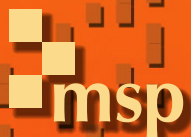


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On the D-module of an isolated singularity

Thomas Bitoun



On the \mathcal{D} -module of an isolated singularity

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Let Z be the germ of a complex hypersurface isolated singularity of equation f , with Z at least of dimension 2. We consider the family of analytic \mathcal{D} -modules generated by the powers of $1/f$ and describe it in terms of the pole order filtration on the de Rham cohomology of the complement of $\{f = 0\}$ in the neighbourhood of the singularity.

1. Introduction

The \mathcal{D} -modules generated by powers of a polynomial (or analytic function) f have been the topic of several noted publications in the last decade, for example, [Bitoun and Schedler 2018; Mustața and Olano 2023; Saito 2021; 2022]. On the one hand, they are elementary objects accessible to beginners in \mathcal{D} -module theory. On the other hand, they relate to analytic invariants and Hodge theory in deep and subtle ways.

This note provides a new, elementary approach to describing these \mathcal{D} -modules in the general isolated singularity case in terms of the pole order filtration on the de Rham cohomology of the algebraic link of the singularity.

Our results include:

- A new approach to, and a new proof of, Vilonen's characterization of the intersection cohomology \mathcal{D} -module [Vilonen 1985, Theorem], presented in Theorem 2.2.4 and Remark 2.2.7.
- A new computation of the length of the \mathcal{D} -module of meromorphic functions (Theorem 2.2.4).
- An elementary description of the Hodge structure of the \mathcal{D} -module of meromorphic functions (Theorem 2.2.4).
- A full solution to the question of the length of the \mathcal{D} -module generated by $1/f$ and of the corresponding Poisson cohomology (see [Bitoun and Schedler 2018; Etingof and Schedler 2014, Conjecture 3.8]) in Corollary 3.0.4.
- Connections between top-forms decompositions with prescribed pole order and the \mathcal{D} -modules generated by a power of $1/f$ via generalizations of Vilonen's theorem, described in Corollary 3.0.1.
- Demonstrating the importance of \mathcal{D} -submodules generated by pieces of the Hodge or pole order filtrations, first considered in [Mustața and Olano 2023] and studied in Corollary 2.2.6 and Theorem 2.2.8.

MSC2020: primary 14F10; secondary 14B05.

Keywords: \mathcal{D} -modules, isolated singularities, Hodge filtration, de Rham cohomology, mixed Hodge structures.

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- Explaining the failure of the conjecture from [Bitoun and Schedler 2018] (equivalent in terms of the Hamiltonian flow to [Etingof and Schedler 2014, Conjecture 3.8]), as first noted in [Mustață and Olano 2023] (see also [Saito 2022]). This is discussed at the end of the introduction.

Finally, we note that our approach has already led to new results; see, e.g., the updates to [Saito 2022].

We now describe the contents in more technical detail. Let f be a complex analytic function in n variables, $n \geq 3$, and assume that $Z := \{f = 0\}$ is reduced and has an isolated singularity at o . Our main tool is the product pairing in the neighbourhood of the singularity between the meromorphic functions with poles along Z and the top regular forms, with values in the n -th de Rham cohomology group H' of the complement of Z .

Let δ_o be the irreducible \mathcal{D} -module supported at o . In Theorem 2.2.4, we show that the pairing can be interpreted as a \mathcal{D} -module map $r : \mathcal{O}(*Z)_o \rightarrow \delta_o \otimes H'$, which is surjective with kernel equal to the intersection homology \mathcal{D} -module \mathcal{L}_o . The latter relies on Vilonen's characterization of that \mathcal{D} -module, of which our result can be viewed as a generalization (e.g., Corollary 3.0.1; see also Remark 2.2.7). The morphism r is especially convenient to study the \mathcal{D} -submodules of $\mathcal{O}(*Z)_o$ generated by powers of $1/f$ or by the pieces of Hodge filtration; see Corollaries 2.2.6 and 3.0.1.

