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On the D-module of an isolated singularity

Thomas Bitoun

Let Z be the germ of a complex hypersurface isolated singularity of equation f, with Z at least of dimension 2. We consider the family of analytic D-modules generated by the powers of $1/f$ and describe it in terms of the pole order filtration on the de Rham cohomology of the complement of $\{f = 0\}$ in the neighbourhood of the singularity.

1. Introduction

The $\mathcal{D}$-modules generated by powers of a polynomial (or analytic function) f have been the topic of several noted publications in the last decade, for example, [Bitoun and Schedler 2018; Mustaţă and Olano 2023; Saito 2021; 2022]. On the one hand, they are elementary objects accessible to beginners in $\mathcal{D}$-module theory. On the other hand, they relate to analytic invariants and Hodge theory in deep and subtle ways.

This note provides a new, elementary approach to describing these $\mathcal{D}$-modules in the general isolated singularity case in terms of the pole order filtration on the de Rham cohomology of the algebraic link of the singularity.

Our results include:

- A new approach to, and a new proof of, Vilonen's characterization of the intersection cohomology $\mathcal{D}$-module [Vilonen 1985, Theorem], presented in Theorem 2.2.4 and Remark 2.2.7.
- A new computation of the length of the $\mathcal{D}$-module of meromorphic functions (Theorem 2.2.4).
- An elementary description of the Hodge structure of the $\mathcal{D}$-module of meromorphic functions (Theorem 2.2.4).
- A full solution to the question of the length of the $\mathcal{D}$-module generated by $1/f$ and of the corresponding Poisson cohomology (see [Bitoun and Schedler 2018; Etingof and Schedler 2014, Conjecture 3.8]) in Corollary 3.0.4.
- Connections between top-forms decompositions with prescribed pole order and the $\mathcal{D}$-modules generated by a power of $1/f$ via generalizations of Vilonen's theorem, described in Corollary 3.0.1.
- Demonstrating the importance of $\mathcal{D}$-submodules generated by pieces of the Hodge or pole order filtrations, first considered in [Mustaţă and Olano 2023] and studied in Corollary 2.2.6 and Theorem 2.2.8.

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- Explaining the failure of the conjecture from [Bitoun and Schedler 2018] (equivalent in terms of the Hamiltonian flow to [Etingof and Schedler 2014, Conjecture 3.8]), as first noted in [Mustaţă and Olano 2023] (see also [Saito 2022]). This is discussed at the end of the introduction.

Finally, we note that our approach has already led to new results; see, e.g., the updates to [Saito 2022].

We now describe the contents in more technical detail. Let f be a complex analytic function in n variables, $n \geq 3$, and assume that $Z := \{f = 0\}$ is reduced and has an isolated singularity at o. Our main tool is the product pairing in the neighbourhood of the singularity between the meromorphic functions with poles along Z and the top regular forms, with values in the n-th de Rham cohomology group H' of the complement of Z.

Let δ_o be the irreducible $\mathcal{D}$-module supported at o. In Theorem 2.2.4, we show that the pairing can be interpreted as a $\mathcal{D}$-module map $r : O(*Z)_o \to \delta_o \otimes H'$, which is surjective with kernel equal to the intersection homology $\mathcal{D}$-module $\mathcal{L}_o$. The latter relies on Vilonen's characterization of that $\mathcal{D}$-module, of which our result can be viewed as a generalization (e.g., Corollary 3.0.1; see also Remark 2.2.7). The morphism r is especially convenient to study the $\mathcal{D}$-submodules of $O(*Z)_o$ generated by powers of $1/f$ or by the pieces of Hodge filtration; see Corollaries 2.2.6 and 3.0.1.

Using the above, [Mustaţă and Olano 2023] implies that the dimension of the first piece F_0H' of the Hodge filtration is the reduced genus g of [Bitoun and Schedler 2018], while the length of $\mathcal{D}(1/f)/\mathcal{L}_o$ is dim P_0H', where P_0H' is the set of classes generated by forms with pole order at most 1 along Z. However it is well known that the pole order filtration is, in general, strictly greater than the former; see, e.g., [Dimca 1991, 5.4 ii; Karpishpan 1991, (b) of Theorem 0.3]. This explains the failure of [Bitoun and Schedler 2018, Conjecture 1.7]. Finally, let us note that even though the natural language of the note is that of analytic D-modules, we deduce results on length in the algebraic case as well; see Corollary 3.0.4.

