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**Semistable representations  
as limits of crystalline representations**

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# Semistable representations as limits of crystalline representations

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We construct an explicit sequence  $V_{k_n, a_n}$  of crystalline representations of exceptional weights converging to a given irreducible two-dimensional semistable representation  $V_{k, \mathcal{L}}$  of  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ . The convergence takes place in the blow-up space of two-dimensional trianguline representations studied by Colmez and Chenevier. The process of blow-up is described in detail in the rigid-analytic setting and may be of independent interest. Also, we recover a formula of Stevens expressing the  $\mathcal{L}$ -invariant as a logarithmic derivative.

Our result can be used to compute the reduction of  $V_{k, \mathcal{L}}$  in terms of the reductions of the  $V_{k_n, a_n}$ . For instance, using the zig-zag conjecture we recover (resp. extend) the work of Breuil and Mézard and Guerberoff and Park computing the reductions of the  $V_{k, \mathcal{L}}$  for weights  $k$  at most  $p - 1$  (resp.  $p + 1$ ), at least on the inertia subgroup. In the cases where zig-zag is known, we are further able to obtain some new information about the reductions for small odd weights. Finally, we explain some apparent violations to local constancy in the weight of the reductions of crystalline representations of small weight.

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## 1. Introduction

Let  $p$  be an odd prime. Let  $E$  be a finite extension of  $\mathbb{Q}_p$  containing  $\sqrt{p}$ . Let  $D_{\text{st}}$  be Fontaine’s functor inducing an equivalence of categories between semistable representations of the Galois group  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$  over  $E$  and admissible filtered  $(\varphi, N)$ -modules over  $E$ . We introduce two kinds of representations using this functor.

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For every integer  $k \geq 2$  and  $a_p \in E$  of positive valuation, there is an irreducible two-dimensional *crystalline* representation  $V_{k,a_p}$  over  $E$  of  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$  with Hodge–Tate weights  $(0, k - 1)$  and  $D_{\text{st}}(V_{k,a_p}^*) = D_{k,a_p}$ , where  $D_{k,a_p} = Ee_1 \oplus Ee_2$  is the filtered  $\varphi$ -module defined by

$$\begin{cases} \varphi(e_1) = p^{k-1}e_2, \\ \varphi(e_2) = -e_1 + a_p e_2 \end{cases} \quad \text{and} \quad \text{Fil}^i D_{k,a_p} = \begin{cases} D_{k,a_p} & \text{if } i \leq 0, \\ Ee_1 & \text{if } 1 \leq i \leq k - 1, \\ 0 & \text{if } i \geq k. \end{cases}$$

Similarly, for every integer  $k \geq 2$  and  $\mathcal{L} \in \mathbb{P}^1(E)$  (called the  $\mathcal{L}$ -invariant), there is a two-dimensional *semistable* representation  $V_{k,\mathcal{L}}$  over  $E$  of  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$  with Hodge–Tate weights  $(0, k - 1)$  and  $D_{\text{st}}(V_{k,\mathcal{L}}^*) = D_{k,\mathcal{L}}$ , where  $D_{k,\mathcal{L}} = Ee_1 \oplus Ee_2$  is the filtered  $(\varphi, N)$ -module defined by

$$\begin{cases} \varphi(e_1) = p^{\frac{1}{2}k}e_1, \\ \varphi(e_2) = p^{\frac{1}{2}(k-2)}e_2 \end{cases} \quad \text{and} \quad \text{Fil}^i D_{k,\mathcal{L}} = \begin{cases} D_{k,\mathcal{L}} & \text{if } i \leq 0, \\ E(e_1 + \mathcal{L}e_2) & \text{if } 1 \leq i \leq k - 1 \text{ and } \mathcal{L} \neq \infty, \\ E(e_1 + e_2) & \text{if } 1 \leq i \leq k - 1 \text{ and } \mathcal{L} = \infty, \\ 0 & \text{if } i \geq k, \end{cases}$$

and

$$\begin{cases} N(e_1) = e_2, \\ N(e_2) = 0 \end{cases} \quad \text{if } \mathcal{L} \neq \infty, \quad \text{and} \quad N = 0 \quad \text{if } \mathcal{L} = \infty.$$

If  $k \geq 3$ , then the semistable representation  $V_{k,\mathcal{L}}$  is irreducible and when  $\mathcal{L} = \infty$  is isomorphic to the representation  $V_{k,a_p}$  with  $a_p = p^{k/2} + p^{(k-2)/2}$ .

This paper studies several relationships between the crystalline representations  $V_{k,a_p}$  and the semistable representations  $V_{k,\mathcal{L}}$  for  $k \geq 3$ .

In particular, we show how information about the reductions of the former representations implies information about the reductions of the latter. In general, computing the reductions of Galois representations has applications to computing deformation rings, to the weight part of Serre’s conjecture, to the Breuil–Mézard conjecture and to modularity lifting theorems.

**1.1. Notation.**

- $p$  is an odd prime and  $\sqrt{p}$  is a fixed square root of  $p$ .
- $E$  is a  $p$ -adic number field, i.e., a finite extension of  $\mathbb{Q}_p$ .
- $v_p$  is the  $p$ -adic valuation normalized such that  $v_p(p) = 1$ .
- $\zeta_{p-1}$  is a fixed primitive  $(p-1)$ -th root of unity in  $\mathbb{Q}_p^*$ .
- $\log$  is the  $p$ -adic logarithm, normalized by setting  $\log(p) = 0$ .
- $\mathcal{T}$  is the rigid-analytic space parametrizing continuous characters of  $\mathbb{Q}_p^*$ .
- $x^i \in \mathcal{T}(\mathbb{Q}_p)$  for  $i \geq 0$  is the character  $\mathbb{Q}_p^* \rightarrow \mathbb{Q}_p^*$  that sends an element to its  $i$ -th power.
- $\chi \in \mathcal{T}(\mathbb{Q}_p)$  is the  $p$ -adic cyclotomic character  $\mathbb{Q}_p^* \rightarrow \mathbb{Q}_p^*$  which maps  $p$  to 1 and which is the identity on  $\mathbb{Z}_p^*$ .
- Characters of the form  $x^i \chi$  for  $i \geq 0$  are called exceptional characters.

- $\mu_\lambda$  is the character of  $\mathbb{Q}_p^*$  sending  $p$  to  $\lambda \in \overline{\mathbb{F}}_p^*$  or  $\overline{\mathbb{Q}}_p^*$  and  $\mathbb{Z}_p^*$  to 1.
- Normalize the map  $\mathbb{Q}_p^* \rightarrow \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)^{\text{ab}}$  of class field theory by sending  $p$  to a geometric Frobenius. We sometimes think of  $\chi$  and  $\mu_\lambda$  as characters of  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ .
- Let  $\Gamma = \text{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ . We also think of  $\chi$  as a character  $\Gamma \xrightarrow{\sim} \mathbb{Z}_p^*$ . From [Section 3.1](#), we fix a topological generator  $\gamma$  of  $\Gamma$  such that  $\chi(\gamma) = \zeta_{p-1}^a(1+p)$  for a fixed integer  $a$ .
- $I_{\mathbb{Q}_p}$  is the inertia subgroup of  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ .
- $\omega$  is the fundamental character of  $I_{\mathbb{Q}_p}$  of level 1; it has a canonical extension to  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ .
- $\omega_2$  is the fundamental character of  $I_{\mathbb{Q}_p}$  of level 2; choose an extension of  $\omega_2$  to  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^2})$  so that for an integer  $c$  with  $p+1 \nmid c$  the representation  $\text{ind}(\omega_2^c)$  obtained by inducing this extension from  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^2})$  to  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$  has determinant  $\omega^c$ .
- $\mathcal{R}_A$  is the Robba ring with coefficients in  $A$  for  $A$  an affinoid algebra.
- $\mathcal{R}_A(\delta)$  for  $\delta \in \mathcal{T}(A)$  is the  $(\varphi, \Gamma)$ -module of rank 1 over  $\mathcal{R}_A$  with action of  $\varphi$  and  $\Gamma$ :

$$\varphi f(T) = \delta(p) f((1+T)^p - 1) \quad \text{and} \quad \gamma f(T) = \delta(\chi(\gamma)) f((1+T)^{\chi(\gamma)} - 1) \quad \text{for } \gamma \in \Gamma.$$

- $k$  is an integer greater than or equal to 2 and  $r = k - 2$ .
- $v_-$  and  $v_+$  are the largest and smallest nonnegative integers, respectively, such that  $v_- < \frac{1}{2}(k-2) < v_+$  for  $k \in [3, p+1]$ .
- $H_0 = 0$  and  $H_l = \sum_{i=1}^l (1/i)$  is the  $l$ -th partial harmonic sum for  $l \geq 1$ ; write  $H_\pm = H_{v_\pm}$ .
- $v = v_p(\mathcal{L} - H_- - H_+)$  is the  $p$ -adic valuation of  $\mathcal{L}$  in a finite extension of  $\mathbb{Q}_p$  shifted by the partial harmonic sums  $H_-$  and  $H_+$ . Note  $v$  equals  $v_p(\mathcal{L})$  if either quantity is negative.
- $\mathbb{P}(V)$  is the projectivization of a vector space  $V$ .
- $\phi$  is Euler's totient function.
- $\phi_n(T)$  for  $n \geq 1$  is the  $p^n$ -th cyclotomic polynomial.

**1.2. Limits of crystalline representations.** Colmez [\[2008\]](#) and Chenevier [\[2013\]](#) have constructed a moduli space of nonsplit trianguline  $(\varphi, \Gamma)$ -modules of rank 2 over the Robba ring (assuming that the quotient of the characters occurring in the triangulation is not of a certain kind). The first goal of this paper is to construct for  $k \geq 3$  an explicit sequence of crystalline representations converging in this space to the (dual of the) semistable representation  $V_{k,\mathcal{L}}$  for a prescribed  $\mathcal{L}$ -invariant  $\mathcal{L}$ . We will use this in conjunction with a local constancy result to study the reductions of semistable representations.

In order to state our result, let us recall the definition of the rigid-analytic space constructed by Colmez and Chenevier. Let  $\mathcal{T}$  be the parameter space for characters of  $\mathbb{Q}_p^*$ . Let  $x : \mathbb{Q}_p^* \rightarrow \mathbb{Q}_p^*$  be the identity character. Let  $\chi : \mathbb{Q}_p^* \rightarrow \mathbb{Z}_p^*$  is the  $p$ -adic cyclotomic character, sending  $p$  to 1 and such that  $\chi|_{\mathbb{Z}_p^*}$  is the identity character. We call the characters  $x^i \chi$  for  $i \geq 0$  *exceptional*. For each  $i \geq 0$ , let  $F_i$  and  $F'_i$  be the closed analytic subvarieties of  $\mathcal{T} \times \mathcal{T}$  such that, for every finite extension  $E$  of  $\mathbb{Q}_p$ , we have

$$F_i(E) = \{(\delta_1, \delta_2) \in \mathcal{T}(E) \times \mathcal{T}(E) \mid \delta_1 \delta_2^{-1} = x^i \chi\},$$

$$F'_i(E) = \{(\delta_1, \delta_2) \in \mathcal{T}(E) \times \mathcal{T}(E) \mid \delta_1 \delta_2^{-1} = x^{-i}\}.$$

Let  $F = \bigcup_{i \geq 0} F_i$  and  $F' = \bigcup_{i \geq 0} F'_i$ . The Colmez–Chenevier space  $\widetilde{\mathcal{T}}_2$  is the blow-up of  $(\mathcal{T} \times \mathcal{T}) \setminus F'$  along  $F$  in the category of rigid-analytic spaces. Our first main theorem is the following:

**Theorem 1.1.** *Let  $k \geq 3$ ,  $r = k - 2$  and  $\mathcal{L} \in \mathbb{P}^1(E)$ . For  $n \geq 1$ , let*

$$(k_n, a_n) = \begin{cases} (k + p^n(p - 1), p^{\frac{1}{2}r}(1 + \frac{1}{2}\mathcal{L}p^n(p - 1))) & \text{if } \mathcal{L} \neq \infty, \\ (k + p^{n^2}(p - 1), p^{\frac{1}{2}r}(1 + p^n)) & \text{if } \mathcal{L} = \infty. \end{cases} \tag{1}$$

Then

$$V_{k_n, a_n}^* \rightarrow V_{k, \mathcal{L}}^*,$$

i.e., the sequence of points in  $\widetilde{\mathcal{T}}_2$  associated to the crystalline representations  $V_{k_n, a_n}^*$  converges to the point in  $\widetilde{\mathcal{T}}_2$  associated to the semistable representation  $V_{k, \mathcal{L}}^*$ .

**Remarks.** (1) The weights  $k_n$  appearing in the theorem are for sufficiently large  $n$  *exceptional* in the sense that they are two more than twice the valuation  $v_p(a_n)$  of  $a_n$  modulo  $(p - 1)$  [Ghate 2021]. This will be of key importance in what follows. In fact, if  $(k_n, a_n)$  is an arbitrary sequence of points such that  $V_{k_n, a_n}^* \rightarrow V_{k, \mathcal{L}}^*$ , then the  $k_n$  are eventually exceptional weights. Indeed, it is not hard to see that  $(k_n, a_n) \rightarrow (k, p^{r/2})$  and  $k_n \equiv k \pmod{p - 1}$  for large  $n$ .

(2) In the case  $\mathcal{L} \neq \infty$ , the sequence of points  $(k_n, a_n)$  in Theorem 1.1 lies on the line  $a_p(l) = p^{r/2}(1 + \frac{1}{2}\mathcal{L}(l - k))$ . Clearly

$$\mathcal{L} = 2 \frac{a'_p(k)}{a_p(k)}. \tag{2}$$

In Section 6.2, we show that (2) holds more generally for an arbitrary sequence  $(k_n, a_n)$  of points on any smooth curve  $a_p(l)$  such that  $V_{k_n, a_n}^* \rightarrow V_{k, \mathcal{L}}^*$  in the blow-up space  $\widetilde{\mathcal{T}}_2$ . The formula (2) is a variant of a classical formula initially proved by Greenberg and Stevens [1993, Theorem 3.18] for elliptic curves with split multiplicative reduction (a weight-2, slope-0 case) and extended by Stevens [2010, Theorem B] (see also [Bertolini, Darmon, and Iovita 2010, Theorem 4]) to higher weights and slopes. Further generalizations were proved by Colmez [2010, Théorème 0.5, Corollaire 0.7], Benois [2010, Theorem 2] and others. This classical formula was a key local ingredient in the proof of the Mazur–Tate–Teitelbaum conjecture for elliptic curves due to Greenberg and Stevens. Our proof of (2) is essentially geometric. Recall the classical picture in algebraic geometry of the blow-up of  $\mathbb{A}^2$  at a point shown in Figure 1.

The strict transform of the curve  $y = f(x)$  passing through the origin  $(0, 0)$  in  $\mathbb{A}^2$  passes through the exceptional divisor  $\mathbb{P}^1$  above the origin at “height” the derivative of  $f$  at 0. The proofs of (2) and Theorem 1.1 are based on an analogous principle in a rigid-analytic setting.

(3) The techniques used to prove Theorem 1.1 can also be used to prove that the limit of a sequence of irreducible two-dimensional crystalline representations of  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$  with Hodge–Tate weights belonging to an interval  $[a, b]$  is also irreducible crystalline with Hodge–Tate weights in  $[a, b]$ , at least if the difference

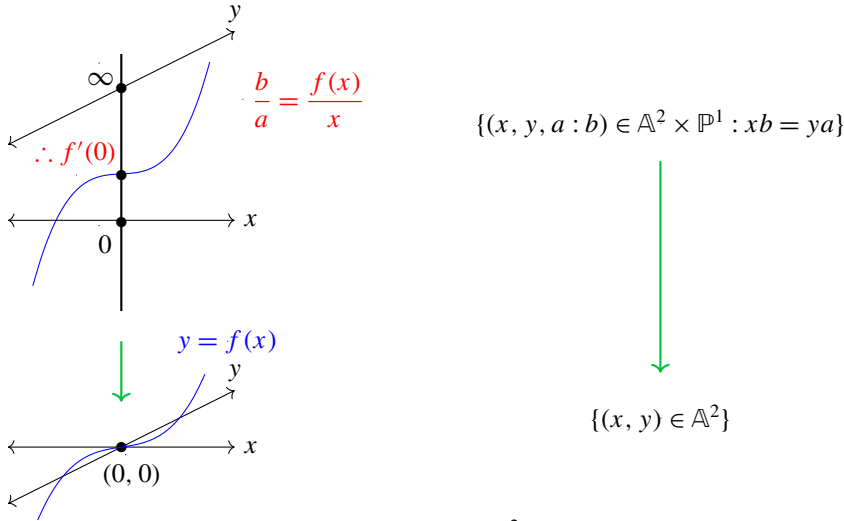


Figure 1. Blow-up of  $\mathbb{A}^2$  at  $(0, 0)$ .

of the Hodge–Tate weights of the representations in the sequence is at least 2 infinitely often. This gives another (geometric) proof of a special case of a general result of Berger [2004, Théorème 1] (see Section 8).

**1.3. Computing the  $\mathcal{L}$ -invariant.** In this section, we now explain the techniques involved in proving Theorem 1.1. The discussion should also serve as an overview of the contents of Sections 2 to 6.

Let  $E$  be a finite extension of  $\mathbb{Q}_p$ . Let  $\mathcal{R}_E$  be the Robba ring over  $E$  consisting of bidirectional power series having coefficients in  $E$  that converge on the elements of  $\overline{\mathbb{Q}_p}$  with valuation in  $]0, M]$  for some  $M > 0$ . For a more precise description, see Section 3.

The  $E$ -valued points of  $\tilde{\mathcal{T}}_2$  are tuples  $(\delta_1, \delta_2, L)$  where  $\delta_1, \delta_2$  are  $E^*$ -valued characters of  $\mathbb{Q}_p^*$  (with  $\delta_1 \delta_2^{-1} \neq x^{-j}$  for any  $j \geq 0$ ) and  $L \in \mathbb{P}^1(E)$  is the  $\mathcal{L}$ -invariant of the  $(\varphi, \Gamma)$ -module associated to the isomorphism class of the nonsplit extension

$$0 \rightarrow \mathcal{R}_E(\delta_1) \rightarrow * \rightarrow \mathcal{R}_E(\delta_2) \rightarrow 0,$$

when  $\delta_1 \delta_2^{-1} = x^i \chi$  for  $i \geq 0$  is an exceptional character, and is taken to be  $\infty$  otherwise (more precisely, in the former case,  $L$  is the  $\mathcal{L}$ -invariant defined by Colmez — see Section 3 — of this extension twisted by  $\delta_2^{-1}$ ).

The functor  $\mathbf{D}_{\text{rig}}$  sets up an equivalence of categories between the category of  $E$ -linear representations of  $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$  and the category of  $(\varphi, \Gamma)$ -modules over  $\mathcal{R}_E$  of slope 0. Let  $k \geq 3$  and  $r = k - 2$ , and, for  $n \geq 1$ , let  $(k_n, a_n)$  be as in (1). By [Berger 2012, Proposition 3.1],  $\mathbf{D}_{\text{rig}}(V_{k_n, a_n}^*)$  is an extension

$$0 \rightarrow \mathcal{R}_E(\mu_{y_n}) \rightarrow \mathbf{D}_{\text{rig}}(V_{k_n, a_n}^*) \rightarrow \mathcal{R}_E(\mu_{1/y_n} \chi^{1-k_n}) \rightarrow 0,$$

where

$$y_n = \frac{1}{2}(a_n + \sqrt{a_n^2 - 4p^{k_n-1}}). \tag{3}$$

This allows us to associate to (the dual of)  $V_{k_n, a_n}$  the point  $(\mu_{y_n}, \mu_{1/y_n} \chi^{1-k_n}, \infty)$  of the blow-up  $\tilde{\mathcal{T}}_2$ . We claim that for  $n \geq 1$  this sequence of points converges in the blow-up to the point

$$\begin{aligned}
 &(\mu_{p^{r/2}}, \mu_{1/p^{r/2}}\chi^{1-k}, -\mathcal{L}) \quad \text{if } \mathcal{L} \neq \infty, \\
 &(\mu_{p^{r/2}}, \mu_{1/p^{r/2}}\chi^{1-k}, \infty) \quad \text{if } \mathcal{L} = \infty.
 \end{aligned}$$

It turns out that the corresponding  $(\varphi, \Gamma)$ -module is étale (for any  $\mathcal{L}$ ) and that the corresponding Galois representation is the semistable representation  $V_{k,\mathcal{L}}^*$  (see the end of Section 6.1). In the  $\mathcal{L} = \infty$  case, the representation  $V_{k,\mathcal{L}}^*$  is in fact crystalline. This proves Theorem 1.1.

It remains to prove the claim. Let  $\tilde{\mathcal{T}}$  be the blow-up of  $\mathcal{T} \setminus \{x^{-j}\}_{j \geq 0}$  at  $\{x^i\chi\}_{i \geq 0}$ . The  $E$ -valued points of  $\tilde{\mathcal{T}}$  are tuples  $(\delta, L)$ , where  $\delta$  is an  $E^*$ -valued character of  $\mathbb{Q}_p^*$  (with  $\delta \neq x^{-j}$  for any  $j \geq 0$ ) and  $L \in \mathbb{P}^1(E)$  is the  $\mathcal{L}$ -invariant (defined by Colmez, see Section 3) of the  $(\varphi, \Gamma)$ -module associated to the isomorphism class of the nonsplit extension

$$0 \rightarrow \mathcal{R}_E(\delta) \rightarrow * \rightarrow \mathcal{R}_E \rightarrow 0,$$

when  $\delta = x^i\chi$  for  $i \geq 0$  is an exceptional character, and is taken to be  $\infty$  otherwise. Now a sequence of points  $(\delta_{1,n}, \delta_{2,n}, L_n)$  in  $\tilde{\mathcal{T}}_2$  converges to a point  $(\delta_1, \delta_2, L)$  in  $\tilde{\mathcal{T}}_2$  if and only if

- $\delta_{1,n}$  and  $\delta_{2,n}$  converge to  $\delta_1$  and  $\delta_2$ , respectively, in  $\mathcal{T}$ , and
- $(\delta_{1,n}\delta_{2,n}^{-1}, L_n)$  converges to  $(\delta_1\delta_2^{-1}, L)$  in  $\tilde{\mathcal{T}}$ .

That is, with respect to the commutative diagram

$$\begin{array}{ccc}
 \tilde{\mathcal{T}}_2 & \longrightarrow & \tilde{\mathcal{T}} \\
 \downarrow & & \downarrow \\
 \mathcal{T} \times \mathcal{T} \setminus F' & \longrightarrow & \mathcal{T} \setminus \{x^{-j}\}_{j \geq 0}
 \end{array}$$

where the bottom map is the restriction of the twisting map  $\mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$  sending  $(\delta_1, \delta_2)$  to  $\delta_1\delta_2^{-1}$ , the top map sends  $(\delta_1, \delta_2, L)$  to  $(\delta_1\delta_2^{-1}, L)$  and the vertical maps are the blow-up maps at  $F$  and  $\{x^i\chi\}_{i \geq 0}$ , respectively, the sequence  $(\delta_{1,n}, \delta_{2,n}, L_n)$  converges to  $(\delta_1, \delta_2, L)$  in  $\tilde{\mathcal{T}}_2$  if and only if the projections of this sequence under the left vertical map and the top horizontal map converge to the corresponding projections of  $(\delta_1, \delta_2, L)$ . Indeed, the top map is obtained from the universal property of the blow-up map on the right (this can be checked on charts using the definition of the map  $g$  constructed after Lemma 2.3), so there is an induced map from  $\tilde{\mathcal{T}}_2$  to the fiber product of  $\mathcal{T} \times \mathcal{T} \setminus F'$  and  $\tilde{\mathcal{T}}$  over  $\mathcal{T} \setminus \{x^{-j}\}_{j \geq 0}$  which is a closed immersion [Schoutens 1995, Proposition 3.1.2], and hence induces an inclusion on points  $\tilde{\mathcal{T}}_2(E)$  to the fiber product of  $(\mathcal{T} \times \mathcal{T} \setminus F')(E)$  and  $\tilde{\mathcal{T}}(E)$  over  $(\mathcal{T} \setminus \{x^{-j}\}_{j \geq 0})(E)$  with closed image, and then this is the definition of convergence in the last fiber product.

An easy check shows that the characters  $\mu_{y_n}$  and  $\mu_{1/y_n}\chi^{1-k_n}$  converge to the characters  $\mu_{p^{r/2}}$  and  $\mu_{1/p^{r/2}}\chi^{1-k}$  respectively. The ratio of these characters is the exceptional character  $\mu_{p^r}\chi^{k-1} = x^r\chi$ . Thus, if the sequence  $(\mu_{y_n}, \mu_{1/y_n}\chi^{1-k_n}, \infty)$  converges in  $\tilde{\mathcal{T}}_2$ , then it converges to a point in the fiber over  $F_r$ . Thus, it remains to check that  $(\mu_{y_n^2}\chi^{k_n-1}, \infty)$  converges in  $\tilde{\mathcal{T}}$  to

$$\begin{aligned}
 &(x^r\chi, -\mathcal{L}) \quad \text{if } \mathcal{L} \neq \infty, \\
 &(x^r\chi, \infty) \quad \text{if } \mathcal{L} = \infty.
 \end{aligned} \tag{4}$$

In order to prove (4), we set up local coordinates  $U_r$  around the point  $x^r \chi$ , describe the blow-up  $\tilde{U}_r$  of this coordinate patch with center at the exceptional character  $x^r \chi$  explicitly (see Section 2 for details) and compute the limit in  $\tilde{U}_r$ . Let  $\zeta_{p-1}$  be a fixed primitive  $(p-1)$ -th root of unity. Associating the tuple  $(\delta(p), \delta(\zeta_{p-1}), \delta(1+p) - 1)$  to a character  $\delta \in \mathcal{T}(\mathbb{Q}_p)$  identifies  $\mathcal{T}(\mathbb{Q}_p)$  with  $\mathbb{Q}_p^* \times \mu_{p-1} \times p\mathbb{Z}_p$ . Under this identification, the exceptional character  $\mu_{p^r} \chi^{k-1}$  goes to the tuple  $(p^r, \zeta_{p-1}^{k-1}, (1+p)^{k-1} - 1)$ . The set  $p^r \mathbb{Z}_p^* \times \{\zeta_{p-1}^{k-1}\} \times p\mathbb{Z}_p$  is a neighborhood of  $\mu_{p^r} \chi^{k-1}$  in  $\mathcal{T}(\mathbb{Q}_p)$ . This leads us to consider the affinoid algebra

$$U_r = \text{Sp } \mathbb{Q}_p \langle S_1, S_2, T_1, T_2, T_3 \rangle / (p^r T_1 - S_1, 1 - T_1 T_2, p T_3 - S_2)$$

as a neighborhood of  $\mu_{p^r} \chi^{k-1}$  in  $\mathcal{T}$  because clearly  $U_r(\mathbb{Q}_p) = p^r \mathbb{Z}_p^* \times p\mathbb{Z}_p$ . The variable  $S_1$  corresponds to the first factor and  $S_2$  to the second factor. From now on, by fixing the tame part of the characters under consideration, we identify  $U_r(E)$  with the subset  $p^r \mathcal{O}_E^* \times \{\zeta_{p-1}^{k-1}\} \times p\mathcal{O}_E$  of  $\mathcal{T}(E)$ , where  $\mathcal{O}_E$  is the ring of integers of  $E$ .