Using the above, [Mustață and Olano 2023] implies that the dimension of the first piece $F_0 H'$ of the Hodge filtration is the reduced genus g of [Bitoun and Schedler 2018], while the length of $\mathcal{D}(1/f)/\mathcal{L}_o$ is $\dim P_0 H'$, where $P_0 H'$ is the set of classes generated by forms with pole order at most 1 along Z . However it is well known that the pole order filtration is, in general, strictly greater than the former; see, e.g., [Dimca 1991, 5.4 ii; Karpishpan 1991, (b) of Theorem 0.3]. This explains the failure of [Bitoun and Schedler 2018, Conjecture 1.7]. Finally, let us note that even though the natural language of the note is that of analytic \mathcal{D} -modules, we deduce results on length in the algebraic case as well; see Corollary 3.0.4.

2. A morphism of \mathcal{D} -modules

2.1. Setup and conventions. Let $n \geq 3$. Let X be a complex analytic manifold of dimension n , let Z be a hypersurface of X that has isolated singularities and let $U \subset X$ be the open complement of Z . By restricting to an open neighbourhood of a singularity we may assume that Z has a unique singularity o , which we do from now on. By a \mathcal{D} -module, we mean a left coherent, analytic D_X -module, and by a D_o -module, we mean a finitely generated left module over the stalk of D_X at the point o . For a holonomic \mathcal{D} -module M , we let $DR^l(M)$ be the l -th cohomology group of the de Rham complex of M . We use the same notation $DR^l(N)$ for N a holonomic D_o -module. We let $\mathcal{O}(*Z)$ be the \mathcal{D} -module of meromorphic functions on X with poles along Z .

2.2. Construction. Let us first recall standard facts.

Let k be a field, let B be an arbitrary k -algebra and let M be a right (resp. left) B -module. Then for an arbitrary k -vector space W , the space of k -linear maps $L(M, W)$ from M to W is a left (resp. right) B -module, where the action is given by $bf(m) := f(mb)$ (resp. $fb(m) := f(bm)$), for all $b \in B$, $m \in M$, $f \in L(M, W)$. In the following lemma, we apply this to the right D_0 -module Ω_o^n for $k = \mathbb{C}$.

Lemma 2.2.1. *Let V be a finite-dimensional complex vector space and let o be a point of X . For Ω_o^n the stalk at o of the sheaf of differential n -forms, the space of linear maps $L(\Omega_o^n, V)$ from the right D_o -module Ω_o^n to V is naturally a D_o -module. Moreover the D_o -submodule $L(\Omega_o^n, V)_o$ of linear maps annihilated by a power of the ideal of o is isomorphic to the D_o -module $\delta_o \otimes_{\mathbb{C}} V$, where δ_o is the irreducible D_o -module supported at o .*

Proof. Only the last part needs further proof. Under Kashiwara’s equivalence [Björk 1993, Lemma 2.6.18], a holonomic left D -module M supported at o corresponds to the finite-dimensional vector space $M/(\mathfrak{m}M)$, where \mathfrak{m} is the ideal of o , and $M \simeq \delta_o \otimes_{\mathbb{C}} M/(\mathfrak{m}M)$. Therefore, the D -module $L(\Omega_o^n, V)_o$ corresponds to the vector space of linear maps from $\Omega_o^n/\mathfrak{m}\Omega_o^n \simeq \mathbb{C}$ to V , which is isomorphic to V . Hence the existence of an isomorphism $L(\Omega_o^n, V)_o \simeq \delta_o \otimes_{\mathbb{C}} V$. \square

We will use the following lemma. For an element λ of $L(\Omega_o^n, V)$, we denote by $\text{Im}(\lambda)$ the image of the corresponding linear map $\Omega_o^n \rightarrow V$.

Lemma 2.2.2. *Let V be a finite-dimensional complex vector space and let N be a D_o -submodule of $L(\Omega_o^n, V)_o$. Assume that for all $v \in V$, there exists an element λ_v of N such that $v \in \text{Im}(\lambda_v)$. Then $N = L(\Omega_o^n, V)_o$.*

Proof. By Lemma 2.2.1 and Kashiwara’s equivalence [Björk 1993, Lemma 2.6.18], $N = L(\Omega_o^n, V')_o$ for a vector subspace V' of V . Thus if $v \in V \setminus V'$, then for all $\lambda \in N = L(\Omega_o^n, V')_o$, $v \notin \text{Im}(\lambda)$. This contradicts the assumption on N . Hence $N = L(\Omega_o^n, V)_o$. \square

In our setup 2.1, we denote by \mathcal{L} the D -module preimage in $\mathcal{O}(\star Z)$ of the intersection cohomology D -module $\mathcal{L}_Z \subseteq \mathcal{O}(\star Z)/\mathcal{O}$ associated with Z . We now recall Vilonen’s description of the intersection homology D -module in terms of residues.