2. A morphism of $\mathcal{D}$-modules

2.1. ***Setup and conventions.*** Let $n \geq 3$. Let X be a complex analytic manifold of dimension n, let Z be a hypersurface of X that has isolated singularities and let $U \subset X$ be the open complement of Z. By restricting to an open neighbourhood of a singularity we may assume that Z has a unique singularity o, which we do from now on. By a D-module, we mean a left coherent, analytic D_X-module, and by a D_o-module, we mean a finitely generated left module over the stalk of D_X at the point o. For a holonomic D-module M, we let $DR^l(M)$ be the l-th cohomology group of the de Rham complex of M. We use the same notation $DR^l(N)$ for N a holonomic D_o-module. We let $\mathcal{O}(\star Z)$ be the D-module of meromorphic functions on X with poles along Z.

2.2. ***Construction.*** Let us first recall standard facts.

Let k be a field, let B be an arbitrary k-algebra and let M be a right (resp. left) B-module. Then for an arbitrary k-vector space W, the space of k-linear maps $L(M, W)$ from M to W is a left (resp. right) B-module, where the action is given by $bf(m) := f(mb)$ (resp. $fb(m) := f(bm)$), for all $b \in B$, $m \in M$, $f \in L(M, W)$. In the following lemma, we apply this to the right D_0-module Ω^n_o for $k = \mathbb{C}$.

Lemma 2.2.1. *Let V be a finite-dimensional complex vector space and let o be a point of X. For Ω_o^n the stalk at o of the sheaf of differential n-forms, the space of linear maps $L(\Omega_o^n, V)$ from the right D_o-module Ω_o^n to V is naturally a D_o-module. Moreover the D_o-submodule $L(\Omega_o^n, V)_o$ of linear maps annihilated by a power of the ideal of o is isomorphic to the D_o-module $\delta_o \otimes_{\mathbb{C}} V$, where δ_o is the irreducible D_o-module supported at o.*

Proof. Only the last part needs further proof. Under Kashiwara's equivalence [Björk 1993, Lemma 2.6.18], a holonomic left D-module M supported at o corresponds to the finite-dimensional vector space $M/(\mathfrak{m}M)$, where $\mathfrak{m}$ is the ideal of o, and $M \simeq \delta_o \otimes_{\mathbb{C}} M/(\mathfrak{m}M)$. Therefore, the D-module $L(\Omega_o^n, V)_o$ corresponds to the vector space of linear maps from $\Omega_o^n/\mathfrak{m}\Omega_o^n \simeq \mathbb{C}$ to V, which is isomorphic to V. Hence the existence of an isomorphism $L(\Omega_o^n, V)_o \simeq \delta_o \otimes_{\mathbb{C}} V$. □

We will use the following lemma. For an element λ of $L(\Omega_o^n, V)$, we denote by $\mathrm{Im}(\lambda)$ the image of the corresponding linear map $\Omega_o^n \to V$.

Lemma 2.2.2. *Let V be a finite-dimensional complex vector space and let N be a D_o-submodule of $L(\Omega_o^n, V)_o$. Assume that for all $v \in V$, there exists an element λ_v of N such that $v \in \mathrm{Im}(\lambda_v)$. Then $N = L(\Omega_o^n, V)_o$.*

Proof. By Lemma 2.2.1 and Kashiwara's equivalence [Björk 1993, Lemma 2.6.18], $N = L(\Omega_o^n, V')_o$ for a vector subspace V' of V. Thus if $v \in V \setminus V'$, then for all $\lambda \in N = L(\Omega_o^n, V')_o$, $v \notin \mathrm{Im}(\lambda)$. This contradicts the assumption on N. Hence $N = L(\Omega_o^n, V)_o$. □

In our setup 2.1, we denote by $\mathcal{L}$ the D-module preimage in $\mathcal{O}(\star Z)$ of the intersection cohomology D-module $\mathcal{L}_Z \subseteq \mathcal{O}(\star Z)/\mathcal{O}$ associated with Z. We now recall Vilonen's description of the intersection homology D-module in terms of residues.

