already implies a bound on  $m_5(Q; h)$ , showing that  $m_5(Q; h) \ll |Q|(q^h)^{13/6}(\phi(Q)/|Q|)^{-27}$ . Our goal is a bound where the power of  $q^h$  is  $2 + \varepsilon$  for all  $\varepsilon > 0$ ; note that [Conjecture 1.1](#) would predict a bound where the power of  $q^h$  is 2. In turn, this will allow us to prove [Corollary 1.5](#), that  $R_5(q^h) \ll q^{(2+\varepsilon)h}$ .

**4.1. Proof of [Theorem 1.4](#).** As in the proof of [Lemma 3.7](#), we begin by bounding

$$m_5(Q; h) \ll |Q| \left( \frac{\phi(Q)}{Q} \right)^5 \sum_{\substack{R|Q \\ R \text{ monic}}} \sum_{\substack{R_i|Q \\ R_i \text{ monic} \\ |R_i| > 1 \\ [R_1, \dots, R_5] = R}} \frac{S(R_1, \dots, R_5)}{\phi(R_1) \cdots \phi(R_5)},$$

where

$$S(R_1, \dots, R_5) = \sum_{\substack{A_i \in \mathcal{R}(R_i) \\ \sum_i A_i/R_i = 0}} \prod_{i=1}^5 E\left(\frac{A_i}{R_i}\right).$$

Our goal is to apply [Lemma 3.6](#) to bound the size of  $S(R_1, \dots, R_5)$ . But, when applying this lemma, we can freely choose which of the  $R_i$ 's plays the roles of  $R_1$  and  $R_2$ . As in the previous section, we will denote our choice by  $\tilde{R}_1$  and  $\tilde{R}_2$ . If any  $R_i$  satisfies  $|R_i| < q^h$ , the choice is immaterial, so assume that  $|R_i| \geq q^h$  for all  $i$ . If there is any triple  $R_i, R_j, R_\ell$  with  $R_i = R_j = R_\ell$ , pick  $\tilde{R}_1 = R_i$  and  $\tilde{R}_2 = R_j$ . In this case  $X_2$  will have no contribution, and  $X_3$  and  $X_1$  will each be  $\ll q^{-h/2}$ , for a total contribution to  $m_5(Q; h)$  from these terms (as in the proof of [Lemma 3.7](#)) of  $\ll |Q|q^{2h}(\phi(Q)/|Q|)^{-27}$ . If there is no such triple, but there exists  $R_i \neq R_j$  with either  $|R_i/(R_i, R_j)| < q^h$ , or  $|R_i/(R_i, R_j)| \geq q^h$  and  $|(R_i, R_j)| \geq q^{h/2}$ , then we choose  $\tilde{R}_1 = R_i$  and  $\tilde{R}_2 = R_j$ . In this case we have  $X_3 = 0$  and  $X_1, X_2$  each contributing  $\ll q^{-h/2}$ , and again the total contribution to  $m_5(Q; h)$  from these terms is  $\ll |Q|q^{2h}(\phi(Q)/|Q|)^{-27}$ . So, it remains to bound what happens in the remaining cases. We first show that in the remaining cases, up to some reordering, certain factors of  $R_2$  and  $R_3$  are bounded.

**Lemma 4.1.** *For fixed square-free  $Q \in \mathbb{F}_q[t]$ , let  $(R_1, \dots, R_5)$  be a tuple of divisors of  $Q$  such that*

- $|R_i| \geq q^h$  for all  $i$ ,
- no three  $R_i$ 's are equal,
- for any  $R_i, R_j$ , either  $R_i = R_j$ , or  $|R_i/(R_i, R_j)| \geq q^h$  and  $|(R_i, R_j)| < q^{h/2}$ , and
- $R_1, R_2$ , and  $R_3$  are all distinct.

Then

- $|R_2/(R_1, R_2)| \geq q^h$ , and
- $|R_3/(R_3, R_1 R_2)| \geq q^{h/2}$ .

Loosely, this lemma states that in the cases that we cannot already bound by the tools of the previous section, prime factors must “spread out” among the first three  $R_i$ 's.

**Remark.** The bound on  $|R_3/(R_3, R_1 R_2)|$  above is worse than the bound on  $|R_2/(R_1, R_2)|$ . In order to apply [Lemma 4.3](#) below, we will need both of them to be at least of size  $q^{h/2}$ , so the bound on  $|R_2/(R_1, R_2)|$  is better than necessary.

However, the fact that these bounds get worse is precisely what prevents us from applying our technique to bound higher moments. If instead we applied the same argument to a 7-tuple  $(R_1, \dots, R_7)$  of divisors of  $Q$ , we would not be able to guarantee that  $|R_4/(R_4, R_1 R_2 R_3)| \geq q^{h/2}$ , even if we weaken the conditions to allow reordering. This threshold is crucial for our argument, which does not generalize to 7-tuples.

*Proof.* The fact that  $|R_2/(R_1, R_2)| \geq q^h$ , follows directly from the third assumption, since  $R_1 \neq R_2$ .

For the second conclusion, let  $R_{123} = \gcd(R_1, R_2, R_3)$  and let  $R_{13} = (R_1, R_3)/\gcd(R_1, R_2, R_3)$  and  $R_{23} = (R_2, R_3)/\gcd(R_1, R_2, R_3)$ , so that  $R_{13}$  is the product of all primes dividing  $R_1$  and  $R_3$  but not  $R_2$ , and vice versa. Then  $(R_3, R_1 R_2) = R_{13} R_{23} R_{123}$ . By assumption,  $|(R_2, R_3)| < q^{h/2}$ , so  $|R_{23} R_{123}| < q^{h/2}$ , and in particular  $|R_{23}| < q^{h/2}$ . Now assume by contradiction that  $|R_3/(R_3, R_1 R_2)| < q^{h/2}$ . Then

$$\left| \frac{R_3}{(R_1, R_3)} \right| = \left| \frac{R_3}{R_{13} R_{123}} \right| = \left| \frac{R_3}{R_{13} R_{23} R_{123}} \right| \cdot |R_{23}| < q^{h/2} \cdot q^{h/2} = q^h,$$

which contradicts the third assumption because  $R_1 \neq R_3$ . □

The following auxiliary lemma provides a standard bound on  $\tau_k$ , the  $k$ -fold divisor function, in the function field setting. We will also use that  $\phi(F) \gg |F|/\log \log |F|$  for all  $F \in \mathbb{F}_q[t]$ .

**Lemma 4.2.** Fix  $k \geq 1$ . Let  $M = \max_{b \geq 1} (\tau_k(t^b))^{1/b}$ . Then

$$\limsup_{\deg F \rightarrow \infty} \frac{\log \tau_k(F) \log \log |F|}{\log |F|} = \log M,$$

and thus, for all  $\varepsilon > 0$ , we have  $\tau_k(F) \ll_\varepsilon |F|^\varepsilon$ .

*Proof.* The proof of the above lemma follows closely along the lines of [Shiu 1980]. We will show one direction of the statement, adapted to our setting; the other direction also follows very closely, so we omit it. Note first that

$$1 \leq (\tau_k(t^b))^{1/b} = \binom{b+k-1}{b}^{1/b} < \left( \frac{(b+k-1)e}{k-1} \right)^{(k-1)/b} \rightarrow 1$$

as  $b \rightarrow \infty$ , so  $M$  exists.

We now show that

$$\limsup_{\deg F \rightarrow \infty} \frac{\log \tau_k(F) \log \log |F|}{\log |F|} \geq \log M.$$

Fix  $b$  such that  $\tau_k(t^b) = M^b$ . Let

$$F = \prod_{\substack{\deg P=d \\ P \text{ irred.}}} P^b,$$

so that  $\tau_k(F) = \prod_{\deg P=d} \tau_k(P^b) = (\tau_k(t^b))^{\pi(d; \mathbb{F}_q)} = M^{b\pi(d; \mathbb{F}_q)}$ . We have that  $\pi(d; \mathbb{F}_q) \sim q^d/d$  as  $d \rightarrow \infty$ , so that

$$\log |F| = bd \log q \pi(d; \mathbb{F}_q) \sim bq^d \log q,$$

and

$$\log \log |F| = d \log q + O(1).$$

Thus as  $d \rightarrow \infty$ ,

$$\log \tau_k(F) = b\pi(d; \mathbb{F}_q) \log M \sim b \log M \cdot \frac{q^d}{d} \sim \frac{\log M \log |F|}{\log \log |F|},$$

so

$$\limsup_{\deg F \rightarrow \infty} \frac{\log \tau_k(F) \log \log |F|}{\log |F|} \geq \log M.$$

As mentioned above, the proof that

$$\limsup_{\deg F \rightarrow \infty} \frac{\log \tau_k(F) \log \log |F|}{\log |F|} \leq \log M$$

also follows the proof in [Shiu 1980] closely, so we omit it. □

The above bound implies that, for all  $\varepsilon > 0$ ,  $\tau_k(F) = |F|^{O(1/\log \log |F|)} = O_\varepsilon(|F|^\varepsilon)$ .