The character  $\mu_{p^r} \chi^{k-1} = x^r \chi$  corresponds to the maximal ideal

$$m = (S_1 - p^r, S_2 - ((1+p)^{k-1} - 1))$$

of  $\mathcal{O}(U_r)$ . The blow-up  $\tilde{U}_r$  of  $U_r$  at the maximal ideal  $m$  turns out to have the following standard description (see (5)):

$$\tilde{U}_r(E) = \{(s_1, s_2, \xi_1 : \xi_2) \in U_r(E) \times \mathbb{P}^1(E) \mid (s_1 - p^r)\xi_2 = (s_2 - ((1+p)^{k-1} - 1))\xi_1\}.$$

For large  $n$ , the points  $(\mu_{y_n^2} \chi^{kn-1}, \infty)$  in  $\tilde{\mathcal{T}}$  lie in  $\tilde{U}_r$ . We prove that the sequence converges in the blow-up  $\tilde{U}_r$  to the point (see Section 6.1)

$$\begin{aligned} & \left( p^r, (1+p)^{k-1} - 1, \mathcal{L} \frac{p^r}{(1+p)^{k-1} \log(1+p)} : 1 \right) && \text{if } \mathcal{L} \neq \infty, \\ & (p^r, (1+p)^{k-1} - 1, 1 : 0) && \text{if } \mathcal{L} = \infty, \end{aligned}$$

where  $\log$  is normalized so that  $\log(p) = 0$ . The proof of (4) then follows immediately from Theorem 5.2, a technical but important formula for the  $\mathcal{L}$ -invariant of a point in the exceptional fiber, noting that the fudge factor there cancels with the extra factor appearing in the third coordinate of the limit point above when  $\mathcal{L} \neq \infty$  and flips the sign.

Theorem 5.2 is proved as follows. Given a point in the exceptional fiber, we convert it to a tangent direction in  $U_r$  at the point  $(p^r, (1+p)^{k-1} - 1)$ , i.e., an element of  $\mathbb{P}(\text{Hom}(m/m^2 \otimes_{\mathbb{Q}_p} E, E))$ . This is done using the map in Proposition 2.5. We then explicitly describe the isomorphism  $\mathbb{P}(\text{Hom}(m/m^2 \otimes_{\mathbb{Q}_p} E, E)) \rightarrow \mathbb{P}(H^1(\mathcal{R}_E(x^r \chi)))$  stated in [Chenevier 2013, Theorem 2.33], using some preparatory material on the cohomology of “big”  $(\varphi, \Gamma)$ -modules in Section 4. The image of the given point in the exceptional fiber under the composition of these two maps yields a cohomology class in  $H^1(\mathcal{R}_E(x^r \chi))$  (up to scalars). The corresponding  $(\varphi, \Gamma)$ -module (up to isomorphism) is referred to as “the  $(\varphi, \Gamma)$ -module” associated to the given point (Definition 5.1). We then represent this cohomology class as an explicit linear combination of the basis elements of  $H^1(\mathcal{R}_E(x^r \chi))$  studied by Benois [2011, Proposition 1.5.4]. The original formula

for the  $\mathcal{L}$ -invariant due to Colmez is, however, in terms of a different basis of  $H^1(\mathcal{R}_E(x^r \chi))$ , namely, the one constructed in [Colmez 2008, Proposition 2.19] (see Section 3). Restating the formula for the  $\mathcal{L}$ -invariant in terms of Benois' basis (Definition 3.6) allows us to give a formula for the  $\mathcal{L}$ -invariant of the given point in the exceptional fiber.

**1.4. Reductions of semistable representations.** Chenevier [2013, Proposition 3.9] proved that points of the space  $\tilde{\mathcal{T}}_2$  lie in families of  $(\varphi, \Gamma)$ -modules. By [Kedlaya and Liu 2010, Theorem 0.2], such a family comes from a family of Galois representations at least affinoid locally around an étale point. Furthermore (see, e.g., the discussion on [Chenevier 2013, p. 1513], which uses results of [Chenevier 2014] on pseudorepresentations), the semisimplification of the reduction of Galois representations living in connected families are isomorphic. This means that if the points in  $\tilde{\mathcal{T}}_2$  corresponding to two Galois representations are close, then the semisimplifications of the reductions of the two Galois representations are the same. Moreover, the dual of the reduction of a lattice in a  $p$ -adic representation is the same as the reduction of the dual lattice in the dual representation. Using these facts, along with Theorem 1.1, we see that if one knows the reductions of the crystalline representations appearing in Theorem 1.1, then one can compute the reduction of  $V_{k,\mathcal{L}}$  for any  $k \geq 3$  and  $\mathcal{L} \neq \infty$  (the case  $k = 2$  is not as interesting since the reduction is always reducible).

More generally, the above method allows one to compute the reduction of any irreducible two-dimensional noncrystalline semistable representation with distinct Hodge–Tate weights. Indeed, suppose  $V$  is such a semistable representation. Twisting by  $\chi^a$  for some integer  $a$ , we may assume that the Hodge–Tate weights of the semistable representation  $V \otimes \chi^a$  are  $(0, k - 1)$  for an integer  $k \geq 2$ . By, for instance, [Guerberoff and Park 2019, Lemma 3.1.2(3)], the filtered  $(\varphi, N)$ -module  $D_{\text{st}}((V \otimes \chi^a)^*)$  is the module  $D(\lambda, \mathcal{L})$  described in [Guerberoff and Park 2019, Example 3.1.1] with  $\lambda = up^{(k-2)/2}$  for some unit  $u$  (since  $r$  there is equal to  $k - 1$ ). Now  $V \otimes \chi^a \otimes \mu_u$  is isomorphic to  $V_{k,\mathcal{L}}$ , as can be seen by comparing the corresponding filtered  $(\varphi, N)$ -modules. By [loc. cit., Lemma 3.1.2(4)], we must have  $k \geq 3$ .

This approach to computing the reduction of semistable representations using crystalline representations is of some importance because the reductions of these two classes of representations are nowadays largely studied by completely different methods: the crystalline case uses the compatibility of reduction between the  $p$ -adic and mod  $p$  local Langlands correspondences, or computes the reduction of the corresponding Wach module, whereas the reductions in the semistable case are determined by studying the reductions of the corresponding strongly divisible modules. In our experience, the former methods, while quite intricate, are not as complicated as the latter method. Thus, in view of the remarks above, the techniques used in the crystalline case may be brought to bear on the study of the reductions of semistable representations. We note, however, that the former method is only available for two-dimensional representations of  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ , whereas the latter method is available in principle for representations of  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$  of any dimension (though in practical terms only for those of small Hodge–Tate weights).

Let us illustrate this with some examples. The reductions of semistable representations have been computed completely for even weights in the range  $[2, p - 1]$  by Breuil and Mézard [2002], and for

odd weights in the same range by Guerberoff and Park [2019] at least on inertia. In [Ghate 2021], the second author made the following conjecture called the zig-zag conjecture<sup>1</sup> describing the reductions of crystalline representations of exceptional weights and half-integral slopes in terms of an alternating sequence of reducible and irreducible mod  $p$  representations.

**Conjecture 1.2** (Zig-zag conjecture). *Say that  $k \equiv k_0 = 2v(a_p) + 2 \pmod{p - 1}$  is an exceptional congruence class of weights for a particular half-integral slope  $\frac{1}{2} \leq v_p(a_p) \in \frac{1}{2}\mathbb{Z} \leq \frac{1}{2}(p - 1)$ . Let  $r = k - 2$  and  $r_0 = k_0 - 2$ . Define two parameters*

$$\tau = v_p\left(\frac{a_p^2 - \binom{r-v_-}{v_+} \binom{r-v_+}{v_-} p^{r_0}}{pa_p}\right) \quad \text{and} \quad t = v_p(k - k_0),$$

where  $v_-$  and  $v_+$  are the largest and smallest integers such that  $v(a_p)$  lies in  $(v_-, v_+)$ . Then, for all weights  $k > k_0$  with  $t$  sufficiently large, the (semisimplification of the) reduction  $\overline{V}_{k,a_p}$  of the crystalline representation  $V_{k,a_p}$  on the inertia subgroup  $I_{\mathbb{Q}_p}$  is given by

$$\overline{V}_{k,a_p}|_{I_{\mathbb{Q}_p}} \sim \left\{ \begin{array}{ll} \text{ind}(\omega_2^{r_0+1}) & \text{if } \tau < t, \\ \omega^{r_0} \oplus \omega & \text{if } \tau = t, \\ \text{ind}(\omega_2^{r_0+p}) & \text{if } t < \tau < t + 1, \\ \omega^{r_0-1} \oplus \omega^2 & \text{if } \tau = t + 1, \\ \text{ind}(\omega_2^{r_0+2p-1}) & \text{if } t + 1 < \tau < t + 2, \\ \omega^{r_0-2} \oplus \omega^3 & \text{if } \tau = t + 2, \\ \vdots & \vdots \\ \text{ind}(\omega_2^{r_0+1+\frac{1}{2}(r_0-2)(p-1)}) & \text{if } t + \frac{1}{2}(r_0 - 4) < \tau < t + \frac{1}{2}(r_0 - 2), \\ \omega^{\frac{1}{2}(r_0+2)} \oplus \omega^{\frac{1}{2}r_0} & \text{if } \tau = t + \frac{1}{2}(r_0 - 2), \\ \text{ind}(\omega_2^{r_0+1+\frac{1}{2}r_0(p-1)}) & \text{if } \tau > t + \frac{1}{2}(r_0 - 2), \end{array} \right\} \text{ and } r_0 \text{ is even}$$

$$\text{or}$$

$$\left\{ \begin{array}{ll} \text{ind}(\omega_2^{r_0+1+\frac{1}{2}(r_0-1)(p-1)}) & \text{if } t + \frac{1}{2}(r_0 - 3) < \tau < t + \frac{1}{2}(r_0 - 1), \\ \omega^{\frac{1}{2}(r_0+1)} \oplus \omega^{\frac{1}{2}(r_0+1)} & \text{if } \tau \geq t + \frac{1}{2}(r_0 - 1) \end{array} \right\} \text{ and } r_0 \text{ is odd.}$$

This conjecture has been verified for some small slopes (see [Buzzard and Gee 2013] for slope  $\frac{1}{2}$ , [Bhattacharya, Ghate, and Rozensztajn 2018] for slope 1 and [Ghate and Rai 2025] for slope  $\frac{3}{2}$ ) even with  $t = 0$ .<sup>2</sup>

This ‘‘crystalline’’ conjecture is intimately connected to the ‘‘semistable’’ results in [Breuil and Mézard 2002, Theorem 1.2; Guerberoff and Park 2019, Theorem 5.0.5]. More precisely, the zig-zag conjecture and Theorem 1.1 can be used to completely recover the description of the reductions of semistable representations in these theorems when  $k \neq 2$ , and even (conjecturally) extend it to the cases  $k = p + 1$  and  $k = p$ , respectively, at least on the inertia subgroup (see Theorem 1.3 below). Moreover, in the

<sup>1</sup>This version is mildly different from [Ghate 2021, Conjecture 1.1] in that there we require  $k$  to be sufficiently far away from some weights which are strictly larger than  $p + 1$ , whereas here  $k$  is required to be sufficiently close to the weights  $3 \leq k_0 \leq p + 1$ .

<sup>2</sup>The condition that  $t$  is sufficiently large is required for larger slopes due to some numerical observations made by Rozensztajn.

odd-weight cases for which the zig-zag conjecture has been proved, we obtain new information about the reductions of semistable representations on the full Galois group  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ .

To elaborate further, let us set up some notation. For an integer  $k$  in the interval  $[3, p + 1]$ , define  $v_-$  and  $v_+$  to be the largest and smallest integers, respectively, such that  $v_- < \frac{1}{2}(k - 2) < v_+$ . For  $l \geq 1$ , let

$$H_l = \sum_{i=1}^l \frac{1}{i}$$

be the  $l$ -th partial harmonic sum and set  $H_0 = 0$ . For convenience, write  $H_- = H_{v_-}$  and  $H_+ = H_{v_+}$ . For any  $\mathcal{L}$  in a finite extension of  $\mathbb{Q}_p$ , let

$$v = v_p(\mathcal{L} - H_- - H_+)$$

be the  $p$ -adic valuation of the  $\mathcal{L}$ -invariant shifted by the partial harmonic sums  $H_-$  and  $H_+$ . Let  $\omega$  and  $\omega_2$  denote the mod  $p$  fundamental characters of levels 1 and 2, respectively, on the inertia group  $I_{\mathbb{Q}_p}$ . The character  $\omega$  may be thought of as a character of  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ ; we choose an extension of  $\omega_2$  to  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^2})$  such that for any integer  $c$  with  $p + 1 \nmid c$ , the representation  $\text{ind}(\omega_2^c)$  obtained by inducing  $\omega_2^c$  from  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^2})$  to  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$  has determinant  $\omega^c$ . Let  $\mu_\lambda$  be the unramified character of  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$  mapping a geometric Frobenius element at  $p$  to  $\lambda \in \overline{\mathbb{F}}_p^*$  or  $\overline{\mathbb{Q}}_p^*$ . Normalizing local class field theory by sending  $p$  to a geometric Frobenius element at  $p$ , we obtain the character  $\mu_\lambda$  of  $\mathbb{Q}_p^*$  taking  $p$  to  $\lambda$  and  $\mathbb{Z}_p^*$  to 1 used in Section 1.3. Our second main theorem is the following:

**Theorem 1.3.** *If the zig-zag conjecture is true, then for any weight  $k$  satisfying  $3 \leq k \leq p + 1$ , we have the following  $(k - 1)$ -fold description of the (semisimplification of the) reduction of the semistable representation  $V_{k,\mathcal{L}}$  for  $\mathcal{L} \in \mathbb{P}^1(\overline{\mathbb{Q}}_p)$  on the inertia subgroup:*

$$\overline{V}_{k,\mathcal{L}}|_{I_{\mathbb{Q}_p}} \sim \left\{ \begin{array}{ll} \left. \begin{array}{l} \omega_2^{k-1} \oplus \omega_2^{p(k-1)} \\ \omega^{k-2} \oplus \omega \\ \omega_2^{k-2+p} \oplus \omega_2^{p(k-2)+1} \\ \omega^{k-3} \oplus \omega^2 \\ \vdots \\ \omega_2^{\frac{1}{2}k+1+p(\frac{1}{2}k-2)} \oplus \omega_2^{\frac{1}{2}k-2+p(\frac{1}{2}k+1)} \\ \omega^{\frac{1}{2}k} \oplus \omega^{\frac{1}{2}(k-2)} \\ \omega_2^{\frac{1}{2}k+p(\frac{1}{2}k-1)} \oplus \omega_2^{\frac{1}{2}k-1+\frac{1}{2}pk} \end{array} \right\} & \left. \begin{array}{l} \text{if } v < 1 - \frac{1}{2}(k - 2), \\ \text{if } v = 1 - \frac{1}{2}(k - 2), \\ \text{if } 1 - \frac{1}{2}(k - 2) < v < 2 - \frac{1}{2}(k - 2), \\ \text{if } v = 2 - \frac{1}{2}(k - 2), \\ \vdots \\ \text{if } -1 < v < 0, \\ \text{if } v = 0, \\ \text{if } v > 0, \end{array} \right\} \text{ and } k \text{ is even} \\ \text{or} \\ \left. \begin{array}{l} \omega_2^{\frac{1}{2}(k+1)+p(\frac{1}{2}(k-3))} \oplus \omega_2^{\frac{1}{2}(k-3)+p(\frac{1}{2}(k+1))} \\ \omega^{\frac{1}{2}(k-1)} \oplus \omega^{\frac{1}{2}(k-1)} \end{array} \right\} & \left. \begin{array}{l} \text{if } -\frac{1}{2} < v < \frac{1}{2}, \\ \text{if } v \geq \frac{1}{2} \end{array} \right\} \text{ and } k \text{ is odd.} \end{array} \right.$$

**Remark.** The theorem holds when  $\mathcal{L} = \infty$  even without assuming the zig-zag conjecture by the work of Fontaine and Edixhoven [1992], since, by convention,  $v_p(\infty) = -\infty$ .

A generalization of the  $\nu < 1 - \frac{1}{2}(k - 2)$  case of [Theorem 1.3](#) has recently been proved for all weights  $k \geq 4$  (and odd primes  $p$ ) by [\[Bergdall, Levin and Liu 2023, Theorem 1.1\]](#). Note that the term  $v_p((k - 2)!)$  in their result vanishes in our setting.

Also [Theorem 1.3](#) above indeed matches with the results in [\[Breuil and Mézard 2002; Guerberoff and Park 2019\]](#) for  $\mathcal{L} \neq \infty$  and for  $k \in [3, p - 1]$ :

**Even  $k \in [3, p - 1]$ :** Breuil and Mézard [\[2002, Theorem 1.2\]](#) have computed the reduction of  $V_{k,\mathcal{L}}$  in terms of the valuations  $v_p(a)$  and  $v_p(\mathcal{L})$ , where

$$a = (-1)^{\frac{1}{2}k} \left( -1 + \frac{k}{2} \binom{k}{2} - 1 \right) (-\mathcal{L} + 2H_{\frac{1}{2}k-1}).$$

Since  $k \in [3, p - 1]$ , we see that  $v_p(\frac{1}{2}k(\frac{1}{2}k - 1)) = 0$ , and therefore

$$\begin{aligned} v_p(a) &= v_p \left( \frac{-1}{\binom{1}{2}k \binom{1}{2}k - 1} + (-\mathcal{L} + 2H_{\frac{1}{2}k-1}) \right) = v_p \left( \frac{1}{\frac{1}{2}k} - \frac{1}{\frac{1}{2}k - 1} - \mathcal{L} + 2 \sum_{i=1}^{\frac{1}{2}k-1} \frac{1}{i} \right) \\ &= v_p \left( -\mathcal{L} + \sum_{i=1}^{\frac{1}{2}k} \frac{1}{i} + \sum_{i=1}^{\frac{1}{2}k-2} \frac{1}{i} \right) = v_p(\mathcal{L} - H_{\frac{1}{2}k} - H_{\frac{1}{2}k-2}) = \nu, \end{aligned}$$

where we have used  $k$  is even in the last equality. Now:

- If  $\nu > 0$ , then [Theorem 1.3](#) yields

$$\bar{V}_{k,\mathcal{L}}|_{I_{\mathbb{Q}_p}} \sim \omega_2^{\frac{1}{2}k+p(\frac{1}{2}k-1)} \oplus \omega_2^{\frac{1}{2}k-1+\frac{1}{2}pk},$$

which agrees with [\[Breuil and Mézard 2002, Theorem 1.2\(ii\)\]](#).

- If  $\nu = 0$ , then [Theorem 1.3](#) yields

$$\bar{V}_{k,\mathcal{L}}|_{I_{\mathbb{Q}_p}} \sim \omega_2^{\frac{1}{2}k} \oplus \omega_2^{\frac{1}{2}(k-2)},$$

which agrees with [\[loc. cit., Theorem 1.2\(i\)\]](#).

- So assume  $\nu < 0$ . Then  $\nu = v_p(\mathcal{L})$ . If  $\nu < 2 - \frac{1}{2}k$ , then [Theorem 1.3](#) yields  $\bar{V}_{k,\mathcal{L}}|_{I_{\mathbb{Q}_p}} \sim \omega_2^{k-1} \oplus \omega_2^{p(k-1)}$ , which agrees with [\[loc. cit., Theorem 1.2\(iii\)\]](#). Now if  $2 - \frac{1}{2}k \leq \nu < 0$  and  $\nu \in \mathbb{Z}$ , then [\[loc. cit., Theorem 1.2\(iii\)\]](#) yields

$$\bar{V}_{k,\mathcal{L}}|_{I_{\mathbb{Q}_p}} \sim \omega_2^{\frac{1}{2}k-\nu} \oplus \omega_2^{\frac{1}{2}k+\nu-1},$$

which is the same as the reduction computed in [Theorem 1.3](#). Finally, if  $2 - \frac{1}{2}k \leq \nu < 0$  and  $\nu \notin \mathbb{Z}$ , then [\[loc. cit., Theorem 1.2\(iii\)\]](#) yields

$$\bar{V}_{k,\mathcal{L}}|_{I_{\mathbb{Q}_p}} \sim \omega_2^{\frac{1}{2}k-\lfloor \nu \rfloor + p(\frac{1}{2}k+\lfloor \nu \rfloor-1)} \oplus \omega_2^{\frac{1}{2}k+\lfloor \nu \rfloor-1+p(\frac{1}{2}k-\lfloor \nu \rfloor)},$$

which matches with the reduction computed in [Theorem 1.3](#).

**Remark.** Since the zig-zag conjecture has already been proved for  $p \geq 5$  and slope 1 in [\[Bhattacharya, Ghate, and Rozensztajn 2018, Theorem 1.1\]](#), even on  $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ , we recover the following result of

Breuil and Mézard when  $k = 4$  (see [Breuil and Mézard 2002, Theorem 1.2]):<sup>3</sup> if  $p \geq 5$  and  $k = 4$ , then

$$\bar{V}_{k,\mathcal{L}} \sim \begin{cases} \text{ind}(\omega_2^3) & \text{if } v < 0, \\ \mu_\lambda \omega^2 \oplus \mu_{\lambda^{-1}} \omega & \text{if } v = 0, \\ \text{ind}(\omega_2^{2+p}) & \text{if } v > 0, \end{cases}$$

where

$$\lambda = \overline{-2\left(\mathcal{L} - \frac{3}{2}\right)}.$$

If all nearby étale points in  $\tilde{\mathcal{T}}_2$  close to a given étale point, which has a lattice with nonsplit reduction, also have lattices with isomorphic reduction (an assumption which is stronger than the reductions being isomorphic up to semisimplification, and which might follow by extending results of [Chenevier 2014] from pseudorepresentations to Galois representations), then one can read off more subtle information about whether the representation  $\bar{V}_{k,\mathcal{L}}$  is peu or très ramifiée (when it is reducible and  $\lambda = \pm 1$ ) from the corresponding information of a sufficiently close crystalline representation  $V_{k_n, a_n}$ , which, in turn, is controlled by the size of  $v_p(u_n - \varepsilon_n)$  in the notation of [Bhattacharya, Ghate, and Rozensztajn 2018, Theorem 1.3]. Indeed, in the middle case of the trichotomy above, we have  $v_p(3 - 2\mathcal{L}) = 0$ , putting us in part (i) of [Breuil and Mézard 2002, Theorem 1.2]. Note that  $\overline{a_n/p} = 1$  and  $\overline{k_n - 2} = 2$ . There are now two cases to consider depending on whether  $\varepsilon_n = \pm 1$ . If  $v_p(\mathcal{L} - 2) = 0$ , then a small check shows that  $\varepsilon_n = 1$ , so by [Bhattacharya, Ghate, and Rozensztajn 2018, Theorem 1.3(1a)], the reduction  $\bar{V}_{k,\mathcal{L}}$  (without semisimplification) is peu ramifiée. If  $v_p(\mathcal{L} - 2) > 0$ , then another small check shows that  $\varepsilon_n = -1$  and that  $v_p(u_n - \varepsilon_n) = v_p(\mathcal{L} - 2)$ . Thus, by [loc. cit., Theorem 1.3(1b)], the reduction  $\bar{V}_{k,\mathcal{L}}$  (without semisimplification) is peu ramifiée if and only if  $v_p(\mathcal{L} - 2) < 1$ , at least if  $\mathcal{L}$  lies in an unramified extension of  $\mathbb{Q}_p$ . Both these conclusions are consistent with the corresponding conclusions in [Breuil and Mézard 2002, Theorem 1.2(i)]!

**Odd  $k \in [3, p - 1]$ :** Guerberoff and Park [2019, Theorem 5.0.5] computed the reduction of  $V_{k,\mathcal{L}}$  in terms of the valuation  $v_p(\mathcal{L} - a(k - 1))$ , where

$$a(j) = H_{j/2} + H_{j/2-1}$$

for  $j \geq 1$ . We clearly have  $a(k - 1) = H_- + H_+$ . Therefore, the regions used to classify the reductions in Theorem 1.3 match with those used in [loc. cit., Theorem 5.0.5]. Now:

- If  $v < 1 - (k - 2)/2$ , then Theorem 1.3 yields

$$\bar{V}_{k,\mathcal{L}}|_{I_{\mathbb{Q}_p}} \sim \omega_2^{k-1} \oplus \omega_2^{p(k-1)},$$

which agrees with the reduction computed in [loc. cit., Theorem 5.0.5(2)].

- Assume  $-\frac{1}{2} - l < v < \frac{1}{2} - l$  for some  $l \in \{0, 1, \dots, \frac{1}{2}(k - 5)\}$ . This region can be written as

$$\left(-l + \frac{1}{2}(k - 3)\right) - \frac{1}{2}(k - 2) < v < \left(-l + \frac{1}{2}(k - 1)\right) - \frac{1}{2}(k - 2).$$

<sup>3</sup>The computation  $\bar{a}(\overline{a_p/p}) = \overline{(-1 + 2(-\mathcal{L} + 2))(\overline{p/p})} = \overline{-2(\mathcal{L} - \frac{3}{2})}$  shows that the reduction agrees with the reduction computed in [Breuil and Mézard 2002, Theorem 1.2].

Theorem 1.3 yields

$$\bar{V}_{k,\mathcal{L}}|_{I_{\mathbb{Q}_p}} \sim \omega_2^{(k-1)+(p-1)(-l+\frac{1}{2}(k-3))} \oplus \omega_2^{p((k-1)+(p-1)(-l+\frac{1}{2}(k-3)))},$$

which agrees with the reduction computed in [Guerberoff and Park 2019, Theorem 5.0.5(1)].

- Assume  $v = -\frac{1}{2} - l$  for some  $l \in \{0, 1, \dots, \frac{1}{2}(k-5)\}$ . This can be written as

$$v = (-l + \frac{1}{2}(k-3)) - \frac{1}{2}(k-2).$$

Theorem 1.3 yields

$$\bar{V}_{k,\mathcal{L}}|_{I_{\mathbb{Q}_p}} \sim \omega^{k-1-(-l+\frac{1}{2}(k-3))} \oplus \omega^{(-l+\frac{1}{2}(k-3))},$$

which agrees with the reduction computed in [loc. cit., Theorem 5.0.5].

- Finally, if  $v \geq \frac{1}{2}$ , then Theorem 1.3 yields

$$\bar{V}_{k,\mathcal{L}}|_{I_{\mathbb{Q}_p}} \sim \omega^{\frac{1}{2}(k-1)} \oplus \omega^{\frac{1}{2}(k-1)},$$

which agrees with the reduction computed in [loc. cit., Theorem 5.0.5].

