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Syzygies of tangent-developable surfaces and K3 carpets via secant varieties

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We give simple geometric proofs of Aprodu, Farkas, Papadima, Raicu and Weyman's theorem on syzygies of tangent-developable surfaces of rational normal curves and Raicu and Sam's result on syzygies of K3 carpets. As a consequence, we obtain a quick proof of Green's conjecture for general curves of genus g over an algebraically closed field $\Bbbk$ with $\operatorname{char}(\Bbbk) = 0$ or $\operatorname{char}(\Bbbk) \geq \lfloor (g-1)/2 \rfloor$. Our approach provides a new way to study tangent-developable surfaces in general. Along the way, we show the arithmetic normality of tangent-developable surfaces of arbitrary smooth projective curves of large degree.

1. Introduction

Let C be a smooth projective curve of genus $g \geq 3$ over the field $\mathbb{C}$ of complex numbers. If $\operatorname{Cliff}(C) \geq 1$, i.e., C is nonhyperelliptic, then K_C is very ample. In this case, Noether's theorem says that the canonical curve $C \subseteq \mathbb{P}^{g-1}$ is projectively normal. Petri's theorem states that if $\operatorname{Cliff}(C) \geq 2$, then the defining ideal $I_{C|\mathbb{P}^{g-1}}$ of C in $\mathbb{P}^{g-1}$ is generated by quadrics. To generalize classical theorems of Noether and Petri, Green [1984, Conjecture 5.1] formulated a very famous conjecture that predicts

$$K_{p,2}(C, K_C) = 0 \quad \text{for } 0 \leq p \leq \operatorname{Cliff}(C) - 1.$$

By the Green–Lazarsfeld nonvanishing theorem [Green 1984, Appendix], we have $K_{p,2}(C, K_C) \neq 0$ for $\operatorname{Cliff}(C) \leq p \leq g-3$. Then Green's conjecture determines the shape of the minimal free resolution of the canonical ring $R(C, K_C) = \bigoplus_{m \in \mathbb{Z}} H^0(C, mK_C)$ (see [Aprodu and Nagel 2010, Remark 4.19]). Although the conjecture is still open, Voisin [2002; 2005] resolved the general curve case in the early 2000s. To prove the generic Green conjecture, it suffices to exhibit one curve of each genus for which the assertion holds. If $S \subseteq \mathbb{P}^g$ is a K3 surface of degree $2g-2$, then its general hyperplane section is a canonical curve $C \subseteq \mathbb{P}^{g-1}$ and $K_{p,2}(S, \mathscr{O}_S(1)) = K_{p,2}(C, K_C)$. Based on the Hilbert scheme interpretation of Koszul cohomology, Voisin accomplished sophisticated cohomology computations on Hilbert schemes of K3 surfaces to show $K_{p,2}(S, \mathscr{O}_S(1)) = 0$. For an introduction to Voisin's work; see [Aprodu and Nagel 2010]. Recently, Kemeny [2021] gave a simpler proof of Voisin's theorem for even-genus case and a streamlined version of her arguments for the odd-genus case.

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Prior to Voisin's work, O'Grady and Buchweitz–Schreyer independently observed that one could use the *tangent-developable surface* $T \subseteq \mathbb{P}^g$ of a rational normal curve of degree g to solve Green's conjecture for general curves of genus g (see [Eisenbud 1992]). Note that $T \subseteq \mathbb{P}^g$ is arithmetically Cohen–Macaulay as for a K3 surface. One can actually view T as a degeneration of a K3 surface (see [Aprodu et al. 2019, Remark 6.6]). A general hyperplane section of T is a canonically embedded g-cuspidal rational curve $\overline{C} \subseteq \mathbb{P}^{g-1}$ of degree $2g-2$, which is a degeneration of a general canonical curve $C \subseteq \mathbb{P}^{g-1}$ with $\mathrm{Cliff}(C) = \lfloor (g-1)/2 \rfloor$. By the upper semicontinuity of graded Betti numbers, $K_{p,2}(\overline{C}, \mathscr{O}_{\overline{C}}(1)) = K_{p,2}(T, \mathscr{O}_T(1)) = 0$ implies $K_{p,2}(C, K_C) = 0$ (see [loc. cit., Section 6]). The required vanishing of $K_{p,2}(T, \mathscr{O}_T(1))$ for confirming the generic Green conjecture was finally established by Aprodu, Farkas, Papadima, Raicu and Weyman [loc. cit.] just a few years ago. Their important result gives not only an alternative proof of the generic Green conjecture but also an extension to positive characteristic. This circle of ideas is surveyed in [Ein and Lazarsfeld 2020]. In this paper, we give a simple geometric proof of the main result of [Aprodu et al. 2019].

Theorem 1.1. *Let $T \subseteq \mathbb{P}^g$ be the tangent-developable surface of a rational normal curve of degree $g \geq 3$ over an algebraically closed field $\Bbbk$ with* $\mathrm{char}(\Bbbk) = 0$ *or* $\mathrm{char}(\Bbbk) \geq (g+2)/2$. *Then*

$$K_{p,2}(T, \mathscr{O}_T(1)) = 0 \quad \text{for } 0 \leq p \leq \lfloor (g-3)/2 \rfloor.$$

The proof of Theorem 1.1 in [Aprodu et al. 2019] goes as follows. We have a short exact sequence

$$0 \longrightarrow \mathscr{O}_T \longrightarrow \nu_* \mathscr{O}_{\widetilde{T}} \longrightarrow \omega_{\mathbb{P}^1} \longrightarrow 0,$$

where $\nu : \widetilde{T} = \mathbb{P}^1 \times \mathbb{P}^1 \to T$ is a resolution of singularities. Then it is elementary to see that

$$K_{p,2}(T, \mathscr{O}_T(1)) = \mathrm{coker}\big(K_{p,1}(T, \nu_* \mathscr{O}_{\widetilde{T}}; \mathscr{O}_T(1)) \xrightarrow{\gamma} K_{p,1}(\mathbb{P}^1, \omega_{\mathbb{P}^1}; \mathscr{O}_{\mathbb{P}^1}(g))\big).$$

Let $U := H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(1))$, $V := D^{p+2}U$, $W := D^{2p+2}U$, and $q := g-p-3$. The authors of [Aprodu et al. 2019] devoted considerable effort to show that γ arises as the composition

$$S^q V \otimes W \xrightarrow{\mathrm{id}_{S^q V} \otimes \Delta} S^q V \otimes \textstyle\bigwedge^2 V \xrightarrow{\delta} \ker(S^{q+1} V \otimes V \xrightarrow{\delta} S^{q+2} V), \tag{1-1}$$

where Δ is the *co-Wahl map* and δ is the *Koszul differential*. To achieve this, they established an explicit characteristic-free Hermite reciprocity for $\mathfrak{sl}_2$-representations, and they carried out complicated algebraic computations. Now, $K_{p,2}(T, \mathscr{O}_T(1))$ is the homology of a complex

$$S^q V \otimes W \xrightarrow{\gamma} S^{q+1} V \otimes V \xrightarrow{\delta} S^{q+2} V.$$

This homology, denoted by $W_q(V, W)$, is the degree-q piece of the *Koszul module* (or *Weyman module*) associated to (V, W). It is enough to prove that

$$W_q(V, W) = 0 \quad \text{for } q \geq p. \tag{1-2}$$

The vanishing result (1-2) was first proved in characteristic zero in [Aprodu et al. 2022] by an application of Bott vanishing, and the argument is extended in [Aprodu et al. 2019] to positive characteristics.

Our strategy to prove Theorem 1.1 is essentially the same as that of [Aprodu et al. 2019], but our geometric approach utilizing the *secant variety* $\Sigma \subseteq \mathbb{P}^g$ of a rational normal curve C of degree g provides a substantial simplification of the proof. The tangent surface T is a Weil divisor on Σ, and $\widetilde{T}$ is a Cartier divisor on B, where $\beta : B \to \Sigma$ is the blow-up of Σ along C with the exceptional divisor $Z = \mathbb{P}^1 \times \mathbb{P}^1$. Letting $M_H := \beta^* M_{\mathscr{O}_\Sigma(1)}$, we realize (1-1) as maps induced in cohomology of vector bundles on B (see (3-1)):

$$H^1\big(\widetilde{T}, \textstyle\bigwedge^{p+2} M_H|_{\widetilde{T}}\big) \xrightarrow{\alpha} H^2\big(B, \textstyle\bigwedge^{p+2} M_H \otimes \omega_B(Z)\big) \xrightarrow{\delta} H^2\big(Z, \omega_{\mathbb{P}^1} \boxtimes \textstyle\bigwedge^{p+2} M_L \otimes \omega_{\mathbb{P}^1}\big).$$

This was previously asked in [Aprodu et al. 2019, last paragraph on p. 666]. There is a rank-2 vector bundle E on $\mathbb{P}^2$ such that $B = \mathbb{P}(E)$. If $\pi : B \to \mathbb{P}^2$ is the canonical fibration, then $Q := \pi(\widetilde{T})$ is a smooth conic and $\widetilde{T} = \mathbb{P}(E|_Q)$. It is easy to check that $\alpha = \mathrm{id}_{S^q V} \otimes \Delta$, where Δ is the dual of the restriction map $H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(p+1)) \to H^0(Q, \mathscr{O}_Q(p+1))$. Put $M_E := \pi_* M_H$ and $\sigma := \pi|_Z$. The map δ is naturally factored as

$$\begin{array}{ccccc}
H^2\big(B, \bigwedge^{p+2} M_H \otimes \omega_B(Z)\big) & \xrightarrow{\mathrm{id}_{S^q V} \otimes \iota} & H^2\big(Z, \sigma^* \bigwedge^{p+2} M_E \otimes (\omega_{\mathbb{P}^1} \boxtimes \omega_{\mathbb{P}^1})\big) & \xrightarrow{m \otimes \mathrm{id}_V} & H^2\big(Z, \omega_{\mathbb{P}^1} \boxtimes \bigwedge^{p+2} M_L \otimes \omega_{\mathbb{P}^1}\big) \\
\| & & \| & & \| \\
S^q V \otimes \bigwedge^2 V & & S^q V \otimes V \otimes V & & S^{q+1} V \otimes V
\end{array}$$

where ι is the canonical injection and m is the multiplication map identified with

$$H^1(\mathbb{P}^{p+2} \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^{p+2}}(q) \boxtimes \omega_{\mathbb{P}^1}(-p-2)) \longrightarrow H^1(\mathbb{P}^{p+2} \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^{p+2}}(q+1) \boxtimes \omega_{\mathbb{P}^1}).$$

Without any lengthy computation, we quickly obtain the key part of [Aprodu et al. 2019] — the descriptions of the maps in (1-1) (see Lemma 3.1). This provides a conceptual explanation of the difficult computation in [loc. cit.] and a geometric understanding of syzygies of T in $\mathbb{P}^g$ as expected in [loc. cit., last paragraph on p. 666]. Next, regarding $V = H^0(\mathbb{P}^{p+2}, \mathscr{O}_{\mathbb{P}^{p+2}}(1))$, we give a direct proof of (1-2) using vector bundles on $\mathbb{P}^{p+2}$. This part of the proof is largely equivalent to the original proof in [Aprodu et al. 2019; 2022].[1] Put $M_V := M_{\mathscr{O}_{\mathbb{P}^{p+2}}(1)}$. Then (1-1) can be identified with

$$W \otimes H^0(\mathbb{P}^{p+2}, \mathscr{O}_{\mathbb{P}^{p+2}}(q)) \longrightarrow \textstyle\bigwedge^2 V \otimes H^0(\mathbb{P}^{p+2}, \mathscr{O}_{\mathbb{P}^{p+2}}(q)) \longrightarrow H^0(\mathbb{P}^{p+2}, M_V(q+1)).$$

Under the assumption $\mathrm{char}(\Bbbk) = 0$ or $\mathrm{char}(\Bbbk) \geq (g+2)/2$, we show that $W \otimes \mathscr{O}_{\mathbb{P}^{p+2}} \to M_V(1)$ is surjective and its kernel is $(p+1)$-regular in the sense of Castelnuovo and Mumford. This implies (1-2). Here the characteristic assumption plays a crucial role (see Remark 3.3).