Here we have a final preparatory lemma before the main proposition leading to the bound on  $m_5(Q; h)$ . In what follows, our main strategy will be carefully isolating factors of the  $R_i$ 's in order to bound the number of terms in our sum. In doing so, we will make use of the following bound.

**Lemma 4.3.** *Let  $Q \in \mathbb{F}_q[t]$  be a square-free polynomial, and let  $n \in \mathbb{N}_{\geq 2}$ . Let  $\mathcal{I} \subseteq \mathbb{F}_q(t)$  be an interval of size  $q^{-h}$ . That is to say, for some rational function  $\alpha \in \mathbb{F}_q(t)$ , let  $\mathcal{I} := \{\beta \in \mathbb{F}_q(t) : |\alpha - \beta| < q^{-h}\}$ . Assume in the following that  $X_i, Y_i \in \mathbb{F}_q[t]$  for all  $i$ . Then, for any  $\varepsilon > 0$ ,*

$$\sum_{\substack{Y_1, \dots, Y_n | Q \\ X_i \in \mathcal{R}(Y_i) \\ \sum_i X_i / Y_i \in \mathcal{I} \\ q^{h/2} \leq |\prod_i Y_i| \leq q^{2h}}} \frac{\mu(\prod_i Y_i)^2}{\prod_i \phi(Y_i)^2} \ll_{n, \varepsilon} q^{-h(1-\varepsilon)}.$$

*Proof.* For given  $X_1, \dots, X_n$  and  $Y_1, \dots, Y_n$ , let  $X$  and  $Y$  be defined so that  $Y = \prod_i Y_i$  and  $X/Y = \sum_i X_i / Y_i$ . Then for all tuples considered in the sum,  $X/Y \in \mathcal{I}$  and  $q^{h/2} \leq |Y| \leq q^{2h}$ . Proceed by counting the number of possibilities for  $X/Y$  satisfying this constraint, which is bounded above by the number of points in  $\mathcal{I}$  with denominator smaller than  $q^{2h}$ , and finally count the number of ways of splitting  $Y$  up into  $Y_1, \dots, Y_n$ . However, we want to also consider the weighting in the sum of  $1/\phi(Y)^2$ , so we start by splitting the sum up into different sizes of  $Y$ , and then applying bounds on  $\phi(Y)$ .

To begin with, we rewrite the sum in terms of  $X$  and  $Y$ . Note that all  $Y_i$  in our sum are relatively prime, because of the Möbius factor. Thus  $Y$  is square-free and  $\phi(Y) = \prod_i \phi(Y_i)$ . Moreover, a choice of  $X, Y$ , and a decomposition  $Y = Y_1 \cdots Y_n$  determines  $X_i$  for each  $i$  by the Chinese remainder theorem. Our sum is thus equal to

$$\sum_{\substack{Y | Q \\ q^{h/2} \leq |Y| \leq q^{2h}}} \sum_{\substack{X \in \mathcal{R}(Y) \\ X/Y \in \mathcal{I}}} \frac{\mu(Y)^2}{\phi(Y)^2} \#\{Y_1, \dots, Y_n : Y_1 \cdots Y_n = Y\}.$$

Now split the sum up according to  $|Y|$ , defining  $m := \deg Y$ . The sum is then equal to

$$\sum_{m=h/2}^{2h} \sum_{\substack{Y | Q \\ |Y|=q^m}} \sum_{\substack{X \in \mathcal{R}(Y) \\ X/Y \in \mathcal{I}}} \frac{\mu(Y)^2}{\phi(Y)^2} \tau_n(Y) \ll_{n, \varepsilon} \sum_{m=h/2}^{2h} (q^m)^{\varepsilon/3} \sum_{\substack{Y | Q \\ |Y|=q^m}} \sum_{\substack{X \in \mathcal{R}(Y) \\ X/Y \in \mathcal{I}}} \frac{\mu(Y)^2 (\log \log |Y|)^2}{|Y|^2},$$

by Lemma 4.2 and the fact that  $\phi(Y)^{-2} \ll (|Y|/\log \log |Y|)^{-2}$ . We can further relax the condition that  $|Y| = q^m$  to the condition that  $|Y| \leq q^m$ . The number of  $X/Y$  with  $|Y| \leq q^m$  in the interval  $\mathcal{I}$  is  $q^{2m-h} + O(1)$ ; since  $m \geq h/2$ , this is  $\ll q^{2m-h}$ . Thus the sum is

$$\ll_{n, \varepsilon} \sum_{m=h/2}^{2h} q^{m(\varepsilon/3)} \frac{(\log \log(q^m))^2}{q^{2m}} q^{2m-h} \ll q^{-h} \sum_{m=h/2}^{2h} q^{m(2\varepsilon/3)} \ll q^{-h(1-\varepsilon)},$$

as desired. □

We now turn to bounding the contribution to the fifth moment  $m_5(Q; h)$  coming from tuples  $(R_1, \dots, R_5)$  satisfying the conclusions of Lemma 4.1.

**Proposition 4.4.** Fix  $h \geq 1$  and let  $Q \in \mathbb{F}_q[t]$  be square-free. Let  $\mathcal{S}$  be the set of tuples  $(R_1, \dots, R_5)$  such that

- $R_i \mid Q$  for all  $i$ ,
- $q^h \leq |R_i| \leq q^{2h}$  for all  $i$ ,
- $|R_2/(R_1, R_2)| \geq q^{h/2}$ , and
- $|R_3/(R_3, R_1 R_2)| \geq q^{h/2}$ .

Then, for all  $\varepsilon > 0$ ,

$$\sum_{(R_1, \dots, R_5) \in \mathcal{S}} \prod_{i=1}^5 \frac{1}{\phi(R_i)} \sum_{\substack{A_i \in \mathcal{R}(R_i) \\ |A_i/R_i| < q^{-h} \\ \sum_{1 \leq i \leq 5} A_i/R_i = 0}} q^{5h} \ll q^{(2+\varepsilon)h} \frac{|Q|}{\phi(Q)}.$$

*Proof.* We begin by sketching an overview of the strategy. For each subset  $I \subseteq [5]$ , let

$$R_I = \prod_{\substack{P \mid R_i \forall i \in I \\ P \nmid R_j \forall i \notin I}} P$$

be the product of the irreducible factors dividing  $R_i$  if and only if  $i \in I$ . Note that these  $R_I$ 's must be pairwise relatively prime.

We start by using the constraint that  $|A_1/R_1| < q^{-h}$ . We will count the total number of rational functions in this interval with denominator of degree at most  $2h$ . For each option of  $A_1/R_1$ , we can decompose  $R_1 = \prod_{I \ni 1} R_I$ , so the number of ways to decompose  $R_1$  into these  $R_I$ -factors is  $\tau_{2k-1-1}(R_1)$ , which we can bound based on the degree of  $R_1$ . We then also get  $A_1/R_1 = \sum_{I \ni 1} A_I/R_I$ , where the  $A_I$ 's are determined by the Chinese remainder theorem.

We will then focus on the constraint that  $|A_2/R_2| < q^{-h}$ . However,  $(R_1, R_2) = \prod_{1,2 \in I} R_I$  has already been fixed, so the same analysis as used for  $R_1$  applies to the remaining factors of  $R_2$ . Crucially,  $R_2/(R_1, R_2)$  remains relatively large by assumption, which will ensure that we save enough by doing this. Finally, the constraint on  $A_3/R_3$ , using our assumption that  $R_3/(R_3, R_1 R_2)$  is large enough, yields savings in the same way.