Theorem 2.2.3 (Vilonen). *An element s of the stalk $\mathcal{O}(\star Z)_o$ is in the stalk $\mathcal{L}_o$ if and only if $\forall \omega' \in \Omega_o^n$, $s\omega'$ is exact, i.e.,*

$$\mathcal{L}_o = \{s \in \mathcal{O}(\star Z)_o \mid \textit{for all } \omega' \in \Omega_o^n,\, s\omega' \in d(\Omega_o^{n-1}(\star Z))\}.$$

Proof. This is a reformulation of [Vilonen 1985, Theorem]; see [Björk 1993, 5.7.21] for a textbook treatment. We include the proof below for the benefit of the reader. Let

$$V := \{s \in \mathcal{O}(\star Z)_o \mid \text{ for all } \omega' \in \Omega_o^n,\, s\omega' \in d(\Omega_o^{n-1}(\star Z))\}.$$

It follows by a special case of the argument given in the proof of Theorem 2.2.4 below that V is a D_o-submodule. We want to prove that $V = \mathcal{L}_o$.

Let us first show that $\mathcal{L}_o \subseteq V$. Note that for $N \subseteq \mathcal{O}(\star Z)_o$ a D_o-submodule, $N \subseteq V$ if and only if the image of $DR^n(N)$ in the de Rham cohomology group $DR^n(\mathcal{O}(\star Z))_o$ vanishes. But $DR^n(\mathcal{L}_o) = DR^n(\mathcal{L})_o = 0$ by [Björk 1993, Lemma 5.7.18]; hence $\mathcal{L}_o \subseteq V$. Let us now show that $\mathcal{L}_o = V$. The quotient $V/\mathcal{L}_o$ is supported at the singularity since it is the case for $\mathcal{O}(\star Z)_o/\mathcal{L}_o$. Therefore $V/\mathcal{L}_o \simeq \delta_o^j$ for some $j \geq 0$. By the long exact sequence of the DR^i's applied to the short exact sequence $0 \to V \to \mathcal{O}(\star Z)_o \to \mathcal{O}(\star Z)_o/V \to 0$, we have that the natural map $DR^n(V) \to DR^n(\mathcal{O}(\star Z)_o)$ is an injection, because $DR^{n-1}(\delta_o) = 0$. Hence $DR^n(V) = 0$ by the definition of V. But using the long exact sequence of the DR^i's associated with the short exact sequence $0 \to \mathcal{L}_o \to V \to \delta_o^j \to 0$, we deduce from $DR^{n-1}(\delta_o) = DR^{n+1}(\mathcal{L}_o) = 0$ and $DR^n(\delta_o) = \mathbb{C}$ that

$$\mathbb{C}^j \simeq \frac{DR^n(V)}{DR^n(\mathcal{L}_o)}.$$

Since the latter vanishes, we must have $j = 0$ and $\mathcal{L}_o = V$. □

Let us now prove the main theorems of this note. Note that the de Rham cohomology $DR^n(\mathcal{O}(\star Z)_o)$ is endowed with a Hodge structure, which we denote by H'.

Theorem 2.2.4. *Under the hypotheses in Section 2.1, the pairing*

$$\mathcal{O}(\star Z)_o \times \Omega_o^n \xrightarrow{B} H', \quad (s, \omega') \mapsto [s\omega'],$$

where $[-]$ is the cohomology class of a form, induces a surjective homomorphism of D_o-modules

$$\mathcal{O}(\star Z)_o \xrightarrow{r} L(\Omega_o^n, H')_o, \quad s \mapsto B(s, -),$$

where o is the singularity. This homomorphism is compatible with the Hodge filtrations, where the Hodge filtration on $L(\Omega_o^n, H')_o$ is the one induced by that of H' under Kashiwara's equivalence for Hodge D-modules. The kernel of r is the D_o-module $\mathcal{L}_o$.

Proof. Let us first show that the map

$$\mathcal{O}(\star Z)_o \to L(\Omega_o^n, DR^n(\mathcal{O}(\star Z)_o)), \quad s \mapsto (\omega' \mapsto [s\omega']),$$

defines a morphism of D_o-modules and takes its values in $L(\Omega_o^n, DR^n(\mathcal{O}(\star Z)_o))_o$.