It is worth noting that the characteristic assumption on the field $\Bbbk$ in Theorem 1.1 cannot be improved. If $2 \leq \mathrm{char}(\Bbbk) \leq (g+1)/2$, then $K_{\lfloor (g-3)/2 \rfloor, 2}(T, \mathscr{O}_T(1)) \neq 0$ (see [Aprodu et al. 2019, Remark 5.17]). Theorem 1.1 implies the generic Green conjecture for general curves of genus g when $\mathrm{char}(\Bbbk) = 0$ or $\mathrm{char}(\Bbbk) \geq (g+2)/2$, but this is not optimal. Raicu and Sam [2022, Theorem 1.5] recently obtained a sharp result that Green's conjecture holds for general curves of genus g when $\mathrm{char}(\Bbbk) = 0$ or $\mathrm{char}(\Bbbk) \geq \lfloor (g-1)/2 \rfloor$. This confirms a conjecture of Eisenbud and Schreyer [2019, Conjecture 1.1]. For the failure of Green's

[1] After writing the paper, the author learned from Claudiu Raicu [2021] that a similar argument proving (1-2) directly on projective spaces is given in his lecture notes based on Robert Lazarsfeld's suggestion.

conjecture in small characteristic, see [Eisenbud and Schreyer 2019; Schreyer 1986]. It has long been known that the generic Green conjecture would follow from the canonical ribbon conjecture [Bayer and Eisenbud 1995]. A *canonical ribbon* is a hyperplane section of a *K*3 *carpet* $X = X(a, b) \subseteq \mathbb{P}^{a+b+1}$ for integers $b \geq a \geq 1$, which is a unique double structure on a rational normal surface scroll $S(a, b) \subseteq \mathbb{P}^{a+b+1}$ of type (a, b) such that $\omega_X = \mathscr{O}_X$ and $h^1(X, \mathscr{O}_X) = 0$ (see [Gallego and Purnaprajna 1997, Theorem 1.3]). The K3 carpet X is a degeneration of a K3 surface of degree $2(a + b)$, and a canonical ribbon is a degeneration of a general canonical curve C of genus $a+b+1$ with $\mathrm{Cliff}(C) = a$. By extending algebraic arguments of [Aprodu et al. 2019], Raicu and Sam [2022, Theorem 1.1] proved $K_{p,2}(X, \mathscr{O}_X(1)) = 0$ for $0 \leq p \leq a-1$ when $\mathrm{char}(\Bbbk) = 0$ or $\mathrm{char}(\Bbbk) \geq a$. This implies the canonical ribbon conjecture and hence the generic Green conjecture (see [loc. cit., Section 6]). However, for settling Eisenbud and Schreyer conjecture, we only need to consider the case $a = \lfloor (g-1)/2 \rfloor$ and $b = \lfloor g/2 \rfloor$, and we recover [loc. cit., Theorem 1.1] for this case here.

Theorem 1.2. *Let $X = X(\lfloor (g-1)/2 \rfloor, \lfloor g/2 \rfloor) \subseteq \mathbb{P}^g$ be a K3 carpet with $g \geq 3$ over an algebraically closed field $\Bbbk$ with $\mathrm{char}(\Bbbk) = 0$ or $\mathrm{char}(\Bbbk) \geq \lfloor (g-1)/2 \rfloor$ and $\mathrm{char}(\Bbbk) \neq 2$. Then*

$$K_{p,2}(X, \mathscr{O}_X(1)) = 0 \quad \text{for } 0 \leq p \leq \lfloor (g-3)/2 \rfloor.$$

In particular, Green's conjecture holds for general curves of genus g over $\Bbbk$.

Recall that Schreyer [1986] verified Green's conjecture for every curve of genus $g \leq 6$ over an algebraically closed field of arbitrary characteristic. In the theorem, if $g \geq 7$, then the condition $\mathrm{char}(\Bbbk) \neq 2$ is redundant. Our proof of Theorem 1.2 is essentially different from that of [Raicu and Sam 2022] but surprisingly the same as that of Theorem 1.1. The key point is that the K3 carpet X in the theorem is a Weil divisor linearly equivalent to the tangent-developable surface T on the secant variety Σ. Thus X is a degeneration of the tangent surface T. The proof of Theorem 1.1 works for X, and the characteristic assumption on $\Bbbk$ for (1-2) with another W can be improved. Consequently, we obtain a very quick proof of the generic Green conjecture and Eisenbud and Schreyer's conjecture.

In view of Theorem 1.1, it is quite natural to study syzygies of tangent-developable surfaces of smooth projective curves of genus $g \geq 1$. However, there was no research in this direction to the best of the author's knowledge. As a first step, we show the arithmetic normality, and compute the Castelnuovo–Mumford regularity.

Theorem 1.3. *Let C be a smooth projective curve of genus $g \geq 1$ over an algebraically closed field of characteristic zero, L be a line bundle on C with $\deg L \geq 4g+3$, and T be the tangent-developable surface of C embedded in $\mathbb{P}^r$ by $|L|$. Then $T \subseteq \mathbb{P}^r$ is arithmetically normal but not arithmetically Cohen–Macaulay, and $H^i(T, \mathscr{O}_T(m)) = 0$ for $i > 0$, $m > 0$ but $H^1(T, \mathscr{O}_T) \neq 0$, $H^2(T, \mathscr{O}_T) \neq 0$. In particular, $\mathrm{reg}\, \mathscr{O}_T = 3$, and $\mathrm{reg}\, \mathscr{I}_{T|\mathbb{P}^r} = 4$.*

To prove the theorem, we develop new techniques based on the methods for secant varieties in [Ein et al. 2020]. Along with our proof of Theorem 1.1, this provides a general framework for syzygies of tangent-developable surfaces. First, we show that the dualizing sheaf ω_T is trivial (Proposition 2.5). The hard part of the proof of Theorem 1.3 is to check the 2-normality of $T \subseteq \mathbb{P}^r$, which is turned out to be equivalent to $H^1(C \times C, (L \boxtimes L)(-3D)) = 0$, where D is the diagonal of $C \times C$. This cohomology vanishing was established by Bertram, Ein and Lazarsfeld [Bertram et al. 1991] when $\deg L \geq 4g+3$. This degree condition is optimal: If $\deg L = 4g+2$, then $T \subseteq \mathbb{P}^r$ is arithmetically normal if and only if C is neither elliptic nor hyperelliptic (see Remark 4.4). We will discuss some conjectures on syzygies of tangent-developable surfaces at the end of Section 4.

After setting notation and presenting basic facts in Section 2, we prove Theorems 1.1 and 1.2 in Section 3 and Theorem 1.3 in Section 4. Throughout the paper, we work over an algebraically closed field $\mathbb{k}$ of arbitrary characteristic unless otherwise stated.

2. Preliminaries

2.1. ***Syzygies.*** Let X be a projective scheme, B be a coherent sheaf on X, and L be a very ample line bundle on X. Put $V := H^0(X, L)$. The *Koszul cohomology* $K_{p,q}(X, B; L)$ is the cohomology of the Koszul-type complex

$$\bigwedge^{p+1} V \otimes H^0(X, B \otimes L^{q-1}) \longrightarrow \bigwedge^{p} V \otimes H^0(X, B \otimes L^{q}) \longrightarrow \bigwedge^{p-1} V \otimes H^0(X, B \otimes L^{q+1}).$$

When $B = \mathscr{O}_X$, we put $K_{p,q}(X, L) := K_{p,q}(X, \mathscr{O}_X; L)$ and $\kappa_{p,q}(X, L) := \dim_{\mathbb{k}} K_{p,q}(X, L)$. Let $S := \bigoplus_{m \geq 0} S^m V$. Then the graded S-module $R(X, B; L) := \bigoplus_{m \in \mathbb{Z}} H^0(X, B \otimes L^m)$ admits a minimal free resolution

$$0 \longleftarrow R(X, B; L) \longleftarrow E_0 \longleftarrow E_1 \longleftarrow \cdots \longleftarrow E_r \longleftarrow 0,$$

where $E_p = \bigoplus_q K_{p,q}(X, B; L) \otimes_{\mathbb{k}} S(-p-q)$. We may think that $K_{p,q}(X, B; L)$ is the space of p-th syzygies of weight q. For a globally generated vector bundle E on X, we denote by M_E the kernel of the evaluation map $H^0(X, E) \otimes \mathscr{O}_X \to E$. The following is well known.

Proposition 2.1 (see [Park 2022, Proposition 2.1]). *Assume that $H^i(X, B \otimes L^m) = 0$ for $i > 0$ and $m > 0$. For $q \geq 2$, we have $K_{p,q}(X, B; L) = H^{q-1}\big(X, \bigwedge^{p+q-1} M_L \otimes B \otimes L\big)$. If furthermore $H^{q-1}(X, B) = H^q(X, B) = 0$, then $K_{p,q}(X, B; L) = H^q\big(X, \bigwedge^{p+q} M_L \otimes B\big)$.*

2.2. ***Castelnuovo–Mumford regularity.*** A coherent sheaf $\mathscr{F}$ on $\mathbb{P}^n$ is said to be *m-regular* if

$$H^i(\mathbb{P}^n, \mathscr{F}(m-i)) = 0$$

for $i > 0$. By Mumford's theorem [Lazarsfeld 2004, Theorem 1.8.3], if $\mathscr{F}$ is m-regular, then $\mathscr{F}$ is $(m+1)$-regular. The *Castelnuovo–Mumford regularity* $\operatorname{reg} \mathscr{F}$ is the minimum m such that $\mathscr{F}$ is m-regular. If $\mathscr{F}$ fits into an exact sequence $\cdots \to \mathscr{F}_2 \to \mathscr{F}_1 \to \mathscr{F}_0 \to \mathscr{F} \to 0$ of coherent sheaves on $\mathbb{P}^n$ and $\mathscr{F}_i$ is $(m+i)$-regular for each $i \geq 0$, then $\mathscr{F}$ is m-regular [loc. cit., Example 1.8.7]. If $\mathscr{F}$ is m-regular and E is an m'-regular vector bundle on $\mathbb{P}^n$, then $\mathscr{F} \otimes E$ is $(m+m')$-regular [loc. cit., Proposition 1.8.9]. We can think of the regularity of a coherent sheaf $\mathscr{G}$ on a closed subscheme $X \subseteq \mathbb{P}^n$ by regarding $\mathscr{G}$ as a sheaf on $\mathbb{P}^n$.

2.3. ***Multilinear algebra.*** Let V be a finite-dimensional vector space over $\Bbbk$. The symmetric group $\mathfrak{S}_n$ naturally acts on $V^{\otimes n}$ by permuting the factors. The *divided power* $D^n V$ is the subspace

$$\{\omega \in V^{\otimes n} \mid \sigma(\omega) = \omega \text{ for all } \sigma \in \mathfrak{S}_n\} \subseteq T^n V,$$

and the *symmetric power* $S^n V$ is the quotient of $V^{\otimes n}$ by the span of $\sigma(\omega) - \omega$ for all $\omega \in V^{\otimes n}$ and $\sigma \in \mathfrak{S}_n$. We have a natural identification $D^n V = (S^n V^\vee)^\vee$. By composing the inclusion of $D^n V$ into $V^{\otimes n}$ with the projection onto $S^n V$, we get a natural map $D^n V \to S^n V$. This is an isomorphism if $\operatorname{char}(\Bbbk) = 0$ or $\operatorname{char}(\Bbbk) > n$, but it may be neither injective nor surjective in general. The *wedge product* $\bigwedge^n V$ is the quotient of $V^{\otimes n}$ by the span of $v_1 \otimes \cdots \otimes v_n$ for all $v_1, \dots, v_n \in V$ with $v_i = v_j$ for some $i \neq j$. We write $v_1 \wedge \cdots \wedge v_n$ for the class of $v_1 \otimes \cdots \otimes v_n$ in the quotient. There is a natural inclusion $\bigwedge^n V \to V^{\otimes n}$ given by $v_1 \wedge \cdots \wedge v_n \mapsto \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma(v_1 \otimes \cdots \otimes v_n)$. This gives a splitting of the quotient map $V^{\otimes n} \to \bigwedge^n V$ if and only if $\operatorname{char}(\Bbbk) = 0$ or $\operatorname{char}(\Bbbk) > n$. We refer to [Aprodu et al. 2019, Section 3] for more details.