We begin by rewriting our sum in terms of the  $R_I$ . For each subset  $I \subseteq [5]$ , and for a fixed  $R_1, \dots, R_5$ , we again define  $R_I$  to be the product of all primes  $P$  so that  $P$  divides  $R_i$  for each  $i \in I$  and  $P$  does not divide  $R_j$  for all  $j \notin I$ . The  $R_I$  are a system of *relative greatest common divisors*; see [Elsholtz and Planitzer 2020] for details. For example,  $R_{\{1,2\}}$  is the product of all primes dividing  $R_1$  and  $R_2$ , but  $(R_{\{1,2\}}, R_j) = 1$  for  $j = 3, 4, 5$ . The polynomials  $R_I$  must satisfy the following properties, implied by the constraints on the  $R_i$ 's:

- Each  $R_I$  divides  $Q$ , and, for each  $I \neq J \subseteq [5]$ ,  $(R_I, R_J) = 1$ .
- Each irreducible polynomial dividing an  $R_i$  must divide at least two of them in order for the sum over  $A_i$  to be nonempty, so  $R_I = 1$  unless  $|I| \geq 2$ . We will always assume that  $|I| \geq 2$ .

- Each choice of  $A_i$  is equivalent to a choice of  $A_{i,I}$  for all subsets  $I$  containing  $i$ , that is,  $A_i/R_i = \sum_{I \ni i} A_{i,I}/R_I$ .
- The quantity  $(A_i, R_i)$  is equal to 1 for all  $i$  if and only if  $(A_{i,I}, R_I)$  is equal to 1 for all  $I, i$ .
- The constraint that, for all  $i$ ,  $|A_i/R_i| < q^{-h}$ , implies that, for each index  $i$ ,

$$\left| \sum_{I \ni i} \frac{A_{i,I}}{R_I} \right| < q^{-h}.$$

- The constraint that  $\sum_{i=1}^5 A_i/R_i = 0$  implies that, for each subset  $I$ ,

$$\sum_{i \in I} A_{i,I} = 0.$$

Finally, define  $\ell_I$  to be the minimum element of a subset  $I \subseteq [5]$ . The requirement that  $(R_1, \dots, R_5) \in S$  implies the following:

- For all  $i$ ,

$$q^h \leq \left| \prod_{I \ni i} R_I \right| \leq q^{2h}.$$

- Since  $R_2/(R_1, R_2) = \prod_{\ell_I=2} R_I$ , and  $R_3/(R_1, R_2, R_3) = \prod_{\ell_I=3} R_I$ ,

$$\left| \prod_{\ell_I=2} R_I \right| \geq q^{h/2} \quad \text{and} \quad \left| \prod_{\ell_I=3} R_I \right| \geq q^{h/2}.$$

The sum under consideration is then

$$\ll q^{5h} \sum_{\substack{R_I | Q \\ I \subseteq [5]}} \frac{\mu(\prod_I R_I)^2}{\prod_I \phi(R_I)^{|I|}} \sum_{\substack{I, i \in I \\ A_{i,I} \in \mathcal{R}(R_I)}} 1.$$

$$\begin{array}{l} q^h \leq | \prod_{I \ni i} R_I | \leq q^{2h} \\ | \prod_{\ell_I=2} R_I | \geq q^{h/2} \\ | \prod_{\ell_I=3} R_I | \geq q^{h/2} \end{array} \quad \forall i, \left| \sum_{I \ni i} \frac{A_{i,I}}{R_I} \right| < q^{-h} \\ \forall I, \sum_{i \in I} A_{i,I} = 0$$

Note first that if  $m_i$  is the maximum element of a subset  $I$ , then  $A_{m_i,I}$  is fully determined by the other  $A_{i,I}$  and the fact that  $\sum_{i \in I} A_{i,I} = 0$ . Then for  $i \in I$  with  $\ell_I < i < m_I$ , we will use the trivial bound on the number of options for  $A_{i,I}$ ; namely that there are at most  $R_I$  choices for  $A_{i,I}$ . For the rest of this bound, we treat  $A_{i,I}$  as fixed when  $\ell_I < i < m_I$ .

We finally consider the number of options for the remaining  $A_{\ell_I,I}$ , where  $\ell_I$  is the smallest element in  $I$ , which is where the savings in the argument will come from. We will proceed by ordering the intervals  $I$  in our sum by  $\ell_I$ ; we will first sum over options for  $A_I$  when  $I = \{4, 5\}$ , with  $\ell_I = 4$ , and then over  $A_{i,I}$  for all  $I$  with  $\ell_I = 3$ , and so on. As we do this, we will need at each step to satisfy the constraints that, for each  $i$ ,

$$\left| \sum_{I \ni i} \frac{A_{i,I}}{R_I} \right| < q^{-h}, \tag{17}$$

where as we split up the sums over different  $A_{i,I}$ 's, some of the values in this sum will be fixed and others will still be free to vary in our sum. But even if some of the terms in the sum above are fixed, the remaining terms are still constrained to lie in some interval of size  $q^{-h}$ , possibly an interval centered at a nonzero rational function. In particular, the constraints in (17) are equivalent to the constraints that, for all  $i$ ,

$$\left| F_i + \sum_{\substack{I \subseteq [5] \\ \ell_I = i}} \frac{A_{i,I}}{R_I} \right| < q^{-h},$$

where  $F_i$  is a fixed rational function determined by the values of  $A_{i,I}$  when  $\ell_I < i < m_I$ . The bounds we use are independent  $F_i$ , only requiring that the size of the interval is  $q^{-h}$ , so we can replace  $F_i$  by 0. This yields

$$\ll q^{5h} \sum_{\substack{R_I | Q \\ I \subseteq [5] \\ q^h \leq |\prod_{I \ni i} R_I| \leq q^{2h} \\ |\prod_{\ell_I=2} R_I| \geq q^{h/2} \\ |\prod_{\ell_I=3} R_I| \geq q^{h/2}}} \frac{\mu(\prod_I R_I)^2}{\prod_I \phi(R_I)^{|I|}} \prod_I \phi(R_I)^{|I|-2} \sum_{\substack{A_{\ell_I,I} \in \mathcal{R}(R_I) \\ I \subseteq [5] \\ \forall i, |\sum_{\ell_J=i} A_{i,J}/R_J| < q^{-h}}} 1. \tag{18}$$

The only terms  $A_{i,I}$  that remain in (18) are of the form  $A_{\ell_I,I}$ , and there is only one term for each subset  $I$ , so to simplify our notation we will write  $A_I := A_{\ell_I,I}$  from now on.

Consider subsets  $I$  with  $\ell_I = 4$ . There is only one of these, namely  $\{4, 5\}$ , so we rewrite the sum as

$$\ll q^{5h} \sum_{\substack{R_I | Q \\ I \subseteq [5], I \neq \{4,5\} \\ q^h \leq |\prod_{I \ni i} R_I| \leq q^{2h} \\ |\prod_{\ell_I=2} R_I| \geq q^{h/2} \\ |\prod_{\ell_I=3} R_I| \geq q^{h/2}}} \frac{\mu(\prod_I R_I)^2}{\prod_I \phi(R_I)^2} \sum_{\substack{A_I \in \mathcal{R}(R_I) \\ I \subseteq [5], I \neq \{4,5\} \\ \forall i, |\sum_{\ell_J=i} A_J/R_J| < q^{-h}}} \sum_{\substack{R_{\{4,5\}} | Q \\ A_{\{4,5\}} \in \mathcal{R}(R_{\{4,5\}})}} \frac{1}{\phi(R_{\{4,5\}})^2}.$$

In the inside sum, we have dropped the additional constraint that  $A_{\{4,5\}}/R_{\{4,5\}}$  must lie in an interval of size  $q^{-h}$ , since ignoring it only increases the size of the sum. For each  $R_{\{4,5\}}$ , there are  $\phi(R_{\{4,5\}})$  choices of  $A_{\{4,5\}}$ , so the inner sum becomes

$$\sum_{R_{\{4,5\}} | Q} \frac{1}{\phi(R_{\{4,5\}})} = \frac{|Q|}{\phi(Q)},$$

since  $Q$  is square-free.

