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Appendix written jointly with Dan Abramovich

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The integral Chow ring of weighted blow-ups

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We give a formula for the Chow rings of weighted blow-ups. Along the way, we also compute the Chow rings of weighted projective stack bundles, a formula for the Gysin homomorphism of a weighted blow-up, and a generalization of the splitting principle. In addition, in the Appendix we compute the Chern class of a weighted blow-up.

1. Introduction

A short introduction to weighted blow-ups. The blow-up is an important operation that is ubiquitous in algebraic geometry. When working with algebraic stacks, there is a natural generalization of the blow-up, called a weighted blow-up. Weighted blow-ups appear naturally in the study of moduli spaces, for example in [Inchiostro 2022, Theorem 2.6] $\overline{\mathcal{M}}_{1,2}$ is obtained as the weighted blow-up of the weighted projective plane $\mathcal{P}(2, 3, 4)$ at a point. This is a particular case of [Arena et al. 2023, Theorem 7.3], where $\overline{\mathcal{M}}_{1,n}$ is obtained as the blow-up of the moduli space of pseudostable curves at the cuspidal locus.

See [Quek and Rydh 2021] for a thorough introduction to weighted blow-ups, or [Arena et al. 2023] for a more condensed version. Intuitively, a weighted blow-up is like an ordinary blow-up, but with positive integer weights on the normal directions at each point of the center. Weighted blow-ups preserve many of the properties of (ordinary) blow-ups such as transforming the center into a divisor or being an isomorphism outside of the center.

For example, the blow-up of $\mathbb{A}^d$ at the origin $\{0\}$ with weights $a_1, \dots, a_d$ replaces the origin with the weighted projective stack $\mathcal{P}(a_1, \dots, a_d)$, which is our exceptional divisor. Moreover, it is isomorphic to $\mathbb{A}^d \smallsetminus \{0\}$ outside $\mathcal{P}(a_1, \dots, a_d)$. Formally, the weights are indicated by using a Rees algebra, as illustrated in the following example.

Example. Suppose we wish to blow up the origin $X = \{0\}$ in $Y = \mathbb{A}^2$ with weights 1 in the x-direction and 2 in the y-direction. This is given by

$$\mathrm{Bl}_X Y := \mathrm{Proj}_Y\Big(\bigoplus I_n\Big) = \Big[\Big(\mathrm{Spec}_Y\Big(\bigoplus I_n\Big) \smallsetminus V(I_+)\Big)/\mathbb{G}_m\Big] \to Y,$$

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where

$$I_0 = k[x, y] \supset I_1 = (x, y) \supset I_2 = (x^2, y) \supset I_3 = (x^3, xy, y^2) \supset \cdots .$$

The weights of x, y are the same as the maximum degree in the graded algebra $\bigoplus I_n$ in which they appear as linear terms, and I_n consists of polynomials in x, y with weight at least n.

We also assume that all weighted blow-ups are regular in the sense of [Quek and Rydh 2021, Definition 5.2.7] or equivalently [Arena et al. 2023, Definition 2.13].

Content of the paper. Let $f : \widetilde{Y} \to Y$ be a regular weighted blow-up of $X \subset Y$ with positive weights $a_1, \dots, a_d$ and let $\widetilde{X}$ be the exceptional divisor. For most of the paper we will assume X, Y are smooth algebraic spaces over a field of characteristic 0. In Section 7 we will generalize to the case of $\mathscr{X}, \mathscr{Y}$ quotient stacks by a linear algebraic group.

Then we have the commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{j} & \widetilde{Y} \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ X & \xrightarrow{i} & Y \end{array}$$

which is not Cartesian, unlike the ordinary blow-up case (an example of this can be found in [Quek and Rydh 2021, Remark 3.2.10]).

In the case of a classical blow-up, a description of the Chow ring $A^*(\widetilde{Y})$ and of its $A^*(Y)$-module structure is given in [Fulton 1998, Exercise 8.3.9] or [Eisenbud and Harris 2016, Proposition 13.12]. The purpose of this paper is to give a similar description for the Chow ring of a weighted blow-up.

We will use the functoriality of Chow rings including pull-backs, pushforwards and the Gysin map $f^!$ with the key property of making the following diagram commute:

$$\begin{array}{ccc} A^*(\widetilde{X}) & \xrightarrow{j_*} & A^*(\widetilde{Y}) \\ {\scriptstyle f^!}\big\uparrow & & {\scriptstyle f^*}\big\uparrow \\ A^*(X) & \xrightarrow{i_*} & A^*(Y) \end{array}$$

The formula for the Chow ring will follow from the exact sequence in the theorem below, generalizing the key sequence in [Fulton 1998, Proposition 6.7(e)].

Theorem 6.1 (key sequence). *Let $X, Y, \widetilde{X}, \widetilde{Y}, f$ be as above. Then we have the exact sequence of Chow groups*

$$A^*(X) \xrightarrow{(f^!, -i_*)} A^*(\widetilde{X}) \oplus A^*(Y) \xrightarrow{j_* + f^*} A^*(\widetilde{Y}) \to 0.$$

Further, if we use rational coefficients, then this becomes a split short exact sequence with g_* left inverse to $(f^!, -i_*)$:

$$0 \to A^*(X, \mathbb{Q}) \xrightarrow{(f^!, -i_*)} A^*(\widetilde{X}, \mathbb{Q}) \oplus A^*(Y, \mathbb{Q}) \xrightarrow{j_* + f^*} A^*(\widetilde{Y}, \mathbb{Q}) \to 0.$$

Note that since our blow-up diagram is not Cartesian, the codomain of $f^!$ is $A^*(X \times_Y \widetilde{Y})$, but $\widetilde{X}$ is the reduction of $X \times_Y \widetilde{Y}$ so we can identify their Chow groups.

Moreover, when working with integral coefficients, the sequence is no longer exact on the left as shown in Example 6.2. Passing to rational coefficients however, allows us to maintain exactness on the left and to define a left inverse of $(f^!, -i_*)$ via g_*. In fact, it is enough to pass to $\mathbb{Z}[1/a_1, \dots, 1/a_d]$-coefficients.

From the sequence, we can get the following description of $A^*(\widetilde{Y})$.

Theorem 6.4 (Chow ring of a weighted blow-up). *If $\widetilde{Y} \to Y$ is a weighted blow-up of Y at a closed subvariety X, then the Chow ring $A^*(\widetilde{Y})$ is isomorphic **as a group** to the quotient*

$$A^*(\widetilde{Y}) \cong \frac{(A^*(X)[t]) \cdot t \oplus A^*(Y)}{\big(((P(t) - P(0))\alpha, -i_*(\alpha)), \forall \alpha \in A^*(X)\big)},$$

with $P(t) = c_{\mathrm{top}}^{\mathbb{G}_m}(\mathcal{N}_X Y)(t)$ (defined below) and $[\widetilde{X}] = -t$.

The multiplicative structure on $A^*(\widetilde{Y})$ is induced by the multiplicative structures on $A^*(X)$ and $A^*(Y)$ and by the pull-back map in the following way:

$$(0, \beta) \cdot (t, 0) = (i^*(\beta)t, 0).$$

Equivalently $A^*(\widetilde{Y})$ can be expressed as a quotient of the fiber product

$$\frac{A^*(Y) \times_{A^*(X)} A^*(X)[t]}{((i_*\alpha, P(t)\alpha), \forall \alpha \in A^*(X))},$$

with $i^* : A^*(Y) \to A^*(X)$ on the left and $A^*(X)[t] \to A^*(X)$ on the right given by evaluating t at 0.

In order to use the key sequence, we need to give a presentation for the Chow ring of the exceptional divisor $\widetilde{X}$.

In the classical case, $\widetilde{X}$ is a projective bundle over X and the Chow ring of a projective bundle can be described via the formula [Eisenbud and Harris 2016, Theorem 9.6]. In the case of a weighted blow-up, the exceptional divisor is a projective stack bundle, i.e., the projectivization of a weighted affine bundle (Definitions 3.2, 3.4). In Section 3 we define the top $\mathbb{G}_m$-equivariant Chern class for a weighted affine bundle E in terms of its homogeneous pieces as

$$c_{\mathrm{top}}^{\mathbb{G}_m}(E) = \prod c_{\mathrm{top}}^{\mathbb{G}_m}(E_i) = \prod_i (c_{n_i}(E_i) + a_i t c_{n_i - 1}(E_i) + \cdots + a_i^{n_i} t^{n_i}),$$

and we give a formula for the integral Chow ring of a projective stack bundle (which was proven for rational coefficients in [Mustaţă and Mustaţă 2012, Lemma 2.10(b)]).

Theorem 3.12 (weighted projective bundle formula). *Let E be a weighted, affine bundle over X of rank n. Let $c_{\mathrm{top}}^{\mathbb{G}_m}(E)(t)$ be its $\mathbb{G}_m$-equivariant top Chern class. Then*

$$A^*(\mathcal{P}(E)) \cong \frac{A^*(X)[t]}{c_{\mathrm{top}}^{\mathbb{G}_m}(E)(t)}.$$

Finally, to have a complete description of the exact sequence in Theorem 6.1, we need the appropriate generalization for the excess intersection formula [Fulton 1998, Theorem 6.3]. Unlike the case of an ordinary blow-up, in the weighted blow-up case we don't have an excess bundle and we describe $f^!$ as the multiplication by a difference quotient of the top $\mathbb{G}_m$-equivariant Chern class of the normal bundle.

Theorem 5.5 (weighted key formula). *Let X, Y, $\widetilde{X}$, $\widetilde{Y}$, f be as above. Let us identify $A^*(\widetilde{X}) \cong A^*(X)[t]/P(t)$ with $P(t) = c_{\mathrm{top}}^{\mathbb{G}_m}(\mathcal{N}_X Y)(t)$. Then we have the following formula for the Gysin homomorphism $f^! : A^*(X) \to A^*(\widetilde{X})$:*

$$f^!(\alpha) = \frac{P(t) - P(0)}{t}\alpha.$$

The proof of our formula for the Gysin homomorphism relies on a generalization of the splitting principle, Theorem 4.9, stated in terms of maps to classifying stacks BT, $B\,\mathrm{GL}_{\boldsymbol{n}}$, $BG_{\boldsymbol{a},\boldsymbol{n}}$.

Theorem 4.9 (the splitting principle). *Let $E \to X$ be a weighted affine bundle defined by a map $X \to BG_{\boldsymbol{a},\boldsymbol{n}}$. Let T be the standard maximal torus in $\mathrm{GL}_{\boldsymbol{n}}$. Then the map $X'' \to X$ in the fiber diagram*

$$\begin{array}{ccc} X'' & \longrightarrow & BT \\ \downarrow & & \downarrow \\ X' & \longrightarrow & B\,\mathrm{GL}_{\boldsymbol{n}} \\ \downarrow & & \downarrow \\ X & \longrightarrow & BG_{\boldsymbol{a},\boldsymbol{n}} \end{array}$$

induces an injection of Chow rings $A^*(X) \hookrightarrow A^*(X'')$ via pull-back.

Here $G_{\boldsymbol{a},\boldsymbol{n}}$ is the structure group of the weighted affine bundle E and $\mathrm{GL}_{\boldsymbol{n}}$ is the structure group of its associated weighted vector bundle. Note that the upper square of the diagram is equivalent to the classical splitting principle in [Fulton 1998, page 51] as in [Totaro 2014, Theorem 2.13].

2. Equivariant intersection theory

From now on Y, X will be smooth quasiseparated algebraic spaces, of finite type over a field k of characteristic 0, with $\mathbb{G}_m$ actions. We will also assume the $\mathbb{G}_m$ action is trivial on X.

Let us first recall the following definitions of equivariant Chow groups for a linear algebraic group G.

Definition 2.1. [Edidin and Graham 1998, Definition-Proposition 1] Let Y be a d-dimensional quasi-separated algebraic space of finite type over a field k, together with a G action. Let g be the dimension of G. The i-th G-equivariant Chow group of Y is defined as

$$A_i^G(Y) := A_{i+l-g}(Y \times U/G),$$

where U is an open subspace of an l-dimensional representation, on which G acts freely and whose complement has codimension greater than $d - i$.