2.4. ***Projective spaces.*** Throughout the paper, we put $U := H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(1))$, and fix a basis $1, x$ of U. The monomials $1, x, \dots, x^d$ form a basis of $S^d U$, and the divided power monomials $x^{(0)}, x^{(1)}, \dots, x^{(d)}$ form a basis of $D^d U$. Let $1, y$ be the dual basis of $U^\vee$ to $1, x$. There is a natural identification $U^\vee = \bigwedge^2 U \otimes U^\vee = U$ sending $1, y$ to $-x, 1$. Then $H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(d)) = S^d U$ and $H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-d-2)) = D^d U$. Note that $M_{\mathscr{O}_{\mathbb{P}^1}(d)} = S^{d-1} U \otimes \mathscr{O}_{\mathbb{P}^1}(-1)$.

We may regard $\mathbb{P}^n$ as the symmetric product of $\mathbb{P}^1$. By permuting the components, $\mathfrak{S}_n$ acts on the ordinary product $(\mathbb{P}^1)^n$, and the line bundle $\mathscr{O}_{\mathbb{P}^1}(d)^{\boxtimes n}$ on $(\mathbb{P}^1)^n$ descends to a line bundle $T_{n, \mathscr{O}_{\mathbb{P}^1}(d)} = \mathscr{O}_{\mathbb{P}^n}(d)$ on $\mathbb{P}^n$ in such a way that $q_n^* T_{n, \mathscr{O}_{\mathbb{P}^1}(d)} = \mathscr{O}_{\mathbb{P}^1}(d)^{\boxtimes n}$, where $q_n : (\mathbb{P}^1)^n \to \mathbb{P}^n$ is the quotient map. Then $H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) = D^n S^d U$. Since $H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1)) = D^n U$, we get $H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) = S^d D^n U$. This gives *Hermite reciprocity* $D^n S^d U = S^d D^n U$. If the action of $\mathfrak{S}_n$ on $(\mathbb{P}^1)^n$ is alternating, then $\mathscr{O}_{\mathbb{P}^1}(d)^{\boxtimes n}$ descends to $N_{n, \mathscr{O}_{\mathbb{P}^1}(d)} = \mathscr{O}_{\mathbb{P}^n}(d-n+1)$ and $H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d-n+1)) = \bigwedge^n S^d U$. This gives another *Hermite reciprocity*

$$\bigwedge^n S^d U = S^{d-n+1} D^n U. \tag{2-1}$$

See [Aprodu et al. 2019, Remark 3.2] and [Park 2022, Subsection 2.3]. Our Hermite reciprocity coincides with the explicit map constructed in [Aprodu et al. 2019, Section 3] (see [Raicu and Sam 2021]), but we will not use this fact.

Let D_n be the image of the injective map $\mathbb{P}^{n-1} \times \mathbb{P}^1 \to \mathbb{P}^n \times \mathbb{P}^1$ given by $(\xi, z) \mapsto (\xi + z, z)$. Note that the effective divisor D_n on $\mathbb{P}^n \times \mathbb{P}^1$ is defined by

$$\sum_{i=0}^{n} (-1)^i (x^0 \wedge \cdots \wedge \widehat{x^i} \wedge \cdots \wedge x^n) \otimes x^i \in \bigwedge^n S^n U \otimes S^n U = H^0(\mathbb{P}^n \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^n}(1) \boxtimes \mathscr{O}_{\mathbb{P}^1}(n)).$$

Consider the short exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^n}(d-1) \boxtimes \mathscr{O}_{\mathbb{P}^1}(-n) \xrightarrow{\cdot D_n} \mathscr{O}_{\mathbb{P}^n}(d) \boxtimes \mathscr{O}_{\mathbb{P}^1} \longrightarrow \mathscr{O}_{\mathbb{P}^{n-1}}(d) \boxtimes \mathscr{O}_{\mathbb{P}^1}(d) \longrightarrow 0. \tag{2-2}$$

Pushing forward to $\mathbb{P}^1$ yields a short exact sequence

$$0 \longrightarrow \bigwedge^n M_{\mathscr{O}_{\mathbb{P}^1}(d+n-1)} \longrightarrow \bigwedge^n S^{d+n-1} U \otimes \mathscr{O}_{\mathbb{P}^1} \longrightarrow \bigwedge^{n-1} M_{\mathscr{O}_{\mathbb{P}^1}(d+n-1)} \otimes \mathscr{O}_{\mathbb{P}^1}(d+n-1) \longrightarrow 0.$$

On the other hand, D_n can be also defined by

$$\sum_{i=0}^{n}(-1)^i x^{(n-i)} \otimes x^i \in D^n U \otimes S^n U = H^0(\mathbb{P}^n \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^n}(1) \boxtimes \mathscr{O}_{\mathbb{P}^1}(n)).$$

Then we see that the map

$$\begin{array}{ccc} H^1(\mathbb{P}^n \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^n}(d-1) \boxtimes \omega_{\mathbb{P}^1}(-n)) & \xrightarrow{\cdot D_n} & H^1(\mathbb{P}^n \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^n}(d) \boxtimes \omega_{\mathbb{P}^1}) \\ \| & & \| \\ S^{d-1} D^n U \otimes D^n U & & S^d D^n U \end{array} \tag{2-3}$$

is given by $f \otimes x^{(i)} \mapsto (-1)^n f x^{(i)}$. We simply regard this as the multiplication map.

2.5. ***Secant varieties.*** We recall the set-up of [Ein et al. 2020; 2021], and we present some preliminary results. Let C be a smooth projective curve of genus $g \geq 0$, and L be a line bundle on C with $\deg L \geq 2g+3$.[2] We assume $\operatorname{char}(\Bbbk) = 0$ whenever $g \geq 1$. We denote by $C^2 = C \times C$ the ordinary product of C and by $C_2 = C^2/\mathfrak{S}_2$ the symmetric product of C. The quotient map $\sigma : C^2 \to C_2$ is given by $(x, y) \mapsto x+y$. If we regard C_2 as the Hilbert scheme of two points on C, then σ is the universal family. For any line bundle A on C, there is a line bundle T_A on C_2 such that $\sigma^* T_A = A \boxtimes A$. Let D be the diagonal of C^2, and $Q := \sigma(D)$. Then $Q \cong C$ unless $g = 0$ and $\operatorname{char}(\Bbbk) = 2$. We may write $Q = 2\delta$ for some divisor δ on C_2. Note that $\sigma^*\delta = D$ and $\sigma_* \mathscr{O}_{C^2} = \mathscr{O}_{C_2}(-\delta) \oplus \mathscr{O}_C$ (see [Ein et al. 2020, Lemma 3.5]). We can write $K_{C_2} = T_{K_C}(-\delta)$. Consider the *tautological bundle* $E := \sigma_*(\mathscr{O}_C \boxtimes L)$ on C_2. We have $\operatorname{rank} E = 2$ and $\det E = T_L(-\delta)$. Let $B := \mathbb{P}(E)$, and $\pi : B \to C_2$ be the canonical fibration. As $H^0(C_2, E) = H^0(C, L)$ and E is globally generated, $|\mathscr{O}_{\mathbb{P}(E)}(1)|$ induces a map $\beta : B \to \mathbb{P}^r = \mathbb{P}H^0(C, L)$. Then $\Sigma := \beta(B)$ is the *secant variety* of C in $\mathbb{P}^r$, and $\beta : B \to \Sigma$ is the blow-up of Σ along C (see [Ein et al. 2021, Theorem 1.1]). Unless $g = 0$ and $\deg L = 3$ (in this case $\Sigma = \mathbb{P}^3$), $\operatorname{Sing} \Sigma = C$ and $\beta : B \to \Sigma$ is a resolution of singularities. Let $Z := \beta^{-1}(C) \cong C^2$. Then $\pi|_Z : Z \to C_2$ is just σ, and $\beta|_Z : Z = C \times C \to C$ is the second projection:

$$\begin{array}{ccccc} Z = C \times C & \hookrightarrow & B = \mathbb{P}(E) & \xrightarrow{\beta} & \Sigma \subseteq \mathbb{P}^r \\ & \searrow^{\sigma} & \downarrow{\scriptstyle \pi} & & \\ & & C_2 & & \end{array}$$

Theorem 2.2 [Ein et al. 2020, Theorems 1.1 and 1.2]. *$\Sigma \subseteq \mathbb{P}^r$ is arithmetically Cohen–Macaulay. If $g = 0$, then Σ has rational singularities and* $\operatorname{reg} \mathscr{O}_\Sigma = 2$. *If $g \geq 1$ and* $\operatorname{char}(\Bbbk) = 0$, *then Σ has normal Du Bois singularities and* $\operatorname{reg} \mathscr{O}_\Sigma = 4$.

Pick $H \in |\mathscr{O}_{\mathbb{P}(E)}(1)|$. We may write $K_B = -2H + \pi^* T_{K_C+L}(-2\delta)$ and $Z = 2H - \pi^* T_L(-2\delta)$. Take $\bar{S} \in |2\delta|$. Let $\widetilde{S} := \pi^{-1}(\bar{S}) = \mathbb{P}(E|_{\bar{S}})$, and $S := \beta(\widetilde{S})$. We are mostly interested in the case $\bar{S} = Q$. In fact, Q is a unique member in $|2\delta|$ when $g \geq 2$. Note that $\dim |2\delta| = 5$ when $g = 0$ and $\dim |2\delta| = 1$ when $g = 1$. Assume that $\operatorname{char}(\Bbbk) \neq 2$ when $g = 0$. Note that $M_E|_Q = N^\vee_{C|\mathbb{P}^r} \otimes L$ and $E|_Q = \mathscr{P}^1(L)$ is the first

[2] It is assumed that $\operatorname{char}(\Bbbk) = 0$ in [Ein et al. 2020; 2021], but everything works when $g = 0$ and $\operatorname{char}(\Bbbk) \geq 0$ (see also [Raicu and Sam 2021]).

jet bundle. By [Kaji 1986, Corollary 1.18], $\mathscr{P}^1(L)$ is the unique nontrivial extension of L by $\omega_C \otimes L$ when $\mathrm{char}(\Bbbk) \nmid \deg L$. Let $\widetilde{T} := \pi^{-1}(Q)$. Then $T := \beta(\widetilde{T})$ is the *tangent-developable surface* of C in $\mathbb{P}^r$, and T has cuspidal singularities along C. Note that $\nu := \beta|_{\widetilde{T}} : \widetilde{T} \to T$ is a resolution of singularities and T is Cohen–Macaulay.

Proposition 2.3. $\deg S = 2 \deg L + 2g - 2$.

Proof. We have $\deg S = (H|_{\widetilde{S}})^2 = (\det E) \cdot \bar{S} = 2 \deg L + 2g - 2$. □

Recall from [Ein et al. 2020, Theorem 5.2] that $\beta_* \mathscr{O}_B(-Z) = \mathscr{I}_{C|\Sigma}$ and $R^1\beta_*\mathscr{O}_B(-Z) = 0$.