For the small weights  $k = 3$  and  $5$ , we may in fact improve on [loc. cit., Theorem 5.0.5] by computing the reductions  $\bar{V}_{k,\mathcal{L}}$  on the full Galois group  $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$  (by giving formulas for  $\lambda$ ). Indeed, since zig-zag has been proved for slopes  $\frac{1}{2}$  and  $\frac{3}{2}$  in [Buzzard and Gee 2013, Theorem A; Ghate and Rai 2025, Theorem 1.1], respectively, we obtain the following theorems.

**Theorem 1.4.** *Let  $p \geq 3$  and  $k = 3$ . We have the following dichotomy for the shape of the semisimplification of the reduction of the semistable representation  $V_{k,\mathcal{L}}$ :*

$$\bar{V}_{k,\mathcal{L}} \sim \begin{cases} \text{ind}(\omega_2^2) & \text{if } v < \frac{1}{2}, \\ \mu_\lambda \omega \oplus \mu_{\lambda^{-1}} \omega & \text{if } v \geq \frac{1}{2}, \end{cases}$$

where

$$\lambda + \frac{1}{\lambda} = \overline{-p^{-\frac{1}{2}}(\mathcal{L} - 1)}.$$

**Theorem 1.5.** *Let  $p \geq 5$  and  $k = 5$ . We have the following tetrachotomy for the shape of the semisimplification of the reduction of the semistable representation  $V_{k,\mathcal{L}}$ :*

$$\bar{V}_{k,\mathcal{L}} \sim \begin{cases} \text{ind}(\omega_2^4) & \text{if } v < -\frac{1}{2} \\ \mu_{\lambda_1} \omega^3 \oplus \mu_{\lambda_1^{-1}} \omega & \text{if } v = -\frac{1}{2}, \\ \text{ind}(\omega_2^{3+p}) & \text{if } -\frac{1}{2} < v < \frac{1}{2}, \\ \mu_{\lambda_2} \omega^2 \oplus \mu_{\lambda_2^{-1}} \omega^2 & \text{if } v \geq \frac{1}{2}, \end{cases}$$

where the constants  $\lambda_i$  are given by

$$\lambda_1 = \overline{-3p^{\frac{1}{2}}(\mathcal{L} - \frac{5}{2})}, \quad \lambda_2 + \frac{1}{\lambda_2} = \overline{2p^{-\frac{1}{2}}(\mathcal{L} - \frac{5}{2})}.$$

**Remark.** We remark that recently the second author noticed that it is possible to reverse the arguments used to prove [Theorem 1.3](#) to give a *proof* of the zig-zag conjecture, at least on inertia and for *all* half-integral slopes  $0 < v_p(a_p) \leq \frac{1}{2}(p - 3)$ , [[Ghate 2022, v1](#)] (these restrictions can be removed [[Ghate 2022, v2](#)] using very recent work of the first two authors [[Chitrao and Ghate 2023](#)] which computes  $\bar{V}_{k,\mathcal{L}}$  directly on  $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$  using the Iwahori mod  $p$  local Langlands correspondence [[Chitrao 2025](#)]).

**1.5. Relation with local constancy.** Berger [[2012](#)] proved the following theorem on the local constancy in the weight of the semisimplification of the reduction of crystalline representations.

**Theorem 1.6** [[Berger 2012, Theorem B](#)]. *Let  $a_p \neq 0$  and  $k > 3v_p(a_p) + \alpha(k - 1) + 1$ , where*

$$\alpha(j) = \sum_{n \geq 1} \left\lfloor \frac{j}{p^{n-1}(p-1)} \right\rfloor.$$

*Then there exists  $m = m(k, a_p)$  such that  $\bar{V}_{k',a_p} = \bar{V}_{k,a_p}$  if  $k' \geq k$  and  $k' - k \in p^{m-1}(p-1)\mathbb{Z}$ .*

This local constancy result does not extend to small weights  $k$  in the sense that the bound in the theorem is sharp. This was noticed in [[Ghate 2021](#)] using the following examples.

- Let  $k = 4$  and  $a_p = p \geq 5$ . Then  $4 \not\geq 3(1) + 1$  and the crystalline representation  $V_{k,a_p}$  does not satisfy Berger’s bound on the weight. Now let  $k' = 4 + p^n(p - 1)$  for large  $n$ . If the above local constancy result were to hold for  $V_{k,a_p}$ , then we would have  $\bar{V}_{k',a_p} \sim \bar{V}_{k,a_p}$ . However,  $\bar{V}_{k,a_p}$  is irreducible as  $k$  belongs to the range  $[2, p + 1]$  treated by Fontaine and Edixhoven [[1992](#)], but  $\bar{V}_{k',a_p} \sim \mu_3\omega^2 \oplus \mu_{3-1}\omega$  is reducible by [[Bhattacharya, Ghate, and Rozenstajn 2018, Theorem 1.1](#)].
- Let  $k = 5$  and  $a_p = p^{3/2}$  for  $p \geq 7$ . Then  $5 \not\geq 3(1.5) + 1$  and the crystalline representation  $V_{k,a_p}$  again does not satisfy the bound on the weight. Now let  $k' = 5 + p^n(p - 1)$ , for large  $n$ . If the above local constancy result were to hold for  $V_{k,a_p}$ , then we would have  $\bar{V}_{k',a_p} \sim \bar{V}_{k,a_p}$ . However,  $\bar{V}_{k,a_p} \sim \text{ind}(\omega_2^4)$  since  $k$  belongs to the Fontaine–Edixhoven range, but  $\bar{V}_{k',a_p} \sim \text{ind}(\omega_2^{3+p})$  by [[Ghate and Rai 2025, Theorem 1.1](#)].

However, we have the following corollary to [Theorem 1.1](#).

**Corollary 1.7.** *For  $k \geq 3$ , the sequence of crystalline representations  $V_{k+p^n(p-1), p^{r/2}}^*$  converges to the semistable representation  $V_{k,\mathcal{L}}^*$  for  $\mathcal{L} = 0$ .*

Therefore, for large  $n$  the reductions of the crystalline representations  $V_{k+p^n(p-1), p^{r/2}}$  in the examples above should be isomorphic to the reduction of the semistable representation  $V_{k,\mathcal{L}}$  with  $\mathcal{L} = 0$ , and not necessarily to the reduction of the crystalline representation  $V_{k,p^{r/2}}$  (though all may be the same, as is the case when  $k = 3$ ). Indeed, one checks that

- if  $k = 4$  and  $p \geq 5$ , then by [[Breuil and Mézard 2002, Theorem 1.2](#)], the reduction of the semistable representation  $V_{4,\mathcal{L}}$  for  $\mathcal{L} = 0$  is  $\mu_3\omega^2 \oplus \mu_{3-1}\omega$ , and,
- if  $k = 5$  and  $p \geq 7$ , then by [[Guerberoff and Park 2019, Theorem 5.0.5](#)] (or, better, by [Theorem 1.5](#)), the reduction of the semistable representation  $V_{5,\mathcal{L}}$  for  $\mathcal{L} = 0$  is  $\text{ind}(\omega_2^{3+p})$ .

Thus, there is no apparent contradiction to local constancy in the weight for the reductions of the crystalline representations above when  $k$  is small if one works in the Colmez–Chenevier space which includes semistable representations of weight  $k$ .

## 2. The blow-up of $U_r$

In this section, we provide details about blow-ups in the rigid-analytic setting which may be of independent interest. This is inspired by [Schoutens 1995], but we carefully work out the details and also do not work over algebraically closed fields. For background on rigid-analytic geometry, see [Bosch, Güntzer, and Remmert 1984]. We also recall the important Proposition 2.5, which for each finite extension  $E$  of  $\mathbb{Q}_p$  establishes a bijection between the  $E$ -valued points of  $\tilde{U}_r$  that lie above  $(p^r, (1+p)^{k-1} - 1)$  and the tangent directions in  $U_r$  at the exceptional point.

**2.1. The blow-up as a rigid-analytic variety.** The blow-up of  $U_r$  consists of a rigid-analytic variety  $\tilde{U}_r$  and a map  $\pi : \tilde{U}_r \rightarrow U_r$  satisfying the properties given in [Schoutens 1995, Definition 1.2.1]. In this subsection, we will construct  $\tilde{U}_r$ .

Recall that  $U_r = \mathrm{Sp} \mathcal{O}(U_r)$ , where

$$\mathcal{O}(U_r) = \mathbb{Q}_p \langle S_1, S_2, T_1, T_2, T_3 \rangle / (p^r T_1 - S_1, 1 - T_1 T_2, p T_3 - S_2).$$

The  $\mathbb{Q}_p$ -valued point  $(p^r, (1+p)^{k-1} - 1)$  corresponds to the maximal ideal  $m = (f_1, f_2)$  of  $\mathcal{O}(U_r)$ , where  $f_1 = S_1 - p^r$  and  $f_2 = S_2 - ((1+p)^{k-1} - 1)$ . By the blow-up of  $U_r$  at  $(p^r, (1+p)^{k-1} - 1)$ , we mean the blow-up at the maximal ideal  $m$ .

We will construct  $\tilde{U}_r$  by a patching argument. Let  $Q_1, Q_2, Q'_1$  and  $Q'_2$  be indeterminates. For  $i = 0, 1, \dots$ , consider the affinoid algebra

$$A_i = \mathcal{O}(U_r) \langle p^i Q_1 \rangle / (f_2 - Q_1 f_1),$$

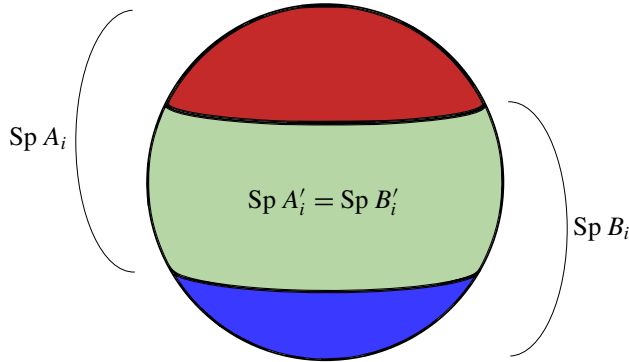
which describes the subset of  $U_r$  where  $|f_2| \leq p^i |f_1|$ . Sending  $p^{i+1} Q_1$  to  $p \cdot p^i Q_1$  and  $X$  to  $p^i Q_1$  induces an isomorphism  $A_{i+1} \langle X \rangle / (pX - p^{i+1} Q_1) \cong A_i$ , showing that  $\mathrm{Sp} A_i$  is the affinoid subdomain (hence an open subvariety) of  $\mathrm{Sp} A_{i+1}$  consisting of maximal ideals  $\mathfrak{m}$  such that  $|Q_1 \bmod \mathfrak{m}| \leq p^i$ . We therefore get a sequence of inclusions  $\mathrm{Sp} A_0 \rightarrow \mathrm{Sp} A_1 \rightarrow \mathrm{Sp} A_2 \rightarrow \dots$  associated to the sequence of affinoid algebra homomorphisms  $\dots \rightarrow A_2 \rightarrow A_1 \rightarrow A_0$ . Using [Bosch, Güntzer, and Remmert 1984, Proposition 9.3.2/1] we paste together the  $\mathrm{Sp} A_i$  for  $i \geq 0$  to get a rigid-analytic variety  $\tilde{V}_1$ . The same proposition also states that  $\{\mathrm{Sp} A_i\}_{i \geq 0}$  is an admissible cover of  $\tilde{V}_1$ . Similarly, for  $i \geq 0$ , we define

$$B_i = \mathcal{O}(U_r) \langle p^i Q_2 \rangle / (f_1 - Q_2 f_2)$$

and glue the corresponding affinoid spaces to get a rigid-analytic variety  $\tilde{V}_2$ . The blow-up  $\tilde{U}_r$  is the rigid-analytic variety obtained by gluing  $\tilde{V}_1$  and  $\tilde{V}_2$  along certain open subvarieties.

We describe these open subvarieties now. For each  $i \geq 0$ , consider the affinoid algebra

$$A'_i = \mathcal{O}(U_r) \langle p^i Q_1, p^i Q'_1 \rangle / (f_2 - Q_1 f_1, 1 - Q_1 Q'_1),$$



**Figure 2.** Exceptional fiber of the blow-up in the rigid setting.

which describes a Laurent subdomain  $\text{Sp } A'_i$  of  $\text{Sp } A_i$  given by the condition  $|Q_1| \geq p^{-i}$ . Since we have an isomorphism  $A'_{i+1}(X, Y)/(pX - p^{i+1}Q_1, pY - p^{i+1}Q'_1) \cong A'_i$  (sending  $X$  to  $p^i Q_1$  and  $Y$  to  $p^i Q'_1$ ), we see that  $\text{Sp } A'_i$  is an affinoid subdomain (and hence an open subvariety) of  $\text{Sp } A'_{i+1}$ . Using [Bosch, Güntzer, and Remmert 1984, Proposition 9.3.2/1], we paste together  $\text{Sp } A'_i$  for  $i \geq 0$  to get a rigid-analytic variety  $\tilde{V}'_1$ . Moreover, using [loc. cit., Proposition 9.3.3/1], we see that the canonical inclusions  $\text{Sp } A'_i \rightarrow \text{Sp } A_i$  induce an inclusion  $\tilde{V}'_1 \hookrightarrow \tilde{V}_1$ , identifying  $\tilde{V}'_1$  as a subvariety of  $\tilde{V}_1$ . Similarly, we construct a subvariety  $\tilde{V}'_2$  of  $\tilde{V}_2$  by gluing all the  $\text{Sp } B'_i$  together, where

$$B'_i = \mathcal{O}(U_r)(p^i Q_2, p^i Q'_2)/(f_1 - Q_2 f_2, 1 - Q_2 Q'_2).$$

It turns out that  $\text{Sp } A'_i$  and  $\text{Sp } B'_i$  are the intersections of  $\text{Sp } A_i$  and  $\text{Sp } B_i$  in the blow-up  $\tilde{U}_r$  that we will soon construct. The spaces above are summarized by Figure 2.

The space  $\tilde{V}_1$  corresponds to the surface of the sphere except the south pole in Figure 2, whereas  $\tilde{V}_2$  corresponds to the surface of the sphere except the north pole.

We claim that  $\tilde{V}'_j$  is an open subvariety of  $\tilde{V}_j$  for  $j = 1, 2$ . Without loss of generality, assume  $j = 1$ . To prove this claim, we need to show that  $\tilde{V}'_1 \cap \text{Sp } A_i$  is an admissible open subset of  $\text{Sp } A_i$  for each  $i \geq 0$ . Write  $\tilde{V}'_1 \cap \text{Sp } A_i = \bigcup_{k=i}^{\infty} (\text{Sp } A'_k \cap \text{Sp } A_i)$ . If  $k \geq i$ , then  $\text{Sp } A'_k \cap \text{Sp } A_i$  is the set of maximal ideals  $\mathfrak{m}$  of  $A_i$  satisfying  $|Q_1 \bmod \mathfrak{m}| \geq p^{-k}$ . Therefore  $\tilde{V}'_1 \cap \text{Sp } A_i$  is the set of maximal ideals  $\mathfrak{m}$  of  $A_i$  such that  $|Q_1 \bmod \mathfrak{m}| > 0$ . In other words, it is the complement of the vanishing set of  $Q_1$  in  $\text{Sp } A_i$ ; i.e., it is a Zariski open subset. Since Zariski open subsets are admissible open, we have proved the claim.

To glue  $\tilde{V}'_1$  and  $\tilde{V}'_2$ , we define an isomorphism  $\phi : \tilde{V}'_1 \rightarrow \tilde{V}'_2$  as follows. Recall that, for  $i \geq 0$ ,

$$\begin{aligned} A'_i &= \mathcal{O}(U_r)(p^i Q_1, p^i Q'_1)/(f_2 - Q_1 f_1, 1 - Q_1 Q'_1), \\ B'_i &= \mathcal{O}(U_r)(p^i Q_2, p^i Q'_2)/(f_1 - Q_2 f_2, 1 - Q_2 Q'_2). \end{aligned}$$

Consider an isomorphism of affinoid algebras  $B'_i \rightarrow A'_i$  given by

$$\begin{aligned} p^i Q_2 &\rightarrow p^i Q'_1, \\ p^i Q'_2 &\rightarrow p^i Q_1. \end{aligned}$$

This isomorphism gives rise to an isomorphism  $\phi_i : \mathrm{Sp} A'_i \rightarrow \mathrm{Sp} B'_i$  of the corresponding affinoid spaces. For  $i, j \geq 0$ , these maps clearly restrict to the same map on  $\mathrm{Sp} A'_i \cap \mathrm{Sp} A'_j$ . Therefore applying [Bosch, Guntzer, and Remmert 1984, Proposition 9.3.3/1], we get an isomorphism of rigid-analytic varieties  $\phi : \tilde{V}'_1 \rightarrow \tilde{V}'_2$ .

Define  $\tilde{U}_r$  to be the rigid-analytic variety obtained by gluing  $\tilde{V}_1$  and  $\tilde{V}_2$  along  $\tilde{V}'_1$  and  $\tilde{V}'_2$ , respectively, using the isomorphism  $\phi$ . This rigid-analytic variety is analogous to the blow-up in the classical algebraic geometry setting.

We see this by computing its  $E$ -valued points for any finite extension  $E$  of  $\mathbb{Q}_p$ . Since  $\tilde{U}_r$  is covered by  $\tilde{V}_1$  and  $\tilde{V}_2$ , we compute  $\tilde{V}_1(E)$  and  $\tilde{V}_2(E)$  first. Recall, for  $i \geq 0$ ,

$$A_i = \mathcal{O}(U_r)\langle p^i Q_1 \rangle / (f_2 - Q_1 f_1).$$

We identify the set of  $E$ -valued points of  $\mathrm{Sp} A_i$  with

$$\{(s_1, s_2, q_1) \in U_r(E) \times E \mid (1 + s_2 - (1 + p)^{k-1}) = q_1(s_1 - p^r), |q_1| \leq p^i\}.$$

The first condition is a consequence of the relation  $f_2 = Q_1 f_1$ . For  $i \geq j$ , the gluing map  $A_i \rightarrow A_j$  induces the canonical inclusion map on  $E$ -valued points  $\mathrm{Sp} A_j(E) \rightarrow \mathrm{Sp} A_i(E)$ . Therefore, the set of  $E$ -valued points of  $\tilde{V}_1$  is the union of all the  $\mathrm{Sp} A_i(E)$ , i.e.,

$$\tilde{V}_1(E) = \{(s_1, s_2, q_1) \in U_r(E) \times E \mid (1 + s_2 - (1 + p)^{k-1}) = q_1(s_1 - p^r)\}.$$

We similarly have

$$\tilde{V}_2(E) = \{(s_1, s_2, q_2) \in U_r(E) \times E \mid (s_1 - p^r) = q_2(1 + s_2 - (1 + p)^{k-1})\}.$$

We can now describe  $\tilde{U}_r(E)$ . If  $P$  is an  $E$ -valued point of  $\tilde{V}_1 \cap \tilde{V}_2$ , it is an  $E$ -valued point of  $\mathrm{Sp} A_i \cap \mathrm{Sp} B_i = \mathrm{Sp} A'_i$  for some  $i \geq 0$ . Being a point of  $\mathrm{Sp} A_i$ , it is of the form  $(s_1, s_2, q_1)$  for some  $s_1, s_2, q_1 \in E$ . Since the gluing map between  $B'_i$  and  $A'_i$  takes  $S_1$  to  $S_1$ ,  $S_2$  to  $S_2$  and  $Q_2$  to the inverse of  $Q_1$  in  $A'_i$ , we see that  $(s_1, s_2, q_1^{-1})$  represents  $P$  as an  $E$ -valued point of  $\mathrm{Sp} B_i$  (the condition  $|q_1^{-1}| \leq p^i$  is satisfied because  $P \in \mathrm{Sp} A'_i$  implies  $|q_1| \geq p^{-i}$ ). We can therefore identify  $(s_1, s_2, q_1) \in \tilde{V}_1(E)$  with the point  $(s_1, s_2, 1 : q_1) \in U_r(E) \times \mathbb{P}^1(E)$  and  $(s_1, s_2, q_2) \in \tilde{V}_2(E)$  with the point  $(s_1, s_2, q_2 : 1) \in U_r(E) \times \mathbb{P}^1(E)$ , so that a point in the intersection goes to the same point in  $U_r(E) \times \mathbb{P}^1(E)$  under both of these identifications. As a result of this discussion, we see that

$$\tilde{U}_r(E) = \{(s_1, s_2, \xi_1 : \xi_2) \in U_r(E) \times \mathbb{P}^1(E) \mid (s_1 - p^r)\xi_2 = (1 + s_2 - (1 + p)^{k-1})\xi_1\} \tag{5}$$

exactly as in the classical algebraic geometry setting.

**2.2. The blow-up map  $\pi : \tilde{U}_r \rightarrow U_r$ .** In this subsection, we define a candidate for the blow-up map  $\pi : \tilde{U}_r \rightarrow U_r$  using [Bosch, Guntzer, and Remmert 1984, Proposition 9.3.3/1].

To define  $\pi$ , we first define its restrictions  $\pi_i$  to  $\tilde{V}_i$  for  $i = 1, 2$ . We first define the map  $\pi_1 : \tilde{V}_1 \rightarrow U_r$ . For each  $i \geq 0$ , consider the map  $\mathrm{Sp} A_i \rightarrow U_r$  associated to the canonical homomorphism

$$\mathcal{O}(U_r) \rightarrow \mathcal{O}(U_r)\langle p^i Q_1 \rangle / (f_2 - Q_1 f_1) = A_i.$$

These maps clearly restrict to the same map on  $\mathrm{Sp} A_i \cap \mathrm{Sp} A_j$  for  $i, j \geq 0$ . So using [Bosch, Güntzer, and Remmert 1984, Proposition 9.3.3/1], we glue these maps together to get a map  $\pi_1 : \tilde{V}_1 \rightarrow U_r$  of rigid-analytic varieties. The other map  $\pi_2 : \tilde{V}_2 \rightarrow U_r$  is constructed similarly.

Now, to get a map  $\pi : \tilde{U}_r \rightarrow U_r$ , we have to check that  $\pi_1$  and  $\pi_2$  agree on the intersection of  $\tilde{V}_1$  and  $\tilde{V}_2$  in  $\tilde{U}_r$ . Since this intersection is equal to  $\tilde{V}'_1 (\cong \tilde{V}'_2)$ , we have to check that the following diagram commutes:

$$\begin{array}{ccc}
 \tilde{V}'_1 & \xrightarrow{\pi_1|_{\tilde{V}'_1}} & U_r \\
 \phi \downarrow & & \nearrow \\
 \tilde{V}'_2 & \xrightarrow{\pi_2|_{\tilde{V}'_2}} & U_r
 \end{array}$$

Since  $\phi$  is obtained by pasting the maps  $\phi_i : \mathrm{Sp} A'_i \rightarrow \mathrm{Sp} B'_i$ , the commutativity of the diagram above is equivalent to the commutativity of the following diagram for all  $i \geq 0$ :

$$\begin{array}{ccc}
 \mathrm{Sp} A'_i & \xrightarrow{\pi_1|_{\mathrm{Sp} A'_i}} & U_r \\
 \phi_i \downarrow & & \nearrow \\
 \mathrm{Sp} B'_i & \xrightarrow{\pi_2|_{\mathrm{Sp} B'_i}} & U_r
 \end{array}$$

The commutativity of the diagram above can be checked by reversing all the arrows and noting that the resulting maps are  $\mathcal{O}(U_r)$ -algebra homomorphisms. This shows that  $\pi_1$  and  $\pi_2$  agree on the intersection of  $\tilde{V}_1$  and  $\tilde{V}_2$  in  $\tilde{U}_r$ . Therefore, we glue  $\pi_1$  and  $\pi_2$  using [loc. cit., Proposition 9.3.3/1] to get a map  $\pi : \tilde{U}_r \rightarrow U_r$ .

**2.3. Proof that  $\pi : \tilde{U}_r \rightarrow U_r$  is the blow-up.** In this subsection, we prove that the map  $\pi : \tilde{U}_r \rightarrow U_r$  defined in the previous subsection is the blow-up of  $U_r$  at the maximal ideal  $m = (f_1, f_2)$ . To prove this, we need to check that  $\pi$  satisfies the two properties stated in [Schoutens 1995, Definition 1.2.1], namely the invertibility of a certain sheaf, and a corresponding universal property.

We first show that the ideal sheaf on  $U_r$  corresponding to the ideal  $m$  of  $\mathcal{O}(U_r)$  extends to an invertible ideal sheaf on  $\tilde{U}_r$  in the sense of [loc. cit., Definition 1.1.1]. Since this is a local criterion (this means that the criterion can be checked over an admissible cover), we need to check that, for all  $i \geq 0$ , the maps  $\pi|_{\mathrm{Sp} A_i} : \mathrm{Sp} A_i \rightarrow U_r$  and  $\pi|_{\mathrm{Sp} B_i} : \mathrm{Sp} B_i \rightarrow U_r$  satisfy the same property. Let us prove this statement for  $\pi|_{\mathrm{Sp} A_i} : \mathrm{Sp} A_i \rightarrow U_r$ . In this case, the extended ideal sheaf corresponds to the ideal  $m A_i$  on  $\mathrm{Sp} A_i$ . Moreover, [loc. cit., Proposition 1.1.4] states that the invertibility of this sheaf is equivalent to the invertibility of the ideal  $m A_{i,m}$  of  $A_{i,m}$  for all maximal ideals  $m$  of  $A_i$ , where  $A_{i,m}$  is the localization of  $A_i$  at  $m$ . To show this, it is enough to prove that  $m A_i$  is generated by a regular element of  $A_i$  because regular elements go to regular elements under flat base change.

Since  $m A_i$  is generated by  $f_1$ , we prove that  $f_1$  is not a zero divisor in  $A_i$  for  $i \geq 0$ . Note that we only need to prove this statement for  $i = 0$  because of the injections  $A_i \hookrightarrow A_0$ .

**Lemma 2.1.**  $f_1$  is not a zero divisor in  $A_0 = \mathcal{O}(U_r)\langle Q_1 \rangle / (f_2 - Q_1 f_1)$ .

*Proof.* Recall that

$$\mathcal{O}(U_r) = \mathbb{Q}_p\langle S_1, S_2, T_1, T_2, T_3 \rangle / (p^r T_1 - S_1, 1 - T_1 T_2, p T_3 - S_2).$$

To prove that  $f_1$  is not a zero divisor in  $A_0$ , it is enough to prove that  $f_1$  is not a zero divisor in  $\mathbb{Q}_p\langle S_1, S_2, Q_1 \rangle / (f_2 - Q_1 f_1)$ . Indeed, since  $\text{Sp } A_0$  is an affinoid subdomain (a Laurent subdomain) of  $\text{Sp } \mathbb{Q}_p\langle S_1, S_2, Q_1 \rangle / (f_2 - Q_1 f_1)$ , the canonical map  $\mathbb{Q}_p\langle S_1, S_2, Q_1 \rangle / (f_2 - Q_1 f_1) \rightarrow A_0$  is flat by [Bosch, Güntzer, and Remmert 1984, Corollary 7.3.2/6]. Therefore, if  $f_1$  is not a zero divisor in the former then, by flatness, it is not a zero divisor in the latter.