In this article we will mostly use this definition in the particular case of a $\mathbb{G}_m$ action. In particular, this leaves us with very convenient choices for representations: V will be an l-dimensional vector space with the standard $\mathbb{G}_m$ action with weight 1 and $U = V \setminus \{0\}$.

Example 2.2. We have $A^*_{\mathbb{G}_m}(X) \cong A^*(X)[t]$. Indeed, since the $\mathbb{G}_m$ action is trivial on X

$$A^i_{\mathbb{G}_m}(X) = A^{\mathbb{G}_m}_{d-i}(X) = A_{d-i+l-1}(X \times U/\mathbb{G}_m) = A^i(X \times \mathbb{P}^{l-1}) = \bigoplus_{k=0}^{i} A^k(X)t^{i-k},$$

with $t = c_1(\mathcal{O}_{\mathbb{P}^{l-1}}(1))$ and the isomorphism follows.

Definition 2.3 [Edidin and Graham 1998, Definition 1]. Let E be a G equivariant vector bundle over Y. The equivariant Chern classes of E are the operators

$$c^G_j : A^G_i(Y) \to A^G_{i-j}(Y), \quad \text{with } c^G_j(E) \cap \alpha = c_j(E \times U/G) \cap \alpha \in A_{i+l-j-g}(Y \times U/G) = A^G_{i-j}(Y).$$

As mentioned in [Molina Rojas and Vistoli 2006, Section 2] many of the standard properties of Chow groups still hold in the equivariant case. Below we collect some that will be used later.

Proposition 2.4. *The following are true*:

(1) *The first Chern class of the tensor product of line bundles is the sum of the first Chern classes of each line bundle*: $c^G_1(L \otimes L') = c^G_1(L) + c^G_1(L')$.

(2) *Let E be a G-equivariant vector bundle over Y and $f : Y' \to Y$ a map such that f^*E has a filtration of G-equivariant vector bundles $f^*E = F_r \supset \cdots \supset F_0 = 0$. Let $E_i = F_i/F_{i-1}$. Then*

$$c^G(f^*E) = \prod_i c^G(E_i).$$

(3) *If Z is a closed G-invariant subscheme of Y, we have the exact sequence*

$$A^*_G(Z) \to A^*_G(Y) \to A^*_G(Y \setminus Z) \to 0.$$

(4) *Let $\pi : E \to Y$ be a G-equivariant vector bundle over Y and $s_0 : Y \to E$ be the zero section. Then the Gysin pull-back map $s_0^* : A^*_G(E) \to A^*_G(Y)$ is an isomorphism equal to the inverse of π^*.*

Proof. We will prove (3). The proofs of the remaining parts are analogous. Let U have dimension l high enough that $A^G_i(Y)$ is defined as $A_{i+l-g}(Y \times U/G)$. Then, also by definition, we have $A^G_i(Z) := A_{i+l-1}(Z \times U/G)$. In particular, $Z \times U/G$ and $Y \times U/G$ are algebraic spaces, and the localization sequence

$$A^*(Z \times U/G) \to A^*(Y \times U/G) \to A^*(Y \times U/G \setminus Z \times U/G) \to 0$$

is exact. Therefore statement (3) holds as well. □

Another proposition, that will be very useful later, is [Molina Rojas and Vistoli 2006, Lemma 2.2], for which we will quote the statement and the proof.

Lemma 2.5 [Molina Rojas and Vistoli 2006, Lemma 2.2]. *Let G be an affine linear group acting on a smooth scheme Y. Let $\pi : E \to Y$ be a G-equivariant vector bundle of rank n. Call $E_0 \subset E$ the complement of the zero section $s : Y \to E$. Then the pull-back homomorphism $\pi|^*_{E_0} : A^*_G(Y) \to A^*_G(E_0)$ is surjective, and its kernel is generated by the top Chern class $c^G_n(E) \in A^n_G(Y)$.*

Proof. Consider the diagram

$$\begin{array}{ccccccc} & & A^*_G(Y) & & & & \\ & \nearrow & \uparrow{\scriptstyle s^*} & \searrow{\scriptstyle \pi|^*_{E_0}} & & & \\ A^*_G(Y) & \xrightarrow{s_*} & A^*_G(E) & \longrightarrow & A^*_G(E_0) & \longrightarrow & 0 \end{array}$$

where the bottom is the localization sequence. Since s^* is an isomorphism, inverse to π^*, we see that $\pi|^*_{E_0}$ is surjective with kernel generated by the image of s^*s_*. By the self-intersection formula, s^*s_* is multiplication by $c^G_n(E)$. □

3. Chow groups of weighted projective stack bundles

In this section we will give a formula for the Chow ring of a weighted projective stack bundle. Weighted projective stack bundles appear as the exceptional divisor of a weighted blow-up. We start by computing the Chern classes of weighted affine bundles, and then show we can apply Lemma 2.5 to them. A similar formula for rational coefficients appears in [Mustaţă and Mustaţă 2012, Theorem 2.10(b)].

Definition 3.1. An affine bundle is a smooth affine morphism $E \to X$ such that E is, locally in the smooth topology, isomorphic to $X \times \mathbb{A}^n$.

Definition 3.2 [Quek and Rydh 2021, Definition 2.1.3]. A weighted affine bundle is a $\mathbb{G}_m$-equivariant affine bundle $E \to X$, where locally in the smooth topology $\mathbb{G}_m$ acts linearly on $\mathbb{A}^n$ with positive weights $a_1, \ldots, a_n \in \mathbb{Z}$.

We note in Remark 4.2 that the structure group is special, which means weighted affine bundles over a scheme are Zariski-locally trivial. This is used to apply [Stacks 2005–, Tag 0GUB] in the proof of Corollary 3.11.

It will sometimes be convenient to emphasize the *distinct* weights of a weighted affine bundle. When we do this we will list the distinct weights as $a_1, \ldots, a_r$, and use n_i to refer to the dimension of the subspace of $\mathbb{A}^n$ where the action has weight a_i. In these cases, we will highlight the fact that the weights are distinct in the relevant statements.

Definition 3.3. A weighted vector bundle is a weighted affine bundle whose underlying $\mathbb{G}_m$ space is a vector bundle. Equivalently, it is a weighted affine bundle with linear transition functions.

Notice that our terminology is slightly different from that of [Quek and Rydh 2021]. What they call twisted/untwisted weighted vector bundles, we call weighted affine/vector bundles respectively. Also note that the $\mathbb{G}_m$ action on a weighted vector bundle must preserve the degree. In particular, the bundle splits into homogeneous vector bundles, where $\mathbb{G}_m$ acts with the same degree.

Definition 3.4 [Quek and Rydh 2021, Definition 2.1.5]. A weighted projective stack bundle over X is the stack-theoretic Proj of a graded algebra corresponding to a weighted affine bundle with strictly positive weights. Precisely, if R is a graded algebra such that $E = \mathrm{Spec}_X(R)$ then $\mathrm{Proj}_X(R) = [\mathrm{Spec}_X(R) \setminus V(R_+)/\mathbb{G}_m]$.

3.1. ***Equivariant Chern classes of a weighted line bundle.*** Let us denote by $L \to X$ a line bundle over X with the trivial action. Let us denote by $L^{(a)}$ the same underlying line bundle, endowed with the weight-a $\mathbb{G}_m$ action. This is a notation we will adopt only for Sections 3.1 and 3.2, but abandon later as the weight of the $\mathbb{G}_m$ action will be clear from context.

Some of the following lemmas are likely already known, but are stated and proven for completeness as we couldn't find specific references.

Lemma 3.5. *Let $L^{(a)}$ be a $\mathbb{G}_m$-equivariant line bundle over X with weight a. Then the first equivariant Chern class of $L^{(a)}$ is $c_1^{\mathbb{G}_m}(L^{(a)}) = c_1(L) + at$ via the identification in Example 2.2.*

Proof. We know that $L^{(a)} = L^{(a)} \otimes \mathcal{O}_X = L \otimes \mathcal{O}_X{}^{(a)}$. In particular,

$$c_1^{\mathbb{G}_m}(L^{(a)}) = c_1^{\mathbb{G}_m}(L \otimes \mathcal{O}_X{}^{(a)}) = c_1^{\mathbb{G}_m}(L) + ac_1^{\mathbb{G}_m}(\mathcal{O}_X{}^{(1)}).$$

Now, since the action on L is trivial, $(L \times U)/\mathbb{G}_m = L \times \mathbb{P}^{l-1}$, and since $A^1(X \times \mathbb{P}^{l-1}) = A^1(X) \oplus A^0(X)t$, we get

$$c_1^{\mathbb{G}_m}(L) = c_1(L \times \mathbb{P}^{l-1}) = c_1(L) \in A^1(X).$$

We only need to prove $c_1(\mathcal{O}_X{}^{(1)}) = t$.

Let us consider the projection to a point P, $f : X \to P$. The map defines a graded ring homomorphism $f^* : A^*_{\mathbb{G}_m}(P) \to A^*_{\mathbb{G}_m}(X)$, i.e., a map $f^* : \mathbb{Z}[t] \to A^*(X)[t]$ defined by $1 \mapsto 1$ and $t \mapsto t$.

Now $\mathcal{O}_X{}^{(1)} = f^*(\mathcal{O}_P{}^{(1)})$ and

$$c_1^{\mathbb{G}_m}(\mathcal{O}_X{}^{(1)}) = c_1^{\mathbb{G}_m}(f^*(\mathcal{O}_P{}^{(1)})) = f^* c_1^{\mathbb{G}_m}(\mathcal{O}_P{}^{(1)}).$$

Therefore it is enough to prove $c_1^{\mathbb{G}_m}(\mathcal{O}_P{}^{(1)}) = t$.

By definition $c_1^{\mathbb{G}_m}(\mathcal{O}_P{}^{(1)}) = c_1(\mathcal{O}_P \times U/\mathbb{G}_m)$ as a bundle over $U/\mathbb{G}_m$, with $U/\mathbb{G}_m = \mathbb{A}^2 \smallsetminus (0,0)/\mathbb{G}_m = \mathbb{P}^1$.

Now, a nonzero section $s : U \to U \times \mathcal{O}_P/\mathbb{G}_m$ is given by $(x_0, x_1) \mapsto (x_0, x_1, x_0)$ and intersects the zero section $(x_0, x_1, 0)$ in $x_0 = 0$, which gives us $\mathcal{O}_{\mathbb{P}^1}(1)$, whose first Chern class is t in $A^1(U \times \mathcal{O}_P/\mathbb{G}_m) = A^1(\mathbb{P}^1)$, as desired. □

3.2. ***Equivariant Chern classes of homogeneous bundles.***

Proposition 3.6 (homogeneous bundles admit a splitting with line bundles). *Let $E^{(a)}$ be a rank-n vector bundle over X with $\mathbb{G}_m$ acting homogeneously with weight a on it. Then there exists $f : X' \to X$ such that $f^*E^{(a)}$ has a filtration*

$$f^*E^{(a)} \supset F_n^{(a)} \supset \cdots \supset F_0^{(a)} = 0$$

with $\mathbb{G}_m$-equivariant line bundle quotients $L_j^{(a)} = F_{j+1}^{(a)}/F_j^{(a)}$ and f^* is injective.

Proof. Let us consider the underlying bundle E. Then by the splitting construction [Fulton 1998, page 51] there is a map $f : X' \to X$ with a filtration $f^*E = F_n \supset \cdots \supset F_0 = 0$ with line bundle quotients.

These bundles naturally have the structure of $\mathbb{G}_m$-equivariant vector bundles with weight 1. By replacing the weight-1 action with a weight-a action we get the desired sequence. □

Corollary 3.7. *Let $E^{(a)}$ be a homogeneous $\mathbb{G}_m$-equivariant vector bundle of rank n with weight a, and E the underlying vector bundle endowed with the trivial $\mathbb{G}_m$ action. Then the top equivariant Chern class in $A^*_{\mathbb{G}_m}(X) = A^*(X)[t]$ is*

$$c_n^{\mathbb{G}_m}(E^{(a)}) = c_n(E) + atc_{n-1}(E) + \cdots + a^n t^n.$$

Proof. Let $f : X' \to X$ as in Proposition 3.6. Then $c_n^{\mathbb{G}_m}(f^*E^{(a)}) = \prod_{i=1}^n c_1^{\mathbb{G}_m}(L_i^{(a)})$. By Lemma 3.5 $c_1^{\mathbb{G}_m}(L_i^{(a)}) = c_1(L_i) + at$.