Lemma 2.4. *$\beta_*\mathscr{O}_B(-\widetilde{S} - Z) = \mathscr{I}_{S|\Sigma}$ and $R^1\beta_*\mathscr{O}_B(-\widetilde{S} - Z) = 0$.*

Proof. As $\beta_*\mathscr{O}_Z(-\widetilde{S}) = 0$, we have $\beta_*\mathscr{O}_B(-\widetilde{S} - Z) = \beta_*\mathscr{O}_B(-\widetilde{S}) = \mathscr{I}_{S|\Sigma}$. For the second assertion, following [loc. cit., proof of Theorem 5.2(2)], we show that $R^1\beta_*\mathscr{O}_B(-\widetilde{S} - Z)_x = 0$ for any $x \in C \subseteq \Sigma$. Let $F := \beta^{-1}(x) \cong C$. By the formal function theorem, it suffices to prove that

$$H^1(F, \mathscr{O}_B(-\widetilde{S} - Z) \otimes \mathscr{O}_B/\mathscr{I}^m_{F|B}) = 0 \quad \text{for } m \geq 1.$$

It is enough to check that

$$H^1(F, \mathscr{O}_B(-\widetilde{S} - Z) \otimes \mathscr{I}^m_{F|B}/\mathscr{I}^{m+1}_{F|B}) = 0 \quad \text{for } m \geq 0.$$

As $(-\widetilde{S} - Z)|_Z = (L \boxtimes -L)(-4D)$ and F is a fiber of the second projection $Z = C \times C \to C$, we have $\mathscr{O}_B(-\widetilde{S} - Z)|_F = L(-4x)$. Note that $\mathscr{I}^m_{F|B}/\mathscr{I}^{m+1}_{F|B} = S^m N^\vee_{F|B}$. Recall from [Ein et al. 2020, Proposition 3.13] that $N^\vee_{F|B} = \mathscr{O}_C \oplus L(-2x)$. The problem is then reduced to verifying that

$$H^1(C, (m+1)L + (-4 - 2m)x) = 0 \quad \text{for } m \geq 0.$$

This vanishing holds since $\deg((m+1)L + (-4-2m)x) \geq (m+1)(2g+3) - 4 - 2m \geq 2g - 1$. □

Consider the following commutative diagram with short exact sequences:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{O}_B(-\widetilde{S} - Z) & \xrightarrow{\cdot Z} & \mathscr{O}_B(-\widetilde{S}) & \longrightarrow & \mathscr{O}_Z(-2D) & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle \cdot\widetilde{S}} & & \downarrow & & \\
0 & \longrightarrow & \mathscr{O}_B(-\widetilde{S} - Z) & \xrightarrow[\cdot \widetilde{S}+Z]{} & \mathscr{O}_B & \longrightarrow & \mathscr{O}_{\widetilde{S}+Z} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathscr{O}_{\widetilde{S}} & = & \mathscr{O}_{\widetilde{S}} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{2-4}$$

By Lemma 2.4, applying β_* to the second-row exact sequence in (2-4), we find $\beta_*\mathscr{O}_{\widetilde{S}+Z} = \mathscr{O}_S$ and $R^1\beta_*\mathscr{O}_{\widetilde{S}+Z} = R^1\beta_*\mathscr{O}_B = H^1(C, \mathscr{O}_C) \otimes \mathscr{O}_C$, and we get a short exact sequence

$$0 \longrightarrow \beta_*\mathscr{O}_B(-\widetilde{S} - Z) \longrightarrow \mathscr{O}_\Sigma \longrightarrow \mathscr{O}_S \longrightarrow 0. \tag{2-5}$$

Note that

$$R^1\beta_*\mathscr{O}_{\widetilde{S}+Z} = R^1\beta_*\mathscr{O}_B = R^1(\beta|_Z)_*\mathscr{O}_Z = R^1(\beta|_Z)_*\mathscr{O}_Z(-D).$$

The kernel of the map $R^1(\beta|_Z)_*\mathscr{O}_Z(-2D) \to R^1(\beta|_Z)_*\mathscr{O}_Z(-D)$ is $\mathscr{O}_D(-D|_D) = \omega_C$. By applying β_* to the rightmost vertical exact sequence in (2-4), we get a short exact sequence

$$0 \longrightarrow \mathscr{O}_S \longrightarrow \beta_*\mathscr{O}_{\widetilde{S}} \longrightarrow \omega_C \longrightarrow 0. \tag{2-6}$$

Proposition 2.5. *The dualizing sheaf ω_S is trivial.*

Proof. Consider two short exact sequences

$$0 \longrightarrow \omega_B \longrightarrow \omega_B(Z) \longrightarrow \omega_Z \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \omega_B \longrightarrow \omega_B(\widetilde{S}) \longrightarrow \omega_{\widetilde{S}} \longrightarrow 0. \tag{2-7}$$

By Theorem 2.2 for $g = 0$ and Grauert–Riemenschneider vanishing for $g \geq 1$, we have $R^1\beta_*\omega_B = 0$. Then $R^1\beta_*\omega_B(Z) = R^1\beta_*\omega_Z = \omega_C$ and $R^1\beta_*\omega_B(\widetilde{S}) = 0$. By taking $-\otimes \mathscr{O}_B(\widetilde{S})$ to the left of (2-7), we see that $R^1\beta_*\omega_B(\widetilde{S}+Z) = 0$. When $g = 0$, we have $\beta_*\omega_B(Z) = \beta_*\omega_B = \omega_\Sigma$ by Theorem 2.2. When $g \geq 1$, Theorem 2.2 and [Kovács et al. 2010, Theorem 1.1] show that $\beta_*\omega_B(Z) = \omega_\Sigma$. As $\omega_{\widetilde{S}}(Z|_{\widetilde{S}}) = \mathscr{O}_{\widetilde{S}}$, applying β_* to the short exact sequence with the consideration of (2-6),

$$0 \longrightarrow \omega_B(Z) \xrightarrow{\cdot\widetilde{S}} \omega_B(\widetilde{S}+Z) \longrightarrow \mathscr{O}_{\widetilde{S}} \longrightarrow 0,$$

we obtain a short exact sequence

$$0 \longrightarrow \omega_\Sigma \longrightarrow \beta_*\omega_B(\widetilde{S}+Z) \longrightarrow \mathscr{O}_S \longrightarrow 0.$$

On the other hand, applying $\mathscr{H}om_\Sigma(-, \omega_\Sigma)$ to (2-5), we get a short exact sequence

$$0 \longrightarrow \omega_\Sigma \longrightarrow \mathscr{H}om_\Sigma(\beta_*\mathscr{O}_B(-\widetilde{S}-Z), \omega_\Sigma) \longrightarrow \omega_S \longrightarrow 0$$

since Σ is Cohen–Macaulay. Notice that

$$\mathscr{H}om_\Sigma(\beta_*\mathscr{O}_B(-\widetilde{S}-Z), \omega_\Sigma) = \mathscr{H}om_\Sigma(\beta_*\mathscr{O}_B(-\widetilde{S}), \omega_\Sigma) = \beta_*\omega_B(\widetilde{S}+Z).$$

Hence we conclude that $\omega_S = \mathscr{O}_S$. □

Remark 2.6. When $g = 0$ and $S = T$ is the tangent-developable surface, the above proposition was shown in [Aprodu et al. 2019, Corollary 5.16]. When $g = 1$ and $\deg L = 5$, applying the Serre construction to the tangent-developable surface T in $\mathbb{P}^4$, Hulek [1986, Chapter VII] gave a new construction of the Horrocks–Mumford bundle on $\mathbb{P}^4$.

2.6. ***Rational curve case.*** Assume that $\operatorname{char}(\Bbbk) \neq 2$. Let $C \subseteq \mathbb{P}^g$ be a rational normal curve of degree $g \geq 3$, and $L := \mathscr{O}_{\mathbb{P}^1}(g)$. When we consider a rational normal curve, g is not the genus but the degree. Note that $C_2 = \mathbb{P}^2$ and $\mathscr{O}_{\mathbb{P}^2}(\delta) = \mathscr{O}_{\mathbb{P}^2}(1)$. We have $T_{\mathscr{O}_{\mathbb{P}^1}(d)} = \mathscr{O}_{\mathbb{P}^2}(d)$, and put $N_{\mathscr{O}_{\mathbb{P}^1}(d)} := T_{\mathscr{O}_{\mathbb{P}^1}(d)}(-\delta) = \mathscr{O}_{\mathbb{P}^2}(d-1)$.

Then $\sigma^* T_{\mathscr{O}_{\mathbb{P}^1}(d)} = \mathscr{O}_{\mathbb{P}^1}(d) \boxtimes \mathscr{O}_{\mathbb{P}^1}(d)$ and $\sigma^* N_{\mathscr{O}_{\mathbb{P}^1}(d)} = \mathscr{O}_{\mathbb{P}^1}(d-1) \boxtimes \mathscr{O}_{\mathbb{P}^1}(d-1)$. Note that $\pi_* \mathscr{O}_B(-Z) = 0$ and $R^1 \pi_* \mathscr{O}_B(-Z) = \mathscr{O}_{\mathbb{P}^2}(-\delta)$. By applying π_* to the short exact sequence

$$0 \longrightarrow \mathscr{O}_B(-Z) \xrightarrow{\cdot Z} \mathscr{O}_B \longrightarrow \mathscr{O}_Z \longrightarrow 0, \tag{2-8}$$

we get a splitting short exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2} \longrightarrow \sigma_* \mathscr{O}_Z \longrightarrow \mathscr{O}_{\mathbb{P}^2}(-\delta) \longrightarrow 0.$$

Then we obtain the following canonically splitting short exact sequence:

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^0(\mathbb{P}^2, T_{\mathscr{O}_{\mathbb{P}^1}(d)}) & \longrightarrow & H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(d) \boxtimes \mathscr{O}_{\mathbb{P}^1}(d)) & \longrightarrow & H^0(\mathbb{P}^2, N_{\mathscr{O}_{\mathbb{P}^1}(d)}) & \longrightarrow 0 \\ & \| & & \| & & \| & \\ & D^2 S^d U & & S^d U \otimes S^d U & & \bigwedge^2 S^d U & \end{array} \tag{2-9}$$

Lemma 2.7. $M_E = S^{g-2} U \otimes \mathscr{O}_{\mathbb{P}^2}(-1)$.

Proof. Recall that there is an injective map $D_2 = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2 \times \mathbb{P}^1$. Let $p_1 : \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^2$ be the first projection. Then $(p_1)|_{D_2} = \sigma$. By applying $p_{1,*}$ to the short exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(-1) \boxtimes \mathscr{O}_{\mathbb{P}^1}(g-2) \xrightarrow{\cdot D_2} \mathscr{O}_{\mathbb{P}^2} \boxtimes \mathscr{O}_{\mathbb{P}^1}(g) \longrightarrow \mathscr{O}_{\mathbb{P}^1} \boxtimes \mathscr{O}_{\mathbb{P}^1}(g) \longrightarrow 0,$$

we get a short exact sequence

$$0 \longrightarrow S^{g-2} U \otimes \mathscr{O}_{\mathbb{P}^2}(-1) \longrightarrow S^g U \otimes \mathscr{O}_{\mathbb{P}^2} \longrightarrow E \longrightarrow 0,$$

where $H^0(\mathbb{P}^2, E) \otimes \mathscr{O}_{\mathbb{P}^2} = S^g U \otimes \mathscr{O}_{\mathbb{P}^2} \to E$ is the evaluation map. Thus the lemma follows. □

Proposition 2.8. (1) $K_B + \widetilde{S} + Z = 0$.

(2) $S \subseteq \mathbb{P}^g$ *is arithmetically Cohen–Macaulay and* reg $\mathscr{O}_S = 3$.