So let us now prove that  $f_1$  is not a zero divisor in  $\mathbb{Q}_p\langle S_1, S_2, Q_1 \rangle / (f_2 - Q_1 f_1)$ . Assume that there exists  $h \in \mathbb{Q}_p\langle S_1, S_2, Q_1 \rangle$  such that  $f_2 - Q_1 f_1$  divides  $f_1 h$ . We know that

$$f_2 - Q_1 f_1 = 1 + S_2 - (1 + p)^{k-1} - Q_1(S_1 - p^r)$$

is  $S_2$ -distinguished of degree 1 (see [loc. cit., Definition 5.2.1/1]). Applying the Weierstrass division theorem (see [loc. cit., Theorem 5.2.1/2]) to  $h$  and  $f_2 - Q_1 f_1$ , we see that there exist unique power series  $q \in \mathbb{Q}_p\langle S_1, S_2, Q_1 \rangle$  and  $r \in \mathbb{Q}_p\langle S_1, Q_1 \rangle$  (note that  $S_2$  does not appear in  $r$  because  $f_2 - Q_1 f_1$  is of degree 1 in  $S_2$ ) such that  $h = q(f_2 - Q_1 f_1) + r$ . The fact that  $f_2 - Q_1 f_1$  divides  $f_1 h$  implies that  $f_2 - Q_1 f_1$  divides  $f_1 r$ . This is possible only if  $r = 0$ , i.e., only if  $f_2 - Q_1 f_1$  divides  $h$ . Therefore  $h = 0$  in  $\mathbb{Q}_p\langle S_1, S_2, Q_1 \rangle / (f_2 - Q_1 f_1)$ .  $\square$

Having checked the first property of blow-ups for  $\pi$ , we now check the second property. Using [Schoutens 1995, Lemma 1.2.4], we note that it is enough to check it for affinoid spaces. In other words, given any affinoid space  $Y = \text{Sp } R$  and a map of rigid-analytic varieties  $f : Y \rightarrow U_r$  such that the ideal sheaf on  $Y$  associated to the ideal  $mR$  is invertible, we prove that there exists a unique map  $g : Y \rightarrow \tilde{U}_r$  such that the following diagram commutes:

$$\begin{array}{ccc} & & \tilde{U}_r \\ & \nearrow g & \downarrow \pi \\ Y & \xrightarrow{f} & U_r \end{array}$$

For  $i = 1, 2$ , define  $\mathfrak{a}_i = (f^*(f_i)R : mR)$  and let  $Y_i = Y \setminus V(\mathfrak{a}_i)$ . We prove two lemmas about the  $Y_i$ .

**Lemma 2.2.** For  $i = 1, 2$ , a point  $y \in Y$  belongs to  $Y_i$  if and only if  $m\mathcal{O}_{Y,y} = f^*(f_i)\mathcal{O}_{Y,y}$ .

*Proof.* We thank one of the referees for pointing out the useful [Atiyah and Macdonald 1969, Corollary 3.15], which shortens the proof. Let  $y$  be a point in  $Y$  and  $\mathfrak{m}_y$  be the corresponding maximal ideal of  $R$ . Fix  $i = 1, 2$ . Using [loc. cit., Corollary 3.15], we see that  $\mathfrak{a}_i R_{\mathfrak{m}_y} = (f^*(f_i)R_{\mathfrak{m}_y} : mR_{\mathfrak{m}_y})$ . Now,

$$y \in Y_i \iff \mathfrak{a}_i R_{\mathfrak{m}_y} = R_{\mathfrak{m}_y} \iff mR_{\mathfrak{m}_y} = f^*(f_i)R_{\mathfrak{m}_y}.$$

Using [Bosch, Güntzer, and Remmert 1984, Proposition 7.3.2/3], we have a map  $R_{m_y} \rightarrow \mathcal{O}_{Y,y}$ , which induces an isomorphism  $\widehat{R}_{m_y} \simeq \widehat{\mathcal{O}}_{Y,y}$ . We conclude that

$$mR_{m_y} = f^*(f_i)R_{m_y} \iff m\mathcal{O}_{Y,y} = f^*(f_i)\mathcal{O}_{Y,y}.$$

Indeed, the forward implication is obtained by extending scalars and the reverse implication follows by extending scalars to the completion  $\widehat{\mathcal{O}}_{Y,y}$  and using the fact that  $R_{m_y} \rightarrow \widehat{R}_{m_y}$  is faithfully flat.  $\square$

Using [Schoutens 1995, Lemma 1.1.2] and the lemma above one easily checks that  $Y = Y_1 \cup Y_2$ . The sets  $Y_1$  and  $Y_2$ , being Zariski open subsets of  $Y$ , are admissible open and form an admissible cover of  $Y$ .

Before proving the next lemma, we remark that, by [loc. cit., Lemma 0.4], we have  $\mathfrak{a}_i\mathcal{O}(Y_i) = \mathcal{O}(Y_i)$ , which implies that  $m\mathcal{O}(Y_i) = f^*(f_i)\mathcal{O}(Y_i)$  for  $i = 1, 2$ .

**Lemma 2.3.** *Let  $i = 1, 2$ . Then,  $f^*(f_i)$  is not a zero divisor in  $\mathcal{O}(Y')$  for any admissible open subset  $Y'$  of  $Y$  contained in  $Y_i$ .*

*Proof.* The proof has three steps. In the first step, we prove that  $f^*(f_i)$  is not a zero divisor in  $\mathcal{O}_{Y,y}$  for any  $y \in Y'$ . In the second step, we prove that the canonical map  $\mathcal{O}(Y') \rightarrow \prod_{y \in Y'} \mathcal{O}_{Y,y}$  is injective. In the third step, we use these two facts to conclude that  $f^*(f_i)$  is not a zero divisor in  $\mathcal{O}(Y')$ . Let  $i = 1$  or  $2$ .

- Using the remark preceding this lemma, we get  $m\mathcal{O}(Y') = f^*(f_i)\mathcal{O}(Y')$ . Extending to the stalks, we get  $m\mathcal{O}_{Y,y} = f^*(f_i)\mathcal{O}_{Y,y}$  for each  $y \in Y'$ . Since the sheaf associated to the ideal  $m\mathcal{O}(Y)$  is invertible, we see that  $m\mathcal{O}_{Y,y}$  is generated by a regular element of  $\mathcal{O}_{Y,y}$  for each  $y \in Y$ . Therefore  $f^*(f_i)$  is not a zero divisor in  $\mathcal{O}_{Y,y}$  for each  $y \in Y'$ .
- Consider the map  $\mathcal{O}(Y') \rightarrow \prod_{y \in Y'} \mathcal{O}_{Y,y}$ . Let  $a$  be an element of  $\mathcal{O}(Y')$  that maps to 0 in  $\mathcal{O}_{Y,y}$  for each  $y \in Y'$ . Choose an admissible cover  $\{U_j\}_{j \in J}$  of  $Y'$  by affinoid subdomains of  $Y$ . For any  $j \in J$ , the image of  $a$  in  $\mathcal{O}_{Y,y}$  is 0 for each  $y \in U_j$ . Using [Bosch, Güntzer, and Remmert 1984, Corollary 7.3.2/4], we see that the restriction of  $a$  to  $\mathcal{O}(U_j)$  is 0 for each  $j \in J$ . Since the cover  $\{U_j\}_{j \in J}$  is admissible, we see that  $a = 0$  in  $\mathcal{O}(Y')$ .
- Suppose there exists a  $b \in \mathcal{O}(Y')$  such that  $f^*(f_i)b = 0$ . This means that  $f^*(f_i)b = 0$  in  $\mathcal{O}_{Y,y}$  for each  $y \in Y'$ . Using the fact that  $f^*(f_i)$  is not a zero divisor in  $\mathcal{O}_{Y,y}$  for  $y \in Y'$ , we see that the image of  $b$  in  $\mathcal{O}_{Y,y}$  is 0. The injectivity of the map  $\mathcal{O}(Y') \rightarrow \prod_{j \in J} \mathcal{O}_{Y,y}$  implies that  $b = 0$  in  $\mathcal{O}(Y')$ . Therefore  $f^*(f_i)$  is not a zero divisor in  $\mathcal{O}(Y')$ .  $\square$

We now construct  $g : Y \rightarrow \widetilde{U}_r$ . In order to do this, we use the above lemmas to define the restrictions of  $g$  to the elements of a refinement of the cover  $Y = Y_1 \cup Y_2$  and then apply the usual patching argument.

As  $f^*(f_2) \in m\mathcal{O}(Y_1) = f^*(f_1)\mathcal{O}(Y_1)$ , there exists a  $q_1 \in \mathcal{O}(Y_1)$  such that  $f^*(f_2) = q_1 f^*(f_1)$ . Similarly, there exists a  $q_2 \in \mathcal{O}(Y_2)$  such that  $f^*(f_1) = q_2 f^*(f_2)$ . Furthermore,  $q_1$  and  $q_2$  are unique because of Lemma 2.3.

Since  $Y_1$  is an admissible open subset of  $Y$ , there exists an admissible cover  $\{Y_{1,m}\}_{m \in I}$  of  $Y_1$  by affinoid subdomains  $Y_{1,m}$  of  $Y$ . Here and just below  $m$  is an element of the indexing set  $I$  and should not be confused with the maximal ideal  $m$  of  $\mathcal{O}(U_r)$  used throughout this paper. For each  $m \in I$ , there

is a restriction map  $\mathcal{O}(Y_1) \rightarrow \mathcal{O}(Y_{1,m})$  and so we can think of  $q_1$  as an element of  $\mathcal{O}(Y_{1,m})$ . For each  $m \in I$ , fix a natural number  $\underline{m}$  such that  $|p^{\underline{m}}q_1| \leq 1$ . Consider the affinoid algebra map  $f^* : \mathcal{O}(U_r) \rightarrow \mathcal{O}(Y)$  and compose it with the restriction map  $\mathcal{O}(Y) \rightarrow \mathcal{O}(Y_{1,m})$ . Extend this composition to a map  $\mathcal{O}(U_r)\langle p^{\underline{m}}Q_1 \rangle \rightarrow \mathcal{O}(Y_{1,m})$  by sending  $p^{\underline{m}}Q_1$  to  $p^{\underline{m}}q_1$ . This is possible because of [Bosch, Güntzer, and Remmert 1984, Corollary 1.4.3/2]. Since we have the relation  $f^*(f_1) = q_1 f^*(f_2)$  in  $\mathcal{O}(Y_{1,m})$ , we see that this extension factors through an affinoid algebra map  $A_{\underline{m}} \rightarrow \mathcal{O}(Y_{1,m})$ . Let  $g_{1,m} : Y_{1,m} \rightarrow \text{Sp } A_{\underline{m}}$  be the corresponding map of affinoid spaces. Similarly, there is an admissible cover  $\{Y_{2,n}\}_{n \in J}$  of  $Y_2$  by affinoid subdomains  $Y_{2,n}$  of  $Y$  and, for each  $n \in J$ , a fixed natural number  $\underline{n}$  such that  $|p^{\underline{n}}q_2| \leq 1$  and a map of affinoid spaces  $g_{2,n} : Y_{2,n} \rightarrow \text{Sp } B_{\underline{n}}$  associated to the map of affinoid algebras  $B_{\underline{n}} \rightarrow \mathcal{O}(Y_{2,n})$  sending  $p^{\underline{n}}Q_2$  to  $p^{\underline{n}}q_2$ . Since  $g_{1,m}^*$  and  $g_{2,n}^*$  are  $\mathcal{O}(U_r)$ -algebra homomorphisms, we see that  $\pi \circ g_{1,m} = f$  on  $Y_{1,m}$  and  $\pi \circ g_{2,n} = f$  on  $Y_{2,n}$  for  $m \in I, n \in J$ .

Note that  $\{Y_{1,m}, Y_{2,n}\}_{m \in I, n \in J}$  is an admissible cover of  $Y$ . To obtain  $g : Y \rightarrow \tilde{U}_r$ , we glue all the maps  $g_{1,m}$  and  $g_{2,n}$ . Without loss of generality, assume  $\underline{m} \geq \underline{m}'$ . Then,  $g_{1,m}$  and  $g_{1,m}'$  agree on  $Y_{1,m} \cap Y_{1,m}'$  because the following diagram commutes:

$$\begin{array}{ccc}
 A_{\underline{m}'} & \xrightarrow{\quad g_{1,m'}^* \quad} & \mathcal{O}(Y_{1,m} \cap Y_{1,m}') \\
 \uparrow & & \nearrow \\
 A_{\underline{m}} & \xrightarrow{\quad g_{1,m}^* \quad} & 
 \end{array}$$

A similar argument shows that  $g_{2,n}$  and  $g_{2,n}'$  agree on  $Y_{2,n} \cap Y_{2,n}'$ .

We check that the maps  $g_{1,m}$  and  $g_{2,n}$  agree on  $Y_{1,m} \cap Y_{2,n}$ . Let  $N \geq \underline{m}, \underline{n}$ . Think of  $g_{1,m}$  as a map from  $Y_{1,m}$  to  $\text{Sp } A_N$  (by composing with the map  $\text{Sp } A_{\underline{m}} \rightarrow \text{Sp } A_N$ ) and  $g_{2,n}$  as a map from  $Y_{2,n}$  to  $\text{Sp } B_N$  (by composing with the map  $\text{Sp } B_{\underline{n}} \rightarrow \text{Sp } B_N$ ). We claim that  $g_{1,m}$  maps  $Y_{1,m} \cap Y_{2,n}$  into  $\text{Sp } A'_N$  and that  $g_{2,n}$  maps  $Y_{1,m} \cap Y_{2,n}$  into  $\text{Sp } B'_N$ . To show this, we first prove that  $q_1$  is the inverse of  $q_2$  in  $\mathcal{O}(Y_{1,m} \cap Y_{2,n})$ . Transfer the relations  $f^*(f_2) = q_1 f^*(f_1)$  and  $f^*(f_1) = q_2 f^*(f_2)$  from  $\mathcal{O}(Y_1)$  and  $\mathcal{O}(Y_2)$ , respectively, to  $\mathcal{O}(Y_{1,m} \cap Y_{2,n})$  by the restriction maps. Substituting the first relation into the second relation, we get  $f^*(f_1) = q_2 q_1 f^*(f_1)$ . Using Lemma 2.3, we cancel  $f^*(f_1)$  to see that  $q_1$  is indeed the inverse of  $q_2$  in  $\mathcal{O}(Y_{1,m} \cap Y_{2,n})$ .

Compose  $g_{1,m}^* : A_N \rightarrow \mathcal{O}(Y_{1,m})$  with the restriction map  $\mathcal{O}(Y_{1,m}) \rightarrow \mathcal{O}(Y_{1,m} \cap Y_{2,n})$ . Extend this composition to a map  $A_N \langle p^N Q'_1 \rangle \rightarrow \mathcal{O}(Y_{1,m} \cap Y_{2,n})$  by sending  $p^N Q'_1$  to  $p^N q_2$ . This is possible because  $|p^N q_2| \leq 1$  since the intersection of two affinoid subdomains of an affinoid space is again an affinoid space and because under the homomorphism  $\mathcal{O}(Y_{2,n}) \rightarrow \mathcal{O}(Y_{1,m} \cap Y_{2,n})$  of affinoid algebras, the image of an element has norm at most the norm of the element itself. Using the fact that  $q_2$  is the inverse of  $q_1$  in  $\mathcal{O}(Y_{1,m} \cap Y_{2,n})$ , we may further extend the above map to  $A'_N$  to obtain the following commutative diagram:

$$\begin{array}{ccc}
 A_N = \mathcal{O}(U_r)\langle p^N Q_1 \rangle / (f_2 - Q_1 f_1) & \xrightarrow{\quad g_{1,m}^* \quad} & \mathcal{O}(Y_{1,m}) \\
 \downarrow & & \downarrow \\
 A'_N = \mathcal{O}(U_r)\langle p^N Q_1, p^N Q'_1 \rangle / (f_2 - Q_1 f_1, 1 - Q_1 Q'_1) & \longrightarrow & \mathcal{O}(Y_{1,m} \cap Y_{2,n})
 \end{array}$$

This shows that the affinoid algebra map  $A_N \rightarrow \mathcal{O}(Y_{1,m} \cap Y_{2,n})$  factors through the map  $A_N \rightarrow A'_N$ . Reversing the arrows, we see that the restriction of  $g_{1,m}$  to  $Y_{1,m} \cap Y_{2,n}$  factors through the inclusion  $\mathrm{Sp} A'_N \rightarrow \mathrm{Sp} A_N$ . In other words,  $g_{1,m}$  maps  $Y_{1,m} \cap Y_{2,n}$  into  $\mathrm{Sp} A'_N$ . Similarly, we can prove that  $g_{2,n}$  maps  $Y_{1,m} \cap Y_{2,n}$  into  $\mathrm{Sp} B'_N$ .

Now we can prove that  $g_{1,m}$  and  $g_{2,n}$  agree on  $Y_{1,m} \cap Y_{2,n}$ . Indeed, this statement is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc} A'_N & \xrightarrow{g_{1,m}^*} & \mathcal{O}(Y_{1,m} \cap Y_{2,n}) \\ \phi_N^* \uparrow & & \uparrow \\ B'_N & \xrightarrow{g_{2,n}^*} & \mathcal{O}(Y_{1,m} \cap Y_{2,n}) \end{array}$$

where the vertical map is the gluing map. This diagram commutes because under the top two maps  $Q_2 \mapsto Q'_1 \mapsto q_2$  and  $Q'_2 \mapsto Q_1 \mapsto q_1$ , and these are exactly the images under the lower map.

By [Bosch, Güntzer, and Remmert 1984, Proposition 9.3.3/1], there exists a map of rigid-analytic spaces  $g : Y \rightarrow \tilde{U}_r$  such that the following diagram commutes:

$$\begin{array}{ccc} & & \tilde{U}_r \\ & \nearrow g & \downarrow \pi \\ Y & \xrightarrow{f} & U_r \end{array}$$

To prove that  $\pi : \tilde{U}_r \rightarrow U_r$  is the blow-up of  $U_r$  at the ideal  $m = (f_1, f_2)$ , it remains to check that  $g$  is unique in making the above diagram commute. Suppose  $g' : Y \rightarrow \tilde{U}_r$  is another candidate. We prove that  $g = g'$ . We need:

**Lemma 2.4.** *For  $i = 1, 2$ , we have  $g'^{-1}(\tilde{V}_i) = Y_i$ .*

*Proof.* Without loss of generality, assume that  $i = 1$ .

• We prove that  $g'^{-1}(\tilde{V}_1) \subseteq Y_1$ . Recall that  $\tilde{V}_1 = \bigcup_{i \geq 0} \mathrm{Sp} A_i$ . Therefore it is enough to prove that  $Y_{1,i} := g'^{-1}(\mathrm{Sp} A_i) \subseteq Y_1$  for each  $i \geq 0$ . (The notation  $Y_{1,i}$  here and  $Y_{2,i}$  below are local to this proof.) Fix  $i \geq 0$ . Consider the following commutative diagram:

$$\begin{array}{ccc} & & \mathrm{Sp} A_i \\ & \nearrow g' & \downarrow \pi \\ Y_{1,i} = g'^{-1}(\mathrm{Sp} A_i) & \xrightarrow{f} & U_r \end{array}$$

We know that  $m A_i = f_1 A_i$ . Extending this ideal from  $A_i$  to  $\mathcal{O}(Y_{1,i})$  using  $g'^*$ , we see that  $m \mathcal{O}(Y_{1,i}) = g'^*(f_1) \mathcal{O}(Y_{1,i}) = f^*(f_1) \mathcal{O}(Y_{1,i})$ . Let  $y \in Y_{1,i}$ . Noting that  $Y_{1,i}$  is an admissible open subset of  $Y$ , we have  $m \mathcal{O}_{Y,y} = f^*(f_1) \mathcal{O}_{Y,y}$ . By Lemma 2.2, we see that  $y \in Y_1$ .

• We prove that  $Y_1 \subseteq g'^{-1}(\tilde{V}_1)$ . This is equivalent to proving  $g'^{-1}(\tilde{U}_r \setminus \tilde{V}_1) \subseteq Y \setminus Y_1$ . Recall that  $\tilde{U}_r$  is covered by  $\tilde{V}_1$  and  $\tilde{V}_2$ . Since  $\tilde{V}_2 = \bigcup_{i \geq 0} \mathrm{Sp} B_i$ , we have

$$\tilde{U}_r \setminus \tilde{V}_1 = \bigcup_{i \geq 0} (\mathrm{Sp} B_i \setminus \tilde{V}_1).$$

Thus, it is enough to prove that  $Y_{2,i} := g'^{-1}(\mathrm{Sp} B_i \setminus \tilde{V}_1) \subseteq Y \setminus Y_1$  for each  $i \geq 0$ . Fix  $i \geq 0$ . We prove that if  $y \in Y_{2,i}$ , then  $y \notin Y_1$ . We do so by proving that the ideal  $m\mathcal{O}_{Y,y}$  of  $\mathcal{O}_{Y,y}$  is not generated by  $f^*(f_1)$  (see Lemma 2.2). So, let  $y \in Y_{2,i}$  map to  $\mathrm{Sp} B_i \setminus \tilde{V}_1$  under  $g'$ . Assume towards a contradiction that  $y \in Y_1$ . By Lemma 2.2, we see that  $m\mathcal{O}_{Y,y}$  is generated by  $f^*(f_1)$ . By the analog of the first bullet point,  $Y_{2,i} \subseteq Y_2$ , so we also have  $m\mathcal{O}_{Y,y} = f^*(f_2)\mathcal{O}_{Y,y}$ . Therefore there exists a unit  $u \in \mathcal{O}_{Y,y}$  such that  $f^*(f_1) = uf^*(f_2)$ . Consider the following commutative diagram:

$$\begin{array}{ccc}
 & & \mathrm{Sp} B_i \\
 & \nearrow g' & \downarrow \pi \\
 Y_{2,i} = g'^{-1}(\mathrm{Sp} B_i \setminus \tilde{V}_1) & \xrightarrow{f} & U_r
 \end{array}$$

Note that  $Y_{2,i}$  is an admissible open subset of  $Y$ . We obtain the relation  $f^*(f_1) = g'^*(Q_2)f^*(f_2)$  in  $\mathcal{O}_{Y,y}$  by pushing the relation  $f_1 = Q_2f_2$  from  $B_i$  to  $\mathcal{O}(Y_{2,i})$  using  $g'^*$  and then from  $\mathcal{O}(Y_{2,i})$  to  $\mathcal{O}_{Y,y}$ . Since  $f^*(f_2)$  is not a zero divisor in  $\mathcal{O}_{Y,y}$ , we see that  $g'^*(Q_2) = u$ , which is a unit of  $\mathcal{O}_{Y,y}$ . We prove that this is a contradiction. We know that  $\mathrm{Sp} B_i \cap \tilde{V}_1$  is the set of maximal ideals  $\mathfrak{m}$  of  $B_i$  satisfying the extra condition  $|Q_2 \bmod \mathfrak{m}| > 0$ . Therefore  $\mathrm{Sp} B_i \setminus \tilde{V}_1$  is the set of maximal ideals  $\mathfrak{m}$  of  $B_i$  satisfying  $|Q_2 \bmod \mathfrak{m}| = 0$ . Since  $g'(y) \in \mathrm{Sp} B_i \setminus \tilde{V}_1$ , we see that  $Q_2$  vanishes at  $g'(y)$ . In other words, if we denote the maximal ideal of  $\mathcal{O}_{\tilde{U}_r, g'(y)}$  by  $\mathfrak{m}_{g'(y)}$ , then we have  $Q_2 \bmod \mathfrak{m}_{g'(y)} = 0$ . If  $\mathfrak{n}_y$  is the maximal ideal of  $\mathcal{O}_{Y,y}$ , then using the commutative diagram

$$\begin{array}{ccc}
 \mathcal{O}_{\tilde{U}_r, g'(y)} & \xrightarrow{g'^*} & \mathcal{O}_{Y,y} \\
 \downarrow & & \downarrow \\
 \mathcal{O}_{\tilde{U}_r, g'(y)} / \mathfrak{m}_{g'(y)} & \longrightarrow & \mathcal{O}_{Y,y} / \mathfrak{n}_y
 \end{array}$$

we get  $g'^*(Q_2) \bmod \mathfrak{n}_y = 0$ . This means that  $g'^*(Q_2)$  is contained in the maximal ideal of  $\mathcal{O}_{Y,y}$ . In other words,  $g'^*(Q_2)$  is not a unit. This is a contradiction. Therefore  $y \notin Y_1$ . □

To show that  $g = g'$ , we prove that their restrictions to  $Y_{1,m}$  and  $Y_{2,n}$  are equal for all  $m \in I$  and  $n \in J$ . We prove that  $g$  and  $g'$  agree on  $Y_{1,m}$ . Using the definition of  $g$ , we see that the restriction of  $g$  to  $Y_{1,m}$  factors as  $Y_{1,m} \rightarrow \mathrm{Sp} A_m \rightarrow \tilde{U}_r$ . Since  $g'$  maps the affinoid space  $Y_{1,m}$  to  $\tilde{U}_r$  and  $\{\mathrm{Sp} A_i, \mathrm{Sp} B_j\}_{i,j \geq 0}$  is an admissible cover of  $\tilde{U}_r$ , we deduce that  $g'$  maps  $Y_{1,m}$  into the union of finitely many  $\mathrm{Sp} A_i$  and  $\mathrm{Sp} B_j$ . By Lemma 2.4, we see that  $g' : Y_{1,m} \rightarrow \tilde{U}_r$  factors as  $Y_{1,m} \rightarrow \mathrm{Sp} A_i \rightarrow \tilde{U}_r$  for some  $i \gg 0$ . We may assume that  $i \geq \underline{m}$ . To check that  $g$  and  $g'$  agree on  $Y_{1,m}$ , it is therefore enough to check that the following diagram commutes:

$$\begin{array}{ccc}
 & & \mathrm{Sp} A_i \\
 & \nearrow g' & \uparrow \\
 Y_{1,m} & & \mathrm{Sp} A_{\underline{m}} \\
 & \searrow g &
 \end{array}$$

In other words, we have to check that the following diagram commutes:

$$\begin{array}{ccc}
 & & A_i \\
 & \swarrow^{g'^*} & \downarrow \\
 \mathcal{O}(Y_{1,m}) & & A_m \\
 & \swarrow^{g^*} & \\
 & & 
 \end{array}$$

Since both  $g^*$  and  $g'^*$  are equal to  $f^*$  on  $\mathcal{O}(U_r)$ , we only need to check that  $g^*(Q_1) = g'^*(Q_1)$ . Pushing the relation  $f_2 = Q_1 f_1$  from  $A_i$  to  $\mathcal{O}(Y_{1,m})$  under  $g'^*$ , we get  $f^*(f_2) = g'^*(Q_1)f^*(f_1)$ . Similarly, pushing the same relation from  $A_m$  to  $\mathcal{O}(Y_{1,m})$  under  $g^*$ , we get  $f^*(f_2) = g^*(Q_1)f^*(f_1)$ . Using Lemma 2.3, we get  $g^*(Q_1) = g'^*(Q_1)$ . This proves the commutativity of the diagram above. In other words,  $g$  and  $g'$  agree on  $Y_{1,m}$ . A similar argument shows that  $g$  and  $g'$  agree on  $Y_{2,n}$ . Therefore  $g = g'$  on  $Y$ . This finally proves that  $\pi : \tilde{U}_r \rightarrow U_r$  is the blow-up of  $U_r$  at the maximal ideal  $(f_1, f_2)$ .