It follows directly from the definitions that the map is $\mathcal{O}$-linear. Moreover, the fact that the class of an exact form in $DR^n(\mathcal{O}(\star Z)_o)$ vanishes implies that r is compatible with the actions of derivations. We may restrict ourselves to verifying it for the action of the partials $(\partial_i)_i$ corresponding to coordinates $(x_i)_i$. Let ω be a volume form in the neighbourhood of o and let $\omega^{(i)}$ be an $(n-1)$-form such that $dx_i \wedge \omega^{(i)} = \omega$. Then $s\omega' = sg\omega$ for some holomorphic function g and $d(sg\omega^{(i)}) = \partial_i(s)\omega + s\partial_i(g)\omega$. That is,

$$\partial_i s \mapsto \big(g\omega \mapsto [\partial_i(s)g\omega] = -[s\partial_i(g)\omega]\big).$$

Hence the map is compatible with the right D_o-module action on Ω_o^n. Therefore we have a morphism of D_o-modules $\mathcal{O}(\star Z)_o \to L(\Omega_o^n, DR^n(\mathcal{O}(\star Z)_o))$. Note that by Theorem 2.2.3, the kernel of this morphism is $\mathcal{L}_o$. But $\mathcal{O}(\star Z)_o/\mathcal{L}_o$ is supported at o; hence r factors through $L(\Omega_o^n, DR^n(\mathcal{O}(\star Z)_o))_o$.

That r is surjective follows directly from Lemma 2.2.2. Finally, the compatibility of r with the Hodge filtrations is a direct consequence of the construction of the Hodge filtration on the de Rham complex. □

Remark 2.2.5. Letting Z_∞ be the Milnor fibre at o, we note that for $H := H^{n-1}(Z_\infty)_1$ the unipotent monodromy part of the cohomology group of the Milnor fibre and N the logarithm of the unipotent part of the monodromy, we have a natural identification of mixed Hodge structures $\gamma : H' \simeq H/(NH)$. This follows from applying DR^n to the short exact sequence $0 \to M_f \simeq \mathcal{L} \to M''_f \simeq \mathcal{O}(\star Z)/\mathcal{O} \to M''_f/M_f \to 0$ of [Saito 2009, Remarks 3.2i] and using the isomorphisms [Saito 2009, 3.2.5 and 3.2.4].

As a direct corollary, we get the following.

Corollary 2.2.6. *The image by r of the $\mathcal{D}$-submodule $\mathcal{D}_o F_l \mathcal{O}(\star Z)_o$ generated by the l-th piece of the Hodge filtration on $\mathcal{O}(\star Z)_o$ is $L(\Omega^n_o, F_l H')_o$, where $F_l H'$ is the l-th piece of the Hodge filtration on H'. Therefore the length of $\mathcal{D}_o F_l \mathcal{O}(\star Z)_o/\mathcal{L}_o$ is* $\dim F_l H'$.

Proof. Since r is a surjective morphism of Hodge D_o-modules by Theorem 2.2.4, we have $r(F_l \mathcal{O}(\star Z)_o) = \Sigma_{i+j\le l} G_i \delta_o \otimes F_j H'$, where G is the good filtration on δ_o induced by the standard generator of δ_o and the usual good filtration of $\mathcal{D}_o$; see, e.g., [Saito 2009, 1.5.3]. But the submodules $\mathcal{D}_o(\Sigma_{i+j\le l} G_i \delta_o \otimes F_j H')$ and $\mathcal{D}_o(G_0 \otimes F_l H') = L(\Omega^n_o, F_l H')_o$ are equal. Indeed the $\mathcal{D}_o$-module structure on $L(\Omega^n_o, H')_o \simeq \delta_0 \otimes H'$ is such that $\mathcal{D}_o$ acts only on the left factor δ_0, which is generated by $G_0\delta_0$. The equality follows. Therefore $r(\mathcal{D}_o F_l \mathcal{O}(\star Z)_o) = \mathcal{D}_o r(F_l \mathcal{O}(\star Z)_o) = \mathcal{D}_o(\Sigma_{i+j\le l} G_i \delta_o \otimes F_j H') = L(\Omega^n_o, F_l H')_o$, as stated. □

Remark 2.2.7. We draw the reader's attention to the fact that Theorem 2.2.4 can be thought of as providing a new proof of Vilonen's theorem. Namely, we know that r is surjective. So if we accept the elementary fact that $\mathcal{O}(\star Z)_o$ is of length $1 + \dim H'$ [Björk 1993, 5.7.17], we must have that the kernel of r is $\mathcal{L}_o$. But the kernel of r exactly matches the description in Vilonen's theorem.