(3) *The Hilbert function of* $S \subseteq \mathbb{P}^g$ *is given by* $H_S(t) = (g-1)t^2 + 2$ *for* $t \geq 1$.

Proof. The assertion (1) follows from the direct computation

$$K_B + \widetilde{S} + Z = (-2H + \pi^* \mathscr{O}_{\mathbb{P}^2}(g-4)) + \pi^* \mathscr{O}_{\mathbb{P}^2}(2) + (2H - \pi^* \mathscr{O}_{\mathbb{P}^2}(g-2)) = 0.$$

As $\beta_* \mathscr{O}_B(-\widetilde{S} - Z) = \beta_* \omega_B = \omega_\Sigma$ by Theorem 2.2, the short exact sequence (2-5) is

$$0 \longrightarrow \omega_\Sigma \longrightarrow \mathscr{O}_\Sigma \longrightarrow \mathscr{O}_S \longrightarrow 0. \tag{2-10}$$

By Theorem 2.2, we readily obtain (2). Now, $H_T(t)$ agrees with the Hilbert polynomial $P_S(t)$ for $t \geq 1$. Note that det $P_S(t) = 2$ and the leading coefficient of $P_T(t)$ is $(\deg S)/2 = g-1$. As $P_S(0) = \chi(\mathscr{O}_S) = 2$ and $P_S(1) = g+1$, we get (3). □

When $S = T$ is the tangent-developable surface, Proposition 2.8(2) was first shown by Schreyer [1983, Proposition 6.1] (see also [Aprodu et al. 2019, Theorem 5.1]). Observe that $S \subseteq \mathbb{P}^g$ has the same Hilbert polynomial with a K3 surface of degree $2g-2$ in $\mathbb{P}^g$.

Suppose that $\bar{S} = Q$. Here $Q = \sigma(D)$ is a smooth conic in $\mathbb{P}^2$. When $\mathrm{char}(\Bbbk) = 0$ or $\mathrm{char}(\Bbbk) \geq (g+2)/2$, [Kaji 1986, Corollary 1.18] implies that $E|_Q = \mathscr{O}_{\mathbb{P}^1}(g-1) \oplus \mathscr{O}_{\mathbb{P}^1}(g-1)$. In this case, $\widetilde{T} = \mathbb{P}^1 \times \mathbb{P}^1$, and $\mathscr{O}_{\widetilde{T}}(H|_{\widetilde{T}}) = \mathscr{O}_{\mathbb{P}^1}(g-1) \boxtimes \mathscr{O}_{\mathbb{P}^1}(1)$. When $\mathrm{char}(\Bbbk) \mid g$, [loc. cit., Corollary 1.18] implies that $\widetilde{T} \neq \mathbb{P}^1 \times \mathbb{P}^1$. However, we always have $Z = \mathbb{P}^1 \times \mathbb{P}^1$. Now, consider the following short exact sequence:

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(d-1)) & \xrightarrow{\cdot Q} & H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(d+1)) & \longrightarrow & H^0(Q, \mathscr{O}_Q(d+1)) & \longrightarrow 0 \\ & \| & & \| & & \| & \\ & \bigwedge^2 S^d U & & \bigwedge^2 S^{d+2} U & & S^{2d+2} U & \end{array} \tag{2-11}$$

Since $\sigma^* Q = 2D$ and the diagonal D of $\mathbb{P}^1 \times \mathbb{P}^1$ is defined by

$$x \otimes 1 - 1 \otimes x \in U \otimes U = H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(1) \boxtimes \mathscr{O}_{\mathbb{P}^1}(1)),$$

the inclusion $H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(d-1)) \to H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(d+1))$ in (2-11) is

$$\bigwedge^2 S^d U \longrightarrow \bigwedge^2 S^{d+2} U, \quad x^i \wedge x^j \longmapsto x^{i+2} \wedge x^j - 2x^{i+1} \wedge x^{j+1} + x^i \wedge x^{j+2}.$$

The surjection $H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(d+1)) \to H^0(Q, \mathscr{O}_Q(d+1))$ in (2-11) is

$$\mu_{d+2} : \bigwedge^2 S^{d+2} U \longrightarrow S^{2d+2} U, \quad x^i \wedge x^j \longmapsto (i-j) x^{i+j-1}, \tag{2-12}$$

which is the *Wahl map* (or the *Gaussian map*). See [Bertram et al. 1991; Wahl 1990] for details on Wahl maps.

Suppose that $\bar{S} = 2\ell$, where $\ell \subseteq \mathbb{P}^2$ is a line meeting Q at two distinct points. We may write $E|_\ell = \mathscr{O}_{\mathbb{P}^1}(a) \oplus \mathscr{O}_{\mathbb{P}^1}(b)$ for some integers $b \geq a \geq 1$. As $\det E = N_L = \mathscr{O}_{\mathbb{P}^2}(g-1)$, we have $a + b = g - 1$. Note that $\sigma^{-1}(\ell) = \mathbb{P}^1$ and $\sigma|_{\sigma^{-1}(\ell)} : \sigma^{-1}(\ell) \to \ell$ is a two-to-one map. By restricting the short exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1}(g-1) \boxtimes \mathscr{O}_{\mathbb{P}^1}(-1) \longrightarrow \sigma^* E \longrightarrow \mathscr{O}_{\mathbb{P}^1} \boxtimes \mathscr{O}_{\mathbb{P}^1}(g) \longrightarrow 0$$

to $\sigma^{-1}(\ell)$, we get a short exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1}(g-2) \longrightarrow \mathscr{O}_{\mathbb{P}^1}(2a) \oplus \mathscr{O}_{\mathbb{P}^1}(2b) \longrightarrow \mathscr{O}_{\mathbb{P}^1}(g) \longrightarrow 0.$$

It follows that $(a, b) = (\lfloor (g-1)/2 \rfloor, \lfloor g/2 \rfloor)$. Then S is a double structure on a rational normal surface scroll $S(a, b) = \mathbb{P}(E|_\ell) \subseteq \mathbb{P}^g$. Recall that $H^1(S, \mathscr{O}_S) = 0$ and $\omega_S = \mathscr{O}_S$. Thus $S = X(\lfloor (g-1)/2 \rfloor, \lfloor g/2 \rfloor) \subseteq \mathbb{P}^g$ is a *K*3 *carpet* (see [Gallego and Purnaprajna 1997, Definition 1.2 and Theorem 1.3][3]). On the other hand, consider the following short exact sequence:

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(d-1)) & \xrightarrow{\cdot 2\ell} & H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(d+1)) & \longrightarrow & H^0(2\ell, \mathscr{O}_{2\ell}(d+1)) & \longrightarrow 0 \\ & \| & & \| & & & \\ & \bigwedge^2 S^d U & & \bigwedge^2 S^{d+2} U & & & \end{array} \tag{2-13}$$

[3]The proof of [Gallego and Purnaprajna 1997, Theorem 1.3] also works in positive characteristic.

We may assume that $\sigma^{-1}(\ell)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ is defined by

$$x \otimes 1 + 1 \otimes x \in U \otimes U = H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(1) \boxtimes \mathscr{O}_{\mathbb{P}^1}(1)).$$

Then the inclusion $H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(d-1)) \to H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(d+1))$ in (2-13) is

$$\textstyle\bigwedge^2 S^d U \longrightarrow \bigwedge^2 S^{d+2} U, \quad x^i \wedge x^j \longmapsto x^{i+2} \wedge x^j + 2x^{i+1} \wedge x^{j+1} + x^i \wedge x^{j+2}.$$

It is a direct summand of the inclusion

$$H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(d) \boxtimes \mathscr{O}_{\mathbb{P}^1}(d)) \xrightarrow{\cdot 2\sigma^{-1}(\ell)} H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(d+2) \boxtimes \mathscr{O}_{\mathbb{P}^1}(d+2))$$

whose cokernel is $S^{2d+2}U \oplus S^{2d+4}U$. The surjection $H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(d+1)) \to H^0(2\ell, \mathscr{O}_{2\ell}(d+1))$ in (2-13) factors through

$$\tau_{d+2} : \textstyle\bigwedge^2 S^{d+2} U \longrightarrow \begin{array}{c} S^{2d+2}U \\ \oplus \\ S^{2d+4}U \end{array}, \quad x^i \wedge x^j \longmapsto \begin{cases} ((-1)^i (i-j) x^{i+j-1}, 0) & \text{if } i \equiv j \pmod 2, \\ (0, (-1)^i x^{i+j}) & \text{if } i \not\equiv j \pmod 2, \end{cases} \tag{2-14}$$

in such a way that $\mathrm{im}(\tau_{d+2})$ injects into $H^0(2\ell, \mathscr{O}_{2\ell}(d+1))$.

3. Proofs of Theorems 1.1 and 1.2

Assume that $\mathrm{char}(\Bbbk) \neq 2$. Let $C \subseteq \mathbb{P}^g$ be a rational normal curve of degree $g \geq 3$, and $L := \mathscr{O}_{\mathbb{P}^1}(g)$. We use the notation in Section 2. From (2-10), we see that

$$K_{p,2}(S, \mathscr{O}_S(1)) = \mathrm{coker}\big(K_{p,2}(\Sigma, K_\Sigma; \mathscr{O}_\Sigma(1)) \xrightarrow{\rho_{p+2}} K_{p,2}(\Sigma, \mathscr{O}_\Sigma(1))\big).$$

Thus $K_{p,2}(S, \mathscr{O}_S(1)) = 0$ if and only if ρ_{p+2} is surjective. Recall from Theorem 2.2 that Σ has rational singularities. As $\beta^* M_{\mathscr{O}_\Sigma(1)} = M_H$, by Proposition 2.1, we find

$$K_{p,2}(\Sigma, K_\Sigma; \mathscr{O}_\Sigma(1)) = H^2\big(\Sigma, \textstyle\bigwedge^{p+2} M_{\mathscr{O}_\Sigma(1)} \otimes \omega_\Sigma\big) = H^2\big(B, \bigwedge^{p+2} M_H \otimes \omega_B\big),$$
$$K_{p,2}(\Sigma, \mathscr{O}_\Sigma(1)) = H^2\big(\Sigma, \textstyle\bigwedge^{p+2} M_{\mathscr{O}_\Sigma(1)}\big) = H^2\big(B, \bigwedge^{p+2} M_H\big).$$

In (2-4), we have $-\widetilde{S} - Z = K_B$, $-\widetilde{S} = K_B + Z$, $-2D = K_Z$. Then ρ_{p+2} fits into the following commutative diagram with exact sequences induced from (2-4):

$$\begin{array}{ccccc} & & H^1\big(\widetilde{S}, \bigwedge^{p+2} M_H|_{\widetilde{S}}\big) & \overset{\gamma_{p+2}}{\searrow} & \\ & & \alpha_{p+2} \big\downarrow & & \\ H^2\big(B, \bigwedge^{p+2} M_H \otimes \omega_B\big) & \hookrightarrow & H^2\big(B, \bigwedge^{p+2} M_H \otimes \omega_B(Z)\big) & \xrightarrow[\delta_{p+2}]{} & H^2\big(Z, \omega_{\mathbb{P}^1} \boxtimes \bigwedge^{p+2} M_L \otimes \omega_{\mathbb{P}^1}\big) \\ & \underset{\rho_{p+2}}{\searrow} & \big\downarrow & & \\ & & H^2\big(B, \bigwedge^{p+2} M_H\big) & & \end{array} \tag{3-1}$$

It is easy to check that ρ_{p+2} is surjective if and only if $\gamma_{p+2} = \delta_{p+2} \circ \alpha_{p+2}$ surjects onto $\mathrm{im}(\delta_{p+2})$. We shall prove the latter for $0 \leq p \leq \lfloor (g-3)/2 \rfloor$ when $S = T$ is the tangent-developable surface and

$S = X(\lfloor (g-1)/2 \rfloor, \lfloor g/2 \rfloor)$ is a K3 carpet. To this end, we first describe $\gamma_{p+2} = \delta_{p+2} \circ \alpha_{p+2}$. Let $q := g-p-3,\ V := D^{p+2}U,\ W := H^1(\bar{S}, \mathscr{O}_{\bar{S}}(-p-2))$.