Therefore $c_n^{\mathbb{G}_m}(f^*E^{(a)}) = c_n(f^*E) + atc_{n-1}(f^*E) + \cdots + a^n t^n$. By the injectivity of f^* we are done. □

3.3. *Chern classes of weighted affine bundles.*

Proposition 3.8. *Let E be a weighted affine bundle over X. Let $0 < a_1 < \cdots < a_r$ be the* ***distinct*** *weights of the $\mathbb{G}_m$ action. Then there exist unique subbundles F_i such that*

$$E \supset F_r \supset \cdots \supset F_1 \supset 0,$$

with well-defined quotients $E_i = F_i/F_{i-1}$ which are homogeneous vector bundles with weights a_i.

Proof. Let $E = \operatorname{Spec}_X(R)$ and $\{U_i\}$ be a cover for X such that $E|_{U_i}$ is the trivial bundle. Then we have graded isomorphisms

$$\alpha_i : R|_{U_i} \to \mathcal{O}_{U_i}[x_{i,1}^{(a_1)}, \ldots, x_{i,n_1}^{(a_1)}, x_{i,1}^{(a_2)}, \ldots, x_{i,n_2}^{(a_2)}, \ldots, x_{i,1}^{(a_r)}, \ldots, x_{i,n_r}^{(a_r)}],$$

with $x_{i,1}^{(a_h)}, \ldots, x_{i,n_h}^{(a_h)}$ having weight a_h. A general transition map

$$\alpha_{ij} = \alpha_j|_{U_{ij}} \circ \alpha_i|_{U_{ij}}^{-1} : \mathcal{O}_{U_ij}[x_{i,1}^{(a_1)}, \ldots, x_{i,n_r}^{(a_{n_r})}] \to \mathcal{O}_{U_{ij}}[x_{j,1}^{(a_1)}, \ldots, x_{j,r}^{(a_{n_r})}]$$

will map $x_{i,l}^{(a_h)}$ to a homogeneous polynomial of degree a_h.

Now let $F_k|_{U_i} := \alpha_i^{-1}(\mathcal{O}_{U_i}[x_{i,1}^{(a_1)}, \ldots, x_{i,n_k}^{(a_k)}])$ be the locus where $\mathbb{G}_m$ acts with weights smaller than or equal to a_k. This defines uniquely r subbundles of E. Moreover the quotients F_k/F_{k-1} are well-defined. Indeed, while these are affine bundles, they are locally isomorphic to vector bundles so we can at least take quotients locally. By construction, taking quotients locally gives us bundles consisting only of the weight-a_k pieces of E, and since the lower-degree pieces have been reduced to 0, we are left with linear transition functions making the F_k/F_{k-1} homogeneous vector bundles of weight a_k, as needed. □

Proposition 3.9. *Let E be a weighted affine bundle over X as in Proposition 3.8. Let $N_X E$ be the (nonweighted) normal bundle of X in E. Then, with $X \hookrightarrow E$ the zero section, we have*

$$N_X E \cong E_1 \oplus \cdots \oplus E_r.$$

Proof. Let I be the ideal sheaf of X in E and let α_{ij} be its transition functions as in Proposition 3.8, with each $x_{i,l}^{(a_h)}$ mapped to a homogeneous polynomial of degree a_h.

When computing the transition functions $\bar{\alpha}_{i,j}$ for $N_X E$, we are taking the quotient by I^2; i.e., $\bar{\alpha}_{i,j}$ will preserve only the linear terms of said polynomials and delete the monomials coming from local coordinates where $\mathbb{G}_m$ acts with lower degree.

In particular, local coordinates on which $\mathbb{G}_m$ acts with a certain degree must be mapped to coordinates in the same degree, and the normal bundle splits into $E'_1 \oplus E'_2 \oplus \cdots \oplus E'_r$, where $\mathbb{G}_m$ acts on the coordinates of E'_i with weight a_i.

But the way we obtained the transition functions for E'_i is equivalent to considering the locus F_i in Proposition 3.8 and quotient by F_{i-1}, so we obtained the desired decomposition. □

Definition 3.10. For an affine bundle E, with E_i as in Propositions 3.8 and 3.9, we define the $\mathbb{G}_m$-equivariant total Chern class of E as $c^{\mathbb{G}_m}(E) = c^{\mathbb{G}_m}(N_X E) = \prod c^{\mathbb{G}_m}(E_i)$.

Corollary 3.11. *Lemma 2.5 also holds in the case where E is a weighted affine bundle.*

Proof. Note that by Definition 3.10 and Proposition 3.9, we have $c_i^G(E) = c_i^G(N_X E)$. The rest of the proof is the same as in Lemma 2.5, noting that $\pi^* : A^*(X) \to A^*(E)$ is still an isomorphism by [Stacks 2005–, Tag 0GUB], and $s^* : A^*(E) \to A^*(X)$ is still its inverse. □

3.4. *The Chow ring of a weighted projective stack bundle.*

Theorem 3.12 (weighted projective bundle formula). *Let E be a weighted, affine bundle over X of rank n. Let $c_{\text{top}}^{\mathbb{G}_m}(E)(t)$ be its $\mathbb{G}_m$-equivariant top Chern class. Then*

$$A^*(\mathcal{P}(E)) \cong \frac{A^*(X)[t]}{c_{\text{top}}^{\mathbb{G}_m}(E)(t)}.$$

Proof. Note that, by definition, $A^*(\mathcal{P}(E)) = A^*([(E \smallsetminus X)/\mathbb{G}_m]) = A^*_{\mathbb{G}_m}(E \smallsetminus X)$, with $E \smallsetminus X$ being E minus the zero section. By Lemma 2.5 we only need to prove that the image of $c_n^{\mathbb{G}_m}(E)$ via the identification in Example 2.2 is $\prod_i (c_{n_i}(E_i) + a_i t c_{n_i-1}(E_i) + \cdots + a_i^{n_i} t^{n_i})$.

By Corollary 3.7, it follows $c_n^{\mathbb{G}_m}(E) = \prod c_{n_i}^{\mathbb{G}_m}(E_i) = \prod_i (c_{n_i}(E_i) + a_i t c_{n_i-1}(E_i) + \cdots + a_i^{n_i} t^{n_i})$ and we are done. □

Below there are some (familiar) special cases.

Example 3.13 (the Chow ring of $\mathcal{P}(a_1, \dots, a_n)$ [Inchiostro 2022, Lemma 4.7]). Let $\mathcal{P}(a_1, \dots, a_n)$ be the weighted projective stack with weights $a_1, \dots, a_n$. We can consider $\mathcal{P}(a_1, \dots, a_n)$ as a weighted projective bundle over a point. In particular, we have that $\mathcal{P}(a_1, \dots, a_n)$ splits into n trivial bundles with weights $a_1, \dots, a_n$. Each of these line bundles will have the first Chern class equal to zero. It follows that

$$A^*(\mathcal{P}(a_1, \dots, a_n)) = \frac{\mathbb{Z}[t]}{a_1 \cdots a_n t^n}.$$

Example 3.14 (the Chow ring of a classical projective bundle [Eisenbud and Harris 2016, Theorem 9.6]). Let E be a vector bundle of rank n over X in the classical sense. In this case, we have $\mathbb{G}_m$ acting homogeneously on the whole space with weight 1. In particular

$$A^*(\mathcal{P}(E)) = \frac{A^*(X)[t]}{c_n(E) + c_{n-1}(E)t + \cdots + t^n}.$$

Example 3.15. As a toric example, this can be recovered as a consequence of [Iwanari 2009, Theorem 2.2]. The exceptional divisor $\widetilde{X}$ of the weighted blow-up of $X = V(x_1, \dots, x_n)$ in $\mathbb{P}^{m+n}$, with (possibly equal) weights $a_1, \dots, a_n$.

In this case, the Chow ring of the base will be $A^*(X) = A^*(\mathbb{P}^m) = \mathbb{Z}[x]/(x^{m+1})$.

The normal cone of X in $\mathbb{P}^{n+m}$, $N_X\mathbb{P}^{n+m}$, will split into the sum of n copies of $\mathcal{O}_{\mathbb{P}^m}(1)$, on each one of which $\mathbb{G}_m$ will act with weight a_i. We will denote the normal cone together with the $\mathbb{G}_m$ action by $N_{a_1,\dots,a_n}\mathbb{P}^{n+m}$.

Now $c_1(\mathcal{O}_{\mathbb{P}^m}(1)) = x \in \mathbb{Z}[x]/x^{m+1}$. Therefore $c_n^{\mathbb{G}_m}(N_X\mathbb{P}^{n+m}) = \prod(x + a_i t)$ and

$$A^*(\widetilde{X}) = A^*(\mathcal{P}(N_{a_1,\dots,a_n}\mathbb{P}^{n+m})) = \frac{\mathbb{Z}[x,t]}{\left(x^{m+1}, \prod_{i=1}^n (x + a_i t)\right)}.$$

4. The splitting principle

The goal of this section is to prove Theorem 4.9, an analog of the splitting principle. We start by proving some facts about structure groups and classifying spaces. Using those results we construct, for any weighted affine bundle $E \to X$, a map $X' \to X$ that allows us to pull back our affine bundle to a vector bundle $E' \to X'$ with $A^*(X') \cong A^*(X)$.

4.1. *Structure groups.* From here on, let $E \to X$ be an affine bundle with fibers isomorphic to an affine space V, $\boldsymbol{n} = (n_1, \dots, n_r)$ be the dimensions of its homogeneous parts and $\boldsymbol{a} = (a_1, \dots, a_r)$ be their *distinct* weights. Let $G_{\boldsymbol{a},\boldsymbol{n}}$ be the group $\mathrm{Aut}_{\mathbb{G}_m}(V)$ of $\mathbb{G}_m$-equivariant automorphisms of V and $\mathrm{GL}_{\boldsymbol{n}} = \prod \mathrm{GL}(n_i)$. Moreover, define $VG_{\boldsymbol{a},\boldsymbol{n}} = [V/G_{\boldsymbol{a},\boldsymbol{n}}]$ and $V\mathrm{GL}_{\boldsymbol{n}} = [V/\mathrm{GL}_{\boldsymbol{n}}]$.

Lemma 4.1. *There is a surjective group homomorphism $G_{\boldsymbol{a},\boldsymbol{n}} \to \mathrm{GL}_{\boldsymbol{n}}$ and a section $\mathrm{GL}_{\boldsymbol{n}} \to G_{\boldsymbol{a},\boldsymbol{n}}$. The kernel of the surjection is a unipotent group $U_{\boldsymbol{a},\boldsymbol{n}}$.*

Proof. In [Quek and Rydh 2021, Section 2.1.7], the authors offer an explicit description of $G_{\boldsymbol{a},\boldsymbol{n}}$. In fact, they decompose $G_{\boldsymbol{a},\boldsymbol{n}}$ in the recursive semidirect product

$$G_{\boldsymbol{a},\boldsymbol{n}} = (\mathrm{GL}_{n_r} \times G_{\boldsymbol{a}',\boldsymbol{n}'}) \ltimes \mathbb{G}_a^{n_r N_r},$$

with $\boldsymbol{a}' = (a_1, \dots, a_{r-1})$, $\boldsymbol{n}' = (n_1, \dots, n_{r-1})$, and N_r the dimension of a_r-th-degree piece of a graded polynomial algebra with free variables $\{x_{i,j} : 1 \le i \le r-1,\ 1 \le j \le n_i\}$, where $x_{i,j}$ is given weight a_i.