Now consider subsets  $I$  with  $\ell_I = 3$ , i.e.,  $\{3, 4\}$ ,  $\{3, 4, 5\}$ , and  $\{3, 5\}$ . We first bookkeep by isolating these terms in the sum, yielding

$$\ll q^{5h} \frac{|Q|}{\phi(Q)} \sum_{\substack{R_I | Q \\ I \subseteq [5], \ell_I < 3 \\ q^h \leq |\prod_{\ell_I=1} R_I| \leq q^{2h} \\ q^{h/2} \leq |\prod_{\ell_I=2} R_I| \leq q^{2h}}} \frac{\mu(\prod_I R_I)^2}{\prod_I \phi(R_I)^2} \sum_{\substack{A_I \in \mathcal{R}(R_I) \\ I \subseteq [5], \ell_I < 3 \\ \forall i, |\sum_{\ell_J=i} A_J/R_J| < q^{-h}}} \sum_{\substack{\ell_I=3 \\ R_I | Q \\ A_I \in \mathcal{R}(R_I) \\ q^{h/2} \leq |\prod_{\ell_I=3} R_I| \leq q^{2h} \\ |\sum_{\ell_I=3} A_I/R_I| < q^{-h}}} \frac{\mu(\prod_{\ell_I=3} R_I)^2}{\prod_{\ell_I=3} \phi(R_I)^2}.$$

We now bound the inner sum using [Lemma 4.3](#). The inner sum comprises three terms  $R_I$ , so apply the lemma with  $n = 3$  to get that the inner sum is  $\ll q^{-h(1+\varepsilon)}$ .

We repeat the process, now considering subsets  $I$  with  $\ell_I = 2$ . Isolating these terms yields

$$\ll q^{4h+\varepsilon h} \frac{|Q|}{\phi(Q)} \sum_{\substack{R_I | Q \\ I \subseteq [5], \ell_I=1 \\ q^h \leq |\prod_{\ell_I=1} R_I| \leq q^{2h}}} \frac{\mu(\prod_{\ell_I=1} R_I)^2}{\prod_{\ell_I=1} \phi(R_I)^2} \sum_{\substack{A_I \in \mathcal{R}(R_I) \\ I \subseteq [5], \ell_I=1 \\ |\sum_{\ell_I=1} A_I/R_I| < q^{-h}}} \sum_{\substack{\ell_I=2 \\ R_I | Q \\ A_I \in \mathcal{R}(R_I) \\ q^{h/2} \leq |\prod_{\ell_I=2} R_I| \leq q^{2h} \\ |\sum_{\ell_I=2} A_I/R_I| < q^{-h}}} \frac{\mu(\prod_{\ell_I=2} R_I)^2}{\prod_{\ell_I=2} \phi(R_I)^2}.$$

Here there are seven  $R_I$  terms and seven  $A_I$  terms in the inner sum, so, again applying [Lemma 4.3](#), the inner sum is  $\ll q^{-h+\varepsilon h}$ . Lastly, we address the terms with  $\ell_I = 1$ :

$$\ll q^{3h} q^{2\varepsilon h} \frac{|Q|}{\phi(Q)} \sum_{\substack{R_I | Q \\ I \subseteq [5], \ell_I=1 \\ q^h \leq |\prod_{\ell_I=1} R_I| \leq q^{2h}}} \frac{\mu(\prod_{\ell_I=1} R_I)^2}{\prod_{\ell_I=1} \phi(R_I)^2} \sum_{\substack{A_I \in \mathcal{R}(R_I) \\ I \subseteq [5], \ell_I=1 \\ |\sum_{\ell_I=1} A_I/R_I| < q^{-h}}} 1.$$

We apply [Lemma 4.3](#) one final time, this time with  $n = 15$ , since there are 15 sets  $I \subseteq [5]$  with  $|I| \geq 2$  and  $\ell_I = 1$ . This yields

$$\ll q^{2h+3\varepsilon h} \frac{|Q|}{\phi(Q)},$$

as desired. □

We are now ready to prove a general bound on  $m_5(Q; h)$ .

**Theorem 4.5.** Fix  $\varepsilon > 0$  and let  $Q \in \mathbb{F}_q[t]$  be square-free. Define  $m_5(Q; h)$  by (15). Then

$$m_5(Q; h) \ll |Q| q^{2h+\varepsilon} \left(\frac{|Q|}{\phi(Q)}\right)^{-4} + |Q| q^{2h} \left(\frac{|Q|}{\phi(Q)}\right)^{27}.$$

*Proof.* Using [Lemma 3.1](#), we can express

$$m_5(Q; h) = |Q| \left(\frac{\phi(Q)}{|Q|}\right)^5 V_5(Q; h),$$

where

$$V_5(Q; h) = \sum_{\substack{R_1, \dots, R_5 | Q \\ |R_i| > 1 \\ R_i \text{ monic}}} \prod_{i=1}^5 \frac{\mu(R_i)}{\phi(R_i)} \sum_{\substack{A_1, \dots, A_5 \in \mathcal{R}(R_i) \\ \sum_i A_i/R_i = 0}} E\left(\frac{A_1}{R_1}\right) \cdots E\left(\frac{A_5}{R_5}\right).$$

Now apply [Lemma 3.6](#) to bound the contribution to  $V_5(Q; h)$  from many tuples  $R_1, \dots, R_5$ . If  $|R_i| < q^h$  for any  $i$ , then these terms contribute 0; assume from now on that  $|R_i| \geq q^h$ . If for any triple  $i, j, k$  we have  $R_i = R_j = R_k$ , then we apply [Lemma 3.6](#) with  $\tilde{R}_1 = R_i$  and  $\tilde{R}_2 = R_j$ ; in this case  $X_2 = 0$  and  $X_1$  and  $X_3$  are  $O(q^{-h/2})$ , so these terms contribute  $O(q^{2h}(|Q|/\phi(Q))^{32})$ . If there exist  $R_i \neq R_j$  such that either  $|R_i/(R_i, R_j)| < q^h$  or  $|(R_i, R_j)| \geq q^{h/2}$ , then we apply [Lemma 3.6](#) with  $\tilde{R}_1 = R_i$  and  $\tilde{R}_2 = R_j$ ; in this case,  $X_3 = 0$ , and  $X_1$  and  $X_2$  are each  $O(q^{-h/2})$ , so these terms contribute  $O(q^{2h}(|Q|/\phi(Q))^{32})$  as well.

Assume now that  $(R_1, R_2, R_3, R_4, R_5)$  does not fall into either of the above cases. Then, for all  $i$ ,  $|R_i| < q^{2h}$ . To see this, assume that  $(R_1, R_2, R_3, R_4, R_5)$  has no  $i, j, k$  with  $R_i = R_j = R_k$ , and that, for all  $R_i \neq R_j$ ,  $|R_i/(R_i, R_j)| \geq q^h$  and  $|(R_i, R_j)| < q^{h/2}$ . Assume, relabeling if necessary, that  $R_1 \geq q^{2h}$ . Since  $R_1 \mid \prod_{j \neq 1} (R_1, R_j)$ , we must have  $|(R_1, R_j)| \geq q^{h/2}$  for some  $j \neq 1$ . This cannot be true for some  $j$  with  $R_j \neq R_1$ , so we have  $R_j = R_1$ . At the same time, there can only be one  $j \neq 1$  with  $R_j = R_1$ , so without loss of generality our tuple must be of the form  $(R_1, R_1, R_3, R_4, R_5)$ . There cannot be an additional equal pair among  $R_3, R_4$ , and  $R_5$ ; if there is (without loss of generality  $R_3 = R_4$ ), then  $R_5 \mid (R_1, R_5)(R_3, R_5)$ , so since  $|R_5| \geq q^h$  either  $|(R_1, R_5)| \geq q^{h/2}$  or  $|(R_3, R_5)| \geq q^{h/2}$ , which along with the lack of equal triples yields a contradiction. Now consider  $R_3$ . Note that  $R_3 \mid (R_1, R_3)(R_4, R_3)(R_5, R_3)$  and  $(R_3/(R_1, R_3)) \mid (R_4, R_3)(R_5, R_3)$ . But by assumption,  $|R_3/(R_1, R_3)| \geq q^h$  and  $|(R_4, R_3)(R_5, R_3)| < (q^{h/2})^2 = q^h$ , which yields a contradiction.