We now consider D_o-submodules generated by an $\mathcal{O}_o$-submodule of $\mathcal{O}(\star Z)_o$. For M an $\mathcal{O}_o$-submodule of $\mathcal{O}(\star Z)_o$, we set $\Omega^n_o(M) = M \otimes_{\mathcal{O}_o} \Omega^n_o$ and let $[\Omega^n_o(M)] \subseteq DR^n(\mathcal{O}(\star Z)_o)$ be the vector subspace of the corresponding classes of forms, namely the classes that can be represented as mw', for some $m \in M$ and $w' \in \Omega^n_o$.

Theorem 2.2.8. *Let M be an $\mathcal{O}_o$-submodule of $\mathcal{O}(\star Z)_o$ and let $D_o M$ be the D_o-submodule of $\mathcal{O}(\star Z)_o$ generated by M. Assume that $\mathcal{O}(\star Z)_o$ and $D_o M$ agree generically on Z. Then the quotient $D_o M/\mathcal{L}_o$ is isomorphic to $L(\Omega^n_o, [\Omega^n_o(M)])_o$. Moreover,*

$$D_o M = \{s \in \mathcal{O}(\star Z)_o \mid \textit{for all } \omega' \in \Omega^n_o, s\omega' \in M \otimes_{\mathcal{O}_o} \Omega^n_o + d(\Omega^{n-1}_o(\star Z))\}.$$

Proof. Since $\mathcal{L}_o$ is the minimal extension and $D_o M$ extends $\mathcal{O}(\star Z)_o$ generically on Z, $D_o M$ contains $\mathcal{L}_o$. It thus makes sense to consider the quotient $D_o M/\mathcal{L}_o$. We claim that $r(D_o M) = L(\Omega^n_o, [\Omega^n_o(M)])_o$, where r is the morphism from Theorem 2.2.4. Indeed, we have $M \subseteq r^{-1}(L(\Omega^n_o, [\Omega^n_o(M)])_o)$. Therefore $D_o M \subseteq r^{-1}(L(\Omega^n_o, [\Omega^n_o(M)])_o)$, because $r^{-1}(L(\Omega^n_o, [\Omega^n_o(M)])_o)$ is a D_o-module containing M, and hence $r(D_o M) \subseteq L(\Omega^n_o, [\Omega^n_o(M)])_o$. We then note that the equality $r(D_o M) = L(\Omega^n_o, [\Omega^n_o(M)])_o$ follows immediately from Lemma 2.2.2. □

3. Some corollaries

In what follows, let us consider the filtration P by order of the pole on the de Rham cohomology $DR^n(\mathcal{O}(\star Z)_o) \simeq H'$. Namely, we let $P_l H'$ be the subspace of the classes that can be represented, via the isomorphism above, by forms having a pole of order at most $l+1$ along Z.

Corollary 3.0.1. *Let f be a local equation of Z, and let $l \geq 0$. We have the following description of $D_o(1/f^{l+1})$, the D_o-submodule of $\mathcal{O}(\star Z)_o$ generated by $1/f^{l+1}$:*

$$D_o \frac{1}{f^{l+1}} = \{s \in \mathcal{O}(\star Z)_o \mid \textit{for all } \omega' \in \Omega_o^n,\, s\omega' \in \Omega_o^n((l+1)Z) + d(\Omega_o^{n-1}(\star Z))\}.$$

It follows that the D_o-module length of the quotient $D_o(1/f^{l+1})/\mathcal{L}_o$ is $\dim_{\mathbb{C}} P_l H'$.