Unraveling the recursion gives

$$\begin{aligned} G_{\boldsymbol{a},\boldsymbol{n}} &= \left(\mathrm{GL}_{n_r} \times \cdots \times ((\mathrm{GL}_{n_1} \times \{1\}) \ltimes \mathbb{G}_a^{n_1 N_1}) \ltimes \cdots\right) \ltimes \mathbb{G}_a^{n_r N_r} \\ &= (\cdots (\mathrm{GL}_{\boldsymbol{n}} \ltimes \mathbb{G}_a^{n_1 N_1}) \ltimes \cdots) \ltimes \mathbb{G}_a^{n_r N_r}. \end{aligned}$$

This expression provides us with the desired surjection and section. The kernel is a successive extension of additive groups and so is unipotent. □

Remark 4.2. Let us recall that a linear algebraic group G is special (in the sense of Serre) when every principal G-bundle is Zariski-locally trivial. The description of $G_{\boldsymbol{a},\boldsymbol{n}}$ above implies that the group $G_{\boldsymbol{a},\boldsymbol{n}}$ is special (as noted in [Quek and Rydh 2021, Remark 2.1.8]).

This was useful in Section 3, while working with weighted affine bundles.

4.2. *Lemmas on classifying spaces.*

Lemma 4.3. *Let G be a linear algebraic group which is a semidirect product of groups $G = L \ltimes U$. Then we get the following Cartesian diagram*:

$$\begin{array}{ccc} \{*\} & \longrightarrow & BL \\ \downarrow & & \downarrow \\ BU & \longrightarrow & BG \end{array}$$

Proof. Consider the following diagram:

$$\begin{array}{ccc} BU \times_{BG} BL & \longrightarrow & BL \\ \downarrow & & \downarrow \\ BU & \longrightarrow & BG \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & BL \end{array}$$

The bottom square is a Cartesian square, coming from the short exact sequence $1 \to U \to G \to L \to 1$. Indeed an object over a scheme S of the fiber product $* \times_{BL} BG$ is a principal G-bundle $P_G \to S$ with a trivialization of the associated L-bundle $(P_G \times L)/G \cong P_G/U \cong S \times L$. But then the preimage of $S \times \{\mathrm{id}\}$ in P_G is a principal U-bundle, that is, an object of BU. Conversely, if we have a principal U-bundle $P_U \to S$ then we have an associated G-bundle $P_G = (P_U \times G)/U$ whose associated L-bundle $P_L \cong (P_G \times L)/G \cong P_G/U$ has a canonical trivialization: $P_G/U \cong ((P_U \times G)/U)/U$, where the second U acts on the right, giving us $P_L \cong (P_U \times L)/U \cong S \times L$. One can check that these correspondences are inverse to each other.

So we have that large square is Cartesian and $\{*\} \times_{BL} BL = BU \times_{BG} BL$. Further, the composite map on the right is the identity on BL, so that $\{*\} \times_{BL} BL = \{*\}$. □

Corollary 4.4. *$B\,\mathrm{GL}_{\boldsymbol{n}} \to BG_{\boldsymbol{a},\boldsymbol{n}}$ is a $U_{\boldsymbol{a},\boldsymbol{n}}$ bundle*; *specifically we have the following Cartesian diagram*:

$$\begin{array}{ccc} U_{\boldsymbol{a},\boldsymbol{n}} & \longrightarrow & B\,\mathrm{GL}_{\boldsymbol{n}} \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & BG_{\boldsymbol{a},\boldsymbol{n}} \end{array}$$

Consequently, *given a morphism $X \to BG_{\boldsymbol{a},\boldsymbol{n}}$ the fiber product $X \times_{BG_{\boldsymbol{a},\boldsymbol{n}}} B\,\mathrm{GL}_{\boldsymbol{n}} \to X$ is a $U_{\boldsymbol{a},\boldsymbol{n}}$ bundle.*

Proof. Applying Lemma 4.3 to $G_{\boldsymbol{a},\boldsymbol{n}}$, $\mathrm{GL}_{\boldsymbol{n}}$, $U_{\boldsymbol{a},\boldsymbol{n}}$ as in Lemma 4.1 and appending on the left the Cartesian diagram coming from the standard presentation of $BU_{\boldsymbol{a},\boldsymbol{n}}$

$$\begin{array}{ccc} U_{\boldsymbol{a},\boldsymbol{n}} & \longrightarrow & \{*\} \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & BU_{\boldsymbol{a},\boldsymbol{n}} \end{array}$$

we get the Cartesian diagram

$$\begin{array}{ccccc} U_{\boldsymbol{a},\boldsymbol{n}} & \longrightarrow & \{*\} & \longrightarrow & B\,\mathrm{GL}_{\boldsymbol{n}} \\ \downarrow & & \downarrow & & \downarrow \\ \{*\} & \longrightarrow & BU_{\boldsymbol{a},\boldsymbol{n}} & \longrightarrow & BG_{\boldsymbol{a},\boldsymbol{n}} \end{array}$$

as desired.

Then $X \times_{BG_{\boldsymbol{a},\boldsymbol{n}}} B\,\mathrm{GL}_{\boldsymbol{n}} \to X$ is the pull-back of a $U_{\boldsymbol{a},\boldsymbol{n}}$ bundle and hence is a $U_{\boldsymbol{a},\boldsymbol{n}}$ bundle. □

Lemma 4.5. *Let L be a subgroup of a group scheme G acting on a scheme V. Then the following diagram is Cartesian*:

$$\begin{array}{ccc} [V/L] & \longrightarrow & BL \\ \downarrow & & \downarrow \\ [V/G] & \longrightarrow & BG \end{array}$$

Proof. An object over a scheme S in $[V/G]\times_{BG} BL$ is a triple (P, Q, α), where $P \to S$ is a G-torsor, together with a G-equivariant map to V

$$\begin{array}{ccc} P & \xrightarrow{\phi} & V \\ \downarrow & & \\ S & & \end{array}$$

$Q \to S$ is an L-torsor, and α is an isomorphism of G-torsors $P \xrightarrow{\alpha} Q \times G/L$.

Given such an object we can construct an object in $[V/L]$ by considering the L-torsor $Q \to S$ together with the map $\psi : Q \to V$ defined as follows:

$$\begin{array}{ccccccc} Q & \longrightarrow & Q\times G/L & \xrightarrow{\alpha^{-1}} & P & \xrightarrow{\phi} & V \\ \downarrow & & & & & & \\ S & & & & & & \end{array}$$

To verify this is indeed an object of $[V/L]$, we need to prove that ψ is L-equivariant. Now, ϕ and α^{-1} are G-equivariant, and in particular L-equivariant. Moreover the quotient map $Q \to Q \times G/L$ maps an element ql to $[ql, e] = [qll^{-1}, le] = [q, l]$. But L acts on $Q \times G/L$ through its inclusion into G; hence $ql \mapsto [q, e]\,l$ as desired.

On the other hand given an L-torsor $Q \to S$ together with an L-equivariant map $\psi : Q \to V$ in $[V/L]$, we can construct the triple $(Q \times G/L, Q, \mathrm{id})$ as an object of $[V/G]\times_{BG} BL$. In order for $Q \times G/L$ to be an object in $[V/G]$, we must equip it with a G-equivariant map $Q \times G/L \to V$ or, equivalently, with a G-equivariant, L-invariant map $Q \times G \to V$. The map $\Phi : Q \times G \to V$ defined by $(q, g) \mapsto \psi(q)g$ is L-invariant with respect to the action $l\cdot(q, g) = (ql, l^{-1}g)$ we are quotienting by; indeed

$$(ql, l^{-1}g) \mapsto \psi(ql)l^{-1}g = \psi(q)ll^{-1}g = \psi(q)g.$$

Moreover Φ is G-equivariant: $\Phi((q, g)\cdot h) = \psi(q)gh = (\psi(q)g)\cdot h$. The verification that the functors defined are indeed inverses is standard and will be omitted. □

4.3. *The splitting principle.*

Proposition 4.6. *Given a weighted affine bundle $E \to X$* (*respectively a weighted vector bundle*), *we have a natural map $X \to BG_{\boldsymbol{a},\boldsymbol{n}}$* (*respectively $X \to B\,\mathrm{GL}_{\boldsymbol{n}}$*) *such that E is the pull-back of $VG_{\boldsymbol{a},\boldsymbol{n}}$* (*respectively $V\,\mathrm{GL}_{\boldsymbol{n}}$*).

Proof. We prove the result for $VG_{\boldsymbol{a},\boldsymbol{n}}$, as the result for $V\,\mathrm{GL}_{\boldsymbol{n}}$ is effectively the same, with the obvious modifications. Let us denote by Isom the sheaf of isomorphisms of affine bundles $\mathrm{Isom}_X(E, V\times X)$

(respectively, the sheaf of isomorphisms of weighted vector bundles). By a straightforward application of the definitions, it can be seen that Isom is a principal $G_{a,n}$ bundle over X and $\mathrm{Isom} \times_X E \cong V \times_{\{*\}} \mathrm{Isom}$.

In particular we get the following Cartesian diagram:

$$\begin{array}{ccc} \mathrm{Isom} & \longrightarrow & \{*\} \\ \downarrow & & \downarrow \\ X & \longrightarrow & BG_{a,n} \end{array}$$

Then we can fit the spaces above in the commutative cube

$$\begin{array}{ccccccc} \mathrm{Isom} \times_X E & & \longrightarrow & & V & & \\ & \searrow & & & & \searrow & \\ \downarrow & & E & & \longrightarrow & & VG_{a,n} \\ & & & & \downarrow & & \\ \mathrm{Isom} & & \longrightarrow & & \{*\} & & \downarrow \\ & \searrow & \downarrow & & & \searrow & \\ & & X & & \longrightarrow & & BG_{a,n} \end{array}$$

where the bottom, back and side squares are fiber squares.

Note that $\mathrm{Isom} \times_X E \to E$ is a principal $G_{a,n}$ bundle. Moreover, the action of $G_{a,n}$ on Isom gives a $G_{a,n}$-equivariant map $\mathrm{Isom} \times_X E \to V$ via the identification of $\mathrm{Isom} \times_X E$ with $V \times \mathrm{Isom}$. This gives us a map of quotient stacks $E \to VG_{a,n}$ which makes the top of the cube Cartesian.

It follows that

$$\begin{array}{ccc} E & \longrightarrow & VG_{a,n} \\ \downarrow & & \downarrow \\ X & \longrightarrow & BG_{a,n} \end{array}$$

is a fiber square, as needed. □

Corollary 4.7. *Let $E \to X$ be a weighted affine bundle, with corresponding map $X \to BG_{a,n}$. Then E', the pull-back of E via the map $X' = X \times_{BG_{a,n}} \mathrm{GL}_n \to X$, is a weighted* ***vector*** *bundle.*

Proof. Consider the following diagram:

$$\begin{array}{ccccccc} E' & & \longrightarrow & & X' & & \\ \downarrow & \searrow & & & \downarrow & \searrow & \\ & & V\,\mathrm{GL}_n & & \longrightarrow & & B\,\mathrm{GL}_n \\ E & & \longrightarrow & & X & & \downarrow \\ & \searrow & \downarrow & & & \searrow & \\ & & VG_{a,n} & & \longrightarrow & & BG_{a,n} \end{array}$$

The back and right squares are Cartesian by construction. By Lemma 4.5 and Proposition 4.6 we have that the front and bottom squares are Cartesian. Any such cube with these sides Cartesian is Cartesian, in particular the top square. Since $E' \to X'$ is the pull-back of the vector bundle $V\,\mathrm{GL}_n \to B\,\mathrm{GL}_n$, it is a vector bundle as desired. □

Lemma 4.8. *Let $X' \to X$ be as in Corollary 4.7. Then the pull-back map of Chow rings $A^*(X) \to A^*(X')$ is an isomorphism.*

Proof. By Corollary 4.4, $X' \xrightarrow{\phi} X$ is a $U_{\boldsymbol{a},\boldsymbol{n}}$-bundle and $U_{\boldsymbol{a},\boldsymbol{n}}$ is a unipotent group. In particular, $U_{\boldsymbol{a},\boldsymbol{n}}$ is a successive extension of the additive group $\mathbb{G}_a$ by itself and being a $U_{\boldsymbol{a},\boldsymbol{n}}$-bundle is equivalent to being a succession of affine bundles; hence by [Stacks 2005–, Tag 0GUB] we obtain an isomorphism of Chow rings $\phi^* : A^*(X) \to A^*(X')$. ☐

Theorem 4.9 (the splitting principle). *Let $E \to X$ be a weighted affine bundle defined by a map $X \to BG_{\boldsymbol{a},\boldsymbol{n}}$. Let T be the standard maximal torus in $\mathrm{GL}_{\boldsymbol{n}}$ and $BT := [*/T]$ its classifying stack. Then the map $X'' \to X$ in the fiber diagram*

$$\begin{array}{ccc} X'' & \longrightarrow & BT \\ \downarrow & & \downarrow \\ X' & \longrightarrow & B\,\mathrm{GL}_{\boldsymbol{n}} \\ \downarrow & & \downarrow \\ X & \longrightarrow & BG_{\boldsymbol{a},\boldsymbol{n}} \end{array}$$

induces an injection of Chow rings $A^*(X) \hookrightarrow A^*(X'')$ via pull-back.