So, the only terms remaining are those with  $|R_i| < q^{2h}$  for all  $i$ , no equal triple, and either  $|(R_i, R_j)| \geq q^{h/2}$  or  $|R_i/(R_i, R_j)| < q^h$  whenever  $R_i \neq R_j$ . By Lemma 4.1,  $(R_1, \dots, R_5)$  satisfies the constraints of Proposition 4.4. By Proposition 4.4, these terms contribute  $O(q^{(2+\varepsilon)h} |Q|/\phi(Q))$  to  $V_5(Q; h)$  for all  $\varepsilon > 0$ . Thus, for all  $\varepsilon > 0$ ,

$$V_5(Q; h) \ll q^{(2+\varepsilon)h} \frac{|Q|}{\phi(Q)} + q^{2h} \left( \frac{|Q|}{\phi(Q)} \right)^{32},$$

so

$$m_5(Q; h) \ll |Q| q^{(2+\varepsilon)h} \left( \frac{|Q|}{\phi(Q)} \right)^{-4} + |Q| q^{2h} \left( \frac{|Q|}{\phi(Q)} \right)^{27}. \quad \square$$

As in the integer case, we particularly want to consider  $Q$  to be the product of irreducible polynomials  $P$  with  $|P| \leq q^{2h}$ . In this case,  $|Q|/\phi(Q) \ll h$ , so that we get the following corollary.

**Corollary 4.6.** Fix  $\varepsilon > 0$  and let  $Q \in \mathbb{F}_q[t]$  be given by

$$Q = \prod_{\substack{P \text{ irred.} \\ |P| \leq q^{2h}}} P.$$

Then

$$m_5(Q; h) \ll_\varepsilon |Q| q^{(2+\varepsilon)h}.$$

**4.2. Proof of Corollary 1.5: bounds on  $R_k(q^h)$ .** In this subsection, we discuss the transition from bounds on  $V_k(Q; h)$ , from Theorem 1.4 and Lemma 3.7, to bounds on sums of singular series in function fields, in order to prove Corollary 1.5. Much of this is similar to the integer case discussion in Section 2.

As in the integer case, for  $\mathcal{D} = \{D_1, \dots, D_k\}$  a set of distinct polynomials in  $\mathbb{F}_q[T]$ , we define the singular series

$$\mathfrak{S}(\mathcal{D}) := \prod_{P \text{ monic, irred.}} \frac{(1 - v_P(\mathcal{D})/|P|)}{(1 - 1/|P|)^k} = \sum_{\substack{R_1, \dots, R_k \\ 1 \leq |R_i|}} \left( \prod_{i=1}^k \frac{\mu(R_i)}{\phi(R_i)} \right) \sum_{\substack{A_1, \dots, A_k \\ A_i \in \mathcal{R}(R_i) \\ \sum_i A_i/R_i = 0}} e \left( \sum_{i=1}^k \frac{A_i D_i}{R_i} \right),$$

where  $\nu_P(\mathcal{D})$  is the number of equivalence classes of  $\mathbb{F}_q[T]/(P)$  occupied by elements of  $\mathcal{D}$ . We also define  $\mathfrak{S}_0(\mathcal{D})$ , given by  $\mathfrak{S}_0(\mathcal{D}) := \sum_{\mathcal{J} \subseteq \mathcal{D}} (-1)^{|\mathcal{D} \setminus \mathcal{J}|} \mathfrak{S}(\mathcal{J})$ , and consider

$$R_k(q^h) := \sum_{\substack{D_1, \dots, D_k \\ D_i \text{ distinct} \\ |D_i| \leq q^h}} \mathfrak{S}_0(\{D_1, \dots, D_k\}). \tag{19}$$

Our results on  $m_k(Q; h)$  (and equivalently  $V_k(Q; h)$ ) imply bounds on these sums of  $k$ -fold singular series, just as in the integer case in Section 2. We set  $Q$  to be the product of all monic irreducible polynomials of degree at most  $2h$ , so that  $|Q|/\phi(Q) \ll_q h$ . Just as in the integer case, we can truncate the expression for  $\mathfrak{S}_0(\mathcal{D})$  to only contain terms dividing  $Q$ , with an acceptable error term. In particular, we get

$$R_k(h) = \sum_{\substack{D_1, \dots, D_k \\ D_i \text{ distinct} \\ |D_i| \leq q^h}} \sum_{\substack{R_1, \dots, R_k \\ |R_i| > 1 \\ R_i | Q}} \prod_{i=1}^k \frac{\mu(R_i)}{\phi(R_i)} \sum_{\substack{A_1, \dots, A_k \\ A_i \in \mathcal{R}(R_i) \\ \sum_i A_i/R_i = 0}} e\left(\sum_{i=1}^k \frac{D_i A_i}{R_i}\right) + O(1).$$

It will again be helpful for us to define the singular series of a  $k$ -tuple  $\mathcal{D} = (D_1, \dots, D_k)$  relative to the modulus  $Q$ . Here the  $k$ -tuple can have repeated elements; since the Euler product is finite, convergence is not a concern. We define

$$\mathfrak{S}(\mathcal{D}; Q) := \prod_{\substack{P | Q \\ P \text{ monic}}} \frac{(1 - \nu_P(\mathcal{D})/|P|)}{(1 - 1/|P|)^k} = \sum_{\substack{R_1, \dots, R_k | Q \\ R_i \text{ monic}}} \left( \prod_{i=1}^k \frac{\mu(R_i)}{\phi(R_i)} \right) \sum_{\substack{A_1, \dots, A_k \\ A_i \in \mathcal{R}(R_i) \\ \sum_i A_i/R_i = 0}} e\left(\sum_{i=1}^k \frac{A_i D_i}{R_i}\right).$$

If  $\mathcal{D}$  has a repeated element, so that  $\mathcal{D} = \{D, D, D_3, \dots, D_k\}$ , then

$$\mathfrak{S}(\mathcal{D}; Q) = \frac{|Q|}{\phi(Q)} \mathfrak{S}(\{D, D_3, \dots, D_k\}; Q),$$

so we can remove repeated elements from  $\mathcal{D}$  at the expense of a factor of  $|Q|/\phi(Q)$ . We define  $\mathfrak{S}_0(\mathcal{D}; Q)$  to be the alternating sum  $\sum_{\mathcal{J} \subseteq \mathcal{D}} (-1)^{|\mathcal{D} \setminus \mathcal{J}|} \mathfrak{S}(\mathcal{J}; Q)$ , so we have

$$R_k(q^h) = \sum_{\substack{D_1, \dots, D_k \\ D_i \text{ distinct} \\ |D_i| \leq q^h}} \mathfrak{S}_0(\{D_1, \dots, D_k\}; Q) + O(1).$$

This is quite close to the quantity  $V_k(Q; h)$ , except with the added constraint that the  $D_i$ 's must be distinct. It suffices to remove this condition. To do so, we put  $\delta_{ij} = 1$  if  $D_i = D_j$  and 0 otherwise, so that

$$\sum_{\substack{D_1, \dots, D_k \\ D_i \text{ distinct} \\ |D_i| \leq q^h}} \mathfrak{S}_0(\{D_1, \dots, D_k\}; Q) = \sum_{\substack{D_1, \dots, D_k \\ |D_i| \leq q^h}} \left( \prod_{1 \leq i < j \leq k} (1 - \delta_{ij}) \right) \mathfrak{S}_0(\{D_1, \dots, D_k\}; Q).$$

We can expand the product and group terms according to which  $D_i$ 's are required to be equal, noting that, for example,  $\delta_{12}\delta_{23} = \delta_{13}\delta_{23}$ . We can also combine terms according to symmetry; the term  $\delta_{12}$  and the term  $\delta_{34}$  will have identical contributions in the final sum.