Lemma 3.1. *The map γ_{p+2} is the composition*

$$S^q V \otimes W \xrightarrow{\alpha_{p+2}} S^q V \otimes \bigwedge^2 V \xrightarrow{\delta_{p+2}} S^{q+1} V \otimes V$$

such that δ_{p+2} is the Koszul differential given by $f \otimes (x^{(i)} \wedge x^{(j)}) \longmapsto f x^{(i)} \otimes x^{(j)} - f x^{(j)} \otimes x^{(i)}$ and $\alpha_{p+2} = \mathrm{id}_{S^q V} \otimes \Delta_{p+2}$, where Δ_{p+2} fits into the short exact sequence[4]

$$0 \longrightarrow H^1(\bar{S}, \mathscr{O}_{\bar{S}}(-p-2)) \xrightarrow{\Delta_{p+2}} H^2(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(-p-2)(-\bar{S})) \xrightarrow{\cdot \bar{S}} H^2(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(-p-2)) \longrightarrow 0.$$

Proof. First, we give a description of α_{p+2}. As $\pi_* M_H = M_E$, we have a short exact sequence

$$0 \longrightarrow \pi^* M_E \longrightarrow M_H \longrightarrow \mathscr{O}_B(-H) \otimes \pi^* N_L \longrightarrow 0. \tag{3-2}$$

Then we get $\pi_* \bigwedge^{p+2} M_H = \bigwedge^{p+2} M_E$ and $R^1 \pi_* \bigwedge^{p+2} M_H = 0$. By restricting (3-2) to $\widetilde{S}$, we also get $(\pi|_{\widetilde{S}})_* \bigwedge^{p+2} M_H|_{\widetilde{S}} = \bigwedge^{p+2} M_E|_{\bar{S}}$ and $R^1(\pi|_{\widetilde{S}})_* \bigwedge^{p+2} M_H|_{\widetilde{S}} = 0$. Recall from Lemma 2.7 that $M_E = S^{g-2}U \otimes \mathscr{O}_{\mathbb{P}^2}(-1)$. Thus we find

$$\begin{aligned} H^1\big(\widetilde{S}, \textstyle\bigwedge^{p+2} M_H|_{\widetilde{S}}\big) &= H^1\big(\bar{S}, \textstyle\bigwedge^{p+2} M_E|_{\bar{S}}\big) = \textstyle\bigwedge^{p+2} S^{g-2}U \otimes H^1(\bar{S}, \mathscr{O}_{\bar{S}}(-p-2)),\\ H^2\big(B, \textstyle\bigwedge^{p+2} M_H \otimes \omega_B(Z)\big) &= H^2\big(\mathbb{P}^2, \textstyle\bigwedge^{p+2} M_E \otimes \mathscr{O}_{\mathbb{P}^2}(-\bar{S})\big) = \textstyle\bigwedge^{p+2} S^{g-2}U \otimes H^2(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(-p-2)(-\bar{S})),\\ H^2\big(B, \textstyle\bigwedge^{p+2} M_H\big) f &= H^2\big(\mathbb{P}^2, \textstyle\bigwedge^{p+2} M_E\big) = \textstyle\bigwedge^{p+2} S^{g-2}U \otimes H^2(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(-p-2)). \end{aligned}$$

By considering the vertical short exact sequence in (3-1), we obtain $\alpha_{p+2} = \mathrm{id}_{\bigwedge^{p+2} S^{g-2}U} \otimes \Delta_{p+2}$. By Hermite reciprocity (2-1), we have $\bigwedge^{p+2} S^{g-2}U = S^q V$.

To describe δ_{p+2}, we restrict (3-2) to Z to get a short exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & \sigma^* M_E & \longrightarrow & \mathscr{O}_{\mathbb{P}^1} \boxtimes M_L & \longrightarrow & (\mathscr{O}_{\mathbb{P}^1} \boxtimes -L) \otimes \sigma^* N_L & \longrightarrow 0 \\ & \| & & \| & & \| & \\ & M_{\mathscr{O}_{\mathbb{P}^1}(g-1)} \boxtimes \mathscr{O}_{\mathbb{P}^1}(-1) & & S^{g-1}U \otimes \mathscr{O}_{\mathbb{P}^1} \boxtimes \mathscr{O}_{\mathbb{P}^1}(-1) & & \mathscr{O}_{\mathbb{P}^1}(g-1) \boxtimes \mathscr{O}_{\mathbb{P}^1}(-1) & \end{array}$$

where the maps are the identity on $\mathscr{O}_{\mathbb{P}^1}(-1)$ and the map $S^{g-1}U \otimes \mathscr{O}_{\mathbb{P}^1} \to \mathscr{O}_{\mathbb{P}^1}(g-1)$ is the evaluation map. The induced map $H^2\big(Z, \sigma^* \bigwedge^{p+2} M_E \otimes \omega_Z\big) \to H^2\big(Z, \omega_{\mathbb{P}^1} \boxtimes \bigwedge^{p+2} M_L \otimes \omega_{\mathbb{P}^1}\big)$ can be written as $\delta'_{p+2} \otimes \mathrm{id}_V$, where

$$\delta'_{p+2} : H^1\big(\mathbb{P}^1, \textstyle\bigwedge^{p+2} M_{\mathscr{O}_{\mathbb{P}^1}(g-1)} \otimes \omega_{\mathbb{P}^1}\big) \longrightarrow \textstyle\bigwedge^{p+2} S^{g-1}U \otimes H^1(\mathbb{P}^1, \omega_{\mathbb{P}^1})$$

and $V = D^{p+2}U = H^1(\mathbb{P}^1, \omega_{\mathbb{P}^1}(-p-2))$. By considering the push-forward of (2-2) to $\mathbb{P}^1$, we can identify δ'_{p+2} with

$$H^1(\mathbb{P}^{p+2} \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^{p+2}}(g-p-3) \boxtimes \omega_{\mathbb{P}^1}(-p-2)) \xrightarrow{\cdot D_{p+2}} H^1(\mathbb{P}^{p+2} \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^{p+2}}(g-p-2) \boxtimes \omega_{\mathbb{P}^1}).$$

[4] When $S = T$, one can easily confirm that the map γ in the Introduction coincides with the map γ_{p+2} by replacing the range by $\mathrm{im}(\delta_{p+2}) = K_{p,1}(\mathbb{P}^1, \omega_{\mathbb{P}^1}; \mathscr{O}_{\mathbb{P}^1}(g))$. In this case, Δ_{p+2} can be thought of as the dual of the Wahl map μ_{p+2}.

This is the map (2-3) with $n=p+2$ and $d=g-p-2=q+1$, so $\delta'_{p+2}: S^qV\otimes V\to S^{q+1}V$ is the multiplication map, where we use Hermite reciprocity (2-1) to have that

$$H^1\big(\mathbb{P}^1,\textstyle\bigwedge^{p+2}M_{\mathscr{O}_{\mathbb{P}^1}(g-1)}\otimes\omega_{\mathbb{P}^1}\big)=\textstyle\bigwedge^{p+2}S^{g-2}U\otimes D^{p+2}U=S^qV\otimes V,$$
$$\textstyle\bigwedge^{p+2}S^{g-1}U\otimes H^1(\mathbb{P}^1,\omega_{\mathbb{P}^1})=\textstyle\bigwedge^{p+2}S^{g-1}U=S^{q+1}V.$$

Now, consider the following commutative diagram with short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi^*M_E\otimes\mathscr{O}_B(-Z) & \xrightarrow{\cdot Z} & \pi^*M_E & \longrightarrow & \sigma^*M_E & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_H\otimes\mathscr{O}_B(-Z) & \xrightarrow[\cdot Z]{} & M_H & \longrightarrow & \mathscr{O}_{\mathbb{P}^1}\boxtimes M_L & \longrightarrow & 0
\end{array}$$

The top row is $(S^{g-2}U\otimes\pi^*\mathscr{O}_{\mathbb{P}^2}(-1))\otimes$ (2-8). The diagram induces the following commutative diagram:

$$\begin{array}{ccc}
H^2\big(B,\pi^*\textstyle\bigwedge^{p+2}M_E\otimes\omega_B(Z)\big) & \xhookrightarrow{\mathrm{id}_{S^qV}\otimes\iota_V} & H^2\big(Z,\sigma^*\textstyle\bigwedge^{p+2}M_E\otimes\omega_Z\big)\\
\| & & \downarrow{\scriptstyle \delta'_{p+2}\otimes\mathrm{id}_V}\\
H^2\big(B,\textstyle\bigwedge^{p+2}M_H\otimes\omega_B(Z)\big) & \xrightarrow[\delta_{p+2}]{} & H^2\big(Z,\omega_{\mathbb{P}^1}\boxtimes\textstyle\bigwedge^{p+2}M_L\otimes\omega_{\mathbb{P}^1}\big)
\end{array}$$

Recalling $\omega_B(Z)=\pi^*\omega_{\mathbb{P}^2}(1)$ and $\bigwedge^{p+2}S^{g-2}U=S^qV$, we have

$$H^2\big(B,\pi^*\textstyle\bigwedge^{p+2}M_E\otimes\omega_B(Z)\big)=S^qV\otimes H^0(\mathbb{P}^2,\mathscr{O}_{\mathbb{P}^2}(p+1))^\vee,$$
$$H^2\big(Z,\sigma^*\textstyle\bigwedge^{p+2}M_E\otimes\omega_Z\big)=S^qV\otimes H^0(Z,\mathscr{O}_{\mathbb{P}^1}(p+2)\boxtimes\mathscr{O}_{\mathbb{P}^1}(p+2))^\vee.$$

By considering (2-8), we see that the upper horizontal injection in the above commutative diagram can be written as $\mathrm{id}_{S^qV}\otimes\iota_V$, where

$$\iota_V: H^2(\mathbb{P}^2,\omega_{\mathbb{P}^2}(-p-1))\longrightarrow H^2\big(Z,(\mathscr{O}_{\mathbb{P}^1}(-p-2)\boxtimes\mathscr{O}_{\mathbb{P}^1}(-p-2))\otimes\omega_Z\big)$$

is the dual of the canonical surjection in (2-9) with $d=p+2$. In other words, the map $\iota_V:\bigwedge^2V\to V\otimes V$ is given by $\iota_V(x^{(i)}\wedge x^{(j)})=x^{(i)}\otimes x^{(j)}-x^{(j)}\otimes x^{(i)}$. Hence we conclude that $\delta_{p+2}=(\delta'_{p+2}\otimes\mathrm{id}_V)\circ(\mathrm{id}_{S^qV}\otimes\iota_V)$: $S^qV\otimes\bigwedge^2V\to S^{q+1}V\otimes V$ is the Koszul differential. □

From now on, we consider $\gamma_{p+2}:W\otimes S^qV\to\mathrm{im}(\delta_{p+2})$ (the target $S^{q+1}V\otimes V$ is replaced by $\mathrm{im}(\delta_{p+2})$). Our aim is to show the surjectivity of γ_{p+2} for $0\le p\le\lfloor(g-3)/2\rfloor$.[5] Viewing $V=H^0(\mathbb{P}^{p+2},\mathscr{O}_{\mathbb{P}^{p+2}}(1))$ and putting $M_V:=M_{\mathscr{O}_{\mathbb{P}^{p+2}}(1)}$, we have a short exact sequence

$$0\longrightarrow\textstyle\bigwedge^2M_V\longrightarrow\bigwedge^2V\otimes\mathscr{O}_{\mathbb{P}^{p+2}}\longrightarrow M_V(1)\longrightarrow 0.$$

Consider the composition

$$W\otimes\mathscr{O}_{\mathbb{P}^{p+2}}\xrightarrow{\Delta_{p+2}\otimes\mathrm{id}_{\mathscr{O}_{\mathbb{P}^{p+2}}}}\textstyle\bigwedge^2V\otimes\mathscr{O}_{\mathbb{P}^{p+2}}\longrightarrow M_V(1).\tag{3-3}$$

[5] The Koszul complex $S^qV\otimes\bigwedge^2V\to S^{q+1}V\otimes V\to S^{q+2}V$ is exact, so $\mathrm{im}(\delta_{p+2})=\ker(S^{q+1}V\otimes V\to S^{q+2}V)$. Thus the surjectivity of γ_{p+2} for $0\le p\le\lfloor(g-3)/2\rfloor$ is equivalent to (1-2): $W_q(V,W)=0$ for $q=g-p-3\ge p\ge 0$.