Proof. By the argument in the proof of [Totaro 2014, Theorem 2.13] we have an injection $A^*(X') \hookrightarrow A^*(X'')$. By composing with the isomorphism in Lemma 4.8, we have the desired map. ☐

5. The Gysin homomorphism induced by a weighted blow-up

The goal for this section is to prove Theorem 5.5, which replaces the excess bundle formula in the case of weighted blow-ups.

The strategy for the proof is to reduce to the special case of the weighted blow-up of $BT = [\{0\}/T]$ in $[\mathbb{A}^d/T]$ induced by zero section, which will be computed in Section 5.3.

The reduction to the special case is performed in two steps: first we reduce to the case of the blow-up of an affine space (Section 5.2), and then we apply the splitting principle Theorem 4.9.

Some caution is needed when defining $f^!$, as we don't always have the needed Cartesian diagram. In Section 5.1 we address the issue as well as setting some notation for the rest of the paper.

5.1. ***Notation.*** Let $\widetilde{Y} \to Y$ be the weighted blow-up of Y centered at X, and let $\widetilde{X}$ be the exceptional divisor.

As observed in [Quek and Rydh 2021, Remark 3.2.10] the commutative square is not always Cartesian

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

and when defining $f^!$ we have to make sure to define it with respect to the fiber square

$$\begin{array}{ccc} X \times_Y \widetilde{Y} & \longrightarrow & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Moreover we have $\widetilde{X} = (X \times_Y \widetilde{Y})_{\mathrm{red}}$ and the diagram below commutes:

$$\begin{array}{ccccc} \widetilde{X} & \longrightarrow & X \times_Y \widetilde{Y} & \xrightarrow{\ j\ } & \widetilde{Y} \\ & \searrow^{g} & \downarrow h & & \downarrow f \\ & & X & \xrightarrow{\ i\ } & Y \end{array}$$

When looking at Chow rings though, we have a natural isomorphism

$$(\mathrm{red})_* : A^*(\widetilde{X}) \to A^*(X \times_Y \widetilde{Y})$$

induced by the reduction map $\mathrm{red} : \widetilde{X} \to X \times_Y \widetilde{Y}$.

In particular, it makes sense to talk about $f^!$ as the composition of $(\mathrm{red})_*^{-1} \circ f^!$. Throughout the rest of the paper, we will refer to it simply as $f^!$.

Lemma 5.1. *The map $f^! : A^*(X) \to A^*(\widetilde{X})$ is of the form $f^!(\alpha) = g^*(\alpha) \cdot \gamma$ for some element $\gamma \in A^*(\widetilde{X})$.*

Proof. In a similar fashion to what we did in Proposition 2.4, we will prove the statement by passing through algebraic spaces.

Precisely, let $\widetilde{X}_U = (\mathcal{N}_X Y \smallsetminus 0) \times U/\mathbb{G}_m$ with U open as in Definition 2.1 inducing isomorphisms of Chow groups for $\widetilde{X}$ of the appropriate degree. In fact, if U is chosen large enough we also get the algebraic space $\widetilde{Y}_U$ with analogous induced isomorphisms of Chow groups for $\widetilde{Y}$.

For the appropriate degrees, the induced maps $f_U : \widetilde{Y}_U \to Y$ with $(\tilde{y}, u) \mapsto f(\tilde{y})$ and $g_U : \widetilde{X}_U \to X$ with $(\tilde{x}, u) \mapsto g(\tilde{x})$ will themselves induce group homomorphisms $f_U^! : A^*(X) \to A^*(\widetilde{X}_U)$ and $g_U^* : A^*(X) \to A^*(\widetilde{X}_U)$.

Now $g_U : \widetilde{X}_U \to X$ is a smooth map and by [Fulton 1998, Theorem 17.4.2] we have isomorphisms

$$A^p(\widetilde{X}_U) \cong A^p(\widetilde{X}_U \xrightarrow{\mathrm{id}} \widetilde{X}_U) \xrightarrow{\cdot [g_U]} A^{p-d}(\widetilde{X}_U \to X).$$

In particular, for the degrees on which $A^p(\widetilde{X}_U) = A^p_{\mathbb{G}_m}(\mathcal{N}_X Y \smallsetminus 0) \cong A^p(\widetilde{X})$, we have that $f_U^! = \gamma_U \cdot [g_U] = \gamma_U \cdot g_U^*$ for some $\gamma_U \in A^p(\widetilde{X}_U)$ is equivalent to saying $f^!(\alpha) = \gamma_U \cdot g^*(\alpha)$ for some element $\gamma_U \in A^p(\widetilde{X})$.

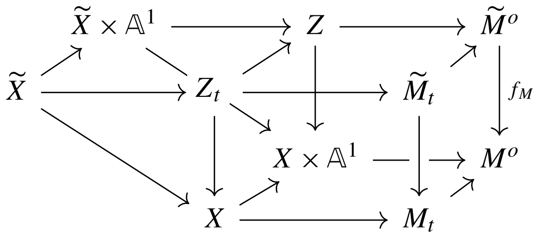
Since the elements γ_U must agree whenever U has high enough dimension, they must coincide. Hence there exists a unique element $\gamma \in A^*(\widetilde{X})$ such that $f^!(\alpha) = \gamma \cdot g^*(\alpha)$. □

5.2. ***Specialization to the weighted normal cone.*** Analogously to [Fulton 1998, Section 5.2], Quek and Rydh [2021, Section 4.3] constructed a deformation to the weighted normal cone of X in Y, which is a weighted affine bundle in our case. We will be using their construction to reduce our argument to the case where Y a weighted affine bundle over X. A similar construction can be found in [Mustaţă and Mustaţă 2012, Section 2.3].

Note that when given a weighted embedding that defines a weighted blow-up of smooth varieties, the weighted normal cone is a weighted affine bundle, which we will denote by $\mathcal{N}_X Y$.

Theorem 5.2. *Let X, Y, $\widetilde{X}$, $\widetilde{Y}$ be as usual. Let $N = \mathcal{N}_X Y$ be the weighted normal affine bundle of X in Y and $f_N : \widetilde{N} \to N$ be the weighted blow-up of the zero section of said bundle, with the same weights as $f : \widetilde{Y} \to Y$. Then the induced maps $f^! : A^*(X) \to A^*(\widetilde{X})$ and $f^!_N : A^*(X) \to A^*(\widetilde{X})$ coincide.*

Proof. Let M^o be the deformation to the weighted normal cone as defined in [Quek and Rydh 2021, Definition 4.3.3] and let $\widetilde{M}^o$ be the weighted blow-up of $X \times \mathbb{A}^1$ in M^o with the same weights as f, i.e., the weighted blow-up induced by the weighted embedding in [loc. cit., Definition 4.3.4,(iv)]. Let M_t and $\widetilde{M}_t$ respectively, be the fibers over t. Let $Z = (X \times \mathbb{A}^1) \times_{M^o} \widetilde{M}^0$. Then we have the following diagram:



By looking at the composition $X \to M_t \to M^o$ in the subdiagram

$$\begin{array}{ccccc} Z_t & \longrightarrow & \widetilde{M}_t & \longrightarrow & \widetilde{M}^o \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle f_M} \\ X & \longrightarrow & M_t & \longrightarrow & M^o \end{array}$$

we see that for $t \neq 0$ we have that $f^!_M : A^*(X) \to A^*(Z_t) = A^*(\widetilde{X})$ is precisely $f^!$, and for $t = 0$ it is precisely $f^!_N$. Now looking at the composition $X \to X \times \mathbb{A}^1 \to M^o$ in the subdiagram

$$\begin{array}{ccccc} Z_t & \longrightarrow & Z & \longrightarrow & \widetilde{M}^o \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle f_M} \\ X & \longrightarrow & X \times \mathbb{A}^1 & \longrightarrow & M^o \end{array}$$

we want to show that $f^!_M : A^*(X) \to A^*(Z_t) = A^*(\widetilde{X})$ is the same for all t.

By [Fulton 1998, Theorem 6.4] the following diagram commutes:

$$\begin{array}{ccc} A^*(Z_t) = A^*(\widetilde{X}) & \xleftarrow[\;i^!_t\;]{} & A^*(Z) = A^*(\widetilde{X} \times \mathbb{A}^1) \\ {\scriptstyle f^!_M}\uparrow & & {\scriptstyle f^!_M}\uparrow \\ A^*(X) & \xleftarrow[\;i^*_t\;]{} & A^*(X \times \mathbb{A}^1) \end{array}$$

But the horizontal maps are isomorphisms, inverse to the pull-back along the products with $\mathbb{A}^1$. Since the horizontal maps are independent of t and the map on the right is independent of t, so is the map on the left. $\square$

5.3. ***The special case of*** $[\mathbb{A}^d/T]$**.** Let us now study the particular case of a point in the affine space over the diagonal action of the torus:

$$\begin{array}{ccccc} \mathcal{P}(a_1,\dots,a_d) & \longrightarrow & 0\times_{\mathbb{A}^d}\mathrm{Bl}_{a_1,\dots,a_d}\mathbb{A}^d & \xrightarrow{\;j\;} & \mathrm{Bl}_{a_1,\dots,a_d}\mathbb{A}^d \\ & \searrow^{g} & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} \\ & & 0 & \xrightarrow{\;i\;} & \mathbb{A}^d \end{array}$$

In order to explicitly give a formula for $f^!$ we need presentations for the equivariant Chow rings $A_T^*(-)$.

Now $A_T^*(0)\cong A_T^*(\mathbb{A}^d)\cong\mathbb{Z}[x_1,\dots,x_d]$. Details about $A_T^*(0)$ can be found in [Edidin and Graham 1998] and in [Iwanari 2009] for equivariant Chow rings of toric stacks.

Let us first observe that, since $\mathcal{P}(a_1,\dots,a_d)$ is the reduction of $0\times_{\mathbb{A}^d}\mathrm{Bl}_{a_1,\dots,a_d}\mathbb{A}^d$, there is an isomorphism of Chow rings $A_T^*(0\times_{\mathbb{A}^d}\mathrm{Bl}_{a_1,\dots,a_d}\mathbb{A}^d)\cong A_T^*(\mathcal{P}(a_1,\dots,a_d))$.

Moreover $\mathrm{Bl}_{a_1,\dots,a_d}\mathbb{A}^d$ is a line bundle over $\mathcal{P}(a_1,\dots,a_d)$; in fact it is the total space of $\mathcal{O}_{\mathcal{P}(a_1,\dots,a_d)}(-1)$, and we have the isomorphism $A_T^*(\mathrm{Bl}_{a_1,\dots,a_d}\mathbb{A}^d)\cong A_T^*(\mathcal{P}(a_1,\dots,a_d))$.

We are left with computing $A_T^*(\mathcal{P}(a_1,\dots,a_d))$.