Let us now proceed with analyzing  $R_5(q^h)$ . After some counting, we get that

$$\sum_{\substack{D_1, \dots, D_5 \\ D_i \text{ distinct} \\ |D_i| \leq q^h}} \mathfrak{S}_0(\{D_1, \dots, D_5\}; Q) = \sum_{\substack{D_1, \dots, D_5 \\ |D_i| \leq q^h}} f((\delta_{i,j})_{i,j \in [5]}) \mathfrak{S}_0(\{D_1, \dots, D_5\}; Q),$$

where

$$f((\delta_{i,j})_{i,j \in [5]}) = 1 - 10\delta_{12} + 20\delta_{12}\delta_{13} + 15\delta_{12}\delta_{34} - 20\delta_{12}\delta_{13}\delta_{45} - 30\delta_{12}\delta_{13}\delta_{14} + 24\delta_{12}\delta_{13}\delta_{14}\delta_{15}.$$

We will consider the contribution from each term in  $f$ . The term 1 gives us precisely  $V_5(Q; h)$ , which we have already analyzed. We can then bound each of the remaining six terms by expanding  $\mathfrak{S}_0$  into a sum of  $\mathfrak{S}$ , removing any repeated terms in the appropriate tuple, and applying Lemma 3.7 to bound  $V_k(Q; h)$  for some  $k < 5$ . These computations are summarized in the following lemma.

**Lemma 4.7.** *Using the notation of this section,*

- (a)  $\sum_{\substack{D_1, D_3, D_4, D_5 \\ |D_i| \leq q^h}} \mathfrak{S}_0(\{D_1, D_1, D_3, D_4, D_5\}; Q) \ll q^{2h}(|Q|/\phi(Q))^9,$
- (b)  $\sum_{\substack{D_1, D_4, D_5 \\ |D_i| \leq q^h}} \mathfrak{S}_0(\{D_1, D_1, D_1, D_4, D_5\}; Q) \ll q^{2h}(|Q|/\phi(Q))^3 + q^h(|Q|/\phi(Q))^{10},$
- (c)  $\sum_{\substack{D_1, D_3, D_5 \\ |D_i| \leq q^h}} \mathfrak{S}_0(\{D_1, D_1, D_3, D_3, D_5\}; Q) \ll q^{2h}(|Q|/\phi(Q))^3 + q^h(|Q|/\phi(Q))^{10},$
- (d)  $\sum_{\substack{D_1, D_4 \\ |D_i| \leq q^h}} \mathfrak{S}_0(\{D_1, D_1, D_1, D_4, D_4\}; Q) \ll q^{2h}(|Q|/\phi(Q))^3 + q^h(|Q|/\phi(Q))^4,$
- (e)  $\sum_{\substack{D_1, D_5 \\ |D_i| \leq q^h}} \mathfrak{S}_0(\{D_1, D_1, D_1, D_1, D_5\}; Q) \ll q^h(|Q|/\phi(Q))^4,$
- (f)  $\sum_{\substack{D_1 \\ |D_1| \leq q^h}} \mathfrak{S}_0(\{D_1, D_1, D_1, D_1, D_1\}; Q) \ll q^h(|Q|/\phi(Q))^4.$

*Proof.* For the sake of brevity we omit most of these computations, which are very similar, but we will show that the term corresponding to  $\delta_{12}$ , in part (a), is  $\ll q^{2h}(|Q|/\phi(Q))^9$ .

Assume we have a tuple  $\mathcal{D} = \{D_1, D_1, D_3, D_4, D_5\}$ , with one repeated term. As mentioned above,

$$\mathfrak{S}(\mathcal{D}; Q) = \frac{|Q|}{\phi(Q)} \mathfrak{S}(\{D_1, D_3, D_4, D_5\}; Q).$$

Expanding  $\mathfrak{S}_0$  and applying this relation shows that

$$\mathfrak{S}_0(\mathcal{D}; Q) = \left( \frac{|Q|}{\phi(Q)} - 2 \right) \mathfrak{S}_0(\{D_1, D_3, D_4, D_5\}; Q) + \left( \frac{|Q|}{\phi(Q)} - 1 \right) \mathfrak{S}_0(\{D_3, D_4, D_5\}; Q),$$

so in this way we can remove repeated elements from our sum. The term we want to bound is

$$\begin{aligned} \sum_{\substack{D_1, D_3, D_4, D_5 \\ |D_i| \leq q^h}} \mathfrak{S}_0(\{D_1, D_1, D_3, D_4, D_5\}; Q) \\ &= \sum_{\substack{D_1, D_3, D_4, D_5 \\ |D_i| \leq q^h}} \left( \frac{|Q|}{\phi(Q)} - 2 \right) \mathfrak{S}_0(\{D_1, D_3, D_4, D_5\}; Q) + \left( \frac{|Q|}{\phi(Q)} - 1 \right) \mathfrak{S}_0(\{D_3, D_4, D_5\}; Q) \\ &= \left( \left( \frac{|Q|}{\phi(Q)} - 2 \right) V_4(Q; h) + q^h \left( \frac{|Q|}{\phi(Q)} - 1 \right) V_3(Q; h) \right) \\ &\ll \left( \frac{|Q|}{\phi(Q)} \right)^3 q^{2h} + \left( \frac{|Q|}{\phi(Q)} \right)^9 q^{2h}, \end{aligned}$$

where in the last step the bounds follow from [Lemma 3.7](#). □

This lemma gives the following corollary.

**Corollary 4.8.** *Let*

$$Q = \prod_{\substack{P \text{ irred.} \\ |P| \leq q^{6h}}} P.$$

For all  $\varepsilon > 0$ ,

$$R_5(q^h) \ll V_5(Q; h) + q^{2h} \left( \frac{|Q|}{\phi(Q)} \right)^9 \ll q^{(2+\varepsilon)h}.$$

Performing the same analysis when  $k = 3$  yields the bound:

**Corollary 4.9.** *Let  $Q$  be as above. Then*

$$R_3(q^h) \ll V_3(Q; h) + q^h \left( \frac{|Q|}{\phi(Q)} \right)^2 \ll q^h \left( \frac{|Q|}{\phi(Q)} \right)^8.$$

## 5. Numerical evidence for odd moments

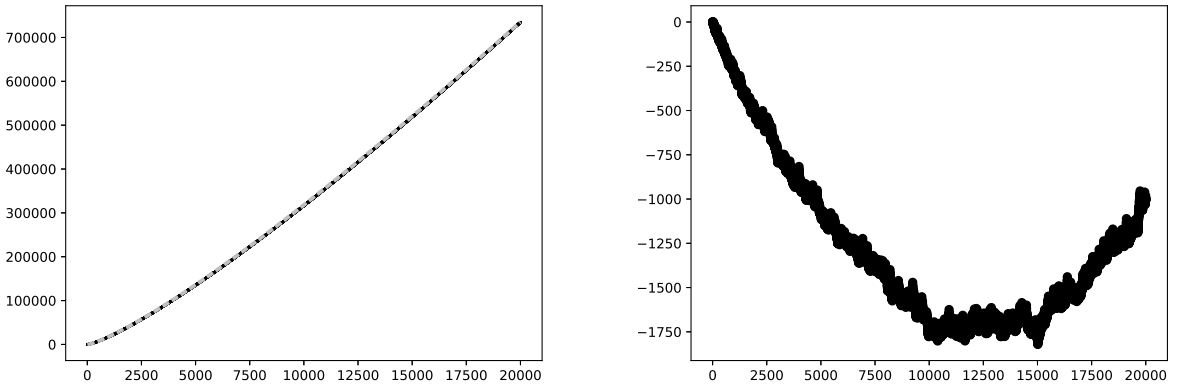
Here we present several charts supporting our conjectures on the sizes of the odd moments. To begin with, we have computed  $\frac{1}{6}R_3(h) = \sum_{1 \leq d_1 < d_2 < d_3 \leq h} \mathfrak{S}_0(\{d_1, d_2, d_3\})$ . In [Figure 1](#), left,  $\frac{1}{6}R_3(h)$  is plotted in black. We expect  $R_3(h)$ , and thus also  $\frac{1}{6}R_3(h)$ , to be of the shape  $Ah(\log h)^2$  for some constant  $A$ . We found an experimental best fit value of  $A = 0.373727$ , and for this  $A$  have plotted  $Ah(\log h)^2$  alongside  $\frac{1}{6}R_3(h)$ , as a dashed gray line.

The fit of the theoretical gray dashed curve is quite close, but there are lower-order fluctuations; in [Figure 1](#), right, we plot the difference between  $\frac{1}{6}R_3(h)$  and  $Ah(\log h)^2$ .