Theorem 2.2.3 (Vilonen). *An element s of the stalk $\mathcal{O}(\star Z)_o$ is in the stalk \mathcal{L}_o if and only if $\forall \omega' \in \Omega_o^n$, $s\omega'$ is exact, i.e.,*

$$\mathcal{L}_o = \{s \in \mathcal{O}(\star Z)_o \mid \text{for all } \omega' \in \Omega_o^n, s\omega' \in d(\Omega_o^{n-1}(\star Z))\}.$$

Proof. This is a reformulation of [Vilonen 1985, Theorem]; see [Björk 1993, 5.7.21] for a textbook treatment. We include the proof below for the benefit of the reader. Let

$$V := \{s \in \mathcal{O}(\star Z)_o \mid \text{for all } \omega' \in \Omega_o^n, s\omega' \in d(\Omega_o^{n-1}(\star Z))\}.$$

It follows by a special case of the argument given in the proof of Theorem 2.2.4 below that V is a D_o -submodule. We want to prove that $V = \mathcal{L}_o$.

Let us first show that $\mathcal{L}_o \subseteq V$. Note that for $N \subseteq \mathcal{O}(\star Z)_o$ a D_o -submodule, $N \subseteq V$ if and only if the image of $DR^n(N)$ in the de Rham cohomology group $DR^n(\mathcal{O}(\star Z)_o)$ vanishes. But $DR^n(\mathcal{L}_o) = DR^n(\mathcal{L})_o = 0$ by [Björk 1993, Lemma 5.7.18]; hence $\mathcal{L}_o \subseteq V$. Let us now show that $\mathcal{L}_o = V$. The quotient V/\mathcal{L}_o is supported at the singularity since it is the case for $\mathcal{O}(\star Z)_o/\mathcal{L}_o$. Therefore $V/\mathcal{L}_o \simeq \delta_o^j$ for some $j \geq 0$. By the long exact sequence of the DR^i ’s applied to the short exact sequence $0 \rightarrow V \rightarrow \mathcal{O}(\star Z)_o \rightarrow \mathcal{O}(\star Z)_o/V \rightarrow 0$, we have that the natural map $DR^n(V) \rightarrow DR^n(\mathcal{O}(\star Z)_o)$ is an injection,

because $DR^{n-1}(\delta_o) = 0$. Hence $DR^n(V) = 0$ by the definition of V . But using the long exact sequence of the DR^i 's associated with the short exact sequence $0 \rightarrow \mathcal{L}_o \rightarrow V \rightarrow \delta_o^j \rightarrow 0$, we deduce from $DR^{n-1}(\delta_o) = DR^{n+1}(\mathcal{L}_o) = 0$ and $DR^n(\delta_o) = \mathbb{C}$ that

$$\mathbb{C}^j \simeq \frac{DR^n(V)}{DR^n(\mathcal{L}_o)}.$$

Since the latter vanishes, we must have $j = 0$ and $\mathcal{L}_o = V$. □

Let us now prove the main theorems of this note. Note that the de Rham cohomology $DR^n(\mathcal{O}(\star Z)_o)$ is endowed with a Hodge structure, which we denote by H' .

Theorem 2.2.4. *Under the hypotheses in Section 2.1, the pairing*

$$\mathcal{O}(\star Z)_o \times \Omega_o^n \xrightarrow{B} H', \quad (s, \omega') \mapsto [s\omega'],$$

where $[-]$ is the cohomology class of a form, induces a surjective homomorphism of D_o -modules

$$\mathcal{O}(\star Z)_o \xrightarrow{r} L(\Omega_o^n, H')_o, \quad s \mapsto B(s, -),$$

where o is the singularity. This homomorphism is compatible with the Hodge filtrations, where the Hodge filtration on $L(\Omega_o^n, H')_o$ is the one induced by that of H' under Kashiwara's equivalence for Hodge D -modules. The kernel of r is the D_o -module \mathcal{L}_o .

Proof. Let us first show that the map

$$\mathcal{O}(\star Z)_o \rightarrow L(\Omega_o^n, DR^n(\mathcal{O}(\star Z)_o)), \quad s \mapsto (\omega' \mapsto [s\omega']),$$

defines a morphism of D_o -modules and takes its values in $L(\Omega_o^n, DR^n(\mathcal{O}(\star Z)_o))_o$.