Lemma 5.3.
$$A_T^*(\mathcal{P}(a_1,\dots,a_d))\cong\frac{\mathbb{Z}[x_1,\dots,x_d,t]}{P(t)},\quad \textit{where } P(t):=\prod_{i=1}^d(x_i+ta_i).$$

Proof. By construction $\mathcal{P}(a_1,\dots,a_d)=[(\mathbb{A}^d\smallsetminus 0)/\mathbb{G}_m]$ and the actions of $\mathbb{G}_m$ and T on $\mathbb{A}^d\smallsetminus 0$ commute. In particular

$$A_T^*(\mathcal{P}(a_1,\dots,a_d))\cong A_T^*([(A^d\smallsetminus 0)/\mathbb{G}_m])\cong A_{T\times\mathbb{G}_m}^*(\mathbb{A}^d\smallsetminus 0).$$

Similarly to the computation above,

$$A_{T\times\mathbb{G}_m}^*(0)\cong A_{T\times\mathbb{G}_m}^*(\mathbb{A}^d)\cong\mathbb{Z}[x_1,\dots,x_d,t],$$

where $x_1,\dots,x_d$ are given by the action of T and t is given by the action of $\mathbb{G}_m$. Finally, the image of the first map in the localization sequence,

$$A_{T\times\mathbb{G}_m}^*(0)\to A_{T\times\mathbb{G}_m}^*(\mathbb{A}^d)\to A_{T\times\mathbb{G}_m}^*(\mathbb{A}^d\smallsetminus 0)\to 0,$$

is generated by $P(t):=\prod_{i=1}^d(x_i+ta_i)$. Indeed the top Chern class of the $T\times\mathbb{G}_m$-equivariant bundle splits along each component of $\mathbb{A}^d$. On the i-th component of $\mathbb{A}^d$ the i-th component of T acts with weight 1 and the other components of T act with weight 0, while $\mathbb{G}_m$ acts with weight a_i.

Therefore $A_T^*(\mathcal{P}(a_1,\dots,a_d))$ and $A_T^*(\mathrm{Bl}_{a_1,\dots,a_d}\mathbb{A}^d)$ are both isomorphic to $\mathbb{Z}[x_1,\dots,x_d,t]/P(t)$. ☐

Theorem 5.4. *Let* $f:[\mathrm{Bl}_{a_1,\dots,a_d}\mathbb{A}^d/T]\to[\mathbb{A}^d/T]$ *be the blow-up of* $[0/T]$ *in* $[\mathbb{A}^d/T]$ *with weights* $a_1,\dots,a_d$*. Then*

$$f^!(1)=\left(\frac{c_{\mathrm{top}}^{\mathbb{G}_m}([\mathbb{A}^n/T])(t)-c_{\mathrm{top}}^{\mathbb{G}_m}([\mathbb{A}^n/T])(0)}{t}\right).$$

Proof. By [Edidin and Graham 1998, Proposition 3] and Lemma 5.1 $f^!$ satisfies $f^*i_* = j_*f^!$, making the following diagram commute:

$$\begin{array}{ccc} A_T^*(\mathcal{P}(a_1,\dots,a_d)) & \xrightarrow{j_*} & A_T^*(\mathrm{Bl}_{a_1,\dots,a_d}\mathbb{A}^d) \\ \uparrow{\scriptstyle f^!(1)\cdot g^*} & & \uparrow{\scriptstyle f^*} \\ A^*(0)_T & \xrightarrow{i_*} & A^*(\mathbb{A}^d)_T \end{array}$$

Since $\mathbb{A}^d$ is a rank-d bundle over 0 and $\mathrm{Bl}_{a_1,\dots,a_d}\mathbb{A}^d$ is the tautological line bundle over $\mathcal{P}(a_1,\dots,a_d)$, the homomorphisms i^* and j^* are isomorphisms of T-equivariant Chow rings, which gives us the following:

$$\begin{array}{ccccc} A_T^*(\mathcal{P}(a_1,\dots,a_d)) & \xrightarrow{j_*} & A_T^*(\mathrm{Bl}_{a_1,\dots,a_d}\mathbb{A}^d) & \xrightarrow{j^*} & A_T^*(\mathcal{P}(a_1,\dots,a_d)) \\ \uparrow{\scriptstyle f^!(1)\cdot g^*} & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle g^*} \\ A^*(0)_T & \xrightarrow{i_*} & A^*(\mathbb{A}^d)_T & \xrightarrow{i^*} & A^*(0)_T \end{array}$$

Now $i^*i_* : A_T^*(0) \to A_T^*(0)$ is just the multiplication by the top equivariant Chern class of the bundle $\mathbb{A}^d$ over 0. Specifically $i^*i_*(\alpha) = \alpha\cdot x_1\cdots x_d$.

Similarly, j^*j_* is the image of the top equivariant Chern class of the bundle $\mathrm{Bl}_{a_1,\dots,a_d}\mathbb{A}^d$ over $\mathcal{P}(a_1,\dots,a_d)$. But $\mathrm{Bl}_{a_1,\dots,a_d}\mathbb{A}^d$ is the total space of $\mathcal{O}_{\mathcal{P}(a_1,\dots,a_d)}(-1)$, so we have that j^*j_* is multiplication by $-t$, where $\mathcal{P}(a_1,\dots,a_d)$ has the presentation of Example 3.13. Therefore we must have

$$f^*i_*(\alpha) = \alpha\cdot x_1\cdots x_d = \alpha\cdot c_{\mathrm{top}}^{\mathbb{G}_m}([\mathbb{A}^d/T])(0) = f^!(1)\alpha(-t) = j_*f^!(\alpha)$$

and since t is not a zero divisor in $A_T^*(\mathcal{P}(a_1,\dots,a_d))$, we must have

$$f^!(1) = \frac{-c_{\mathrm{top}}^{\mathbb{G}_m}([\mathbb{A}^d/T])(0)}{t} = \frac{c_{\mathrm{top}}^{\mathbb{G}_m}([\mathbb{A}^d/T])(t) - c_{\mathrm{top}}^{\mathbb{G}_m}([\mathbb{A}^d/T])(0)}{t},$$

as needed. □

5.4. ***A formula for the Gysin homomorphism.***

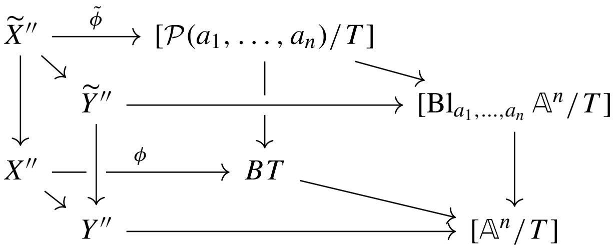
Theorem 5.5. *Let X, Y, $\widetilde{X}$, $\widetilde{Y}$, f be as usual. Let us identify $A^*(\widetilde{X}) \cong A^*(X)[t]/P(t)$ with $P(t) = c_{\mathrm{top}}^{\mathbb{G}_m}(\mathcal{N}_XY)(t)$. Then we have the following formula for the Gysin homomorphism $f^! : A^*(X) \to A^*(\widetilde{X})$:*

$$f^!(\alpha) = \frac{P(t)-P(0)}{t}\alpha.$$

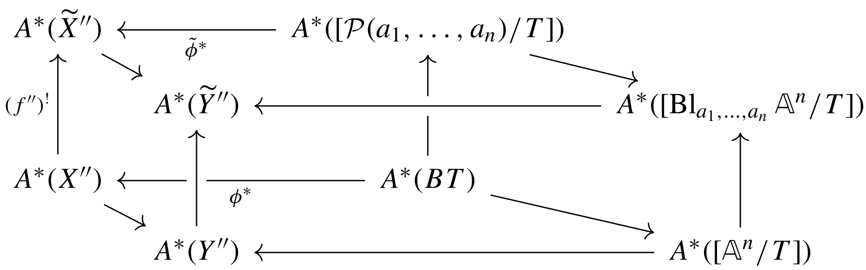
Proof. With the presentation of $A^*(\widetilde{X})$ above, the map g^* is the natural inclusion of $A^*(X)$ in $A^*(X)[t]/P(t)$ and, by Lemma 5.1 we only need to show

$$f^!(1) = \frac{P(t)-P(0)}{t}.$$

By Theorem 5.2 we can assume that Y is a weighted affine bundle over X. By the splitting principle in Theorem 4.9 it is enough to prove the equality for the pull-back X''. Since weighted blow-ups commute with base change, the blow-up $f'' : \widetilde{Y}'' \to Y''$ sits in the commutative diagram



which induces the following commutative diagram of Chow groups:



Since equivariant Chern classes commute with pull-backs and Y'' is a vector bundle over X'', by Theorem 5.4 the following holds:

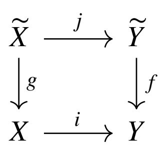
$$(f'')^!(1) = (f'')^!(\phi^*(1)) = \tilde{\phi}^*\left(\frac{c_{\text{top}}^{\mathbb{G}_m}([\mathbb{A}^n/T])(t) - c_{\text{top}}^{\mathbb{G}_m}(\mathbb{A}^n/T)(0)}{t}\right)$$
$$= \left(\frac{c_{\text{top}}^{\mathbb{G}_m}(\tilde{\phi}^*[\mathbb{A}^n/T])(t) - c_{\text{top}}^{\mathbb{G}_m}(\tilde{\phi}^*[\mathbb{A}^n/T])(0)}{t}\right) = \left(\frac{c_{\text{top}}^{\mathbb{G}_m}(Y'')(t) - c_{\text{top}}^{\mathbb{G}_m}(Y'')(0)}{t}\right),$$

which is the desired difference quotient. □

6. The Chow ring of a weighted blow-up

In this section we generalize the key sequence in [Fulton 1998, Proposition 6.7(e)] and then use it to compute a formula for the Chow ring of a weighted blow-up.

Let us recall the notation. Let $f : \widetilde{Y} \to Y$ be the weighted blow-up of Y at X. Let $\widetilde{X}$ be the exceptional divisor. Then we have the commutative diagram



and the map $f^!$ (computed in 5.5) is

$$f^!(\alpha) = \frac{P(t) - P(0)}{t}\alpha,$$

$P(t) = c_{\text{top}}^{\mathbb{G}_m}(N_X Y)(t)$ and f^* is the Gysin homomorphism defined as in [Vistoli 1989, Definition 3.10].

6.1. *The Grothendieck sequence.*

Theorem 6.1 (key sequence). *Let X, Y, $\widetilde{X}$, $\widetilde{Y}$, f, i, j be as above. Then we have the exact sequence of Chow groups*

$$A^*(X) \xrightarrow{(f^!,-i_*)} A^*(\widetilde{X}) \oplus A^*(Y) \xrightarrow{j_*+f^*} A^*(\widetilde{Y}) \to 0.$$

Further, if we use rational coefficients, then this becomes a split short exact sequence with g_* left inverse to $(f^!, -i_*)$:

$$0 \to A^*(X, \mathbb{Q}) \xrightarrow{(f^!,-i_*)} A^*(\widetilde{X}, \mathbb{Q}) \oplus A^*(Y, \mathbb{Q}) \xrightarrow{j_*+f^*} A^*(\widetilde{Y}, \mathbb{Q}) \to 0.$$

Proof. To prove exactness let us look at the double complex of higher Chow groups given by localization sequence as in [Bloch 1986, Theorem 3.1]

$$\begin{array}{ccccccccc} \cdots A^*(U,1) & \xrightarrow{\tilde{\delta}_1} & A^*(\widetilde{X}) & \xrightarrow{j_*} & A^*(\widetilde{Y}) & \longrightarrow & A^*(U) & \longrightarrow & 0 \\ \| & & \uparrow{\scriptstyle f^!} & & \uparrow{\scriptstyle f^*} & & \| & & \\ \cdots A^*(U,1) & \xrightarrow{\delta_1} & A^*(X) & \xrightarrow{i_*} & A^*(Y) & \longrightarrow & A^*(U) & \longrightarrow & 0 \end{array}$$

where $U \cong \widetilde{Y} \setminus \widetilde{X} \cong Y \setminus X$.

Since both of the complexes are exact, the total complex

$$\cdots A^*(U,1) \oplus A^*(X) \to A^*(\widetilde{X}) \oplus A^*(Y) \to A^*(\widetilde{Y}) \oplus A^*(U) \to A^*(U) \to 0$$

is also exact.

Let us prove that the map $A^*(\widetilde{X}) \oplus A^*(Y) \xrightarrow{j_*+f^*} A^*(\widetilde{Y})$ is surjective. Let α be any cycle in $A^*(\widetilde{Y})$, $\bar{\alpha}$ be the restriction of α to $A^*(U)$, and $\beta \in A^*(Y)$ be any cycle that restricts to $\bar{\alpha}$ in $A^*(U)$. By commutativity, $\alpha - f^*(\beta)$ restricts to 0 in $A^*(U)$ and must be in the image of j_*. So α is in the image of $j_* + f^*$. Therefore the complex

$$\cdots A^*(U,1) \oplus A^*(X) \to A^*(\widetilde{X}) \oplus A^*(Y) \to A^*(\widetilde{Y}) \to 0$$

is still exact.