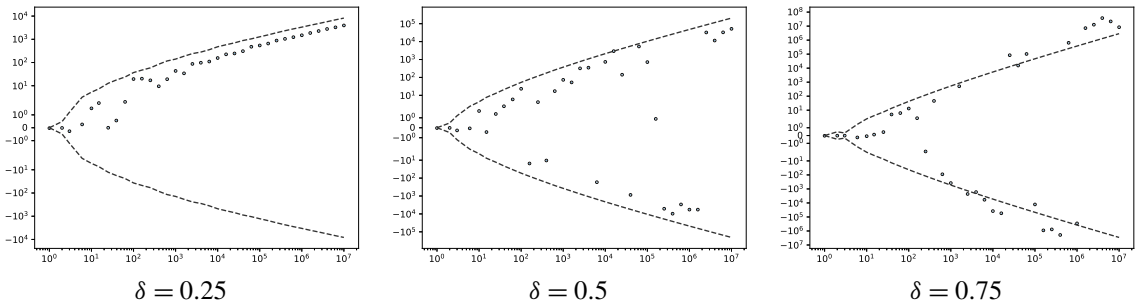
Our analysis above includes relatively little discussion about the moments of the distribution of primes themselves. We have computed several third, fifth, and seventh moments of the distribution of primes, shown in [Figures 2–4](#). Specifically, we have computed

$$\tilde{M}_k(N; N^\delta) = \frac{1}{N} \sum_{n=N}^{2N} (\psi(n + N^\delta) - \psi(n) - N^\delta)^k$$

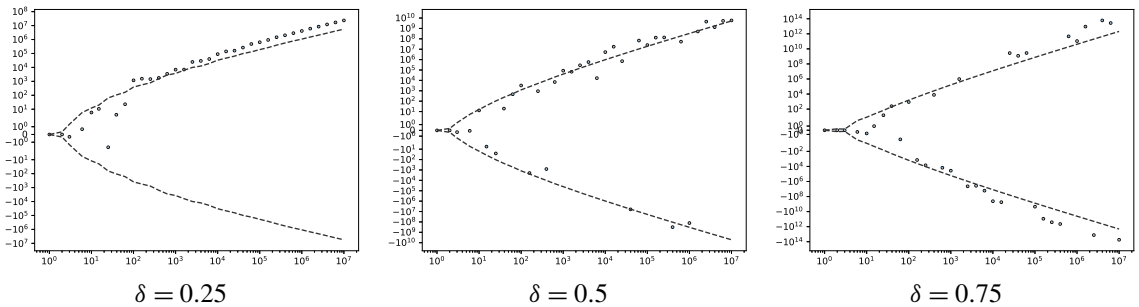
for each of  $\delta = 0.25, 0.5$  and  $0.75$ , and for each of  $k = 3, 5, 7$ . For a fixed  $\delta$  and  $k$ , we plot  $\tilde{M}_k(N; N^\delta)$  for values of  $N$  ranging from 1 to  $10^7$ , and growing exponentially.



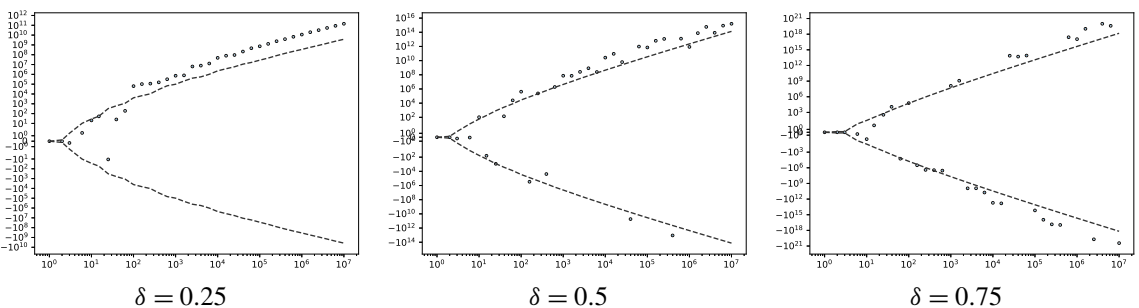
**Figure 1.** Left:  $\frac{1}{6}R_3(h)$  for  $3 \leq h \leq 20000$ . Right:  $\frac{1}{6}R_3(h) - Ah(\log h)^2$  for  $3 \leq h \leq 20000$ .



**Figure 2.** Plots of the third moment  $M_3(N; N^\delta)$  for  $N \leq 10^7$ .



**Figure 3.** Plots of the fifth moment  $M_5(N; N^\delta)$  for  $N \leq 10^7$ .



**Figure 4.** Plots of the seventh moment  $M_7(N; N^\delta)$  for  $N \leq 10^7$ .

Each of the plots in Figures 2–4 is drawn with both  $x$ - and  $y$ -axes on a logarithmic scale. We expect the  $k$ -th moment to be of size approximately  $O(H^{(k-1)/2}(\log(N/H))^{(k+1)/2})$ , where  $H = N^\delta$ , so to give a sense of size, for each plot,  $N^{\delta(k-1)/2}(\log N^{1-\delta})^{(k+1)/2}$  is plotted in dashed red. We have also plotted the reflection of the red dashed curve across the  $x$ -axis, since the odd moments are frequently negative.

The fit of the red line is reasonably good in all cases, but not perfect. In every case here we seem to see that the odd moments are more frequently positive than negative, but still take on negative values. For  $\delta = 0.25$ , the odd moments seem to be positive for sufficiently large  $N$ ; it is possible that this effect occurs for all sufficiently large  $N$ , where the threshold depends on  $k$  and  $\delta$ .

### 6. Toy models and open problems

Throughout, we have studied the sum

$$R_k(h) = \sum_{\substack{q_1, \dots, q_k \\ 1 < q_i}} \left( \prod_{i=1}^k \frac{\mu(q_i)}{\phi(q_i)} \right) \sum_{\substack{a_1, \dots, a_k \\ 1 \leq a_i \leq q_i \\ (a_i, q_i) = 1 \\ \sum a_i / q_i \in \mathbb{Z}}} \prod_{i=1}^k E\left(\frac{a_i}{q_i}\right),$$

where  $E(\alpha) = \sum_{m=1}^h e(m\alpha)$ . The sums  $E(\alpha)$  approximately detect when  $\|\alpha\| \leq 1/h$ ; the analogous sum in the function field case precisely detects when  $\alpha$  has small degree. As a result, much of our understanding boils down to answering the following key question.

**Question 6.1.** *Let  $\delta > 0$  and let  $Q > 1/\delta$ . What is*

$$\#\left\{q_1, \dots, q_k \in [Q, 2Q], a_i \bmod q_i : \left\| \frac{a_i}{q_i} \right\| \leq \delta, \sum_i \frac{a_i}{q_i} \in \mathbb{Z} \right\}?$$

We conjecture that the answer to this question is as follows.

**Conjecture 6.2.** *Let  $\delta > 0$  and let  $Q > 1/\delta$ . Let  $S$  be the size of the set in [Question 6.1](#). Then for any  $\varepsilon > 0$ ,*

$$S \ll \begin{cases} Q^{k+\varepsilon} \delta^{k/2}, & k \text{ even,} \\ Q^{k+\varepsilon} \delta^{(k+1)/2}, & k \text{ odd.} \end{cases}$$

As we discussed in the [Introduction](#), Montgomery and Vaughan [1986] considered the related problem of moments of reduced residues modulo  $q$ . Their work depends on the following answer to [Question 6.1](#) above.

**Theorem 6.3.** *Let  $S$  be the size of the set in [Question 6.1](#). Then*

$$S \ll \begin{cases} \delta^{k/2} \sum_{\substack{Q \leq r_i \leq 2Q \\ 1 \leq i \leq k/2}} \frac{r_1^2 \cdots r_{k/2}^2}{\text{lcm}(r_i)} + \delta^{k/2-1/7k} \sum_{\substack{Q \leq r_i \leq 2Q \\ 1 \leq i \leq k}} \frac{r_1 \cdots r_k}{\text{lcm}(r_i)}, & k \text{ even,} \\ \delta^{k/2-1/7k} \sum_{\substack{Q \leq r_i \leq 2Q \\ 1 \leq i \leq k}} \frac{r_1 \cdots r_k}{\text{lcm}(r_i)}, & k \text{ odd.} \end{cases}$$

The proof of the above theorem is identical to the proof in [Montgomery and Vaughan 1986]. This agrees with Conjecture 6.2 for the case when  $k$  is even, but gives a weaker bound when  $k$  is odd.