**2.4. Points in the exceptional fiber as tangent directions.** The following standard fact allows us to realize  $E$ -valued points of the fiber over the exceptional point  $(p^r, (1 + p)^{r+1} - 1)$  as tangent directions in  $U_r$  at this point. Recall that the maximal ideal  $m$  of  $\mathcal{O}(U_r)$  corresponding to this exceptional point is the ideal generated by  $f_1 = S_1 - p^r$  and  $f_2 = S_2 - ((1 + p)^{k-1} - 1)$ .

**Proposition 2.5.** *There is a bijection*

$$\pi^{-1}(p^r, (1 + p)^{r+1} - 1)(E) \rightarrow \mathbb{P}(\text{Hom}(m/m^2 \otimes_{\mathbb{Q}_p} E, E))$$

between the  $E$ -valued points of the fiber over the point  $(p^r, (1 + p)^{k-1} - 1)$  and the elements of the projectivization of the tangent space  $\text{Hom}(m/m^2 \otimes_{\mathbb{Q}_p} E, E)$  over  $E$ , which sends a point  $(p^r, (1 + p)^{k-1} - 1, a : b) \in \tilde{U}_r(E)$  to the class in  $\mathbb{P}(\text{Hom}(m/m^2 \otimes_{\mathbb{Q}_p} E, E))$  represented by  $v$ , where

$$v(\bar{f}_1 \otimes 1) = a \quad \text{and} \quad v(\bar{f}_2 \otimes 1) = b,$$

with bar denoting image modulo  $m^2$ .

*Proof.* Let  $P = (p^r, (1 + p)^{k-1} - 1, a : b)$  be an  $E$ -valued point in the fiber above the exceptional point. Assume  $b \neq 0$ . Write  $P = (p^r, (1 + p)^{k-1} - 1, a/b : 1)$ . From the construction of  $\tilde{U}_r$ , we see that  $P \in \tilde{V}_2(E)$ . Therefore  $P$  is an  $E$ -valued point of  $\text{Sp } B_i$  for some  $i \geq 0$ . Let  $g_P$  be the homomorphism  $B_i \rightarrow E$  associated with  $P$ . The discussion at the end of Section 2.1 implies that  $g_P$  is defined by  $g_P(S_1) = p^r$ ,  $g_P(S_2) = (1 + p)^{k-1} - 1$  and  $g_P(Q_2) = a/b$ . Let  $m_P$  be the kernel of  $g_P$ . The map  $\mathcal{O}(U_r) \rightarrow B_i$  induces an  $E$ -linear surjection

$$(m/m^2) \otimes_{\mathbb{Q}_p} E \rightarrow f_2 B_i / f_2 B_i m_P \otimes_{B_i, g_P} E,$$

given by

$$\begin{aligned}
 \bar{f}_1 \otimes 1 &\mapsto \bar{f}_2 \otimes (a/b), \\
 \bar{f}_2 \otimes 1 &\mapsto \bar{f}_2 \otimes 1,
 \end{aligned}$$

where the bars over the elements on the left denote their images modulo  $m^2$  and the bars over the elements on the right denote their images modulo  $f_2 B_i m_p$ , and where we have used  $\bar{f}_1 \otimes 1 = \bar{f}_2 \bar{Q}_2 \otimes 1 = \bar{f}_2 \otimes (a/b)$  in the codomain. The codomain of this map can be identified with  $E$  up to multiplication by a nonzero scalar. The class of the above map in  $\mathbb{P}(\text{Hom}(m/m^2 \otimes_{\mathbb{Q}_p} E, E))$  is the same as that of the map  $v$ , where  $v(\bar{f}_1 \otimes 1) = a$  and  $v(\bar{f}_2 \otimes 1) = b$ . A similar construction works if  $a \neq 0$ . One checks immediately that the resulting assignment  $P \mapsto v$  is well-defined and a bijection.  $\square$

### 3. Explicit bases of $H^1(\mathcal{R}_{\mathbb{Q}_p}(x^r \chi))$ and $\mathcal{L}$ -invariants

Let  $\Gamma = \text{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$  and think of the cyclotomic character  $\chi$  as a character  $\chi : \Gamma \rightarrow \mathbb{Z}_p^*$ . In this section, we study two explicit bases of the first Fontaine–Herr cohomology group  $H^1(\mathcal{R}_{\mathbb{Q}_p}(x^r \chi))$  and the corresponding formulas for the  $\mathcal{L}$ -invariant. Here  $\mathcal{R}_{\mathbb{Q}_p}(x^r \chi)$  is the Robba ring over  $\mathbb{Q}_p$  in the variable  $T$ , with the standard actions of  $\varphi$  and  $\Gamma$  twisted by  $x^r \chi$ .

Colmez [2008, Proposition 2.19] has constructed two power series  $G(|x|, r + 1)$  and  $G'(|x|, r + 1)$  in  $\mathcal{R}_{\mathbb{Q}_p}$ , which he uses to define a basis of  $H^1(\mathcal{R}_{\mathbb{Q}_p}(x^r \chi))$ . In the first subsection, we explicitly compute  $G(|x|, r + 1)$ . In the second subsection, we give a partial description of  $G'(|x|, r + 1)$ . In the third subsection, we state Colmez’s formula for the  $\mathcal{L}$ -invariant of a nonzero element of  $H^1(\mathcal{R}_{\mathbb{Q}_p}(x^r \chi))$  expressed as a linear combination in this basis. In the same subsection, we describe another basis of  $H^1(\mathcal{R}_{\mathbb{Q}_p}(x^r \chi))$  studied by Benois [2011, Proposition 1.5.4]. Finally, we find the change of basis matrix between these two bases and restate the formula for the  $\mathcal{L}$ -invariant in terms of Benois’ basis (Definition 3.6).

We first recall that the Fontaine–Herr cohomology groups  $H^i(D)$  of a  $(\varphi, \Gamma)$ -module  $D$  over  $\mathcal{R}_{\mathbb{Q}_p}$  for  $i = 0, 1, 2$  are defined to be the cohomology groups of the complex

$$0 \rightarrow D \rightarrow D \oplus D \rightarrow D \rightarrow 0, \tag{6}$$

where the second map is  $x \mapsto ((\varphi - 1)x, (\gamma - 1)x)$  and the third map is  $(y, z) \mapsto (\gamma - 1)y - (\varphi - 1)z$  for  $\gamma$  a fixed topological generator of  $\Gamma$ . We note that this complex is as in [Chenevier 2013, Section 2.1] and is a bit different from the one in [Colmez 2008, Section 2.1].

We now recall some standard facts about Robba rings. For  $M > 0$ , let  $\mathcal{E}_E^{[0, M]}$  be the set of bidirectional power series in the variable  $T$  with coefficients in  $E$  that converge on the elements of  $\overline{\mathbb{Q}_p}$  with valuation belonging to  $(0, M]$ . More precisely, it is the set of the series  $\sum_{n=-\infty}^{\infty} a_n T^n$  satisfying

$$\lim_{n \rightarrow \pm\infty} v_p(a_n) + ns \rightarrow \infty \quad \text{for all } 0 < s \leq M.$$

For each  $0 < s \leq M$ , there is a valuation  $v_p(\cdot, s)$  on  $\mathcal{E}_E^{[0, M]}$  defined by

$$v_p\left(\sum_{n=-\infty}^{\infty} a_n T^n, s\right) = \inf\{v_p(a_n) + ns \mid n \in \mathbb{Z}\}.$$

Any sequence in  $\mathcal{E}_E^{[0, M]}$  that is Cauchy with respect to  $v_p(\cdot, s)$  for each  $0 < s \leq M$  is convergent in  $\mathcal{E}_E^{[0, M]}$  (see [Kedlaya 2006, Definition 2.5.1]). Moreover, choosing a sequence  $0 < r_l \leq M$  converging to 0 with  $r_1 = M$ , we get a countable family of valuations  $v_p(\cdot, r_l)$  on  $\mathcal{E}_E^{[0, M]}$ . Note that if  $r_l \leq s \leq M$

for some  $l \geq 1$ , then  $v_p(f(T), s) \geq \inf\{v_p(f(T), r_l), v_p(f(T), M)\}$ . Therefore to check if a sequence in  $\mathcal{E}_E^{[0, M]}$  is convergent, we only need to check that it converges to a common limit with respect to the valuations  $v_p(\cdot, r_l)$  for  $l \geq 1$ . In particular,  $\mathcal{E}_E^{[0, M]}$  is a Fréchet space with respect to the valuations  $v_p(\cdot, r_l)$ . For the space  $\mathcal{E}_E^{[0, 1/(p-1)]}$ , we set  $r_l = 1/\phi(p^l)$ , where  $\phi$  is Euler's totient function.

The Robba ring over  $E$  is defined to be  $\mathcal{R}_E = \bigcup_{M>0} \mathcal{E}_E^{[0, M]}$ . It is endowed with commuting actions of an operator  $\varphi$  and the group  $\Gamma$  via  $\varphi T = (1 + T)^p - 1$  and  $\gamma T = (1 + T)^{\chi(\gamma)} - 1$  for  $\gamma \in \Gamma$ , respectively.

We recall some facts from [Lazard 1962]. Let  $\phi_n(T)$  for  $n \geq 1$  be the  $p^n$ -th cyclotomic polynomial defined by

$$\phi_n(T) = \frac{(1 + T)^{p^n} - 1}{(1 + T)^{p^{n-1}} - 1} = 1 + (1 + T)^{p^{n-1}} + \dots + (1 + T)^{(p-1)p^{n-1}}. \tag{7}$$

The polynomials  $\phi_n(T)$  are  $(1/\phi(p^n))$ -extremal in the sense of [loc. cit., Definition 2.7]. This means that  $v_p(\phi_n(T), 1/\phi(p^n))$  is attained at the constant and leading terms:

$$v_p(a_0) = v_p\left(\phi_n(T), \frac{1}{\phi(p^n)}\right) = v_p(a_{p^{n-1}(p-1)}) + \frac{p^{n-1}(p-1)}{\phi(p^n)} = 1, \tag{8}$$

where  $\phi_n(T) = a_0 + a_1T + \dots + a_{p^{n-1}(p-1)}T^{p^{n-1}(p-1)}$ , with  $a_0 = p$  and  $a_{p^{n-1}(p-1)} = 1$ . For  $n \geq 1$ , we also have

$$\varphi(\phi_n(T)) = \phi_{n+1}(T). \tag{9}$$

Let  $t = \log(1 + T) \in \mathcal{E}^{[0, 1/(p-1)]}$ . The following formula relates  $t$  and the cyclotomic polynomials:

$$t = T \prod_{n \geq 1} \frac{\phi_n(T)}{p}. \tag{10}$$

We also have

$$\varphi(t) = pt \tag{11}$$

and

$$\gamma(t) = \chi(\gamma)t \tag{12}$$

for all  $\gamma \in \Gamma$ .

Let  $r \geq 0$ . By [Colmez 2008, Proposition 2.16(i)], there is an isomorphism

$$\mathcal{E}^{[0, \frac{1}{p-1}]} / t^{r+1} \rightarrow \prod_{n \geq 1} \mathcal{E}^{[0, \frac{1}{p-1}]} / \phi_n^{r+1}. \tag{13}$$

By [loc. cit., Proposition 2.16(ii)], for  $n \geq 1$ , there is a  $\Gamma$ -equivariant isomorphism

$$\iota_n : \mathcal{E}^{[0, \frac{1}{p-1}]} / \phi_n^{r+1} \rightarrow \mathbb{Q}_p(\zeta_{p^n})[t] / t^{r+1}$$

obtained by sending  $T$  to  $\zeta_{p^n} e^{t/p^n} - 1$ . Thus the  $\Gamma$ -equivariant homomorphism

$$\mathcal{E}^{[0, \frac{1}{p-1}]} / t^{r+1} \rightarrow \prod_{n \geq 1} \mathbb{Q}_p(\zeta_{p^n})[t] / t^{r+1} \tag{14}$$

induced by the maps  $\iota_n$  for  $n \geq 1$  is an isomorphism.

For the rest of the paper, we fix a topological generator  $\gamma$  of  $\Gamma$  such that  $\chi(\gamma) = \zeta_{p-1}^a(1+p)$  for some fixed integer  $a$ .

**3.1. An explicit description of  $G(|x|, r + 1)$ .** Let  $r \geq 1$ . The first basis vector of  $H^1(\mathcal{R}_{\mathbb{Q}_p}(x^r \chi))$  constructed by Colmez is represented by the element

$$c_1 = (t^{-(r+1)}(p^{-1}\varphi - 1)G(|x|, r + 1), t^{-(r+1)}(\gamma - 1)G(|x|, r + 1))$$

in  $\mathcal{R}_{\mathbb{Q}_p}(x^r \chi) \oplus \mathcal{R}_{\mathbb{Q}_p}(x^r \chi)$  (for the untwisted action of  $\varphi$  and  $\gamma$ ); see [Colmez 2008, Proposition 2.19]. Here  $G(|x|, r + 1)$  is any power series  $f(T)$  in  $\mathcal{E}^{[0,1/(p-1)]}$  mapping to  $\prod_{n \geq 1} 1/p^n$  under the map (13) (see [loc. cit., Section 2.6]). In other words,  $G(|x|, r + 1)$  satisfies the system of congruences

$$\begin{aligned} f(T) &\equiv \frac{1}{p} \pmod{\phi_1(T)^{r+1}}, \\ f(T) &\equiv \frac{1}{p^2} \pmod{\phi_2(T)^{r+1}}, \\ &\vdots \\ f(T) &\equiv \frac{1}{p^n} \pmod{\phi_n(T)^{r+1}}, \\ &\vdots \end{aligned}$$

The goal of this section is to write down  $G(|x|, r + 1)$  explicitly.

**Definition 3.1.** For each  $n \geq 1$  and  $r \geq 1$ , define

$$G_{n,r+1}(T) := \left[ 1 - \left( \frac{\phi_n(T)}{p} \right)^{r+1} \right]^{r+1} \frac{1}{p^n} \prod_{i > n} \left[ 1 - \left( 1 - \frac{\phi_i(T)}{p} \right)^{r+1} \right]^{r+1}.$$

To check  $G_{n,r+1}(T)$  is well-defined, we need to show that the infinite product converges. We prove that for any  $n, r \geq 1$  the product  $\prod_{i > n} (1 - (1 - (\phi_i(T)/p))^{r+1})$  converges.

**Lemma 3.2.** For any  $c = 1, 2, \dots, p - 1$  and any  $j = 1, 2, \dots, cp^{i-1}$ , we have

$$v_p\left(\binom{cp^{i-1}}{j}\right) = i - 1 - v_p(j).$$

*Proof.* We leave this as an exercise for the reader. □

**Lemma 3.3.** For any  $i \geq 1$ , we have

$$v_p\left(1 - \frac{\phi_i(T)}{p}, \frac{1}{\phi(p^i)}\right) = 0$$

and, for  $l \geq 1$ , we have

$$v_p\left(1 - \frac{\phi_i(T)}{p}, \frac{1}{\phi(p^l)}\right) > i - l - 1.$$

*Proof.* Since  $\phi_i(T)$  is  $(1/\phi(p^i))$ -extremal, we see that by (8),  $v_p(\phi_i(T), 1/\phi(p^i)) = 1$ . Since all the terms except the constant terms of the polynomials  $\phi_i(T)$  and  $\phi_i(T) - p$  are the same, we get  $v_p(\phi_i(T) - p, 1/\phi(p^i)) = 1$ . Therefore  $v_p(1 - \phi_i(T)/p, 1/\phi(p^i)) = 0$ .

Write  $\phi_i(T) - p = a_1T + a_2T^2 + \dots + a_{\phi(p^i)}T^{\phi(p^i)}$ . By expanding the powers of  $(1 + T)$  in  $\phi_i(T)$ , we see that

$$a_j = \binom{p^{i-1}}{j} + \binom{2p^{i-1}}{j} + \dots + \binom{(p-1)p^{i-1}}{j}.$$

By Lemma 3.2,

$$v_p(a_j) + \frac{j}{\phi(p^l)} \geq i - 1 - v_p(j) + \frac{p^{v_p(j)}}{\phi(p^l)} = i - 1 - v_p(j) + \frac{p^{v_p(j)-(l-1)}}{p-1} > i - 1 - v_p(j) + v_p(j) - (l-1) = i - l.$$

Therefore

$$v_p\left(1 - \frac{\phi_i(T)}{p}, \frac{1}{\phi(p^l)}\right) = v_p\left(\phi_i(T) - p, \frac{1}{\phi(p^l)}\right) - 1 > i - l - 1. \quad \square$$

We now prove that the product  $\prod_{i>n} (1 - (1 - (\phi_i(T)/p))^{r+1})$  converges in  $\mathcal{E}^{[0, 1/(p-1)]}$  for  $n, r \geq 1$ . Consider the polynomials  $1 - (1 - (\phi_i(T)/p))^{r+1}$  as elements of  $\mathcal{E}^{[0, \infty]}$ . By [Lazard 1962, Proposition 4.11], the product  $\prod_{i>n} (1 - (1 - \phi_i(T)/p)^{r+1})$  converges in  $\mathcal{E}^{[0, \infty]}$  if and only if

$$v_p\left(\left(1 - \frac{\phi_i(T)}{p}\right)^{r+1}, s\right) \rightarrow \infty \quad \text{as } i \rightarrow \infty \tag{15}$$

for each  $s > 0$ . Fix  $s > 0$  and pick  $l_s \geq 1$  such that  $1/\phi(p^{l_s}) < s$ . As  $(1 - \phi_i(T)/p)^{r+1}$  contains no negative power of  $T$ , we have

$$v_p\left(\left(1 - \frac{\phi_i(T)}{p}\right)^{r+1}, s\right) \geq (r+1)v_p\left(1 - \frac{\phi_i(T)}{p}, \frac{1}{\phi(p^{l_s})}\right) > (r+1)(i - l_s - 1),$$

by Lemma 3.3, whence (15) holds. Therefore the product  $\prod_{i>n} (1 - (1 - (\phi_i(T)/p))^{r+1})$  converges in  $\mathcal{E}^{[0, \infty]} \subseteq \mathcal{E}^{[0, 1/(p-1)]}$ .

Our candidate for  $G(|x|, r+1)$  is  $\sum_{n=1}^{\infty} G_{n,r+1}(T)$ . To see that the infinite sum is well-defined, we prove the following theorem.

**Theorem 3.4.** *For any positive integer  $l$ , the sequence  $G_{n,r+1}(T)$  converges to 0 with respect to the valuation  $v_p(\cdot, 1/\phi(p^l))$ .*

*Proof.* Fix  $l \geq 1$ . We need to show

$$v_p\left(G_{n,r+1}(T), \frac{1}{\phi(p^l)}\right) \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Consider

$$\begin{aligned} &v_p\left(G_{n,r+1}(T), \frac{1}{\phi(p^l)}\right) \\ &= (r+1)v_p\left(1 - \left(\frac{\phi_n(T)}{p}\right)^{r+1}, \frac{1}{\phi(p^l)}\right) - n + (r+1) \sum_{i>n} v_p\left(1 - \left(1 - \frac{\phi_i(T)}{p}\right)^{r+1}, \frac{1}{\phi(p^l)}\right). \end{aligned}$$

Note that

$$1 - \left(\frac{\phi_n(T)}{p}\right)^{r+1} = \left(1 - \frac{\phi_n(T)}{p}\right) \left(1 + \frac{\phi_n(T)}{p} + \dots + \left(\frac{\phi_n(T)}{p}\right)^r\right).$$

Now if  $n \geq l$ , then

$$v_p\left(\phi_n(T), \frac{1}{\phi(p^l)}\right) = 1.$$

Indeed, since  $\phi_n(T) = a_0 + a_1T + \dots + a_{\phi(p^n)}T^{\phi(p^n)}$  is  $(1/\phi(p^n))$ -extremal, we have

$$v_p(a_i) + \frac{i}{\phi(p^l)} \geq v_p(a_i) + \frac{i}{\phi(p^n)} \geq v_p(a_0) = 1 \quad \text{for any } i.$$

Hence

$$v_p\left(\phi_n(T), \frac{1}{\phi(p^l)}\right) = v_p(a_0) = 1.$$

Thus

$$v_p\left((\phi_n(T)/p)^j, \frac{1}{\phi(p^l)}\right) = 0 \quad \text{for any } j \geq 0.$$

Therefore

$$v_p\left(1 - \left(\frac{\phi_n(T)}{p}\right)^{r+1}, \frac{1}{\phi(p^l)}\right) \geq v_p\left(1 - \frac{\phi_n(T)}{p}, \frac{1}{\phi(p^l)}\right) > n - l - 1,$$

where the last inequality is a consequence of [Lemma 3.3](#). On the other hand, writing

$$1 - \left(1 - \left(\frac{\phi_i(T)}{p}\right)\right)^{r+1} = a'_0 + a'_1T + \dots + a'_{(r+1)\phi(p^i)}T^{(r+1)\phi(p^i)},$$

we observe that  $a'_0 = 1$  and  $a'_1T + \dots + a'_{(r+1)\phi(p^i)}T^{(r+1)\phi(p^i)} = -(1 - (\phi_i(T)/p))^{r+1}$ . By [Lemma 3.3](#),

$$v_p\left(-\left(1 - \frac{\phi_i(T)}{p}\right)^{r+1}, \frac{1}{\phi(p^l)}\right) > 0 \quad \text{if } i > l.$$

We also know that  $v_p(a'_0) = 0$ . Therefore

$$v_p\left(1 - \left(1 - \frac{\phi_i(T)}{p}\right)^{r+1}, \frac{1}{\phi(p^l)}\right) = 0.$$

Using these estimates together, we see that, for  $n \geq l$ ,

$$v_p\left(G_{n,r+1}(T), \frac{1}{\phi(p^l)}\right) > (r+1)(n-l-1) - n = nr - (r+1)l - (r+1).$$

Letting  $n \rightarrow \infty$ , we see that  $v_p(G_{n,r+1}(T), 1/\phi(p^l)) \rightarrow \infty$ , as desired. Therefore the sequence  $G_{n,r+1}(T)$  converges to 0 with respect to the valuation  $v_p(\cdot, 1/\phi(p^l))$  for each  $l > 0$ .  $\square$

Using the theorem above, we see that the series  $G(|x|, r+1) = \sum_{n=1}^{\infty} G_{n,r+1}(T)$  converges. It is easily checked, using  $\phi_i(T) \equiv p \pmod{\phi_n(T)}$  for  $i > n$ , that it is a solution to the system of congruences that we started with:

$$G(|x|, r+1) \equiv \frac{1}{p^n} \pmod{\phi_n(T)^{r+1}} \quad \text{for all } n \in \mathbb{N}.$$

Therefore, we have written down  $G(|x|, r+1)$  explicitly.

**3.2. A partial description of  $G'(|x|, r + 1)$ .** Let  $r \geq 1$ . The second basis vector of  $H^1(\mathcal{R}_{\mathbb{Q}_p}(x^r \chi))$  constructed by Colmez is represented by the element

$$c_2 = (t^{-(r+1)}(p^{-1}\varphi - 1)(\log T - G'(|x|, r + 1)), t^{-(r+1)}(\gamma - 1)(\log T - G'(|x|, r + 1)))$$

in  $\mathcal{R}_{\mathbb{Q}_p}(x^r \chi) \oplus \mathcal{R}_{\mathbb{Q}_p}(x^r \chi)$  (for the untwisted action of  $\varphi$  and  $\gamma$ ); see [Colmez 2008, Proposition 2.19]. Here  $G'(|x|, r + 1)$  is an element of  $\mathcal{E}^{[0, 1/(p-1)]}$  mapping to  $\prod_{n \geq 1} \log(\zeta_{p^n} e^{t/p^n} - 1)$  under the map (14) (see [loc. cit., Section 2.7]).

We wish to compute  $G'(|x|, r + 1)$  explicitly, just as we did for  $G(|x|, r + 1)$ . We are only able to determine  $G'(|x|, r + 1)$  modulo  $t$  but this is sufficient for our purposes.

Consider the element

$$g'(T) = \frac{\log \gamma}{\gamma - 1} \log T - \log T - \frac{\log \chi(\gamma)t}{\chi(\gamma) - 1} \in \mathcal{E}^{[0, \frac{1}{p-1}]}. \tag{16}$$

In the proof of his Theorem 1.5.7, Benois [2011] relates  $g'(T)$  (which he calls  $y$ ) to the element  $d_1 = -t^{-1} \nabla_0(\log T) + (\chi(\gamma) - 1)^{-1}$  of  $\mathcal{R}_{\mathbb{Q}_p}[\log T, 1/t]$  using the equation

$$d_1 = -\log(\chi(\gamma))^{-1} t^{-1}(\log T + g'(T)), \tag{17}$$

where

$$\nabla_0 = \frac{1}{\log \chi(\gamma)} \frac{\log \gamma}{\gamma - 1}.$$

We claim that  $\iota_n(-g'(T)) \equiv \log(\zeta_{p^n} e^{t/p^n} - 1) \pmod{t}$ . Indeed,

$$\begin{aligned} \iota_n(g'(T)) &= \iota_n\left(\frac{\log \gamma}{\gamma - 1} \log T - \log T - \frac{\log \chi(\gamma)t}{\chi(\gamma) - 1}\right) \\ &\equiv \frac{\log \gamma}{\gamma - 1} \log(\zeta_{p^n} - 1) - \log(\zeta_{p^n} - 1) \pmod{t} \\ &\equiv \frac{1 + \gamma + \dots + \gamma^{p^{n-1}(p-1)-1}}{p^{n-1}(p-1)} \frac{\log \gamma^{p^{n-1}(p-1)}}{\gamma^{p^{n-1}(p-1)} - 1} \log(\zeta_{p^n} - 1) - \log(\zeta_{p^n} - 1) \pmod{t}. \end{aligned}$$

Since  $\gamma^{p^{n-1}(p-1)} \log(\zeta_{p^n} - 1) = \log(\zeta_{p^n} - 1)$ , we get

$$\begin{aligned} \iota_n(g'(T)) &\equiv \frac{1 + \gamma + \dots + \gamma^{p^{n-1}(p-1)-1}}{p^{n-1}(p-1)} \log(\zeta_{p^n} - 1) - \log(\zeta_{p^n} - 1) \pmod{t} \\ &\equiv \frac{1}{p^{n-1}(p-1)} \log(N_{\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p}(\zeta_{p^n} - 1)) - \log(\zeta_{p^n} - 1) \pmod{t} \\ &\equiv -\log(\zeta_{p^n} - 1) \pmod{t} \\ &\equiv -\log(\zeta_{p^n} e^{t/p^n} - 1) \pmod{t}. \end{aligned}$$

Since the images of  $-g'(T)$  and  $G'(|x|, r + 1)$  under the isomorphism (14) are equal (with  $r$  there equal to zero!), there exists  $g''(T) \in \mathcal{E}^{[0, 1/(p-1)]}$  such that

$$G'(|x|, r + 1) = -g'(T) + t g''(T). \tag{18}$$

This determines  $G'(|x|, r + 1)$  explicitly modulo  $t$ .