Proof. Apply Theorem 2.2.8 to $M = \mathcal{O}_o/f^{l+1}$. It then follows that the quotient $D_o(1/f^{l+1})/\mathcal{L}_o = D_o(\mathcal{O}_o/f^{l+1})/\mathcal{L}_o$ is isomorphic to

$$L\Big(\Omega_o^n, \Big[\Omega_o^n\Big(\frac{\mathcal{O}_o}{f^{l+1}}\Big)\Big]\Big)_o = L(\Omega_o^n, P_l H')_o.$$

Since the latter is isomorphic to $\delta_o \otimes_{\mathbb{C}} P_l H'$ by Lemma 2.2.1, the length assertion is proved. □

Therefore we deduce the following properties, first proved in [Mustaţă and Olano 2023, Theorems 1.1 and 1.3], from those of the pole order filtration.

Corollary 3.0.2. *Recall the hypotheses in Section 2.1.*

(1) *The D_o-module length of the quotient $D_o(1/f^{l+1})/\mathcal{L}_o$ is at least $\dim_{\mathbb{C}} F_l H'$.*

(2) *If Z is quasihomogeneous, then the inequality from 1 is an equality.*

Proof. Since the D_o-module length of the quotient $D_o(1/f^{l+1})/\mathcal{L}_o$ is $\dim_{\mathbb{C}} P_l H'$ by Corollary 3.0.1, assertions (1) and (2) follow from Theorems (b) and (a) of [Karpishpan 1991], respectively. □

Remark 3.0.3. We note that, conversely, any result on the length of $D_o(1/f^{l+1})/\mathcal{L}_o$ transfers by Corollary 3.0.1 to a statement about the pole order filtration. For example, [Saito 2022, Theorem 1] describes those lengths in terms of the Gauss–Manin connection (compare with [Karpishpan 1991, Theorem (c)]). Moreover, [Mustaţă and Olano 2023, §5; Saito 2022, 3.2 Example I] provide new examples where the Hodge filtration is strictly contained in the pole order filtration.

We also obtain results for algebraic D-modules. Let $n \geq 3$ and let g be a complex polynomial in n variables defining a reduced irreducible hypersurface Y with an isolated singularity at the origin, i.e., $|Y^{\text{sing}}| = 1$. Then for all $l \geq 0$, we denote by $D^{\text{alg}}(1/g^{l+1})$ the left D^{alg}-submodule of $R[1/g]$ generated by $1/g^{l+1}$, where R is the ring of complex polynomials in n variables and D^{alg} is the n-th Weyl algebra $A_n(\mathbb{C})$. We let IC be the D^{alg}-module preimage in $R[1/g]$ of the intersection cohomology D^{alg}-module IC_Y.

Corollary 3.0.4. *The quotient D^{alg}-module $D^{\text{alg}}(1/g^{l+1})/(IC)$ is of length $\dim_{\mathbb{C}} P_l H^n_{dR}(B \setminus Y)$, where P_l is the pole order filtration of the de Rham cohomology of the complement of Y in a small analytic ball B centred at the origin.*

Proof. Using the notation of Section 2.1, the analytification functor $D_{\mathbb{C}^n} \otimes_{D^{\text{alg}}} -$ is an equivalence between the category of regular holonomic D^{alg}-modules and a full subcategory of regular $D_{\mathbb{C}^n}$-modules [Brylinski 1986, Proposition 7.8]. It follows directly from the definition that the analytification of the regular holonomic D^{alg}-module $R[1/g]$ is the sheaf of meromorphic functions $\mathcal{O}(\star Y^{\text{an}})$, and the analytification of

$D^{\text{alg}}(1/g^{l+1})$ is $D_{\mathbb{C}^n}(1/g^{l+1})$. Moreover, because the analytification is an equivalence, the minimality of IC and $\mathcal{L}$ force them to correspond to each other under the analytification functor. But, as $D_{\mathbb{C}^n}(1/g^{l+1})/\mathcal{L}$ is supported at the origin, its length is the same as that of its stalk at the origin. But the natural map from $H^n_{dR}(B \setminus Y)$ to $H' = DR^n(\mathcal{O}(\star Y^{\text{an}}))_o$ is an isomorphism for a small enough analytical ball B around o, and it is compatible with the pole order filtration. Therefore the assertion follows from Corollary 3.0.1. □

Remark 3.0.5. While the corollary is presented with the constraint $|Y^{\text{sing}}| = 1$ for clarity, the assertion can be extended to polynomials g with multiple isolated singularities. This would involve introducing a summation over all singularities.

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