We can also consider generalizations of Question 6.1. For example, instead of specifying that  $\|a_i/q_i\| \leq \delta$ , we may ask that it lie in any specified interval.

**Question 6.4.** Let  $Q > 1/\delta$  and let  $I_1, \dots, I_k$  be  $k$  intervals in  $[0, 1]$  with  $|I_j| \geq \delta$  for all  $j$ . What is

$$\#\left\{q_1, \dots, q_k \in [Q, 2Q], a_i \bmod q_i : \left\| \frac{a_i}{q_i} \right\| \in I_i, \sum_i \frac{a_i}{q_i} \in \mathbb{Z} \right\}?$$

Answers to these questions would give us more refined understanding of sums of singular series. The conjectures above are related to sums over  $\mathfrak{S}(\{h_1, \dots, h_k\})$ , where each  $h_i$  lies in the same interval  $[0, h]$ . We can instead ask about sums of singular series restricted to arbitrary intervals, or along arithmetic progressions. We state the following questions using smooth cutoff functions as opposed to intervals.

**Question 6.5.** Let  $\Phi_1, \dots, \Phi_k$  be smooth functions with compact support on  $\mathbb{R}$ , and let  $H \in \mathbb{R}_{>0}$ . What is

$$\sum_{h_1, \dots, h_k \in \mathbb{Z}} \mathfrak{S}_0(\{h_1, \dots, h_k\}) \Phi_1\left(\frac{h_1}{H}\right) \cdots \Phi_k\left(\frac{h_k}{H}\right)?$$

**Question 6.6.** Let  $\Phi_1, \dots, \Phi_k$  be smooth functions with compact support on  $\mathbb{R}$ , and let  $H \in \mathbb{R}_{>0}$ . For arithmetic progressions  $a_1 \bmod q_1, \dots, a_k \bmod q_k$ , what is

$$\sum_{\substack{h_1, \dots, h_k \in \mathbb{Z} \\ h_i \equiv a_i \pmod{q_i}}} \mathfrak{S}_0(\{h_1, \dots, h_k\}) \Phi_1\left(\frac{h_1}{H}\right) \cdots \Phi_k\left(\frac{h_k}{H}\right)?$$

Question 6.5 addresses the correlations of  $\psi(x+h) - \psi(x)$  and  $\psi(x+h_1+h) - \psi(x+h_1)$ ; in other words, the correlations of the number of primes in intervals in different places. Question 6.6 addresses the correlations of the number of primes in distinct arithmetic progressions. For both of these questions, the main term ought to come from diagonal terms where  $h_1 = h_2$ , for example, thus collapsing the weight function, whereas the error term ought to arise from off-diagonal contributions.

In the case when  $k = 2$ , Question 6.6 has been widely studied in the context of prime number races. The “Shanks–Rényi prime number race” is the following problem: Let  $\pi(x; q, a)$  denote the number of primes  $p \leq x$  with  $p \equiv a \pmod{q}$ . Then, for any  $n$ -tuple  $(a_1, \dots, a_n)$  of equivalence classes mod  $q$  that are relatively prime to  $q$ , will we have the ordering

$$\pi(x; q, a_1) > \pi(x; q, a_2) > \cdots > \pi(x; q, a_n)$$

for infinitely many integers  $x$ ? Many aspects of this question have been studied; see for example the expositions of [Granville and Martin 2006; Ford and Konyagin 2002].

Ford, Harper, and Lamzouri [2019] showed that, although any ordering appears infinitely often, for  $n$  large with respect to  $q$ , the prime number races among orderings can exhibit large biases. They rely on the fact that counts of primes in distinct progressions have negative correlations, which they arrange to produce a bias. This analysis is also connected to the work of Lemke Oliver and Soundararajan [2016],

who use averages of two-term singular series in arithmetic progressions to show bias in the distribution of consecutive primes. It is plausible that a more precise understanding of the questions above would lead to an extension of the work of Lemke Oliver and Soundararajan.

## References

- [Chan 2006] T. H. Chan, “A note on primes in short intervals”, *Int. J. Number Theory* **2**:1 (2006), 105–110. [MR](#) [Zbl](#)
- [Cramér 1936] H. Cramér, “On the order of magnitude of the difference between consecutive prime numbers”, *Acta Arith.* **2**:1 (1936), 23–46. [Zbl](#)
- [Elsholtz and Planitzer 2020] C. Elsholtz and S. Planitzer, “The number of solutions of the Erdős–Straus equation and sums of  $k$  unit fractions”, *Proc. Roy. Soc. Edinburgh Sect. A* **150**:3 (2020), 1401–1427. [MR](#) [Zbl](#)
- [Ford and Konyagin 2002] K. Ford and S. Konyagin, “Chebyshev’s conjecture and the prime number race”, pp. 67–91 in *IV International Conference “Modern problems of number theory and its applications”: Current problems, Part II* (Tula, 2001), edited by V. N. Chubarikov and G. I. Arkhipov, Mosk. Gos. Univ. im. Lomonosova, Mekh.-Mat. Fak., Moscow, 2002. [MR](#) [Zbl](#)
- [Ford, Harper, and Lamzouri 2019] K. Ford, A. J. Harper, and Y. Lamzouri, “Extreme biases in prime number races with many contestants”, *Math. Ann.* **374**:1-2 (2019), 517–551. [MR](#) [Zbl](#)
- [Gallagher 1976] P. X. Gallagher, “On the distribution of primes in short intervals”, *Mathematika* **23**:1 (1976), 4–9. [MR](#) [Zbl](#)
- [Granville and Lumley 2023] A. Granville and A. Lumley, “Primes in short intervals: heuristics and calculations”, *Exp. Math.* **32**:2 (2023), 378–404. [MR](#) [Zbl](#)
- [Granville and Martin 2006] A. Granville and G. Martin, “Prime number races”, *Amer. Math. Monthly* **113**:1 (2006), 1–33. [MR](#) [Zbl](#)
- [Hayes 1966] D. R. Hayes, “The expression of a polynomial as a sum of three irreducibles”, *Acta Arith.* **11** (1966), 461–488. [MR](#) [Zbl](#)
- [Keating and Rudnick 2014] J. P. Keating and Z. Rudnick, “The variance of the number of prime polynomials in short intervals and in residue classes”, *Int. Math. Res. Not.* **2014**:1 (2014), 259–288. [MR](#)
- [Lemke Oliver and Soundararajan 2016] R. J. Lemke Oliver and K. Soundararajan, “Unexpected biases in the distribution of consecutive primes”, *Proc. Natl. Acad. Sci. USA* **113**:31 (2016), E4446–E4454. [MR](#)
- [Montgomery and Soundararajan 2002] H. L. Montgomery and K. Soundararajan, “Beyond pair correlation”, pp. 507–514 in *Paul Erdős and his mathematics, I* (Budapest, 1999), edited by G. Halász et al., Bolyai Soc. Math. Stud. **11**, János Bolyai Math. Soc., Budapest, 2002. [MR](#) [Zbl](#)
- [Montgomery and Soundararajan 2004] H. L. Montgomery and K. Soundararajan, “Primes in short intervals”, *Comm. Math. Phys.* **252**:1-3 (2004), 589–617. [MR](#) [Zbl](#)
- [Montgomery and Vaughan 1986] H. L. Montgomery and R. C. Vaughan, “On the distribution of reduced residues”, *Ann. of Math. (2)* **123**:2 (1986), 311–333. [MR](#) [Zbl](#)
- [Shiu 1980] P. Shiu, “The maximum orders of multiplicative functions”, *Quart. J. Math. Oxford Ser. (2)* **31**:122 (1980), 247–252. [MR](#) [Zbl](#)

Communicated by Andrew Granville

Received 2021-10-08    Revised 2024-05-14    Accepted 2024-06-15

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

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ANT peer review and production are managed by EditFLOW<sup>®</sup> from MSP.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

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# Algebra & Number Theory

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