It follows directly from the definitions that the map is \mathcal{O} -linear. Moreover, the fact that the class of an exact form in $DR^n(\mathcal{O}(\star Z)_o)$ vanishes implies that r is compatible with the actions of derivations. We may restrict ourselves to verifying it for the action of the partials $(\partial_i)_i$ corresponding to coordinates $(x_i)_i$. Let ω be a volume form in the neighbourhood of o and let $\omega^{(i)}$ be an $(n-1)$ -form such that $dx_i \wedge \omega^{(i)} = \omega$. Then $s\omega' = sg\omega$ for some holomorphic function g and $d(sg\omega^{(i)}) = \partial_i(s)\omega + s\partial_i(g)\omega$. That is,

$$\partial_i s \mapsto (g\omega \mapsto [\partial_i(s)g\omega] = -[s\partial_i(g)\omega]).$$

Hence the map is compatible with the right D_o -module action on Ω_o^n . Therefore we have a morphism of D_o -modules $\mathcal{O}(\star Z)_o \rightarrow L(\Omega_o^n, DR^n(\mathcal{O}(\star Z)_o))$. Note that by Theorem 2.2.3, the kernel of this morphism is \mathcal{L}_o . But $\mathcal{O}(\star Z)_o/\mathcal{L}_o$ is supported at o ; hence r factors through $L(\Omega_o^n, DR^n(\mathcal{O}(\star Z)_o))_o$.

That r is surjective follows directly from Lemma 2.2.2. Finally, the compatibility of r with the Hodge filtrations is a direct consequence of the construction of the Hodge filtration on the de Rham complex. □

Remark 2.2.5. Letting Z_∞ be the Milnor fibre at o , we note that for $H := H^{n-1}(Z_\infty)_1$ the unipotent monodromy part of the cohomology group of the Milnor fibre and N the logarithm of the unipotent part of the monodromy, we have a natural identification of mixed Hodge structures $\gamma : H' \simeq H/(NH)$. This

follows from applying DR^n to the short exact sequence $0 \rightarrow M_f \simeq \mathcal{L} \rightarrow M'_f \simeq \mathcal{O}(\star Z)/\mathcal{O} \rightarrow M''_f/M_f \rightarrow 0$ of [Saito 2009, Remarks 3.2i] and using the isomorphisms [Saito 2009, 3.2.5 and 3.2.4].

As a direct corollary, we get the following.

Corollary 2.2.6. *The image by r of the \mathcal{D} -submodule $\mathcal{D}_o F_l \mathcal{O}(\star Z)_o$ generated by the l -th piece of the Hodge filtration on $\mathcal{O}(\star Z)_o$ is $L(\Omega_o^n, F_l H')_o$, where $F_l H'$ is the l -th piece of the Hodge filtration on H' . Therefore the length of $\mathcal{D}_o F_l \mathcal{O}(\star Z)_o/\mathcal{L}_o$ is $\dim F_l H'$.*

Proof. Since r is a surjective morphism of Hodge D_o -modules by Theorem 2.2.4, we have $r(F_l \mathcal{O}(\star Z)_o) = \sum_{i+j \leq l} G_i \delta_o \otimes F_j H'$, where G is the good filtration on δ_o induced by the standard generator of δ_o and the usual good filtration of \mathcal{D}_o ; see, e.g., [Saito 2009, 1.5.3]. But the submodules $\mathcal{D}_o(\sum_{i+j \leq l} G_i \delta_o \otimes F_j H')$ and $\mathcal{D}_o(G_o \otimes F_l H') = L(\Omega_o^n, F_l H')_o$ are equal. Indeed the \mathcal{D}_o -module structure on $L(\Omega_o^n, H')_o \simeq \delta_o \otimes H'$ is such that \mathcal{D}_o acts only on the left factor δ_o , which is generated by $G_o \delta_o$. The equality follows. Therefore $r(\mathcal{D}_o F_l \mathcal{O}(\star Z)_o) = \mathcal{D}_o r(F_l \mathcal{O}(\star Z)_o) = \mathcal{D}_o(\sum_{i+j \leq l} G_i \delta_o \otimes F_j H') = L(\Omega_o^n, F_l H')_o$, as stated. \square

Remark 2.2.7. We draw the reader's attention to the fact that Theorem 2.2.4 can be thought of as providing a new proof of Vilonen's theorem. Namely, we know that r is surjective. So if we accept the elementary fact that $\mathcal{O}(\star Z)_o$ is of length $1 + \dim H'$ [Björk 1993, 5.7.17], we must have that the kernel of r is \mathcal{L}_o . But the kernel of r exactly matches the description in Vilonen's theorem.