Moreover, the image of the map $A^*(U,1) \oplus A^*(X) \xrightarrow{(-\tilde{\delta}_1+f^!,-i_*)} A^*(\widetilde{X}) \oplus A^*(Y)$ coincides with the one of $A^*(X) \xrightarrow{(f^!,-i_*)} A^*(\widetilde{X}) \oplus A^*(Y)$. Indeed, let $(\tilde{x}, y) = (-\tilde{\delta}_1(u) + f^!(x), -i_*(x))$ and let $x' = x + \delta_1(u)$. Then $f^!(x') = f^!(x) - f^!(\delta_1(u)) = f^!(x) - \tilde{\delta}_1(u) = \tilde{x}$ and $i_*(x') = i_*(x) - i_*(\delta_1(u)) = i_*(x) = y$.

It follows that $\ker(j_* + f^*) = \operatorname{Im}(f^!, -i_*)$ and that

$$A^*(X) \xrightarrow{(f^!,-i_*)} A^*(\widetilde{X}) \oplus A^*(Y) \xrightarrow{j_*+f^*} A^*(\widetilde{Y}) \to 0$$

is exact.

Lastly, if we use rational coefficients then there is a left inverse of $(f^!, -i^*)$ given by $(\alpha, \beta) \mapsto g_*(\alpha)$. Indeed, let $x \in A^*(X)$. Then $g_*(f^!(x)) = g_*(\gamma \cdot g^*(x)) = g_*(\gamma) \cdot x$, with γ the difference quotient $(P(t) - P(0))/t$ as in Theorem 5.5. Now γ is a degree-$(n-1)$ polynomial in t, of which only the leading term $a_1 \cdots a_n t^{n-1}$ will survive the pushforward. We only need to show that $g_*(t^{n-1}) = 1/(a_1 \cdots a_n)$.

It is enough to verify this when X is a point and $\widetilde{X}$ is the weighted projective stack $\mathcal{P}(a_1, \dots, a_n)$. Notice that $a_i t = x_i$, where the x_i are the fundamental classes of the usual coordinate divisors, so $a_1 \cdots a_{n-1} t^{n-1} = x_1 \cdots x_{n-1}$, which is the fundamental class of a stacky point isomorphic to $B\mu_{a_n}$ and so pushes forward to $1/a_n$; thus $g_*(t^{n-1}) = 1/(a_1 \cdots a_n)$. □

Example 6.2. To see why the sequence with integer coefficients is not short exact, let us consider X an elliptic curve in $Y = \mathbb{P}^2$. Let $\widetilde{Y}$ be the blow-up of Y at X with weight 2. Let $P, Q \in X$ be distinct points of order 2 and consider the difference $[P] - [Q] \in A^*(X)$. When pushed forward to $A^*(Y)$ via i_*, all points are rationally equivalent; hence $i_*([P] - [Q]) = 0$. On the other hand, $f^!$ is multiplication by 2, so $f^!([P]) = f^!([Q]) = 0$. But $[P] - [Q]$ is nonzero, so $(f^!, -i_*)$ is not injective.

Remark 6.3. Note that, when looking at the double complex in the proof of Theorem 6.1 and taking the total complex, one defines a long exact sequence of higher Chow groups. The isomorphisms of higher Chow groups

$$\alpha_i : A^*(\widetilde{Y} \smallsetminus \widetilde{X}, i) \to A^*(Y \smallsetminus X, i)$$

allow us to delete $A^*(\widetilde{Y} \smallsetminus \widetilde{X})$ and $A^*(Y \smallsetminus X)$ from the complex via a diagram chase analogous to the ones in the proof. Then we can obtain the long exact sequence

$$\cdots \to A^*(X, i) \to A^*(\widetilde{X}, i) \oplus A^*(Y, i) \to A^*(\widetilde{Y}, i) \to A^*(X, i-1) \to \cdots .$$

6.2. *The Chow ring of a weighted blow-up.*

Theorem 6.4 (Chow ring of a weighted blow-up). *If $\widetilde{Y} \to Y$ is a weighted blow-up of Y at a closed subvariety X, then the Chow ring $A^*(\widetilde{Y})$ is isomorphic **as a group** to the quotient*

$$A^*(\widetilde{Y}) \cong \frac{(A^*(X)[t]) \cdot t \oplus A^*(Y)}{(((P(t) - P(0))\alpha, -i_*(\alpha)), \forall \alpha \in A^*(X))},$$

with $P(t) = c_{\mathrm{top}}^{\mathbb{G}_m}(\mathcal{N}_X Y)(t)$ and $[\widetilde{X}] = -t$.

The multiplicative structure on $A^*(\widetilde{Y})$ is induced by the multiplicative structures on $A^*(X)$ and $A^*(Y)$ and by the pull-back map in the following way:

$$(0, \beta) \cdot (t, 0) = (i^*(\beta)t, 0).$$

Equivalently $A^*(\widetilde{Y})$ can be expressed as a quotient of the fiber product

$$\frac{A^*(Y) \times_{A^*(X)} A^*(X)[t]}{((i_*\alpha, P(t)\alpha), \forall \alpha \in A^*(X))},$$

with $i^* : A^*(Y) \to A^*(X)$ on the left and $A^*(X)[t] \to A^*(X)$ on the right given by evaluating t at 0.

Proof. The exact sequence in Theorem 6.1 gives us an isomorphism of *groups*

$$A^*(\widetilde{Y}) \cong \frac{A^*(\widetilde{X}) \oplus A^*(Y)}{((f^!(\alpha), -i_*(\alpha)), \forall \alpha \in A^*(X))}.$$

If we use Theorem 3.12 to rewrite $A^*(\widetilde{X})$ and also add an explicit factor of $[\widetilde{X}]$ to represent how $A^*(\widetilde{X})$ is mapped into $A^*(\widetilde{Y})$, then as a *group* we can rewrite $A^*(\widetilde{Y})$ as

$$\frac{((A^*(X)[t])\cdot[\widetilde{X}])\oplus A^*(Y)}{((c_{\text{top}}^{\mathbb{G}_m}(\mathcal{N}_XY)(t)[\widetilde{X}],0),(f^!(\alpha)[\widetilde{X}],-i^*(\alpha))\forall\alpha\in A^*(X))}.$$

Notice that $t[\widetilde{X}]=-[\widetilde{X}]^2$ (since t is the class of $\mathcal{O}_{\widetilde{X}}(1)$) so that there is an isomorphism between the ring presented above and the ring presented without the symbol $[\widetilde{X}]$ given by replacing $[\widetilde{X}]$ with $-t$.

Now we need to determine the ring structure. Since much of the ring structure is inherited from that of $A^*(Y)$ and $A^*(\widetilde{X})$, what remains is just to determine how to multiply elements coming from $A^*(Y)$ with those coming from $A^*(\widetilde{X})$. Consider the usual commutative square:

$$\begin{array}{ccc}\widetilde{X} & \longrightarrow & \widetilde{Y}\\ \downarrow & & \downarrow\\ X & \longrightarrow & Y\end{array}$$

Intersecting some class $\beta\in A^*(Y)$ with $\widetilde{X}$ amounts to pulling it back to $A^*(\widetilde{X})$. By commutativity we have $g^*(i^*(\beta))=j^*(f^*(\beta))$ and by pushforward we obtain $(0,\beta)\times(t,0)=(i^*(\beta)t,0)$.

Finally, notice also that $c_{\text{top}}^{\mathbb{G}_m}(\mathcal{N}_XY)(t)[\widetilde{X}]$ is now redundant, as for $\alpha=1$ we have

$$\begin{aligned}(t)(f^!(\alpha)(-t)-i_*(\alpha))&=(t)\left(\frac{c_{\text{top}}^{\mathbb{G}_m}(\mathcal{N}_XY)(t)-c_{\text{top}}(\mathcal{N}_XY)}{t}(-t)-i_*(1)\right)\\&=t(-c_{\text{top}}^{\mathbb{G}_m}(\mathcal{N}_XY)(t)+c_{\text{top}}(\mathcal{N}_XY)-i_*(1))=(-t)\cdot c_{\text{top}}^{\mathbb{G}_m}(\mathcal{N}_XY)(t).\end{aligned}$$

The last equality comes from $t\cdot i_*(1)=g^*(i^*(i_*(1)))=c_{\text{top}}(\mathcal{N}_XY)$.

Putting everything together, we have that $A^*(\widetilde{Y})$ is the group

$$A^*(\widetilde{Y})\cong\frac{(A^*(X)[t])\cdot t\oplus A^*(Y)}{(((P(t)-P(0))\alpha,-i_*(\alpha)),\forall\alpha\in A^*(X))},$$

with the desired multiplication. □

Corollary 6.5. *If $i^*:A^*(Y)\to A^*(X)$ is surjective, then this formula simplifies to resemble a formula of Keel* [1992, Theorem 1, page 571]

$$A^*(\widetilde{Y})\cong\frac{A^*(Y)[t]}{(t\cdot\ker(i^*),\,Q(t))},$$

where $Q(t)=c_{\text{top}}^{\mathbb{G}_m}(\mathcal{N}_XY)(t)-c_{\text{top}}^{\mathbb{G}_m}(\mathcal{N}_XY)(0)+[X]$.

Proof. We first prove that we can suppress $A^*(X)$ from the presentation in Theorem 6.4, i.e., that the elements of the form $(\alpha\cdot t,0)$ with $\alpha\in A^*(X)$ can be described as pairs of the form $(0,\beta)\cdot(t,0)$ for some $\beta\in A^*(Y)$. Let β such that $i^*(\beta)=\alpha$. By the multiplicative-structure condition of Theorem 6.4 gives

$$(\alpha\cdot t,0)=(i^*(\beta)\cdot t,0)=(t,0)$$

as desired.

In particular, the condition $t \cdot (\beta - i^*(\beta)) \forall \beta \in A^*(Y)$ reduces to $t \cdot \ker(i^*)$.

Finally, $t \cdot f^!(\alpha) + i_*(\alpha) = (c_{\mathrm{top}}^{\mathbb{G}_m}(\mathcal{N}_X Y)(t) - c_{\mathrm{top}}(\mathcal{N}_X Y))\alpha + i_*(\alpha)$ and $i_*(\alpha) = i_*(i^*(\beta)) = [X] \cdot \beta$ for some $\beta \in A^*(Y)$. So $t \cdot f^! + i_*$ is multiplication by $(c_{\mathrm{top}}^{\mathbb{G}_m}(\mathcal{N}_X Y)(t) - c_{\mathrm{top}}(\mathcal{N}_X Y) + [X])$, which is precisely the $Q(t)$ desired. □

6.3. *An example: the Chow ring of $\overline{\mathcal{M}}_{1,2}$.* The Chow ring $A^*(\overline{\mathcal{M}}_{1,2})$ has been computed in [Di Lorenzo et al. 2024; Inchiostro 2022]. The latter uses the construction of $\overline{\mathcal{M}}_{1,2}$ as the weighted blow-up of $\mathcal{P}(2, 3, 4)$. We give yet another computation of the ring, using the same blow-up construction.