**3.3. Another basis of  $H^1(\mathcal{R}_{\mathbb{Q}_p}(x^r \chi))$ .** Let  $r \geq 1$ . In this section, we consider a different basis  $\{\bar{\alpha}_{r+1}, \bar{\beta}_{r+1}\}$  of  $H^1(\mathcal{R}_{\mathbb{Q}_p}(x^r \chi))$  described in [Benois 2011, Proposition 1.5.4] which is more suitable for computations. Here bar denotes class in  $H^1(\mathcal{R}_{\mathbb{Q}_p}(x^r \chi))$ . By [loc. cit., Corollary 1.5.5], for any class in  $H^1(\mathcal{R}_{\mathbb{Q}_p}(x^r \chi))$  represented by  $(a, b) \in \mathcal{R}_{\mathbb{Q}_p}(x^r \chi) \oplus \mathcal{R}_{\mathbb{Q}_p}(x^r \chi)$ , we have

$$\overline{(a, b)} = \lambda' \cdot \bar{\alpha}_{r+1} + \mu' \cdot \bar{\beta}_{r+1}, \quad \text{where } \lambda' = \text{res}(at^r dt) \text{ and } \mu' = \text{res}(bt^r dt). \quad (19)$$

We want to compute the change of basis relation between Colmez's basis and Benois' basis. This allows us to state the formula for the  $\mathcal{L}$ -invariant of a nonzero cohomology class in terms of certain residues.

**Proposition 3.5.** *Let*

$$\begin{aligned} c_1 &= (t^{-(r+1)}(p^{-1}\varphi - 1)G(|x|, r+1), t^{-(r+1)}(\gamma - 1)G(|x|, r+1)), \\ c_2 &= (t^{-(r+1)}(p^{-1}\varphi - 1)(\log T - G'(|x|, r+1)), t^{-(r+1)}(\gamma - 1)(\log T - G'(|x|, r+1))) \end{aligned}$$

be elements of  $\mathcal{R}_{\mathbb{Q}_p}(x^r \chi) \oplus \mathcal{R}_{\mathbb{Q}_p}(x^r \chi)$  representing Colmez's basis of  $H^1(\mathcal{R}_{\mathbb{Q}_p}(x^r \chi))$ . Then

$$\bar{c}_1 = -\frac{p-1}{p} \cdot \bar{\alpha}_{r+1}, \quad \bar{c}_2 = \log(\chi(\gamma)) \cdot \bar{\beta}_{r+1}.$$

*Proof.* We prove the proposition using (19). We therefore need to compute the four residues

$$\begin{aligned} \text{res}(t^{-1}(p^{-1}\varphi - 1)G(|x|, r+1) dt), & \quad \text{res}(t^{-1}(\gamma - 1)G(|x|, r+1) dt), \\ \text{res}(t^{-1}(p^{-1}\varphi - 1)(\log T - G'(|x|, r+1)) dt), & \quad \text{res}(t^{-1}(\gamma - 1)(\log T - G'(|x|, r+1)) dt). \end{aligned}$$

We shall use the following formulas: for any  $f(T) \in \mathcal{R}_{\mathbb{Q}_p}$ , we have

$$\text{res}(\varphi(f(T)) dt) = \text{res}(f(T) dt), \quad (20)$$

$$\text{res}(\gamma(f(T)) dt) = \chi(\gamma)^{-1} \text{res}(f(T) dt). \quad (21)$$

• Recall that  $G(|x|, r+1) = \sum_{n=1}^{\infty} G_{n,r+1}(T)$ , where

$$G_{n,r+1}(T) = \left[ 1 - \left( \frac{\phi_n(T)}{p} \right)^{r+1} \right]^{r+1} \frac{1}{p^n} \prod_{i>n} \left[ 1 - \left( 1 - \frac{\phi_i(T)}{p} \right)^{r+1} \right]^{r+1}.$$

Using equation (9), we see that  $(p^{-1}\varphi)G_{n,r+1}(T) = G_{n+1,r+1}(T)$ . Therefore

$$(p^{-1}\varphi - 1)G(|x|, r+1) = -G_{1,r+1}(T).$$

Hence  $\text{res}(t^{-1}(p^{-1}\varphi - 1)G(|x|, r+1) dt) = -\text{res}(t^{-1}G_{1,r+1}(T) dt)$ . Now

$$\begin{aligned} \text{res}(t^{-1}G_{1,r+1}(T) dt) &= \text{res}\left( t^{-1} \left[ 1 - \left( \frac{\phi_1(T)}{p} \right)^{r+1} \right]^{r+1} \frac{1}{p} \prod_{i>1} \left[ 1 - \left( 1 - \frac{\phi_i(T)}{p} \right)^{r+1} \right]^{r+1} dt \right) \\ &\stackrel{(10)}{=} \text{res}\left( T^{-1} \left( \frac{\phi_1(T)}{p} \right)^{-1} \left[ 1 - \left( \frac{\phi_1(T)}{p} \right)^{r+1} \right]^{r+1} \frac{1}{p} f(T) dt \right), \end{aligned}$$

where

$$f(T) = \prod_{i>1} \left( \frac{\phi_i(T)}{p} \right)^{-1} \left[ 1 - \left( 1 - \frac{\phi_i(T)}{p} \right)^{r+1} \right]^{r+1}.$$

An argument similar to those above shows that  $f(T) \in \mathcal{E}^{[0,1/(p-1)]}$ . Moreover, since the value of  $\phi(T)/p$  at  $T = 0$  is 1, we see that  $f(T) \in 1 + T\mathbb{Q}_p[[T]]$ . Using (9), we see that there exists  $g(T) \in 1 + T\mathbb{Q}_p[[T]] \cap \mathcal{E}^{[0,1/(p-1)]}$  such that  $f(T) = \varphi(g(T))$ . We therefore have

$$\text{res}(t^{-1}G_{1,r+1}(T) dt) = \text{res}\left(T^{-1} \left( \frac{\phi_1(T)}{p} \right)^{-1} \left[ 1 - \left( \frac{\phi_1(T)}{p} \right)^{r+1} \right]^{r+1} \frac{1}{p} \varphi(g(T)) dt\right).$$

Adding and subtracting 1 from the term  $[1 - (\phi_1(T)/p)^{r+1}]^{r+1}$ , we get

$$\text{res}(t^{-1}G_{1,r+1}(T) dt)$$

$$\stackrel{(7)}{=} \text{res}\left(\frac{1}{(1+T)^p - 1} \varphi(g(T)) dt\right) + \text{res}\left(T^{-1} \left( \frac{\phi_1(T)}{p} \right)^{-1} \left( \left[ 1 - \left( \frac{\phi_1(T)}{p} \right)^{r+1} \right]^{r+1} - 1 \right) \frac{1}{p} \varphi(g(T)) dt\right).$$

The first term is equal to  $\text{res}(\varphi(g(T)/T) dt)$ , which by (20), equals  $\text{res}(g(T)/T dt) = 1$ , since the constant term of  $g(T)$  is 1. For the second term, we see that  $\phi_1(T)/p$  divides  $[1 - (\phi_1(T)/p)^{r+1}]^{r+1} - 1$  as polynomials. Also, the constant term of

$$\left( \frac{\phi_1(T)}{p} \right)^{-1} \left( \left[ 1 - \left( \frac{\phi_1(T)}{p} \right)^{r+1} \right]^{r+1} - 1 \right) \varphi(g(T))$$

is  $-1$ . Hence, the second residue is  $-1/p$ . Therefore,

$$\text{res}(t^{-1}(p^{-1}\varphi - 1)G(|x|, r + 1) dt) = \frac{1}{p} - 1.$$

- We prove  $\text{res}(t^{-1}(\gamma - 1)G(|x|, r + 1) dt) = 0$ , by showing that, for all  $n \geq 1$ ,

$$\text{res}(t^{-1}(\gamma - 1)G_{n,r+1}(T) dt) = 0.$$

Again by (9), we see that  $(p^{-1}\varphi)G_{n,r+1}(T) = G_{n+1,r+1}(T)$ . Therefore

$$\begin{aligned} \text{res}(t^{-1}(\gamma - 1)G_{n,r+1}(T) dt) &= \text{res}(t^{-1}(\gamma - 1)(p^{-1}\varphi)^{n-1}G_{1,r+1}(T) dt) \\ &\stackrel{(11)}{=} \text{res}(\varphi^{n-1}t^{-1}(\gamma - 1)G_{1,r+1}(T) dt) \\ &\stackrel{(12)}{=} \text{res}(\varphi^{n-1}(\chi(\gamma)\gamma - 1)(t^{-1}G_{1,r+1}(T)) dt), \end{aligned}$$

which vanishes by (20) and (21).

- Using (18), we get  $\log T - G'(|x|, r + 1) = \log T + g'(T) - tg''(T)$ , whereas rearranging the terms in (17), we get  $\log T + g'(T) = -\log(\chi(\gamma))td_1$ . These two equations imply

$$\log T - G'(|x|, r + 1) = -\log(\chi(\gamma))td_1 - tg''(T).$$

Therefore we have

$$t^{-1}(p^{-1}\varphi - 1)(\log T - G'(|x|, r + 1)) \stackrel{(11)}{=} -\log(\chi(\gamma))(\varphi - 1)d_1 - (\varphi - 1)g''(T).$$

By the discussion on [Benois 2011, p. 1604, p. 1601],  $\text{res}(((\varphi - 1)d_1) dt) = 0$ . By (20), we get

$$\text{res}(t^{-1}(p^{-1}\varphi - 1)(\log T - G'(|x|, r + 1)) dt) = 0.$$

• Similarly, we have

$$t^{-1}(\gamma - 1)(\log T - G'(|x|, r + 1)) \stackrel{(12)}{=} -\log(\chi(\gamma))(\chi(\gamma)\gamma - 1)d_1 - (\chi(\gamma)\gamma - 1)g''(T).$$

Using  $(\chi(\gamma)\gamma - 1)d_1 = -1/T$  (see the proof of [loc. cit., Theorem 1.5.7(iiib)]) and using (21), we get

$$\text{res}(t^{-1}(\gamma - 1)(\log T - G'(|x|, r + 1)) dt) = \log \chi(\gamma).$$

Using (19), we see that  $\bar{c}_1$  is  $-((p - 1)/p) \cdot \bar{\alpha}_{r+1}$  and  $\bar{c}_2$  is  $\log(\chi(\gamma)) \cdot \bar{\beta}_{r+1}$ . □

Let us state the formula for the  $\mathcal{L}$ -invariant using [Colmez 2008, Definition 2.20]. Let  $(a, b)$  represent a nonzero element of  $H^1(\mathcal{R}_{\mathbb{Q}_p}(x^r \chi))$ . Writing  $\overline{(a, b)} = \lambda \cdot \bar{c}_1 + \mu \cdot \bar{c}_2$ , where the bar over an element denotes its class in  $H^1(\mathcal{R}_{\mathbb{Q}_p}(x^r \chi))$ , Colmez defines the  $\mathcal{L}$ -invariant associated to the class of  $(a, b)$  to be  $((p - 1)/p)(\lambda/\mu) \in \mathbb{P}^1(\mathbb{Q}_p)$ .<sup>4</sup> Using Proposition 3.5, we can restate the formula for the  $\mathcal{L}$ -invariant in terms of Benois' basis.

**Definition 3.6.** If  $\overline{(a, b)} = \lambda' \cdot \bar{\alpha}_{r+1} + \mu' \cdot \bar{\beta}_{r+1} \neq 0$ , where  $\overline{(a, b)}$  is the class of  $(a, b)$  in  $H^1(\mathcal{R}_{\mathbb{Q}_p}(x^r \chi))$ , then the  $\mathcal{L}$ -invariant associated to this class is

$$\mathcal{L} = -\log(\chi(\gamma)) \cdot \frac{\lambda'}{\mu'}.$$

#### 4. Cohomology of $m\mathcal{R}_{\mathcal{O}(U_r)}(\delta_{U_r})$

The aim of this section is to find a generator of  $H^1(m\mathcal{R}_{\mathcal{O}(U_r)}(\delta_{U_r}))$  for the tautological character  $\delta_{U_r} : \mathbb{Q}_p^* \rightarrow \mathcal{O}(U_r)^*$ , which will be defined in the first subsection. This generator is an essential ingredient in the proof of Theorem 5.2.

**4.1. The tautological character  $\delta_{U_r} : \mathbb{Q}_p^* \rightarrow \mathcal{O}(U_r)^*$ .** Define  $\delta_{U_r}$  by the equations

$$\begin{aligned} \delta_{U_r}(p) &= S_1, \\ \delta_{U_r}(\zeta_{p-1}) &= \zeta_{p-1}^{r+1}, \\ \delta_{U_r}(1 + p) &= 1 + S_2. \end{aligned} \tag{22}$$

This character is tautological in the following sense. If  $\delta$  is an  $E$ -valued character of  $\mathbb{Q}_p^*$  and if  $(\delta(p), \delta(\zeta_{p-1}), \delta(1 + p) - 1) \in U_r(E)$ , then we have the following commutative diagram:

<sup>4</sup>The sign is the opposite of the one in [Colmez 2008, Definition 2.20]. We believe that this choice of sign is correct. Indeed, consider the image of  $p(1 + p)$  under the Kummer map  $\kappa : \mathbb{Q}_p^* \rightarrow H^1(\mathbb{Q}_p, \mathbb{Q}_p(1))$ . By the discussion in [Benois 2011, Section 1.5.6], we see that with the original choice of sign, the  $\mathcal{L}$ -invariant associated to the image of  $\kappa(p(1 + p))$  under the canonical isomorphism  $H^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) \simeq H^1(\mathcal{R}_{\mathbb{Q}_p}(x|x|))$  is  $-\log(1 + p)$ . However, using Tate's formula, the  $\mathcal{L}$ -invariant associated to  $\kappa(p(1 + p))$  is equal to  $\log(p(1 + p))/(v_p(p(1 + p))) = \log(1 + p)$ . The original sign is also incompatible with our results in the Introduction.

$$\begin{array}{ccc}
 & & \mathcal{O}(U_r)^* \\
 & \nearrow \delta_{U_r} & \downarrow \\
 \mathbb{Q}_p^* & \xrightarrow{\delta} & E^*
 \end{array}$$

where the vertical map is induced by the affinoid algebra map  $\mathcal{O}(U_r) \rightarrow E$  associated to the  $E$ -valued point  $(\delta(p), \delta(\zeta_{p-1}), \delta(1+p) - 1)$  of  $U_r$ .

We prove that  $\delta_{U_r}$  is bien plaçé in the sense of [Chenevier 2013, Definition 2.31].

**Lemma 4.1.** *The character  $\delta_{U_r}$  is bien plaçé.*

*Proof.* • Given any  $i \in \mathbb{Z}$ , we have to check that  $1 - \delta_{U_r}(p)p^i = 1 - S_1 p^i$  is not a zero divisor in  $\mathcal{O}(U_r)$ . But this follows immediately from the fact that  $\mathbb{Q}_p\langle S_1, S_2 \rangle$  is a domain and  $\mathbb{Q}_p\langle S_1, S_2 \rangle \rightarrow \mathcal{O}(U_r)$  is flat [Bosch, Güntzer, and Remmert 1984, Corollary 7.3.2/6].

• Given any  $i \geq 0$ , we have to check that the image of

$$1 - \delta_{U_r}(\chi(\gamma))\chi(\gamma)^{1-i} = 1 - \zeta_{p-1}^{a(r+2-i)}(1 + S_2)(1 + p)^{1-i}$$

in the quotient  $\mathcal{O}(U_r)/(1 - \delta_{U_r}(p)p^{-i})$  is not a zero divisor. This again follows from the facts that

$$\mathbb{Q}_p\langle S_1, S_2 \rangle / (1 - \delta_{U_r}(p)p^{-i}) \rightarrow \mathcal{O}(U_r) / (1 - \delta_{U_r}(p)p^{-i})$$

is flat [loc. cit., Corollary 7.3.2/6], that the above element is nonzero on the left (easy to check) and that the ring on the left is a domain (indeed, if  $f(S_1, S_2)g(S_1, S_2) = (1 - S_1 p^{-i})h(S_1, S_2)$ , then applying the Weierstrass division theorem [loc. cit., Corollary 5.2.1/2], we have  $f = q(1 - S_1 p^{-i}) + r$  and  $g = q'(1 - S_1 p^{-i}) + r'$  for some  $q, q' \in \mathbb{Q}_p\langle S_1, S_2 \rangle$  and  $r, r' \in \mathbb{Q}_p\langle S_2 \rangle$ , so that  $(1 - S_1 p^{-i}) \mid r r'$  and hence one of  $r$  or  $r'$  must be zero). □

**4.2. Computation of  $H^1(m\mathcal{R}_{\mathcal{O}(U_r)}(\delta_{U_r}))$ .** In this section, we make explicit some facts in [Chenevier 2013] about the first Fontaine–Herr cohomology groups of certain “big”  $(\varphi, \Gamma)$ -modules.

Let  $\mathcal{R}_{\mathcal{O}(U_r)}(\delta_{U_r})$  be the Robba ring over  $\mathcal{O}(U_r)$  in the variable  $T$ , with the standard actions of  $\varphi$  and  $\Gamma$  twisted by  $\delta_{U_r}$ :  $\varphi T = \delta_{U_r}(p)((1 + T)^p - 1)$  and  $\gamma T = \delta_{U_r}(\chi(\gamma))((1 + T)^{\chi(\gamma)} - 1)$ . Let  $\psi$  be the usual left inverse of  $\varphi$ , twisted by  $\delta_{U_r}(p^{-1})$ . Let  $\mathcal{R}_{\mathcal{O}(U_r)}^+(\delta_{U_r})$  be the power series in  $\mathcal{R}_{\mathcal{O}(U_r)}(\delta_{U_r})$  consisting of nonnegative powers of  $T$ .

We work with two different versions of the Fontaine–Herr cohomology groups for  $(\varphi, \Gamma)$ -modules  $D$  over the Robba ring  $\mathcal{R}_{\mathcal{O}(U_r)}$ , namely, the  $(\varphi, \Gamma)$ -version  $H^1_{(\varphi, \Gamma)}(D)$  and the  $(\psi, \Gamma)$ -version  $H^1_{(\psi, \Gamma)}(D)$ . The former groups are defined as usual as the cohomology groups of the complex (6). For the definition of the latter groups, one replaces  $\varphi$  by  $\psi$  in the maps in the complex (6). There is a map  $\eta$  from the  $(\varphi, \Gamma)$ -complex to the  $(\psi, \Gamma)$ -complex

$$\begin{array}{ccccccc}
 0 & \longrightarrow & D & \longrightarrow & D \oplus D & \longrightarrow & D \longrightarrow 0 \\
 & & \downarrow \eta_0 & & \downarrow \eta_1 & & \downarrow \eta_2 \\
 0 & \longrightarrow & D & \longrightarrow & D \oplus D & \longrightarrow & D \longrightarrow 0
 \end{array}$$

where

$$\eta_0(x) = x, \quad \eta_1(x, y) = (-\psi(x), y) \quad \text{and} \quad \eta_2(x) = -\psi(x).$$

If  $\gamma - 1$  is bijective on  $D^{\psi=0}$ , then  $\eta$  induces an isomorphism on cohomology:  $H^1_{(\varphi, \Gamma)}(D) \simeq H^1_{(\psi, \Gamma)}(D)$  (see [Chenevier 2013, Section 2]).

By [loc. cit., Theorem 2.33], the cohomology group  $H^1_{(\varphi, \Gamma)}(\mathcal{R}_{\mathcal{O}(U_r)}(\delta_{U_r}))$  is a free module of rank 1 over  $\mathcal{O}(U_r)$ . First, we compute an  $\mathcal{O}(U_r)$ -generator of  $H^1_{(\varphi, \Gamma)}(\mathcal{R}_{\mathcal{O}(U_r)}(\delta_{U_r}))$ . By [loc. cit., Theorem 2.33], the inclusion  $m\mathcal{R}_{\mathcal{O}(U_r)}(\delta_{U_r}) \rightarrow \mathcal{R}_{\mathcal{O}(U_r)}(\delta_{U_r})$  induces an isomorphism

$$H^1_{(\varphi, \Gamma)}(m\mathcal{R}_{\mathcal{O}(U_r)}(\delta_{U_r})) \rightarrow H^1_{(\varphi, \Gamma)}(\mathcal{R}_{\mathcal{O}(U_r)}(\delta_{U_r})).$$

Second, we use this isomorphism to lift the  $\mathcal{O}(U_r)$ -generator of  $H^1_{(\varphi, \Gamma)}(\mathcal{R}_{\mathcal{O}(U_r)}(\delta_{U_r}))$  to an  $\mathcal{O}(U_r)$ -generator of  $H^1_{(\varphi, \Gamma)}(m\mathcal{R}_{\mathcal{O}(U_r)}(\delta_{U_r}))$  (see Definition 4.5).

Let  $D = \mathcal{R}_{\mathcal{O}(U_r)}(\delta_{U_r})$ . In Section 4.1, we showed that  $\delta_{U_r}$  is bien placé. By [loc. cit., Proposition 2.32] we have the isomorphisms

$$C(D^+)/(\gamma - 1) \longrightarrow C(D)/(\gamma - 1) \longleftarrow D^{\psi=1}/(\gamma - 1) \longrightarrow H^1_{(\psi, \Gamma)}(D),$$

where  $C(D) = (1 - \varphi)D^{\psi=1}$ ,  $D^+ = \mathcal{R}^+_{\mathcal{O}(U_r)}(\delta_{U_r})$  and  $C(D^+) = (1 - \varphi)(D^+)^{\psi=1}$ , and where we recall that the actions of  $\varphi$  and  $\psi$  are the usual ones twisted by  $\delta_{U_r}$ . The first map is induced by the canonical injection of  $D^+$  into  $D$ . The second map is induced by  $f(T) \mapsto (1 - \varphi)f(T)$ . The third map is induced by  $f(T) \mapsto (0, f(T))$ . We emphasize that the cohomology group appearing in the display above is in the sense of  $(\psi, \Gamma)$ -modules.

We wish to find an  $\mathcal{O}(U_r)$ -generator of  $H^1_{(\psi, \Gamma)}(D)$ . We do so by finding an  $\mathcal{O}(U_r)$ -generator of  $C(D^+)/(\gamma - 1)$  and then using the isomorphisms above to get an  $\mathcal{O}(U_r)$ -generator of  $H^1_{(\psi, \Gamma)}(D)$ . To find an  $\mathcal{O}(U_r)$ -generator of  $C(D^+)/(\gamma - 1)$ , we prove a couple of lemmas.

**Lemma 4.2.**  $C(D^+) = (D^+)^{\psi=0}$ .

*Proof.* By definition,  $C(D^+)$  is the image of the map  $1 - \varphi : (D^+)^{\psi=1} \rightarrow (D^+)^{\psi=0}$ . Applying [Chenevier 2013, Lemma 2.9(vi)] with  $\lambda = S_1$  and  $N = 0$  ( $r \geq 1 \Rightarrow |S_1| < 1$ ), we see that this map is a bijection. The lemma follows.  $\square$

**Lemma 4.3.**  $S_1(1 + T)$  generates  $(D^+)^{\psi=0}/(\gamma - 1)$  as a free  $\mathcal{O}(U_r)$ -module of rank 1.

*Proof.* In order to prove this, we apply [loc. cit., Proposition 2.14]. We know that  $D^+$  is a free  $\mathcal{R}^+_{\mathcal{O}(U_r)}$ -module of rank 1, say with basis  $w$ . To check that  $D^+$  is  $\Gamma$ -bounded in the sense of [loc. cit., Section 1.8], we choose the following model of  $\mathcal{O}(U_r)$ :

$$\mathcal{A} = \mathbb{Z}_p \langle S_1, S_2, T_1, T_2, T_3 \rangle / (p^r T_1 - S_1, 1 - T_1 T_2, p T_3 - S_2).$$

Then for  $\gamma' \in \Gamma$  we have  $\text{Mat}(\gamma') = [\delta_{U_r}(\chi(\gamma'))] \in M_1(\mathcal{A}[[T]])$ , so  $D^+$  is  $\Gamma$ -bounded. Using [loc. cit., Proposition 2.14], we see that  $\{S_1(1 + T)w\}$  is a basis of  $(D^+)^{\psi=0}$  over  $\mathcal{R}^+_{\mathcal{O}(U_r)}(\Gamma)$  (see [loc. cit., Section 2.12] for the definition of the last ring). Now  $\mathcal{R}^+_{\mathcal{O}(U_r)}(\Gamma)/(\gamma - 1) \cong \mathcal{O}(U_r)$ . Therefore  $(D^+)^{\psi=0}/(\gamma - 1)$  is a free  $\mathcal{O}(U_r)$ -module of rank 1 generated by  $S_1(1 + T)w$ .  $\square$

For the remainder of section, we drop  $w$  when we talk about the  $\mathcal{O}(U_r)$ -generator  $S_1(1 + T)w$  of  $(D^+)^{\psi=0}/(\gamma - 1)$ . Using these lemmas, we see that the quotient  $C(D^+)/(\gamma - 1)$  is generated by  $S_1(1 + T)$  as a free  $\mathcal{O}(U_r)$ -module. Pushing this generator to  $C(D)/(\gamma - 1)$  under the first isomorphism  $C(D^+)/(\gamma - 1) \rightarrow C(D)/(\gamma - 1)$ , we see that  $S_1(1 + T)$  is an  $\mathcal{O}(U_r)$ -generator of  $C(D)/(\gamma - 1)$ . Now we want a generator of  $D^{\psi=1}/(\gamma - 1)$ .