We now consider D_o -submodules generated by an \mathcal{O}_o -submodule of $\mathcal{O}(\star Z)_o$. For M an \mathcal{O}_o -submodule of $\mathcal{O}(\star Z)_o$, we set $\Omega_o^n(M) = M \otimes_{\mathcal{O}_o} \Omega_o^n$ and let $[\Omega_o^n(M)] \subseteq DR^n(\mathcal{O}(\star Z)_o)$ be the vector subspace of the corresponding classes of forms, namely the classes that can be represented as mw' , for some $m \in M$ and $w' \in \Omega_o^n$.

Theorem 2.2.8. *Let M be an \mathcal{O}_o -submodule of $\mathcal{O}(\star Z)_o$ and let $D_o M$ be the D_o -submodule of $\mathcal{O}(\star Z)_o$ generated by M . Assume that $\mathcal{O}(\star Z)_o$ and $D_o M$ agree generically on Z . Then the quotient $D_o M/\mathcal{L}_o$ is isomorphic to $L(\Omega_o^n, [\Omega_o^n(M)])_o$. Moreover,*

$$D_o M = \{s \in \mathcal{O}(\star Z)_o \mid \text{for all } \omega' \in \Omega_o^n, s\omega' \in M \otimes_{\mathcal{O}_o} \Omega_o^n + d(\Omega_o^{n-1}(\star Z))\}.$$

Proof. Since \mathcal{L}_o is the minimal extension and $D_o M$ extends $\mathcal{O}(\star Z)_o$ generically on Z , $D_o M$ contains \mathcal{L}_o . It thus makes sense to consider the quotient $D_o M/\mathcal{L}_o$. We claim that $r(D_o M) = L(\Omega_o^n, [\Omega_o^n(M)])_o$, where r is the morphism from Theorem 2.2.4. Indeed, we have $M \subseteq r^{-1}(L(\Omega_o^n, [\Omega_o^n(M)])_o)$. Therefore $D_o M \subseteq r^{-1}(L(\Omega_o^n, [\Omega_o^n(M)])_o)$, because $r^{-1}(L(\Omega_o^n, [\Omega_o^n(M)])_o)$ is a D_o -module containing M , and hence $r(D_o M) \subseteq L(\Omega_o^n, [\Omega_o^n(M)])_o$. We then note that the equality $r(D_o M) = L(\Omega_o^n, [\Omega_o^n(M)])_o$ follows immediately from Lemma 2.2.2. \square

3. Some corollaries

In what follows, let us consider the filtration P by order of the pole on the de Rham cohomology $DR^n(\mathcal{O}(\star Z)_o) \simeq H'$. Namely, we let $P_l H'$ be the subspace of the classes that can be represented, via the isomorphism above, by forms having a pole of order at most $l + 1$ along Z .

Corollary 3.0.1. *Let f be a local equation of Z , and let $l \geq 0$. We have the following description of $D_o(1/f^{l+1})$, the D_o -submodule of $\mathcal{O}(\star Z)_o$ generated by $1/f^{l+1}$:*

$$D_o \frac{1}{f^{l+1}} = \{s \in \mathcal{O}(\star Z)_o \mid \text{for all } \omega' \in \Omega_o^n, s\omega' \in \Omega_o^n((l+1)Z) + d(\Omega_o^{n-1}(\star Z))\}.$$

It follows that the D_o -module length of the quotient $D_o(1/f^{l+1})/\mathcal{L}_o$ is $\dim_{\mathbb{C}} P_l H'$.