We start by recalling the following:

Theorem 6.6 [Inchiostro 2022, Theorem 2.6]. *There exists an isomorphism $\overline{\mathcal{M}}_{1,2} \cong \mathrm{Bl}_Z^{(4,6)} \mathcal{P}(2, 3, 4)$, where $\mathrm{Bl}_Z^{(4,6)} \mathcal{P}(2, 3, 4)$ is the weighted blow-up of the point $Z = [s^2 : s^3 : 0]$ in $\mathcal{P}(2, 3, 4)$ with weights $(4, 6)$.*

Given this, $A^*(\overline{\mathcal{M}}_{1,2})$ becomes a straightforward computation,

Proposition 6.7 [Inchiostro 2022, Theorem 4.12].

$$A^*(\overline{\mathcal{M}}_{1,2}) \cong \frac{\mathbb{Z}[y, t]}{(ty, 24(t^2 + y^2))}.$$

Proof. First, since $i : Z \to \mathcal{P}(2, 3, 4)$ is just the inclusion of a point, we have that i is surjective, and since $A^*(\mathcal{P}(2, 3, 4)) \cong \mathbb{Z}[y]/(24y^3)$ we know the kernel is (y). By Corollary 6.5 we then have

$$A^*(\overline{\mathcal{M}}_{1,2}) \cong \frac{A^*(\mathcal{P}(2, 3, 4))[t]}{(ty, Q(t))} \cong \frac{\mathbb{Z}[y, t]}{(24y^3, ty, Q(t))},$$

where $Q(t)$ restricts to $c_{\mathrm{top}}^{\mathbb{G}_m}(\mathcal{N}_Z Y)$ and has constant term $[Z]$. As $\mathcal{N}_Z Y$ splits into trivial line bundles, we see $c_{\mathrm{top}}^{\mathbb{G}_m}(\mathcal{N}_Z Y) = (4t)(6t)$. Writing $Z = V(x_3) \cap V(x_1^3 - x_2^2)$, we see $[Z] = (4y)(6y)$, so $Q(t) = 24t^2 + 24y^2$. Lastly, the term $24y^3$ is now redundant and we have

$$A^*(\overline{\mathcal{M}}_{1,2}) \cong \frac{\mathbb{Z}[y, t]}{(ty, 24(t^2 + y^2))}.$$ □

7. Generalization to quotient stacks

Let us now consider the case of $\mathscr{Y} = [Y/G]$, where Y is an algebraic space and G is a linear algebraic group; hence it is possible to define the G-equivariant Chow ring $A_G^*(Y)$ as in [Edidin and Graham 1998].

A weighted embedding of $\mathscr{X}$ in $\mathscr{Y}$ defines a weighted embedding of X in Y via pull-back and, since the quotient maps are smooth, we have $\widetilde{\mathscr{Y}} \cong [\widetilde{Y}/G]$ and $\widetilde{\mathscr{X}} \cong [\widetilde{X}/G]$.

Theorem 7.1. *The theorems in Sections 3, 5, and 6 hold for $f : \widetilde{\mathscr{Y}} \to \mathscr{Y}$ weighted blow-up of $\mathscr{Y}$ at $\mathscr{X}$.*

Proof. Let us prove that $A^*(\widetilde{\mathscr{X}}) \cong A^*(\mathscr{X})[t]/P(t)$ as in Theorem 3.12; the proof for the other results will be almost identical.

For any p let U be as in Definition 2.1 of dimension high enough such that $A^q(X_U) \cong A_G^q(X) \cong A^q(\mathscr{X})$ with $X_U := X \times U/G$, up to degree p.

Since X_U is an algebraic space, by Theorem 3.12

$$A^*(\widetilde{X}_U) \cong A^*(X_U)[t]/P_U(t).$$

Note that $P_U(t) = c_{\text{top}}^{\mathbb{G}_m}(\mathcal{N}_{X_U} Y_U)$ is the pull-back of $P(t) = c_{\text{top}}^{\mathbb{G}_m}(\mathcal{N}_{\mathscr{X}} \mathscr{Y})$, which is a finite-degree polynomial. In particular for large enough p, $P_U(t)$ does not depend on U and it is exactly $P(t)$.

Since $\widetilde{X}_U$ is open inside a vector bundle, we have isomorphisms $A^q(\widetilde{X}_U) \cong A^q(\widetilde{\mathscr{X}})$ up to degree p. Since Theorem 3.12 holds up to degree p, for any p we have the desired isomorphism of Chow rings. □

Appendix: Chern class of weighted blow-up

A1. ***The goal.*** Consider a smooth subvariety X of a smooth variety Y, with blow-up $\widetilde{Y}$ and exceptional divisor, as in the following standard diagram:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\ j\ } & \widetilde{Y} \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ X & \xrightarrow{\ i\ } & Y \end{array}$$

Fulton [1998, Theorem 15.4] provided a formula for the total Chern class $c(\widetilde{Y}) := c(T_{\widetilde{Y}})$ in terms of the blow-up data. The purpose of this note is to revisit that formula and generalize it to the case of a weighted blow-up. Since smoothness is important in these considerations, our weighted blow-ups are always stack-theoretic.

A2. ***Setup and formula.*** In our setup, X and Y are still smooth varieties, and X is the support of a weighted center with weighted normal bundle N of rank $d = \operatorname{codim}(X \subset Y)$. The weighted normal bundle is a weighted affine bundle with total Chern class we denote by $c(N) \in A^*(X)$ and total $\mathbb{G}_m$-equivariant Chern class $c^{\mathbb{G}_m}(N) = Q(t) \in A^*_{\mathbb{G}_m}(X) = A^*(X)[t]$, where t is the equivariant parameter corresponding to the standard character of $\mathbb{G}_m$. In particular $Q(0) = c(N)$.

We recall from Theorem 6.4 in the main text that

$$A^*(\widetilde{Y}) = (A^*(Y) \oplus t A^*(\widetilde{X}))/I,$$

where

$$I = \big(i_*(\alpha) \oplus -(Q(t) - Q(0))\alpha \mid \alpha \in A^*(X)\big).$$

We denote by

$$q : A^*(Y) \oplus t A^*(X)[t] \to (A^*(Y) \oplus t A^*(\widetilde{X}))/I = A^*(\widetilde{Y})$$

the natural quotient map.

Theorem A.1. *We have*

$$\frac{c(\widetilde{Y})}{f^* c(Y)} = q\left(\frac{(1-t)Q(t)}{Q(0)}\right).$$

We note that the right-hand side is of the form $q(1 \oplus t \cdot R(t))$, with some $R(t) \in A^*(X)[[t]]$.

The formula was proved for Chow groups with rational coefficients by Anca and Andrei Mustaţa [2012, Proposition 2.12]. Our proof in essence verifies that their arguments carry over integrally.

A3. *Approach.* Our approach combines the equivariant methods used in the main text to study and compute Chow rings of weighted projective stack bundles and weighted blow-ups, combined with ideas in Aluffi's paper [2010] and lecture [2011], especially the user-friendly presentation of the formula in Aluffi's lecture. While Aluffi provides a proof only for complete intersections, the methods of Theorem 6.4 allow us to reduce the general case to a situation where Aluffi's proof applies.

A3.1. *The quotient class.* One first notes that the class $c(\widetilde{Y})/(f^*c(Y))$ appearing on the left-hand side has properties enabling flexible treatment:

Lemma A.2. *The class $c(\widetilde{Y})/(f^*c(Y))$ is of the form $q(1 \oplus t \cdot R(t))$, with some $R(t) \in A^*(X)[\![t]\!]$, and is functorial for smooth morphisms $Y' \to Y$ and closed embeddings $Y' \to Y$ that meet X transversely.*

Proof. To see that it has this form, consider the localization sequence

$$A^*(\widetilde{X}) \to A^*(\widetilde{Y}) \to A^*(\widetilde{Y} \smallsetminus \widetilde{X}) \to 0.$$

Since $c(\widetilde{Y})$ and $f^*c(Y)$ must pull back to the same class on $A^*(\widetilde{Y} \smallsetminus \widetilde{X})$, and their ratio pulls back to 1. In particular we see that $c(\widetilde{Y})/(f^*c(Y)) - 1$ must be in the image of $A^*(\widetilde{X})$, which means $c(\widetilde{Y})/(f^*c(Y))$ is of the desired form.

To see functoriality, consider the following diagram:

$$\begin{array}{ccc} \widetilde{Y}' & \xrightarrow{\tilde{h}} & \widetilde{Y} \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{h} & Y \end{array}$$

We must show

$$\frac{c(\widetilde{Y}')}{f'^*c(Y')} = \tilde{h}^* \frac{c(\widetilde{Y})}{f^*c(Y)},$$

but this is equivalent to

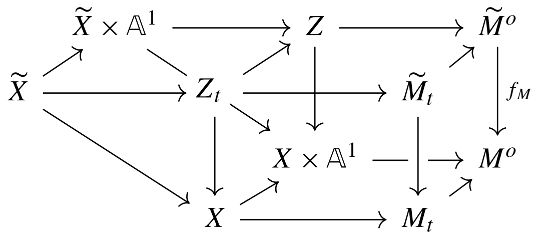
$$\frac{c(\widetilde{Y}')}{\tilde{h}^*c(\widetilde{Y})} = f'^* \frac{c(Y')}{h^*c(Y)}.$$

This is true when h is smooth because the relative tangent bundle of h is compatible with pull-back under f, and true when h is a closed embedding since the normal bundle of h is compatible with pull-back under f. □

A3.2. *Degeneration to the weighted normal bundle.* Applying the lemma to the degeneration to the weighted normal bundle we obtain:

Lemma A.3. *It suffices to prove the theorem, namely to compute $R(t)$ and $c(\widetilde{Y})/(f^*c(Y))$, when $Y = \mathcal{N}_X Y$.*

Proof. Recall the diagram from Theorem 5.2



where $M_t \cong Y$ for $t \neq 0$ and $M_0 = \mathcal{N}_X Y$.

By the previous lemma, the expression $c(\widetilde{M}_t)/(f^* c(M_t))$ can be pulled back from $\widetilde{M}^o$ along the embedding corresponding to t and is determined by a class on $\widetilde{X}$. However, neither $\widetilde{X}$ nor $\widetilde{M}^o$ depend on t so it is enough to compute things when $t = 0$, that is, for $\mathcal{N}_X Y$. □

A3.3. *The universal case.* By Theorem 4.9, the homomorphism $A^*(BG_{\boldsymbol{a},\boldsymbol{n}}) \to A^*(BT)$ is injective. Therefore:

Lemma A.4. *It suffices to prove the theorem when $X = BT$ and $Y = [V/T]$. Equivalently, it suffices to prove the theorem T-equivariantly when X is a point, the origin on $Y = \mathbb{A}^d$.*

Proof. This follows from functoriality and Theorem 4.9 since the maps $X \to BG_{\boldsymbol{a},\boldsymbol{n}}$ and $BT \to BG_{\boldsymbol{a},\boldsymbol{n}}$ are smooth. □

A3.4. *The toric case of affine space.* Finally, let X be the origin of $Y = \mathbb{A}^d$. Let

$$A_T^*(0) \cong A_T^*(\mathbb{A}^d) \cong \mathbb{Z}[x_1, \ldots, x_d]$$

and

$$A_T^*(\mathcal{P}(a_1, \ldots, a_d)) \cong A_T^*(\mathrm{Bl}_{(a_1,\ldots,a_d)} \mathbb{A}^d) \cong \frac{\mathbb{Z}[x_1, \ldots, x_d, t]}{\left(\prod (x_i + a_i t)\right)}$$

as in Section 5.3.

Proposition A.5. *We have $c^T(Y) = Q(0)$ and $c^T(\widetilde{Y}) = q((1-t)Q(t))$.*

Proof. This is essentially the same argument as [Aluffi 2006, Theorem 4.2].

Let $D = \Sigma \widetilde{X}_i + \widetilde{X}$ be the sum of all the irreducible toric divisors on $\widetilde{Y}$. By repeating the argument in [Fulton 1993, Proposition p. 87], we have the exact sequence

$$0 \to \Omega^1_{\widetilde{Y}} \to \Omega^1_{\widetilde{Y}}(\log D) \to \left(\bigoplus_{i=1}^{d} \mathcal{O}_{X_i}\right) \oplus \mathcal{O}_{\widetilde{X}} \to 0,$$

and $\Omega^1_{\widetilde{Y}}(\log D)$ is trivial. By Whitney's formula,

$$c^T(\Omega^1_{\widetilde{Y}}) = \frac{1}{c^T(\mathcal{O}_{\widetilde{X}}) c^T\left(\bigoplus \mathcal{O}_{\widetilde{X}_i}\right)} = (1+t) \prod_i (1 - a_i t - x_i).$$

By taking the dual we obtain

$$c^T(T\widetilde{Y}) = (1-t)\prod_i (1+a_i t + x_i) = Q(t).$$

The same argument works to prove $c^T(Y) = Q(0)$. □

The theorem follows.

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