Lemma 3.2. *Assume that* $\operatorname{char}(\Bbbk) = 0$ *or* $\operatorname{char}(\Bbbk) \geq \lfloor (g-1)/2 \rfloor$. *If the composition* (3-3) *is surjective, then* γ_{p+2} *is surjective for* $0 \leq p \leq \lfloor (g-3)/2 \rfloor$.

Proof. We can form a short exact sequence

$$0 \longrightarrow K \longrightarrow W \otimes \mathscr{O}_{\mathbb{P}^{p+2}} \xrightarrow{(3\text{-}3)} M_V(1) \longrightarrow 0, \tag{3-4}$$

where K is a vector bundle with $\operatorname{rank} K = p+1$ and $\det K = \mathscr{O}_{\mathbb{P}^{p+2}}(-p-1)$. Notice that γ_{p+2} is the $k = q$ case of the map

$$\gamma'_k : W \otimes H^0(\mathbb{P}^{p+2}, \mathscr{O}_{\mathbb{P}^{p+2}}(k)) \longrightarrow H^0(\mathbb{P}^{p+2}, M_V(k+1)).$$

As $q = g-p-3 \geq p$, it suffices to show the surjectivity of γ'_k for $k \geq p$. To this end, consider the dual of (3-4):

$$0 \longrightarrow M_V^\vee(-1) \longrightarrow W^\vee \otimes \mathscr{O}_{\mathbb{P}^{p+2}} \longrightarrow K^\vee \longrightarrow 0.$$

Since $M_V^\vee(-1)$ is 1-regular and $W^\vee \otimes \mathscr{O}_{\mathbb{P}^{p+2}}$ is 0-regular, it follows that $K^\vee$ is 0-regular. Then $(K^\vee)^{\otimes p}$ is 0-regular, and so is $\bigwedge^p K^\vee$ because $\operatorname{char}(\Bbbk) = 0$ or $\operatorname{char}(\Bbbk) \geq \lfloor (g-1)/2 \rfloor \geq p+1$. Thus $K = \bigwedge^p K^\vee \otimes \det K$ is $(p+1)$-regular, and hence, γ'_k is surjective for $k \geq p$. □

It only remains to prove that the composition (3-3) is surjective. The second map in (3-3) is just the globalization of the map

$$\textstyle\bigwedge^2 V \longrightarrow V_h, \quad x^{(i)} \wedge x^{(j)} \longmapsto a_j x^{(i)} - a_i x^{(j)},$$

where V_h is the kernel of a nonzero linear functional $h := \sum_{i=0}^{p+2} a_i y^i \in V^\vee = S^{p+2} U^\vee$. We now regard $h = \sum_{i=0}^{p+2} b_i x^i \in S^{p+2} U$, where $b_i = (-1)^i a_{p+2-i}$. The surjectivity of (3-3) is equivalent to the injectivity of the composition of the dual maps

$$V_h^\vee \longrightarrow \textstyle\bigwedge^2 V^\vee \xrightarrow{\Delta_{p+2}^\vee} W^\vee. \tag{3-5}$$

The image of $V_h^\vee$ in $\bigwedge^2 V^\vee$ is spanned by $v_j := x^j \wedge h$ for $j = 0, \dots, d-1, d+1, \dots, p+2$, where $d := \deg h$. Recall that $\Delta_{p+2}^\vee$ is the restriction map $H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(p+1)) \to H^0(\bar{S}, \mathscr{O}_{\bar{S}}(p+1))$.

Proof of Theorem 1.1. We consider the case $\bar{S} = Q$. In this case, $\Delta_{p+2}^\vee = \mu_{p+2}$ (see (2-12)). We have $\Delta_{p+2}^\vee(v_j) = \sum_{i=0}^{d} (j-i) b_i x^{i+j-1}$. As $\operatorname{char}(\Bbbk) = 0$ or $\operatorname{char}(\Bbbk) \geq (g+2)/2 \geq p+3$, we get $\deg(\Delta_{p+2}^\vee(v_j)) = d+j-1$. This implies that

$$\Delta_{p+2}^\vee(v_0), \dots, \Delta_{p+2}^\vee(v_{d-1}), \Delta_{p+2}^\vee(v_{d+1}), \dots, \Delta_{p+2}^\vee(v_{p+2}) \text{ are linearly independent.}$$

Thus the composition (3-5) is injective. □

Proof of Theorem 1.2. We consider the case $\bar{S} = 2\ell$. In this case, $\Delta_{p+2}^\vee$ factors through τ_{p+2} (see (2-14)). Write $\tau_{p+2}(v_j) = (u_j, w_j)$. By Proposition 2.8 (2), we only need to deal with the case $p \geq 1$. As $\operatorname{char}(\Bbbk) = 0$ or $\operatorname{char}(\Bbbk) \geq \lfloor (g-1)/2 \rfloor \geq (p+3)/2$, we have

$$\begin{aligned} &\deg u_j = d+j-1 \quad \text{and} \quad \deg w_j \leq d+j-1 \quad \text{when } d \equiv j \pmod 2, \\ &\deg u_j \leq d+j-2 \quad \text{and} \quad \deg w_j = d+j \qquad\;\; \text{when } d \not\equiv j \pmod 2. \end{aligned}$$

This implies

$$\tau_{p+2}(v_0), \ldots, \tau_{p+2}(v_{d-1}), \tau_{p+2}(v_{d+1}), \ldots, \tau_{p+2}(v_{p+2}) \text{ are linearly independent.}$$

Thus the composition (3-5) is injective. □

Remark 3.3. The characteristic assumption in Theorem 1.1 is used to prove the injectivity of (3-5). If $k := \mathrm{char}(\Bbbk) \le p+2$, then (3-5) is not injective for $h = 1$. Indeed, $x^k \wedge 1 \neq 0$ in $\bigwedge^2 V^\vee$ is sent to 0 in $W^\vee$. The characteristic assumption in Theorem 1.2 is used in Lemma 3.2.

Remark 3.4. By the duality theorem, $K_{p,q}(S, \mathscr{O}_S(1)) = K_{g-p-2,3-q}(S, \mathscr{O}_S(1))^\vee$. In the situation of Theorem 1.1 or Theorem 1.2, we have

$$K_{p,1}(S, \mathscr{O}_S(1)) = 0 \quad \text{for } p \ge \lfloor g/2 \rfloor.$$

As $K_{p+1,1}(S, \mathscr{O}_S(1)) = \ker(\rho_{p+2}) = \ker(\gamma_{p+2})$ and γ_{p+2} is surjective for $0 \le p \le \lfloor (g-3)/2 \rfloor$, we can compute $\kappa_{p,1}(S, \mathscr{O}_S(1))$ for $0 \le p \le \lfloor (g-1)/2 \rfloor$. Note that $K_{p,0}(S, \mathscr{O}_S(1)) \neq 0$ if and only if $p = 0$. In this case, $\kappa_{0,0}(S, \mathscr{O}_S(1)) = 1$. Thus we can completely determine all graded Betti numbers $\kappa_{p,q}(S, \mathscr{O}_S(1))$.

4. Proof of Theorem 1.3

Assume that $\mathrm{char}(\Bbbk) = 0$. Let $C \subseteq \mathbb{P}^r$ be a smooth projective curve of genus $g \ge 1$ embedded by $|L|$, where L is a line bundle on C with $\deg L \ge 2g+3$. We use the notation in Section 2. Recall from Theorem 2.2 that $\Sigma \subseteq \mathbb{P}^r$ is arithmetically Cohen–Macaulay and $H^3(\Sigma, \mathscr{O}_\Sigma(m)) = 0$ for $m > 0$. By considering (2-5) and Lemma 2.4, we see that

$$\begin{aligned} H^i(T, \mathscr{O}_T(m)) = H^{i+1}(B, \mathscr{O}_B(mH - \widetilde{T} - Z)) \quad &\text{for } i \ge 1,\ m \ge 1, \\ T \subseteq \mathbb{P}^r \text{ is } m\text{-normal} \quad \iff \quad &H^1(B, \mathscr{O}_B(mH - \widetilde{T} - Z)) = 0. \end{aligned} \tag{4-1}$$

Note that $mH - \widetilde{T} - Z = (m-2)H + \pi^* T_L(-4\delta)$. Thus $R^1\pi_*\mathscr{O}_B(mH - \widetilde{T} - Z) = 0$ for $m \ge 1$. When $m = 1$, we have $H^j(B, \mathscr{O}_B(H - \widetilde{T} - Z)) = 0$ for $j > 0$ since $\pi_*\mathscr{O}_B(H - \widetilde{T} - Z) = 0$. When $m = 2$, we have $H^j(B, \mathscr{O}_B(2H - \widetilde{T} - Z)) = H^j(C_2, T_L(-4\delta))$ for $j > 0$. Recall that $\sigma_*\mathscr{O}_Z = \mathscr{O}_{C_2} \oplus \mathscr{O}_{C_2}(-\delta)$. Then we have $\sigma_*(L \boxtimes L)(-kD) = T_L(-(k+1)\delta) \oplus T_L(-k\delta)$.

Lemma 4.1. *$H^i(T, \mathscr{O}_T(m)) = 0$ for $i > 0, m > 0$, and $h^1(T, \mathscr{O}_T) = 2g$, $h^2(T, \mathscr{O}_T) = 1$. In particular,* $\mathrm{reg}\, \mathscr{O}_T = 3$.

Proof. By (2-6) and Proposition 2.5, $h^1(T, \mathscr{O}_T) = 2g$ and $h^2(T, \mathscr{O}_T) = 1$. It suffices to prove that $H^i(T, \mathscr{O}_T(m)) = 0$ for $i = 1,\ m = 1, 2$ and $i = 2,\ m = 1$. Indeed, this implies $\mathrm{reg}\, \mathscr{O}_T = 3$. By (4-1), we need to check that $H^j(B, \mathscr{O}_B(mH - \widetilde{T} - Z)) = 0$ for $j = 2,\ m = 1, 2$ and $j = 3,\ m = 1$. As the required vanishing is trivial when $m = 1$, it is enough to show that $H^2(C^2, (L \boxtimes L)(-3D)) = 0$. Observe that $R^1 p_*(L \boxtimes L)(-3D) = 0$, where $p : C \times C \to C$ is a projection map. Then

$$H^2(C^2, (L \boxtimes L)(-3D)) = H^2(C, p_*(L \boxtimes L)(-3D)) = 0.$$ □

As $T \subseteq \mathbb{P}^r$ is linearly normal and $h^1(T, \mathscr{O}_T) = 2g$, a general hyperplane section of the tangent-developable surface $T \subseteq \mathbb{P}^r$ is obtained from an isomorphic projection of an $(r+g)$-cuspidal curve of geometric genus g canonically embedded in $\mathbb{P}^{r+2g-1}$.