**Lemma 4.4.** *The element  $y = S_1 \sum_{n=0}^{\infty} S_1^n (1 + T)^{p^n}$  is the preimage of  $S_1(1 + T)$  under the isomorphism  $1 - \varphi : D^{\psi=1}/(\gamma - 1) \rightarrow C(D)/(\gamma - 1)$ .*

*Proof.* Note  $y \in D^{\psi=1}$  (for the twisted  $\psi$ ). To see this, write (for the twisted  $\varphi$ )

$$S_1 \sum_{n=0}^{\infty} S_1^n (1 + T)^{p^n} = \varphi \left( S_1 \sum_{n=0}^{\infty} S_1^n (1 + T)^{p^n} \right) + (1 + T)\varphi(1).$$

From the definition of  $\psi$  (see [Chenevier 2013, Section 2.1]), we see that  $y \in D^{\psi=1}$ . Clearly  $(1 - \varphi)y = S_1(1 + T)$ . □

Finally, consider the third isomorphism  $D^{\psi=1}/(\gamma - 1) \rightarrow H^1_{(\psi, \Gamma)}(D)$  induced by sending  $f(T)$  to  $(0, f(T))$ . Using this isomorphism, we see that the class of  $(0, y)$  is a generator of  $H^1_{(\psi, \Gamma)}(D)$  as a free  $\mathcal{O}(U_r)$ -module. We repeat that this cohomology group is in the sense of  $(\psi, \Gamma)$ -modules. In order to get a generator of the  $(\varphi, \Gamma)$ -cohomology group, we note that  $D$  is a rank-1 module so it is tautologically trianguline, so  $\gamma - 1$  is bijective on  $D^{\psi=0}$  by [loc. cit., Corollary 2.5], so

$$H^1_{(\varphi, \Gamma)}(D) \xrightarrow{(-\psi, \text{id})} H^1_{(\psi, \Gamma)}(D)$$

is an isomorphism of cohomology groups, by the discussion at the beginning of this section. Since  $y \in D^{\psi=1}$ , we have  $(\varphi - 1)y \in D^{\psi=0}$ . Since  $\gamma - 1$  is bijective on  $D^{\psi=0}$ , there exists a unique  $x \in D^{\psi=0}$  such that  $(\gamma - 1)x = (\varphi - 1)y$ . Therefore  $(x, y)$  represents a class in  $H^1_{(\varphi, \Gamma)}(D)$ . Moreover, the class of  $(x, y)$  maps to the class of  $(0, y)$  under the isomorphism above. Therefore, the class of  $(x, y)$  generates  $H^1_{(\varphi, \Gamma)}(D)$  as a free  $\mathcal{O}(U_r)$ -module.

For the rest of this paper, every cohomology group that we write will be the  $(\varphi, \Gamma)$ -version, and we drop the subscript “ $(\varphi, \Gamma)$ ” from the notation.

We conclude this section by describing an  $\mathcal{O}(U_r)$ -generator of  $H^1(m\mathcal{R}_{\mathcal{O}(U_r)}(\delta_{U_r}))$ .

Using the isomorphism  $H^1(mD) \rightarrow H^1(D)$ , we see that the class of  $(x, y)$  is represented by elements of  $mD$ : there exists a  $d \in D$  such that  $x - (\varphi - 1)d, y - (\gamma - 1)d \in mD$ .

**Definition 4.5.** We denote by  $e$  the class of  $(x - (\varphi - 1)d, y - (\gamma - 1)d)$  in  $H^1(mD)$ .

Thus  $e$  is an explicit generator of the free  $\mathcal{O}(U_r)$ -module  $H^1(m\mathcal{R}_{\mathcal{O}(U_r)}(\delta_{U_r}))$  of rank 1.

### 5. The $\mathcal{L}$ -invariant of the limit point

Recall that the fiber in the blow-up  $\tilde{U}_r$  of  $U_r$  over the exceptional point  $(p^r, (1 + p)^{r+1} - 1)$  corresponding to the maximal ideal  $m = (f_1, f_2)$  is parametrized by  $\mathbb{P}^1(\mathbb{Q}_p)$ . The main theorem in this section gives us

a formula for the  $\mathcal{L}$ -invariant (in the sense of Definition 3.6) of the  $(\varphi, \Gamma)$ -module associated to a general  $E$ -valued point  $(p^r, (1+p)^{r+1} - 1, a : b)$  in the exceptional fiber. Fix such a point  $(p^r, (1+p)^{r+1} - 1, a : b)$ . Let  $v \in \text{Hom}(m/m^2 \otimes_{\mathbb{Q}_p} E, E)$  be a tangent vector representing the tangent direction associated with this point by Proposition 2.5. Recall that  $D = \mathcal{R}_{\mathcal{O}(U_r)}(\delta_{U_r})$ . Consider the specialization map  $v^* : E \otimes_{\mathbb{Q}_p} mD \rightarrow \mathcal{R}_E(x^r \chi)$  induced by the composition of maps

$$mD \rightarrow m \otimes D \rightarrow m/m^2 \otimes_{\mathbb{Q}_p} D/mD \xrightarrow{v} E \otimes_{\mathbb{Q}_p} \mathcal{R}_{\mathbb{Q}_p}(x^r \chi) = \mathcal{R}_E(x^r \chi)$$

given by

$$fg \mapsto f \otimes g \mapsto \bar{f} \otimes \bar{g} \mapsto v(\bar{f})\bar{g}$$

for all  $f \in m, g \in D$ , where the first map is the inverse of the multiplication map  $m \otimes D \rightarrow mD$  (which is an isomorphism since  $\mathcal{R}_{\mathcal{O}(U_r)}$  is flat over  $\mathcal{O}(U_r)$  by [Chenevier 2013, Lemma 1.3(v)]), and the second map is the map obtained by going mod  $m$  on each factor. In particular, we have

$$v^*(f_1g_1 + f_2g_2) = v(\bar{f}_1)\bar{g}_1 + v(\bar{f}_2)\bar{g}_2,$$

where  $g_1, g_2 \in D$  are arbitrary,  $\bar{g}_1, \bar{g}_2$  are their images in  $D/mD$ , and  $\bar{f}_1, \bar{f}_2$  are the images of  $f_1 = S_1 - p^r$  and  $f_2 = 1 + S_2 - (1+p)^{r+1}$  in  $m/m^2$ , respectively.

The map  $v^*$  yields a specialization map  $H^1(v^*) : E \otimes_{\mathbb{Q}_p} H^1(mD) \rightarrow H^1(\mathcal{R}_E(x^r \chi))$  induced by

$$H^1(v^*)(\eta_1, \eta_2) = (v^*(\eta_1), v^*(\eta_2))$$

for  $(\eta_1, \eta_2) \in mD \oplus mD$  representing a cohomology class in  $H^1(mD)$ . Recall that  $e$  is represented by

$$(x - (\delta_{U_r}(p)\varphi - 1)d, y - (\delta_{U_r}(\chi(\gamma))\gamma - 1)d) \in mD \oplus mD$$

and is an  $\mathcal{O}(U_r)$ -generator of  $H^1(mD)$ , where now  $\varphi$  and  $\gamma$  are the usual untwisted operators (see Definition 4.5). Here we have rewritten  $e$  in terms of these untwisted operators since this explicit formula for  $e$  will be needed below. We make the following definition.

**Definition 5.1.** The  $(\varphi, \Gamma)$ -module associated<sup>5</sup> to the point  $(p^r, (1+p)^{r+1} - 1, a : b)$  is the image of  $e$  in  $H^1(\mathcal{R}_E(x^r \chi))$  under  $H^1(v^*)$ .

Note that the generator  $e$  and the tangent vector  $v$  are only well-defined up to scalars, but the isomorphism class of this  $(\varphi, \Gamma)$ -module is unchanged under multiplication by scalars. The theorem below gives us a formula for the  $\mathcal{L}$ -invariant of the  $(\varphi, \Gamma)$ -module associated to  $(p^r, (1+p)^{r+1} - 1, a : b)$  in the exceptional fiber in terms of  $a$  and  $b$ . Note that the  $\mathcal{L}$ -invariant only depends on the projective image of the cohomology class  $H^1(v^*)(e)$ .

**Theorem 5.2.** *The  $\mathcal{L}$ -invariant of the  $(\varphi, \Gamma)$ -module associated to the  $E$ -valued point in the exceptional fiber with coordinates  $(p^r, (1+p)^{r+1} - 1, a : b)$  is*

$$\mathcal{L} = -\frac{(1+p)^{r+1} \log(1+p)}{p^r} \cdot \frac{a}{b} \in \mathbb{P}^1(E).$$

<sup>5</sup>There is a more conceptual definition of this  $(\varphi, \Gamma)$ -module in terms of a cover  $U_i$  of  $\tilde{U}_r$  as the module  $(D_i)_z$ , with notation as in the proof of [Chenevier 2013, Proposition 3.9], but this definition coincides with the more computational one given here by [loc. cit., Theorem 2.33] (see the remarks at the end of that proof).

*Proof.* We compute the  $\mathcal{L}$ -invariant of the  $(\varphi, \Gamma)$ -module given by  $H^1(v^*)(e)$  using [Definition 3.6](#). For convenience of notation, let  $\text{Res}(f(T)) = \text{res}(f(T) dt)$  for  $f(T) \in \mathcal{R}_E, \mathcal{R}_{\mathcal{O}(U_r)}$ . Write

$$H^1(v^*)(e) = \lambda_v \cdot \bar{\alpha}_{r+1} + \mu_v \cdot \bar{\beta}_{r+1},$$

where, by [\(19\)](#),

$$\lambda_v = \text{Res}(t^r v^*(x - (\delta_{U_r}(p)\varphi - 1)d)), \quad \mu_v = \text{Res}(t^r v^*(y - (\delta_{U_r}(\chi(\gamma))\gamma - 1)d)).$$

To compute these residues, we first note that the following square commutes:

$$\begin{array}{ccc} mD \otimes_{\mathbb{Q}_p} E & \xrightarrow{v^*} & \mathcal{R}_E(x^r \chi) \\ \downarrow \text{Res}(t^r \_) & & \downarrow \text{Res}(t^r \_) \\ m \otimes_{\mathbb{Q}_p} E & \xrightarrow{v} & E \end{array}$$

where we write  $v$  again for the map  $m \otimes E \xrightarrow{\bar{v}} m/m^2 \otimes E \xrightarrow{v} E$ . Indeed, given any element  $f_1 g_1 + f_2 g_2 \in mD$ , we have

$$\begin{aligned} \text{Res}(t^r v^*(f_1 g_1 + f_2 g_2)) &= \text{Res}(t^r (v(\bar{f}_1)\bar{g}_1 + v(\bar{f}_2)\bar{g}_2)) \\ &= v(\bar{f}_1) \text{Res}(t^r \bar{g}_1) + v(\bar{f}_2) \text{Res}(t^r \bar{g}_2) \\ &= v(\text{Res}(t^r g_1) f_1 + \text{Res}(t^r g_2) f_2) \\ &= v(\text{Res}(t^r (f_1 g_1 + f_2 g_2))). \end{aligned}$$

Now we prove that  $\text{Res}(t^r x) = 0$  and  $\text{Res}(t^r y) = 0$ . We use the following formulas, which are similar to [\(20\)](#) and [\(21\)](#). For any  $f(T) \in \mathcal{R}_{\mathcal{O}(U_r)}$ , we have

$$\text{Res}(\varphi(f(T))) = \text{Res}(f(T)), \tag{23}$$

$$\text{Res}(\gamma(f(T))) = \chi(\gamma)^{-1} \text{Res}(f(T)). \tag{24}$$

Using the definition of  $x$  and  $y$  (see the discussion before [Definition 4.5](#)), we get

$$(\delta_{U_r}(\chi(\gamma))\gamma - 1)x = (\delta_{U_r}(p)\varphi - 1)y = -S_1(1 + T),$$

by [Lemma 4.4](#). Multiplying by  $t^r$  and taking residues, by [\(12\)](#) and [\(11\)](#), we get

$$\begin{aligned} \text{Res}((\delta_{U_r}(\chi(\gamma))\chi(\gamma)^{-r}\gamma - 1)t^r x) &= \text{Res}(-t^r S_1(1 + T)) = 0, \\ \text{Res}((\delta_{U_r}(p)p^{-r}\varphi - 1)t^r y) &= \text{Res}(-t^r S_1(1 + T)) = 0. \end{aligned}$$

Applying [\(24\)](#) and [\(23\)](#) to the terms on the left, we get  $\text{Res}(t^r x) = 0 = \text{Res}(t^r y)$ .

Thus, using the commutativity of the diagram above, we get

$$\begin{aligned} \lambda_v &= v(\text{Res}(t^r (x - (\delta_{U_r}(p)\varphi - 1)d))) \\ &= -v(\text{Res}(t^r (\delta_{U_r}(p)\varphi - 1)d)) \\ &\stackrel{(11)}{=} -v(\text{Res}((\delta_{U_r}(p)p^{-r}\varphi - 1)t^r d)). \end{aligned}$$

Using (23), we get

$$\begin{aligned}\lambda_v &= -v((\delta_{U_r}(p)p^{-r} - 1) \operatorname{Res}(t^r d)) \\ &\stackrel{(22)}{=} -v((S_1 - p^r)p^{-r} \operatorname{Res}(t^r d)) \\ &= -v(f_1 p^{-r} \operatorname{Res}(t^r d)) = -p^{-r} v(f_1 \operatorname{Res}(t^r d)).\end{aligned}$$

The expression in the brackets belongs to  $m$ . Recall that  $v : m \otimes E \rightarrow E$  is the composition of the maps  $\bar{\cdot} : m \rightarrow m/m^2$  and  $v : m/m^2 \otimes E \rightarrow E$ . We therefore get

$$\lambda_v = -p^{-r} \overline{\operatorname{Res}(t^r d)} \cdot v(\bar{f}_1) = -p^{-r} \overline{\operatorname{Res}(t^r d)} \cdot a.$$

Similarly,

$$\begin{aligned}\mu_v &= v(\operatorname{Res}(t^r (y - (\delta_{U_r}(\chi(\gamma))\gamma - 1)d))) \\ &= -v(\operatorname{Res}(t^r (\delta_{U_r}(\chi(\gamma))\gamma - 1)d)) \\ &\stackrel{(12)}{=} -v(\operatorname{Res}((\delta_{U_r}(\chi(\gamma))\chi(\gamma)^{-r}\gamma - 1)t^r d)).\end{aligned}$$

Using (24), we get

$$\mu_v = -v((\delta_{U_r}(\chi(\gamma))\chi(\gamma)^{-(r+1)} - 1) \operatorname{Res}(t^r d)).$$

Since  $\chi(\gamma) = \zeta_{p-1}^a(1+p)$ , for some  $a$ , we get

$$\begin{aligned}\mu_v &\stackrel{(22)}{=} -v((\zeta_{p-1}^{a(r+1)}(1+S_2)\zeta_{p-1}^{-a(r+1)}(1+p)^{-(r+1)} - 1) \operatorname{Res}(t^r d)) \\ &= -v(f_2(1+p)^{-(r+1)} \operatorname{Res}(t^r d)) \\ &= -(1+p)^{-(r+1)} v(f_2 \operatorname{Res}(t^r d)).\end{aligned}$$

The expression inside the brackets is an element of  $m$ . As above, we get

$$\mu_v = -(1+p)^{-(r+1)} \overline{\operatorname{Res}(t^r d)} \cdot v(\bar{f}_2) = -(1+p)^{-(r+1)} \overline{\operatorname{Res}(t^r d)} \cdot b.$$

Both  $\lambda_v$  and  $\mu_v$  cannot be equal to 0 simultaneously because  $H^1(v^*)(e) \neq 0$ . So  $\overline{\operatorname{Res}(t^r d)} \neq 0$ . By Definition 3.6 (or directly by [Benois 2011, Proposition 2.3.7]), the  $\mathcal{L}$ -invariant of  $H^1(v^*)(e)$  is

$$-\log(\chi(\gamma)) \cdot \frac{\lambda_v}{\mu_v} = -\frac{(1+p)^{r+1} \log(1+p)}{p^r} \cdot \frac{a}{b}. \quad \square$$

## 6. Proof of Theorem 1.1 and generalizations

**6.1. Proof of Theorem 1.1.** Let  $k \geq 3$  and let  $r = k - 2$ . Let  $(k_n, a_n)$  for  $n \geq 1$  be as in (1). These quantities depend on  $\mathcal{L} \in \mathbb{P}^1(\overline{\mathbb{Q}}_p)$ . We prove that the sequence of crystalline representations  $V_{k_n, a_n}^*$  converges to the semistable representation  $V_{k, \mathcal{L}}^*$ .

We recall some facts from Section 1.3. Recall that the ordered pair of characters  $(\delta_{1,n}, \delta_{2,n})$  is associated to the representation  $V_{k_n, a_n}^*$  and the sequence of characters  $\delta_{1,n} \delta_{2,n}^{-1} = \mu_{y_n^2} \chi^{k_n-1}$ , where  $y_n$  is as in (3), converges to the exceptional character  $x^r \chi$ . Moreover, the character  $\mu_{y_n^2} \chi^{k_n-1}$  corresponds to the following point of  $\tilde{U}_r$ :

$$(y_n^2, (1+p)^{k_n-1} - 1, y_n^2 - p^r : (1+p)^{k_n-1} - (1+p)^{k-1}). \quad (25)$$

We compute the limit of the above points in  $\tilde{U}_r$  as  $n \rightarrow \infty$  using the following lemmas.

**Lemma 6.1.** *We have*

$$\lim_{n \rightarrow \infty} \frac{y_n^2 - p^r}{p^n(p-1)} = \begin{cases} \mathcal{L}p^r & \text{if } \mathcal{L} \neq \infty, \\ 2p^r/(p-1) & \text{if } \mathcal{L} = \infty. \end{cases}$$

*Proof.* Assume  $\mathcal{L} \neq \infty$ . We have

$$y_n^2 - p^r = \left( \frac{a_n + \sqrt{a_n^2 - 4p^{k_n-1}}}{2} \right)^2 - p^r = a_n^2 \left( \frac{1 + \sqrt{1 - 4p^{k_n-1}a_n^{-2}}}{2} \right)^2 - p^r.$$

For large  $n$ , the valuation of  $a_n = p^{r/2} + \frac{1}{2}\mathcal{L}p^{n+r/2}(p-1)$  is equal to  $\frac{1}{2}r$ . Therefore for large  $n$ , the expression inside the second radical sign is an element of  $1 + p^{1+p^n(p-1)}\mathcal{O}_E$ , where  $\mathcal{O}_E$  is the ring of integers of  $E$ . Since taking square roots is an automorphism of this group, we see that  $\sqrt{1 - 4p^{k_n-1}a_n^{-2}} \in 1 + p^{1+p^n(p-1)}\mathcal{O}_E$ . Therefore we can write

$$\frac{1 + \sqrt{1 - 4p^{k_n-1}a_n^{-2}}}{2} = 1 + up^{1+p^n(p-1)}$$

for some  $u \in \mathcal{O}_E$ . Substituting this in the previous equation, we get

$$\begin{aligned} y_n^2 - p^r &= (p^r + \mathcal{L}p^{n+r}(p-1) + \frac{1}{4}\mathcal{L}^2p^{2n+r}(p-1)^2)(1 + 2up^{1+p^n(p-1)} + u^2p^{2+2p^n(p-1)}) - p^r \\ &= p^r + \mathcal{L}p^{n+r}(p-1) + p^{2n}(u') - p^r \end{aligned}$$

for some  $u'$  whose valuation is bounded below. So

$$\lim_{n \rightarrow \infty} \frac{y_n^2 - p^r}{p^n(p-1)} = \mathcal{L}p^r.$$

The limit in the case  $\mathcal{L} = \infty$  is computed similarly. □

**Lemma 6.2.** *We have*

$$\lim_{n \rightarrow \infty} \frac{(1+p)^{k_n-1} - (1+p)^{k-1}}{p^n(p-1)} = \begin{cases} (1+p)^{k-1} \log(1+p) & \text{if } \mathcal{L} \neq \infty, \\ 0 & \text{if } \mathcal{L} = \infty. \end{cases}$$

*Proof.* Assume  $\mathcal{L} \neq \infty$ . Since

$$(1+p)^{k_n-1} - (1+p)^{k-1} = (1+p)^{k-1}[(1+p)^{p^n(p-1)} - 1],$$

we see that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{(1+p)^{k_n-1} - (1+p)^{k-1}}{p^n} &= (1+p)^{k-1} \lim_{n \rightarrow \infty} \frac{(1+p)^{p^n(p-1)} - 1}{p^n} \\ &= (1+p)^{k-1} \lim_{n \rightarrow \infty} \frac{\{1 + [(1+p)^{(p-1)} - 1]\}^{p^n} - 1}{p^n} \\ &\stackrel{(10)}{=} (1+p)^{k-1} \log(1+p)^{p-1}. \end{aligned}$$

The limit in the case  $\mathcal{L} = \infty$  is proved similarly. □

Write the sequence (25) as

$$\left( y_n^2, (1+p)^{kn-1} - 1, \frac{y_n^2 - p^r}{p^n(p-1)} : \frac{(1+p)^{kn-1} - (1+p)^{k-1}}{p^n(p-1)} \right).$$

Using Lemmas 6.1 and 6.2, we see that the sequence above converges to the point

$$\begin{cases} (p^r, (1+p)^{k-1} - 1, \mathcal{L}p^r : (1+p)^{k-1} \log(1+p)) & \text{if } \mathcal{L} \neq \infty, \\ (p^r, (1+p)^{k-1} - 1, 1 : 0) & \text{if } \mathcal{L} = \infty. \end{cases}$$

Assume  $\mathcal{L} \neq \infty$ . By Theorem 5.2, the  $\mathcal{L}$ -invariant of the  $(\varphi, \Gamma)$ -module associated to this limit point is  $-\mathcal{L}$ . By [Colmez 2008, Théorème 0.5(i)], this  $(\varphi, \Gamma)$ -module is also étale since  $(\mu_{p^{r/2}}, \mu_{1/p^{r/2}}\chi^{1-k}, -\mathcal{L}) \in \mathcal{S}_* \setminus \mathcal{S}_*^{\text{nc1}}$  in the notation of [loc. cit.] since  $\frac{1}{2}r - \frac{1}{2}r = 0$ ,  $\frac{1}{2}r > 0$  and  $\frac{1}{2}r \neq k - 1$ . The corresponding Galois representation is  $V(\mu_{p^{r/2}}, \mu_{1/p^{r/2}}\chi^{1-k}, -\mathcal{L})$  in the notation of [loc. cit.]. Comparing the filtered  $(\varphi, N)$ -module associated to  $V_{k,\mathcal{L}}^*$  given in the Introduction with the one associated to  $V(\mu_{p^{r/2}}, \mu_{1/p^{r/2}}\chi^{1-k}, -\mathcal{L})$  using [loc. cit., Section 4.6] (more specifically, [loc. cit., Proposition 4.18] with  $a = 1 - k$ ,  $b = 0$ ,  $\alpha = \mu_{p^{k/2}}$ , and  $\mathcal{L}$  replaced by  $-\mathcal{L}$ ), we see that  $V(\mu_{p^{r/2}}, \mu_{1/p^{r/2}}\chi^{1-k}, -\mathcal{L}) \simeq V_{k,\mathcal{L}}^*$ . Thus the sequence of crystalline representations  $V_{k_n, a_n}^*$  converges to the semistable representation  $V_{k,\mathcal{L}}^*$  for  $\mathcal{L} \in E$ .

Now assume we are in the  $\mathcal{L} = \infty$  case. By Theorem 5.2, the  $\mathcal{L}$ -invariant associated to the above limit point is  $\infty$ . The corresponding Galois representation is  $V(\mu_{p^{r/2}}, \mu_{1/p^{r/2}}\chi^{1-k}, \infty)$  in the notation of [loc. cit.]. Comparing the filtered  $\varphi$ -module associated to  $V_{k,\infty}^*$  in the Introduction with the one associated to  $V(\mu_{p^{r/2}}, \mu_{1/p^{r/2}}\chi^{1-k}, \infty)$  in [loc. cit., Section 4.5] (more precisely, take  $a = 1 - k$ ,  $b = 0$ ,  $\beta = \mu_{p^{r/2}}$  and  $\alpha = \mu_{p^{k/2}}$  in the second isomorphism in [loc. cit., Proposition 4.13]), we see that  $V(\mu_{p^{r/2}}, \mu_{1/p^{r/2}}\chi^{1-k}, \infty) \simeq V_{k,\infty}^*$ . (Alternatively, the corresponding Galois representation is  $V(\mu_{p^{r/2}}, \mu_{1/p^{r/2}}\chi^{1-k})$  in the notation of [Berger 2012] and, by Proposition 3.1 of that work, this last representation is isomorphic to the crystalline representation  $V_{k, a_p}^*$  with  $a_p = p^{k/2} + p^{k/2-1}$ , which as mentioned in the Introduction, is isomorphic to  $V_{k,\infty}^*$ .) Thus, the sequence of crystalline representations  $V_{k_n, a_n}^*$  again converges to the (crystalline) representation  $V_{k,\infty}^*$ .

**6.2.  $\mathcal{L}$ -invariants as logarithmic derivatives.** In this subsection, all  $\mathcal{L}$ -invariants are finite.

We prove formula (2) from the Introduction showing that  $\mathcal{L}$  is twice the logarithmic derivative of  $a_p$ . More precisely, we prove that if  $a_p : \mathbb{Z}_p \rightarrow E$  is a differentiable function of  $l$  with  $a_p(k) = p^{r/2}$  for  $r = k - 2$  and  $k \geq 3$ , then the crystalline representations  $V_{l, a_p(l)}^*$  converge in  $\tilde{\mathcal{T}}_2$  to the semistable representation  $V_{k,\mathcal{L}}^*$  with

$$\mathcal{L} = 2a_p(k)^{-1}a'_p(k)$$

as  $l$  tends to  $k$  in the  $p$ -adic topology through integers  $l \equiv k \pmod{p-1}$  with  $l \neq k$ . The condition  $l \equiv k \pmod{p-1}$  is necessary because  $V_{l, a_p(l)}^*$  converges to  $V_{k,\mathcal{L}}^*$  implies that (the tame part of)  $\det V_{l, a_p(l)}^* = \chi^{1-l}$  converges to (the tame part of)  $\det V_{k,\mathcal{L}}^* = \chi^{1-k}$ .

This formula is a variant of a classical formula due to [Greenberg and Stevens 1993, Theorem 3.18; Stevens 2010, Theorem B; Bertolini, Darmon, and Iovita 2010, Theorem 4; Colmez 2010, Théorème 0.5, Corollaire 0.7; Benois 2010, Theorem 2] and others (see Remark 6.6). Our proof of the formula seems

new. It uses some elementary  $p$ -adic analysis (see the two lemmas below) and [Theorem 5.2](#), which in turn uses some geometry (the blow-up space  $\tilde{\mathcal{T}}_2$ ) and some algebra (the interpretation of this space in terms of trianguline  $(\varphi, \Gamma)$ -modules over the Robba ring).