Proof. Apply Theorem 2.2.8 to $M = \mathcal{O}_o/f^{l+1}$. It then follows that the quotient $D_o(1/f^{l+1})/\mathcal{L}_o = D_o(\mathcal{O}_o/f^{l+1})/\mathcal{L}_o$ is isomorphic to

$$L\left(\Omega_o^n, \left[\Omega_o^n\left(\frac{\mathcal{O}_o}{f^{l+1}}\right)\right]\right)_o = L(\Omega_o^n, P_l H')_o.$$

Since the latter is isomorphic to $\delta_o \otimes_{\mathbb{C}} P_l H'$ by Lemma 2.2.1, the length assertion is proved. □

Therefore we deduce the following properties, first proved in [Mustařa and Olano 2023, Theorems 1.1 and 1.3], from those of the pole order filtration.

Corollary 3.0.2. *Recall the hypotheses in Section 2.1.*

- (1) *The D_o -module length of the quotient $D_o(1/f^{l+1})/\mathcal{L}_o$ is at least $\dim_{\mathbb{C}} F_l H'$.*
- (2) *If Z is quasihomogeneous, then the inequality from 1 is an equality.*

Proof. Since the D_o -module length of the quotient $D_o(1/f^{l+1})/\mathcal{L}_o$ is $\dim_{\mathbb{C}} P_l H'$ by Corollary 3.0.1, assertions (1) and (2) follow from Theorems (b) and (a) of [Karpishpan 1991], respectively. □

Remark 3.0.3. We note that, conversely, any result on the length of $D_o(1/f^{l+1})/\mathcal{L}_o$ transfers by Corollary 3.0.1 to a statement about the pole order filtration. For example, [Saito 2022, Theorem 1] describes those lengths in terms of the Gauss–Manin connection (compare with [Karpishpan 1991, Theorem (c)]). Moreover, [Mustařa and Olano 2023, §5; Saito 2022, 3.2 Example I] provide new examples where the Hodge filtration is strictly contained in the pole order filtration.

We also obtain results for algebraic D -modules. Let $n \geq 3$ and let g be a complex polynomial in n variables defining a reduced irreducible hypersurface Y with an isolated singularity at the origin, i.e., $|Y^{\text{sing}}| = 1$. Then for all $l \geq 0$, we denote by $D^{\text{alg}}(1/g^{l+1})$ the left D^{alg} -submodule of $R[1/g]$ generated by $1/g^{l+1}$, where R is the ring of complex polynomials in n variables and D^{alg} is the n -th Weyl algebra $A_n(\mathbb{C})$. We let IC be the D^{alg} -module preimage in $R[1/g]$ of the intersection cohomology D^{alg} -module IC_Y .

Corollary 3.0.4. *The quotient D^{alg} -module $D^{\text{alg}}(1/g^{l+1})/(IC)$ is of length $\dim_{\mathbb{C}} P_l H_{dR}^n(B \setminus Y)$, where P_l is the pole order filtration of the de Rham cohomology of the complement of Y in a small analytic ball B centred at the origin.*

Proof. Using the notation of Section 2.1, the analytification functor $D_{\mathbb{C}^n} \otimes_{D^{\text{alg}}} -$ is an equivalence between the category of regular holonomic D^{alg} -modules and a full subcategory of regular $D_{\mathbb{C}^n}$ -modules [Brylinski 1986, Proposition 7.8]. It follows directly from the definition that the analytification of the regular holonomic D^{alg} -module $R[1/g]$ is the sheaf of meromorphic functions $\mathcal{O}(\star Y^{\text{an}})$, and the analytification of

$D^{\text{alg}}(1/g^{l+1})$ is $D_{\mathbb{C}^n}(1/g^{l+1})$. Moreover, because the analytification is an equivalence, the minimality of IC and \mathcal{L} force them to correspond to each other under the analytification functor. But, as $D_{\mathbb{C}^n}(1/g^{l+1})/\mathcal{L}$ is supported at the origin, its length is the same as that of its stalk at the origin. But the natural map from $H_{dR}^n(B \setminus Y)$ to $H' = DR^n(\mathcal{O}(\star Y^{\text{an}}))_o$ is an isomorphism for a small enough analytical ball B around o , and it is compatible with the pole order filtration. Therefore the assertion follows from Corollary 3.0.1. \square

Remark 3.0.5. While the corollary is presented with the constraint $|Y^{\text{sing}}| = 1$ for clarity, the assertion can be extended to polynomials g with multiple isolated singularities. This would involve introducing a summation over all singularities.

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thomas.bitoun@ucalgary.ca

University of Calgary, Calgary, AB, Canada

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