Lemma 4.2. *Suppose* $\deg L \geq 3g+2$. *Then* 2*-normality of* $T \subseteq \mathbb{P}^r$ *is equivalent to*

$$H^1(C \times C, (L \boxtimes L)(-3D)) = 0.$$

Proof. By (4-1), 2-normality of $T \subseteq \mathbb{P}^r$ is equivalent to $H^1(C_2, T_L(-4\delta)) = 0$. By [Bertram et al. 1991, Theorem 1(i)], $H^1(C^2, (L \boxtimes L)(-2D)) = 0$. It follows that $H^1(C_2, T_L(-3\delta)) = 0$. Then $H^1(C_2, T_L(-4\delta)) = 0$ if and only if $H^1(C^2, (L \boxtimes L)(-3D)) = 0$. □

Proposition 4.3. $K_{p,3}(T, \mathscr{O}_T(1)) = 0$ *for* $0 \leq p \leq r-3$, *and* $\kappa_{r-2,3}(T, \mathscr{O}_T(1)) = 1$.

Proof. By Proposition 2.1 and Lemma 4.1, $K_{p,3}(T, \mathscr{O}_T(1)) = H^2\big(T, \bigwedge^{p+2} M_{\mathscr{O}_T(1)} \otimes \mathscr{O}_T(1)\big)$. Note that $\omega_T = \mathscr{O}_T$ (Proposition 2.5) and $\bigwedge^{p+2} M^\vee_{\mathscr{O}_T(1)} = \bigwedge^{r-p-2} M_{\mathscr{O}_T(1)} \otimes \mathscr{O}_T(1)$. By Serre duality, we have

$$H^2\big(T, \textstyle\bigwedge^{p+2} M_{\mathscr{O}_T(1)} \otimes \mathscr{O}_T(1)\big) = H^0\big(T, \textstyle\bigwedge^{p+2} M^\vee_{\mathscr{O}_T(1)} \otimes \mathscr{O}_T(-1)\big)^\vee = H^0\big(T, \textstyle\bigwedge^{r-p-2} M_{\mathscr{O}_T(1)}\big)^\vee.$$

Then the proposition immediately follows. □

Proof of Theorem 1.3. By Lemma 4.1, it suffices to show that $T \subseteq \mathbb{P}^r$ is m-normal for $m = 1, 2, 3$. Indeed, this together with $\operatorname{reg} \mathscr{O}_T = 3$ implies $\operatorname{reg} \mathscr{I}_{T|\mathbb{P}^r} = 4$, and thus, $T \subseteq \mathbb{P}^r$ is m-normal for $m \geq 1$. In view of (4-1), $T \subseteq \mathbb{P}^r$ is trivially 1-normal. For the 2-normality, we assume that $\deg L \geq 4g+3$. Then [Bertram et al. 1991, Theorem 1.7(i)] says that $H^1(C \times C, (L \boxtimes L)(-3D)) = 0$, so the 2-normality of $T \subseteq \mathbb{P}^r$ follows from Lemma 4.2. Now, 3-normality of $T \subseteq \mathbb{P}^r$ is equivalent to $K_{0,3}(T, \mathscr{O}_T(1)) = 0$, but this is a special case of Proposition 4.3. □

Remark 4.4. In Theorem 1.3, the degree condition $\deg L \geq 4g+3$ is only used to show that $T \subseteq \mathbb{P}^r$ is 2-normal. If $\deg L = 4g+2$, then by Lemma 4.2 and [Bertram et al. 1991, Theorem 1.7(ii), (iii)], $T \subseteq \mathbb{P}^r$ is 2-normal (and arithmetically normal) if and only if $g \geq 3$ and C is nonhyperelliptic. More generally, using the results in [Bertram et al. 1991, Section 1] and [Pareschi 1995, Theorem 3.8(b)], one can prove that if $\deg L \geq 4g+3-\operatorname{Cliff}(C)$, then $T \subseteq \mathbb{P}^r$ is arithmetically normal.

Corollary 4.5. *Suppose that* $\deg L \geq 4g+3$. *Then the Hilbert function of* $T \subseteq \mathbb{P}^r$ *is given by* $H_T(t) = (\deg L + g - 1)t^2 + 2 - 2g$ *for* $t \geq 1$. *In particular*, $\kappa_{1,1}(T, \mathscr{O}_T(1)) = (r-2)(r-3)/2 - 6g$.

Proof. By Theorem 1.3, $H_T(t)$ coincides with the Hilbert polynomial $P_T(t)$ for $t \geq 1$. We know that $\deg P_T(t) = 2$ and the leading coefficient of $P_T(t)$ is $(\deg T)/2 = \deg L + g - 1$. As $P_T(0) = \chi(\mathscr{O}_T) = 2-2g$ and $P_T(1) = h^0(C, L) = \deg L - g + 1$, we easily get the assertion. The last assertion follows from $\kappa_{1,1}(T, \mathscr{O}_T(1)) = h^0(\mathbb{P}^r, \mathscr{I}_{T|\mathbb{P}^r}(2)) = \binom{r+2}{2} - H_T(2)$. □

Assume that $\deg L \geq 4g+3$. Notice that (see Proposition 4.3)

$$\begin{aligned} K_{p,0}(T, \mathscr{O}_T(1)) \neq 0 \quad &\Longleftrightarrow \quad p = 0 \qquad (\kappa_{0,0}(T, \mathscr{O}_T(1)) = 1), \\ K_{p,3}(T, \mathscr{O}_T(1)) \neq 0 \quad &\Longleftrightarrow \quad p = r-2 \ \ (\kappa_{r-2,3}(T, \mathscr{O}_T(1)) = 1). \end{aligned}$$

It would be exceedingly interesting to study vanishing/nonvanishing behaviors of $K_{p,1}(T, \mathscr{O}_T(1))$ and $K_{p,2}(T, \mathscr{O}_T(1))$ as the positivity of L grows. For a possible generalization of Theorem 1.1 (see also Remark 3.4), one may hope the following.

Conjecture 4.6. $K_{p,1}(T, \mathscr{O}_T(1)) = 0$ *for* $p \geq \lfloor (\deg L - 3g)/2 \rfloor$.

We have $\kappa_{p,1}(T, \mathscr{O}_T(1)) \leq \kappa_{p,1}(C, L)$. Thus $K_{p,1}(T, \mathscr{O}_T(1)) = 0$ at least for $p \geq r-1$ (or for $p \geq r - \mathrm{gon}(C) + 1$ by [Rathmann 2016, Theorem 1.1]). We find

$$\kappa_{r-2,2}(T, \mathscr{O}_T(1)) = 2g(r+1), \quad \kappa_{r-1,2}(T, \mathscr{O}_T(1)) = 2g.$$

If Conjecture 4.6 holds, then we can compute $\kappa_{p,2}(T, \mathscr{O}_T(1))$ for $\lfloor (\deg L - 3g - 2)/2 \rfloor \leq p \leq r-1$.

Concerning vanishing of $K_{p,2}(T, \mathscr{O}_T(1))$, in view of Theorems 1.1 and 1.3, one may ask whether

$$K_{p,2}(T, \mathscr{O}_T(1)) = 0 \quad \text{for } 0 \leq p \leq \lfloor (\deg L - 4g - 3)/2 \rfloor. \tag{4-2}$$

Suppose that $g = 1$. Observe that T has the same Hilbert polynomial with an abelian surface of degree $2\deg L$ and T is a degeneration of an abelian surface with polarization $(1, \deg L)$ (see [Gross and Popescu 1998, Theorem 3.1]). Notice that (4-2) is similar to Gross and Popescu's conjecture [1998, conjecture on p. 375]. However, Example 4.8(1) shows that (4-2) is not true even for $g = 1$ and $p = 1$. The correct expectation might be the following.

Conjecture 4.7. *Suppose that* $g = 1$. *Then* $K_{p,2}(T, \mathscr{O}_T(1)) = 0$ *for* $0 \leq p \leq \lfloor (\deg L - 7)/3 \rfloor$.

This conjecture would imply that a general $(1, d)$-polarized abelian surface satisfies the property N_p for $0 \leq p \leq \lfloor (d-7)/3 \rfloor$.

In general, it is tempting to guess that

$$K_{p,2}(T, \mathscr{O}_T(1)) = 0 \quad \text{for } 0 \leq p \leq \lfloor (\deg L - 4g - 3)/(g+2) \rfloor, \tag{4-3}$$

but it is hard to make a precise prediction at this moment.

Example 4.8. Let $q_1, \dots, q_m$ be minimal generators of the ideal $I_{C|\mathbb{P}^r} \subseteq \boldsymbol{k}[x_0, \dots, x_r]$. Then $\widetilde{T}$ is defined by q_i and $\sum_{j=0}^{r} (\partial q_i / \partial x_j) y_j$ for $1 \leq i \leq m$ in $\mathbb{P}^r \times \mathbb{P}^r$, and $T = q(\widetilde{T}) \subseteq \mathbb{P}^r$, where $q : \mathbb{P}^r \times \mathbb{P}^r \to \mathbb{P}^r$ is the second projection. Using Macaulay2, we can compute the Betti table of T in $\mathbb{P}^r$ whenever r is reasonably not so big.

(1) Let $C := Z(z_0^3 - z_0 z_2^2 - z_1^2 z_2) \subseteq \mathbb{P}^2$ be an elliptic curve. There is a point $x \in C$ such that $\mathscr{O}_C(1) = \mathscr{O}_C(3x)$. Consider the embedding $C \subseteq \mathbb{P}^{d-1}$ given by $|\mathscr{O}_C(dx)|$ for $d \geq 3$. The computation of the Betti table of the tangent-developable surface T of C in $\mathbb{P}^{d-1}$ confirms Conjectures 4.6 and 4.7 when $7 \leq d \leq 16$. For these cases, the predictions are sharp. The Betti tables of T in $\mathbb{P}^{d-1}$ for $d = 9, 10$ are as follows:

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	9	3	-	-	-	-	-
2	-	6	81	171	165	81	18	2
3	-	-	-	-	-	-	1	-

	0	1	2	3	4	5	6	7	8
0	1	-	-	-	-	-	-	-	-
1	-	15	20	-	-	-	-	-	-
2	-	-	70	252	350	260	105	20	2
3	-	-	-	-	-	-	-	1	-

(2) Let $C \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth projective curve of genus 2 defined by

$$x^2 \otimes (x^3 + 1) + 1 \otimes (x^2 - x) \in S^2U \otimes S^3U = H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(2) \boxtimes \mathscr{O}_{\mathbb{P}^1}(3)).$$

Consider the embedding $C \subseteq \mathbb{P}^{11}$ given by $|(\mathscr{O}_{\mathbb{P}^1}(3) \boxtimes \mathscr{O}_{\mathbb{P}^1}(2))|_C|$. Note that $\deg C = 13$. The Betti table of the tangent-developable surface T of C in $\mathbb{P}^{11}$ is the following:

	0	1	2	3	4	5	6	7	8	9	10
0	1	-	-	-	-	-	-	-	-	-	-
1	-	24	48	-	-	-	-	-	-	-	-
2	-	-	153	864	1848	2304	1827	928	288	48	4
3	-	-	-	-	-	-	-	-	-	1	-

We can confirm Conjecture 4.6 and (4-2) for this case, and we see that the expectation (4-3) is not sharp when $g \geq 2$.

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Presentations of Galois groups of maximal extensions with restricted ramification 835
YUAN LIU

Motivic distribution of rational curves and twisted products of toric varieties 883
LOÏS FAISANT

Smooth cuboids in group theory 967
JOSHUA MAGLIONE and MIMA STANOJKOVSKI

Malle's conjecture for fair counting functions 1007
PETER KOYMANS and CARLO PAGANO

Syzygies of tangent-developable surfaces and K3 carpets via secant varieties 1029
JINHYUNG PARK