Recall that for integer  $l \geq 2$ , the  $(\varphi, \Gamma)$ -module  $\mathbf{D}_{\text{rig}}(V_{l, a_p(l)}^*)$  is an extension of  $\mathcal{R}_E(\mu_{1/y(l)}\chi^{1-l})$  by  $\mathcal{R}_E(\mu_{y(l)})$ , where

$$y(l) = \frac{a_p(l) + \sqrt{a_p(l)^2 - 4p^{l-1}}}{2}.$$

Let  $\delta_1(l) = \mu_{y(l)}$  and  $\delta_2(l) = \mu_{1/y(l)}\chi^{1-l}$ . Since  $l \equiv k \pmod{p-1}$ , the characters  $\delta_1(l)\delta_2(l)^{-1}$  converge to the exceptional character  $x^r\chi$  as  $l \rightarrow k$ . Therefore the characters  $\delta_1(l)\delta_2(l)^{-1}$  eventually belong to  $\tilde{U}_r$ . Moreover,  $\delta_1(l)\delta_2(l)^{-1}$  corresponds to the following point of  $\tilde{U}_r$ :

$$(y(l)^2, (1+p)^{l-1} - 1, y(l)^2 - p^r : (1+p)^{l-1} - (1+p)^{k-1}). \tag{26}$$

We compute the limit of these points in  $\tilde{U}_r$  as  $l \rightarrow k$  using the following two lemmas.

**Lemma 6.3.** *We have*

$$\lim_{l \rightarrow k} \frac{y(l)^2 - p^r}{l - k} = 2a_p(k)a'_p(k).$$

*Proof.* The statement generalizes that of [Lemma 6.1](#) in the case  $\mathcal{L} \neq \infty$  and we give a slightly different proof. Note that

$$\begin{aligned} y(l)^2 - p^r &= \frac{2a_p(l)^2 + 2a_p(l)\sqrt{a_p(l)^2 - 4p^{l-1}} - 4p^{l-1}}{4} - p^r \\ &= \frac{a_p(l)^2 - p^r}{2} + \frac{a_p(l)^2\sqrt{1 - 4a_p(l)^{-2}p^{l-1}} - p^r}{2} - p^{l-1}. \end{aligned}$$

Therefore

$$\frac{y(l)^2 - p^r}{l - k} = \frac{a_p(l)^2 - p^r}{2(l - k)} + \frac{a_p(l)^2 - p^r}{2(l - k)}\sqrt{1 - 4a_p(l)^{-2}p^{l-1}} + p^r \frac{\sqrt{1 - 4a_p(l)^{-2}p^{l-1}} - 1}{2(l - k)} - \frac{p^{l-1}}{l - k}.$$

By the definition of the derivative, the first and the second summands in the equation above converge to  $a_p(k)a'_p(k)$  as  $l \rightarrow k$ . The last summand clearly converges to 0. The third summand also converges to 0. Indeed, as  $l \rightarrow k$ , the valuation of  $a_p(l)$  becomes  $\frac{1}{2}r$ . Therefore for such  $l \geq k$  we can write

$$\sqrt{1 - 4a_p(l)^{-2}p^{l-1}} = \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} (-4a_p(l)^{-2}p^{l-1})^i.$$

Using the fact that  $p^{l-1}/l - k$  converges to 0 as  $l \rightarrow k$ , we see that

$$\lim_{l \rightarrow k} \frac{\sqrt{1 - 4a_p(l)^{-2}p^{l-1}} - 1}{2(l - k)} = 0.$$

Putting everything together, we see that

$$\lim_{l \rightarrow k} \frac{y(l)^2 - p^r}{l - k} = 2a_p(k)a'_p(k). \quad \square$$

**Lemma 6.4.** *We have*

$$\lim_{l \rightarrow k} \frac{(1+p)^{l-1} - (1+p)^{k-1}}{l-k} = (1+p)^{k-1} \log(1+p).$$

*Proof.* This is the same as Lemma 6.2 in the case  $\mathcal{L} \neq \infty$ , so we give a slightly different proof. Write

$$\frac{(1+p)^{l-1} - (1+p)^{k-1}}{l-k} = (1+p)^{k-1} \frac{[(1+p)^{l-k} - 1]}{l-k} = (1+p)^{k-1} \left[ \sum_{i=1}^{\infty} \frac{1}{i} \binom{l-k-1}{i-1} p^i \right].$$

Since  $\binom{-1}{i-1} = (-1)^{i-1}$ , taking the limit as  $l \rightarrow k$  we obtain

$$\lim_{l \rightarrow k} \frac{(1+p)^{l-1} - (1+p)^{k-1}}{l-k} = (1+p)^{k-1} \log(1+p). \quad \square$$

Now rewrite the point (26) in the blow-up as

$$\left( y(l)^2, (1+p)^{l-1} - 1, \frac{y(l)^2 - p^r}{l-k} : \frac{(1+p)^{l-1} - (1+p)^{k-1}}{l-k} \right).$$

Using Lemmas 6.3 and 6.4, we see that as  $l \rightarrow k$ , the points above converge to

$$(p^r, (1+p)^{k-1} - 1, 2a_p(k)a'_p(k) : (1+p)^{k-1} \log(1+p)).$$

By Theorem 5.2, we see that the  $\mathcal{L}$ -invariant of the  $(\varphi, \Gamma)$ -module associated to this limit point is  $-2a_p(k)^{-1}a'_p(k)$ . Working as at the end of Section 6.1, we see that the corresponding Galois representation is  $V(\mu_{p^{r/2}}, \mu_{1/p^{r/2}} \chi^{1-k}, -\mathcal{L})$  with  $\mathcal{L} = 2a_p(k)^{-1}a'_p(k)$ , so the crystalline representations  $V_{l,a_p(l)}^*$  converge to the semistable representation  $V_{k,\mathcal{L}}^*$  with  $\mathcal{L} = 2a_p(k)^{-1}a'_p(k)$ .

**Remark 6.5.** We thank one of the referees for pointing out the following simplification to the computation of the limits in Lemmas 6.1 and 6.3. Assume as above that  $a_p : \mathbb{Z}_p \rightarrow E$  is a differentiable function of  $l$  with  $a_p(k) = p^{r/2}$  with  $r = k - 2$  and  $k \geq 3$ . If we replace the crystalline representations  $V_{l,a_p(l)}^*$  by  $V_{l,a_p(l)+p^{l-1}/a_p(l)}^*$ , then the limit computation in Lemma 6.3 becomes easier. Indeed, the  $y(l)$  for the crystalline representation  $V_{l,a_p(l)+p^{l-1}/a_p(l)}^*$  is equal to  $a_p(l)$ . Therefore, for  $l$  close to  $k$  and  $l \equiv k \pmod{p-1}$ , the point in  $\tilde{\mathcal{T}}$  associated to  $V_{l,a_p(l)+p^{l-1}/a_p(l)}^*$  belongs to  $\tilde{U}_r$  and is given by

$$(a_p(l)^2, (1+p)^{l-1} - 1, a_p(l)^2 - p^r : (1+p)^{l-1} - (1+p)^{k-1}).$$

To evaluate the limit of these points in the third coordinate, we have to evaluate the limit

$$\lim_{\substack{l \rightarrow k \\ l \equiv k \pmod{p-1}}} \frac{a_p(l)^2 - p^r}{l-k}.$$

But, this limit is just the derivative of  $a_p(l)^2$  at  $l = k$  and so is immediately equal to  $2a_p(k)a'_p(k)$ . Similarly, for the limits in Lemma 6.1. In principle, since the limit of the above sequence is the same as that of the original sequence, the crystalline representations  $V_{l,a_p(l)+p^{l-1}/a_p(l)}$  can also be used to compute the reduction of the semistable representation  $V_{k,\mathcal{L}}$ . Note that  $\bar{V}_{l,a_p(l)+p^{l-1}/a_p(l)} \cong \bar{V}_{l,a_p(l)}$  for  $l$  close to  $k$  by [Berger 2012, Theorem A].

Also, Lemmas 6.2 and 6.4 follow immediately by differentiating the function  $f : \mathbb{Z}_p \rightarrow \mathbb{Q}_p$  defined by  $f(x) = (1 + p)^{x-1}$  at  $x = k$ .

**Remark 6.6** (relation to work of Greenberg and Stevens, etc.). Our formula (2) is slightly different from the classical formula due to [Greenberg and Stevens 1993; Stevens 2010; Colmez 2010; Bertolini, Darmon, and Iovita 2010; Benois 2010] and others. We thank D. Benois for pointing this out.

Let us explain the difference. In the classical setting, one starts with a newform  $f$  of weight  $k \geq 2$  for the subgroup  $\Gamma_0(Np)$ , where  $(N, p) = 1$  (for simplicity, we assume that the nebentypus at  $N$  is trivial) with  $U_p$ -eigenvalue  $\alpha_p(f) = p^{(k-2)/2}$  (as opposed to  $-p^{(k-2)/2}$  for simplicity). Then one takes the Hida family  $F$  if  $f$  is ordinary (equivalently  $k = 2$ ), or the Coleman family  $F$  if  $f$  has positive slope (equivalently  $k > 2$ ), passing through  $f$  with  $U_p$ -eigenvalue  $\alpha_p$ , which is an analytic function of the weight. Note that  $\alpha_p(k) = \alpha_p(f)$ . The classical formula states that the  $\mathcal{L}$ -invariant of the form  $f$  is

$$\mathcal{L}(f) = -2 \frac{\alpha'_p(k)}{\alpha_p(k)}.$$

Incidentally, the above authors use various definitions of the  $\mathcal{L}$ -invariant of the form  $f$  but these are all equal (see [Colmez 2005] for a survey comparing these alternative definitions).

For  $l \equiv k \pmod{p-1}$ , the nebentypus of the weight  $l$  member  $F_l$  of the family  $F$  is trivial. Since forms living in a Hida or Coleman family have the same slope and since the slope of the  $U_p$ -eigenvalue of a  $\Gamma_0(Np)$ -newform of weight  $l$  is equal to  $\frac{1}{2}(l-2)$ , we see that  $f = F_k$  is the unique classical member in the family  $F$  of weight  $l \equiv k \pmod{p-1}$  that is  $p$ -new. In fact, any classical member  $F_l$  with  $l \equiv k \pmod{p-1}$  and  $l \neq k$  arises as a  $p$ -stabilization of a form  $\tilde{F}_l$  that is only  $N$ -new. The  $U_p$ -eigenvalue  $\alpha_p(l)$  of  $F_l$  is a root of

$$x^2 - a_p(l)x + p^{l-1},$$

where  $a_p(l)$  is the  $T_p$ -eigenvalue of the form  $\tilde{F}_l$ . The local Galois representation

$$\rho_{F_l}|_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \simeq \rho_{\tilde{F}_l}|_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$$

is isomorphic to the crystalline representation  $V_{l, a_p(l)}$ . Moreover,  $\rho_f|_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$  is isomorphic to the semistable representation  $V_{k, \mathcal{L}}$  with  $\mathcal{L} = -\mathcal{L}(f)$ . The change in sign is because the  $\mathcal{L}$ -invariant in the filtration on  $D_{k, \mathcal{L}}$  given in the Introduction is the negative of the one in the filtration on the filtered module in [Mazur 1994]. Therefore in the classical setup,

$$\mathcal{L} = -\mathcal{L}(f) = 2 \frac{\alpha'_p(k)}{\alpha_p(k)}.$$

In our setup, we have a smooth function  $a_p : \mathbb{Z}_p \rightarrow E$  such that  $a_p(k) = p^{(k-2)/2}$ . We prove that the sequence of crystalline representations  $V_{l, a_p(l)}^*$  converge to the semistable representation  $V_{k, \mathcal{L}}^*$ , with

$$\mathcal{L} = 2 \frac{\alpha'_p(k)}{a_p(k)}.$$

Note that there does not seem to be a common framework within which one may compare these two formulas. Since the functions  $a_p(l)$  and  $\alpha_p(l)$  associated to the Hida or Coleman families above are related by the formula

$$a_p(l) = \alpha_p(l) + \frac{p^{l-1}}{\alpha_p(l)},$$

the  $a_p(l)$  above cannot be interpolated by a smooth function  $a_p : \mathbb{Z}_p \rightarrow E$ . Indeed,  $p^{l-1}$  is not even defined on the whole of  $\mathbb{Z}_p$ . However, we can obtain the classical formula above using the trick in [Remark 6.5](#). Indeed, suppose  $\alpha_p : \mathbb{Z}_p \rightarrow E$  is a differentiable function of  $l$  with  $\alpha_p(k) = p^{(k-2)/2}$  with  $k \geq 3$ . Consider the crystalline representations  $V_{l, \alpha_p(l) + p^{l-1}/\alpha_p(l)}^*$  for integer  $l \equiv k \pmod{p-1}$  with  $l \neq k$ . The  $y(l)$  for these representations is  $\alpha_p(l)$ . Therefore, for  $l$  close to  $k$  and  $l \equiv k \pmod{p-1}$ , the point in  $\tilde{\mathcal{T}}$  associated to  $V_{l, \alpha_p(l) + p^{l-1}/\alpha_p(l)}^*$  belongs to  $\tilde{U}_r$  and is given by

$$(\alpha_p(l)^2, (1+p)^{l-1} - 1, \alpha_p(l)^2 - p^r : (1+p)^{l-1} - (1+p)^{k-1}).$$

Arguing as in [Remark 6.5](#) (with  $a_p(l)$  there replaced by  $\alpha_p(l)$ ), we see that as  $l$  tends to  $k$  the limit of the above sequence of points is

$$(p^r, (1+p)^{k-1} - 1, 2\alpha_p(k)\alpha'_p(k) : (1+p)^{k-1} \log(1+p)).$$

Using [Theorem 5.2](#), we see that the  $\mathcal{L}$ -invariant of the  $(\varphi, \Gamma)$ -module associated to the limit point is  $-\mathcal{L}$  with

$$\mathcal{L} = 2 \frac{\alpha'_p(k)}{\alpha_p(k)}.$$

In the notation of [\[Colmez 2008\]](#), the corresponding Galois representation is  $V(\mu_{p^{r/2}}, \mu_{1/p^{r/2}}, -\mathcal{L})$ , which is isomorphic to  $V_{k, \mathcal{L}}^*$  as explained at the end of [Section 6.1](#).

### 7. Proof of [Theorem 1.3](#)

In this section, we use [Theorem 1.1](#) along with local constancy for the reduction and the zig-zag conjecture to compute the reductions of semistable representations  $V_{k, \mathcal{L}}$  with weights  $k$  in the range  $[3, p+1]$  for  $p$  odd.

Consider the crystalline representation  $V_{k_n, a_n}$ , with  $(k_n, a_n)$  as in [\(1\)](#). Since  $v_p(a_n) = \frac{1}{2}r$  for  $r = k - 2$ , for  $n$  sufficiently large, the weight  $k_n$  is an *exceptional* weight with respect to the half-integral slope  $0 < v = \frac{1}{2}r \leq \frac{1}{2}(p-1)$  in the sense of [\[Ghate 2021, Section 1\]](#), namely  $k_n \equiv 2v + 2 \pmod{p-1}$  for  $n$  large. Therefore, we can apply [Conjecture 1.2](#) to compute the reduction of  $V_{k_n, a_n}$  for large  $n$ . The zig-zag conjecture specifies the reduction in terms of an alternating sequence of irreducible and reducible mod  $p$  representations depending on the relative size of a rational parameter  $\tau$  with respect to (certain integer shifts of) another integer parameter  $t$ .

For  $V_{k_n, a_n}$ , the formula for  $\tau$  (see [\[Ghate 2021, \(1.1\)\]](#)), which we call  $\tau_n$ , is

$$\tau_n = v_p \left( \frac{a_n^2 - \binom{k_n-2-v_-}{v_+} \binom{k_n-2-v_+}{v_-} p^r}{p a_n} \right),$$

where  $v_-$  and  $v_+$  are the largest and the smallest integers, respectively, such that  $v_- < v < v_+$ . Let us simplify this. We first treat the case  $\mathcal{L} \neq \infty$ . We have

$$\tau_n = v_p \left( \frac{p^r \left(1 + \frac{1}{2} p^n (p-1) \mathcal{L}\right)^2 - \binom{k+p^n(p-1)-2-v_-}{v_+} \binom{k+p^n(p-1)-2-v_+}{v_-} p^r}{p^{1+\frac{1}{2}r} \left(1 + \frac{1}{2} p^n (p-1) \mathcal{L}\right)} \right).$$

For large  $n$ , we have  $v_- + v_+ = k - 2$ , so

$$\begin{aligned} \binom{k+p^n(p-1)-2-v_-}{v_+} &= \frac{(1 + p^n(p-1))(2 + p^n(p-1)) \cdots (v_+ + p^n(p-1))}{v_+!} \\ &= 1 + p^n(p-1)H_+ + \text{terms involving } p^{2n} \end{aligned}$$

for the harmonic sum<sup>6</sup>  $H_+ = \sum_{i=1}^{v_+} 1/i$ . Similarly,

$$\binom{k+p^n(p-1)-2-v_+}{v_-} = 1 + p^n(p-1)H_- + \text{terms involving } p^{2n}$$

for  $H_- = \sum_{i=1}^{v_-} (1/i)$  (if  $v_- \geq 1$ ;  $H_- = 0$  if  $v_- = 0$ ). Substituting, we get

$$\begin{aligned} \tau_n &= r + v_p \left( \left(1 + \frac{1}{2} \mathcal{L} p^n (p-1)\right)^2 - [1 + p^n(p-1)(H_- + H_+) + \text{terms involving } p^{2n}] \right. \\ &\quad \left. - v_p \left( p^{1+\frac{1}{2}r} \left(1 + \frac{1}{2} \mathcal{L} p^n (p-1)\right) \right) \right) \\ &= r + v_p \left( \mathcal{L} p^n (p-1) - (H_- + H_+) p^n (p-1) + \text{terms involving } p^{2n} \right) - \left(1 + \frac{1}{2} r\right) - v_p \left(1 + \frac{1}{2} \mathcal{L} p^n (p-1)\right). \end{aligned}$$

Thus, for large  $n$ ,

$$\tau_n = \frac{1}{2}r - 1 + n + v_p(\mathcal{L} - H_- - H_+)$$

if  $\mathcal{L} \neq H_- + H_+$  (and  $\tau_n \geq \frac{1}{2}r - 1 + 2n + c$  for some  $c \in \mathbb{Q}$  independent of  $n$  if  $\mathcal{L} = H_- + H_+$ ).

As mentioned above, the zig-zag conjecture involves another parameter  $t$  which for  $V_{k_n, a_n}$  we call  $t_n$ . We have

$$t_n = v_p(k_n - 2 - r) = n.$$

These formulas for  $\tau_n$  and  $t_n$  show that for large  $n$  the parameter  $\tau_n - t_n$  lies, independently of  $n$ , in one of the intervals (which may possibly be a point) that appear in the statement of [Ghate 2021, Conjecture 1.1]. This interval is determined by the size of  $v = v_p(\mathcal{L} - H_- - H_+)$  (and is the rightmost one when  $v = \infty$ ). The conjecture accordingly specifies the exact shape of the reductions of the crystalline representations  $V_{k_n, a_n}$  for large  $n$  in terms of the size of  $v$ . Since taking duals commutes with taking reduction, we obtain the reductions of the  $V_{k_n, a_n}^*$  for large  $n$  in terms of the size of  $v$ . Using Theorem 1.1 and local constancy for the reduction, we get the reduction of the limiting semistable representation  $V_{k, \mathcal{L}}^*$  in terms of the size of  $v$ . Finally, taking duals again, we obtain Theorem 1.3 for  $\mathcal{L}$  finite.

Now assume we are in the case  $\mathcal{L} = \infty$ . Then a computation similar to the one above shows that  $\tau_n = \frac{1}{2}r - 1 + n$  and  $t_n = n^2$ , so that for large  $n$  we have  $\tau_n < t_n$  (and so  $\tau_n - t_n$  is in the leftmost interval

<sup>6</sup>Thus, the provenance of the harmonic sums that are prevalent in all the computation of the reductions of semistable representations in the literature can now be traced back to the  $p$ -adic expansions of the binomial coefficients appearing in the zig-zag conjecture.

appearing in the conjecture). Using [Ghate 2021, Conjecture 1.1] and Theorem 1.1 again, we see that  $\bar{V}_{k,\infty} \sim \text{ind}(\omega_2^{k-1})$ , at least on the inertia subgroup  $I_{\mathbb{Q}_p}$ . Thus Theorem 1.3 also holds for  $\mathcal{L} = \infty$  as  $v = -\infty$  (but as remarked after the theorem, the result in this case is classical by [Edixhoven 1992] and even holds without assuming zig-zag!).

## 8. Bounded Hodge–Tate weights

In this section, we use the techniques of this paper to give a proof of the fact that the limit of a sequence of two-dimensional (irreducible) crystalline representations  $V_n$  for  $n \geq 1$  with Hodge–Tate weights in an interval  $[a, b]$  such that the difference of the Hodge–Tate weights is at least 2 infinitely often is also (irreducible) crystalline with Hodge–Tate weights in the interval  $[a, b]$ . However, note that Berger [2004, Théorème 1] has proved more generally that the limit of subquotients of crystalline representations with bounded Hodge–Tate weights belonging to  $[a, b]$  is also crystalline with Hodge–Tate weights in  $[a, b]$ .

Twisting by a fixed power of the cyclotomic character, we may assume that infinitely often the Hodge–Tate weights of  $V_n$  are  $(0, k_n - 1)$  with  $k_n \geq 3$ . This sequence of integers  $k_n$  is not to be confused with the one defined in (1). Then there exists a sequence of unramified characters  $\mu_n$  for  $n \geq 1$  such that  $V_n \simeq V_{k_n, a_n} \otimes \mu_n$  for some  $a_n$  with  $v_p(a_n) > 0$ . For each  $n \geq 1$ , consider the ordered pair of characters  $(\delta_{1,n}, \delta_{2,n})$  associated with  $V_n^*$  under triangulation. We have  $\delta_{1,n} = \mu_{y_n} \cdot \mu_n^{-1}$  and  $\delta_{2,n} = \mu_{1/y_n} \chi^{1-k_n} \cdot \mu_n^{-1}$ , where  $y_n$  is as in (3). The assumption that the sequence  $V_n$  converges means that the associated sequence of points in  $\tilde{\mathcal{T}}_2$  converge to a point in  $\tilde{\mathcal{T}}_2$ . Say that the sequence  $(\delta_{1,n}, \delta_{2,n})$  converges to  $(\delta_1, \delta_2)$  in  $(\mathcal{T} \times \mathcal{T}) \setminus F'$ .

First assume that  $\delta_1 \delta_2^{-1}$  is not of the form  $x^r \chi$  for  $r \geq 0$ . Then, by convention, the  $\mathcal{L}$ -invariant associated to the limit point is  $\infty$ . Using the convergence of (the unramified parts of)  $\delta_{1,n} \delta_{2,n}$  and  $\delta_{1,n} \delta_{2,n}^{-1}$ , we see that  $\mu_n^{-2}$  converges to  $\mu_{\alpha^2}$  for some  $\alpha \in \bar{\mathbb{Q}}_p^*$  and  $\mu_{y_n^2}$  converges to  $\mu_{y^2}$  for some  $y \in \bar{\mathbb{Q}}_p^*$ . Therefore  $\delta_{1,n} = \mu_{y_n} \cdot \mu_n^{-1}$  converges to  $\delta_1 = \mu_{\pm\alpha y}$ . Similarly,  $\delta_{2,n}$  converges to  $\delta_2 = \mu_{\pm\alpha/y} \chi^{1-k}$  for some  $k$ . So the sequence  $V_n^*$  converges to  $V(\mu_{\pm\alpha y}, \mu_{\pm\alpha/y} \chi^{1-k}, \infty)$ , for some  $k$ , in the notation of [Colmez 2008]. Now  $k_n$  is a bounded sequence of integers converging to  $k$ . Therefore  $k_n = k$  for large  $n$ . In particular,  $k \geq 3$ . But  $V(\mu_{\pm\alpha y}, \mu_{\pm\alpha/y} \chi^{1-k}, \infty)$  is equal to the crystalline representation  $V_{k, y+p^{k-1}/y}^* \otimes \mu_{\pm\alpha}$ . Taking duals, we see that the sequence  $V_n$  converges to the crystalline representation  $V_{k, y+p^{k-1}/y} \otimes \mu_{\pm\alpha}^{-1}$ . Untwisting by the fixed power of the cyclotomic character, the original sequence  $V_n$  also converges to a crystalline representation with Hodge–Tate weights in  $[a, b]$ .

Now assume that  $\delta_1 \delta_2^{-1} = x^r \chi$  for some  $r \geq 0$ . This means that  $\delta_{1,n} \delta_{2,n}^{-1} = \mu_{y_n^2} \chi^{k_n-1}$  converges to  $x^r \chi$  in the neighborhood  $U_r$  of  $x^r \chi$  in  $\mathcal{T}$ . Let  $k = r + 2$ . Since  $k_n$  is a bounded sequence of integers, the convergence implies that  $k_n = k$  for all but finitely many  $n$ . In particular, we have  $k \geq 3$ . The point corresponding to the character  $\mu_{y_n^2} \chi^{k-1}$  in the blow-up  $\tilde{U}_r$  of  $U_r$  is

$$(y_n^2, (1+p)^{k-1} - 1, y_n^2 - p^r : 0) = (y_n^2, (1+p)^{k-1} - 1, 1 : 0),$$

at least if  $y_n^2 \neq p^r$  (if  $y_n^2 = p^r$ , then it is also the last point since this is the only crystalline point in the fiber above  $x^r \chi$ ). Since  $y_n^2$  converges to  $p^r$ , this sequence converges to the point  $(p^r, (1+p)^{k-1} - 1, 1 : 0)$

in  $\tilde{U}_r$ . By [Theorem 5.2](#), we see that the  $\mathcal{L}$ -invariant associated with this limit point is  $\infty$ . So the limit of the sequence  $V_n^*$  is  $V(\delta_1, \delta_2, \infty)$  in the notation of [\[Colmez 2008\]](#).

Now  $\delta_1 \delta_2^{-1} = x^r \chi = \mu_{p^{r/2}} \cdot (\mu_{1/p^{r/2}} \chi^{1-k})^{-1}$  implies that there is a character  $\delta$  such that  $\delta_1 = \delta \cdot \mu_{p^{r/2}}$  and  $\delta_2 = \delta \cdot \mu_{1/p^{r/2}} \chi^{1-k}$ . Using [\[loc. cit., Proposition 4.4\(i\)\]](#), we see that  $(\delta_1 \delta_2)(p)$  is a unit. Hence  $\delta(p)$  is a unit. Consequently,  $V(\delta_1, \delta_2, \infty) \simeq V(\mu_{p^{r/2}}, \mu_{1/p^{r/2}} \chi^{1-k}, \infty) \otimes \delta$ . Therefore the limit of the crystalline representations  $V_n^*$  is the crystalline representation  $V_{k,\infty}^* \otimes \delta$ . Taking duals, we see that the limit of the  $V_n$  is  $V_{k,\infty} \otimes \delta^{-1}$ . Note that the  $\delta_{1,n} = \mu_{y_n} \cdot \mu_n^{-1}$  are unramified characters converging to  $\delta_1 = \delta \cdot \mu_{p^{r/2}}$ . This forces  $\delta$  to be unramified and hence crystalline. Therefore  $V_{k,\infty} \otimes \delta^{-1}$  is crystalline. The last representation has Hodge–Tate weights  $(0, k - 1)$ . Untwisting by the fixed power of the cyclotomic character, we see that the Hodge–Tate weights of the limit representation of the original sequence again lie in  $[a, b]$ .

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